Quadratic vector fields with univalued solutions in dimension 3 and higher

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Champs de vecteurs quadratiques avec solutions univaluées en dimension 3 et supérieure

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Résumé

Il est vraiment remarquable le fait que parmi les exemples connus de champs de vecteurs quadratiques semicomplets, il est toujours possible de trouver des coordonnées linéaires où le champ de vecteurs correspondant a tous—ou "presque tous"— ses coefficients dans l'ensemble des nombres réels. En effet, les coefficients sont très souvent entiers.

L'espace des champs quadratiques en $\mathbb{C}^3$, à équivalence linéaire près, est une famille de dimension complexe 9. Le résultat principal de cette thèse établi que les degrés de liberté pour déterminer les coefficients d'un champ de vecteurs semicomplet (sous des hypothèses génériques très faibles) est au plus 3. Autrement dit, il y a 3 paramètres à partir desquels tous les autres coefficients peuvent être obtenus dans un sens naturel. En particulier, si ces 3 coefficients sont réels, alors tous les coefficients sont réels.

Nous commençons par considérer un champ quadratique générique $\mathcal{Z}$ en $\mathbb{C}^n$, homogène et qui n'est pas un multiple du champ de vecteurs radial. Le premier pas dans notre travail sera de construire une forme canonique pour le champ de vecteurs X induit sur $\mathbb{C}P(n-1)$; Cette forme canonique est invariante sous l'action d'un groupe particulier de symétries.

Lorsque $n = 3$, nous pouvons améliorer notre approche en étudiant les singularités non pas sur le diviseur exceptionnel mais sur l'hyperplan à l'infini $\Delta \cong \mathbb{C}P(2)$. Dans ce contexte la dynamique du feuilletage devient assez simple alors que les singularités ont tendance à devenir dégénérées. L'avantage est que l'on peut travailler avec des singularités dégénérées avec la technique des éclatements successifs. Ceci aboutit a des expressions simples pour les valeurs propres directement en terme des coefficients de X.

Abstract

It is a remarkable fact that among the known examples of quadratic semicomplete vector fields on $\mathbb{C}^3$, it is always possible to find linear coordinates where the corresponding vector field has all—or "almost all"—coefficients in the real numbers. Indeed, the coefficients are very often integral.

The space of quadratic vector fields on $\mathbb{C}^3$, up to linear equivalence, is a complex 9-dimensional family. The main result of this thesis establishes that the degree of freedom in determining the coefficients of a semicomplete vector field (under very mild generic assumptions) is at most 3. In other words, there are 3 parameters from which all remaining parameters are determined. Moreover if these 3 parameters are real, then so is the vector field.

We start by considering a generic quadratic vector field $\mathcal{Z}$ on $\mathbb{C}^n$ that is homogeneous and is not a multiple of the radial vector field. The first step in our work will be to construct a canonical form for the induced vector field X on $\mathbb{C}P(n-1)$. This canonical form will be invariant under the action of a specific group of symmetries.

When $n = 3$, we then push further our approach by studying the singularities not lying on the exceptional divisor but at the hyperplane at infinity $\Delta \cong \mathbb{C}P(2)$. In this setting the dynamics of the foliation turn out to be quite simple while the singularities tend to be degenerated. The advantage is that we can deal with degenerated singularities with the technique of successive blow-ups. This leads to simple expressions for the eigenvalues directly in terms of the coefficients of X.

Contents

Introduction

A homogeneous, polynomial vector field of degree 2 on $\mathbb{C}^n$ is called a *quadratic vector field*. This thesis is devoted to the study of quadratic vector fields on $\mathbb{C}^n$, albeit with emphasis on $\mathbb{C}^3$, having complex solutions that admit a maximal domain of definition in $\mathbb{C}$. In this sense, the material is clearly related to problems first studied by Painlevé, Chazy and their followers about solutions of differential equations.

A vector field that admits a maximal domain of definition is said to be semi-complete. We refer the reader to Section 1.2 for the accurate definition of what is meant by a maximal domain of definition. This means in particular that every solution of the vector field in question is single-valued as a complex function defined on some open set of $\mathbb{C}$. For example the solution of the differential equation $dy/dt = 1/t$ is given by the logarithm which is a multivalued function and therefore fails to have a maximal domain of definition in our sense. One of the most important properties of semicomplete vector fields is that if $U' \subset U$ and X is semicomplete on U, then the restriction of X to U' is semicomplete on U'. In particular, we can talk about semicomplete germs of vector fields (see [23]).

Let V_n be the space of quadratic homogeneous vector fields on $\mathbb{C}^n$. A vector field $X \in V_n$ has generically $2^n - 1$ radial orbits, i.e., lines on $\mathbb{C}^n$ passing through the origin that are invariant by X. The foliation induced by the blow-up of X at the origin of $\mathbb{C}^n$ has $2^n - 1$ singular points (one for each radial orbit) on the exceptional divisor ($\cong \mathbb{CP}(n-1)$) which is left invariant by the foliation. Every one of these singular points have n eigenvalues: one transverse to the divisor (say λ) given by the radial orbit and other $n-1$ tangent to the divisor (say λ_i, $i \in \{1, ..., n-1\}$). We define the eigenvalues of the vector field associated to one radial orbit ρ as λ_i/λ , $i \in \{1, ..., n-1\}$. A very interesting property—valid for generic quadratic vector fields with isolated singularity—is that these eigenvalues happen to be integral as soon as the vector field X is semicomplete (see for example Guillot [8]).

Nowadays there are many rather non-trivial examples of quadratic semicomplete vector fields on $\mathbb{C}^3$ (see for example Guillot [8] and Guillot [11]). It is a remarkable fact that, among all these examples, it is always possible to find linear coordinates where the corresponding vector field has all—or "almost all"—coefficients in the real numbers. Indeed, the coefficients are very often integral.

A first attempt at explaining this phenomenon would be to consider the fact that the eigenvalues of the singular points are integral, as pointed out above. However, it is very hard to connect the coefficients of a quadratic vector field with the spectrum of the singular points so as to be able to conclude things from the real/integral character of the spectrum.

In more technical terms, the Baum-Bott map associates to a quadratic vector field a type of information directly connected with its spectrum. The naive point of view in the above question would be to try to make sense of an "inverse map" for the Baum-Bott map.

Naturally this question is very hard, and technically impossible, since, for example, for quadratic vector fields on $\mathbb{C}^3$ the Baum-Bott map is known to have degree at most 240 see [16]. Yet, the bulk of this work is essentially devoted to try to make sense of this inverse in the sense that we try to conclude—as far as possible—that a quadratic semicomplete vector field has real/integral coefficients.

Recall first that the space of quadratic vector fields on $\mathbb{C}^3$, up to linear equivalence, is a complex 9-dimensional family. Our main result establishes that the degree of freedom in determining the coefficients of a semicomplete vector field (under very mild generic assumptions) is at most 3. In other words, there are 3 parameters from which all remaining parameters are determined. Moreover if these 3 parameters are real, then so is the vector field.

More precisely, consider a generic quadratic vector field $\mathcal{Z}$ on $\mathbb{C}^3$ that is homogeneous and is not a multiple of the radial vector field $u\partial/\partial u + v\partial/\partial v + w\partial/\partial w$. Let $\widetilde{\mathbb{C}^3}$ be the blow-up at the origin of $\mathbb{C}^3$. We denote by X the pull-back of $\mathcal{Z}$ by the blow-up map. In affine coordinates (x, y, z) of $\widetilde{\mathbb{C}^3}$ the vector field X has the form $X = zY$ with $Y = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y + zH(x,y)\partial/\partial z$ and

$$\begin{aligned} P(x,y) &= ax + by + cx^2 + dxy + ey^2 + x(Ax^2 + Bxy + Cy^2) \\ Q(x,y) &= a'x + b'y + c'x^2 + d'xy + e'y^2 + y(Ax^2 + Bxy + Cy^2)\ , \\ H(x,y) &= D + Ex + Fy - (Ax^2 + Bxy + Cy^2)\ . \end{aligned}$$

Our main result can be stated as follows:

Theorem 2.2.8. *Assume that X is semicomplete and that Conditions I through IV are satisfied. Then, in suitable affine coordinates for the blow-up of C^3, the vector field X takes on the form $X = zY$ with $Y = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y + zH(x,y)\partial/\partial z$ where*

$$\begin{aligned} P(x,y) &= \gamma_1 + ax + d^2 r_1 y + cx^2 + dxy + x^2 y \\ Q(x,y) &= \gamma_2 + c^2 r_2 x + b'y + cxy + dy^2 + xy^2\ , \\ H(x,y) &= D + cr_3 x + dr_4 y - xy\ , \end{aligned}$$

where r_i is a rational number ($i = 1, \ldots, 4$) and where $c, d \neq 0$. Furthermore γ_1 and γ_2 are (explicit) linear combinations over $\mathbb{Q}$ of the coefficients a, b', c, and d. Similarly, unless b' is already determined (linearly over $\mathbb{Q}$ from the coefficients a, c, d) then D must also be an explicit linear combinations over $\mathbb{Q}$ of the coefficients a, b', c, and d.

Actually, as a by-product of the proof of Theorem 2.2.8 we actually obtain an extra—nonlinear—relation between the coefficients a, b', c, d.

$$\hat{r} = \frac{a + b' + 3c + 3d + 4}{(a + 2c + d + 2)(b' + c + 2d + 2) - (c^2 r_2 + c + 1)(d^2 r_1 + d + 1)} \in \mathbb{Q}\,.$$

Up to assuming that the rational number $\hat{r}$ on the left side is different from $(b' + c + 2d + 2)^{-1}$, we can "solve" this relation for a. In other words, unless $b' + c + 2d + 2$ is a rational number—which of course provides directly a new relation between b', c, and d—we obtain in particular the following corollary:

Corollary. *If the 3 coefficients b', c and d are real, then all the coefficients of P, Q and H , as given above, are real as well.*

We have also found some interesting results by adding the assumption that the restriction to the exceptional divisor of the foliation associated with X leaves a line invariant. This means that the initial quadratic vector field $\mathcal{Z}$ leaves some plane through the origin invariant. Naturally we also assume that the remaining conditions of Theorem 2.2.8 are satisfied.

To be more accurate, we consider the blow-up X of $\mathcal{Z}$ in suitable affine coordinates under the normal form (2.13). Recalling that $X = zY$, we assume the following holds:

(1) We have $a' = 0$ so that the axis $\{y = 0\}$ is invariant by Y. Also we require $b \neq 0$ so that the axis $\{x = 0\}$ *is not invariant* by Y.

(2) The restriction to the exceptional divisor of the foliation associated with X has at least 5 singular points.

(3) $B \neq 0$.

These conditions basically mean that we are willing to consider the least favorable case that can be encountered once the existence of an invariant line is ensured. They serve to keep the discussion focused on the main difficulties of the problem. Our aim is to investigate how close to a vector field having only real coefficients the condition of an invariant line leads us.

Our work in this direction can be summarized by the following theorem:

Theorem 2.2.16. *Under the generic assumptions derived from the above, the coefficients of the vector field Y in (1) satisfy the following conditions:*

1. *The coefficient c is an algebraic number.*

2. *The coefficient d lies in a finite extension of $\mathbb{Q}(c)$ of degree at most* 3.

3. *The coefficient b is the rational function over $\mathbb{Q}$ of c and d indicated in Lemma 2.2.11.*

4. *The coefficient a is the rational function over $\mathbb{Q}$ of c and d indicated in Remark 2.2.13.*

5. *All the remaining coefficients are rational functions over $\mathbb{Q}$ of c and d as also indicated in Remark 2.2.13.*

Recall that a semicomplete vector field on $\mathbb{C}^3$ has seven pairs of integral eigenvalues. We denote these seven pairs by (λ_i, μ_i) and make $\xi_i = \lambda_i \mu_i$, we have the following relation given by Guillot [7]

$$\sum_{i=1}^{7} \frac{1}{\xi_i} = 1 \tag{1}$$

These particular kind of Diophantine equations are known as "Egyptian fractions". Equation 1 has a quite big number of solutions (in the order of millions) so the initial impulse of solving it is maybe not the best approach. Furthermore, even with a solution at hand it would be difficult to give an explicit expression of the actual family of vector fields having $\xi_i = \lambda_i \mu_i$ as eigenvalues, or even verify if it is in fact semicomplete.

This problem is relevant in our context because if the coefficients of the vector field in question are real/integral we could restrict ourselves to the real projective plane $\mathbb{R}P(2)$ and apply the real Poincaré-Hopf for real line fields so as to substantially simplify equation 1.

Let $\mathcal{Z}$ be a generic quadratic vector field on $\mathbb{C}^n$ which is homogeneous and *is not* a multiple of the radial vector field

$$\mathfrak{E} = x_1 \partial/\partial x_1 + \cdots + x_n \partial/\partial x_n \, .$$

Consider the blow-up $\pi : \widetilde{\mathbb{C}}^n \to \mathbb{C}^n$ of $\mathbb{C}^n$ at the origin along with the corresponding lift $X = \pi^* \mathcal{Z}$of the vector field $\mathcal{Z}$ (i.e. X is the blow-up of $\mathcal{Z}$).

The first step in our work will be to construct a canonical form for the induced vector field X on $\mathbb{CP}(n-1)$. This canonical form will be invariant under the action of a specific group of symmetries.

When $n = 3$, our approach consists of studying the singularities not lying on the exceptional divisor but at the hyperplane at infinity $\Delta \in \mathbb{CP}(2)$. In this setting the dynamics of the foliation turn out to be quite simple while the singularities tend to be degenerated. The advantage is that we can deal with degenerated singularities with the technique of successive blow-ups. This leads to simple expressions for the eigenvalues directly in terms of the coefficients of X. In other words, there is a duality regarding the singular foliation induced by X on the exceptional divisor and the one induced on Δ: on the exceptional divisor the dynamics of the foliation tend to be more complicated but with simple singularities while the foliation induced on Δ has a simple dynamic but the singularities tend to be degenerated.

In the first chapter of this thesis we give some basic definitions and results on singular holomorphic foliations and polynomial vector fields in $\mathbb{C}^n$. We also present the fundamentals of the theory of semicomplete vector fields. Further details concerning semicomplete vector fields can be found in [23], [5], and [12]. The simplest of these properties states that the restriction of a *complete* vector field on M to every open set $U \subset M$ is semicomplete on U. More generally if $U' \subset U$ and X is semicomplete on U, then the restriction of X to U' is semicomplete on U'. In particular, we can talk about semicomplete germs of vector fields (see [23]).

Another simple observation involving homogeneous polynomial vector fields asserts that these vector fields are semicomplete on some neighborhood of the origin in $\mathbb{C}^n$ if and only if they are semicomplete on the whole $\mathbb{C}^n$.

The first section of Chapter 2 will be devoted to the construction of a normal form of the vector field X which is given in Lemma 2.1.1. In the particular case $n = 3$, Lemma 2.1.1 implies that in affine coordinates (x, y, z), the vector field X is given by $X = zY$ with $Y = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y + zH(x, y)\partial/\partial z$ and

$$\begin{aligned} P(x,y) &= ax + by + cx^2 + dxy + Bx^2y \\ Q(x,y) &= a'x + b'y + d'xy + e'y^2 + Bxy^2 \ , \\ H(x,y) &= D + Ex + Fy - Bxy \ . \end{aligned}$$

In the second section of Chapter 2 we focus on the particular case of semicomplete vector fields on $\mathbb{C}^3$. We will study the eigenvalues of the singular points of X that lie on the plane at infinity. From these eigenvalues we retrieve some simple relations between the coefficients of X. We prove the main result of this thesis given by Theorem 2.2.8 stated above. We also explore the case where the quadratic vector field $\mathcal{Z}$ leaves some plane through the origin invariant. In other words the foliation associated to the blow-up X of $\mathcal{Z}$ leaves a line invariant. This is also the setting of Guillot [11], albeit the methods and the aims differ slightly. Our main results in this direction—for a generic case—partially explains the integral "or almost integral" coefficients that he obtains in his impressive list of normal forms.

Finally, the last section of this thesis is independent of most of the preceding material. In this section we give a rather direct interpretation of the vector field considered by Ohyama in [20], [21].

$$
\begin{aligned}
w' + x' + y' &= wx + xy + yw, \\
w' + y' + z' &= wy + yz + zw, \\
w' + x' + z' &= wx + xz + zw, \\
x' + y' + z' &= xy + yz + xz
\end{aligned}
$$

in terms of the original Darboux-Halphen vector field associated to the system of differential equations

$$
\begin{aligned}
x' + y' &= 2xy, \\
y' + z' &= 2yz, \\
x' + z' &= 2xz
\end{aligned}
$$

In particular, it will be shown how solutions for Ohyama vector field are explicitly given in terms of solutions of Darboux-Halphen vector field. High dimensional generalizations of Ohyama vector fields are also possible from the perspective of this section.

Preliminaries

1.1 Basics of foliations and vector fields

Definitions and the general facts about singular holomorphic foliations on complex projective spaces provided in this paragraph are well known as the reader can check from a variety of sources including [1], [18], [8], [15], [3], [25]. The formulations given here are intended only to summarize this standard material and help the reader to follow the discussion conducted in this paper.

Bar explicit mention in contrary, all holomorphic foliations considered in this work are singular and of dimension 1. The phrase *foliation by Riemann surfaces* is also used to mean a (singular) foliation of dimension 1. A general definition adapted to our purposes is as follows.

