Derived integral models and its applications to the study of certain derived rigid analytic moduli spaces

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THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par l'Université Toulouse 3 - Paul Sabatier

Présentée et soutenue par

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Le 28 juin 2019

Modèles intégres dérivés et ses applications à l'étude des certain espaces des modules rigides analytiques dérivés

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Ecole doctorale : **EDMITT - Ecole Doctorale Mathématiques, Informatique et Télécommunications de Toulouse**

Spécialité : **Mathématiques et Applications**

Unité de recherche :
IMT : Institut de Mathématiques de Toulouse

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Acknowledgments

First of all and above all, I would like to thank to my advisor, Bertrand Toën. Not only he guided me through this last three with immense devotion, but also shared many thoughts, ideas and knowledge, from which I much profited. His vision and deep knowledge of mathematics have had a great impact in each ways I pursue new ideas in mathematics. Not only, I must thank Bertrand for all the guidance, but also as a source of inspiration.

I also want to express my deep gratitude towards Marco Robalo and Mauro Porta. I met Marco for a quiet few years from now and since then he has been an important mathematical figure for me. He seems to always have reasonable advices and suggestions for the younger. From Mauro I learned a huge amount of mathematics. He grasps at length the most delicate technical details. Furthermore, his kindness sharing it with me helped me profoundly. This has been the case since I was a master student in Paris. Moreover, Mauro's enthusiasm for mathematics is enormous and contagious.

Carlos Simpson has been a great source of mathematical inspiration. His works on so many fields of mathematics and his vision of mathematics had a tremendous impact on the contents of the present thesis. I received great advice from Carlos when I visited him in Nice, a couple of years ago. Moreover, Carlos accepted to be a reporter of the current thesis. For all this, I am tremendously thankful.

Bhargav Bhatt has been a key figure in my mathematical background. Not only did I navigate through much of his work, which is always so interesting and full of content, but I also greatly profited from discussing personally with him. Bhargav always showed patiente and willingness to discuss mathematics with me. Having the possibility to engage in his way of thinking was a great opportunity to me as a mathematician. Moreover, Bhargav kindly accepted to write a report for the thesis. For all these reasons, I must express my deep gratitude to him.

I feel honored to have Carlos Simpson, Vadim Schechtman, Anna Cadoret and Tony Yu Yue as members of the jury of my thesis. Accordingly, I would like to thank them here.

I great benefited from the companion of many people in Toulouse. Firstly, I would like to Joseph Tapia, from which I learned so much. He has always been available to discuss mathematics and learn more about it. His vast knowledge in so many different disciplines of the subject had a great impact on me. He has been always humble towards me, as a student. Consequently I felt that my ideas and my development, as a mathematician, are appreciated. I would also like to thank to Joan Milles, Michel Vaquié, Marcello Bernadara and my PhD brothers Elena, Massimo and Leyth. It has been a great pleasure to discuss with Dario Beraldo and Tasos Moulinos. I am thankful not only for the mathematics but also from the cultural experience which this past three years have provided me.

I must also express my gratitude towards many mathematicians that I have met, around the globe, in the last three years. Not only these meetings provided me with immense knowledge, but also allowed me to be into contact with inspiring minds. I want thus to thank to Jacob Lurie, Gabriele Vezzosi, Mathieu Anel, Anthony Blanc, Samuel Bach Matthew Morrow, Peter Scholze, Benjamin Hennion, Adheel Kahn, Joost Nuiten, Ricardo Campos, Jean-Baptiste Teyssier, Pierre Schapira, François Petit, Ariane Mézard, Laurent Fargues, Dustin Clausen Vincent Daruvar, Emiliano Ambrosi, Valerio Melani, Brian Conrad, Benoit Stroh, Benoit Cadorel, Alberto Vezzani, David Gepner, Carl Wang-Erickson, Vincent Lafforgue, Marco d'Addezio, Marc Hoyois, Ben Antieu,

Apart from the mathematical community, I had the luck to meet many great people. People whose way of living had an enormous influence in the developing of myself as an individual. I first want to thank to all my old friends from my hometown. Paulo Ilhéu, Joao Nuno, Joao Ferreira, Miguel Sabino, Margarida Santos, Marta Azenha, Lea Lopes, Pedro Duraes, Luís, Zézinho, Diogo Guedes. I also share great memories from the friends I met in Paris: Rodrigo Codorniu, Mattia Cavicchi, Nuno Marinho, Pedro Damas, Mariana Fabiao, Anita Novais, Fernando and Isabel, Mário Pereira, Bruno Vilela, Madalena Caetano, Sofia Balsinha, Rita Pereira, Eduardo. Thank you for such great moments

I also had the great opportunity the opportunity to contribute from the great companion of all my friends in Toulouse. Guilherme Sousa, Beatriz Catarino, Marta Bruxelas, Pedro Velez, Rita Tojo.

Barbosa and Titá had always been a source of love and protection. They always supported me and accepted me as a son, since I was a child. I feel so grateful for all these years and so much more.

I could not feel other than deeply grateful to have met Catarina O'Neill, whose love and tenderness brought so much light and meaning to my life. You have always supported and appreciated my mathematical life. And above all, you always showed me the bright side of the moon. I am deeply thankful for such warm.

Lastly, I want to thank to my parents that always sacrifice so much for me and for my development. So many times without asking nothing in their favor. I am deeply touched by your devotion, as parents, towards me, as a

son.

Your influence made kept me moving forward. To you, obrigado por tudo!

And as I close this section, I would like to say a few more words. The universe is vast and our understanding limited. Nonetheless, the inspiration we obtain by coming across extraordinary people and engaging in such vast intellectual activity is invaluable. I have been so lucky to make part of this journey.

Résumé

Dans cette thèse, on étude différents aspects de la théorie de la géométrie dérivée rigide analytique. D'abord, on étude et généralise le théorèome classique de localisation de Raynaud au cadre dérivé. Muni d'une théorie des modéles formels, développé dans cette thése, on étude ses applications à l'étude des certains espaces de modules dérivés. Certains exemples correspondent bien au champ d'Hilbert rigide analytique dérivé et le champ des représentations continues des groupes fondamentales des variétés lisses sur un corps fini. La structure dérivée sur ce dernier nos permet de comprendre totalement la théorie de déformations des représentations galoisiennes. Enfin, on montre que ce dernier admet une structure sympléctique dérivé naturel. Ce dernier résultat s'appuye dans le théorème de HKR en géométrie analytique qui on prouve en collaboration avec F. Petit et M. Porta.

Liste des résultats principaux

Soit X un schéma propre et lisse sur un corps algébriquement clos. On est intéressé à l'étude des systémes locaux ℓ-adiques étales sur X, d'un certain rang. En effet, d'après les travaux de V. Drinfeld et plus récemment de V. Lafforgue autour de la correspondance de Langlands pour les corps de fonctions on sait que la famille de tels objets est fortément réliée aux formes automorphes. Il semble donc naturel d'étudier le foncteur de modules qui paramètre des systémes locaux ℓ-adiques sur X ou, de manière équivalente, des représentations continues ℓ-adiques du groupe fondamentale étale, $\pi_1^{\text{ét}}(X)$. Tel foncteur est noté

$$\mathrm{LocSys}_{\ell,n}(X) \colon \mathrm{Afd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \to \mathrm{Grpd},$$

où $\mathrm{Afd}_{\mathbb{Q}_\ell}^{\mathrm{op}}$ denote la catégorie des $\mathbb{Q}_\ell$-algèbres affinoides et Grpd la catégorie des groupoides, et il associe à chaque $\mathbb{Q}_\ell$-algébre affinoide

$$A \in \mathrm{Afd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \mapsto \mathrm{LocSys}_{\ell,n}(X)(A) \in \mathrm{Grpd},$$

où $\mathrm{LocSys}_{\ell,n}(X)(A)$ correspondant au groupoide des systèmes locaux ℓ-adiues étales sur X. Le principale but de cette thèse est l'étude des propriétés géometriques de $\mathrm{LocSys}_{\ell,n}(X)$. En particulier, on prouve le théréome:

Theorem 1. *Soit X un schéma propre et lisse sur un corps algébriquement clos. Alors le foncteur*

$$\mathrm{LocSys}_{\ell,n}(X) \in \mathrm{Fun}\big(\mathrm{Afd}_{\mathbb{Q}_\ell}^{\mathrm{op}}, \mathrm{Grpd}\big)$$

est représentable par un champ $\mathbb{Q}_\ell$-analytique géométrique. En plus, $\mathrm{LocSys}_{\ell,n}(X)$ *admet un atlas lisse par un espace rigide $\mathbb{Q}_\ell$-analytique*

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \to \mathrm{LocSys}_{\ell,n}(X),$$

où $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ *paramètre des systèmes locaux, sur X, munis d'une trivialisation. En plus,* $\mathrm{LocSys}_{\ell,n}(X)$ *admet une structure dérivée naturelle tel que si $\rho \in \mathrm{LocSys}_{\ell,n}(X)(\overline{\mathbb{Q}}_\ell)$ alors le* complexe cotangent analytique *est donné par*

$$\mathbb{L}^{\mathrm{an}}_{\mathrm{LocSys}_{\ell,n}(X),\rho} \simeq \mathrm{C}^*_{\text{ét}}(X, \mathrm{Ad}\rho)^\vee[-1],$$

où $\mathrm{Ad}(\rho) := \rho \otimes \rho^\vee$ c'est la répresentation adjointe de ρ.

Un autre résultat important concernant la géométrie de $\mathrm{LocSys}_{\ell,n}(X)$ est l'existence d'une forme sympléctique décalée sur $\mathrm{LocSys}_{\ell,n}(X)$:

Theorem 2. *Soit X un schéma propre et lisse sur un corps algébriquement clos de dimension d. Le champ $\mathbb{Q}_\ell$-analytique* $\mathrm{LocSys}_{\ell,n}(X)$ *admet une forme sympléctique $(2d-2)$-décalée naturelle ω. En plus, la $(2d-2)$-forme sur* $\mathrm{LocSys}_{\ell,n}(X)$ *est induite de la dualité de Poincaré en cohomologie étale.*

Pour prouver l'existence d'une telle structure dérivée naturelle sur $\mathrm{LocSys}_{\ell,n}(X)$ on a eu besoin de généraliser le théorème classique de localisation de Raynaud au cadre dérivé. Plus précisement, on a prouvé

Theorem 3. *Soit k un corps non-archimédien dont la valuation est de rang 1 et k° son anneau des entiers. Soit* dAn_k *la ∞-catégorie des espaces k-analytiques dérivés et* dfSch *l'∞-catégorie des schémas formels dérivés, de type (topologiquement) fini, sur k°. Alors il existe un foncteur de* rigidification

$$(-)^{\mathrm{rig}} \colon \mathrm{dfSch}_{k^\circ} \to \mathrm{dAn}_k$$

dont la restriction aux schémas formels discrets coincide avec le foncteur de rigidification de Raynaud usuel. En plus, on a une équivalence d'∞-catégories

$$(-)^{\mathrm{rig}} \colon \mathrm{dfSch}_{k^\circ}[S^{-1}] \to \mathrm{dAn}_k$$

où S denote la classe des éclatements dérivés admissibles et génériquement strong *dans* dfSch_{k°.

Le thérèome de Raynaud *dérivé* a trouvé jusqu'à maintenant certaines applications importantes. Un exemple c'est le prochain résultat prouvé en collaboration avec Mauro Porta:

Theorem 4. *Soit $X \in \mathrm{dAn}_k$ et $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}$ tel que $(\mathfrak{X})^{\mathrm{rig}} \simeq X$ dans dAn_k. Alors on a une suite exacte d'∞-catégories stables*

$$\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X}) \to \mathrm{Coh}^+(\mathfrak{X}) \to \mathrm{Coh}^+(X),$$

où $\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$ denote la sous-catégorie pleine de $\mathrm{Coh}^+(\mathfrak{X})$ engendré par les modules presque-parfaits sur $\mathfrak{X}$ qui sont supportés dans la fibre spéciale, $\mathfrak{X}_{\mathrm{sp}}$, de $\mathfrak{X}$.

Enfin, on a prouvé un analogue rigide k-analytique et aussi $\mathbb{C}$-analytique du théorème de HKR structuré, qui était prouvé par B. Toën et G. Vezzosi. Ce théorème fait partir d'un travail en collaboration avec F. Petit et M. Porta:

Theorem 5. *Soit k le corps des nombres complexes, $\mathbb{C}$, ou un corps non-archimédien de caractéristique 0 de valuation non-triviale. Soit X un espace k-analytique dérivé. Alors il existe une équivalence des espaces k-analytiques dérivés*

$$X_{X \times X} X \simeq \mathrm{T}X[-1],$$

compatibles avec la projection vers X.

Contents

Chapter 1

Introduction

1.1 Summary

In this thesis, we study different aspects of derived k-analytic geometry. Namely, we extend the theory of classical formal models for rigid k-analytic spaces to the derived setting. Having a theory of derived formal models at our disposal we proceed to study certain applications such as the representability of derived Hilbert stack in the derived k-analytic setting. We construct a moduli stack of derived k-adic representations of profinite spaces and prove its geometricity as a derived k-analytic stack. Under certain hypothesis we show the existence of a natural shifted symplectic structure on it. Our main applications is to study pro-étale k-adic local systems on smooth schemes in positive characteristic. Finally, we study at length an analytic analogue (both over the field of complex numbers $\mathbb{C}$ and over a non-archimedean field k) of the *structured* algebraic HKR, proved by Toen and Vezzosi.

