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THÈSE POUR OBTENIR LE GRADE DE DOCTEUR DE L'UNIVERSITÉ DE MONTPELLIER

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Des graphes planaires vers des dimensions supérieures

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Décembre 2019

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Introduction

Graphs. Graph theory has been invented in order to study networks. In this theory we focus on the connections between some kind of objects and not directly on the objects. Applications arise in a large panel of domains. But maybe one that is massively used directly every day, is when you look for an itinerary on an online map. The user asks to the software what is the best path to go from point A to point B in the transport network. It may seem an easy problem for the user of the software, but behind it, there are years of research in graph theory for making it possible and efficient. In this network, the objects are locations in the world and we consider that two locations are connected if there is a road between them. Moreover, asking the best path in this network is not the only question. You can also ask “what is the fastest way to visit all the towns of a region?”. In addition to the previous questions, the transport networks have raised numerous problems in graph theory which have become very important with the development of computer science.

But transport networks are not the only networks that can be modeled with graphs. An application which has gained a lot of interest recently, thanks to the development of the internet and to the storage capacity of computers is the study of social networks: the objects here are the people in the world, and we consider that two persons are connected if they know each other (in some way). In this kind of network, you can try to detect communities or to know who are the central persons. Applications are numerous: you can study for example the spreading of a rumor.

We finally give another application in biology where we can study the spreading of a disease. For instance, you can consider a region of the world and a biologic network where the objects are some points in the region where some species exists. We consider that two points are connected if there is a movement of a part of the population from one point to the other point. In that case, we say that the network is *oriented* because the connections are not symmetric as a movement from a point A to a point B does not imply a movement from B to A . Furthermore, in this network the connections are said to be *weighted* by the proportion of moving.

In the terminology of graph theory, networks are called *graphs*, objects are called *vertices* and connections are called *edges*. As we have seen, every network is different and can be modeled by different graph models (for example oriented graphs, or weighted graphs). Among these different types of graphs, we focus in this thesis on one particular type: planar graphs.

Planar graphs. A graph is planar if it is possible to draw it in the plane such that two edges can only intersect at endpoints. A very important example is the road network if there is no road passing over another one (with the help of a bridge or a tunnel). This is for instance the case in the problem of the seven bridges of Königsberg solved by Euler in 1736 and which may be considered as the first use of graph theory.

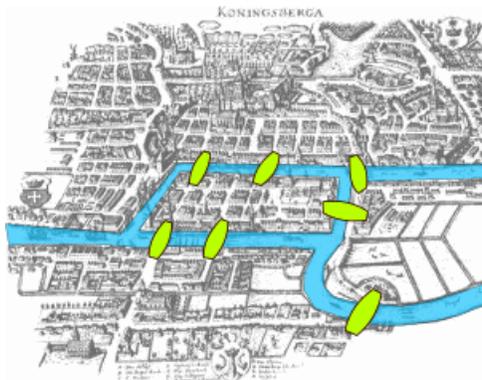


Figure 1: The seven bridges of the Königsberg problem.

Another fundamental problem is the problem of coloring a planar graph. Consider a cartography of the french metropolitan departments map where you want to color the departments such that no two adjacent departments have the same color. How much colors will you have to use? The solution is four as you can see on Figure 2 and this result is known since Appel and Haken proved it in 1976 [26]. Planar graphs are also related to the notion of meshes used in computer-aided design in order to draw 3D model of objects [81].

In addition to their practical applications, planar graphs are mathematically important because they form a big class of graphs which has a lot of good properties and a lot of useful tools which make them a good starting point to tackle a new problem. For instance, the celebrated graph minor theorem from Robertson and Seymour uses planar graphs in the beginning of the proof [97].

Due to the importance of planar graphs in practical and in theoretical aspects, tools have been developed for this class of graphs as we will see.

Dushnik-Miller dimension. Among the useful properties and tools that planar graphs have, there is the Dushnik-Miller dimension whose definition requires the notion of posets. A *poset* is an abstract set of elements where the elements are arbitrarily ordered by assuming that some elements are greater, lesser or incomparable than some other elements. While in a poset, two elements are not necessarily comparable, a *linear order* is a poset such that every pair of elements are comparable. The *Dushnik-Miller dimension*, also known as the *order dimension*, of a poset is an integer which measures the default of linearity of the poset in some sense (as a linear poset has Dushnik-Miller dimension 1 and

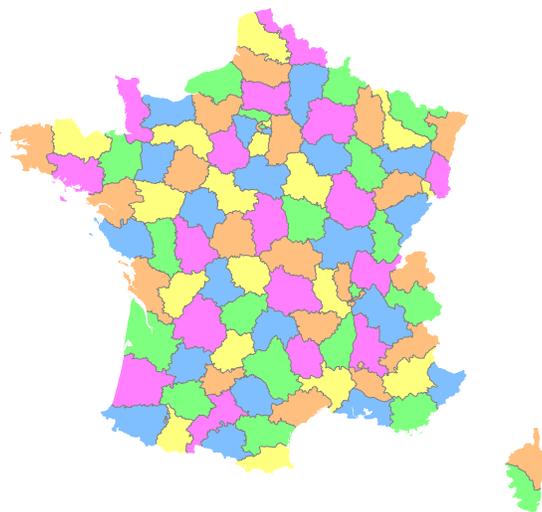


Figure 2: A four coloration of the departments the metropolitan France map.

a poset which is not linear has Dushnik-Miller dimension strictly bigger than 1).

Dushnik-Miller dimension can be applied to graphs as follows. We can define the *incidence poset* of a graph which captures the fact that a vertex is incident or not to an edge. The Dushnik-Miller dimension of a graph G can be defined as the Dushnik-Miller dimension of the incidence poset of G . A theorem of Schnyder [100] states that a graph is planar if and only if Dushnik-Miller dimension of this graph is at most 3. As a consequence of this theorem, the notion of planarity can be seen as a combinatorial property while it is defined as a topological one.

According to the previous theorem of Schnyder, planar graphs correspond to graphs whose incidence poset has Dushnik-Miller dimension at most 3. The Dushnik-Miller dimension of a graph could be therefore considered as a measure of the geometrical complexity of this graph. A natural question, as raised by Trotter in [104], is to find a generalization of the theorem of Schnyder to higher dimensions. This question was tackled by Ossona de Mendez [92] and Felsner and Kappes, who found partial results in this direction [67]. In this thesis we investigate if it is possible to answer this question with the help of two different geometric classes of graphs. The first one is a particular class of intersection graphs.

Intersection graphs. Intersection graphs are defined in the following way. Consider some geometric objects in a geometric space and consider the intersection graph of these objects where a vertex corresponds to an object and an edge corresponds to an intersection between two objects. This kind of graphs have been widely studied with variations on the considered objects and on the way that these objects intersect. When the objects possibly have an “interior

intersection” with other objects, we use the term X -intersection graph, where X is a description of the allowed geometrical objects. When the objects have no interior intersection with other objects, then the term X -contact graph is preferred. We now give some examples of intersection graphs and of some contact graphs which have been proved to be practically important.

Examples of intersection graphs can be obtained by considering unit disks in the plane. We can form the intersection graph of these disks (see Figure 3). We say that a graph is a *unit disk graph* if it is the intersection graph of unit disks in the plane. This kind of graphs are used in communication networks where a disk corresponds to the range of an antenna [77].

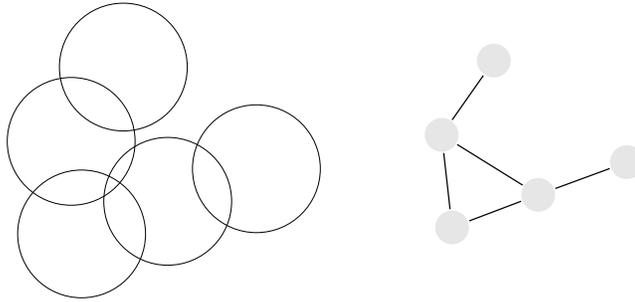


Figure 3: An example of a unit disk arrangement and its associated intersection graph.

Examples of contact graphs can be obtained by considering circles in the plane such that no two disks have an interior intersection. Such an arrangement of circles is called a *circle packing* (see Figure 4). As there are no interior intersection, we can consider the contact graph of this circle packing. The celebrated circle packing theorem by Koebe [16] states that a graph is planar if and only if it is the contact graph of a circle packing. This theorem was recently used to re-prove the planar separator theorem [90] and to find drawings of planar graphs with bounded angular resolution [86].

The second type of geometric graphs class used in the goal of answering the previous question on the generalization of the Theorem of Schnyder is the class of TD-Delaunay graphs.

TD-Delaunay graphs. TD-Delaunay graphs are a variant of the well-known class of Delaunay graphs. Given points in the plane, the Delaunay graph of this set of points is defined as follows: two points are considered to be adjacent if there exists a disk containing these two points but no other point in its interior (see Figure 5a for an example). Delaunay graphs have plenty of geometrical properties which made them of particular interest in algorithms generating triangular meshes [42]. The class of TD-Delaunay graphs is defined in a similar way by replacing disks by equilateral triangles in the previous definition (see Figure 5b for an example). They were introduced by Chew and Drysdale [37] and are used by Chew [43] in the problem of finding sparse spanners.

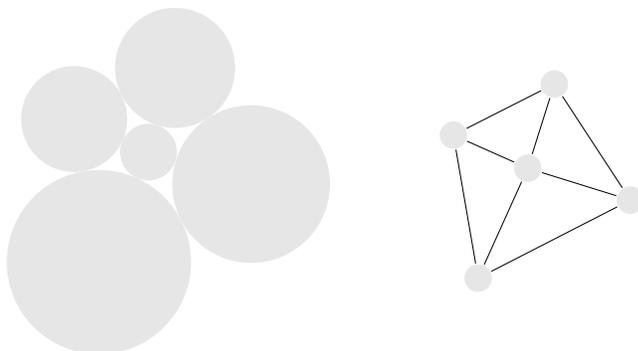
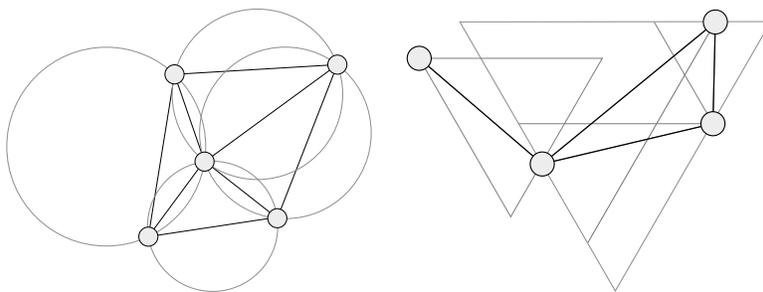


Figure 4: An example of a circle packing and its associated contact graph.



(a) An example of a Delaunay graph where the construction circles are in grey.

(b) An example of a TD-Delaunay graph where the construction triangles are in grey.

Figure 5: Examples of Delaunay and TD-Delaunay graphs.

Organization of the thesis. In the first chapter, we investigate a generalization of Schnyder’s theorem for higher Dushnik-Miller dimension. Three results are proved in this chapter. The first result is that TD-Delaunay complexes do not characterize Dushnik-Miller dimension (contradicting a conjecture). The second result is a generalization of some topological properties of some simplicial complexes linked to the Dushnik-Miller dimension. The third result investigates a new class of contact graphs, called *stair packings*, which are contact graphs of shapes similar to “stairs” when drawn in the plane, and studies the link between these graphs and the Dushnik-Miller dimension.

In the last two chapters, results on planar graphs are proved. We hope that these results could be generalized in some sense as it the case of the results proven in the first chapter.

The second chapter deals with \perp -intersection graphs and \perp -contact graphs, which are intersection graphs of shapes similar to “ \perp ” drawn in the plane. The proven results in this chapter are characterizations of these graphs in terms of planarity and could be generalized to higher dimensions.

In the third chapter we introduce the new notion of *Möbius stanchion systems* defined on planar graphs. This notion is connected to the problem of finding a drawing of a planar graph on any closed surface, like a sphere or a “donut”, such that there is only one *face*.

Acknowledgments.

Introduction

Graphes. La théorie des graphes a été développée pour étudier les nombreux réseaux utilisés dans le monde scientifique. Dans cette théorie on ne s'intéresse pas directement à des objets mais au système que forment les connexions entre ces différents objets. On peut trouver des applications dans de nombreux domaines. Parmi ces applications, celle qui consiste à chercher son chemin sur une carte en ligne est massivement utilisée tous les jours. L'utilisateur demande au logiciel quel est le plus court chemin entre un point A et un point B du réseau de transport. Du point de vue de l'utilisateur ce problème peut paraître facile mais des années de recherches en théorie des graphes ont été nécessaires pour résoudre ce problème en un temps concis. Dans ce réseau particulier, les objets sont par exemple des villes et on considère que deux villes sont connectées entre elles s'il y a une route directe qui relie l'une à l'autre. Résoudre la question du plus court chemin n'est pas l'unique question intéressante que l'on peut se poser sur le réseau de transport. On peut aussi chercher le chemin le plus rapide pour visiter toutes les villes d'une région. En plus de ces questions, le réseau de transport a soulevé de nombreux problèmes de théorie des graphes qui sont devenus de plus en plus important avec l'avènement de l'informatique.

Mais les réseaux de transports ne sont pas les seuls réseaux qui peuvent être modélisés avec des graphes. L'étude des graphes sociaux est un nouveau domaine d'application qui est apparu récemment grâce au développement de l'internet et à l'augmentation des capacités de stockage des ordinateurs. Dans ce type de graphes les objets sont les êtres humains et on considère que deux personnes sont reliées entre elles si elles se connaissent en un certain sens que le modèle doit définir. Dans ce type de réseau, on peut essayer par exemple de chercher à discerner des communautés ou à trouver les personnes centrales. Les applications sont nombreuses : on peut par exemple étudier la diffusion de rumeurs.

En biologie aussi on peut utiliser les graphes. C'est par exemple le cas quand on s'intéresse au développement d'une maladie virale dans la population humaine. Un second exemple consiste à considérer une région du monde où vit une espèce animale. On peut supposer que deux localisations de cette région sont connectées dans ce modèle s'il y a un mouvement d'une partie de la population entre ces deux localisations. Dans ce cas, on dit que le réseau est orienté car les connexions ne sont pas symétriques : le déplacement d'une partie de la population se fait d'un point A vers un point B et n'implique pas un déplacement

dans l'autre sens. De plus dans ce réseau les connexions peuvent être *pondérés* par l'importance du déplacement.

Dans la terminologie de la théorie des graphes, les réseaux sont appelés *graphes*, les objets *sommets* et les connexions *arêtes*. Comme on l'a vu, chaque réseau est différent et peut être modélisé par différents modèles de graphes (comme les graphes orientés ou pondérés). Parmi ces types de graphes, on s'intéresse particulièrement dans cette thèse à un type particulier : les graphes planaires.

Les graphes planaires. Un graphe est dit *planaire* si on peut le dessiner dans le plan de sorte que les arêtes ne se croisent qu'à leurs extrémités. Un exemple de graphe planaire très important est le réseau de transport, si l'on suppose qu'il n'y ait pas de route qui passe au-dessus d'une autre par l'intermédiaire d'un pont ou d'un tunnel. C'est par exemple le cas dans le problème des sept ponts de Königsberg qui a été résolu par Euler en 1736 et qui peut être considéré comme la première utilisation de la théorie des graphes.

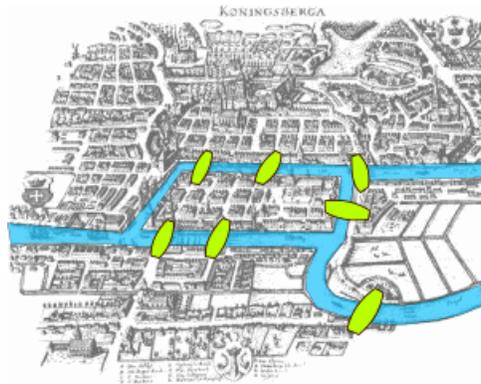


Figure 6: Le problème des sept ponts de Königsberg.

Un autre problème fondamental est le problème de coloration d'un graphe planaire. Considérons une carte de la France métropolitaine où l'on veut colorier les départements de sorte que deux départements adjacents n'aient pas la même couleur. Combien de couleurs faut-il utiliser ? Appel et Haken [26] ont résolu ce problème en 1976 et ont démontré que le nombre de couleurs nécessaires est quatre comme le montre la Figure 7. Les graphes planaires sont aussi reliés à la notion de maillage qui est utilisée en dessin assisté par ordinateur pour dessiner des modèles 3D d'objets [81].

En plus de leurs applications pratiques, les graphes planaires ont un intérêt mathématique car ils forment une vaste classe de graphes possédant de nombreuses propriétés et des outils dédiés qui en font un bon point de départ pour s'attaquer à un problème. C'est par exemple le cas du célèbre théorème sur les mineurs de graphes de Robertson et Seymour qui utilise les graphes planaires au début de sa démonstration [97].

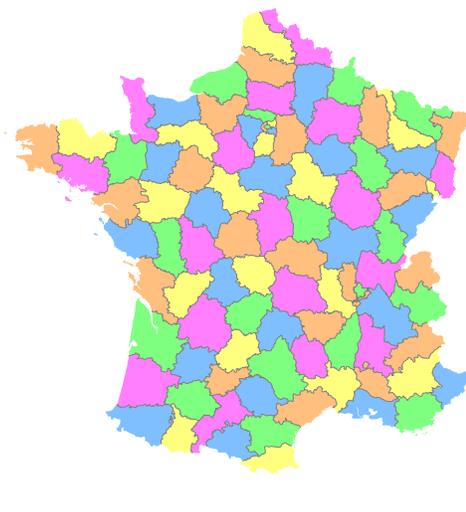


Figure 7: Une coloration de la carte des départements métropolitains français avec quatre couleurs.

Du fait de l'importance des graphes planaires en pratique et en théorie, des outils ont été développés pour cette classe de graphes comme on va le voir.

La dimension de Dushnik-Miller. Parmi les propriétés et outils utiles que possèdent les graphes planaires, il y a la dimension de Dushnik-Miller dont la définition nécessite la notion d'ordre. Un *ordre* est la donnée d'un ensemble abstrait d'éléments où ces éléments sont arbitrairement classés en supposant que certains sont plus grands ou plus petits que d'autres; deux éléments peuvent aussi être supposés incomparables. Alors que dans un ordre en général deux éléments ne sont pas forcément comparables, un *ordre total* (ou ordre linéaire) est un ordre où chaque paire d'éléments est comparable. La *dimension de Dushnik-Miller*, aussi connue sous le nom de *dimension d'ordre*, est un entier qui mesure le défaut de linéarité d'un ordre en un certain sens : un ordre sera de dimension Dushnik-Miller 1 si et seulement si cet ordre est linéaire.

La dimension de Dushnik-Miller peut être appliquée aux graphes en considérant son *ordre d'incidence*. Cet ordre particulier capture le fait qu'un sommet soit incident ou non à une arête. La dimension de Dushnik-Miller d'un graphe est ainsi définie comme la dimension de Dushnik-Miller de son ordre d'incidence. Le théorème de Schnyder [100] assure qu'un graphe est planaire si et seulement s'il est de dimension de Dushnik-Miller au plus 3. Une conséquence de ce théorème est que la notion topologique de planarité peut être vue comme une propriété combinatoire du graphe.

Comme le montre le théorème précédent de Schnyder, la dimension de Dushnik-Miller pourrait mesurer la complexité géométrique d'un graphe. Une question naturelle soulevée par Trotter dans [104] consiste à trouver une généralisation

du théorème de Schnyder pour des dimensions supérieures. Cette question a été abordée par Ossona de Mendez [92] et Felsner et Kappes [67] qui ont trouvé des résultats partiels dans ce sens. Dans cette thèse on étudie la possibilité de répondre à cette question à l'aide de deux classes de graphes géométriques. La première est un cas particulier de graphes d'intersection.

Les graphes d'intersection. Les graphes d'intersection sont définis de la manière suivante. En considérant des objets géométriques dans un espace géométrique particulier on peut définir le graphe d'intersection de ces objets par un graphe où un sommet correspond à un objet et une arête correspond à une intersection entre deux objets. Ce type de graphes a été largement étudié tout comme les variations qui peuvent être apportées aux objets considérés ou à la manière dont ces objets s'intersectent. Quand les objets ont possiblement une intersection intérieure avec d'autres objets, on préférera utiliser le terme de graphe d'intersection de X où X est une description des objets géométriques autorisés. Quand les objets n'ont pas d'intersection intérieure, on préférera le terme de graphe de contact de X . On donne maintenant des exemples de graphes d'intersection et de graphes de contact qui sont importants en pratique.

Des exemples de graphes d'intersection peuvent être obtenus en considérant des disques unités dans le plan. À partir de ces disques on peut former le graphe d'intersection (voir Figure 8). On dit qu'un graphe est un *graphe de disques unités* s'il s'agit du graphe d'intersection de disques unités dans le plan. Ce type de graphes est par exemple utilisé dans les réseaux de communications où les disques représentent le rayon d'action d'une antenne [77].

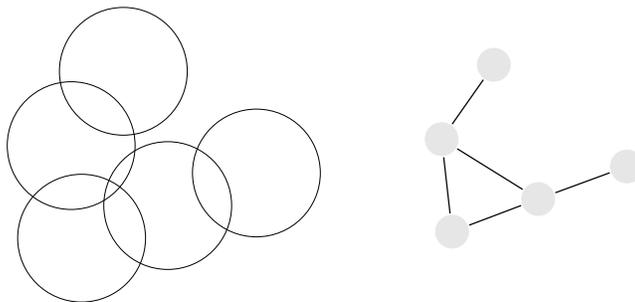


Figure 8: Un exemple d'arrangement de disques unités et son graphe d'intersection associé.

En considérant des disques dans le plan qui ne s'intersectent pas en un point intérieur, on obtient ce que l'on appelle un *empilement de disques* (voir Figure 9). Comme il n'y a pas d'intersection intérieure, on peut considérer le graphe de contact de cet arrangement. Le célèbre théorème d'empilement de disques de Koebe [16] assure qu'un graphe est planaire si et seulement si c'est le graphe de contact d'un empilement de disques. Ce théorème a été récemment utilisé pour reprouver le théorème du séparateur planaire [90] et pour trouver des plongements de graphes planaires avec un angle de résolution borné [86].

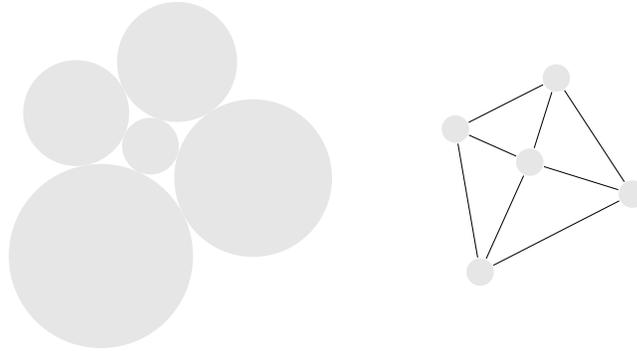
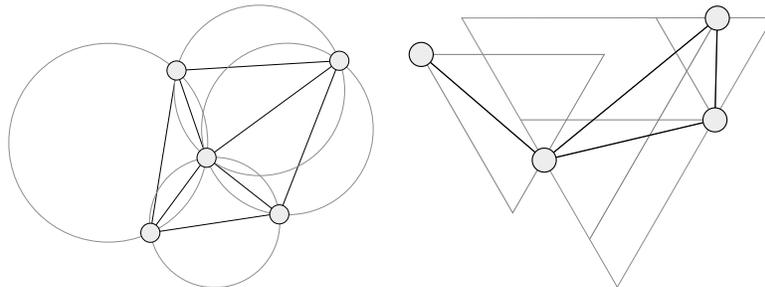


Figure 9: Un exemple d'empilement de disques et son graphe de contact associé.

Le second type de graphes géométriques utilisé pour répondre à la question du paragraphe précédent sur la généralisation du théorème de Schnyder est la classe des graphes de TD-Delaunay.

Les graphes de TD-Delaunay. Les graphes de TD-Delaunay sont des variantes des graphes de Delaunay. Étant donné des points dans le plan, le graphe de Delaunay de cet ensemble de points est défini comme suit : deux points sont considérés comme adjacents s'il existe un disque contenant ces deux points mais aucun autre point à l'intérieur (voir l'exemple sur la Figure 10a). Les graphes de Delaunay ont de nombreuses propriétés géométriques qui confèrent à ces graphes un intérêt particulier dans les algorithmes générant des maillages triangulaires [42]. La classe des graphes de TD-Delaunay est définie de manière similaire en remplaçant dans la définition précédente les disques par des triangles équilatéraux (voir l'exemple sur la Figure 10b). Ils ont été définis par Chew et Drysdale [37] et ont été utilisés par Chew [43] dans le problème consistant à chercher des spanners creux.



(a) Un exemple de graphe de Delaunay où les disques de construction sont en gris.

(b) Un exemple de graphe de TD-Delaunay où les triangles de construction sont en gris.

Figure 10: Exemples de graphes de Delaunay et de TD-Delaunay.

Organisation de la thèse. Dans le premier chapitre, on s'intéresse à la généralisation du théorème de Schnyder aux dimensions supérieures. On démontre dans ce chapitre trois résultats dans ce sens. Le premier assure que les complexes de TD-Delaunay ne caractérisent pas la dimension de Dushnik-Miller (ce qui contredit une conjecture de Mary [87] et de Evans *et al.* [48]). Le deuxième résultat généralise des propriétés topologiques de certains complexes simpliciaux connectés à la dimension de Dushnik-Miller. Le troisième résultat concerne une nouvelle classe de graphes de contact, appelé *graphes de contacts d'escaliers*, où les contacts se font entre des formes ressemblant à des escaliers dans le plan. Ce dernier résultat fait le lien entre de tels graphes et la dimension de Dushnik-Miller.

Dans les deux derniers chapitres, on démontre des résultats sur les graphes planaires.

Le deuxième chapitre concerne plus spécifiquement les graphes d'intersection de \perp et les graphes de contact de \perp qui sont des graphes d'intersection de formes ressemblant à des " \perp " dessinés dans le plan. On montre dans ce chapitre que l'on peut représenter certains types de graphes planaires comme des graphes de contacts ou d'intersection de \perp . Ces résultats pourraient être généralisés en dimension supérieure.

Dans le troisième chapitre on introduit la notion de *systèmes de Möbius stanchions* qui est définie sur les graphes planaires. Cette notion est connectée au problème consistant à trouver un plongement d'un graphe planaire sur une surface fermée, comme une sphère ou un tore, de sorte qu'il n'y ait qu'une face.

General definitions

Graphs and planarity

Given a finite set V and a subset E of the pairs of V , we define the *graph* $G = (V, E)$. Elements of V are called *vertices* and elements of E are called *edges*. An edge e is *incident* to a vertex v if $v \in e$. Two edges e and e' are said to be *adjacent* if there exists a vertex v such that v is incident to both of them. Two vertices are said to be *adjacent* if E contains the pair consisting in this pair of two vertices. The *degree* of a vertex is the number of edges incident to this vertex. Given F a subset of the vertices of a graph G , the graph $G \setminus F$ obtained after the removal of the vertices of F consists in the graph whose vertex set is $V \setminus F$ and whose edge set is the edges of G not incident to some vertices of F . A *path* is sequence e_1, \dots, e_k of edges such that e_i is adjacent to e_{i+1} for every $i \in \llbracket 1, k-1 \rrbracket$. We say that a graph G is *connected* if for every pair of vertices x and y , there exists a path e_1, \dots, e_k of G such that e_1 is incident to x and e_k is incident to y . Otherwise we say that the graph is *disconnected*. A graph is *k-connected* if and only if any removal of $k-1$ vertices does not disconnect the graph. A graph is said to be a *tree* if it is connected and if $|V| = |E| + 1$. Given a graph G , a *spanning tree* of G is a subgraph T of G such that T is a tree and such that the vertices of T and G are the same. An *orientation* of a graph consists in giving a direction to each edge. Given a vertex v and an incident edge e , we say that e is *in-going* (resp. *out-going*) if e goes to v (resp. come from v).

Planarity. A graph is said to be *planar* if it is possible to draw it in the plane such that edges only intersect at endpoints. A *plane graph* is a graph given with an embedding of the graph in the plane. A *face* of a plane graph is a connected component of $\mathbb{R}^2 \setminus G$. The infinite face is called the *outer face*, while the other faces are called *inner faces*. The *dual* of a plane graph G , noted G^* , is a graph whose vertex set consisting in the faces of G and where two faces are considered to be connected if there exists an edge of G which separates these two faces. A *near-triangulation* is a plane graph such that every inner face is triangular. A *triangulation* is a near-triangulation such that the outer face is triangular. In a plane graph G , a *chord* is an edge not bounding the outer face but that links two vertices of the outer face. A *separating triangle* of G is a cycle of length three such that both regions delimited by this cycle (the inner and the outer region) contain some vertices.

We recall the following theorem:

Theorem 1 (folklore). *A triangulation is 4-connected if and only if it contains no separating triangle.*

Intersection graphs and contact graphs

The representation of graphs by contact or intersection of predefined shapes in the plane is a broad subject of research since the work of Koebe on the

representation of planar graphs by contacts of circles [16]. In particular, the class of planar graphs has been widely studied in this context.

Definition 2. *Given a shape¹ X , an X -intersection representation is a collection of X -shaped geometrical objects in the plane. The X -intersection graph described by such a representation has one vertex per geometrical object, and two vertices are adjacent if and only if the corresponding objects intersect.*

Definition 3. *In the case where the shape X defines objects that are homeomorphic to a segment (resp. to a disc), an X -contact representation is an X -intersection representation such that if an intersection occurs between two objects, then it occurs at a single point that is the endpoint of one of them (resp. it occurs on their boundary). We say that a graph G is an X -contact graph if it is the X -intersection graph of an X -contact representation.*

The case of shapes homeomorphic to discs has been widely studied; see for example the literature for triangles [12, 10], homothetic triangles [9, 7], axis parallel rectangles [103], squares [5, 6], hexagons [11], convex bodies [8], or axis aligned polygons [4].

¹We do not provide a formal definition of *shape*, but a shape characterizes a family of connected geometric objects in the plane.

Chapter 1

Dushnik-Miller dimension of some geometric complexes

In this chapter we focus on combinatorial properties of two types of geometrically defined simplicial complexes through the lens of Dushnik-Miller dimension. The first one is the class of TD-Delaunay complexes which is a generalization of the class of TD-Delaunay graphs in the plane which is itself a variation of the well-known class of Delaunay graphs. The second one concerns stair packing contact complexes. This new kind of contact complexes is directly connected to the Dushnik-Miller dimension.

1.1 Introduction

1.1.1 Basic notions

We now recall the following basic notions of order theory. Given a set V , a *poset* (V, \leq) is a binary relation \leq on V satisfying the following properties for every $a, b, c \in V$:

- $a \leq a$ (reflexivity);
- if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity);
- if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry).

A *linear order* on a set V , is a poset on V such that for every pair (a, b) of elements of V , $a \leq b$ or $b \leq a$. Given a poset (V, \leq) , a *linear extension* is a linear order \leq' on V such that for every pair $(a, b) \in V^2$, $a \leq b \Rightarrow a \leq' b$. When the set V is finite, there always exists a linear extension of a poset on V . The *digraph of the poset* (V, \leq) is the digraph whose vertex set is V and where there

is an arc from x to y when $x \leq y$ for every pair (x, y) of elements of V . The *transitive reduction* of a digraph D is a subgraph D' with the fewest arcs such that for every pair (x, y) of elements of V , there exists a directed path from x to y in D if and only if there exists a directed path from x to y in D' . Note that the transitive reduction of a digraph is unique. The *Hasse diagram* of a poset is the transitive reduction of the digraph of the poset. Given a linear order (V, \leq) and a finite subset F of V , we define $\max_{\leq}(F)$ as the maximum of the set F according to the linear order \leq .