Definition 1.1.1. Let M be a complex manifold of dimension n. A singular holomorphic foliation $\mathcal{F}$ on M consists of a covering $\{(U_i, \varphi_i)\}$ of M by coordinate charts together with a collection of holomorphic vector fields Z_i satisfying the following conditions:

- For every i, Z_i is a holomorphic vector field defined on $\varphi_i(U_i) \subset \mathbb{C}^n$ with singular set of codimension at least 2.
- Whenever $U_i \cap U_j \neq \emptyset$, we have $\varphi_i^* Z_i = g_{ij} \varphi_j^* Z_j$ for some nowhere vanishing holomorphic function g_{ij} defined on $U_i \cap U_j$.

There immediately follows that the singular set of any holomorphic foliation has codimension at least two. Thus singular points of holomorphic foliations on complex surfaces are necessarily isolated. It is also well known that in the case of algebraic manifolds, every holomorphic foliation is naturally associated with a global meromorphic vector field.

Now consider a homogeneous polynomial vector field $\mathcal{Z}$ on $\mathbb{C}^{n+1}$, i.e. $\mathcal{Z}$ is given by

$$P_1(x_1, \ldots, x_{n+1})\partial/\partial x_1 + \cdots + P_{n+1}(x_1, \ldots, x_{n+1})\partial/\partial x_{n+1}$$

where $P_1, \ldots, P_{n+1}$ are homogeneous polynomials of degree d. Unless otherwise mentioned, we always assume that $\mathcal{Z}$ *is not* a multiple of the *radial vector field*

$$\mathfrak{E} = x_1\partial/\partial x_1 + \cdots + x_{n+1}\partial/\partial x_{n+1} \, .$$

Recall also that $\mathbb{C}P(n)$ is the space of radial lines through the origin in $\mathbb{C}^{n+1}$. Equivalently, it is the orbit space of the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$ defined by $(\lambda, (x_1, \ldots, x_{n+1})) \mapsto (\lambda x_1, \ldots, \lambda x_{n+1})$. A point in $\mathbb{C}P(n)$ can then be represented in homogeneous coordinates by $[x_1, \ldots, x_{n+1}]$ with at least one of the entries x_i different from zero ($i = 1, \ldots, n+1$).

Since $\mathcal{Z}$ is homogeneous, its *direction* is well defined over lines passing through the origin. In fact, for every $\lambda \in \mathbb{C}^*$ we have

$$\Lambda^*\mathcal{Z} = \lambda^{d-1}\mathcal{Z} \tag{1.1}$$

where $\Lambda(x_1, \ldots, x_{n+1}) = (\lambda x_1, \ldots, \lambda x_{n+1})$. In other words, for $(x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$ and $\lambda \in \mathbb{C}^*$, the vectors $Z(x_1, \ldots, x_{n+1})$ and $Z(\lambda x_1, \ldots, \lambda x_{n+1})$ are parallel. Now given a point $(x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$, the vector $\mathcal{Z}(x_1, \ldots, x_{n+1})$ can be projected in the tangent space of $\mathbb{C}P(n)$ at the point $[x_1, \ldots, x_{n+1}]$ and the image of this projection is different from zero unless $\mathcal{Z}$ is parallel to the radial vector field $\mathfrak{E}$ at the point $(x_1, \ldots, x_{n+1})$. Furthermore, the direction associated with the projected vector is well defined thanks to Equation 1.1. Owing to condition (1) above, there follows that $\mathcal{Z}$ defines a line field on $\mathbb{C}P(n)$ away from a proper analytic set $\mathcal{A} \subset \mathbb{C}P(n)$. In turn, this line field provides us with a singular holomorphic foliation $\mathcal{D}$ on $\mathbb{C}P(n)$ in the sense of Definition 1.1.1: indeed the above mentioned line field extends to a regular line field at "generic" points of possible codimension 1 components of $\mathcal{A}$ cf. [3]: this phenomenon is often referred to as the *saturation* of the initial line field.

The converse to the above statement also holds: every holomorphic foliation on $\mathbb{C}P(n)$ in the sense of Definition 1.1.1 can be obtained out of a homogeneous polynomial vector field on $\mathbb{C}^{n+1}$. Furthermore, the vector field in question is unique up to a multiplicative function and up to the addition of a multiple of the vector field $\mathfrak{E}$. This allows one to define the *degree* of a foliation $\mathcal{D}$ on $\mathbb{C}P(n)$ as the minimum of the degrees of the homogeneous vector fields on $\mathbb{C}^{n+1}$ inducing $\mathcal{D}$ on $\mathbb{C}P(n)$.

The foliation induced by a homogeneous vector field on $\mathbb{C}P(n)$ and the foliation associated with this same vector field on $\mathbb{C}^{n+1}$ can essentially be merged together into a single foliation as follows. Consider the (one-point) blow-up $\pi : \widetilde{\mathbb{C}}^{n+1} \to \mathbb{C}^{n+1}$ of $\mathbb{C}^{n+1}$ at the origin and note that the exceptional divisor $E = \pi^{-1}(0)$ is naturally isomorphic to $\mathbb{C}P(n)$. In fact, $\widetilde{\mathbb{C}}^{n+1}$ is a line bundle over $E = \pi^{-1}(0) \simeq \mathbb{C}P(n)$ and natural coordinates for $\widetilde{\mathbb{C}}^{n+1}$ will be described in the next section. For the time being, note that the pull-back $\widetilde{\mathcal{Z}}$ of $\mathcal{Z}$ by π turns out to be a holomorphic vector field defined on all of $\widetilde{\mathbb{C}}^{n+1}$. In addition, the vector field $\widetilde{\mathcal{Z}}$ vanishes identically over the exceptional divisor E provided that $d \geq 2$. Denote by $\mathcal{F}$ the foliation associated to $\widetilde{\mathcal{Z}}$ on $\widetilde{\mathbb{C}}^{n+1}$. Clearly the exceptional divisor is not contained in the singular set of $\mathcal{F}$ since the latter has codimension at least 2. The exceptional divisor $E = \pi^{-1}(0)$, however, is left invariant by $\mathcal{F}$ since $\mathcal{Z}$ is not a multiple of the radial vector field. Thus, we can consider the restriction $\mathcal{F}_E$ of $\mathcal{F}$ to $E = \pi^{-1}(0)$. It is easy to check that foliation $\mathcal{F}_E$ on $E = \pi^{-1}(0) \simeq \mathbb{C}P(n)$ coincides with the foliation $\mathcal{D}$ induced by $\mathcal{Z}$ on $\mathbb{C}P(n)$ up to the following minor issue: the singular set of $\mathcal{F}$ - viewed as foliation on $\widetilde{\mathbb{C}}^{n+1}$ - may possess components of codimension 2 contained in $E = \pi^{-1}(0)$. Considered as subsets of $E = \pi^{-1}(0)$ these components are of codimension 1 so that the *restriction* $\mathcal{F}_E$ of $\mathcal{F}$ to E can be extended as a regular foliation at "generic" points of the sets in question: it is again the saturation phenomenon already mentioned above. The resulting extension leads to the foliation that we referred to as being induced by $\mathcal{Z}$ which naturally satisfies the conditions in Definition 1.1.1. In the sequel, in terms of notation, the foliation induced by $\mathcal{Z}$ may alternatively be referred to as the foliation induced by $\widetilde{\mathcal{Z}}$ or by $\mathcal{F}$ on $E = \pi^{-1}(0)$.

There is also an alternate construction of foliations on $\mathbb{C}P(n)$ that is directly based on *polynomial vector fields* defined on $\mathbb{C}^n$. In this case, it is convenient to think of $\mathbb{C}P(n)$ as a compactification of $\mathbb{C}^n$. Using the standard affine atlas for $\mathbb{C}P(n)$, the initial polynomial vector field becomes rational in the other coordinates: it can, however, be made into a polynomial (holomorphic) vector field by multiplying this rational vector field by the least common multiple of all denominators. Thus, in the end, we obtain a polynomial vector field for each of the standard affine coordinates of $\mathbb{C}P(n)$ and, whereas in general these vector fields do not glue together to define a vector field on $\mathbb{C}P(n)$, they do satisfy the condition in Definition 1.1.1 and hence yield a foliation $\mathcal{F}$ on $\mathbb{C}P(n)$. Here, again, the converse of this construction holds: every foliation on $\mathbb{C}P(n)$ can be obtained by means of some polynomial vector field on $\mathbb{C}^n$. Note, however, that the degree of $\mathcal{F}$ does not necessarily coincides with the degree of the polynomial vector fields in question; cf. below.

Let us close this paragraph with some additional comments about the degree of foliations on $\mathbb{C}P(n)$. First the preceding discussion implies that the space of degree d holomorphic foliation on $\mathbb{C}P(n)$ is modeled by a quasi-projective variety of dimension

$$(d+n+1)\frac{(d+n-1)!}{d!(n-1)!}-1.$$

There is therefore a natural sense in speaking about a generic foliation of degree d on $\mathbb{C}P(n)$. In this direction, it is well known that a generic foliation of degree d has only isolated singular points, with non-zero eigenvalues, cf. for example the discussion in [8] (the definition of eigenvalues of a foliation at a singular point can be found in [8], [15] for example). A simple application of Bézout theorem then implies that the number of these isolated singular points is

$$\frac{d^{n+1}-1}{d-1}\,.$$

In particular, a generic foliation $\mathcal{F}$ of degree 2 on $\mathbb{C}P(2)$ has exactly 7 singular points. Note also that a generic foliation has the maximum number of singular points among all foliations with the same degree. More precisely, since the dimension of $\mathbb{C}P(2)$ is 2, every foliation on $\mathbb{C}P(2)$ has only isolated singularities and the number of these singular points is always bounded by $(d^3-1)/(d-1)$.

A useful geometric interpretation of the degree of a foliation on $\mathbb{C}P(n)$ can be obtained as follows. Fix a foliation $\mathcal{F}$ on $\mathbb{C}P(n)$ and consider a hyperplane H in $\mathbb{C}P(n)$. Assume that H is generic in the sense that it is not invariant by $\mathcal{F}$. Let $S \subset H$ be the *tangency set* between $\mathcal{F}$ and H, namely S is the set defined by

$$S = \{p \in H \; ; \quad T_p\mathcal{F} \subseteq H\} \, ,$$

where $T_p\mathcal{F}$ is the tangent space of $\mathcal{F}$ at p. In other words, if p is a regular point of $\mathcal{F}$ then $T_p\mathcal{F}$ is nothing but the tangent line to L at p, where L is the leaf of $\mathcal{F}$ through p. In turn, $T_p\mathcal{F}$ is reduced to the origin is p is a singular point of $\mathcal{F}$. It is immediate to check that S is a codimension 1 algebraic set of H. Since H can naturally be identified with $\mathbb{C}P(n-1)$, the tangency set S can be viewed as a codimension 1 projective variety in $\mathbb{C}P(n-1)$. The degree of $\mathcal{F}$ as previously defined then agrees with the degree of S as a a codimension 1 projective variety in $\mathbb{C}P(n-1)$.

1.2 Semicomplete vector fields

Unlike foliations, vector fields may have codimension 1 zero-sets. Also, *meromorphic vector fields* arise naturally in contexts where all underlying foliations are holomorphic. For example, we have seen that the (singular) foliation associated with a polynomial vector field X on $\mathbb{C}^n$ can *holomorphically* be extended to a (singular) foliation on all of $\mathbb{C}P(n)$. The same does not apply, however, to the vector field X whose extension to $\mathbb{C}P(n)$ is, in general, meromorphic with poles on the hyperplane at infinity.

From a local point of view, every meromorphic vector field X can be written as $X = fY$ where f and Y satisfy the following conditions:

- Y is a holomorphic vector field whose singular set has codimension at least 2.
- f is a meromorphic function.

Note that the function f (locally) determines both the divisors of zeros and of poles of the vector field X. The singular foliation $\mathcal{F}$ defined by the local orbits of Y is then called the *foliation associated with* X. Since the decomposition $X = fY$ is unique up to a multiplicative invertible function, all these notions are well defined.

Consider a (meromorphic) vector field X as above and let $\mathcal{F}$ denote its associated foliation. If L is a regular leaf of $\mathcal{F}$ that is not contained in the zero set of X (neither in the pole divisor of X as well), then L is naturally equipped with an Abelian form, denoted by dT, which is induced by the restriction of X to L. More precisely, dT is a foliated 1-form defined by the condition that dT evaluated at the vector field X must be constant equal to 1. This (foliated) 1-form dT will be called the *time-form* induced by X (on $\mathcal{F}$ or on L). The information encoded in the vector field X is actually equivalent to the data encoded in the pair constituted by $\mathcal{F}$ and by the time-form dT.

Recall that a holomorphic vector field on a manifold M is called *complete* if it gives rise to a $\mathbb{C}$-action on M. The notion of completeness can be extended to meromorphic vector fields by saying that a meromorphic vector field is complete if its restriction to the complement of its pole divisor provides a complete (holomorphic) vector field.

Next, let X be a meromorphic vector field defined on some complex manifold M. We begin by recalling the definition of semicompleteness borrowed from [23] and from [12].

Definition 1.2.1. A holomorphic vector field X defined on a complex manifold M is said to be semicomplete on M if for every $p \in M$ there exists a connected domain $U_p \subset \mathbb{C}$ with $0 \in U_p$ and a map $\phi_p : U_p \to M$ such that:

- $\phi_p(0) = p$ and $d\phi_p(t)/dt|_{t=t_0} = X(\phi_p(t_0))$.
- For every sequence $\{t_i\} \subset U_p$ such that $\lim_{i\to\infty} t_i \in \partial U_p$ the sequence $\{\phi_p(t_i)\}$ escapes from every compact subset of M.

A meromorphic vector field X on a complex manifold M is said to be semicomplete on M if its restriction to the open set where X is holomorphic is semicomplete in the above mentioned sense.

Several basic properties of semicomplete vector fields can be found in [23], [5], and [12]. The simplest of these properties states that the restriction of a *complete* vector field on M to every open set $U \subset M$ is semicomplete on U. More generally if $U' \subset U$ and X is semicomplete on U, then the restriction of X to U' is semicomplete on U'. In particular, we can talk about semicomplete germs of vector fields (see [23]).

Another simple observation involving *homogeneous* polynomial vector fields asserts that these vector fields are semicomplete on some neighborhood of the origin in $\mathbb{C}^n$ if and only if they are semicomplete on the whole $\mathbb{C}^n$.

Let us now proceed to explain a couple of less immediate results that will be used in the course of this paper. First, let X be a meromorphic vector field with associated foliation $\mathcal{F}$. Fix a leaf L of $\mathcal{F}$, regular for X, and denote by dT the time-form induced by X on L. Consider also an open path $c : [0,1] \to L$ - avoiding possible poles of dT - so that the integral

$$\int_c dT$$

is well defined. According to [23], a simple useful necessary condition for the vector field X to be semicomplete is that none of the above defined integrals can be equal to zero (and we emphasize the assumption that c is an open path). Another property that is particularly important for us is the fact that the set of semicomplete vector fields on a fixed domain U is closed for the compact-open topology, see [5]. The main use of this property made in this paper is summarized by Lemma 1.2.2 below which essentially already appears in [5].

Let Z be a vector field defined on a neighborhood of the origin in $\mathbb{C}^n$ and having the form

$$Z = x_1^{k_1} \dots x_n^{k_n} Y \tag{1.2}$$

where Y is a holomorphic vector field and where $k_i \in \mathbb{Z}$ for every $i = 1, \dots, n$. Being holomorphic, the vector field Y can be expanded in Taylor series to yield $Y = \sum_{j=d}^{\infty} Y_j$, where each Y_j is a homogeneous polynomial vector field of degree j, and where d is the smallest (non-negative) integer j for which Y_j is not identically zero. The homogeneous vector field Y_d will be referred to as the *first non-zero homogeneous component of the Taylor series of* Y. The integer d is called the *order of* Y *at the origin*. With this terminology, we can state:

Lemma 1.2.2. *Assume that* Z *as in Formula (1.2) is a meromorphic semicomplete vector field on some neighborhood of the origin. Then the (rational) vector field* $x_1^{k_1} \dots x_n^{k_n} Y_d$ *is semicomplete on all of* $\mathbb{C}^n$.

Proof. Set $K = d-1+\sum_{i=1}^{n} k_i$ and consider $\lambda \in \mathbb{C}^*$. Let Λ denote the map defined by $\Lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$. If U is a neighborhood of the origin where the vector field Z is semicomplete, there follows that the pull-back $\Lambda^* Z$ is defined and semicomplete on $\Lambda^{-1}(U)$, and then on U provided that $|\lambda|$ is small enough. Since a constant multiple of a semicomplete vector field is still semicomplete, it follows that the vector fields $Z_\lambda = \lambda^{-K} \Lambda^* Z$ are semicomplete on U for every $\lambda \in \mathbb{C}^*$ small. A direct computation however shows that the vector fields Z_λ have the form

$$Z_\lambda = x_1^{k_1} \dots x_n^{k_n} \left[Y_d + \sum_{j=d+1}^{\infty} \lambda^{j-d} Y_j \right] .$$

Thus, as $\lambda \to 0$, the semicomplete vector fields Z_λ converge to $x_1^{k_1} \dots x_n^{k_n} Y_d$ on compact parts of U. From the closedness of the set of semicomplete vector fields, we conclude that $x_1^{k_1} \dots x_n^{k_n} Y_d$ is semicomplete on U, see [5]. Since, in addition,

$x_1^{k_1} \ldots x_n^{k_n} Y_d$ is a homogeneous vector field, it follows that $x_1^{k_1} \ldots x_n^{k_n} Y_d$ is actually semicomplete on all of $\mathbb{C}^n$. $\square$

Lemma 1.2.2 points out at the interest of knowing when a linear vector field multiplied by a monomial of the form $x_1^{k_1} \ldots x_n^{k_n}$ is semicomplete. Lemma 1.2.3 below provides an accurate answer to this question.