1.2 Introduction

1.2.1 Introduction

Let X be a smooth and proper scheme over an algebraically closed field. One usually is interested in studying ℓ-adic étale local systems on X. For example, after the works of V. Drinfeld and more recently V. Lafforgue on geometric Langlands correspondence for function fields one can relate such arithmetic objects to automorphic forms on X. Therefore, it seems natural to study the moduli functor parametrizing étale ℓ-adic local systems on X. Such moduli can be described as a functor

$$\mathrm{LocSys}_{\ell,n}(X)\colon \mathrm{Afd}_k^{\mathrm{op}} \to \mathrm{Grpd}$$

given on objects by the formula

$$A \in \mathrm{Afd}_k^{\mathrm{op}} \mapsto \mathrm{LocSys}_{\ell,n}(X)(A) \in \mathrm{Grpd}$$

where $\mathrm{LocSys}_{\ell,n}(X)(A) \in \mathrm{Grpd}$ denotes the groupoid of étale A-adic étale local systems on X, $\mathrm{Afd}_k^{\mathrm{op}}$ denotes the category of affinoid k-algebras (k a finite extension of $\mathbb{Q}_\ell$) and Grpd denotes the category of groupoids. A main motivation of the current thesis was the study of the geometric properties of $\mathrm{LocSys}_{\ell,n}(X)$. In particular, we prove the following theorem:

Theorem 1.2.1.1. *Let X be a smooth and proper scheme. Then the functor* $\mathrm{LocSys}_{\ell,n}(X) \in \mathrm{Fun}\big(\mathrm{Afd}_k^{\mathrm{op}}, \mathrm{Grpd}\big)$ *is representable by a geometric k-analytic stack. It admits a smooth atlas by a k-analytic space*

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \to \mathrm{LocSys}_{\ell,n}(X),$$

where $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ *parametrizes framed étale ℓ-adic local systems on X. Furthermore,* $\mathrm{LocSys}_{\ell,n}(X)$ *admits a natural derived enhancement. Given $\rho \in \mathrm{LocSys}_{\ell,n}(X)(\overline{\mathbb{Q}}_\ell)$, the analytic cotangent complex of* $\mathrm{LocSys}_{\ell,n}(X)$ *at ρ is naturally equivalent to*

$$\mathbb{L}^{\mathrm{an}}_{\mathrm{LocSys}_{\ell,n}(X),\rho} \simeq C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)^\vee[-1] \in \mathrm{Mod}_{\overline{\mathbb{Q}}_\ell},$$

where $\mathrm{Ad}(\rho) := \rho \otimes \rho^\vee$ denotes the adjoint representation of ρ.

Another important result concerning the geometry of $\mathrm{LocSys}_{\ell,n}(X)$ is the existence of a natural shifted symplectic structure on $\mathrm{LocSys}_{\ell,n}(X)$:

Theorem 1.2.1.2. *Let X be a proper and smooth scheme of dimension d. The moduli k-analytic stack* $\mathrm{LocSys}_{\ell,n}(X)$ *admits a canonical $2-2d$-shifted symplectic structure, whose underlying $2-2d$-form is induced by Poincaré duality for étale cohomology (with derived coefficients).*

In order to endow $\mathrm{LocSys}_{\ell,n}(X)$ with a derived structure we had to develop new techniques to address the existence of formal models for derived k-analytic spaces. More precisely, we generalized to derived setting a well known theorem of Raynaud stating that the category of k-analytic spaces can be obtained as a localization of the category of formal models over k°, satisfying certain finiteness conditions:

Theorem 1.2.1.3. *Let* dAn_k *denote the* ∞*-category of derived* k*-analytic spaces and* dfSch_{k° *the* ∞*-category of (admissible) derived* k°*-adic schemes. Then there exists a derived rigidification functor*

$$(-)^{\mathrm{rig}} \colon \mathrm{dfSch}_{k^\circ} \to \mathrm{dAn}_k$$

which coincides with the usual rigidification functor for ordinary k°*-adic schemes. Moreover, the derived rigidification functor induces an equivalence*

$$(-)^{\mathrm{rig}} \colon \mathrm{dfSch}_{k^\circ}[S^{-1}] \to \mathrm{dAn}_k$$

of ∞*-categories, where* S *denotes the class of generically strong admissible blow-ups in* dfSch_{k°.

The above theorem has found many different applications. We have already mentioned the construction of the derived structure on $\mathrm{LocSys}_{\ell,n}(X)$. Other such examples concern descent results for derived ∞-categories of almost perfect modules on $X \in \mathrm{dAn}_k$. One important application of the theory of formal models for derived k-analytic spaces is the following theorem proved in a joint work with M. Porta:

Theorem 1.2.1.4. *Let* X *be a quasi-separated and proper* k*-analytic space. Then the derived Hilbert stack associated to* X

$$\mathrm{RHilb}(X) \in \mathrm{dSt}\big(\mathrm{An}_k, \tau_{\text{ét}}\big)$$

is representable by a derived k*-analytic stack.*

We can also show the existence of dualizing sheaves for derived k-analytic spaces, which as far as the knowledge of the author is concerned it is an original result. A main ingredient in the proof of the existence of a shifted symplectic form on $\mathrm{LocSys}_{\ell,n}(X)$ one needs a rigid k-analytic version of the algebraic HKR theorem, proved by B. Toën and G. Vezzosi. This is a joint work in progress with F. Petit and M. Porta.

Theorem 1.2.1.5. *Let* k *be the field* $\mathbb{C}$ *of complex numbers or a non-archimedean field of characteristic* 0 *with a non-trivial valuation. Let* X *be a* k*-analytic space. Then there is an equivalence of derived* k*-analytic spaces*

$$X \times_{X \times X} X \simeq \mathrm{T}X[-1]$$

compatible with the canonical projection to X.

1.3 Motivations

1.3.1 Non-abelian Hodge Theory

Let X be a topological space. The moduli scheme of $\mathbb{C}$-local systems on X, denoted $\mathrm{LocSys}_{\mathbb{C},n}(X)$, has been studied extensively in classical algebraic geometry. It can be defined by means of a moduli functor

$$\mathrm{LocSys}_{\mathbb{C},n}(X)^{\mathrm{framed}} \colon \mathrm{Aff}_{\mathbb{C}} \to \mathrm{Set}$$

given on objects by the formula

$$A \in \mathrm{Aff}_{\mathbb{C}} \mapsto \mathrm{Hom}_{\mathrm{grp}}\big(\pi_1^{\mathrm{top}}(X), \mathrm{GL}_n(A)\big) \in \mathrm{Set},$$

where $\mathrm{Hom}_{\mathrm{grp}}$ denotes the set of group homomorphisms. The moduli space $\mathrm{LocSys}_{\mathbb{C},n}(X)^{\mathrm{framed}}$ admits a canonical action of the general linear group scheme $\mathrm{GL}_n \in \mathrm{Sch}_{\mathbb{C}}$ via conjugation. We can form the corresponding universal categorical quotient, which we shall denote $\mathrm{LocSys}_{\mathbb{C},n}(X)$. It is then possible to show via geometric invariant theory, that $\mathrm{LocSys}_{\mathbb{C},n}(X)$ is representable by a scheme of finite type over $\mathbb{C}$, [Sim94a, §1]. By construction, $\mathrm{LocSys}_{\mathbb{C},n}(X)$ parametrizes semisimple rank n group representations of the topological fundamental group $\pi_1^{\mathrm{top}}(X)$.

It is a well known fact that there exists a natural bijection between the set of rank n representation of $\pi_1^{\mathrm{top}}(X)$

$$\rho \colon \pi_1^{\mathrm{top}}(X) \to \mathrm{GL}_n(\mathbb{C})$$

and the set of isomorphism classes of rank n local systems on X. The latter set can be identified with the underlying set of the cohomological group $H^1\big(X, \mathrm{GL}_n(\mathbb{C})\big)$. For this reason, one can interpret the moduli $\mathrm{LocSys}_{\ell,n}(X)$ as a *non-abelian* analogue of singular cohomology $H^1\big(X, \mathbb{C}\big)$ on X with $\mathbb{C}$-coefficients.

Suppose now that X is a complex smooth projective variety. One can canonically equip the set of $\mathbb{C}$-points of X with the structure of a topological manifold. As a consequence, to X we can associate an homotopy type

$$X(\mathbb{C}) \in \mathcal{S}.$$

Concretely, the topological manifold $X(\mathbb{C})$ corresponds to the underlying topological space of the analytification X^{an} of X. One often refers to $X(\mathbb{C}) \in \mathcal{S}$ as the *Betti realization of* X. We can thus consider the moduli scheme

$$\mathrm{LocSys}_{\mathbb{C},n}(X) \coloneqq \mathrm{LocSys}_{\mathbb{C},n}\big(X(\mathbb{C})\big),$$

which parametrizes rank n-local systems on the Betti realization $X(\mathbb{C})$ of $X \in \mathrm{Sch}_{\mathbb{C}}$. Let us introduce two other main ingredients in non-abelian Hodge Theory:

Definition 1.3.1.1. Let X be a projective and smooth variety over the field $\mathbb{C}$ of complex numbers. We define the moduli stack $\mathrm{Conn}_{\mathbb{C},n}(X) \colon \mathrm{Aff}_{\mathbb{C}} \to \mathcal{S}$ given on objects by the formula

$$A \in \mathrm{Aff}_{\mathbb{C}} \mapsto \mathrm{Conn}_{\mathbb{C},n}(X)(A) \in \mathcal{S}$$

where $\mathrm{Conn}_{\mathbb{C},n}(X)(A)$ denotes the space of rank n flat connections with A-coefficients over X. We can also consider the moduli stack $\mathrm{Higgs}(X) \colon \mathrm{Aff}_{\mathbb{C}} \to \mathcal{S}$ parametrizing rank n Higgs bundles with A-coefficients on X, see [Sim94a, §1, p. 15]. Moreover, C. Simpson in his seminal work on non-abelian Hodge theory proved that the analytifications $\mathrm{Conn}^{\mathrm{an}}_{\mathbb{C},n}(X)$ and $\mathrm{Higgs}^{\mathrm{an}}(X)$ are homeomorphic. This last result can be interpreted as a non-abelian analogue of the degeneration of the Hodge to de Rham spectral sequence.

Remark 1.3.1.2. We can extend $\mathrm{LocSys}_{\mathbb{C},n}(X)$ to a moduli stack via the formula

$$\mathrm{LocSys}_{\mathbb{C},n}(X) \colon \mathrm{Aff}_{\mathbb{C}} \to \mathcal{S}$$

given on objects by the formula

$$A \in \mathrm{Aff}_{\mathbb{C}} \mapsto \mathrm{Map}_{\mathcal{S}}\big(X(\mathbb{C}), \mathrm{BGL}_n(A)\big) \in \mathcal{S}. \tag{1.3.1.1}$$

The above definition defines a more general object than the universal categorical quotient of $\mathrm{LocSys}^{\mathrm{framed}}_{\mathbb{C},n}(X)$. Indeed, the latter parametrizes semisimple representations of the fundamental group $\pi_1^{\mathrm{top}}(X)$ whereas the former parametrizes the *space* of all group representations of $\pi_1^{\mathrm{top}}(X)$. Moreover, the formula (1.3.1.1) can be easily generalized to the derived setting. We can define the derived enhancement of $\mathrm{LocSys}_{\mathbb{C},n}(X)$ as the derived mapping stack

$$\mathrm{LocSys}_{\ell,n}(X) \coloneqq \mathrm{Map}_{\mathrm{dSt}}\big(X(\mathbb{C}), \mathrm{BGL}_n(-)\big) \in \mathrm{dSt}\big(\mathrm{dAff}_{\mathbb{C}}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big),$$

where $\mathrm{dSt}\big(\mathrm{dAff}_{\mathbb{C}}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$ denotes the ∞-category of geometric stacks with respect to the algebraic geometric context $\big(\mathrm{dAff}_{\mathbb{C}}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$. Similarly, both the moduli $\mathrm{Conn}_{\mathbb{C},n}(X)$ and $\mathrm{Higgs}(X)$ can be upgraded to derived $\mathbb{C}$-Artin stacks via the equivalences of stacks

$$\mathrm{Conn}_{\mathbb{C},n}(X) \simeq \mathrm{Map}_{\mathrm{dSt}}\big(X_{\mathrm{dR}}, \mathrm{BGL}_n(-)\big), \quad \mathrm{Higgs}(X) \simeq \mathrm{Map}_{\mathrm{dSt}}\big(X_{\mathrm{Dolb}}, \mathrm{BGL}_n(-)\big).$$

We refer the reader to [Por17b, §3] for the definition of the notions X_{dR} and X_{Dolb}.

Theorem 1.3.1.3. [Por17b, Theorem 6.11] *Let X be a projective smooth complex variety. The Riemann-Hilbert correspondence induces an equivalence*

$$\mathrm{LocSys}^{\mathrm{an}}_{\mathbb{C},n}(X) \simeq \mathrm{Conn}^{\mathrm{an}}_{\mathbb{C},n}(X),$$

in the ∞-category $\mathrm{dSt}\big(\mathrm{dAn}_{\mathbb{C}}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$, where $\mathrm{LocSys}^{\mathrm{an}}_{\mathbb{C},n}(X)$ and $\mathrm{Conn}^{\mathrm{an}}_{\mathbb{C},n}(X)$ denote the analytification of both $\mathrm{LocSys}_{\mathbb{C},n}(X)$ and $\mathrm{Conn}_{\mathbb{C},n}(X)$, respectively.

This can be regarded as a vast generalization of the usual comparison isomorphism between de Rham and Betti cohomologies for smooth and proper varieties over $\mathbb{C}$. At the heart of the proof of the above equivalence lies the Riemann-Hilbert correspondence between the categories of rank n local systems on X and rank n flat connexions on X.

Question 1.3.1.4. *What can be said for smooth and proper schemes over a algebraically closed fields of characteristic $p > 0$. More precisely, is there any non-abelian analogues of p-adic Hodge theory?*

One main ingredient in our previous discussion was the existence of the homotopy type $X(\mathbb{C}) \in \mathcal{S}$ of $\mathbb{C}$-points of X. Unfortunaly, in positive characteristic the sole analogue of $X(\mathbb{C})$ is the étale homotopy type of X, which we shall denote $\mathrm{Sh}^{\text{ét}}(X)$. In this case, $\mathrm{Sh}^{\text{ét}}(X)$ is not a homotopy type but instead $\mathrm{Sh}^{\text{ét}}(X) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$, i.e. $\mathrm{Sh}^{\text{ét}}(X)$ is a profinite space or profinite homotopy type.