We will also use the notion of abstract simplicial complexes which generalizes the notion of graphs. An *abstract simplicial complex* Δ with vertex set V is a set of subsets of V which is closed by inclusion (*i.e.* $\forall Y \in \Delta, X \subseteq Y \Rightarrow X \in \Delta$). An element of Δ is called a *face*. A maximal element of Δ according to the inclusion order is called a *facet*.

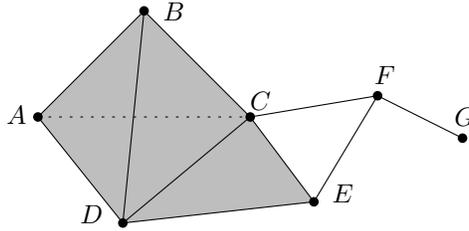


Figure 1.1: An abstract simplicial complex whose facets are $\{A, B, C, D\}, \{C, D, E\}, \{C, F\}, \{E, F\}$ and $\{F, G\}$.

The dimension of a face F is defined as $|F| - 1$. A simplicial complex is said to be *pure* if all its facets have the same dimension.

Given a simplicial complex Δ , we define the *inclusion poset* of Δ as the poset (Δ, \subseteq) defined on the faces of Δ where $X \leq Y$ for any pair $(X, Y) \in \Delta^2$ such that $X \subseteq Y$.

The following lemma will be used to study the Dushnik-Miller dimension.

Lemma 4. *Let \leq' be a linear order on a finite set V . The binary relation \leq defined on $\text{Subsets}(V)$ by*

$$F \leq G \iff \begin{cases} F \subseteq G, \text{ or} \\ F \not\subseteq G \text{ and } G \not\subseteq F \text{ and } \max_{\leq'}(F \setminus G) <' \max_{\leq'}(G \setminus F) \end{cases}$$

is a linear extension of the inclusion poset on $\text{Subsets}(V)$.

Proof. The relation \leq is clearly reflexive. Let us show that it is also antisymmetric. Let F and G be two faces of Δ such that $F \leq G$ and $G \leq F$. Suppose by contradiction that $F \neq G$. Suppose first that $F \subseteq G$. As $G \leq F$, then $G \subseteq F$ or $\max_{\leq'}(G \setminus F) <' \max_{\leq'}(F \setminus G)$. The latter is impossible as $F \setminus G = \emptyset$, and the former implies that $F = G$, a contradiction. Thus $F \not\subseteq G$. Similarly, we have that $G \not\subseteq F$. As $F \leq G$ and $G \leq F$, we deduce that $\max_{\leq'}(F \setminus G) <' \max_{\leq'}(G \setminus F)$

and that $\max_{\leq'}(G \setminus F) <' \max_{\leq'}(F \setminus G)$ which is a contradiction. We conclude that $F = G$ and that \leq is antisymmetric.

Let us show that the relation \leq is transitive. Let F , G and H be three faces of Δ such that $F \leq G \leq H$. Suppose first that $F \subseteq G$ and $G \subseteq H$. Then $F \subseteq H$ and $F \leq H$.

Suppose now that $F \not\subseteq G$ and $G \subseteq H$. As $F \leq G$, we deduce that $G \not\subseteq F$. If $F \subseteq H$, then $F \leq H$. Otherwise $F \not\subseteq H$. If $H \subseteq F$, then we have $G \subseteq F$, as $G \subseteq H$, which is a contradiction. Thus $H \not\subseteq F$. As $G \subseteq H$, we have that $F \setminus H \subseteq F \setminus G$ and $G \setminus F \subseteq H \setminus F$. Thus

$$\max_{\leq'}(F \setminus H) \leq' \max_{\leq'}(F \setminus G) <' \max_{\leq'}(G \setminus F) \leq' \max_{\leq'}(H \setminus F)$$

We conclude that $F \leq H$. The case where $F \subseteq G$ and $G \not\subseteq H$ is similar.

It remains to study the case where $F \not\subseteq G$ and $G \not\subseteq H$. If $F \subseteq H$, then $F \leq H$. Otherwise $F \not\subseteq H$. Suppose by contradiction that we have not the inequality $\max_{\leq'}(F \setminus H) <' \max_{\leq'}(H \setminus F)$. We define $\max_{\leq'}(\emptyset)$ as an element such that $\max_{\leq'}(\emptyset) <' \min_{\leq'}(V)$. We define the following elements:

$$\begin{cases} a = \max_{\leq'}(F \setminus (G \cup H)) \\ b = \max_{\leq'}((F \cap G) \setminus H) \\ c = \max_{\leq'}(G \setminus (F \cup H)) \\ d = \max_{\leq'}((G \cap H) \setminus F) \\ e = \max_{\leq'}(H \setminus (F \cup G)) \\ f = \max_{\leq'}((H \cap F) \setminus G) \end{cases}$$

Because of the different hypotheses we have

$$\begin{cases} \max(a, f) <' \max(c, d) \\ \max(b, c) <' \max(f, e) \\ \max(d, e) <' \max(a, b) \end{cases}$$

Suppose that $c \leq' d$. In this case, $a <' d$ and $f <' d$. As $d <' \max(a, b)$, then $d <' b$. As $b <' \max(f, e)$ and $f <' d <' b$, then $b <' e$. As $e <' \max(a, b)$ and $a <' d <' b$, then $e <' b$, a contradiction. The case where $d \leq' c$ is symmetrical.

We conclude that the binary relation \leq on $\text{Subsets}(V)$ is reflexive, antisymmetric and transitive. Therefore \leq is an order. As \leq' is a linear order, we deduce that \leq is also a linear order. Furthermore if $F \subseteq G$, then $F \leq G$. We conclude that \leq is a linear extension of the inclusion poset on $\text{Subsets}(V)$. \square

1.1.2 Delaunay graphs and their variants

Original Delaunay graphs are defined as follows.

Definition 5. *Given a set of points \mathcal{P} in \mathbb{R}^2 such that no quadruplet of \mathcal{P} lies on one circle, the Delaunay graph $D(\mathcal{P})$ of \mathcal{P} is defined as the graph with vertex*

set \mathcal{P} and where the vertices are connected as follows. Two points are connected if and only if there exists a closed disk containing these two points but no other point in the interior of this disk.

An example of a Delaunay graph is given in Figure 1.2. An interesting property of these graphs is the following:

Theorem 6. *Given a set of points \mathcal{P} in \mathbb{R}^2 such that no quadruplet of \mathcal{P} lies on one circle, the graph $D(\mathcal{P})$ is a near-triangulation.*

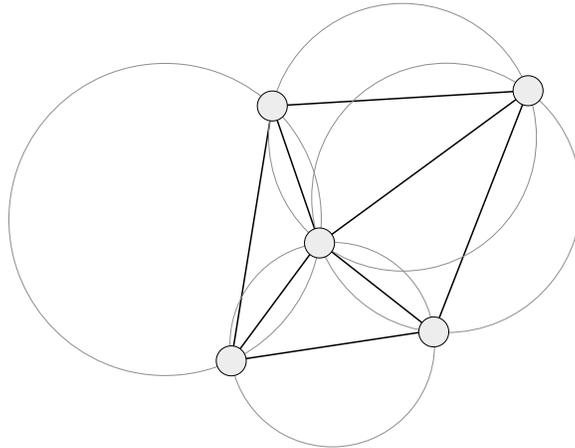


Figure 1.2: An example of a Delaunay graph. Construction circles are in grey.

The class of Delaunay graphs can be generalized with the use of *convex distance functions*.

Definition 7. *Given a compact convex subset S of \mathbb{R}^d and a point c in the interior of S , we define the convex distance function f , also called *Minkowski distance function*, between two points p and q as the minimal scaling factor $f(p, q) = \lambda$ such that after rescaling S by λ and translating it so as to center it on p , then it contains also q .*

The convex distance function associated to the unit disk in \mathbb{R}^d where c is the center of this disk is the Euclidean distance. Any distance associated to a norm in \mathbb{R}^d is a convex distance function by taking S as the unit disk in \mathbb{R}^d according to this norm and c as the center of this disk. The reciprocal of the previous statement is false, as convex distance functions do not satisfy the symmetry property of distances in general. By taking S an equilateral triangle in the plane and c the center of this triangle, we get the convex distance function called *triangular distance* which is abbreviated TD.

Definition 8. *Given a set of points \mathcal{P} in \mathbb{R}^2 and a convex distance function f , for a point $x \in \mathcal{P}$, we define its f -Voronoi cell as the set of points y of the plane*

such that x is among the nearest points of P , according to the convex distance function f , to y , i.e. the set of points $\{y: f(x,y) \leq f(x',y) \ \forall x' \in \mathcal{P}\}$. We define the f -Delaunay graph of a point set \mathcal{P} as the graph with vertex set \mathcal{P} such that two points are connected if and only if their f -Voronoi cells intersect.

Delaunay graphs are particular cases of f -Delaunay graph by taking f as the Euclidean distance. The TD-Delaunay class of graphs is therefore defined as the class of f -Delaunay graphs where f is the triangular distance. Note that TD-Delaunay graphs also admit an alternative definition similar to Definition 5:

Definition 9. Given a set \mathcal{P} of points of \mathbb{R}^2 in general position, let the TD-Delaunay graph of \mathcal{P} , denoted by $\text{TDD}(\mathcal{P})$, be the graph with vertex set \mathcal{P} defined as follows. A subset $F \subseteq \mathcal{P}$ is a face of $\text{TDD}(\mathcal{P})$ if and only if there exists an equilateral triangle T with one horizontal edge at the top such that $T \cap \mathcal{P} = F$ and such that no point of \mathcal{P} is in the interior of T .

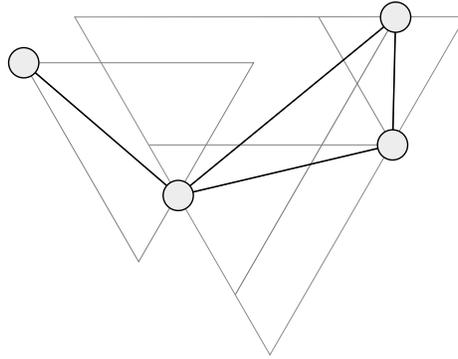


Figure 1.3: An example of a TD-Delaunay graph. Construction triangles are in grey.

Bonichon *et al.* [31] observed the following property of TD-Delaunay graphs.

Theorem 10 (Bonichon *et al.* [31]). *A graph G is planar if and only if G is a subgraph of a TD-Delaunay graph.*

TD-Delaunay graphs can be generalized to higher dimensions by taking the triangular distance in \mathbb{R}^d according to a regular d -simplex. As a graph is planar if and only if G is of Dushnik-Miller dimension, whose definition will be given latter, at most 3, there is maybe a link between the notions of Dushnik-Miller dimension and TD-Delaunay complexes. In [87] and [48] it was independently asked whether Dushnik-Miller dimension at most d complexes are exactly the subcomplexes of TD-Delaunay complexes of \mathbb{R}^{d-1} .

Spanners

Spanners were introduced by Chew [41] and are used in motion planning and network design. This notion motivated the introduction of the previous variants of Delaunay graphs.

Definition 11. Given points in the plane, a plane spanner is a subgraph of the complete graph on these points which is planar when joining adjacent points with segments.

The stretch of a plane spanner is the maximum ratio of the distance in the graph between two vertices where edges are weighted accordingly to their length and the Euclidean distance between these two points.

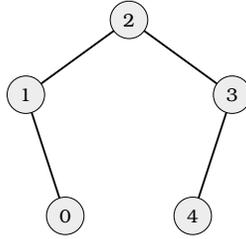


Figure 1.4: An example of a plane spanner. The stretch of this spanner is 4 as the distance between vertices 0 and 4 is 4 times larger than the Euclidean distance between these two vertices.

Given points in the plane, a natural question which has attracted a lot of attention is to find a plane spanner which minimizes the stretch. We define the stretch of a class of spanners as the maximum stretch among them. Chew [41] found the first class of plane spanners with finite stretch which consists in the class of L_1 -Delaunay graphs. Chew [41] also conjectured that the class of L_2 -Delaunay graphs (classical Delaunay graphs) has a finite stretch. This question initiated a series of papers about this topic which dropped the upper bound from 5.08 to 1.998 [32, 45, 106]. The stretch of the class of TD-Delaunay graphs is 2 according to [43].

In higher dimensions, finding a class of graphs with finite stretch is an active domain of research. We conjecture that the stretch of the class of TD-Delaunay graphs in \mathbb{R}^d is d . A lower bound can be easily obtained by putting a point in each “corner” of a regular d -simplex in \mathbb{R}^d .

1.1.3 Dushnik-Miller dimension and representations

The notion of *Dushnik-Miller dimension* of a poset has been introduced by Dushnik and Miller [46]. It is also known as the *order dimension* of a poset. See [104] for a comprehensive study of this topic.

Definition 12. The Dushnik-Miller dimension of a poset (V, \leq) is the minimum number d such that (V, \leq) is the intersection of d linear extensions of (V, \leq) . This means that there exists d linear extensions $(V, \leq_1), \dots, (V, \leq_d)$ of (V, \leq) such that for every $x, y \in V$, $x \leq y$ if and only if $x \leq_i y$ for every $i \in \llbracket 1, d \rrbracket$. In particular if x and y are incomparable with respect to \leq , then there exists i and j such that $x \leq_i y$ and $y \leq_j x$.

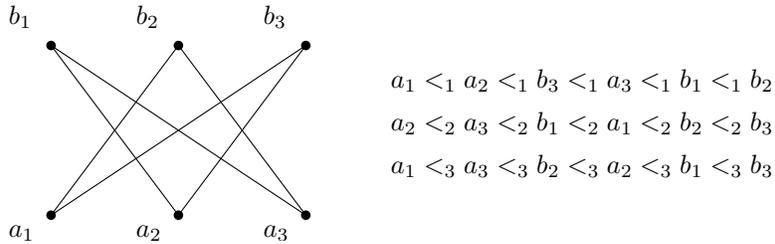


Figure 1.5: A poset \leq defined by its Hasse diagram and 3 linear extensions \leq_1, \leq_2 and \leq_3 of \leq showing that \leq is of Dushnik-Miller dimension at most 3.

The notion of Dushnik-Miller dimension can be applied to a simplicial complex as follows.

Definition 13. Let Δ be a simplicial complex. We define the Dushnik-Miller dimension of Δ , denoted $\dim_{\text{DM}}(\Delta)$, as the Dushnik-Miller dimension of the inclusion poset of Δ .

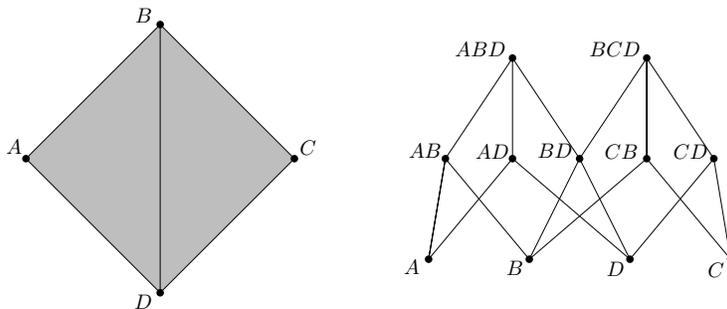


Figure 1.6: A simplicial complex with facets $\{A, B, D\}$ and $\{B, C, D\}$, and its inclusion poset.

Low dimensions are well known. We have $\dim_{\text{DM}}(\Delta) = 1$ if and only if Δ is a union of single vertices. We have $\dim_{\text{DM}}(\Delta) \leq 2$ if and only if Δ is a union of paths. There are complexes with arbitrarily high Dushnik-Miller dimension: $\dim_{\text{DM}}(\Delta) = d$ if Δ is a d -simplex and for any integer n , $\dim_{\text{DM}}(K_n) = \Theta(\log \log n)$ where K_n denotes the complete graph on n vertices. The following theorem shows that the topological notion of planarity can be understood as a combinatorial property thanks to the Dushnik-Miller dimension.

Theorem 14 (Schnyder [100]). *A graph G is planar if and only if G is of Dushnik-Miller dimension at most 3.*

Dushnik-Miller dimension is connected to other sparse classes of graphs [79, 78, 80]. For example, posets whose cover graph (given by the Hasse diagram) has treewidth 2 have bounded Dushnik-Miller dimension. It is also known that

posets with height at most h (i.e. the maximal length of a chain in the poset), and whose cover graph belongs to some bounded expansion class C , have their Dushnik-Miller dimension bounded by some constant $K(h, C)$. Examples of classes of graphs with bounded expansion are classes of graphs with bounded maximum degree and classes of graphs excluding subdivision of a fixed graph [91].

Representations

Representations have been introduced by Scarf [98]. It is a tool for dealing with Dushnik-Miller dimension. In the concept of representations, only vertices have to be ordered while in the definition Dushnik-Miller dimension, all the faces of the complex have to be ordered.

Representations define particular simplicial complexes called *supremum sections*. These simplicial complexes are important as they appear in commutative algebra: Bayer *et al.* [33] studied monomial ideals which are linked to supremum sections by what they call Scarf complexes. They are used by Felsner *et al.* [67] in order to study orthogonal surfaces. Furthermore, they also appear in spanning-tree-decompositions and in the box representations problem as shown by Evans *et al.* [48].

Definition 15. A d -representation R on a set V is a set of d linear orders \leq_1, \dots, \leq_d on V . Given a linear order \leq on a set V , an element $x \in V$, and a set $F \subseteq V$, we say that x dominates F in \leq , and we denote it by $F \leq x$, if $f \leq x$ for every $f \in F$. Given a d -representation R , an element $x \in V$, and a set $F \subseteq V$, we say that x dominates F (in R) if x dominates F in some order $\leq_i \in R$. As in [92], we define $\Sigma(R)$ as the set of subsets F of V such that every $v \in V$ dominates F . The set $\Sigma(R)$ is called the supremum section of R .

Note that $\emptyset \in \Sigma(R)$ for any representation R . An element $x \in V$ is a vertex of $\Sigma(R)$ if $\{x\} \in \Sigma(R)$. Note that sometimes an element $x \in V$ is not a vertex of $\Sigma(R)$. Actually, the definition of d -representations provided here is slightly different from the one in [100] and [92]. There, the authors ask for the intersection of the d orders to be an antichain. With this property, every element of V is a vertex of $\Sigma(R)$. Note that simply removing the elements of V that are not vertices of $\Sigma(R)$ yields a representation in the sense of [92, 100]. Remark also that if F is a face of a supremum section of a representation R , then each of the elements of F dominates F at least once.

Proposition 16. For any d -representation $R = (\leq_1, \dots, \leq_d)$ on a set V , $\Sigma(R)$ is a simplicial complex.

Proof. For any $F \in \Sigma(R)$, let X be any subset of F , and let v be any element of V . Since $F \in \Sigma(R)$, there exists $\leq_i \in R$ such that $F \leq_i v$. Particularly, $X \leq_i v$. Thus $X \in \Sigma(R)$ and we have proven that $\Sigma(R)$ is a simplicial complex. \square

The following is an example of a 3-representation on $\{1, 2, 3, 4, 5\}$ where each line in the table corresponds to a linear order whose elements appear in increasing order from left to right:

$$\begin{array}{c|ccccc} \leq_1 & 1 & 2 & 5 & 4 & 3 \\ \leq_2 & 3 & 2 & 1 & 4 & 5 \\ \leq_3 & 5 & 4 & 3 & 2 & 1 \end{array}$$

The corresponding complex $\Sigma(R)$ is given by the facets $\{1, 2\}$, $\{2, 3, 4\}$ and $\{2, 4, 5\}$. For example $\{1, 2, 3\}$ is not in $\Sigma(R)$ as 2 does not dominate $\{1, 2, 3\}$ in any order. The following theorem shows that representations and Dushnik-Miller dimension are equivalent notions.

Theorem 17 (Ossona de Mendez [92]). *Let Δ be a simplicial complex with vertex set V . Then $\dim_{\text{DM}}(\Delta) \leq d$ if and only if there exists a d -representation R on V such that $\Delta \subseteq \Sigma(R)$.*

Proof. Suppose that there exists d linear orders \leq_1, \dots, \leq_d on Δ such that the inclusion poset of Δ is the intersection of the orders \leq_1, \dots, \leq_d . For every $i \in \llbracket 1, d \rrbracket$, we define \leq'_i as the restriction of \leq_i to the vertices of Δ . We define the d -representation $R = (\leq'_1, \dots, \leq'_d)$ on V . Let us show that $\Delta \subseteq \Sigma(R)$. Let F be a face of Δ and suppose that $F \notin \Sigma(R)$. There exists $x \in V$ such that x does not dominate F in any order \leq'_i . For every $i \in \llbracket 1, d \rrbracket$, we define f_i as the maximum of F in the order \leq'_i . Thus $x <'_i f_i$ for every i . We define $G = \{f_1, \dots, f_d\}$. As a subset of $F \in \Delta$, G belongs to Δ . Since $f_i \in G$ for every $i \in \llbracket 1, d \rrbracket$, then $\{f_i\} <_i G$. Thus $\{x\} <_i G$ for every $i \in \llbracket 1, d \rrbracket$. As the inclusion poset of Δ is the intersection of the orders \leq_1, \dots, \leq_d , then $x \in G$ which contradicts the definition of G . We conclude that $F \in \Sigma(R)$ and that $\Delta \subseteq \Sigma(R)$.

Suppose that there exists a d -representation $R = (\leq'_1, \dots, \leq'_d)$ on V such that $\Delta \subseteq \Sigma(R)$. For every $i \in \llbracket 1, d \rrbracket$ we define the binary relation \leq_i on Δ as follows. For any F and $G \in \Delta$, $F \leq_i G$ if and only if $F \subseteq G$ or when $F \not\subseteq G$ and $G \not\subseteq F$ and $\max_{\leq'_i}(F \setminus G) <'_i \max_{\leq'_i}(G \setminus F)$. According to Lemma 4, the relation \leq_i is a linear extension of the inclusion poset of Δ . Let us now show that the intersection of these linear extensions is the inclusion poset of Δ .

Let $F \neq G \in \Delta$ such that for every $i \in \llbracket 1, d \rrbracket$, $F \leq_i G$. Suppose by contradiction that $F \not\subseteq G$ and let x be a vertex of $F \setminus G$. As G is a face of $\Sigma(R)$, there exists $i \in \llbracket 1, d \rrbracket$ such that $\max_{\leq'_i} G <'_i x$. Therefore $\max_{\leq'_i}(G \setminus F) <'_i \max_{\leq'_i}(F \setminus G)$. This contradicts the fact that $F \leq_i G$. We conclude that Δ is of Dushnik-Miller dimension at most d . \square

Among all representations, *standard* representations are of particular interest. We will see that, for example, the facets of a supremum section of a standard 3-representation are the inner faces of a planar triangulation.

Definition 18. *A d -representation R on a set V with at least d elements, is standard if every element $v \in V$ is a vertex of $\Sigma(R)$, and if for any order, its maximal vertex is among the $d - 1$ smallest elements in all the other orders of R . An element which is a maximal element of an order of R is said to be an exterior element of R . The other elements are said to be interior elements of R .*

For example the following 3-representation on $\{1, 2, 3, 4, 5\}$ is standard.

$$\begin{array}{c|ccccc} \leq_1 & 2 & 3 & 4 & 5 & 1 \\ \leq_2 & 1 & 3 & 5 & 4 & 2 \\ \leq_3 & 1 & 2 & 4 & 5 & 3 \end{array}$$

In this representation the elements 4 and 5 are interior elements and the elements 1, 2, 3 are exterior elements.

1.1.4 State of the art

In this subsection we describe some properties that are satisfied by simplicial complexes of fixed Dushnik-Miller dimension. A natural question is to give a geometric interpretation of complexes of Dushnik-Miller dimension at most d . For example, union of paths are the graphs of Dushnik-Miller dimension at most 2. According to Theorem 14 of Schnyder, there is a combinatorial characterization of planar graphs: they are those of Dushnik-Miller dimension at most 3. The question of characterizing classes of simplicial complexes of larger dimension is open. The following conjecture, using TD-Delaunay complexes (a generalization of TD-Delaunay graphs that will be formally defined later), was made:

Conjecture 19 (Mary [87] and Evans *et al.* [48]). *The class of Dushnik-Miller dimension at most d complexes is the class of TD-Delaunay complexes in \mathbb{R}^{d-1} .*

The result holds for $d = 2$ and $d = 3$ and it is known that TD-Delaunay complexes of \mathbb{R}^{d-1} have Dushnik-Miller dimension at most d .

We now give a survey of results connected to the notion of Dushnik-Miller dimension for higher dimensions.

The class of graphs of Dushnik-Miller dimension at most 4

The class of graphs of Dushnik-Miller dimension at most 4 is rather rich. Extremal questions in this class of graphs have been studied: Felsner and Trotter [72] showed that these graphs can have a quadratic number of edges. Furthermore, in order to solve a question about conflict free coloring [51], Chen *et al.* [44] showed that many graphs of Dushnik-Miller dimension 4 only have independent sets of size at most $o(n)$. This result also implies that there is no constant k such that every graph of Dushnik-Miller dimension at most 4 is k -colorable. Therefore, graphs of Dushnik-Miller dimension at most 4 seem difficult to be characterized.

Dushnik-Miller dimension and polytopes

We recall that a set X of points of \mathbb{R}^d is said to be *affinely independent* if no point of X is included in the affine space generated by the other points of X . The *convex hull* of a set X of points of \mathbb{R}^d is defined as the intersection of all the convex subsets of \mathbb{R}^d containing X . A *simplex* of \mathbb{R}^d is the convex hull of a

set of affinely independent points. We denote by $\text{Conv}(X)$, the convex hull of a set X of points of \mathbb{R}^d .

The following definition is required for understanding the Theorem 21 of Scarf which motivated the study of representations by Scarf [98].

Definition 20. *A d -polytope is the convex hull of a finite set S of points in \mathbb{R}^d . A subset $F \subseteq S$ is said to be a face if $F = S$ or if there exists an affine hyperplane H such that $F \subseteq H$ and $S \setminus F$ is strictly included in one of the half spaces defined by H . In the second case, F is said to be a proper face. The inclusion poset of a polytope is the set of the proper faces of the polytope ordered by inclusion.*

Theorem 21 (Scarf [98] as phrased by Felsner and Kappes [67]). *Let R be a standard d -representation. The simplicial complex $\Sigma(R)$ is isomorphic to the face complex of a simplicial d -polytope with one facet removed.*

Theorem 22 (Brightwell and Trotter [34]). *The Dushnik-Miller dimension of the inclusion poset of a 3-polytope is 4. When removing one of the faces, the Dushnik-Miller dimension of the inclusion poset drops to 3.*

Straight line embeddings

A generalization of the notion of planarity for simplicial complexes is the notion of straight line embedding. As in the planar case, we do not want that two disjoint faces intersect.

Definition 23. *Let Δ be a simplicial complex with vertex set V . A straight line embedding of Δ in \mathbb{R}^d is a mapping $f : V \rightarrow \mathbb{R}^d$ such that*

- $\forall X \in \Delta$, $f(X)$ is a set of affinely independent points of \mathbb{R}^d ,
- $\forall X, Y \in \Delta$, $\text{Conv}(f(X)) \cap \text{Conv}(f(Y)) = \text{Conv}(f(X \cap Y))$.

The following theorem shows that the Dushnik-Miller dimension in higher dimensions also captures some geometrical properties.

Theorem 24 (Ossona de Mendez [92]). *Any simplicial complex of Dushnik-Miller dimension at most $d + 1$ has a straight line embedding in \mathbb{R}^d .*

For $d = 2$, this theorem states that if a simplicial complex has dimension at most 3 then it is planar. Brightwell and Trotter [34] proved that the converse also holds (for $d = 2$)¹. For higher d , the converse is false: take for example the complete graph K_n which has a straight line embedding in \mathbb{R}^3 (and therefore in \mathbb{R}^d for $d \geq 3$) and which has Dushnik-Miller dimension arbitrarily large.

¹Note that in a straight line embedding in \mathbb{R}^2 every triangle is finite, and it is thus impossible to have a straight line embedding for a spherical complex like an octahedron or any polyhedron.

Shellability

Shellability of simplicial complexes is a topological notion used for example in planarity problems [69, 71]. Shellable simplicial complexes have got topological properties which makes this notion important. The idea is to remove facets of a pure simplicial complex one by one in a “good” order.

Definition 25. *A pure simplicial complex Δ of dimension d is shellable if its $(d-1)$ -faces can be ordered F_1, \dots, F_s in such a way that for every $1 < j \leq s$, the simplicial complex $(\bigcup_{i=1}^{j-1} \text{Subsets}(F_i)) \cap \text{Subsets}(F_j)$ is pure of dimension $d-2$.*

According to the following theorem, supremum sections of standard representations give example of shellable simplicial complexes.

Theorem 26 (Ossona de Mendez [92]). *Let R be a standard representation. Then $\Sigma(R)$ is shellable.*

As shellability is only defined on pure simplicial complexes, it is impossible to extend Theorem 26 to any supremum sections as they are not necessarily pure.

Contact complexes of boxes

The intersection graphs of interior disjoint axis-parallel rectangles in the plane have been studied. Such a contact system of rectangles is said to be proper if at most three rectangles intersect on a point. A tiling is a contact system of axis-parallel rectangles contained in the square $[0, 1] \times [0, 1]$ such that every point of the square belongs to at least one of the rectangles. See Figure 1.7 for an example of a box tiling.

Thomassen [103] found a characterization of the intersection graphs of proper tilings, exactly as the strict subgraphs of 4-connected planar triangulations. A natural question which arises is to determine the Dushnik-Miller dimension of intersection complexes of interior disjoint axis-parallel boxes in \mathbb{R}^d . Francis and Gonçalves proved that if the boxes form a “framed tiling” of \mathbb{R}^d such that no $d+2$ boxes intersect at some point, then the associated intersection complex has Dushnik-Miller dimension at most $d+1$ [66].

Cp-orders of orthogonal surfaces

Orthogonal surfaces were introduced by Scarf in [98]. They are defined as follows. Given a finite set of points V in \mathbb{R}^d , the orthogonal surface S_V is the topological boundary of the set $\langle V \rangle = \{x \in \mathbb{R}^d : \exists v \in V : v_i \leq x_i \forall i \in [1, d]\}$. When the considered points do not share coordinates, the points are said to be in general position and the orthogonal surface is said to be *generic*. For any subset U of V , we define the point p^U of \mathbb{R}^d by $p_i^U = \max_{x \in U} x_i$ for every $i \in [1, d]$. A *characteristic point* of a generic orthogonal surface is a point p^U which lies on the orthogonal surface. When equipping the set of these characteristic points with the *dominance order* (where a point $x \in \mathbb{R}^d$ is considered to

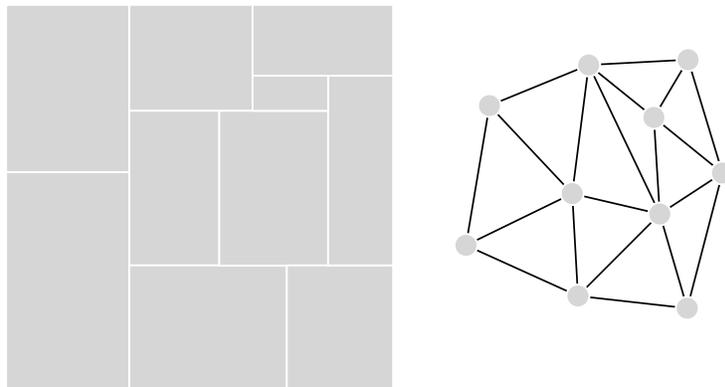


Figure 1.7: Example of a box tiling such that no set of 4 rectangles intersect on a point and its associated intersection graph.

be smaller than another point $y \in \mathbb{R}^d$ if and only if $x_i \leq y_i$ for every $i \in [1, d]$, we get the *cp-order* of a generic orthogonal surface. Scarf showed that cp-orders of some particular generic orthogonal surfaces are isomorphic to the face-lattice of some simplicial d -polytope with one facet removed. Two questions arise from this theorem. First, a realization problem: what are the polytopes whose face-lattice can be seen as a cp-order of some orthogonal surface? Secondly, how to generalize this result to non-generic orthogonal surface? Felsner and Kappes [67] found partial results on these two questions.