Let then Z be a vector field having the form

$$Z = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} (x_1 \partial/\partial x_1 + \lambda_2 x_2 \partial/\partial x_2 + \cdots + \lambda_n x_n \partial/\partial x_n) \tag{1.3}$$

where $k_1, \ldots, k_n$ are integers and where $\lambda_2, \ldots, \lambda_n$ are complex numbers.

Lemma 1.2.3. *Assume that $\lambda_2 \ldots \lambda_n \neq 0$. Then a vector field Z as in (1.3) is semicomplete if and only if one of the conditions below is fulfilled:*

1. $k_1 + k_2\lambda_2 + \cdots + k_n\lambda_n = 0$;
2. $k_1 + k_2\lambda_2 + \cdots + k_n\lambda_n \neq 0$ *but* $\lambda_2, \ldots, \lambda_n$ *are all rational numbers of the form* $\lambda_i = a_i/b_i$, *with* $a_i, b_i \in \mathbb{Z}$. *Moreover, the integers* a_i, b_i, $i = 2, \ldots, n$, *must also satisfy the equation*

$$k_1 \mathrm{lcm}\,(b_2, \ldots, b_n) + k_2 a_2 \frac{\mathrm{lcm}\,(b_2, \ldots, b_n)}{b_2} + \cdots + k_3 a_n \frac{\mathrm{lcm}\,(b_2, \ldots, b_n))}{b_n} = \pm 1\,,$$

where $\mathrm{lcm}\,(b_2, \ldots, b_n)$ *stands for the least common multiple of* $b_2, \ldots, b_n$.

To prove Lemma (1.2.3), let $\mathcal{F}$ denote the foliation on $\mathbb{C}^n$ associated with the vector field Z. Fixed a point $(x_1^0, \ldots, x_n^0) \in \mathbb{C}^n$, we let L_0 denote the leaf of $\mathcal{F}$ containing $(x_1^0, \ldots, x_n^0)$. We also consider the map $\Psi : \mathbb{C} \to L_0$ defined by

$$\Psi(T) = ((x_1^0)e^T, (x_2^0)e^{\lambda_2 T}, \ldots, (x_n^0)e^{\lambda_n T})\,.$$

Clearly Ψ is a covering map from $\mathbb{C}$ to L_0 so that we fix a fundamental domain $\Xi \subseteq \mathbb{C}$ for this covering. The restriction of Z to L_0 can be pulled-back to $\mathbb{C}$ with coordinate T to yield an one-dimensional holomorphic vector field $\Psi^* Z_{|L_0}$ satisfying

$$\Psi^* Z_{|L_0} = (x_1^0)^{k_1} \ldots (x_n^0)^{k_n} \exp[(k_1 + k_2\lambda_2 + \cdots + k_n\lambda_n)T]\partial/\partial T\,.$$

In particular the restriction of Z to L_0 is semicomplete on L_0 if and only if the above vector field $\Psi^* Z_{|L_0}$ is semicomplete on Ξ. Since multiplication by a constant does not change the semicomplete character of a vector field, there follows that Z is semicomplete if and only if for every point $(x_1^0, \ldots, x_n^0)$ the vector field

$$Y = e^{(k_1+k_2\lambda_2+\cdots+k_n\lambda_n)T}\partial/\partial T \tag{1.4}$$

is semicomplete on Ξ.

Proof of Lemma 1.2.3. We assume in the sequel that Z is semicomplete and consider the vector field Y in (1.4). Note that Y is semicomplete on all of $\mathbb{C}$ provided that $k_1 + k_2\lambda_2 + \cdots + k_n\lambda_n = 0$ which accounts for the first case in our statement. Hence, we assume in what follows that $k_1 + k_2\lambda_2 + \cdots + k_n\lambda_n \neq 0$. Note that there are two cases to be considered according to whether or not all of $\lambda_2, \ldots, \lambda_n$ are rational numbers.

Consider first the case where at least one λ_i, say λ_2, is not rational. Then the covering map $\Psi : \mathbb{C} \to L_0$ is actually a diffeomorphism, i.e. L_0 is simply connected. Hence $\Xi = \mathbb{C}$ so that the vector field Y in (1.4) is semicomplete on all of $\mathbb{C}$. The desired contradiction then follows from the claim below.

Claim. The vector field Y in (1.4), with $k_1 + k_2\lambda_2 + \cdots + k_n\lambda_n \neq 0$, is never semicomplete on all of $\mathbb{C}$.

Proof of the claim. Just note that the time-form dT induced on $\mathbb{C}$ by Y is nothing but $e^{-(k_1+k_2\lambda_2+\cdots+k_n\lambda_n)T}dT$ and its integral over the path $c : [0, 1] \to \mathbb{C}$ given by

$$c(t) = \frac{2\pi it}{k_1 + k_2\lambda_2 + \cdots + k_n\lambda_n} \tag{1.5}$$

equals zero. The path c being clearly open (embedded) in $\mathbb{C}$, the claim follows at once. $\square$

In view of what precedes, it only remains to consider the case in which all of $\lambda_2, \ldots, \lambda_n$ are rational numbers. We then set $\lambda_i = a_i/b_i$ for relatively prime integers a_i, b_i, $i = 2, \ldots, n$. The least common multiple of all the denominators b_i will be denoted by $\mathsf{M} = \mathrm{lcm}\,(b_2, \ldots, b_n)$.

With the preceding notation, it is clear that the map $\Psi : \mathbb{C} \to L_0$ is periodic with period $2\pi\imath\mathsf{M}$. In other words, the fundamental domain Ξ is characterized by

$$\Xi = \{T \in \mathbb{C}\,;\ 0 \leq \Im(T) < 2\pi\mathsf{M}\}\,,$$

where $\Im(T)$ stands for the imaginary part of $T \in \mathbb{C}$.

Next consider again the path $c : [0,1] \to \mathbb{C}$ defined in (1.5). The integral of the time-form dT over c is still zero since we still have $k_1 + k_2\lambda_2 + \cdots + k_n\lambda_n \neq 0$. Therefore the path c cannot be contained in Ξ. However $c(t)$ can alternatively be written as

$$c(t) = \frac{2\pi i \mathsf{M} t}{k_1\mathsf{M} + k_2 a_2 \mathsf{M}/b_2 + \cdots + k_n a_n \mathsf{M}/b_n}\,,$$

with the advantage that the denominator in the above fraction is clearly an integer relatively prime with M. The condition for c not to be contained in Ξ is therefore to have $|k_1\mathsf{M} + k_2 a_2\mathsf{M}/b_2 + \cdots + k_n a_n \mathsf{M}/b_n| \leq 1$. Since this denominator does not vanish by assumption, there follows that it has to be either 1 or -1 provided that Y (or equivalently Z) is semicomplete. This accounts for the second possibility mentioned in the statement.

Conversely, when $|k_1\mathsf{M} + k_2 a_2\mathsf{M}/b_2 + \cdots + k_n a_n \mathsf{M}/b_n| \leq 1$ the path c is mapped by Ψ into a loop in the leaf L_0 and the restriction of Y to Ξ becomes semicomplete. The proof of the lemma is finished. $\square$

Normal Forms

2.1 Parameterizing quadratic foliations on $\mathbb{C}^n$

In this section we will discuss quadratic vector fields on $\mathbb{C}^n$ and some simple corresponding normal forms for their blow-ups. Let us begin by making accurate the conditions that are assumed to hold throughout this section. Consider a quadratic vector field $\mathcal{Z}$ on $\mathbb{C}^n$ which *is not* a multiple of the radial vector field

$$\mathfrak{E} = x_1 \partial/\partial x_1 + \cdots + x_n \partial/\partial x_n \, .$$

The singular holomorphic foliation on $\mathbb{C}^n$ associated with $\mathcal{Z}$ will be denoted by $\mathcal{F}_{\mathcal{Z}}$. Since $\mathcal{Z}$ is homogeneous, it also induces a foliation $\mathcal{D}$ on $\mathbb{CP}(n-1)$. Throughout this work, $\mathcal{D}$ is assumed to satisfy the following condition:

Condition I: the foliation $\mathcal{D}$ of $\mathbb{CP}(n-1)$ has degree exactly 2.

An immediate consequence of Condition I is that the (polynomial) components of $\mathcal{Z}$ have only constant common factors. To further clarify our setting, consider the blow-up $\widetilde{\mathbb{C}}^n$ of $\mathbb{C}^n$ centered at the origin and let $\pi : \widetilde{\mathbb{C}}^n \to \mathbb{C}^n$ stand for the corresponding projection. Recall that $\widetilde{\mathbb{C}}^n$ is naturally identified with the tautological line bundle over $\mathbb{C}P(n-1)$. A standard set of coordinates defining a projective atlas for $\widetilde{\mathbb{C}}^n$ consists of n coordinate open sets $U^{(j)}$, $j = 1, \ldots, n$, each of them isomorphic to $\mathbb{C}^n$ with coordinates $(u_1^{(j)}, \ldots, u_{n-j}^{(j)}, \widehat{u}_{n-j+1}^{(j)}, u_{n-j+2}^{(j)}, \ldots, u_n^{(j)})$. The exceptional divisor - given by the pre-image of the origin by π and denoted by $\pi^{-1}(0)$ - intersects the open set $U^{(j)}$ in the hyperplane $\{\widehat{u}_{n-j+1}^{(j)} = 0\}$. Furthermore the restriction of π to $U^{(j)}$ becomes

$$\pi(u_1^{(j)}, \ldots, \widehat{u}_{n-j+1}^{(j)}, \ldots, u_n^{(j)}) = \widehat{u}_{n-j+1}^{(j)}(u_1^{(j)}, \ldots, u_{n-j}^{(j)}, 1, u_{n-j+2}^{(j)}, \ldots, u_n^{(j)}) \,. \qquad (2.1)$$

From Formula (2.1), it is easy to work out all the identifications among the coordinate sets $U^{(j)}$. For example, we have

$$\begin{array}{rcl} \widehat{u}_{n-j+1}^{(j)} & = & u_{j-n+1}^{(1)}\widehat{u}_n^{(1)} \\ u_n^{(j)} & = & 1/u_{n-j+1}^{(1)} \\ u_i^{(j)} & = & u_i^{(1)}/u_{j-n+1}^{(1)} \end{array} \qquad (2.2)$$

for $i \neq n-j+1$ and $i \neq n$.

Going back to the quadratic vector field $\mathcal{Z}$, let $(x_1, \ldots, x_n)$ denote the coordinates of $\mathbb{C}^n$. Next set

$$\mathcal{Z} = P_1\partial/\partial x_1 + \cdots + P_n\partial/\partial x_n \qquad (2.3)$$

where $P_1, \ldots, P_n$ are homogeneous polynomials of degree 2 in $(x_1, \ldots, x_n)$. Consider the blow-up $\pi : \widetilde{\mathbb{C}}^n \to \mathbb{C}^n$ of $\mathbb{C}^n$ at the origin along with the corresponding lift $X = \pi^*\mathcal{Z}$ of the vector field $\mathcal{Z}$ (i.e. X is the blow-up of $\mathcal{Z}$). We also let $\mathcal{F} = \pi^*\mathcal{F}_{\mathcal{Z}}$, where $\mathcal{F}_{\mathcal{Z}}$ stands for the foliation on $\mathbb{C}^n$ associated with $\mathcal{Z}$. Clearly $\mathcal{F}$ coincides with the foliation associated to X.

To avoid the use of subscripts/superscripts and abridge notation, in the sequel we set $(u_1^{(1)}, \ldots, u_{n-1}^{(1)}, \widehat{u}_n^{(1)}) = (u_1, \ldots, u_{n-1}, w)$. In the coordinates $(u_1, \ldots, u_{n-1}, w)$, the vector field X takes on the form

$$X = w\left[(P_1 - u_1P_n)\frac{\partial}{\partial u_1} + \cdots + (P_{n-1} - u_{n-1}P_n)\frac{\partial}{\partial u_{n-1}} + wP_n\frac{\partial}{\partial w}\right] \qquad (2.4)$$

where all the polynomials $P_1, \ldots, P_n$ are evaluated at the point $(u_1, \ldots, u_{n-1}, 1)$.

Setting $Y_{u_i} = P_i - u_iP_n$ for $i = 1, \ldots, n-1$ and $Y_w = wP_n$, the foliation $\mathcal{F}$ associated with X is also given by the local orbits of the vector field $Y = (1/w)X$.

Clearly we have

$$Y = Y_{u_1} \frac{\partial}{\partial u_1} + \cdots + Y_{u_{n-1}} \frac{\partial}{\partial u_{n-1}} + Y_w \frac{\partial}{\partial w} \, . \tag{2.5}$$

In particular the exceptional divisor $E = \pi^{-1}(0)$ is left invariant by $\mathcal{F}$ since, by assumption, the initial vector field $\mathcal{Z}$ is not a multiple of the Radial vector field $\mathfrak{C}$. In fact, the restriction $\mathcal{F}|_E$ of $\mathcal{F}$ to $E = \pi^{-1}(0) \simeq \mathbb{CP}(n-1)$ is identified with the foliation induced on $\mathbb{CP}(n-1)$ by the homogeneous vector field $\mathcal{Z}$. Since $E \simeq \pi^{-1}(0)$ is locally given by $\{w = 0\}$, $\mathcal{F}|_E$ is determined by the vector field

$$Y|_{\{w=0\}} = Y_{u_1} \frac{\partial}{\partial u_1} + \cdots + Y_{u_{n-1}} \frac{\partial}{\partial u_{n-1}} \, . \tag{2.6}$$

At this point, the condition that $\mathcal{F}|_E = \mathcal{D}$ must be a degree 2 foliation of $\mathbb{CP}(n-1)$ can explicitly be reformulated as the following pair of conditions:

1. At least one among the polynomials $Y_{u_1}, \ldots, Y_{u_{n-1}}$ has degree 2 or 3;

2. In the ring of polynomials, the greatest common divisor of $Y_{u_1}, \ldots, Y_{u_{n-1}}$ is a constant.

Note that item 2. above ensures that the singular set of $\mathcal{F}|_E$ in the domain of the affine coordinates $(u_1, \ldots, u_{n-1})$ has codimension at least 2 *inside* $E \simeq \pi^{-1}(0)$. Hence the foliation $\mathcal{F}|_E$ is automatically saturated. In other words, $\mathcal{F}_E$ actually coincides with $\mathcal{D}$ and in the sequel we will refer to this foliation by $\mathcal{F}_E$, thus dropping the notation $\mathcal{D}$. An immediate by-product of the previous observation is that the singular points of $\mathcal{F}_E$ are in one-to-one correspondence with radial lines in $\mathbb{C}^n$ that are invariant under $\mathcal{Z}$.

Recall that in the coordinates $(u_1, \ldots, u_{n-1}, w)$, the exceptional divisor coincides with the hyperplane $\{w = 0\}$. Furthermore we can assume without loss of generality that the x_n-axis of $\mathbb{C}^n$ is invariant by $\mathcal{Z}$ since there always exists at least one singular point for $\mathcal{F}_E$. In other words, we can assume that origin of the coordinates $(u_1, \ldots, u_{n-1})$ is a singular point of $\mathcal{F}_E$. With this assumption, the vector field Y of Formula (2.5) is characterized by the following explicit form:

- For $i \in \{1, \dots, n-1\}$, the component $Y_{u_i} = Y_{u_i}(u_1, \dots, u_{n-1})$ has the form

$$Y_{u_i} = R_i(u_1, \dots, u_{n-1}) + u_i Q(u_1, \dots, u_{n-1})$$

  where:

  (a) R_i is a degree 2 polynomial on the variables $u_1, \dots, u_{n-1}$ with constant term equal to zero;

  (b) Q is a homogeneous polynomial of degree 2 which does not depend on the index i.

- The component Y_w of the vector field Y is given by

$$Y_w = w[D + E_1 u_1 + \cdots + E_{n-1} u_{n-1} - Q(u_1, \dots, u_{n-1})] \,,$$

  where all the coefficients $D, E_1, \dots, E_{n-1}$ belong to $\mathbb{C}$.

The remainder of this section is devoted to obtaining a simple normal form for the vector field Y which holds under a very mild assumption.

In order to abridge notation, let $(e_1, \dots, e_{n-1})$ stand for the canonical basis of $\mathbb{C}^{n-1}$. The polynomials R_i and Q can then be written as follows.

$$\begin{aligned} R_i &= a^i_{e_1} u_1 + \cdots + a^i_{e_{n-1}} u_{n-1} + a^i_{e_1+e_1} u_1 u_1 + a^i_{e_1+e_2} u_1 u_2 + \cdots + a^i_{e_{n-1}+e_{n-1}} u_{n-1} u_{n-1} \\ &= \sum_{j=1}^{n-1} a^i_{e_j} u_j + \sum_{j=1}^{n-1} \sum_{k \geq j}^{n-1} a^i_{e_j+e_k} u_j u_k \,, \end{aligned} \tag{2.7}$$

for $i = 1, \dots, n-1$. Moreover we have

$$Q = \beta_{e_1+e_1} u_1 u_1 + \beta_{e_1+e_2} u_1 u_2 + \cdots + \beta_{e_{n-1}+e_{n-1}} u_{n-1} u_{n-1} = \sum_{j=1}^{n-1} \sum_{k \geq j}^{n-1} \beta_{e_j+e_k} u_j u_k \,. \tag{2.8}$$

Finally, if we let $D = \beta_0$ and $E_i = \beta_{e_i}$ (for $i = 1, \dots, n-1$), the different components of Y become

$$\begin{aligned} Y_{u_i} &= \left[\sum_{j=1}^{n-1} a^i_{e_j} u_j + \sum_{j=1}^{n-1}\sum_{k\geq j}^{n-1} a^i_{e_j+e_k} u_j u_k\right] + u_i \left[\sum_{j=1}^{n-1}\sum_{k\geq j}^{n-1} \beta_{e_j+e_k} u_j u_k\right] . \\ Y_w &= w \left[\beta_0 + \sum_{j=1}^{n-1} \beta_{e_i} u_i - \sum_{j=1}^{n-1}\sum_{k\geq j}^{n-1} \beta_{e_j+e_k} u_j u_k\right] . \end{aligned} \tag{2.9}$$

From now on the following condition is assumed to hold:

Condition II: the foliation $\mathcal{F}_E$ possesses (at least) n singular points not sitting in a same projective hyperplane of $\mathbb{C}P(n-1)$.