It has long been understood that there are deficiencies with studying study continuous representations

$$\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\mathbb{C})$$

The main reason is the fact that any such ρ must necessarily factor via a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. Nonetheless, in positive characteristic, one usually studies rank n ℓ-adic étale lisse sheaves on X. Therefore, the objects of our interest correspond to *continuous representations*

$$\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell). \tag{1.3.1.2}$$

Ultimately, one is also interested in studying continuous representations of homotopy types

$$\rho\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n(\overline{\mathbb{Q}}_\ell)$$

However, we do not have a clue of what shall mean a continuous representations of homotopy types. We present the reader with a list of properties that we would like such continuous representations satisfy:

(i) Let $A \in \mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}$ be a derived $\mathbb{Q}_\ell$-algebra. There should exist a space of continuous representations

$$\rho\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n(A). \tag{1.3.1.3}$$

Moreover, if $A \simeq \pi_0(A)$. There should be a natural equivalence between the space of such objects (1.3.1.3) and the space of continuous representations

$$\rho\colon \mathrm{B}\pi_1^{\text{ét}}(X) \to \mathrm{BGL}_n(A).$$

(ii) For each continuous representation $\rho\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n(A)$ one should be able to a find a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\rho_U"] \arrow[d] & \mathrm{BGL}_n(B) \arrow[d] \\ \mathrm{Sh}^{\text{ét}}(X) \arrow[r, "\rho"] & \mathrm{BGL}_n(A) \end{tikzcd}$$

where $B \in \mathcal{C}\mathrm{Alg}_{\mathbb{Z}_\ell}$ is of no ℓ-torsion and one has furthermore an equivalence $B \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \simeq A \in \mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}$. Moreover $U \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})_{/\mathrm{Sh}^{\text{ét}}(X)}$ denotes a profinite space such that one has a fiber sequence

$$Y \to U \to \mathrm{Sh}^{\text{ét}}(X),$$

with $Y \in \mathcal{S}^{\mathrm{fc}}$ is a finite constructible space.

(iii) Let $A \in \mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}$ be a derived $\mathbb{Q}_\ell$-algebra. A continuous representation

$$\rho\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n(A)$$

should induce, via extension of scalars, a continuous representation

$$\overline{\rho}\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n(B \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell)$$

where $B \in \mathcal{C}\mathrm{Alg}_{\mathbb{Z}_\ell}$ is as in (ii). We require furthermore that such continuous $\overline{\rho}$ factor through a finite quotient $X_i \in \mathcal{S}^{\mathrm{fc}}$ of $\mathrm{Sh}^{\text{ét}}(X) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$.

In order to facilitate our exposition we adopt the following convention for the mapping space of continuous representations, which we have not yet defined:

Notation 1.3.1.5. Given $A \in \mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}$ we shall denote by

$$\mathrm{Map}_{\mathrm{cont}}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{BGL}_n(A)\big) \in \mathcal{S}$$

the space of continuous representations $\rho\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n(A)$.

Suppose we have a reasonable definition of a notion continuous representation as in (1.3.1.2). Then we can define the moduli functor $\mathrm{LocSys}_{\ell,n}(X)\colon \mathrm{dAfd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \to \mathcal{S}$ as given by

$$Z := \big(\mathcal{Z}, \mathcal{O}_Z\big) \in \mathrm{dAfd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \mapsto \mathrm{Map}_{\mathrm{cont}}\Big(\mathrm{Sh}^{\text{ét}}(X), \Gamma(Z)\Big) \in \mathcal{S}$$

where $\Gamma(Z)$ denotes the global sections ring of Z, defined as $\pi_*\big(\mathcal{O}_Z\big) \in \mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}$.

The derived moduli stack $\mathrm{LocSys}_{\ell,n}(X)$ is an ℓ-adic analogue of the moduli of rank n complex local systems for complex varieties, $\mathrm{LocSys}_{\mathbb{C},n}(X)$.

Remark 1.3.1.6. It would be desirable that certain results of p-adic Hodge for proper smooth schemes over $\mathbb{Q}_p$ admit non-abelian Hodge theoretical analogues. And in such case, the moduli $\mathrm{LocSys}_{p,n}(X)$ should play a role. However, there are serious obstructions for a complete p-adic analogue as we shall see in later sections.

1.3.2 Deligne Comptage

In his seminal work [Dri81] V. Drinfeld proved a counting formula for rank 2 ℓ-adic lisse sheaves on a smooth and proper curve X, up to λ-torsion. Drinfeld's formula uses in an essential way his work on the Langlands correspondence, [Dri80]. More recently, in the work of [Yu18] such formula was generalized to higher ranks and to the open case by considering fixed monodromy at infinity.

In [Del15], P. Deligne conjectured that the counting problem of rank n ℓ-adic étale lisse sheaves on a smooth variety X over a finite field $\mathbb{F}_q$ could be stated equivalently as a Grothendieck-Lefschetz trace formula on a suitable moduli space of ℓ-adic local systems. In order to understand his assertion recall that Galois descent induces a bijection between the set of ℓ-adic lisse sheaves on X and the set of ℓ-adic lisse sheaves over the base change

$$\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$$

which admit a $\mathbb{Z}_\ell$-lattice stable under pullback along the geometric Frobenius endomorphism of X,

$$\mathrm{Frob}\colon \overline{X} \to \overline{X}.$$

More precisely, given $\mathcal{F}$ a $\mathbb{Q}_\ell$-adic sheaf on X we can consider it as an ℓ-adic sheaf on $\overline{X}$, via pullback along the canonical map $f\colon \overline{X} \to X$. Moreover, the ℓ-adic sheaf $\mathcal{F}$ satisfies

$$\mathrm{Frob}^*\mathcal{F} \simeq \mathcal{F}$$

in the category of ℓ-adic lisse sheaves on $\overline{X}$. The crucial observation of P. Deligne is the fact that Drinfeld's formula is reminiscent of a Grothendieck-Lefschetz trace formula with respect to the pair $\big(\mathrm{LocSys}_{\ell,n}(\overline{X}), F\big)$, where $\mathrm{LocSys}_{\ell,n}(\overline{X})$ should correspond to the moduli of ℓ-adic continuous representations of the étale fundamental group $\pi_1^{\text{ét}}(\overline{X})$ and

$$F\colon \mathrm{LocSys}_{\ell,n}(\overline{X}) \to \mathrm{LocSys}_{\ell,n}(\overline{X})$$

denotes the endomorphism of $\mathrm{LocSys}_{\ell,n}(X)$ given on objects by the formula

$$\mathcal{F} \in \mathrm{LocSys}_{\ell,n}(\overline{X}) \mapsto \mathrm{Frob}^*\mathcal{F} \in \mathrm{LocSys}_{\ell,n}(\overline{X}).$$

One would then like to confirm that $\mathcal{F}$ admits a finite number of fixed points and such number could be computed by means of a trace formula for $\big(\mathrm{LocSys}_{\ell,n}(X), F\big)$. However, in [Del15] the author does not give any hint on how to construct $\mathrm{LocSys}_{\ell,n}(\overline{X})$ except for certain complex analogies, inspired mainly by non-abelian Hodge theory. Therefore, in order to prove Deligne's conjecture one would have to show the following statements:

(i) There exists a natural candidate $\mathrm{LocSys}_{\ell,n}(\overline{X})$ for the moduli of ℓ-adic continuous representations of $\pi_1^{\text{ét}}(\overline{X})$. Moreover, such candidate should have sufficiently geometric properties, such as being representable by an algebraic or $\mathbb{Q}_\ell$-analytic stack. In particular, one should have a complete understanding of its deformation theory around a closed point

$$\rho \colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell).$$

(ii) There exists a reasonable cohomological theory on $\mathrm{LocSys}_{\ell,n}(\overline{X})$, which we shall denote by

$$C^*_\bullet(\mathrm{LocSys}_{\ell,n}(\overline{X})).$$

(iii) The automorphism $F \colon \mathrm{LocSys}_{\ell,n}(\overline{X}) \to \mathrm{LocSys}_{\ell,n}(\overline{X})$ admits a finite set of fixed points and a trace formula holds with respect to the triplet $\big(\mathrm{LocSys}_{\ell,n}(\overline{X}),\ C^*_\bullet(\mathrm{LocSys}_{\ell,n}(\overline{X})),\ F\big)$.

Moreover, it would be interesting if one is able to prove the above conjectural statements by purely geometric means without need to pass to the automorphic setting.

In this thesis we will answer positively to (i) and (ii). We construct $\mathrm{LocSys}_{\ell,n}(X)$ directly as a $\mathbb{Q}_\ell$-analytic stack, whose proof follows roughly the same lines as in the complex case. We show thereafter that it is possible to enhance $\mathrm{LocSys}_{\ell,n}(X)$ with a natural derived structure. Such derived structure allow us to consider derived de Rham cohomology on $\mathrm{LocSys}_{\ell,n}(X)$, $C^*_{\mathrm{dR}}\big(\mathrm{LocSys}_{\ell,n}(X)\big)$. However, we will show that $\mathrm{LocSys}_{\ell,n}(X)$ has too many connected components. Indeed, the moduli stack $\mathrm{LocSys}_{\ell,n}(X)$ admits one connected component for each residual representation

$$\rho \colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\mathbb{F}_{\ell^r}).$$

Therefore, our results cannot applied directly to show Deligne's conjecture, however we will indicate some possible future avenues in later chapters.

1.3.3 Theory of formal models for ordinary k-analytic spaces

Let us illustrate a possible route to solve the question posed in Theorem 2.1.0.15. We start by recall the main results concerning the existence of formal models for k-analytic spaces:

Theorem 1.3.3.1 (§8 [Bos05]). *Let $X \in \mathrm{An}_k$ denote a quasi-paracompact and quasi-separated k-analytic space. Then there exists an admissible formal k°-scheme $\mathfrak{X} \in \mathrm{fSch}_{k^\circ}$ such that one has an equivalence*

$$\mathfrak{X}^{\mathrm{rig}} \simeq X,$$

in the category An_k.

Theorem 1.3.3.1 can be stated equivalently as the essential image of the functor $(-)^{\mathrm{rig}}$ coincides with the full subcategory of An_k spanned by quasi-paracompact and quasi-separated k-analytic spaces. In particular, Theorem 1.3.3.1 implies that whenever $X = \mathrm{Sp}(A)$ is k-affinoid, we can find an affine formal model of the form $\mathrm{Spf}(A_0) \in \mathrm{fSch}_{k^\circ}$ such that A_0 is an *admissible* k°-algebra. In other words, A_0 is topologically of finite presentation and $\mathfrak{m}$-torsion free, where $\mathfrak{m}$ denotes the maximal ideal of k°.

Let us illustrate how a derived analogue of Theorem 1.3.3.1 is helpful to treat the question of Theorem 2.1.0.15. Suppose that k is a finite extension of $\mathbb{Q}_\ell$. For every $A \in \mathrm{AnRing}_k^{\mathrm{sm}}$ we can find a formal model $A_0 \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{sm}}$, i.e. there exists A_0 verifying the following conditions:

(i) The ordinary commutative ring $\pi_0(A_0)$ admits an adic topology compatible with the one on k°;

(ii) One has an equivalence

$$A_0 \otimes_{k^\circ} k \simeq A^{\mathrm{alg}}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_k$.

In particular, thanks to [Lur16, Remark 8.1.2.4] the derived k°-algebra A_0 can be realized as an inverse limit

$$A_0 \simeq \lim_{n\geq 0} A_{0,n}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}$, where $A_{0,n}$ denotes the pushout diagram

$$\begin{tikzcd} A_0[t] \arrow[r, "t\mapsto \ell^n"] \arrow[d, "t\mapsto 0"] & A_0 \arrow[d] \\ A_0 \arrow[r] & A_{0,n} \end{tikzcd}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}$. In this case, the classifying space $\mathrm{BGL}_n(A_0) \in \mathcal{S}$ can be realized as an object in the ∞-category $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ of profinite spaces. Namely, one can consider the object

$$\mathrm{BGL}_n(A_0) \coloneqq \{\mathrm{BGL}_n(A_{0,m})\}_m \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}). \tag{1.3.3.1}$$

Moreover, as the transition maps in (1.3.3.1) are compatible with the group structures for different m it follows that $\mathrm{BGL}_n(A_0) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ can be promoted to an object in $\mathrm{Mon}^{\mathrm{grp}}_{\mathbb{E}_1}\big(\mathrm{Pro}(\mathcal{S})\big)$.

Remark 1.3.3.2. Suppose that one is provided with a functorial assignement

$$A_0 \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \mapsto \mathrm{BGL}_n(A_0) \in \mathrm{Mon}^{\mathrm{grp}}_{\mathbb{E}_1}(\mathcal{S}^{\mathrm{fc}}),$$

where $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ denotes the ∞-category of derived adic k°-algebras. In this case, one could expect to define a functor

$$F\colon \mathrm{AnRing}^{\mathrm{sm}}_k \to \mathcal{S}$$

given on objects by the formula

$$\big(A \to k\big) \in \mathrm{AnRing}^{\mathrm{sm}}_k \mapsto \mathrm{Map}_{\mathrm{cont}}\left(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{BAut}(A^n)\right)_{\mathrm{Map}_{\mathrm{cont}}\left(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{BAut}(k^n)\right)} \{\rho\} \in \mathcal{S} \tag{1.3.3.2}$$

where $\mathrm{Map}_{\mathrm{cont}}\big(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{BAut}(A^n)\big)$ denotes the colimit

$$\mathrm{Map}_{\mathrm{cont}}\left(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{BAut}(A^n)\right) \coloneqq \operatorname*{colim}_{A_0} \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}))}\left(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{BAut}(A_0^n)\right).$$

over A_0, varying through the ∞-category of formal models for $A \in \mathrm{AnRing}^{\mathrm{sm}}_k$. The formula displayed in (1.3.3.2) is already a good approximation of a formal moduli problem classifying continuous deformations of

$$\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(k).$$

However, there are still certain issues which render the formula (1.3.3.2) problematic. First, the étale homotopy group $\pi_1^{\text{ét}}(X)$ does not classify ℓ-adic lisse sheaves on $X_{\text{ét}}$ with derived coefficients. One should instead replace the profinite group $\pi_1^{\text{ét}}(X)$ with the étale homotopy type of X, $\mathrm{Sh}^{\text{ét}}(X)$. Fortunately, the formula in (1.3.3.2) is sufficiently general so that we can replace $\mathrm{B}\pi_1^{\text{ét}}(X) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ with $\mathrm{Sh}^{\text{ét}}(X) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ or any other profinite space without concern.