1.1.5 Organization of the chapter

In section 2, we prove technical lemmas on representations that are used in the other sections. These lemmas are necessary because any simplicial complex can have different representations. In this section, we prove that the different representations of a same simplicial complex are connected via elementary operations.

In section 3, we study the Dushnik-Miller dimension of TD-Delaunay complexes. We disprove Conjecture 19 stating that the class of TD-Delaunay complexes in \mathbb{R}^d corresponds to the class of Dushnik-Miller at most $d+1$ complexes. We provide a counter-example by considering a dual statement.

Section 4 deals with the topology of supremum sections. As we will see in section 3 and 5, supremum sections are connected to some geometric configurations. That is why, it is of particular interest to study their topology. *Collapsibility* is a topological notion introduced by Whitehead [107] which resembles shellability and which is defined on any simplicial complex. We prove in this section that any supremum section is collapsible thanks to the discrete Morse theory developed by Forman [68].

In section 5, we study the Dushnik-Miller dimension of contact complexes of *stairs*, whose shapes in the plane resemble “stairs”. In this study, we consider

three different arrangements of such stairs which give different results. Particularly, we show that any Dushnik-Miller dimension at most $d + 1$ complex is a contact complex of *stairs*.

1.2 Technical lemmas

In this section we prove technical lemmas on representations that will be used in the next section.

Lemma 27 (Ossona de Mendez [92]). *Let R be a standard d -representation on V . Let X be a face of dimension strictly smaller than $d - 1$ which contains an interior element of R . Then X is included in two different faces of dimension $\dim(X) + 1$.*

Proof. For any $i \in \llbracket 1, d \rrbracket$, let $x_i = \max_{\leq_i}(X)$ and let $M_i = \max_{\leq_i}(V)$. Without loss of generality, suppose that x_1 is an interior element of R .

As $|X| < d$, there exists $i < j$ which are indices from $\llbracket 1, d \rrbracket$ such that $x_i = x_j = u$. By definition of x_i and x_j , we have $x_1 \leq_i x_i$ and $x_1 \leq_j x_j$. As x_1 is an interior element, then $M_j <_i x_1$ and $M_i <_j x_1$. We conclude that $u \neq M_i$ and $u \neq M_j$. Furthermore for every k different from i , $M_k <_i x_1 \leq_i x_i = u$ and for every k different from j , $M_k <_j x_1 \leq_j x_j = u$ and then $u \neq M_k$ for every $k \in \llbracket 1, d \rrbracket$. We conclude that u is not an exterior element and that $u = x_i <_i M_i$.

Note that $X \not\leq_k M_i$ for any $k \neq i$ as x_1 is an interior element. We can therefore define a as the least element according to \leq_i such that $X <_i a$ and such that $X \not\leq_k a$ for every $k \neq i$. In the same way we define b as the least element according to \leq_j such that $X <_j b$ and such that $X \not\leq_k b$ for every $k \neq j$. We define $Y = X \sqcup \{a\}$.

Let us show that Y is a face of $\Sigma(R)$. First remark that by definition of a , $\max_{\leq_1} X = \max_{\leq_1} Y$ and that every element of X still dominates Y (because $x_i = x_j$ is still dominating Y in the order \leq_j). Furthermore a dominates Y in the order \leq_i .

Let u be an element not in Y . As X is a face of $\Sigma(R)$, there exists $k \in \llbracket 1, d \rrbracket$ such that $X <_k u$. Suppose that there exists $k' \neq i$ such that $X <_{k'} u$, then $Y <_{k'} u$. Otherwise $k = i$, and $X \not\leq_k u$. By minimality of a , we have $a <_k u$. Thus u dominates Y in the order \leq_k . We conclude that Y is a face of $\Sigma(R)$.

Furthermore, by symmetry $X \sqcup \{b\}$ is also a face of $\Sigma(R)$. \square

Lemma 28. *Let R be a standard d -representation on V and let X be a face of $\Sigma(R)$ which is included in $\{M_1, \dots, M_d\}$ where $M_i = \max_{\leq_i} V$ for every $i \in \llbracket 1, d \rrbracket$. The face X is included in one $(d - 1)$ -face. Furthermore, if $X \subseteq \{M_1, \dots, M_d\}$ is a $(d - 2)$ -face, then X is included in exactly one $(d - 1)$ -face.*

Proof. If there are no interior elements, then the only facet of $\Sigma(R)$ is the face $\{M_1, \dots, M_d\}$ and the proof is clear.

Otherwise, suppose that there exists interior elements. There exists $i \in \llbracket 1, d \rrbracket$ such that $M_i \notin X$ as $\{M_1, \dots, M_d\}$ is not a face. We define a as the least element of V according to the order \leq_i which is not an exterior element. We define

$Y = \{M_1, \dots, M_d, a\} \setminus M_i$. Let us show that Y is a face of $\Sigma(R)$. Every element of Y is dominating Y . As $\max_{\leq_i} Y = a$, every element not in Y is dominating Y in \leq_i . We conclude that Y is a $(d-1)$ -face of $\Sigma(R)$ which includes X .

Suppose now that X is a $(d-2)$ -face. Without loss of generality we can suppose that $X = \{M_2, \dots, M_d\}$. Let Y be a $(d-1)$ -face of $\Sigma(R)$ which contains X . Then there exists $b \in V$ different from M_2, \dots, M_d such that $Y = X \sqcup \{b\}$. We note a the least element of V according to \leq_i which is not an exterior element. As a must dominate Y , then $Y \leq_1 a$. We deduce that $b \leq_1 a$. As b cannot be M_1 (otherwise a could not dominate Y), then b is an interior element. By minimality of a , we deduce that $a \leq_1 b$ and we conclude that $a = b$. We conclude that X is included in exactly one $(d-1)$ -face. \square

Corollary 29. *For every standard d -representation R on V , $\Sigma(R)$ is pure of dimension $d-1$. That means that every face of $\Sigma(R)$ is included in a $(d-1)$ -face.*

Lemma 30. *Let R be a d -representation on V such that every element of V is a vertex of $\Sigma(R)$. The representation R is standard if and only if there exist vertices M_1, \dots, M_d such that in $\Sigma(R)$ every face belongs to at least one $(d-1)$ -face, every $(d-2)$ -face belongs to at least two $(d-1)$ -faces except the $(d-2)$ -faces $F_i = \{M_1, \dots, M_d\} \setminus M_i$ which belong to only one $(d-1)$ -face.*

Proof. (\implies) Clear from Lemma 27 and Lemma 28.

(\impliedby) Suppose that there exist vertices M_1, \dots, M_d such that every face of $\Sigma(R)$ belongs to at least one $(d-1)$ -face, and such that every $(d-2)$ -face of $\Sigma(R)$ belongs to at least two $(d-1)$ -faces except the $(d-2)$ -faces $F_i = \{M_1, \dots, M_d\} \setminus M_i$ which belong to only one $(d-1)$ -face.

For any $i \in \llbracket 1, d \rrbracket$ we define F'_i as a $(d-1)$ -face minimizing $\max_{\leq_i}(F'_i)$ in \leq_i among the other $(d-1)$ -faces. Let us denote $f'_{i,j}$ the element of F'_i that is maximal in \leq_j . As each of the d elements of F'_i dominates it at least once, $F'_i = \{f'_{i,1}, \dots, f'_{i,d}\}$. Let us show that $F'_i \setminus \{f'_{i,i}\}$, which is a $(d-2)$ -face, does not belong to any other $(d-1)$ -face. Indeed, if there exists a vertex $x \neq f'_{i,i}$ such that $F'_i \setminus \{f'_{i,i}\} \cup \{x\}$ is a $(d-1)$ -face then $F'_i <_i x$ by definition of F'_i , and $f'_{i,i}$ cannot dominate $F'_i \setminus \{f'_{i,i}\} \cup \{x\}$ (as $f'_{i,i} <_i x$ and $f'_{i,i} <_j f'_{i,j}$ for any $j \neq i$), a contradiction.

Suppose that there exists i and $j \neq i$ such that $F'_i \setminus \{f'_{i,i}\} = F'_j \setminus \{f'_{j,j}\}$. We define $X = F'_i \setminus \{f'_{i,i}\}$. As X is included in F'_i and in F'_j , and as it belongs to only one $(d-1)$ -face F'_i , we have that $F'_i = F'_j$ and thus that $f'_{i,i} = f'_{j,j}$. Then in the $(d-1)$ -face $F'_i = F'_j$, the vertex $f'_{i,i} = f'_{j,j}$ thus dominates F'_i in two orders, $<_i$ and $<_j$, and one of the remaining $d-1$ elements of F'_i cannot dominate it, a contradiction. We conclude that the faces $F'_i \setminus \{f'_{i,i}\}$ are distinct and are in bijection with the faces F_i . By symmetry of the F'_i s, we can assume that $F_i = F'_i \setminus \{f'_{i,i}\}$ and thus that $f'_{i,i} = M_i$ for every i .

We now show that the M_i 's are the maxima of the representation. Let $i \in \llbracket 1, d \rrbracket$ and $j \neq i$. Since $M_i \in F_j$, none of the $(d-1)$ -faces is dominated by M_i in the order \leq_j . Indeed if it was the case then we would have a $(d-1)$ -face F'' such that $F'' \leq_j M_i <_j f'_{j,j} (= M_j)$ which contradicts the definition of F'_j . Thus $F'' \leq_i M_i$ for every $(d-1)$ -face F'' . For every element x , as x is included

in at least one $(d-1)$ -face, we have that $x \leq_i M_i$. We conclude that M_i is the maximum of \leq_i .

We now show that M_i is among the $d-1$ smallest elements of the order \leq_j for every $j \neq i$. Since the face F_j contains M_i for every $i \neq j$, none of the elements in $V \setminus F_j$ dominates F_j in \leq_i . Thus each of the $|V| - (d-1)$ elements of $V \setminus F_j$ dominates F_j in the order \leq_j . The element M_i is thus among the $d-1$ smallest elements in order \leq_j . The representation R is thus standard. \square

Λ -equivalent representations

The following lemmas are technical and will be useful for the proof of Theorem 44 dealing with TD-Delaunay complexes. The goal here is to study the impact on the representation of an elementary transposition of two consecutive elements in one order of the representation. In the following R is a d -representation (\leq_1, \dots, \leq_d) on a set V and R' is a d -representation $(\leq'_1, \dots, \leq'_d)$ on the same set V .

Definition 31. *Two d -representations R and R' are Σ -equivalent if $\Sigma(R) = \Sigma(R')$. Two such representations are Λ -equivalent if, furthermore, for any $\{x, y\} \in \Sigma(R) = \Sigma(R')$, $x \leq_i y$ if and only if $x \leq'_i y$.*

Note that in the case where R and R' are Σ -equivalent, for any $i \in \llbracket 1, d \rrbracket$, the orders \leq_i and \leq'_i may differ on the comparison of two elements x and y if $\{x, y\} \notin \Sigma(R) = \Sigma(R')$.

Given a d -representation R , an (\leq_i) -increasing xy -path is a path $(x = a_0, a_1, a_2, \dots, a_k = y)$ in $\Sigma(R)$ such that $a_j \leq_i a_{j+1}$ for every $0 \leq j < k$. Given two Λ -equivalent d -representations R and R' , note that a path P is (\leq_i) -increasing (in $\Sigma(R)$) if and only if it is (\leq'_i) -increasing (in $\Sigma(R')$). Note also that for any face $F \in \Sigma(R)$, $\max_{\leq_i} \{x \in F\} = \max_{\leq'_i} \{x \in F\}$.

Definition 32. *Let $R = (\leq_1, \dots, \leq_d)$ be a d -representation on V . Let x and y be two distinct vertices of $\Sigma(R)$ (i.e. $\{x\}$ and $\{y\} \in \Sigma(R)$) such that $\{x, y\} \notin \Sigma(R)$ and such that x and y are consecutive in an order \leq_i . We define $R' = (\leq'_1, \dots, \leq'_d)$ the d -representation on V obtained after the permutation of x and y in the order i . We say that we obtained R' by an elementary transposition.*

Lemma 33. *Two d -representations on V , $R = (\leq_1, \dots, \leq_d)$ and $R' = (\leq'_1, \dots, \leq'_d)$, are Λ -equivalent if and only if R' can be obtained from R after a sequence of elementary transpositions.*

Proof. Let us first prove that if R' is obtained from R by an elementary transposition of x and y in the order \leq_1 (with $x \leq_1 y$), then these two representations are Λ -equivalent.

We first prove that these two representations are Σ -equivalent. Towards a contradiction and by symmetry of R and R' , let us consider a face $F \in \Sigma(R)$ such that $F \notin \Sigma(R')$. There exists therefore a vertex $z \in V$ which does not dominate F in R' . As z dominates F in R , and as $\leq_i \equiv \leq'_i$ for every $i \neq 1$, we thus have that $F \leq_1 z$ and $F \not\leq'_1 z$. This implies that $z = y$ and $x \in F$. As x

and y are not adjacent, we have that $y \notin F$. Furthermore, y only dominates F in order \leq_1 of R . We denote by f_i the maximum vertex of F in order \leq_i , then $f_1 = x$ and $y <_i f_i$ for every $i \neq 1$.

As $\{x, y\} \notin \Sigma(R)$, there exists an element $w \in V$ such that $w <_i \max_{\leq_i}(x, y)$ for every i . Thus either $w = x$ or $w = y$, contradicting the fact that $\{y\} \in \Sigma(R)$ and $\{x\} \in \Sigma(R)$, or $w <_i f_i$ for every i (in particular for $i = 1$ because x and y are consecutive), contradicting the fact that $F \in \Sigma(R)$. So $F \in \Sigma(R')$, and $\Sigma(R) \subseteq \Sigma(R')$. By symmetry of the elementary transpositions we also have that $\Sigma(R') \subseteq \Sigma(R)$. Thus $\Sigma(R) = \Sigma(R')$. It is now easy to check that for any edge $\{a, b\} \in \Sigma(R)$ and for any i , $a \leq_i b$ if and only if $a \leq'_i b$. These two representations are thus Λ -equivalent.

Let us now prove that for any two Λ -equivalent representations R and R' , the representation R' can be obtained from R after a sequence of elementary transpositions. Let i be the smallest integer such that $\leq_i \neq \leq'_i$. Let also x be the highest element in \leq'_i that is ranked differently in \leq_i . Finally let y be the element just above x in \leq_i . As $y \leq'_i x$ we have that xy is not an edge and we can thus perform an elementary transposition in \leq_i between x and y . This operation brings x closer to its rank in \leq'_i . It is clear that continuing this process eventually leads to R' . \square

The following lemma tells us how much a vertex x can be lifted in an order \leq_i of a d -representation R , while remaining Λ -equivalent.

Lemma 34. *Let R be a d -representation on V , let x be a vertex of $\Sigma(R)$ (i.e. $\{x\} \in \Sigma(R)$), and let \leq_i be any order of R . There exists a d -representation R' Λ -equivalent to R such that:*

- $\leq'_j \equiv \leq_j$ for every $j \neq i$,
- $x \leq'_i y$ if and only if there exists an (\leq'_i) -increasing xy -path in $\Sigma(R')$ (thus if and only if there exists an (\leq_i) -increasing xy -path in $\Sigma(R)$), and
- $a \leq'_i b \leq'_i x$ implies that $a \leq_i b$.

Proof. We proceed by induction on n , the number of couples (y, z) such that $x \leq_i y <_i z$, such that there exists an (\leq_i) -increasing xy -path, and such that there is no (\leq_i) -increasing xz -path. In the initial case, $n = 0$, as for every vertex z such that $x \leq_i z$ there exists an (\leq_i) -increasing xz -path, we are done and $R' = R$.

If $n > 0$, consider such a couple (y, z) with the property that y and z are consecutive in \leq_i (by taking z as the lowest element in \leq_i such that there is no (\leq_i) -increasing xz -path). Note that $\{y, z\} \notin \Sigma(R)$ as otherwise, extending an (\leq_i) -increasing xy -path with the edge yz one obtains an (\leq_i) -increasing xz -path. By Lemma 33, the d -representation R'' obtained by permuting y and z is Λ -equivalent to R and such that $\leq'_j \equiv \leq_j$ for every $j \neq i$, such that for any two vertices a and b without (\leq'_i) -increasing xa -path nor xb -path, $a \leq'_i b$ if and only if $a \leq_i b$, and has only $n - 1$ couples (y', z') .

We can thus apply the induction hypothesis to R'' and we obtain that there exists a d -representation R' which is Λ -equivalent to R'' (and thus Λ -equivalent to R) such that $\leq_j'' \equiv \leq_j' \equiv \leq_j$ for every $j \neq i$, such that $x \leq_i'' y$ if and only if there exists an (\leq_i'') -increasing xy -path in $\Sigma(R')$, and such that $a \leq_i'' b \leq_i'' x$ implies that $a \leq_i' b$, which implies $a \leq_i b$ (as there is no (\leq_i') -increasing xa -path nor xb -path). \square

The following lemma tells us that every vertex is connected to every face of a supremum section by an \leq_i -increasing path for some i .

Lemma 35. *Let R be a d -representation on V . For any face $F \in \Sigma(R)$ and any vertex x of $\Sigma(R)$, there exists an (\leq_i) -increasing $f_i x$ -path for some order $\leq_i \in R$, where f_i is the maximal vertex of F in order \leq_i .*

Proof. We proceed by induction on n , the number of orders $\leq_i \in R$ such that $F \leq_i x$. If $n = 0$ then $F \notin \Sigma(R)$, a contradiction. So the lemma holds by vacuity. If $n > 0$, consider an order $\leq_i \in R$ such that $F \leq_i x$. By Lemma 34 (applied to f_i in \leq_i) either $\Sigma(R)$ contains an (\leq_i) -increasing $f_i x$ -path, and we are done, or there exists a d -representation R' which is Λ -equivalent to R such that $\leq_j' \equiv \leq_j$ for every $j \neq i$, and such that $x <_i' f_i$. In this case, we apply the induction hypothesis on R' . Indeed, $F \in \Sigma(R')$ and R' has only $n - 1$ orders \leq_j' such that $F \leq_j' x$. Note that as R and R' are Λ -equivalent, the maximal vertices of F in \leq_j' and \leq_j are the same for every j . The induction thus provides us an (\leq_j') -increasing $f_j x$ -path and this path is also (\leq_j) -increasing in $\Sigma(R)$. \square

Lemma 36. *Let R be a d -representation on V . For any vertex set $F \notin \Sigma(R)$, there exists a vertex x which does not dominate F , and such that for every $\leq_i \in R$ there exists an (\leq_i) -increasing $x f^i$ -path for some vertex $f^i \in F$.*

Proof. We proceed by induction on the number n of elements which do not dominate F . If $n = 0$ then $F \in \Sigma(R)$, a contradiction. So the lemma holds by vacuity. If $n > 0$, consider any such element x which does not dominate F . By Lemma 34 either there are (\leq_i) -increasing $x f^i$ -paths for every $\leq_i \in R$ and we are done, or there exists a d -representation R' which is Λ -equivalent to R such that $F \leq_i' x$ for some order $\leq_i' \in R'$, such that $a \leq_i' b \leq_i' x$ implies that $a \leq_i b$, and such that $\leq_j' \equiv \leq_j$ for every $j \neq i$.

In this case, we apply the induction hypothesis on R' . Indeed, as any element not dominating F in R' does not dominate F in R , and as x is not dominating F in R' , we have at most $n - 1$ elements that do not dominate F in R' . By induction hypothesis there exists a vertex x such that for every $\leq_i' \in R'$, $\Sigma(R')$ has an (\leq_i') -increasing $x f^i$ -path for some vertex $f^i \in F$, and this path is also (\leq_i) -increasing in $\Sigma(R)$. \square

1.3 TD-Delaunay complexes

The goal of this section is to give an explicit counter-example to Conjecture 19. We start by defining formally the class of TD-Delaunay complexes. In the second

subsection we introduce the notion of *multi-flows* that will be used in the next subsection where we disprove Conjecture 19. Finally in the forth subsection, we study the connections of our work with Rectangular Delaunay complexes.

1.3.1 TD-Delaunay complexes

The coordinates of a point $x \in \mathbb{R}^d$ will be denoted by (x_1, \dots, x_d) . For any integer d , let H_d be the $(d-1)$ -dimensional hyperplane of \mathbb{R}^d defined by $\{x \in \mathbb{R}^d : x_1 + \dots + x_d = 0\}$. Given $c = (c_1, \dots, c_d) \in \mathbb{R}^d$, we define a *regular simplex* S_c of H_d by setting $S_c = \{u \in H_d : u_i \leq c_i, \forall i \in \llbracket 1, d \rrbracket\}$. A regular simplex S_c is non-empty if $\sum_{i=1}^d c_i \geq 0$. For $c = (1, \dots, 1) = \mathbf{1}$ we call $S_{\mathbf{1}}$ the *canonical regular simplex*. In this context, a point set $\mathcal{P} \subseteq H_d$ is in *general position* if for any two vertices $x, y \in \mathcal{P}$, $x_i \neq y_i$ for every $i \in \llbracket 1, d \rrbracket$. The *interior* \dot{S}_c of a regular simplex S_c is defined by $\dot{S}_c = \{u \in H_d : u_i < c_i, \forall i \in \llbracket 1, d \rrbracket\}$.

We recall that a *positive homothety* h of \mathbb{R}^d is an affine transformation of \mathbb{R}^d defined by $h(M) = \alpha M + (1 - \alpha)\Omega$ where $\alpha \geq 0$ and $\Omega \in \mathbb{R}^d$. We say that a subset A of \mathbb{R}^d is *positively homothetic* to another subset B of \mathbb{R}^d if there exists a positive homothety h such that $A = h(B)$.

Proposition 37. *The regular simplices of H_d are the subsets of H_d positively homothetic to $S_{\mathbf{1}}$.*

Proof. Let U be a subset of H_d positively homothetic to $S_{\mathbf{1}}$. Let us show that there exists $c \in \mathbb{R}^d$ such that $U = S_c$ and $\sum_{i=1}^d c_i \geq 0$. There exists a positive homothety h of \mathbb{R}^d of ratio α and center $\Omega \in H_d$ such that $h(S_{\mathbf{1}}) = U$. We define $c = (c_1, \dots, c_d) \in \mathbb{R}^d$ by $c_i = \alpha + (1 - \alpha)\Omega_i$. We show that $h(S_{\mathbf{1}}) = S_c$. If $\alpha = 0$ it is clear that $h(S_{\mathbf{1}}) = \{\Omega\}$ and that $S_c = \{\Omega\}$. We can therefore suppose that $\alpha > 0$.

Let $u \in S_{\mathbf{1}}$. Then $u_i \leq 1$ for every $i \in \llbracket 1, d \rrbracket$. Then $h(u)_i = \alpha u_i + (1 - \alpha)\Omega_i \leq \alpha + (1 - \alpha)\Omega_i = c_i$. Furthermore $h(u) \in H_d$ as $h(u) \in U$. Then $h(u) \in S_c$ and thus $h(S_{\mathbf{1}}) \subseteq S_c$.

Let $v \in S_c$. Then $v_i \leq c_i$ for every $i \in \llbracket 1, d \rrbracket$. Because $\alpha \neq 0$, we define u such that $h(u) = v$, that is such that $\alpha u_i + (1 - \alpha)\Omega_i = v_i$. As $0 = \sum_{i=1}^d v_i = \alpha \sum_{i=1}^d u_i + (1 - \alpha) \sum_{i=1}^d \Omega_i = \sum_{i=1}^d u_i$, we deduce that $u \in H_d$. Furthermore for every $i \in \llbracket 1, d \rrbracket$, $\alpha u_i + (1 - \alpha)\Omega_i = v_i \leq \alpha + (1 - \alpha)\Omega_i$. Thus $u_i \leq 1$. So, $u \in S_{\mathbf{1}}$.

We conclude that $h(S_{\mathbf{1}}) = S_c$. We deduce that every subset of H_d which is positively homothetic to $S_{\mathbf{1}}$ is a regular simplex.

Let S_c be a regular simplex of H_d with $c \in \mathbb{R}^d$ such that $\sum_{i=1}^d c_i \geq 0$. We look for $\alpha \geq 0$ and $\Omega \in H_d$ such that $c = \alpha + (1 - \alpha)\Omega$. Suppose that such an $\alpha \geq 0$ and an Ω exist. Then $c_i = \alpha + (1 - \alpha)\Omega_i$ for every $i \in \llbracket 1, d \rrbracket$. Then $\sum_{i=1}^d c_i = d\alpha$. Thus $\alpha = (\sum_{i=1}^d c_i)/d \geq 0$. If $\alpha \neq 1$, then $\Omega_i = (c_i - \alpha)/(1 - \alpha)$. Otherwise if $\alpha = 1$, the point Ω can be defined as any point of H_d . It is easy to check that this gives the desired α and Ω and that they are well defined even if $\alpha = 0$ or 1 . We conclude that S_c is positively homothetic to $S_{\mathbf{1}}$ in H_d . \square

Let us now define TD-Delaunay simplicial complexes by extending the notion of TD-Delaunay graph defined by Chew and Drysdale [37].

Definition 38. *Given a set \mathcal{P} of points of H_d ($\subsetneq \mathbb{R}^d$) in general position, let the TD-Delaunay complex of \mathcal{P} , denoted by $\text{TDD}(\mathcal{P})$, be the simplicial complex with vertex set \mathcal{P} defined as follows. A subset $F \subseteq \mathcal{P}$ is a face of $\text{TDD}(\mathcal{P})$ if and only if there exists a regular simplex S such that $S \cap \mathcal{P} = F$ and such that no point of \mathcal{P} is in the interior of S .*

Let $F \subseteq \mathcal{P}$, we define $c^F \in \mathbb{R}^d$ by $c_i^F = \max_{x \in F} x_i$. Remark that $\sum_{i=1}^d c_i^F \geq 0$ because for every $x \in F$, $\sum_{i=1}^d c_i^F \geq \sum_{i=1}^d x_i = 0$.

Lemma 39. *Given a set \mathcal{P} of points of H_d in general position and a subset $F \subseteq \mathcal{P}$, $F \in \text{TDD}(\mathcal{P})$ if and only if S_{c^F} does not contain any point of \mathcal{P} in its interior.*

Proof. Suppose that $F \in \text{TDD}(\mathcal{P})$. Then there exists a regular simplex S_c such that $S \cap \mathcal{P} = F$ and such that no point of \mathcal{P} is in the interior of S . As S_c contains F , for every $x \in F$ and every $i \in \llbracket 1, d \rrbracket$, $x_i \leq c_i$. Thus $c_i^F \leq c_i$ for every i and $S_{c^F} \subseteq S_c$. Therefore S_{c^F} does not contain any point of \mathcal{P} in its interior, because otherwise S_c would contain some.

Suppose that S_{c^F} does not contain any point of \mathcal{P} in its interior. Let $x \in F$. By definition of c^F , $x_i \leq c_i^F$ for every i . So $x \in S_{c^F}$ and S_{c^F} contains F . Let $x \in \mathcal{P} \setminus F$. If x is in S_{c^F} , then x is not in the interior of S_{c^F} . So there exists $i \in \llbracket 1, d \rrbracket$ such that $x_i = c_i^F$. But there exists $y \in F$ (different from x) such that $y_i = c_i^F$. This contradicts the fact that the points of \mathcal{P} are in general position. Therefore $S_{c^F} \cap \mathcal{P} = F$ and $F \in \text{TDD}(\mathcal{P})$. \square

Proposition 40. *For any point set \mathcal{P} in general position in H_d ($\subsetneq \mathbb{R}^d$), $\text{TDD}(\mathcal{P})$ is an abstract simplicial complex.*

Proof. Consider any $F \in \text{TDD}(\mathcal{P})$, and any non-empty subset $G \subsetneq F$. Then $c_i^G \leq c_i^F$ for every i . So $S_{c^G} \subseteq S_{c^F}$. The regular simplex S_{c^G} does not contain any point of \mathcal{P} in its interior otherwise S_{c^F} would contain some. Thus because of the previous lemma, $G \in \text{TDD}(\mathcal{P})$ and we conclude that $\text{TDD}(\mathcal{P})$ is an abstract simplicial complex. \square

See Figure 1.3 for an example of a TD-Delaunay complex in the plane.

Consider a point set \mathcal{P} of \mathbb{R}^d in general position. We define the orders \leq_i on \mathcal{P} as $x \leq_i y$ if and only if $x_i \leq y_i$, in other words if and only if $x_i - y_i \leq 0$. First, note that as the points are in general position these orders are linear. Note also that the values x_i for $x \in \mathcal{P}$ and $i \in \llbracket 1, d \rrbracket$ form a solution of a linear system of inequalities (with inequalities of the form $x_i - y_i \leq 0$). In the following we connect TD-Delaunay complexes to representations through systems of inequalities. To do so we consider these d orders as a d -representation denoted by $R(\mathcal{P})$. If $\mathcal{P} \subset H_d$ this d -representation is closely related to $\text{TDD}(\mathcal{P})$.

Theorem 41. *Given a point set \mathcal{P} of $H_d \subsetneq \mathbb{R}^d$ in general position, we have that $\text{TDD}(\mathcal{P}) = \Sigma(R(\mathcal{P}))$. Thus, any TD-Delaunay complex of $H_d \simeq \mathbb{R}^{d-1}$ has Dushnik-Miller dimension at most d .*

Proof. Let us first prove that $\text{TDD}(\mathcal{P}) \subseteq \Sigma(R(\mathcal{P}))$, by showing that for any $F \in \text{TDD}(\mathcal{P})$ we have that $F \in \Sigma(R(\mathcal{P}))$. By definition there exists $c \in \mathbb{R}^d$ such that S_c contains exactly the points F , and they lie on its border. For every i , we denote by f_i the maximum among the elements of F with respect to \leq_i . Towards a contradiction we suppose that $F \notin \Sigma(R(\mathcal{P}))$. Thus there exists a vertex z of \mathcal{P} such that z does not dominate F in any order of $R(\mathcal{P})$. Thus $z <_i f_i$ for every i . Therefore $z_i < (f_i)_i$ for every i . But $(f_i)_i \leq c_i$ because $f_i \in S_c$ thus $z_i < c_i$. Hence $z \in \overset{\circ}{S}_c$ contradicting the choice of c .

Let us now prove that $\Sigma(R(\mathcal{P})) \subseteq \text{TDD}(\mathcal{P})$, by showing that for any $F \in \Sigma(R(\mathcal{P}))$ we have that $F \in \text{TDD}(\mathcal{P})$. Consider any non-empty face $F \in \Sigma(R(\mathcal{P}))$ (the case of the empty face is trivial) and suppose towards a contradiction that $F \notin \text{TDD}(\mathcal{P})$. According to Lemma 39, there exists $x \in \mathcal{P}$ such that $x \in \overset{\circ}{S}_{c^F}$. Thus $x_i < c_i^F$ for every i . For every i , we define f_i as the maximum among the elements of F with respect to \leq_i . Thus $x_i < c_i^F = (f_i)_i$, and $x <_i f_i$ for every i , which contradicts the fact that $F \in \Sigma(R(\mathcal{P}))$. \square

In the following we show that actually the reciprocal of the previous theorem, which is Conjecture 19, does not hold, already for $d = 4$. To do so, in the following we characterize which representations R are such that $\Sigma(R)$ is a TD-Delaunay complex. Actually we are first going to characterize which d -representations R correspond to some point set \mathcal{P} of H_d such that $R = R(\mathcal{P})$. By definition of $R(\mathcal{P})$, for any distinct $x, y \in \mathcal{P}$ and for any order $\leq_i \in R(\mathcal{P})$ we have that $x \leq_i y$, if and only if

$$y_i - x_i > 0 \tag{1.1}$$

Furthermore as we consider a point set \mathcal{P} of H_d we have that $\sum_{1 \leq i \leq d} x_i = 0$, which gives:

$$x_d = - \sum_{1 \leq i < d} x_i \tag{1.2}$$

In the following we consider a d -representation R , and we define the system of inequalities obtained by taking Inequality (1.1) for every $i \in \llbracket 1, d \rrbracket$ but only for the pairs $\{x, y\} \in \Sigma(R)$, and by replacing the d^{th} coordinates by the right hand of Equation (1.2).

