Topic in mean field games theory & applications in economics and quantitative finance

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Topics in Mean Field Games Theory & Applications in Economics and Quantitative Finance

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To my parents and to my wife Rim for her endless support

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Abstract

Mean Field Game (MFG) systems describe equilibrium configurations in differential games with infinitely many infinitesimal interacting agents. This thesis is articulated around three different contributions to the theory of Mean Field Games. The main purpose is to explore the power of this theory as a modeling tool in various fields, and to propose original approaches to deal with the underlying mathematical questions.

The first chapter presents the key concepts and ideas that we use throughout the thesis: we introduce the MFG problem, and we briefly explain the asymptotic link with N-Player differential games when $N \to \infty$. Next we present our main results and contributions, that are explained more in details in the subsequent chapters.

In Chapter 2, we explore a Mean Field Game model with myopic agents. In contrast to the classical MFG models, we consider less rational agents which do not anticipate the evolution of the environment, but only observe the current state of the system, undergo changes and take actions accordingly. We analyze the resulting system of coupled PDEs and provide a rigorous derivation of that system from N-Player stochastic differential games models. Next, we show that our population of agents can self-organize and converge exponentially fast to the well-known *ergodic MFG equilibrium*.

Chapters 3 and 4 deal with a MFG model in which producers compete to sell an exhaustible resource such as oil, coal, natural gas, or minerals. In Chapter 3, we propose an alternative approach based on a variational method to formulate the MFG problem, and we explore the deterministic limit (without fluctuations of demand) in a regime where resources are renewable or abundant. In Chapter 4 we address the rigorous link between the Cournot MFG model and the N-Player Cournot competition when N is large.

In Chapter 5, we introduce a MFG model for the optimal execution of a multi-asset portfolio. We start by formulating the MFG problem, then we compute the optimal execution strategy for a given investor knowing her/his initial inventory and we carry out several simulations. Next, we analyze the influence of the trading activity on the observed intraday pattern of the covariance matrix of returns and we apply our results in an empirical analysis on a pool of 176 US stocks.

Résumé

Les systèmes de jeux à champ moyen (MFG) décrivent des configurations d'équilibre dans des jeux différentiels avec un nombre infini d'agents infinitésimaux. Cette thèse s'articule autour de trois contributions différentes à la théorie des jeux à champ moyen. Le but principal est d'explorer des applications et des extensions de cette théorie, et de proposer de nouvelles approches et idées pour traiter les questions mathématiques sous-jacentes.

Le premier chapitre introduit en premier lieu les concepts et idées clés que nous utilisons tout au long de la thèse. Nous introduisons le problème MFG et nous expliquons brièvement le lien asymptotique avec les jeux différentiels à N-joueurs lorsque $N \to \infty$. Nous présentons ensuite nos principaux résultats et contributions.

Le Chapitre 2 explore un modèle MFG avec un mode d'interaction non anticipatif (joueurs myopes). Contrairement aux modèles MFG classiques, nous considérons des agents moins rationnels qui n'anticipent pas l'évolution de l'environnement, mais observent uniquement l'état actuel du système, subissent les changements et prennent des mesures en conséquence. Nous analysons le système couplé d'EDP résultant de ce modèle, et nous établissons le lien rigoureux avec le jeu correspondant à N-Joueurs. Nous montrons que la population d'agents peut s'auto-organiser par un processus d'autocorrection et converger exponentiellement vite vers une configuration d'équilibre MFG bien connue.

Les Chapitres 3 et 4 concernent l'application de la théorie MFG pour la modélisation des processus de production et commercialisation de produits avec ressources épuisables (e.g. énergies fossiles). Dans le le Chapitre 3, nous proposons une approche variationnelle pour l'étude du système MFG correspondant et analysons la limite déterministe (sans fluctuations de la demande) dans un régime où les ressources sont renouvelables ou abondantes. Nous traitons dans le Chapitre 4 l'approximation MFG en analysant le lien asymptotique entre le modèle de Cournot à N-joueurs et le modèle de Cournot MFG lorsque N est grand.

Enfin, le Chapitre 5 considère un modèle MFG pour l'exécution optimale d'un portefeuille d'actifs dans un marché financier. Nous explicitons notre modèle MFG et analysons le système d'EDP résultant, puis nous proposons une méthode numérique pour calculer la stratégie d'execution optimale pour un agent étant donné son inventaire initial et présentons plusieurs simulations. Par ailleurs, nous analysons l'influence de l'activité de trading sur la variation Intraday de la matrice de covariance des rendements des actifs. Ensuite, nous vérifions nos conclusions et calibrons notre modèle en utilisant des données historiques des transactions pour un pool de 176 actions américaines.

Contents

CHAPTER 1

General Introduction

The theory of Mean Field Games is a young branch of Dynamic Games that aims at modeling and analyzing complex decision processes involving a large number of agents, which have individually a small influence on the overall system, and are influenced by the behavior of other agents. Examples of such systems might be financial exchanges, social media, or large flows of pedestrians. The theory was introduced about ten years ago in series of seminal papers, by Lasry and Lions [86–88], Caines et al. [24, 25], and in lectures by Pierre-Louis Lions at the Collège de France, which were video-taped and made available on the internet [91]. Since its inception, the Mean Field Games (MFGs for short) theory has expanded tremendously, and has become an important tool in the study of dynamical and equilibrium behavior of large systems.

Mean Field Games describe the evolution of a stochastic differential game with a continuum of indistinguishable players, and where the choice of any "atomic" player is affected by other players through a global mean field effect. The "mean field" terminology is borrowed from physics and refers to the fact that the influence of all other players is aggregated in a single averaged effect. In terms of partial differential equations, a MFG model is typically described by a system of a transport or Fokker-Plank equation for the distribution of the agents, coupled with a Hamilton-Jacobi-Bellman (HJB) equation governing the game value function of an "atomic" player.

The starting point of Mean Field Games models is stochastic differential games with N players and symmetric interactions. Those models arise in many applications such as in engineering, economics, social science, finance, and management science. However, it is well known that the computation of equilibria in these games is typically a very challenging task either analytically or computationally, even for one period deterministic games. The rationale of MFGs is to search for simplifications by considering the asymptotic behavior in the limit $N \to \infty$ of large population, so that the information on the system is captured only through the statistical distribution – or density – of players in the space of possible states. This strategy allows to reduce the initial N-body problem into a one-body problem which simplifies the modeling and considerably reduces the cost of computations. Thus, Mean Field Games models can be seen as an alternative approximation to N-Player games at the asymptotic regime $N \to \infty$, which allows to obtain some insight into the behavior of the system at a relatively low cost. This partially explains the considerable interest aroused by the theory for many applications.

This thesis deals with three different applications of the theory of Mean Field Games, where the use of the MFG framework has revealed interesting facts related to the strategic behavior of a large number of "rational" agents. Each of these models raises many mathematical challenges which are deeply investigated in this manuscript. Several questions have been treated in this work, while others remain open.

The main purpose of this short introductory chapter, is to introduce the reader to the key concepts and ideas of Mean Field Games Theory, and to explain our main results and contributions. We start by formalizing the Mean Field Game problem and extending it to a general framework where the individual players also interact through their controls. These models have been referred to as extended Mean Field Games[1] in the literature, and will be used in different parts of this work. Next, we explain, rigorously and in a very simple framework, the link between mean field games and the corresponding N-Player games. Namely, we show that MFG models allow to build approximate Nash equilibria for N-Player stochastic differential games. Finally, we conclude this introductory chapter by presenting and explaining our main results and contributions.

1. The Mean Field Game Problem

Set $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ to be a complete filtered probability space supporting a d-dimensional Wiener process $(W_t)_{t\geqslant 0}$ with respect to $\mathbb{F}$, the filtration $\mathbb{F}$ satisfying the usual conditions. Consider a stochastic differential game with a *continuum* of indistinguishable players, where any representative – "atomic" – player is controlling its private state X_t in $\mathbb{R}^d$ at time $t \in [0,T]$, by taking an action α_t in a closed convex subset $A \subset \mathbb{R}^d$. We assume that the dynamics of the state of players is driven by Itô's stochastic differential equations of the form:

$$dX_t = \alpha_t \, dt + \sqrt{2\sigma}\, dW_t, \quad t \in [0,T],$$

where throughout this part σ is a fixed positive constant. For any individual player, the choice of a strategy $(\alpha_t)_{0\leqslant t\leqslant T}$ is driven by the desire to minimize an expected cost over the period $[0,T]$, which is influenced by the state of other players. Therefore, any individual agent needs to anticipate the state of other players over the time period $[0,T]$ in order to set an effective action. The *Mean Field Game equilibrium* is reached when player's anticipation matches reality.

1.1. The Optimal Control Problem. Given an initial distribution of players m_0 in $\mathcal{P}(\mathbb{R}^d)$, the Mean Field Game problem is articulated in the following way:

(1.i) *Anticipating and optimizing:* for each anticipated flow of deterministic measures $\mathbf{m} = (m(t))_{0\leqslant t\leqslant T}$ on $\mathbb{R}^d$, solve the standard stochastic optimal control problem:

$$\inf_{\alpha \in \mathbb{A}} \mathcal{J}^{\mathbf{m}}(\alpha) \ \text{ with } \ \mathcal{J}^{\mathbf{m}}(\alpha) := \mathbb{E}\left[\int_0^T \mathsf{L}(X_s^{\mathbf{m}}, \alpha_s) + \mathsf{F}(X_s^{\mathbf{m}}; m(s))\, ds + \mathsf{G}(X_T^{\mathbf{m}}; m(T))\right], \tag{1.1}$$

subject to

$$dX_t^{\mathbf{m}} = \alpha_t \, dt + \sqrt{2\sigma}\, dW_t, \quad X_0^{\mathbf{m}} \sim m_0. \tag{1.2}$$

(1.ii) *Equilibrium:* find a flow $\mathbf{m} = (m(t))_{0\leqslant t\leqslant T}$ such that $\mathcal{L}(\hat{X}_t^{\mathbf{m}}) = m(t)$ for every $t \in [0,T]$, where $\hat{X}^{\mathbf{m}}$ is a solution to the above optimal control problem.

[1]The name "Mean Field Games of Controls" is also widely used in the literature.

In all this chapter, $\mathbb{A}$ denotes the set of admissible controls; that is the set of $\mathbb{F}$-progressively measurable A-valued stochastic processes, satisfying the admissibility condition:

$$\mathbb{E}\left[\int_0^T |\alpha_s|^2 \, ds\right] < \infty.$$

Given the structure of the cost functional in (1.1), the influence of the system on one's actions is captured through the coupling functions F and G. We shall assume that the dependence of the coupling functions on the measure variable is nonlocal, i.e. for any x in $\mathbb{R}^d$, we see $m \to F(x; m)$, $m \to G(x; m)$ as a regularizing maps on the set of probability measures. The first step in (1.i)-(1.ii) provides the best response of a generic player given the anticipated statistical distribution of the states of the other players. The second step solves a specific fixed point problem corresponding to equilibrium: anticipations of the agents turns out to be correct. Therefore, the MFG equilibrium can be interpreted as a Nash equilibrium configuration for a continuum of indistinguishable agents with mean field interactions. Notice that we make the strong assumption that the agents share a common belief on the future behavior of the density of agents.

1.2. The Analytic Approach. The MFG problem can also be formulated by using an analytical approach. In fact, the first step in (1.i)-(1.ii) consists in solving a standard stochastic optimal control problem by fixing a flow of measures $\mathbf{m} = (m(t))_{0 \leqslant t \leqslant T}$. A natural route is to express the value function $(t, x) \to u(t, x)$ of the optimization problem (1.1) as the solution of the corresponding Hamilton-Jacobi-Bellman equation. Moreover, the matching problem (1.ii) is resolved by coupling the HJB equation with a Kolmogorov equation, which is intended to identify the flow $\mathbf{m} = (m(t))_{0 \leqslant t \leqslant T}$ with the flow of marginal distributions of the optimal states. Thus, the resulting coupled system of PDEs can be written as:

$$\begin{cases} -\partial_t u - \sigma \Delta u + H(x, Du) = F(x; m) & \text{in } (0, T) \times \mathbb{R}^d \\ \\ \partial_t m - \sigma \Delta m - \operatorname{div}(m H_p(x, Du)) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\ \\ m(0) = m_0, \quad u(T, x) = G(x; m(T)) & \text{in } \mathbb{R}^d \end{cases} \tag{1.3}$$

where $H(x, p) := \sup_{\nu \in A} \{-p.\nu - L(x, \nu)\}$, and H_p denotes the partial derivative of H with respect to the second variable. The first equation of (1.3) is the HJB equation of the stochastic control problem (1.i) when the flow $\mathbf{m} = (m(t))_{0 \leqslant t \leqslant T}$ is fixed. The second equation is the Kolmogorov – or Fokker-Planck – equation giving the time evolution of the flow $\mathbf{m} = (m(t))_{0 \leqslant t \leqslant T}$ dictated by the dynamics (1.2) of the state of the system, once we have implemented the optimal feedback function. Notice that the first equation is a backward equation to be solved from a terminal condition, while the second equation is forward in time, starting from an initial condition.

Following the seminal works [**86**, **88**], the MFG system (1.3) is usually addressed in a periodic framework; that is, functions L, F, G are assumed to be $\mathbb{Z}^d$-periodic with respect to the variable 'x', and system (1.3) is complemented with periodic boundary conditions. In this context, we consider the d-dimensional torus $\mathbb{T}^d$ as the set of possible states, i.e.

$Q = \mathbb{T}^d$. The analysis of system (1.3) in the periodic setting, is addressed in many works (cf. [33,86,88] among many others). The existence of classical solutions is obtained under a wide range of sufficient conditions on H, F and G, and uniqueness follows by assuming uniform convexity of the Hamiltonian with respect to the second variable, and the so called *Lasry-Lions monotonicity conditions*:

$$\int_Q \big(F(x;m) - F(x;m')\big)\, d(m-m')(x) \geqslant 0, \qquad \forall m, m' \in \mathcal{P}(Q); \tag{1.4a}$$

$$\int_Q \big(G(x;m) - G(x;m')\big)\, d(m-m')(x) \geqslant 0, \qquad \forall m, m' \in \mathcal{P}(Q). \tag{1.4b}$$

The interpretation of the above monotonicity conditions is that the players dislike congested regions and prefer configurations in which they are scattered. We refer the reader to [43, Vol I, Section 3.4] for a detailed presentation of the notion of monotonicity and several examples.

1.3. Stationary MFGs. Other classes of MFG problems have been studied in the literature, which corresponds to different cost structures. Among the most classical ones is the case of a long time averaged cost; namely:

$$\mathcal{J}^{\mathbf{m}}(\alpha) := \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T L(X_s^{\mathbf{m}}, \alpha_s) + F(X_s^{\mathbf{m}}; m(s))\, ds\right]. \tag{1.5}$$

In this case, the MFG system of partial differential equations is stationary, and takes the following form [12,59,86,87],

$$\begin{cases} -\sigma\Delta\bar{u} + H(x, D\bar{u}) + \bar{\lambda} = F(x;\bar{m}) \quad \text{in } Q = \mathbb{T}^d, \\ \\ -\sigma\Delta\bar{m} - \operatorname{div}(\bar{m}H_p(x, D\bar{u})) = 0 \quad \text{in } Q, \\ \\ \bar{m} \geqslant 0, \quad \int_Q \bar{m} = 1, \quad \int_Q \bar{u} = 0. \end{cases} \tag{1.6}$$

Here we consider a periodic setting, the unknowns are $(\bar{\lambda}, \bar{u}, \bar{m})$, where $\bar{\lambda} \in \mathbb{R}$ is the so-called ergodic constant, and the condition $\int_Q \bar{u} = 0$ is in force in order to ensure uniqueness. The solution to the first equation in (1.6) can be interpreted as the equilibrium value function of a "small" player whose cost depends on the density $\bar{m}$ of the other players, while the second equation characterizes the distribution of players at the equilibrium. It is well known (see e.g. [12,86,87]) that there exists a solution $(\bar{\lambda}, \bar{u}, \bar{m})$ in $\mathbb{R} \times \mathcal{C}^2(Q) \times W_s^1(Q)$ for all $1 \leqslant s < \infty$ to (1.6), under a wide range of sufficient conditions on H, F and initial data. Moreover, uniqueness holds under the monotonicity condition (1.4a) on F, by assuming that H is uniformly convex with respect to the second variable.

Another well known example of stationary MFG systems, is related to the case where players aim to minimize a discounted infinite-horizon cost functional, namely:

$$\mathcal{J}^{\mathbf{m}}(\alpha) := \mathbb{E}\left[\int_0^\infty e^{-\rho s}\{L(X_s^{\mathbf{m}}, \alpha_s) + F(X_s^{\mathbf{m}}; m(s))\}\, ds\right], \tag{1.7}$$

where $\rho > 0$. In the periodic setting, one can obtain a stationary solution to the MFG problem by solving (see e.g. [12], among others):

$$\text{(1.8)} \qquad \begin{cases} -\sigma\Delta\bar{v} + H(x, D\bar{v}) + \rho\bar{v} = F(x; \bar{\mu}) \quad \text{in } Q = \mathbb{T}^d, \\ \\ -\sigma\Delta\bar{\mu} - \operatorname{div}(\bar{\mu}H_p(x, D\bar{v})) = 0 \quad \text{in } Q, \\ \\ \bar{\mu} \geqslant 0, \quad \displaystyle\int_Q \bar{\mu} = 1. \end{cases}$$

It is also well known (cf. [6, 10, 12]) that, under several technical conditions on H and F, there exists a solution $(\bar{v}, \bar{\mu}) \in \mathcal{C}^2(Q) \times W^1_s(Q)$ for all $1 \leqslant s < \infty$ to (1.8). Moreover, if H has a linear growth, i.e.

$$|H(x, p)| \leqslant C(1 + |p|)$$

for some constant $C > 0$, system (1.6) is obtained as a limit of system (1.8) when $\rho \to 0$. Namely, it holds that

$$\text{(1.9)} \qquad \left(\rho\int_Q \bar{v},\ \bar{v} - \int_Q \bar{v},\ \bar{\mu}\right) \longrightarrow (\bar{\lambda}, \bar{u}, \bar{m}) \qquad \text{in } \mathbb{R} \times \mathcal{C}^2(Q) \times L^\infty(Q) \quad \text{as } \rho \to 0.$$

Both systems (1.6) and (1.8) describe a stationary MFG equilibrium.

2. Extended Mean Field Games: Interaction Through the Controls

In MFGs presented so far, the players interact through their distribution in the space of possible states. We extend that framework to the case where players are not only influenced by the state of competitors, but also by their chosen controls. In this part, we present an extended Mean Field Game problem by using a more general framework. Two specific examples will be addressed in Parts II and III.

2.1. The Optimal Control Problem. Given an initial distribution of players' states m_0 in $\mathcal{P}(\mathbb{R}^d)$, the extended Mean Field Game problem is articulated in the following way:

(2.i) *Anticipating and optimizing:* for each anticipated flow of deterministic measures $\boldsymbol{\mu} = (\mu(t))_{0\leqslant t\leqslant T}$ on $\mathbb{R}^d \times A$, solve the standard stochastic optimal control problem:

$$\inf_{\alpha\in\mathbb{A}} \mathcal{J}^{\boldsymbol{\mu}}(\alpha) \ \text{ with } \mathcal{J}^{\boldsymbol{\mu}}(\alpha) := \mathbb{E}\left[\int_0^T \mathfrak{L}(s, X^{\boldsymbol{\mu}}_s, \alpha_s; \mu(s))\, ds + G(X^{\boldsymbol{\mu}}_T; m(T))\right],$$

subject to

$$dX^{\boldsymbol{\mu}}_t = \alpha_t\, dt + \sqrt{2\sigma}\, dW_t, \quad X^{\boldsymbol{\mu}}_0 \sim m_0,$$

where $m(t)$ denotes the first marginal of $\mu(t)$ on $\mathbb{R}^d$, for all $t \in [0, T]$.

(2.ii) *Equilibrium:* find a flow $\boldsymbol{\mu} = (\mu(t))_{0\leqslant t\leqslant T}$ such that $\mathcal{L}(\hat{X}^{\boldsymbol{\mu}}_t, \hat{\alpha}^{\boldsymbol{\mu}}_t) = \boldsymbol{\mu}(t)$ for every $t \in [0, T]$, if $\hat{\alpha}^{\boldsymbol{\mu}} \in \mathbb{A}$ is a minimizer of $\mathcal{J}^{\boldsymbol{\mu}}$ with $\hat{X}^{\boldsymbol{\mu}}$ as optimal path.

Note that G plays the same role as in Section 1: the terminal condition depends only on the terminal state of a typical agent and the terminal distribution of the states. As in Section 1, we shall assume that the dependence of the coupling functions $\mathfrak{L}$, G on the measure variable is nonlocal.

2.2. The Analytic Approach. Following Section 1.2, we denote by $(t,x) \to u(t,x)$ the value function of the optimization problem (2.i). When the flow $\boldsymbol{\mu} = (\mu(t))_{0\leqslant t\leqslant T}$ is fixed, u is a viscosity solution to the following HJB equation:

$$-\partial_t u(t,x) - \sigma\Delta u(t,x) + \mathfrak{H}(t,x,Du(t,x);\mu(t)) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \quad u(x,T) = G(x;m(T)),$$

where

$$\mathfrak{H}(t,x,p;\mu(t)) = \sup_{\nu\in A}\{-p.\nu - \mathfrak{L}(t,x,\nu;\mu(t))\}.$$

Moreover, $\hat{\alpha}^{\boldsymbol{\mu}}(t,x) = -D_p\mathfrak{H}(t,x,Du(t,x);\mu(t))$ is – at least formally – the optimal drift for the agent at position 'x' and at time 't'. Thus, the population density $(m(t))_{0\leqslant t\leqslant T}$ is expected to evolve according to the Kolmogorov equation:

$$\partial_t m - \sigma\Delta m - \operatorname{div}(mD_p\mathfrak{H}(t,x,Du(t,x);\mu(t))) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d, \quad m(0) = m_0.$$

Hence, for any $t \in [0,T]$, the law of $\mathcal{L}(\hat{X}^{\boldsymbol{\mu}}_t, \hat{\alpha}^{\boldsymbol{\mu}}_t)$ appears as the pushed forward image of the law of $\hat{X}^{\boldsymbol{\mu}}_t$, i.e.

$$\mathcal{L}(\hat{X}^{\boldsymbol{\mu}}_t, \hat{\alpha}^{\boldsymbol{\mu}}_t) = (\mathbb{I}_d, \hat{\alpha}^{\boldsymbol{\mu}}(t,.))\sharp\mathcal{L}(\hat{X}^{\boldsymbol{\mu}}_t).$$

Consequently, the equilibrium condition reads:

$$\mu(t) = (\mathbb{I}_d, \hat{\alpha}^{\boldsymbol{\mu}}(t,.))\sharp m(t). \tag{1.10}$$

To summarize, the Extended MFG problem takes the following analytic form:

$$\begin{cases} -\partial_t u - \sigma\Delta u + \mathfrak{H}(t,x,Du;\mu) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d \\ \\ \partial_t m - \sigma\Delta m - \operatorname{div}(mD_p\mathfrak{H}(t,x,Du;\mu)) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d \\ \\ m(0) = m_0, \quad u(T,x) = G(x;m(T)) \quad \text{in } \mathbb{R}^d \\ \\ \mu(t) = (I_d, \hat{\alpha}(t,.))\sharp m(t) \quad \text{in } (0,T) \end{cases} \tag{1.11}$$

Apart from the particular structure of the coupling, the new feature in comparison to the standard MFG of Section 1 is the relationship in (1.10), which provides in equilibrium an implicit expression for the flow $\boldsymbol{\mu} = (\mu(t))_{0\leqslant t\leqslant T}$ of the state and the control joint distributions, in terms of the flow $\mathbf{m} = (m(t))_{0\leqslant t\leqslant T}$ of the marginal distributions of the state. We will discuss later several specific examples of extended MFGs from an analytical standpoint (c.f. Parts II and III).

Extended Mean Field Games were initiated by Gomes and Voskanyan in [**67**] and addressed in various works (cf. [**19**, **39**, **65**, **66**, **68**] among many others). We refer to [**39**] for a complete explanation and a detailed analysis of the PDE system (1.11). In that paper, the authors prove well-posedness for system (1.11) under a wide range of sufficient conditions on $\mathfrak{H}$, G and initial data.

3. Application to Games with Finitely many Players

As we pointed out earlier, Mean Field Games can be seen as an approximation of N-Player games when N is large. In particular, a MFG equilibrium is expected to be an approximation of a Nash equilibrium configuration in the corresponding N-Player game as $N \to \infty$. In order to illustrate this feature, we provide a proof to this fact in a very simple framework: standard MFGs, regular functions and periodic boundary conditions. This result was first noticed by Caines et al. [**24**, **25**] and further developed in several other works (see e.g. [**42**, **81**] among many others). We will later on show this result in a more challenging framework (cf. Chapter 4): extended MFGs, with less regular functions and absorbing boundary conditions. We aim to explain how the optimal feedback strategies which are computed from the MFG system (1.3), provide an approximate Nash equilibrium to the corresponding N-Player game. The precise sense of approximate Nash equilibria will be specified later.

For simplicity, we consider a periodic setting ($Q = \mathbb{T}^d$) and we suppose that H is smooth, globally Lipschitz continuous and satisfies the coercivity condition:

$$C^{-1}\frac{I_d}{1+|p|} \leqslant D^2_{pp}H(x,p) \leqslant CI_d, \quad \forall (x,p) \in Q \times \mathbb{R}^d.$$

In addition, we suppose that F, G, are continuous on $Q \times \mathcal{P}(Q)$, fulfil the conditions (1.4a)-(1.4b), and $F(.;m)$, $G(.;m)$ are bounded respectively in $\mathcal{C}^{1+\alpha}$, $\mathcal{C}^{2+\alpha}$, uniformly with respect to $m \in \mathcal{P}(Q)$, for some $\alpha \in (0,1)$. Under the above conditions, it is well-known (cf. [**33**, **86**, **88**]) that the MFG system (1.3) has a unique solution such that $u \in \mathcal{C}^{1,2}(Q_T)$ and $m \in \mathcal{C}([0,T];\mathcal{P}(Q))$, where $Q_T := (0,T) \times Q$. Throughout the rest of this chapter, (u,m) denotes the unique solution to the MFG system (1.3).

At first, we claim that the feedback strategy $\hat{\alpha}(t,x) := -D_pH(x, Du(t,x))$ is optimal for the optimal stochastic control problem (1.1).

LEMMA 1.1. *Let $(\hat{X}_t)_{0\leqslant t\leqslant T}$ be the solution to the stochastic differential equation*

$$d\hat{X}_t = \hat{\alpha}(t,\hat{X}_t)\,dt + \sqrt{2\sigma}\,dB_t, \quad \hat{X}_0 \sim m_0.$$

Define $\hat{\alpha}_t := \hat{\alpha}(t,\hat{X}_t)$, $m(t) := \mathcal{L}(\hat{X}_t)$, and $\mathbf{m} = (m(t))_{0\leqslant t\leqslant T}$. Then, it holds that

$$\inf_{\alpha\in\mathbb{A}} \mathcal{J}^{\mathbf{m}}(\alpha) = \mathcal{J}^{\mathbf{m}}(\hat{\alpha}) = \int_Q u(0,x)\,dm_0(x).$$

PROOF. This is a verification Theorem whose the proof is standard. One only needs to check that the candidate solution is indeed optimal by using Itô's rule and the equation satisfied by the value function u. We refer the reader to Lemma 4.18 for a similar approach. □

Let us now address the N-Player version of the mean field game problem (1.3). Consider a system of N agents, where any agent i chooses a strategy α^i in $\mathbb{A}$ in order to control her/his private state. The private state of any player i is driven by the following stochastic differential equations (SDE):

$$dX^i_t = \alpha^i_t\,dt + \sqrt{2\sigma}\,dB^i_t, \quad X^i_0 \sim m_0,$$

where $(B^i)_{1\leqslant i\leqslant N}$ is a family N independent[2] $\mathbb{F}$-Wiener processes. We suppose that the initial condition satisfies the usual assumptions so that the resulting system of SDEs is well-posed in the classical sense. The expected total cost to player i is:

$$\mathcal{J}_i^N(\alpha_1, ..., \alpha_N) = \mathbb{E}\left[\int_0^T L(X_s^i, \alpha_s^i) + F\left(X_s^i; \hat{\nu}_{\mathbf{X}_s}^{i,N}\right) ds + G\left(X_T^i; \hat{\nu}_{\mathbf{X}_T}^{i,N}\right)\right],$$

where

$$\hat{\nu}_{\mathbf{X}_t}^{i,N} := \frac{1}{N-1}\sum_{j\neq i}\delta_{X_t^j},$$

and δ is the Dirac mass. The influence of other players on the i^{th} player's actions, is captured through the coupling functions F and G. Observe that the players are indistinguishable and the game is symmetric with respect to other players influence. Moreover, when N is large, the influence of any individual player becomes negligible.

Now let us fix $\hat{\alpha}_t^i := \hat{\alpha}(t, \hat{X}_t^i)$ where

$$d\hat{X}_t^i = \hat{\alpha}(t, \hat{X}_t^i)\, dt + \sqrt{2\sigma}\, dB_t^i, \quad \hat{X}_0^i \sim m_0.$$

This corresponds to the situation where all the players implement the MFG feedback control strategy $\hat{\alpha}$. The following result states that in the above configuration the N-Player system is "almost" in Nash equilibrium, and quantifies the error when the coupling functions F and G enjoy more regularity.

PROPOSITION 1.2 (ϵ-Nash equilibrium). *Suppose that F and G are Lipschitz continuous on $Q \times \mathcal{P}(Q)$. Then the symmetric strategy $(\hat{\alpha}^1, ..., \hat{\alpha}^N)$ is an ϵ-Nash equilibrium in the game $\mathcal{J}_1^N, ..., \mathcal{J}_N^N$; namely, there exists $C > 0$ such that*

$$\mathcal{J}_i^N(\hat{\alpha}^1, ..., \hat{\alpha}^N) \leqslant \mathcal{J}_i^N((\hat{\alpha}_j)_{j\neq i}, \alpha^i) + CN^{-1/(d+4)} \tag{1.12}$$

for any $i \in \{1, ..., N\}$ and $\alpha^i \in \mathbb{A}$.

PROOF. The problem being symmetrical, it is enough to show that

$$\mathcal{J}_1^N(\hat{\alpha}^1, ..., \hat{\alpha}^N) \leqslant \mathcal{J}_1^N((\hat{\alpha}^j)_{j\neq 1}, \alpha) + CN^{-1/(d+4)} \tag{1.13}$$

for any $\alpha \in \mathbb{A}$.

Note that for any $t \in [0, T]$, $(\hat{X}_t^1, ..., \hat{X}_t^N)$ are independent and identically distributed with law $m(t)$. Thus, we can use the following estimate on product measures due to Horowitz and Karandikar (see e.g. [103, Theorem 10.2.1]):

$$\mathbb{E}\left[\mathbf{d}_1(\nu_{\mathbf{X}_t}^{i,N}, m(t))\right] \leqslant C_d N^{-1/(d+4)},$$

where $\mathbf{d}_1(.,.)$ is the Kantorowich-Rubinstein distance[3]. Hence, by using Lipschitz continuity of F and G with respect to the measure variable we obtain:

$$\mathbb{E}\left[\int_0^T \sup_{x\in Q}\left|F(x, \nu_{\mathbf{X}_s}^{i,N}) - F(x, m(s))\right| ds\right] + \mathbb{E}\left[\sup_{x\in Q}\left|G(x, \nu_{\mathbf{X}_T}^{i,N}) - G(x, m(T))\right|\right] \leqslant C(d, T)N^{-1/(d+4)}. \tag{1.14}$$

[2]When the noise processes are correlated the analysis is much more challenging, cf. [33, 43].

[3]We refer the reader to Notation Section for a precise definition.

Now, let us consider an admissible control $(\alpha_t)_{0\leqslant t\leqslant T}$, and define $(X_t^1)_{0\leqslant t\leqslant T}$ by:

$$dX_t^1 = \alpha_t\, dt + \sqrt{2\sigma}\, dB_t^1, \quad X_0^1 \sim m_0.$$

By virtue of (1.14), we have

$$\begin{aligned}
\mathcal{J}_1^N((\hat{\alpha}^j)_{j\neq 1}, \alpha^1) &= \mathbb{E}\left[\int_0^T L\left(X_s^1, \alpha_s^1\right) + F\left(X_s^1, \nu_{\mathbf{X}_t}^{1,N}\right) ds + G\left(X_T^1, \nu_{\mathbf{X}_T}^{1,N}\right)\right] \\
&\geqslant \mathbb{E}\left[\int_0^T L(X_s^1, \alpha_s^1) + F(X_s^1, m(s))\, ds + G(X_T^1, m(T))\right] - CN^{-1/(d+4)} \\
&\geqslant \mathcal{J}_1^N(\hat{\alpha}^1, ..., \hat{\alpha}^N) - CN^{-1/(d+4)}
\end{aligned}$$

where the last inequality follows from the optimality of $\hat{\alpha}$ (c.f. Lemma 1.1). The proof is complete. □

We refer the reader to [15,43,73,86] and references therein, for further background on Mean Field Game theory and applications.

4. Outline of the Thesis

This thesis is articulated around three different contributions to the theory of Mean Field Games. The main purpose of this thesis is to explore the power of this theory as a modeling tool in various fields, and to propose new approaches and answers to deal with the underlying mathematical issues and questions.

In the first part of this dissertation, we introduce a Mean Field Game model with a new kind of interaction between the agents. The main idea is to drop the assumption of perfect anticipation, and to introduce a more interactive mechanism of decision making. In fact, in the classical MFG model of Section 1, agents are supposed to be "very rational", anticipating the exact evolution of the system on the whole time window $[0, T]$ and acting accordingly. In addition, they are assumed to share the same belief on the system's evolution. In contrast to that model, we consider less rational agents which do not anticipate the evolution of the environment (*myopic agents*), but only observe the current state of the system, undergo changes, and take actions accordingly. The actions are chosen in order to obtain the best future cost given the current situation of the system. We prove that such an interactive process can give rise to a fast self-organizing process toward a stable equilibrium configuration.

Chapter 2 is organized as follows: We start by formalizing the MFG problem with myopic players. We consider a specific cost structure, and provide a wide range of sufficient conditions that ensures the existence and uniqueness of classical solutions to the corresponding PDEs system. Next, we explain in a rigorous way the link between our MFG model and the corresponding N-Player game model in the limit of large games $N \to \infty$, by using a coupling argument. Moreover, we prove that the population of non-anticipating agents self-organizes and converges toward a stationary MFG equilibrium, when the initial distribution of the players is sufficiently close to the equilibrium. This result is proved for a coupling function F satisfying the monotonicity condition (1.4a), and a quadratic Hamiltonian function. Finally, we provide several numerical experiments which show in particular that the self-organizing process operates in more general cases. This raises many open questions for future research.

The second part of this dissertation deals with a specific economic model which belongs to the class of extended Mean Field Games. This model describes a mean field game in which producers compete to sell an exhaustible resource such as oil, coal, natural gas, or minerals. It models the dynamics of a continuum – or a density – of firms, producing comparable goods, strategically setting their production rate in order to maximise profit, and leaving the market as soon as they deplete their capacities. These models has been widely addressed in the Mathematical Economics literature recently (cf. [**49**,**50**,**73**,**92**]). In particular, the authors of [**50**] use the MFG framework to discuss the sharp oil prices drop in 2014. This class of models is known as *"Bertrand & Cournot Mean Field Games"* in the PDE MFG literature. From a mathematical standpoint, the Bertrand & Cournot MFG system consists in a system of a backward Hamilton-Jacobi-Bellman (HJB) equation to model a representative firm's value function, coupled with a forward Fokker-Planck equation to model the evolution of the distribution of the active firms' states. The exhaustibility condition gives rise to absorbing boundary conditions at $x = 0$.

The corresponding N-Player version of Bertrand and Cournot games is a classic of the Economics literature (see e.g. [**77**] and references therein), and it is well-known that the "Dynamic Games" approach produces a N-body problem which is extremely difficult to solve either analytically or numerically, especially when N is large. In this specific case with exhaustible resources, the situation is even worse because of the nonstandard boundary conditions which are obtained (cf. [**77**, Section 3.1]). The MFG approach has proven to be a better alternative as a modeling tool and various efficient numerical methods have been proposed in several works, in order to compute "approximate" market equilibria (cf. [**49**, **50**, **92**]). Nevertheless, many challenging mathematical questions remain open, especially for the rigorous link between Bertrand and Cournot MFGs and the corresponding N-Player games. The main purpose of the second part of this manuscript is to provide some answers to these questions.

In Chapter 3, we explore several mathematical features of Bertrand & Cournot MFGs. Our starting point is the result of Bensoussan and Graber in [**70**], where the authors show the existence of smooth solutions to the MFG system and uniqueness under a certain restriction. In Chapter 3, we improve this result by showing uniqueness with no restriction. The rest of Chapter 3 deals with a variant of the Bertrand & Cournot MFG model by considering a reflecting boundary conditions at $x = 0$. This situation can correspond to the case where reserves are exogenously and infinitesimally replenished. We investigate the new system of coupled PDEs and show that this system can be written as an optimality condition of a convex minimization problem. Next, we use this variational interpretation to prove existence and uniqueness of a weak solution to the corresponding first order system at the deterministic limit $\sigma \to 0$. Our analysis shows that the variational interpretation holds true for the original MFG system (with absorbing BCs at $x = 0$); in fact, the original PDE problem is also a system of optimality for the same minimization problem, with an adapted class of admissible solutions. In contrast, the analysis of the deterministic limit $\sigma \to 0$ in the case of absorbing BCs at $x = 0$ remains an open problem.

The main purpose of Chapter 4 is to explain the large population limit when $N \to \infty$, in the context of Bertrand & Cournot MFGs. It is well known that in the continuum mean field setting, Bertrand and Cournot games are identical (cf. Section 1 and [49]). This equivalence does not hold for the corresponding N-Player models, and therefore the analysis of the large population limit should be treated separately in order to take into account the specificity of each model. In Chapter 4, we focus on Cournot competition and we establish the rigorous link between the Cournot MFG and the N-Player dynamic Cournot competition. We consider that producers are constrained to choose a non-negative rate of production in order to manage their production capacity and generate profit. The constraint on the rate of production is natural form a modeling standpoint, and produces a coupled system of PDEs which is analogous to [92] and with less regular Hamiltonian function in comparison to [49, 70]. We start Chapter 4 by proving well-posedness for the resulting MFG system with initial measure data. The main ingredients are suitable a priori estimates in Hölder spaces and compactness results borrowed from [102]. Our analysis completes that which is found in Chapter 3 and [70] by treating the case of a less regular Hamiltonian function and initial measure data. Next, we show that feedback strategies which are computed from the Mean Field Game system provide ϵ-Nash equilibria to the corresponding N-Player Cournot competition, for large values of N. This is done by showing tightness of the empirical process in the so-called Skorokhod M1 topology, which is defined for distribution-valued processes. This result shows that the Cournot MFG model is indeed an approximation to the corresponding N-Player Cournot game when N is large, and therefore strengthens numerical methods which are based on the MFG approximation. To the best of the author's knowledge, this is the first analysis of the limit of large population $N \to \infty$ in the context of extended MFGs with absorbing boundary condition.

The last part of this thesis deals with an application of the Mean Field Games theory to Quantitative Finance. Chapter 5 goes beyond the optimal trading Mean Field Game model introduced by Pierre Cardaliaguet and Charles-Albert Lehalle in [39], by extending it to portfolios of correlated instruments. This leads to several original contributions: first that hedging strategies naturally stem from optimal liquidation schemes on portfolios. Second we show the influence of trading flows on naive estimates of intraday volatility and correlations. Focussing on this important relation, we exhibit a closed form formula expressing standard estimates of correlations as a function of the underlying correlations and the initial imbalance of large orders, via the optimal flows of our mean field game between traders. To support our theoretical findings, we use a real dataset of 176 US stocks from January to December 2014 sampled every 5 minutes to assess the influence of trading activity on the observed correlations. One of our theoretical findings backed by our empirical analysis is that the well know intraday shape of the volatility is far from uniform with respect to the intraday traded flows: given the absolute value of the traded flows is small, this intraday seasonality flattens out.

We start Chapter 5 by formulating the problem of optimal execution of a multi-asset portfolio inside a Mean Field Game. We derive the MFG system of PDEs and prove uniqueness of solutions to that system for a general Hamiltonian function. Then we construct a regular solution in the quadratic framework, that is considered in all the rest of that chapter. Next, we provide a convenient numerical scheme to compute the solution of the MFG system, and present several examples of an agent's optimal trading path, and the average trading path of the population. Furthermore, we address the question of assessing the influence of orders execution on the dependence structure of asset returns. We show that in the context of a multi-asset portfolio, the strategic interaction between the agents leads to a nontrivial relationship between the order flows, which in turn generates a non-trivial impact on the intraday covariance/correlation matrix of asset returns, especially at the beginning of the trading period (cf. Section 3). Namely, we show that the "observed" intraday covariance matrix of asset returns is the sum of the "fundamental" covariance matrix, and an excess covariance generated by the trading activity of the crowd. Finally, we carry out numerical simulations to illustrate this fact, and compare our findings with an empirical analysis on a pool of 176 US stocks.

We conclude this Chapter with a more detailed exposition of our results.

4.1. Quasi-Stationary Mean Field Games. Our first contribution is presented in Chapter 2 and deals with self-organizing phenomena in Mean Field Games with myopic players. For simplicity, we work in a periodic setting in order to avoid issues related to boundary conditions or conditions at infinity. Therefore, we consider functions as defined on $Q := \mathbb{T}^d$ (the d-dimensional torus).

Given an initial distribution of players' states m_0 in $\mathcal{P}(Q)$, and $\rho > 0$, the Mean Field Game problem with myopic players is articulated in the following way:

(1) *Observing and scheduling:* at any time $t \geqslant 0$, a representative player observes the global distribution of the players' states $m(t)$ and solves

$$\inf_{\alpha_t \in \mathbb{A}} \mathcal{J}^{m(t)}(\alpha_t), \tag{1.15}$$

with $\mathcal{J}^{m(t)}$ corresponding to:

$$\mathcal{J}_\rho^{m(t)}(\alpha_t) := \mathbb{E}\left[\int_t^\infty e^{-\rho s} L(\mathcal{X}_s^t, \alpha_t(\mathcal{X}_s^t)) + F(\mathcal{X}_s^t; m(t))\, ds \,\middle|\, \mathcal{F}_t\right]; \tag{1.16a}$$

or

$$\mathcal{J}_\infty^{m(t)}(\alpha_t) := \liminf_{\tau \to +\infty} \frac{1}{\tau}\mathbb{E}\left[\int_t^\tau L(\mathcal{X}_s^t, \alpha_t(\mathcal{X}_s^t)) + F(\mathcal{X}_s^t; m(t))\, ds \,\middle|\, \mathcal{F}_t\right]; \tag{1.16b}$$

where the state of a representative player evolves according to:

$$dX_t = \alpha_t(X_t)\, dt + \sqrt{2\sigma}\, dW_t \quad t \geqslant 0, \quad X_0 \sim m_0;$$

and for any $t \geqslant 0$ the process $(\mathcal{X}_s^t)_{s \geqslant t}$ models the scheduled – fictitious – future evolution of the player given her/his state at time t:

$$d\mathcal{X}_s^t = \alpha_t(\mathcal{X}_s^t)\, ds + \sqrt{2\sigma'}\, dB_{s-t} \quad s > t, \quad \mathcal{X}_t^t = X_t;$$

(2) *Equilibrium:* the flow $(m(t))_{0 \leqslant t \leqslant T}$ satisfies $m(t) = \mathcal{L}(\hat{X}_t)$ for every $t \in [0, T]$, where

$$d\hat{X}_t = \hat{\alpha}_t(\hat{X}_t)\, dt + \sqrt{2\sigma}\, dW_t \quad t \geqslant 0, \quad X_0 \sim m_0,$$

and $\hat{\alpha}_t$ is a minimizer of (1.15) for any $t \in [0, T]$.

Here $(W_t)_{t\geqslant 0}$ and $(B_t)_{t\geqslant 0}$ are two independent $\mathbb{F}$-Wiener processes, and $\mathbb{A}$ is a suitably chosen set of admissible actions (cf. Section 3). At each instant, agents set a strategy which optimizes their expected future cost by assuming their environment as immutable. As the system evolves, the players observe the evolution of the system and adapt to their new environment without anticipating.

From an analytic standpoint, we obtain the following systems of coupled partial differential equations which corresponds to (1.16a) and (1.16b) respectively:

$$
\begin{cases} -\sigma'\Delta v + H(x, Dv) + \rho v = F(x, \mu(t)) \quad \text{in } (0,T)\times Q \\ \\ \partial_t \mu - \sigma\Delta\mu - \operatorname{div}(\mu H_p(x, Dv)) = 0 \quad \text{in } (0,T)\times Q \\ \\ \mu(0) = m_0 \geqslant 0 \quad \text{in } Q, \quad < m_0 >= 1; \end{cases} \tag{1.17}
$$

and

$$
\begin{cases} -\sigma'\Delta u + H(x, Du) + \lambda(t) = F(x, m(t)) \quad \text{in } (0,T)\times Q \\ \\ \partial_t m - \sigma\Delta m - \operatorname{div}(m H_p(x, Du)) = 0 \quad \text{in } (0,T)\times Q \\ \\ m(0) = m_0 \geqslant 0 \quad \text{in } Q, \quad < m_0 >= 1, \quad < u >= 0 \quad \text{in } (0,T); \end{cases} \tag{1.18}
$$

where $\sigma', T > 0$, H is the Legendre-Fenchel transform of the function L, and all functions are assumed $\mathbb{Z}^d$-periodic. Note that (λ, u) (resp. v) depends on time only through m (resp. μ). The parameters σ' and σ are respectively: the noise level related to the prediction process (the assessment of the future evolution), and the noise level associated to the evolution of the players. The first equations in (1.18) and (1.17) give the "evolution" of the game value of a "small" player, and express the adaptation of players choices to the environment evolution. The evolution of μ and m expresses the actual evolution of the population density. We refer to Section 1 and Section 3 for more detailed explanations.

We start by proving well-posedness for systems (1.17), (1.18) under several sufficient conditions on H and F, by considering a smooth initial probability density m_0.

THEOREM 1.3. *Under suitable assumptions (c.f. Section 2), there exists a unique solution* (v, μ) *(resp.* (λ, u, m)*) to the problem* (1.17) *(resp.* (1.18)*), such that:*

- $(v, \mu) \in \mathcal{C}^{1/2}\left([0,T]; \mathcal{C}^2(Q)\right) \times \mathcal{C}^{1,2}(\overline{Q_T})$;
- $(\lambda, u, m) \in \mathcal{C}^{1/2}\left([0,T]\right) \times \mathcal{C}^{1/2}\left([0,T]; \mathcal{C}^2(Q)\right) \times \mathcal{C}^{1,2}\left(\overline{Q_T}\right)$.

The proofs rely on continuous dependence estimates for Hamilton-Jacobi-Bellman equations [**93**], the small-discount approximation and the non-local coupling which provides compactness and regularity. One should note that in contrast to most MFG systems, the uniqueness of solutions to systems (1.18) and (1.17) does not require the monotonicity condition (1.4a) nor the convexity of H with respect to the second variable, because of the forward-forward structure of the systems (cf. Section 2).

Next, we provide a detailed derivation of systems (1.17) and (1.18) from N-Player stochastic differential games models. Consider a game with N indistinguishable players whose states are driven by:

$$dX_t^i = \alpha_t^i(X_t^i)\,dt + \sqrt{2\sigma}\,dW_t^i, \quad X_0^i = V^i, \quad i = 1, \dots, N, \tag{1.19}$$

where $\alpha_t^1, \dots, \alpha_t^N \in \mathbb{A}$, $(W_t^1)_{t\geqslant 0}, \dots, (W_t^N)_{t\geqslant 0}$ are N independent $\mathbb{F}$-Wiener processes, and $(V^i)_{1\leqslant i\leqslant N}$ are i.i.d random variables with law m_0. We suppose that any player i aims to minimize the cost functional:

$$J_\rho^i\left(t, V, \bar{\alpha}_t^1, \dots, \bar{\alpha}_t^N\right) := \mathbb{E}\left[\int_t^\infty e^{-\rho s} L(\mathcal{X}_{s,t}^i, \alpha_t^i(\mathcal{X}_{s,t}^i)) + F\left(\mathcal{X}_{s,t}^i; \hat{\nu}_{\mathbf{X}_t}^{i,N}\right) ds \;\Big|\; \mathcal{F}_t\right], \tag{1.20}$$

or

$$J_\infty^i\left(t, V, \alpha_t^1, \dots, \alpha_t^N\right) := \liminf_{\tau\to+\infty} \frac{1}{\tau}\mathbb{E}\left[\int_t^\tau L(\mathcal{X}_{s,t}^i, \alpha_t^i(\mathcal{X}_{s,t}^i)) + F\left(\mathcal{X}_{s,t}^i; \hat{\nu}_{\mathbf{X}_t}^{i,N}\right) ds \;\Big|\; \mathcal{F}_t\right]. \tag{1.21}$$

Here

$$\hat{\nu}_{\mathbf{X}_t}^{i,N} := \frac{1}{N-1}\sum_{j\neq i} \delta_{X_t^j},$$

and the scheduled-fictitious trajectories $(\mathcal{X}_{s,t}^1)_{s>t}, \dots, (\mathcal{X}_{s,t}^N)_{s>t}$ are driven by

$$\begin{cases} d\mathcal{X}_{s,t}^i = \alpha_t^i(\mathcal{X}_{s,t}^i)\,ds + \sqrt{2\sigma'}\,d\mathcal{B}_{s-t,t}^i & s > t, \\ \mathcal{X}_{t,t}^i = X_t^i, \quad i = 1, \dots, N, \end{cases} \tag{1.22}$$

where $\left\{(\mathcal{B}_{s,t}^1)_{s\geqslant 0}, \dots, (\mathcal{B}_{s,t}^N)_{s\geqslant 0}\right\}_{t\geqslant 0}$ is a family of standard Brownian motions. For any $t \geqslant 0$, the process $(\mathcal{B}_{s-t,t}^i)_{s>t}$ represents the noise related to the scheduling of the i^{th} player, and is assumed to be independent from $W_t^1, \dots, W_t^N$.

Let $(X_t^1)_{t\geqslant 0}, \dots, (X_t^N)_{t\geqslant 0}$ (resp. $(Z_t^1)_{t\geqslant 0}, \dots, (Z_t^N)_{t\geqslant 0}$) be the trajectories associated to an equilibrium configuration with respect to $(J_\rho^i)_{1\leqslant i\leqslant N}$ (resp. $(J_\infty^i)_{1\leqslant i\leqslant N}$). The exact definition of equilibrium in this specific context is provided in Section 3. Then, the following hold:

THEOREM 1.4. *For any* $t \in [0, T]$, *it holds that:*

$$\lim_N \max_{1\leqslant i\leqslant N} \mathbf{d}_1\left(\mathcal{L}\left(X_t^i\right), m(t)\right) = 0;$$
$$\lim_N \max_{1\leqslant i\leqslant N} \mathbf{d}_1\left(\mathcal{L}\left(Z_t^i\right), \mu(t)\right) = 0;$$
$$\lim_N \left\|u[m(t)] - \mathbb{E}u\left[\nu_{X_t}^N\right]\right\|_\infty = 0;$$
$$\lim_N \left|\lambda[m(t)] - \mathbb{E}\lambda\left[\nu_{X_t}^N\right]\right| = 0; \quad \textit{and}$$
$$\lim_N \left\|v[\mu(t)] - \mathbb{E}v\left[\nu_{Z_t}^N\right]\right\|_\infty = 0.$$

The proof of Theorem 1.4 relies on the standard coupling arguments **[107]**, and continuous dependence estimates for HJB equations **[93]**.

Next, we show that the myopic population self-organizes exponentially fast toward the stationnary MFG equilibria (1.6), (1.8), in the case where $H(x, p) = |p|^2/2$ and the coupling F satisfies the monotonicity condition (1.4a).

THEOREM 1.5. *Under the above conditions, there exists* $R_0, \epsilon > 0$ *such that if*

$$\|m_0 - \bar{m}\|_2 \leqslant R_0 \ \textit{and} \ \ \rho < \epsilon,$$

then the following hold for some constants $K, \delta > 0$*:*

$$|\lambda(t) - \bar{\lambda}| + \|u(t) - \bar{u}\|_{\mathcal{C}^2} + \|m(t) - \bar{m}\|_2 \leqslant Ke^{-\delta t};$$

and

$$\|\nu(t) - \bar{\nu}\|_{\mathcal{C}^2} + \|\mu(t) - \bar{\mu}\|_2 \leqslant Ke^{-\delta t}; \quad \textit{for any } t \geqslant 0.$$

Theorem 1.5 reveals that the population of myopic agents "learns" a stationary MFG equilibrium through the process of observation and self-correction. In particular, it shows that the system can exhibit a large scale structure even if the cohesion between the agents is only maintained by interactions between neighbours. The proof of Theorem 1.5 relies on some algebraic observations which are pointed out in [**37**] and which are specific to quadratic Hamiltonians. Therefore, the convergence remains an open problem for more general cases and it is delegated to a future work.

We conclude Chapter 2 by carrying out several numerical experiments. We provide a suitable numerical scheme inspired by [**3**] to simulate the long time behavior of solutions to system (1.18) for various examples. The results suggests that Theorem 1.5 holds under less restrictive conditions.

4.2. A Variational Approach for Bertrand & Cournot MFGs. As we already pointed out, the Bertrand & Cournot MFG system consists in a system of a backward HJB equation to model a representative firm's value function, coupled with a forward Fokker-Planck equation to model the evolution of the distribution of the active firms' capacity. Namely, the Bertrand & Cournot MFG system reads:

$$\begin{cases} \partial_t u + \sigma \partial_{xx} u - ru + q_{u,m}^2 = 0 & \text{in } Q_T, \\ \partial_t m - \sigma \partial_{xx} m - \partial_x \left(q_{u,m} m \right) = 0 & \text{in } Q_T, \end{cases} \tag{1.23a}$$

where the production function $q_{u,m}$ is given by:

$$q_{u,m}(t,x) := \frac{1}{2}\left(1 - \kappa \int_0^\ell q_{u,m}(t,y) m(t,y)\, dy - \partial_x u(t,x)\right), \tag{1.23b}$$

and $\kappa > 0$ is a coefficient that quantifies the substitutability of the produced goods. Note that the function $q_{u,m}$ is defined as a fixed point in the latter expression. As long as we consider Bertrand & Cournot MFGs we suppose that production capacity of any player belongs to $[0, \ell]$, where $\ell > 0$ is a limit capacity that is unreachable for any producer. Therefore, Q_T refers to $(0, T) \times (0, \ell)$ in this context. We refer the reader to Chapter 3 for a detailed explanation of the MFG model (1.23a)-(1.23b).

Firms disappear a soon as they deplete their capacity and can no longer generate revenue. This fact is expressed through absorbing boundary conditions for m and u at $x = 0$, and for simplicity we consider reflection boundary conditions at $x = \ell$. Namely, the problem (1.23a)-(1.23b) is complemented with the following boundary conditions :

$$
\begin{cases}
m(t,0)=0, \quad u(t,0)=0, \quad \partial_x u(t,\ell)=0 \quad \text{in } (0,T), \\
\\
m(0)=m_0, \quad u(T,.)=u_T, \quad \text{in } [0,\ell], \\
\\
\sigma\partial_x m + q_{u,m} m = 0 \quad \text{in } (0,T)\times\{\ell\}.
\end{cases} \tag{1.23c}
$$

where u_T is a positive smooth and non-decreasing function satisfying compatibility conditions on the boundary, and m_0 is a smooth probability density satisfying compatibility conditions as well.

We start by improving the result of [70], by showing the uniqueness of a smooth solution to (1.23a)-(1.23c) with no restriction on the parameters.

THEOREM 1.6. *There exists a unique classical solution to system* (1.23a)-(1.23c).

The authors of [70] consider the Bertrand formulation of the problem:

$$
q_{u,m} = \frac{1}{2}\left(\frac{2}{2+\kappa\eta(t)} + \frac{\kappa}{2+\kappa\eta(t)}\int_0^\ell \partial_x u(t,y)m(t,y)\,dy - \partial_x u\right), \quad \eta(t) = \int_0^\ell m(t,y)\,dy,
$$

which is equivalent to the Cournot formulation (1.23b)[4], but less convenient for the proof of uniqueness.

When all players participate at all resource levels, it is possible to replace the absorbing boundary conditions (1.23c) by a reflecting ones. This situation is also considered in [77] for N-Player dynamic Cournot competition. Reflecting boundary conditions could also correspond to a situation where reserves are exogenously and infinitesimally replenished. In the rest of Chapter 3, we suppose that system (1.23a)-(1.23b) is endowed with the following boundary conditions:

$$
\begin{cases}
\partial_x u(t,0)=\partial_x u(t,\ell)=0, \quad \text{in } (0,T), \\
\\
m(0)=m_0, \quad u(T,.)=u_T, \quad \text{in } [0,\ell], \\
\\
\sigma\partial_x m + q_{u,m} m = 0 \quad \text{in } (0,T)\times\{0,\ell\}.
\end{cases} \tag{1.24}
$$

We start by proving well-posedness for the new system (1.23a), (1.23b), (1.24).

THEOREM 1.7. *There exists a unique classical solution to system* (1.23a), (1.23b), (1.24).

The arguments of the proof in this case are very similar to Theorem 1.6.