Secondly and more importantly, formula (1.3.3.2) does not classify all deformation of ρ, instead only those continuous deformation of ρ which are power bounded. However, in general, there are many continuous deformations of ρ which do not factor through power bounded matrices, even in the ordinary case. Nonetheless, (1.3.3.2) is simple enough to hint us a reasonable notion of continuity in the derived setting. We will delve this question in further detail in §4. Also, (1.3.3.2) motivates an analogous statement of Theorem 1.3.3.1 in the derived setting.

At this point, the avid reader might object that the formula displayed in (1.3.3.2) is not necessarily functorial on $A \in \mathrm{AnRing}^{\mathrm{sm}}_k$. In order to show functoriality of (1.3.3.2) one needs to generalize the following fundamental result due to Raynaud:

Theorem 1.3.3.3 (Theorem 8.4.3 [Bos05]). *The ordinary rigidification functor* $(-)^{\mathrm{rig}} \colon \mathrm{fSch}_{k^\circ} \to \mathrm{An}_k$ *factors through the localization* $\mathrm{fSch}_{k^\circ}[S^{-1}]$*, where* S *denotes the class of admissible ups on* fSch_{k°*. Moreover,* $(-)^{\mathrm{rig}}$ *induces an equivalence of catgories*

$$\mathrm{fSch}_{k^\circ}[S^{-1}] \simeq \mathrm{An}'_k$$

where An'_k *denotes the full subcategory of* An_k *spanned by quasi-paracompact and quasi-separated* k*-analytic spaces.*

If we are able to generalize **??** 1 to the derived setting then we would be able, via a formal reasoning, to show that the assignment in (1.3.3.2) is functorial. The generalization of both **??** 1 and Theorem 1.3.3.1 are now proven facts which make part of the current thesis project which we detail in this introduction.

1.4 Derived k°-adic and derived k-analytic geometries

1.4.1 Derived Raynaud localization theorem

A fundamental ingredient in both Theorem 1.3.3.1 and **??** 1 consists of the rigidification functor

$$(-)^{\mathrm{rig}} \colon \mathrm{fSch}_{k^\circ} \to \mathrm{An}_k \tag{1.4.1.1}$$

which associates to a formal k°-scheme $\mathfrak{X} \in \mathrm{fSch}_{k^\circ}$ its *rigidification* $\mathfrak{X}^{\mathrm{rig}} \in \mathrm{An}_k$. Henceforth, in order to state derived analogues of both Theorem 1.3.3.1 and **??** 1 one would need the following derived analogues:

(i) An ∞-category of derived k°-adic schemes, dfSch_{k° which have been introduced in [Lur16, §8];

(ii) An ∞-category of derived k-analytic spaces, dAn_k, introduced in [PY16a];

(iii) A *derived rigidification functor* $(-)^{\mathrm{rig}} \colon \mathrm{dfSch}_{k^\circ} \to \mathrm{dAn}_k$ which restricts to (1.4.1.1) on ordinary k°-adic schemes.

Even though items (i) and (ii) have been treated extensively in the literature, it is not clear how to directly define a derived rigidification functor

$$(-)^{\mathrm{rig}} \colon \mathrm{dfSch}_{k^\circ} \to \mathrm{dAn}_k.$$

A major obstable results from the fact that the ∞-category dfSch_{k° is defined in [Lur16, §8] based on the theory of locally ringed ∞-topoi. More precisely, derived formal Deligne-Mumford stacks correspond to $(\mathcal{X}, \mathcal{O})$ where $\mathcal{X}$ is an ∞-topos and $\mathcal{O} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ is a local derived k°-algebra on $\mathcal{X}$ such that the ordinary commutative ring sheaf $\pi_0(\mathcal{O})) \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ is equipped with an adic topology compatible with the adic topology on k°. On the other hand, the ∞-category dAn_k is defined in terms of $\mathcal{T}_{\mathrm{an}}(k)$-structured spaces. Unfortunately, no direct comparison exists between *adic locally ringed ∞-topoi* and $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topoi. A possible way to deal with this difficulty is to redefine derived k°-adic geometry via a structured ∞-topoi approach, where we consider structured ∞-topoi with respect to a suitable k°*-adic pregeometry*, $\mathcal{T}_{\mathrm{ad}}(k^\circ)$. After adopting such a viewpoint, one is equipped for free with a transformation of pregeometries

$$(-)^{\mathrm{rig}} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{T}_{\mathrm{an}}(k)$$

induced from the classical rigidification functor. Moreover, by a formal reasoning one can prove that such transformation of pregeometries provide us with a candidate for a derived rigidification functor $\mathrm{dfSch}_{k^\circ} \to \mathrm{dAn}_k$. For this reason, we adopt the structured spaces point of view for derived k°-adic geometry.

Definition 1.4.1.1. Let $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ denote the full subcategory of fSch_{k° spanned by those ordinary affine formal k°-schemes $\mathrm{Spf}(A)$ such that $\mathrm{Spf}(A)$ is étale over some affine n-space, $\mathfrak{A}^n_{k^\circ}$. We equipp $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ with the étale topology. The class of admissible morphims on $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ is defined as the class of étale morphisms in $\mathcal{T}_{\mathrm{ad}}(k^\circ)$.

Definition 1.4.1.2. We defined the ∞-category of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$*-structured spaces* as the ∞-category of couples $(\mathcal{X}, \mathcal{O})$ such that $\mathcal{X} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ is an ∞-topos and $\mathcal{O} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{X}$ is a local $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structure on $\mathcal{X}$.

Notation 1.4.1.3. Let $\mathcal{X}$ be an ∞-topos. The ∞-category of local structures on $\mathcal{X}$ is denoted by $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$.

As in the setting of derived k-analytic geometry, one has a well defined, up to contractible indeterminacy, *underlying algebra functor*

$$(-)^{\mathrm{alg}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}),$$

given on objects by the formula

$$\mathcal{O} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \mapsto \mathcal{O}(\mathfrak{A}^1_{k^\circ}) \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}).$$

Moreover, this functor can be promoted to a functor whose target consists of derived k°-algebras on $\mathcal{X}$:

Lemma 1.4.1.4. *The underlying algebra functor $(-)^{\mathrm{alg}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ can be naturally promoted to a well defined, up to contractible indeterminacy, functor*

$$(-)^{\mathrm{ad}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}),$$

which is given on objects by the formula

$$\mathcal{O} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \mapsto \mathcal{O}(\mathfrak{A}^1_{k^\circ}) \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$$

where $\mathcal{O}(\mathfrak{A}^1_{k^\circ}) \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ is equipped with natural adic topology, induced by the sequence of ideals $\{\mathcal{I}_n\}_n$ of $\pi_0(\mathcal{O}^{\mathrm{alg}})$ which correspond to the kernel of the canonical ring homomorphisms

$$\pi_0(\mathcal{O}^{\mathrm{alg}}) \to \pi_0(\mathcal{O}^{\mathrm{alg}} \otimes_{k^\circ} k^\circ_n)$$

for each $n \geq 1$.

The following fundamental result implies that specifying local $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structures on $\mathcal{X}$ is roughly equivalent as specifying a derived k°-adic locally ring on $\mathcal{X}$:

Theorem 1.4.1.5. *Let $\mathcal{X}$ be an ∞-topos with enough geometric points. Then the functor*

$$(-)^{\mathrm{ad}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$$

induces an equivalence of the ∞-categories of topologically almost of finite presentation objects

$$(-)^{\mathrm{ad}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})^{\mathrm{taft}} \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})^{\mathrm{taft}}.$$

Remark 1.4.1.6. Theorem 1.4.1.5 can be interpreted as a rectification type statement. Indeed, specifying a local $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structure on $\mathcal{X}$ consists in specifying a functor $\mathcal{O} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{X}$ satisfying the admissibility conditions of Theorem 2.1.0.2. A priori, one would expect that the required amount of higher coherence data for $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structures should be considerably more complex than the higher coherence data specifying the underlying algebra object $\mathcal{O}^{\mathrm{alg}}$. Theorem 1.4.1.5 imply that the difference can be measured by the given of an adic topology on the ordinary ring object $\pi_0(\mathcal{O}^{\mathrm{alg}})$.

Moreover, morphisms between local structures correspond to morphisms of functors $\mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{X}$ which satisfy the local condition in Theorem 2.1.0.2. Theorem 1.4.1.5 imply that these amount to the same higher coherent data as specifying continuous adic morphisms between derived commutative k°-algebras on $\mathcal{X}$. The latter morphisms correspond to morphisms in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ satisfying a continuity condition which can be verified directly at the level of π_0.

Definition 1.4.1.7. Let $\mathsf{X} \coloneqq (\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ be a $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured ∞-topos. We say that X is a *derived k°-adic Deligne-Mumford stack* if the k°-adic locally ringed ∞-topos $\mathsf{X}^{\mathrm{ad}} \coloneqq (\mathcal{X}, \mathcal{O}^{\mathrm{ad}})$ is a derived formal Deligne-Mumford stack in the sense of [Lur16, Definition 8.1.3.1].

As in [Lur16, §8] we can define a Spf-construction which will prove to be very useful for us:

Proposition 1.4.1.8. *Let $\mathrm{Spf} \colon \big(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}\big)^{\mathrm{op}} \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ be the Spf-construction. Then Spf is fully faithful and its essential image consists of those pairs $\big(\mathcal{X}, \mathcal{O}\big)$ such that $\big(\mathcal{X}, \mathcal{O}^{\mathrm{ad}}\big)$ is equivalent to an affine derived k°-adic Deligne-Mumford stack as in* [Lur16, §8].

Moreover, as in the derived k-analytic setting we can show that the category of ordinary k°-adic Deligne-Mumford stacks can be realized as a full subcategory of dfSch°_k via the following construction:

Construction 1.4.1.9. Let $\mathsf{X} \in \mathrm{fDM}_{k^\circ}$ denote an ordinary k°-adic Deligne-Mumford stack. Consider the étale site $\mathsf{X}_{\text{ét}}$ and the hypercompletion $\mathcal{X} := \mathrm{Shv}_{\text{ét}}(\mathsf{X})$ of the 1-localic ∞-topos of étale sheaves on X. We can define a $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structure, $\mathcal{O} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{X}$ on X as follows: given $V \in \mathcal{T}_{\mathrm{ad}}(k^\circ)$ we associate it the sheaf $\mathcal{O}(V)$ given on objects by the formula

$$U \in \mathsf{X}_{\text{ét}} \mapsto \mathrm{Map}_{\mathrm{fDM}_{k^\circ}}(U, V).$$

Moreover, the $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$-sheaf $\mathcal{O}(\mathfrak{A}^1_{k^\circ})$ corresponds to the usual sheaf of k°-adic global section on X.

Fortunately, we are now in position to define a rigidification functor: consider the transformation of pregeometries $(-)^{\mathrm{rig}} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{T}_{\mathrm{an}}(k)$ given by restricting the usual rigidification functor to the category $\mathcal{T}_{\mathrm{ad}}(k^\circ)$. Precomposition along $(-)^{\mathrm{rig}} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{T}_{\mathrm{an}}(k)$ induces a functor

$$(-)^+ \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \tag{1.4.1.2}$$

[Lur11c, Theorem 2.1] implies that the functor $(-)^+$ displayed in (1.4.1.2) admits a right adjoint

$$(-)^{\mathrm{rig}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$$

which is the natural candidate for a derived rigidification functor. Indeed one has the following results:

Proposition 1.4.1.10. *The functor $(-)^{\mathrm{rig}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ sends derived k°-adic Deligne-Mumford stacks to derived k-analytic spaces, i.e. more specifically $(-)^{\mathrm{rig}}$ restricts to a well defined functor of ∞-categories*

$$(-)^{\mathrm{rig}} \colon \mathrm{dfDM}_{k^\circ} \to \mathrm{dAn}_k.$$

The following proposition implies that the restriction of $(-)^{\mathrm{rig}} \colon \mathrm{dfDM}_{k^\circ} \to \mathrm{dAn}_k$ to the category of ordinary k°-adic schemes coincide with the usual rigidification functor.

Proposition 1.4.1.11. *Let $\mathsf{X} \in \mathrm{dfDM}_{k^\circ}$ be a derived k°-adic Deligne-Mumford stack which is equivalent to an ordinary k°-adic scheme. Then $\mathsf{X}^{\mathrm{rig}}$ coincides with usual rigidification functor for ordinary k°-adic schemes.*

The following two results are direct generalizations of both Theorem 1.3.3.1 and **??** 1 to the derived setting:

Theorem 1.4.1.12. *Let $X \in \mathrm{dAn}_k$ be a quasi-paracompact and quasi-separated derived k-analytic space. Then there exists $\mathsf{X} \in \mathrm{dfDM}^\circ_k$ such that one has an equivalence*

$$\mathsf{X}^{\mathrm{rig}} \simeq X$$

in the ∞-category dAn_k.

Theorem 1.4.1.13 (Derived Raynaud localization theorem)**.** *The rigidification functor*

$$(-)^{\mathrm{rig}} \colon \mathrm{dfSch}_{k^\circ} \to \mathrm{dAn}_k$$

factors through the localization ∞-category $\mathrm{dfSch}_{k^\circ}[S^{-1}]$ of dfSch_{k°, where S denotes the class of admissible blow ups and generally strong morphisms. Moreover, it induces an equivalence of ∞-categories

$$\mathrm{dfSch}_{k^\circ}[S^{-1}] \simeq \mathrm{dAn}'_k$$

where dAn'_k denotes the full subcategory of dAn_k spanned by quasi-paracompact and quasi-separated derived k-analytic spaces.

As a corollary we obtain:

Corollary 1.4.1.14. *Let $f \colon X \to Y$ be a morphism in the ∞-category dAn_k. Then we can find a morphism $\mathsf{f} \colon \mathsf{X} \to \mathsf{Y}$ in dfDM_{k° such that one has an equivalence*

$$(\mathsf{f})^{\mathrm{rig}} \simeq f$$

in the ∞-category of morphisms $\mathrm{dAn}_k^{\Delta^1}$.