Definition 42 (TD-Delaunay system). *Let R be a d -representation on a vertex set V , and consider the edge set E of $\Sigma(R)$ defined by $E = \{X \in \Sigma(R) : |X| = 2\}$. We define the matrix A_R of $\mathcal{M}_{E \times \llbracket 1, d \rrbracket, V \times \llbracket 1, d-1 \rrbracket}(\mathbb{R})$ where the coefficients, $a_{(e,i),(v,j)}$ of A_R are indexed by an edge $e \in E$, a vertex $v \in V$, and two indices $i \in \llbracket 1, d \rrbracket$ and $j \in \llbracket 1, d-1 \rrbracket$.*

$$a_{(e=\{x,y\},i),(v,j)} = \begin{cases} +1 & \text{if } i = j, v \in e \text{ and } v = \max_{\leq_j}(x, y) \\ -1 & \text{if } i = j, v \in e \text{ and } v = \min_{\leq_j}(x, y) \\ +1 & \text{if } i = d, v \in e \text{ and } v = \min_{\leq_d}(x, y) \\ -1 & \text{if } i = d, v \in e \text{ and } v = \max_{\leq_d}(x, y) \\ 0 & \text{otherwise} \end{cases}$$

The TD-Delaunay system of the representation R is the following linear system of inequalities :

$$A_R X > 0$$

for some vector $X \in \mathbb{R}^{V \times [1, d-1]}$.

Example 43. We consider the following 3-representation R on $V = \{a, b, c\}$:

$$\begin{array}{l|l} \leq_1 & b \quad c \quad a \\ \leq_2 & a \quad c \quad b \\ \leq_3 & a \quad b \quad c \end{array}$$

The complex $\Sigma(R)$ has one facet $\{a, b, c\}$ and contains 3 edges: ab, bc and ac . The matrix of the TD-Delaunay system of R is:

$$A_R = \begin{array}{l} (bc, 1) \\ (ac, 1) \\ (ab, 1) \\ (bc, 2) \\ (ac, 2) \\ (ab, 2) \\ (bc, 3) \\ (ac, 3) \\ (ab, 3) \end{array} \begin{pmatrix} (a, 1) & (b, 1) & (c, 1) & (a, 2) & (b, 2) & (c, 2) \\ & -1 & 1 & & & \\ 1 & & -1 & & & \\ 1 & -1 & & & & \\ & & & & 1 & -1 \\ & & & -1 & & 1 \\ & & & -1 & 1 & \\ & 1 & -1 & & 1 & -1 \\ 1 & & -1 & 1 & & -1 \\ 1 & -1 & & 1 & -1 & \end{pmatrix}$$

The system $A_R X > 0$ where $X \in \mathbb{R}^{V \times [1, d-1]}$ is equivalent to the following linear system, where v_i denotes $X_{(v, i)}$.

$$\left\{ \begin{array}{l} b_1 < c_1 \\ c_1 < a_1 \\ b_1 < a_1 \\ c_2 < b_2 \\ a_2 < c_2 \\ a_2 < b_2 \\ c_1 + c_2 < b_1 + b_2 \\ c_1 + c_2 < a_1 + a_2 \\ b_1 + b_2 < a_1 + a_2 \end{array} \right.$$

Note that setting $a_3, b_3,$ and c_3 to $-a_1 - a_2, -b_1 - b_2,$ and $-c_1 - c_2$ respectively, the last three equations imply that $b_3 < c_3, a_3 < c_3$ and $a_3 < b_3$.

Theorem 44. For any abstract simplicial complex Δ with vertex set V , Δ is a TD-Delaunay complex of $H_d \simeq \mathbb{R}^{d-1}$ if and only if there exists a d -representation R on V such that $\Delta = \Sigma(R)$ and such that the corresponding TD-Delaunay system has a solution.

Proof. (\Rightarrow) This follows from Theorem 41 and from the fact that the coordinates of any point set \mathcal{P} form a solution to the TD-Delaunay system defined by $R(\mathcal{P})$.

(\Leftarrow) Consider now a d -representation R on a set V such that the TD-Delaunay system of R has a solution $X \in \mathbb{R}^{V \times \llbracket 1, d-1 \rrbracket}$ and for any $v \in V$ let us define a point $v \in \mathbb{R}^d$ by setting $v_i = X_{(v,i)}$ for $i \in \llbracket 1, d-1 \rrbracket$, and $v_d = -v_1 - \dots - v_{d-1}$. This implies that all these points belong to H_d . We define \mathcal{P} as the set of points defined previously. It will be clear from the context when we refer to an element of V or to the corresponding point of \mathbb{R}^d . As in the linear system the inequalities are strict, one can slightly perturb the position of the vertices in order to obtain points in general position that still fulfill the system constraints. Recall that by construction, for any edge uv of $\Sigma(R)$ and any $i \in \llbracket 1, d \rrbracket$, $u_i < v_i$ if and only if $u <_i v$. This implies that if $\Sigma(R)$ has an (\leq_i) -increasing xy -path then $x_i \leq y_i$.

Let us first prove that $\text{TDD}(\mathcal{P}) \subseteq \Sigma(R)$. Consider a face $F \in \text{TDD}(\mathcal{P})$ and suppose towards a contradiction that $F \notin \Sigma(R)$. As $F \in \text{TDD}(\mathcal{P})$, there exists $c \in \mathbb{R}^d$ such that S_c contains exactly the points F , and they lie on its border. As $F \notin \Sigma(R)$, then by Lemma 36, there exists $x \in V$ which does not dominate F , and such that for every $\leq_i \in R$ there exists an (\leq_i) -increasing xf^i -path for some vertex $f^i \in F$. Therefore $x_i \leq (f^i)_i \leq c_i$. As x does not dominate F and as the points are in general position one of these inequalities is strict and we conclude that x lies in the interior of S_c , a contradiction.

Let us now prove that $\Sigma(R) \subseteq \text{TDD}(\mathcal{P})$. Consider any non-empty face $F \in \Sigma(R)$. For every $i \in \llbracket 1, d \rrbracket$, we denote by f_i the maximum of F in the order \leq_i and we define $c \in \mathbb{R}^d$ (and S_c) by setting $c_i = (f_i)_i$. First note that for any vertex $u \in F$ and any $i \in \llbracket 1, d \rrbracket$, as uf_i is an edge, we have that $c_i = (f_i)_i \geq u_i$. We hence have that $\sum_{i=1}^d c_i \geq \sum_{i=1}^d u_i = 0$. As $F \in \Sigma(R)$, for every $u \in F$ there exists an $i \in \llbracket 1, d \rrbracket$ such that $u = f_i$. Therefore $u_i = (f_i)_i = c_i$ and as $u_j \leq c_j$ for every j (because either $u = f_j$ or $\{u, f_j\} \in \Sigma(R)$ and then $u_j \leq (f_j)_j = c_j$), we have that u is on the border of S . According to Lemma 35, for every $u \notin F$, there exists an (\leq_i) -increasing $f_i u$ -path in $\Sigma(R)$, for some order $\leq_i \in R$. Therefore $c_i = (f_i)_i < u_i$ and $u \notin S_c$. Thus $F \in \text{TDD}(\mathcal{P})$ and $\Sigma(R) \subseteq \text{TDD}(\mathcal{P})$. \square

1.3.2 Multi-flows

We disprove Conjecture 19 using Theorem 44 by exhibiting a simplicial complex Δ , such that $\Delta = \Sigma(R)$ for some 4-representation R , and such that none of the Σ -equivalent 4-representations R' admits a solution to its TD-Delaunay system. A common tool to prove that a system of inequalities has no solution is a theorem from Gordan [73] which is a variant of Farkas' lemma.

Lemma 45 (Gordan). *For any $m \times n$ real matrix A , either*

- $Ax > 0$ admits a solution $x \in \mathbb{R}^n$, or
- ${}^t Ay = 0$ admits a non-zero solution $y \in (\mathbb{R}_{\geq 0})^n$.

Furthermore both cases are exclusive.

In the following we show that this lemma defines a dual notion of a TD-Delaunay solution, we call it a *multi-flow*. To define it, we first need to recall some notions about flows. Let $G = (V, A)$ be a digraph with vertex set V and arc set $A \subseteq V \times V$. A flow on G is a function of $\varphi : A \rightarrow \mathbb{R}_{\geq 0}$. Let the divergence $\text{div}_{\varphi}(x)$ of a vertex x be given by $\text{div}_{\varphi}(x) = \sum_{(y,x) \in A} \varphi(y,x) - \sum_{(x,y) \in A} \varphi(x,y)$.

Definition 46 (Multi-flow). *Let $R = \{\leq_1, \dots, \leq_d\}$ be a d -representation on V . For $i \in \llbracket 1, d \rrbracket$, $G^i(R)$ will denote the digraph with vertex set V and arc set A^i where $A^i = \{(x,y) \in V \times V : \{x,y\} \in \Sigma(R), x \leq_i y\}$. A multi-flow is a collection of d flows $\varphi_1, \dots, \varphi_d$ respectively on each digraph $G^1(R), \dots, G^d(R)$ such that $\text{div}_{\varphi_i}(v) = \text{div}_{\varphi_d}(v)$, for every $v \in V$ and every $i \in \llbracket 1, d \rrbracket$.*

We can now state the main result of this section.

Proposition 47. *Let R be a d -representation on a set V . Then either*

- *the corresponding TD-Delaunay system admits a solution, or*
- *R admits a non-zero multi-flow (i.e. a multi-flow with some $\varphi_i(uv) > 0$).*

Furthermore both cases are exclusive.

Proof. According to Theorem 44, the corresponding TD-Delaunay system admits a solution if and only if $A_R x > 0$ admits a solution $x \in \mathbb{R}^{V \times \llbracket 1, d-1 \rrbracket}$. According to Lemma 45, it thus remains to show that ${}^t A_R y = 0$ admits a non-zero solution $y \in (\mathbb{R}_{\geq 0})^{E \times \llbracket 1, d \rrbracket}$ if and only if R admits a non-zero multi-flow.

Suppose now that the system ${}^t A_R y = 0$ admits a non-zero solution $y \in (\mathbb{R}_{\geq 0})^{E \times \llbracket 1, d \rrbracket}$. Then $y_{(e,i)} \geq 0$ for all $i \in \llbracket 1, d \rrbracket$ and all $e \in E$. For any $i \in \llbracket 1, d \rrbracket$, we define the flow φ_i on $G^i(R)$ by setting $\varphi_i(a) = y_{(e,i)}$ for every $a \in A^i$, where e is the edge of E corresponding to the arc a . Since ${}^t A_R y = 0$, the following equations hold for every $v \in V$ and every $j \in \llbracket 1, d-1 \rrbracket$,

$$\begin{aligned} \sum_{(e,i) \in E \times \llbracket 1, d \rrbracket} a_{(e,i),(v,j)} y_{(e,i)} &= 0 \\ \sum_{(e,i) \in E \times \llbracket 1, d-1 \rrbracket} a_{(e,i),(v,j)} y_{(e,i)} &= - \sum_{(e,i) \in E \times \{d\}} a_{(e,i),(v,j)} y_{(e,i)} \end{aligned}$$

Since $a_{(e,i),(v,j)} = 0$ whenever $i \neq j$ and $i \neq d$,

$$\sum_{e \in E} a_{(e,j),(v,j)} y_{(e,j)} = - \sum_{e \in E} a_{(e,d),(v,j)} y_{(e,d)}$$

By definition of A_R ,

$$\sum_{\substack{e=\{u,v\} \in E \\ \text{s.t. } u \leq_j v}} y_{(e,j)} - \sum_{\substack{e=\{u,v\} \in E \\ \text{s.t. } v \leq_j u}} y_{(e,j)} = \sum_{\substack{e=\{u,v\} \in E \\ \text{s.t. } u \leq_d v}} y_{(e,d)} - \sum_{\substack{e=\{u,v\} \in E \\ \text{s.t. } v \leq_d u}} y_{(e,d)}$$

Finally by definition of φ_i ,

$$\begin{aligned} \sum_{(u,v) \in A^j} \varphi_j(u,v) - \sum_{(v,u) \in A^j} \varphi_j(v,u) &= \sum_{(u,v) \in A^d} \varphi_d(u,v) - \sum_{(v,u) \in A^d} \varphi_d(v,u) \\ \operatorname{div}_{\varphi_j}(v) &= \operatorname{div}_{\varphi_d}(v) \end{aligned}$$

We conclude that $(\varphi_1, \dots, \varphi_d)$ is a non-zero multi-flow of R .

To prove the converse statement suppose that R admits a non-zero multi-flow $(\varphi_1, \dots, \varphi_d)$. We define $y \in \mathbb{R}^{E \times [1, d]}$ by setting $y_{(e,i)} = \varphi_i(a)$, for any $e \in E$ and $i \in [1, d]$, and where a is the arc of $G^i(R)$ corresponding to the edge e . Clearly $y \in (\mathbb{R}_{\geq 0})^{E \times [1, d]}$, and the multi-flow being non-zero, y is non-zero. As $\operatorname{div}_{\varphi_j}(v) = \operatorname{div}_{\varphi_d}(v)$ for every $v \in V$ and every $j \in [1, d-1]$, one can deduce (by reversing the above calculus) that ${}^t A_R y = 0$. We thus have that y is a non-zero solution to ${}^t A_R y = 0$. \square

1.3.3 A counter-example to Conjecture 19

Theorem 48. *Let R be the following 4-representation on the set V defined as $\{a, b, c, d, e, f, g, h\}$:*

$$\begin{array}{l|cccccccc} \leq_1 & b & c & d & e & g & f & h & a \\ \leq_2 & a & c & d & e & h & f & g & b \\ \leq_3 & a & b & d & f & g & e & h & c \\ \leq_4 & a & b & c & f & h & e & g & d \end{array}$$

The simplicial complex $\Delta = \Sigma(R)$ has Dushnik-Miller dimension 4 but it is not a TD-Delaunay complex of $H_4 \simeq \mathbb{R}^3$.

Proof. Let us first show that any Σ -equivalent 4-representation R' is R up to permutations of the orders and up to a permutation of the smallest 3 elements in each order.

Let R' be a 4-representation on V such that $\Sigma(R') = \Delta$. As R is standard, then because of Lemma 30 there exists vertices M_1, \dots, M_d satisfying the properties described in Lemma 30. Thus the same vertices satisfy also these properties for $\Sigma(R') = \Sigma(R)$. We deduce from the same Lemma that R' is standard with maximal elements a, b, c , and d . Without loss of generality we assume that these elements are maximal in $\leq'_1, \leq'_2, \leq'_3$, and \leq'_4 , respectively. As $\{e, b, c, d\} \in \Delta$ and as none of a, e, f, g, h dominates this face in \leq'_2, \leq'_3 , or \leq'_4 , e is necessarily the fourth smallest element in \leq'_1 . We similarly deduce that e is also the fourth smallest element in \leq'_2 (using the face $\{a, c, d, e\}$ of Δ), and that f is the fourth smallest element in \leq'_3 and \leq'_4 (using the faces $\{a, b, d, f\}$ and $\{a, b, c, f\}$ respectively). As f dominates $\{b, e, g\} \in \Delta$, we have that $g \leq'_1 f$. We similarly deduce that $h \leq'_2 f$, $g \leq'_3 e$, and $h \leq'_4 e$ (using faces $\{h, e, a\}$, $\{g, d, f\}$, and $\{c, h, f\}$). As h dominates $\{c, e, f\} \in \Delta$, we have that $f \leq'_1 h$. We similarly deduce that $f \leq'_2 g$, $e \leq'_3 h$, and $e \leq'_4 g$ (using faces $\{d, e, f\}$, $\{a, e, f\}$, and $\{b, e, f\}$).

This implies that for any such R' the subdigraphs of $G^i(R')$ induced by the vertices $e, f, g,$ and h are the same as in $G^i(R)$. These digraphs are depicted in Figure 1.8, as well as a multi-flow of R' . Vertices $a, b, c,$ and d are not drawn as the flows on their incident edges are null. Their divergences are thus 0. The divergences of the vertices e and f is -1 and it is $+1$ for g and f .

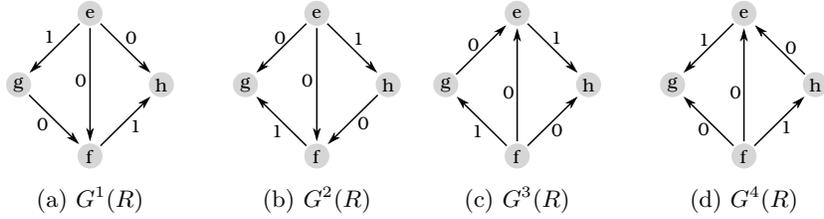


Figure 1.8: Definition of the multiflow on $G(R)$.

Thus, as every 4-representation R' which is Σ -equivalent to R has a multi-flow, Proposition 47 and Theorem 44 imply that Δ is not a TD-Delaunay complex of $H_4 \simeq \mathbb{R}^3$. \square

1.3.4 Link with Rectangular Delaunay complexes

Rectangular Delaunay graphs have been studied independently by Felsner [65] and by Chen *et al.* [44]. In the following, \mathcal{P} denotes a finite set of points of \mathbb{R}^2 such that no two points share the same vertical or horizontal coordinate.

Definition 49. We define the R-Delaunay complex $\text{RD}(\mathcal{P})$ of the point set \mathcal{P} as the simplicial complex whose vertex set is \mathcal{P} such that a subset F of \mathcal{P} forms a face if there exists an axis-parallel rectangle R such that $R \cap \mathcal{P} = F$ and such that R does not contain any point of \mathcal{P} in its interior. The R-Delaunay graph of a point set \mathcal{P} is the graph defined by the faces of size one and two in $\text{RD}(\mathcal{P})$.

See an example on Figure 1.9.

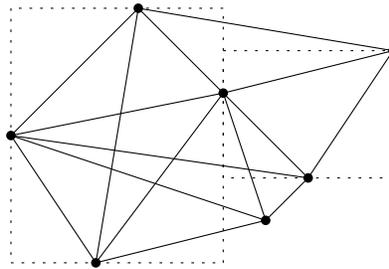


Figure 1.9: An example of a R-Delaunay graph where two construction rectangles are drawn in dashed segments.

R-Delaunay graphs have interesting properties. Felsner [65] showed that those graphs can have a quadratic number of edges. In order to solve a question

motivated by a frequency assignment problem in cellular telephone networks and related to conflict-free colorings [51], Chen *et al.* showed the following result.

Theorem 50 (Chen *et al.* [44]). *Given points \mathcal{P} in the unit square selected randomly and uniformly, the probability that the largest independent set of $RD(\mathcal{P})$ is $O(n \log^2 \log(n) / \log(n))$ tends to 1.*

This implies that R-Delaunay graphs can have arbitrarily large chromatic number.

Given a point set \mathcal{P} , we define the horizontal order \leq_1 on \mathcal{P} as follows: for every two points $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathcal{P}$, $x \leq_1 y$ if and only if $x_1 \leq y_1$. Then we define \leq_2 as the reverse of the horizontal order. We define the vertical order \leq_3 on \mathcal{P} as previously: here $x \leq_3 y$ if and only if $x_2 \leq y_2$; and we define \leq_4 as the reverse of the vertical order.

One can show that using these four orders one obtains a representation R such that $RD(\mathcal{P}) = \Sigma(R)$ and actually any 4-representation, where two pairs of orders are the reverse from each other, defines an R-Delaunay complex. Fel'sner [65] showed this for graphs but his proof easily extends to simplicial complexes. In his words, R-Delaunay complexes are exactly the complexes of dimension $[3 \updownarrow 4]$. This implies that R-Delaunay complexes form a subclass of the simplicial complexes with Dushnik-Miller dimension at most 4. The following theorem refines this by showing that it is also a subclass of TD-Delaunay complexes of $H_4 \simeq \mathbb{R}^3$.

Theorem 51. *The class of R-Delaunay complexes is included in the class \overline{TDD}_4 of TD-Delaunay complexes of $H_4 \simeq \mathbb{R}^3$.*

Proof. We want to show that $RD(\mathcal{P})$ is a TD-Delaunay complex. According to Theorem 44, it is enough to show that the TD-Delaunay system of inequalities defined by R , where R is the previously defined 4-representation, has a solution. By Proposition 47, this is equivalent to showing that there is no non-zero multi-flow $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ on R .

Consider x the rightmost point: it is the greatest point according to order \leq_1 . So x is a sink in $G^1(R)$ and $\text{div}_{\varphi_1}(x) \geq 0$ for any flow φ_1 on $G^1(R)$. As \leq_2 is order \leq_1 reversed, x is also the minimum in order \leq_2 . So x is a source in $G^2(R)$ and $\text{div}_{\varphi_2}(x) \leq 0$ for any flow φ_2 on $G^2(R)$. So if $\text{div}_{\varphi_1}(x) = \text{div}_{\varphi_2}(x)$ we have that this divergence is null, as well as φ_1 and φ_2 on the arcs incident to x . By induction on the number of vertices, we show that all the divergences defined by φ_1 (and φ_2) are null. Note that a flow φ defined on an acyclic digraph (such as $G^1(R), G^2(R), G^3(R)$, and $G^4(R)$) with divergence zero at every vertex, is null everywhere (*i.e.* for each arc a , $\varphi(a) = 0$). Therefore R has no non-zero multi-flow, and the system of inequalities defined by R admits a solution and $RD(\mathcal{P}) = \Sigma(R)$. The simplicial complex $RD(\mathcal{P})$ is thus a TD-Delaunay complex of $H_4 \simeq \mathbb{R}^3$. \square

Note that these classes are distinct as $K_5 \in \overline{TDD}_4$ is not an R-Delaunay graph. Another way to prove the above theorem is by showing that in \mathbb{R}^3 , any rectangle with sides parallel to the axis and drawn on the plane defined by $z = 0$,

is the intersection between this plane and a regular simplex homothetic to the simplex with vertices $(-2, 0, +\sqrt{2})$, $(0, -2, -\sqrt{2})$, $(+2, 0, +\sqrt{2})$, $(0, +2, -\sqrt{2})$. Such a proof would also be simple but it would involve a few calculus that were avoided in the proof above.

Note that the arguments in the proof of Theorem 51 show that any d -representation R , where two orders are the reverse from each other, is such that its TD-Delaunay system has a non-zero solution and thus $\Sigma(R)$ is a TD-Delaunay complex of $H_d \simeq \mathbb{R}^{d-1}$.

1.4 Collapsibility of supremum sections

The goal of this section is to generalize Theorem 26 about the shellability of standard supremum sections to every supremum sections. As there exists supremum sections which are not shellable, for instance the simplicial complex whose facets are $\{a, b, c\}$ and $\{c, d, e\}$, we replace shellability with collapsibility which is a similar notion.

A collapse is a topological operation on simplicial complexes, and more generally on CW-complexes, introduced by Whitehead [107] in order to define a simple homotopy equivalence which is a refinement of the homotopy equivalence. A complex is said to be collapsible if it collapses to a point. The discrete Morse theory introduced by Forman [68] is based on this notion and has numerous applications in applied mathematics and computer science. Homotopy equivalence is a topological notion of topological spaces introduced to classify topological spaces. Roughly speaking, two spaces are said to be homotopy equivalent if there exists a continuous deformation from one to the other. A topological space is said to be contractible if it is homotopy equivalent to a point. Collapsible spaces form an important subclass of contractible spaces. While contractibility is algorithmically undecidable by a result of Novikov [105], the subclass of collapsible spaces is algorithmically recognizable. More precisely Tancer [102] showed that it is NP-complete to decide whether a simplicial complex is collapsible. Furthermore, every 1-dimensional contractible complex is collapsible but the house with two rooms [30] (see Figure 1.10) and the dunce hat [109] show that there are complexes which are contractible but not collapsible. Finally, the conjecture of Zeeman [109], which implies the Poincaré conjecture, states that for every finite contractible 2-dimensional CW-complex K , the space $K \times [0, 1]$ is collapsible.

1.4.1 Notations

Definition 52. *Let Δ be a simplicial complex. We say that a face F of Δ is a free face of Δ if it is non-empty, non-maximal and contained in only one facet of Δ .*

Let Δ and Γ be two simplicial complexes. We say that Δ collapses to Γ if there exists k simplicial complexes $\Delta_1, \dots, \Delta_k$ and a free face F_i of Δ_i for every $i \in \llbracket 1, k-1 \rrbracket$ such that $\Delta_1 = \Delta$, $\Delta_{i+1} = \Delta_i \setminus \{F \in \Delta_i : F_i \subseteq F\}$ for

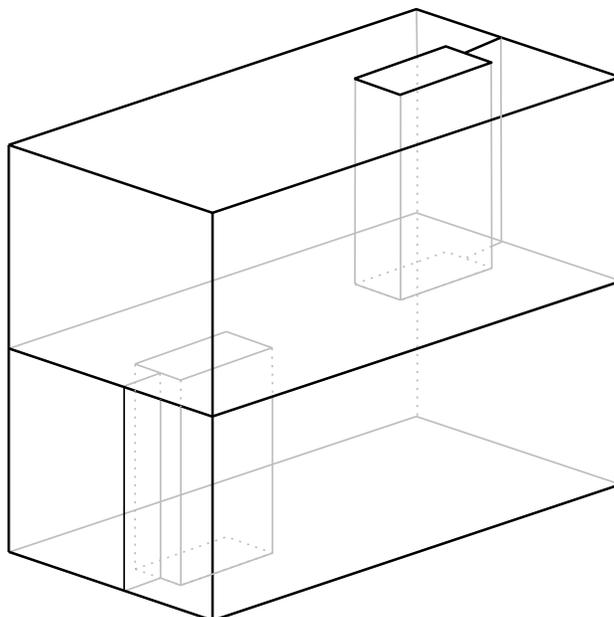


Figure 1.10: The house with two rooms of Bing: an example of a contractible topological space which is not collapsible.

every $i \in \llbracket 1, k - 1 \rrbracket$ and $\Delta_k = \Gamma$. We say that Δ is collapsible if it collapses to a point.

See [84] for a comprehensive study of this topic.

Definition 53. Let (V, \leq) be a poset and let M be a matching of the Hasse diagram of \leq . For an arc a of the Hasse diagram of \leq , we denote $d(a)$ and $u(a)$ the elements of V such that $a = (d(a), u(a))$ and $d(a) < u(a)$. A matching M of the Hasse diagram of \leq is said to be acyclic if, when reversing the orientation of the arcs of M , the obtained digraph remains acyclic.

It is known that if \leq is the poset of inclusion of a simplicial complex and M is a matching of the Hasse diagram of \leq then M is acyclic if and only if there is no sequence of arcs m_1, \dots, m_n of M such that $(d(m_i), u(m_{i+1}))$ is in the Hasse diagram for all $i \in \llbracket 1, n - 1 \rrbracket$ as well as $(d(m_n), u(m_1))$ [40].

Theorem 54 (Chari [40]). Let Δ be a simplicial complex. If the Hasse diagram of the inclusion poset of Δ admits a complete (i.e. perfect) acyclic matching, then Δ is collapsible.

Note that complete acyclic matchings are also called Morse matchings.

We now use Theorem 54 to show that any supremum section is collapsible by finding a complete acyclic matching on the Hasse diagram of its inclusion poset.

1.4.2 Supremum sections are collapsible

We will prove the following theorem:

Theorem 55. *Let R be a representation on a set V . Then $\Sigma(R)$ is collapsible.*

The proof relies on an induction on the dimension of the representation R . Let R be a d -representation on a set V . We denote $R' = (\leq_1, \dots, \leq_{d-1})$ the $(d-1)$ -representation on V obtained from R by deleting the order \leq_d .

Lemma 56. *The simplicial complex $\Sigma(R')$ is a subcomplex of $\Sigma(R)$.*

Proof. Let F be a face of $\Sigma(R')$. As every element x of V dominates F in at least one of the orders $\leq_1, \dots, \leq_{d-1}$, the element x also dominates F in at least one of the orders \leq_1, \dots, \leq_d . We conclude that $F \in \Sigma(R)$. \square

Example 57. *Consider the following 3-representation on $\{a, b, c, d, e\}$.*

$$\begin{array}{c|ccccc} \leq_1 & a & b & e & d & c \\ \leq_2 & c & b & a & d & e \\ \leq_3 & e & d & c & b & a \end{array}$$

See Figure 1.11 to see what $\Sigma(R')$ is for the example.

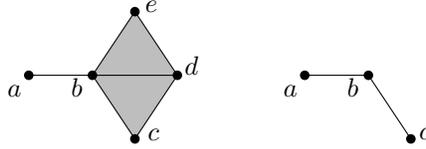


Figure 1.11: The simplicial complex $\Sigma(R)$ where grey triangles correspond to faces with 3 elements and the simplicial complex $\Sigma(R')$ where R' is the representation obtained from R by deleting the order \leq_3 . R is the representation from Example 57.

Lemma 58. *The function ψ defined by*

$$\begin{array}{ccc} \psi & : & \Sigma(R) \setminus \Sigma(R') \rightarrow V \\ & & F \mapsto \min_{\leq_d} \{x \in V : F \not\leq_i x \ \forall i \in \llbracket 1, d-1 \rrbracket\} \end{array}$$

is well-defined.

Proof. Let F be a face of $\Sigma(R) \setminus \Sigma(R')$. We denote $f_i = \max_{\leq_i} F$ for every $i \in \llbracket 1, d \rrbracket$. As $F \notin \Sigma(R')$, there exists an element $x \in V$ such that $\forall i \in \llbracket 1, d-1 \rrbracket, x <_i f_i$. So the minimum is taken in a non-empty set. \square

We define the sets $A = \{F \in \Sigma(R) \setminus \Sigma(R') : \psi(F) \in F\}$ and $A^c = \{F \in \Sigma(R) \setminus \Sigma(R') : \psi(F) \notin F\}$ which is the complementary of A in $\Sigma(R) \setminus \Sigma(R')$. The goal is to find a complete acyclic matching between A and A^c .

Lemma 59. For every $F \in A^c$, we have $\max_{\leq_d} F < \psi(F)$, $\psi(F \cup \{\psi(F)\}) = \psi(F)$, and $F \cup \{\psi(F)\} \in A$.

For every $F \in A$, we have $F \setminus \{\psi(F)\} \in A^c$, and $\psi(F \setminus \{\psi(F)\}) = \psi(F)$.

Proof. Let F be in A^c , we denote $F' = F \cup \{\psi(F)\}$. For every $i \in \llbracket 1, d \rrbracket$, we denote f_i (resp. f'_i) the maximum of F (resp. F') in the order \leq_i . By definition of ψ , $\psi(F) <_i f_i$ for every $i \in \llbracket 1, d-1 \rrbracket$ and $f'_i = f_i$. Furthermore, $\max_{\leq_d} F = f_d <_d \psi(F)$, otherwise $\psi(F)$ would not dominate F .

Let us show that $F' \in \Sigma(R) \setminus \Sigma(R')$. Suppose that $F' \notin \Sigma(R)$. Then there would exist an a such that a does not dominate F' in any order. Thus $a <_i f_i (= f'_i)$ for every $i \in \llbracket 1, d-1 \rrbracket$ and $a <_d \psi(F)$ which contradicts the minimality of $\psi(F)$. We deduce that $F' \in \Sigma(R)$. As $\psi(F) <_i f'_i$ for every $i \in \llbracket 1, d-1 \rrbracket$, $\psi(F)$ does not dominate F' in R' , $F' \notin \Sigma(R')$. Thus $F' \in \Sigma(R) \setminus \Sigma(R')$.

If $\psi(F') <_d \psi(F)$ then we would have $\psi(F') <_i f'_i$ for every $i \in \llbracket 1, d \rrbracket$ as $f'_d = \psi(F)$, contradicting that $F' \in \Sigma(R)$. As $F \subseteq F'$, we have $\psi(F') \leq_d \psi(F)$. We deduce that $\psi(F') = \psi(F) = f'_d \in F'$.

Finally, we conclude that $F \cup \{\psi(F)\} \in A$, $\psi(F \cup \{\psi(F)\}) = \psi(F)$ and $f_d < \psi(F)$.

The second property can be proved in the same manner. \square

Lemma 60. The Hasse diagram of the inclusion poset of $\Sigma(R) \setminus \Sigma(R')$ admits a complete acyclic matching.