Next, we show that system (1.23a), (1.23b), (1.24) is a system of optimality to the following convex minimization problem:

$$
\min_{(m,q)\in\mathcal{K}} J(m,q),
$$

[4]This will be clear in Chapter 3.

such that

$$J(m, q) = \int_0^T \int_0^\ell e^{-rt} \left(q^2(t, x) - q(t, x)\right) m(t, x)\,dx\,dt \\ + \frac{\kappa}{2} \int_0^T e^{-rt} \left(\int_0^\ell q(t, y) m(t, y)\,dy \right)^2 dt - \int_0^\ell e^{-rT} u_T(x) m(T, x)\,dx. \tag{1.25}$$

The class $\mathcal{K}$ on which the minimization problem is considered is defined as follows: We say that $(m, q) \in \mathcal{K}$, if $m \in L^1([0, T] \times [0, \ell])_+$, $q \in L^2([0, T] \times [0, \ell])$, and m is a weak solution to the Fokker-Planck equation

$$\partial_t m - \sigma \partial_{xx} m - \partial_x(qm) = 0, \quad m(0) = m_0, \tag{1.26}$$

equipped with Neumann boundary conditions; where weak solutions to (1.26) are defined as in [**102**]:

- the integrability condition $mq^2 \in L^1([0, T] \times [0, \ell])$ holds, and
- (1.26) holds in the sense of distributions–namely, for all $\phi \in C_c^\infty([0, T) \times [0, \ell])$ such that $\partial_x \phi(t, 0) = \partial_x \phi(t, \ell) = 0$, for each $t \in (0, T)$, we have
$$\int_0^T \int_0^\ell (-\partial_t \phi - \sigma \partial_{xx} \phi + q \partial_x \phi) m \ \,dx\,dt = \int_0^\ell \phi(0) m_0\,dx.$$

Our claim is stated as follows:

THEOREM 1.8. *Let (u, m) be a solution to (1.23a), (1.23b), (1.24). Set*

$$q = \frac{1}{2} \left(\frac{2}{2 + \kappa} + \frac{\kappa}{2 + \kappa} \int_0^\ell \partial_x u(t, y) m(t, y)\,dy - \partial_x u \right).$$

Then (m, q) is a minimizer for problem (1.25), that is, $J(m, q) \leqslant J(\tilde{m}, \tilde{q})$ for all $(\tilde{m}, \tilde{q})$ in $\mathcal{K}$. Moreover, if $\log m_0 \in L^1([0, \ell])$ then the minimizer is unique.

The key argument in the proof of Theorem 1.8 is the change of variable $(m, w) := (m, mq)$ (cf. Section 3) that is used in [**13**] and several works which cite that paper. The proof of Theorem 1.8 shows that an analogous result to Theorem 1.8 holds for system (1.23a)-(1.23c) (cf. Remark 3.9). Namely, System (1.23a)-(1.23c), is also a system of optimality for the same minimization problem, except this time with Dirichlet boundary conditions at $x = 0$ imposed on the Fokker-Planck equation (1.26).

We conclude Chapter 3 by addressing the deterministic limit $\sigma \to 0$ for the problem (1.23a), (1.23b), (1.24).

THEOREM 1.9. *Assume that $\sigma = 0$. Then there exists a unique pair (u, m) which solves the problem (1.23a), (1.23b), (1.24) in the following sense:*

(1) $u \in W_2^1([0, T] \times [0, \ell]) \cap L^\infty(0, T; W_\infty^1(0, \ell))$ *is a continuous solution of the Hamilton-Jacobi equation*

$$\partial_t u - ru + \frac{1}{4}(f(t) - \partial_x u)^2 = 0, \; u(T, x) = u_T(x), \tag{1.27}$$

equipped with Neumann boundary conditions, in the viscosity sense;

(2) $m \in L^1 \cap \mathcal{C}([0,T];\mathcal{P}([0,\ell]))$ *satisfies the continuity equation*

$$\partial_t m - \frac{1}{2}\partial_x\{(f(t) - \partial_x u)m\} = 0, \; m(0) = m_0, \tag{1.28}$$

equipped with Neumann boundary conditions, in the sense of distributions; and

(3) $f(t) = \left(\frac{2}{2+\kappa} + \frac{\kappa}{2+\kappa}\int_0^\ell \partial_x u(t,y)m(t,y)\,dy\right)$ *for a.e.* $t \in [0,T]$.

As usual, existence is obtained by deriving suitable apriori estimates for the solution of (1.23a), (1.23b), (1.24) with $\sigma > 0$, and then taking the limit $\sigma \to 0$ by using compactness arguments. Compactness estimates for m are derived by using the fact that it is the minimizer for an optimization problem. The proof of uniqueness relies on results for transport equations with a non-smooth vector field.

4.3. Approximate Equilibria for N-Player Dynamic Cournot Competition. Very little is known so far on the rigorous link between the so called Bertrand and Cournot MFG models and the corresponding N-Player Bertrand and Cournot stochastic differential games. Indeed, the classical theory cannot be applied to this specific case for two main reasons: on the one hand, because of the absorbing boundary conditions; and on the other hand, because these models belong to the class of extended Mean Field Games. This has motivated the analysis of Chapter 4, in which we address rigorously this question for Cournot competition.

We consider a continuum of firms where each firm *is constrained* to choose a non-negative production rate in order to manage its production capacity and to generate profit. In this case the MFG problem is (1.23a), (1.23c) where the function $q_{u,m}$ takes now the following form:

$$q_{u,m}(t,x) := \frac{1}{2}\left(1 - \kappa\int_0^\ell q_{u,m}(t,y)m(t,y)\,dy - \partial_x u(t,x)\right)^+, \tag{1.29}$$

where $w^+ = (w + |w|)/2$. In comparison to the previous situation the function $q_{u,m}$ is less regular. Nevertheless, we will prove that $\partial_x u$ is always non-negative so that $q_{u,m}$ remains bounded for every $(t,x) \in [0,T] \times [0,\ell]$. This remark plays a crucial role in our analysis because it provides the stability and compactness which is needed to construct a suitable solution to system (1.23a),(1.23c), (1.29).

Our first result is the following:

THEOREM 1.10. *There exists a unique solution* (u,m) *to system* (1.23a),(1.23c), (1.29) *starting from* $m_0 \in \mathcal{P}([0,\ell])$, *such that* $\mathrm{supp}(m_0) \subset (0,\ell]$.

By a solution to (1.23a),(1.23c), (1.29) we mean a couple (u,m) where the equation for u holds in the classical sense, while the equation for m holds in the weak sense (cf. Section 2). The proof of uniqueness is essentially the same as in Theorem 1.6 and Theorem 1.7. The only difference is the special form of $q_{u,m}$ which makes the computations a little more tricky. Existence relies on suitable a priori estimates in Hölder spaces and compactness results borrowed from **[102]**.

Next, we address the link between Cournot MFGs and N-Player dynamic Cournot competition. Our main result states that the feedback strategies which are computed from the MFG system (1.23a),(1.23c), (1.29) allows to build ε-Nash equilibria for N-Player Cournot competition for large enough N. Let us explain briefly the N-Player dynamic Cournot game. Given a common time horizon $T > 0$, consider N indistinguishable agents where the reserves state of any agent i is modeled by a stopped stochastic process $\left(X^i_{t\wedge\tau^i}\right)_{t\geqslant 0}$. Here the diffusions $(X^1_t)_{t\geqslant 0}, \dots, (X^N_t)_{t\geqslant 0}$ are driven by the following Skorokhod system:

$$
\begin{cases}
dX^i_t = -q^i_t\, dt + \sqrt{2\sigma}\, dW^i_t - d\xi^{X^i}_t, \\
\\
X^i_0 \sim m_0, \quad i = 1, \dots, N, \\
\\
\tau^i := \inf\left\{t \geqslant 0 \,:\, X^i_t \leqslant 0\right\} \wedge T,
\end{cases}
\tag{1.30}
$$

where $(\xi^X_t)_{t\geqslant 0}$ is the *local time* associated to the reflected diffusion $(X_t)_{t\geqslant 0}$ in $x = \ell$; and $(W^1_t)_{t\geqslant 0}, \dots, (W^N_t)_{t\geqslant 0}$ are N independent $\mathbb{F}$-Wiener processes which models demand fluctuations. The reserves level of any player i can not exceed ℓ, and is gradually depleted according to a non-negative controlled rate of production $\left(q^i_t\right)_{t\in[0,T]}$. The stopping condition indicates that a firm can no longer replenish its reserves once they are exhausted. By assuming a linear demand schedule, the profit functional of any producer i is given by (c.f. Section 3.1):

$$
\mathcal{J}^{i,N}_c(q^1, \dots, q^N) := \mathbb{E}\left\{\int_0^T e^{-rs}\left(1 - \kappa\bar{q}^i_s - q^i_s\right) q^i_s \mathbb{1}_{s<\tau^i}\, ds + e^{-rT} u_T(X^i_{\tau^i})\right\},
$$

where

$$
\bar{q}^i_t = \frac{1}{N-1}\sum_{j\neq i} q^j_t \mathbb{1}_{t<\tau^j}, \quad \text{for} \quad 0 \leqslant t \leqslant T,
$$

and $u_T(0) = 0$. Our main result is the following:

THEOREM 1.11. *For any $N \geqslant 1$ and $i \in \{1, \dots, N\}$, let us consider the following Skorokhod problem:*

$$
\begin{cases}
d\hat{X}^i_t = -q_{u,m}(t, \hat{X}^i_t)\, dt + \sqrt{2\sigma}\, dW^i_t - d\xi^{\hat{X}^i}_t \\
\hat{X}^i_0 \sim m_0, \quad i = 1, \dots, N,
\end{cases}
\tag{1.31}
$$

and set $\hat{q}^i_t := q_{u,m}(t, \hat{X}^i_t)$. Then for any $\varepsilon > 0$, the strategy profile $(\hat{q}^1, \dots, \hat{q}^N)$ is admissible (c.f. Section 3), and provides an ε-Nash equilibrium to the game $\mathcal{J}^{1,N}_c, \dots, \mathcal{J}^{N,N}_c$ for large N. Namely: $\forall\varepsilon > 0$, $\exists N_\varepsilon \geqslant 1$ such that

$$
\forall N \geqslant N_\varepsilon, \forall i = 1, \dots, N, \quad \mathcal{J}^{i,N}_c\left(q^i; (\hat{q}^j)_{j\neq i}\right) \leqslant \varepsilon + \mathcal{J}^{i,N}_c\left(\hat{q}^1, \dots, \hat{q}^N\right),
\tag{1.32}
$$

for any admissible strategy q_i.

The crucial step in the proof of Theorem 1.11 is the analysis of the large population limit $N \to \infty$ for the empirical process:

$$
\hat{\nu}^N_t := \frac{1}{N}\sum_{k=1}^N \delta_{\hat{X}^k_t} \mathbb{1}_{t<\hat{\tau}^k}, \quad \forall t \in [0, T],
$$

where

$$\hat{\tau}^i := \inf\left\{t \geqslant 0 \,:\, \hat{X}^i_t \leqslant 0\right\} \wedge T.$$

We prove a tailor-made (weak) law of large numbers by working in a suitable function space. Namely, we view the empirical process $\hat{\nu}^N$ as a random variable on the space $D_{\mathcal{S}'_{\mathbb{R}}}$ of all *càdlàg* (right continuous and has left-hand limits) functions, mapping $[0, T]$ into the space of tempered distributions. This function space is addressed in [**89**], where the authors extend the so called Skorokhod M1 topology on that space, and provide a convenient characterization of tightness. The topological space $(D_{\mathcal{S}'_{\mathbb{R}}}, \mathrm{M1})$ is also used in [**76**] for the analysis of the mean field limit for a *stochastic* McKean-Vlasov equation with absorbing boundary conditions. By using the machinery of [**76**, **89**] we prove the following Lemma which is a crucial step toward the proof of Theorem 1.11:

LEMMA 1.12. *As $N \to \infty$, the empirical process $\hat{\nu}^N$ converges in law toward the deterministic flow m on $(D_{\mathcal{S}'_{\mathbb{R}}}, \mathrm{M1})$.*

The proof of Lemma 1.12 is organized as follows: we start by showing the existence of sub-sequences $(\hat{\nu}^{N'})$ that converges in law to some limiting process ν^* (c.f. Proposition 4.14). Then, we show that ν^* is a sub-probability measure that is supported on $[0, \ell]$ and satisfying the same equation as m (c.f. Lemma 4.17). Finally, we invoke the uniqueness of weak solutions to the Fokker-Planck equation to deduce full weak convergence toward the deterministic flow m.

4.4. Optimal Portfolio Trading Within a Crowded Market. We conclude this thesis with Chapter 5, where we use the Mean Field Game framework to model the interaction of a continuum of heterogeneous traders seeking to execute large market orders to manage a multi-asset portfolio. Our model is an extension of the Cardaliaguet-Lehalle model [**39**] to the case of multi-asset portfolios. For the sake of simplicity, we will not explain the model here, and we will only present our main results and findings. We refer to Section 2 for a detailed and complete explanation of the Mean Field Game model.

The Mean Field Game system of PDEs associated to our model takes the following form:

$$\begin{cases} \dfrac{\gamma^a}{2}\mathbf{q}\cdot\Sigma\mathbf{q} = \partial_t u^a + \mathbb{A}\boldsymbol{\mu}\cdot\mathbf{q} + \displaystyle\sum_{i=1}^d V_i H_i\left(\partial_{q_i} u^a(t,\mathbf{q})\right) \quad \text{in } (0,T)\times\mathbb{R}, \\[2em] \partial_t m + \displaystyle\sum_{i=1}^d V_i \partial_{q_i}\left(m\dot{H}_i\left(\partial_{q_i} u^a(t,\mathbf{q})\right)\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}\times D, \\[2em] \mu^i_t = \displaystyle\int_{(q,a)} V_i\dot{H}_i\left(\partial_{q_i} u^a(t,\mathbf{q})\right) m(t, d\mathbf{q}, da) \quad \text{in } [0,T], \\[2em] m(0, d\mathbf{q}, da) = m_0(d\mathbf{q}, da), \quad u^a_T = -\mathbf{A}^a\mathbf{q}\cdot\mathbf{q}, \end{cases} \tag{1.33}$$

where D is a closed subset of $\mathbb{R}$, the matrix $\mathbb{A} := \mathrm{diag}(\alpha_1, ..., \alpha_d)$ characterizes the magnitude of the permanent market impact, γ^a is the risk aversion coefficient for investor a, Σ is the fundamental covariance matrix of assets returns, V_i is the "typical" daily volume for asset i, H_i is a function which characterizes liquidity, $\dot{H}_i$ denotes the first derivative of H_i, and $\mathbf{A}^{\mathbf{a}}$ is a diagonal matrix of penalization of large final inventories. Here μ^i is the average speed of trading on asset i at the equilibrium, m characterizes the distribution of agents in $\mathbb{R} \times D$, while u^a is the value function of a representative investor. We refer the reader to Section 2.1 for a detailed explanation of the MFG problem.

At first, we address the uniqueness of solutions to system (1.33). We refer the reader to Section 2.1 for a precise definition of what we mean by a solution to (1.33). For now we can keep in mind that the equation for u^a holds in the classical sense for a.e. $a \in D$, while the equation for m holds in the sense of distributions.

PROPOSITION 1.13. *Under assumptions of Section 2.1, the Mean Field Game system (1.33) has at most one solution.*

Next, we show that system (1.33) is well-posed in the case where

$$H_i(p) = |p|^2/4\eta_i, \quad \eta_i > 0, \quad i = 1, ..., N. \tag{1.34}$$

Assumption (1.34) is in force throughout all the rest of Chapter 5. In that particular case, the solution to (1.33) is constructed by using a suitable *ansatz* (cf. (5.13)) which reduces the problem into a simpler system of coupled ODEs. Nevertheless, due to the forward-backward structure of our system, we need a smallness condition on $\mathbb{A}$ in order to construct a solution. This assumption is also considered in **[39]**, and is not problematic from a modeling standpoint since $|\mathbb{A}|$ is generally small in applications (cf. Section 2.3).

THEOREM 1.14. *Under suitable assumptions on data (c.f. Section 2.2), there exists $\alpha_0 > 0$ such that, for $|\mathbb{A}| \leqslant \alpha_0$ the Mean Field Game system (1.33) has a unique solution.*

By solving the MFG system (1.33) we are able to characterize the optimal trading strategy of an individual agent given her/his initial inventory. Indeed, we show that the optimal feedback strategy at the MFG equilibrium has the following form:

$$\begin{aligned} \mathbf{v}_a^*(t, \mathbf{q}) &= 2\mathbb{V}\mathbb{H}_a(t)\mathbf{q} + 2\mathbb{V}\int_t^T \exp\left\{\int_t^w 2\mathbb{H}_a(s)\mathbb{V}\, ds\right\} \mathbb{A}\boldsymbol{\mu}_w\, dw \\ &=: \mathbf{v}_a^{1,*}(t, \mathbf{q}) + \mathbf{v}_a^{2,*}(t, \boldsymbol{\mu}), \end{aligned}$$

where $\mathbb{V} := \mathrm{diag}\left(\frac{V_1}{4\eta_1}, ..., \frac{V_d}{4\eta_d}\right)$ and $(\mathbb{H}_a(t))_{t\in[0,T]}$ is a $\mathbb{R}^{d\times d}$-valued process characterizing the optimal trading speed of investor a (c.f. Section 2.2). This expression shows that the optimal execution strategy of an individual agent is the sum of the classical Almgren-Chriss strategy $\mathbf{v}_a^{1,*}$ (cf. the Introduction of Chapter 5 and references therein) and an additional component $\mathbf{v}_a^{2,*}$ which adjusts the speed based on the anticipated future average trading (mean field) on the remainder of the trading window $[t, T]$.

In Section 2.3 we address the case of identical preferences (i.e. D is reduced to a single point) and provide a convenient numerical scheme to compute the solution of the MFG system. We present several examples of an agent's optimal trading path, and the average trading path of the population (cf. Section 2.3). The simulated examples illustrate some specific trading strategies to the case of multi-asset portfolio, such as Arbitrage Strategies (cf. Figure 1(d)) and Hedging strategies (cf. Figure 1(c)).

In the second part of Chapter 5, we use our model to investigate the influence of large orders execution on the observed covariance matrix of asset returns. We place ourselves from the point of view of an external observer which aims to estimate the intraday covariance matrix of asset returns using historical market data.

At first, we suppose that the assets prices $(S_t^1, ..., S_t^d)_{t\in[0,T]}$ evolve according to a specific dynamics; namely, we suppose that

$$dS_t^i = \sigma_i \, dW_t^i + \alpha_i \mu_t^i \, dt, \quad i = 1, ..., d; \tag{1.35}$$

where $\alpha_1, ..., \alpha_d$ are nonnegative scalars modeling the magnitude of the permanent market impact, $\sigma_1, ..., \sigma_d > 0$, $(W_t^1, ..., W_t^d)_{t\geqslant 0}$ are d correlated Wiener processes, and we denote by Σ the covariance matrix of the d-dimensional process $(\sigma_1 W_t^1, ..., \sigma_d W_t^d)_{t\in[0,T]}$. Here $(\mu_t^1, ..., \mu_t^d)_{t\in[0,T]}$ are the optimal trading flows associated the the Mean Field Game system (1.33) with identical preferences (c.f. Section 2.2). We suppose that the same game occurs every day with different initial distribution inventories. In other words, we suppose that m_0 is a $\mathcal{P}(\mathbb{R})$-valued random variable which takes a given realization on each day. For simplicity we assume that the observed covariance matrix between t_k and t_{k+1} is estimated by using the following naive estimator:

$$C^{i,j}_{[t_k,t_{k+1}]} := \frac{1}{N-1}\sum_{l=1}^{N}\left(\delta S^{i,k,l} - \widehat{\delta S}^{i,k}\right)\left(\delta S^{j,k,l} - \widehat{\delta S}^{j,k}\right), \tag{1.36}$$

where $\delta S^{n,k,l}$ is the increment of the price of asset n in bin $k+1$ of day l, and $\widehat{\delta S}^{n,k} = N^{-1}\sum_{l=1}^{N}\delta S^{n,k,l}$. In this case, the intraday covariance matrix of asset returns can be computed explicitly:

PROPOSITION 1.15. *Suppose that* m_0 *is independent from* $(\mathbf{W}_t)_{t\in[0,T]}$, *then for any* i, j *in* $\{1, ..., d\}$ *and* $(t_k)_k \subset [0, T]$, *it holds that:*

$$C^{i,j}_{[t_k,t_{k+1}]} = (t_{k+1} - t_k)\Sigma_{i,j} + \alpha_i\alpha_j \frac{\eta_i\eta_j}{4V_iV_j}\Lambda_k^{i,j} + \epsilon_N,$$

where $\epsilon_N \to 0$ *as* $N \to \infty$, *and* $(\Lambda^{i,j,k})_{1\leqslant i,j\leqslant d}$ *is* $d\times d$ *real matrice which depends on* m_0 *and the MFG optimal execution strategy (cf. Proposition 5.4).*

Proposition 1.15 shows that the realized covariance matrix is the sum of the fundamental covariance matrix and an excess realized covariance that is generated endogenously by the execution impact of the crowd of investors. In addition, we show that the realized covariance can deviate significantly from fundamentals when: the market impact is large, the considered assets are highly non-liquid, the risk aversion coefficient γ is high, and/or when the standard deviation of m_0 is large. We carry out several numerical experiments in order to illustrate this fact.

Next, we conduct an empirical analysis of the covariance matrix of asset returns by considering a pool of $d = 176$ US stocks. The data consists of five-minute binned trades and quotes information from January 2014 to December 2014. We show that the average intraday volatility, and the average intraday covariance between stocks, exhibits the well-known “left-slanted smile” shape, which is consistent with our model. Moreover, by conditioning our estimations to low trade imbalances (relatively small orders), the average intraday volatility and covariance patterns flattens out (cf. Figures 4(a) and 4(b)), which fits well the findings of our theoretical analysis. Finally, we propose a toy model based approach to calibrate our MFG model on data.

Part I

Self-Organization in Mean Field Games

CHAPTER 2

Quasi-Stationary Mean Field Games

This work is published in "Applied Mathematics and Optimization" under the title "On Quasi-Stationary Mean Field Games Models", except the last section on numerical experiments

1. Introduction

In this chapter, we introduce a Mean Field Game model with a non-anticipating decision-making mechanism. Our agents anticipate no evolution, undergo changes in their environment and adjust their actions given the available information. This framework is well suited to a context in which agents have poor – or no – visibility on the evolution of the system and only manage the ongoing situation. We introduce the MFG model, then we explain the link with the corresponding N-Player game. Finally, we analyze the formation of equilibria configurations for this type of interaction systems and assess the rate at which these systems converge towards these equilibria.

Let us consider a continuum of indistinguishable players whose states are driven by a stochastic differential equation. The players are non-anticipating – myopic – and take actions given a picture of the environment at time t, aiming to get the best possible future cost $(s > t)$. From a mathematical standpoint, a generic player chooses at any time t a drift vector field $\alpha_t(.)$ that has a suitable regularity. This action is chosen in order to minimize a cost functional which depends on the current – observed – state of the system, and on a evaluation of the future path $(s > t)$ of the player given her/his choice at time t. Thus, choosing the optimal $\alpha_t(.)$ at time t, amounts to schedule optimally the future evolution of the player, by fixing the state of the system at time t. Players follow their planned evolution and adjust their drift according to the observed moves of opponents. The ongoing process of adapting the drift describes a process of self-correction. We should note that this process intrinsically implies the existence of two time scales: a fast time scale which is linked to the optimization of the expected future cost; and a slow time scale linked to the actual evolution of the system. In this work, we shall consider two kinds of cost functionals: a long time average cost functional, and a long run discounted cost functional.

For simplicity, we work in a periodic setting in order to avoid issues related to boundary conditions or conditions at infinity. Therefore functions are assumed to be $\mathbb{Z}^d$-periodic with respect to the state variable 'x', and considered as defined on $Q := \mathbb{T}^d$ (the d-dimensional Torus). Given an initial distribution of players' states m_0 in $\mathcal{P}(Q)$, and $\rho > 0$, the Mean Field Game problem with myopic players is articulated in the following way:

(1) *Observing and scheduling:* at any time $t \geqslant 0$, a representative player observes the global distribution of the players' states $m(t)$ and solves

$$\inf_{\alpha_t \in \mathbb{A}} \mathcal{J}^{m(t)}(\alpha_t), \tag{2.1}$$

with $\mathcal{J}^{m(t)}$ corresponding to:

$$\mathcal{J}^{m(t)}_{\rho}(\alpha_t) := \mathbb{E}\left[\int_t^{\infty} e^{-\rho s} L(\mathcal{X}^t_s, \alpha_t(\mathcal{X}^t_s)) + F(\mathcal{X}^t_s; m(t))\, ds \;\middle|\; \mathcal{F}_t\right]; \tag{2.2a}$$

or

$$\mathcal{J}^{m(t)}_{\infty}(\alpha_t) := \liminf_{\tau \to +\infty} \frac{1}{\tau}\mathbb{E}\left[\int_t^{\tau} L(\mathcal{X}^t_s, \alpha_t(\mathcal{X}^t_s)) + F(\mathcal{X}^t_s; m(t))\, ds \;\middle|\; \mathcal{F}_t\right]; \tag{2.2b}$$

where the state of a representative player evolves according to:

$$dX_t = \alpha_t(X_t)\, dt + \sqrt{2\sigma}\, dW_t \quad t \geqslant 0, \quad X_0 \sim m_0;$$

and for any $t \geqslant 0$ the process $(\mathcal{X}^t_s)_{s \geqslant t}$ models the predicted – fictitious – future evolution of the player given her/his state at time t:

$$d\mathcal{X}^t_s = \alpha_t(\mathcal{X}^t_s)\, ds + \sqrt{2\sigma'}\, dB_{s-t} \quad s > t, \quad \mathcal{X}^t_t = X_t;$$

(2) *Equilibirium:* the flow $(m(t))_{0 \leqslant t \leqslant T}$ satisfies $m(t) = \mathcal{L}(\hat{X}_t)$ for every $t \in [0, T]$, where

$$d\hat{X}_t = \hat{\alpha}_t(X_t)\, dt + \sqrt{2\sigma}\, dW_t \quad t \geqslant 0, \quad X_0 \sim m_0,$$

and $\hat{\alpha}_t$ is a minimizer of (2.1) for any $t \in [0, T]$.

Here $\sigma, \sigma' > 0$, $(W_t)_{t \geqslant 0}$ and $(B_t)_{t \geqslant 0}$ are two independent Wiener processes, and $\mathbb{A}$ is the set of *admissible* vector fields which will be explained precisely in Section 3. Moreover, $\mathcal{F}_t := \sigma\{X_0, W_u, u \leqslant t\}$ is the information available to the players at time t. The exact regularity of functions L and F will be specified latter.

Note that 't' is a slow time scale which is related to the evolution of the population, while 's' is a fast time scale which is related to the scheduling. Given the structure of the cost functionals (2.2a), (2.2b), the agents take into account the actual picture of the system through the coupling function F, and build a long-run strategy according to an evaluation of their future path emanating from that choice. As the distribution of the players $m(t)$ evolves in time, players schedule an effective response by adjusting their α_t.

From an analytic standpoint, we obtain the following systems of coupled partial differential equations (cf. Proposition 2.14) which corresponds to (2.2a) and (2.2b) respectively:

$$\begin{cases} -\sigma'\Delta v + H(x, Dv) + \rho v = F(x, \mu(t)) & \text{in } (0,T) \times Q \\ \partial_t \mu - \sigma\Delta\mu - \operatorname{div}(\mu H_p(x, Dv)) = 0 & \text{in } (0,T) \times Q \\ \mu(0) = m_0 \geqslant 0 \quad \text{in } Q, & \displaystyle\int_Q m_0 = 1; \end{cases} \tag{2.3a}$$

and

$$
\begin{cases}
-\sigma'\Delta u + H(x, Du) + \lambda(t) = F(x, m(t)) \quad \text{in } (0,T)\times Q \\[2em]
\partial_t m - \sigma\Delta m - \operatorname{div}(mH_p(x, Du)) = 0 \quad \text{in } (0,T)\times Q \\[2em]
m(0) = m_0 \geqslant 0 \quad \text{in } Q, \quad \int_Q m_0 = 1, \quad \int_Q u = 0,
\end{cases}
\tag{2.3b}
$$

The first equations in (2.3b) and (2.3a) give the "evolution" of the game value function of an "atomic" player, and express the adaptation of players choices to the environment evolution. The evolution of μ and m expresses the actual evolution of the population density. We refer to Section 3 for more detailed explanations and the derivation of these systems from N-Player games. We shall see that for any time t, $(\lambda(t), u(t))$ (resp. $\nu(t)$) characterizes *a local Nash equilibrium* related to a long time average cost (2.2b) (resp. a discounted cost (2.2a)). Note that the long time averaging with respect to the fast scale 's' provides a "stationary" structure for the first equation in (2.3a)-(2.3b), which depends on the slow time scale 't' because of the process of self-correction. Because of this particular structure we shall say that the MFG systems (2.3a)-(2.3b) are *quasi-stationary*. The main purpose of this chapter is to provide some insight on the behaviour of multi-agent systems with myopic interaction by analyzing the MFG systems (2.3a)-(2.3b).

In contrast to most MFG systems, the uniqueness of solutions to systems (2.3a) and (2.3b) does not require the monotonicity condition (1.4a) nor the convexity of H with respect to the second variable. This fact is essentially related to the forward-forward structure of the systems. We also show that the small-discount approximation (1.9) holds for quasi-stationary models under the same conditions as for the stationary ones. Under the monotonicity condition (1.4a), we prove in Section 4 that for a quadratic Hamiltonian, a solution (λ, u, m) to (2.3b) converges exponentially fast in some sense to the unique equilibrium $(\bar{\lambda}, \bar{u}, \bar{m})$ of (1.6) as $t \to +\infty$, provided that $m_0 - \bar{m}$ is sufficiently small and $\sigma = \sigma'$. An analogous result holds also for systems (1.8)-(2.3a) when the discount rate ρ is small enough. This asymptotic behavior is interpreted by the emergence of a *self-organizing phenomenon* and *a phase transition* in the system. Note that this entails in particular that our systems can exhibit a large scale structure even if the cohesion between the agents is only maintained by interactions between neighbors. The techniques used to prove this asymptotic results rely on some algebraic properties pointed out in [**37**] specific to the quadratic Hamiltonian. On the other hand, one can not use the usual duality arguments to show convergence for general data. Therefore the convergence remains an open problem for more general cases.

Similar asymptotic results were established for the MFG system in [**37**, **38**] for local and nonlocal coupling. Long time convergence of forward-forward MFG models is also discussed in [**1**, **64**]. Self-Organizing and phase transition in Mean Field Games were addressed in [**97**–**99**], for applications in neuroscience, biology, economics, and engineering. For an overview on collective motions and self-organization phenomena in mean field models, we refer to [**58**] and the references therein. The derivation of the Mean Field

Games system was addressed in [59, 86, 87] for the ergodic case (long time average cost). More general cases were analyzed in the important recent paper [33] on the master equation and its application to the convergence problem in Mean Field Games. The reader will notice in Section 3 that the analysis of the mean-field limit in our case is very similar to that of the McKean-Vlasov equation. Therefore the proof of convergence is less technical than in [33] and is based on the usual coupling arguments (see e.g. [96, 100, 107], among others). MFG models with myopic players are briefly addressed in [1] for applications to urban settlements and residential choice. However, the sense given to "myopic players" is different from the one we are considering in this work: indeed, "myopic players" in [1] corresponds to individuals which compute their cost functional taking only into account their very close neighbours, while in this manuscript "myopic players" refers to individuals which anticipate nothing and only undergo the evolution of their environment. In [53], the authors introduce a model for the study of crowds dynamics, that is very similar to the one addressed in this chapter: in Section 2.2.2, the authors consider a situation where at any time pedestrians build the optimal path to destination, based on the observed state of the system. Although the approaches are different, the two models have many similarities.

Local Nash equilibria for mean field systems of rational agents were also considered in [55–57]. The authors use the "Best Reply Strategy approach" to derive a kinetic equation and provide applications to the evolution of wealth distribution in a conservative [56] and non-conservative [57] economy. The link between Mean Field Games and the "Best Reply Strategy approach" is analyzed in [54].

This chapter is organized as follows: In Section 2, we give sufficient conditions for the existence and uniqueness of classical solutions for systems (2.3a)-(2.3b). The proofs rely on continuous dependence estimates for Hamilton-Jacobi-Bellman equations [93], the small-discount approximation, and the non-local coupling which provides compactness and regularity. Section 3 is devoted to a detailed derivation of systems (2.3a) and (2.3b) from N-Player stochastic differential games models. In Section 4, we prove that a solution (λ, u, m) to (2.3b) converges exponentially fast in some sense to the unique solution $(\bar{\lambda}, \bar{u}, \bar{m})$ of (1.6) as $t \to +\infty$. We prove this result under the monotonicity condition (1.4a), for a quadratic Hamiltonian, when $m_0 - \bar{m}$ is sufficiently small and $\sigma = \sigma'$. We also show that an analogous result holds for systems (1.8)-(2.3a) when the discount rate ρ is small enough. We conclude this chapter by carrying out several numerical experiments. We provide a suitable numerical scheme inspired by [3] to simulate the long time behavior of solutions to system (1.18) for various examples.

Throughout all this chapter, $\gamma \in (0, 1)$ is a fixed parameter.

2. Analysis of the Quasi-Stationary MFG Systems

This section is devoted to the analysis of systems (2.3a) and (2.3b). A detailed derivation of these systems from a N-Player differential game will be given in Section 3.

We shall use the following conditions:

($\mathcal{H}1$) the operator $m \to F(.;m)$ is defined from $\mathcal{P}(Q)$ into $\mathrm{Lip}(Q) := \mathcal{C}^{0+1}(Q)$, and satisfies

$$\sup_{m\in\mathcal{P}(Q)} \|F(.;m)\|_{\mathrm{Lip}} < \infty; \tag{2.4}$$

($\mathcal{H}2$) the Hamiltonian $H : Q \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is locally Lipschitz continuous, and $\mathbb{Z}^d$-periodic with respect to the first variable;

($\mathcal{H}3$) H_p exists and is locally Lipschitz continuous;

($\mathcal{H}4$) $D_x H_p$ and H_{pp} exist and are locally Lipschitz continuous;

($\mathcal{H}5$) there exists a constant $\kappa_F > 0$ such that, for any $m, m' \in \mathcal{P}(Q)$,

$$\|F(.;m) - F(.;m')\|_\infty \leqslant \kappa_F\, \mathbf{d}_1(m, m');$$

($\mathcal{H}6$) m_0 is a probability measure, absolutely continuous with respect to the Lebesgue measure, and its density m_0 belongs to $\mathcal{C}^{2+\gamma}(Q)$.

The Hamiltonian H satisfies one of the following sets of conditions:

$\mathfrak{C}1.$ H grows at most linearly in p, i.e., there exists $\kappa_H > 0$, such that

$$|H(x,p)| \leqslant \kappa_H\,(1+|p|)\,, \quad \forall x \in Q,\ \forall p \in \mathbb{R}^d;$$

$\mathfrak{C}2.$ H is superlinear in p uniformly in x, i.e.,

$$\inf_{x\in Q} |H(x,p)|/|p| \to +\infty \quad \text{as } |p| \to +\infty,$$

and there exists $\theta \in (0,1)$, $\kappa > 0$, such that a.e $x \in Q$ and $|p|$ large enough,

$$\langle D_x H, p\rangle + \theta . H^2 \geqslant -\kappa |p|^2. \tag{2.5}$$

Condition ($\mathfrak{C}1.$) arises naturally in control theory when the controls are chosen in a bounded set, whereas under condition ($\mathfrak{C}2.$) the control variable of each player can take any orientation in states space and can be arbitrary large with a large cost. As it is pointed out in [**12**, **86**, **87**], the condition (2.5) is interpreted as a condition on the oscillations of H and plays no role when $d = 1$.

A triplet (λ, u, m) is a classical solution to (2.3b), if $m : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$ is continuous, of class $\mathcal{C}^2$ in space, and of class $\mathcal{C}^1$ in time, $u : (0,T)\times\mathbb{R}^d \longrightarrow \mathbb{R}$ is of class $\mathcal{C}^2$ in space, and (λ, u, m) satisfies (2.3b) in the classical sense. Similarly, a couple (ν, μ) is a classical solution to (2.3a), if $\mu : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$ is continuous, of class $\mathcal{C}^2$ in space, of class $\mathcal{C}^1$ in time, $\nu : (0,T)\times\mathbb{R}^d \longrightarrow \mathbb{R}$ is of class $\mathcal{C}^2$ in space and (ν, μ) satisfies (2.3a) in the classical sense.

In this section, we give an existence and uniqueness result of classical solutions for system (2.3a) and (2.3b) under condition ($\mathfrak{C}1.$). In addition, we show that system (2.3b) is also well-posed under condition ($\mathfrak{C}2.$).

We start by dealing with the case where the Hamiltonian has a linear growth (condition ($\mathfrak{C}1.$)). Let us consider the *quasi-stationary approximate problem* (2.3a). We start by analyzing the first equation in (2.3a).

LEMMA 2.1. *Under assumptions ($\mathcal{H}1$), ($\mathcal{H}2$) and ($\mathfrak{C}1.$), for any $\mu \in \mathcal{P}(Q)$ and $\rho > 0$, the problem*

$$-\sigma'\Delta\nu + H(x, D\nu) + \rho\nu = F(x;\mu) \quad \textit{in } Q \tag{2.6}$$

has a unique solution $\nu_\rho[\mu] \in \mathcal{C}^{2+\gamma}(Q)$. *In addition, there exists constants* $\kappa_0 > 0$ *and* $\theta \in (0, 1]$, *such that for any* $\mu \in \mathcal{P}(Q)$ *and* $\rho > 0$, *the following estimates hold*

$$\|\rho\nu_\rho[\mu]\|_\infty \leqslant \|F\|_\infty + \kappa_H, \tag{2.7a}$$

$$\|\nu_\rho[\mu] - \langle \nu_\rho[\mu] \rangle \|_{\mathcal{C}^{2+\theta}} \leqslant \kappa_0. \tag{2.7b}$$

PROOF. The proof of existence and uniqueness for equation (2.6) relies on regularity results and a priori estimates from elliptic theory. A detailed proof to this result is given in [**12**, Theorem 2.6] in a more general framework. By looking at the extrema of $\nu[\mu]$, one easily gets (2.7a). The second bound is proved by contradiction using the strong maximum principle. The details of the proof are given in [**12**, Theorem 2.5]. Condition (2.4) ensures that the constant κ_0 does not depend on μ. □

REMARK 2.2. Note that the well-posedness of equation (2.6) still holds under the following condition on H (the so-called *natural growth condition*),

$$\exists \kappa'_H > 0, \qquad |H(x, p)| \leqslant \kappa'_H \left(1 + |p|^2\right), \quad \forall x \in Q, \forall p \in \mathbb{R}^d;$$

which is less restrictive than ($\mathfrak{C}1.$).

We now state a continuous dependence estimate due to Marchi [**93**], which plays a crucial role.

LEMMA 2.3. *Assume* ($\mathcal{H}1$)-($\mathcal{H}3$), *and* ($\mathfrak{C}1.$). *For any* $\mu, \mu' \in \mathcal{P}(Q)$ *and* $\rho > 0$, *we have that*

$$\|\nu_\rho[\mu] - \nu_\rho[\mu']\|_\infty \leqslant \rho^{-1}\|F(.; \mu) - F(.; \mu')\|_\infty. \tag{2.8a}$$

Moreover, for any $M > 0$, *there exists a constant* $\chi_M > 0$, *such that for any* $\rho \in (0, M)$ *and* $\mu, \mu' \in \mathcal{P}(Q)$, *the following holds*

$$\|w_\rho[\mu] - w_\rho[\mu']\|_{C^2} \leqslant \chi_M\|F(.; \mu) - F(.; \mu')\|_\infty, \tag{2.8b}$$

where $w_\rho = \nu_\rho - \langle \nu_\rho \rangle$.

PROOF. Note that

$$\nu^\pm := \nu_\rho[\mu'] \pm \rho^{-1}\|F(.; \mu) - F(.; \mu')\|_\infty,$$

are respectively a super- and a sub-solution to equation (2.6) with the coupling term $F(.; \mu)$. Thus, estimate (2.8a) follows thanks to the comparison principle.

The proof of (2.8b) is similar to [**93**, Theorem 2.2]. Nevertheless we give a proof to this result because in this particular framework we do not need to fulfill all the conditions of [**93**]. We shall proceed by contradiction assuming that there exists sequences (ρ_n) in $(0, M)$, $(\mu_n), (\mu'_n) \in \mathcal{P}(Q)$, such that for any $n \geqslant 0$,

$$c_n \geqslant n\|F(.; \mu_n) - F(.; \mu'_n)\|_\infty, \tag{2.9}$$

where $c_n := \|w_{\rho_n}[\mu_n] - w_{\rho_n}[\mu'_n]\|_{\mathcal{C}^2}$, and $\lim_n \rho_n = 0$. Note that the function

$$W_n := c_n^{-1}\left(w_{\rho_n}[\mu_n] - w_{\rho_n}[\mu'_n]\right)$$

satisfies the following equation

$$R_n - \sigma'\Delta W_n + f_n.DW_n = 0,$$

where

$$R_n := \rho_n W_n + c_n^{-1}\rho_n \left\langle \nu_{\rho_n}[\mu_n] - \nu_{\rho_n}[\mu_n'] \right\rangle + c_n^{-1}\left(F(.;\mu_n) - F(.;\mu_n')\right),$$

and

$$f_n(x) := \int_0^1 H_p\left(x, sDw_{\rho_n}[\mu_n] + (1-s)Dw_{\rho_n}[\mu_n']\right)\, ds.$$

Using (2.9) and (2.8a), one checks that $\lim_n \|R_n\|_\infty = 0$. In addition, $(\mathcal{H}3)$ and (2.7b) entails that $\|f_n\|_{\mathrm{Lip}}$ is uniformly bounded. Moreover, invoking standard regularity theory for linear elliptic equations (see e.g. [**63**]), the sequence (W_n) is uniformly bounded in $\mathcal{C}^{2+\theta'}(Q)$ for some $\theta' \in (0,1]$. We infer that (f_n, W_n) converge uniformly to some (f, W) in $\mathcal{C}(Q) \times \mathcal{C}^2(Q)$ which satisfies

$$-\sigma' \Delta W + f.DW = 0, \quad \|W\|_{\mathcal{C}^2} = 1, \quad \int_Q W = 0.$$

Since W is periodic, we deduce from the strong maximum principle that W must be constant; this provides the desired contradiction. □

COROLLARY 2.4. *Under $(\mathcal{H}5)$ and assumptions of Lemma 2.3, for any $\rho > 0$ there exists a constant $\kappa_\rho > 0$ such that,*

$$\|\nu_\rho[\mu] - \nu_\rho[\mu']\|_{\mathcal{C}^2} \leqslant \kappa_\rho\, \mathbf{d}_1(\mu, \mu') \tag{2.10a}$$

for any $\mu, \mu' \in \mathcal{P}(Q)$.

We shall give now an existence and uniqueness result for system (2.3a).

THEOREM 2.5. *Under conditions $(\mathcal{H}1)$-$(\mathcal{H}6)$ and $(\mathfrak{C}1.)$, there exists a unique classical solution (ν, μ) in $\mathcal{C}^{1/2}\left([0,T]; \mathcal{C}^2(Q)\right) \times \mathcal{C}^{1,2}(\overline{Q_T})$ to the problem (2.3a).*

PROOF. *Existence :* For a constant $\delta > 0$ large enough to be chosen below, let $\mathbb{X}_\delta$ be the set of maps $\mu \in \mathcal{C}([0,T]; \mathcal{P}(Q))$ such that

$$\sup_{s \neq t} \frac{\mathbf{d}_1(\mu(t), \mu(s))}{|t-s|^{1/2}} < \delta.$$

Note that $\mathbb{X}_\delta$ is compact thanks to Ascoli's Theorem, and the compactness of $(\mathcal{P}(Q), \mathbf{d}_1)$. We aim to prove our claim using Schauder's fixed point theorem (see e.g. [**105**, p. 25]). Set for any $(x, \nu) \in Q \times \mathcal{P}(Q)$,

$$\Psi(x;\nu) := H_p(x, D\nu_\rho[\nu](x)).$$

Note that Ψ and $D_x\Psi$ are uniformly bounded thanks to $(\mathcal{H}3)$, $(\mathcal{H}4)$, and the uniform bound (2.7b). We define an operator

$$\mathfrak{T} : \mathbb{X}_\delta \to \mathbb{X}_\delta,$$

such that, for a given $\nu \in \mathbb{X}_\delta$, $\mathfrak{T}\nu := \mu$ is the solution to the following "McKean-Vlasov" equation

$$\partial_t \mu - \sigma\Delta\mu - \mathrm{div}(\mu\Psi(x;\nu(t))) = 0, \qquad \mu(0) = m_0. \tag{2.11}$$

Let us check that $\mathfrak{T}$ is well defined. Note that the above equation can be written as

$$\partial_t \mu - \sigma\Delta\mu - \langle D\mu, \Psi(x;\nu(t))\rangle - \mu\mathrm{div}\left(\Psi(x;\nu(t))\right) = 0.$$

Using assumption ($\mathcal{H}3$) and estimate (2.10a), we have for any $t \neq s$ and $x \in Q$

$$|\Psi(x;\nu(t)) - \Psi(x;\nu(s))| \leqslant C_{\rho,H_p}\,\mathbf{d}_1(\nu(t),\nu(s)),$$

so that

$$\sup_{x\in Q}\sup_{t\neq s}\frac{|\Psi(x;\nu(t)) - \Psi(x;\nu(s))|}{|t-s|^{1/2}} \leqslant C_{\delta,\rho,H_p} < \infty.$$

In the same way, one checks that functions $(x,t) \to \Psi(x;\nu(t))$ and $(x,t) \to \operatorname{div}[\Psi(x;\nu(t))]$ are in $\mathcal{C}^{\gamma'/2,\gamma'}(\overline{Q_T})$, where $\gamma' = \min(\gamma,\theta)$, thanks to Lemma 2.1 and ($\mathcal{H}4$). Here γ and θ are the Hölder exponents appearing in ($\mathcal{H}6$) and (2.7b) respectively. We infer that problem (2.11) has a unique solution $\mu \in \mathcal{C}^{1+\gamma'/2,2+\gamma'}(\overline{Q_T})$ which satisfies

$$\|\mu\|_{\mathcal{C}^{1+\gamma'/2,2+\gamma'}} \leqslant C_{\|\Psi\|_{\mathcal{C}^1}}\|m_0\|_{\mathcal{C}^{2+\gamma'}}, \tag{2.12}$$

owing to existence and uniqueness theory for parabolic equations in Hölder spaces [85, Theorem IV.5.1 p. 320]. Furthermore, using classical properties of Fokker-Planck equation (see Lemma A.1), it follows that

$$\mathbf{d}_1(\mu(t),\mu(s)) \leqslant C_T\,(1+\|\Psi\|_\infty)\,|t-s|^{1/2}.$$

Therefore $\mu \in \mathbb{X}_\delta$ for big enough δ, since $\|\Psi\|_\infty$ and C_T does not dependent on ν nor on δ. In particular, the operator $\mathfrak{T}$ is well defined form $\mathbb{X}_\delta$ into $\mathbb{X}_\delta \cap \mathcal{C}^{1+\gamma'/2,2+\gamma'}(\overline{Q_T})$.

Let us check now that $\mathfrak{T}$ is continuous. Given a sequence $\nu_n \to \nu$ in $\mathbb{X}_\delta$, let

$$\mu_n := \mathfrak{T}\nu_n, \quad \text{and} \quad \mu := \mathfrak{T}\nu.$$

Invoking Ascoli's Theorem, estimate (2.12) and the uniqueness of the solution to (2.11), it holds that

$$\lim_n \|\mu_n - \mu\|_{\mathcal{C}^{1,2}} = 0.$$

The convergence is then easily proved to be in $\mathcal{C}([0,T],\mathcal{P}(Q))$. Thus, by Schauder fixed point theorem the map $\mathfrak{T} : \mathbb{X}_\delta \to \mathbb{X}_\delta$ has a fixed point $\mu \in \mathcal{C}^{1,2}(\overline{Q_T})$ and $(\nu_\rho[\mu],\mu)$ is a classical solution to (2.3a). In addition, estimate (2.10a) entails that $\nu_\rho[\mu] \in \mathcal{C}^{1/2}\left([0,T];\mathcal{C}^2(Q)\right)$.

Uniqueness : Let (ν,μ) and (ν',μ') be two solutions to the system (2.3a), $w := \nu - \nu'$ and $\nu := \mu - \mu'$. One checks that

$$\begin{cases} \partial_t \nu - \sigma\Delta\nu - \operatorname{div}(\nu H_p(x,D\nu)) = \operatorname{div}\left(\mu'\left(H_p(x,D\nu) - H_p(x,D\nu')\right)\right), \\ \nu_{t=0} = 0. \end{cases}$$

By standard duality techniques, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\nu(t)\|_2^2 + \sigma\|D\nu(t)\|_2^2 = -\left(\int_Q \nu D\nu.H_p(x,D\nu) + \int_Q \mu' D\nu.\left(H_p(x,D\nu) - H_p(x,D\nu')\right)\right),$$

so that

$$\frac{1}{2}\frac{d}{dt}\|\nu(t)\|_2^2 + \frac{\sigma}{2}\|D\nu(t)\|_2^2 \leqslant C\left(\|G\|_\infty\|\nu(t)\|_2^2 + \int_Q |\mu' Dw|^2\right). \tag{2.13}$$

From (2.10a) and (2.12), we infer that

$$\int_Q |\mu' Dw|^2 \leqslant C\|m_0\|^2_{\mathcal{C}^{2+\gamma}}\,\mathbf{d}_1(\mu,\mu')^2 \leqslant C\|m_0\|^2_{\mathcal{C}^{2+\gamma}}\|\nu(t)\|_2^2.$$

Plugging this into (2.13) provides

$$\|\nu(t)\|_2^2 \leqslant C \int_0^t \|\nu(s)\|_2^2 \, ds,$$

which implies that $\nu \equiv 0$, and so $w \equiv 0$ thanks to (2.10a). The proof is complete. $\square$

Let us now deal with system (2.3b). We shall start by proving the well-posedness for the first equation in (2.3b) and by giving a continuous dependence estimate.

LEMMA 2.6. *Under conditions (H1)-(H3), (H5) and ($\mathfrak{C}$1.), for any measure $m \in \mathcal{P}(Q)$, there exists a unique solution $(\lambda[m], u[m]) \in \mathbb{R} \times \mathcal{C}^2(Q)$ to the problem*

$$-\sigma'\Delta u + H(x, Du) + \lambda = F(x; m) \quad \text{in } Q, \quad < u >= 0. \tag{2.14}$$

Moreover, for any $m, m' \in \mathcal{P}(Q)$, the following estimates hold

$$\left|\lambda[m] - \lambda[m']\right| \leqslant \kappa_F \, \mathbf{d}_1(m, m'), \tag{2.15a}$$

$$\left\|u[m] - u[m']\right\|_{\mathcal{C}^2} \leqslant \chi_1 \kappa_F \, \mathbf{d}_1(m, m'). \tag{2.15b}$$

PROOF. It is well known (see e.g. [6, 10]) that for a given $m \in \mathcal{P}(Q)$, there exists a unique periodic solution $(\lambda[m], u[m])$ in $\mathbb{R} \times \mathcal{C}(Q)$ to (2.14). Regularity of the solution, and estimates (2.15a), (2.15b) follow from Lemma 2.3, and small-discount approximation techniques (see e.g. [6, 10, 12]). $\square$

REMARK 2.7. It is possible to show more regularity for the maps $m \to \lambda[m]$, $m \to u[m]$ under additional regularity assumptions on F and H. For instance, if $H_{pp} \geqslant \kappa_e I_d$ for some $\kappa_e > 0$, and F satisfies

$$\sup_{m \neq m'} \mathbf{d}_1(m, m')^{-1} \left\| \frac{\delta F}{\delta m}(., m, .) - \frac{\delta F}{\delta m}(., m', .) \right\|_{\mathcal{C}^0 \times \mathcal{C}^1} < \infty,$$

then $u[.]$ and $\lambda[.]$ are of class $\mathcal{C}^1$ in $\mathcal{P}(Q)$. We refer to [33] for the definition of derivatives in $\mathcal{P}(Q)$ and notations. In addition, we have that

$$\frac{\delta u}{\delta m}(m)(\nu) := w(m, \nu) \quad \text{and} \quad \frac{\delta \lambda}{\delta m}(m)(\nu) := \delta(m, \nu)$$

for any m in $\mathcal{P}(Q)$ and any signed measure ν on Q, where (δ, w) is the solution to the following problem

$$-\sigma'\Delta w + H_p\left(x, Du[m]\right).Dw + \delta = \frac{\delta F}{\delta m}(m)(\nu) \quad \text{in } Q, \text{ and } \int_Q w = 0.$$

One has also an analogous result for the map $\nu_\rho[.]$ defined in Lemma 2.1. We omit the details and invoke [33, Proposition 3.8] for a similar approach.

We prove now well-posedness for system (2.3b).

THEOREM 2.8. *Under assumptions (H1)-(H6) and ($\mathfrak{C}$1.), there exists a unique classical solution (λ, u, m) in $\mathcal{C}^{1/2}([0,T]) \times \mathcal{C}^{1/2}\left([0,T]; \mathcal{C}^2(Q)\right) \times \mathcal{C}^{1,2}\left(\overline{Q_T}\right)$ to system (2.3b). This result holds if one replaces condition ($\mathfrak{C}$1.) by condition ($\mathfrak{C}$2.).*

PROOF. The proof of existence relies on small-discount approximation techniques. We give here an adaptation of these techniques for the quasi-stationary case. The crucial point in this proof is estimates (2.7a) and (2.7b).

Assume first that H satisfies condition ($\mathfrak{C}1.$). Let (ν^ρ, μ^ρ) be the unique classical solution to (2.3a), and set $w^\rho := \nu^\rho - \langle \nu^\rho \rangle$. Invoking (2.7a) and (2.7b), we have

$$\begin{cases} -\sigma' \Delta w^\rho + H(x, Dw^\rho) + \rho w^\rho = F(x; \mu^\rho(t)) - \rho < \nu^\rho > \quad \text{in } Q_T, \\ \\ \partial_t \mu^\rho - \sigma \Delta \mu^\rho - \operatorname{div}\left(H_p(x, D\nu^\rho)\mu^\rho\right) = 0 \quad \text{in } Q_T, \quad \mu^\rho(0) = m_0 \quad \text{in } Q, \\ \\ \sup_{0 \leqslant t \leqslant T} \|w^\rho(t)\|_{C^{2+\theta}} \leqslant \kappa_0, \qquad \sup_{0 \leqslant t \leqslant T} \|\rho \nu^\rho[\mu(t)]\|_\infty \leqslant \|F\|_\infty + \kappa_H. \end{cases} \tag{2.16}$$

On the other hand, recall that according to [85, Theorem IV.5.1 p. 320] it holds that

$$\|\mu^\rho\|_{\mathcal{C}^{1+\gamma'/2, 2+\gamma'}} \leqslant C_1 \|m_0\|_{\mathcal{C}^{2+\gamma'}}, \tag{2.17}$$

where $\gamma' = \min(\gamma, \theta)$, and the constant $C_1 > 0$ is independent of ρ thanks to (2.7b). Hence, one can extract a subsequence $\rho_n \to 0$ such that for any $t \in [0, T]$

$$(\rho_n \langle \nu^{\rho_n}(t) \rangle, w^{\rho_n}(t), \mu^{\rho_n}) \to (\lambda(t), u(t), m) \quad \text{in } \mathbb{R} \times \mathcal{C}^2(Q) \times \mathcal{C}^{1,2}(\overline{Q_T}) \quad \text{as } n \to \infty, \tag{2.18}$$

where (λ, u, m) is a classical solution to (2.3b). In addition, for any $t, s \in [0, T]$, estimates (2.15a) and (2.15b) provide

$$\|u[m(t)] - u[m(s)]\|_{C^2} \leqslant \chi_1 \kappa_F \, \mathbf{d}_1(m(t), m(s)),$$

and

$$|\lambda[m(t)] - \lambda[m(s)]| \leqslant \kappa_F \, \mathbf{d}_1(m(t), m(s)).$$

Thus, $u[m] \in \mathcal{C}^{1/2}\left([0, T]; \mathcal{C}^2(Q)\right)$ and $\lambda[m] \in \mathcal{C}^{1/2}\left([0, T]\right)$. The proof of uniqueness is identical to Theorem 2.5. Hence, the proof of well-posedness under ($\mathfrak{C}1.$) is complete.

If we suppose that H satisfies only ($\mathfrak{C}2.$), then by virtue of (2.4) one can derive the following uniform bound using Bernstein's method (see [86, 87] and [12, Theorem 2.1]):

$$\exists \kappa_B > 0, \ \forall \nu \in \mathcal{P}(Q)^N, \qquad \|Du[\nu]\|_\infty \leqslant \kappa_B. \tag{2.19}$$

Thus, by a suitable truncation of H one reduces the problem to the previous case. $\square$

REMARK 2.9. All the results of this section hold true if one replaces the elliptic parts of the equations with a more general operator $\mathfrak{L}$ of the following form:

$$\mathfrak{L} := -\operatorname{Tr}\left(\psi(x) D^2\right),$$

where ψ is $\mathbb{Z}^d$-periodic, $\|\psi\|_{\text{Lip}} < \infty$, and there exists $\kappa_\psi > 0$ such that $\psi(x) \geqslant \kappa_\psi I_d$.

3. N-Player Games & Mean Field Limit

We provide in this section a rigorous interpretation for the quasi-stationary systems (2.3a) and (2.3b) in terms of N-Player stochastic differential games. We shall start by writing systems of equations for N players, then we pass to the limit when the number of players goes to infinity assuming that all the players are identical. Throughout this section, we employ the notations introduced in Lemma 2.1 and Lemma 2.6.

3.1. The N-Player Stochastic Differential Game Model. We consider a game of N-players where at each time agents choose their strategy

- assuming no evolution in their environment;
- according to an evaluation of their future situation emanating from the choice.

Observing the evolution of the system, players adjust their strategies without anticipating. More precisely, each player observe the state of the system at time t and chooses the best drift vector field $\alpha_t(.)$ which optimize her/his future evolution $(s > t)$. The player adapts and corrects her/his choice as the system evolves. This situation amounts to resolving at each moment an optimization problem which consists in finding the vector field (strategy) which guarantees the best future cost. Our agents are myopic: they anticipate no evolution and only undergo changes in their environment.

Let us now give a mathematical formalism to our model. Let $(W^j)_{1\leqslant j\leqslant N}$ be a family of N independent Brownian motions in $\mathbb{R}^d$ over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(A^i)_{1\leqslant i\leqslant N}$ be closed subsets of $\mathbb{R}^d$. We suppose that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to fulfill the assumptions that will be formulated in this section. Let $V := (V^1, ..., V^N)$ be a vector of i.i.d random variables with values in $\mathbb{R}^d$ that are independent of $(W^j)_{1\leqslant j\leqslant N}$ and let

$$\mathcal{F}_t := \sigma\left\{V^j, W^j_u,\ \ 1\leqslant j\leqslant N, \quad u\leqslant t\right\}$$

be the information available to the players at time t. We suppose that $\mathcal{F}_t$ contains the $\mathbb{P}$-negligible sets of $\mathcal{F}$.