The results on the existence of derived formal models, namely Theorem 6.2.3.15 and Theorem 3.4.4.10 have found applications so far. In the next chapter we cover certain of these applications:

1.4.2 Derived k-analytic Hilbert moduli stack

The contents of this chapter were proven in a joint work with M. Porta.

Let X be a proper variety. We can associate to X its Hilbert scheme, denoted $\mathrm{Hilb}(X)$. $\mathrm{Hilb}(X)$ is defined via its functor of points. Roughly, $\mathrm{Hilb}(X)$ parametrizes closed subschemes of X flat over the base. More precisely, $\mathrm{Hilb}(X)$ represents the functor

$$\mathrm{Hilb}(X) \colon \mathrm{Sch}_k \to \mathrm{Set},$$

which associates to a scheme $S \in \mathrm{Sch}_k$ the set of closed subschemes of $X \times S$ which are flat over S. When $X = \mathbb{P}^n$ we recover the usual Hilbert scheme $\mathrm{Hilb}(n)$ parametrizing closed subschemes of the projective n-space $\mathbb{P}^n$. The moduli scheme $\mathrm{Hilb}(X)$ plays an important role in many representability statements, including an important role in geometric invariant theory. It would thus be desirable to extend the construction of $\mathrm{Hilb}(X)$ to the k-analytic setting.

This was achieved in [CG16], in the ordinary setting. The authors prove that given a separated k-analytic space X, one can associate it a k-analytic Hilbert space, $\mathrm{Hilb}^{\mathrm{an}}(X)$ which parametrizes flat families of closed subschemes of X.

However, the requirement of flatness in the above definition is restrictive, both in the algebraic and k-analytic settings. One would like to not only parametrize flat families of closed subschemes of X but *all* families of closed subschemes of X. A possible way to deal with this issue is to consider a natural derived enhancement of $\mathrm{Hilb}(X)$, namely the derived Hilbert stack $\mathrm{RHilb}(X)$. The representability of $\mathrm{RHilb}(X)$ as a geometric stack has been establish in the context of derived algebraic geometry via the Lurie-Artin representability theorem [Lur12a, Theorem 3.2.1].

[PY17a, Theorem 7.1] provides a derived k-analytic analogue of Lurie-Artin representability theorem. Therefore, one could hope that the derived k-analytic Hilbert space $\mathrm{RHilb}^{\mathrm{an}}(X)$ could be shown to be representable directly using [PY17a, Theorem 7.1].

However, via this approach we are allowed to prove the existence of an analytic cotangent complex of $\mathrm{RHilb}^{\mathrm{an}}(X)$ only at points $f \colon S \to \mathrm{RHilb}^{\mathrm{an}}(X)$ corresponding to families of closed subschemes of $j \colon Z \hookrightarrow X \times S$ which are of finite presentation, in the derived setting. However, not all points of $\mathrm{RHilb}^{\mathrm{an}}(X)$ satisfy this condition, we are typically interested with families which are *almost* of finite presentation.

Definition 1.4.2.1. We denote by $\mathrm{dSt}\big(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$ the ∞-category of geometric stacks with respect to the derived k-analytic geometric context. We refer to objects of $\mathrm{dSt}\big(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$ simply as *derived k-analytic stacks.*

Using the techniques developped in [Ant18b] together with [PY17a, Theorem 7.1] we are then able to prove the following main result:

Theorem 1.4.2.2. *Let X be a separated ordinary k-analytic space. The derived k-analytic Hilbert stack $\mathrm{RHilb}^{\mathrm{an}}(X)$ is representable by a derived k-analytic stack.*

In order to prove Theorem 1.4.2.2 the authors had to generalize certain results related to the existence of formal models for certain classes of morphisms between derived k-analytic spaces. Namely, we generalized to the derived setting a classical result of Bosch-Lutkebohmer concerning liftings of flat morphisms $f \colon X \to Y$ to flat morphisms of formal models.

Theorem 1.4.2.3. *Let $f \colon X \to Y$ be a flat morphism of derived k-analytic spaces. Then there exists a flat morphism $\mathsf{f} \colon \mathsf{X} \to \mathsf{Y}$ of derived k°-adic Deligne-Mumford stacks such that there exists an equivalence*

$$f \simeq \mathsf{f}^{\mathrm{rig}}$$

in the ∞-category of morphisms $\mathrm{dAn}_k^{\Delta^1}$.

The main difficulty proving Theorem 1.4.2.3 was to verify the strong condition for derived morphisms at the formal level. More precisely, the main obstruction to apply Theorem 3.4.4.10 directly was to verify the condition

$$\pi_i\big(\mathsf{f}^* \mathcal{O}_{\mathsf{Y}}\big) \simeq \pi_i\big(\mathcal{O}_{\mathsf{X}}\big)$$

for $i > 0$, where $\mathsf{f} \colon \mathsf{X} \to \mathsf{Y}$ is a generic formal model for $f \colon X \to Y$. This was accomplished by a reasoning at the level of Postnikov towers plus proving refined results concerning liftings of formal models for almost coherent modules on X. At the heart of such liftings results is the following descent result:

Theorem 1.4.2.4. *Let* $\mathrm{dfSch}_{k^\circ}^{\mathrm{taft,qcqs}}$ *denote the* ∞*-category of qcqs topologically almost of finite presentation derived* k°*-adic schemes. Then the functor*

$$\mathrm{Coh}^+_{\mathrm{loc}} \colon \big(\mathrm{dfSch}_{k^\circ}^{\mathrm{taft,qcqs}}\big)^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{st}}$$

is a hypercomplete sheaf for the Zariski topology.

Theorem 1.4.2.4 is a generalization of [HPV16a, Theorem 7.3] to the case where the base is assumed to be derived. With Theorem 1.4.2.4 at hand one is allowed to prove the following important results:

Proposition 1.4.2.5. *Let* $X \in \mathrm{dAn}_k$ *be a derived* k*-analytic space. Let* $\mathcal{F} \in \mathrm{Coh}^{\mathrm{b}}(X)$ *be a bounded almost perfect module over* X*. Then for any formal model* $\mathsf{X} \in \mathrm{dfDM}_{k^\circ}$ *of* X*, there exists* $\mathcal{G} \in \mathrm{Coh}^{\mathrm{b}}(\mathsf{X})$ *such that* $\mathcal{G}^{\mathrm{rig}} \simeq \mathcal{F}$ *in the* ∞*-category* $\mathrm{Coh}^+(X)$*. Moreover, the full subcategory of* $\mathrm{Coh}^{\mathrm{b}}(\mathsf{X}) \times_{\mathrm{Coh}^{\mathrm{b}}(X)} \mathrm{Coh}^{\mathrm{b}}(X)_{/\mathcal{F}}$ *spanned by formal models for* $\mathcal{F}$ *is filtered.*

Proposition 1.4.2.6. *Let* $X \in \mathrm{dAn}_k$ *be a derived* k*-analytic space. Let* $f \colon \mathcal{F} \to \mathcal{G}$ *be a morphism of almost coherent modules in the* ∞*-category* $\mathrm{Coh}^+(X)$*. Then for every formal model* $\mathsf{X} \in \mathrm{dfDM}_{k^\circ}$ *of* X *and a choice of formal models* $\mathcal{F}'$ *and* $\mathcal{G}'$ *for* $\mathcal{F}$ *and* $\mathcal{G}$*, respectively, there exists a sufficiently large* $n \geq 0$ *such that the multiplication* $t^n f \colon \mathcal{F} \to \mathcal{G}$ *admits a formal model of the form*

$$g \colon \mathcal{F}' \to \mathcal{G}'$$

in the ∞*-category* $\mathrm{Coh}^+(\mathsf{X})$*.*

As an application of Theorem 1.4.2.3 we obtain the following:

Proposition 1.4.2.7. *Let* $f \colon X \to Y$ *be a flat and proper morphism of derived* k*-analytic spaces. Then we have:*

(i) The functor $f^* \colon \mathrm{Coh}^+(Y) \to \mathrm{Coh}^+(X)$ *admits a left adjoint*

$$f_+ \colon \mathrm{Coh}^+(X) \to \mathrm{Coh}^+(Y).$$

(ii) Let $\mathsf{f} \colon \mathsf{X} \to \mathsf{Y}$ *be a formal model for* f*, whose existence is guaranteed by Theorem 1.4.1.14. Define* $\omega_{X/Y} \coloneqq \omega_{\mathsf{X}/\mathsf{Y}}^{\mathrm{rig}}$*, the* analytic dualizing sheaf. *Then given* $\mathcal{F} \in \mathrm{Coh}^+(X)$ *we have a canonical equivalence*

$$f_+(\mathcal{F}) \simeq f_*(\mathcal{F} \otimes \omega_{X/Y})$$

in the ∞*-category* $\mathrm{Coh}^+(Y)$*.*

(iii) The functor $f^! \colon \mathrm{Coh}^+(Y) \to \mathrm{Coh}^+(Y)$ *given on objects by the formula*

$$\mathcal{F} \in \mathrm{Coh}^+(Y) \mapsto f^!(\mathcal{F}) \coloneqq f^*(\mathcal{F} \otimes \omega_{X/Y}) \in \mathrm{Coh}^+(X)$$

is a right adjoint for the functor $f_* \colon \mathrm{Coh}^+(X) \to \mathrm{Coh}^+(Y)$*.*

1.5 Moduli of continuous ℓ-adic representations of a profinite group

1.5.1 Moduli of continuous ℓ-representations

Part of the present thesis was to study at length the moduli functor $\mathrm{LocSys}_{\ell,n}(X)$ parametrizing continuous ℓ-adic representations of $\pi_1^{\text{ét}}(X)$, where X is a proper and smooth variety over an algebraically closed field of positive characteristic $p > 0$.

In this context, the formal moduli problem considered in Theorem 2.1.0.15 should correspond to the formal neighborhood of the moduli $\mathrm{LocSys}_{\ell,n}(X)$, described in Section 1.3.2. Furthermore, Theorem 3.4.4.10 allow us to define general continuous representations

$$\rho \colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n\big(\Gamma(Z)\big),$$

where $Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}$ is not necessarily the spectrum of an Artinian derived $\mathbb{Q}_\ell$-algebra. Notice that when Z is discrete we can describe and study the moduli problem associated to $\mathrm{LocSys}_{\ell,n}(X)$ directly. Consider the following moduli functor $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \colon \mathrm{Afd}_k^{\mathrm{op}} \to \mathrm{Set}$ given on objects via the formula

$$A \in \mathrm{Afd}_k^{\mathrm{op}} \mapsto \mathrm{Hom}_{\mathrm{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{BGL}_n(A)\right) \in \mathrm{Set},$$

where $\mathrm{Hom}_{\mathrm{cont}}\left(\pi_1^{\text{ét}}(X), \mathrm{BGL}_n(A)\right)$ denotes the set of continuous group homomorphisms $\pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A)$. Here we consider $\mathrm{GL}_n(A)$ as a topological group via the canonical topology on A induced by a choice of norm compatible with the non-trivial valuation on k. We have the following fundamental result:

Theorem 1.5.1.1. *Let X be a smooth and proper scheme over an algebraically closed field. The functor*

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \colon \mathrm{Afd}_k^{\mathrm{op}} \to \mathrm{Set}$$

defined above is representable by a strict k-analytic space.

The strategy of the proof of Theorem 1.5.1.1 follows closely the scheme of Simpson's proof the representability of the moduli of discrete group homomorphisms for a discrete group G, see [Sim94a, §1]. However, the continuous case is more involved as both the topologies on $\pi_1^{\text{ét}}(X)$ and on $\mathrm{GL}_n(A)$ differ. More precisely, $\pi_1^{\text{ét}}(X)$ is a profinite group whereas the topology on $\mathrm{GL}_n(A)$ is far from being profinite. Nonetheless, every formal model A_0 over k° for A provides an open subgroup

$$\mathrm{GL}_n(A_0) \subseteq \mathrm{GL}_n(A)$$

which is a pro-group, i.e. the topology on $\mathrm{GL}_n(A_0)$ is induced via the canonical isomorphism of groups

$$\mathrm{GL}_n(A_0) \simeq \lim_{m \geq 0} \mathrm{GL}_n(A_{0,m}).$$

In this case, every continuous group homomorphism $\rho \colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A)$ admits an open subgroup $U \leq \pi_1^{\text{ét}}(X)$ such that $\rho_{|U} \colon U \to \mathrm{GL}_n(A)$ factors through the inclusion $\mathrm{GL}_n(A_0) \subseteq \mathrm{GL}_n(A)$. Fortunately, the study of group homomorphisms

$$\rho_{|U} \colon U \to \mathrm{GL}_n(A_0) \tag{1.5.1.1}$$

is easier than our original problem. Therefore, it is useful to study the moduli functor $F_{\pi_1^{\text{ét}}(X)} \colon \mathcal{C}\mathrm{Alg}_{k^\circ}^{\heartsuit,\mathrm{ad}} \to \mathrm{Set}$ parametrizing continuous group homomorphisms

$$\tilde{\rho} \colon U \to \mathrm{GL}_n(A_0), \quad A_0 \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\heartsuit,\mathrm{ad}}.$$

Nevertheless, the topology on $\mathrm{GL}_n(A_0)$ is almost never profinite, except when A_0 is a finite extension of k°, therefore some care is needed when describing the above functor by means of algebraic data. Even though $F_{\pi_1^{\text{ét}}(X)}$ is not representable, Theorem 1.5.1.1 implies that k-analytic analogue $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ is so.

Remark 1.5.1.2. The k-analytic space $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ admits a natural action of the *analytification* of the general linear group scheme, $\mathbf{GL}_n^{\mathrm{an}}$. This action can be described via the morphism of k-analytic spaces

$$\mathbf{GL}_n^{\mathrm{an}} \times \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \to \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X), \tag{1.5.1.2}$$

defined by the formula

$$\left((g, \rho) \in \mathbf{GL}_n^{\mathrm{an}}(A) \times \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)(A)\right) \mapsto \left(g\rho g^{-1} \in \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)(A)\right).$$

One would like to define $\mathrm{LocSys}_{\ell,n}(X)$ as the quotient k-analytic space obtained as the quotient of $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ by the conjugation action of $\mathbf{GL}_n^{\mathrm{an}}$ described in Theorem 1.5.1.2. However, in the k-analytic setting there is no solid theory of geometric invariant theory as opposed to the context of algebraic geometry. For this reason we prefer to adopt the language of k-analytic stacks for a reasonable definition of $\mathrm{LocSys}_{\ell,n}(X)$.