See Figure 1.12 to see an example of a complete acyclic matching.

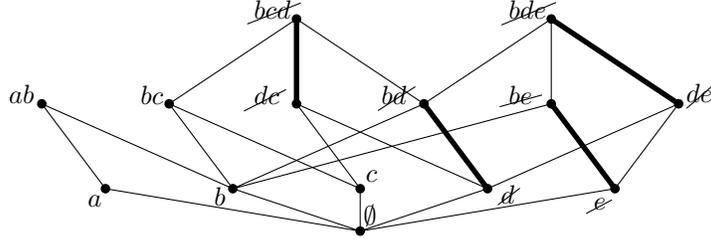


Figure 1.12: The Hasse diagram of $\Sigma(R)$, where R is the representation from Example 57, where the crossed-out faces are the faces of $\Sigma(R) \setminus \Sigma(R')$ and where the fat edges correspond to a complete acyclic matching of $\Sigma(R) \setminus \Sigma(R')$.

Proof. We define the function $\varphi : A^c \rightarrow A$ defined by $\varphi(F) = F \cup \{\psi(F)\}$ for every $F \in A^c$. Let us show that φ is a bijection. To do so, we define the function $\eta : A \rightarrow A^c$ by $\eta(F) = F \setminus \{\psi(F)\}$ where $F \in A$. Lemma 59 implies that η is well defined, that $\eta \circ \varphi = \text{id}_{A^c}$, and that $\varphi \circ \eta = \text{id}_A$. Thus φ is a bijection and φ defines a complete matching $M = \{(F, \varphi(F)) : F \in A^c\}$ between A^c and A .

Suppose that M is not acyclic: there exists a sequence m_1, \dots, m_n of arcs of M where $m_i = (F_i, F_i \cup \{\psi(F_i)\})$ for a face $F_i \in A^c$ for every $i \in \llbracket 1, n \rrbracket$, and such that $(F_i, F_{i+1} \cup \{\psi(F_{i+1})\})$ is in the Hasse diagram for every $i \in \llbracket 1, n-1 \rrbracket$ as well as $(F_n, F_1 \cup \{\psi(F_1)\})$. As for every $i \in \llbracket 1, n-1 \rrbracket$, $F_i \subseteq F_{i+1} \cup \psi(F_{i+1})$ and

$|F_i|+1 = |F_{i+1} \cup \psi(F_{i+1})|$ (because if (A, B) is an arc of the Hasse diagram $H(R)$, then $|A| + 1 = |B|$), we deduce that $\psi(F_{i+1}) \in F_i$. Therefore $\psi(F_{i+1}) <_d \psi(F_i)$ for every $i \in \llbracket 1, n-1 \rrbracket$ and thus $\psi(F_n) <_d \psi(F_1)$. As $(F_1 \cup \{\psi(F_1)\}, F_n)$ is in the Hasse diagram, we show in the same way that $\psi(F_1) <_d \psi(F_n)$ which contradicts the fact that $\psi(F_n) <_d \psi(F_1)$. We conclude that M is a perfect acyclic matching of $\Sigma(R) \setminus \Sigma(R')$. \square

We can now prove Theorem 55.

Proof. We prove the result by induction on the number of orders. Let $R = (\leq_1)$ be a 1-representation on V . We denote m_1 the minimum on V in \leq_1 . Let F be a face of $\Sigma(R)$ which contains an element x different from m_1 . Then m_1 does not dominate F in R as $m_1 <_1 x$. The set $\{m_1\}$ is a face of $\Sigma(R)$ as every element of V dominates $\{m_1\}$ in the order \leq_1 . Thus $\Sigma(R) = \{\emptyset, \{m_1\}\}$ and $H(R)$ admits a complete acyclic matching $(\emptyset, \{m_1\})$. The base case is therefore true.

Let $d \geq 2$, we now suppose that the result is true for any $(d-1)$ -representation on V . Let $R = (\leq_1, \dots, \leq_d)$ be a d -representation on V . We denote R' the $(d-1)$ -representation on V obtained from R by deleting the order \leq_d . We define K as $\Sigma(R) \setminus \Sigma(R')$. By Lemma 60, the Hasse diagram of the inclusion poset of K admits a complete acyclic matching M_1 .

By induction hypothesis, $H(R')$ admits a complete acyclic matching M_2 . Thus $M_1 \cup M_2$ is a complete matching of $H(R)$. Furthermore, $H(R)$ is the union of $H(R')$ and the Hasse diagram of the inclusion poset of K with some arcs between K and $\Sigma(R')$. If an arc between K and $\Sigma(R')$ is oriented from $\Sigma(R')$ to K , then there would be a face F of $\Sigma(R')$ that would contain a face G of K . As $\Sigma(R')$ is closed by inclusion, then G is also in $\Sigma(R')$ which contradicts the definition of K . Therefore the arcs between K and $\Sigma(R')$ are oriented from K to $\Sigma(R')$ and we deduce that $M_1 \cup M_2$ is a perfect acyclic matching of $H(R)$. We conclude by induction. \square

1.5 Contact complexes of stair systems

We have seen that the class of TD-Delaunay complexes in \mathbb{R}^d does not correspond to the class of Dushnik-Miller dimension at most $d+1$ complexes. In the same way we propose here a way to represent geometrically Dushnik-Miller dimension in terms of contacts of *stairs*. We propose two different definitions which give different results. The first allows us to characterize supremum sections in two different ways. The second definition gives a way to represent any Dushnik-Miller dimension at most $d+1$ complexes as contact complexes of stairs in \mathbb{R}^d .

As initiated by Scarf [98], Felsner *et al.* [67] studied Dushnik-Miller dimension by the means of orthogonal surfaces.

Definition 61. For any finite set F of points in \mathbb{R}^d , we define the point $p^F \in \mathbb{R}^d$ as follows:

$$p_i^F = \max_{x \in F} x_i$$

for every $i \in \llbracket 1, d \rrbracket$.

Definition 62. Let $p \in \mathbb{R}^d$, we define

$$I_p = \{z \in \mathbb{R}^d : p_i \leq z_i \forall i \in \llbracket 1, d \rrbracket\} \quad \overset{\circ}{I}_p = \{z \in \mathbb{R}^d : p_i < z_i \forall i \in \llbracket 1, d \rrbracket\}$$

In other words, I_p denotes the (closed) positive orthant of \mathbb{R}^d whose corner is p and $\overset{\circ}{I}_p$ denotes its open version. Given a subset A of \mathbb{R}^d , A^c denotes the complementary of A in \mathbb{R}^d . Given a set V , we denote by $\text{Subsets}(V)$ the set of subsets of V . Given a set of points \mathcal{P} of \mathbb{R}^d , we say that \mathcal{P} is in *general position* if no two points p and p' of \mathcal{P} share the same coordinate (i.e. $p_i \neq p'_i \quad \forall i \in \llbracket 1, d \rrbracket$). In the rest of this section, point sets will always be finite and in general position.

Truncated stair systems

Definition 63. A truncated stair system $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ is given by a set \mathcal{P} of points of \mathbb{R}^d and a function $\mathcal{F} : \mathcal{P} \rightarrow \text{Subsets}(\mathcal{P})$. Let $x \in \mathcal{P}$, we define the (closed) truncated stair $S(x, \mathcal{P}, \mathcal{F})$ and the open truncated stair $\overset{\circ}{S}(x, \mathcal{P}, \mathcal{F})$ (or simply $S(x)$ and $\overset{\circ}{S}(x)$ when clear from the context) of x according to the truncated stair system $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ by

$$S(x, \mathcal{P}, \mathcal{F}) = I_x \bigcap_{z \in \mathcal{F}(x)} (I_z)^c \quad \overset{\circ}{S}(x, \mathcal{P}, \mathcal{F}) = \overset{\circ}{I}_x \bigcap_{z \in \mathcal{F}(x)} (I_z)^c$$

Remark that $S(x, \mathcal{P}, \mathcal{F})$ is closed and that $\overset{\circ}{S}(x, \mathcal{P}, \mathcal{F})$ is open for every $x \in \mathcal{P}$.

In the following we consider truncated stair systems where the truncated stairs do not overlap (see an example in Figure 1.13).

Definition 64. We say that a truncated stair system $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ is a truncated stair packing if for every pair (x, y) of distinct elements of \mathcal{P} , we have $\overset{\circ}{S}(x, \mathcal{P}, \mathcal{F}) \cap \overset{\circ}{S}(y, \mathcal{P}, \mathcal{F}) = \emptyset$.

Given a truncated stair packing \mathcal{S} , we define the contact complex $\Delta(\mathcal{S})$ of \mathcal{S} as the abstract simplicial complex on the vertex set \mathcal{P} where $F \subseteq \mathcal{P}$ is a face of $\Delta(\mathcal{S})$ if $\bigcap_{x \in F} S(x, \mathcal{P}, \mathcal{F}) \neq \emptyset$.

Definition 65. We say that a truncated stair packing $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ is a truncated stair tiling if for every $x \in \mathcal{P}$, $I_x \subseteq \bigcup_{x \in \mathcal{P}} S(x, \mathcal{P}, \mathcal{F})$.

Lemma 66. Let $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ be a truncated stair packing and $F \subseteq \mathcal{P}$. Then $F \in \Delta(\mathcal{S})$ if and only if $p^F \in \bigcap_{x \in F} S(x, \mathcal{P}, \mathcal{F})$.

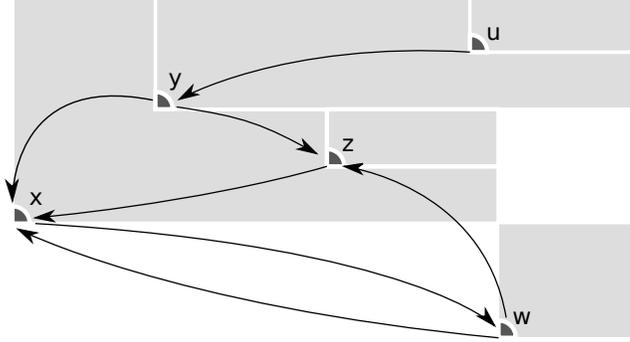


Figure 1.13: An example of a truncated stair packing which is not a truncated stair tiling. An arc from y to x means that $y \in \mathcal{F}(x)$.

Proof. As $F \in \Delta(\mathcal{S})$, there exists $p \in \bigcap_{x \in F} S(x, \mathcal{P}, \mathcal{F})$. For every $x \in F$, $p \in S(x, \mathcal{P}, \mathcal{F})$ and thus $p \in I_x$. Therefore $p_i^F \leq p_i$ for every $i \in \llbracket 1, d \rrbracket$.

Let $x \in F$. Let us show that $p^F \in S(x, \mathcal{P}, \mathcal{F})$. As $x_i \leq p_i^F$ for every i , then $p^F \in I_x$. Let $z \in \mathcal{F}(x)$. As $p \in S(x, \mathcal{P}, \mathcal{F})$, then $p \in \mathring{I}_z^c$. Thus there exists $i \in \llbracket 1, d \rrbracket$ such that $p_i \leq z_i$. As $p_i^F \leq p_i$, we conclude that $p_i^F \leq z_i$ and that $p^F \in \mathring{I}_z^c$. Thus $p^F \in S(x, \mathcal{P}, \mathcal{F})$. We conclude that $p^F \in \bigcap_{x \in F} S(x, \mathcal{P}, \mathcal{F})$. \square

Ordered truncated stair systems

Ordered truncated stair systems are particular truncated stair systems where the function \mathcal{F} is defined by a linear order on \mathcal{P} .

Definition 67. Let \mathcal{P} be a set of points of \mathbb{R}^d and a linear order \leq on \mathcal{P} . We define the function $\mathcal{F}_\leq : \mathcal{P} \rightarrow \text{Subsets}(\mathcal{P})$ by $\mathcal{F}_\leq(x) = \{z \in \mathcal{P} : z < x\}$ for every $x \in \mathcal{P}$. We say that a truncated stair system $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ is an ordered truncated stair system if there exists an order \leq on \mathcal{P} such that $\mathcal{S} = (\mathcal{P}, \mathcal{F}_\leq)$.

Remark that ordered truncated stair systems are particular cases of truncated stair systems. This kind of truncated stair systems have good properties and we will see that they are connected to supremum sections.

Lemma 68. Any ordered truncated stair system is a truncated stair tiling.

Proof. Let $\mathcal{S} = (\mathcal{P}, \leq)$ be an ordered truncated stair system. Let us first show that \mathcal{S} is a truncated stair packing. Let x and y be two distinct elements of \mathcal{P} . Without loss of generality we can suppose that $y < x$. Towards a contradiction suppose that $\mathring{S}(x) \cap \mathring{S}(y) \neq \emptyset$ and let $z \in \mathring{S}(x) \cap \mathring{S}(y)$. As $y < x$, then $z \in (I_y)^c$. As $z \in \mathring{S}(y)$, then $z \in \mathring{I}_y$. Thus $z \in \mathring{I}_y \cap (I_y)^c$ which is impossible. We conclude that $\mathring{S}(x) \cap \mathring{S}(y) = \emptyset$ and that \mathcal{S} is a truncated stair packing.

Let us now show that \mathcal{S} has the tiling property. Let $y \in \mathcal{P}$ and $w \in I_y$. Let us show that $w \in \bigcup_{x \in \mathcal{P}} S(x)$.

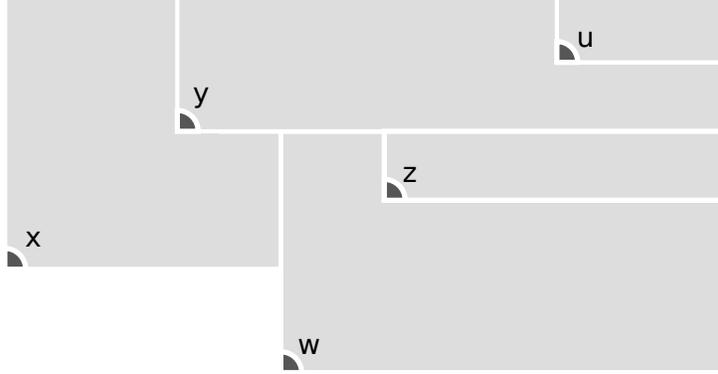


Figure 1.14: An example of an ordered truncated stair system where the points are ordered in increasing order: u, y, z, w, x .

We define x as the least element of \mathcal{P} according to the order \leq such that $w \in I_x$. The element x is well defined as $w \in I_y$. Suppose by contradiction that $w \notin S(x)$. As $S(x) = I_x \cap_{u < x} (\mathring{I}_u)^c$, then there exists $u < x$ such that $w \in \mathring{I}_u$. Thus $u < x$ and $w \in I_u$ which contradicts the minimality of x . Therefore $w \in S(x)$ and $w \in \cup_{x \in \mathcal{P}} S(x)$. We conclude that \mathcal{S} is a truncated stair tiling. \square

1.5.1 Dushnik-Miller dimension of ordered truncated stair systems

Lemma 69. *Let \mathcal{P} be a set of points of \mathbb{R}^d and let $R = (\leq_1, \dots, \leq_{d+1})$ be a $(d+1)$ -representation on \mathcal{P} . We consider the truncated stair system $\mathcal{S} = (\mathcal{P}, \mathcal{F}_{\leq_{d+1}})$. If for every p and $q \in \mathcal{P}$ and every $i \in \llbracket 1, d \rrbracket$, $p_i \leq q_i \iff p \leq_i q$, then $\Sigma(R) = \Delta(\mathcal{S})$.*

Proof. Let $F \in \Sigma(R)$. Let us check that $p^F \in \bigcap_{x \in F} S(x)$ (see Definition 61 for p^F). Let $x \in F$. As $x_i \leq \max_{u \in F} u_i = p_i^F$ for every i , then $p^F \in I_x$. Let $z \in \mathcal{P}$ such that $z <_{d+1} x$. Suppose by contradiction that $p^F \notin (\mathring{I}_z)^c$. Then $p^F \in \mathring{I}_z$ and $z_i < p_i^F = \max_{u \in F} u_i$ for all $i \in \llbracket 1, d \rrbracket$. Thus for every i , there exists $u \in F$ such that $z <_i u$. As $z <_{d+1} x$, we conclude that z does not dominate F in R which contradicts the fact that $F \in \Sigma(R)$. Therefore $p^F \in (\mathring{I}_z)^c$ and $p^F \in S(x)$. We conclude that $\bigcap_{x \in F} S(x)$ is not empty and thus $F \in \Delta(\mathcal{S})$.

Consider a set $F \subseteq \mathcal{P}$ such that $\bigcap_{x \in F} S(x)$ is not empty and let p be a point in $\bigcap_{x \in F} S(x)$. Towards a contradiction, consider a point $z \in \mathcal{P}$ that does not dominate F in R . In that case, there exists an $x \in F$ such that $z <_{d+1} x$. As $p \in S(x)$ and $z <_{d+1} x$, then $p \in (\mathring{I}_z)^c$. Then there exists $i \in \llbracket 1, d \rrbracket$ such that $z_i \geq p_i$. For every $y \in F$, $p \in I_y$ and thus $p_i \geq y_i$. We conclude that $z_i \geq \max_{u \in F} u_i$. Thus $z \geq_i u$ for every $u \in F$ and z dominates F in the order \leq_i , a contradiction. We conclude that $F \in \Sigma(R)$. \square

Lemma 69, allows us to we prove the following theorem.

Theorem 70. *Let Δ be a simplicial complex on a vertex set V . Then Δ is the intersection complex of an ordered truncated stair system of \mathbb{R}^d if and only if there exists a $(d + 1)$ -representation R on V such that $\Delta = \Sigma(R)$.*

Proof. Suppose that there exists an embedding \mathcal{P} of V in \mathbb{R}^d in general position and a linear order \leq on \mathcal{P} such that $\Delta = \Delta(\mathcal{S})$ where $\mathcal{S} = (\mathcal{P}, \mathcal{F}_{\leq})$. For every $i \in \llbracket 1, d \rrbracket$, we define the linear order \leq_i on V as $p \leq_i q \iff p_i \leq q_i$ (which is well defined because the points are in general position). We define the $(d + 1)$ -representation R on V by $R = (\leq_1, \dots, \leq_d, \leq)$. According to Lemma 69, $\Sigma(R) = \Delta(\mathcal{S}) = \Delta$.

Suppose that there exists a $(d + 1)$ -representation R on V such that $\Delta = \Sigma(R)$. We define \mathcal{P} an embedding of V in \mathbb{R}^d as follows: for every x in V we define x_i as the position of x in the linear order \leq_i in increasing order. Thus for every p and $q \in \mathcal{P}$ and every $i \in \llbracket 1, d \rrbracket$, $p_i \leq q_i \iff p \leq_i q$. Then according to Lemma 69, $\Sigma(R) = \Delta(\mathcal{S}) = \Delta$ where $\mathcal{S} = (\mathcal{P}, \mathcal{F}_{\leq_{d+1}})$. \square

We now introduce a notion of dimension to study the sets $\bigcap_{x \in F} S(x)$.

Definition 71. *Let $\mathcal{S} = (\mathcal{P}, \leq)$ be an ordered truncated stair system and let F be a face of $\Delta(\mathcal{S})$. We define the dimension of $\bigcap_{x \in F} S(x)$ as the maximum number of coordinates in which a point q of this set is different from p^F , i.e. $\max_{q \in \bigcap_{x \in F} S(x)} |\{i \in \llbracket 1, d \rrbracket : q_i \neq p_i^F\}|$.*

We can equivalently define it as the maximum dimension of a box contained in the intersection $\bigcap_{x \in F} S(x)$.

Lemma 72. *Let \mathcal{P} be a set of points of \mathbb{R}^d , \leq a linear order on \mathcal{P} and $F \in \Delta(\mathcal{S})$ where $\mathcal{S} = (\mathcal{P}, \mathcal{F}_{\leq})$. Then $\dim(\bigcap_{x \in F} S(x)) = d - |F| + 1$.*

Proof. Let $p \in \bigcap_{x \in F} S(x)$ different from p^F . We note f^* the maximum of F in the order \leq . For every $y \in F$, since $p \in S(y)$, $p \in I_y$. Furthermore, as $p \in S(f^*)$, $p \in (\mathring{I}_y)^c$ if $y \neq f^*$. Thus for every i , $y_i \leq p_i$, and there exists a $i \in \llbracket 1, d \rrbracket$ such that $p_i \leq y_i$. Thus there exists $i \in \llbracket 1, d \rrbracket$ such that $p_i = y_i$. We define $I(y)$ as the set of values i such that $p_i = y_i$. As the points are in general position, the sets $I(y)$ are disjoint.

We define $I = \{i \in \llbracket 1, d \rrbracket : p_i = p_i^F\}$. As $p_i^F \leq p_i$ for every i , we deduce that $I = \{i : \exists y \in F : p_i = y_i\}$. Thus $I = \bigsqcup_{y \in F} I(y)$ and $|I| \geq |F| - 1$. Thus $\dim(\bigcap_{x \in F} S(x)) \leq d - |F| + 1$

Let $y \in F$ different from f^* . We define the $d + 1$ -representation R in the same way as in the proof of Theorem 70. Thus $\Delta(\mathcal{P}, \leq) = \Sigma(R)$ and $\leq_{d+1} \equiv \leq$. As $F \in \Sigma(R)$, then y must dominate F in one order. As $y <_{d+1} f^*$, then y dominates F in an order $i_0 \in \llbracket 1, d \rrbracket$. We can thus define $\varphi(y) = i_0 \in \llbracket 1, d \rrbracket$ and we define $J = \{\varphi(y) : y \in F \setminus \{f^*\}\}$. Consider a sufficiently small $\varepsilon > 0$. We define $\varepsilon^J \in \mathbb{R}^d$ as

$$\varepsilon_i^J = \begin{cases} 0 & \text{if } i \in J \\ \varepsilon & \text{otherwise} \end{cases}$$

We consider the point $p = p^F + \varepsilon^J$ which coincides with p^F on exactly $|F| - 1$ coordinates.

Let us show that $p \in \bigcap_{x \in F} S(x)$. Let $y \in F$. Then $p \in I_y$, as $y_i \leq p_i^F \leq p_i$ for every $i \in \llbracket 1, d \rrbracket$. Let $z < y$ and suppose that $z \in F$. Then $z \neq f^*$ and we can consider $i = \varphi(z)$. Thus $F \leq_i z$ by definition of i . Thus $p_i = p_i^F = z_i$ and $p \in (\overset{\circ}{I}_z)^c$. Suppose now that $z \notin F$. As $p^F \in \bigcap_{x \in F} S(x)$, then there exists i such that $p_i^F \leq z_i$. As points are in general position, z_i cannot be equal to p_i^F thus $p_i^F < z_i$. For a sufficiently small ε , we have $p_i \leq p_i^F + \varepsilon < z_i$. We conclude that $p \in (\overset{\circ}{I}_z)^c$ and thus $p \in \bigcap_{x \in F} S(x)$. Thus $\dim(\bigcap_{x \in F} S(x)) \geq d - |F| + 1$ \square

1.5.2 Dushnik-Miller dimension of truncated stair tilings

The properties of ordered truncated stair tilings now allow us to study truncated stair tilings. We will need to define the following *removal property*.

Definition 73. *Let $(\mathcal{P}, \mathcal{F})$ be a truncated stair system. The truncated stair system $(\mathcal{P} \setminus \{x\}, \mathcal{F}')$ obtained after the removal of a point x of \mathcal{P} is defined as follows:*

$$\mathcal{F}'(y) = \begin{cases} \mathcal{F}(y) \cup \mathcal{F}(x) \setminus \{x\} & \text{if } x \in \mathcal{F}(y) \\ \mathcal{F}(y) & \text{otherwise} \end{cases}$$

We say that the truncated stair tiling $(\mathcal{P}, \mathcal{F})$ has the removal property if any truncated stair system obtained after any sequence of removal is still a truncated stair tiling.

Lemma 74. *Let \mathcal{P} be a set of points of \mathbb{R}^d and \leq be a linear order on \mathcal{P} . Then the truncated stair system $(\mathcal{P}, \leq) = (\mathcal{P}, \mathcal{F}_{\leq})$ has the removal property.*

Proof. We prove this lemma by induction on the number of elements of \mathcal{P} . The property is true if \mathcal{P} is reduced to a single point.

Suppose by induction that the property is true for any (\mathcal{P}', \leq') such that \mathcal{P}' is of size $n - 1$. Let (\mathcal{P}, \leq) be an ordered truncated stair system of size n .

Let $x \in \mathcal{P}$ be an element that we are going to remove. We note \mathcal{F}' the function obtained from \mathcal{F}_{\leq} after removing x and \leq' the restriction of the order \leq on $\mathcal{P} \setminus \{x\}$. Let us show that $(\mathcal{P} \setminus \{x\}, \mathcal{F}') = (\mathcal{P} \setminus \{x\}, \leq')$. For any $y \in \mathcal{P} \setminus \{x\}$, we have to show that $\mathcal{F}'(y) = \mathcal{F}_{\leq'}(y)$. If $x \notin \mathcal{F}_{\leq}(y)$, then according to the previous definition we have $\mathcal{F}'(y) = \mathcal{F}_{\leq}(y)$. By definition of \mathcal{F}_{\leq} , we have $\mathcal{F}_{\leq}(y) = \{u \in \mathcal{P} : u < y\}$. As $x \notin \mathcal{F}(y)$, then $y < x$ and thus $\{u \in \mathcal{P} : u < y\} = \{u \in \mathcal{P} \setminus \{x\} : u < y\}$. We conclude that $\mathcal{F}'(y) = \{u \in \mathcal{P} \setminus \{x\} : u < y\}$ in this case. Otherwise $x \in \mathcal{F}(y)$ and thus $x < y$. Therefore $\mathcal{F}'(y) = \mathcal{F}(y) \cup \mathcal{F}(x) \setminus \{x\} = \{u \in \mathcal{P} : u < y \text{ and } u \neq x\}$ as $\mathcal{F}(x) \subseteq \mathcal{F}(y)$ and $\mathcal{F}'(y) = \{u \in \mathcal{P} \setminus \{x\} : u < y\}$. In any case we have $\mathcal{F}'(y) = \{u \in \mathcal{P} \setminus \{x\} : u < y\}$ and thus $(\mathcal{P} \setminus \{x\}, \mathcal{F}') = (\mathcal{P} \setminus \{x\}, \leq')$.

By Lemma 68, $(\mathcal{P} \setminus \{x\}, \mathcal{F}')$ is a truncated stair tiling and by induction it has the removal property. We conclude that $(\mathcal{P}, \mathcal{F})$ also has the removal property. \square

Lemma 75. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{F})$ be a truncated stair tiling which has the removal property. Then there exists a partial order \leq on \mathcal{P} such that $\Delta(\mathcal{S}) = \Delta(\mathcal{S}_{\leq})$ where $\mathcal{S}_{\leq} = (\mathcal{P}, \mathcal{F}_{\leq})$.*

Proof. We define a binary relation \leq on \mathcal{P} defined by $x \leq y$ if and only if $x \in \mathcal{F}(y)$. Let us show that this relation is acyclic. Suppose that there exists a cycle C in this relation. We define (C, \mathcal{F}') the truncated stair system obtained after the removal of all the elements not in C . By the removal property of $(\mathcal{P}, \mathcal{F})$, the truncated stair system (C, \mathcal{F}') is a truncated stair tiling. We define the point p by setting

$$p_i = 1 + \max_{u \in C} u_i$$

for every $i \in \llbracket 1, d \rrbracket$. As (C, \mathcal{F}') is a truncated stair tiling and as $p \in I_x$ for every $x \in C$, then there exists $x \in C$ such that $p \in S(x, C, \mathcal{F}')$. As C is a cycle, there exists $y \in C$ such that $y \in \mathcal{F}'(x)$. Thus $p \in (\dot{I}_y)^c$: there exists $i \in \llbracket 1, d \rrbracket$ such that $p_i \leq y_i$. This contradicts the definition of p . We conclude that the relation \leq is acyclic.

Let \leq' be the transitive closure of \leq . Let us show that $S(y, \mathcal{P}, \mathcal{F}) = S(y, \mathcal{P}, \leq')$ for every $y \in \mathcal{P}$. Remark that by construction, for every $y \in \mathcal{P}$, $\mathcal{F}(y) \subseteq \{x \in \mathcal{P} : x <' y\}$. Therefore $S(y, \mathcal{P}, \leq') \subseteq S(y, \mathcal{P}, \mathcal{F})$. Let $y \in \mathcal{P}$ and suppose for a contradiction that $S(y, \mathcal{P}, \leq') \subsetneq S(y, \mathcal{P}, \mathcal{F})$. We show now that in that case, the set $\dot{S}(y, \mathcal{P}, \mathcal{F}) \setminus S(y, \mathcal{P}, \leq')$ is not empty. As $S(y, \mathcal{P}, \leq') \subsetneq S(y, \mathcal{P}, \mathcal{F})$, there exists $p \in S(y, \mathcal{P}, \mathcal{F}) \setminus S(y, \mathcal{P}, \leq')$.

As $p \in S(y, \mathcal{P}, \mathcal{F})$, then $p \in I_y$: for every $i \in \llbracket 1, d \rrbracket$, $y_i \leq p_i$. We define $I = \{i \in \llbracket 1, d \rrbracket : y_i = p_i\}$ and $J = \{j \in \llbracket 1, d \rrbracket : y_j < p_j\}$. As $p \neq y$ (otherwise $p \in S'(y)$), the set J is not empty. For every $t \in [0, +\infty[$, we define p^t as

$$p_i^t = \begin{cases} p_i + t & \text{if } i \in I \\ p_i - t & \text{otherwise} \end{cases}$$

for every $i \in \llbracket 1, d \rrbracket$. We define $t_0 > 0$ arbitrarily small. Let us show that for every $t \in]0, t_0[$ we have $p^t \in \dot{S}(y)$.

Note that $y_i < p_i^t$ for every i . Let $x \in \mathcal{F}(y)$. As $p \in S(y, \mathcal{P}, \mathcal{F})$, there exists k such that $p_k \leq x_k$. If $k \in I$, then $y_k = p_k < x_k$ (because x and y do not share any coordinate). Then $p_k^t = p_k + t = y_k + t < x_k$. If $k \in J$, then $y_k < p_k \leq x_k$. Then $p_k^t = p_k - t < x_k$. We conclude that $p^t \in \dot{S}(y, \mathcal{P}, \mathcal{F})$. As the truncated stair system (\mathcal{P}, \leq') is a truncated stair tiling, there exists $x < y$ such that $p \in \dot{I}_x$ whereas $p \notin S(y, \mathcal{P}, \leq')$. Thus $x_i < p_i$ for every i and $x_i < p_i^t$ for every i . We conclude that $p^t \in \dot{S}(y, \mathcal{P}, \mathcal{F}) \setminus S(y, \mathcal{P}, \leq')$.

Consider a point $w \in \dot{S}(y, \mathcal{P}, \mathcal{F}) \setminus S(y, \mathcal{P}, \leq')$. As (\mathcal{P}, \leq') is a truncated stair tiling and as $y \in \bigcup_{x \in \mathcal{P}} S(x, \mathcal{P}, \leq')$, then $w \in \bigcup_{x \in \mathcal{P}} S(x, \mathcal{P}, \leq')$. Thus there exists $x \in \mathcal{P}$ such that $w \in S(x, \mathcal{P}, \leq')$. As $w \notin S(y, \mathcal{P}, \leq')$, we deduce that $x \neq y$. Moreover, as $S(x, \mathcal{P}, \leq') \subseteq S(x, \mathcal{P}, \mathcal{F})$, we deduce that $S(x, \mathcal{P}, \mathcal{F}) \cap \dot{S}(y, \mathcal{P}, \mathcal{F})$ is not empty which is impossible as $(\mathcal{P}, \mathcal{F})$ is a truncated stair packing. We conclude that $S(y, \mathcal{P}, \mathcal{F}) = S(y, \mathcal{P}, \leq')$ for every $y \in \mathcal{P}$ and thus that $\Delta(\mathcal{S}) = \Delta(\mathcal{S}_{\leq})$ where $\mathcal{S}_{\leq} = (\mathcal{P}, \mathcal{F}_{\leq})$. \square

The removal property is important in this proof as there exist truncated stair tilings which do not have the removal property. It is for instance the case of the following truncated stair system:

$$\begin{cases} x &= (3, 0, 1) \\ y &= (1, 3, 0) \\ z &= (0, 1, 3) \\ w &= (2, 2, 2) \end{cases}$$

where the function \mathcal{F} is defined by $\mathcal{F}(x) = \{y, w\}$, $\mathcal{F}(y) = \{z, w\}$, $\mathcal{F}(z) = \{x, w\}$ and $\mathcal{F}(w) = \emptyset$. This truncated stair tiling is drawn in Figure 1.15.