Consider a system driven by the following stochastic differential equations

$$dX^i_t = \alpha^i_t(X^i_t)\,dt + \sqrt{2\sigma_i}\,dW^i_t, \quad X^i_0 = V^i, \quad i = 1, ..., N. \tag{2.20}$$

For any $t \geqslant 0$, the i-th player choses α^i_t in the set of *admissible strategies* denoted by $\mathbb{A}^i$, that is, the set of $\mathbb{Z}^d$-periodic processes α^i defined on Ω, indexed by $\mathbb{R}^d$ with values in A^i, such that

$$\sup_{\omega\in\Omega} \|\alpha^i(\omega, .)\|_{\mathrm{Lip}} < \infty, \tag{2.21}$$

and $(\alpha^i_t)_{t\in[0,T]}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. The reason of considering condition (2.21) will be clear in (2.23) below. At each time $t \geqslant 0$, player i faces an optimization problem for choosing $\alpha^i_t(.) \in \mathbb{A}^i$ which insures the best future cost.

These instant choices give rise to a global (in time) strategies $(\alpha^1_t, ..., \alpha^N_t)_{t\geqslant 0}$ which does not necessarily guarantee the well-posedness of equations (2.20) in a suitable sense. Hence we need to introduce the following definitions:

DEFINITION 2.10. Let $T > 0$ and $i = 1, ..., N$. We say that the i-th equation of (2.20) is *well-posed* on $[0, T]$, if there exists a process X^i, unique a.s, with continuous sample paths on $[0, T]$ which satisfies the following properties:

(i) $(X^i)_{t\in[0,T]}$ is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted;
(ii) $\mathbb{P}\left[X^i_0 = V^i\right] = 1$;
(iii) $\mathbb{P}\left[\int_0^t \left|\alpha^i_s(X^i_s)\right|\,ds < \infty\right] = 1\ \ \forall t \in [0, T]$;

(iv) for any $t \in [0,T]$, the following holds

$$X_t^i = V^i + \int_0^t \alpha_s^i(X_s^i)\, ds + \sqrt{2\sigma_i} W_t^i \quad \text{a.s}\,.$$

System (2.20) is well-posed if all equations are.

DEFINITION 2.11. Let $T > 0$ and $i = 1, ..., N$. We say that the global strategy $(\alpha_t^i)_{t \geqslant 0}$ is *feasible* on $[0,T]$, if the i-th equation of (2.20) is well-posed on $[0,T]$.

Note that in contrast to standard optimal control situations, the optimal global strategy is not a solution to a global (in time) optimization problem, but it is the history of all the choices made during the game. The agents plan and correct their plans as the game evolves, and the global strategy is achieved through this process of planning and self-correction.

The case of a long time average cost. Consider the case where the i-th player seeks to minimize the following long time average cost:

$$J_\infty^i\left(t, V, \alpha_t^1, ..., \alpha_t^N\right) := \liminf_{\tau \to +\infty} \frac{1}{\tau} \mathbb{E}\left[\int_t^\tau L^i(\mathcal{X}_{s,t}^i, \alpha_t^i(\mathcal{X}_{s,t}^i)) + F^i(\mathcal{X}_{s,t}^i; X_t^{-i})\, ds \,\middle|\, \mathcal{F}_t\right], \tag{2.22}$$

where $L^i : \mathbb{R}^d \times A^i \to \mathbb{R}$ and $F^i : \mathbb{R}^d \times \mathbb{R}^{d(N-1)} \to \mathbb{R}$ are continuous and $\mathbb{Z}^d$- periodic with respect to the first variable. At any time $t \geqslant 0$, the process $(\mathcal{X}_{s,t}^i)_{s>t}$ represents the possible future trajectory of player i, related to the chosen strategy (vector field) $\alpha_t^i \in \mathbb{A}^i$. In other words, $(\mathcal{X}_{s,t}^i)_{s>t}$ is what is likely to happen (in the future $s > t$) if player i plays α_t^i at the instant t. Mathematically, we consider that $(\mathcal{X}_{s,t}^i)_{s>t}$ are driven by the following (fictitious) stochastic differential equations

$$\begin{cases} d\mathcal{X}_{s,t}^i = \alpha_t^i(\mathcal{X}_{s,t}^i)\, ds + \sqrt{2\sigma_i'}\, d\mathcal{B}_{s-t,t}^i & s > t, \\ \mathcal{X}_{t,t}^i = X_t^i, \quad i = 1, ..., N, \end{cases} \tag{2.23}$$

where $\{(\mathcal{B}_{\cdot,t}^i)_{1 \leqslant i \leqslant N}\}_{t \geqslant 0}$ is a family of standard Brownian motions, and for any $t \geqslant 0$, the process $(\mathcal{B}_{s-t,t}^i)_{s>t}$ represents the noise related to the future prediction (or guess) of the i-th player. For simplicity, we assume that for any $i \in \{1, ..., N\}$, $t \geqslant 0$, and $s > t$,

$$\mathcal{B}_{s-t,t}^i \ \text{ is independent from } \ \mathcal{F}_t. \tag{2.24}$$

Observe that system (2.23) is well-posed in the strong sense, and that the definition of $(\mathcal{X}_{s,t}^i)_{s>t}$ introduces a fast (instantaneous) scale 's' related to the projection in future, which is different from the real (slow) scale 't'.

The cost functional (2.22) is an evaluation of the future cost of player i, given the information available at time t. The cost structure expresses the fact that agents are myopic: they anticipate no future change and act as if the system will remain immutable. As they adjust, they undergo changes and do not anticipate them.

We now give a definition of Nash equilibrium for our game.

DEFINITION 2.12. We say that a vector of global strategies $(\hat{\alpha}_t^1, ..., \hat{\alpha}_t^N)_{t \geqslant 0}$ is a *Nash equilibrium* of the N-person game on $[0,T]$, for the initial position $V = (V_1, ..., V_N)$, if for any $i = 1, ..., N$,

$$(\hat{\alpha}_t^i)_{t \geqslant 0} \ \text{ is feasible on } [0,T],$$

and

$$\hat{\alpha}_t^i = \arg\max_{\alpha^i \in \mathbb{A}^i} J_\infty^i \left(t, V, \hat{\alpha}_t^1, ..., \hat{\alpha}_t^{i-1}, \alpha^i, \hat{\alpha}_t^{i+1}, ..., \hat{\alpha}_t^N\right) \quad \text{a.s} \quad \forall t \in [0,T].$$

In other words, a Nash equilibrium on $[0,T]$ is the history of local Nash equilibria, which is feasible on [0,T].

Next we provide a verification result that produces a Nash equilibrium for the N-Player game associated to the cost functional (2.22). Let us introduce the following notation for empirical measures:

$$\hat{\nu}_Y^M := \frac{1}{M}\sum_{i=1}^{M} \delta_{Y_i}, \qquad \forall Y = (Y_i) \in \mathbb{R}^{Md}.$$

For any $i = 1, ..., N$, we suppose that F^i depends only on $x \in Q$ and on the empirical density of the other variables. Namely, for any $x \in Q$ and $Y = (Y^1, ..., Y^{N-1}) \in \mathbb{R}^{d(N-1)}$,

$$F^i(x; Y) := F^i\left(x;\ \hat{\nu}_Y^{N-1}\right).$$

Set for $(x,p) \in Q \times \mathbb{R}^d$,

$$H^i(x,p) := \sup_{\alpha \in A^i} \left\{-p.\alpha - L^i(x,\alpha)\right\}.$$

Throughout this section, we assume that assumptions of Theorem 2.8 hold for H^i and F^i, and that the supremum is achieved at a unique point $\bar{\alpha}^i$ in the definition of H^i, for all (x,p), so that

$$H_p^i(x,p) = -\bar{\alpha}^i(x,p) := \arg\max_{\alpha \in A^i} \left\{-p.\alpha - L^i(x,\alpha)\right\}. \tag{2.25}$$

We also employ the notations introduced in Lemma 2.6: namely, for any $\pi \in \mathcal{P}(Q)$, we denote by $(\lambda^i[\pi], u^i[\pi])$ the unique solution to

$$-\sigma_i' \Delta u^i + H^i(x, Du^i) + \lambda^i = F^i(x;\pi) \quad \text{in } Q, \quad < u >= 0.$$

REMARK 2.13. It is possible to consider a more general form for the drift in system (2.23). For instance, one can replace α by the following (more general) affine form:

$$f^i(x,\alpha) := g^i(x) + G^i(x)\alpha,$$

where $G^i \in \mathrm{Lip}(Q)^{d\times d}$ and $g^i \in \mathrm{Lip}(Q)^d$. Then

$$H^i(x,p) = -p.g^i + \sup_{\alpha \in A^i} \left\{-p.G^i(x)\alpha - L^i(x,\alpha)\right\}.$$

If L^i is Lipschitz in x, uniformly as α varies in any bounded subset, and asymptotically super-linear, i.e.

$$\lim_{|\alpha| \to +\infty} \inf_{x \in Q} L^i(x,\alpha)/|\alpha| = +\infty,$$

then the supremum in the definition of H^i is attained. Uniqueness of the supremum holds if L^i is strictly convex with respect to the second variable.

The following result characterizes a Nash equilibrium on $[0,T]$ associated to the cost functional (2.22).

PROPOSITION 2.14.

(1) *The following system of equations is well-posed on* $[0,T]$,

$$d\bar{X}^i_t = \bar{\alpha}^i\left(\bar{X}^i_t, Du^i\left[\hat{\nu}^{N-1}_{\bar{X}^{-i}_t}\right](\bar{X}^i_t)\right)dt + \sqrt{2\sigma_i}\,dW^i_t, \quad \bar{X}^i_0 = V^i, \quad i = 1, ..., N. \tag{2.26}$$

(2) *Let for* $x \in Q$ *and* $t \in [0,T]$

$$\bar{\alpha}^i_t(x) := \bar{\alpha}^i\left(x, Du^i\left[\hat{\nu}^{N-1}_{\bar{X}^{-i}_t}\right](x)\right), \quad i = 1, ..., N.$$

The vector $(\bar{\alpha}^1_t, ..., \bar{\alpha}^N_t)_{t\geqslant 0}$ *defines a Nash equilibrium on* $[0,T]$ *for any initial data.*

(3) *The following holds*

$$\lambda^i\left[\hat{\nu}^{N-1}_{\bar{X}^{-i}_t}\right] = \liminf_{\tau\to+\infty}\frac{1}{\tau}\mathbb{E}\left[\int_t^\tau L^i(\bar{\mathcal{X}}^i_{s,t}, \bar{\alpha}^i_t(\bar{\mathcal{X}}^i_{s,t})) + F^i\left(\bar{\mathcal{X}}^i_{s,t}; \hat{\nu}^{N-1}_{\bar{X}^{-i}_t}\right) ds \,\middle|\, \mathcal{F}_t\right],$$

where $(\bar{\mathcal{X}}^i_{s,t})_{s>t}$ *are obtained by solving*

$$\begin{cases} d\bar{\mathcal{X}}^i_{s,t} = \bar{\alpha}^i_t(\bar{\mathcal{X}}^i_{s,t})\,ds + \sqrt{2\sigma'_i}\,d\mathcal{B}^i_{s-t,t}, & s > t,\\ \bar{\mathcal{X}}^i_{t,t} = \bar{X}^i_t, \quad i = 1, ..., N. \end{cases}$$

PROOF. Assertion (1) is a consequence of the regularity results of Lemma 2.6, while assertions (2) and (3) follows by standard verification arguments (see e.g. [12, Theorem 3.4], among many others). For any $\tau > t \geqslant 0$ and $i \in \{1, .., N\}$ one has

$$u^i(\bar{\mathcal{X}}^i_{\tau,t}) = u^i(\bar{\mathcal{X}}^i_{t,t}) + \int_t^\tau Du^i(\bar{\mathcal{X}}^i_{s,t}).\bar{\alpha}^i_t(\bar{\mathcal{X}}^i_{s,t})\,ds + \int_t^\tau \sigma'_i\Delta u^i(\bar{\mathcal{X}}^i_{s,t})\,ds + \sqrt{2\sigma'_i}\int_t^\tau Du^i(\bar{\mathcal{X}}^i_{s,t})\,d\mathcal{B}^i_{s-t,t},$$

where here $u^i \equiv u^i\left[\hat{\nu}^{N-1}_{\bar{X}^{-i}_t}\right]$ in order to simplify the presentation. Owing to (2.25) one gets

$$\begin{aligned} u^i(\bar{\mathcal{X}}^i_{\tau,t}) &= u^i(\bar{\mathcal{X}}^i_{t,t}) + \int_t^\tau \left(-H^i(\bar{\mathcal{X}}^i_{s,t}, Du^i(\bar{\mathcal{X}}^i_{s,t})) + \sigma'_i\Delta u^i(\bar{\mathcal{X}}^i_{s,t})\right) ds \\ &\quad + \sqrt{2\sigma'_i}\int_t^\tau Du^i(\bar{\mathcal{X}}^i_{s,t})\,d\mathcal{B}^i_{s-t,t} - \int_t^\tau L^i(\bar{\mathcal{X}}^i_{s,t}, \bar{\alpha}^i_t(\bar{\mathcal{X}}^i_{s,t}))\,ds \\ &= u^i(\bar{\mathcal{X}}^i_{t,t}) - \int_t^\tau \left\{L^i(\bar{\mathcal{X}}^i_{s,t}, \bar{\alpha}^i_t(\bar{\mathcal{X}}^i_{s,t})) + F^i(\bar{\mathcal{X}}^i_{s,t}; \bar{X}^{-i}_t)\right\} ds \\ &\quad + (\tau - t)\lambda_i + \sqrt{2\sigma'_i}\int_t^\tau Du^i(\bar{\mathcal{X}}^i_{s,t})\,d\mathcal{B}^i_{s-t,t}. \end{aligned}$$

Hence, from (2.24) we infer that

$$\begin{aligned} \tau^{-1}\mathbb{E}\left[u^i(\bar{\mathcal{X}}^i_{\tau,t}) \mid \mathcal{F}_t\right] &= (1 - t\tau^{-1})\lambda_i + \tau^{-1}\mathbb{E}\left[u^i(\bar{\mathcal{X}}^i_{t,t}) \mid \mathcal{F}_t\right] \\ &\quad - \tau^{-1}\mathbb{E}\left[\int_t^\tau L^i(\bar{\mathcal{X}}^i_{s,t}, \bar{\alpha}^i_t(\bar{\mathcal{X}}^i_{s,t})) + F^i\left(\bar{\mathcal{X}}^i_{s,t}; \hat{\nu}^{N-1}_{\bar{X}^{-i}_t}\right) ds \,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

Note that estimate (2.15b) provides a uniform bound on $u^i[.]$. Thus, by taking the limit in the last expression one gets

$$\lambda_i\left[\hat{\nu}^{N-1}_{\bar{X}^{-i}_t}\right] = \liminf_{\tau\to+\infty}\frac{1}{\tau}\mathbb{E}\left[\left[\int_t^\tau L^i(\bar{\mathcal{X}}^i_{s,t},\bar{\alpha}^i_t(\bar{\mathcal{X}}^i_{s,t})) + F^i\left(\bar{\mathcal{X}}^i_{s,t};\hat{\nu}^{N-1}_{\bar{X}^{-i}_t}\right)\, ds \,\Big|\, \mathcal{F}_t\right]\right].$$

On the other hand, one easily checks that $(\bar{\alpha}^1_t,\dots,\bar{\alpha}^N_t)_{t\geqslant 0}$ is a Nash equilibrium for any initial data $V=(V^1,\dots,V^N)$ owing to (2.25). □

REMARK 2.15. Note that the problem structure decouples the "fictitious" dynamics (2.23), and allows to compute the controls.

The case of a discounted cost functional. Set $\rho^1,\dots,\rho^N>0$. We consider now the case where the i-th player seeks to minimize the following discounted cost functional:

$$J^i_{\rho_i}\left(t,V,\bar{\alpha}^1_t,\dots,\bar{\alpha}^N_t\right) := \mathbb{E}\left[\left[\int_t^\infty e^{-\rho^i s}L^i(\mathcal{X}^i_{s,t},\alpha^i_t(\mathcal{X}^i_{s,t})) + F^i\left(\mathcal{X}^i_{s,t};\hat{\nu}^{N-1}_{X^{-i}_t}\right)\, ds \,\Big|\, \mathcal{F}_t\right]\right], \tag{2.27}$$

where all the functions are defined in the same way as in the previous case, with analogous notations and assumptions. One checks that a similar result to Proposition 2.14 holds, i.e. that the following problem:

$$d\bar{Z}^i_t = \bar{\alpha}^i\left(\bar{Z}^i_t, Dv^i_\rho\left[\hat{\nu}^{N-1}_{\bar{Z}^{-i}_t}\right](\bar{Z}^i_t)\right)dt + \sqrt{2\sigma_i}\,dW^i_t,\quad \bar{Z}^i_0 = V^i,\quad i=1,\dots,N, \tag{2.28}$$

characterizes a Nash equilibrium on $[0,T]$ associated to the cost functional (2.27).

3.2. The Large Population Limit. We address now the convergence problem when the number of players goes to infinity, assuming that all the players are indistinguishable.

Assume that: $A^i=A$; $\rho_i=\rho$; $\sigma_i=\sigma$; $\sigma'_i=\sigma'$; $F^i=F$; $H^i=H$; and $\bar{\alpha}^i=\bar{\alpha}$ so that $L^i=H^*$ for all $1\leqslant i\leqslant N$, where H^* is the *Legendre transform* of H with respect to the p variable. We suppose also that

$$\mathcal{L}(V^i) = m_0 \in C^{2+\gamma}(Q)\ \text{ for any }\ i=1,\dots,N.$$

For simplicity we shall use the notations $X_t := (X^1_t,\dots,X^N_t)$ and $Z_t := (Z^1_t,\dots,Z^N_t)$ instead of $\bar{X}_t := (\bar{X}^1_t,\dots,\bar{X}^N_t)$ and $\bar{Z}_t := (\bar{Z}^1_t,\dots,\bar{Z}^N_t)$. Under the above assumptions, systems (2.26) and (2.28) are rewritten respectively on the following form:

$$\begin{cases} dX^i_t = -H_p\left(X^i_t, Du\left[\hat{\nu}^{N-1}_{X^{-i}_t}\right](X^i_t)\right)dt + \sqrt{2\sigma}\,dW^i_t, & 0\leqslant t\leqslant T,\\ X^i_0 = V^i,\quad i=1,\dots,N; \end{cases} \tag{2.29}$$

and

$$\begin{cases} dZ^i_t = -H_p\left(Z^i_t, Dv_\rho\left[\hat{\nu}^{N-1}_{Z^{-i}_t}\right](Z^i_t)\right)dt + \sqrt{2\sigma}\,dW^i_t, & 0\leqslant t\leqslant T,\\ Z^i_0 = V^i,\quad i=1,\dots,N. \end{cases} \tag{2.30}$$

Our main result in this section says that at the mean field limit $N\to\infty$, one recovers the quasi-stationary systems (2.3b) and (2.3a), which respectively correspond to (2.29) and (2.30). Note that systems (2.3b) and (2.3a) can be rewritten on the form of Mckean Vlasov equations:

$$\begin{cases} \partial_t m - \sigma\Delta m - \operatorname{div}\left(m H_p\left(x, Du[m(t)]\right)\right) = 0 \quad \text{in } Q_T, \\ m(0) = m_0 \quad \text{in } Q, \end{cases} \tag{2.3b}$$

and

$$\begin{cases} \partial_t \mu - \sigma\Delta \mu - \operatorname{div}\left(\mu H_p\left(x, D\nu_\rho[\mu(t)]\right)\right) = 0 \quad \text{in } Q_T, \\ \mu(0) = m_0 \quad \text{in } Q. \end{cases} \tag{2.3a}$$

Thus, one can use the usual coupling arguments (see e.g. [**96**, **100**, **107**]) to deduce the convergence. The main theorem of this section is the following:

THEOREM 2.16. *For any* $t \in [0, T]$, *it holds that:*

$$\lim_N \max_{1\leqslant i\leqslant N} \mathbf{d}_1\left(\mathcal{L}\left(X_t^i\right), m(t)\right) = 0;$$
$$\lim_N \max_{1\leqslant i\leqslant N} \mathbf{d}_1\left(\mathcal{L}\left(Z_t^i\right), \mu(t)\right) = 0;$$
$$\lim_N \left\| u[m(t)] - \mathbb{E}u\left[\hat{\nu}^N_{X_t}\right]\right\|_\infty = 0;$$
$$\lim_N \left|\lambda[m(t)] - \mathbb{E}\lambda\left[\hat{\nu}^N_{X_t}\right]\right| = 0; \quad \textit{and}$$
$$\lim_N \left\| \nu_\rho[\mu(t)] - \mathbb{E}\nu_\rho\left[\hat{\nu}^N_{Z_t}\right]\right\|_\infty = 0.$$

The analysis of the limit transition $N \to +\infty$ is essentially based on continuous dependence estimates, and therefore the mean field analysis is identical for both systems. Thus, we shall give the details only for system (2.29).

Let us introduce the following artificial systems:

$$\begin{cases} dY^i = -H_p\left(Y_t^i, Du\left[m(t)\right](Y_t^i)\right)\, dt + \sqrt{2\sigma}\, dW_t^i, \quad 0 \leqslant t \leqslant T, \\ \\ Y_0^i = V^i, \quad i = 1, ..., N; \end{cases} \tag{2.31}$$

and

$$\begin{cases} d\tilde{X}^i = -H_p\left(\tilde{X}_t^i, Du\left[\hat{\nu}^N_{X_t}\right](\tilde{X}_t^i)\right)\, dt + \sqrt{2\sigma}\, dW_t^i, \quad 0 \leqslant t \leqslant T, \\ \\ \tilde{X}_0^i = V^i, \quad i = 1, ..., N. \end{cases} \tag{2.32}$$

Observe that systems (2.31)-(2.32) are well-posed, and that the uniqueness of the solution to (2.3b) provides that

$$\mathcal{L}\left(Y_t^1, ..., Y_t^N\right) = \otimes_{i=1}^N m(t).$$

On the other hand, note that

$$\mathcal{L}\left(\tilde{X}_t^{\xi(1)}, ..., \tilde{X}_t^{\xi(N)}\right) = \mathcal{L}\left(\tilde{X}_t^1, ..., \tilde{X}_t^N\right) \tag{2.33}$$

is fulfilled for any permutation ξ, and any $t \in [0, T]$. In addition, one checks that

$$\max_{1\leqslant i\leqslant N} \sup_{0\leqslant t\leqslant T} \mathbb{E}\left|X_t^i - \tilde{X}_t^i\right| \leqslant \frac{C_T}{N-1} \tag{2.34}$$

holds thanks to the continuous dependence estimate (2.15b), since

$$\sup_{0\leqslant t\leqslant T}\max_{1\leqslant i\leqslant N} \mathbf{d}_1\left(\hat{\nu}^N_{X_t}, \hat{\nu}^{N-1}_{X_t^{-i}}\right) \leqslant \frac{C}{N-1}.$$

Next we compare the trajectories of (2.31) and (2.32), and show that they are increasingly close on $[0,T]$ when $N \to +\infty$.

PROPOSITION 2.17. *Under assumptions of this section, it holds that*

$$\max_{1\leqslant i\leqslant N}\sup_{0\leqslant t\leqslant T} \mathbb{E}\left|\tilde{X}^i_t - Y^i_t\right| \leqslant C_T N^{-1/(d+8)}.$$

PROOF. For any $i \in \{1,\dots,N\}$ and $t \in [0,T]$, one has

$$\begin{aligned}\frac{d}{dt}\left[\tilde{X}^i_t - Y^i_t\right] &= H_p\left(Y^i_t, Du[m(t)](Y^i_t)\right) - H_p\left(\tilde{X}^i_t, Du\left[\hat{\nu}^N_{X_t}\right](\tilde{X}^i_t)\right)\\ &= \mathcal{T}_1(t) + \mathcal{T}_2(t) + \mathcal{T}_3(t) + \mathcal{T}_4(t),\end{aligned}$$

where

$$\begin{aligned}\mathcal{T}_1(t) &:= H_p(Y^i_t, Du[m(t)](Y^i_t)) - H_p(Y^i_t, Du\left[\hat{\nu}^N_{Y_t}\right](Y^i_t)),\\ \mathcal{T}_2(t) &:= H_p(Y^i_t, Du\left[\hat{\nu}^N_{Y_t}\right](Y^i_t)) - H_p(Y^i_t, Du\left[\hat{\nu}^N_{X_t}\right](Y^i_t)),\\ \mathcal{T}_3(t) &:= H_p(Y^i_t, Du\left[\hat{\nu}^N_{X_t}\right](Y^i_t)) - H_p(\tilde{X}^i_t, Du\left[\hat{\nu}^N_{X_t}\right](Y^i_t)),\end{aligned}$$

and

$$\mathcal{T}_4(t) := H_p(\tilde{X}^i_t, Du\left[\hat{\nu}^N_{X_t}\right](Y^i_t)) - H_p(\tilde{X}^i_t, Du\left[\hat{\nu}^N_{X_t}\right](\tilde{X}^i_t)).$$

Using the continuous dependence estimate (2.15b) one gets

$$|\mathcal{T}_2(t)| \leqslant \frac{C}{N}\sum_{j=1}^N |\tilde{X}^j_t - Y^j_t|, \quad \text{and} \quad |\mathcal{T}_1(t)| \leqslant C\,\mathbf{d}_1(m(t), \hat{\nu}^N_{Y_t}).$$

On the other hand, the following holds

$$|\mathcal{T}_3(t) + \mathcal{T}_4(t)| \leqslant C|\tilde{X}^i_t - Y^i_t|.$$

The key step is the estimation the non-local term $\mathbb{E}\,\mathbf{d}_1(m, \hat{\nu}^N_Y)$; we use the following estimate due to Horowitz and Karandikar (see [**103**, Theorem 10.2.7]):

$$\mathbb{E}\,\mathbf{d}_1(m(t), \hat{\nu}^N_{Y_t}) \leqslant \kappa_d N^{-1/(d+8)} \quad \forall t \in [0,T],$$

where the constant $\kappa_d > 0$ depends only on d. Using the symmetry of the joint probability law (2.33) and the last estimate, we infer that

$$\mathbb{E}\left|\tilde{X}^i_t - Y^i_t\right| \leqslant C\int_0^t \left(\frac{1}{N^{1/(d+8)}} + \mathbb{E}\left|\tilde{X}^i_s - Y^i_s\right|\right) ds,$$

which concludes the proof. □

Recall the following definition and characterizations of chaotic measures [**107**].

DEFINITION 2.18. Let π^N be a symmetric joint probability measure on Q^N and π in $\mathcal{P}(Q)$. We say that π^N is π-chaotic if for any $k \geqslant 1$ and any continuous functions $\phi_1, \dots, \phi_k$ on Q one has

$$\lim_N \int \prod_{l=1}^k \phi_l \, d\pi^N = \prod_{l=1}^k \int \phi_l \, d\pi.$$

LEMMA 2.19. *Let $\mathbb{X}_N$ be a sequence of random variables on Q^N whose the joint probability law π^N is symmetric, and $\pi \in \mathcal{P}(Q)$. Then the following assertions are equivalent:*

(i) *π^N is π-chaotic;*
(ii) *the empirical measure $\hat{\nu}^N_{\mathbb{X}_N}$ converges in law toward the deterministic measure π;*
(iii) *for any continuous function ϕ on Q, it holds that*

$$\lim_N \mathbb{E}\left|\int \phi \, d\left(\hat{\nu}^N_{\mathbb{X}_N} - \pi\right)\right| = 0.$$

Combining Lemma 2.19 and Proposition 2.17, we deduce *the propagation of chaos* for system (2.32).

PROPOSITION 2.20. *For any $t \in [0,T]$, if $m^N(t)$ is the joint probability law of $\tilde{X}_t := (\tilde{X}^j_t)_{1\leqslant j\leqslant N}$, then $m^N(t)$ is $m(t)$-chaotic.*

PROOF. Let ϕ be a Lipschitz continuous function on Q. From Proposition 2.17, we have that

$$\mathbb{E}\left|\int \phi \, d(\hat{\nu}^N_{\tilde{X}_t} - \hat{\nu}^N_{Y_t})\right| \leqslant \frac{\|\phi\|_{\mathrm{Lip}}}{N} \sum_{k=1}^{N} \mathbb{E}\left|\tilde{X}^i_t - Y^i_t\right| \leqslant \frac{\|\phi\|_{\mathrm{Lip}} C_T}{N^{1/(d+8)}}.$$

Invoking the fact that $\mathcal{L}(Y_t) = \otimes_{i=1}^N m(t)$ and Lemma 2.19, it holds that

$$\lim_N \mathbb{E}\left|\int \phi \, d(m(t) - \hat{\nu}^N_{Y_t})\right| = 0.$$

The claimed result follows from

$$\mathbb{E}\left|\int \phi \, d(\hat{\nu}^N_{\tilde{X}_t} - m(t))\right| \leqslant \mathbb{E}\left|\int \phi \, d(\hat{\nu}^N_{\tilde{X}_t} - \hat{\nu}^N_{Y_t})\right| + \mathbb{E}\left|\int \phi \, d(m(t) - \hat{\nu}^N_{Y_t})\right|.$$

□

We are now in position to prove Theorem 2.16.

PROOF OF THEOREM 2.16. Observe that for any two random variables $\mathbb{X}$, $\mathbb{Y}$, one has

$$\mathbf{d}_1\left(\mathcal{L}(\mathbb{X}), \mathcal{L}(\mathbb{Y})\right) \leqslant \mathbb{E}\left|\mathbb{X} - \mathbb{Y}\right|.$$

Hence, combining Proposition 2.17 and estimate (2.34) we obtain that

$$\lim_N \max_{1\leqslant i\leqslant N} \mathbf{d}_1\left(\mathcal{L}\left(X^i_t\right), m(t)\right) = 0.$$

On the other hand, we have

$$\lim_N \left|\lambda[m(t)] - \mathbb{E}\lambda\left[\hat{\nu}^N_{\tilde{X}_t}\right]\right| = 0 \quad \text{and} \quad \lim_N \left\|u[m(t)] - \mathbb{E}u\left[\hat{\nu}^N_{\tilde{X}_t}\right]\right\|_\infty = 0,$$

thanks to Proposition 2.20 and Lemma 2.19. In fact, pointwise convergence is a consequence of assertion (ii) in Lemma 2.19, and the convergence is actually uniform since $u\left[\mathcal{P}(Q)\right]$ is compact in $\mathcal{C}(Q)$. We conclude the proof for (λ, u, m) by invoking (2.34) and the continuous dependence estimates (2.15a)-(2.15b). The results for (ν, μ) follows using similar steps as for (λ, u, m). □

REMARK 2.21. Note that the two main arguments in the proof of Theorem 2.16 are the continuous dependence estimate, and symmetry with respect to states of the other players.

4. Exponential Convergence to the Ergodic MFG Equilibrium

We prove in this section the exponential convergence of the quasi-stationnary system (2.3b) to the ergodic equilibrium assuming that $\sigma' = \sigma$ and $H(x,p) = |p|^2/2$. The proofs rely on algebraic properties of the equations, the continuous dependence estimates (Lemma 2.14), and the monotonicity condition (1.4a). Throughout this section we suppose that assumptions $(\mathcal{H}1)$,$(\mathcal{H}5)$, and $(\mathcal{H}6)$ are fulfilled. In addition, we assume that the coupling F satisfies the monotonicity condition:

$$\forall m, m' \in \mathcal{P}(Q), \qquad \int_Q \big(F(x;m) - F(x;m')\big)\, d(m-m')(x) \geqslant 0. \tag{1.4a}$$

For the sake of simplicity we set $\sigma = \sigma' = 1$.

In this framework the quasi-stationary MFG system (2.3b) takes the following form,

$$\begin{cases} \lambda(t) - \Delta u + \dfrac{1}{2}|Du|^2 = F(x; m(t)) \quad \text{in } (0,\infty)\times Q, \\ \\ \partial_t m - \Delta m - \operatorname{div}(mDu) = 0 \quad \text{in } (0,\infty)\times Q, \\ \\ m(0) = m_0 \quad \text{in } Q, \quad \displaystyle\int_Q u = 0 \quad \text{in } (0,\infty). \end{cases} \tag{2.35}$$

System (2.35) has a unique global (in time) classical solution thanks to Theorem 2.8. Consider the following ergodic Mean Field Games problem:

$$\begin{cases} \bar\lambda - \Delta \bar u + \dfrac{1}{2}|D\bar u|^2 = F(x; \bar m) \quad \text{in } Q, \\ \\ -\Delta \bar m - \operatorname{div}(\bar m D\bar u) = 0 \quad \text{in } Q, \\ \\ \bar m \geqslant 0 \quad \text{in } Q, \quad \displaystyle\int_Q \bar m = 1, \quad \int_Q \bar u = 0. \end{cases} \tag{2.36}$$

Under the monotonicity condition (1.4a), uniqueness holds for system (2.36). In all this section $(\bar\lambda, \bar u, \bar m)$ denotes the unique solution to (2.36). Observe that $\bar m \equiv e^{-\bar u}/\int_Q e^{-\bar u}$, so that the following holds

$$1/\bar\kappa \leqslant \bar m \leqslant \bar\kappa \tag{2.37}$$

for some constant $\bar\kappa > 0$.

The main result of this section is the following:

THEOREM 2.22. *There exists* $R_0 > 0$ *such that if*

$$\|m_0 - \bar m\|_2 \leqslant R_0,$$

then the following holds for some constants $K, \delta > 0$*:*

$$|\lambda(t) - \bar\lambda| + \|u(t) - \bar u\|_{\mathcal{C}^2} + \|m(t) - \bar m\|_2 \leqslant K e^{-\delta t} \quad \textit{for any } t \geqslant 0.$$

This convergence result reveals that our decision-making mechanism lead to the emergence of a Mean Field Games equilibrium, under the conditions mentioned above. This can also be interpreted as a phase transition from a non-equilibrium state to an equilibrium state (see also [97, 99]). Agents reach this equilibrium by adjusting and self-correcting. We believe that this convergence result holds true in more general cases. For instance, one can show that an analogous convergence result holds for system (2.3a) when the discount rate ρ is small enough (c.f. Remark 2.25).

Let (λ, u, m) be the solution to (2.35), and set

$$\delta := \lambda - \bar{\lambda}, \quad w := u - \bar{u}, \quad \text{and } \pi := m - \bar{m}. \tag{2.38}$$

The triplet (σ, w, π) is a solution to the following system of equations:

$$\begin{cases} \delta(t) - \Delta w + \langle D\bar{u}, Dw \rangle + \dfrac{1}{2}|Dw|^2 = F(x; \bar{m} + \pi(t)) - F(x; \bar{m}) & \text{in } (0, \infty) \times Q, \\ \partial_t \pi - \Delta \pi - \operatorname{div}(\pi D\bar{u}) - \operatorname{div}(\bar{m} Dw) - \operatorname{div}(\pi Dw) = 0 & \text{in } (0, \infty) \times Q, \\ \pi(0) = m_0 - \bar{m} \quad \text{in } Q, \quad \displaystyle\int_Q w = 0 & \text{in } (0, \infty). \end{cases} \tag{2.39}$$

The following preliminary Lemma states the dependence of w and σ on π in the first equation of (2.39).

LEMMA 2.23. *Let ϖ be a probability measure on Q which is absolutely continuous with respect to the Lebesgue measure, and such that*

$$\varpi = \bar{m} + \pi,$$

where $\pi \in L^2(Q)$. Then there exists a unique periodic solution $(\delta[\pi], w[\pi])$ in $\mathbb{R} \times \mathcal{C}^2(Q)$ to the following problem:

$$\begin{cases} \delta - \Delta w + \langle D\bar{u}, Dw \rangle + \dfrac{1}{2}|Dw|^2 = F(x; \varpi) - F(x; \bar{m}) & \text{in } (0, \infty) \times Q \\ \displaystyle\int_Q w = 0 & \text{in } (0, \infty). \end{cases} \tag{2.40}$$

Moreover, the following estimates hold

$$|\delta[\pi]| \leqslant C\|\pi\|_2, \tag{2.41a}$$

$$\|w[\pi]\|_{\mathcal{C}^2} \leqslant C'\|\pi\|_2. \tag{2.41b}$$

PROOF. Existence and uniqueness of regular solutions to such problems are discussed in Section 2. Estimates (2.41a)-(2.41b) are a direct consequence of the uniqueness and the continuous dependence estimates (2.15a)-(2.15b). □

Next we give the following technical Lemma.

LEMMA 2.24. *There exists a constant $\kappa > 0$ such that*

$$\|\pi/\bar{m}\|_2 \leqslant \kappa\|D(\pi/\bar{m})\|_2,$$

for any $\pi \in V^{1,2}(Q) := \left\{\pi \in W_2^1(Q) \;:\; \int_Q \pi = 0\right\}$.

PROOF. As usual, the result is obtained by contradiction. In fact, if our claim is not satisfied one can find a sequence $(\pi_n) \in V^{1,2}(Q)$ such that for any $n \geqslant 1$,

$$\|\pi_n/\bar{m}\|_2 = 1 \quad \text{and} \quad \frac{1}{n} \geqslant \|D(\pi_n/\bar{m})\|_2. \tag{2.42}$$

By Sobolev embeddings, $(\pi_n/\bar{m})_n$ converges (up to a subsequence) to some $\bar{\pi}$ in $L^2(Q)$. Using (2.42) it follows that $\bar{\pi}$ is constant, i.e. $\bar{\pi} \equiv C$. Moreover,

$$C = \int_Q \bar{m}\bar{\pi} = \lim_n \int_Q \pi_n = 0;$$

this provides the desired contradiction owing to (2.42). $\square$

Combining these elements one can prove the main Theorem of this section.

PROOF OF THEOREM 2.22. Let (δ, w, π) be a smooth solution to (2.39). Recall that

$$D\bar{m} = -\frac{e^{-\bar{u}}D\bar{u}}{\int_Q e^{-\bar{u}}} = -\bar{m}D\bar{u},$$

so that

$$D\left(\frac{\pi}{\bar{m}}\right) = \frac{D\pi + \pi D\bar{u}}{\bar{m}}, \tag{2.43}$$

and

$$\operatorname{div}(\bar{m}Dw) = \bar{m}\Delta w + \langle D\bar{m}, Dw\rangle = -\bar{m}\left(-\Delta w + \langle D\bar{u}, Dw\rangle\right). \tag{2.44}$$

We infer that

$$\operatorname{div}(\bar{m}Dw) = -\bar{m}\left(-\delta(t) - \frac{1}{2}|Dw|^2 + F(x;\bar{m}+\pi(t)) - F(x;\bar{m})\right),$$

which provides in particular

$$\partial_t\pi - \Delta\pi - \operatorname{div}(\pi D\bar{u}) + \bar{m}\left(-\delta(t) - \frac{1}{2}|Dw|^2 + F(x;\bar{m}+\pi(t)) - F(x;\bar{m})\right) - \operatorname{div}(\pi Dw) = 0.$$

Hence, using (2.43), (2.41b) and the monotonicity of F, one has

$$\begin{aligned}
\frac{d}{dt}\int_Q \frac{\pi^2}{2\bar{m}} &= \int_Q \frac{\pi}{\bar{m}}\Big[\Delta\pi + \operatorname{div}(\pi D\bar{u}) + \operatorname{div}(\pi D w) \\
&\qquad -\bar{m}\left(-\delta(t) - \frac{1}{2}|Dw|^2 + F(x;\bar{m}+\pi(t)) - F(x;\bar{m})\right)\Big] \\
&= \int_Q \left\{-\frac{|D\pi+\pi D\bar{u}|^2}{\bar{m}} + \frac{\pi|Dw|^2}{2} - \frac{\pi\langle D\pi, Dw\rangle}{\bar{m}} - \frac{\pi^2\langle D\bar{u}, Dw\rangle}{\bar{m}}\right\} \\
&\qquad -\int_Q \pi\left(F(x;\bar{m}+\pi(t)) - F(x;\bar{m})\right) \\
&\leqslant \int_Q -\bar{m}\left|D\left(\frac{\pi(t)}{\bar{m}}\right)\right|^2 + C\|\pi(t)\|_2^3 + 1/\bar{\kappa}\|D\pi(t)\|_2\|\pi(t)\|_2^2. \\
&\leqslant -1/\bar{\kappa}\int_Q \left|D\left(\frac{\pi(t)}{\bar{m}}\right)\right|^2 + C\left(\|\pi(t)\|_2 + \|D\pi(t)\|_2\right)\|\pi(t)\|_2^2.
\end{aligned} \tag{2.45}$$

Using Lemma 2.24, one easily checks that

$$\|\pi(t)\|_2 + \|D\pi(t)\|_2 \leqslant C\left\|D\left(\frac{\pi(t)}{\bar{m}}\right)\right\|_2. \tag{2.46}$$

Thus the following holds

$$\frac{d}{dt}\|\pi(t)^2/\bar{m}\|_1 \leqslant -1/C_0\|\pi(t)^2/\bar{m}\|_1 + M\|\pi(t)^2/\bar{m}\|_1^2, \tag{2.47}$$

for some $C_0, M > 0$, thanks to (2.37) and Young's inequality. For any $R_0 < \frac{1}{\sqrt{\bar{\kappa}MC_0}}$, the last differential inequality entails that

$$\|\pi(t)^2/\bar{m}\|_1 \leqslant \frac{1/MC_0}{1+\left(\frac{1}{MC_0\|\pi(0)^2/\bar{m}\|_1} - 1\right)e^{t/C_0}} \quad \text{for any } t \geqslant 0.$$

Estimates of Lemma 2.23 conclude the proof. □

REMARK 2.25. One notices that the previous proof can be adapted to show that (2.3a) converges exponentially fast to (1.8) when the discount rate ρ is small enough, under the same assumptions of Theorem 2.22. In fact, setting $\tilde{\pi} := \mu - \bar{\mu}$, the same arguments leading to (2.47) also provide

$$\frac{d}{dt}\|\tilde{\pi}(t)^2/\bar{\mu}\|_1 \leqslant -1/\tilde{C}_0\|\tilde{\pi}(t)^2/\bar{\mu}\|_1 + \tilde{M}\|\tilde{\pi}(t)^2/\bar{\mu}\|_1^2 + C\rho\|\tilde{\pi}(t)^2/\bar{\mu}\|_1.$$

Therefore, the same conclusion holds when ρ is small enough.

REMARK 2.26. In practice, this convergence results can help to understand the emergence of highly-rational equilibria in situations with myopic decision-making mechanisms. For instance in [53, Section 2.2.2] the authors consider a decision-making mechanism for pedestrian dynamics that is very similar to the mechanism described in this chapter. Theorem 2.22 can apply for this kind of models, in the case of a quadratic running cost, and a monotonous coupling function (pedestrians dislike congested areas) which satisfies ($\mathcal{H}1$) and ($\mathcal{H}5$).

5. Numerical Experiments

In the previous section, we saw that, at least under certain conditions, the population of myopic agents self-organizes exponentially fast toward a highly-rational MFG equilibrium. The main purpose of this section is to explore the limits of Theorem 2.22 by using numerical simulation. We address only system (2.3b) since system (2.3a) is approximatively equivalent when ρ is small.

Consider the following example of coupling functions:

$$F(x; m) = A\psi(x) + B\phi(x) \int_Q \phi(y) m(y)\, dy, \tag{2.48}$$

where $A, B > 0$, ψ, ϕ are non-negative smooth functions. One easily checks that F fulfills $(\mathcal{H}1)$,$(\mathcal{H}5)$, and (1.4a). The shape of the functions ψ, ϕ and the relative magnitude of the coefficients A, B characterize the relative preference of the agents. In all this section, we consider a periodic setting, and for simplicity we take $Q = [0, 1]^2$, $0 \leqslant \phi \leqslant 1$ to be a smooth function that vanishes on the boundary of Q, $\Psi = 1-\phi$, and $\sigma = \sigma' = 1$. Namely, we take ϕ to be a non-negative function which is maximal at $(1/2, 1/2)$ and which is supported in Q.

We start by presenting the numerical method of [**3**] to approximate the solution for the ergodic MFG system (1.6). Next, we build an adapted numerical scheme to approximate the solutions of the quasi-stationary MFG system (2.3b), and we carry out several numerical experiments to observe the asymptotic behaviour of the solutions for various examples of H and different values of $\|m_0 - \bar{m}\|_2$.

5.1. Simulation of the Stationary System. We discretize Q into a uniform grid $(x_{i,j})$, $1 \leqslant i, j \leqslant N_h$, with a mesh step $h = N_h^{-1}$, where N_h is a positive integer. The values of $\bar{u}$, $\bar{m}$ at $x_{i,j}$ are approximated by $u_{i,j}$, $m_{i,j}$, and the value of $\bar{\lambda}$ is approximated by λ according to the following scheme:

$$\begin{cases} \lambda - (\Delta_h u)_{i,j} + H_h\left(x_{i,j}, [D_h u]_{i,j}\right) = F_h(x_{i,j}; m), \\[2ex] (\Delta_h m)_{i,j} + T_{i,j}(m, u) = 0, \\[2ex] \displaystyle\sum_{1\leqslant i,j\leqslant N_h} u_{i,j} = 0, \qquad \sum_{1\leqslant i,j\leqslant N_h} m_{i,j} = h^{-2}, \quad 1 \leqslant i, j \leqslant N_h, \end{cases} \tag{2.49}$$

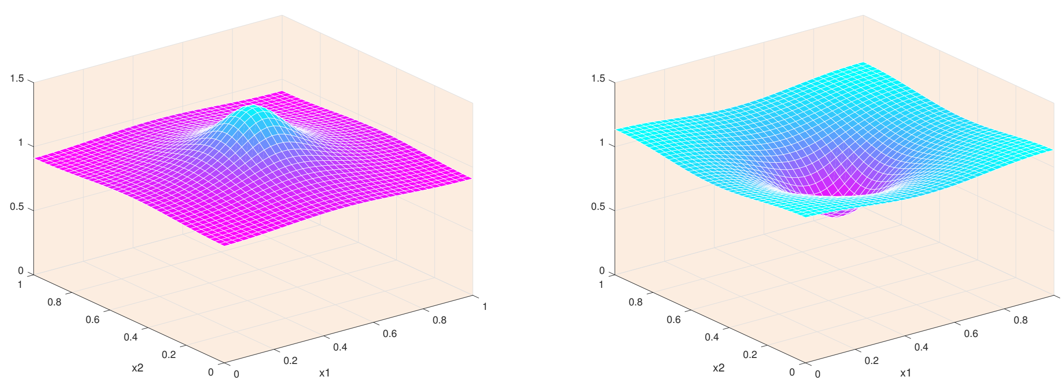
where

$$(\Delta_h w)_{i,j} = -h^{-2}\left(4w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1}\right);$$
$$[D_h w]_{i,j} = \left(D_1 w_{i,j}, D_1 w_{i-1,j}, D_2 w_{i,j}, D_2 w_{i,j-1}\right);$$
$$D_1 w_{i,j} = h^{-1}\left(w_{i+1,j} - w_{i,j}\right), \text{ and } D_2 w_{i,j} = h^{-1}\left(w_{i,j+1} - w_{i,j}\right);$$
$$H_h\left(x, p_1, p_2, p_3, p_4\right) = H\left(x, \sqrt{|p_1^-|^2 + |p_3^-|^2 + |p_2^+|^2 + |p_4^+|^2}\right);$$
$$F_h(x_{i,j}; m) = A(1 - \phi(x_{i,j})) + h^2 B \sum_{1\leqslant k,\ell\leqslant N_h} m_{i,j}\phi(x_{k,\ell})\phi(x_{i,j});$$

and

$$\begin{aligned}T_{i,j}(u,m) = &-m_{i-1,j}\partial_{p_1}H_h\left(x_{i-1,j},[D_hu]_{i-1,j}\right) + m_{i+1,j}\partial_{p_2}H_h\left(x_{i+1,j},[D_hu]_{i+1,j}\right)\\ &- m_{i,j-1}\partial_{p_3}H_h\left(x_{i,j-1},[D_hu]_{i,j-1}\right) + m_{i,j+1}\partial_{p_4}H_h\left(x_{i,j+1},[D_hu]_{i,j+1}\right)\\ &\quad m_{i,j}\left\{\partial_{p_1}H_h\left(x_{i,j},[D_hu]_{i,j}\right) + \partial_{p_3}H_h\left(x_{i,j},[D_hu]_{i,j}\right)\right\}\\ &\qquad - m_{i,j}\left\{\partial_{p_2}H_h\left(x_{i,j},[D_hu]_{i,j}\right) + \partial_{p_4}H_h\left(x_{i,j},[D_hu]_{i,j}\right)\right\}.\end{aligned}$$

We refer the reader to [**3**] for more explanations, and a detailed analysis of the numerical scheme (2.49). In our simulation, we use Newton's fixed point scheme to compute the solution of system (2.49). The results are shown in Figures 1(a) - 1(b).

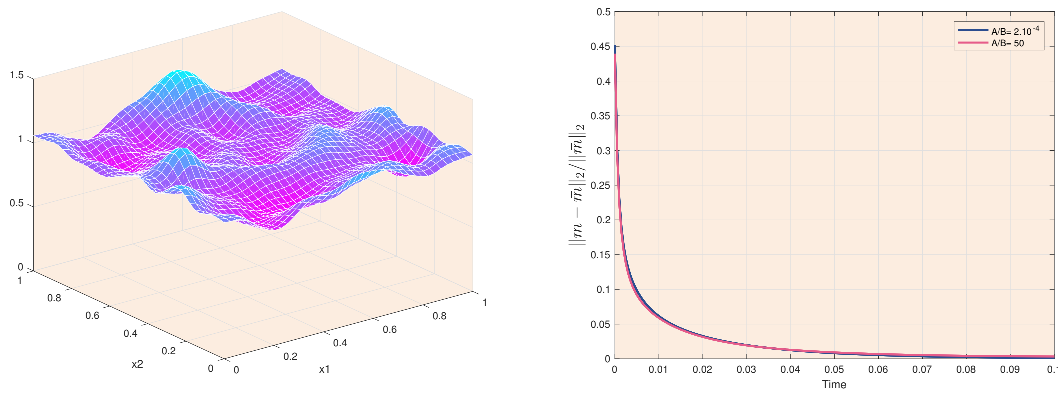


(a) Equilibrium configuration for $A/B = 50$

(b) Equilibrium configuration for $A/B = 2.10^{-4}$

FIGURE 1. Simulated examples of the ergodic mass $\bar{m}$ for $H(x,p) = |p|^2/2$ and different values of A/B.

As expected, the agents are congested around the centre $(1/2, 1/2)$ when $A > B$, whereas they are more distant from the centre when $A << B$. As it is already pointed out, the equilibrium configuration that is computed in Figures 1(a) - 1(b) corresponds to a situation where agents have anticipated the right distribution of other agents and optimized their state accordingly. By virtue of [**36**], we know that such a situation occurs when the population of agents is "experienced", and have learned from past experiences in the same game. Indeed, the authors of [**36**] show under some restrictions, that a given population can learn the MFG equilibria by repeating the same game a sufficient number of times. Our work shows that the population can reach the MFG equilibrium configuration through an alternative process. In fact, Theorem 2.22 shows that even with a "non-experienced" population, the crowd of players self-organizes during the game and reaches exponentially fast the MFG equilibrium by observing the other players states and acting accordingly.



(a) The considered initial distribution m_0 (b) The evolution of $\|m - \bar{m}\|_2$

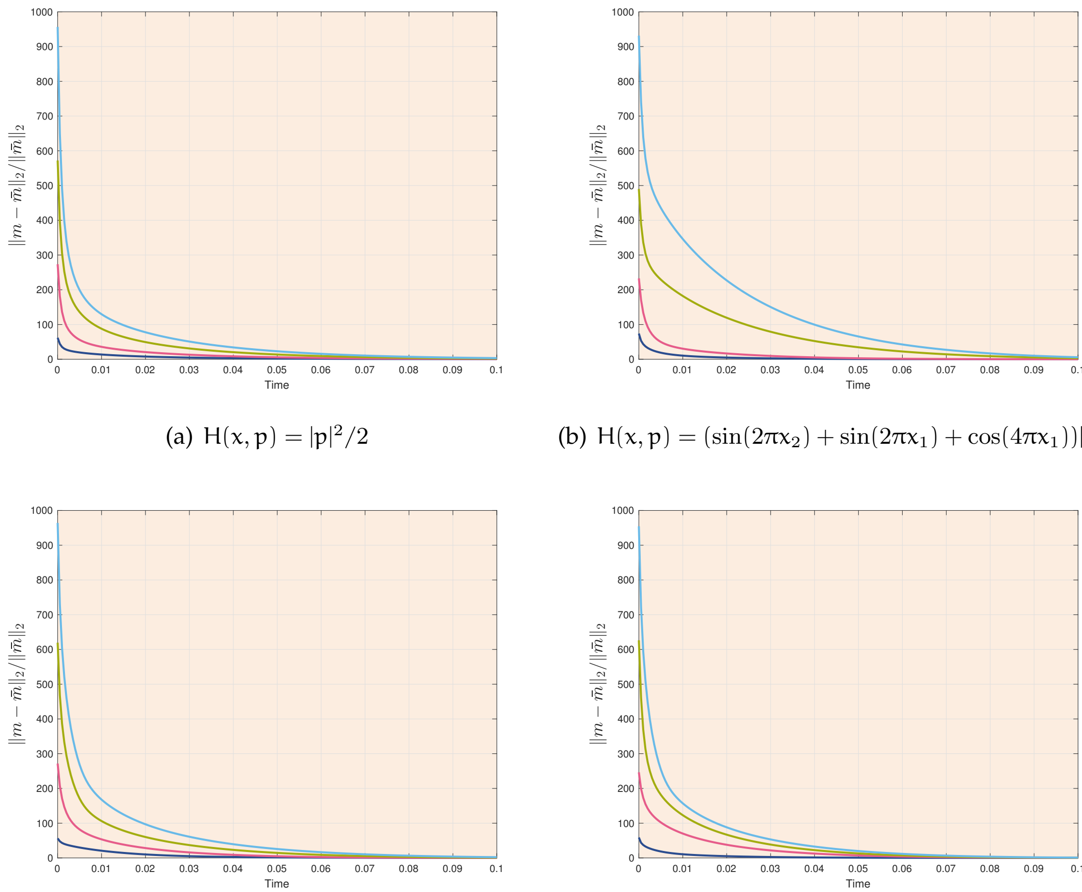
FIGURE 2. An example of m_0 and the simulated evolution of $\|m - \bar{m}\|_2$ for $H(x, p) = |p|^2/2$.

5.2. Simulation of the Quasi-Stationary System. In order to simulate examples of the self-organizing phenomena, we start by computing an approximate solution to the Quasi-Stationary MFG system (2.3b). We discretize Q into a uniform grid $(x_{i,j})_{1\leqslant i,j\leqslant N_h}$ with a the mesh step $h = N_h^{-1}$, and given some $T > 0$ we discretize uniformly $[0, T]$ into N_T+1 distinct points. Here N_T is a positive integer. The values of u, m at $(kT/N_T, x_{i,j})$ are approximated by $u^k_{i,j}$, $m^k_{i,j}$, and the value of λ at kT/N_T is approximated by λ^k according to the following scheme:

$$
\left\{
\begin{aligned}
&\lambda^k - (\Delta_h u^k)_{i,j} + H_h\left(x_{i,j}, \left[D_h u^k\right]_{i,j}\right) = F_h(x_{i,j}; m^k),\\
&m^{k+1}_{i,j} - m^k_{i,j} - T/N_T\left\{(\Delta_h m^{k+1})_{i,j} + T_{i,j}(m^k, u^k)\right\} = 0,\\
&m^0_{i,j} = m_0(x_{i,j}), \qquad \sum_{1\leqslant i,j\leqslant N_h} u^k_{i,j} = 0, \qquad \sum_{1\leqslant i,j\leqslant N_h} m^k_{i,j} = h^{-2},\\
&k = 0, ..., N_T, \quad 1 \leqslant i, j \leqslant N_h,
\end{aligned}
\right.
\tag{2.50}
$$

where the finite difference operators Δ_h, D_h and the discrete functions H_h, F_h, $(T_{i,j})_{i,j}$ are defined as for (2.49). Given m^k, $k = 0, ..., N_T$, we start by computing the root (λ^k, u^k) of the discrete HJB equation by using Newton's method, then m^{k+1} is generated according to the second equation of (2.50). The computation is accelerated by using (λ^k, u^k) as an initial guess in the computation of (λ^{k+1}, u^{k+1}).

In order to test (2.50) we generate a random probability density on the square Q (c.f. Figure 2(a)) and simulate the evolution of system (2.3b), in the case of a quadratic



(a) $H(x, p) = |p|^2/2$

(b) $H(x, p) = (\sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1))|p|^3/3$

(c) $H(x, p) = |p|^3/3$

(d) $H(x, p) = (\sin(2\pi x_2)+\sin(2\pi x_1)+\cos(4\pi x_1))|p|^3/3+(1 + \cos(2\pi x_1))|p|^2/2$

FIGURE 3. Simulated examples of the evolution of $\|m - \bar{m}\|_2$. We choose $A = 2.10^{-1}$ and $B = 10^3$.

Hamiltonian, for a sufficiently large time. Figure 2(b) shows that the system converges exponentially fast toward the ergodic MFG equilibrium. One observes that the speed of convergence does not depend on the choice of the parameter A/B.

Now, we want to explore the limits of Theorem 2.22. Namely, we simulate the long time behavior of the system for various examples of H and large values of $\|m_0-\bar{m}\|_2/\|\bar{m}\|_2$. From a practical standpoint, generating a probability density m_0 so that $\|m_0-\bar{m}\|_2$ is very large is a tedious task. In fact, because of the constraint $m_0 - \bar{m} \geqslant -\bar{m}$, choosing large values for N_h is necessary to obtain the desired large magnitude of $\|m_0 - \bar{m}\|_2$. Therefore, we allow m_0 to take negative values in order to reduce the computation cost. From a theoretical standpoint, this has no impact since the proof of Theorem 2.22 involve the square of $m_0 - \bar{m}$.

Figures 3(a), 3(b), 3(c) and 3(d) show the evolution of $\|m - \bar{m}\|_2$ for various examples of H and different values of $\|m_0 - \bar{m}\|_2$, in the case where $A = 2.10^{-1}$ and $B = 10^3$. The simulation shows that the self-organizing effect still holds even for large values of $\|m_0 - \bar{m}\|_2$ and more general Hamiltonian functions H. Nevertheless, one can note some differences in the speed of convergence, especially when $\|m_0 - \bar{m}\|_2$ is very large.