The main result of this section is then the following simple lemma.

Lemma 2.1.1. *Assume that Condition II is satisfied. Then the initial coordinates $(x_1, \ldots, x_n)$ of $\mathbb{C}^n$ can be chosen so that the components R_i, $i = 1, \ldots, n-1$, and Q of the resulting vector field Y (determined by Formulas (2.7), (2.8), and (2.9) satisfy the following relations:*

a) $a^i_{2e_j} = 0$ whenever $j \neq i$;

b) $\beta_{2e_j} = 0$ for all $j = 1, \ldots, n-1$.

Proof. We assume that Condition II is satisfied and consider n singular points of $\mathcal{F}_E$ not lying in a hyperplane of $\mathbb{C}P(n-1)$. To each of these points, we naturally associate a radial line in $\mathbb{C}^n$ passing through the origin. The (initial) coordinates $(x_1, \ldots, x_n)$ can be chosen so that these lines coincide with the coordinate axes, which means that the quadratic vector field $\mathcal{Z}$ leave each one of the coordinate axes invariant. Since the x_n-axis is invariant, it is clear that Y has the form indicated in Formula (2.9). It remains to show that in these coordinates, we also have $a^i_{2e_j} = 0$ whenever $j \neq i$ and $\beta_{2e_j} = 0$ for all $j = 1, \ldots, n-1$.

For this, recall first that $\mathcal{Z} = P_1 \partial/\partial x_1 + \cdots + P_n \partial/\partial x_n$. Since the x_1 axis is invariant by the foliation induced by $\mathcal{Z}$, there follows that $P_i(x_1, 0, \ldots, 0) = 0$ for all $i = 2, \ldots, n-1$. In turn, the vanishing of $P_i(x_1, 0, \ldots, 0)$ amounts to saying that the coefficient of x_1^2 in the expression of P_i equals zero (for every $i = 2, \ldots, n$).

More precisely, if we set

$$P_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} \sum_{k \geq j}^{n} p^i_{b_j + b_k} x_j x_k \,,$$

where $(b_1, \ldots, b_n)$ stands for the canonical basis of $\mathbb{C}^n$, we conclude that $p^i_{2b_1} = 0$ for all $i = 2, \ldots, n$. Now, recalling that $Y_{u_i}(u_1, \ldots, u_{n-1})$ is given by $P_i(u_1, \ldots, u_{n-1}, 1) - u_i P_n(u_1, \ldots, u_{n-1}, 1)$ and that $Y_w = wP_n(u_1, \ldots, u_{n-1}, 1)$, it is immediate to check that $p^i_{2b_1}$ is the coefficient of u_1^2 at Y_{u_i}, for $i = 1, \ldots, n-1$, while $p^n_{2e_1}$ is the coefficient of u_1^2 at Q. Therefore

$$\beta_{2e_1} = 0 \quad \text{and} \quad a^i_{2e_1} = 0, \;\; \forall \, i = 2, \ldots, n-1 \,.$$

The lemma now follows from repeating the above argument for the remaining x_j-axes, $j = 2, \ldots, n-1$. □

Remark 2.1.2. Note that the choice of coordinates $(x_1, \ldots, x_n)$ for $\mathbb{C}^n$ leading to the normal form indicated above is not unique: in fact, recalling that a generic quadratic foliation in $\mathbb{C}^n$ has $2^{n-1} - 1$ singular points on $\mathbb{C}P(n-1)$, the mentioned normal form has a finite group of symmetries arising from the choice of the $n-1$ "non-aligned" singular points. Furthermore, once coordinates $(x_1, \ldots, x_n)$ satisfying the desired conditions are fixed, every change of coordinates preserving the axes, i.e. having the form $(x_1, \ldots, x_n) \mapsto (\lambda_1 x_1, \ldots, \lambda_n x_n)$, will also satisfy the required conditions.

A simple way to rephrase Lemma 2.1.1 consists of saying that the vector field Y of Formulas (2.5) and (2.9) is such that the term u_i^2 appears only at Y_{u_i}, for $i = 1, \ldots, n-1$ and that the polynomial Q in Equation (2.8) only contains products of variables with different indices.

Remark 2.1.3. For reference, in the particular case $n = 3$, Lemma 2.1.1 implies that the vector field Y of Formulas (2.5) and (2.9) is given by

$$\begin{aligned} Y_{u_1} &= a^1_{(1,0)} u_1 + a^1_{(0,1)} u_2 + a^1_{(2,0)} u_1^2 + a^1_{(1,1)} u_1 u_2 + \beta_{(1,1)} u_1^2 u_2 \,; \\ Y_{u_2} &= a^2_{(1,0)} u_1 + a^2_{(0,1)} u_2 + a^2_{(1,1)} u_1 u_2 + a_{(0,2)} u_2^2 + \beta_{(1,1)} u_1 u_2^2 \,; \\ Y_w &= w(\beta_{(0,0)} + \beta_{(1,0)} u_1 + \beta_{(0,1)} u_2 - \beta_{(1,1)} u_1 u_2) \,. \end{aligned} \tag{2.10}$$

provided that $\mathcal{F}_E$ admits three non-aligned singular points in $\mathbb{CP}(2)$.

For reference, we also explicitly state the case $n = 4$ where the previous normal form is reduced to

$$
\begin{aligned}
Y_{u_1} &= a^1_{(1,0,0)}u_1 + a^1_{(0,1,0)}u_2 + a^1_{(0,0,1)}u_3 + a^1_{(2,0,0)}u_1^2 + a^1_{(1,1,0)}u_1u_2 + a^1_{(1,0,1)}u_1u_3 + \\
&\quad + a^1_{(0,1,1)}u_2u_3 + u_1(\beta_{(1,1,0)}u_1u_2 + \beta_{(1,0,1)}u_1u_3 + \beta_{(0,1,1)}u_2u_3)\,; \\
Y_{u_2} &= a^2_{(1,0,0)}u_1 + a^2_{(0,1,0)}u_2 + a^2_{(0,0,1)}u_3 + a^2_{(0,2,0)}u_2^2 + a^2_{(1,1,0)}u_1u_2 + a^2_{(1,0,1)}u_1u_3 + \\
&\quad + a^2_{(0,1,1)}u_2u_3 + u_2(\beta_{(1,1,0)}u_1u_2 + \beta_{(1,0,1)}u_1u_3 + \beta_{(0,1,1)}u_2u_3)\,; \qquad (2.11) \\
Y_{u_3} &= a^3_{(1,0,0)}u_1 + a^3_{(0,1,0)}u_2 + a^3_{(0,0,1)}u_3 + a^3_{(0,0,2)}u_3^2 + a^3_{(1,1,0)}u_1u_2 + a^3_{(1,0,1)}u_1u_3 + \\
&\quad + a^3_{(0,1,1)}u_2u_3 + u_3(\beta_{(1,1,0)}u_1u_2 + \beta_{(1,0,1)}u_1u_3 + \beta_{(0,1,1)}u_2u_3)\,; \\
Y_w &= w[\beta_{(0,0,0)} + \beta_{(1,0,0)}u_1 + \beta_{(0,1,0)}u_2 + \beta_{(0,0,1)}u_2 - \beta_{(1,1,0)}u_1u_2 - \beta_{(1,0,1)}u_1u_3 + \\
&\quad + \beta_{(0,1,1)}u_2u_3]\,.
\end{aligned}
$$

Let us close this section with some additional simple remarks on the normal form provided by Lemma 2.1.1. The observations in question are related to the existence non-uniqueness of linear coordinates leading to the mentioned normal form, an issue already pointed out in Remark 2.1.2.

First, it is convenient to make clear how a linear change of coordinates T in the variables $(x_1, \ldots, x_n)$ of $\mathbb{C}^n$ impacts the form of the vector field/foliation in the (affine) variables $(u_1, \ldots, u_{n-1}, w)$. For this, let T stand for the linear isomorphism of $\mathbb{C}^n$ represented by the matrix $\{\alpha^i_j\}$, where $\alpha^i_j \in \mathbb{C}$, for $i, j = 1, \ldots, n$. More precisely, in the the initial coordinates $(x_1, \ldots, x_n)$, we have

$$T(x_1, \ldots, x_n) = (\alpha^1_1 x_1 + \cdots + \alpha^1_n x_n, \alpha^2_1 x_1 + \cdots + \alpha^2_n x_n, \ldots, \alpha^n_1 x_1 + \cdots + \alpha^n_n x_n)\,.$$

Naturally T can be lifted to an automorphism $\widetilde{T}$ of $\widetilde{\mathbb{C}}^n$. It is immediate to check that in the above mentioned coordinates $(u_1, \ldots, u_{n-1}, w)$, the lifted automorphism $\widetilde{T}$ becomes

$$\widetilde{T} = \left(\frac{\sum_{i=1}^{n-1} \alpha^1_i u_i + \alpha^1_n}{\sum_{i=1}^{n-1} \alpha^n_i u_i + \alpha^n_n}, \ldots, \frac{\sum_{i=1}^{n-1} \alpha^{n-1}_i u_i + \alpha^{n-1}_n}{\sum_{i=1}^{n-1} \alpha^n_i u_i + \alpha^n_n}, \left(\sum_{i=1}^{n-1} \alpha^n_i u_i + \alpha^n_n \right) w \right)\,. \qquad (2.12)$$

The preceding formulas will be useful in the discussion below. Going back to the normal form provided by Lemma 2.1.1, its invariance properties are summarized by Lemma 2.1.4 below:

Lemma 2.1.4. *Assume that the vector field* Y *has the form indicated in Lemma 2.1.1 with respect to the affine coordinates* $(u_1, \ldots, u_{n-1}, w)$*. Then the following holds:*

(1) up to dropping the condition that $0 \in \mathbb{C}^{n-1}$ *is a singular point of* $\mathcal{F}_E$*, the normal form is invariant by translations in the* $(u_1, \ldots, u_{n-1})$*-coordinates, i.e. by transformations* $\widetilde{T}$ *taking on the form* $\widetilde{T}(u_1, \ldots, u_{n-1}, w) = (u_1 + c_1, \ldots, u_{n-1} + c_{n-1}, w)$*, with* $c_1, \ldots, c_{n-1}$ *in* $\mathbb{C}$*;*

(2) the normal form is also invariant under re-scalings of the coordinates $(u_1, \ldots, u_{n-1})$*, i.e. it is invariant under transformations* $\widetilde{T}$ *taking on the form* $\widetilde{T}(u_1, \ldots, u_{n-1}, w) = (\lambda_1 u_1, \ldots, \lambda_{n-1} u_{n-1}, w)$*, with* $\lambda_1, \ldots, \lambda_{n-1}$ *in* $\mathbb{C}^*$*;*

(3) the normal form is well defined with respect to the standard atlas of $\widetilde{\mathbb{C}}^n$ *in the sense that* Y *has the normal form presented in Lemma 2.1.1 with respect to the* $(u_1, \ldots, u_{n-1}, w)$*-coordinates if and only if it has the same form with respect to all the other affine coordinates in the standard atlas of* $\widetilde{\mathbb{C}}^n$*.*

Proof. The verification of all assertions is rather straightforward. Condition (3) can be traced back to Lemma 2.1.1 since this lemma shows that the normal form in question is equivalent to the invariance of all the coordinate axes under $\mathcal{Z}$. Clearly the same argument applies also to the other standard affine coordinates of $\widetilde{\mathbb{C}}^n$.

Condition (2) can also be reduced to Lemma 2.1.1 since the linear change of coordinates T in the initial coordinates $(x_1, \ldots, x_n)$ giving rise to $\widetilde{T}(u_1, \ldots, u_{n-1}, w) = (\lambda_1 u_1, \ldots, \lambda_{n-1} u_{n-1}, w)$ has diagonal form (cf. Remark 2.1.2 and Formula (2.12)). Finally, Condition (1) can also be checked by direct inspection or, alternatively, we may realize that the corresponding linear map $T = \{\alpha_j^i\}$ equals the identity matrix plus a matrix whose non-zero entries must be in the positions (i, n) for $i = 1, \ldots, n-1$ (i.e. the elements of the last column up to the element in the diagonal position): the matrix T leaves thus invariant the axes $x_1, \ldots, x_{n-1}$, albeit not the x_n-axis. The proof of Lemma 2.1.1 then allows us to conclude the statement of Condition (1). □

The proof of Lemma 2.1.4 also yields the following corollary:

Corollary 2.1.5. *With the notation of Lemma 2.1.4, assume that Y takes on the normal form provided by Lemma 2.1.1 in the $(u_1, \ldots, u_{n-1}, w)$-coordinates. Assume also that $(u_1^0, \ldots, u_{n-1}^0, 0)$ is a singular point of $\mathcal{F}_E$ in the mentioned affine coordinates. Then the normal form in question is invariant under the translation $(u_1, \ldots, u_{n-1}, w) = (u_1 + u_1^0, \ldots, u_{n-1} + u_{n-1}^0, w)$.* □

We finish this section with a statement that will also be useful later on.

Lemma 2.1.6. *Consider again the vector field Y in the normal form given by Lemma 2.1.1. Then $\beta_0 = 0$ if and only if the x_n-axis is entirely constituted by singular points of the vector field $\mathcal{Z}$ in $\mathbb{C}^n$. Similarly, $a^i_{2e_i} = -\beta_{e_i}$ if and only if the x_i-axis is entirely constituted by singular points of $\mathcal{Z}$, for $i = 1, \ldots, n-1$.*

Proof. The restriction of the vector field $X = wY$ to the w-axis can naturally be identified to the restriction of $\mathcal{Z}$ to the x_n-axis. It is then clear that $\beta_0 = 0$ if and only if $\mathcal{Z}$ is singular all along the x_n-axis.

To show that $a^i_{2e_i} = -\beta_{e_i}$ if and only if $\mathcal{Z}$ is singular all along the x_i-axis, we should consider the polynomials P_i appearing as components of $\mathcal{Z}$, $i = 1, \ldots, n$. Fix then $i \in \{1, \ldots, n-1\}$. Note that the restriction of P_j to the x_i-axis vanishes identically for every $j \neq i$ since the axis x_i is invariant by $\mathcal{Z}$. More generally, the restriction of $\mathcal{Z}$ to the x_i-axis is given by $p^i_{(2e_i,0)} x_i^2 \partial/\partial x_i$. Hence the x_i-axis is contained in the singular set of $\mathcal{Z}$ if and only if $p^i_{(2e_i,0)} = 0$.

To conclude the proof of the lemma, first note that the coefficient β_{e_i} is nothing but the coefficient of the monomial $x_i x_n$ in P_n, for $i = 1, \ldots, n-1$. Similarly, the coefficient $a^i_{2e_i}$ is obtained as the difference between the monomial x_i^2 of P_i and the monomial $x_i x_n$ in P_n. In other words, we have

$$a^i_{2e_i} = p^i_{2b_i} - p^n_{b_i} = p^i_{2b_i} - \beta_{e_i} \, .$$

Thus $p^i_{(2e_i,0)} = 0$ if and only if $a^i_{2e_i} = -\beta_{e_i}$ and this establishes the lemma. □

2.2 Blow-ups and normal forms in dimension 3

In this section we consider the case $n = 3$. Our purpose is to refine the parametrization of quadratic vector fields discussed in the previous section while, also, investigating the possible existence of local obstruction to semicompleteness sitting in the divisor of poles. As far as the former is concerned, we will obtain an essentially canonical normal form that is well adapted to our purposes. Unfortunately, our discussion will also show that no additional local obstruction to the semicompleteness of our vector fields can be obtained at infinity.

Set then $n = 3$ and recall that X stands for vector field on $\widetilde{\mathbb{C}}^3$ which is obtained as the blow-up of the quadratic vector field $\mathcal{Z}$ on $\mathbb{C}^3$ which, in turn, *is not* a multiple of the radial vector field $\mathfrak{E}$.

Finally we still want to keep Condition I and Condition II of Section 2.1. Condition II, however, will be made slightly stronger. First recall that the blow-up $\widetilde{\mathbb{C}}^3$ of $\mathbb{C}^3$ at the origin can naturally be viewed a a line bundle over $\mathbb{C}P(2)$. Now we state:

Condition III: the foliation $\mathcal{F}_E$ possesses (at least) 3 singular points not sitting in a same projective line of $\mathbb{C}P(2)$. Furthermore, none of the fibers of the line bundle $\widetilde{\mathbb{C}}^3 \to \mathbb{C}P(2)$ sitting over one of these singular points is fully constituted by singular points of X.

Unless we explicitly say otherwise, Conditions I, II, and III are assumed to hold throughout the section.

The starting point of the discussion is the representation of X in standard affine coordinates for $\widetilde{\mathbb{C}}^3$. Since $n = 3$, we may abridge the notation used in the previous section and eliminate most superscripts. With this in mind, standard affine coordinates for $\widetilde{\mathbb{C}}^3$ will be denoted by (u_1, v_1, w_1) where the exceptional divisor is locally given by $\{w_1 = 0\}$. The remaining two standard coordinates will be denoted by (u_2, v_2, w_3) and (u_3, v_3, w_3), see below.