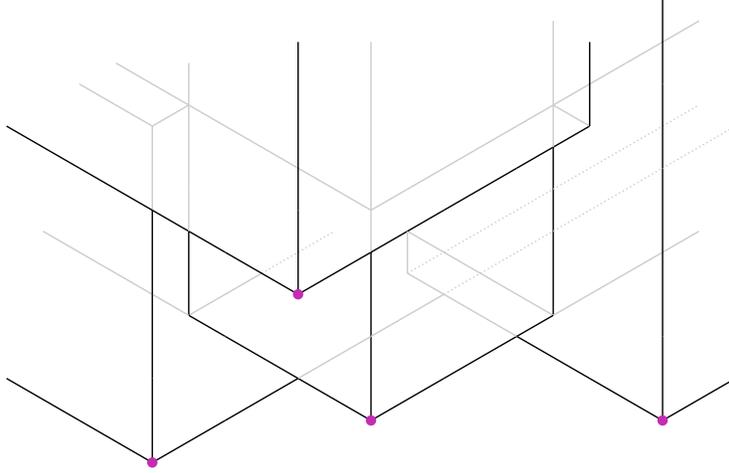


Figure 1.15: A truncated stair packing which has not the removal property.

Now, Lemma 74, Lemma 68 and Theorem 70, allow us to prove the following theorem.

Theorem 76. *A simplicial complex Δ on a vertex set V is the contact complex of a truncated stair tiling of \mathbb{R}^d which has the removal property if and only if there exists a $(d + 1)$ -representation R on V such that $\Delta = \Sigma(R)$.*

Proof. Suppose that there exists a $(d + 1)$ -representation R on V such that $\Delta = \Sigma(R)$. Then by Theorem 70, there exists an embedding \mathcal{P} in \mathbb{R}^d of the points of V and a linear order \leq on V such that $\Delta = \Delta(\mathcal{P}, \leq)$. According to Lemma 74 and Lemma 68, (\mathcal{P}, \leq) is a truncated stair tiling which has the removal property. Thus Δ is the contact complex of a truncated stair tiling of \mathbb{R}^d which has the removal property.

Suppose now that Δ is the contact complex of a truncated stair tiling $(\mathcal{P}, \mathcal{F})$ of \mathbb{R}^d which has the removal property. Then by Lemma 75 and Theorem 70, there exists a $(d + 1)$ -representation R on V such that $\Delta = \Sigma(R)$. \square

1.5.3 Dushnik-Miller dimension and stair systems

We introduce here a variant of the definition of stair systems. This one is more geometric as it does not involve an order or a function as in the previous definitions.

Definition 77. Let $y \in \mathbb{R}^d$, we define J_y and \mathring{J}_y the subsets of \mathbb{R}^d as follows

$$J_y = \{u \in \mathbb{R}^d : u_i \leq y_i \forall i \in [1, d]\} \quad \mathring{J}_y = \{u \in \mathbb{R}^d : u_i < y_i \forall i \in [1, d]\}$$

Definition 78. A stair system \mathcal{S} is given by a set \mathcal{P} of points of \mathbb{R}^d and a function $C : \mathcal{P} \rightarrow \text{Subsets}(\mathbb{R}^d)$ such that $C(x)$ is a finite set for every $x \in \mathcal{P}$. Let $x \in \mathcal{P}$, we define the (closed) stair $R(x, \mathcal{P}, C)$ and the open stair $\mathring{R}(x, \mathcal{P}, C)$ (or simply $R(x)$ and $\mathring{R}(x)$ when clear from the context) of x according to the stair system $\mathcal{S} = (\mathcal{P}, C)$ by

$$R(x, \mathcal{P}, C) = \bigcup_{y \in C(x)} I_x \cap J_y \quad \mathring{R}(x, \mathcal{P}, C) = \bigcup_{y \in C(x)} \mathring{I}_x \cap \mathring{J}_y$$

We say that (\mathcal{P}, C) is a stair packing if the sets $\mathring{R}(x, \mathcal{P}, C)$ are disjoint.

An example of a stair packing is given in Figure 1.16.

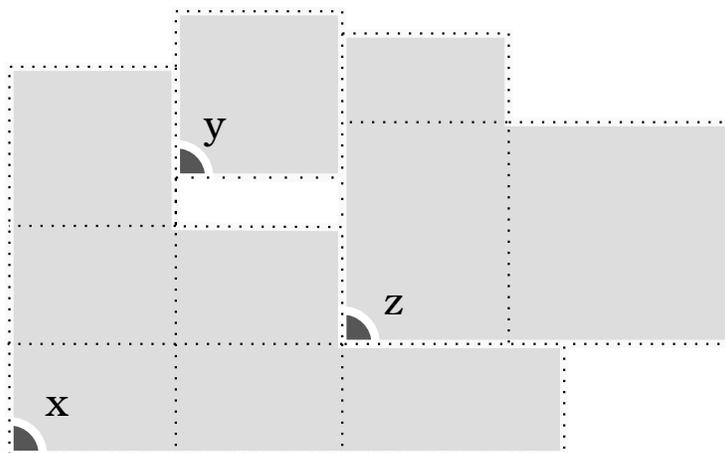


Figure 1.16: An example of a stair packing.

Definition 79. Let (\mathcal{P}, C) be a stair packing. We define the contact complex of \mathcal{P} , denoted $\Delta(\mathcal{P}, C)$ as follows: a subset F of \mathcal{P} is a face of $\Delta(\mathcal{P}, C)$ if and only if $\bigcap_{x \in F} R(x)$ is not empty.

Remark that $\Delta(\mathcal{P}, C)$ is a simplicial complex. Furthermore remark that this definition generalizes the previous definition of a stair system.

Theorem 80. *Let Δ be a simplicial complex of Dushnik-Miller dimension at most $d + 1$. Then Δ is the contact complex of a stair system in \mathbb{R}^d .*

Proof. There exists a $d + 1$ -representation $R = (\leq_1, \dots, \leq_{d+1})$ on V , the vertex set of Δ , such that $\Delta \subseteq \Sigma(R)$. According to Lemma 69, there exists an embedding \mathcal{P} of V such that \mathcal{P} is in general position and such that $\Sigma(R) = \Delta(\mathcal{P}, \leq_{d+1})$.

Let us define a stair system on \mathcal{P} as follows. For every vertex $x \in \mathcal{P}$, we define

$$R(x) = \bigcup_{F \in \Delta: x \in F} I_x \cap J_{p^F}$$

Let us show that (\mathcal{P}, C) is a stair packing. Let $x <_{d+1} y$ be two distinct points of \mathcal{P} . Suppose by contradiction that $\overset{\circ}{R}(x) \cap \overset{\circ}{R}(y)$ is not empty. Thus there exists $p \in \overset{\circ}{R}(x) \cap \overset{\circ}{R}(y)$. Therefore, there exists $G \in \Delta$ such that $y \in G$ and $p \in I_x \cap J_{p^G}$. Then $x_i < p_i \leq p_i^G$ for every $i \in \llbracket 1, d \rrbracket$. Thus for every i , there exists $u \in G$ such that $x_i < u_i$ and $x <_i u$. As $x <_{d+1} y$, it contradicts the fact that G is a face of $\Sigma(R)$ (as $\Delta \subseteq \Sigma(R)$). Therefore $\overset{\circ}{R}(x) \cap \overset{\circ}{R}(y)$ is empty. We conclude that (\mathcal{P}, C) is a stair packing.

Let us now show that $\Delta = \Delta(\mathcal{P}, C)$. Let $F \in \Delta$ and let us show that $p^F \in \bigcap_{x \in F} R(x)$. Indeed, for every $x \in F$, $p^F \in I_x$ and $p^F \in J_{p^F}$. Thus F is in $\Delta(\mathcal{P}, C)$ and $\Delta \subseteq \Delta(\mathcal{P}, C)$.

Let $F \in \Delta(\mathcal{P}, C)$. Suppose by contradiction that $F \notin \Delta$. Let x be the maximum of F according to the order \leq_{d+1} . As $F \in \Delta(\mathcal{P}, C)$, there exists $p \in \bigcap_{u \in F} R(u)$. Thus $u_i \leq p_i$ for every $u \in F$ and every $i \in \llbracket 1, d \rrbracket$. As $p \in R(x)$, there exists $H \in \Delta$ such that $x \in H$ and such that $p \in I_x \cap J_{p^H}$. Thus $p_i^F \leq p_i \leq p_i^H$ for every $i \in \llbracket 1, d \rrbracket$. As $H \in \Delta$ and as $F \notin \Delta$, we deduce that $F \not\subseteq H$. Then there exists $y \in F \setminus H$ which is different from x . As $p_i^F \leq p_i^H$ for every i , then for every i there exists $h \in H$ such that $y_i < h_i$ and $y <_i h$. Furthermore $y <_{d+1} x$ as x is supposed to be the maximum of F according to the order \leq_{d+1} . This contradicts that $H \in \Delta \subseteq \Sigma(R)$. We conclude that $F \in \Delta$ and thus that $\Delta = \Delta(\mathcal{P}, C)$. \square

While in the case of ordered stair systems, the stairs are of dimension d (according to Lemma 2.18), the stairs in \mathbb{R}^d obtained from the previous theorem are not necessarily of dimension d . A question arises: is any simplicial complex of Dushnik-Miller dimension at most $d + 1$ the contact complex of a stair system in \mathbb{R}^d where all the stairs are of dimension d ?

1.6 Conclusion

We have proved different results related to the Dushnik-Miller dimension. Nevertheless there still remains some open questions.

Open questions on TD-Delaunay complexes. TD-Delaunay complexes of \mathbb{R}^3 have shown to be of interest as sparse spanners with small stretch [31]. It is likely that this extends to higher dimensional spaces. A question is for example whether graphs of Dushnik-Miller dimension 4 give better spanners than TD-Delaunay complexes in \mathbb{R}^3 .

Furthermore, as the class of TD-Delaunay complexes of $H_d \simeq \mathbb{R}^{d-1}$ is strictly included in the class of complexes of Dushnik-Miller dimension d , it may be interesting to study the problem of the straight line embedding for this restricted class. Indeed Schnyder showed that a planar graph with n vertices can be embedded in the plane with integer coordinates between 0 and n [100]. Ossona de Mendez showed that a complex of Dushnik-Miller dimension d can be drawn in \mathbb{R}^{d-1} without crossings [92]. Nevertheless the last embedding uses exponential coordinates and it is still an open problem to reduce the sizes of these embeddings. Is it possible to reduce this size when dealing with TD-Delaunay complexes?

Open questions on the topology of supremum sections. As a consequence of Theorem 55, which states that any supremum section is collapsible, the homology of any supremum section is trivial. For example, we deduce that for any representation R , the Euler characteristic of $\Sigma(R)$ is 0. Therefore the simplicial complex consisting of a 3-cycle cannot be a supremum section.

A natural question that arises is: does the class of collapsible simplicial complexes corresponds to the class of supremum sections? We conjecture that it is not the case. In other words, there should exist a simplicial complex which is collapsible but which is not a supremum section.

Remark that Kappes [82] showed that any cp-order of a rigid non-degenerate orthogonal surface has a complete acyclic matching. As orthogonal surfaces are connected to representations, there is may be a link between these results.

Open questions on stair systems. According to Theorem 80, we conclude that the class of Dushnik-Miller dimension at most $d + 1$ complexes is included in the class of contact complexes of stair packings in \mathbb{R}^d . The reciprocal of this theorem is a question that arises naturally. Finally, the case where the point set \mathcal{P} is not in general position, in all the three results, seems interesting but more involved.

Stair systems and orthogonal surfaces share some notions. It would be interesting to understand the connections between them. For example, the question of the generalization to points not in general position is a common problem.

In the goal of dealing with points not in general position, Felsner and Kappes introduced the notion of rigidity which is defined in \mathbb{R}^3 as follows [67]. An orthogonal surface in \mathbb{R}^3 is said to be rigid if for every $u, v \in V$ which generate a “characteristic point”, there is no $w \in V$ such that $w_i \leq \max(u_i, v_i)$. Thanks to this property, they prove that the cp-orders of rigid suspended orthogonal surfaces in \mathbb{R}^3 correspond to the face-lattices of 3-polytopes with one facet removed. The goal was consequently to define a notion of rigidity in \mathbb{R}^d and to generalize the previous theorem to higher dimension. While they succeed in defining rigidity in \mathbb{R}^d , the problem of generalizing the previous theorem remains open. It would be interesting to try the same approach for stairs complexes with points not in general position.

Chapter 2

L-intersection and L-contact graphs

In this chapter, we focus on intersection and contact representations of planar graphs with objects that are homeomorphic to segments. The more general representations of this type are the intersection or contact representations of curves. Those are called *string* representations. It is known that every planar graph has a string-intersection representation [16]. However, if one forbids tangent curves, this representation may contain pairs of curves that cross several times. One may thus take an additional parameter into account, namely the maximal number of crossings of any two of the considered curves: a *1-string* representation of a graph is a string representation where every two curves intersect at most once. The question of finding a 1-string representation of planar graphs has been solved by Chalopin *et al.* in the positive [39], and additional parameters are now studied, like order-preserving representations [52].

Segment intersection graphs are in turn a specialization of the class of 1-string graphs. A $\{\perp, |, -\}$ -*contact representation* is a system consisting in some \perp 's, some vertical segments $|$, and some horizontal segments $-$, such that if an intersection occurs between two of these objects, then the intersection is an endpoint of one of the two objects. It is known that bipartite planar graphs are $\{|, -\}$ -contact graphs [22, 19] (*i.e.*, segment contact graphs with vertical or horizontal segments). De Castro *et al.* [20] showed that triangle-free planar graphs are segment contact graphs with only three different slopes. De Fraysseix and Ossona de Mendez [18] then proved that a larger class of planar graphs are segment intersection graphs. Finally, Chalopin *et al.* extended this result to general planar graphs [38], which was conjectured by Scheinerman in his PhD thesis [15].

Definition 81 ([57, 55]). *A graph is said to be a VPG-graph (for Vertex-Path-Grid) if it has a contact or intersection representation in which each vertex is assigned to a path of vertical and horizontal segments.*

See an example of a VPG graph on Figure 2.1.

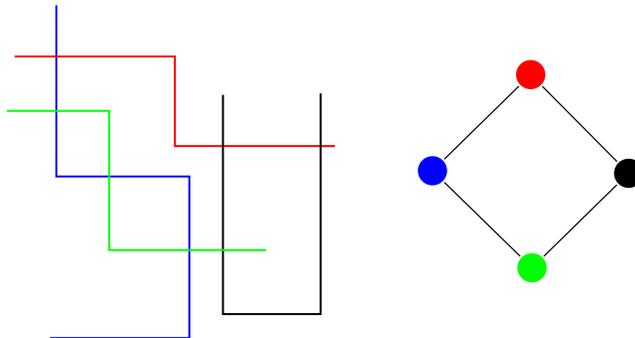


Figure 2.1: Example of a VPG graph.

Asinowski *et al.* [24] showed that the class of VPG-graphs is equivalent to the class of graphs admitting a string-representation.

Definition 82 (Asinowski *et al.* [24]). *A graph is said to be a B_k -VPG graph if it has an intersection representation in which each vertex is assigned to a path of vertical and horizontal segments with at most k bends.*

The inclusion of some classes of graphs in B_k -VPG graphs has been studied in [64]. It is known that B_k -VPG $\subsetneq B_{k+1}$ -VPG, and that the recognition of graphs of B_k -VPG is an NP-complete problem [54]. These classes have interesting algorithmic properties (see [53] for approximation algorithms for independence and domination problems in B_1 -VPG graphs), but most of the literature deals with their combinatorial properties.

Chaplick and Ueckerdt [36] proved that planar graphs are B_2 -VPG graphs. This result was recently improved by Biedl and Derka [62], as they showed that planar graphs have a 1-string B_2 -VPG representation.

Various classes of graphs have been shown to have 1-string B_1 -VPG representations, such as planar partial 3-trees [59] and Halin graphs [21]. In these representations, each vertex is assigned to a path formed by at most one horizontal and one vertical segment. There are different types of such paths. For example, the \perp shape defines paths where the vertical segment is above and to the left of the horizontal one. Interestingly, it has been shown that the class of segment contact graphs is equivalent to the one of B_1 -VPG contact graphs [14]. This implies in particular that triangle-free planar graphs are B_1 -VPG contact graphs. This has been improved by Chaplick *et al.* [36] as they showed that triangle-free planar graphs are in fact $\{\perp, \ulcorner, |, -\}$ -contact graphs (that is without using the shapes \lrcorner and \urcorner). In the following, we will always precise when $|$ or $-$ shapes are allowed. This is particularly important as for example some $\{\perp, |, -\}$ -contact graphs like the octaedron are not \perp -contact graphs.

The restriction of B_1 -VPG to \perp -intersection or \perp -contact graphs has been much studied (see for example [64]) and it has been shown that they are in

relation with other structures such as Schnyder realizers, canonical orders or edge labelings [35]. The same authors also proved that the recognition of \perp -contact graphs can be done in quadratic time, and that this class is equivalent to the one restricted to equilateral \perp shapes. The \perp -contact graphs where the corners lie on a straight line are called *monotone* or *linear* \perp -contact graphs. Those graphs have been recently studied further, in particular in relation with MPT (Max-Point Tolerance) graphs [56, 63].

The two main results of this chapter are the following:

Theorem 83. *Every triangle-free planar graph is an $\{\perp, |, -\}$ -contact graph.*

Theorem 84. *Every planar graph is an \perp -intersection graph.*

Both results were conjectured in [36]¹. In both cases, one cannot restrict the representation to $|$ and $-$ shaped paths. Indeed, any $\{|, -\}$ -intersection representation of a triangle-free planar graph or any $\{|, -\}$ -contact representation of a planar graph defines a vertex partition of the graph into two unions of paths (one induced by the vertical paths and the other induced by the horizontal ones), but such partition is not always possible [3].

If we assume that any point of the plane is on at most two strings, as a triangle free string contact graph (and thus a $\{\perp, |, -\}$ -contact graph) with n vertices has at most $2n$ edges and as a triangle-free planar graph may have up to $2n - 4$ edges, Theorem 83 cannot be extended to much denser graphs. However, for planar Laman graphs (a large family of planar graphs with at most $2n - 3$ edges and which are B_1 -VPG graphs [64]), the question of whether these graphs have a $\{\perp, |, -\}$ -contact representation is open, up to our knowledge. The question whether triangle-free planar graphs are $\{\perp, |\}$ -contact graphs is also open. Theorem 84 implies that planar graphs are in (1-string) B_1 -VPG, improving the results of Biedl and Derka [62] stating that planar graphs are in (1-string) B_2 -VPG. Since an $\{\perp, \lrcorner, |, -\}$ -intersection representation can be turned into a segment intersection representation [88], this also directly provides a rather simple proof of the fact that planar graphs are segment intersection graphs [38].

The common ingredient of our results is a kind of planar graphs that we call *2-sided near-triangulations*. In Section 2.1, we present the 2-sided near-triangulations, allowing us to provide a new decomposition of planar 4-connected triangulations (see [60, 17] for other decompositions of 4-connected triangulations). This decomposition is simpler than the one provided by Whitney [17] that is used in [38]. In Section 2.2, we define thick \perp -contact representations (*i.e.*, \perp -contact representations in which the \perp are thick) with specific properties. We then show that every 2-sided near-triangulation admits such a representation. This result is used in Section 2.3 to prove Theorem 83. Then in Section 2.4 we use 2-sided near-triangulations to prove Theorem 84.

¹Actually, Theorem 83 has been proved in the master thesis (written in German) of B. Kappelle in 2015 [1] but never published.

2.1 2-sided near-triangulations

In this chapter we consider plane graphs with neither loops nor multiple edges. Given a vertex v on the outer face, the *inner neighbors* of v are the neighbors of v that are not on the outer face. We define here 2-sided near-triangulations (see Figure 2.2) whose structure will be useful in the inductions of the proofs of Theorem 83 and Theorem 84.

Definition 85. A 2-sided near-triangulation is a 2-connected near-triangulation T without separating triangles (see the introduction chapter for a definition of separating triangles), such that going clockwise on its outer face, the vertices are denoted $a_1, a_2, \dots, a_p, b_q, \dots, b_2, b_1$, with $p \geq 1$ and $q \geq 1$, and such that there is neither a chord $a_i a_j$ nor $b_i b_j$ (that is an edge $a_i a_j$ or $b_i b_j$ such that $|i - j| > 1$).

Remark that 4-connected triangulations being the triangulations without separating triangles, 4-connected triangulations are 2-sided near-triangulations.

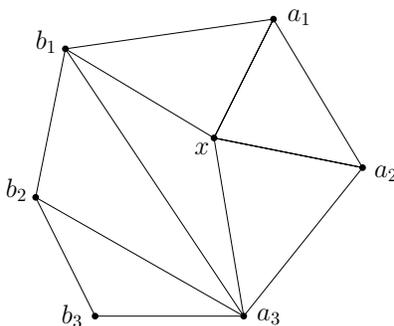


Figure 2.2: Example of a 2-sided near-triangulation.

The structure of the 2-sided near-triangulations allows us to describe the following decomposition:

Lemma 86. Given a 2-sided near-triangulation T with at least 4 vertices, one can always perform one of the following operations:

- (**a_p -removal**) This operation applies if $p > 1$, if a_p has no neighbor b_i with $i < q$, and if none of the inner neighbors of a_p has a neighbor b_i with $i < q$. This operation consists in removing a_p from T , and in denoting b_{q+1}, \dots, b_{q+r} the new vertices on the outer face in anti-clockwise order, if any. This yields a 2-sided near-triangulation T' (see Figure 2.3a).
- (**b_q -removal**) This operation applies if $q > 1$, if b_q has no neighbor a_i with $i < p$, and if none of the inner neighbors of b_q has a neighbor a_i with $i < p$. This operation consists in removing b_q from T , and in denoting a_{p+1}, \dots, a_{p+r} the new vertices on the outer face in clockwise order, if any. This yields a 2-sided near-triangulation T' . This operation is strictly symmetric to the previous one.

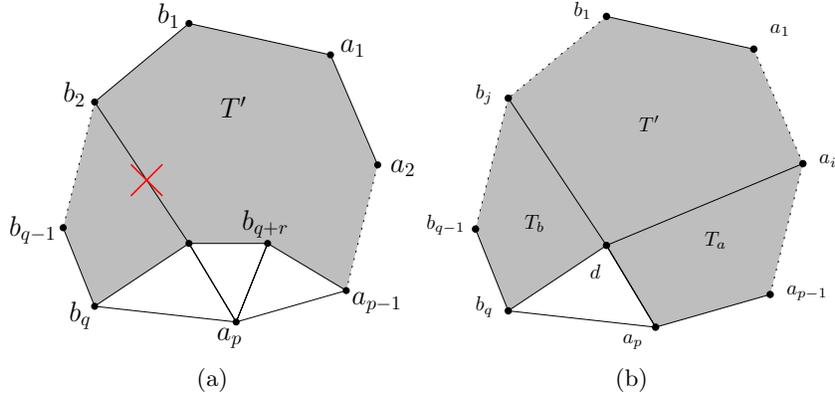


Figure 2.3: Illustrations of (a) the a_p -removal operation and (b) the cutting operation.

- **(cutting)** This operation applies if $p > 1$, if $q > 1$ and the unique common neighbor of a_p and b_q , denoted d , has a neighbor a_i with $i < p$, and a neighbor b_j with $j < q$. The index i (resp. j) is the minimal index such that d is adjacent to a_i (resp. b_j). This operation consists in cutting T into three 2-sided near-triangulations T' , T_a and T_b (see Figure 2.3b):
 - T' is the 2-sided near-triangulation contained in the cycle formed by vertices $(a_1, \dots, a_i, d, b_j, \dots, b_1)$, and the vertex d is renamed a_{i+1} .
 - T_a (resp. T_b) is the 2-sided near-triangulation contained in the cycle (a_i, \dots, a_p, d) (resp. (d, b_q, \dots, b_j)), where the vertex d is denoted b_1 (resp. a_1).

Proof. Suppose that a_p has no neighbor b_i with $i < q$ and that none of the inner neighbors of a_p has a neighbor b_i with $i < q$. We denote b_{q+1}, \dots, b_{q+r} the inner neighbors of a_p in anti-clockwise order. Let T' be the graph obtained by removing a_p and its incident edges from T . It is clear that T' is a near-triangulation, and that it has no separating triangle (otherwise T would have one too). Furthermore, as there is no chord incident to a_p , and as T' has at least three vertices, its outer face is bounded by a cycle and T' is thus 2-connected. As T is a 2-sided near-triangulation, T' has no chord $a_i a_j$, with $i, j < p$, or $b_i b_j$ with $i, j \leq q$. From our assumption, the inner neighbors of a_p have no neighbors b_k with $k < q$, thus there is no chord $b_i b_j$ with $i \leq q$ and $q < j$. Finally, we claim that there is no chord $b_i b_j$ in T' with $q \leq i < j$. Supposing otherwise would mean that the vertices a_p , b_i , and b_j would form a triangle with at least one vertex inside, b_{i+1} , and at least one vertex outside, a_{p-1} : it would be a separating triangle, a contradiction. Therefore T' is a 2-sided near-triangulation.

The proof for the b_q -removal operation is analogous to the previous case.

Suppose that we are not in the first case nor in the second one. Let us first show that $p > 1$ and $q > 1$. Assume, towards a contradiction, that $p = 1$. Then as T is 2-connected, it has at least three vertices on the outer face and $q \geq 2$. In such a case one can always perform the b_q -removal operation, a contradiction.

Let us now show that a_p is not adjacent to a vertex b_i with $i < q$. Assume by contradiction that a_p is adjacent to a vertex b_i with $i < q$. Then by planarity, b_q (with $q > 1$) has no neighbor a_i with $i < p$, and has no inner neighbor adjacent to a vertex a_i with $i < p$. In such a case one can always perform the b_q -removal operation, a contradiction. Symmetrically, we deduce that b_q is not adjacent to a vertex a_i with $i < p$.

Vertices a_p and b_q have one common neighbor d such that $da_p b_q$ is an inner face. Note that as there is no chord incident to a_p or b_q , then d is not on the outer face. Since the outer face is not triangular, they have no other common neighbor y , otherwise there would be a separating triangle $ya_p b_q$ (separating d from both vertices a_1 and b_1).

As we are not in the first case nor in the second case, we have that a_p (resp. b_q) has (at least) one inner neighbor adjacent to a vertex b_i with $i < q$ (resp. a_i with $i < p$). By planarity, d is the only inner neighbor of a_p (resp. b_q) adjacent to a vertex b_j with $j < q$ (resp. a_i with $i < p$). We can thus apply the cutting operation. We define i as the minimal i such that d is adjacent to a vertex a_i and j as the minimal j such that d is adjacent to a vertex b_j .

We now show that T' , T_a and T_b are 2-sided near-triangulations. Consider first T' . It is clear that it is a near-triangulation without separating triangles. It remains to show that there are no chords $a_i a_j$ or $b_i b_j$. By definition of T' , the only possible chord would have $d = a_{i+1}$ as an endpoint, but the existence of an edge da_k with $k < i$ would contradict the minimality of i . Thus T' is a 2-sided near-triangulation.

By definition, T_a is also a near-triangulation containing no separating triangles. Moreover, there is no chord $a_k a_l$ with $i \leq k \leq l - 2$ as there are no such chords in T . Therefore T_a is a 2-sided near-triangulation. Similarly, T_b is also a 2-sided near-triangulation. \square

2.2 Thick \perp -contact representation

A *thick* \perp is an \perp shape where the two segments are turned into thick rectangles (see Figure 2.4). Here we do not allow thick $|$ or $-$, so going clockwise around a thick \perp from the bottom-right corner, we call its sides *bottom*, *left*, *top*, *vertical interior*, *horizontal interior*, and *right*. We draw them in the integer grid, that is such that their bend points have integer coordinates, and we ask the two rectangles to be of thickness one, and of length at least two. A thick \perp is described by four coordinates a, b, c, d such that $a + 1 < b$ and $c + 1 < d$. It is thus the union of two boxes: $([a, a+1] \times [c, d]) \cup ([a, b] \times [c, c+1])$. If not specified, the *corner* of a thick \perp denotes its bottom-left corner (with coordinates (a, c)).

Definition 87. A convenient thick \perp -contact representation (*CTLCR for short*)

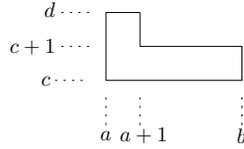
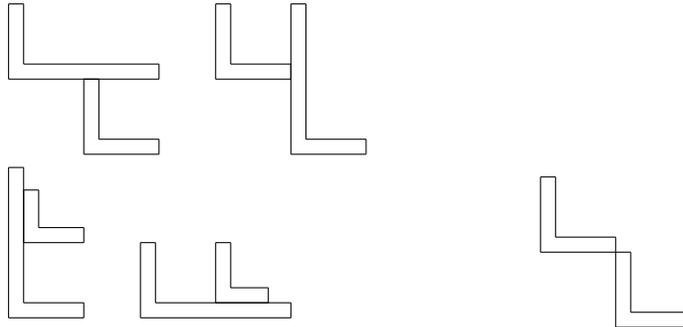


Figure 2.4: A thick \perp .

is a contact representation of thick \perp (which implies that the thick \perp interiors are disjoint) with the following properties:

- Two thick \perp intersect on exactly one side of a thick \perp or on a point (Figure 2.5 lists the allowed ways two thick \perp can intersect). If the intersection is a point, then it is the bottom right corner of one thick \perp and the top left corner of the other one.
- If the bottom (resp. left) side of a thick \perp x is contained in the horizontal (resp. vertical) interior side of a thick \perp y , then the bottom (resp. left) side of y is not contained in the horizontal (resp. vertical) interior side of a thick \perp z (see Figure 2.6).



(a) Allowed segment intersections. From left to right and top to bottom: the intersection is the top, right, left, bottom side of a thick \perp .

(b) Allowed point intersections. The intersection is a point at the bottom right corner of a thick \perp and at the top left corner of a thick \perp .

Figure 2.5: Allowed intersections in a CTLCR.

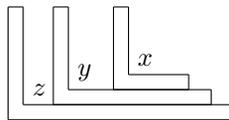


Figure 2.6: Forbidden configurations in a CTLCR.

Remark that in a CTLCR the removal of any thick \perp still leads to a CTLCR. We now show that every 2-sided near-triangulation has a CTLCR (see Figure 2.8 for an illustration).

Theorem 88. *Every 2-sided near-triangulation T has a CTLCR with the following properties, for some integers X and Y :*

- *Every corner of a thick \perp is included in the non-positive quadrant $\{(x, y) : x \leq 0, y \leq 0\}$.*
- *The thick \perp of a_1 has the bottom-most corner and has coordinates $(0, -Y)$. Every vertex a_i is represented by a thick \perp whose corner has coordinates $(x, -Y)$ with $2 - X \leq x < 0$. Furthermore their horizontal interior side does not contain any other side.*
- *The thick \perp of b_1 has the left-most corner and has coordinates $(-X, 0)$. Every vertex b_i is represented by a thick \perp whose corner has coordinates $(-X, y)$ with $2 - Y \leq y < 0$. Furthermore their vertical interior side does not contain any other side.*
- *$Y + X \leq 3n - 3$, where n is the number of vertices in T .*

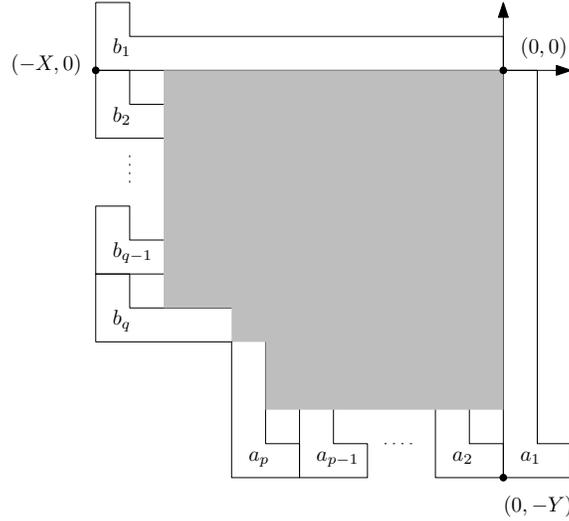


Figure 2.7: Typical CTLCR obtained from Theorem 88.