Part II

Bertrand & Cournot Mean Field Games

CHAPTER 3

A Variational Approach for Bertrand & Cournot Mean-Field Games

Joint work with P. Jameson Graber, published in "PDE Models of Multi-Agent Phenomena" (2018), vol. 28 of the Springer INdAM Series, under the title "Variational mean field games for market competition".

1. Introduction

The main purpose of this introductory section is to provide a general introduction to Bertrand & Cournot Mean Field Games, and to introduce the questions which are addressed in this chapter.

Bertrand & Cournot MFGs is a family of models introduced by Guéant, Lasry, and Lions [73] as well as by Chan and Sircar in [49, 50] to describe a *mean field game* in which producers compete to sell an exhaustible resource such as oil, coal, natural gas, or minerals. Here we view the producers as a continuum of rational agents whose state is given by a density function m in the space of possible reserves, and any individual producer must solve an optimal control problem in order to maximize profit.

Let us explain more precisely the Bertrand & Cournot MFG model. Let t be time, and x be the producer's reserves so that the space of possible states is $Q = \mathbb{R}^+$. Given an initial distribution of reserves $m_0 \in \mathcal{P}(Q)$, the reserves dynamics of a representative producer is given by the following stochastic differential equation:

$$X_t = X_0 - \int_0^t q_s \mathbb{1}_{X_s>0}\, ds + \sqrt{2\sigma}\int_0^t \mathbb{1}_{X_s>0} dW_s, \quad X_0 \sim m_0,$$

where $\sigma > 0$ and $(W_t)_{t\geqslant 0}$ is a standard Brownian motion. Reserves level $(X_t)_{t\geqslant 0}$ decreases at a controlled production rate $(q_t)_{t\geqslant 0}$, and also has random increment which models production uncertainties and/or the fluctuation of market demand [49]. The indicator function $\mathbb{1}_{X_t>0}$ introduces a stopping condition which means that production must be shut down, whenever reserves run out, $X_t = 0$. Given a common horizon $T > 0$, a representative producer who starts with initial reserves level $X_t = x \in \mathbb{R}^+$, at time $t \in [0, T]$, has the following profit functional:

$$\mathcal{J}_{BC}(t, x, \mathbf{m}) := \mathbb{E}\left\{\int_t^T e^{-r(s-t)} p_s q_s \mathbb{1}_{X_s>0}\, ds + e^{-r(T-t)} u_T(X_T)\mathbb{1}_{X_s>0} \;\middle|\; X_t = x\right\},$$

where $r > 0$ is a discount rate, u_T is the profit at the end of the trading period, and $(p_t)_{t\geqslant 0}$ is the market price. The indicator function $\mathbb{1}_{X_t>0}$ introduces a stopping condition which means that producers can no longer earn revenue as soon as they deplete their reserves. The difference between Bertrand [18] and Cournot [52] point of view is related to the choice of the control variable. Indeed, in Cournot competition firms choose their rate of production $(q_t)_{t\geqslant 0}$ and the market price $(p_t)_{t\geqslant 0}$ is obtained through the supply-demand equilibrium. While in a Bertrand model, firms set prices $(p_t)_{t\geqslant 0}$ and receive demand $(q_t)_{t\geqslant 0}$ accordingly.

In our work we suppose a linear demand schedule and we suppose that the produced goods are differentiated. In this case, Bertrand and Cournot competition are given by the following:

- Cournot: given the rate of production $q(t,x)$ that is chosen by a representative producer with reserves x at time t, the received price is given by:

$$p(t,x) = 1 - \kappa \int_Q q(t,y)m(t,y)\,dy - q(t,x), \tag{3.1}$$

  where $\kappa > 0$ is a substitutability coefficient in proportion to which abundant total production will put downward pressure on all the prices.
- Bertrand: given the price $p(t,x)$ that is chosen by a representative producer with reserves x at time t, the received demand is given by:

$$q(t,x) = \frac{1}{1+\kappa\eta(t)} + \frac{\kappa}{1+\kappa\eta(t)} \int_Q p(t,y)m(t,y)\,dy - p(t,x), \tag{3.2}$$

  where $\eta(t) = \int_{\mathbb{R}^+} m(t,u)\,du$. The coefficient $\kappa > 0$ measures products substitutability, so that $\kappa \to 0$ corresponds to a monopoly situation with no competition pressure on the chosen price, while $k \to \infty$ corresponds to a highly competitive market.

We refer the reader to [49, 50] and references therein for further explanations on the economic model. An example of Cournot competition might be oil, coal and natural gas in the energy market, while in a Bertrand model might be competition between food producers where consumers have preference for one type of food, but reduce their demand for it depending on the average price of substitutes.

Now we formalize the optimization problem form Cournot's standpoint. Let us define the value function: $u_C(t,x) = \sup_q \mathcal{J}_{BC}(t,x,\mathbf{m})$, where p is given by (3.1). The optimal production rate $q_C^*(t,x)$ satisfies the first order condition:

$$q_C^*(t,x) = \frac{1}{2}\left(1 - \kappa \int_Q q_C^*(t,y)m(t,y)\,dy - \partial_x u_C(t,x)\right), \tag{3.3}$$

and the corresponding price $p_C^*(t,x)$ is given by:

$$p_C^*(t,x) = \frac{1}{2}\left(1 - \kappa \int_Q q_C^*(t,y)m(t,y)\,dy + \partial_x u_C(t,x)\right).$$

In the same way we define the value function for Bertrand competition: $u_B(t,x) = \sup_p \mathcal{J}_{BC}(t,x,\mathbf{m})$, where q is given by (3.2). The optimal price $p_B^*(t,x)$ satisfies the first order condition:

$$p_B^*(t,x) = \frac{1}{2}\left(\frac{1}{1+\kappa\eta(t)} + \frac{\kappa}{1+\kappa\eta(t)} \int_Q p_B^*(t,y)m(t,y)\,dy + \partial_x u_B(t,x)\right),$$

while the corresponding demand $q_B^*(t,x)$ is given by:

$$q_B^*(t,x) = \frac{1}{2}\left(\frac{1}{1+\kappa\eta(t)} + \frac{\kappa}{1+\kappa\eta(t)}\int_Q p_B^*(t,y)m(t,y)\,dy - \partial_x u_B(t,x)\right). \tag{3.4}$$

By integrating (3.4) with respect to $m(t,.)$, and after a little algebra one recovers the following identity

$$\frac{1}{1+\kappa\eta(t)} + \frac{\kappa}{1+\kappa\eta(t)}\int_Q p_B^*(t,y)m(t,y)\,dy = 1 - \kappa\int_Q q_B^*(t,y)m(t,y)\,dy,$$

which entails

$$q_B^*(t,x) = \frac{1}{2}\left(1 - \kappa\int_Q q_B^*(t,y)m(t,y)\,dy - \partial_x u_B(t,x)\right). \tag{3.5}$$

By noting the similarity between (3.5) and (3.3), we deduce that *Cournot and Bertrand MFG models are equivalent*, in the sense that they result in the same equilibrium prices and quantities. Therefore, we will note for simplicity $q_{u,m} := q_B^* = q_C^*$, and $u = u_B = u_C$ throughout this chapter. Moreover, by using the identities above, note that $q_{u,m}$ can be explained; namely:

$$q_{u,m} = \frac{1}{2}\left(\frac{2}{2+\kappa\eta(t)} + \frac{\kappa}{2+\kappa\eta(t)}\int_Q \partial_x u(t,y)m(t,y)\,dy - \partial_x u(t,x)\right). \tag{3.6a}$$

Consequently, an analytic approach leads to the following system of coupled PDEs:

$$\begin{cases} \partial_t u + \sigma\partial_{xx}u - ru + q_{u,m}^2 = 0, & 0<t<T,\ x\in Q \\ \partial_t m - \sigma\partial_{xx}m - \partial_x\{m q_{u,m}\} = 0, & 0<t<T,\ x\in Q \\ m(0,.) = m_0, \quad u(T,.) = u_T, & x\in \bar{Q}, \end{cases} \tag{3.6b}$$

where the first HJB equation governs the value function a representative producer at time t with reserves x, the second equation is the Fokker-Planck equation describing the evolution of the distribution of active producers, and u_T is a smooth non-decreasing function with $u_T(0) = 0$. Rather than taking $Q = \mathbb{R}^+$, we suppose that $Q := (0,\ell)$ where $\ell > 0$ is an upper limit on the capacity of any given producer. This assumption is in force throughout this manuscript and is more convenient for the analysis of Bertrand & Cournot PDE system (3.6b). From a modeling standpoint, this assumption is expected to have a weak impact when one chooses ℓ sufficiently large in comparison to the upper bound of the support of m_0. We suppose that the PDE system (3.6b) is endowed with the following boundary conditions:

$$\begin{cases} m(t,0) = u(t,0) = \partial_x u(t,\ell) = 0, & 0\leqslant t\leqslant T \\ \sigma\partial_x m(t,\ell) + m(t,\ell)q_{u,m}(t,\ell) = 0, & 0\leqslant t\leqslant T. \end{cases} \tag{3.6c}$$

The absorbing condition at $x = 0$ expresses the fact that producers disappears as soon as they deplete their capacity and can no longer generate revenue. In particular, the density of players is a non-increasing function [70]. Moreover, we consider reflecting boundary conditions at $x = \ell$ in order to be sure that all players' reserves are below ℓ.

The analysis of system (3.6a)-(3.6c) for smooth data is addressed in [70], where the authors construct a smooth solution to that system and prove uniqueness for small κ. In Section 2, we improve this result by showing uniqueness with no restriction. We refer to [49, 92] for numerical methods and simulations.

The major part of this chapter deals with the following coupled system of partial differential equations:

$$\left\{\begin{array}{lll} \text{(i)} & \partial_t u + \sigma\partial_{xx}u - ru + q_{u,m}^2 = 0, & 0 < t < T, \quad 0 < x < \ell \\ \text{(ii)} & \partial_t m - \sigma\partial_{xx}m - \partial_x\{m q_{u,m}\} = 0, & 0 < t < T, \quad 0 < x < \ell \\ \text{(iii)} & m(0,x) = m_0(x), \quad u(T,x) = u_T(x), & 0 \leqslant x \leqslant \ell \\ \text{(iv)} & \partial_x u(t,0) = \partial_x u(t,\ell) = 0, & 0 \leqslant t \leqslant T \\ \text{(v)} & \sigma\partial_x m(t,x) + q_{u,m}(t,x)m(t,x) = 0, & 0 \leqslant t \leqslant T, \quad x \in \{0,\ell\} \end{array}\right. \tag{3.7}$$

where $q_{u,m}$ is given by (3.6a). By contrast with system (3.6a)-(3.6c), we explore a reflecting boundary condition at $x = 0$. In terms of the model, we assume that players do not leave the game during the time period $[0, T]$ so that the number of producers in the market remains constant. This corresponds to a regime where all players participate at all resource levels. This situation is also considered in [77] for N-Player dynamic Cournot competition. Reflecting boundary conditions can also correspond to a situation where reserves are exogenously and infinitesimally replenished. In this particular case, the density of players is a probability density for all the times, i.e. $\eta(t) = 1$ for any $0 \leqslant t \leqslant T$, which considerably simplifies the analysis of the system of equations. In particular, it holds that

$$q_{u,m} = \frac{1}{2}\left(\frac{2}{2+\kappa} + \frac{\kappa}{2+\kappa}\int_0^\ell \partial_x u(t,y)m(t,y)\,dy - \partial_x u(t,x)\right).$$

Let us now outline our main results: Inspired by [70], we show in Section 2 that there exists a unique classical solution to system (3.7). Because of the change in boundary conditions, many of the arguments becomes considerably simpler and stronger results are possible. We show in Section 3 that (3.7) has an interpretation as a system of optimality for a convex minimization problem. Although this feature has been noticed and exploited for mean field games with congestion penalization (see [14] for an overview), here we show that it is also true for certain extended mean field games (cf. [69]). Finally, in Section 4 we give an existence result for the first order case where $\sigma = 0$, using a "vanishing viscosity" argument by collecting a priori estimates from Sections 2 and 3.

As long as system (3.7) is considered, we suppose that the following assumptions are in force:

- u_T and m_0 are function in $C^{2+\gamma}([0,\ell])$ for some $\gamma > 0$.
- u_T and m_0 satisfy compatible boundary conditions : $\dot{u}_T(0) = \dot{u}_T(\ell) = 0$ and $m_0(0) = \dot{m}_0(0) = m_0(\ell) = \dot{m}_0(\ell) = 0$.
- m_0 is a probability density, and $u_T \geqslant 0$.

2. Analysis of the PDE System

In this section we give a proof of existence and uniqueness for system (3.7). Note that most results of this section are an adaptation of those of [**70**, section 2]. However, unlike the case addressed in [**70**], we provide uniform bounds on u and u_x which do not depend on σ. We start by providing some a priori bounds on solutions to (3.7), then we prove existence and uniqueness using the Leray-Schauder fixed point theorem.

Let us start with some basic properties of the solutions.

PROPOSITION 3.1. *Let (u, m) be a pair of smooth solutions to (3.7). Then, for all $t \in [0, T]$, $m(t)$ is a probability density, and*

$$u(t, x) \geqslant 0 \quad \forall t \in [0, T], \forall x \in [0, \ell]. \tag{3.8}$$

Moreover, for some constant $C > 0$ depending on the data, we have

$$\int_0^T \int_0^\ell m \partial_x u^2 \leqslant C. \tag{3.9}$$

PROOF. Using (3.7)(ii) and (3.7)(v), one easily checks that $m(t)$ is a probability density for all $t \in [0, T]$. Moreover, the arguments used to prove (3.8) and (3.9) in [**70**] hold also for the system (3.7). □

LEMMA 3.2. *Let (u, m) be a pair of smooth solution to (3.7), then*

$$\|u\|_\infty + \|\partial_x u\|_\infty \leqslant C, \tag{3.10}$$

where the constant $C > 0$ does not depend on σ. In particular we have that

$$\forall t \in [0, T], \qquad \left| \int_0^\ell \partial_x u(t, y) m(t, y)\, dy \right| \leqslant C, \tag{3.11}$$

where $C > 0$ does not depend on σ.

PROOF. As in [**70**, Lemma 2.3, Lemma 2.7], the result is a consequence of using the maximum principle for suitable functions. We give a proof highlighting the fact that C does not depend on σ. Set

$$f(t) := \frac{2}{2+\kappa} + \frac{\kappa}{2+\kappa} \int_0^\ell \partial_x u(t, y) m(t, y)\, dy,$$

so that

$$-\partial_t u - \sigma \partial_{xx} u + r u \leqslant \frac{1}{2} \left(f^2(t) + \partial_x u^2 \right).$$

Owing to Proposition 3.1, $f \in L^2(0, T)$. Moreover, if

$$w := \exp\left\{ \frac{1}{2\sigma} \left(u + \frac{1}{2} \int_0^t f(s)^2\, ds \right) \right\} - 1,$$

then we have

$$-w_t - \sigma w_{xx} \leqslant 0.$$

In particular w satisfies the maximum principle, and $w \leqslant \mu$ everywhere, where

$$\mu = \max_{0 \leqslant x \leqslant \ell} \exp\left\{ \frac{1}{2\sigma} \left(u_T + \frac{1}{2} \int_0^T f(s)^2\, ds \right) \right\} - 1.$$

Whence, $0 \leqslant u \leqslant 2\sigma \ln(1+\mu)$, so that

$$\|u\|_\infty \leqslant \|u_T\|_\infty + \frac{1}{2}\int_0^T f(s)^2\,ds.$$

On the other hand, we have that

$$\max_{\Gamma_T} |\partial_x u| \leqslant \|\dot{u}_T\|_\infty, \quad \Gamma_T := ([0,T]\times\{0,\ell\}) \cup (\{T\}\times[0,\ell]),$$

so by using the maximum principle for the function $w(t,x) = \partial_x u(t,x)e^{-rt}$, we infer that

$$\|\partial_x u\|_\infty \leqslant e^{rT}\|\dot{u}_T\|_\infty.$$

□

REMARK 3.3. Unlike in [70], where more sophisticated estimates are performed, the estimation of the nonlocal term $\int_0^\ell \partial_x u(t,y)m(t,y)\,dy$ follows easily in this case, owing to (3.10) and the fact that m is a probability density.

PROPOSITION 3.4. *There exists a constant $C > 0$ depending on σ and data such that, if (u,m) is a smooth solution to (3.7), then for some $0 < \alpha < 1$,*

$$\|u\|_{\mathcal{C}^{1+\alpha/2,2+\alpha}(\overline{Q_T})} + \|m\|_{\mathcal{C}^{1+\alpha/2,2+\alpha}(\overline{Q_T})} \leqslant C. \tag{3.12}$$

PROOF. See [70, Proposition 2.8]. □

We now prove the main result of this section.

THEOREM 3.5. *There exists a unique classical solution to (3.7).*

PROOF. The proof of existence is the same as in [70, Theorem 3.1] and relies on Leray-Schauder fixed point theorem. Let (u_1,m_1) and (u_2,m_2) be two solutions of (3.7), and set $u = u_1 - u_2$ and $m = m_1 - m_2$. Define

$$G_i := \frac{1}{2}\left(\frac{2}{2+\kappa} + \frac{\kappa}{2+\kappa}\int_0^\ell \partial_x u_i(t,y)m_i(t,y)\,dy - \partial_x u_i\right).$$

Note that G_i can be written

$$G_i = \frac{1}{2}\left(1 - \kappa\bar{G}_i - \partial_x u_i\right), \quad \text{where} \quad \bar{G}_i := \int_0^\ell G_i(t,y)m_i(t,y)\,dy.$$

Integration by parts yields

$$\left[e^{-rt}\int_0^\ell u(t,y)m(t,y)\,dy\right]_0^T = \int_0^T e^{-rt}\int_0^\ell (G_2^2 - G_1^2 - G_1\partial_x u)m_1 + (G_1^2 - G_2^2 + G_2\partial_x u)m_2\,dy\,dt. \tag{3.13}$$

The left-hand side of (3.13) is zero. As for the right-hand side, we check that

$$G_2^2 - G_1^2 - G_1\partial_x u = (G_2 - G_1)^2 + \kappa G_1(\bar{G}_1 - \bar{G}_2)$$

and, similarly,

$$G_1^2 - G_2^2 + G_2\partial_x u = (G_2 - G_1)^2 - \kappa G_2(\bar{G}_1 - \bar{G}_2).$$

Then (3.13) becomes

$$0 = \int_0^T e^{-rt} \int_0^\ell (G_1 - G_2)^2 (m_1 + m_2)\,dx\,dt + \kappa \int_0^T e^{-rt} (\bar{G}_1 - \bar{G}_2)^2\,dt. \tag{3.14}$$

It follows that $\bar{G}_1 \equiv \bar{G}_2$. Then by uniqueness for parabolic equations with quadratic Hamiltonians [85, Chapter V], it follows that $u_1 \equiv u_2$. From uniqueness for the Fokker-Planck equation it follows that $m_1 \equiv m_2$. □

We conclude this part by improving the result of [70], using the idea of the proof of Theorem 3.5. (The proof is in fact much simpler than in [70].)

THEOREM 3.6. *There exists a unique classical solution to system* (3.6a)-(3.6c).

PROOF. Existence was given in [70]. For uniqueness, let $(u_1, m_1), (u_2, m_2)$ be two solutions, and define $u = u_1 - u_2$, $m = m_1 - m_2$, and

$$G_i = \frac{1}{2}\left(\frac{2}{2+\kappa\eta_i(t)} + \frac{\kappa}{2+\kappa\eta_i(t)}\int_0^\ell \partial_x u_i(t,y) m_i(t,y) dy - \partial_x u_i\right),$$
$$\eta_i(t) := \int_0^\ell m_i(t,y)\,dy.$$

Note that G_i can also be written

$$G_i = \frac{1}{2}(1 - \kappa\bar{G}_i - \partial_x u_i), \quad \text{where} \quad \bar{G}_i := \int_0^\ell G_i(t,y) m_i(t,y)\,dy.$$

Then integrating by parts as in the proof of Theorem 3.5, we obtain

$$0 = \int_0^T e^{-rt} \int_0^\ell (G_1 - G_2)^2 (m_1 + m_2)\,dx\,dt + \kappa \int_0^T e^{-rt} (\bar{G}_1 - \bar{G}_2)^2\,dt. \tag{3.15}$$

We conclude as before. □

3. Optimal Control of Fokker-Planck Equation

The purpose of this section is to prove that (3.7) is a system of optimality for a convex minimization problem. It was first noticed in the seminal paper by Lasry and Lions [86] that systems of the form (1.3) have a formal interpretation in terms of optimal control. Since then this property has been made rigorous and exploited to obtain well-posedness in first-order [32,34,40] and degenerate cases [35]; see [14] for a nice discussion. However, all of these references consider the case of congestion penalization, which results in an a priori summability estimate on the density. There is no such penalization in (3.7). Hence, the optimality arguments used in [32], for example, appear insufficient in the present case to prove existence and uniqueness of solutions to the first order system. Furthermore, it is very difficult in the present context to formulate the dual problem, which in the aforementioned works was an essential ingredient in proving existence of an adjoint state. Nevertheless, aside from its intrinsic interest, we will see in Section 4 that optimality gives us at least enough to pass to the limit as $\sigma \to 0$.

Consider the optimization problem of minimizing the objective functional

$$J(m,q) = \int_0^T \int_0^\ell e^{-rt} \left(q^2(t,x) - q(t,x)\right) m(t,x)\,dx\,dt \\ + \frac{\kappa}{2}\int_0^T e^{-rt} \left(\int_0^\ell q(t,y)m(t,y)\,dy\right)^2 dt - \int_0^\ell e^{-rT} u_T(x) m(T,x)\,dx \tag{3.16}$$

for (m,q) in the class $\mathcal{K}$, defined as follows. Let $m \in L^1([0,T]\times[0,\ell])$ be non-negative, let $q \in L^2([0,T]\times[0,\ell])$, and assume that m is a weak solution to the Fokker-Planck equation

$$\partial_t m - \sigma\partial_{xx} m - \partial_x\{qm\} = 0, \quad m(0) = m_0, \tag{3.17}$$

equipped with Neumann boundary conditions, where weak solutions are defined as in [**102**]; namely:

- the integrability condition $mq^2 \in L^1([0,T]\times[0,\ell])$ holds, and
- (3.17) holds in the sense of distributions–namely, for all $\phi \in \mathcal{C}_c^\infty([0,T)\times[0,\ell])$ such that $\partial_x\phi(t,0) = \partial_x\phi(t,\ell) = 0$ for each $t \in (0,T)$, we have
$$\int_0^T \int_0^\ell (-\partial_t\phi - \sigma\partial_{xx}\phi + q\partial_x\phi) m \;dx\,dt = \int_0^\ell \phi(0) m_0\,dx.$$

Then we say that $(m,q) \in \mathcal{K}$. We refer the reader to [**102**] for properties of weak solutions of (3.17), namely that they are unique and that they coincide with renormalized solutions and for this reason have several useful properties. One property which will be of particular interest to us is the following lemma:

LEMMA 3.7 (Proposition 3.10 in [**102**]). *Let $(m,q) \in \mathcal{K}$, i.e. let m be a weak solution of the Fokker-Planck equation (3.17). Then $\|m(t)\|_{L^1([0,\ell])} = \|m_0\|_{L^1([0,\ell])}$ for all $t \in [0,T]$. Moreover, if $\log m_0 \in L^1([0,\ell])$, then*

$$\|\log m(t)\|_{L^1([0,\ell])} \leqslant C(\|\log m_0\|_{L^1([0,\ell])} + 1) \;\; \forall t \in [0,T], \tag{3.18}$$

where C depends on $\|q\|_{L^2}$ and $\|m_0\|_{L^1}$. In particular, if $\log m_0 \in L^1([0,\ell])$ and (m,q) in $\mathcal{K}$, then $m > 0$ a.e.

THEOREM 3.8. *Let (u,m) be a solution of (3.7). Set*

$$q = \frac{1}{2}\left(\frac{2}{2+\kappa} + \frac{\kappa}{2+\kappa}\int_0^\ell \partial_x u(t,y) m(t,y)\,dy - \partial_x u\right).$$

Then (m,q) is a minimizer for problem (3.16), that is, $J(m,q) \leqslant J(\tilde m,\tilde q)$ for all $(\tilde m,\tilde q)$ satisfying (3.17). Moreover, if $\log m_0 \in L^1([0,\ell])$ then the minimizer is unique.

PROOF. It is useful to keep in mind that the proof is based on the convexity of J following a change of variables. By abuse of notation we might write

$$J(m,w) = \int_0^T \int_0^\ell e^{-rt}\left(\frac{w^2(t,x)}{m(t,x)} - w(t,x)\right) dx\,dt \\ + \frac{\kappa}{2}\int_0^T e^{-rt}\left(\int_0^\ell w(t,y)\,dy\right)^2 dt - \int_0^\ell e^{-rT} u_T(x) m(T,x)\,dx,$$

cf. the change of variables used in [13] and several works which cite that paper. However, in this context we prefer a direct proof.

Using the algebraic identity

$$\tilde{q}^2\tilde{m} - q^2 m = 2q(\tilde{q}\tilde{m} - qm) - q^2(\tilde{m} - m) + \tilde{m}(\tilde{q} - q)^2,$$

we have

(3.19)

$$\begin{aligned}
J(\tilde{m}, \tilde{q}) - J(m, q) = \frac{\kappa}{2}\int_0^T e^{-rt}\left(\int_0^\ell \tilde{q}\tilde{m} - qm \, dy\right)^2 dt - \int_0^\ell e^{-rT}u_T(x)(\tilde{m} - m)(T, x)\, dx \\
+ \kappa\int_0^T e^{-rt}\left(\int_0^\ell \tilde{q}\tilde{m} - qm \, dy\right)\left(\int_0^\ell qm \, dy\right) dt \\
+ \int_0^T\int_0^\ell e^{-rt}\left((qm - \tilde{q}\tilde{m}) + 2q(\tilde{q}\tilde{m} - qm) - q^2(\tilde{m} - m) + \tilde{m}(\tilde{q} - q)^2\right) dx\, dt.
\end{aligned}$$

Now using the fact that u is a smooth solution of

$$\partial_t u + \sigma\partial_{xx}u - ru + q^2 = 0, \quad u(T) = 0, \quad \partial_x u|_{0,\ell} = 0 \tag{3.20}$$

and since

$$\partial_t(\tilde{m} - m) - \sigma\partial_{xx}(\tilde{m} - m) - \partial_x(\tilde{q}\tilde{m} - qm) = 0, \quad (\tilde{m} - m)(0) = 0$$

in the sense of distributions, it follows that

$$\begin{aligned}
\int_0^T\int_0^\ell e^{-rt}q^2(\tilde{m} - m)\, dx\, dt + \int_0^\ell e^{-rT}u_T(x)(\tilde{m} - m)(T, x)\, dx \\
= -\int_0^T\int_0^\ell e^{-rt}(\tilde{q}\tilde{m} - qm)\partial_x u \, dx\, dt.
\end{aligned}$$

Putting this into (3.19) and rearranging, we have

$$\begin{aligned}
J(\tilde{m}, \tilde{q}) - J(m, q) = \int_0^T\int_0^\ell e^{-rt}(qm - \tilde{q}\tilde{m})\left(1 - 2q - \kappa\int_0^\ell qm \, dy - \partial_x u\right) dx\, dt \\
+ \int_0^T\int_0^\ell e^{-rt}\tilde{m}(\tilde{q} - q)^2\, dx\, dt + \frac{k}{2}\int_0^T e^{-rt}\left(\int_0^\ell \tilde{q}\tilde{m} - qm \, dx\right)^2 dt.
\end{aligned} \tag{3.21}$$

To conclude that $J(\tilde{m}, \tilde{q}) \geqslant J(m, q)$, it suffices to prove that

$$1 - 2q - \kappa\int_0^\ell qm \, dy - \partial_x u = 0. \tag{3.22}$$

Recall the definition

$$q = \frac{1}{2}\left(\frac{2}{2 + \kappa} + \frac{\kappa}{2 + \kappa}\int_0^\ell \partial_x u(t, y)m(t, y)\, dy - \partial_x u\right).$$

Integrate both sides against m and rearrange to get

$$\int m\partial_x u \, dy = 1 - (\kappa + 2)\int qm \, dy.$$

Plugging this into the definition of q proves (3.22). Thus (m, q) is a minimizer.

On the other hand, suppose $\log m_0 \in L^1([0,\ell])$ and that $(\tilde{m}, \tilde{q})$ is another minimizer. Then (3.21) implies that

$$\int_0^T \int_0^\ell e^{-rt}\tilde{m}(\tilde{q}-q)^2 \,dx\,dt + \frac{\kappa}{2}\int_0^T e^{-rt}\left(\int_0^\ell \tilde{q}\tilde{m} - qm \,\,dx\right)^2 dt = 0. \tag{3.23}$$

Now by Lemma 3.7, we have $\tilde{m} > 0$ a.e. Therefore (3.23) implies $\tilde{q} = q$. By uniqueness for the Fokker-Planck equation, we conclude that $\tilde{m} = m$ as well. The proof is complete. □

REMARK 3.9 (Extension to system (3.6a)-(3.6c)). A similar argument shows that System (3.6a)-(3.6c), with Dirichlet boundary conditions on the left-hand side, is also a system of optimality for the same minimization problem, except this time with Dirichlet boundary conditions (on the left-hand side) imposed on the Fokker-Planck equation. We omit the details.

4. First-Order Case

In this section we use a vanishing viscosity method to prove that (3.7) has a solution even when we plug in $\sigma = 0$. We need to collect some estimates which are uniform in σ as $\sigma \to 0$. From now on we will assume $0 < \sigma \leqslant 1$, and whenever a constant C appears it does not depend on σ.

LEMMA 3.10. $\|\partial_t u\|_2 \leqslant C$.

PROOF. We first prove that $\sigma\|\partial_{xx}u\|_2 \leqslant C$. For this, multiply

$$\partial_{xt}u - r\partial_x u + \sigma\partial_{xxx}u - q_{u,m}\partial_{xx}u = 0 \tag{3.24}$$

by $\partial_x u$ and integrate by parts. We get, after using Young's inequality and (3.10),

$$4\sigma^2\int_0^T\int_0^\ell \partial_{xx}u^2\,dx\,dt \leqslant 4\int_0^T\int_0^\ell (q_{u,m}\partial_x u)^2\,dx\,dt + 4\sigma\int_0^\ell \dot{u}_T(x)^2\,dx \leqslant C,$$

as desired.

Then the claim follows from (3.7)(i) and Lemma 3.2. □

LEMMA 3.11. $\|u\|_{C^{1/3}} \leqslant C$.

PROOF. Since $\|\partial_x u\|_\infty \leqslant C$ it is enough to show that u is $1/3$-Hölder continuous in time. Let $t_1 < t_2$ in $[0,T]$ be given. Set $\eta > 0$ to be chosen later. We have, by Hölder's inequality,

$$\begin{aligned}
|u(t_1,x) - u(t_2,x)| &\leqslant C\eta + \frac{1}{\eta}\int_{x-\eta}^{x+\eta}|u(t_1,\xi) - u(t_2,\xi)|\,d\xi \\
&\leqslant C\eta + \frac{1}{\eta}\int_{x-\eta}^{x+\eta}\int_{t_1}^{t_2}|\partial_t u(s,\xi)|\,ds\,d\xi \\
&\leqslant C\eta + \frac{1}{\eta}\|\partial_t u\|_2\sqrt{2\eta|t_2-t_1|} \leqslant C\eta + C|t_2-t_1|^{1/2}\eta^{-1/2}.
\end{aligned} \tag{3.25}$$

Setting $\eta = |t_2 - t_1|^{1/3}$ proves the claim. □

To prove compactness estimates for m, we will first use the fact that it is the minimizer for an optimization problem. Let us reintroduce the optimization problem from Section 3 with $\sigma \geqslant 0$ as a variable. We first define the convex functional

$$\Psi(m, w) := \begin{cases} \frac{|w|^2}{m} & \text{if } m \neq 0, \\ 0 & \text{if } w = 0, m = 0, \\ +\infty & \text{if } w \neq 0, m = 0. \end{cases} \tag{3.26}$$

Now we rewrite the functional J, with a slight abuse of notation, as

$$\begin{aligned} J(m, w) = \int_0^T \int_0^\ell e^{-rt} \left(\Psi(m(t,x), w(t,x)) - w(t,x)\right) \, dx \, dt \\ + \frac{\kappa}{2} \int_0^T e^{-rt} \left(\int_0^\ell w(t,y) \, dy\right)^2 dt - \int_0^\ell e^{-rT} u_T(x) m(T,x) \, dx, \end{aligned} \tag{3.27}$$

and consider the problem of minimizing over the class $\mathcal{K}_\sigma$, defined here as the set of all pairs $(m, w) \in L^1((0,T) \times (0,\ell))_+ \times L^1((0,T) \times (0,\ell); \mathbb{R})$ such that

$$\partial_t m - \sigma \partial_{xx} m - \partial_x w = 0, \; m(0) = m_0 \tag{3.28}$$

in the sense of distributions. By Proposition 3.8, for every $\sigma > 0$, J has a minimizer in $\mathcal{K}_\sigma$ given by $(m, w) = (m, q_{u,m} m)$ where (u, m) is the solution of System (3.7). Since (m, w) is a minimizer, we can derive a priori bounds which imply, in particular, that $m(t)$ is Hölder continuous in the Kantorovich-Rubinstein distance on the space of probability measures, with norm bounded uniformly in σ. We recall that the Kantorovich-Rubinstein metric on $\mathcal{P}(\Omega)$, the space of Borel probability measures on Ω, is defined by

$$\mathbf{d}_1(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\Omega \times \Omega} |x - y| \, d\pi(x, y),$$

where $\Pi(\mu, \nu)$ is the set of all probability measures on $\Omega \times \Omega$ whose first marginal is μ and whose second marginal is ν. Here we consider $\Omega = (0, \ell)$.

LEMMA 3.12. *There exists a constant* C *independent of* σ *such that*

$$\||w|^2/m\|_{L^1((0,T)\times(0,\ell))} \leqslant C.$$

As a corollary, m *is* $1/2$*-Hölder continuous from* $[0, T]$ *into* $\mathcal{P}((0,\ell))$*, and there exists a constant (again denoted* C*) independent of* σ *such that*

$$\mathbf{d}_1(m(t_1), m(t_2)) \leqslant C|t_1 - t_2|^{1/2}. \tag{3.29}$$

PROOF. To see that $\||w|^2/m\|_{L^1((0,T)\times(0,\ell))} \leqslant C$, use $(m_0, 0) \in \mathcal{K}$ as a comparison. By the fact that $J(m, w) \leqslant J(m_0, 0)$ we have

$$\begin{aligned} \int_0^T \int_0^\ell e^{-rt} \frac{|w|^2}{2m} \, dx \, dt + \frac{\kappa}{2} \int_0^T e^{-rt} \left(\int_0^\ell w \, dx\right)^2 dt \\ \leqslant \int_0^\ell e^{-rT} u_T(m(T) - m_0) \, dx + \frac{1}{2} \int_0^T \int_0^\ell e^{-rt} m \, dx \, dt \leqslant C. \end{aligned}$$

The Hölder estimate (3.29) follows from [35, Lemma 3.1]. ∎

We also have compactness in L^1, which comes from the following lemma.

LEMMA 3.13. *For every* $K \geqslant 0$, *we have*

$$\int_{m(t)\geqslant 2K} m(t)\,dx \leqslant 2\int_0^\ell (m_0-K)^+\,dx \tag{3.30}$$

for all $t \in [0,T]$.

PROOF. Let $K \geqslant 0$ be given. We define the following auxiliary functions:

$$\phi_{\alpha,\delta}(s) := \begin{cases} 0 & \text{if } s \leqslant K, \\ \frac{1}{6}(1+\alpha)\alpha\delta^{\alpha-2}(s-K)^3 & \text{if } K \leqslant s \leqslant K+\delta, \\ \frac{1}{6}(1+\alpha)\alpha\delta^{\alpha+1} + \frac{1}{2}(1+\alpha)\alpha\delta^{\alpha}(s-K) + (s-K)^{1+\alpha} & \text{if } s \geqslant K+\delta, \end{cases} \tag{3.31}$$

where $\alpha, \delta \in (0,1)$ are parameters going to zero. For reference we note that

$$\dot\phi_{\alpha,\delta}(s) = \begin{cases} 0 & \text{if } s \leqslant K, \\ \frac{1}{2}(1+\alpha)\alpha\delta^{\alpha-2}(s-K)^2 & \text{if } K \leqslant s \leqslant K+\delta, \\ \frac{1}{2}(1+\alpha)\alpha\delta^{\alpha} + (1+\alpha)(s-K)^{\alpha} & \text{if } s \geqslant K+\delta, \end{cases} \tag{3.32}$$

and

$$\ddot\phi_{\alpha,\delta}(s) = \begin{cases} 0 & \text{if } s \leqslant K, \\ (1+\alpha)\alpha\delta^{\alpha-2}(s-K) & \text{if } K \leqslant s \leqslant K+\delta, \\ (1+\alpha)\alpha(s-K)^{\alpha-1} & \text{if } s \geqslant K+\delta. \end{cases} \tag{3.33}$$

Observe that $\ddot\phi_{\alpha,\delta}$ is continuous and non-negative. Multiply (3.7)(ii) by $\dot\phi_{\alpha,\delta}(m)$ and integrate by parts. After using Young's inequality we have

$$\int_0^\ell \phi_{\alpha,\delta}(m(t))\,dx \leqslant \int_0^\ell \phi_{\alpha,\delta}(m_0)\,dx + \frac{\|q_{u,m}\|_\infty^2}{4\sigma}\int_0^t\int_0^\ell \ddot\phi_{\alpha,\delta}(m)m^2\,dx\,dt. \tag{3.34}$$

Since $\ddot\phi_{\alpha,\delta}(s) \leqslant (1+\alpha)\alpha\delta^{-2}$, after taking $\alpha \to 0$ we have

$$\int_0^\ell \phi_\delta(m(t))\,dx \leqslant \int_0^\ell \phi_\delta(m_0)\,dx, \tag{3.35}$$

where $\phi_\delta(s) = (s-K)\chi_{[K+\delta,\infty)}(s)$. Now letting $\delta \to 0$ we see that

$$\int_0^\ell (m(t)-K)^+\,dx \leqslant \int_0^\ell (m_0-K)^+\,dx, \tag{3.36}$$

where $s^+ := (s+|s|)/2$ denotes the positive part. Whence

$$\int_0^\ell (m_\sigma(t)-K)^+\,dx \leqslant \int_0^\ell (m_0-K)^+\,dx, \tag{3.37}$$

which also implies (3.30). □

We also have a compactness estimate for the function $t \mapsto \int_0^\ell \partial_x u(t,y)m(t,y)\,dy$.

LEMMA 3.14. $\sigma\left(\int_0^T\int_0^\ell \frac{|\partial_x m|^2}{m+1}\,dx\,dt\right)^{1/2} \leqslant C.$

PROOF. Multiply the Fokker-Planck equation by $\log(m+1)$ and integrate by parts. After using Young's inequality, we obtain

$$\begin{aligned}\frac{\sigma^2}{2}\int_0^T\int_0^\ell \frac{|\partial_x m|^2}{m+1}\,dx\,dt &\leqslant 2\sigma\int_0^\ell \left((m_0+1)\log(m_0+1)-m_0\right)\,dx + \|q_{u,m}\|_\infty^2\int_0^T\int_0^\ell \frac{m^2}{m+1}\\ &\leqslant \int_0^\ell \left((m_0+1)\log(m_0+1)-m_0\right)\,dx + \|q_{u,m}\|_\infty^2\int_0^T\int_0^\ell m\,dx\,dt \leqslant C.\end{aligned}$$

□

LEMMA 3.15. *Let $\zeta\in\mathcal{C}_c^\infty((0,\ell))$. Then the function*

$$t\mapsto \int_0^\ell m(t,y)\zeta(y)\partial_x u(t,y)\,dy$$

is $1/2$-Hölder continuous, and in particular,

$$\left|\left[\int_0^\ell m(t,y)\zeta(y)\partial_x u(t,y)\,dy\right]_{t_1}^{t_2}\right| \leqslant C_\zeta|t_1-t_2|^{1/2} \tag{3.38}$$

where C_ζ is a constant that depends on ζ but not on σ.

PROOF. Integration by parts yields

$$\begin{aligned}&\left[e^{-rt}\int_0^\ell m(t,y)\zeta(y)\partial_x u(t,y)\,dy\right]_{t_1}^{t_2} \qquad (3.39)\\ &= -2\sigma\int_{t_1}^{t_2} e^{-rs}\int_0^\ell \partial_x m(s,y)\dot\zeta(y)\partial_x u(s,y)\,dy\,ds - \sigma\int_{t_1}^{t_2} e^{-rs}\int_0^\ell m(s,y)\ddot\zeta(y)\partial_x u(s,y)\,dy\,ds\\ &\quad -\frac12\int_{t_1}^{t_2}\left\{\left(\frac{2}{2+\kappa}+\frac{\kappa}{2+\kappa}\int_0^\ell \partial_x u(s)m(s)\right)\int_0^\ell \dot\zeta m(s)\partial_x u(s) - \int_0^\ell \dot\zeta m(s)\partial_x u^2(s)\right\}\,ds.\end{aligned}$$

On the one hand,

$$\left|\sigma\int_{t_1}^{t_2} e^{-rs}\int_0^\ell m(s,x)\ddot\zeta(x)\partial_x u(s,x)\,dx\,ds\right| \leqslant \frac{\|\partial_x u\|_\infty\|\ddot\zeta\|_\infty}{2}|t_1-t_2| \leqslant C\|\ddot\zeta\|_\infty|t_1-t_2|,$$

and

$$\begin{aligned}&\left|\int_{t_1}^{t_2}\left\{\left(\frac{2}{2+\kappa}+\frac{\kappa}{2+\kappa}\int_0^\ell m(s)\partial_x u(s)\right)\int_0^\ell \dot\zeta m(s)\partial_x u(s) - \int_0^\ell \dot\zeta m(s)\partial_x u^2(s)\right\}\,ds\right|\\ &\qquad\qquad \leqslant C\|\dot\zeta\|_\infty\|\partial_x u\|_\infty^2|t_1-t_2|.\end{aligned}$$

On the other hand, by Hölder's inequality and Lemma 3.14 we get

$$\left|2\sigma\int_{t_1}^{t_2} e^{-rs}\int_0^\ell \partial_x m(s,x)\partial_x u(s,x)\dot{\zeta}(x)\,dx\,ds\right|$$
$$\leqslant \|\partial_x u\|_\infty \|\dot{\zeta}\|_\infty 2\sigma\left(\int_{t_1}^{t_2}\int_0^\ell \frac{|\partial_x m|^2}{m+1}\,dx\,ds\right)^{1/2}\left(\int_{t_1}^{t_2}\int_0^\ell (m+1)\,dx\,ds\right)^{1/2}$$
$$\leqslant C\|\dot{\zeta}\|_\infty(\ell+1)^{1/2}|t_1-t_2|^{1/2}.$$

□

COROLLARY 3.16. *The function*

$$t \mapsto \int_0^\ell m(t,x)\partial_x u(t,x)\,dx$$

is uniformly continuous with modulus of continuity independent of σ.

PROOF. Let $\delta \in (0,\ell)$ and fix $\zeta \in \mathcal{C}_c^\infty((0,\ell))$ be such that $0 \leqslant \zeta \leqslant 1$ and $\zeta \equiv 1$ on $[\delta, \ell-\delta]$. Notice that for any $t_1, t_2 \in [0,T]$

$$\left|\left[\int_0^\ell m(t,x)(1-\zeta(x))\partial_x u(t,x)\,dx\right]_{t_1}^{t_2}\right| \leqslant \|\partial_x u\|_\infty \int_{[0,\ell]\setminus[\delta,\ell-\delta]} [m(t_1,x)+m(t_2,x)]\,dx. \tag{3.40}$$

Now by Lemma 3.13 we have

$$\int_{[0,\ell]\setminus[\delta,\ell-\delta]} m(t,x)\,dx \tag{3.41}$$
$$\leqslant \int_{\{m(t)<2K\}\cap[0,\ell]\setminus[\delta,\ell-\delta]} m(t,x)\,dx + \int_{\{m(t)\geqslant 2K\}} m(t,x)\,dx \leqslant 4K\delta + 2\int_0^\ell (m_0-K)^+\,dx$$

for all $t \in [0,T]$. Combine (3.40) and (3.41) with Lemmas 3.15 and 3.2 to get

$$\left|\left[\int_0^\ell m(t,x)\partial_x u(t,x)\,dx\right]_{t_1}^{t_2}\right| \leqslant C_\zeta|t_1-t_2|^{1/2} + CK\delta + C\int_0^\ell (m_0-K)^+\,dx \quad \forall t_1,t_2 \in [0,T]. \tag{3.42}$$

Let $\eta > 0$ be given. Set K large enough such that $C\int_0^\ell (m_0-K)^+\,dx < \eta/3$, then pick δ small enough that $CK\delta < \eta/3$. Finally, fix ζ as described above. Equation (3.42) implies that if $|t_1-t_2| < \eta^2/(9C_\zeta^2)$, we have $\left|\left[\int_0^\ell m(t,x)\partial_x u(t,x)\,dx\right]_{t_1}^{t_2}\right| < \eta$. Thus the function $t \mapsto \int_0^\ell m(t,x)\partial_x u(t,x)\,dx$ is uniformly continuous, and since none of the constants here depend on σ, the modulus of continuity is independent of σ. □

We are now in a position to prove an existence result for the first-order system.

THEOREM 3.17. *Suppose that* $\sigma = 0$, *then there exists a unique pair* (u, m) *which solves System* (3.7) *in the following sense:*

(1) $u \in W_2^1([0,T]\times[0,\ell]) \cap L^\infty(0,T;W_\infty^1(0,\ell))$ *is a continuous solution of the Hamilton-Jacobi equation*

$$\partial_t u - ru + \frac{1}{4}(f(t)-\partial_x u)^2 = 0,\ u(T,x) = u_T(x), \tag{3.43}$$

equipped with Neumann boundary conditions, in the viscosity sense;

(2) $m \in L^1 \cap \mathcal{C}([0,T];\mathcal{P}([0,\ell]))$ *satisfies the continuity equation*

$$\partial_t m - \frac{1}{2}\partial_x((f(t)-\partial_x u)m) = 0,\ m(0) = m_0, \tag{3.44}$$

equipped with Neumann boundary conditions, in the sense of distributions; and

(3) $f(t) = \frac{2}{2+\kappa} + \frac{\kappa}{2+\kappa}\int_0^\ell m(t,x)\partial_x u(t,x)\,dx$ *a.e.*

PROOF. *Existence:* Collecting Lemmas 3.2, 3.10 3.11, 3.12, 3.13, and Corollary 3.16, we can construct a sequence $\sigma_n \to 0^+$ such that if (u^n, m^n) is the solution corresponding to $\sigma = \sigma_n$, we have

- $u^n \to u$ uniformly, so that $u \in C([0,T]\times[0,\ell])$, and also weakly in $W^{1,2}([0,T]\times[0,\ell])$;
- $\partial_x u^n \rightharpoonup \partial_x u$ weakly* in L^∞;
- $m^n \to m$ in $\mathcal{C}([0,T];\mathcal{P}([0,\ell]))$, so that $m(t)$ is a well-defined probability measure for every $t \in [0,T]$, $m^n \rightharpoonup m$ weakly in $L^1([0,T]\times[0,\ell])$, and $m^n(T) \rightharpoonup m(T)$ weakly in $L^1([0,\ell])$;
- $m^n\partial_x u^n \rightharpoonup w$ weakly in L^1; and
- $f_n(t) := \frac{2}{2+\kappa} + \frac{\kappa}{2+\kappa}\int_0^\ell m^n(t,y)\partial_x u^n(t,y)\,dy \to f(t)$ in $\mathcal{C}([0,T])$.

Since $u^n \to u$ and $f_n \to f$ uniformly, by standard arguments, we have that (3.43) holds in a viscosity sense. Moreover, since $\partial_x u^n \rightharpoonup \partial_x u$ weakly* in L^∞, we also have

$$\partial_t u - ru + \frac{1}{4}(f(t)-\partial_x u)^2 \leqslant 0 \tag{3.45}$$

in the sense of distributions, i.e. for all $\phi \in \mathcal{C}^\infty([0,T]\times[0,\ell])$ such that $\phi \geqslant 0$, we have

$$\begin{aligned}\int_0^\ell e^{-rT}u_T(x)\phi(T,x)\,dx - \int_0^\ell e^{-rT}u(0,x)\phi(0,x)\,dx \\ - \int_0^T\int_0^\ell e^{-rt}u(t,x)\partial_t\phi(t,x)\,dx\,dt + \frac{1}{4}\int_0^T\int_0^\ell (f(t)-\partial_x u(t,x))^2\phi(t,x)\,dx\,dt \leqslant 0.\end{aligned} \tag{3.46}$$

(This follows from the convexity of $\partial_x u \mapsto \partial_x u^2$.)

Since $m^n \rightharpoonup m$ and $m^n\partial_x u^n \rightharpoonup w$ weakly in L^1, it also follows that

$$\partial_t m - \frac{1}{2}\partial_x(f(t)m - w) = 0,\ m(0) = m_0 \tag{3.47}$$

in the sense of distributions. For convenience we define $v := \frac{1}{2}(f(t)m - w)$. Extend the definition of (m, v) so that $m(t,x) = m(T,x)$ for $t \geqslant T$, $m(t,x) = m_0(x)$ for $t \leqslant 0$, and $m(t,x) = 0$ for $x \notin [0,\ell]$; and so that $v(t,x) = 0$ for $(t,x) \notin [0,T]\times[0,\ell]$. Now let $\xi_\delta(t,x)$ be a standard convolution kernel (i.e. a $\mathcal{C}^\infty$, positive function whose support is contained in a ball of radius δ and such that $\iint \xi^\delta(t,x)\,dx\,dt = 1$). Set $m_\delta = \xi_\delta * m$ and $v_\delta = \xi_\delta.$

Then m_δ, v_δ are smooth functions such that $\partial_t m_\delta = \partial_x v_\delta$ in $[0,T]\times[0,\ell]$; moreover m_δ is positive. Using m_δ as a test function in (3.46) we get

$$\int_0^\ell e^{-rT}u_T(x)m_\delta(T,x)\,dx - \int_0^\ell e^{-rT}u(0,x)m_\delta(0,x)\,dx \\ + \int_0^T\int_0^\ell e^{-rt}v_\delta\partial_x u\,dx\,dt + \frac{1}{4}\int_0^T\int_0^\ell (f(t)-\partial_x u)^2 m_\delta\,dx\,dt \leqslant 0.$$

Using the continuity of $m(t)$ in $\mathcal{P}([0,\ell])$ from Lemma 3.12, we see that

$$\lim_{\delta\to 0^+}\int_0^\ell e^{-rT}u_T(x)m_\delta(T,x)\,dx = \int_0^\ell e^{-rT}u_T(x)m(T,x)\,dx,$$

and $\lim_{\delta\to0^+}\int_0^\ell e^{-rT}u(0,x)m_\delta(0,x)\,dx = \int_0^\ell e^{-rT}u(0,x)m_0(x)\,dx$. Since $m_\delta \to m$ and $v_\delta \to v$ in L^1, we have

$$\int_0^\ell e^{-rT}u_T(x)m(T,x)\,dx - \int_0^\ell e^{-rT}u(0,x)m_0(x)\,dx \\ + \int_0^T\int_0^\ell e^{-rt}v\partial_x u\,dx\,dt + \frac{1}{4}\int_0^T\int_0^\ell (f(t)-\partial_x u)^2 m\,dx\,dt \leqslant 0,$$

or

$$\int_0^\ell e^{-rT}u_T(x)m(T,x)\,dx - \int_0^\ell e^{-rT}u(0,x)m_0(x)\,dx \\ + \int_0^T\int_0^\ell e^{-rt}\left(\frac{1}{4}m\partial_x u^2 - \frac{1}{2}w\partial_x u\right)\,dx\,dt + \frac{1}{4}\int_0^T\int_0^\ell f^2(t)m\,dt \leqslant 0. \tag{3.48}$$

Recall the definition of $\Psi(m,w)$ from (3.26). From (3.48) we have

$$\int_0^\ell e^{-rT}u_T(x)m(T,x)\,dx - \int_0^\ell e^{-rT}u(0,x)m_0(x)\,dx \\ + \frac{1}{4}\int_0^T\int_0^\ell f^2(t)m\,dt \leqslant \frac{1}{4}\int_0^T\int_0^\ell e^{-rt}\Psi(m,w)\,dx\,dt. \tag{3.49}$$

On the other hand, for each n we have

$$\int_0^\ell e^{-rT}u_T(x)m^n(T,x)\,dx - \int_0^\ell e^{-rT}u^n(0,x)m_0(x)\,dx \\ +\frac{1}{4}\int_0^T\int_0^\ell f_n^2(t)m^n\,dt = \frac{1}{4}\int_0^T\int_0^\ell e^{-rt}m^n\partial_x u^2\,dx\,dt = \frac{1}{4}\int_0^T\int_0^\ell e^{-rt}\Psi(m^n, m^n\partial_x u^n)\,dx\,dt. \tag{3.50}$$

Since $(m^n, m^n\partial_x u^n) \rightharpoonup (m,w)$ weakly in $L^1\times L^1$, it follows from weak lower semicontinuity that

$$\int_0^\ell e^{-rT}u_T(x)m(T,x)\,dx - \int_0^\ell e^{-rT}u(0,x)m_0(x)\,dx \\ + \frac{1}{4}\int_0^T\int_0^\ell f^2(t)m\,dt \geqslant \frac{1}{4}\int_0^T\int_0^\ell e^{-rt}\Psi(m,w)\,dx\,dt. \tag{3.51}$$

From (3.48), (3.49), and (3.51) it follows that

$$\int_0^T \int_0^\ell e^{-rt}(\Psi(m, w) + m\partial_x u^2 - 2w\partial_x u)\,dx\,dt = 0,$$

where $\Psi(m, w) + m\partial_x u^2 - 2w\partial_x u$ is a non-negative function, hence zero almost everywhere. We deduce that $w = m\partial_x u$ almost everywhere.

Finally, by weak convergence we have for a.e $t \in [0, T]$,

$$\begin{aligned} f(t) = \frac{2}{2+\kappa} + \frac{\kappa}{2+\kappa} \lim_{n\to\infty} \int_0^\ell m^n(t,x)\partial_x u^n(t,x)\,dx \\ = \frac{2}{2+\kappa} + \frac{\kappa}{2+\kappa}\int_0^\ell w(t,x)\,dx \\ = \frac{2}{2+\kappa} + \frac{\kappa}{2+\kappa}\int_0^\ell m(t,x)\partial_x u(t,x)\,dx \end{aligned}$$

Which entails the existence part of the Theorem.

Uniqueness: The proof of uniqueness is essentially the same as for the second order case, the only difference is the lack of regularity which makes the arguments much more subtle invoking results for transport equations with a non-smooth vector field. Let (u_1, m_1) and (u_2, m_2) be two solutions of system (3.7) in the sense given above, and let us set $u := u_1 - u_2$ and $m = m_1 - m_2$. We use a regularization process to get the energy estimate (3.14). Then we get that $u_1 \equiv u_2$ and $\int_0^\ell m_1\partial_x u_1 = \int_0^\ell m_2\partial_x u_2$ in $\{m_1 > 0\}\cup\{m_2 > 0\}$, so that m_1 and m_2 are both solutions to

$$\partial_t m - \frac{1}{2}\partial_x((f_1(t) - \partial_x u_1)m) = 0, \;\; m(0) = m_0,$$

where $f_1(t) := \frac{2}{2+\kappa} + \frac{\kappa}{2+\kappa}\int_0^\ell m_1(t,x)\partial_x u_1(t,x)\,dx$. In orded to conclude that $m_1 \equiv m_2$, we invoke the following Lemma:

LEMMA 3.18. *Assume that v is a viscosity solution to*

$$\partial_t v - rv + \frac{1}{4}(f_1(t) - \partial_x v)^2 = 0, \;\; v(T,x) = u_T(x),$$

then the transport equation

$$\partial_t m - \frac{1}{2}\partial_x((f_1(t) - \partial_x v)m) = 0, \;\; m(0) = m_0$$

possesses at most one weak solution in L^1.

The proof of Lemma 3.18 (see e.g. [**31**, Section 4.2]) relies on semi-concavity estimates for the solutions of Hamilton-Jacobi equations [**30**], and Ambrosio superposition principle [**7**,**8**]. □

CHAPTER 4

Approximate Equilibria for N-Player Dynamic Cournot Competition

Joint work with P. Jameson Graber, accepted for publication in "ESAIM: Control, Optimisation and Calculus of Variations".

1. Introduction

In the previous chapter, we introduced briefly the Bertrand & Cournot Mean Field Game model using the framework of [**49**], and we addressed several mathematical features of this model by completing the analysis which is found in [**70**].

As we already pointed out, the Bertrand and Cournot Mean Field Game model is intended to be an approximation of the N-Player dynamic Bertrand and Cournot competition respectively. However, very little is known so far on the rigorous link between the MFGs models and the N-Player dynamic games models in this context. Indeed, the classical theory cannot be applied to this specific case for two main reasons: on the one hand, because of the absorbing boundary conditions; and on the other hand, because the MFGs models belongs to the class of extended Mean Field Games (c.f. Section 2 and [**15**,**39**,**65**,**66**]). This has motivated the present work, in which we analyse rigorously this question in the specific context of Cournot competition.

We investigate the Large Population approximation for N-Player dynamic Cournot game with linear price schedule, and exhaustible resources. In this context, the producers' state variable is the reserves level, and the strategic variable is the rate of production. Producers disappear from the market as soon as they deplete their reserves, and the remaining active producers set continuously a non-negative rate of production, in order to manage their remaining reserves and maximize sales profit. Market demand is assumed to be linear so that the received price by any representative producer is given by (3.1).

From Cournot's standpoint, we can constraint the producers to choose a non-negative rate of production. In fact, choosing a negative rate of production is irrelevant from a modeling standpoint. Therefore, by using notations of Chapter 3-Section 1, the game value function u in this context is:

$$u(t,x) := \sup_{q \geqslant 0} \mathcal{J}_{BC}(t,x,\mathbf{m}).$$

Hence, the optimal production rate $q_{u,m}(t,x)$ is given by:

$$q_{u,m}(t,x) := \frac{1}{2}\left(1 - \kappa \int_Q q_{u,m}(t,y) m(t,y)\, dy - \partial_x u(t,x)\right)^+, \tag{4.1}$$

and the corresponding price $p_{u,m}(t,x)$ is obtained by plugging (4.1) in (3.1). Note that in contrast to (3.6a), the optimal feedback production rate is less regular in this context and can not be explained as a functional of u and m as in (3.6a). Nevertheless, we will prove that the function $q_{u,m}$ enjoys several features which play a key role in our analysis.