The starting point of this section is the representation of the vector field X in standard affine coordinates for $\widetilde{\mathbb{C}}^3$. More precisely X is given in (u_1, v_1, w_1)-coordinates by $X = w_1 Y$ where Y is the holomorphic vector field corresponding to Formula (2.5) and whose components $Y_{(u)}$, $Y_{(v)}$, and $Y_{(w)}$ are given by Formula (2.9) (recall that $n = 3$). In particular the set of zeros of Y has codimension at least 2 and hence the local orbits of Y define the foliation $\mathcal{F}$ associated with X (and with Y).

We consider then the vector field X as a vector field on $\mathbb{C}^3$ equipped with coordinates (u_1, v_1, w_1). Since the semicomplete character of vector fields is invariant by blow-ups, the initial quadratic vector field $\mathcal{Z}$ is semicomplete if and only if X is semicomplete on $\mathbb{C}^3$. In fact, most of the discussion of this section will be conducted on $\mathbb{C}^3$ thus, to further simplify notation, the coordinates (u_1, v_1, w_1) will be denoted by the more familiar letters (x, y, z). Similarly the components of the resulting vector field Y will be denoted by P, Q, and H.

To summarize the preceding, it suffices to keep in mind that for most of this section the vector field X is considered as a holomorphic vector field defined on $\mathbb{C}^3$ with coordinates (x, y, z). Furthermore, X has the form $X = zY$ with $Y = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y + zH(x, y)\partial/\partial z$ and

$$\begin{aligned} P(x,y) &= ax + by + cx^2 + dxy + Bx^2y \\ Q(x,y) &= a'x + b'y + d'xy + e'y^2 + Bxy^2 , \\ H(x,y) &= D + Ex + Fy - Bxy . \end{aligned} \tag{2.13}$$

Consider now X defined on $\mathbb{C}^3$ as in (2.13). The plane $\{z = 0\}$ is invariant by both X and Y which is of course reminiscent from the invariance of the exceptional divisor in $\widetilde{\mathbb{C}}^3$. Restricting Y to $\{z = 0\}$ we re-obtain the restriction to the foliation $\mathcal{F}$ associated with X on $\widetilde{\mathbb{C}}^3$ to the exceptional divisor. A non-trivial observation, however, consists of noting that the higher degree homogeneous component of the polynomial vector field Y is the vector field $Y^{(3)}$ given by

$$Y^{(3)} = Bxy \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} \right] .$$

Since $Y^{(3)}$ *is not* a multiple of the radial vector field, the foliation $\mathcal{F}$ extends to a holomorphic foliation on $\mathbb{C}P(3) = \mathbb{C}^3 \cup \Delta$ which must leave the plane at infinity Δ invariant. Moreover, the restriction of $\mathcal{F}$ to Δ is naturally identified with the foliation induced on $\mathbb{C}P(2)$ by the *homogeneous* vector field $Y^{(3)}$. This foliation is therefore of degree 1 with projective lines as leaves and its global dynamics is basically evident. On the other hand, the singular points of $\mathcal{F}$ lying in Δ tend to be more degenerate. The idea of resolving these singularities in the hope of finding further obstructions to the semicomplete character of X is therefore a natural one.

Consider then X and Y as in (2.13). As a polynomial vector field on $\mathbb{C}^3$, X (resp. Y) admits a meromorphic extension to $\mathbb{C}P(3)$ viewed as a compactification of $\mathbb{C}^3$. Moreover, as already pointed out, the foliation $\mathcal{F}$ associated with X (and with Y) can be viewed as a (singular) holomorphic foliation on all of $\mathbb{C}P(3)$. Finally the plane at infinity $\Delta = \mathbb{CP}(3) \setminus \mathbb{C}^3$ is left invariant by $\mathcal{F}$.

Recall that the standard atlas for $\mathbb{C}P(3)$ consists of 4 copies of $\mathbb{C}^3$ whose coordinates will be denoted by (x_i, y_i, z_i), $i = 0, 1, 2, 3$ along with suitable identifications. Again to abridge notation, we set $(x_0, y_0, z_0) = (x, y, z)$, i.e. the first copy of $\mathbb{C}^3$ is identified with the previously discussed $\mathbb{C}^3$. In the sequel, we will mostly consider the copy of $\mathbb{C}^3$ which is equipped with (x_1, y_1, z_1) such that $(1/x_1, y_1/x_1, z_1/x_1) = (x, y, z)$. In (x_1, y_1, z_1)-coordinates the vector field X takes on the form $X = X_1 = x_1^{-3} z_1 Y_1$ where the vector field Y_1 is given by

$$Y_1 = P_1(x_1, y_1)\partial/\partial x_1 + Q_1(x_1, y_1)\partial/\partial y_1 + z_1 H_1(x_1, y_1)\partial/\partial z_1$$

with

$$\begin{aligned}
P_1(x_1, y_1) &= -x_1[bx_1^2 y_1 + ax_1^2 + dx_1 y_1 + cx_1 + By_1] \, , \\
Q_1(x_1, y_1) &= x_1[-bx_1 y_1^2 + (b' - a)x_1 y_1 + (e' - d)y_1^2 + a' x_1 + (d' - c)y_1] \, , \\
H_1(x_1, y_1) &= -bx_1^2 y_1 + (D - a)x_1^2 + (F - d)x_1 y_1 + (E - c)x_1 - 2By_1 \, .
\end{aligned}$$

Note that the hyperplane at infinity Δ is given in coordinates (x_1, y_1, z_1) by $\Delta = \{x_1 = 0\}$. Furthermore, the singular points of Y_1 contained in Δ are given by $2Bz_1 y_1 = 0$. This means that the y_1-axis and the z_1-axis both consist entirely of singular points of Y_1 or, equivalently, of the foliation $\mathcal{F}$ associated with X in coordinates (x_1, y_1, z_1). Furthermore, a (generic) point in the y_1-axis having the form $(x_1, y_1, z_1) = (0, \alpha, 0)$, is a *simple* singularity of $\mathcal{F}$ in the sense that $\mathcal{F}$ possesses at least one eigenvalue different from zero at this point. In addition, for all $\alpha \neq 0$, the eigenvalues of $\mathcal{F}$ at $(x_1, y_1, z_1) = (0, \alpha, 0)$ are respectively $1, 0$, and 2 (recall that these eigenvalues are defined only up to a multiplicative constant).

Remark 2.2.1. Consider again a singular point p of the form $p = (0, \alpha, 0)$ where $\mathcal{F}$ has eigenvalues $1, 0$, and 2. It is easy to check that $\mathcal{F}$ possesses a (smooth) separatrix S_α at p which is tangent to the axis x_1 and contained in the plane $\{z_1 = 0\}$. Whereas the vector field $X = X_1 = x_1^{-3} z_1 Y_1$ vanishes identically on the plane $\{z_1 = 0\}$, and hence on S_α, the affine structure induced by X on S_α, in the sense of [12], can be considered. A simple calculation shows that the order of the affine structure at $p \in S_\alpha$ is zero so that the affine structure is actually regular at p. Owing to a central result on the holonomy-monodromy map considered in [12] (Fundamental Lemma in Section 3 of [12]), there follows that the local holonomy map associated with S_α must coincide with the identity. At this point it might be natural to wonder if the condition of having this holonomy map equal to the identity does not provide constraints on the coefficients of the vector field Y_1 and, hence Y as given by Formula (2.13). Unfortunately this does not happen: the reader will check that for every vector field Y as in (2.13) and every separatrix S_α as above, the resulting (local) holonomy map can effectively be computed and it turns out to coincide with the identity.

Contrasting with the case of the y_1-axis, all singularities of $\mathcal{F}$ lying in the z_1-axis are *degenerate* in the sense that all eigenvalues of $\mathcal{F}$ are equal to zero at these points. To better understand the structure of these degenerate singular points, we shall apply the technique of blowing-ups designed to simplify the singularities. This said, let us consider the cylindrical blow-up along the z_1-axis (also referred to as the blow-up *centered at z_1-axis*). The space associated with this blow-up consists of two copies of $\mathbb{C}^3$ with respective coordinates (x_1, τ_1, z_1) and (θ_1, y_1, z_1) with the identification $(x_1, \tau_1, z_1) \simeq (\theta_1, y_1, z_1)$ if and only if $\tau_1 = 1/\theta_1$ and $y_1 = \tau_1 x_1$. In coordinates (x_1, τ_1, z_1), the blow-up map π_τ becomes $\pi_\tau : (x_1, \tau_1, z_1) \mapsto (x_1, \tau_1 x_1, z_1)$. The blow-up (transform) of the vector field $X = X_1$ has the form $\widetilde{X}_1 = x_1^{-3} z_1 \widetilde{Y}_1^*$ with

$$\widetilde{Y}_1^* = \widetilde{P}_1^*(x_1, \tau_1)\partial/\partial x_1 + \widetilde{Q}_1^*(x_1, \tau_1)\partial/\partial \tau_1 - z_1 \widetilde{H}_1^*(x_1, \tau_1)\partial/\partial z_1 \tag{2.14}$$

where

$$\begin{aligned}
\widetilde{P}_1^*(x_1,\tau_1) &= -x_1(bx_1^3\tau_1 + dx_1^2\tau_1 + ax_1^2 + Bx_1\tau_1 + cx_1)\ , \\
\widetilde{Q}_1^*(x_1,\tau_1) &= e'x_1^2\tau_1^2 + b'x_1^2\tau_1 + Bx_1\tau_1^2 + d'x_1\tau_1 + a'x_1\ , \\
\widetilde{H}_1^*(x_1,\tau_1) &= -(bx_1^3\tau_1 + (d-F)x_1^2\tau_1 + (a-D)x_1^2 + 2Bx_1\tau_1 + (c-E)x_1)\ .
\end{aligned}$$

The above formulas show that, in the present case, the vector field $\widetilde{Y}_1^*$ is again divisible by x_1. Thus the vector field $\widetilde{X}_1$ becomes $\widetilde{X}_1 = x_1^{-2}z_1\widetilde{Y}_1$ with

$$\widetilde{Y}_1 = \widetilde{P}_1(x_1,\tau_1)\partial/\partial x_1 + \widetilde{Q}_1(x_1,\tau_1)\partial/\partial \tau_1 - z_1\widetilde{H}_1(x_1,\tau_1)\partial/\partial z_1 \tag{2.15}$$

where:

$$\begin{aligned}
\widetilde{P}_1 &= -x_1(bx_1^2\tau_1 + dx_1\tau_1 + ax_1 + B\tau_1 + c)\ , \\
\widetilde{Q}_1 &= e'x_1\tau_1^2 + b'x_1\tau_1 + B\tau_1^2 + d'\tau_1 + a'\ , \\
\widetilde{H}_1 &= -(bx_1^2\tau_1 + (d-F)x_1\tau_1 + (a-D)x_1 + 2B\tau_1 + (c-E))\ .
\end{aligned}$$

In particular, the singular points of $\widetilde{\mathcal{F}}$ (or equivalently of $\widetilde{Y}_1$) on $\{x_1 = z_1 = 0\}$ are determined by the equation

$$B\tau_1^2 + d'\tau_1 + a'\ . \tag{2.16}$$

As a complement, we may also check on the structure of the vector field $\widetilde{X}_1$ in the coordinates (θ_1, y_1, z_1) for the blow-up centered at the axis z_1 where the blow-up map π_θ is given by $\pi_\theta(\theta_1, y_1, z_1) = (\theta_1 y_1, y_1, z_1)$. The vector field $\widetilde{X}_1$ then becomes $\widetilde{X}_1 = z_1\theta_1^{-3}y_1^{-2}\widetilde{Y}_1$ with

$$\widetilde{Y}_1 = \widetilde{P}_1(\theta_1,y_1)\partial/\partial \theta_1 + \widetilde{Q}_1(\theta_1,y_1)\partial/\partial y_1 - z_1\widetilde{H}_1(\theta_1,y_1)\partial/\partial z_1 \tag{2.17}$$

where:

$$
\begin{aligned}
\widetilde{P}_1(\theta_1, y_1) &= -\theta_1(b'\theta_1^2 + a'\theta_1^2 + e'y_1\theta_1 + d'\theta_1 + Cy_1 + B)\ , \\
\widetilde{Q}_1(\theta_1, y_1) &= \theta_1 y_1(by_1^2\theta_1 + (a-b')y_1\theta_1 + ey_1^2 - a'\theta_1 + (d-e')y_1 + (c-d'))\ , \\
\widetilde{H}_1(\theta_1, y_1) &= -(by_1^2\theta_1^2 + (a-D)y_1\theta_1^2 + ey_1^2\theta_1 + (d-F)y_1\theta_1 \\
&\quad +(c-E)\theta_1 + 2B)\ . \qquad (2.18)
\end{aligned}
$$

Again we note that the origin $(0,0,0)$ of the coordinates (θ_1, y_1, z_1) is a singular point of $\widetilde{Y}_1$ with eigenvalues respectively given by 1, 0, and 2. This is naturally reminiscent from the discussion in Remark 2.2.1 and does not yield additional information on the coefficients of X.

We are now in position to establish the main results of this section.

Proposition 2.2.2. *Let X and Y be as in (2.13) (Conditions I, II, and III being verified). and $\widetilde{X_1} = x_1^{-2} z_1 \widetilde{Y_1}$ with $\widetilde{Y_1}$ as in (2.15). Then the singular points of $\widetilde{X_1}$ on the τ_1-axis are the solutions of the equation $B\tau_1^2 + d'\tau_1 + a' = 0$. Furthermore, we have:*

- *If the discriminant ${d'}^2 - 4a'B = 0$ then there is only one singular point and the expression $2(-d' + c - E)/(-d' + 2c)$ is an integer.*
- *If the discriminant ${d'}^2 - 4a'B \neq 0$ then there are two different singular points and the numbers*

  ı)
  $$\frac{2(c - E - d' + (-1)^j\sqrt{{d'}^2 - 4a'B})}{2c - d' + (-1)^j\sqrt{{d'}^2 - 4a'B}} \quad \text{and}$$

  ıı)
  $$\frac{-2\sqrt{{d'}^2 - 4a'B}}{2c - d' + (-1)^j\sqrt{{d'}^2 - 4a'B}}$$

are rationals with the respective forms m_j/q_j, n_j/p_j $(j = 1, 2)$.

Finally, the integers n_j, m_j, p_j, q_j must also satisfy the equation

$$-\,2\ \mathrm{lcm}(p_j, q_j) + m_j \frac{\mathrm{lcm}\,(p_i, q_j)}{q_j} = \pm 1, \qquad (2.19)$$

for $j = 1, 2$ and where $\mathrm{lcm}(p, q)$ *stands for the least common multiple of p, q.*

Proof. Consider the vector field $\widetilde{X}_1 = x_1^{-2} z_1 \widetilde{Y}_1$, with $\widetilde{Y}_1$ as in (2.15), and recall that the singular points of $\widetilde{Y}_1$ sitting in the axis $\{x_1 = z_1 = 0\}$ are given by the solutions of $B\tau^2 + d'\tau + a' = 0$, see Equation (2.16).

Let α be such that $B\alpha^2 + d'\alpha + a' = 0$. We want to consider the linear part of $\widetilde{Y}_1$ at $(0, \alpha, 0)$ or, equivalently, the linear part of the origin of the vector field obtained as pull-back of $\widetilde{Y}_1$ by the (linear) change of coordinates $(x_1, \tau_1, z_1) \mapsto (x_1, t+\alpha, z_1)$. In any case, the linear vector field in question is given by

$$W = -x_1(B\alpha+c)\frac{\partial}{\partial x_1} + [x_1(e'\alpha^2+b'\alpha+C\alpha^3)+t(d'+2B\alpha)]\frac{\partial}{\partial t} - z_1(2B\alpha+c-E)\frac{\partial}{\partial z_1}\,. \tag{2.20}$$

On the other hand, the function $x_1^{-2} z_1$ (of $\widetilde{X}_1 = x_1^{-2} z_1 \widetilde{Y}_1$) remains unchanged by the indicated change of coordinates. Since, moreover, $\widetilde{X}_1$ is semicomplete, Lemma 1.2.2 ensures that the vector field $x_1^{-2} z_1 W$ is semicomplete as well.

Now there follows immediately that the eigenvalues of W are $\lambda_1 = 1$, $\lambda_2 = -(d' + 2B\alpha)/(B\alpha + c)$, and $\lambda_3 = (2B\alpha + c - E)/(B\alpha + c)$. Furthermore, even though the linear vector field in (2.20) is not in diagonal form, we can still apply Lemma 1.2.3 since the multiplicative function has no factor of the form $\tau_1^{n_2}$ (i.e. $n_2 = 0$).

In view of Lemma 1.2.3, there are two different cases to be considered according to whether or not $n_1 + n_2\lambda_2 + n_3\lambda_3 = 0$. Here, however, note that the condition $n_1 + n_2\lambda_2 + n_3\lambda_3 = 0$ is equivalent to $c + E = 0$. In turn, $c + E$ is exactly the coefficient of x^2 in $P_1(x, y, z)$ of the quadratic vector field $\mathcal{Z}$ of (2.3), where $n = 3$, cf. Lemma 2.1.6. This shows that this case cannot occur in view of Condition III. In other words, we necessarily have $n_1 + n_2\lambda_2 + n_3\lambda_3 \neq 0$.

Next assume first that the equation $B\alpha^2 + d'\alpha + a' = 0$ has a double root. Then $d'^2 - 4a'B = 0$ and $d' + 2B\alpha = 0$ so that $\lambda_2 = 0$. Indeed, substituting $-d'/2B$ for α, we conclude that $\lambda_3 = 2(-d' + c - E)/(-d' + 2c)$ which, in addition, must be an integer thanks to Lemma 1.2.3 along with the condition $n_1 + n_2\lambda_2 + n_3\lambda_3 \neq 0$.