Proof. We proceed by induction on the number of vertices. The theorem clearly holds for the 2-sided near-triangulation with three vertices. Let T be a 2-sided near-triangulation; it can thus be decomposed using one of the three operations described in Lemma 86. We go through the three operations successively.

(a_p -removal) Let T' be the 2-sided near-triangulation resulting from an a_p -removal operation on T . By the induction hypothesis, T' has a CTLCR

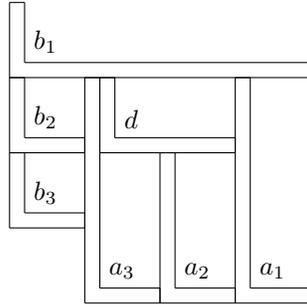


Figure 2.8: One of the CTLCR of the near-triangulation from Figure 2.2.

with the required properties for parameters X' and Y' (see Figure 2.9a). We can now modify this CTLCR slightly in order to obtain a CTLCR of T (thus adding a thick \perp corresponding to vertex a_p). Move the corners of the thick \perp corresponding to vertices b_1, \dots, b_q three units to the left. Since their vertical interior side do not contain any other side, one can do this without modifying the rest of the representation. Then one can add the thick \perp of a_p such that it touches the thick \perp of vertices b_q and a_{p-1} (as depicted in Figure 2.9b). One can easily check that the obtained representation is a CTLCR of T and satisfies all the requirements. Note in particular that in this case the vertical interior side of a_p can contain some sides, the left sides of b_{q+1}, \dots, b_{q+r} , but in the induction these thick \perp are such that their vertical interior side do not contain any other side, so we avoid the configuration depicted in Figure 2.6.

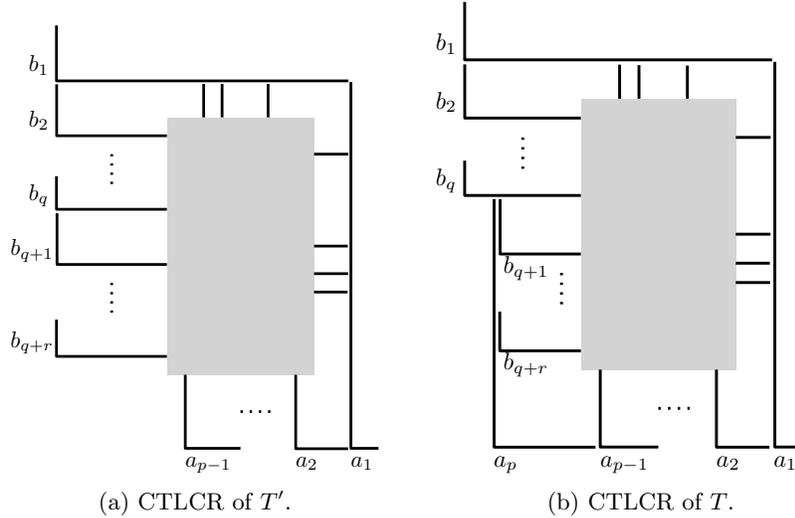


Figure 2.9: The $(a_p\text{-removal})$ operation for a CTLCR. Here, the grey region contains the corners of the inner vertices.

(b_q -removal) This case is symmetric to the previous one.

(cutting) Let T' , T_a and T_b be the three 2-sided near-triangulations resulting from the cutting operation described in Lemma 86. By induction hypothesis, each of them has a CTLCR satisfying the requirements of Theorem 88, with parameters X', Y', X_a, Y_a, X_b , and Y_b respectively. We are going to modify the CTLCR of T' in order to include the ones of T_a and T_b , as they are given by the induction (see Figure 2.10).

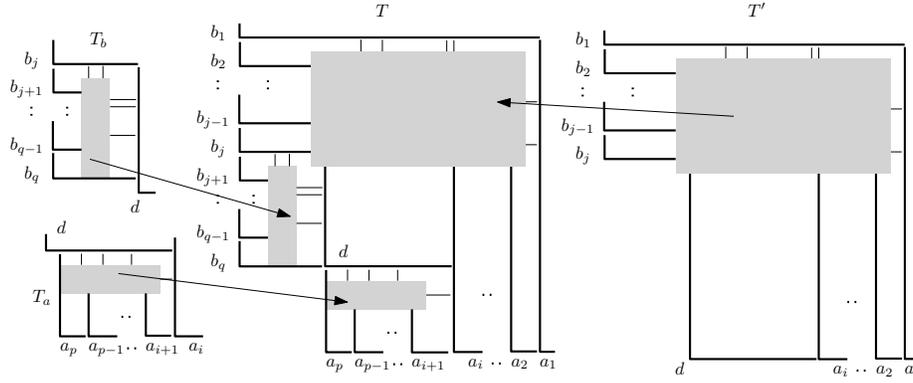


Figure 2.10: The (cutting) operation for a CTLCR.

Given the CTLCR of T' , move the corners of the thick \perp corresponding to vertices b_1, \dots, b_j by at most $(X_b - 2)$ units to the left, so that the x -coordinates of the corners of b_j and $d = a_{i+1}$ differ by exactly X_b . Move the corner of $d = a_{i+1}$ downward by at most $Y_b - 2$ units, so that the y -coordinates of the corners of b_i and $d = a_{i+1}$ differ by exactly Y_b . As these y -coordinates originally already differ by at least 2, this move is of at most $(Y_b - 2)$ units. Move the corners of a_1, \dots, a_i by at most $(Y_b - 2) + Y_a$ units downward, so that the y -coordinates of the corners of $d = a_{i+1}$ and a_i differ by exactly Y_a . Again, since these thick \perp have their vertical or horizontal interior sides that do not contain any other side, one can do this without modifying the rest of the representation. Now, one has to modify the representation in order to prolong the horizontal part of the thick \perp of $d = a_{i+1}$ so that its bottom side reaches length X_a . This can be done by cutting the CTLCR along a vertical line ℓ that crosses the thick \perp of $d = a_{i+1}$, by moving the left part at most $(X_a - 2)$ units to the left (as this side has already length at least 2), and for any thick \perp of the left side touching ℓ by prolonging its horizontal part to the right, until reaching the right part of the CTLCR. Now the corners of the thick \perp 's of b_j , $d = a_{i+1}$ and a_i are well placed so that one can add the CTLCR of T_a below the thick \perp of d and the one of T_b on its left (see Figure 2.10). One can easily check that the obtained representation satisfies all the requirements.

Let us now check that these operations preserve the fact that $Y + X \leq 3n - 3$. Indeed, for the a_p -removal $Y = Y'$ and $X = X' + 3$ while T has one more vertex than T' . The case of b_q -removal is identical. For the cutting operation, we

have that $X \leq X_a + X_b + X' - 4$ and that $Y \leq Y_a + Y_b + Y' - 2$. Thus $Y + X \leq (Y_a + X_a) + (Y_b + X_b) + (Y' + X') - 6 \leq 3n - 3 \times 3 + 4 \times 3 - 6 \leq 3n - 3$ because d is counted twice too much and b_j and a_i are counted both one time too much. \square

Note that the last item of the theorem implies that such a CTLCR fits into a grid of width W and height H with $W + H \leq 3n + 1$. Actually, allowing two thick \perp shapes to intersect on two segments (that is, allowing the corner of any thick $\perp x$ to be at the intersection of the vertical and horizontal interior sides of another thick $\perp y$), one can reach $W + H \leq 2n + 3$. It is not clear whether allowing thick $|$ or $-$ would decrease this bound much further.

Similar representations with axis aligned polygons have been studied in the literature. For example, the problem of representing a planar graph G with such contact representations where the area of each polygon is prescribed by some weight function $w(v)$, defined on $V(G)$, has various applications. It has been showed [4] that finding such a representation is always possible for Hamiltonian triangulations even if restricting to thick $]$, \perp , \lceil , or $|$ shaped polygons, where the thickness of each part is not necessarily one. In the same paper the authors also consider this problem for 4-connected triangulations when restricting to thick \perp , \lceil , or $|$ shaped polygons (in relation with the so-called one-legged Hamiltonian cycles), but this remains open. We wonder whether our approach (through 2-sided near-triangulations) could lead to a positive answer, but the $\{\perp, |, -\}$ -contact representations described in the following section lead to a representation with thick \perp , $|$, and $-$ shaped polygons for triangle-free planar graphs (leaving some holes in the representation).

2.3 $\{\perp, |, -\}$ -contact representations for triangle-free planar graphs

We can now use the CTLCR to prove Theorem 83. We need the following lemma as a tool.

Lemma 89. *For any plane triangle-free graph G , there exists a 4-connected triangulation T containing G as an induced subgraph.*

Proof. The main idea of the construction of T is to insert vertices and edges in every face of G (even for the exterior face).

For the sake of clarity, vertices of G are called *black* and vertices of $T \setminus G$ are called *red*. The new graph T contains G as an induced subgraph, along with other vertices and edges. More precisely, for every face of G , let $P = \{v_0, e_0, v_1, e_1, \dots\}$ be the list of vertices and edges along the face boundary (see Figure 2.11), where e_i is the edge between vertices v_i and v_{i+1} ; there can be repetitions of vertices or edges. For each face of G , given the list P , the graph T contains a vertex v'_i for each occurrence of a vertex $v_i \in P$, a vertex e'_i for each occurrence of an edge $e_i \in P$, and an additionnal vertex t . Each vertex

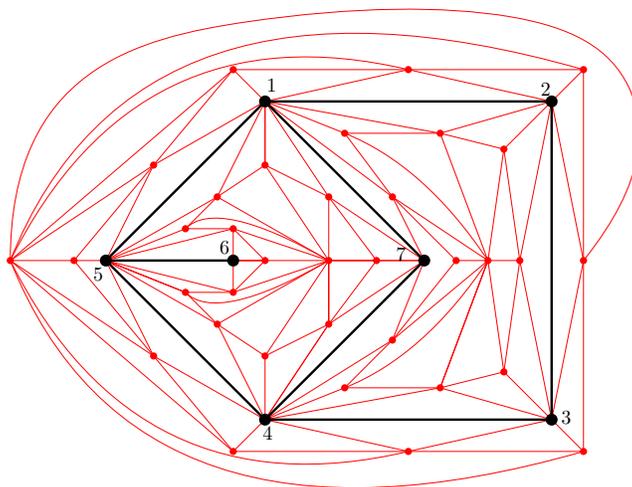


Figure 2.11: A planar triangle-free graph G (in black) and a 4-connected triangulation containing it as an induced subgraph (adding red vertices and red edges). The boundary lists of the two inner faces of G are respectively $\{1, (1, 2), 2, (2, 3), 3, (3, 4), 4, (4, 7), 7, (7, 1)\}$, $\{1, (1, 7), 7, (7, 4), 4, (4, 5), 5, (5, 6), 6, (6, 5), 5, (5, 1)\}$. The outer face is $\{1, (1, 2), 2, (2, 3), 3, (3, 4), 4, (4, 5), 5, (5, 1)\}$.

v'_i is connected to e'_i and e'_{i+1} (with subscripts addition done modulo the size of the face), each vertex v_i is connected to v'_i , e'_{i-1} and e'_i , and the vertex t is connected to all vertices v'_i and e'_i (see Figures 2.11 and 2.12 for examples).

The new graph T is a triangulation, and we now show that it is 4-connected, *i.e.* has no separating triangle. Suppose that there is a separating triangle in the new graph. There are four cases depending on the colors of the edges of this triangle:

- The separating triangle contains three black edges. It is impossible since G is triangle-free.
- The separating triangle contains exactly one red edge. One of its endpoints must be a red vertex. But a red vertex is incident to only red edges, a contradiction.
- The separating triangle contains exactly two red edges. Then their common endpoint is a red vertex, and the triangle is made of two vertices v_i and v_{i+1} , together with the vertex e'_i . All these triangles are faces, a contradiction.
- The separating triangle contains three red edges. Since for each face, the red vertices (vertices v'_i , e'_i and t) induce a wheel graph centered on t , with at least 8 peripheral vertices (vertices v'_i and e'_i), this separating triangle

has at least one black vertex. As two adjacent black vertices are linked by a black edge, this separating triangle has exactly one black vertex. As the two red vertices are two adjacent v'_i or e'_j vertices, we have that those are v'_i and e'_j , for some i and for $j = i$ or $i + 1$. Such a triangle is not separating, a contradiction.

This concludes the proof of the lemma. □

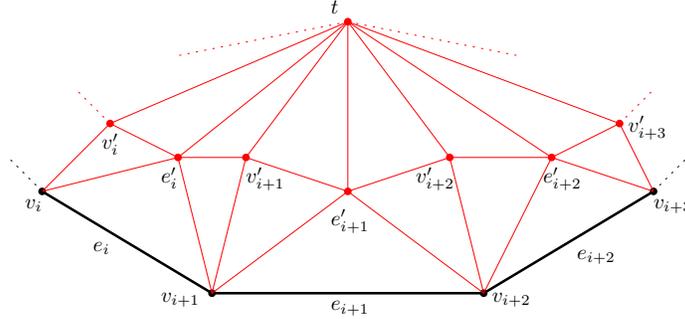


Figure 2.12: Zoom on the new connections.

We can now prove Theorem 83, which asserts that every triangle-free planar graph has an $\{\perp, |, -\}$ -contact representation.

Proof of Theorem 83. Consider a triangle-free planar graph G . According to Lemma 89, there exists a 4-connected triangulation T containing G as an induced subgraph. Denoting a_1, b_2, b_1 the three exterior vertices of T in clockwise order, one sees that T is a 2-sided near-triangulation. By Theorem 88, T has a CTLCR and removing every thick \perp corresponding to a vertex of $T \setminus G$ leads to a CTLCR of G .

If a thick \perp x has its bottom side included in the horizontal interior side of another thick \perp y , then x does not contain any other side on its horizontal interior side. Furthermore, x does not intersect anyone on its right side nor on its bottom right corner. Indeed, if there was such an intersection with a thick \perp z , then y and z would also intersect, contradicting the fact that G is triangle-free (see Figure 2.13). One can thus replace the thick \perp of x by a thick $|$.

Similarly, if a thick \perp x has its left side included in the vertical interior side of a thick \perp y , we can replace the thick \perp of x by a thick $-$.

Note that now the intersections are on segments of length 1, or on a point, between the bottom right corner of a thick \perp or $-$, and the top left corner of a thick \perp or $|$. Then, we replace each thick \perp , $|$, and $-$ by thin ones as depicted in Figure 2.14. It is clear that we obtain a $\{\perp, |, -\}$ -contact representation whose contact graph is G . This concludes the proof. □

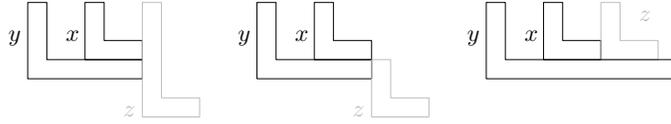


Figure 2.13: If a thick \sqsubset x has its bottom side included in the horizontal interior of a thick \sqsubset y , then x has no intersection with a thick \sqsubset z on its right side and on its bottom right corner.

An example of the process is shown in Figure 2.15. Note that any $\{\sqsubset, |, -\}$ -contact representation with n paths fits into a grid of width and height at most n . Indeed, any horizontal (resp. vertical) line of the grid that does not contain any of the n_h horizontal (resp. n_v vertical) subpaths, nor contain any contact point between two vertical (resp. horizontal) subpaths can be deleted. As the number of contact points between two vertical (resp. horizontal) subpaths where the path on top (resp. on the right) is $|$ shaped (resp. $-$ shaped) is at most $n - n_h$ (resp. $n - n_v$), we end up with at most n horizontal (resp. vertical) lines. It is open to know whether this bound can be improved. Let G_t be the (bipartite) planar graph obtained by taking t cubes C_1, \dots, C_t and gluing C_i and C_{i+1} on a square, for all $1 \leq i < t$, such that C_i and C_{i+2} remain vertex disjoint, for all $1 \leq i < t - 1$. We believe that for this graph, with $n = 4t + 4$ vertices, any $\{\sqsubset, |, -\}$ -contact representation requires a grid of width and height at least $n/2$.

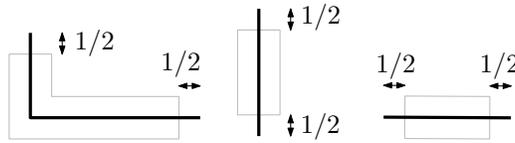


Figure 2.14: Replacing thick \sqsubset , $|$, and $-$ by thin ones.

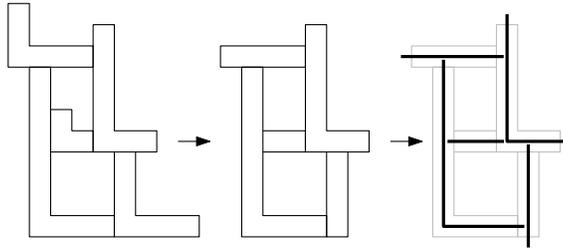


Figure 2.15: Given a CTLCR of a triangle-free graph G , we first replace some thick \sqsubset by thick $|$ and thick $-$, and then replace every thick shape by a thin one according to Figure 2.14.

2.4 The \perp -intersection representations

An \perp -intersection representation of a graph G is a representation of G such that vertices are represented by \perp -shaped paths in a grid, that intersect if and only if the vertices are adjacent in G . Using Theorem 88, one can prove that every 4-connected triangulation has such a representation. To allow us to work on every triangulation (not only the 4-connected ones) we need to enrich our \perp -intersection representations with the notion of *anchors*².

There are two types of anchors (see Figure 2.16). A *horizontal anchor* is a set $([x_1, x_3] \times y_1) \cup (x_1 \times [y_1, y_2]) \cup (x_2 \times [y_1, y_2])$ where $x_1 < x_2 < x_3$ and $y_1 < y_2$. The *middle corner* of such a horizontal anchor is the point (x_2, y_1) . A *vertical anchor* is a set $(x_1 \times [y_1, y_3]) \cup ([x_1, x_2] \times y_1) \cup ([x_1, x_2] \times y_2)$ where $x_1 < x_2$ and $y_1 < y_2 < y_3$. The *middle corner* of such a vertical anchor is the point (x_1, y_2) . In addition to middle corners, every anchor has a *main corner* which is the point of coordinates (x_1, y_1) for both types of anchors. An anchor can thus be seen as a union of three segments, or as the union of two \perp paths.

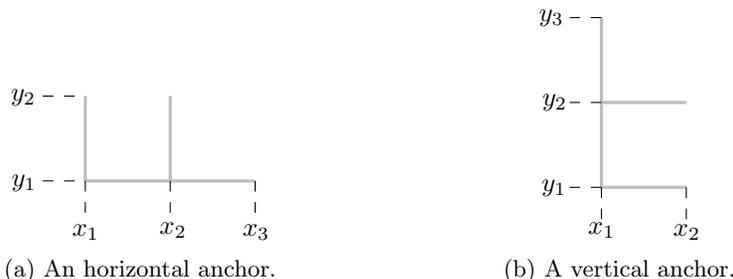


Figure 2.16: The two types of anchors (horizontal and vertical).

Consider a near-triangulation T , and any inner face abc of T . Note that if the \perp paths of a , b and c do not intersect at a common point or segment, then their horizontal (resp. vertical) subpaths lie on three different lines. They thus form a rectangle whose top side belongs to the \perp path with the up-most corner, whose right side belongs to the \perp path with the right-most corner, and whose other sides belong to the third \perp path. Given an \perp -intersection representation of T , an *anchor for abc* is an anchor intersecting the \perp paths of a , b and c and no other path, and such that the middle corner is in the rectangle described by a , b and c as depicted in Figure 2.17.

Definition 90. A full \perp -intersection representation (*abbreviated FLIR*) of a near-triangulation T is an \perp -intersection representation of T together with a set of pairwise disjoint anchors, one for each inner face of T .

Let us now prove that every 2-sided near-triangulation admits an FLIR.

Proposition 91. Every 2-sided near-triangulation has an FLIR such that among the corners of the \perp paths and of the anchors:

²The notion was introduced in [64] under the name of *private region*.

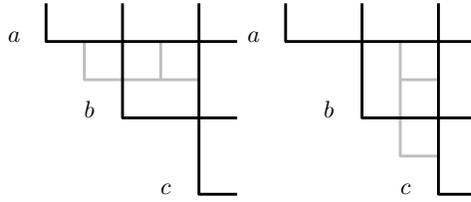


Figure 2.17: The two possible anchors for the \perp paths corresponding to a triangle abc .

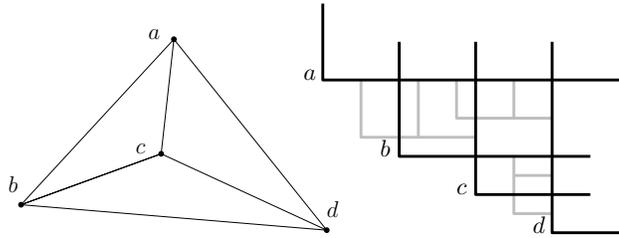


Figure 2.18: Example of a triangulation and a corresponding FLIR (the anchors are drawn in grey).

- from left to right, the first corners are those of vertices b_1, b_2, \dots, b_q and the last one is the corner of vertex a_1 , and
- from bottom to top, the first corners are those of vertices a_1, a_2, \dots, a_p and the last one is the corner of vertex b_1 .

As the \perp of a_i and a_{i+1} (resp. b_i and b_{i+1}) intersect, the FLIR is rather constrained. This is illustrated in Figure 2.19, where the grey region contains the corners of the inner vertices, and the corners of the anchors.

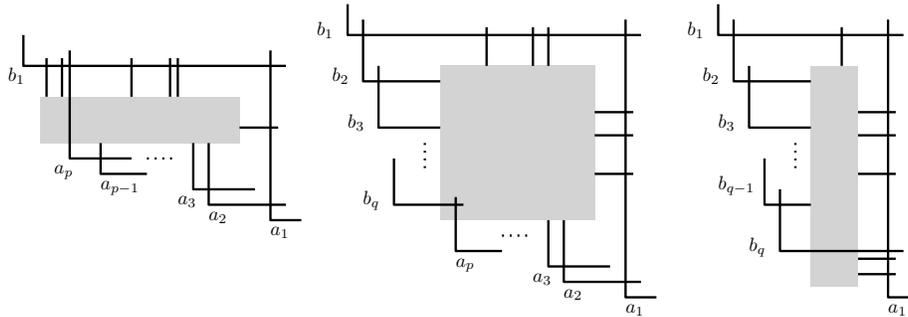


Figure 2.19: Illustration of Proposition 91 when $p > 1$ and $q > 1$, when $p > 1$ and $q = 1$, and when $p = 1$ and $q > 1$.

Proof. We proceed by induction on the number of vertices.

The result clearly holds for the 2-sided near-triangulation with three vertices, no matter if $p = 1$ and $q = 2$, or $p = 2$ and $q = 1$. Let T be a 2-sided near-triangulation with at least four vertices. By Lemma 86 we consider one of the following operations on T :

(a_p -removal) Consider the FLIR of T' obtained by induction and see in Figure 2.20 how one can add an \perp for a_p and an anchor for each inner face $a_p b_j b_{j+1}$ with $q \leq j < q+r$ and for the inner face $a_p a_{p-1} b_{q+r}$. One can easily check that the obtained representation verifies all the requirements of Proposition 91.

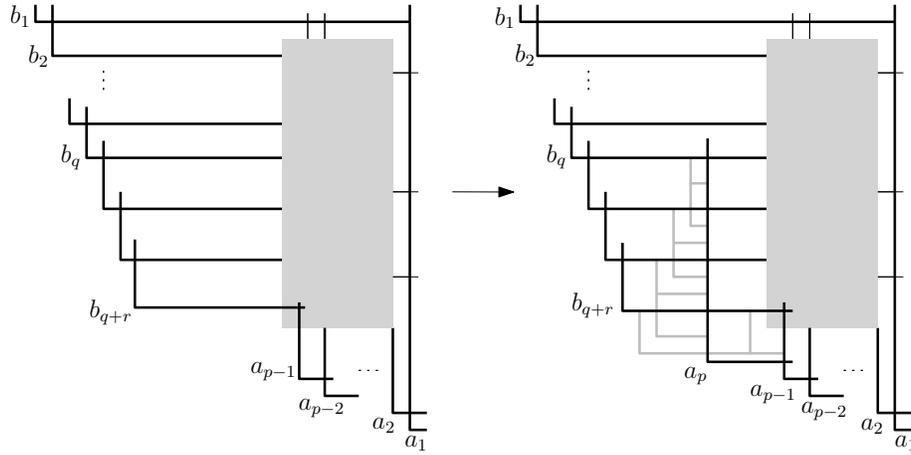


Figure 2.20: The (a_p -removal) operation.

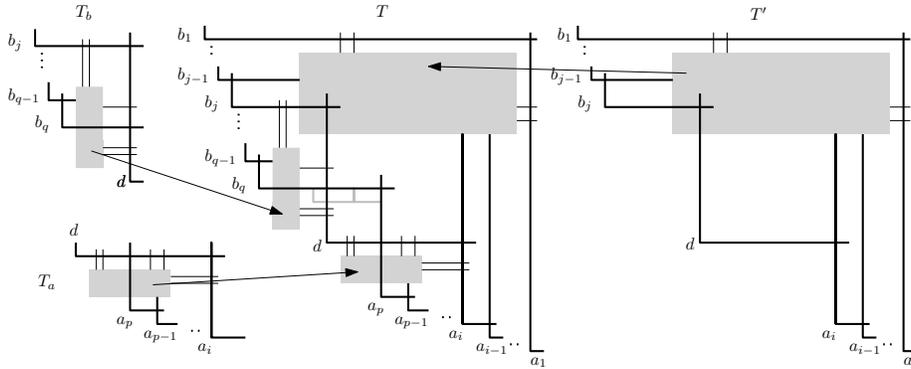


Figure 2.21: The (cutting) operation.

(b_q -removal) This case is symmetric to the previous one.

(cutting) Consider the FLIRs of T' , T_a and T_b . Figure 2.21 depicts how to combine them, and how to add an anchor for $da_p b_q$, in order to get the FLIR

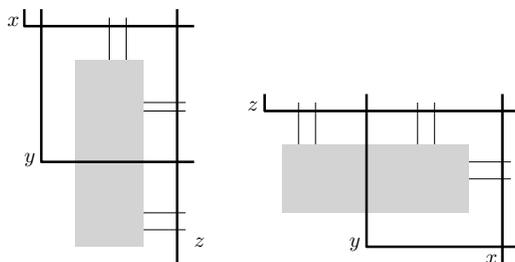


Figure 2.22: Illustration of Proposition 92, and the FLIR obtained after reflection with respect to a line of slope 1.

of T . One can easily check that the obtained representation verifies all the requirements of Proposition 91. \square

We now prove Theorem 84 which asserts that every planar graph is a \perp -intersection graph. It is well known that every planar graph is an induced subgraph of some triangulation (see [39] for a proof similar to the one of Lemma 89). Thus, given a planar graph G , one can build a triangulation T for which G is an induced subgraph. If one can create an FLIR of T , then it remains to remove the \perp paths corresponding to vertices of $T \setminus G$ along with the anchors in order to get an \perp -intersection representation of G . In order to prove Theorem 84, we thus only need to show that every triangulation admits an FLIR. Namely, proving the following proposition completes the proof of Theorem 84.

Proposition 92. *Every triangulation T with outer-vertices x, y, z has an FLIR such that among the corners of the \perp paths and of the anchors:*

- *the corner of x is the top-most and left-most,*
- *the corner of y is the second left-most, and*
- *the corner of z is the bottom-most and right-most.*

Note that in this proposition there is no constraint on x, y, z , so by renaming the outer vertices, other FLIRs can be obtained.

Another way to obtain more FLIRs is by applying a reflection with respect to a line of slope 1. In such an FLIR (see Figure 2.22) among the corners of the \perp paths and of the anchors:

- the corner of x is the bottom-most and right-most,
- the corner of y is the second bottom-most, and
- the corner of z is the top-most and left-most.

This reflection operation is used later in the proof of Proposition 92.

Proof. We proceed by induction on the number of vertices in T . Let T be a triangulation with outer vertices x, y, z .

If T is 4-connected, then it is also a 2-sided near-triangulation. By Proposition 91 and by renaming the outer-vertices x to b_1 , y to b_2 and z to a_1 , T has an FLIR with the required properties.

If T is not 4-connected, then it has a separating triangle formed by vertices a, b and c . We call T_{in} and T_{out} the triangulations obtained from T by removing the vertices outside and inside abc respectively.

By the induction hypothesis, T_{out} has an FLIR verifying Proposition 92 (considering the outer vertices to be x, y, z in the same order as in T). Without loss of generality we can suppose that the \perp paths of a, b and c appear in the following order: the top-most and left-most is b , the second left-most is c and the bottom-most is a . There are two cases according to the type of the anchor of the inner face abc .

If the anchor of abc in the FLIR of T_{out} is vertical (see Figure 2.23a), then applying the induction hypothesis on T_{in} with b, c, a as outer vertices considered in that order, T_{in} has an FLIR as depicted on the Figure 2.23b. Figure 2.23c depicts how to include the FLIR of $T_{in} \setminus \{a, b, c\}$ in the close neighborhood of the anchor of abc . As abc is not a face of T , the close neighborhood of its anchor is indeed available for this operation.

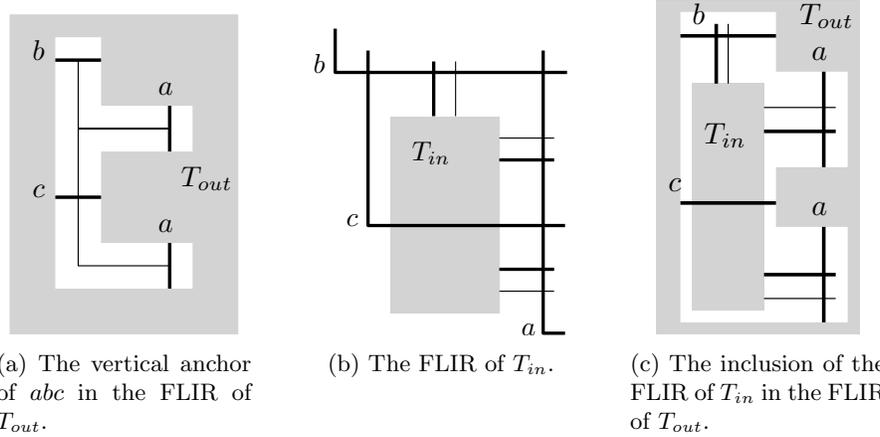


Figure 2.23: FLIR inclusion in the case of a vertical anchor.

Now suppose that the anchor of abc in the FLIR of T_{out} is horizontal (see Figure 2.24a). By application of the induction hypothesis on T_{in} with a, c, b as outer vertices considered in that order, then T_{in} has an FLIR as depicted on the Figure 2.24b. By a reflection of slope 1, T_{in} has an FLIR such that b is the top-most and left-most, c is the second left-most and a is bottom-most (see Figure 2.24c). Similarly to the previous case, we include this last FLIR of $T_{in} \setminus \{a, b, c\}$ in the one from T_{out} (see Figure 2.24d).