We shall start by studying the Cournot Mean Field Game system (3.6b),(3.6c),(4.1). We prove existence and uniqueness of regular solutions to that system by deriving suitable a priori estimates. We shall assume that the initial data is a probability measure that is supported on $(0,\ell]$, which entails that all producers start with positive reserves. Our analysis completes that which is found in Chapter 3 and [**70**], by treating the case of a less regular production function $q_{u,m}$ and initial measure data. Next, we prove that the feedback control given by the solution of the Mean Field Game system, provides an ε-Nash equilibrium (cf. Definition 4.9) to the corresponding N-Player Cournot game, where the error ε is arbitrary small for large enough N. We refer the reader to Section 3 for a definition of ε-Nash equilibria. This result shows that the Cournot MFG model is indeed a good approximation to the dynamic Cournot competition with finitely many players, and reinforces numerical methods based on the MFG approach. As in the classical theory, the key argument in the proof of this result is a suitable law of large numbers. In our context, the main mathematical challenge comes from the fact that agents interact through the boundary behaviour, and are coupled by means of their chosen production strategies. To prove a tailor-made law of large numbers, we employ a compactness method borrowed from [**76**, **89**], by showing tightness of the empirical process in the space of distribution valued *càdlàg* processes, endowed with Skorokhod's M1 topology [**89**]. In contrast to the classical tools used so far, this method does not provide an exact quantification of the error ε, which is its main downside. Nevertheless, this approach has proven to be convenient for studying systems with absorbing boundary conditions. We also believe that it could be extended to the case of a systemic common noise, just as [**89**] contains an analysis of a *stochastic* McKean-Vlasov equation. However, we do not address this case here, finding the analysis of the stochastic HJB/FP-system somewhat out of reach under our assumptions on the data (Cf. [**33**, Section 4] and the hypotheses found there).

For background on Skorokhod's topologies for real valued processes, we refer the reader to [**108**] and references therein. The M1 topology is extended to the space of tempered distributions, and to more general spaces in [**89**]. The fact that the feedback MFG control provides ε-Nash equilibria for the corresponding differential games with a large (but finite) number of players, was first noticed by Caines et al. [**24**, **25**] and further developed in several works (see e.g.[**42**, **81**] among others). A simple class of MFGs with absorbing boundary conditions is also addressed in [**26**], where the authors also show that a solution of the mean field game equations induces approximate Nash equilibria for the corresponding N-player games. Cournot games with exhaustible resources and finite number of agents is investigated by Harris et al. in [**77**], and the corresponding MFG models were studied in [**49**, **50**, **73**, **92**] with different variants, and numerical simulations.

This chapter is organized as follows: In Section 2 we introduce the Cournot Mean Field Game system, prove existence and uniqueness of regular solutions to that system by deriving suitable Hölder estimates. In Section 3 we explain the corresponding N-Player Cournot game, and show that the feedback control that is computed from the MFG system, is an ε-Nash equilibrium to the N-Player game. For that purpose, we start by showing the weak convergence of the empirical process with respect to the M1 topology, then we deduce the main result by recalling the interpretation of the MFG system in terms of games with a continuum of agents and "mean field" interactions.

Preliminaries Throughout this chapter we fix $\ell > 0$, define $Q := (0,\ell)$, and $Q_T := (0,T) \times (0,\ell)$. For a subset $\mathcal{D} \subset \overline{Q_T}$, we define $W^{1,2}_s(\mathcal{D})$ to be the space of elements of $L^s(\mathcal{D})$ having weak derivatives of the form $\partial_t^j \partial_x^k$ with $2j + k \leqslant 2$, endowed with the following norm:

$$\|w\|_{W^{1,2}_s} := \sum_{2j+k\leqslant 2} \|\partial_t^j \partial_x^k w\|_{L^s}.$$

Moreover, we set $\mathcal{C}_0(\mathcal{D})$ to be the space of all continuous functions on $\mathcal{D}$ that vanish at infinity ($\mathcal{C}_0(\mathcal{D}) = \mathcal{C}(\mathcal{D})$ when $\mathcal{D}$ is compact).

The space of $\mathbb{R}$-valued Radon measures on $\mathcal{D}$ is denoted $\mathfrak{M}(\mathcal{D})$, which we identify with $\mathcal{C}_0(\mathcal{D})^*$ endowed with weak* topology, and $\mathcal{P}(\mathcal{D})$, $\tilde{\mathcal{P}}(\mathcal{D})$ are respectively the convex subset of probability measures on $\mathcal{D}$, and the convex subset of sub-probability measures: that is the set of positive radon measures μ, s.t. $\mu(\mathcal{D}) \leqslant 1$. For any measure $\mu \in \mathfrak{M}(\mathcal{D})$, we denote by $\mathrm{supp}(\mu)$ the support of μ.

For simplicity, we fix a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, and suppose that is rich enough to fulfil the assumptions that will be formulated in this part, and we denote by u_T a smooth function on $\bar{Q}$ such that the first derivative of u_T denoted by $\dot{u}_T$ fulfils :

$$\dot{u}_T \geqslant 0 \quad \text{and} \quad u_T(0) = \dot{u}_T(\ell) = 0. \tag{$\mathcal{H}$ 1}$$

Let us recall a few basic facts on stochastic differential equation with reflecting boundary in a half-line. Given a random variable V that is supported on $(-\infty, \ell]$, we look for a pair of a.s. continuous and adapted processes $(X_t)_{t\geqslant 0}$ and $(\xi^X_t)_{t\geqslant 0}$ such that:

$$\begin{aligned} &X_t = V + \int_0^t b(s, X_s)\, ds + \sqrt{2\sigma} W_t - \int_0^t \mathbb{1}_{\{X_s = \ell\}}\, d\xi^X_s \in (-\infty, \ell], \\ &\xi^X_t = \int_0^t \mathbb{1}_{\{X_s=\ell\}}\, d\xi^X_s, \\ &X_0 = V, \;\; \xi^X_0 = 0, \quad \text{and } \xi^X \text{ is nondecreasing,} \end{aligned} \tag{4.2a}$$

where $(W_t)_{t\geqslant 0}$ is a $\mathbb{F}$-Wiener process that is independent of V. The random process $(X_t)_{t\geqslant 0}$ is the *reflected diffusion*, $(\xi^X_t)_{t\geqslant 0}$ is the *local time*, and the above set of equations is called the *Skorokhod problem*. Throughout this chapter, we shall write problem (4.2a) in the following simple form:

$$dX_t = b(t, X_t)\, dt + \sqrt{2\sigma}\, dW_t - d\xi^X_t, \quad X_0 = V.$$

Suppose that the function b is bounded, and satisfies for some $K > 0$ the following condition:

$$|b(t,x) - b(t,y)| \leqslant K|x - y| \tag{4.2b}$$

for all $t \in [0,T]$, and $x, y \in (-\infty, \ell]$. Then, it is well-known (see e.g. [9, 61]) that under these conditions, problem (4.2a) has a unique solution on $[0,T]$. Moreover, this solution is given explicitly by:

$$X_t := \Gamma_t(Y), \quad \xi_t^X := Y_t - \Gamma_t(Y); \tag{4.2c}$$

where the process $(Y_t)_{t\in[0,T]}$ is the solution to

$$Y_t = V + \int_0^t b(s, \Gamma_s(Y))\,ds + \sqrt{2\sigma}W_t, \tag{4.2d}$$

and where Γ is the so called *Skorokhod map*, that is given by

$$\Gamma_t(Y) := Y_t - \sup_{0\leqslant s\leqslant t} (\ell - Y_s)^- .$$

Furthermore, notice that

$$\xi_t^X - \xi_{t+h}^X \geqslant \inf_{\nu\in[0,h]}(Y_t - Y_{t+\nu}) \tag{4.2e}$$

for any $t \in [0,T)$ and $h \in (0, T-t)$. In fact, when $\xi_t^X < \xi_{t+h}^X$, then

$$0 < \xi_{t+h}^X := \sup_{0\leqslant s\leqslant t+h} (\ell - Y_s)^- = \sup_{t\leqslant s\leqslant t+h} (\ell - Y_s)^- = (Y_{\nu_0} - \ell)$$

for some $t \leqslant \nu_0 \leqslant t+h$. Therefore

$$\begin{aligned}\xi_t^X - \xi_{t+h}^X &= \sup_{0\leqslant s\leqslant t} (\ell - Y_s)^- - \sup_{0\leqslant s\leqslant t+h} (\ell - Y_s)^- \\ &\geqslant (Y_t - \ell) - (Y_{\nu_0} - \ell) \geqslant \inf_{\nu\in[0,h]}(Y_t - Y_{t+\nu}).\end{aligned}$$

This entails (4.2e) since the last inequality still holds when $\xi_t^X = \xi_{t+h}^X$.

Now we consider a boundary value problem for the *Fokker-Planck* equation. Let b in $L^2(Q_T)$, $m_0 \in \mathcal{P}(\bar{Q})$, and consider the following Fokker-Planck equation

$$\begin{cases} \partial_t m - \sigma\partial_{xx} m - \partial_x(bm) = 0 & \text{in } Q_T \\ m(0) = m_0 & \text{in } Q, \end{cases} \tag{4.3a}$$

complemented with the following mixed boundary conditions:

$$m(t,0) = 0, \quad \text{and} \quad \sigma\partial_x m(t,\ell) + b(t,\ell)m(t,\ell) = 0 \quad \text{on } (0,T). \tag{4.3b}$$

Then we define a *weak solution* to (4.3a)-(4.3b) to be a function $m \in L^1(Q_T)_+$ such that $m|b|^2$ in $L^1(Q_T)$, and

$$\int_0^T\int_0^\ell m(-\partial_t\phi - \sigma\partial_{xx}\phi + b\partial_x\phi)\,dx\,dt = \int_0^\ell \phi(0,.)\,dm_0 \tag{4.3c}$$

for every $\phi \in \mathcal{C}_c^\infty([0,T)\times\overline{Q})$ satisfying

$$\phi(t,0) = \partial_x\phi(t,\ell) = 0, \ \ \forall t \in (0,T). \tag{4.3d}$$

This is the definition given by Porretta in [102]. The only difference is that here we consider mixed boundary conditions and measure initial data.

When $m_0 \in L^1(Q)_+$, the problem (4.3a) endowed with periodic, Dirichlet or Neumann boundary conditions has several interesting features that were pointed out in [102, Section 3]. In particular, they are unique [**102**, Corollary 3.5] and enjoy some extra regularity [**102**, Proposition 3.10]. Note that these results still hold in the case of mixed boundary conditions (4.3b). Throughout this chapter, we shall use the results of [**102**, Section 3] for (4.3a)-(4.3b).

In the case where b is bounded, we shall use the fact that (4.3a)-(4.3b) admits a unique weak solution, for any $m_0 \in \mathcal{P}(\bar{Q})$. In fact, one can construct a solution by considering a suitable approximation of m_0, and then use the compactness results of [**102**, Proposition 3.10] in order to pass to the limit in $L^1(Q_T)$. The uniqueness is obtained by considering the dual equation, and using the same steps as for [**102**, Corollary 3.5] (we refer the reader to Proposition A.2).

2. Analysis of Cournot MFG System

This section is devoted to the analysis of the following coupled system of parabolic partial differential equations:

$$
\begin{cases}
\partial_t u + \sigma \partial_{xx} u - ru + q_{u,m}^2 = 0 \quad \text{in } Q_T, \\[2ex]
\partial_t m - \sigma \partial_{xx} m - \partial_x \{q_{u,m} m\} = 0 \quad \text{in } Q_T, \\[2ex]
m(t,0) = 0, \quad u(t,0) = 0, \quad \partial_x u(t,\ell) = 0 \quad \text{in } (0,T), \\[2ex]
m(0) = m_0, \quad u(T,x) = u_T(x), \quad \text{in } [0,\ell], \\[2ex]
\sigma \partial_x m + q_{u,m} m = 0 \quad \text{in } (0,T) \times \{\ell\},
\end{cases}
\tag{4.4}
$$

where the function $q_{u,m}$ involved in the system is given by:

$$
q_{u,m}(t,x) := \frac{1}{2}\left(1 - \kappa \bar{q}(t) - \partial_x u(t,x)\right)^+, \quad \text{where} \quad \bar{q}(t) := \int_0^\ell q_{u,m}(t,x) m(t,x)\, dx,
\tag{4.5}
$$

and $\kappa > 0$. We focus in Section 2 on the mathematical analysis of the PDE system (4.4).

Let us assume that:

$$
m_0 \in \mathcal{P}(\bar{Q}), \ \text{ and } \ \operatorname{supp}(m_0) \subset (0,\ell].
\tag{$\mathcal{H}2$}
$$

We shall say that a pair (u,m) is a solution to (4.4), if

(i) $u \in \mathcal{C}^{1,2}(Q_T)$, $u, \partial_x u \in \mathcal{C}(\overline{Q_T})$;
(ii) $m \in \mathcal{C}([0,T]; \mathfrak{M}(\bar{Q})) \cap L^1(Q_T)_+$, and $\|m(t)\|_{L^1} \leqslant 1$ for every $t \in (0,T]$;
(iii) the equation for u holds in the classical sense, while the equation for m holds in the weak sense (4.3c).

2.1. Preliminary Estimates. We start by giving an alternative convenient expression for the production rate function $q_{u,m}$. We aim to write $q_{u,m}$ as a functional of u_x, m and the market price function $p_{u,m}$, that is given by (3.1), namely:

$$
p_{u,m}(t,x) := 1 - (q_{u,m}(t,x) + \kappa \bar{q}(t)).
\tag{4.6}
$$

The latter expression means that the price $p_{u,m}(t,x)$ received by an atomic player with reserves x at time t, is a linear and nonincreasing function, of the player's production rate $q_{u,m}(t,x)$, and the aggregate production rate across all producers $\bar{q}(t)$.

We use a convenient *ad-hoc* Bertrand formulation for our problem. For any $\mu \in \mathfrak{M}(\bar{Q})$, we define

$$a(\mu) := \frac{1}{1+\kappa\eta(\mu)}; \quad c(\mu) := 1 - a(\mu); \quad \eta(\mu) := \int_0^\ell d|\mu| \tag{4.7}$$

and set

$$\overline{p}(t) := \frac{1}{\eta(m(t))}\int_0^\ell p_{u,m}(t,x)m(t,x)\,dx. \tag{4.8a}$$

By integrating (4.6) with respect to m and after a little algebra one recovers the following identity

$$a\left(m(t)\right) + c\left(m(t)\right)\overline{p}(t) = 1 - \kappa\bar{q}(t),$$

which entails

$$p_{u,m}(t,x) = a\left(m(t)\right) + c\left(m(t)\right)\overline{p}(t) - q_{u,m}(t,x), \tag{4.8b}$$

and

$$q_{u,m}(t,x) = \frac{1}{2}\left\{a\left(m(t)\right) + c\left(m(t)\right)\overline{p}(t) - u_x(t,x)\right\}^+. \tag{4.8c}$$

One can see this formulation as a *Bertrand and Cournot equivalence* (c.f. Chapter 3). For convenience, we shall often use (4.8c) as a definition for $q_{u,m}$.

In contrast to the MFG system studied in Chapter 3 and [**49**, **70**], $p_{u,m}$ has no explicit formula and is only defined as a fixed point through (4.8a)-(4.8c). The following Lemma makes that statement clear and point out a few facts on the market price function.

LEMMA 4.1. *Let $u \in L^\infty\left(0,T;\mathcal{C}^1(\bar{Q})\right)$, $m \in L^1(Q_T)_+$, and $\kappa > 0$. Then the market price function $p_{u,m}$ is well-defined through (4.8a)-(4.8c), belongs to $L^\infty(0,T;\mathcal{C}(\bar{Q}))$, and satisfies*

$$-\|\partial_x u\|_\infty \leqslant p_{u,m} \leqslant 1. \tag{4.9}$$

Moreover, if $\partial_x u$ is non-negative, then $p_{u,m}$ is non-negative as well.

PROOF. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x - \frac{1}{2}(x-y)^+$. Note that f is 1-Lipschitz in the first variable, and $\frac{1}{2}$-Lipschitz in the second. For any $p, w \in \mathbb{X} := L^\infty(0,T;\mathcal{C}(\bar{Q}))$, define

$$\mathfrak{L}(m,p)(t) := a\left(m(t)\right) + c\left(m(t)\right)\overline{p}(t), \quad \text{where} \quad \overline{p}(t) := \frac{1}{\eta(m(t))}\int_0^\ell p(t,x)m(t,x)\,dx,$$

and

$$\Lambda(w,m,p)(t,x) := f\left(\mathfrak{L}(m,p)(t), w(t,x)\right).$$

We note the following inequalities for future reference:

$$\left|\mathfrak{L}(m,p)(t)-\mathfrak{L}(m,p')(t)\right| \leqslant \frac{\kappa}{1+\kappa}\left\|p(t,\cdot)-p'(t,\cdot)\right\|_{\infty}, \tag{4.10a}$$

$$\left\|\Lambda(w,m,p)(t,\cdot)-\Lambda(w,m,p')(t,\cdot)\right\|_{\infty} \leqslant \frac{\kappa}{1+\kappa}\left\|p(t,\cdot)-p'(t,\cdot)\right\|_{\infty}, \tag{4.10b}$$

$$\left|\Lambda(w,m,p)(t,x)-\Lambda(w',m,p)(t,x)\right| \leqslant \frac{1}{2}\left|w(t,x)-w'(t,x)\right|, \tag{4.10c}$$

$$\left|\Lambda(w,m,p)(t,x)-\Lambda(w,m',p)(t,x)\right| \leqslant \left|\mathfrak{L}(m,p)(t)-\mathfrak{L}(m',p)(t)\right|. \tag{4.10d}$$

We aim to use Banach fixed point Theorem to show that

$$p = a(m) + c(m)\bar{p} - \frac{1}{2}\{a(m)+c(m)\bar{p}-\partial_x u\}^{+} \tag{4.11}$$

has a unique solution $p_{u,m} \in \mathbb{X}$, which satisfies (4.9). For any $p \in \mathbb{X}$, let us set

$$\psi(p) := \Lambda(\partial_x u, m, p) = a(m) + c(m)\bar{p} - \frac{1}{2}\{a(m)+c(m)\bar{p}-\partial_x u\}^{+}.$$

Observe that $\psi(\mathbb{X}) \subset \mathbb{X}$, and $p \leqslant 1$ entails $\psi(p) \leqslant 1$. Moreover, if we suppose that $p \geqslant -\|\partial_x u\|_\infty$, then it holds that

$$\psi(p) \geqslant -c(m)\|\partial_x u\|_\infty,$$

so that $\psi(p) \geqslant -\|\partial_x u\|_\infty$, since $c(m) < 1$. On the other hand, by appealing to (4.10b) we have

$$\|\psi(p_1)-\psi(p_2)\|_{\mathbb{X}} \leqslant \frac{\kappa}{1+\kappa}\|p_1-p_2\|_{\mathbb{X}} \quad \forall p_1, p_2 \in \mathbb{X}.$$

Therefore by invoking Banach fixed point Theorem, and the estimates above we deduce the existence of a unique solution $p_{u,m} \in \mathbb{X}$ to problem (4.11) satisfying (4.9).

When $\partial_x u$ is non-negative, note that $p \geqslant 0$ entails $\psi(p) \geqslant 0$, so that the same fixed point argument yields $p_{u,m} \geqslant 0$. ☐

Next, we collect some facts related to the Fokker-Planck equation (4.3a)-(4.3b).

LEMMA 4.2 (regularity of η). *Let m be a weak solution to (4.3a)-(4.3b), starting from some m_0 satisfying ($\mathcal{H}2$). Suppose that b is bounded, and satisfies (4.2b). Then the map $t \to \eta(t) := \eta(m(t))$ is continuous on $[0,T]$.*

Moreover, if in addition m_0 belongs to $L^1(Q)$, then we have:

(i) the function $t \to \eta(t)$ is locally Hölder continuous on $(0,T]$; namely, there exists $\gamma > 0$ such that

$$|\eta(t_1)-\eta(t_2)| \leqslant C(t_0,\|b\|_\infty)\,|t_1-t_2|^{\gamma} \quad \forall t_1,t_2 \in [t_0,T] \tag{4.12a}$$

for all $t_0 \in (0,T)$;

(ii) for any $\alpha > 0$ and $\phi \in \mathcal{C}^{\alpha}(\bar{Q})$, there exists $\beta > 0$ such that

$$\left|\int_0^{\ell}\phi(x)\,(m(t_1,x)-m(t_2,x))\,dx\right| \leqslant C(t_0,\|b\|_\infty,\|\phi\|_{\mathcal{C}^\alpha})|t_1-t_2|^{\beta} \quad \forall t_1,t_2 \in [t_0,T] \tag{4.12b}$$

for all $t_0 \in (0,T)$.

REMARK 4.3. This lemma shows that $t \mapsto m(t)$ is locally Hölder continuous in time in $(0, T]$ with respect to the $(\mathcal{C}^\alpha)^*$ topology; this is useful later to get equicontinuity for construction of a fixed point (cf. Section 2.3). Our method of proof does not allow us to show Hölder continuity on all of $[0, T]$, because it is based on heat kernel estimates, which degenerate as $t \to 0$ (cf. Equation (4.21)). However, we find it nontrivial to construct a counterexample.

PROOF. The proof requires several steps and lies on the probabilistic interpretation of m which we recall briefly here, and use in other parts of this chapter.

Step 1 (probabilistic interpretation): Consider the reflected diffusion process governed by

$$dX_t = -b(t, X_t)\,dt + \sqrt{2\sigma}\,dW_t - d\xi_t^X, \quad X_0 \sim m_0, \tag{4.13a}$$

where X_0 is $\mathcal{F}_0$-measurable, $(W_t)_{t\in[0,T]}$ is a $\mathbb{F}$-Wiener process that is independent of X_0, and set

$$\tau := \inf\{t \geqslant 0 : X_t \leqslant 0\} \wedge T. \tag{4.13b}$$

By virtue of the regularity assumptions on b, equation (4.13a) is well-posed in the classical sense. Furthermore, since the process $(\xi_t^X)_{t\geqslant 0}$ is monotone, $(X_t)_{t\in[0,T]}$ is a continuous semimartingale. Hence, by means of Itô's rule and the optional stopping theorem, we have for any test function $\phi \in \mathcal{C}_c^\infty([0,T)\times\overline{Q})$ satisfying (4.3d):

$$\mathbb{E}\left[\phi(0, X_0)\right] = \mathbb{E}\left[\int_0^\tau \left(-\partial_t\phi(v, X_v) - \sigma\partial_{xx}\phi(v, X_v) + \partial_x\phi(v, X_v)b(v, X_v)\right)\,dv\right].$$

and thus the law of X_t is a weak solution to the Fokker-Planck equation. The function b being bounded, one sees that

$$\mathbb{E}\left[\int_0^T b(s, X_s)^2\,ds\right] < \infty.$$

Therefore, by virtue of the uniqueness for (4.3a)-(4.3b) (cf. Proposition A.2), we obtain:

$$\int_A m(t,x)\,dx = \mathbb{P}(t < \tau; X_t \in A) \tag{4.13c}$$

for every Borel set $A \in \bar{Q}$ and for a.e. $t \in (0, T)$.

Step 2: Now, let us show that $t \to \mathbb{P}(t < \tau)$ is right continuous on $[0, T]$. In fact, we have for any $\epsilon > 0$ and $t \in [0, T]$

$$\begin{aligned} \mathbb{P}(t<\tau) - \mathbb{P}(t+h<\tau) &= \mathbb{P}(t+h \geqslant \tau; t < \tau) \\ &\leqslant \mathbb{P}(t+h \geqslant \tau; X_t \geqslant \epsilon) + \mathbb{P}(t<\tau; X_t < \epsilon). \end{aligned} \tag{4.14a}$$

On the one hand, for every $t \in [0, T]$

$$\lim_{\epsilon\to 0^+} \mathbb{P}(t<\tau; X_t<\epsilon) \leqslant \lim_{\epsilon\to 0^+} \mathbb{P}\left(0 < X_t < \epsilon\right) = 0, \tag{4.14b}$$

thanks to the bounded convergence theorem. On the other hand

$$\begin{aligned}\mathbb{P}(t+h \geqslant \tau; X_t \geqslant \epsilon) &\leqslant \mathbb{P}\left(\inf_{\nu\in[t,t+h]} X_\nu - X_t \leqslant -\epsilon\right)\\ &\leqslant \mathbb{P}\left(\inf_{\nu\in[0,h]} \sqrt{2\sigma}(W_{t+\nu}-W_t) + (\xi^X_t - \xi^X_{t+h}) \leqslant -\epsilon + h\|b\|_\infty\right),\end{aligned}$$

where we have used the fact that the local time is nondecreasing and b is bounded. Furthermore, by using (4.2e), it holds that

$$\xi^X_t - \xi^X_{t+h} \geqslant \inf_{\nu\in[0,h]}(Y_t - Y_{t+\nu}) \geqslant \sqrt{2\sigma}\inf_{\nu\in[0,h]}(W_t - W_{t+\nu}) - h\|b\|_\infty.$$

Therefore

$$\mathbb{P}(t+h \geqslant \tau; X_t \geqslant \epsilon) \leqslant \mathbb{P}\left(\sup_{\nu\in[0,h]} B_\nu - \inf_{\nu\in[0,h]} B_\nu \geqslant \frac{\epsilon - 2h\|b\|_\infty}{\sqrt{2\sigma}}\right),$$

where $(B_t)_{t\geqslant 0}$ is a Wiener process.

Now, choose $\epsilon = \epsilon(h) := h^{1/2}\log(1/h)$. We have $\epsilon(h) \to 0$ as $h \to 0^+$, and by using Markov's inequality and the distribution of the maximum of Brownian motion we get:

$$\mathbb{P}(t+h \geqslant \tau; X_t \geqslant \epsilon) \leqslant \frac{2\sqrt{2\sigma}}{\epsilon(h) - 2h\|b\|_\infty}\mathbb{E}|B_h| \leqslant \frac{2\sqrt{2\sigma}}{\log(1/h) - 2\|b\|_\infty h^{1/2}}. \tag{4.14c}$$

Thus $0 \leqslant \mathbb{P}(t < \tau) - \mathbb{P}(t+h < \tau) \to 0$ as $h \to 0^+$.

Step 3 (Hölder estimates): Now, we prove (4.12a)-(4.12b). At first, note that (4.13c) entails

$$\int_0^\ell \phi(x) m(t,x)\,dx = \mathbb{E}\left[\phi(X_t)\mathbb{1}_{t<\tau}\right] \tag{4.15}$$

for a.e. $t \in (0,T)$ and for any $\phi \in \mathcal{C}(\bar{Q})$. Actually (4.15) holds for every $t \in [0,T]$, since the RHS and LHS of (4.15) are both right continuous on $[0,T]$, and m_0 is supported on $(0,\ell]$. Indeed, on the one hand $t \to \int_0^\ell \phi(x)m(t,x)\,dx$ is continuous on $[0,T]$ for any continuous function ϕ on $\bar{Q}$, since $m \in \mathcal{C}([0,T]; L^1(Q))$ (cf. [**102**, Theorem 3.6]). On the other hand, for any $\phi \in \mathcal{C}(\bar{Q})$

$$\begin{aligned}&\mathbb{E}\left|\phi(X_{t+h})\mathbb{1}_{t+h<\tau} - \phi(X_t)\mathbb{1}_{t<\tau}\right|\\ &\qquad\leqslant \|\phi\|_\infty(\mathbb{P}(t<\tau) - \mathbb{P}(t+h<\tau)) + \mathbb{E}\left|\phi(X_{t+h}) - \phi(X_t)\right|,\end{aligned} \tag{4.16}$$

so that

$$\lim_{h\to 0^+}\mathbb{E}\left|\phi(X_{t+h})\mathbb{1}_{t+h<\tau} - \phi(X_t)\mathbb{1}_{t<\tau}\right| = 0$$

thanks to (4.14a)-(4.14c), and the bounded convergence theorem.

Now, let us fix $\epsilon > 0$ and define $\phi_\epsilon = \phi_\epsilon(x)$ to be a smooth cut-off function on $[0,\ell]$, which satisfies the following conditions:

$$0 \leqslant \phi_\epsilon \leqslant 1;\quad 0 \leqslant \dot{\phi}_\epsilon \leqslant 2/\epsilon;\quad \phi_\epsilon\mathbb{1}_{[0,\epsilon]} = 0;\quad \phi_\epsilon\mathbb{1}_{[2\epsilon,\ell]} = 1. \tag{4.17}$$

As a first step, we aim to derive an estimation of the concentration of mass at the origine. Namely, we want to show that for an arbitrary $k > 1$,

$$\int_0^\ell (1-\phi_\epsilon(x))m(t,x)\,dx \leqslant C(k,\|b\|_\infty)\left(1-e^{-\pi^2 t/4\ell^2}\right)^{-1/2k}\epsilon^{1/2k} \quad \forall t\in(0,T]. \tag{4.18}$$

Given (4.15), this is equivalent to showing that

$$\mathbb{E}\left[(1-\phi_\epsilon(X_t))\mathbb{1}_{t<\tau}\right] \leqslant C(k,\|b\|_\infty)\left(1-e^{-\pi^2 t/4\ell^2}\right)^{-1/2k}\epsilon^{1/2k} \quad \forall t\in(0,T] \tag{4.19}$$

holds for any $k > 1$. Apply Girsanov's Theorem with the following change of measure:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{-(2\sigma)^{-1/2}\int_0^t b(s,X_s)\,dW_s - \frac{\sigma^{-1}}{4}\int_0^t b(s,X_s)^2\,ds\right\} =: \Psi_t.$$

Under $\mathbb{Q}$, the process $(X_t)_{t\in[0,T]}$ is a reflected Brownian motion at ℓ, with initial condition X_0, thanks to (4.2c). Moreover, by virtue of Hölder inequality, we have for every $k > 1$:

$$\begin{aligned}
&\mathbb{E}_\mathbb{P}\left[(1-\phi_\epsilon(X_t))\mathbb{1}_{t<\tau}\right] \\
&\quad= \mathbb{E}_\mathbb{Q}\left[\Psi_t^{-1}(1-\phi_\epsilon(X_t))\mathbb{1}_{t<\tau}\right] \leqslant \mathbb{E}_\mathbb{Q}[\Psi_t^{-k'}]^{1/k'}\mathbb{E}_\mathbb{Q}\left[(1-\phi_\epsilon(X_t))^k\mathbb{1}_{t<\tau}\right]^{1/k} \\
&\qquad\qquad \leqslant \mathbb{E}_\mathbb{P}[\Psi_t^{1-k'}]^{1/k'}\mathbb{E}_\mathbb{Q}\left[(1-\phi_\epsilon(X_t))^k\mathbb{1}_{t<\tau}\right]^{1/k} \\
&\qquad\qquad \leqslant \mathbb{E}_\mathbb{P}\left[\exp\left\{C(k,\sigma)\int_0^t b(s,X_s)^2\,ds\right\}\right]^{1/2k'}\mathbb{E}_\mathbb{Q}\left[(1-\phi_\epsilon(X_t))^k\mathbb{1}_{t<\tau}\right]^{1/k}.
\end{aligned}$$

Indeed, one checks that

$$\mathbb{E}_\mathbb{P}[\Psi_t^{1-k'}]^{1/k'} \leqslant \mathbb{E}_\mathbb{P}\left[Z_t\right]^{1/2k'}\mathbb{E}_\mathbb{P}\left[\exp\left\{2\left(\frac{1-k'}{\sigma}\right)^2\int_0^t b(s,X_s)^2\,ds\right\}\right]^{1/2k'},$$

where $(Z_t)_{t\geqslant 0}$ is a super-martingale. Using the fact that b is bounded, we obtain

$$\mathbb{E}_\mathbb{P}\left[(1-\phi_\epsilon(X_t))\mathbb{1}_{t<\tau}\right] \leqslant C(k,\|b\|_\infty)\mathbb{E}_\mathbb{Q}\left[(1-\phi_\epsilon(X_t))^k\mathbb{1}_{t<\tau}\right]^{1/k}. \tag{4.20}$$

Now

$$\mathbb{E}_\mathbb{Q}\left[(1-\phi_\epsilon(X_t))^k\mathbb{1}_{t<\tau}\right] = \int_0^\ell (1-\phi_\epsilon(x))^k w(t,x)\,dx,$$

where w solves

$$\partial_t w = \frac{1}{2}\partial_{xx}w,\ w(t,0)=0,\ \partial_x w(t,\ell)=0, w|_{t=0}=m_0.$$

We can compute w via Fourier series, namely

$$w(t,x) = \sum_{n\geqslant 1} A_n e^{-\lambda_n^2 t/2}\sin(\lambda_n x), \quad A_n := \frac{2}{\ell}\int_0^\ell \sin(\lambda_n y)\,dm_0(y), \quad \lambda_n := \frac{(2n-1)\pi}{2\ell}.$$

Note that

$$\int_0^\ell (1-\phi_\epsilon(x))^k w(t,x)\,dx \leqslant (2\epsilon)^{1/2}\|w(t,\cdot)\|_{L^2}$$
$$\leqslant \epsilon^{1/2}\left(\sum_{n\geqslant 1}\ell|A_n|^2 e^{-\lambda_n^2 t}\right)^{1/2} \quad \text{(Parseval)}$$
$$\leqslant \epsilon^{1/2}\left(\frac{4}{\ell(1-e^{-\pi^2 t/4\ell^2})}\right)^{1/2}.$$

So (4.20) now yields

$$\mathbb{E}_{\mathbb{P}}\left[(1-\phi_\epsilon(X_t))\mathbb{1}_{t<\tau}\right] \leqslant C(k,\|b\|_\infty)\left(\frac{4}{\ell(1-e^{-\pi^2 t/4\ell^2})}\right)^{1/2k}\epsilon^{1/2k} \tag{4.21}$$

which is (4.19). This in turn implies (4.18).

Furthermore, note that for any $1<s<3/2$,

$$\|m(t_1)-m(t_2)\|_{W_s^{-1}} \leqslant C\left(\|m_0\|_{L^1}, \|m|b|^2\|_{L^1}\right)|t_1-t_2|^{1-1/s} \quad \forall t_1,t_2\in[0,T], \tag{4.22}$$

where $W_s^{-1}(Q)$ is the dual space of $W_0^{1,s'}(Q) := \left\{v\in W_{s'}^1(Q) : v(0)=0\right\}$. This claim follows from [**102**, Proposition 3.10(iii)], where we obtain the estimate

$$\begin{aligned}\|m\|_{L^\infty(0,T;L^1(Q))} + \|\partial_x m\|_{L^s(Q_T)} + \|m\|_{L^\nu(Q_T)} + \|\partial_t m\|_{L^s(0,T;W_s^{-1}(Q))}\\ \leqslant C\left(\|m_0\|_{L^1}, \|m|b|^2\|_{L^1}\right)\end{aligned} \tag{4.23}$$

for any s up to $3/2$ and ν up to 3. In particular, (4.22) follows from the estimate on $\|\partial_t m\|_{L^s(0,T;W_s^{-1}(Q))}$. Now, fix $0<t_1\leqslant t_2\leqslant T$, and let ϕ_ϵ be the cut-off function that is defined in (4.17). Based on the specifications of (4.17), observe that

$$\|\phi_\epsilon\|_{W_{s'}^1} \leqslant C\epsilon^{-1/s}.$$

Since ϕ_ϵ satisfies Neumann boundary conditions at $x=\ell$ and Dirichlet at $x=0$, it is a valid test function and we can appeal to the estimates above to obtain for any $k>1$,

$$\begin{aligned}|\eta(t_1)-\eta(t_2)| &= \left|\int_0^\ell \{(1-\phi_\epsilon(x))+\phi_\epsilon(x)\}(m(t_1,x)-m(t_2,x))\,dx\right|\\
&\leqslant \int_0^\ell |1-\phi_\epsilon(x)||m(t_1,x)-m(t_2,x)|\,dx + \left|\int_0^\ell \phi_\epsilon(x)(m(t_1,x)-m(t_2,x))\,dx\right|\\
&\leqslant \int_0^\ell (1-\phi_\epsilon(x))(m(t_1,x)+m(t_2,x))\,dx + \left|\int_0^\ell \phi_\epsilon(x)(m(t_1,x)-m(t_2,x))\,dx\right|\\
&\leqslant C(k,\|b\|_\infty)\left(1-e^{-\pi^2 t_1/4\ell^2}\right)^{-1/2k}\epsilon^{1/2k} + \|\phi_\epsilon\|_{W_{s'}^1}\|m(t_1)-m(t_2)\|_{W_s^{-1}}\\
&\leqslant C(k,\|b\|_\infty)\left(1-e^{-\pi^2 t_1/4\ell^2}\right)^{-1/2k}\epsilon^{1/2k} + C\epsilon^{-1/s}|t_1-t_2|^{1-1/s}.\end{aligned} \tag{4.24}$$

Given $0<\gamma<(s-1)/(s+2)$, we take $\epsilon=|t_1-t_2|^{s(1-\gamma)-1}$ and then set $k=\frac{s(1-\gamma)-1}{2\gamma}>1$ to obtain (4.12a).

Finally, let $\phi \in \mathcal{C}^\alpha(\bar{Q})$ for some $\alpha > 0$, an let $t_0 \in (0, T)$. In view of (4.15), we have for every $t_1, t_2 \in [t_0, T]$,

$$\left|\int_0^\ell \phi(x)(m(t_1,x) - m(t_2,x))\,dx\right| \leqslant \mathbb{E}\,|\phi(X_{t_1})\mathbb{1}_{t_1<\tau} - \phi(X_{t_2})\mathbb{1}_{t_2<\tau}|$$
$$\leqslant \|\phi\|_{\mathcal{C}^\alpha}\left(|\eta(t_1) - \eta(t_2)| + \mathbb{E}\,|X_{t_1} - X_{t_2}|^\alpha\right).$$

Hence, by using (4.12a) and the Burkholder-Davis-Gundy inequality [**104**, Thm IV.42.1], we deduce the desired result:

$$\left|\int_0^\ell \phi(x)(m(t_1,x) - m(t_2,x))\,dx\right| \leqslant C(t_0, \|b\|_\infty)\|\phi\|_{\mathcal{C}^\alpha}\,|t_1 - t_2|^\beta\,,$$

for some $\beta > 0$.

Step 4 (general data): Now, we suppose that m_0 is a probability measure satisfying ($\mathcal{H}2$), and not necessarily an element of $L^1(Q)$. Let us choose a sequence $(m_0^n) \subset L^1(Q)_+$, which converges weakly (in the sense of measures) to m_0, such that

$$\|m_0^n\|_{L^1} \leqslant \int_0^\ell dm_0 \leqslant 1, \tag{4.25}$$

and let m^n to be the weak solution to (4.3a)-(4.3b) starting from m_0^n. The function b being bounded, we can use [**102**, Proposition 3.10] to extract a subsequence of (m^n), which converges to m in $L^1(Q_T)$. Owing to (4.12a), the sequence $\eta^n := \eta(m^n)$ is equicontinuous. Hence, one can extract further a subsequence to deduce that η is continuous on $(0, T]$. Combining this conclusion with the fact that $t \to \mathbb{P}(t < \tau)$ is right continuous on $[0, T]$ and (4.13c), we deduce in particular that

$$\eta(t) = \mathbb{P}(t < \tau), \quad \forall t \in (0, T]. \tag{4.26}$$

Now, since m_0 is supported on $(0, \ell]$ one has $\eta(m_0) = \eta(0) = \mathbb{P}(0 < \tau) = 1$, which in turn entails that η is continuous on $[0, T]$ thanks to (4.14a)-(4.14c) and (4.26). The proof is complete. □

REMARK 4.4. When m_0 satisfies ($\mathcal{H}2$) and does not necessarily belong to $L^1(Q)$, the probabilistic characterisation (4.15) still holds for every $t \in [0, T]$. In fact, using the same approximation techniques as in Lemma 4.2- Step 4, and appealing to (4.12b) and (4.13c), it holds that

$$\int_0^\ell \phi(x)m(t,x)\,dx = \mathbb{E}\left[\phi(X_t)\mathbb{1}_{t<\tau}\right]$$

for every $t \in [0, T]$, $\alpha > 0$ and $\phi \in \mathcal{C}^\alpha(\bar{Q})$. Thus, (4.15) ensues by using density arguments.

2.2. A Priori Estimates. Now, we collect several a priori estimates for system (4.4).

LEMMA 4.5. *Suppose that (u, m) satisfies the system (4.4) such that $m \in L^1(Q_T)_+$, and u belongs to $W_s^{1,2}(Q_T)$ for large enough $s > 1$. Then, we have:*

(i) the maps u and $\partial_x u$ are non-negative; in particular

$$0 \leqslant q_{u,m} \leqslant 1/2; \tag{4.27}$$

(ii) there exists $\theta > 0$ *and a constant* $c_0 > 0$ *such that*

$$\|u\|_{C^\theta(\overline{Q_T})}, \|\partial_x u\|_{C^\theta(\overline{Q_T})} \leqslant c_0 \tag{4.28}$$

where c_0 *depends only on* T *and data. In addition, we have*

$$\|\partial_{xx} u\|_{C^\theta(Q')}, \leqslant c_1(Q', \theta) \ \ \forall Q' \subset\subset (0,T) \times (0,\ell];$$

If in addition m_0 *belongs to* $L^1(Q)$*, then there exists a Hölder exponent* $\theta > 0$ *such that*

$$\|p_{u,m}\|_{C^\theta([t_0,T]\times[0,\ell])} \leqslant c_2(t_0,\theta), \quad \forall t_0 \in (0,T),$$

and

$$\|\partial_t u\|_{C^\theta(Q')} \leqslant c_2(Q',\theta) \ \ \forall Q' \subset\subset (0,T) \times (0,\ell].$$

PROOF. For large enough $s > 1$, we know that $u, \partial_x u \in \mathcal{C}(\overline{Q_T})$ thanks to Sobolev-Hölder embeddings. In view of

$$-\partial_t u - \sigma \partial_{xx} u + ru \geqslant 0,$$

one easily deduces that $u \geqslant e^{-rT} \min_x u_T$, which entails in particular that $u \geqslant 0$ thanks to ($\mathcal{H}$ 1). Thus, the minimum is attained at $u(t,0) = 0$, so that $\partial_x u(t,0) \geqslant 0$ for all $t \in [0,T]$. Differentiating the first equation in (4.4) we have that $\partial_x u$ is a generalised solution (cf. [85, Chapter III]) of the following parabolic equation:

$$\partial_{xt} u + \sigma \partial_{xxx} u - r \partial_x u - q_{u,m} \partial_{xx} u = 0.$$

By virtue of the maximum principle [85, Theorem III.7.1] we infer that $\partial_x u \geqslant 0$, since $\partial_x u(t,0)$, $\partial_x u(t,\ell)$ and $\dot{u}_T$ are all non-negative functions. Therefore (4.27) follows straightforwardly from (4.8c) thanks to Lemma 4.1.

Note that u solves a parabolic equation with bounded coefficients. Since compatibility conditions of order zero are fulfilled thanks to ($\mathcal{H}$ 1), then from [85, Theorem IV.9.1] we have an estimate on u in $W^{1,2}_k(Q_T)$ for arbitrary $k > 1$, namely

$$\|u\|_{W^{1,2}_k(Q_T)} \leqslant C\left(\|q_{u,m}\|_{L^k(Q_T)} + \|u_T\|_{W_k^{2-\frac{2}{k}}(Q_T)}\right) \leqslant C\left(\|q_{u,m}\|_{L^\infty(Q_T)} + \|u_T\|_{W_k^{2-\frac{2}{k}}(Q_T)}\right). \tag{4.29}$$

This estimate depends only on T, k and data, thanks to (4.27). We deduce (4.28) thanks to Sobolev-Hölder embeddings.

Now, let $\phi \in \mathcal{C}_c^\infty((0,T) \times (0,+\infty))$. Observe that $w = \phi \partial_x u$ satisfies

$$\partial_t w + \sigma \partial_{xx} w - rw - q_{u,m} \partial_x w = \partial_t \phi \partial_x u + 2\sigma \partial_x \phi \partial_{xx} u + \sigma \partial_{xx} \phi \partial_x u - q_{u,m} \partial_x \phi \partial_x u.$$

For any $k > 1$, the right-hand side is bounded in $L^k(Q_T)$ with a constant that depends only on ϕ, and previous estimates. Since w has homogeneous boundary conditions, we deduce from [85, Theorem IV.9.1] that $\|\partial_x w\|_{C^\theta(\overline{Q_T})}$ is bounded by a constant depending only on the norm of ϕ and previous estimates. The local Hölder estimate on $\partial_{xx} u$ then follows.

Let $p(t,x) = p_{u,m}(t,x)$. Recall that $p(t,x) = f(\mathfrak{L}(m,p)(t), \partial_x u(t,x))$ where $f(x,y) := x - \frac{1}{2}(x-y)^+$ (cf. Lemma 4.1). Since f is 1-Lipschitz in the first variable and $\frac{1}{2}$-Lipschitz in the second, we deduce that

$$|p(t_1,x_1) - p(t_2,x_2)| \leqslant |\mathfrak{L}(m,p)(t_1) - \mathfrak{L}(m,p)(t_2)| + \frac{1}{2}|\partial_x u(t_1,x_1) - \partial_x u(t_2,x_2)|. \tag{4.30}$$

In particular, for each t,

$$|p(t,x_1) - p(t,x_2)| \leqslant \frac{1}{2}|\partial_x u(t,x_1) - \partial_x u(t,x_2)| \tag{4.31}$$

which, by (4.28), implies that $p(t,\cdot)$ is Hölder continuous for every t.

Now, we further assume that $m_0 \in L^1(Q)_+$ to use (4.12a)-(4.12b). We shall use the following function which is introduced in Lemma 4.1:

$$\ell(m,p)(t) = a\,(m(t)) + c\,(m(t))\,\overline{p}(t), \quad \text{where} \quad \overline{p}(t) = \frac{1}{\eta(m(t))}\int_0^L p(t,x)m(t,x)\,dx.$$

Fix $t_0 \in (0,T)$ and for t_1, t_2 in $[t_0,T]$ write

$$\begin{aligned} \mathfrak{L}(m,p)(t_1) - \mathfrak{L}(m,p)(t_2) &= a(m(t_1)) - a(m(t_2)) \\ &+ \kappa(a(m(t_1)) - a(m(t_2)))\int_0^\ell p(t_1,.)\,dm(t_1) \\ &+ \kappa a(m(t_2))\int_0^\ell p(t_1,.)\,d(m(t_1) - m(t_2)) \\ &+ \kappa a(m(t_2))\int_0^\ell (p(t_1,.) - p(t_2,.))\,dm(t_2), \end{aligned} \tag{4.32}$$

where we have used the fact that $c(m) = \kappa a(m)\eta(m)$. Observe that $\eta \to \frac{1}{1+\kappa\eta}$ is κ-Lipschitz in the η variable, and recall that $p(t_1,\cdot)$ is Hölder continuous. Moreover, by virtue of (4.28) we know that $q_{u,m}$ satisfies (4.2b). Therefore, using the upper bound on $a(m), c(m)$ and (4.12a)-(4.12b) we infer that

$$|\mathfrak{L}(m,p)(t_1) - \mathfrak{L}(m,p)(t_2)| \leqslant C|t_1 - t_2|^\beta + \frac{\kappa}{1+\kappa}\|p(t_1,\cdot) - p(t_2,\cdot)\|_\infty. \tag{4.33}$$

Note that the constant in (4.33) depend only on c_0 and κ thanks to (4.27), (4.28) and Lemma 4.1. Using now (4.33) in (4.30), and choosing θ small enough, we deduce

$$\frac{1}{1+\kappa}\|p(t_1,\cdot)-p(t_2,\cdot)\|_\infty \leqslant C|t_1-t_2|^\beta + \frac{1}{2}\|\partial_x u(t_1,\cdot)-\partial_x u(t_2,\cdot)\|_\infty \leqslant C|t_1-t_2|^\theta. \tag{4.34}$$

Putting together (4.31) and (4.34) we infer that p has a Hölder estimate, whereupon by (4.33) so does $\mathfrak{L}(m,p)$. Thus $q_{u,m}$ also has a Hölder estimate, and so does $\partial_t u$ by the HJB equation satisfied by u. □

2.3. Well-Posedness. We are now in position to prove the main result of this section:

THEOREM 4.6. *There exists a unique solution (u,m) to system (4.4).*

PROOF OF THEOREM 4.6. The proof requires several steps, the key arguments being precisely the estimates collected in Lemmas 4.1-4.5.

Step 1 (data in L^1): We suppose that m_0 is an element of $L^1(Q)$ satisfying ($\mathcal{H}$ 1). Define $\mathbb{X}$ to be the space of couples (φ,ν), such that φ and $\partial_x\varphi$ are globally continuous on $\overline{Q_T}$, and ν belongs to $L^1(Q_T)_+$. The functional space $\mathbb{X}$ endowed with the norm:

$$\|(\varphi,\nu)\|_{\mathbb{X}} := \|\varphi\|_\infty + \|\partial_x\varphi\|_\infty + \|\nu\|_{L^1}$$

is a Banach space. Consider the map $\mathbb{T} : (\varphi, \nu, \lambda) \in \mathbb{X} \times [0,1] \to (w, \mu)$ where (w, μ) are given by the following parametrized system of coupled partial differential equations:

$$\text{(4.35)} \qquad \begin{cases} (i) \quad \partial_t w + \sigma \partial_{xx} w - rw + \lambda^2 q_{\varphi,\nu}^2 = 0 \quad \text{in } Q_T, \\ \\ (ii) \quad \partial_t \mu - \sigma \partial_{xx} \mu - \partial_x \{\lambda q_{\varphi,\nu} \mu\} = 0 \quad \text{in } Q_T, \\ \\ (iii) \quad \mu(t,0) = 0, \quad w(t,0) = 0, \quad \partial_x w(t,\ell) = 0 \quad \text{in } [0,T], \\ \\ (iv) \quad \mu(0) = \lambda m_0, \quad w(T,x) = \lambda u_T(x) \quad \text{in } [0,\ell], \\ \\ (v) \quad \sigma \partial_x \mu + \lambda q_{\varphi,\nu} \mu = 0 \quad \text{in } [0,T] \times \{\ell\}. \end{cases}$$

By virtue of Lemma 4.1, the map $q_{\varphi,\nu}$ is well-defined for any $(\varphi, \nu) \in \mathbb{X}$, and satisfy

$$\text{(4.36)} \qquad |q_{\varphi,\nu}| \leqslant C(1 + \|\partial_x \varphi\|_\infty).$$

In view of [85, Theorem IV.9.1], the function w exists and is bounded in $W_s^{1,2}(Q_T)$ for any $s > 1$, by a constant which depends on $\|\partial_x \varphi\|_\infty$ and data. Note that the required compatibility conditions hold owing to ($\mathcal{H}$ 1). Although [85, Theorem IV.9.1] is stated for Dirichlet boundary conditions, its proof is readily adapted to Neumann or mixed boundary conditions as in the present context; cf. the discussion in the first paragraph of [85, Section IV.9]). We deduce that

$$\|w\|_{\mathcal{C}^\alpha} + \|\partial_x w\|_{\mathcal{C}^\alpha} \leqslant C(T, \ell, u_T, \|\partial_x \varphi\|_\infty)$$

for some $\alpha > 0$. On the other hand, it is well known (see e.g. [85, Chapter III]) that for any $(\varphi, \nu) \in \mathbb{X}$, equation (4.35)(ii) has a unique weak solution μ. Therefore, $\mathbb{T}$ is well-defined. Let us now prove that $\mathbb{T}$ is continuous and compact. Suppose $(\varphi_n, \nu_n, \lambda_n)$ is a a bounded sequence in $\mathbb{X} \times [0,1]$ and let $(w_n, \mu_n) = \mathbb{T}(\varphi_n, \nu_n, \lambda_n)$. To prove compactness, we show that, up to a subsequence, (w_n, μ_n) converges to some (w, μ) in $\mathbb{X}$. Since $\partial_x \varphi_n$ is uniformly bounded, by virtue of [102, Proposition 3.10], the sequence μ_n is relatively compact in $L^1(Q_T)_+$, thanks to (4.36) (cf. (4.37) below where more details are given). Since w_n and $\partial_x w_n$ are uniformly bounded in $\mathcal{C}^\alpha(\overline{Q}_T)$, by the Ascoli-Arzelà Theorem and uniform convergence of the derivative there exists some w such that $w, \partial_x w$ are continuous in $\overline{Q_T}$ and, passing to a subsequence, $w_n \to w$ and $\partial_x w_n \to \partial_x w$ uniformly, where in fact $w_n \rightharpoonup w$ weakly in $W_s^{1,2}(Q_T)$ for any $s > 1$. This is what we wanted to show.

To prove continuity, we assume $(\varphi_n, \nu_n, \lambda_n) \to (\varphi, \nu, \lambda)$ in $\mathbb{X} \times [0,1]$. It is enough to show that, after passing to a subsequence, $\mathbb{T}(\varphi_n, \nu_n, \lambda_n) \to \mathbb{T}(\varphi, \nu, \lambda)$. By the preceding argument, we can assume $\mathbb{T}(\varphi_n, \nu_n, \lambda_n) \to (w, \mu)$. We can also use estimates (4.10b)-(4.10d) to deduce that $q_{\varphi_n,\nu_n} \to q_{\varphi,\nu}$ a.e. (cf. the proof of Equation (4.40) below), and since q_{φ_n,ν_n} is uniformly bounded we can also assert $q_{\varphi_n,\nu_n} \to q_{\varphi,\nu}$ in L^s for any $s \geqslant 1$. Then we deduce that (w, μ) is a solution of (4.35) for the given (φ, ν, λ). Therefore, $(w, \mu) = \mathbb{T}(\varphi, \nu, \lambda)$, as desired.

Now, let $(u, m) \in \mathbb{X}$ and $\lambda \in [0,1]$ so that $(u, m) = \mathbb{T}(u, m, \lambda)$. Then (u, m) satisfies assumptions of Lemma 4.5 with $m_0, u_T, q_{u,m}$ replaced by $\lambda m_0, \lambda u_T$ and $\lambda q_{u,m}$, respectively. Since the bounds of Lemma 4.5 carry through uniformly in $\lambda \in [0,1]$ we infer that

$$\|(u,m)\|_{\mathbb{X}} \leqslant 1 \vee c_0,$$

where $c_0 > 0$ is the constant of Lemma 4.5. In addition, for $\lambda = 0$ we have $\mathbb{T}(u, m, 0) = (0, 0)$. Therefore, by virtue of Leray-Schauder fixed point Theorem (see e.g. [**63**, Theorem 11.6]), we deduce the existence of a solution (u, m) in $\mathbb{X}$ to system (4.4).

Step 2 (measure data): We deal now with general m_0, i.e. a probability measure that is supported on $(0, \ell]$. Let $(m_0^n) \subset L^1(Q)_+$ be a sequence of functions, which converges weakly (in the sense of measures) to m_0, and such that

$$\|m_0^n\|_{L^1} \leqslant \int_0^\ell dm_0 \leqslant 1, \text{ and } \operatorname{supp}(m_0^n) \subset (0, \ell].$$

For any $n \geqslant 1$, define (u^n, m^n) to be a solution in $\mathbb{X}$ to system (4.4) starting from m_0^n.

In view of [**102**, Proposition 3.10 (iii)] and (4.27), the corresponding solutions m^n to the non-local Fokker-Planck equation lie in a relatively compact set of $L^1(Q_T)$. Moreover, it holds that

$$m^n \geqslant 0 \text{ and } \sup_{0 \leqslant t \leqslant T} \|m^n(t)\|_{L^1} \leqslant \int_0^\ell dm_0. \tag{4.37}$$

Passing to a subsequence we have $m^n \to m$ in $L^1(Q_T)$, $m^n(t) \to m(t)$ in $L^1(Q)$ for a.e. t in $(0, T)$, and $m^n \to m$ for a.e. (t, x) in Q_T. It follows that $m \in L^1(Q_T)_+$ and

$$\|m(t)\|_{L^1} \leqslant 1 \quad \text{for a.e. } t \in (0, T). \tag{4.38}$$

In addition, we know that $q_{u,m}$ fulfils the assumptions of Lemma 4.2. Thus $t \to \|m(t)\|_{L^1}$ is continuous on $(0, T]$, so that (4.38) holds for avery $t \in (0, T]$. Furthermore, we can appeal to the probabilistic characterisation (4.15), thanks to Remark 4.4, to get

$$\begin{aligned}\left|\int_0^\ell \phi(x)(m(t+h,x) - m(t))\,dx\right| &\leqslant \mathbb{E}\,|\phi(X_{t+h})\mathbb{1}_{t+h<\tau} - \phi(X_t)\mathbb{1}_{t<\tau}| \\ &\leqslant \|\phi\|_\infty|\eta(t) - \eta(t+h)| + \mathbb{E}\,|\phi(X_{t+h}) - \phi(X_t)|\end{aligned}$$

for every $\phi \in \mathcal{C}(\bar{Q})$, and $t \in [0, T]$. Now owing to Lemma 4.2, η is continuous on $[0, T]$. Hence, by taking the limit in the last estimation we infer that

$$\lim_{h \to 0} \int_0^\ell \phi(x)(m(t+h,x) - m(t))\,dx = 0$$

thanks to the bounded convergence theorem. Consequently the map $t \to m(t)$ is continuous on $[0, T]$ with respect to the strong topology of $\mathfrak{M}(\bar{Q})$.

On the other hand, by Lemma 4.5 we have that u^n, $\partial_x u^n$ are uniformly bounded in $\mathcal{C}^\theta(\overline{Q}_T)$, and $\partial_t u^n$, $\partial_{xx} u^n$ are uniformly bounded in $\mathcal{C}^\theta(Q')$ for each $Q' \subset\subset (0, T) \times (0, \ell]$. Thus, up to a subsequence we obtain that $u, \partial_x u \in \mathcal{C}(\overline{Q_T})$, and

$$u^n \to u \in \mathcal{C}^{1,2}((0, T) \times (0, \ell]) \tag{4.39}$$

where the convergence is in the $\mathcal{C}^{1,2}$ norm on arbitrary compact subsets of $(0, T) \times (0, \ell]$.