Definition 1.5.1.3. We denote by $\mathrm{LocSys}_{\ell,n}(X)$ the geometric realization of the simplicial objects

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \left(\mathbf{GL}_n^{\mathrm{an}}\right)^2 \times \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathbf{GL}_n^{\mathrm{an}} \times \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \rightrightarrows \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \tag{1.5.1.3}$$

computed in the ∞-category of pre-sheaves on Afd_k, denoted $\mathrm{PShv}\big(\mathrm{Afd}_k\big) \coloneqq \mathrm{Fun}\left(\mathrm{Afd}_k^{\mathrm{op}}, \mathcal{S}\right)$.

Remark 1.5.1.4. The moduli functor $\mathrm{LocSys}_{\ell,n}(X)\colon \mathrm{Afd}_k^{\mathrm{op}} \to \mathcal{S}$ satisfies descent with respect to the étale topology on $(\mathrm{Afd}_k, \tau_{\text{ét}})$. Furthermore, it follows by construction that given $A \in \mathrm{Afd}_k$ the space

$$\mathrm{LocSys}_{\ell,n}(X)(A) \in \mathcal{S}$$

is equivalent to the 1-groupoid of continuous representations $\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A)$. Therefore, $\mathrm{LocSys}_{\ell,n}(X)$ parametrizes continuous ℓ-adic group representations of $\pi_1^{\text{ét}}(X)$ or equivalently rank n pro-étale local systems on $X_{\text{ét}}$ as the following result illustrates:

Proposition 1.5.1.5. *Let X be a smooth scheme over an algebraically closed field. Then one has a natural equivalence of spaces*

$$\mathrm{LocSys}_{\ell,n}(X)(A) \simeq \mathrm{Loc}_{n,\text{pro-ét}}(X)(A),$$

where the right hand side denotes the 1*-groupoid of rank n pro-étale A-linear local systems on X.*

Theorem 1.5.1.1 entails through a formal reasoning the following main result:

Theorem 1.5.1.6. *The moduli functor* $\mathrm{LocSys}_{\ell,n}(X)\colon \mathrm{Afd}_k^{\mathrm{op}} \to \mathcal{S}$ *is representable by a geometric stack with respect to the geometric context* $\big(\mathrm{Afd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$.

Notation 1.5.1.7. We refer to geometric stacks with respect with the geometric context $\big(\mathrm{Afd}_k, \tau_{\acute{e}t}, \mathrm{P}_{\mathrm{sm}}\big)$ simply as *k-analytic stacks.*

Theorem 1.5.1.6 provides a positive answer to condition (i) in §1.5. However, we have a very few understanding of the geometry of $\mathrm{LocSys}_{\ell,n}(X)$ and a reasonable cohomology theory for $\mathrm{LocSys}_{\ell,n}(X)$ is still lacking. We will try to amend these questions by constructing a natural derived structure on $\mathrm{LocSys}_{\ell,n}(X)$.

1.5.2 Derived enhancement of $\mathrm{LocSys}_{\ell,n}(X)$

Let X be a geometrically connected proper and smooth scheme over an algebraically closed field. The moduli stack $\mathrm{LocSys}_{\ell,n}$ parametrizes varying families of rank n ℓ-adic pro-étale local systems on X. Moreover, Theorem 1.5.1.6 states that $\mathrm{LocSys}_{\ell,n}(X)$ is representable by a k-analytic stack. In this section we will further attempt to answer the following question:

Question 1.5.2.1. *Does* $\mathrm{LocSys}_{\ell,n}(X)$ *admits an analytic cotangent complex which classifies deformations of pro-étale local systems? And if so, can we compute it explicitly?*

Theorem 1.5.2.1 is basically a question on the existence of a canonical derived structure on $\mathrm{LocSys}_{\ell,n}(X)$. In order to attempt to answer to Theorem 1.5.2.1 one needs to allow derived coefficients in the definition of $\mathrm{LocSys}_{\ell,n}(X)$. More precisely, we nedd to extend the functor $\mathrm{LocSys}_{\ell,n}(X)$ to a functor

$$\mathrm{LocSys}_{\ell,n}(X)\colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S},$$

so one needs a reasonable defintion of continuous representations with derived coefficients.

This questions has already been partially dealt in Theorem 2.1.0.15. However, to fully answer this question one needs to make a considerable technical detour on the theory of enriched ∞-categories. We start by observing that the étale fundamental group is too poor, in general, to classify ℓ-adic lisse sheaves on $X_{\text{ét}}$ with derived coefficients. One should consider instead the étale homotopy type $\mathrm{Sh}^{\text{ét}}(X)$ of X and parametrize *continuous ℓ-adic representations of* $\mathrm{Sh}^{\text{ét}}(X)$.

Let $Z \in \mathrm{dAfd}_k$, thanks to Theorem 3.4.4.10 there exists a formal model $\mathrm{Spf}\,(A_0)$ for Z. One could try to define continuous representations of homotopy types

$$\rho\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n\big(\Gamma(Z)\big)$$

as in Theorem 2.1.0.15 by considering the colimit over all such formal models for Z, i.e. by defining

$$\mathrm{Map}_{\mathrm{cont}}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{BGL}_n\big(\Gamma(Z)\big)\big) := \operatorname*{colim}_{A_0} \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}))}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{BGL}_n(A_0)\big) \in \mathcal{S}. \tag{1.5.2.1}$$

By considering $\mathrm{BGL}_n(A_0)$ as a group-like pro-object in the ∞-category $\mathcal{S}$ via the equivalence

$$A_0 \simeq \lim_{m \geq 0} A_{0,m}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}$. Theorem 3.4.4.10 implies that such an association is functorial in $Z \in \mathrm{dAfd}_k$. Even though formula (1.5.2.1) is a good approximation for the space of continuous ℓ-adic representations of $\mathrm{Sh}^{\text{ét}}(X)$ it is certainly not a correct definition. In order to give a more reasonable construction one need to make a technical detour on the theory of $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$-enriched ∞-categories.

Construction 1.5.2.2. Let $Z \in \mathrm{Afd}_k$ be a derived k-affinoid space and consider the ∞-category $\mathrm{Perf}(Z)$ of perfect complexes on Z. Fix a formal model $\mathrm{Spf}(A_0) \in \mathrm{dfSch}_{k^\circ}$ for Z. The ∞-categories $\mathrm{Perf}(Z)$ and $\mathrm{Perf}(A_0)$ are related. The derived rigidification theorem introduced in §3 induces a rigidification functor at the level of ∞-categories of perfect modules

$$(-)^{\mathrm{rig}} \colon \mathrm{Perf}(A_0) \to \mathrm{Perf}(Z).$$

Moreover, as a first approximation we can think of $\mathrm{Perf}(Z)$ as the idempotent completion of the Verdier quotient $\mathrm{Perf}(A_0)/\mathrm{Perf}_{\mathrm{nil}}(A_0)$ where $\mathrm{Perf}_{\mathrm{nil}}(A_0) \subseteq \mathrm{Perf}(A_0)$ denotes the full subcategory spanned by $\mathfrak{m}$-torsion perfect A_0-modules. In other words, $\mathrm{Perf}(Z)$ is roughly equivalent to the Verdier quotient $\mathrm{Perf}(A_0)/\mathrm{Perf}_{\mathrm{nil}}(A_0)$ computed in the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{st,id\text{-}comp}}$.

The crucial observation is that the ∞-category $\mathrm{Perf}(A_0)$ is naturally enriched over the ∞-category $\mathrm{Pro}\big(\mathrm{Sp}_\ell\big)$ of pro-objects on the ∞-category of *ℓ-nilpotent spectra*, $\mathrm{Sp}_\ell := \mathrm{Sp} \otimes_S S/\ell$, where $S \in \mathrm{Sp}$ denotes the sphere spectrum. By a formal argument, detailed in [Ant17a], we can then consider $\mathrm{Perf}(Z)$ as an ∞-category naturally enriched over the ∞-category of ind-pro-spaces, $\mathrm{Ind}\big(\mathrm{Pro}(\mathcal{S})\big)$. Moreover, the formula

$$Z \in \mathrm{Afd}_k^{\mathrm{op}} \mapsto \mathrm{Perf}(Z) \in \mathcal{E}\mathcal{C}\mathrm{at}_\infty,$$

is functorial in $Z \in \mathrm{Afd}_k^{\mathrm{op}}$. Furthermore, there is a canonical inclusion functor

$$\mathrm{Pro}\big(\mathcal{S}^{\mathrm{fc}}\big) \hookrightarrow \mathrm{Pro}\big(\mathcal{S}\big) \hookrightarrow \mathcal{E}\mathcal{C}\mathrm{at}_\infty,$$

where $\mathcal{E}\mathcal{C}\mathrm{at}_\infty$ denotes the ∞-category of $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$-enriched ∞-categories. Therefore, it makes sense to consider the ∞-category of enriched functors

$$\mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(Z)\big) \in \mathcal{C}\mathrm{at}_\infty. \tag{1.5.2.2}$$

Moreover, the association displyed in (1.5.2.2) is functorial in Z. We can thus define a functor

$$\mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(-)\big) \colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$$

given on objects by the formula

$$Z \in \mathrm{Afd}_k^{\mathrm{op}} \mapsto \mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(Z)\big) \in \mathcal{E}\mathcal{C}\mathrm{at}_\infty.$$

As X is geometrically connected, the profinite space $\mathrm{Sh}^{\text{ét}}(X) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ is connected. For this reason, the mapping space

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})}\big(*, \mathrm{Sh}^{\text{ét}}(X)\big) \in \mathcal{S}$$

is contractible. As a consequence, there exists a unique, up to contractible indeterminacy, morphism

$$\iota \colon * \to \mathrm{Sh}^{\text{ét}}(X)$$

in the ∞-category $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$. Precomposition along ι induces a functor of ∞-categories

$$\mathrm{ev} \colon \mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(Z)\big) \to \mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(*, \mathrm{Perf}(Z)\big) \simeq \mathrm{Perf}(Z) \in \mathcal{C}\mathrm{at}_\infty.$$

The important observation is illustrated by the following lemma:

Lemma 1.5.2.3. *Let $Z \in \mathrm{Afd}_k^{\mathrm{op}}$ be a derived k-affinoid space which we suppose further to be n-truncated for a given integer $n \geq 1$. Let $M \in \mathrm{Perf}(Z)$ be a perfect module on Z and let. Then the fiber of the functor*

$$\mathrm{ev}\colon \mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(Z)\big) \to \mathrm{Perf}(Z)$$

at $M \in \mathrm{Perf}(Z)$ is naturally equivalent to the mapping space

$$\mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathcal{T}\mathrm{op}_{\mathrm{na}})}\big(\Omega \mathrm{Sh}^{\text{ét}}(X), \mathcal{E}\mathrm{nd}(M)\big) \in \mathcal{S}$$

where $\mathcal{E}\mathrm{nd}(M)$ denotes the ind-pro-endomorphism space of M.

Remark 1.5.2.4. Theorem 1.5.2.3 implies that $\mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(-)\big)\colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ parametrizes continuous representations of $\mathrm{Sh}^{\text{ét}}(X)$ with values in $\mathbb{E}_1$-monoid objects in ind-pro-endomorphisms spaces. That is to say, we take into account both the profinite structure on $\mathrm{Sh}^{\text{ét}}(X)$ and the ind-pro-structure on $\mathrm{End}(M)$.

Definition 1.5.2.5. We define the moduli functor of *perfect continuous ℓ-adic representations of* $\mathrm{Sh}^{\text{ét}}(X)$ as

$$\mathrm{PerfSys}_\ell(X) \coloneqq (-)^{\simeq} \circ \mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(-)\big)$$

which lives naturally in the ∞-category $\mathrm{Fun}\big(\mathrm{dAfd}_k^{\mathrm{op}}, \mathcal{S}\big)$.

Definition 1.5.2.6. Let $\mathrm{RLocSys}_{\ell,n}(X)\colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S}$ denote the substack of $\mathrm{PerfSys}_\ell(X)$ spanned by rank n objects, i.e.

$$\mathrm{RLocSys}_{\ell,n}(X) \coloneqq \mathrm{ev}^{-1}\big(\mathcal{C}_n(Z)\big),$$

where $\mathcal{C}_n(Z) \subseteq \mathrm{Perf}(Z)$ denotes the full subcategory spanned by rank n objects in $\mathrm{Perf}(Z)$.

Whenever Z is discrete and $M \simeq \Gamma(Z)^n$ we recover the space of continuous representations

$$\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n\big(A\big), \quad A \simeq \Gamma(Z).$$

Lemma 1.5.2.7. *Let $Z \coloneqq \mathrm{Sp}(A) \in \mathrm{Afd}_k$ be an ordinary k-affinoid space. Then we have a natural equivalence of spaces*

$$\mathrm{LocSys}_{\ell,n}(X)(A) \simeq \mathrm{RLocSys}_{\ell,n}(X)(A).$$

Notation 1.5.2.8. Following our convention, we will denote $\mathrm{RLocSys}_{\ell,n}(X)$ by $\mathrm{LocSys}_{\ell,n}(X)$.

As promised we state an explicit computation of the cotangent complex of $\mathrm{LocSys}_{\ell,n}(X)$

Proposition 1.5.2.9. *Let $Z \in \mathrm{dAfd}_k$ and let $\rho \in \mathrm{LocSys}_{\ell,n}(X)(Z)$ be a continuous ℓ-adic representation. Then $\mathrm{LocSys}_{\ell,n}(X)$ admits an analytic cotangent complex at ρ and we have the following natural equivalence*

$$\mathbb{L}^{\mathrm{an}}_{\mathrm{LocSys}_{\ell,n}(X),\rho} \simeq C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)^\vee[-1]$$

in the derived ∞-category Mod_k.