Assume now that the equation $B\alpha^2 + d'\alpha + a' = 0$ has two distinct roots given by $(-d+\sqrt{\Delta})/2B$ and by $(-d-\sqrt{\Delta})/2B$, where $\Delta = d'^2 - 4a'B = 0$. Substituting

$(-d' + \sqrt{\Delta})/2B$ for α, the eigenvalues of W become $\lambda_1 = 1$,

$$\lambda_2 = -\frac{2\sqrt{\Delta}}{-d' + 2c + \sqrt{\Delta}} \quad \text{and} \quad \lambda_3 = \frac{2(-d' + c - E + \sqrt{\Delta})}{-d' + 2c + \sqrt{\Delta}} .$$

Owing to Lemma 1.2.3, both λ_2 and λ_3 are rational. Furthermore, setting $\lambda_2 = n/p$ and $\lambda_3 = m/q$, Lemma 1.2.3 yields equation (2.19) since $n_1 + n_2\lambda_2 + n_3\lambda_3 \neq 0$. This proves the second part of the statement for $j = 2$. To derive the claims corresponding to the case $j = 1$, just repeat the same arguments with the root $\alpha = (-d - \sqrt{\Delta})/2B$. The proof of the proposition is complete. $\square$

Next note that the above discussion involving the vector field $X = X_1 = x_1^{-3} z_1 Y_1$ defined on coordinates (x_1, y_1, z_1) satisfying $(1/x_1, y_1/x_1, z_1/x_1) = (x, y, z)$ can be reproduced in coordinates (x_2, y_2, z_2) where $(x_2/y_2, 1/y_2, z_2/y_2) = (x, y, z)$. The vector field X then takes on the form $X = X_2 = y_2^{-3} z_2 Y_2$ with the vector field Y_2 being given by

$$Y_2 = P_2(x_2, y_2)\partial/\partial x_2 + Q_2(x_2, y_2)\partial/\partial y_2 + z_2 H_2(x_2, y_2)\partial/\partial z_2$$

where:

$$\begin{aligned}
P_2(x_2, y_2) &= -y_2[a' x_2^2 y_2 + (d' - c)x_2^2 + (b' - a)x_2 y_2 - b y_2^2 \\
&\quad +(e' - d)x_2 + B y_2] , \\
Q_2(x_2, y_2) &= -y_2[a' x_2 y_2^2 + d' x_2 y_2 + b' y_2^2 + B x_2 + e' y_2] , \\
H_2(x_2, y_2) &= -a' x_2 y_2^2 + (E - d')x_2 y_2 + (D - b')y_2^2 - 2B x_2 - 2B y_2 \quad (2.21)
\end{aligned}$$

In particular the vector field Y_2 has the normal form indicated in (2.13). Thus Proposition 2.2.2 can be applied to X_2, Y_2 so that we obtain the following:

Corollary 2.2.3. *Consider vector fields X_2 and Y_2 with $X_2 = y_2^{-3} z_2 Y_2$ and where Y_2 is as in (2.21). Then we have:*

- *If the discriminant $d^2 - 4bB = 0$ then there is only one singular point and the expression $(d - 2e')/(e' + F)$ is an integer.*
- *If the discriminant $d^2 - 4bB \neq 0$ then there are two different singular points*

and the numbers

ı)
$$\frac{2(e' - d - F + \sqrt{d^2 - 4bB})}{2e' - d + \sqrt{d^2 - 4bB}} \quad \text{and}$$

ıı)
$$\frac{-2\sqrt{d^2 - 4bB}}{2e' - d + \sqrt{d^2 - 4bB}}$$

are rationals with respective forms m'_j/q'_j, n'_j/p'_j $(j = 1, 2)$. Furthermore the integers n'_j, m'_j, p'_j, q'_j also satisfy the equation

$$-2 \ \mathrm{lcm}(p'_j, q'_j) + m'_j \frac{\mathrm{lcm}\,(p'_j, q'_j)}{q'_j} = \pm 1, \tag{2.22}$$

where $\mathrm{lcm}(p', q')$ *stands for the least common multiple of p', q'.* □

Example 2.2.4. Consider the family of Halphen vector fields on $\mathbb{C}^3$ parameterized by

$$\begin{aligned} H(\alpha_1, \alpha_2, \alpha_3) &= [\alpha_1 x_0^2 + (1 - \alpha_1)(x_0 y_0 + x_0 z_0 - y_0 z_0)] \frac{\partial}{\partial x_0} + \\ &+ [\alpha_2 y_0^2 + (1 - \alpha_2)(x_0 y_0 - x_0 z_0 + y_0 z_0)] \frac{\partial}{\partial y_0} + \\ &+ [\alpha_3 z_0^2 + (1 - \alpha_3)(-x_0 y_0 + x_0 z_0 + y_0 z_0)] \frac{\partial}{\partial z_0} \end{aligned}$$

where the parameters α_i, $i = 1, 2, 3$ are complex numbers. The univaluedness character of the solutions of the above vector fields was first considered by G. Halphen. The topic was recently thoroughly explained by A. Guillot in [9].

Consider the blow-up of $\mathbb{C}^3$ at the origin and coordinates (x, y, z) where the blow-up map becomes $(x, y, z) \mapsto (xz, yz, z) = (x_0, y_0, z_0)$. The blow up $\widetilde{H}$ of the vector field H is given by $\widetilde{H} = H_1(x, y)\partial/\partial x + H_2(x, y)\partial/\partial y + zH_3(x, y)\partial/\partial z$ where

$$\begin{aligned} H_1 &= -x^2 y(\alpha_3 - 1) + x^2(\alpha_1 + \alpha_3 - 1) + xy(\alpha_3 - \alpha_1) - x(\alpha_1 + \alpha_3 - 1) + y(\alpha_1 - 1)\,; \\ H_2 &= -xy^2(\alpha_3 - 1) + xy(\alpha_3 - \alpha_2) + y^2(\alpha_2 + \alpha_3 - 1) + x(\alpha_2 - 1) - y(\alpha_2 + \alpha_3 - 1)\,; \\ H_3 &= xyz(\alpha_3 - 1) - xz(\alpha_3 - 1) - yz(\alpha_3 - 1) + z\alpha_3\,. \end{aligned}$$

Note that $\widetilde{H}$ appears in the normal form presented in (2.13). In particular, we have $d' = \alpha_3 - \alpha_2$, $a' = \alpha_2 - 1$, and $B = -(\alpha_3 - 1)$ so that the discriminant $(d')^2 - 4a'B$ equals $(\alpha_3 + \alpha_2 - 2)^2$. If we assume that the initial vector field H is semicomplete, then Proposition 2.2.2 implies that the numbers

$$\frac{\alpha_1 + 2\alpha_2 + 2\alpha_3 - 4}{\alpha_1 + \alpha_2 + \alpha_3 - 2} \quad \text{and} \quad \frac{\alpha_2 + \alpha_3 - 2}{\alpha_1 + \alpha_2 + \alpha_3 - 2}$$

($j = 2$) are rationals of the form m/q and n/p, where m, n, p, q are integers. In particular, we have $2 - m/q = \alpha_1/(\alpha_1 + \alpha_2 + \alpha_3 - 2)$. In turn, formula (2.19) from Proposition 2.2.2 yields

$$\operatorname{lcm}(p, q) \left[-2 + \frac{m}{q} \right] = \pm 1 \, .$$

Note that this equation is possible only if $\operatorname{lcm}(p, q) = q$, i.e. if p divides q. Also we must have $-2q + m = \pm 1$. Thus

$$2 - \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3 - 2} = 2 \pm \frac{1}{q}$$

and so $(\alpha_1 + \alpha_2 + \alpha_3 - 2)/\alpha_1 = \pm q \in \mathbb{Z}$. Next consider relation *ii*)

$$\begin{aligned} \frac{n}{p} &= \frac{\alpha_2 + \alpha_3 - 2}{\alpha_1 + \alpha_2 + \alpha_3 - 2} \\ &= 1 - \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3 - 2} = 1 - \frac{1}{q} \end{aligned}$$

Since p divides q and $q = nq/p + 1$, it follows that $m_1 = (\alpha_1 + \alpha_2 + \alpha_3 - 2)/\alpha_1$ is an integer strictly greater than 1. Using the other two standard affine coordinates for the blow-up of $\mathbb{C}^3$ at the origin, we similarly conclude that $m_2 = (\alpha_1 + \alpha_2 + \alpha_3 - 2)/\alpha_2$ and $m_3 = (\alpha_1 + \alpha_2 + \alpha_3 - 2)/\alpha_3$ must be integers greater than 1 as well.

Remark 2.2.5. There follows from the preceding that the condition of having m_1, m_2, m_3 integers greater than 1 is a necessary condition for the Halphen vector field to be semicomplete. The fact that these conditions are also sufficient is harder to prove, see [9].

Recall that we are interested in knowing the extent to which the coefficients of the vector field Y in formula (2.13) are reals/rationals. For this purpose, we might ignore formulas (2.19) and (2.22) and summarize the content of Proposition 2.2.2 and of Corollary 2.2.3 as follows:

Lemma 2.2.6. *The fact that the numbers indicated in items i) and ii) appearing in Proposition 2.2.2 and in Corollary 2.2.3 must be rationals, is equivalent to say the the four numbers below are rationals:*
a) $\dfrac{d'-2c}{c+E}$ b) $\dfrac{d'-2c}{\sqrt{d'^2-4a'B}}$ c) $\dfrac{d-2e'}{e'+F}$ d) $\dfrac{d-2e'}{\sqrt{d^2-4bB}}$

Proof. Condition b) is immediately obtained from condition ii) in Proposition 2.2.2. Now use condition b) and condition i) (Proposition 2.2.2) to conclude that $(c+E)/\sqrt{d'^2-4a'B}$ is rational so that condition a) follows as the the quotient of the rational numbers. The procedure to derive conditions c) and d) from conditions i) and ii) in Corollary 2.2.3 is analogous. □

Recall that the vector field X, as the blow-up of the quadratic vector field $\mathcal{Z}$ on $\mathbb{C}^3$, is globally defined on $\widetilde{\mathbb{C}}^3$ (the blow up of $\mathbb{C}^3$ at the origin). The above conditions on the coefficients of the vector field X (or Y) were obtained by using the normal form (2.13) of X in affine coordinates (u_1, v_1, w_1). Since this normal form is preserved by the standard affine atlas of $\widetilde{\mathbb{C}}^3$ (cf. Lemma 2.1.4), Proposition 2.2.2 can also be applied to the expression of the vector field X in the remaining coordinates (u_2, v_2, w_2) and (u_3, v_3, w_3).

In view of the transition maps described in Section 3, let $X = v_2 Y$ be the representation of the vector field X in affine coordinates (u_2, v_2, w_2) with $Y = Y_{(u)}\partial/\partial u_2 + Y_{(w)}\partial/\partial w_2 + Y_{(v)}\partial/\partial v_2$ as follows:

$$\begin{aligned} Y_{(u)} &= (d-e')u_2 + bw_2 + (c-d')u_2^2 + (a-b')u_2w_2 - a'u_2^2w_2\,; \\ Y_{(w)} &= -Bu_2 - e'w_2 - d'u_2w_2 - b'w_2^2 - a'u_2w_2^2\,; \\ Y_{(v)} &= v_2[(e'+F) + (d'+E)u_2 + (b'+D)w_2 + a'u_2w_2]\,. \end{aligned} \tag{2.23}$$

Note that this is, in fact, the normal form of the vector field X as presented in (2.13). Then we can apply Proposition 2.2.2 to $X = v_2Y$ and proceed as in

Lemma 2.2.6 to conclude that the following numbers are rational:

a') $\dfrac{d'-2c}{c+E}$ b') $\dfrac{d'-2c}{\sqrt{d'^2-4a'B}}$ e) $\dfrac{a+b'}{D}$ f) $\dfrac{a+b'}{\sqrt{(a-b')^2+4a'b}}$

Of course the first two numbers above were already known to be rational.

Finally, the representation $X = u_3 Y$ of X in affine coordinates (u_3, v_3, w_3) where $Y = Y_{(w)}\partial/\partial w_3 + Y_{(v)}\partial/\partial v_3 + Y_{(u)}\partial/\partial u_3$ is as follows:

$$\begin{aligned} Y_{(w)} &= -cw_3 - Bv_3 - aw_3^2 - dv_3w_3 - bv_3w_3^2\,; \\ Y_{(v)} &= a'w_3 + (d'-c)v_3 + (b'-a)v_3w_3 + (e'-d)v_3^2 - bv_3^2w_3\,; \\ Y_{(u)} &= u_3[(c+E) + (d+F)v_3 + (a+D)w_3 + bv_3w_3]\,. \end{aligned} \tag{2.24}$$

At this point, the use of Proposition 2.2.2 in these coordinates will only provide us with redundant information.

The previous discussion is summarized in the following theorem.

Theorem 2.2.7. *Consider the vector field X obtained as the blow up at the origin of the semicomplete quadratic vector field $\mathcal{Z}$ on $\mathbb{C}^3$. Assume that the vector field $X = w_1 Y$ is in the normal form (2.13) Then the following expressions represent rational numbers:*

a) $\dfrac{d'-2c}{c+E}$ b) $\dfrac{d'-2c}{\sqrt{d'^2-4a'B}}$ c) $\dfrac{d-2e'}{e'+F}$

e) $\dfrac{d-2e'}{\sqrt{d^2-4bB}}$ f) $\dfrac{a+b'}{D}$ g) $\dfrac{a+b'}{\sqrt{(a-b')^2+4a'b}}$

The above formulas can further be simplified. Let us go back to the normal form (2.13) along with the corresponding affine (x, y, z)-coordinates. In particular, we have $X = zY$. Now recall that transformations of the form

$$(x, y, z) \mapsto (\lambda_1 x, \lambda_2 y, z)$$

does preserve the normal form. Moreover, all the expressions in the coefficients of X, Y given by Theorem 2.2.7 remain invariant by these change of coordinates. The setting, however, changes slightly if consider translations of the form $(x, y, z) \mapsto$

$(x+k_1, y+k_2, z)$. As pointed out in Section 3, this type of transformation preserves the normal form (2.13) except by the fact that the origin of these coordinates no longer needs to be a singular point of Y. This said, all the six numbers $d' - 2c$, $c+E$, $d'^2 - 4a'B$, $d-2e'$, $e'-F$, and $d^2 - 4bB$ remain unchanged under this change of coordinates. In particular, do does the four first numbers in Theorem 2.2.7.

On the other hand, the coefficients c, d, d', and e' change as follows

$$c \mapsto c + Bk_2 \ , \quad d \mapsto d + 2Bk_1 \ , \quad d' \mapsto d' + 2Bk_2 \ , \quad \text{and} \quad e' \mapsto e' + Bk_1 \, .$$

Thus we have:

$$\frac{c}{d'} = \frac{c + Bk_2}{d' + 2Bk_2} \quad \text{and} \quad \frac{e'}{d} = \frac{e' + Bk_1}{d + 2Bk_1}$$

up to assuming that $d' \neq 0$ and $d \neq 0$.

To make things accurate, let us add one more condition to normal form (2.13)

Condition IV: In normal form (2.13), we have $B \neq 0$.

This assumption hardly impact the generality of the discussion, since $B = 0$ would imply that the "line at infinity" is invariant. Moreover, up to permuting the order of the invariant axes x, y, and z for the initial quadratic vector field, if we never obtain $B \neq 0$, it would mean that $b = a' = 0$ in normal form (2.13), in addition to $B = 0$. The ensuing discussion would then be formally included in the discussion below (for more details, see the next section).

Since $B \neq 0$, we can find k_1 and k_2 such that $c = d'$ and $d = e'$ while, at the same time, making sure that $cd \neq 0$. Moreover the remaining coefficients are such that the six numbers $d' - 2c$, $c+E$, $d'^2 - 4a'B$, $d-2e'$, $e'-F$, and $d^2 - 4bB$ remain unchanged. Hence in these new coordinates (still denoted by x, y, and z), we have $X = zY$ with $Y = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y + zH(x,y)\partial/\partial z$ and

$$\begin{aligned} P(x,y) &= \gamma_1 + ax + by + cx^2 + dxy + Bx^2y \\ Q(x,y) &= \gamma_2 + a'x + b'y + d'xy + e'y^2 + Bxy^2 \ , \\ H(x,y) &= D + Ex + Fy - Bxy \, . \end{aligned} \tag{2.25}$$

with $c = d'$, $d = e'$, and $c\,d \neq 0$. Furthermore, in view of Theorem 2.2.7 and of the fact that the numbers $d' - 2c$, $c+E$, $d'^2 - 4a'B$, $d-2e'$, $e'-F$, and $d^2 - 4bB$

have not changed, we also conclude that the numbers

$$\frac{d'-2c}{c+E}\,,\quad \frac{d'-2c}{\sqrt{d'^2-4a'B}}\,,\quad \frac{d-2e'}{e'+F}\,,\quad \text{and}\quad \frac{d-2e'}{\sqrt{d^2-4bB}} \tag{2.26}$$

are all rational.

Finally all the four numbers in (2.26) as well as the quotients c/d' and e'/d are invariant by transformations of the form $(x,y,z)\mapsto(\lambda_1 x,\lambda_2 y,z)$. Thus we can use these maps to ensure that the point $(1,1,0)$ is a singular point of Y. Taking everything in account we have proved the main result of this section, namely:

Theorem 2.2.8. *Assume that X is semicomplete and that Conditions I through IV are satisfied. Then, in suitable affine coordinates for the blow-up of C^3, the vector field X takes on the form $X = zY$ with $Y = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y + zH(x,y)\partial/\partial z$ where*

$$\begin{aligned}
P(x,y) &= \gamma_1 + ax + d^2 r_1 y + cx^2 + dxy + x^2 y \\
Q(x,y) &= \gamma_2 + c^2 r_2 x + b'y + cxy + dy^2 + xy^2\,, \\
H(x,y) &= D + cr_3 x + dr_4 y - xy\,,
\end{aligned} \tag{2.27}$$

where r_i is a rational number ($i = 1,\dots,4$) and where $c, d \neq 0$. Furthermore γ_1 and γ_2 are (explicit) linear combinations over $\mathbb{Q}$ of the coefficients a, b', c, and d. Similarly, unless b' is already determined (linearly over $\mathbb{Q}$ from the coefficients a, c, d) then D must also be an explicit linear combinations over $\mathbb{Q}$ of the coefficients a, b', c, and d.