To show that the Hamilton-Jacobi equation holds in a classical sense and the Fokker-Planck equation holds in the sense of distributions, it remains to show that

$$q_{u^n,m^n} \to q_{u,m} \text{ a.e.} \tag{4.40}$$

at least on a subsequence. Set $p^n = p_{u^n,m^n} = \Lambda(\partial_x u^n, m^n, p^n)$ and $p = p_{u,m} = \Lambda(\partial_x u, m, p)$, with Λ defined in Lemma 4.1. Using (4.10b)-(4.10d) we get

$$\begin{aligned}(4.41)\quad \|p^n(t,\cdot)-p(t,\cdot)\|_\infty &\leqslant \|\Lambda(\partial_x u^n, m^n, p^n)(t,\cdot) - \Lambda(\partial_x u, m^n, p^n)(t,\cdot)\|_\infty \\ &+\|\Lambda(\partial_x u, m^n, p^n)(t,\cdot)-\Lambda(\partial_x u, m^n, p)(t,\cdot)\|_\infty+\|\Lambda(\partial_x u, m^n, p)(t,\cdot)-\Lambda(\partial_x u, m, p)(t,\cdot)\|_\infty \\ &\leqslant \frac{1}{2}\|\partial_x u^n - \partial_x u\|_\infty + \frac{\kappa}{1+\kappa}\|p^n(t,\cdot)-p(t,\cdot)\|_\infty + |\mathfrak{L}(m^n,p)(t) - \mathfrak{L}(m,p)(t)|\end{aligned}$$

which means

$$\|p^n(t,\cdot)-p(t,\cdot)\|_\infty \leqslant \frac{1+\kappa}{2}\|\partial_x u^n - \partial_x u\|_\infty + (1+\kappa)|\mathfrak{L}(m^n,p)(t) - \mathfrak{L}(m,p)(t)|. \tag{4.42}$$

Noting that (up to a subsequence) $m^n(t) \to m(t)$ in $L^1(Q)$ a.e., we use the fact that $a(m), c(m), \eta(m)$ are all continuous with respect to this metric to deduce that

$$|\mathfrak{L}(m^n,p)(t) - \mathfrak{L}(m,p)(t)| \to 0 \;\text{ a.e. } t \in (0,T) \tag{4.43}$$

from which we conclude that

$$\|p^n(t,\cdot)-p(t,\cdot)\|_\infty \to 0 \;\text{ a.e. } t \in (0,T). \tag{4.44}$$

Now from (4.44) and (4.10a) we have

$$|\mathfrak{L}(m,p^n)(t) - \mathfrak{L}(m,p)(t)| \to 0 \;\text{ a.e. } t \in (0,T). \tag{4.45}$$

Combining (4.43) and (4.45) we see that $\mathfrak{L}(m^n,p^n) \to \mathfrak{L}(m,p)$ a.e. We deduce (4.40) from the definition (4.8c). Therefore (u^n, m^n) converges to some (u,m) which is a solution to (4.4) with initial data m_0.

Step 3 (uniqueness): Let $(u_i, m_i), i = 1,2$ be two solutions of (4.4). We set

$$G_i := q_{u_i,m_i} \;\text{ and }\; \bar{G}_i := \int_0^\ell q_{u_i,m_i}(t,y)\,dm_i(t).$$

From (4.5), we know that

$$G_i = \frac{1}{2}\left(1 - \kappa\bar{G}_i - \partial_x u_i\right)^+. \tag{4.46}$$

Let $u = u_1 - u_2, m = m_1 - m_2, G = G_1 - G_2, \bar{G} = \bar{G}_1 - \bar{G}_2$. Using $(t,x) \to e^{-rt}u(t,x)$ as a test function in the equations satisfied by m_1, m_2, with some algebra yields

$$\begin{aligned}(4.47)\quad 0 &= \int_0^T e^{-rt}\int_0^\ell (G_2^2 - G_1^2 - G_1\partial_x u)m_1 + (G_1^2 - G_2^2 + G_2\partial_x u)m_2\,dx\,dt \\ &= \int_0^T e^{-rt}\int_0^\ell (G_1 - G_2)^2(m_1+m_2)\,dx\,dt + \int_0^T e^{-rt}\int_0^\ell (2G + \partial_x u)(G_2 m_2 - G_1 m_1)\,dx\,dt.\end{aligned}$$

Now since $G_2 = 0$ on the set where $1 - \kappa\bar{G}_2(t) - \partial_x u_2 < 0$, we can write

$$\begin{aligned}(2G + \partial_x u)G_2 &= \left(\left(1 - \kappa\bar{G}_1 - \partial_x u_1\right)^+ - \left(1 - \kappa\bar{G}_2(t) - \partial_x u_2\right) + \partial_x u_1 - \partial_x u_2\right) G_2\\ &= \left(-\kappa\bar{G} + \left(1 - \kappa\bar{G}_1 - \partial_x u_1\right)^-\right) G_2.\end{aligned}$$

Similarly we can write

$$\begin{aligned}(2G + \partial_x u)G_1 &= \left(\left(1 - \kappa\bar{G}_1 - \partial_x u_1\right) - \left(1 - \kappa\bar{G}_2(t) - \partial_x u_2\right)^+ + \partial_x u_1 - \partial_x u_2\right) G_1\\ &= \left(-\kappa\bar{G} - \left(1 - \kappa\bar{G}_2 - \partial_x u_2\right)^-\right) G_1.\end{aligned}$$

Thus we compute

$$\begin{aligned}\int_0^\ell (2G + \partial_x u)(G_2 m_2 - G_1 m_1)\,dx\,dt = \kappa\bar{G}^2 + \int_0^\ell \left(1 - \kappa\bar{G}_1 - \partial_x u_1\right)^- G_2 m_2\,dx\,dt\\ + \int_0^\ell \left(1 - \kappa\bar{G}_2 - \partial_x u_2\right)^- G_1 m_1\,dx\,dt \geqslant \kappa\bar{G}^2.\end{aligned}$$

So from (4.47) we conclude

$$\int_0^T e^{-rt}\int_0^\ell (G_1 - G_2)^2(m_1 + m_2)\,dx\,dt + \kappa\int_0^T e^{-rt}(\bar{G}_1 - \bar{G}_2)^2\,dt = 0. \tag{4.48}$$

In particular, $\bar{G}_1 \equiv \bar{G}_2$. We can then appeal to uniqueness for the Hamilton-Jacobi equation to get $u_1 \equiv u_2$ (cf. [85, Chapter V]). By (4.46), this entails that $G_1 \equiv G_2$, and so $m_1 \equiv m_2$ by uniqueness for the Fokker-Planck equation. □

3. Application of the MFG Approach

In this section, we present the N-Player Cournot game with limited resources, and build an approximation of Nash equilibria to that game when N is large, by means of the Mean Field Game system (4.4). Namely, we show that the optimal feedback strategies, computed from the MFG system (4.4), provides an ε-Nash equilibria for the N-Player Cournot game, where the error ε is arbitrarily small as $N \to \infty$.

Throughout this section (u, m) is the solution to (4.4) starting from some probability measure m_0 satisfying ($\mathcal{H}$ 1), and the function $q_{u,m}$ is given by (4.8c).

3.1. Cournot Game with Linear Demand and Exhaustible Resources. We start by introducing the N-Player Cournot game. Consider a market with N producers of a given good, whose strategic variable is the rate of production and where raw materials are in limited supply. Concretely, one can think of energy producers that use exhaustible resources, such as oil, to produce and sell energy. Firms disappear from the market as soon as they deplete their reserves of raw materials.

Let us formalize this model in precise mathematical terms. Let $\left(W^j\right)_{1\leqslant j\leqslant N}$ be a family of N independent $\mathbb{F}$-Wiener processes on $\mathbb{R}$, and consider the following system of Skorokhod problems:

$$\begin{cases} dX_t^i = -q_t^i\,dt + \sqrt{2\sigma}\,dW_t^i - d\xi_t^{X^i}, \\ X_0^i = V_i, \quad i = 1, \dots, N. \end{cases} \tag{4.49}$$

Here $(V_1, \ldots, V_N)$ is a vector of i.i.d and $\mathcal{F}_0$-measurable random variables with law m_0, such that $V_1, \ldots, V_N$ are independent of $W^1, \ldots, W^N$ respectively. Let us fix a common horizon $T > 0$, and set

$$\tau^i := \inf\left\{t \geqslant 0 \,:\, X_t^i \leqslant 0\right\} \wedge T.$$

The stopped random process $\left(X^i_{t\wedge\tau^i}\right)_{t\in[0,T]}$ models the reserves level of the i^{th} producer on the horizon T, which is gradually depleted according to a non-negative controlled rate of production $\left(q_t^i\right)_{t\in[0,T]}$. The stopping condition indicates that a firm can no longer replenish its reserves once they are exhausted. The Wiener processes in (4.49) model the idiosyncratic fluctuations related to production. We consider ℓ to be an upper bound on the reserves level of any player. This latter assumption is also considered in Chapter 3 and [70], and is taken into account by considering reflected dynamics in (4.49). Since the rate of production is always non-negative, note that reflection has practically no effect when ℓ is large compared to the initial reserves.

REMARK 4.7 (State constraints). Instead of reflecting boundary conditions, one could insist upon a hard state constraint of the form $X_t^i \leqslant \ell$. Some recent work on MFG with state constraints suggests this is possible [27–29], provided one correctly interprets the resulting system of PDE. In this work we take a more classical approach, for which probabilistic tools are more readily available.

The producers interact through the market. We assume that demand is linear, so that the price p^i received by the firm i reads:

$$p_t^i = 1 - (q_t^i + \kappa \bar{q}_t^i), \quad \text{where} \quad \bar{q}_t^i = \frac{1}{N-1}\sum_{j\neq i} q_t^j \mathbb{1}_{t<\tau^j}, \quad \text{for} \quad 0 \leqslant t \leqslant T. \tag{4.50}$$

Here $\kappa > 0$ expresses the degree of market interaction, in proportion to which abundant total production will put downward pressure on all the prices. Note that only firms with nonempty reserves at $t \in [0, T]$ are taken into account in (4.50). The other firms are no longer present on the market. The producer i chooses the production rate q^i in order to maximize the following discounted profit functional:

$$\mathcal{J}_c^{i,N}(q^1, \ldots, q^N) := \mathbb{E}\left\{\int_0^T e^{-rs}\left(1 - \kappa\bar{q}_s^i - q_s^i\right) q_s^i \mathbb{1}_{s<\tau^i}\, ds + e^{-rT} u_T(X^i_{\tau^i})\right\}.$$

Observe that firms can no longer earn revenue as soon as they deplete their reserves. We refer to [50, 77] for further explanations on the economic model and applications.

We denote by $\mathbb{A}_c$ the set of admissible controls for any player; that is the set of *Markovian feedback controls*, i.e. $q_t^i = q^i\left(t, X_t^1, \ldots, X_t^N\right)$; such that $(q_t^i)_{t\in[0,T]}$ is positive, satisfies

$$\mathbb{E}\left[\int_0^T |q_s^i|^2 \mathbb{1}_{s<\tau^i}\, ds\right] < \infty,$$

and the i^{th} equation of (4.49) is well-posed in the classical sense. Restriction to Markovian controls rules out equilibria with undesirable properties such as non-credible threats (cf. [62, Chapter 13]).

Now, we give a definition of Nash equilibria to this game:

DEFINITION 4.8 (Nash equilibrium). A strategy profile $\left(q^{1,*}, \ldots, q^{N,*}\right)$ in $\prod_{i=1}^{N} \mathbb{A}_c$ is a *Nash equilibrium* of the N-Player Cournot game, if for any $i = 1, \ldots, N$ and $q^i \in \mathbb{A}_c$

$$\mathcal{J}_c^{i,N}\left(q^i; (q^{j,*})_{j\neq i}\right) \leqslant \mathcal{J}_c^{i,N}\left(q^{1,*}, \ldots, q^{N,*}\right).$$

In words, a Nash equilibrium is a set of admissible strategies such that each player has taken an optimal trajectory in view of the competitors' choices.

The existence of Nash equilibria for the N-Player Cournot game with exhaustible resources is addressed in [77]. In particular, the authors show the existence of a unique Nash equilibrium in the static (one period) case, and study numerically a specific duopoly example by using a convenient asymptotic expansion. In general, the analysis of equilibria for N-Player Cournot games is a challenging task both analytically and numerically, especially when N is large. In the case of exhaustible resources, the dynamic programming principle generates an even more complex PDE system because of the nonstandard boundary conditions which are obtained (cf. [77, Section 3.1]).

To remedy this problem several works have rather considered a Mean-Field model [49,50,73,77,92] as an approximation to the initial N-Player game, when N is large. More precisely, we introduce the following:

DEFINITION 4.9 (ε-Nash equilibrium). Let $\varepsilon > 0$, and let $(\hat{q}^1, \ldots, \hat{q}^N)$ be an admissible strategy profile (i.e. an element of $\prod_{i=1}^{N} \mathbb{A}_c$). We say $(\hat{q}^1, \ldots, \hat{q}^N)$ provides an *ε-Nash equilibrium* to the game $\mathcal{J}_c^{1,N}, \ldots, \mathcal{J}_c^{N,N}$ provided that, for any $i = 1, \ldots, N$ and $q^i \in \mathbb{A}_c$,

$$\mathcal{J}_c^{i,N}\left(q^i; (\hat{q}^j)_{j\neq i}\right) \leqslant \varepsilon + \mathcal{J}_c^{i,N}\left(\hat{q}^1, \ldots, \hat{q}^N\right).$$

In words, an ε-Nash equilibrium is a set of admissible strategies such that each player has taken an *almost* optimal trajectory in view of the competitors' choices, where ε measures the distance from optimality.

The main purpose of this section is to construct an ε-Nash equilibrium by using the MFG system (4.4). Namely, our main result is the following:

THEOREM 4.10. *For any $N \geqslant 1$ and $i \in \{1, \ldots, N\}$ let*

$$\begin{cases} d\hat{X}_t^i = -q_{u,m}(t, \hat{X}_t^i)\, dt + \sigma\, dW_t^i - d\xi_t^{\hat{X}^i} \\ X_0^i = V_i, \end{cases} \tag{4.51}$$

and set $\hat{q}_t^i := q_{u,m}(t, \hat{X}_t^i)$. Then for any $\varepsilon > 0$, the strategy profile $(\hat{q}^1, \ldots, \hat{q}^N)$ is admissible, i.e. belongs to $\prod_{i=1}^{N} \mathbb{A}_c$, and provides an ε-Nash equilibrium to the game $\mathcal{J}_c^{1,N}, \ldots, \mathcal{J}_c^{N,N}$ for large N. Namely: $\forall \varepsilon > 0,\ \exists N_\varepsilon \geqslant 1$ such that

$$\forall N \geqslant N_\varepsilon, \forall i = 1, \ldots, N, \quad \mathcal{J}_c^{i,N}\left(q^i; (\hat{q}^j)_{j\neq i}\right) \leqslant \varepsilon + \mathcal{J}_c^{i,N}\left(\hat{q}^1, \ldots, \hat{q}^N\right), \tag{4.52}$$

for any admissible strategy $q_i \in \mathbb{A}_c$.

The rest of this section is devoted to the proof of Theorem 4.10.

3.2. Tailor-Made Law of Large Numbers. Let us set

$$\hat{\tau}^i := \inf\left\{t \geqslant 0 \ :\ \hat{X}_t^i \leqslant 0\right\} \wedge T,$$

and define the following process:

$$\hat{\nu}_t^N := \frac{1}{N}\sum_{k=1}^{N} \delta_{\hat{X}_t^k} \mathbb{1}_{t<\hat{\tau}^k}, \quad \forall t \in [0,T], \tag{4.53}$$

where δ_x denotes the Dirac delta measure of the point $x \in \mathbb{R}$. Observe that the above definition makes sense because the stochastic dynamics $(\hat{X}^1, ..., \hat{X}^N)$ exists in the strong sense owing to Lemma 4.5. In particular, the strategy profile $(\hat{q}^1, ..., \hat{q}^N)$ defined in Theorem 4.10 belongs to $\prod_{i=1}^N \mathbb{A}_c$. Moreover, by using the probabilistic characterization (4.13c), note that for any measurable and bounded function ϕ on $\bar{Q}$ we have

$$\mathbb{E}\left[\int_0^\ell \phi \, d\hat{\nu}_t^N\right] = \int_0^\ell \phi \, dm(t), \quad \text{for a.e. } t \in (0,T). \tag{4.54}$$

The above identity is not strong enough to show Theorem 4.10 and we need a stronger condition (cf. (4.71)). Therefore, we need to work harder in order to get more information on the asymptotic behavior of the empirical process (4.53) when $N \to \infty$.

We aim to prove that the *empirical process* $(\hat{\nu}^N)_{N\geqslant 1}$ converges in law to the deterministic measure m in a suitable function space, by using arguments borrowed from [**76**, **89**]. For this, we start by showing the existence of sub-sequences $(\hat{\nu}^{N'})$ that converges in law to some limiting process ν^*. Then, we show that ν^* belongs to $\tilde{\mathcal{P}}(\bar{Q})$ and satisfies the same equation as m. Finally, we invoke the uniqueness of weak solutions to the Fokker-Planck equation to deduce full weak convergence toward m.

The crucial step consists in showing that the sequence of the laws of $(\hat{\nu}^N)_{N\geqslant 1}$ is relatively compact on a suitable topological space. This is where the machinery of [**89**] is convenient. In order to use the analytical tools of that paper, we view the empirical process as a random variable on the space of *càdlàg* (right continuous and has left-hand limits) functions, mapping $[0,T]$ into the space of tempered distributions. This function space is denoted $D_{\mathcal{S}'_\mathbb{R}}$ and is endowed with the so called Skorokhod's M1 topology. Note that there are no measurability issues owing to [**89**, Proposition 2.7]. Moreover, by virtue of [**101**], the process $(\hat{\nu}_t^N)_{t\in[0,T]}$ has a version that is *càdlàg* in the strong topology of $\mathcal{S}'_\mathbb{R}$ for every $N \geqslant 1$, since $\hat{\nu}_t^N(\phi) := \int_\mathbb{R} \phi \, d\hat{\nu}_t^N$ is a real-valued *càdlàg* process, for every $\phi \in \mathcal{S}_\mathbb{R}$ and $N \geqslant 1$. We refer the reader to [**89**] for the construction of $(D_{\mathcal{S}'_\mathbb{R}}, \mathrm{M1})$, and to [**108**] for general background on Skorokhod's topologies. We shall denote by $(D_\mathbb{R}, \mathrm{M1})$ the space of $\mathbb{R}$-valued *càdlàg* functions mapping $[0,T]$ to $\mathbb{R}$, endowed with Skorokhod's M1 topology.

The main strengths of working with the M1 topology in our context, are based on the following facts:

- tightness on $(D_{\mathcal{S}'_\mathbb{R}}, \mathrm{M1})$ implies the relative compactness on $(D_{\mathcal{S}'_\mathbb{R}}, \mathrm{M1})$ thanks to [**89**, Theorem 3.2]);
- the proof of tightness on $(D_{\mathcal{S}'_\mathbb{R}}, \mathrm{M1})$ is reduced through the canonical projection to the study of tightness in $(D_\mathbb{R}, \mathrm{M1})$, for which we have suitable characterizations [**89**, **108**];

- bounded monotone real-valued processes are automatically tight on $(D_{\mathbb{R}}, M1)$; this is an important feature, that enables to prove tightness of the sequence of empirical process laws, by using a suitable decomposition.

It is also important to note that this approach could be generalized to deal with the case of a systemic noise, by using a martingale approach as in [76, Lemma 5.9]. We do not deal with that case in this chapter.

More generally, one can replace $\mathcal{S}'_{\mathbb{R}}$ by any dual space of a *countably Hilbertian nuclear space* (cf. [89] and references therein). Although the class $\mathcal{S}'_{\mathbb{R}}$ seems to be excessively large for our purposes, we recover measure-valued processes by means of Riesz representation theorem (cf. [76, Proposition 5.3] for an example in the same context).

Throughout this part, *we shall use the symbol $\Rightarrow$ to denote convergence in law*. The key technical lemma of this section is the following:

LEMMA 4.11. *As $N \to \infty$, we have $\hat{\nu}^N \Rightarrow m$ on $(D_{\mathcal{S}'_{\mathbb{R}}}, M1)$, i.e. for every continuous bounded real-valued function Ψ on $(D_{\mathcal{S}'_{\mathbb{R}}}, M1)$, it holds that*

$$\lim_N \mathbb{E}\left[\Psi\left(\hat{\nu}^N\right)\right] = \Psi(m).$$

The bulk of this section is devoted to the proof of Lemma 4.11. The proof of Theorem 4.10 is completed in Section 3.3.

Tightness. At first, we aim to prove the tightness of $(\hat{\nu}^N)_{N\geqslant 1}$ on the space $(D_{\mathcal{S}'_{\mathbb{R}}}, M1)$; that is, for every $\phi \in \mathcal{S}_{\mathbb{R}}$ and for all $\varepsilon > 0$, there exists a compact subset K of $(D_{\mathbb{R}}, M1)$ such that:

$$\mathbb{P}\left(\hat{\nu}^N(\phi) \in K\right) > 1 - \varepsilon \quad \text{for all } \; N \geqslant 1.$$

For that purpose, we shall use a convenient characterization of tightness in $(D_{\mathbb{R}}, M1)$ (cf. [108, Theorem 12.12.3]).

We start by controlling the concentration of mass at the origin:

LEMMA 4.12. *For every $t \in [0, T]$, we have*

$$\sup_{N\geqslant 1} \mathbb{E}\hat{\nu}^N_t(0, \varepsilon) \to 0, \quad \textit{as } \varepsilon \to 0.$$

PROOF. Let us fix $\varepsilon > 0$. Note that, for every $t \in [0, T]$

$$\mathbb{E}\hat{\nu}^N_t(0, \varepsilon) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{P}\left(\hat{X}^i_t \in (0, \varepsilon); t < \hat{\tau}^i\right).$$

Thus, on the one hand

$$\sup_{N\geqslant 1} \mathbb{E}\hat{\nu}^N_0(0, \varepsilon) = \int_0^{\varepsilon} dm_0 \to 0, \quad \text{as } \varepsilon \to 0,$$

owing to the dominated convergence theorem. On the other hand, we have for every $t \in (0, T]$

$$\sup_{N\geqslant 1} \mathbb{E}\hat{\nu}^N_t(0, \varepsilon) \leqslant \sup_{N\geqslant 1} N^{-1}\sum_{i=1}^{N} \mathbb{E}\left[(1 - \phi_\varepsilon(\hat{X}^i_t))\mathbb{1}_{t<\hat{\tau}^i}\right] \tag{4.55}$$

where ϕ_ε is the cut-off function defined in (4.17). Thus, by virtue of (4.21) we obtain

$$\sup_{N\geqslant 1} \mathbb{E}\hat{\nu}_t^N(0,\varepsilon) \leqslant C(\ell, t, \|q_{u,m}\|_\infty)\varepsilon^{1/4},$$

which entails the desired result. □

The second ingredient is the control of the mass loss increment:

LEMMA 4.13. *For every* $t \in [0,T]$ *and* $\lambda > 0$

$$\lim_{h\to 0} \limsup_N \mathbb{P}\left(\left|\eta\left(\hat{\nu}_t^N\right) - \eta\left(\hat{\nu}_{t+h}^N\right)\right| \geqslant \lambda\right) = 0,$$

where the map $\mu \to \eta(\mu)$ *is defined in* (4.7).

PROOF. The proof is inspired by [76, Proposition 4.7]. Let $\varepsilon, h > 0$ and $t \in [0,T]$, we have

$$\begin{aligned}(4.56)\quad & \mathbb{P}\left(\eta\left(\hat{\nu}_t^N\right) - \eta\left(\hat{\nu}_{t+h}^N\right) \geqslant \lambda\right) \\ &\qquad \leqslant \mathbb{P}\left(\hat{\nu}_t^N(0,\varepsilon) \geqslant \lambda/2\right) + \mathbb{P}\left(\eta\left(\hat{\nu}_t^N\right) - \eta\left(\hat{\nu}_{t+h}^N\right) \geqslant \lambda; \hat{\nu}_t^N(0,\varepsilon) < \lambda/2\right).\end{aligned}$$

The reason why we use the latter decomposition will be clear in (4.57). Owing to Markov's inequality and Lemma 4.12, one has

$$\limsup_N \mathbb{P}(\hat{\nu}_t^N(0,\varepsilon) \geqslant \lambda/2) \leqslant 2\lambda^{-1} \sup_N \mathbb{E}\hat{\nu}_t^N(0,\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0.$$

Now we deal with the second part in estimate (4.56). Define $\mathcal{I}_t$ to be the following random set of indices:

$$\mathcal{I}_t := \left\{1 \leqslant i \leqslant N \,:\, \hat{X}_t^i \geqslant \varepsilon\right\};$$

then, we have

$$\begin{aligned}&\mathbb{P}\left(\eta\left(\hat{\nu}_t^N\right) - \eta\left(\hat{\nu}_{t+h}^N\right) \geqslant \lambda; \hat{\nu}_t^N(0,\varepsilon) < \lambda/2\right) \\ &\qquad \leqslant \sum_{\#\mathcal{I}\geqslant N(1-\lambda/2)} \mathbb{P}\left(\eta\left(\hat{\nu}_t^N\right) - \eta\left(\hat{\nu}_{t+h}^N\right) \geqslant \lambda \mid \mathcal{I}_t = \mathcal{I}\right)\mathbb{P}(\mathcal{I}_t = \mathcal{I}),\end{aligned}$$

where $\#\mathcal{I}$ denotes the number of elements of $\mathcal{I} \subseteq \{1,2,\dots,N\}$. Thus, we reduce the problem to the estimation of the dynamics increments; using the same steps as for (4.14c) we have

$$\begin{aligned}(4.57)\quad & \mathbb{P}\left(\eta\left(\hat{\nu}_t^N\right) - \eta\left(\hat{\nu}_{t+h}^N\right) \geqslant \lambda \mid \mathcal{I}_t = \mathcal{I}\right) \\ &\qquad \leqslant \mathbb{P}\left(\#\left\{i \in \mathcal{I} \,:\, \inf_{s\in[t,t+h]} \hat{X}_s^i - \hat{X}_t^i \leqslant -\varepsilon\right\} \geqslant \lambda N/2 \,\middle|\, \mathcal{I}_t = \mathcal{I}\right) \\ &\qquad\qquad \leqslant \mathbb{P}\left(\#\left\{i \in \mathcal{I} \,:\, \sup_{s\in[0,h]} B_s^i - \inf_{s\in[0,h]} B_s^i \geqslant \frac{\varepsilon - h}{\sqrt{2\sigma}}\right\} \geqslant \lambda N/2\right),\end{aligned}$$

where we have used the uniform bound on $q_{u,m}$ of Lemma 4.5, and where $(B^i)_{1\leqslant i\leqslant N}$ is a family of independent Wiener processes. By symmetry, this final probability depends only on $\#\mathcal{I}$, so that the right hand side above is maximized when $\mathcal{I} = \{1, ..., N\}$. We infer that

$$\mathbb{P}\left(\eta\left(\hat{\nu}_t^N\right) - \eta\left(\hat{\nu}_{t+h}^N\right) \geqslant \lambda; \hat{\nu}_t^N(0,\varepsilon) < \lambda/2\right) \leqslant \mathbb{P}\left(\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\left\{\sup_{s\in[0,h]} B_s^i - \inf_{s\in[0,h]} B_s^i \geqslant \frac{\varepsilon - h}{\sqrt{2\sigma}}\right\}} \geqslant \lambda/2\right).$$

In the same way as for (4.14c), we choose $\varepsilon(h) = h^{1/2}\log(1/h)$ so that $\lim_{h\to 0^+}\varepsilon(h) = 0$, and use Markov's inequality to get

$$\mathbb{P}\left(\eta\left(\hat{\nu}_t^N\right) - \eta\left(\hat{\nu}_{t+h}^N\right) \geqslant \lambda; \hat{\nu}_t^N(0,\varepsilon) < \lambda/2\right) \leqslant \frac{2\sqrt{\sigma}}{\lambda(\log(1/h) - h^{1/2})}.$$

This entails the desired result by taking the limit $h \to 0^+$.

Now we deal with the case of a left hand limit. Let $t \in (0, T]$ and $h \mapsto \varepsilon(h)$ as defined above. Using a similar decomposition as before, we have for small enough $h > 0$

$$\begin{aligned}&\mathbb{P}\left(\eta\left(\hat{\nu}_{t-h}^N\right) - \eta\left(\hat{\nu}_t^N\right) \geqslant \lambda\right)\\ &\qquad \leqslant \mathbb{P}\left(\hat{\nu}_{t-h}^N(0,\varepsilon) \geqslant \lambda/2\right) + \mathbb{P}\left(\eta\left(\hat{\nu}_{t-h}^N\right) - \eta\left(\hat{\nu}_t^N\right) \geqslant \lambda; \hat{\nu}_{t-h}^N(0,\varepsilon) < \lambda/2\right).\end{aligned}$$

Appealing to Markov's inequality, estimate (4.55), and estimate (4.21) of Section 2, we have for small enough $h > 0$

$$\mathbb{P}\left(\hat{\nu}_{t-h}^N(0,\varepsilon) \geqslant \lambda/2\right) \leqslant 2\lambda^{-1}\mathbb{E}\hat{\nu}_{t-h}^N(0,\varepsilon) \leqslant 2C\lambda^{-1}\left(1 - e^{-\pi^2 t/8\ell^2}\right)^{-1/4}\varepsilon^{1/4},$$

whence

$$\lim_{h\to 0^+}\limsup_N \mathbb{P}\left(\hat{\nu}_{t-h}^N(0,\varepsilon(h)) \geqslant \lambda/2\right) = 0.$$

On the other hand, we show by using the same steps as in (4.57) that

$$\mathbb{P}\left(\eta\left(\hat{\nu}_{t-h}^N\right) - \eta\left(\hat{\nu}_t^N\right) \geqslant \lambda; \hat{\nu}_{t-h}^N(0,\varepsilon) < \lambda/2\right) \leqslant \frac{2\sqrt{\sigma}}{\lambda(\log(1/h) - h^{1/2})}.$$

This entails the desired result by taking the limit $h \to 0^+$. ☐

We are now in position to show tightness on $(D_{\mathcal{S}'_{\mathbb{R}}}, M1)$.

PROPOSITION 4.14 (Tightness). *The sequence of the laws of* $(\hat{\nu}^N)_{N\geqslant 1}$ *is tight on the space* $(D_{\mathcal{S}'_{\mathbb{R}}}, M1)$.

PROOF. We present a brief sketch to explain the main arguments, and refer to [**76**, Proposition 5.1] for a similar proof.

Thanks to [**89**, Theorem 3.2], it is enough to show that the sequence of the laws of $\left(\hat{\nu}^N(\phi)\right)_{N\geqslant 1}$ is tight on $(D_{\mathbb{R}}, M1)$ for any $\phi \in \mathcal{S}_{\mathbb{R}}$. To prove this, one can use the conditions of [**108**, Theorem 12.12.3], which can be rewritten in a convenient form by virtue of [**11**]. From [**89**, Proposition 4.1] , we are done if we achieve the two following steps:

(1) find $\alpha, \beta, c > 0$, such that

$$\mathbb{P}\left(H_{\mathbb{R}}\left(\hat{\nu}_{t_1}^N(\phi), \hat{\nu}_{t_2}^N(\phi), \hat{\nu}_{t_3}^N(\phi)\right) \geqslant \lambda\right) \leqslant c\lambda^{-\alpha}|t_3 - t_1|^{1+\beta},$$

for any $N \geqslant 1$, $\lambda > 0$ and $0 \leqslant t_1 < t_2 < t_3 \leqslant T$, where

$$H_{\mathbb{R}}(x_1, x_2, x_3) := \inf_{0\leqslant\gamma\leqslant 1}|x_2 - (1-\gamma)x_1 - \gamma x_3| \quad \text{for } x_1, x_2, x_3 \in \mathbb{R};$$

(2) show that

$$\lim_{h\to 0^+}\lim_N \mathbb{P}\left(\sup_{t\in(0,h)}|\hat\nu_t^N(\phi)-\hat\nu_0^N(\phi)|+\sup_{t\in(T-h,T)}|\hat\nu_T^N(\phi)-\hat\nu_t^N(\phi)|\geqslant\lambda\right)=0.$$

The key step is to consider the following decomposition [89, Proposition 4.2]:

$$\bar\nu_t^N(\phi):=\frac{1}{N}\sum_{k=1}^N\phi(\hat X^k_{t\wedge\hat\tau^k})=\hat\nu_t^N(\phi)+\phi(0)\mathcal{E}_t^N, \tag{4.58}$$

where

$$\mathcal{E}_t^N:=1-\eta\left(\hat\nu_t^N\right)$$

is the exit rate process, which quantifies the fraction of firms out of market. Since $\left(\mathcal{E}_t^N\right)_{t\in[0,T]}$ is monotone increasing we have

$$\inf_{0\leqslant\gamma\leqslant 1}\left|\mathcal{E}^N_{t_2}-(1-\gamma)\mathcal{E}^N_{t_1}-\gamma\mathcal{E}^N_{t_3}\right|=0,$$

so that

$$H_{\mathbb{R}}\left(\hat\nu^N_{t_1}(\phi),\hat\nu^N_{t_2}(\phi),\hat\nu^N_{t_3}(\phi)\right)\leqslant\left|\bar\nu^N_{t_1}(\phi)-\bar\nu^N_{t_2}(\phi)\right|+\left|\bar\nu^N_{t_2}(\phi)-\bar\nu^N_{t_3}(\phi)\right|.$$

Thus, by virtue of Markov's inequality

$$\begin{aligned}\mathbb{P}&\left(H_{\mathbb{R}}\left(\hat\nu^N_{t_1}(\phi),\hat\nu^N_{t_2}(\phi),\hat\nu^N_{t_3}(\phi)\right)\geqslant\lambda\right)\\&\leqslant 8\lambda^{-4}\left(\mathbb{E}\left|\bar\nu^N_{t_1}(\phi)-\bar\nu^N_{t_2}(\phi)\right|^4+\mathbb{E}\left|\bar\nu^N_{t_2}(\phi)-\bar\nu^N_{t_3}(\phi)\right|^4\right).\end{aligned}$$

Therefore, we deduce requirement (1) from the following estimate:

(4.59) $\forall s,t\in[0,T]$,

$$\mathbb{E}\left|\bar\nu_t^N(\phi)-\bar\nu_s^N(\phi)\right|^4\leqslant\|\phi\|^4_{\mathcal{C}^1}\frac{1}{N}\sum_{k=1}^N\mathbb{E}|\hat X^k_{t\wedge\hat\tau^k}-\hat X^k_{s\wedge\hat\tau^k}|^4\leqslant C\|\phi\|^4_{\mathcal{C}^1}|t-s|^2;$$

where we have used Hölder's inequality and the Burkholder-Davis-Gundy inequality [104, Thm IV.42.1].

The second requirement is also obtained by using the latter estimate, decomposition (4.58), and Lemma 4.13. In fact, we have

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in(0,h)}|\hat\nu_t^N(\phi)-\hat\nu_0^N(\phi)|\geqslant\lambda\right)\\&\qquad\leqslant\mathbb{P}\left(\sup_{t\in(0,h)}|\bar\nu_t^N(\phi)-\bar\nu_0^N(\phi)|\geqslant\lambda/2\right)+\mathbb{P}\left(|\phi(0)|\mathcal{E}_h^N\geqslant\lambda/2\right),\end{aligned}$$

so that the desired result follows thanks to (4.59), and Lemma 4.13. By the same way, we deal with the second term $\mathbb{P}\left(\sup_{t\in(T-h,T)}|\hat\nu_T^N(\phi)-\hat\nu_t^N(\phi)|\geqslant\lambda\right)$. □

Full convergence. We arrive now at the final ingredient for the proof of Lemma 4.11. Let us set

$$\mathcal{C}^{test} := \left\{\phi \in \mathcal{C}_c^\infty([0,T)\times\bar{Q}) \;\middle|\; \phi(t,0) = \partial_x\phi(t,\ell) = 0, \;\; \forall t \in (0,T)\right\}.$$

We start by deriving an equation for $(\hat{\nu}_t^N)_{t\in[0,T]}$.

PROPOSITION 4.15. *For every* $N \geqslant 1$ *and* $\phi \in \mathcal{C}^{test}$, *it holds that*

$$\int_0^\ell \phi(0,.)\,d\hat{\nu}_0^N = \int_0^T\int_0^\ell (-\partial_t\phi - \sigma\partial_{xx}\phi + q_{u,m}\partial_x\phi)\,d\hat{\nu}_s^N\,ds + I_N(\phi) \quad a.s.,$$

where

$$I_N(\phi) := -\frac{\sqrt{2\sigma}}{N}\sum_{k=1}^N\int_0^T \partial_x\phi\left(s,\hat{X}_s^k\right)\mathbb{1}_{s<\hat{\tau}^k}\,dW_s^k.$$

PROOF. Let us consider $\phi \in \mathcal{C}^{test}$. First observe that for any $k \in \{1,...,N\}$, and $t \in [0,T]$

$$\hat{X}^k_{t\wedge\hat{\tau}^k} = V_k - \int_0^t \hat{q}_s^k\mathbb{1}_{s<\tau^{\hat{q}^k}}\,ds + \sqrt{2\sigma}W^k_{t\wedge\tau^{\hat{q}^k}} - \xi_t^{\hat{X}^k}.$$

Hence, for any $k \in \{1,...,N\}$, the random process $\left(\hat{X}^k_{t\wedge\hat{\tau}^k}\right)_{t\in[0,T]}$ is a continuous semimartingale, and by applying Itô's rule we have:

$$\begin{aligned}\phi(T,\hat{X}^k_{\hat{\tau}^k}) - \phi(0,V_k) &+ \int_0^T \partial_x\phi\left(s,\hat{X}^k_{s\wedge\hat{\tau}^k}\right)d\xi_s^{\hat{X}^k}\\ &= \int_0^T\left\{\sigma\partial_{xx}\phi(s,\hat{X}_s^k) - q_{u,m}(s,\hat{X}_s^k)\partial_x\phi(s,\hat{X}_s^k)\right\}\mathbb{1}_{s<\hat{\tau}^k}\,ds\\ &\qquad + \int_0^T \partial_t\phi\left(s,\hat{X}^k_{s\wedge\hat{\tau}^k}\right)ds + \sqrt{2\sigma}\int_0^T\partial_x\phi\left(s,\hat{X}_s^k\right)\mathbb{1}_{s<\hat{\tau}^k}\,dW_s^k.\end{aligned}$$

By using the boundary conditions satisfied by ϕ, and noting that $\partial_t\phi(t,0) = 0$ for any $t \in (0,T)$, we deduce that

$$\begin{aligned}-\phi(0,V_k) &- \sqrt{2\sigma}\int_0^T\partial_x\phi\left(s,\hat{X}_s^k\right)\mathbb{1}_{s<\hat{\tau}^k}\,dW_s^k\\ &= \int_0^T\left\{\partial_t\phi\left(s,\hat{X}_s^k\right) + \sigma\partial_{xx}\phi\left(s,\hat{X}_s^k\right) - q_{u,m}(s,\hat{X}_s^k)\partial_x\phi\left(s,\hat{X}_s^k\right)\right\}\mathbb{1}_{s<\hat{\tau}^k}\,ds\end{aligned}$$

The desired result follows by summing over $k \in \{1,...,N\}$ and multiplying by N^{-1}. ☐

By virtue of [89, Theorem 3.2], the tightness of the sequence of laws of $(\hat{\nu}^N)_{N\geqslant1}$ ensures that this sequence is relatively compact on $(D_{\mathcal{S}'_{\mathbb{R}}}, M1)$. Consequently, Proposition 4.14 entails the existence of a subsequence (still denoted $(\hat{\nu}^N)_{N\geqslant1}$) such that

$$\hat{\nu}^N \Rightarrow \hat{\nu}^*, \qquad \text{on } (D_{\mathcal{S}'_{\mathbb{R}}}, M1).$$

Thanks to [89, Proposition 2.7 (i)],

$$\forall\phi\in\mathcal{S}_{\mathbb{R}}, \quad \hat{\nu}^N(\phi) \Rightarrow \hat{\nu}^*(\phi), \qquad \text{as } N\to\infty, \qquad \text{on } (D_{\mathbb{R}}, M1).$$

To avoid possible confusion about multiple distinct limit points, we will denote $\hat{\nu}^*$ any limiting processes that realizes one of these limiting laws. First, we note that $\hat{\nu}^*$ is a $\tilde{\mathcal{P}}(\bar{Q})$-valued process:

PROPOSITION 4.16. *For every* $t \in [0,T]$, $\hat{\nu}^*_t$ *is almost surely supported on* $\bar{Q}$ *and belongs to* $\tilde{\mathcal{P}}(\bar{Q})$.

PROOF. This follows from the Portmanteau theorem [20] and the Riesz representation theorem. We omit the details and refer to [76, Proposition 5.3]. □

Next, we recover the partial differential equation satisfied by the process $(\hat{\nu}^*_t)_{t\in[0,T]}$.

LEMMA 4.17. *For every* $\phi \in \mathcal{C}^{test}$, *it holds that*

$$\int_0^\ell \phi(0,.)\,dm_0 + \int_0^T\int_0^\ell \left(\partial_t\phi + \sigma\partial_{xx}\phi - q_{u,m}\partial_x\phi\right)\,d\hat{\nu}^*_s\,ds = 0 \quad a.s.$$

PROOF. Let us consider $\phi \in \mathcal{C}^{test}$ and set:

$$\mu(\phi) := \int_0^\ell \phi(0,.)\,dm_0 + \int_0^T\int_0^\ell \left(\partial_t\phi + \sigma\partial_{xx}\phi - q_{u,m}\partial_x\phi\right)\,d\hat{\nu}^*_s\,ds;$$

and

$$\mu_N(\phi) := \int_0^\ell \phi(0,.)\,dm_0 + \int_0^T\int_0^\ell \left(\partial_t\phi + \sigma\partial_{xx}\phi - q_{u,m}\partial_x\phi\right)\,d\hat{\nu}^N_s\,ds.$$

Owing to Proposition 4.15 we have

$$\mu_N(\phi) = I_N(\phi) + \int_0^\ell \phi(0,.)\,d(m_0 - \hat{\nu}^N_0).$$

Note that

$$\mathbb{E}I_N(\phi)^2 \leqslant C\|\partial_x\phi\|^2_\infty N^{-1}.$$

Hence, by appealing to Horowitz-Karandikar inequality (see e.g. [103, Theorem 10.2.1]) we deduce that

$$\mathbb{E}\mu_N^2(\phi) \leqslant C\|\partial_x\phi\|^2_\infty N^{-2/5}.$$

Consequently, to conclude the proof it is enough to show that

$$\mu_N(\phi) \Rightarrow \mu(\phi) \quad \text{as } N \to \infty.$$

Let $\mathcal{A}$ be the set of elements in $D_{\mathcal{S}'_{\mathbb{R}}}$ that take values in $\tilde{\mathcal{P}}(\bar{Q})$, and consider a sequence $(\psi^N) \subset \mathcal{A}$ which converges to some ψ in $\mathcal{A}$ with respect to the M1 topology. Let $\mathbf{q}_{u,m}$ be a continuous function on $[0,T]\times\mathbb{R}$, which satisfies the following conditions:

(4.60a)

$$\mathbf{q}_{u,m}{}_{|_{\bar{Q}_T}} \equiv q_{u,m}; \quad \|\mathbf{q}_{u,m}\|_\infty = \|q_{u,m}\|_\infty; \quad \forall t\in[0,T],\ \operatorname{supp}\mathbf{q}_{u,m}(t,.) \subset (-\ell, 2\ell).$$

We also define the sequence

$$\mathbf{q}^n_{u,m}(t,x) := \left(\mathbf{q}_{u,m}(t,.) * \xi_n\right)(x), \quad n \geqslant 1, \tag{4.60b}$$

where $\xi_n(x) := n\xi(nx)$ is a compactly supported mollifier on $\mathbb{R}$.

We have

$$J := \left| \int_0^T \int_0^\ell q_{u,m} \partial_x \phi \, d\psi_s^N \, ds - \int_0^T \int_0^\ell q_{u,m} \partial_x \phi \, d\psi_s \, ds \right|$$
$$= \left| \int_0^T \int_{\mathbb{R}} \mathbf{q}_{u,m} \partial_x \phi \, d\psi_s^N \, ds - \int_0^T \int_{\mathbb{R}} \mathbf{q}_{u,m} \partial_x \phi \, d\psi_s \, ds \right|$$
$$\leqslant 2 \|\partial_x \phi\|_\infty \left\| \mathbf{q}_{u,m}^n - \mathbf{q}_{u,m} \right\|_\infty$$
$$+ \left| \int_0^T \int_{\mathbb{R}} \mathbf{q}_{u,m}^n \partial_x \phi \, d(\psi_s^N - \psi_s) \, ds \right| =: J_1 + J_2.$$

Since $\mathbf{q}_{u,m}^n(s,.)\partial_x\phi(s,.) \in \mathcal{S}_{\mathbb{R}}$ for any $s \in [0,T]$, then J_2 vanishes as $\psi^N \to \psi$. On the other hand, note that J_1 also vanishes as $n \to +\infty$ so that we obtain $\lim_N J = 0$. Moreover, one easily checks that

$$\int_0^T \int_0^\ell F \, d\psi_s^N \, ds \to \int_0^T \int_0^\ell F \, d\psi_s \, ds, \quad F \equiv \partial_t \phi, \partial_{xx}\phi \quad \text{as} \quad N \to +\infty.$$

Therefore, by virtue of the continuous mapping theorem, we obtain that $\mu_N(\phi) \Rightarrow \mu(\phi)$, which concludes the proof. □

We are now in position to prove Lemma 4.11.

PROOF OF LEMMA 4.11. From Lemma 4.17, we know that $d\boldsymbol{\nu}^* = d\hat{\nu}_t^* \, dt$ and $d\mathbf{m} = dm(t)\, dt$ both satisfy (almost surely) the same Fokker-Planck equation in the sense of measures (cf. Appendix 2). By invoking the uniqueness of solutions to that equation (cf. Proposition A.2), we deduce that $\hat{\nu}^* \equiv m$ almost surely. Since all converging subsequences converge weakly toward m, we infer that $\hat{\nu}^N \Rightarrow m$, on $(D_{\mathcal{S}'_{\mathbb{R}}}, M1)$. □

3.3. Large Population Approximation. By virtue of the analytical tools of the previous section, we are now in position to show Theorem 4.10. We start by recalling an important fact related to the Mean Field Game system (4.4), then we prove Theorem 4.10.

The mean-field problem. In this part, we recall briefly the interpretation of system (4.4) in terms of games with a continuum of players a "mean field" interactions. We refer the reader to Chapter 3 and [**49**,**70**,**73**,**92**] for more background. Let us consider a continuum of agents, producing and selling comparable goods. At time $t = 0$, all the players have a positive capacity $x \in (0, \ell]$, and are distributed on $(0, \ell]$ according to m_0.

The remaining capacity (or reserves) of any atomic producer with a production rate $(\rho)_{t \geqslant 0}$ depletes according to

$$dX_t^\rho = -\rho_t \mathbb{1}_{t<\tau^\rho} \, dt + \sqrt{2\sigma} \mathbb{1}_{t<\tau^\rho} \, dW_t - d\xi_t^{X^\rho},$$

where

$$\tau^\rho := \inf\{t \geqslant 0 : X_t^\rho \leqslant 0\} \wedge T,$$

and $(W_t)_{t\in[0,T]}$ is a $\mathbb{F}$-Wiener process. A generic player which anticipates the total production $\bar{q} = \int_0^\ell q_{u,m} \, dm$, expects to receive the price

$$p := 1 - (\kappa \bar{q} + \rho)$$

and solves the following optimization problem:

$$\max_{\rho\geqslant 0}\mathcal{J}_c(\rho) := \max_{\rho\geqslant 0}\mathbb{E}\left\{\int_0^T e^{-rs}\left(1-\kappa\bar{q}_s-\rho_s\right)\rho_s\mathbb{1}_{s<\tau^\rho}\,ds + e^{-rT}u_T\left(X^\rho_{\tau^\rho}\right)\right\}, \tag{4.61}$$

The maximum in (4.61) is taken over all $\mathbb{F}$-adapted and non-negative processes $(\rho_t)_{t\in[0,T]}$, satisfying

$$\mathbb{E}\left[\int_0^T |\rho_s|^2\mathbb{1}_{s<\tau^\rho}\,ds\right] < \infty$$

and $(X^\rho_t)_{t\in[0,T]}$ exists in the classical sense. We claim that the feedback MFG strategy $q_{u,m}$ is optimal for the stochastic optimal control problem (4.61):

LEMMA 4.18. *Let $\rho^*_t := q_{u,m}(t, X^{\rho^*}_t)$, then it holds that:*

$$\max_{\rho\geqslant 0}\mathcal{J}_c(\rho) = \mathcal{J}_c(\rho^*) = \int_0^\ell u(0,.)\,dm_0. \tag{4.62}$$

PROOF. This kind of verification results is standard: one checks that the candidate optimal control is indeed the maximum using the equation satisfied by u; which is the value function. Let ρ be an admissible control ($\mathbb{F}$-adapted and satisfying the constraints). Since the local time is monotone, then X^ρ is a semimartingale and with the use of Itô's rule we obtain

$$\begin{aligned}
&\mathbb{E}\left[e^{-rT}u_T\left(X^\rho_{\tau^\rho}\right)\right] = \\
&\mathbb{E}\left[u(0,X^\rho_0) + \int_0^{\tau^\rho} e^{-rs}\left\{\partial_t u(s,X^\rho_s) - ru(s,X^\rho_s) - \rho_s\partial_x u(s,X^\rho_s) + \sigma\partial_{xx}u(s,X^\rho_s)\right\}ds\right] \\
&\qquad\qquad = \mathbb{E}\left[u(0,X^\rho_0) - \int_0^{\tau^\rho} e^{-rs}\left\{q^2_{u,m}(s,X^\rho_s) + \rho_s\partial_x u(s,X^\rho_s)\right\}ds\right],
\end{aligned}$$

where we have used the boundary value problem satisfied by u and the fact that $\partial_t u$, $\partial_x u$, $\partial_{xx}u$ are continuous on $(0,T)\times(0,\ell]$ (cf. (4.39)).

By using definition (4.5), note that

$$q^2_{u,m} = \frac{1}{4}|(1-\kappa\bar{q}-\partial_x u)\vee 0|^2 = \sup_{\rho\geqslant 0}\rho(1-\kappa\bar{q}-\rho-\partial_x u) = q_{u,m}(1-\kappa\bar{q}-q_{u,m}-\partial_x u).$$

Therefore

$$\mathbb{E}\left[e^{-rT}u_T\left(X^\rho_{\tau^\rho}\right)\right] \leqslant \mathbb{E}\left[u(0,X^\rho_0) - \int_0^{\tau^\rho} e^{-rs}\rho_s(1-\kappa\bar{q}-\rho_s)\,ds\right],$$

so that

$$\int_0^\ell u(0,.)\,dm_0 = \mathbb{E}\left[u(0,X^\rho_0)\right] \geqslant \mathbb{E}\left[\int_0^{\tau^\rho} e^{-rs}(1-\kappa\bar{q}-\rho_s)\rho_s\,ds + e^{-rT}u_T\left(X^\rho_{\tau^\rho}\right)\right].$$

By virtue of Lemma 4.5, we know that the process $\left(X_t^{\rho^*}\right)_{t\in[0,T]}$ exists in the strong sense. Replacing ρ by ρ^* in the above computations, inequalities become equalities and we easily infer that

$$\mathcal{J}_c(\rho^*) = \int_0^\ell u(0,.)\,dm_0.$$

Thus (4.62) is proved. □

Proof of Theorem 4.10. We start by collecting the following technical result:

LEMMA 4.19. *Fix $n \geqslant 1$, define $\mathcal{A}$ to be all elements in $D_{\mathcal{S}'_{\mathbb{R}}}$ that take values in $\tilde{\mathcal{P}}(\bar{Q})$, and let Ψ_m (resp. $\Psi^n_{\mathbf{q}}$) be the map defined from $D_{\mathcal{S}'_{\mathbb{R}}}$ into $D_{\mathcal{S}'_{\mathbb{R}}}$ (resp. from $\mathcal{A}$ into $D_{\mathbb{R}}$) such that*

$$\Psi_m(\nu)(t) := \nu(t) - m(t) \quad \textit{and} \quad \Psi^n_{\mathbf{q}}(\nu)(t) := \left|\int_{\mathbb{R}} \mathbf{q}^n_{u,m}(t,.)\,d\nu(t)\right|.$$

Then $\Psi_m, \Psi^n_{\mathbf{q}}$ are continuous with respect to the M1 topology.

PROOF. Throughout the proof, we shall use notations of [89,108].

Step 1 (continuity in $\mathcal{S}'_{\mathbb{R}}$): By virtue of Theorem 4.6, we know that $t \to m(t)$ is continuous on $[0,T]$ with respect to the strong topology of $\mathcal{S}'_{\mathbb{R}}$. Let $\phi \in \mathcal{S}'_{\mathbb{R}}$, we aim to compute the modulus of continuity of $t \to \int_{\mathbb{R}} \phi\,dm(t)$. For this, we shall appeal to the probabilistic characterization (4.15), thanks to Remark 4.4. We have for any $h > 0$

$$\left|\int_{\mathbb{R}} \phi\,d(m(t+h) - m(t))\right| \leqslant \mathbb{E}\,|\phi(X_{t+h})\mathbb{1}_{t+h<\tau} - \phi(X_t)\mathbb{1}_{t<\tau}| \tag{4.63}$$
$$\leqslant C\|\phi\|_{\mathcal{C}^1}\left(\mathbb{P}(t<\tau) - \mathbb{P}(t+h<\tau) + \mathbb{E}\,|X_{t+h} - X_t|\right).$$

Following the same steps as for (4.14a)-(4.14c), and using Burkholder-Davis-Gundy inequality, we obtain for small enough $h > 0$

$$\left|\int_{\mathbb{R}} \phi\,d(m(t+h) - m(t))\right| \leqslant C\|\phi\|_{\mathcal{C}^1}\omega_m(h),$$

where

$$\omega_m(h) := h^{1/2} + \left(\log(1/h) - h^{1/2}\right)^{-1} + \sup_{s\in[0,T]} \int_0^\ell (1 - \phi_{h^{1/2}\log(1/h)}(x))m(s,x)\,dx,$$

and ϕ_ϵ is the cut-off function defined in (4.17). In order to get $\lim_{h\to 0^+}\omega_m(h) = 0$, we need to prove that

$$\lim_{h\to 0^+}\sup_{s\in[0,T]} \int_0^\ell (1 - \phi_{h^{1/2}\log(1/h)}(x))m(s,x)\,dx = 0.$$

This ensues easily from Dini's Lemma, by choosing the sequence $(\phi_\epsilon)_{\epsilon>0}$ to be monotonically increasing.

Step 2 (continuity of Ψ_m): Let $\epsilon > 0$, $x, y \in D_{\mathcal{S}'_{\mathbb{R}}}$, B be any bounded subset of $\mathcal{S}_{\mathbb{R}}$, and $\lambda_x := (z_x, t_x), \lambda_y := (z_y, t_y)$ be a parametric representations of the graphs of x and y respectively, such that

$$g_B(\lambda_x, \lambda_y) := \sup_{s\in[0,1]} p_B(z_x(s) - z_y(s)) \vee |t_x(s) - t_y(s)| \leqslant \epsilon,$$

where $p_B(\nu) := \sup_{x\in B}|\nu(x)|$. Note that λ_x, λ_y depend on ϵ, but we do not use the subscript ϵ in order to simplify the notation. We have

$$g_B(\lambda_x,\lambda_y) \geqslant \sup_{s\in[0,1]} p_B\left(z_x(s)-m(t_x(s))-z_y(s)+m(t_y(s))\right) \vee |t_x(s)-t_y(s)| \\ - \sup_{s\in[0,1]} \max p_B(m(t_x(s))-m(t_y(s))) \vee |t_x(s)-t_y(s)|\,.$$

Since the map $t \to m(t) \in \mathcal{S}'_{\mathbb{R}}$ is continuous, observe that

$$\lambda'_\nu:\ s \to (z_\nu(s)-m(t_\nu(s)), t_\nu(s))\,,\quad \nu \equiv x, y$$

is a parametric representation of the graph

$$\gamma'_\nu := \left\{(w,t) \in \mathcal{S}'_{\mathbb{R}} \times [0,T] : w \in \left[\nu(t^-)-m(t), \nu(t)-m(t)\right]\right\},\quad \nu \equiv x, y.$$

Consequently

(4.64)
$$\mathbf{d}_{B,M1}\left(\Psi_m(x),\Psi_m(y)\right) \leqslant g_B(\lambda_x,\lambda_y) + \sup_{s\in[0,1]} p_B(m(t_x(s))-m(t_y(s))) \vee |t_x(s)-t_y(s)| \\ \leqslant 2\epsilon + \sup_{s\in[0,1]} p_B(m(t_x(s))-m(t_y(s))).$$

Hence, by using the estimation of Step 1, we infer that:

$$\mathbf{d}_{B,M1}\left(\Psi_m(x),\Psi_m(y)\right) \leqslant C(B)\omega_m(\epsilon), \tag{4.65}$$

which in turn implies that Ψ_m is continuous.