Theorem 1.5.2.9 implies that deformations of ρ are classified by the étale homology complex $C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)^\vee[-1]$. Moreover, Porta-Yu Yue representability theorem, [PY17a, Theorem 7.1] implies the following:

Theorem 1.5.2.10. *The derived moduli stack $\mathrm{LocSys}_{\ell,n}(X)\colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S}$ is representable by a geometric stack with respect to the geometric context $\big(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$, i.e. $\mathrm{LocSys}_{\ell,n}(X)$ is representable by derived k-analytic stack.*

Construction 1.5.2.11. Let $\overline{\rho}\colon \pi_1^{\text{ét}}(X) \to \mathrm{BGL}_n(\mathbb{F}_{\ell^r})$ be a continuous representation. We can consider at the formal moduli problem $\mathrm{Def}_{\overline{\rho}}\colon \mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_{\mathbb{F}_{\ell^r}} \to \mathcal{S}$ given on objects by the formula

$$A \in \mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_{\mathbb{F}_{\ell^r}} \mapsto \mathrm{LocSys}_{\ell,n}(X)(A) \times_{\mathrm{LocSys}_{\ell,n}(X)(\mathbb{F}_{\ell^r})} \{\overline{\rho}\} \in \mathcal{S}.$$

The functor $\mathrm{Def}_{\overline{\rho}}$ paramatrizes continuous deformations of $\overline{\rho}$ with values in small derived $\mathbb{F}_{\ell^r}$-algebras. Such functor satisfies the conditions of Lurie-Schlessinger Theorem. Therefore it is pro-representable by a Noetherian derived $W(\mathbb{F}_{\ell^r})$-algebra, $A_{\overline{\rho}}$, augmented over $\mathbb{F}_{\ell^r}$ and complete with respect with the maximal ideal

$$\mathfrak{m}_{\overline{\rho}} := \ker\big(\pi_0(A_{\overline{\rho}}) \to \mathbb{F}_{\ell^r}\big).$$

Let $k := \mathrm{Frac}(W(\mathbb{F}_{\ell^r}))$. By construction, we can consider $\mathrm{Def}_{\overline{\rho}}$ as a functor defined on the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. Moreover, such functor is representable by $\mathrm{Spf}(A_{\overline{\rho}})$, which is a locally admissible derived k°-adic scheme. In this case, we can take its rigidification

$$\mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}} \in \mathrm{dAn}_k,$$

which is a derived k-analytic stack. We have a canonical morphism $\mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}} \to \mathrm{LocSys}_{\ell,n}(X)$ which is roughly described by sending a continuous deformation of $\overline{\rho}$ to its corresponding continuous ℓ-adic representation. Varying $\overline{\rho}\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\mathbb{F}_{\ell^r})$ we obtain a canonical morphism

$$\coprod_{\overline{\rho}} \mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}} \to \mathrm{LocSys}_{\ell,n}(X) \tag{1.5.2.3}$$

in the ∞-category $\mathrm{dSt}\big(\mathrm{dAfd}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$ of derived k-analytic stacks.

The derived structure on $\mathrm{LocSys}_{\ell,n}(X)$ allow us to prove the following geometric result:

Proposition 1.5.2.12. *The canonical morphism displayed in* (1.5.2.3) *is an étale admissible inclusion of sub-analytic derived k-analytic stacks.*

One could ask if the morphism (1.5.2.3) is an equivalence of geometric stacks. The following example illustrates that this is not is the case in general:

Example 1.5.2.13. Let $G = \mathbb{Z}_p$ and $A = \mathbb{Q}_p\langle T\rangle$ the Tate algebra in one variable. Consider the continuous representation $\rho\colon \mathbb{Z}_p \to \mathbb{Q}_p\langle T\rangle$ determined by the association

$$1 \in \mathbb{Z}_p \mapsto \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}.$$

Then $\rho \in \mathrm{LocSys}_{p,2}(G)(\mathbb{Q}_p\langle T\rangle)$ but does not belong to the disjoint union $\coprod_{\overline{\rho}} \mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}} \subseteq \mathrm{LocSys}_{\ell,n}(G)$.

However, as Theorem 1.5.2.12 suggests the derived k-analytic stack $\mathrm{LocSys}_{\ell,n}(X)$ is highly disconnected. It would be desirable to have a way to glue together the formal neighborhoods $\mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}}$ together. One could state it more precisely as a conjecture:

Conjecture 1.5.2.14. *There exists a (possibly ind-)derived k-analytic stack $\mathrm{LocSys}^{\mathrm{gl}}_{\ell,n}(X)$ and a morphism of derived k-analytic stacks*

$$\pi\colon \mathrm{LocSys}_{\ell,n}(X) \to \mathrm{LocSys}^{\mathrm{gl}}_{\ell,n}(X)$$

such that π is an equivalence at closed points and it induces an equivalence of cotangent complexes at closed points. Moreover, the moduli stack $\mathrm{LocSys}^{\mathrm{gl}}_{\ell,n}(X)$ is equipped with an endomorphism, F, which is compatible with $\mathrm{Frob}^*\colon \mathrm{LocSys}_{\ell,n}(X) \to \mathrm{LocSys}_{\ell,n}(X)$, i.e. we have a commutative diagram

$$\begin{tikzcd} \mathrm{LocSys}_{\ell,n}(X) \arrow[r, "\mathrm{Frob}^*"] \arrow[d, "\pi"] & \mathrm{LocSys}_{\ell,n}(X) \arrow[d, "\pi"] \\ \mathrm{LocSys}^{\mathrm{gl}}_{\ell,n}(X) \arrow[r, "F"] & \mathrm{LocSys}^{\mathrm{gl}}_{\ell,n}(X) \end{tikzcd}$$

in the ∞-category $\mathrm{dSt}\big(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$. Moreover, $\mathrm{LocSys}^{\mathrm{gl}}_{\ell,n}(X)$ is almost of finite presentation and in particular it admits finitely many connected components.

1.5.3 Open case

When X is assumed to be a smooth scheme over $k = \overline{k}$ of positive characteristic $p \neq \ell$ its étale fundamental group $\pi_1^{\text{ét}}(X)$ is not topologically finitely generated (except if we assume X proper). In this case we have a short exact sequence of profinite groups

$$1 \to \pi_1^w(X) \to \pi_1^{\text{ét}}(X) \to \pi_1^{\text{t}}(X) \to 1$$

where $\pi_1^w(X)$ denotes the wild fundamental group of X a pro-p profinite group and $\pi_1^{\text{t}}(X)$ its tame fundamental group which is topologically finitely generated. The tame fundamental group parametrizes tamely ramified at infinity coverings of X. We cannot expect the full stack $\mathrm{LocSys}_{\ell,n}(X)$ is representable as the profinite group $\pi_1^{\text{ét}}(X)$ is too big. However, by bounding the ramification at infinity we can consider the substack

$$\mathrm{LocSys}_{\ell,n,\Gamma}(X) \hookrightarrow \mathrm{LocSys}_{\ell,n}(X)$$

which parametrizes continuous representations $\rho \colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n(A)$ such that the restriction $\rho_{|\pi_1^w(X)}$ factors through a finite quotient $q \colon \pi_1^w(X) \to \Gamma$. We have the following result

Proposition 1.5.3.1. *The moduli stack* $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$ *is representable by a derived k-analytic stack.*

1.5.4 Shifted symplectic structure

Let X be a smooth and proper scheme over a field $k = \overline{k}$ of positive characteristic $p > 0$. As X is proper, Poincaré duality for étale cohomology implies that we have a non-degenerate bilinear pairing

$$C^*_{\text{ét}}\big(X, \mathbb{Q}_\ell\big) \otimes C^*_{\text{ét}}\big(X, \mathbb{Q}_\ell\big)^\vee \to \mathbb{Q}_\ell[-2d],$$

where $d = \dim X$. When X is non-proper we should replace étale cohomology with étale cohomology with support. Thanks to the projection formula, given a continuous representation

$$\rho \colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A), \quad A \in \mathrm{Afd}_k,$$

one still obtains a non-degenerate pairing of the form

$$C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big) \otimes C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)^\vee \to \mathbb{Q}_\ell[-2d], \tag{1.5.4.1}$$

where $\mathrm{Ad}(\rho) \coloneqq \rho \otimes \rho^\vee$ denotes the adjoint representation associated to ρ. It is possible to give a more conceptual construction of the pairing introduced above as follows: the ∞-category $\mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(A)\big)$ is a rigid symmetric monoidal, that is every object is dualizable in $\mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(A)\big)$. Therefore, given ρ as above one has a natural trace morphism

$$\mathrm{tr} \colon \mathrm{Ad}(\rho) \coloneqq \rho \otimes \rho^\vee \to 1,$$

where 1 denotes a unit for the symmetric monoidal structure on $\mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(A)\big)$. As $\mathrm{Ad}(\rho) \in \mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(A)\big)$ is an $\mathbb{E}_1$-monoid object we have a canonical multiplication morphism

$$\mathrm{mult} \colon \mathrm{Ad}(\rho) \otimes \mathrm{Ad}(\rho) \to \mathrm{Ad}(\rho). \tag{1.5.4.2}$$

Both (1.5.4.1) and (1.5.4.2) imply the existence of a canonical morphism of the form

$$\mathrm{Map}\big(1, \mathrm{Ad}(\rho)\big) \otimes \mathrm{Map}\big(1, \mathrm{Ad}(\rho)\big) \xrightarrow{\mathrm{can}} \mathrm{Map}\big(1, \mathrm{Ad}(\rho) \otimes \mathrm{Ad}(\rho)\big) \xrightarrow{\mathrm{mult}} \mathrm{Map}\big(1, \mathrm{Ad}(\rho)\big) \xrightarrow{\mathrm{tr}} \mathrm{Map}\big(1, 1\big).$$

We can identify the above composite with the canonical map displayed in (1.5.4.1). Define $\mathcal{O} \colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{C}\mathrm{Alg}_k$ as the sheaf on the étale site $\big(\mathrm{dAfd}_k, \tau_{\text{ét}}\big)$ given on objects by the formula

$$Z \in \mathrm{dAfd}_k \mapsto \mathcal{O}(Z) \coloneqq \Gamma(Z) \in \mathcal{C}\mathrm{Alg}_k.$$

The canonical map $C^*_{\text{ét}}\big(X, \mathbb{Q}_\ell\big) \to \mathbb{Q}_\ell[-2d]$ in the derived ∞-category $\mathrm{Mod}_{\mathbb{Q}_\ell}$ induces by the projection formula on ℓ-adic cohomology a canonical morphism

$$C^*_{\text{ét}}\big(X, A\big) \to A[2d]$$

We obtain thus a non-degenerated pairing at the level of the cotangent complex of $\mathrm{LocSys}_{\ell,n}(X)$ at ρ

$$\omega_\rho \colon \mathbb{L}^{\mathrm{an}}_{\mathrm{LocSys}_{\ell,n}(X),\rho} \otimes \mathbb{L}^{\mathrm{an}}_{\mathrm{LocSys}_{\ell,n}(X),\rho} \to \mathcal{O}\big(\operatorname{Sp}(A)\big)$$

The results of [Toë18, §3] imply that the pairing

$$\omega \colon \bigwedge^2 \mathbb{L}^{\mathrm{an}}_{\mathrm{LocSys}_{\ell,n}(X)} \to \mathcal{O}$$

is closed, i.e. it can be realized as an element in cyclic homology $\mathrm{HC}\big(\mathrm{LocSys}_{\ell,n}(X)\big) \in \mathrm{Mod}_k$. We obtain thus the following important result:

Theorem 1.5.4.1. *The derived k-analytic stack $\mathrm{LocSys}_{\ell,n}(X) \in \mathrm{dSt}\big(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$ admits a canonical shifted symplectic structure $\omega \in \mathrm{HC}\big(\mathrm{LocSys}_{\ell,n}(X)\big)$ of degree $2-2d$. Moreover, given $\rho \in \mathrm{LocSys}_{\ell,n}(X)(Z)$ with $Z \in \mathrm{dAfd}_k$ the underlying non-degenerate pairing is equivalent to the Poincaré pairing*

$$C^*_{\text{ét}}(X, \mathrm{Ad}(\rho))[1] \otimes C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big) \to \Gamma(Z)[2-2d]$$

in ℓ-adic cohomology.

Corollary 1.5.4.2. *Let $\rho \in \mathrm{LocSys}_{\ell,n}(X)(Z)$, then the shifted symplectic form $\omega \in \mathrm{HC}\big(\mathrm{LocSys}_{\ell,n}(X)\big)$ induces an equivalence*

$$\mathbb{T}^{\mathrm{an}}_{\mathrm{LocSys}_{\ell,n}(X)} \simeq \mathbb{L}^{\mathrm{an}}_{\mathrm{LocSys}_{\ell,n}(X)}[2-2d]$$

between the tangent and cotangent complexes on $\mathrm{LocSys}_{\ell,n}(X)$.

1.6 Analytic HKR theorem

The results in this section were first study in a joint collaboration work between M. Porta and F. Petit. I thank both of them for letting me take part on the project.

1.6.1 Main results

Let k be a field of characteristic 0. In the setting of derived algebraic geometry the structured HKR theorem was first proved in [TV15]. More precisely, the HKR theorem states that there is an equivalence of ∞-categories

$$S^1\text{-}\mathcal{C}\mathrm{Alg}_k \simeq \mathcal{C}\mathrm{Alg}_{k[\epsilon]}$$

where the left hand side denotes the ∞-category of derived k-algebras equipped with an action of the circle $S^1 \in \mathcal{S}$, whereas the right hand side denotes the ∞-category of derived $k[\epsilon] := k \oplus k[1]$-algebras. As a consequence one has the following global results:

Theorem 1.6.1.1 ([TV15]). *Let X be a derived algebraic scheme over a field k of characteristic 0. Then one has an equivalence of derived algebraic stacks*

$$X \times_{X \times X} X \simeq \mathrm{T}X[-1]$$

where the left hand side denotes the derived fiber product of X with itself over $X \times X$ via the diagonal map and the right hand side denotes the -1-shifted tangent bundle on X. Moreover the above equivalence is compatible with the canonical projection to X.