As T_{in} and T_{out} cover T , and intersect only on the triangle abc , and as every

inner face of T is an inner face in T_{in} or in T_{out} , these constructions clearly verify Proposition 92. This concludes the proof of the proposition. \square

Note that as for $\{\perp, |, -\}$ -contact representations any \perp -intersection representation with n paths fits into a grid of width and height at most n . Here also, it is open to know whether this bound can be improved. We believe that for the graph G_t described in the previous section any \perp -intersection representation requires a grid of width and height at least $n/2$.

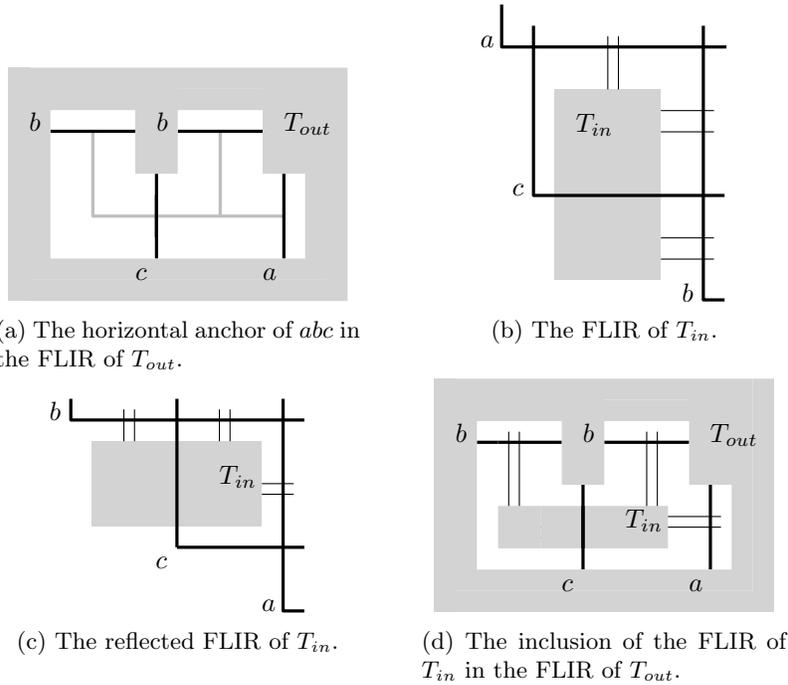


Figure 2.24: FLIR inclusion in the case of a horizontal anchor.

2.5 Conclusion

We hope that 2-sided triangulations will be useful for proving other results on planar graphs. Secondly we wonder if it is possible to generalize the results in \mathbb{R}^d for any d . A convenient definition of an \perp in \mathbb{R}^3 should be find. A generalization could be seen through the work about stair packings. Indeed thick- \perp can be seen as stairs with only one bend in \mathbb{R}^2 . Therefore the question is: what is a stair with only one bend in \mathbb{R}^3 ? Given a good definition of such stairs, it would be interesting to have an analogue of what has been proved in this chapter. More precisely, it may be possible to represent any Dushnik-Miller dimension at most 4 complex Δ satisfying what we call the Helly property, *i.e.* $F \in \Delta$ if and only

if for every pair e of elements of F is in Δ , as the contact complex of stairs with at most one bend in \mathbb{R}^3 .

Chapter 3

Möbius stanchion systems

Stanchions are the objects used to design waiting lines in airports or in cinemas. This kind of sticks are connected to each other thanks to strips. We suppose here that the system of stanchions and strips forms a plane graph. Consider the problem of painting both sides of every strips connecting the stanchions without lifting up the paintbrush and going from one belt to the next one around a stanchion. To do so, you will paint the strips of one face at a time. Therefore you will need to lift the paintbrush as many times as the number of faces of the corresponding plane graph.

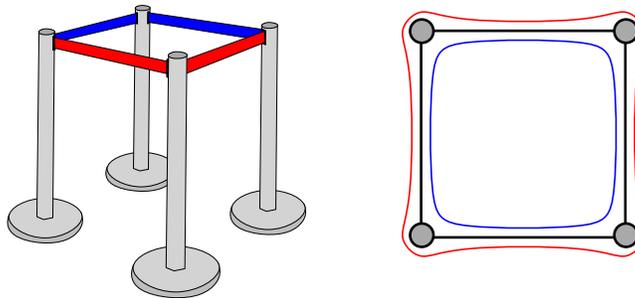


Figure 3.1: A system of 4 stanchions and 4 strips.

As a lazy graph theorist you want to twist the strips between stanchions in a Möbius fashion such that you do not need to lift up your paintbrush. We call such a twist a Möbius stanchion system (or MSS for short) and we investigate the space of all the MSSs of a planar graph. Our main result is that all the MSSs are connected by a series of two elementary operations.

These notions are connected to the study by Širáň and Škoviera [99] of unicellular embeddings of signed graphs and to the study of the Penrose polynomial by Aigner [25].

This problem lies at the crossroad of the following topics:

Unicellular embeddings of plane graphs. Questions on unicellular embeddings have been studied by several researchers.

Definition 93. *A unicellular map is the embedding of a connected graph in a surface in such a way that the complement of the graph is a topological disk.*

Bernardi and Chapuy [29] are counting the number of unicellular maps on non-orientable surfaces with a prescribed number of edges. Edmonds [47], Ringel [95] and Stahl [101] showed that every connected graph admits a unicellular embedding on some closed surface.

In our terminology, an edge twist corresponds to put a crosscap on the edge of the embedding of a planar graph. A Möbius stanchion system gives therefore an embedding of a planar graph on a surface, which is possibly non-orientable, such that the embedding is unicellular.

Unicellular embeddings of signed graphs. There is also an interpretation of our work in terms of the theory of signed graphs. See [108] for a comprehensive study of signed graphs.

Definition 94. *A signed graph is a graph such that every edge is labelled $+1$ or -1 . Negating an edge of a signed graph consists in replacing the label of this edge by its opposite. A closed walk of a signed graph is said to be balanced if it contains an even number of negative edges. Otherwise it is called unbalanced.*

As in the classical theory of graphs, we can embed signed graphs on surfaces.

Definition 95. *An embedding of a signed graph on a topological surface S is an embedding $i : G \rightarrow S$ of the underlying unsigned graph on the surface S such that for every closed walk C on the graph, the embedding $i(C)$ of the closed walk is orientation preserving in S if and only if C is balanced. A unicellular embedding of a signed graph G is an embedding such that there is exactly one face.*

A natural question is to ask whether a signed graph admits a unicellular embedding. Širáň and Škoviera [99] found a combinatorial characterization of such signed graphs. In our work, we start with an unsigned plane graph G (equipped with an embedding on the sphere) and we look for signed graphs \tilde{G} , whose unsigned graph is G , such that by putting a crosscap on the sphere on every negative edges of \tilde{G} , the embedding of G becomes a unicellular embedding of \tilde{G} on the new surface. We call this kind of signed graphs Möbius stanchion systems and we show that they are connected via two elementary operations which consists in negating certain edges and negating simultaneously certain pair of edges.

The Penrose polynomial. The Penrose polynomial, defined implicitly in [93], is an analogue of Tutte's polynomial defined only for planar graphs. A formal definition of the Penrose polynomial will be given later. In his study of the polynomial, Aigner [25] developed equivalent notions of what we call

Möbius stanchion systems. In his terminology, edges can be “changing” or “non-changing”. This will correspond to our “twisted” or “non-twisted” edges. As a corollary of his study, he found a combinatorial characterization of Möbius stanchion systems. But his goal is different from our goal. Here, in this work, we study the MSSs of a planar graph, which are particular subgraphs of the initial graph, and their connections, while Aigner is studying the properties of the Penrose polynomial.

Furthermore the notion of the Penrose of polynomial has been extended to binary matroids [27] and to embedded graphs [50]. Links to the Tutte polynomial have been shown in [83].

Organization of the chapter. We start by defining formally Möbius stanchion systems. Then in the second section we investigate the space of the MSSs of a given planar graph and how MSSs are connected each together via elementary operations.

3.1 Möbius stanchion systems

In the following, G is a connected plane multi-graph and we denote G^* its dual.

Definition 96. *A Möbius twisting is a subgraph of G , indicating which edges are twisted.*

Given a Möbius twisting M , M^* denotes the subgraph of G^* with all the vertices of G^* and whose edges are the dual edges of M .

Definition 97. *We define the painting walk $P(G, M) = (V_P, E_P)$ of G under the Möbius twisting M as the following graph:*

- *For each angle α created by two consecutive edges e and f around a vertex x , we add a vertex $x_\alpha \in V_P$.*
- *For any edge $e \in G$, there are four angle vertices in V_P associated with e . If $e \notin M$, we connect the angle vertices on the same faces. Otherwise, we say that the edge is twisted and we connect the angle vertices of distinct faces and vertices.*

Since $P(G, M)$ is 2-regular, the graph $P(G, M)$ is a union of cycles. Examples are given in the Figure 3.2.

Definition 98. *Given a Möbius stanchion M , we denote by $c(M)$ the number of connected components of $P(G, M)$.*

We investigate when the painting walk is just a single cycle, that is when $c(M) = 1$. In this case we say that M is a *Möbius stanchion system* (or MSS for short) of G .

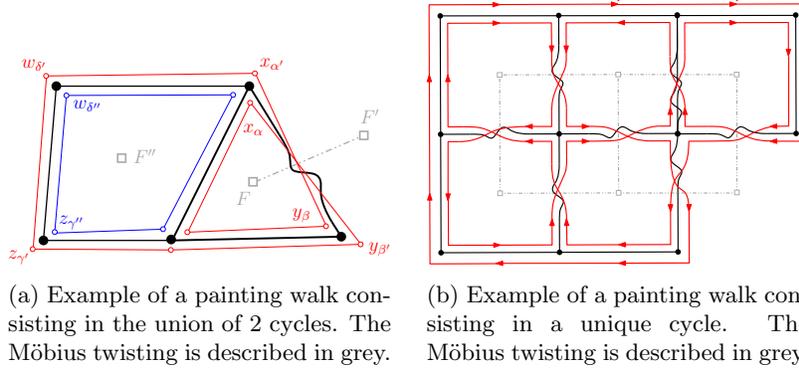


Figure 3.2: Examples of Möbius twistings.

The Penrose polynomial. Finally, we translate the Theorem of Aigner according to our language. We denote by $Z(G)$ the set of the Eulerian subgraphs of G which is called the *cycle space* of G and we denote by $K(G)$ the family of edge sets that span the cuts of the graph which is called the *cut space* of G . Note that $Z(G)$ and $K(G)$ are vector spaces on the finite field \mathbb{F}_2 . As a consequence of a theorem of Aigner [25], we deduce a combinatorial characterization of MSSs.

Theorem 99 (Aigner). *Let M be a Möbius twisting of G . Then*

$$c(M) = 1 + \dim\{C \in Z(G) : M \cap C \in K(G)\}$$

where we see M as a subset of the edges of G consisting in the twisted edges of G .

As a corollary we have the following characterization of MSSs:

Corollary 100. *Let M be a Möbius twisting of G . Then M is an MSS if and only if*

$$\{C \in Z(G) : M \cap C \in K(G)\} = \{\emptyset\}$$

In his article, Aigner studied the following polynomial:

Definition 101. *The Penrose polynomial of G is defined by*

$$P(G, \lambda) = \sum_{M \subseteq E} (-1)^{|M|} \lambda^{c(M)}$$

This polynomial is important because it is an analogue of Tutte's polynomial for planar graphs: it reveals coloring properties of the graph as shown by the following theorem.

Theorem 102 (Penrose). *Let G be a 3-regular connected plane graph. Then the number of 3 colorings of G is equal to*

$$P(G, 3) = \left(-\frac{1}{4}\right)^{\frac{n}{2}} P(G, -2)$$

where n is the number of vertices of G .

3.2 The space of MSSs

3.2.1 Preliminary remarks

Notice that if no edge is twisted, then the painting walk is the union of precisely f cycles, where f is the number of faces of G . Twisting an edge allows at most two cycles to be merged, hence in order to obtain a single cycle, we must have connected all the cycles. In other words:

Lemma 103. *For every MSS M of G , then M^* is connected.*

Proof. Suppose that M^* is not connected. There exists N a subgraph of G such that $N^* \subsetneq M^*$ is a connected component of M^* . Consider C the border of the union of the faces contained in N^* . Let C be the set of edges with an incident face in N^* and one not in N^* . Since N^* is a connected component, an edge on C separates a face in N^* from a face not in N^* . Therefore no edge of C is twisted. Thus starting from any angle vertex on a face of N^* , we can only visit angle vertices on faces of N^* and the painting walk contains at least two cycles, a contradiction. \square

On the other hand, one can prove by induction that:

Lemma 104. *For any Möbius twisting M of G such that M^* is a spanning tree of G^* , then M is an MSS of G .*

Proof. Let us prove a stronger result: any Möbius twisting T of G such that T^* is a tree is an MSS of the vertices contained in T^* , *i.e.* the angle vertices contained in the faces in T^* form a single cycle of the painting walk. In particular, this will imply the desired result. We will proceed by induction on the number of vertices of T^* .

If the tree T^* consists of an isolated vertex F then the painting walk contains a cycle that corresponds exactly to the face F .

Consider now a Möbius twisting T of G such that T^* is a spanning tree and contains more than two vertices. Necessarily T^* has a leaf F which corresponds to an edge e of G . By induction $T - e$ is a MSS of the faces of $T^* - F$. The face F is connected with the remaining of the tree by only one twisted edge in T^* . This will add the face F to the MSS of the faces of $T^* - F$ as shown in Figure 3.3, resulting in a MSS of the faces of T^* . \square

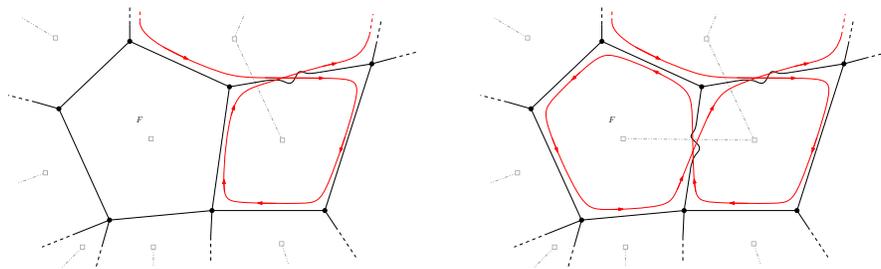


Figure 3.3: Leaf induction of Lemma 104

By classical duality results on planar graph, we have that M^* is a spanning tree of G^* if and only if M is the complement of a spanning tree of G (a.k.a a *cospanning tree* or *minimal feedback edge set*). Lemmas 103 and 104 characterize the minimal MSSs in terms of number of twisted edges. Indeed the minimum number in an MSS is $f - 1$, and this lower bound is achieved by all cospanning trees.

Theorem 105. *Any MSS of G contains at least $f - 1$ edges. Furthermore cospanning trees of G are the minimal MSSs of G containing exactly $f - 1$ edges.*

However, some MSSs are not cospanning trees, as the one in Figure 3.2 for instance. Can we characterize the MSSs with more than $f - 1$ edges? Toward this goal we will define two operations acting on the space of MSSs.

3.2.2 Preserving operations

The first natural operation to consider is the twisting of an edge of G .

Definition 106 (Twisting of an edge). *Let M be a Möbius twisting of G and let e be an edge of G . We denote by $M \sim e$ the Möbius twisting of G consisting of $M \setminus e$ if $e \in M$ and $M \cup e$ if $e \notin M$.*

Notice that the twisting of an edge is an involution, *i.e.* $(M \sim e) \sim e = M$. Some twisting of edges will preserve MSSs. Indeed, for any cospanning tree T , an edge $e \in T$ cannot be removed since T is minimal. In order to characterize the preserving twistings of M , we fix an arbitrary orientation of the cycle $P(G, M)$ and we distinguish two types of edges:

Definition 107. *Let M be a MSS of G . Let e be an edge of G . If the cycle $P(G, M)$ goes along e in opposite directions, we say that e is a two-ways edge, otherwise we say that e is a one-way edge.*

Notice that this definition is independent of the orientation of the cycle, and applies for both twisted and non-twisted edges (see Figure 3.4).

One can show by enumeration of all possible cases that two-ways edges are exactly the ones preserving the MSS under the twisting operation.

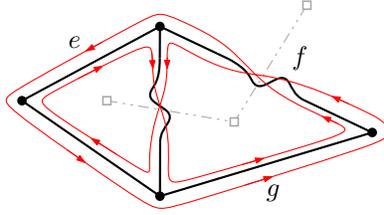


Figure 3.4: Example of a Möbius twisting where e is a two-ways edge and f and g are one-way edges.

Lemma 108 (Simple twisting). *Let M be a MSS of G , and e an edge of G . Then $M \sim e$ is a MSS of G if and only if e is a two-ways edge.*

We call the twisting operation of a two-ways edge a *simple twisting*.

Proof. Consider x and y the vertices of G such that $e = (xy)$ and let $x_\alpha, x_{\alpha'}, y_\beta$ and $y_{\beta'}$ be the corresponding vertices of the painting walk $P(G, M)$ where x_α and y_β lie on the same face of G . Without loss of generality, we can suppose that starting from x_α and following the orientation of the cycle, we immediately reach y_β or $y_{\beta'}$, depending on whether the edge e is in M or not. Using the fact that M is a MSS of G , we conclude that the cyclical appearance ordering of the vertices $x_\alpha, x_{\alpha'}, y_\beta$ and $y_{\beta'}$ on the cycle $P(G, M)$ can only be one of the four following cyclical orderings:

- i) $[x_\alpha, y_\beta, y_{\beta'}, x_{\alpha'}]$: then e is a two-ways edge
- ii) $[x_\alpha, y_{\beta'}, y_\beta, x_{\alpha'}]$: then e is a two-ways edge
- iii) $[x_\alpha, y_\beta, x_{\alpha'}, y_{\beta'}]$: then e is a one-way edge
- iv) $[x_\alpha, y_{\beta'}, x_{\alpha'}, y_\beta]$: then e is a one-way edge

Cases *i)* and *ii)* are dual under the twisting of e , proving that if e is a two-ways edge, then $M \sim e$ is a MSS of G . Twisting edge e in cases *iii)* and *iv)* will create two disjoint cycles in the painting walk, thus the operation would not preserve the MSS, proving the converse. \square

Starting from a cospanning tree, one can possibly apply the simple twisting operation to obtain new MSSs with more than $f - 1$ twisted edges. However, some MSSs are still unreachable using only this operation. In the right example of Figure 3.2 all the twisted edges of this MSS are one-way edges and thus cannot be untwisted. We now define another preserving operation that twists pairs of one-way edges, under some conditions on the way the two edges are connected.

Definition 109. *Let M be a Möbius twisting of G . Let xy be an edge of G and let $x_\alpha, x_{\alpha'}, y_\beta, y_{\beta'}$ be the corresponding angle vertices. We define the two parts of xy as the two sets $\{x_\alpha, y_\beta\}, \{x_{\alpha'}, y_{\beta'}\}$ if xy is not twisted and as the two sets $\{x_\alpha, y_{\beta'}\}, \{x_{\alpha'}, y_\beta\}$ otherwise.*

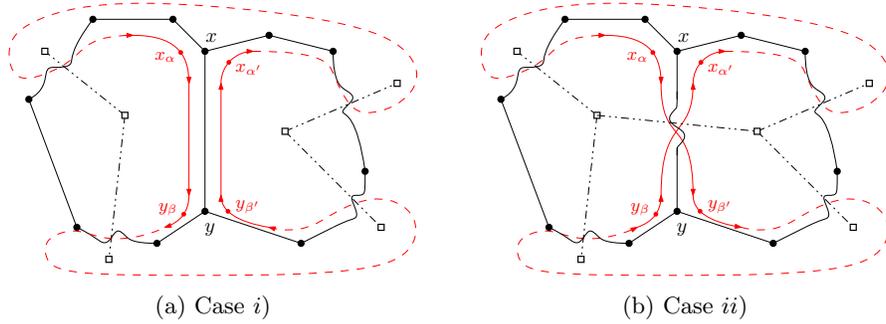


Figure 3.5: Cases *i*) and *ii*) are dual under the twisting operation.

Lemma 110 (Good and bad connections). *Let M be a MSS of G . Let e and f be two one-way edges of G . Then there are only two types of connection between both edges. More precisely, we denote by e^+ and e^- (resp f^+ and f^-) the two parts of the painting walk around e (resp. f). Modulo the inversion of the notation of the parts, the parts are crossed in one of the following cyclical order:*

- i) $[e^+, f^+, e^-, f^-]$*
- ii) $[e^+, f^+, f^-, e^-]$*

We refer to case *i*) as a good connection and to case *ii*) as a bad connection, and we say that the pair (e, f) is well-connected (resp. badly-connected) in case *i*) (resp. in case *ii*)).

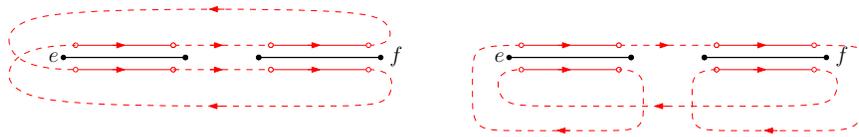


Figure 3.6: On the left: a pair of good connected one-way edges. On the right: a pair of bad connected one-way edges

Proof. Suppose that both e^+ and e^- are followed by parts of f . Then up to the inversion of the notation of f^+ and f^- , we are in case *i*).

If not, both parts of e and both parts of f appear consecutively along $P(G, M)$. By inverting e^+ and e^- , we can suppose that e^- is followed directly by e^+ . By inverting f^+ and f^- , we can suppose that we have in cyclical order $[e^+, f^+, f^-, e^-]$, which is case *ii*). \square

Again, notice that this definition is independent of the orientation of the cycle and valid for both twisted and non-twisted edges. One can show by enumeration of the two possible cases that the pairs of edges that form a good

connection are exactly the ones that can be simultaneously twisted while preserving the fact that the Möbius twisting is a MSS:

Lemma 111 (Double twisting). *Let M be a MSS of G and (e, f) be a pair of one-way edges. Then $M \sim e \sim f$ is a MSS of G if and only if e and f form a good connection.*

Proof. Using Lemma 110, we only have two types of connection to check:

- Good connection: Figure 3.7 and Figure 3.8 show that the double twisting preserves the MSS, no matter which edges is twisted first.
- Bad connection: the twisting of e will result in the creation of two cycles in the painting walk, and one of them does not contain any part of f . Thus the twisting of f does not act on this isolated cycle and the resulting painting walk has at least two cycles, breaking the MSS.

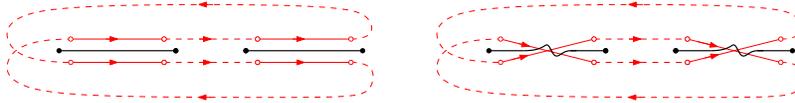


Figure 3.7: Two good connections dual under the double twisting.

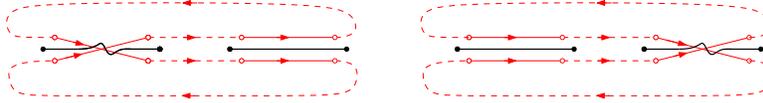


Figure 3.8: The other two good connections are also dual.

□

We are now ready to study the structure of the MSSs of a plane graph, using the two preserving operations defined in this section: the simple and the double twisting.

3.2.3 Reachable MSSs

In this section, we prove that these two preserving operations suffice to reach the MSSs. More precisely, we define the following set:

Definition 112. *Let M be a MSS of G . We define $\mathcal{R}(G, M)$ as the set of MSSs of G reachable from M with simple and double twistings.*

We first prove that any MSS is either a cospanning tree, or reachable from a cospanning tree:

Lemma 113. *Let M be a MSS of G . There exists a cospanning tree T of G such that $M \in \mathcal{R}(G, T)$, or equivalently $T \in \mathcal{R}(G, M)$.*

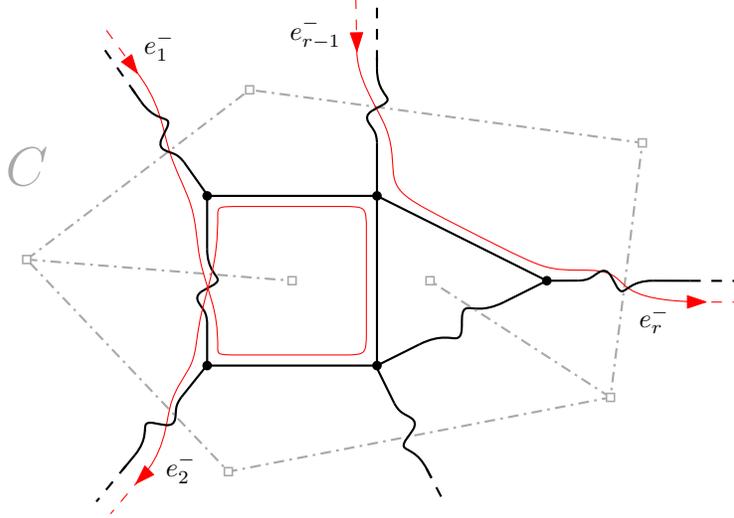


Figure 3.9: The cycle C of M^* and the first edge sides crossed by the painting walk.

Proof. We proceed by induction on the number m of edges of M . According to Lemma 103, M^* is connected and therefore $m \geq f - 1$ where f is the number of faces of G . If $m = f - 1$, then M is a cospanning tree and the result holds. Suppose now that $m > f - 1$.

If M contains a two-way edge (resp. a pair of well-connected one-way edges), we could apply Lemma 108 (resp. Lemma 111) to obtain the result by induction hypothesis. In fact, we claim that one of these two scenarii must hold.

For contradiction, we suppose that all the edges in M are one-way edges and that we have no pair of good connected edges in M . Since M^* has size at least f , this implies that M^* contains at least one cycle C .

We follow the painting walk $P(G, M)$ according to an arbitrary direction and we consider an arbitrary starting edge side of the painting walk. As M is an MSS, every edge sides is crossed by the painting walk. For any edge e , we note e^- the part of the edge e that is crossed first by the painting walk and e^+ the one that is crossed second. We note e_1, \dots, e_r the first crossed edges of the cycle C before the painting walk crosses again one of these edges. Therefore, there exists $i \in [1, r]$ such that the edge sides $e_1^-, e_2^-, \dots, e_r^-, e_i^+$ are consecutive in the following cyclic order:

$$[e_1^-, e_2^-, \dots, e_r^-, e_i^+]$$

(see Figure 3.9 for an example). As e_r is a one-way edge and as G is planar, we can suppose without loss of generality that both parts of e_r are going outside C . It is consequently impossible that e_r^+ follows e_r^- and we deduce that $i \neq r$.

We conclude that we have the following cyclic order:

$$[e_i^-, e_r^-, e_i^+, e_r^+]$$

which contradicts the fact that the pair of twisted edges (e_i, e_r) is not in good connection. Hence it is impossible that M does not contain a two-ways edge nor a well-connected pair of one-way edges. This concludes the proof of the lemma. \square

Lemma 114 (Propp [94]). *Let G be a plane graph. Consider the graph of the spanning trees of G where two spanning trees are connected if and only they differ on two edges. Then this graph is connected and is of diameter at most $n(n-1)/2$ where n is the number of vertices of G .*

Lemma 115. *For any two cospinning trees S and T of G , $T \in \mathcal{R}(G, S)$.*

Proof. According to Lemma 114, we can assume that S^* and T^* are two spanning trees differing on only two edges that we denote by $e \in S$ and $f \in T$. Suppose that $P = S + f = T + e$ is also a MSS. Then by Lemma 108, both e and f are two-ways edges of P , and we can therefore go from S to T by the simple twistings of e and f .

Otherwise, f is a one-way edge with respect to the the MSS S . Lemma 103 shows that e is also a one-way edge with respect to the MSS S since it cannot be removed. By Lemma 111, e and f form a good connection since S and T are MSSs, and we can therefore perform a double twisting to go from S to T . \square

As a direct corollary of the previous results, we get:

Theorem 116. *For any MSS M of G , $\mathcal{R}(G, M)$ is the set of all MSSs of G .*

We define the graph $\mathcal{R}(G)$ of the MSSs connected by the two elementary operations. Its vertices are the MSSs of G . Two MSSs are connected if and only if there is a simple twist or a double twist which can be performed from one to get the other one. The previous result asserts that $\mathcal{R}(G)$ is connected. There is at most 2^m vertices in $\mathcal{R}(G)$, where m is the number of edges of G , but we now show that this graph is rather “compact” in the sense that its diameter is in $O(n^2)$.

Theorem 117. *The diameter of $\mathcal{R}(G)$ is at most $f(f-1)/2 + 2m$ where $f = 2 + m - n$ is the number of faces of G .*

Proof. If two cospinning trees differ on two edges then these edges form a one-way pair of good connected edges and thus they are connected in $\mathcal{R}(G)$ by a double twist. Reciprocally if two cospinning trees are connected by a double twist then they differ on two edges. Thus the cospinning trees graph is the induced subgraph of $\mathcal{R}(G)$ restricted to cospinning trees. According to Lemma 113, any MSS of G is at distance at most m from a cospinning tree of G . From Lemma 114, we deduce the conclusion. \square

3.3 Conclusion

We wonder if Theorem 116 is still true when the graph is embedded on any orientable surface. Indeed, the arguments given in the proof are not true on the torus but may be adapted to fit on any orientable surface.

Another question is related to the number of MSSs. Minimal solutions are cospanning trees and thus their number can be computed in polynomial time. Therefore we ask the following question: does there exist a polynomial time algorithm for computing the number of MSSs of a plane graph?

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Abstract. In this thesis, we look for generalizations of some properties of planar graphs to higher dimensions by replacing graphs by simplicial complexes.

In particular we study the Dushnik-Miller dimension which measures how a partial order is far from being a linear order. When applied to simplicial complexes, this dimension seems to capture some geometric properties. In this idea, we disprove a conjecture asserting that any simplicial complex of Dushnik-Miller dimension at most $d + 1$ can be represented as a TD-Delaunay complex in \mathbb{R}^d , which is a variant of the well known Delaunay graphs in the plane. We show that any supremum section, particular simplicial complexes related to the Dushnik-Miller dimension, is collapsible, which means that it is possible to reach the single point by removing in a certain order the faces of the complex. We introduce the notion of stair packings and we prove that the Dushnik-Miller dimension is connected to contact complexes of such packings.

We also prove new results on planar graphs. The two following theorems about representations of planar graphs are proved: any planar graph is an \perp -intersection graph and any triangle-free planar graph is an $\{\perp, |, -\}$ -contact graph. We introduce and study a new notion on planar graphs called Möbius stanchion systems which is related to questions about unicellular embeddings of planar graphs.

Résumé. Dans cette thèse on cherche à généraliser certaines propriétés des graphes planaires aux dimensions supérieures en remplaçant les graphes par des complexes simpliciaux.

En particulier on étudie la dimension de Dushnik-Miller qui mesure à quel point un ordre partiel ressemble à un ordre total. Appliquée aux complexes simpliciaux, cette dimension semble capturer des propriétés géométriques. Concernant ce sujet, on infirme une conjecture assurant que n'importe quel complexe simplicial de dimension de Dushnik-Miller au plus $d + 1$ peut être représenté par un complexe de TD-Delaunay dans \mathbb{R}^d , qui est une variante des graphes de Delaunay dans le plan. On montre que toute section supremum, qui est un complexe simplicial particulier relié à la dimension de Dushnik-Miller, est "collapsible", c'est-à-dire que l'on peut atteindre un point unique en retirant dans le bon ordre les faces du complexe. On introduit la notion d'empilements d'escaliers et on démontre que la dimension de Dushnik-Miller est reliée aux complexes de contacts de tels empilements.

On démontre aussi de nouveaux résultats sur les graphes planaires. Les deux résultats suivants sur la représentabilité des graphes planaires sont démontrés : tout graphe planaire est le graphe d'intersection de \perp et tout graphe planaire sans triangle est le graphe de contact de $\{\perp, |, -\}$. On introduit et étudie une nouvelle notion sur les graphes planaires que l'on appelle "Möbius stanchion systems" qui sont reliés à des questions sur les plongements unicellulaires des graphes planaires.