Step 3 (continuity of $\Psi^n_{\mathbf{q}}$): Let us fix $n \geqslant 1$. Note that $\mathbf{q}^n_{u,m}$ maps $[0,T]$ into $\mathcal{S}_{\mathbb{R}}$, and the following holds:

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\left|x^\alpha\partial_x^\beta\mathbf{q}^n_{u,m}(t,x)\right| \leqslant C(\ell,\alpha)n^\beta\int_{\mathbb{R}}\left|\partial_x^\beta\xi(y)\right|\,dy,\quad \forall\alpha,\beta\in\mathbb{N}. \tag{4.66}$$

Owing to (4.66), we have $\mathbf{q}^n_{u,m}([0,T]) \subset B_n$, where B_n is a bounded subset of $\mathcal{S}_{\mathbb{R}}$. Let $\epsilon > 0$, $x, y \in \mathcal{A}$, and $\lambda_x := (z_x, t_x), \lambda_y := (z_y, t_y)$ be a parametric representations of the graphs of x and y respectively such that

$$g_{B_n}(\lambda_x,\lambda_y) \leqslant \epsilon.$$

We have

$$g_{B_n}(\lambda_x,\lambda_y) \geqslant \sup_{s\in[0,1]}\left|\int_0^\ell \mathbf{q}^n_{u,m}(t_x(s),.)\,d(z_x(s)-z_y(s))\right| \vee |t_x(s)-t_y(s)| \\ \geqslant \sup_{s\in[0,1]}\left|\int_0^\ell \mathbf{q}^n_{u,m}(t_x(s),.)\,dz_x(s) - \int_0^\ell \mathbf{q}^n_{u,m}(t_y(s),.)\,dz_y(s)\right| \vee |t_x(s)-t_y(s)| \\ - \sup_{s\in[0,1]}\left|\int_0^\ell \left(\mathbf{q}^n_{u,m}(t_x(s),.)-\mathbf{q}^n_{u,m}(t_y(s),.)\right)\,dz_y(s)\right| \vee |t_x(s)-t_y(s)|\,.$$

Thus, it holds that

$$\sup_{s\in[0,1]}\left|\int_0^{\ell}\mathbf{q}^n_{u,m}(t_x(s),.)\,dz_x(s)-\int_0^{\ell}\mathbf{q}^n_{u,m}(t_y(s),.)\,dz_y(s)\right|\vee|t_x(s)-t_y(s)|$$
$$\leqslant 2\epsilon+\sup_{s\in[0,1]}\left|\int_0^{\ell}\left(\mathbf{q}^n_{u,m}(t_x(s),.)-\mathbf{q}^n_{u,m}(t_y(s),.)\right)\,dz_y(s)\right|\leqslant 2\epsilon+\omega^n_2(\epsilon).$$

where ω^n_2 is the continuity modulus of $\mathbf{q}^n_{u,m}$. By noting that

$$\lambda''_v:\ s\to\left(\int_0^{\ell}\mathbf{q}^n_{u,m}(t_v(s),.)\,dz_v(s),t_v(s)\right),\quad v\equiv x,y$$

is a parametric representation of the graph

$$\gamma''_v:=\left\{(w,t)\in\mathcal{S}'_{\mathbb{R}}\times[0,T]:w\in\left[\int_0^{\ell}\mathbf{q}^n_{u,m}(t^-,.)\,dv(t^-),\int_0^{\ell}\mathbf{q}^n_{u,m}(t,.)\,dv(t)\right]\right\},\quad v\equiv x,y,$$

we deduce that

$$\mathbf{d}_{M1}\left(\Psi^n_{\mathbf{q}}(x),\Psi^n_{\mathbf{q}}(y)\right)\leqslant 2\epsilon+\omega^n_2(\epsilon).$$

The proof is complete. □

Let us now explain the proof of Theorem 4.10. We shall proceed by contradiction, assuming that (4.52) does not hold. Then there exists $\varepsilon_0>0$, a sequence of integers N_k such that $\lim_k N_k=+\infty$, and sequences $(i_k)\subset\{1,...,N_k\}$, $(q^{i_k})\subset\mathbb{A}_c$, such that

$$\mathcal{J}^{i_k,N_k}_c\left(q^{i_k};(\hat{q}^j)_{j\neq i_k}\right)>\varepsilon_0+\mathcal{J}^{i_k,N_k}_c\left(\hat{q}^1,...,\hat{q}^N\right),\ \ \forall k\geqslant 0.\tag{4.67}$$

We derive a contradiction by estimating the difference between $\mathcal{J}^{i_k,N_k}_c$ and the mean field objective $\mathcal{J}_c$. Using Lemma 4.11, we will show that this difference goes to zero.

Let us set for any $k\geqslant 0$,

$$\begin{cases} dX^{i_k}_t:=-q^{i_k}_t\,dt+\sqrt{2\sigma}\,dW^{i_k}_t-d\xi^{X^{i_k}}_t,\ \ X^{i_k}_0=V_{i_k},\\ \tau^{i_k}:=\inf\{t\geqslant 0\ :\ X^{i_k}_t\leqslant 0\}\wedge T,\end{cases}$$

and define

$$\mathbb{Z}^k_{1,T}:=\int_0^T q^{i_k}_s\mathbb{1}_{s<\tau^{i_k}}\,ds,\quad\text{ and }\quad\mathbb{Z}^k_{2,T}:=\int_0^T\left|q^{i_k}_s\right|^2\mathbb{1}_{s<\tau^{i_k}}\,ds.$$

Recall that all elements of $\mathbb{A}_c$ are non-negative, so that $\mathbb{Z}^k_{1,T}\geqslant 0$ for any $k\geqslant 0$. We start by collecting estimates on $\left(\mathbb{Z}^{i_k}_{1,T}\right)_{k\geqslant 0}$ and $\left(\mathbb{Z}^{i_k}_{2,T}\right)_{k\geqslant 0}$. Observe that for any $t\in[0,T]$,

$$X^{i_k}_{t\wedge\tau^{i_k}}=V_k-\int_0^t q^{i_k}_s\mathbb{1}_{s<\tau^{i_k}}\,ds+\sqrt{2\sigma}W^{i_k}_{t\wedge\tau^{i_k}}-\xi^{X^{i_k}}_t,\quad\forall k\geqslant 0.$$

Since the local time is nondecreasing, we infer that

$$0\leqslant\mathbb{Z}^k_{1,T}\leqslant V_{i_k}-X^{i_k}_{\tau^{i_k}}+\sqrt{2\sigma}W^{i_k}_{\tau^{i_k}},\quad\forall k\geqslant 0$$

holds almost surely. By means of the optional stopping theorem, we deduce that

$$\sup_{k\geqslant 0}\mathbb{E}\left[\mathbb{Z}^k_{1,T}\right]\leqslant\ell.\tag{4.68}$$

Moreover, recall that

$$\mathcal{J}_c^{i_k,N_k}\left(q^{i_k};(\hat{q}^j)_{j\neq i_k}\right)=\mathbb{E}\left\{\int_0^T e^{-rs}\left(1-\kappa\overline{\hat{q}_s}^{i_k}-q_s^{i_k}\right)q_s^{i_k}\mathbb{1}_{s<\tau^{i_k}}\,\mathrm{d}s+e^{-rT}u_T(X^{i_k}_{\tau^{i_k}})\right\},$$

where for any $k\geqslant 0$

$$\overline{\hat{q}_s}^{i_k}=\frac{1}{N_k-1}\sum_{j\neq i_k}q_{u,m}(s,\hat{X}_s^j)\mathbb{1}_{s<\hat{\tau}^j}.$$

Thus, for any $k\geqslant 0$

$$e^{-rT}\mathbb{E}\left[\mathbb{Z}_{2,T}^k\right]\leqslant\|u_T\|_\infty+\mathbb{E}\left\{\int_0^T e^{-rs}\left|1-\kappa\overline{\hat{q}_s}^{i_k}\right|q_s^{i_k}\mathbb{1}_{s<\tau^{i_k}}\,\mathrm{d}s\right\}-\mathcal{J}_c^{i_k,N_k}\left(q^{i_k};(\hat{q}^j)_{j\neq i_k}\right).$$

By virtue of (4.67) and the uniform bound on $q_{u,m}$ that is given in (4.27), we deduce that

$$e^{-rT}\mathbb{E}\left[\mathbb{Z}_{2,T}^k\right]\leqslant 2\|u_T\|_\infty+(\kappa+1)\sup_{k\geqslant 0}\mathbb{E}\left[\mathbb{Z}_{1,T}^k\right]+C(\kappa,T),$$

so that

$$\sup_{k\geqslant 0}\mathbb{E}\left[\mathbb{Z}_{2,T}^k\right]\leqslant C(T,\kappa,\|u_T\|_\infty,\ell). \tag{4.69}$$

On the other hand, we have for any $k\geqslant 0$,

$$\begin{aligned}
&\mathcal{J}_c^{i_k,N_k}\left(q^{i_k};(\hat{q}^j)_{j\neq i_k}\right)\\
&\quad\leqslant\mathbb{E}\left\{\int_0^T e^{-rs}\left(1-\kappa\int_0^\ell q_{u,m}(s,.)\,\mathrm{d}\hat{\nu}_s^{N_k}-q_s^{i_k}\right)q_s^{i_k}\mathbb{1}_{s<\tau^{i_k}}\,\mathrm{d}s+e^{-rT}u_T(X^{i_k}_{\tau^{i_k}})\right\}\\
&\qquad\qquad+\kappa\left(\frac{N_k}{N_k-1}-1\right)+\frac{\kappa}{N_k}\sup_{k\geqslant 0}\mathbb{E}\left[\mathbb{Z}_{1,T}^{i_k}\right].
\end{aligned}$$

Thus, for any $k\geqslant 0$

$$\begin{aligned}
&\mathcal{J}_c^{i_k,N_k}\left(q^{i_k};(\hat{q}^j)_{j\neq i_k}\right)-\mathcal{J}_c(q^{i_k})-CN_k^{-1}\\
&\qquad\leqslant\kappa\mathbb{E}\left[\int_0^T e^{-rs}q_s^{i_k}\mathbb{1}_{s<\tau^{q^{i_k}}}\left|\int_{\mathbb{R}}\mathbf{q}_{u,m}(s,.)\,\mathrm{d}\left(m(s)-\hat{\nu}_s^{N_k}\right)\right|\mathrm{d}s\right]\\
&\qquad\leqslant\kappa\mathbb{E}\left[\int_0^T e^{-rs}q_s^{i_k}\mathbb{1}_{s<\tau^{q^{i_k}}}\left|\int_{\mathbb{R}}\mathbf{q}^n_{u,m}(s,.)\,\mathrm{d}\left(m(s)-\hat{\nu}_s^{N_k}\right)\right|\mathrm{d}s\right]\\
&\qquad\qquad+\kappa\mathbb{E}\left[\int_0^T e^{-rs}q_s^{i_k}\mathbb{1}_{s<\tau^{q^{i_k}}}\,\mathrm{d}s\right]\left\|\mathbf{q}^n_{u,m}-\mathbf{q}_{u,m}\right\|_\infty,
\end{aligned}$$

where $\mathcal{J}_c$ is defined in Lemma 4.18, and $\mathbf{q}_{u,m},\mathbf{q}^n_{u,m}$ are given by (4.60a)-(4.60b).

Let us fix $\varepsilon>0$. Since $\left(\mathbf{q}^n_{u,m}\right)_{n\geqslant 1}$ converges uniformly toward $\mathbf{q}_{u,m}$ on $[0,T]\times\mathbb{R}$, we can choose n large enough and independently of $k\geqslant 0$ so that

$$\begin{aligned}
&\mathcal{J}_c^{i_k,N_k}\left(q^{i_k};(\hat{q}^j)_{j\neq i_k}\right)-\mathcal{J}_c(q^{i_k})\\
&\qquad\leqslant\kappa\mathbb{E}\left[\mathbb{Z}_{2,T}^k\right]^{1/2}\mathbb{E}\left[\int_0^T\left|\int_{\mathbb{R}}\mathbf{q}^n_{u,m}(s,.)\,\mathrm{d}\left(\hat{\nu}_s^{N_k}-m(s)\right)\right|^2\mathrm{d}s\right]^{1/2}+\kappa\varepsilon\mathbb{E}\left[\mathbb{Z}_{1,T}^k\right]+CN_k^{-1}.
\end{aligned}\tag{4.70}$$

Appealing to Lemma 4.11, Lemma 4.19 and the continuous mapping theorem we have

$$\lim_{N} \mathbb{E}\left[\int_0^T \left|\int_{\mathbb{R}} \mathbf{q}^n_{u,m}(s,.)\, d\left(\hat{\nu}_s^{N_k} - m(s)\right)\right|^2 ds\right] = 0. \tag{4.71}$$

Thus, by combining (4.68), (4.69), and (4.70):

$$\mathcal{J}_c^{i_k,N_k}\left(q^{i_k};(\hat{q}^j)_{j\neq i_k}\right) - \mathcal{J}_c(q^{i_k}) \leqslant C(T,\kappa,\|u_T\|_\infty,\ell)\varepsilon$$

for big enough $k \geqslant 0$. Whence, by means of Lemma 4.18:

$$\mathcal{J}_c^{i_k,N_k}\left(q^{i_k};(\hat{q}^j)_{j\neq i_k}\right) \leqslant C(T,\kappa,\|u_T\|_\infty,\ell)\varepsilon + \mathcal{J}_c(\rho^*)$$

for big enough $k \geqslant 0$. In the same manner, one can show that

$$\mathcal{J}_c(\rho^*) \leqslant C\varepsilon + \mathcal{J}_c^{i_k,N_k}\left(\hat{q}^1,...,\hat{q}^N\right)$$

holds for big enough $k \geqslant 0$. Hence, going back to (4.67) and using the above estimates, we obtain

$$\varepsilon_0 < C(T,\kappa,\|u_T\|_\infty,\ell)\varepsilon.$$

We deduce the desired contradiction by choosing ε suitably small.

Part III

Optimal Execution Mean Field Games

CHAPTER 5

A Mean Field Game of Portfolio Trading And Its Consequences On Perceived Correlations

Joint work with Charles-Albert Lehalle, submitted to "SIAM Journal on Financial Mathematics".

1. Introduction

Optimal execution deals with the optimization of a trading path from a given initial position to zero in a given time window. This problem is regularly faced by traders or brokers, when large institutional investors [1] decide to buy or to sell a large number of shares or contracts on the market. As the number of assets is significantly larger than the average size of a "normal" trade, it is probably not a good idea to try to execute all the assets in a one single transaction, since the willingness to buy or sell in the market (*liquidity*) is limited. Hence, the agent execution strategy often boils down to breaking up the large order (*parent order*) into small orders (*child orders*), and try to execute each one of these child orders over a period of time. The execution process has to be spread out over time to avoid large execution costs due to the limited liquidity, but fast enough to minimize adverse price movements over the course of the execution process.

Optimal liquidation emerged as an academic field with two seminal papers: one [5] focussed on the balance between trading fast (to minimize the uncertainty of the obtained price) and trading slow (to minimize the "*market impact*", i.e. the detrimental influence of the trading pressure on price moves) for one representative instrument; while the other [17] focussed on a portfolio of tradable instruments, shedding light on the interplay with correlations of price returns and market impact. The last twenty years have seen a lot of proposals to sophisticate the single instrument case (see these reference books [23, 47, 71] for typical models and references) but very few on extending it to portfolios of multiple assets (with the notable exception of [75]). Moreover, the usual framework for optimal execution is the one of one large privileged agent facing a "mean-field" or a "background noise" made of the sum of behaviours of other market participants, and academic literature seldom tackles the strategic interaction of many market participants seeking to execute large orders.

More recently, game theory has been introduced in this field. First around cases with few agents, like in [106], and then by [39, 60, 79] relying on Mean Field Games (MFG) to get rid of the combinatorial complexity of games with few players, considering a lot of agents, such that their aggregated behaviour reduces to a "*anonymous mean field of liquidity*", shared by all of them.

[1] Such as pension funds, hedge funds, mutual funds, and sovereign funds.

In this paper, we clearly start within the framework and results obtained by [39] and extend them to the case of a portfolio of tradable instruments. Our agents are the same as in this paper: optimal traders seeking to buy and sell positions given at the start of the day. That for, they rely on the stochastic control problem well defined for one instrument in [47], which result turns to be deterministic because of its linear-quadratic nature: minimize the cost of the trading under risk-averse conditions and a terminal cost. This framework can be compared to the one used by [17] in their section on portfolio, with a diagonal matrix for the market impact, and in a game played by a continuum of agents. Note that in all these papers, including ours, the time scale is large enough to not take into account orderbook dynamics, and small enough to be used by traders and dealing desks; our typical terminal time goes from one hour to several days, and time steps have to be read in minutes. In their paper, Cardaliaguet and Lehalle have shown how a continuum of such agents with heterogenous preferences can emulate a mix of typical brokers (having a large risk aversion and terminal cost), and opportunistic traders (with a low risk aversion). It will be the same for us. But while their paper only addresses the strategic behavior of investors on a one single financial instrument this one handles the case of a portfolio of correlated assets. In the real applications, a financial instrument is rarely traded on its own; most investors construct diversified or hedged portfolios or index trackers by simultaneously buying and selling a large number of assets.

This has motivated the present work in which we introduce an extension of the initial Cardaliaguet-Lehalle framework to the case of a multi-asset portfolio. On the one hand, this extension allows to cover a new type of trading strategies, such as Program Trading (executing large baskets of stocks), Arbitrage Strategies (which aims to benefit from discrepancies in the dynamics of two or more assets), Hedging Strategies (where a round trip on a second asset – typically a very liquid one – can be used to partially hedge the price risk in the execution process of a given asset), and Index Tracking (i.e. following the composition given by a formula, like in factor investing, or simply following the market capitalization of a list of instruments). On the other hand, it enables us to understand the dependence structure between the market orders flows at "equilibrium", and assess their influence on standard estimates of the covariance (or correlation) matrix of asset returns. These questions were independently raised by some authors and studied in seldom empirical and theoretical works (see e.g. [16, 21, 51, 78, 95] and references therein).

Following the seminal paper [39], we assume that the market impact is either instantaneous or permanent, and that the public prices – of all assets – are influenced by the permanent market impact of all market participants. Conversely, since the agents are affected by the public prices, they aim to anticipate the "market mean field" (i.e. the market trend due to the market impact of the mean field of all agents) by using all the information they have in order to minimize their exposition to the other agents' impact. As explained in [39] this leads to a Nash equilibrium configuration of MFG type, in which all agents anticipate the average trading speed of the population and adjust their execution accordingly. We refer the reader to Section 2 for a more detailed explanation of the Mean Field Game model. In the context of a MFG with multi-asset portfolio, the strategic interaction between the agents during a trading day leads to a non-trivial relationship between the assets' order flows, which in turn generates a non-trivial impact on the intraday covariance (or correlation) matrix of asset returns. In Section 3, we provide an exact formula for the excess covariance matrix of returns that is endogenously generated by the trading activity, and we show that the magnitude of this effect is more significant when the market impact is large. This means for a highly crowded market, illiquid products or large initial orders (cf. Section 3). These results can be related to the ones of [**51**], except that in this paper we do not focus our attention on distressed sells only; we are able to capture the influence of the usual variations of trading flows to deformations of the naive estimate of the covariance matrix of a portfolio of assets that are simultaneously traded. We also carry out several numerical simulations and apply our results in an empirical analysis which is conducted on a database of market data from January to December 2014 for a pool of 176 US stocks. At first, we exhibit the theoretical relation between the intraday covariance matrix of net traded flows and the standard intraday covariance matrix is increasing, then we use this relation to estimate some parameters of our model, including the market impact coefficients (cf. Section 3). Next, we normalize the covariance matrix of returns to compute the intraday median diagonal pattern (across diagonal terms), and the intraday median off-diagonal pattern (across off-diagonal terms) (cf. Section 3.3), as a way of characterizing the typical intraday evolution for diagonal and off-diagonal terms. It allows us to obtain empirically the well-known intraday pattern of volatility that is in line with our model, and we show that it flattens out as the typical size of transactions diminishes. In such a case the empirical volatility is close to its "fundamental" value (cf. Figure 4). Finally, we propose a toy model based approach to calibrate our MFG model on data.

This paper is structured as follows: in Section 2 we formulate the problem of optimal execution of a multi-asset portfolio inside a Mean Field Game. We derive a MFG system of PDEs and prove uniqueness of solutions to that system for a general Hamiltonian function. Then we construct a regular solution in the quadratic framework, which will be considered throughout the rest of the paper. Next, we provide a convenient numerical scheme to compute the solution of the MFG system, and present several examples of an agent's optimal trading path, and the average trading path of the population. Section 3 is devoted to the analysis of the crowd's trading impact on the intraday covariance matrix of returns. At the MFG equilibrium configuration, we derive a formula for the impact of assets' order flows on the dependence structure of asset returns. Next, we carry out numerical simulations to illustrate this fact, and apply our results in an empirical analysis on a pool of 176 US stocks.

2. Optimal Portfolio Trading Within The Crowd

2.1. The Mean Field Game Model. Consider a continuum of investors (agents), that are indexed by a parameter a. Each agent has to trade a portfolio corresponding to instructions given by a portfolio manager. Think about a continuum of brokers or dealing desks executing large orders given by their clients. The portfolios are made of desired positions in a universe of d different stocks (or any financial assets). The initial position of any agent a is denoted by $\mathbf{q}_0^a := (q_0^{1,a}, ..., q_0^{d,a})$. For any i, when the initial inventory $q_0^{i,a}$ is positive, it means the agent has to sell this number of shares (or contracts) whereas when it is negative, the agent has to sell this amount. Given a common horizon $T > 0$, we suppose that all the investors have to sell or buy within the trading period $[0, T]$. This means the agent has to sell this number of shares (or contracts) whereas when it is negative, the agent has to buy this amount.

The intraday position of each investor a is modeled by a $\mathbb{R}^d$-valued process $(\mathbf{q}_t^a)_{t\in[0,T]}$ which has the following dynamics:

$$d\mathbf{q}_t^a = \mathbf{v}_t^a \, dt, \quad \mathbf{q}^a(0) = \mathbf{q}_0^a.$$

The investor controls its trading speed $(\mathbf{v}_t^a)_t := (v_t^{1,a}, ..., v_t^{d,a})_t$ through time, in order to achieve its trading goal. Following the standard optimal liquidation literature, we assume that, for each stock, the dynamics of the mid-price can be written as:

$$dS_t^i = \sigma_i \, dW_t^i + \alpha_i \mu_t^i \, dt, \quad i = 1, ..., d; \tag{5.1}$$

where $\sigma_i > 0$ is the arithmetic volatility of the i^{th} stock, and $\alpha_1, ..., \alpha_d$ are nonnegative scalars modeling the magnitude of the permanent market impact. Here $(W_t^1, ..., W_t^d)_{t\geqslant 0}$ are d correlated Wiener processes, and the process $(\boldsymbol{\mu}_t)_{t\in[0,T]} := (\mu_t^1, ..., \mu_t^d)_t$ corresponds to the average trading speed of all investors across the portfolio of assets. Throughout, we shall denote by Σ the covariance matrix of the d-dimensional process $(\mathbf{W}_t)_{t\in[0,T]} := (\sigma_1 W_t^1, ..., \sigma_d W_t^d)_{t\in[0,T]}$ and suppose that Σ *is not singular.*

The performance of any investor a is related to the amount of cash generated throughout the trading process. Given the price vector $(\mathbf{S}_t)_{t\in[0,T]} := (S_t^1, ..., S_t^d)_{t\in[0,T]}$, we shall assume that the amount of cash $(X_t^a)_{t\in[0,T]}$ on the account of the trader a is given by:

$$X_t^a = -\int_0^t \mathbf{v}_s^a \cdot \mathbf{S}_s \, ds - \sum_{i=1}^d \int_0^t V_i L_i \left(\frac{v_s^{i,a}}{V_i} \right) ds,$$

where the positive scalars $V_1, ..., V_d$ denotes the magnitude of daily market liquidity (in practice the average volume traded each day can be used as a proxy for this parameter) of each asset. Here $L_1, ..., L_d$ are the execution cost functions (similar to the ones of [75]), modelling the instantaneous component of market impact, which takes part in the average cost of trading. The family of functions $L_i : \mathbb{R} \to \mathbb{R}$ are assumed to fulfil the following set of assumptions:

- $L_i(0) = 0$;
- L_i is strictly convex and nonnegative;
- L_i is asymptotically super-linear, i.e. $\lim_{|p|\to+\infty} \frac{L_i(p)}{|p|} = +\infty$.

The initial Cardaliaguet-Lehalle model [39], corresponds to $d = 1$, and a quadratic liquidity function of the form $L(p) = \kappa|p|^2$.

In this chapter, we consider a reward function that is similar to [39], and corresponding to *Implementation Shortfall (IS) orders*. In this specific case the reward function of any investor a is given by:

(5.2)

$$U^a(t,x,s,\mathbf{q};\boldsymbol{\mu}) := \sup_{\mathbf{v}} \mathbb{E}_{x,s,\mathbf{q}}\left(X_T^a + \mathbf{q}_T^a\cdot(\mathbf{S}_T - \mathbf{A}^a\mathbf{q}_T^a) - \frac{\gamma^a}{2}\int_t^T \mathbf{q}_s^a\cdot\Sigma\mathbf{q}_s^a\,ds\right),$$

where $\mathbf{A}^a := \mathbf{diag}(A_1^a, ..., A_d^a)$, $A_i^a > 0$, and γ^a is a non-negative scalar which quantifies the investor's tolerance toward market risk. That is when $\gamma^a = 0$ the investor is indifferent about holding inventories through time, while when γ^a is large the investor attempt to liquidate as quick as possible. The quadratic term $\mathbf{q}_T^a\cdot(\mathbf{S}_T - \mathbf{A}^a\mathbf{q}_T^a)$ penalizes non-zero terminal inventories. One should note that the expression of the profit functional (5.2) is derived by considering that agents are risk-averse with CARA utility function. We omit the details and refer the reader to [71, Chapter 5].

The Hamilton-Jacobi equation associated to (5.2) is

$$\begin{aligned}0 = \partial_t U^a - \frac{\gamma^a}{2}\mathbf{q}\cdot\Sigma\mathbf{q} + \frac{1}{2}\mathrm{Tr}\left(\Sigma D_s^2 U^a\right) + \mathbb{A}\boldsymbol{\mu}\cdot\nabla_s U^a \\ + \sup_{\mathbf{v}}\left\{\mathbf{v}\cdot\nabla_{\mathbf{q}}U^a - \left(\mathbf{v}\cdot\mathbf{s} + \sum_{i=1}^d V_i L_i\left(\frac{\nu^i}{V_i}\right)\right)\nabla_x U^a\right\},\end{aligned}$$

with the terminal condition

$$U^a(T,x,s,\mathbf{q};\boldsymbol{\mu}) = x + \mathbf{q}\cdot(s - \mathbf{A}^a\mathbf{q}).$$

In all this chapter we set $\mathbb{A} := \mathbf{diag}(\alpha_1, ..., \alpha_d)$. Due to the simplifications that we will obtain afterwards, we suppose that $\boldsymbol{\mu} = (\boldsymbol{\mu}_t)_{t\in[0,T]}$ is a deterministic process, so that the HJB equation above is deterministic. When $\boldsymbol{\mu}$ is a random process, that is adapted to the natural filtration of $(\mathbf{W}_t)_{t\in[0,T]}$, we obtain a stochastic backward HJB equation which requires a specific treatment (cf. [33]).

Following the approach of [39], we consider the following ansatz:

$$U^a(t,x,s,\mathbf{q};\boldsymbol{\mu}) = x + \mathbf{q}\cdot s + u^a(t,\mathbf{q};\boldsymbol{\mu}),$$

which entails the following HJB equation for u^a:

$$\frac{\gamma^a}{2}\mathbf{q}\cdot\Sigma\mathbf{q} = \partial_t u^a + \mathbb{A}\boldsymbol{\mu}\cdot\mathbf{q} + \sup_{\mathbf{v}}\left\{\mathbf{v}\cdot\nabla_q u^a - \sum_{i=1}^d V_i L_i\left(\frac{\nu^i}{V_i}\right)\right\}, \tag{5.3}$$

endowed with the terminal condition:

$$u_T^a = -\mathbf{A}^a\mathbf{q}\cdot\mathbf{q}.$$

For any $i = 1, ..., d$, let H_i be the Legendre-Fenchel transform of the function L_i that is given by

$$H_i(p) := \sup_\rho p\rho - L_i(\rho).$$

Since the maps $(L_i)_{1\leqslant i\leqslant d}$ are strictly convex, $(H_i)_{1\leqslant i\leqslant d}$ are functions of class $\mathcal{C}^1$, and the optimal feedback strategies associated to (5.3) are given by

$$\nu^{i,a}(t,\mathbf{q}) := V_i\dot{H}_i\left(\partial_{q_i}u^a(t,\mathbf{q})\right),$$

where $\dot{H}_i$ denotes the first derivative of H_i. Therefore, the Mean Field Game system associated to the above problem reads:

$$
\left\{
\begin{aligned}
&\frac{\gamma^a}{2}\mathbf{q}\cdot\Sigma\mathbf{q} = \partial_t u^a + \mathbb{A}\boldsymbol{\mu}\cdot\mathbf{q} + \sum_{i=1}^{d} V_i H_i\left(\partial_{q_i} u^a(t,\mathbf{q})\right)\\
&\partial_t m + \sum_{i=1}^{d} V_i \partial_{q_i}\left(m\dot{H}_i\left(\partial_{q_i} u^a(t,\mathbf{q})\right)\right) = 0\\
&\mu_t^i = \int_{(q,a)} V_i \dot{H}_i\left(\partial_{q_i} u^a(t,\mathbf{q})\right) m(t,\, d\mathbf{q},\, da)\\
&m(0,\, d\mathbf{q},\, da) = m_0(\, d\mathbf{q},\, da),\quad u_T^a = -\mathbf{A}^a\mathbf{q}\cdot\mathbf{q}.
\end{aligned}
\right.
\tag{5.4}
$$

The Mean Field Game system (5.4) describes a Nash equilibrium configuration, with infinitely many well-informed market investors: any individual player anticipates the right average trading flow on the trading period $[0,T]$, and computes his optimal strategy accordingly. Observe that we make a strong assumption by supposing that the considered group of investors has a precise knowledge of market mean field. In reality this knowledge is only partial and/or approximate.

Well-posedness for system (5.4) is investigated in **[39]** within the general framework of *Mean Field Games of Controls*. In this work, we provide simpler arguments to deal with the specific cases of our study. We shall suppose that $(H_i)_{1\leqslant i\leqslant d}$ are of class $\mathcal{C}^2$ and satisfy the following condition:

$$
\forall i = 1,...,d,\ \forall p\in\mathbb{R},\quad C_0^{-1}\leqslant \ddot{H}_i(p)\leqslant C_0,
\tag{5.5}
$$

for some $C_0>0$, and m_0 is a probability density *with a finite second order moment*. Moreover, we suppose that the investors' index varies in a closed subset $D\subset\mathbb{R}$.

We say that $(u^a, m)_{a\in D}$ is a solution to the MFG system (5.4) if the following hold:

- $u^a\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$, for a.e $a\in D$, and m in $\mathcal{C}([0,T];L^1(\mathbb{R}\times D))$;
- the equation for u^a holds in the classical sens, while the equation for m holds in the sense of distribution;
- for any $t\in[0,T]$,

$$
\int_{\mathbb{R}\times D}|\mathbf{q}|\,dm(t,\,d\mathbf{q},\,da)<\infty,\ \text{ and }\ |\nabla_q u^a(t,\mathbf{q})|\leqslant C_1(1+|\mathbf{q}|),
\tag{5.6}
$$

  for some $C_1>0$.

Let us start with the following remark on the uniqueness of solutions to (5.4).

PROPOSITION 5.1. *Under the above assumptions, system (5.4) has at most one solution.*

PROOF. Let $(u_1^a, m_1)_{a\in D}$ and $(u_2^a, m_2)_{a\in D}$ be two solutions to (5.4), and set $\bar{u}^a := u_1^a - u_2^a$, $\bar{m} := m_1 - m_2$. At first, let us assume that m_1, m_2 are smooth so that the computations below holds. By using system (5.4), we have:

(5.7)
$$\frac{d}{dt}\int_{(q,a)} \bar{u}^a \bar{m} = -\int_{(q,a)} \bar{m}\left\{\sum_{i=1}^{d} V_i \left(H_i\left(\partial_{q_i} u_1\right) - H_i\left(\partial_{q_i} u_2\right)\right) + \mathbb{A}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\cdot \mathbf{q}\right\}$$
$$-\int_{(q,a)} \bar{u}\left\{\sum_{i=1}^{d} V_i \left(\partial_{q_i}\left(m_1 \dot{H}_i\left(\partial_{q_i} u_1\right)\right) - \partial_{q_i}\left(m_2 \dot{H}_i\left(\partial_{q_i} u_2\right)\right)\right)\right\},$$

where $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ correspond respectively to $(u_1^a, m_1)_{a\in D}$ and $(u_2^a, m_2)_{a\in D}$.

On the one hand, note that

$$\int_{(q,a)} \bar{m}\mathbb{A}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\cdot \mathbf{q} = \frac{1}{2}\frac{d}{dt}\mathbb{A}\bar{\mathbf{E}}\cdot\bar{\mathbf{E}}, \quad \text{where} \quad \bar{\mathbf{E}}(t) := \int_{(q,a)} \mathbf{q}\, d\bar{m}(t).$$

This follows from

$$\frac{d}{dt}\bar{\mathbf{E}} = \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2,$$

which is in turn obtained from system (5.4) after an integration by parts.

On the other hand, by virtue of (5.5) we have

$$\begin{aligned}
&\sum_{i=1}^{d} V_i \int \left(\bar{m}\left(H_i\left(\partial_{q_i} u_1\right) - H_i\left(\partial_{q_i} u_2\right)\right) - \partial_{q_i}\bar{u}\left(m_1 \dot{H}_i\left(\partial_{q_i} u_1\right) - m_2 \dot{H}_i\left(\partial_{q_i} u_2\right)\right)\right)\\
&= -\sum_{i=1}^{d} V_i \int \left(m_1\left(H_i\left(\partial_{q_i} u_2\right) - H_i\left(\partial_{q_i} u_1\right) - \dot{H}_i\left(\partial_{q_i} u_1\right)\partial_{q_i}(u_2 - u_1)\right)\right)\\
&-\sum_{i=1}^{d} V_i \int \left(m_2\left(H_i\left(\partial_{q_i} u_1\right) - H_i\left(\partial_{q_i} u_2\right) - \dot{H}_i\left(\partial_{q_i} u_2\right)\partial_{q_i}(u_1 - u_2)\right)\right)\\
&\leqslant -\min_{1\leqslant i\leqslant d} V_i \int_{(q,a)} \frac{(m_1 + m_2)}{2C_0}\left|\nabla_{\mathbf{q}} u_1 - \nabla_{\mathbf{q}} u_2\right|^2.
\end{aligned}$$

Therefore, (5.7) provides

$$\min_{1\leqslant i\leqslant d} V_i \int_0^T \int_{(q,a)} \left|\nabla_{\mathbf{q}} u_1(s) - \nabla_{\mathbf{q}} u_2(s)\right|^2 \, d(m_1 + m_2)\, ds + \frac{C}{2}\mathbb{A}\bar{\mathbf{E}}(T)\cdot\bar{\mathbf{E}}(T) = 0. \tag{5.8}$$

By using a standard regularization process, identity (5.8) holds true for any solutions $(u_1^a, m_1)_{a\in D}$ and $(u_2^a, m_2)_{a\in D}$ of (5.4). Thus, one can use this identity to deduce that $\nabla_q u_1 \equiv \nabla_q u_2$ on $\{m_1 > 0\}\cup\{m_2 > 0\}$, so that m_1, m_2 solve the same transport equation:

$$\partial_t \nu + \sum_{i=1}^{d} V_i \partial_{q_i}\left(\nu \dot{H}_i\left(\partial_{q_i} u_1^a(t,\mathbf{q})\right)\right) = 0, \quad \nu_{t=0} = m_0.$$

This entails $m_1 \equiv m_2$ and so $u_1 \equiv u_2$, by virtue of our regularity assumptions. □

2.2. Quadratic Liquidity Functions. In practice the liquidity function is often chosen as strictly convex power function of the form: $L(p) = \eta|p|^{1+\phi} + \omega|p|$, with $\eta, \phi, \omega > 0$. The additional term $\omega|p|$ captures proportional costs such as the bid-ask spread, taxes, fees paid to brokers, trading venues and custodians [71]. The quadratic case ($\phi = 1$) – that is also considered in [39] – is particularly interesting because it induces some considerable simplifications and allows to compute the solutions at a relatively low cost. Throughout the rest of this chapter, we suppose that the liquidity functions take the following simple form:

$$L^i(p) = \eta_i|p|^2 \quad \text{where} \quad \eta_i > 0, \quad i = 1, ..., d. \tag{5.9}$$

Following the approach of [39], we start by setting $\bar{m}_0(\,da) := \int_q m_0(\,d\mathbf{q},\,da)$. We shall suppose that

$$\bar{m}_0(a) \neq 0, \qquad \text{for a.e } a \in D, \tag{5.10}$$

and that investors do not change their preference parameter a over time. Thus, we always have $\int_q m(t,\,d\mathbf{q},\,da) = \bar{m}_0(\,da)$, so that we can disintegrate m into

$$m(t,\,d\mathbf{q},\,da) = m^a(t,\,d\mathbf{q})\bar{m}_0(\,da),$$

where $m^a(t,\,d\mathbf{q})$ is a probability measure in $\mathbf{q}$ for $\bar{m}_0$-almost any a. Let us now define the following process which plays an important role in our analysis:

$$\mathbf{E}^a(t) := \int_q \mathbf{q}\, m^a(t,\,d\mathbf{q}) \quad \forall t \in [0,T], \;\; \text{for a.e } a \in D,$$

and we shall denote by $E^{a,1}, ..., E^{a,d}$ the components of $\mathbf{E}^a$. By virtue of the PDE satisfied by m, observe that $\mathbf{E}^a$ satisfies the following:

$$\begin{aligned} \dot{\mathbf{E}}^a(t) &= \int_q \mathbf{q}\, \partial_t m^a(t,\,d\mathbf{q}) \\ &= \int_q \left(\frac{V_i}{2\eta_i}\partial_{q_i} u^a(t,\mathbf{q})\right)_{1\leqslant i\leqslant d} m^a(t,\,d\mathbf{q}), \end{aligned} \tag{5.11}$$

so that

$$\boldsymbol{\mu}_t = \int_a \dot{\mathbf{E}}^a(t)\, d\bar{m}_0(a). \tag{5.12}$$

Due to the existence of linear and quadratic terms in the equation satisfied by u^a, we expect the solution to have the following form:

$$u^a(t,\mathbf{q}) = h_a(t) + \mathbf{q}' \cdot \mathcal{H}_a(t) + \frac{1}{2}\mathbf{q}' \cdot \mathbb{H}_a(t) \cdot \mathbf{q} \tag{5.13}$$

where $h_a(t)$ is $\mathbb{R}$-valued function, $\mathcal{H}_a(t) := (\mathcal{H}_a^i(t))_{1\leqslant i\leqslant d}$ is $\mathbb{R}^d$-valued function, and the map $\mathbb{H}_a(t) := (\mathbb{H}_a^{i,j}(t))_{1\leqslant i,j\leqslant d}$ take values in the set of $\mathbb{R}^{d\times d}$-symmetric matrices. Inserting (5.13) in the HJB equation of (5.4) and collecting like terms in $\mathbf{q}$ leads to the following coupled system of BODEs:

$$
\begin{cases}
\dot{h}_a = -\mathbb{V}\mathcal{H}_a \cdot \mathcal{H}_a \\
\dot{\mathcal{H}}_a = -\mathbb{A}\boldsymbol{\mu} - 2\mathbb{H}_a\mathbb{V}\mathcal{H}_a \\
\dot{\mathbb{H}}_a = -2\mathbb{H}_a\mathbb{V}\mathbb{H}_a + \gamma^a\Sigma \\
h_a(T) = 0, \;\; \mathcal{H}_a(T) = 0, \;\; \mathbb{H}_a(T) = -2\mathbf{A}^a,
\end{cases}
\tag{5.14}
$$

where $\mathbb{V} := \operatorname{diag}\left(\frac{V_1}{4\eta_1}, ..., \frac{V_d}{4\eta_d}\right)$. In order to solve completely (5.14) we need to know $\boldsymbol{\mu}$, or the process $\dot{\mathbf{E}}^a$ thanks to (5.12). Thus, one needs an additional equation to completely solve the problem.

By virtue of (5.11), we have

$$
\dot{\mathbf{E}}^a = 2\mathbb{V}\mathcal{H}_a + 2\mathbb{V}\mathbb{H}_a\mathbf{E}^a.
\tag{5.15}
$$

By combining this equation with system (5.14) one obtains the following FBODE:

$$
\begin{cases}
\ddot{\mathbf{E}}^a = -2\mathbb{V}\mathbb{A}\displaystyle\int_a \dot{\mathbf{E}}^a \, d\bar{m}_0(a) + 2\gamma_a\mathbb{V}\Sigma\mathbf{E}^a \\
\mathbf{E}^a(0) = E_0^a := \displaystyle\int_q \mathbf{q} m_0(\mathbf{q}, a)/\bar{m}_0(a) \\
\dot{\mathbf{E}}^a(T) + 4\mathbb{V}\mathbf{A}^a\mathbf{E}^a(T) = 0.
\end{cases}
\tag{5.16}
$$

This system is a generalized form of the one that is studied in [**39**], and summarizes the whole market mean field. Observe that the permanent market impact acts as a friction term while the market risk terms act as a pushing force toward a faster execution. The investors heterogeneity is taken into account in the first derivative term, which means that the contribution of all the market participants to the average trading flow is already anticipated by all agents.

System (5.16) is our starting point to solve the MFG system (5.4) in the quadratic case. Due to the forward-backward structure of system (5.16), we need a smallness condition on $\mathbb{A}$ in order to construct a solution. This assumption is also considered in [**39**], and is not problematic from a modeling standpoint since $|\mathbb{A}|$ is generally small in applications (cf. Section 3.3). Let us present the construction of solutions to system (5.16).

PROPOSITION 5.2. *Suppose that* $\mathbf{A}^a, \gamma_a \in L^\infty(D)$, *then there exists* $\alpha_0 > 0$ *such that, for* $|\mathbb{A}| \leqslant \alpha_0$, *the following hold:*

(i) there exists a unique process $\mathbf{E}^a$ *in* $L^1_{\bar{m}_0}(D; \mathcal{C}^1([0,T]))$ *which solves system* (5.16)*;*

(ii) there exists a constant $C_2 > 0$*, such that*

$$
\sup_{0 \leqslant w \leqslant T} |\boldsymbol{\mu}_w| \leqslant C_2 \left(1 + \int_a |E_0^a| \, d\bar{m}_0\right) e^{C_2 T},
\tag{5.17}
$$

where $(\boldsymbol{\mu}_t)_{t\in[0,T]}$ *is given by* (5.12).

PROOF. At first, note that the solution $\mathbb{H}_a$ to the matrix Riccati equation in (5.14) exists on $[0,T]$, is unique, depends only on data, and satisfies (see e.g. [**83**])

$$
-2\mathbf{A}^a - T\gamma^a\Sigma \leqslant \mathbb{H}_a \leqslant 0,
\tag{5.18}
$$

where the order in the above inequality should be understood in the sense of positive symmetric matrices. Moreover, note that $\Sigma\mathbb{V}$ and $\mathbb{V}\Sigma$ are both diagonalizable with non-negative eigenvalues. Thus by using the ODE satisfied by $\mathbb{H}_a$, we know that $\mathbb{H}_a\mathbb{V}$ and $\mathbb{V}\mathbb{H}_a$ are both diagonalizable with a constant change of basis matrix. In particular, it holds that

$$\left[\mathbb{H}_a(t)\mathbb{V}, \int_t^w \mathbb{H}_a(u)\mathbb{V}\, du\right] = \left[\mathbb{V}\mathbb{H}_a(t), \int_t^w \mathbb{V}\mathbb{H}_a(u)\, du\right] = 0 \tag{5.19}$$

for any $0 \leqslant t, w \leqslant T$, where the symbol $[B, A]$ denotes the Lie Bracket: $[B, A] = BA - AB$.

Given $\mathbb{H}_a$, we aim to construct $\dot{\mathbf{E}}^a$ in $L^1_{\bar{m}_0}(D; \mathcal{C}([0, T]))$ by solving a fixed point relation, and then deduce $\mathbf{E}^a$. For that purpose, we start by deriving a fixed point relation for $\dot{\mathbf{E}}^a$. By virtue of (5.19), observe that any solution $\mathbf{E}^a$ to (5.16) fulfills (5.15) with (see e.g. [94])

$$\mathcal{H}_a(t) = \int_t^T \exp\left\{\int_t^w 2\mathbb{H}_a(s)\mathbb{V}\, ds\right\} \mathbb{A} \int_a \dot{\mathbf{E}}^a(w)\, d\bar{m}_0(a)\, dw,$$

so that

$$\begin{aligned}\mathbf{E}^a(t) &= \exp\left\{\int_0^t 2\mathbb{V}\mathbb{H}_a(w)\, dw\right\} E_0^a \\ &+ 2\mathbb{V}\int_0^t \exp\left\{\int_\tau^t 2\mathbb{V}\mathbb{H}_a(w)\, dw\right\} \int_\tau^T \exp\left\{\int_\tau^w 2\mathbb{H}_a(s)\mathbb{V}\, ds\right\} \mathbb{A} \int_a \dot{\mathbf{E}}^a(w)\, d\bar{m}_0(a)\, dw\, d\tau.\end{aligned}$$

By combining this relation with (5.15), we deduce that $\dot{\mathbf{E}}^a$ satisfies the following fixed point relation:

$$\begin{aligned}\mathbf{x}^a(t) = \Phi_{\mathbb{A}}(\mathbf{x}^a)(t) &:= 2\mathbb{V}\mathbb{H}_a(t)\exp\left\{\int_0^t 2\mathbb{V}\mathbb{H}_a(w)\, dw\right\} E_0^a \\ &+4\mathbb{V}\mathbb{H}_a(t)\mathbb{V}\int_0^t \exp\left\{\int_\tau^t 2\mathbb{V}\mathbb{H}_a(w)\, dw\right\} \int_\tau^T \exp\left\{\int_\tau^w 2\mathbb{H}_a(s)\mathbb{V}\, ds\right\} \mathbb{A} \int_a \mathbf{x}^a(w)\, d\bar{m}_0(a)\, dw\, d\tau \\ &+ 2\mathbb{V}\int_t^T \exp\left\{\int_t^w 2\mathbb{H}_a(w)\mathbb{V}\, dw\right\} \mathbb{A} \int_a \mathbf{x}^a(w)\, d\bar{m}_0(a)\, dw.\end{aligned} \tag{5.20}$$

Conversely, one checks that if $\mathbf{x}^a$ is a solution to the fixed point relation (5.20), for a.e. $a \in D$, then $\mathbf{E}^a(t) = E_0^a + \int_0^t \mathbf{x}^a(s)\, ds$ is a solution to system (5.16).

To solve the fixed point relation (5.20), one just uses Banach fixed point Theorem on $\Phi_{\mathbb{A}} : \mathbb{X} \to \mathbb{X}$, where $\mathbb{X} := L^1_{\bar{m}_0}(D; \mathcal{C}([0, T]))$. It is clear that $\Phi_{\mathbb{A}}$ is a contraction for $|\mathbb{A}|$ small enough: indeed, given $\mathbf{x}, \mathbf{y} \in \mathbb{X}$, it holds that:

$$|\Phi_{\mathbb{A}}(\mathbf{x}^a)(t) - \Phi_{\mathbb{A}}(\mathbf{y}^a)(t)| \leqslant C|\mathbb{A}|\, \|\mathbf{x} - \mathbf{y}\|_{\mathbb{X}}$$

where $C > 0$ depends only on T, $\|\gamma\|_\infty$, $\|\mathbf{A}\|_\infty$, $|\mathbb{V}|$ and $|\Sigma|$. Thus, given the solution $\mathbf{x}^a$ to (5.20), the function $\mathbf{E}^a(t) = E_0^a + \int_0^t \mathbf{x}^a(s)\, ds$ solves (5.16), and belongs to $L^1_{\bar{m}_0}(D; \mathcal{C}([0, T]))$ given that m_0 have a finite first order moment. Estimate (5.17) ensues from Grönwall's Lemma. $\square$

We are now in position to solve the MFG system (5.4) in the case of quadratic liquidity functions (5.9).

THEOREM 5.3. *Under* (5.9), (5.10), *and assumptions of Proposition 5.2, the Mean Field Game system* (5.4) *has a unique solution.*

PROOF. Since (5.16) is solvable thanks to Proposition 5.2, we can now solve completely system (5.14) and deduce $u^a(t, \mathbf{q}; \boldsymbol{\mu})$ thanks to (5.13). In fact, owing to (5.19) we know that (cf. [94]):

$$\begin{cases} \mathcal{H}_a(t) = \displaystyle\int_t^T \exp\left\{\int_t^w 2\mathbb{H}_a(s)\mathbb{V}\, ds\right\} \mathbb{A}\boldsymbol{\mu}_w\, dw \\ h_a(t) = \displaystyle\int_t^T \mathbb{V}\mathcal{H}_a(w) \cdot \mathcal{H}_a(w)\, dw, \end{cases}$$

so that the function $u^a(t, \mathbf{q}; \boldsymbol{\mu})$, that is given by (5.13), is $\mathcal{C}^{1,2}([0, T] \times \mathbb{R})$. Furthermore, by virtue of (5.17)-(5.18), note that

$$|\nabla_q u^a(t, \mathbf{q})| \leqslant C(1 + |\mathbf{q}|) \tag{5.21}$$

for some constant $C > 0$ which depends only on T and data.

Now, as u^a is regular and satisfies (5.21), we know that the transport equation

$$\partial_t m^a + \sum_{i=1}^{d} \frac{V_i}{2\eta_i} \partial_{q_i} \left(m^a \partial_{q_i} u^a(t, \mathbf{q})\right) = 0, \quad m^a(0,\, d\mathbf{q}) = m_0(\, d\mathbf{q},\, da)/\bar{m}_0(a)$$

has a unique weak solution $m^a \in \mathcal{C}([0, T]; L^1(\mathbb{R}))$ for a.e $a \in D$, so that $m := m^a \bar{m}_0$ solves, in the weak sense, the following Cauchy problem:

$$\partial_t m + \sum_{i=1}^{d} \frac{V_i}{2\eta_i} \partial_{q_i} \left(m \partial_{q_i} u^a(t, \mathbf{q})\right) = 0, \quad m(0,\, d\mathbf{q},\, da) = m_0(\, d\mathbf{q},\, da).$$

In addition, one easily checks that m belongs to $\mathcal{C}([0, T]; L^1(\mathbb{R} \times D))$.

By invoking the uniqueness of solutions to (5.16), we have

$$\mathbf{E}^a(t) = \int_q \mathbf{q} m^a(t, q)\, d\mathbf{q} \quad \text{for a.e } a \in D.$$

Thus through the same computations as in (5.11) we obtain

$$\mu_t^i = \int_{(q,a)} \frac{V_i}{2\eta_i} \partial_{q_i} u^a(t, \mathbf{q}) m^a(t, \mathbf{q}) \bar{m}_0(a)\, da\, d\mathbf{q}, \;\; i = 1, ..., d,$$

so that $(u^a, m)_{a \in D}$ solves the MFG system (5.4).

By virtue of Proposition 5.1, any constructed solution is unique. So to conclude the proof, it remains to show that:

$$\int_{\mathbb{R}\times D} |\mathbf{q}| m(t, \mathbf{q}, a)\, d\mathbf{q}\, da < \infty.$$

For that purpose, let us set

$$\Psi_\varepsilon(t) := \int_{\mathbb{R}\times D} |\mathbf{q}|^2 \left(1 + |\mathbf{q}/\varepsilon|^2\right)^{-1} m(t, \mathbf{q}, a)\, d\mathbf{q}\, da, \quad \forall \varepsilon > 0.$$

One easily checks that $\Psi_\varepsilon(t) < \infty$ for every $t \in [0, T]$ and $\varepsilon > 0$. After differentiating Ψ_ε and integrating by parts, we obtain the following ODE that is satisfied by Ψ_ε:

$$\Psi_\varepsilon(t) = \Psi_\varepsilon(0) + 4\int_0^t \int_{\mathbb{R}\times D} (\mathbb{V}\nabla_{\mathbf{q}} u^a \cdot \mathbf{q}) \left(1 + |\mathbf{q}/\varepsilon|^2\right)^{-2} m(w, \mathbf{q}, a)\, d\mathbf{q}\, da\, dw.$$

Hence, by virtue of (5.21) it holds that

$$|\Psi_\varepsilon(t)| \leqslant |\Psi_\varepsilon(0)| + A + B\int_0^t |\Psi_\varepsilon(w)|\, dw,$$

where A, B are positive constants. Now, we use Grönwall's Lemma and take $\varepsilon \to \infty$ by invoking Fatou's Lemma and the fact that m_0 has a finite second order moment. This leads to

$$\int_{\mathbb{R}\times D} |\mathbf{q}|^2 m(t, \mathbf{q}, a)\, d\mathbf{q}\, da < \infty \quad \forall t \in [0, T],$$

which in turn entails the desired result. □

2.3. Stylized Facts & Numerical Simulations. Let us now comment our results and highlight several stylized facts of the system. By virtue of (5.13), the optimal trading speed $\mathbf{v}_a^*$ is given by:

$$\begin{aligned} \mathbf{v}_a^*(t, \mathbf{q}) &= 2\mathbb{V}\mathbb{H}_a(t)\mathbf{q} + 2\mathbb{V}\mathcal{H}_a(t) \\ &= 2\mathbb{V}\mathbb{H}_a(t)\mathbf{q} + 2\mathbb{V}\int_t^T \exp\left\{\int_t^w 2\mathbb{H}_a(s)\mathbb{V}\, ds\right\} \mathbb{A}\boldsymbol{\mu}_w\, dw \\ &=: \mathbf{v}_a^{1,*}(t, \mathbf{q}) + \mathbf{v}_a^{2,*}(t; \boldsymbol{\mu}). \end{aligned} \tag{5.22}$$

The above expression shows that the optimal trading speed is divided into two distinct parts $\mathbf{v}_a^{1,*}, \mathbf{v}_a^{2,*}$. The first part $\mathbf{v}_a^{1,*}$ corresponds to the classical Almgren-Chriss solution in the case of a complexe portfolio (cf. [**71**]). The second part $\mathbf{v}_a^{2,*}$ adjusts the speed based on the anticipated future average trading on the remainder of the trading window $[t, T]$. Since the matrix $\mathcal{H}_a$ is negative, note that the strategy gives more weight to the current expected average trading. Moreover, the contribution of the corrective term decreases as we approach the end of the trading horizon. The correction term aims to take advantage of the anticipated market mean field.

Let us set

$$\mathbb{G}_a(t, w) := \exp\left\{\int_t^w 2\mathbb{H}_a(s)\mathbb{V}\, ds\right\} \mathbb{A}. \tag{5.23}$$

Note that the matrix $\mathbb{G}_a$ is not necessarily symmetric and could have a different structure than $\mathbb{H}_a$. In view of the market price dynamics, the trading speed expression shows that an action of an individual investor or trader on asset i could have a direct impact on the price of asset j, at least when the two assets are fundamentally correlated, i.e. $\Sigma_{i,j} \neq 0$. This phenomenon of *cross impact* is related to the fact that other traders already anticipates the market mean field and aim to take advantage from that information, especially when asset j is more liquid than asset i (or vice versa). Thus, if an investor is trading as the crowd is expecting her/him to trade, then she/he is more likely to get a *"cross-impact"* through the action of the other traders. This fact is empirically addressed in [**16**,**78**].

Another expression of the optimal trading speed can also be derived thanks to (5.15). In fact, we have that:

$$\mathbf{v}_a^*(t,q) = \dot{\mathbf{E}}^a + 2\mathbb{V}\mathbb{H}_a(t)(\mathbf{q} - \mathbf{E}^a). \tag{5.24}$$

The above formulation shows that an individual investor should follow the market mean field but with a correction term which depends on the situation of her/his inventory relative to the population average inventory.

In order to simplify the presentation, we ignore from now on investors heterogeneity and assume that market participants have identical preferences. Under this assumption, system (5.16) simply reads:

$$\begin{cases} \ddot{\mathbf{E}} = -2\mathbb{V}\mathbb{A}\dot{\mathbf{E}} + 2\gamma\mathbb{V}\Sigma\mathbf{E} \\ \mathbf{E}(0) = E_0, \quad \dot{\mathbf{E}}(T) + 4\mathbb{V}\mathbf{A}\mathbf{E}(T) = 0. \end{cases} \tag{5.25}$$

Given a discretization step $\delta t = N^{-1}$, the solution of (5.25) is approached by a sequence $(x_k, y_k)_{0 \leqslant k \leqslant N}$ according to the following implicit scheme:

$$\begin{cases} x_0 = E_0 \\ x_k - x_{k-1} - \delta t y_{k-1} = 0, \quad k = 1, \dots, N \\ y_k - y_{k-1} - \delta t \left(2\gamma\mathbb{V}\Sigma x_k - 2\mathbb{V}\mathbb{A}y_k\right) = 0, \quad k = 1, \dots, N \\ 4\mathbb{V}\mathbf{A}x_N + y_N = 0. \end{cases}$$

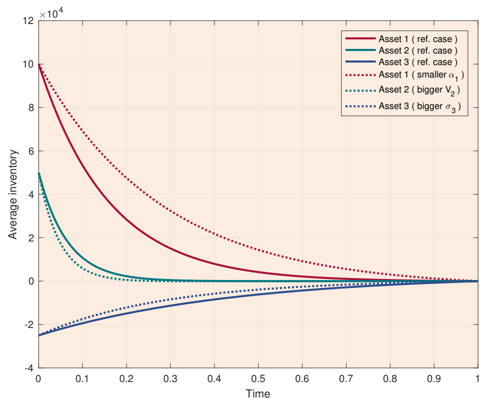
Hence, computing an approximate solution to system (5.25) reduces to solving a straightforward linear system. One checks that under conditions of Proposition 5.2, the above numerical scheme converges and is stable.

Now, we can present some examples by using the above numerical method. We consider a portfolio containing three assets (Asset 1, Asset 2, Asset 3) with the following characteristics:

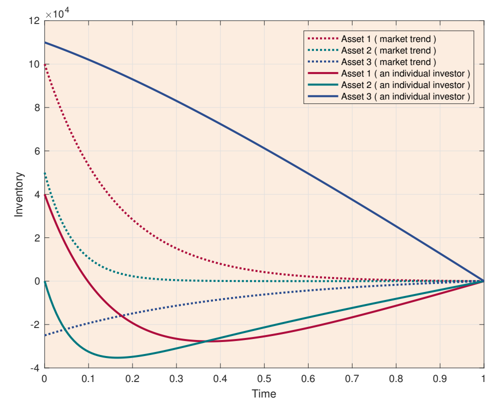
- $\sigma_1 = \sigma_3 = 0.3$ $\$.\mathrm{day}^{-1/2}.\mathrm{share}^{-1}$, $\sigma_2 = 1$ $\$.\mathrm{day}^{-1/2}.\mathrm{share}^{-1}$;
- $V_1 = 2,000,000$ $\mathrm{share}.\mathrm{day}^{-1}$, $V_2 = V_3 = 5,000,000$ $\mathrm{share}.\mathrm{day}^{-1}$;
- $\eta_1 = \eta_2 = 0.1$ $\$.\mathrm{share}^{-1}$, $\eta_3 = 0.4$ $\$.\mathrm{share}^{-1}$, $A_1 = A_2 = 2.5$ $\$.\mathrm{day}^{-1}.\mathrm{share}^{-1}$;
- $\alpha_1 = \alpha_2 = 8 \times 10^{-4}$ $\$.\mathrm{share}^{-1}$, $\alpha_3 = 6 \times 10^{-4}$ $\$.\mathrm{share}^{-1}$.