Proof. Consider $X = zY$ with Y as in (2.25). Next, up to performing a suitable translation in the variables (x,y) of (2.25), we can assume without loss of generality that $c = d'$, $d = e'$, and $c\,d \neq 0$. Furthermore, up to further changing coordinates by a map of the form $(x,y,z)\mapsto(\lambda_1 x,\lambda_2 y,z)$, with $\lambda_1,\lambda_2\in\mathbb{C}^*$, we can also assume that $(1,1,0)$ is a singular point of Y. Here the reader will note that this second change of coordinates does not disrupt the previously established condition $c = d'$, $d = e'$, and $c\,d \neq 0$.

Since $(d'-2c)/(c+E)$ has to be a rational number, the fact that $c = d'$

implies that E is a rational multiple of c, i.e. $E = cr_3$ for some $r_3 \in \mathbb{Q}$. Similarly, $(d-2e')/(e'+F)$ is rational and $d = e'$ so that we must have $F = dr_4$ for a certain rational number r_4.

Next consider the rational numbers

$$\frac{d'-2c}{\sqrt{d'^2-4a'B}} \quad \text{and} \quad \frac{d-2e'}{\sqrt{d^2-4bB}} \, .$$

Plugging in the first number the conditions $B = 1$ and $c = d'$ leads to the conclusion that $a'B/c^2$ must be a rational number. Likewise, the rational nature of $(d-2e')/\sqrt{d^2-4bB}$ yields $bB/d^2 \in \mathbb{Q}$. Now, since the coefficients of X, Y are considered only up to a multiplicative constant, we can set once and for all $B = 1$ (since $B \neq 0$). Thus we must have $b = d^2 r_1$ and $a' = c^2 r_2$ for suitable rational numbers r_1, r_2.

Now since $(1, 1, 0)$ is a singular point of Y, we obtain

$$\gamma_1 = -a - d^2 r_1 - c - d - 1 \quad \textit{and} \quad \gamma_2 = -c^2 r_2 - b' - c - d - 1 \, .$$

Finally to conclude that D is also as indicated in the statement, we proceed as follows. First we can suppose that $D + cr_3 + dr_4 - 1 \neq 0$, otherwise there is nothing to be proved. Next consider the linear part of Y at the point $(1, 1, 0)$ which is given by the vector field

$$\begin{aligned} &\Big[(a+2c+d+2)x + (d^2 r_1 + d + 1)y\Big] \frac{\partial}{\partial x} + \\ &+ \Big[(c^2 r_2 + c + 1)x + (b' + c + 2d + 2)y\Big] \frac{\partial}{\partial y} + \\ &+ z(D + cr_3 + dr_4 - 1) \frac{\partial}{\partial z} \, . \end{aligned} \tag{2.28}$$

Since $X = zY$ is a semicomplete vector field, there follows that z times the above (linear) vector field must be semicomplete as well. In turn, this means that the eigenvalues of the (linear) vector field in question are integral multiples of the eigenvalue associated with the $\partial/\partial z$-direction, cf. Section 2 or [8]. The sum of the eigenvalues being the trace of the matrix, there follows that the quotient

$$\frac{a + b' + 3c + 3d + 4}{D + cr_3 + dr_4 - 1}$$

represents an integer unless $D+cr_3+dr_4-1 = 0$. Thus either $a+b'+3c+3d+4 = 0$ or D is given as a linear combination over $\mathbb{Q}$ of a, b', c, and d. The theorem follows at once. $\square$

2.2.1 When there are invariant planes

In this short section, we shall add the assumption that the initial quadratic vector field $\mathcal{Z}$ leaves some plane through the origin invariant. As equivalent formulation, the restriction to the exceptional divisor of the foliation associated with the blow-up of $\mathcal{Z}$ leaves a line invariant. Naturally we also assume that the remaining conditions of Theorem 2.2.8 are satisfied.

To be more accurate, we consider the blow-up X of $\mathcal{Z}$ in suitable affine coordinates under the normal form (2.13). Recalling that $X = zY$, we assume the following holds:

(1) We have $a' = 0$ so that the axis $\{y = 0\}$ is invariant by Y. Also we require $b \neq 0$ so that the axis $\{x = 0\}$ *is not invariant* by Y.

(2) The restriction to the exceptional divisor of the foliation associated with X has at least 5 singular points.

(3) $B \neq 0$.

These conditions basically mean that we are willing to consider the least favorable case that can be encountered once the existence of an invariant line is ensured. They serve to keep the discussion focused on the main difficulties of the problem.

Since $B \neq 0$, the line at infinity Δ is not invariant by the restriction to the exceptional divisor of the foliation associated with X. Note that this foliation is naturally induced by Y in the affine coordinates used in normal form (2.13). Since Δ contains already two singular points of the mentioned foliation, namely the intersection points of Δ with the axes $\{y = 0\}$ and $\{x = 0\}$, it follows that Δ cannot contain a third singular point. Indeed, since the degree of the foliation in question is 2, every line containing three singular points of it must be invariant, see Section 2. We also remind the reader that, similarly, an invariant line cannot contain more than 3 singular points (cf. Section 2).

In view of condition (2), there follows that Y has at least 3 singular points in the domain of the affine (x, y)-coordinates (with $z = 0$). In particular Y has at least one singular point which *does not lie in* the invariant line.

Now, using maps of the form $(x, y, z) \mapsto (\lambda_1 x + k_1, \lambda_2 y + k_2, z)$ as in the previous section, the vector fields X and Y can be brought to the normal form of Theorem 2.2.8, namely we have $X = zY$, with $Y = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y + zH(x, y)\partial/\partial z$, where

$$\begin{aligned}
P(x,y) &= \gamma_1 + ax + d^2 r_1 y + cx^2 + dxy + x^2 y \ , \\
Q(x,y) &= \gamma_2 + c^2 r_2 x + b' y + cxy + dy^2 + xy^2 \ , \\
H(x,y) &= D + cr_3 x + dr_4 y - xy \ .
\end{aligned} \tag{2.29}$$

Furthermore the vector field Y leaves some horizontal line $\{y = A\}$ invariant. Since there is a singular point of Y restricted to $\{z = 0\}$ which does not lie in the mentioned invariant line, by repeating the procedure in the proof of Theorem 2.2.8, we can assume this point to be $(1, 1)$ unless it lies in the axis $\{x = 0\}$. Again to focus on the main issue of the problem, we add two additional assumptions to the previous ones:

(4) The point $(1, 1) \simeq (1, 1, 0)$ is singular for Y. Also $c \neq 0$.

(5) We have $a + b' + 3c + 3d + 4 \neq 0$ so that D is given as a linear combination over $\mathbb{Q}$ of a, b', c, and d.

The above conditions (1) – (5) will be referred to in this section as *generic invariant line conditions*. The aim of this section is to investigate how close to a vector field having only real coefficients the condition of an invariant line leads us. We start with a simple lemma:

Lemma 2.2.9. *The rational number r_2 is such that $\sqrt{1 - 4r_2}$ is again rational.*

Proof. Recall from the proof of Theorem 2.2.8 that $d' = c$ and $B = 1$. Also the number

$$\frac{d' - 2c}{\sqrt{d'^2 - 4a'B}}$$

is known to be rational. Naturally a' is nothing but $c^2 r_2$ so that plugging in $d' = c$ and $B = 1$, we conclude that

$$\frac{1}{\sqrt{1 - 4r_2}}$$

is rational and the lemma follows. $\square$

Another simple by-product of the proof of Theorem 2.2.8 is as follows. Consider again the linear vector field in (2.28). As mentioned, the eigenvalues of this vector field must be related by integral quotients. In particular the eigenvalues of the matrix M below

$$M = \begin{bmatrix} a+2c+d+2 & d^2r_1+d+1 \\ c^2r_2+c+1 & b'+c+2d+2 \end{bmatrix}$$

have rational ratio. Therefore the quotient between the trace of M and its determinant is rational itself. In other words, we have

$$\hat{r} = \frac{a+b'+3c+3d+4}{(a+2c+d+2)(b'+c+2d+2)-(c^2r_2+c+1)(d^2r_1+d+1)} \in \mathbb{Q}\,. \quad (2.30)$$

Thus we actually obtain an extra—nonlinear—relation between the coefficients a, b', c, d. Up to assuming that the rational number $\hat{r}$ on the left side of (2.30) is different from $(b'+c+2d+2)^{-1}$, we can "solve" this relation for a. In other words, unless $b'+c+2d$ is a rational number—which of course provides directly a new relation between b', c, and d—we can state the following:

Lemma 2.2.10. *Assuming the rational number $\hat{r}$ is different from $(b'+c+2d+2)^{-1}$, the coefficient a can be expressed under the form*

$$a = \frac{\mathrm{Pol}_4(c,d,b)}{\mathrm{Pol}_1(c,d,b)}$$

where Pol_4 (resp. Pol_1) is a polynomial of degree 4 (resp. 1) on b', c, d with rational coefficients. $\square$

To begin to exploit the fact that the line $\{y = A\}$ is invariant by Y, we note that this condition is equivalent to saying that $Q(x, A)$ is identically zero (independently of x). Alternatively, we must have

$$Q(x,y) = (y-A)((xy-dy+\alpha x+\beta)$$

where A, α, and β are coefficients in $\mathbb{C}$. This yields the following over-determined system of equations:

$$\alpha - A = c\,; \quad -A\alpha = c^2 r_2\,, \quad \beta - Ad = b'\,, \quad \text{and} \quad -A\beta = \gamma_2\,.$$

The first two equations yield $A^2+AC+c^2r_2=0$. Owing to Lemma 2.2.9, it follows that A is a rational multiple of c. We set $A=cs_1$ with $s_1 \in \mathbb{Q}$.

We assume through the remainder of the section that $s_1 \neq 0$ since, otherwise, the corresponding discussion would be greatly simplified. Since $A = cs_1$, there follows that $\alpha c(1+s_1)$ and $\beta = b'+cds_1$. However, we know that $-\gamma_2 = c^2r_2+b'+c+d+1$ since Y has a singular point at $(1,1) \simeq (1,1,0)$. Thus the last equation provides:

$$cs_1(b'+cds_1) = c^2r_2 + b' + c + d + 1\,.$$

Once again we can eliminate one parameter. For example either $cs_1 = 1$ which means c is rational, or we obtain:

Lemma 2.2.11. *With the preceding notation, we have*

$$b' = \frac{c^2r_2 + c^2ds_1^2 + c + d + 1}{cs_1 - 1}\,. \tag{2.31}$$

□

In either event, we now have only two parameters that forces the entire vector field to have rational/real coefficients once they are rational/real themselves. To continue the discussion, let us assume that $cs_1 \neq 1$ so that Lemma 2.2.11 holds.

At this point, it is convenient to summarize the previous results as follows:

Proposition 2.2.12. *Assume that all the "generic" conditions considered above hold. Then every coefficient of the vector field Y in (2.29) can be expressed as a rational function over $\mathbb{Q}$ of the coefficients c and d.* □

Remark 2.2.13. The statement of Proposition 2.2.12 can be made slightly more precise. Indeed, b' is explicitly given by Formula (2.31) while a was obtained from Equation (2.30) under the assumption that $\hat{r}$ is different from $(b'+c+2d+2)^{-1}$. Substituting in Equation (2.30) Formula (2.31) for b', we conclude that a has the form

$$a = \frac{\mathrm{Pol}_5(c,d)}{\mathrm{Pol}_3(c,d)}$$

where Pol_5 (resp. Pol_3) is a polynomial over $\mathbb{Q}$ of degree 5 (resp. 3) on c, d. Since all the remaining coefficients of the vector field Y are either constants or given by linear combination with rational coefficients of a, b', c, and d, we conclude that every such coefficient is given as a quotient of the form

$$\frac{\mathrm{Pol}_6(c,d)}{(cs_1 - 1)\mathrm{Pol}_3(c,d)} \tag{2.32}$$

where now Pol_6 is a polynomial over $\mathbb{Q}$ of degree 6 on the variables c, d.

Still more accurate information is provided by carefully looking at our formulas. Considering Formula (2.31), we see that the denominator does not depend on d while the degree of the numerator with respect to d equals 1. Similarly, we conclude in the preceding formula for a that:

- The degree with respect to d of Pol_5 is at most 2.
- The degree with respect to d of Pol_3 is at most 1.

Thus, for a general coefficient of Y, Formula (2.32) is such that

(a) The degree with respect to d of Pol_6 is at most 2.

(b) The degree with respect to d of Pol_3 is at most 1 and so is the degree with respect to d of $(cs_1 - 1)\mathrm{Pol}_3$.

(c) The denominator $(cs_1 - 1)\mathrm{Pol}_3$ of the preceding expression is common to all the coefficients of Y.

To close this section and the discussion, let us try to sharpen Proposition 2.2.12 by looking at the singular points of Y sitting in the invariant line $\{y = cs_1\}$. Denote by τ_1, τ_2 the position of these singular points, i.e. in (x, y, z) coordinates they are given by $(\tau_1, cs_1, 0)$ and $(\tau_2, cs_1, 0)$. Then τ_1, τ_2 are solution of the equation

$$(cs_1 + c)x^2 + (dcs_1 + a)x + \gamma_1 + d^2 r_1 cs_1 = 0\,. \tag{2.33}$$

Furthermore the eigenvalues of Y at the point $(\tau_i, cs_1, 0)$ coincide with the eigenvalues of the matrix M_i given by

$$M_i = \begin{bmatrix} D + dr_4cs_1 + cr_3\tau_i & 0 & 0 \\ 0 & a + 2c\tau_i + dcs_1\tau_i + 2cs_1\tau_i & d^2r_1 + d\tau_i + \tau_i^2 \\ 0 & 0 & b' + c\tau_i + 2dcs_1 + 2cs_1\tau_i \end{bmatrix}$$

which is upper triangular. Therefore we once again obtain:

$$\frac{a + 2c\tau_i + dcs_1\tau_i + 2cs_1\tau_i}{D + dr_4cs_1 + cr_3\tau_i} = m_i \quad \text{and} \quad \frac{b' + c\tau_i + 2dcs_1 + 2cs_1\tau_i}{D + dr_4cs_1 + cr_3\tau_i} = n_i \qquad (2.34)$$

for suitable integers m_i, n_i, $i = 1, 2$.

Remark 2.2.14. The possible values of the integers m_i are determined by the classification of quadratic semicomplete vector fields in dimension 2, cf. [5]. Indeed, the restriction of the quadratic vector field $\mathcal{Z}$ to the invariant plane fixed in the beginning of the section must provide a vector field in the corresponding classification.

In particular we note that m_1 can be assumed to be different from m_2 as a consequence of the classification in [5] unless the restriction of $\mathcal{Z}$ to the invariant plane in question is conjugate to the vector field $x(x - 2y)\partial/\partial x + y(y - 2x)\partial/\partial y$ in which case, we always have $m_i = -3$.

Similarly the sum of n_i (including the corresponding value at infinity) equals 1 as a consequence from the index theorem in [2]. Again it is not hard to work out the cases in which n_i does not vary and hence are all equal to 3.

Whereas this type of information will not really be needed in the sequel, it allows us to justify the "generic conditions" assumed to hold throughout this section. Most importantly, it explains why the polynomials in two variables $R_1(x, y)$ and $R_2(x, y)$ introduced before the proof of Theorem 2.2.16 below are "generically" distinct.

Now we have:

Lemma 2.2.15. *The coefficient d lies in a cubic extension of $\mathbb{Q}(c)$.*

Proof. Consider Equation (2.34) for $i = 1, 2$. If for some i, we have

$$2c + dcs_1 + 2cs_1 - m_i cr_3 = 0$$

then d lies in $\mathbb{Q}(c)$ since we have assumed $cs_1 \neq 0$. Thus we can assume that $2c + dcs_1 + 2cs_1 - m_i cr_3 \neq 0$, $i = 1, 2$. We can then express τ_i as

$$\tau_i = \frac{m_i D + m_i dcr_4 s_1 - a}{2c + dcs_1 + 2cs_1 - m_i cr_3} \, .$$

Analogous formulas can be obtained with n_i instead of m_i but, at this level, it does not lead to any additional information. Since $\tau_1 + \tau_2 = (dcs_1 + a)/(cs_1 + c)$, we conclude that

$$\frac{dcs_1 + a}{cs_1 + c} = \sum_{i=1}^{2} \frac{m_i D + m_i dcr_4 s_1 - a}{2c + dcs_1 + 2cs_1 - m_i cr_3} \, . \tag{2.35}$$

Now note that $m_i D + m_i dcr_4 s_1 - a$ still has the form $\mathrm{Pol}_6/[(cs_1 - 1)\mathrm{Pol}_3]$ where the degree with respect to d of Pol_6 (resp. Pol_3) is at most 2 (resp. 1), cf. Remark 2.2.13 items (a), (b), and (c). Now equation (2.35) shows that d is solution of a polynomial equation with degree at most 3 and coefficients in $\mathbb{Q}[c]$. The proof of the lemma is completed. $\square$

To close this section, we will prove that c itself lies in a number field (finite extension of $\mathbb{Q}$), under a very minor generic assumption (whose failure to be satisfied would again lead to further simpler equations involving c and d).