In Figure 1(a)-1(d), we consider a market with the initial average inventories $E_0^1 = 100,000$, $E_0^2 = 50,000$, and $E_0^3 = -25,000$ shares, for Asset 1, Asset 2, and Asset 3 respectively. In this example, we suppose that the correlation between the price increments of Asset 1 and Asset 2 is 80 %, and we set $\gamma = 5 \times 10^{-5}$ $\$^{-1}$ except for Figure 1(c).

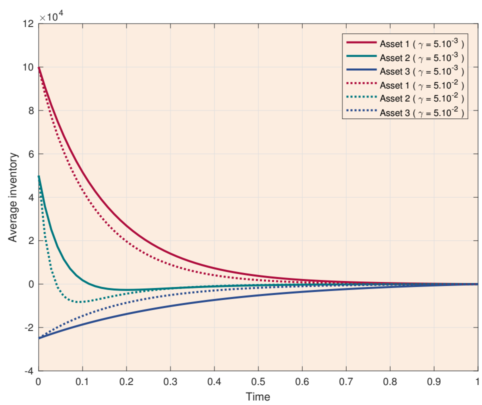
Figure 1(a) shows that changing the permanent market impact prefactors $(\alpha_k)_{1 \leqslant k \leqslant 3}$ has a significant influence on the average execution speed. This fact was pointed out in [39], and is essentially related to the fact that the higher the permanent market impact parameter the more the anticipated influence of the other market participants become important. Namely, when α_k is large, traders anticipate a more significant pressure on the price of Asset k, and adjust their trading speed. On the other hand, dynamics of Asset 2 shows that the higher the market liquidity the faster is the execution. This is expected since the more liquid the faster assets are traded. Finally, dynamics of Asset 3 shows that traders accelerate their execution on volatile asset. It corresponds to a natural reaction due to risk aversion; a trader will try to reduce his exposure to the *more risky* (hence volatile) assets in priority.



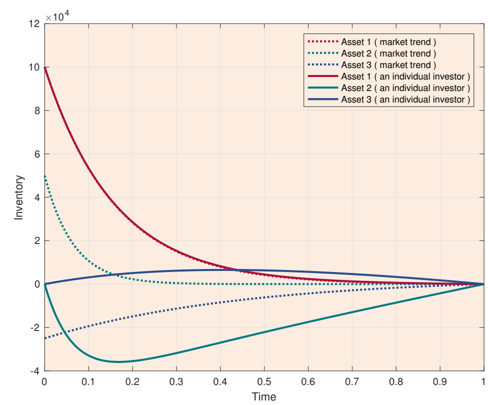
(a) Market mean field with different parameters



(b) Optimal trading of an individual investor with: $q_0^1 = 40,000$, $q_0^2 = 0$, and $q_0^3 = 110,000$



(c) Market mean field with high risk aversion



(d) Optimal trading of an individual investor with: $q_0^1 = 100,000$, and $q_0^2 = q_0^3 = 0$

FIGURE 1. Simulated examples of the dynamics of $\mathbf{E}$ and optimal trading curves of an individual investor. The dashed lines in Figure 1(a) correspond to: $\alpha_1 = 6 \times 10^{-4}$ $\$.share^{-1}$, $V_2 = 7,000,000$ $share.day^{-1}$, and $\sigma_3 = 5$ $\$.day^{-1/2}.share^{-1}$.

Figure 1(c) illustrates the behavior of the crowd of investors with an increasing risk aversion (higher γ). In the two presented scenarios, one can observe that Asset 2 is liquidated very quickly, then a short position is built (around $t = 0.05$ for $\gamma = 5 \times 10^{-2}$ $\$^{-1}$) and it is finally progressively unwound. This exhibits the emergence of a Hedging Strategy: indeed, since Asset 1 and Asset 2 are highly correlated, investors can slow down the execution process for the less liquid asset (Asset 1) to reduce the transaction costs, by using the more liquid asset (Asset 2) to hedge the market risk associated to Asset 1. The trader has an incentive to use such a strategy as soon as the cost of the roundtrip in Asset 2 is smaller than the corresponding reduction of the risk exposure (seen from its reward function $U^{a}(t, x, \mathbf{s}, \mathbf{q}; \boldsymbol{\mu})$ defined by equality (5.2)).

Now, we provide examples of individual players' optimal strategies. We consider two examples: an individual investor with initial inventory $q_0^1 = 40,000$, $q_0^2 = 0$, and $q_0^3 = 110,000$ in Figure 1(b); and an individual investor with initial inventory $q_0^1 = 100,000$, and $q_0^2 = q_0^3 = 0$ in Figure 1(d).

In Figure 1(d) the considered investor starts from $q_0^1 = E_0^1$. Hence, by virtue of (5.24) her liquidation curve follows exactly the market mean field. Moreover, the investor takes advantage of the anticipated evolution of the market by building favorable positions on Asset 2 and Asset 3: building a short (resp. a long) position on Asset 2 (resp. Asset 3), and buying (resp. selling) back in order to take advantage of price drop (resp. raise) induced by the massive liquidation (resp. purchase). The trading strategies on Asset 2 and Asset 3 are related to the term $\mathbf{v}^{2,*}$ in (5.22). This strategy can be described as a "Liquidity Arbitrage Strategy".

Figure 1(b) shows two interesting facts: on the one hand, the individual player builds a short position on Asset 1 after achieving her goal (complete liquidation) in order to take advantage of the market selling pressure; on the other hand, by taking into account the market buying pressure on Asset 1, the investor slows down her liquidation to reduce execution costs since she anticipates no sustainable price decline.

3. The Dependence Structure of Asset Returns

The main purpose of this section is to analyze the impact of large transactions on the observed covariance matrix between asset returns, by using the Mean Field Game framework of Section 2. For that purpose, we assume a simple model where a continuum of players trade a portfolio of assets on each day, and where the initial distribution of inventories across the investors m_0 changes randomly from one day to another according to some given law of probability. We assume that the price dynamics is given by (5.1), and we consider the problem of estimating the covariance matrix of asset returns given a large dataset of intraday observations of the price. For the sake of simplicity, we ignore investors heterogeneity and assume that market participants have identical preferences. Next, we compare our findings with an empirical analysis on a pool of 176 US stocks sampled every 5 minutes over year 2014 and calibrate our model to market data.

Throughout this section, we denote by $\langle X^2 \rangle$ the variance of X, and $\langle X, Y \rangle$ the covariance between X and Y, for any two random variables X, Y. Moreover, we will call a "bin" a slice of 5 minutes. We focused on continuous trading hours because the mechanism of call auctions (i.e. opening and closing auctions is specific). Since US markets open from 9h30 to 16h, our database has 78 bins per day. They will be numbered from 1 to M and indexed by k.

3.1. Estimation using Intraday Data. We suppose that E_0 is a random variable with a given realization on each trading period $[0, T]$, where $T = 1$ day (trading day); and we consider the problem of estimating the covariance matrix of asset returns given the following observations of the price:

$$\left\{\left(\mathbf{S}^n_{t_{1,1}},...,\mathbf{S}^n_{t_{1,M}}\right),\left(\mathbf{S}^n_{t_{2,1}},...,\mathbf{S}^n_{t_{2,M}}\right),....,\left(\mathbf{S}^n_{t_{N,1}},...,\mathbf{S}^n_{t_{N,M}}\right)\right\},\quad n=1,...,d$$

where $\mathbf{S}^n_{t_{\ell,k}}$ is the price of asset n in bin k of day ℓ. We suppose that $t_{\ell,1}=0$, $t_{\ell,M}=T$, for any $1\leqslant\ell\leqslant N$, and $t_{\ell,k}=t_{\ell',k}=t_k$ for any $1\leqslant k\leqslant M$, $1\leqslant\ell,\ell'\leqslant N$.

For simplicity, we suppose that the covariance matrix of asset returns between t_k and t_{k+1} is estimated form data by using the following "naive" estimator :

$$C^{i,j}_{[t_k,t_{k+1}]}:=\frac{1}{N-1}\sum_{l=1}^{N}\left(\delta S^{i,k,l}-\overline{\delta S}^{i,k}\right)\left(\delta S^{j,k,l}-\overline{\delta S}^{j,k}\right),\tag{5.26}$$

where $\delta S^{n,k,l}=S^n_{t_{l,k+1}}-S^n_{t_{l,k}}$ and $\overline{\delta S}^{n,k}=N^{-1}\sum_{l=1}^N\delta S^{n,k,l}$, $n=i,j$. We define the correlation matrix as follows:

$$R^{i,j}_{[t_k,t_{k+1}]}:=\frac{C^{i,j}_{[t_k,t_{k+1}]}}{\left(C^{i,i}_{[t_k,t_{k+1}]}C^{j,j}_{[t_k,t_{k+1}]}\right)^{1/2}}.\tag{5.27}$$

Suppose that the price dynamics is given by (5.1), then the following proposition provides an exact computation of $C^{i,j}_{[t_k,t_{k+1}]}$.

PROPOSITION 5.4. *Assume that E_0 is independent from the process $(\mathbf{W}_t)_{t\in[0,T]}$, then for any $1\leqslant k\leqslant M-1$ and $1\leqslant i,j\leqslant d$, the following hold:*

$$C^{i,j}_{[t_k,t_{k+1}]}=(t_{k+1}-t_k)\Sigma_{i,j}+\alpha_i\alpha_j\frac{\eta_i\eta_j}{4V_iV_j}\Lambda^{i,j}_k+\epsilon_N,\tag{5.28}$$

where $\epsilon_N\to 0$ as $N\to\infty$,

$$\Lambda^{i,j}_k:=\sum_{1\leqslant\ell,\ell'\leqslant d}\left\langle\theta^{i,\ell}_k,\theta^{j,\ell'}_k\right\rangle+\sum_{1\leqslant\ell,\ell'\leqslant d}\left\langle\pi^{i,\ell}_k,\theta^{j,\ell'}_k\right\rangle+\sum_{1\leqslant\ell,\ell'\leqslant d}\left\langle\theta^{i,\ell}_k,\pi^{j,\ell'}_k\right\rangle+\sum_{1\leqslant\ell,\ell'\leqslant d}\left\langle\pi^{i,\ell}_k,\pi^{j,\ell'}_k\right\rangle,$$

and

$$\pi^{n,\ell}_k:=\int_{t_k}^{t_{k+1}}\mathbb{H}^{n,\ell}(s)E^\ell(s)\,ds,\quad\theta^{n,\ell}_k:=\int_{t_k}^{t_{k+1}}\int_s^T\mathbb{G}^{n,\ell}(s,w)\mu^\ell(w)\,dw\,ds.$$

PROOF. Use the exact expression of the price dynamics (5.1), the law of large numbers, and the independence between E_0 and $(\mathbf{W}_t)_{t\in[0,T]}$ to obtain:

$$C^{i,j}_{[t_k,t_{k+1}]}=\epsilon_N+(t_{k+1}-t_k)\Sigma_{i,j}+\alpha_i\alpha_j(N-1)^{-1}\sum_{l=1}^N\int_{t_k}^{t_{k+1}}\left(\mu^{i,l}_s-\bar\mu^i_s\right)\,ds\int_{t_k}^{t_{k+1}}\left(\mu^{j,l}_{s'}-\bar\mu^j_{s'}\right)\,ds',\tag{5.29}$$

where $\bar\mu^n_u=N^{-1}\sum_{l=1}^N\mu^{n,l}_u$, and $\boldsymbol{\mu}^l,\mathbf{E}^l$ are respectively the realizations of $\boldsymbol{\mu},\mathbf{E}$ in day l. Now, owing to (5.22)-(5.23), we know that

$$\boldsymbol{\mu}^l_t=2\mathbb{V}\mathbb{H}(t)\mathbf{E}^l(t)+2\mathbb{V}\int_t^T\mathbb{G}(t,w)\boldsymbol{\mu}^l_w\,dw=:\boldsymbol{\nu}^{1,l}(t)+\boldsymbol{\nu}^{2,l}(t).$$

Thus by setting

$$\tilde{\boldsymbol{\nu}}_k^{n,l} := \int_{t_k}^{t_{k+1}} \left(\boldsymbol{\nu}^{n,l}(s) - N^{-1} \sum_{l=1}^{N} \boldsymbol{\nu}^{n,l}(s) \right) \, ds, \quad n = 1, 2,$$

we deduce that

$$\int_{t_k}^{t_{k+1}} \left(\mu_s^{i,\ell} - \bar{\mu}_s^i \right) \, ds \int_{t_k}^{t_{k+1}} \left(\mu_{s'}^{j,\ell} - \bar{\mu}_{s'}^j \right) \, ds' = \left(\tilde{\nu}_k^{1,l,i} + \tilde{\nu}_k^{2,l,i} \right) \left(\tilde{\nu}_k^{1,l,j} + \tilde{\nu}_k^{2,l,j} \right).$$

The desired result ensues by noting the existence of estimation noises $\epsilon_N^1, \epsilon_N^2, \epsilon_N^3$ and ϵ_N^4, such that:

$$\begin{aligned}
(N-1)^{-1} \sum_{l=1}^{N} \tilde{\nu}_k^{1,l,i} \tilde{\nu}_k^{1,l,j} &= \frac{\eta_i \eta_j}{4V_i V_j} \sum_{1 \leqslant \ell, \ell' \leqslant d} \left\langle \pi_k^{i,\ell}, \pi_k^{j,\ell'} \right\rangle + \epsilon_N^1; \\
(N-1)^{-1} \sum_{l=1}^{N} \tilde{\nu}_k^{2,l,i} \tilde{\nu}_k^{2,l,j} &= \frac{\eta_i \eta_j}{4V_i V_j} \sum_{1 \leqslant \ell, \ell' \leqslant d} \left\langle \theta_k^{i,\ell}, \theta_k^{j,\ell'} \right\rangle + \epsilon_N^2; \\
(N-1)^{-1} \sum_{l=1}^{N} \tilde{\nu}_k^{1,l,i} \tilde{\nu}_k^{2,l,j} &= \frac{\eta_i \eta_j}{4V_i V_j} \sum_{1 \leqslant \ell, \ell' \leqslant d} \left\langle \pi_k^{i,\ell}, \theta_k^{j,\ell'} \right\rangle + \epsilon_N^3; \\
(N-1)^{-1} \sum_{l=1}^{N} \tilde{\nu}_k^{1,l,j} \tilde{\nu}_k^{2,l,i} &= \frac{\eta_i \eta_j}{4V_i V_j} \sum_{1 \leqslant \ell, \ell' \leqslant d} \left\langle \theta_k^{i,\ell}, \pi_k^{j,\ell'} \right\rangle + \epsilon_N^4.
\end{aligned}$$

The proof is complete. □

REMARK 5.5. One can easily derive an analogous result for $\left(C_{[0,T]}^{i,j} \right)_{1 \leqslant i,j \leqslant d}$. Namely, it holds that:

$$C_{[0,T]}^{i,j} = T\Sigma_{i,j} + \alpha_i \alpha_j \frac{\eta_i \eta_j}{4V_i V_j} \Lambda^{i,j} + \epsilon_N, \tag{5.30}$$

where $\epsilon_N \to 0$ as $N \to \infty$,

$$\Lambda^{i,j} := \sum_{1 \leqslant \ell, \ell' \leqslant d} \left\langle \theta^{i,\ell}, \theta^{j,\ell'} \right\rangle + \sum_{1 \leqslant \ell, \ell' \leqslant d} \left\langle \pi^{i,\ell}, \theta^{j,\ell'} \right\rangle + \sum_{1 \leqslant \ell, \ell' \leqslant d} \left\langle \theta^{i,\ell}, \pi^{j,\ell'} \right\rangle, + \sum_{1 \leqslant \ell, \ell' \leqslant d} \left\langle \pi^{i,\ell}, \pi^{j,\ell'} \right\rangle,$$

and

$$\pi^{n,\ell} := \int_0^T \mathbb{H}^{n,\ell}(s) E^\ell(s) \, ds, \quad \theta^{n,\ell} := \int_0^T \int_s^T \mathbb{G}^{n,\ell}(s, w) \mu^\ell(w) \, dw \, ds.$$

Identities (5.28) and (5.30) show that the realized covariance matrix is the sum of the fundamental covariance and an *excess realized covariance matrix* generated by the impact of the crowd of institutional investors' trading strategies. Note on the one hand that the diagonal terms $C^{i,i}$ are always deviated from fundamentals because of the contribution of $\langle (\pi^{i,i})^2 \rangle$ and $\langle (\theta^{i,i})^2 \rangle$. On the other hand, since $\mathbb{H}$ and $\mathbb{G}$ inherit a structure similar to Σ, the excess of realized covariance in the off-diagonal terms is non-zero as soon as one – or both – of the conditions below is satisfied:

- there exists $i_0 \neq j_0$ such that $\Sigma_{i_0,j_0} \neq 0$;
- there exists $i_0 \neq j_0$ such that $\left\langle E_0^{i_0}, E_0^{j_0} \right\rangle \neq 0$.

Moreover, (5.28) and (5.30) show that the excess realized covariance can deviate significantly from fundamentals when: *the market impact is large (high crowdedness), the considered assets are highly liquid (small η_i/V_i), the risk aversion coefficient γ is high, and/or when the standard deviation of E_0 is large*. In addition, since the contribution of $\theta_k^{n,\ell}$ and $\pi_k^{n,\ell}$ vanishes as we approach the end of the trading horizon, observe that

$$C^{i,j}_{[t_k,t_{k+1}]} \sim (t_{k+1} - t_k)\Sigma_{i,j}, \quad \text{as } t_{k+1} \to T, \tag{5.31}$$

which means that one converges to market fundamentals at the end of the trading period. This is due to the fact that, in our model, all traders have high enough risk aversions so that their trading speeds go to zero close to the terminal time T.

By virtue of (5.28), one can also explain the realized correlation matrix in terms of the fundamental correlations $\rho_{i,j} := \Sigma_{i,j}/(\Sigma_{i,i}\Sigma_{j,j})^{1/2}$. Namely, it holds that:

(5.32)

$$\begin{aligned} R^{i,j}_{[t_k,t_{k+1}]} &= \rho_{i,j}\left(\frac{(t_{k+1}-t_k)^2\Sigma_{i,i}\Sigma_{j,j}}{C^{i,i}_{[t_k,t_{k+1}]}C^{j,j}_{[t_k,t_{k+1}]}}\right)^{1/2} + \frac{\alpha_i\alpha_j\eta_i\eta_j\Lambda_k^{i,j}}{4V_iV_j\left(C^{i,i}_{[t_k,t_{k+1}]}C^{j,j}_{[t_k,t_{k+1}]}\right)^{1/2}} + \epsilon_N \\ &=: \rho_{i,j}A_k^{i,j} + B_k^{i,j} + \epsilon_N \end{aligned}$$

for any $1 \leqslant i < j \leqslant d$. This expression shows that the deviation of the realized correlation from fundamentals is a linear map. The numerator of the multiplicative part $A_k^{i,j}$ does not depend on the off-diagonal terms of $\mathbb{H}$ while it is the case for the additive part $B_k^{i,j}$.

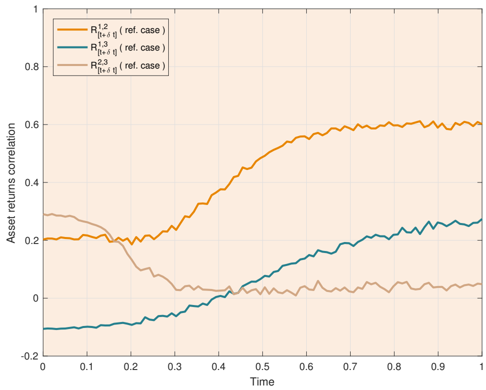
3.2. Numerical Simulations. In this part, we conduct several numerical experiments in order to illustrate the influence of trading activity on the structure of the covariance/correlation matrix of asset returns.

We consider the example of Section 2.3 by choosing $\rho_{1,2} = 60\%$, $\rho_{1,3} = 30\%$ and $\rho_{2,3} = 5\%$. For simplicity, we suppose that E_0 is a centered Gaussian random vector with a covariance matrix Γ that is given by:

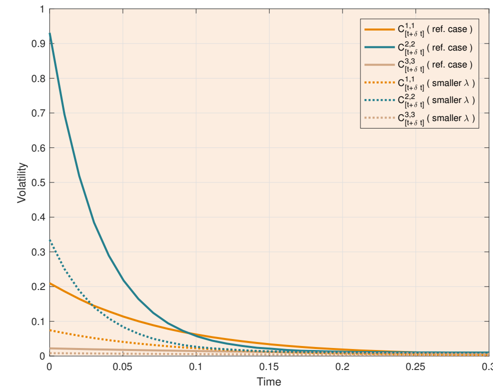
$$\Gamma := \lambda^2.\begin{pmatrix} 1 & 0.2 & -0.1 \\ 0.2 & 1 & 0.3 \\ -0.1 & 0.3 & 1 \end{pmatrix},$$

where $\lambda = 10,000\ share$. We fix a time step $\delta t = 10^{-2}\ day$ ($\sim 4\ min$), set $t_{k+1} - t_k = \delta t$, and estimate $\left(C^{i,j}_{[t_k,t_k+\delta t]}\right)_{\substack{1\leqslant i\leqslant j\leqslant 3\\ 1\leqslant k\leqslant M-1}}$ and $\left(R^{i,j}_{[t_k,t_k+\delta t]}\right)_{\substack{1\leqslant i< j\leqslant 3\\ 1\leqslant k\leqslant M-1}}$ by generating a sample of $N = 10,000$ observations using the numerical method of Section 2.3.

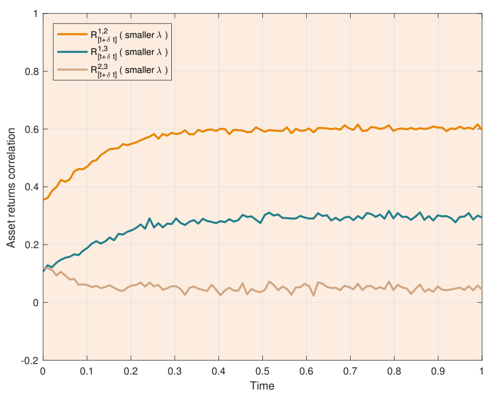
Figures 2(a)-2(d) show that the observed covariance and correlation matrices are significantly deviated from fundamentals and especially at the beginning of the trading day. Figures 2(b)-2(d) also illustrates the sensitivity of the deviation relative to the change of the standard deviation of initial inventories: as λ diminishes, the influence of trading



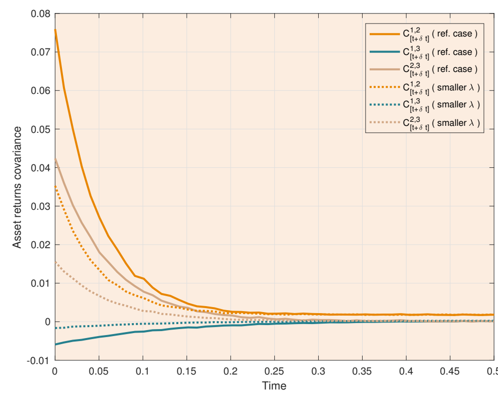
(a) Intraday correlation



(b) Intraday variance for $\lambda = 10^4$ and $\lambda = 6.10^3$



(c) Intraday correlation for $\lambda = 10^3$



(d) Intraday covariance for $\lambda = 10^4$ and $\lambda = 6.10^3$

FIGURE 2. Simulated examples of intraday covariance and correlation matrices using (5.1).

activity is lower and the covariance and correlation matrices converge toward fundamentals.

On the other hand, we observe that the beginning of the trading period is dominated by the dependence structure of initial inventories. This is due to the domination of the additive terms $(B^{i,j}_k)_{1\leqslant i<j\leqslant 3}$; in fact, given the relative high magnitude of denominator terms, $(A^{i,j}_k)_{1\leqslant i<j\leqslant 3}$ are very small when $t_k \to 0$. Furthermore, we note that all the observed quantities converge toward fundamentals at the end of the trading period, which is in line with (5.31).

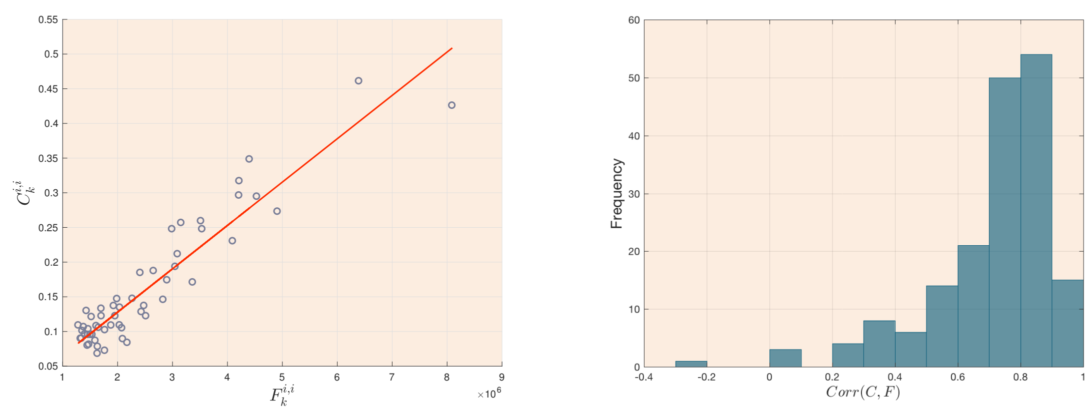
3.3. Empirical Application. Now, we carry out an empirical analysis on a pool of $d = 176$ stocks. The data consists of five-minute binned trades ($\delta t = 5$ min) and quotes information from January 2014 to December 2014, extracted from the primary market of each stock (NYSE or NASDAQ). *We only focus on the beginning of the continuous trading session removing 30 min after the open and the last 90 min before the close,* in order to avoid the particularities of trading activity in these periods and target close strategies. Thus, the number of days is $N = 252$ and the number of bins per day is $M = 55$. Days will be labelled by $l = 1, ..., N$, bins by $k = 1, ..., M$, and for simplicity we note $C_k^{i,j}$ instead of $C^{i,j}_{[t_k-\delta t, t_k]}$ for any $1 \leqslant i, j \leqslant d$.

Our goal is to empirically assess the the influence of trading activity on the intraday covariance matrix of asset returns, and then compare the obtained models with our previous theoretical observations. Given our analysis in Sections 3.1 and 3.2, we expect an excess in the observed covariance matrix of asset returns and especially at the beginning of the trading period. Moreover, we expect the magnitude of this effect to be an increasing function of the typical size of market orders as it is noticed in Figures 2(b) and 2(d).

Market Impact. Let us start by assessing the relationship between the intraday variance of asset returns and the intraday variance of net exchanged flows $(F_k^{i,i})_{1 \leqslant i \leqslant d}$, that is defined by:

$$F_k^{i,i} := \frac{1}{N-1} \sum_{l=1}^{N} \left(\nu_{k,l}^i - \bar{\nu}_k^i\right) \left(\nu_{k,l}^i - \bar{\nu}_k^i\right)$$

for any $1 \leqslant i \leqslant d$ and $k = 1, ..., M$; where $\nu_{k,l}^i$ is the net sum of exchanged volumes between $t_k - \delta t$ and t_k for stock i in day l, and $\bar{\nu}_k^i = N^{-1} \sum_{l=1}^N \nu_{k,l}^i$ (i.e. $\bar{\nu}_k^i$ is an estimate of the expectation of $\nu_{k,l}^i$ regardless of the day). As a by-product, we obtain estimates for the permanent market impact factors. Though ν does not represent exactly the same quantity as the variable μ of Section 3.1, both variables are an indicator of market order flows and for simplicity we shall use ν as a proxy for μ.



(a) GOOG (b) Histogram of Corr (C, F)

FIGURE 3. Dependence structure between $(C_k^{i,i})_{1 \leqslant k \leqslant M}$ and $(F_k^{i,i})_{1 \leqslant k \leqslant M}$. Figure 3(a) displays the relationship between $(C_k^{i,i})_{1 \leqslant k \leqslant M}$ and $(F_k^{i,i})_{1 \leqslant k \leqslant M}$ for GOOG. Figure 3(b) exhibits the histogram of correlations denoted Corr (C, F).

Figure 3(a) shows a strong positive correlation between $(C_k^{i,i})_{1\leqslant k\leqslant M}$ and $(F_k^{i,i})_{1\leqslant k\leqslant M}$ for GOOG. Figure 3(b) shows that this is true for almost all the stocks and reinforces our findings in Sections 3.1 and 3.2. Furthermore, as (5.29) suggests, we suppose a linear relationship between $(C_k^{i,i})_{1\leqslant k\leqslant M}$ and $(F_k^{i,i})_{1\leqslant k\leqslant M}$; thus for every $1 \leqslant i \leqslant d$ we fit the following regression:

$$C^{i,i} = \epsilon + \delta t \cdot \Sigma + \alpha^2 \cdot F^{i,i} \tag{5.33}$$

where ϵ is the error term (assumed normal), the coefficient Σ is related to the "fundamental" covariance matrix of asset returns and the square root of the coefficient α^2 is homogeneous to the market impact factor (cf. (5.29)). In Table 1 we show estimates of α, Σ and the correlation between $(C_k^{i,i})_{1\leqslant k\leqslant M}$ and $(F_k^{i,i})_{1\leqslant k\leqslant M}$ (denoted $\mathrm{Corr}\,(C,F)$) for several examples. In particular, we obtain estimates for the permanent market impact $\widehat{\alpha}$.

	AAPL	BMRN	GOOG	INTC
$\widehat{\alpha}$ (bp)	**0.8**	**8.43**	**2.5**	**0.01**
$\widehat{\alpha^2}$	6.41×10^{-11}	7.11×10^{-9}	6.25×10^{-8}	1.79×10^{-12}
std.	(4.15×10^{-12})	(3.98×10^{-10})	(3.17×10^{-9})	(1.58×10^{-13})
p-value	0.01%	0.01%	0.01%	0.01%
$\widehat{\Sigma}$	**0.16**	−0.01	0.15	5.5×10^{-3}
std.	(0.05)	(0.05)	(0.49)	(2×10^{-4})
p-value	0.01%	60%	75%	2%
$\mathrm{Corr}\,(C,F)$	90%	92%	94%	84%

TABLE 1. Estimates for α, Σ and the realized correlation between $(C_k^{i,i})_{1\leqslant k\leqslant M}$ and $(F_k^{i,i})_{1\leqslant k\leqslant M}$ for Nasdaq stocks. For each estimate the standard deviation (std.) is shown in parentheses and the p-value is given in the third row. Numbers in **bold** are significant at a level of at least 99%.

The Typical Intraday Pattern. Next, we are interested in the intraday evolution of the diagonal and off-diagonal terms of the covariance matrix of returns, and in the way this evolution is affected when the typical size of trades diminishes. For that purpose, we compute the intraday covariance matrix of returns for our pool of US stocks and we normalize each term $(C_k^{i,j})_{1\leqslant k\leqslant M}$ by its daily average, then we consider the median value of diagonal terms and off-diagonal terms as a way of characterizing the evolution of a typical diagonal term and a typical off-diagonal term respectively. The impact of the relative size of orders on the intraday patterns is assessed by conditioning our estimations.

More exactly, we start by defining the matrix of *trade imbalances* for each stock n in order to be able to compare the relative size of trades. Namely, for any n, k, l, we set:

$$w^n_{k,l} := \frac{\nu^n_{k,l}}{\underset{1\leqslant l\leqslant N}{\mathrm{mean}} \sum_k |\nu^n_{k,l}|},$$

where $\mathrm{mean}_{n\in A}\{x_n\}$ denotes the average of (x_n) as n varies in A. This mean is an estimate of the expectation of the sum of the absolute values of $\nu^n_{k,l}$ over a day; it can be seen as a renormalizing constant, enabling us to mix different stocks on Figure 4.

Next, we define the *conditioned intraday covariance matrix* $\left(C^{i,j}_k(\lambda)\right)_{\substack{1\leqslant i,j\leqslant d\\ 1\leqslant k\leqslant M}}$ for every $\lambda \geqslant 0$ as follows:

$$C^{i,j}_k(\lambda) := \frac{1}{\#\mathcal{E}^{i,j}_k(\lambda) - 1} \sum_{l\in\mathcal{E}^{i,j}_k(\lambda)} \left(\delta S^{i,k,l} - \overline{\delta S}^{i,j,k}_\lambda\right)\left(\delta S^{j,k,l} - \overline{\delta S}^{j,i,k}_\lambda\right), \tag{5.34}$$

where:

- the set $\mathcal{E}^{i,j}_k(\lambda)$ corresponds to a conditioning: it contains only days for which this 5 min bins (indexed by k) for this pair of stocks (indexed by (i,j), note that we can have $i = j$) is such that the renormalized net volumes are (in absolute value) below λ. It is strictly defined as follows:
$$\mathcal{E}^{i,j}_k(\lambda) := \left\{1 \leqslant l \leqslant N \;:\; |w^i_{k,l}| \leqslant \lambda \;\text{ and }\; |w^j_{k,l}| \leqslant \lambda\right\};$$
- $\delta S^{n,k,l}$ is the price increment defined as for (5.26) and is computed from the historic stock prices;
- $\overline{\delta S}^{i,j,k}_\lambda$ is the average price increment over selected days, given by: $\overline{\delta S}^{i,j,k}_\lambda = \left(\sum_{l\in\mathcal{E}^{i,j}_k(\lambda)} \delta S^{i,k,l}\right) / \left(\#\mathcal{E}^{i,j}_k(\lambda)\right)$;
- $\#\mathcal{E}^{i,j}_k(\lambda)$ denotes the number of elements of $\mathcal{E}^{i,j}_k(\lambda)$: the number of selected days. Note that the stricter the conditioning (i.e. the smaller λ), the less days in the selection, and hence the smaller $\#\mathcal{E}^{i,j}_k(\lambda)$.

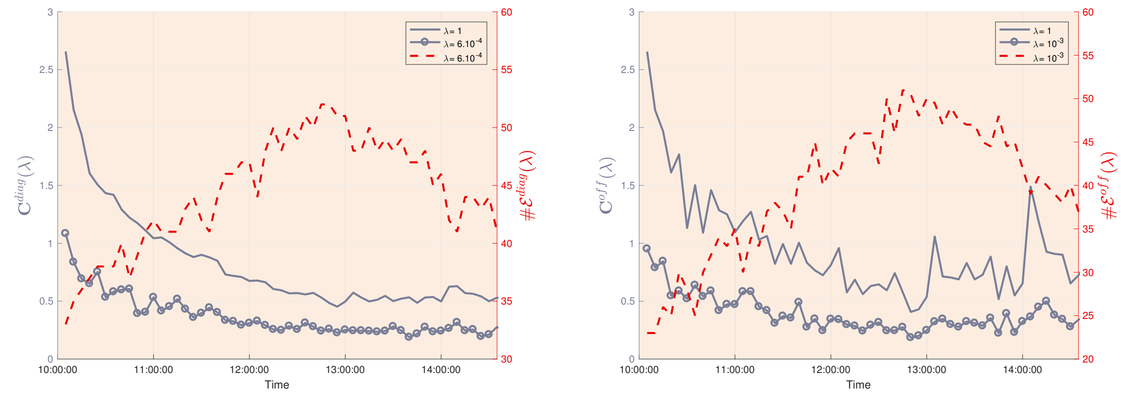
Here $\left(C^{i,j}_k(\lambda)\right)_{\substack{1\leqslant i,j\leqslant d\\ 1\leqslant k\leqslant M}}$ represents the intraday covariance matrix of returns conditioned on trade imbalances between $-\lambda$ and λ. In all our examples, the coefficient λ is chosen to have enough days in the selection (for obvious statistical significance reasons), i.e. so that $\#\mathcal{E}^{i,j}_k(\lambda) \gg 1$ for any $1 \leqslant i,j \leqslant d$ and $1 \leqslant k \leqslant M$.

Now we define the median diagonal pattern $\mathbf{C}^{diag}(\lambda) := \left(C^{diag}_k(\lambda)\right)_{1\leqslant k\leqslant M}$ and the median off-diagonal pattern $\mathbf{C}^{off}(\lambda) := \left(C^{off}_k(\lambda)\right)_{1\leqslant k\leqslant M}$ as follows:

$$C^{diag}_k(\lambda) := \underset{1\leqslant i\leqslant d}{\mathrm{median}} \left\{ C^{i,i}_k(\lambda) / \underset{1\leqslant k\leqslant M}{\mathrm{mean}} \left\{ C^{i,i}_k(1) \right\} \right\}$$

and

$$C^{off}_k(\lambda) := \underset{1\leqslant i<j\leqslant d}{\mathrm{median}} \left\{ C^{i,j}_k(\lambda) / \underset{1\leqslant k\leqslant M}{\mathrm{mean}} \left\{ C^{i,j}_k(1) \right\} \right\},$$



(a) The median diagonal pattern (squared volatility)

(b) The median off-diagonal pattern

FIGURE 4. Plots of the median diagonal pattern $\mathbf{C}^{\mathrm{diag}}(\lambda)$ and the median off-diagonal pattern $\mathbf{C}^{\mathrm{off}}(\lambda)$ for diverse values of λ. The secondary axis corresponds to the number of observations for each 5 minutes bin after the conditioning.

for any $k = 1, ..., M$ and $\lambda \geqslant 0$. Here the notation $\mathrm{median}_{n \in A}\{x_n\}$ denotes the median value of (x_n) as n varies in A. One should note that the choice of the normalization constant (i.e. the mean over bins of $C_k^{i,j}(1)$) will allow us to compare the different curves with respect to the reference case, i.e. without conditioning. In fact, it turns out that $\lambda = 1$ removes all conditionings: 1 is above the maximum value of our renormalized flows. Moreover, we set $\#\mathcal{E}_k^{\mathrm{diag}}(\lambda) := \underset{1 \leqslant i \leqslant d}{\mathrm{median}} \left\{\#\mathcal{E}_k^{i,i}(\lambda)\right\}$ and $\#\mathcal{E}_k^{\mathrm{off}}(\lambda) := \underset{1 \leqslant i < j \leqslant d}{\mathrm{median}} \left\{\#\mathcal{E}_k^{i,j}(\lambda)\right\}$.

We take medians instead of means to have robust estimates of the expectations. We do not want our estimates to be polluted by few days of potential erratic market data, that could for instance be due to trading interruptions.

Figures 4(a) and 4(b) displays representations of $\mathbf{C}^{\mathrm{diag}}(\lambda)$ and $\mathbf{C}^{\mathrm{off}}(\lambda)$ for various values of λ. Observe that $\mathbf{C}^{\mathrm{diag}}(1)$ and $\mathbf{C}^{\mathrm{off}}(1)$ exhibits a pattern that is very similar to our simulation in Figures 2(b) and 2(d), especially between the beginning of the trading period and $13:00$. Indeed, the observed quantities are high at the beginning of the trading period, lower as the day progresses until it reaches a minimum around $13:00$, followed by a slight increase until market close. The general shape of these curves (left-slanted smile) is well-known (see e.g. [**47**] and references therein).

Our core observation is that: *given the absolute value of the net flows are small, this average curve flattens out, even at the beginning of the day*. At our knowledge, it is the first time that this conditioning is mentioned, and it is perfectly in line with our simulated Figures 2(b) and 2(d). This suggests the slopes of the "averaged volatility curves" comes essentially from the days during which there is a large positive or negative imbalance of large orders, that are "optimally" executed. We believe that this analysis should be completed by using a larger data set.

A Toy Model Calibration. Now, we use historical data to fit our MFG model to some examples of traded stocks. For that purpose, we use a very simple approach by reducing as much as possible the number of parameters:

($\mathcal{S}1$) We suppose that E_0 is a centered Gaussian random vector with a covariance matrix Γ. Moreover, as suggested by (5.12) , we use $\mathrm{Corr}\left(\sum_k \nu_k^i, \sum_k \nu_k^j\right)$ as a proxy for $\mathrm{Corr}\left(E_0^i, E_0^j\right)$, and which is in turn estimated from data by using the standard estimator :

$$\frac{1}{N-1}\sum_{l=1}^{N}\left(\sum_k \nu_{k,l}^i - \overline{\sum_k \nu_{k,l}^i}\right)\left(\sum_k \nu_{k,l}^j - \overline{\sum_k \nu_{k,l}^j}\right),$$

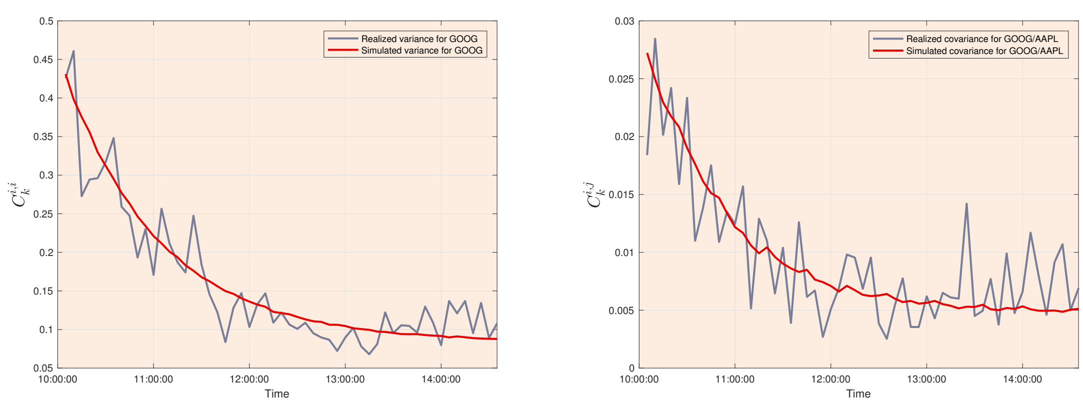
where $\overline{\sum_k \nu_{k,l}^i} = N^{-1}\sum_l \sum_k \nu_{k,l}^i$.

($\mathcal{S}2$) As suggested by Figures 4(a)-4(b) we choose

$$\delta t \Sigma_{i,j} = 0.2 \times \underset{1 \leqslant k \leqslant M}{\mathrm{mean}} \left\{C_k^{i,j}(1)\right\},$$

and we shift upward the simulated curves by $\delta = 0.3 \times \underset{1 \leqslant k \leqslant M}{\mathrm{mean}} \left\{C_k^{i,j}(1)\right\}$;

($\mathcal{S}3$) Finally, we fix the penalization parameters $A_i = A = 10$, and choose $k_i := V_i/\eta_i$, γ, and $\Gamma_{i,i}$ by minimizing the L^2-error between the simulated curves and real curves.



(a) GOOG (b) GOOG/AAPL

FIGURE 5. Comparison between the simulated curves and the real curves for two examples. Figure 5(a) corresponds to $i \equiv j \equiv$ GOOG and Figure 5(b) corresponds to $(i, j) \equiv$ (GOOG, AAPL).

Figures 5(a)-5(b) show illustrative examples by considering the two-stocks portfolio: Asset 1 $\equiv$ GOOG; Asset 2 $\equiv$ AAPL. For that example, the parameters of our model are presented in Table 2.

Estimated using the regression (5.33) **of Section 3.3 and** **($\mathcal{S}$1)-($\mathcal{S}$2)**		
$\mathrm{Corr}\left(E_0^1, E_0^2\right) = 20\%$,	$\alpha_1 = 2.5 \times 10^{-4}$,	$\sigma_1 = 1.55$,
$\rho_{1,2} = 0.5\%$,	$\alpha_2 = 7.9 \times 10^{-5}$,	$\sigma_2 = 0.43$,
Calibrated on curves of Figure 5(a) and 5(b)		
$\Gamma_{1,1} = 3.6 \times 10^9$,	$\Gamma_{2,2} = 2.02 \times 10^9$,	$\gamma = 10^{-3}$,
$k_1 = 2 \times 10^7$,	$k_2 = 8 \times 10^8$.	

TABLE 2. The MFG model parameters for the two-stocks portfolio: Asset $1 \equiv \mathrm{GOOG}$; Asset $2 \equiv \mathrm{AAPL}$.

Here $\Gamma_{1,2}$, α_1, α_2, σ_1, σ_2, $\rho_{1,2}$ are estimated from data (cf. Table 1 and Figures 4(a)-4(b)), while $\Gamma_{1,1}$, $\Gamma_{2,2}$, γ, k_1, k_2 are computed by minimizing the L^2-error between the simulated curves and real curves. Following this approach, one requires $2d+1$ parameters to fit a portfolio of d stocks (i.e. $d(d+1)/2$ curves).

APPENDIX A

On the Fokker-Planck Equation

1. Estimates in the Kantorowich-Rubinstein Distance

Let $V : [0,T] \times Q \to \mathbb{R}$ be a given bounded vector field, which is continuous in time and Hölder continuous in space, and we consider the following Fokker-Planck equation:

$$\begin{cases} \partial_t m - \sigma \Delta m - \operatorname{div}(mV) = 0 & \text{in } (0,T) \times Q, \\ m(0) = m_0 & \text{in } Q; \end{cases} \tag{A.1}$$

and the following stochastic differential equation:

$$d\mathbb{X}_t = V(t, \mathbb{X}_t)\, dt + \sqrt{2\sigma}\, dB_t \quad t \in (0,T], \quad \mathbb{X}_0 = Z_0, \tag{A.2}$$

where (B_t) is a standard d-dimensional Brownian motion over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $Z_0 \in L^1(\Omega)$ is random and independent of (B_t). We suppose that all functions are $\mathbb{Z}^d$-periodic and that system (A.1) is complemented with periodic boundary conditions. Under these assumptions, there is a unique solution to (A.2) and the following hold:

LEMMA A.1. *If* $m_0 = \mathcal{L}(Z_0)$ *then,* $m(t) = \mathcal{L}(\mathbb{X}_t)$ *is a weak solution to* (A.1) *and there exists a constant* $C_T > 0$ *such that, for any* $t, s \in [0,T]$,

$$\mathbf{d}_1(m(t), m(s)) \leqslant C_T (1 + \|V\|_\infty) |t-s|^{1/2}.$$

PROOF. The first assertion is a straightforward consequence of Itô's formula. On the other hand, for any 1-Lipschitz continuous function ϕ and any $t \geqslant s$, one has

$$\begin{aligned} \int_{\mathbb{T}^d} \phi(x)\, d(m(t) - m(s))(x) &\leqslant \mathbb{E}|\phi(\mathbb{X}_t) - \phi(\mathbb{X}_s)| \leqslant \mathbb{E}|\mathbb{X}_t - \mathbb{X}_s| \\ &\leqslant \mathbb{E}\left[\int_s^t |V(u, \mathbb{X}_u)|\, du + \sqrt{2\sigma} |B_t - B_s| \right] \\ &\leqslant \|V\|_\infty (t-s) + \sqrt{2\sigma(t-s)}. \end{aligned}$$

□

2. Boundary Conditions and Uniqueness for Solutions

In this part, we show that problem (4.3a)-(4.3b) admits at most one weak solution in a wide class of positive Radon measures. We believe that this result is well-known, and we explain the proof for lack of precise reference.

Let us start by generalizing the notion of weak solution that is given in (4.3c). For any $m_0 \in \mathcal{P}([0,\ell])$, $\ell > 0$, we define a *measure-valued weak solution* to (4.3a)-(4.3b) to be a measure $\mathbf{m}$ on $\overline{Q}_T := [0,T] \times [0,\ell]$ of the type

$$d\mathbf{m} = dm(t)\,dt,$$

with $m(t) \in \tilde{\mathcal{P}}([0,\ell])$ (a sub-probability measure on $[0,\ell]$) for all $t \in [0,T]$, and $t \to m(t,A)$ measurable on $[0,T]$ for any Borel set $A \subset [0,\ell]$; such that

$$\|b\|^2_{L^2_{\mathbf{m}}} := \int_0^T \int_0^\ell |b|^2\,d\mathbf{m} < \infty$$

and

$$\int_0^T \int_0^\ell (-\partial_t \phi - \sigma \partial_{xx}\phi + b\partial_x \phi)\,d\mathbf{m} = \int_0^\ell \phi(0,.)\,dm_0 \tag{A.3}$$

for every $\phi \in \mathcal{C}^{test}$. We claim that such a solution is unique:

PROPOSITION A.2. *There is at most one measure-valued weak solution to* (4.3a)-(4.3b).

PROOF. Our approach is similar to [**102**, Section 3.1]. Let $\mathbf{m}$ be a measure-valued weak solution to (4.3a)-(4.3b), and consider the following dual problem:

$$\begin{cases} -\partial_t w - \sigma\partial_{xx} w + \mathbf{b}\partial_x w = \psi \quad \text{in } Q_T, \\ w(t,0) = \partial_x w(t,\ell) = 0 \quad \text{in } (0,T), \\ w(T,x) = 0 \quad \text{in } Q, \end{cases} \tag{A.4}$$

where $\psi, \mathbf{b} \in \mathcal{C}^\infty(\overline{Q}_T)$. Let w be a smooth solution to (A.4). Since w^2 is smooth, we have:

$$\int_0^T \int_0^\ell \left\{-\partial_t(w^2) - \sigma\partial_{xx}(w^2) + b\partial_x(w^2)\right\}\,d\mathbf{m} = \int_0^\ell w^2(0,.)\,dm_0.$$

By (A.4) we thus have

$$\int_0^T \int_0^\ell w(\psi - \mathbf{b}\partial_x w)\,d\mathbf{m} - \sigma \int_0^T \int_0^\ell |\partial_x w|^2\,d\mathbf{m} + \sigma \int_0^T \int_0^\ell b w \partial_x w\,d\mathbf{m} = \int_0^\ell w^2(0,.)\,dm_0,$$

so that

$$\frac{\sigma}{2} \int_0^T \int_0^\ell |\partial_x w|^2\,d\mathbf{m} \leqslant C\left(\|w\|^2_\infty \int_0^T \int_0^\ell |b - \mathbf{b}|^2\,d\mathbf{m} + \|\psi\|_\infty \|w\|_\infty\right).$$

Hence, from the maximum principle:

$$\int_0^T \int_0^\ell |\partial_x w|^2\,d\mathbf{m} \leqslant C\|\psi\|^2_\infty \left(1 + \|b - \mathbf{b}\|^2_{L^2_{\mathbf{m}}}\right). \tag{A.5}$$

Now, let $\mathbf{m}_1, \mathbf{m}_2$ be two measure-valued weak solutions to (4.3a)-(4.3b). We know that

$$b \in L^2_{\mathbf{m}_1}(Q_T) \cap L^2_{\mathbf{m}_2}(Q_T).$$

Thus, $b \in L^2_{\mathbf{m}}(Q_T)$, where $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$. Let b^ϵ be a sequence of smooth functions converging to b in $L^2_{\mathbf{m}}(Q_T)$. Since $\mathbf{m}$ is regular, note that such a sequence exists by density of smooth functions in $L^2_{\mathbf{m}}(Q_T)$. The measures $\mathbf{m}_1, \mathbf{m}_2$ being positive, b^ϵ converges toward b in $L^2_{\mathbf{m}_1}(Q_T) \cap L^2_{\mathbf{m}_2}(Q_T)$ as well. Now, let us consider w^ϵ to be a solution to the dual problem that is obtained by replacing $\mathbf{b}$ by $\mathfrak{b}^\epsilon$ in (A.4). By using w^ϵ as a test function, we obtain

$$\text{(A.6)}\quad \int_0^T\int_0^\ell \psi\,\mathrm{d}(\mathbf{m}_1-\mathbf{m}_2) = \int_0^T\int_0^\ell (\mathfrak{b}-\mathfrak{b}^\epsilon)\partial_x w^\epsilon\,\mathrm{d}\mathbf{m}_2 - \int_0^T\int_0^\ell (\mathfrak{b}-\mathfrak{b}^\epsilon)\partial_x w^\epsilon\,\mathrm{d}\mathbf{m}_1 =: I_2^\epsilon - I_1^\epsilon.$$

By virtue of (A.5), we have for $j=1,2$:

$$\|\partial_x w^\epsilon\|_{L^2_{\mathbf{m}_j}} \leqslant C\|\psi\|_\infty\left(1+\|\mathfrak{b}-\mathfrak{b}^\epsilon\|_{L^2_{\mathbf{m}_j}}\right) \leqslant C,$$

so that

$$\left|I_j^\epsilon\right| \leqslant \|\partial_x w^\epsilon\|_{L^2_{\mathbf{m}_j}}\|\mathfrak{b}-\mathfrak{b}^\epsilon\|_{L^2_{\mathbf{m}_j}} \leqslant C\|\mathfrak{b}-\mathfrak{b}^\epsilon\|_{L^2_{\mathbf{m}_j}} \to 0, \quad \text{as } \epsilon\to 0.$$

Consequently, for any smooth function ψ

$$\int_0^T\int_0^\ell \psi\,\mathrm{d}(\mathbf{m}_1-\mathbf{m}_2) = 0,$$

which entails $\mathbf{m}_1 \equiv \mathbf{m}_2$ and concludes the proof. $\square$

Notation

Basic Notation.

- The usual inner product on $\mathbb{R}^d$ is denoted by $x.y$ or $< x, y >$.
- For $x \in \mathbb{R}^d$, $|x|$ is the usual Euclidian norm.
- For any vector $\mathbf{x} := (x_j)_{1 \leqslant j \leqslant d}$, we set $\mathbf{x}^{-i} = (x_j)_{j \neq i}$.
- For any $x, y \in \mathbb{R}$ we set the following notation for the minimum and maximum, respectively:

$$x \wedge y := \frac{1}{2}\left(x + y - |x - y|\right), \quad \text{and} \quad x \vee y := \frac{1}{2}\left(x + y + |x - y|\right).$$

- We denote by $\mathbb{T}^d$ the d-dimensional torus.
- We denote by $\mathbb{R}^+$ the open set of positive real numbers.
- Unless otherwise stated, Q is considered to be a bounded domain in $\mathbb{R}^d$, and $\bar{Q}$ is the topological closure of Q.
- For any $T > 0$, Q_T is the cylinder $(0, T) \times Q$; i.e. the set of points (t, x) such that $t \in (0, T)$ and $x \in Q$.

Probabilistic Notation.

- $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ is a complete filtered probability space.
- $\mathcal{L}(X)$ denotes the law of X.
- $\mathbb{E}[X]$ is the expectation of X (with the respect to the "standard" probability measure $\mathbb{P}$).
- $\mathbb{E}_{\mathbb{Q}}[X]$ is the expectation of X with respect to another probability measure $\mathbb{Q}$.
- We use $X \sim \mu$ to define a random variable X such that $\mathcal{L}(X) = \mu$.
- $\left\langle X^2 \right\rangle$ denotes the variance of X, and $\langle X, Y \rangle$ the covariance between X and Y, for any two random variables X, Y.

Notation for Functions.

- We define the indicator function by:

$$\mathbb{1}_A := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

- For any $\mathbb{R}$-valued function w we define the positive and negative parts of w, respectively:

$$w^+ := \frac{1}{2}(|w| + w), \quad \text{and} \quad w^- := \frac{1}{2}(|w| - w).$$

- For any function $w : \mathbb{R} \to \mathbb{R}^d$, $\dot{w}$ and $\ddot{w}$ denote respectively the first and second derivative of w.
- For any distribution μ, we denote by $\mathrm{supp}(\mu)$ the support of μ.

- **Function spaces**
  - $(L^s(Q), \|.\|_s)$, $1 \leqslant s < \infty$, is the set of s-summable Lebesgue measurable functions on Q.
  - $(L^\infty(Q), \|.\|_\infty)$ is the set of a.e bounded, and Lebesgue measurable functions on Q.
  - $L^s(Q)_+$ is the set of elements $w \in L^s(Q)$ such that $w(x) \geqslant 0$ for a.e. $x \in Q$.
  - $W_s^k(Q)$, $k \in \mathbb{N}$, $1 \leqslant s \leqslant \infty$, is the Sobolev space of functions having a weak derivatives up to order k which are s-summable on Q.
  - $\mathcal{C}(Q)$ is the space of continuous functions on Q.
  - $\mathcal{C}^\theta(Q)$ is the space of Hölder continuous functions with exponent θ on Q.
  - $\mathcal{C}^{k+\theta}(Q)$, $k \in \mathbb{N}$, $\theta \in (0, 1]$, the set of functions having k-th order derivatives which are θ-Hölder continuous.
  - $\mathcal{C}_c^\infty(Q)$ is the set of smooth functions whose support is a compact included in Q.
  - $\mathcal{C}_0(Q)$ is the space of all continuous functions on Q that vanish at infinity ($\mathcal{C}_0(Q) = \mathcal{C}(Q)$ when Q is compact).
  - $\mathcal{S}_\mathbb{R}$ denotes the space of rapidly decreasing functions, and $\mathcal{S}'_\mathbb{R}$ the space of tempered distributions.
  - $\mathcal{C}^{1,2}(Q_T)$ is the set of all functions on Q which are locally continuously differentiable in t and twice locally continuously differentiable in x.
  - $W_s^{1,2}(Q_T)$ is the space of elements of $L^s(Q)$ having weak derivatives of the form $\partial_t^j \partial_x^k$ with $2j + k \leqslant 2$, endowed with the following norm:
  $$\|w\|_{W_s^{1,2}} := \sum_{2j+k\leqslant 2} \|\partial_t^j \partial_x^k w\|_{L^s}.$$
  - $\mathcal{C}^{\theta/2,\theta}(\bar{Q}_T)$, $\theta > 0$, is the parabolic Hölder space, endowed with the norm $\|.\|_{C^{\theta/2,\theta}}$, as defined in [85].
  - For any Lipschitz continuous function w, we may use the following notation:
  $$\|w\|_{\mathrm{Lip}} := \sup_{x \neq y} \frac{|w(x) - w(y)|}{|x - y|}.$$
  - $\mathfrak{M}(Q)$ is the space of $\mathbb{R}$-valued Radon measures on Q, and $\mathcal{P}(Q)$, $\tilde{\mathcal{P}}(Q)$ are respectively the convex subset of probability measures on Q, and the convex subset of sub-probability measures: that is the set of positive radon measures μ, s.t. $\mu(Q) \leqslant 1$. The set $\mathcal{P}(Q)$ is endowed with the Kantorowich-Rubinstein distance, that is given by:
  $$\mathbf{d}_1(\pi, \pi') := \inf_{\nu \in \Pi(\pi, \pi')} \int_{Q \times Q} |x - y| \, d\nu(x, y), \quad \forall \pi, \pi' \in \mathcal{P}(Q)$$
  where $\Pi(\pi, \pi')$ is the set of all probability measures on $\Omega \times \Omega$ whose first marginal is π and whose second marginal is π'.

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AUTORISATION DE SOUTENANCE

Vu les dispositions de l'arrêté du 25 mai 2016,

Vu la demande du directeur de thèse

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est autorisé à soutenir une thèse pour l'obtention du grade de **DOCTEUR**

Ecole doctorale InfoMaths

Fait à Ecully, le 19 mars 2019

P/Le directeur de l'E.C.L.
La directrice des Etudes

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