First note that in the proof of Lemma 2.2.15 we have constructed a polynomial in two variables $R_1(x, y)$ with coefficients in $\mathbb{Q}$ such that $R_1(c, d) = 0$ (besides the degree of R_1 with respect to y is at most 3). Next note that the construction carried out in proof in question can be repeated by using the equalities involving n_i in (2.34). This leads us to another polynomial $R_2(x, y)$ satisfying $R_2(c, d) = 0$. Let us then assume that R_1 is different from R_2, the fact that this assumption is "generic" is justified by the discussion in Remark 2.2.14.

Owing to Lemma 2.2.15, the transcendence degree of $\mathbb{Q}(c, d)$ over $\mathbb{Q}$ is at most 1. In fact, c, d are related by the polynomial equation $R_1(c, d) = 0$. Assume aiming at a contradiction that the transcendence degree of $\mathbb{Q}(c, d)$ is actually 1. By virtue of Lúroth classical theorem, $\mathbb{Q}(c, d)$ is an algebraic extension of a pure transcendent field $\mathbb{Q}(T)$. In other words, c and d are algebraic expressions in the indeterminate T. Substituting these expressions for c and d in the equation R_2, however, yields a non-trivial algebraic equation satisfied by T: indeed, R_2 is by assumption independent of R_1 (or in geometric terms, the Riemann surfaces defined by the equations $R_1(x, y) = 0$ and $R_2(x, y) = 0$ have no common irreducible component). Therefore T must itself be algebraic and a contradiction immediately arises. In this way, we have then proved that $\mathbb{Q}(c, d)$ is an algebraic extension of $\mathbb{Q}$. In particular, $\mathbb{Q}(c) = \mathbb{Q}[c]$.

The material in this section is then summarized by the following theorem:

Theorem 2.2.16. *Under the previous generic assumptions, the coefficients of the vector field Y in (2.29) satisfy the following conditions:*

1. *The coefficient c is an algebraic number.*
2. *The coefficient d lies in a finite extension of $\mathbb{Q}(c)$ of degree at most* 3.
3. *The coefficient b is the rational function over $\mathbb{Q}$ of c and d indicated in Lemma 2.2.11.*
4. *The coefficient a is the rational function over $\mathbb{Q}$ of c and d indicated in Remark 2.2.13.*
5. *All the remaining coefficients are rational functions over $\mathbb{Q}$ of c and d as also indicated in Remark 2.2.13.*

2.3 Ohyama and Darboux-Halphen systems

Throughout this section, the words “vector field” and “system of differential equations” are used as synonymous. Whereas we might simply stick with the terminology of vector fields, the phrase “system of differential equations” occurs more often in the classical literature making up the background of this section. For this reason, we decided to use both forms according to our convenience.

Among quadratic vector fields on $\mathbb{C}^3$ there is a particularly interesting 3-parameters family $H(\alpha_1, \alpha_2, \alpha_3)$ which is presented in [9] as

$$\begin{aligned} H(\alpha_1, \alpha_2, \alpha_3) &= [\alpha_1 x_0^2 + (1-\alpha_1)(x_0 y_0 + x_0 z_0 - y_0 z_0)]\frac{\partial}{\partial x_0} \\ &\quad + [\alpha_2 y_0^2 + (1-\alpha_2)(x_0 y_0 - x_0 z_0 + y_0 z_0)]\frac{\partial}{\partial y_0} \\ &\quad + [\alpha_3 z_0^2 + (1-\alpha_3)(-x_0 y_0 + x_0 z_0 + y_0 z_0)]\frac{\partial}{\partial z_0} \qquad (2.36) \end{aligned}$$

where the parameters α_i belong to $\mathbb{C}$. These vector fields were studied by Halphen who provided explicit solutions for them in terms of logarithmic derivatives of Jacobi Theta-functions, see [13], [14]. A much more recent geometric/dynamics study of these equations can be found in [9]. In fact, Halphen has first considered the vector field $H(0, 0, 0)$ in the above family, which in turn, had already been considered by Darboux. The same system was independently solved by Brioschi so that, in modern literature, this vector field is referred to as *Halphen equation*, *Darboux-Halphen System* or yet *Darboux-Brioschi-Halphen System*. It reads explicitly as

$$\begin{aligned} x' + y' &= 2xy, \\ y' + z' &= 2yz, \\ x' + z' &= 2xz \qquad (2.37) \end{aligned}$$

Solutions for the general vector field $H(\alpha_1, \alpha_2, \alpha_3)$ were then derived by Halphen in his second note [14] (see also for [9] for a general discussion). Darboux-Halphen vector field appears often in Physics but also in number theory: the system is very closely related to Ramanujan's system whose solutions are his famous "P, Q, and R (quasi-modular) functions.

Naturally, it is an interesting question to look for analogues of Darboux-Halphen system in higher dimensions. This problem was considered by Ohyama in [20], [21]. There the author applied Jacobi's method to find a non-linear dynamical system from a linear differential equation via modular forms, as a result Ohyama obtained the following vector field

$$\begin{aligned} w' + x' + y' &= wx + xy + yw, \\ w' + y' + z' &= wy + yz + zw, \\ w' + x' + z' &= wx + xz + zw, \\ x' + y' + z' &= xy + yz + xz \end{aligned} \tag{2.38}$$

Note that an alternative construction of the vector field (2.38) was given by Guillot in [10]. The purpose of this section is to give a rather direct interpretation of the vector field (2.38) in terms of the original Darboux-Halphen vector field (2.37). In particular, it will be shown how solutions for Ohyama vector field are explicitly given in terms of solutions of Darboux-Halphen vector field. High dimensional generalizations of Ohyama vector fields are also possible from the perspective of this section.

To begin with, note that the system (2.38) is equivalent to the vector field $D(x, y, z, w) = 1/3(P(x, y, z, w)\partial/\partial x + Q(x, y, z, w)\partial/\partial y + R(x, y, z, w)\partial/\partial z + S(x, y, z, w)\partial/\partial w)$ where

$$\begin{aligned} P(x, y, z, w) &= (2wx - wy + 2xy - wz + 2xz - yz), \\ Q(x, y, z, w) &= (-wx + 2wy + 2xy - wz - xz + 2yz), \\ R(x, y, z, w) &= (-wx - wy - xy + 2wz + 2xz + 2yz), \\ S(x, y, z, w) &= (2wx + 2wy - xy + 2wz - xz - yz) \end{aligned} \tag{2.39}$$

The starting point of our discussion is the Lie-theoretic interpretation of Halphen vector fields provided in [9]. Let Z denote the vector field

$$Z = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + \frac{\partial}{\partial w}\,.$$

The radial vector field $\mathfrak{E}$ of $\mathbb{C}^4$ is accordingly denoted by

$$\mathfrak{E} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}\,.$$

Then Ohyama's vector field in (2.39) satisfies the functional equation $[Z, D] = 2\mathfrak{E}$. As it happens in the general case of vector fields in Halphen family (2.36), the triple of vector fields consisting of D, Z, and $\mathfrak{E}$ generates a Lie algebra of vector fields isomorphic to the Lie algebra of $\mathrm{PSL}\,(2,\mathbb{C})$. Following [24], we will exploit this Lie algebra to construct a codimension 2 reduction of the Halphen vector fields. It will turn out that, in the case of Ohyama vector field D, the resulting codimension 2 system is completely integrable. The remainder of the section is devoted to this construction and its consequences.

As previously done, we first consider the blow up $\widetilde{\mathbb{C}}^4$ of $\mathbb{C}^4$ at the origin. The blow-up map π is given by $(x,t,u,v) \mapsto (x, xt, xu, xv) = (x,y,z,w)$ so that the exceptional divisor coincides with the plane $\{x = 0\}$ in the affine coordinates (x,t,u,v). Furthermore the pull-back of the vector field D by π takes on the form $\pi^*D(x,t,u,v) = -x/3(P_1\partial/\partial x + Q_1\partial/\partial t + R_1\partial/\partial u + S_1\partial/\partial v)$ with

$$\begin{aligned}
P_1 &= x(-2u - 2v - 2t + uv + tu + tv),\\
Q_1 &= (1-t)(u + v - 2t + uv + tu + tv),\\
R_1 &= (1-u)(t - 2u + v + uv + tu + tv),\\
S_1 &= (1-v)(t + u - 2v + uv + tu + tv)
\end{aligned} \tag{2.40}$$

Since D is quadratic, its blow-up π^*D induces a foliation on the exceptional divisor $E \cong \mathbb{CP}(3)$. Formula (2.40) shows that, in the affine coordinates (x,t,u,v), the foliation in question is associated with the vector field

$$D_1(t,u,v) = Q_1(t,u,v)\partial/\partial t + R_1(t,u,v)\partial/\partial u + S_1(t,u,v)\partial/\partial v\,.$$

It is straightforward to note that the point $(x,t,u,v) = (0,1,1,1)$ is a radial singularity for the foliation associated with the vector field $\pi^* D$ on $\widetilde{\mathbb{C}}^4$. In other words, at $(0,1,1,1)$ all the eigenvalues of this foliation coincide (and since these are defined only up to a multiplicative constant, they can be set equal to 1). At the same time, the point $(0,1,1,1) \cong (1,1,1)$ is the basis of the pencil $\mathcal{P}$ of projective lines in $E \cong \mathbb{CP}(3)$ induced by the vector field Z. Since $[Z,D]$ is parallel to $\mathfrak{E}$, the general observation in [24] hints that the foliation associated with D should project to a well defined foliation on the space of lines of $\mathcal{P}$, itself isomorphic to $\mathbb{CP}(2)$. To check that this is, indeed, the case and also to identify the corresponding (projected) foliation, it essentially suffices to blow-up the above mentioned radial singular point.

For notational convenience, we first perform the translation $(x,T,U,V) \mapsto (x,T+1,U+1,V+1) = (t,u,v)$ so as to bring to the origin the point to be blown-up. Then we perform a new blow-up at the origin of $\mathbb{C}^4$ in the new coordinates (x,T,U,V). The lift of the vector field (2.40) by the blow-up map $(x,T,Z,W) \mapsto (Tx,T,TZ,TW) = (x,T,U,V)$ has the form $XT^2[P_2\partial/\partial X + Q_2\partial/\partial T + R_2\partial/\partial Z + S_2\partial/\partial W)]$ with

$$
\begin{aligned}
P_2 &= -X(-W - Z - \frac{2}{3}TW - \frac{2}{3}TZ - \frac{2}{3}TWZ), \\
Q_2 &= 1 + TW + TZ + \frac{1}{3}T^2W + \frac{1}{3}T^2Z + \frac{1}{3}T^2WZ, \\
R_2 &= Z - Z^2, \\
S_2 &= W - W^2
\end{aligned}
\tag{2.41}
$$

Note that the coordinates $(Z,W) \cong (0,0,Z,W)$ define natural affine coordinates for $\mathbb{CP}(2)$ viewed as the space of lines in $\mathcal{P}$. Formula (2.41) then shows that, indeed, foliation associated with D projects to a well defined foliation on $\mathbb{CP}(2)$ identified with the space of lines in $\mathcal{P}$. Furthermore, in natural affine coordinates (Z,W) for $\mathbb{CP}(2)$, the projected foliation is given by the vector field

$$
D_2(Z,W) = (Z - Z^2)\partial/\partial Z + (W - W^2)\partial/\partial W\,.
$$

The vector field D_2 admits $I(Z,W) = (WZ - W)/(ZW - Z)$ as a global meromorphic first integral on $\mathbb{CP}(2)$. This means that the solutions (leaves) of D_2 are given by the algebraic equation

$$(WZ - W)/(ZW - Z) = k \tag{2.42}$$

where k stands for a constant in $\mathbb{C} \cup \{\infty\}$. For a fixed value of k, the corresponding solution of D_2 can be parameterized by $Z \mapsto (Z, kZ/(kZ - Z + 1)) = (Z, W)$. Except for $k = 1$, the image of the map $Z \mapsto (Z, kZ/(kZ - Z + 1))$ is contained in a degree 2 rational curve in $\mathbb{CP}(2)$. When $k = 1$, the image is contained in the projective line induced by $\{Z = W\}$. In any event, to each fixed $k \in \mathbb{C} \cup \{\infty\}$, there corresponds a vector field in $\mathbb{C}^3$ obtained by restricting the blow-up of D to the "cylinder" over the rational curve arising from Equation (2.42).

More precisely, let σ be defined by $\sigma(x, T, Z) = (x, T, Z, W(Z))$, where $W(Z) = kZ/(kZ - Z + 1)$. The image of Σ is a (embedded) open manifold of dimension 3 in $\mathbb{C}^4$ equipped with coordinates (x, T, Z, W) which is left invariant by the vector field $XT^2[P_2\partial/\partial X + Q_2\partial/\partial T + R_2\partial/\partial Z + S_2\partial/\partial W)]$ of (2.41). Thus the restriction of this vector field (the above indicated transform of D) to the image of σ can be pulled-back by σ itself to yield a vector field on $\mathbb{C}^3$ endowed with coordinates (x, T, Z). To compute this pull-back, note that the left-inverse $J_l(\sigma)^{-1}$ of the Jacobian matrix of σ has the following expression where $a_{14},\ a_{24},\ a_{34}$ are arbitrary complex numbers.

$$J_l(\sigma)^{-1} = \begin{bmatrix} 1 & 0 & -\frac{a_{14}k}{(1-Z+kZ)^2} & a_{14} \\ 0 & 1 & -\frac{a_{24}k}{(1-Z+kZ)^2} & a_{24} \\ 0 & 0 & 1 - \frac{a_{34}k}{(1-Z+kZ)^2} & a_{34} \end{bmatrix}$$

Therefore, the pull-back of the vector field (2.41) by σ has the form $D_\sigma(X, T, Z) = XT^2/(3 - 3Z + 3kZ)[P_\sigma\partial/\partial X + Q_\sigma\partial/\partial T + R_\sigma\partial/\partial Z)$ with

$$\begin{aligned} P_\sigma &= -XZ(3 + 3k + (3k - 3)Z + (2k + 2)T + (4k - 2)TZ), \\ Q_\sigma &= 3 + (3k - 3)Z + (3k + 3)TZ + (3k - 3)TZ^2 + (k + 1)T^2Z + (2k - 1)T^2Z^2, \\ R_\sigma &= Z(3 + (3k - 6)Z - (3k - 3)Z^2)\,. \end{aligned} \tag{2.43}$$

The reader will also note that R_σ is equal to $(3 - 3Z + 3kZ)(Z - Z^2)$

The above formula for the vector field D_σ will be used below to provide a direct connection between solutions of D and solutions of the original Darboux-Halphen system. For the time being, however, it is convenient to first take a closer look at the original Darboux-Halphen system.

Recall that the vector field associated with the Darboux-Halphen system (2.37) on $\mathbb{C}^3$ is given by

$$H = (xy + xz - yz)\partial/\partial x + (xy - xz + yz)\partial/\partial y + (-xy + xz + yz)\partial/\partial z\,.$$

Again we blow-up $\mathbb{C}^3$ at the origin. In coordinates (x, t, u) where the blow-up map is given by $(x, t, u) \mapsto (x, xt, xu) = (x, y, z)$, the blow-up of the vector field H becomes

$$H_1 = x[x(-tu + t + u)\partial/\partial x + (t - 1)(u - t + tu)\partial/\partial t + (u - 1)(t - u + tu)\partial/\partial u]\,.$$

The preceding discussion can then be repeated in the present context. In particular, the regular vector field $\partial/\partial x + \partial/\partial y + \partial/\partial z$ still induces a pencil of projective lines on the exceptional divisor - isomorphic to $\mathbb{CP}(2)$ - whose basis locus is the point $(0, 1, 1) \cong (0, 0)$. Incidentally, this point is a radial singularity for the foliation associated with H_1. We then proceed as before. Namely, we translate the radial singularity to the origin and then blow-it up. In other words, we consider the translation $t = T + 1$ and $u = U + 1$ and the subsequently blow-up map $\pi : (x, T, Z) \mapsto (Tx, T, TZ) = (x, t, u)$. The transform $H_2 = \pi^* H_1$ then becomes

$$H_2 = T^2 X(-XZ(2+2T)\partial/\partial X + (1+2TZ+T^2 Z)\partial/\partial T + (2Z-2Z^2)\partial/\partial Z) \quad (2.44)$$

Comparing the foliations associated to D_2 and to H_2, we see that they coincide in the direction $\partial/\partial Z$. Indeed, the actual vector fields D_2 and H_2 coincide, up to the constant factor 2, in the direction $\partial/\partial Z$. Moreover, the coordinate Z can explicitly be given as a uniform function of the complex time. The central point of the discussion is therefore to show find the solution for the variable T, in terms of the solution for Z. Indeed, once this has been done, the solution for the variable X

can immediately be derived: in the direction of $\partial/\partial X$ the vector fields in question provide (homogeneous) linear equations for X in terms of the (already known) functions T and Z.

Summarizing the preceding paragraph, we are led to compare the Riccati equations given respectively by the vector fields

$$\mathcal{R}_1 = (1 + 2TZ + T^2Z)\partial/\partial T + (2Z - 2Z^2)\partial/\partial Z \tag{2.45}$$

and

$$\begin{aligned}\mathcal{R}_2 \quad = \quad & \frac{3 + (3k-3)Z + (3k+3)TZ + (3k-3)TZ^2 + (k+1)T^2Z + (2k-1)T^2Z^2}{(3 + 3(k-1)Z)}\partial/\partial T \\ & +(Z - Z^2)\partial/\partial Z\,. \end{aligned}\tag{2.46}$$

The remainder of this section is devoted to proving that the above equations are bimeromorphically equivalent so as to establish an explicit correspondence between their solutions.

The proof of the last assertion is, however, a straightforward application of the birational theory of Riccati equations.

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