It would be desirable to have an analytic analogue of the above result. In a joint work with M. Porta and F. Petit the authors prove:

Theorem 1.6.1.2. *Let k be the field $\mathbb{C}$ of complex numbers or a non-archimedean field of characteristic 0 with a non-trivial valuation. Let X be a k-analytic space. Then there is an equivalence of derived k-analytic spaces*

$$X \times_{X \times X} X \simeq \mathrm{T}X[-1]$$

compatible with the canonical projection to X.

Suppose X is a derived k-affinoid space. Let $A \coloneqq \Gamma(X, \mathcal{O}_X)$. Then Theorem 1.6.1.2 implies that we have an equivalence of simplicial algebras

$$A \widehat{\otimes}_{A \widehat{\otimes}_k A} A \simeq \mathrm{Sym}^{\mathrm{an}}_A\big(\mathbb{L}^{\mathrm{an}}_A[1]\big)$$

where $\mathrm{Sym}^{\mathrm{an}}_A$ denotes the analytification relative to A of the algebraic Sym_A. Theorem 1.6.1.2 is a consequence of the following more general result:

Theorem 1.6.1.3. *There are ∞-categories $k[\epsilon]\text{-}\mathrm{AnRing}_k$ of* mixed analytic rings *and $S^1\text{-}\mathrm{AnRing}_k$ of S^1-equivariant analytic rings. These ∞-categories are equivalent compatibly with their forgetful functors to AnRing_k.*

The ∞-category $S^1\text{-}\mathrm{AnRing}_k$ is defined as

$$S^1\text{-}\mathrm{AnRing}_k \coloneqq \mathrm{Fun}\big(\mathrm{B}S^1, \mathrm{AnRing}_k\big).$$

By a formal argument, the ∞-category $S^1\text{-}\mathrm{AnRing}_k$ is canonically monadic over AnRing_k. Let us denote the associated monad by T_{S^1}. However, the construction of the ∞-category $k[\epsilon]\text{-}\mathrm{AnRing}_k$ is more involved. We need thus to assume that there exists an ∞-category $k[\epsilon]\text{-}\mathrm{AnRing}_k$ equipped with a functor

$$U_\epsilon \colon k[\epsilon]\text{-}\mathrm{AnRing}_k \to \mathrm{AnRing}_k$$

such that U_ϵ is conservative, commutes with sifted colimits and it admits a left adjoint

$$\mathrm{DR} \colon \mathrm{AnRing}_k \to k[\epsilon]\text{-}\mathrm{AnRing}_k$$

such that for every $A \in \mathrm{AnRing}_k$ there exists a canonical equivalence

$$U_\epsilon\big(\mathrm{DR}(A)\big) \simeq \mathrm{Sym}^{\mathrm{an}}_A\big(\mathbb{L}^{\mathrm{an}}[1]\big).$$

In particular, U_ϵ exhibits $k[\epsilon]\text{-}\mathrm{AnRing}_k$ as monadic over AnRing_k. Let us denote the corresponding monad by T_ϵ.

The *structured HKR theorem* can be stated as:

Theorem 1.6.1.4 (Structured analytic HKR theorem)**.** *The monads T_{S^1} and T_ϵ are equivalent as monads over AnRing_k. In particular, there exists an equivalence of ∞-categories*

$$S^1\text{-}\mathrm{AnRing}_k \simeq k[\epsilon]\text{-}\mathrm{AnRing}_k$$

compatible with the forgetful functors to AnRing_k.

Remark 1.6.1.5. Theorem 1.6.1.4 implies both Theorem 1.6.1.2 and Theorem 1.6.1.3.

1.7 Main results

For the reader's convenience we list the main results presented in the present thesis. The reader can find a more precise formulation of these in the body of the text.

Theorem 1.7.0.1. *Let $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ denote the k°-adic pregeometry and $\mathcal{X}$ an ∞-topos. Then the exists a canonical functor*

$$\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$$

which is an equivalence when restricted to topologically almost of finite presentation local $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structures on $\mathcal{X}$.

Theorem 1.7.0.2. *There exists a rigidification functor*

$$(-)^{\mathrm{rig}} \colon \mathrm{dfDM}_{k^\circ} \to \mathrm{dAn}_k$$

which coincides with the usual Raynaud's rigidification functor, when restricted to discrete objects. Moreover, the functor $(-)^{\mathrm{rig}} \colon \mathrm{dfDM}_{k^\circ} \to \mathrm{dAn}_k$ *is compatible with* n*-truncations.*

Theorem 1.7.0.3. *Let* $X \in \mathrm{dAn}_k$ *be a derived* k*-analytic space. Then there exists* $\mathsf{X} \in \mathrm{dfDM}_{k^\circ}$ *a derived* k°*-adic Deligne-Mumford stack such that one has an equivalence*

$$\mathsf{X}^{\mathrm{rig}} \simeq X$$

in the ∞*-category* dAn_k*.*

Theorem 1.7.0.4. *The rigidification functor*

$$(-)^{\mathrm{rig}} \colon \mathrm{dfDM}_{k^\circ} \to \mathrm{dAn}_k$$

is a localization functor. More precisely, it induces an equivalence

$$\mathrm{dfSch}[S^{-1}] \simeq \mathrm{dAn}'_k$$

in the ∞*-category* $\mathcal{C}\mathrm{at}_\infty$*. Here* $\mathrm{dfSch} \subseteq \mathrm{dfDM}_{k^\circ}$ *denotes the full subcategory spanned by admissible derived* k°*-adic schemes,* S *the saturated class of generically strong morphisms and* $\mathrm{dAn}'_k \subseteq \mathrm{dAn}_k$ *the full subcategory spanned by quasi-paracompact and quasi-separated derived* k*-analytic spaces.*

Theorem 1.7.0.5. *Let* $X \in \mathrm{dAn}_k$ *be a quasi-compact and quasi-separated derived* k*-analytic space and* $\mathsf{X} \in \mathrm{dfDM}_{k^\circ}$ *a formal model for* X*. Then we have an equivalence of stable* ∞*-categories*

$$\mathrm{Coh}^+(X) \simeq \mathrm{Coh}^{+,\circ}(\mathsf{X})$$

where $\mathrm{Coh}^{+,\circ}(\mathsf{X})$ *denotes the Verdier quotient of the diagram*

$$\mathrm{Coh}^+_{\mathrm{nil}}(\mathsf{X}) \to \mathrm{Coh}^+(\mathsf{X}) \to \mathrm{Coh}^+(X)$$

computed in the ∞*-category* $\mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty$*.*

Theorem 1.7.0.6. *Let* X *be a quasi-compact and quasi-separated derived* k*-analytic stack. Then the derived Hilbert stack* $\mathrm{RHilb}(X)$ *is representable by a derived* k*-analytic stack.*

Theorem 1.7.0.7. *Let* X *be a smooth over an algebraically closed field. Then there exists a moduli functor*

$$\mathrm{LocSys}_{\ell,n}(X) \colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S}$$

which is given on objects by the formula

$$Z \in \mathrm{dAfd}_k^{\mathrm{op}} \mapsto \mathrm{Map}_{\mathrm{cont}}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{BGL}_n(\Gamma(Z))\big)$$

where $\mathrm{Sh}^{\text{ét}}(X) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ *denotes the étale homotopy type of* X *and* $\mathrm{Map}_{\mathrm{cont}}$ *denotes the space of morphisms*

$$\rho \colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n(\Gamma(Z))$$

which preserve the canonical topologies on both $\mathrm{Sh}^{\text{ét}}(X)$ *and* $\mathrm{BGL}_n(\Gamma(Z))$*. Moreover, given a finite quotient* $\pi_1^w(X) \to \Gamma$ *of the wild fundamental group of* X*, there exists a derived stack*

$$\mathrm{LocSys}_{\ell,n,\Gamma}(X) \in \mathrm{dSt}\big(\mathrm{dAfd}_k, \tau_{\text{ét}}\big)$$

which parametrizes

$$\rho \colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n(\Gamma(Z))$$

whose restriction to $\pi_1^w(X)$ *factor through the quotient morphism* $\pi_1^w(X) \to \Gamma$*.*

Theorem 1.7.0.8. *Let X be a smooth and proper scheme over an algebraically closed field. Then the derived stack $\mathrm{LocSys}_{\ell,n}(X) \in \mathrm{dSt}(\mathrm{dAfd}_k^{\mathrm{op}}, \tau_{\text{ét}})$ is representable by a derived geometric stack. Moreover, given $\rho \in \mathrm{LocSys}_{\ell,n}(X)$ its analytic cotangent complex at ρ is given by*

$$\mathbb{L}^{\mathrm{an}}_{\mathrm{LocSys}_{\ell,n}(X),\rho} \simeq C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)^{\vee}[-1] \in \mathrm{Mod}_k. \tag{1.7.0.1}$$

In the case where X is a smooth non-proper scheme over an algebraically closed field of characteristic $p > 0$ (different than the residual characteristic of k), then the moduli

$$\mathrm{LocSys}_{\ell,n,\Gamma}(X) \in \mathrm{dSt}\big(\mathrm{dAfd}_k, \tau_{\text{ét}}\big)$$

is representable by a derived geometric stack. Moreover, the formula displayed in (1.7.0.1) *holds for the analytic cotangent complex of* $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$.

Theorem 1.7.0.9. *Let X be a proper and smooth scheme over an algebraically closed field. Then the moduli stack $\mathrm{LocSys}_{\ell,n}(X)$ admits a natural shifted symplectic structure $\omega \in \mathcal{A}^{2,\mathrm{cl}}(\mathrm{LocSys}_{\ell,n}(X))$. Moreover, given $\rho \in \mathrm{LocSys}_{\ell,n}(X)(Z)$ the underlying 2-form on ρ coincides with the Poincaré duality morphism*

$$\omega_\rho \colon C^*_{\text{ét}}(X, \mathrm{Ad}(\rho))[1] \otimes C^*_{\text{ét}}(X, \mathrm{Ad}(\rho))[1] \to \Gamma(Z)[2-2d],$$

where $d = \dim(X)$.

Theorem 1.7.0.10. *Let X be a derived k-analytic space. Then one has an equivalence of derived k-analytic stacks*

$$X \times_{X \times X} X \simeq \mathbb{T}X[-1],$$

compatible with the projection to X. In particular, if we assume further that X is a derived k-affinoid space and we let $A \coloneqq \Gamma(X, \mathcal{O}_X)$ we have an equivalence of derived k-algebras

$$A \widehat{\otimes}_{A \widehat{\otimes} A} A \simeq \mathrm{Sym}^{\mathrm{an}}_A(\mathbb{L}^{\mathrm{an}}[1]).$$

Theorem 1.7.0.11. *There are ∞-categories $k[\epsilon]$-AnRing_k and S^1-AnRing_k of mixed derived k-analytic rings and S^1-equivariant derived k-analytic rings, respectively. Moreover, these ∞-categories are monadic and comonadic over AnRing_k and there exists an equivalence of ∞-categories*

$$S^1\text{-}\mathrm{AnRing}_k \simeq k[\epsilon]\text{-}\mathrm{AnRing}_k$$

which fits into a commutative diagram

$$\begin{tikzcd} S^1\text{-}\mathrm{AnRing}_k \arrow[rr] \arrow[dr] & & k[\epsilon]\text{-}\mathrm{AnRing}_k \arrow[dl] \\ & \mathrm{AnRing}_k & \end{tikzcd}$$

of monads over AnRing_k.

1.8 Notations and Conventions

We shall denote k a non-archimedean field equipped with a non-trivial valuation, k° its ring of integers and sometimes we will use the letter $t \in k^\circ$ to denote a uniformizer for k. We denote An_k the category of strict k-analytic spaces and Afd_k the full subcategory spanned by strict k-affinoid spaces and we adopt the convention that whenever we mention k-affinoid or k-analytic space we mean strict k-affinoid and strict k-analytic space, respectively. We denote fSch_{k° the category of quasi-separated formal schemes over the formal spectrum $\mathrm{Spf}(k^\circ)$, where we consider k° equipped with its canonical topology induced by the valuation on k. In order to make clear that we consider formal schemes over $\mathrm{Spf}(k^\circ)$, we shall often employ the terminology k°-adic scheme to refer to formal scheme over $\mathrm{Spf}\ k^\circ$.

Let $n \geq 1$, we shall make use of the following notations:

$$\mathbb{A}^n_k \coloneqq \operatorname{Spec} k[T_1, \dots, T_m], \quad \mathfrak{A}^n_{k^\circ} \coloneqq \operatorname{Spf}(k^\circ \langle T_1, \dots, T_m \rangle)$$

and

$$\mathbf{A}^n_k \coloneqq \left(\mathbb{A}^n_k\right)^{\mathrm{an}}, \quad \mathbf{B}^n_k \coloneqq \operatorname{Sp}(k\langle T_1, \dots T_m \rangle),$$

where $(-)^{\mathrm{an}}$ denotes the usual analytification functor $(-)^{\mathrm{an}} \colon \mathrm{Sch}_k \to \mathrm{An}_k$, see [Ber93a]. We denote by $\mathbf{GL}_n^{\mathrm{an}}$ the analytification of the usual general linear group scheme over k, which associates to every k-affinoid algebra $A \in \mathrm{Afd}_k$ the general linear group $\mathrm{GL}_n(A)$ with A-coefficients.

In this thesis we extensively use the language of ∞-categories. Most of the times, we reason model independently, however whenever needed we prove ∞-categorical results using the theory of quasi-categories and we follows closely the notations in [Lur09b]. We use caligraphic letters $\mathcal{C}$, $\mathcal{D}$ to denote ∞-categories. We denote $\mathcal{C}\mathrm{at}_\infty$ the ∞-category of (small) ∞-categories. We will denote by $\mathcal{S}$ the ∞-category of spaces, $\mathcal{S}^{\mathrm{fc}}$ the ∞-category of finite constructible space, see [Lur09a, §3.1]. Let $\mathcal{C}$ be an ∞-category, we denote by $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ the corresponding ∞-categories of ind-objects and pro-objects on $\mathcal{C}$, respectively. When $\mathcal{C} = \mathcal{S}^{\mathrm{fc}}$, the ∞-category $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ is referred as the ∞-category of *profinite spaces*.

Let R be a derived commutative ring. We will denote by $\mathcal{C}\mathrm{Alg}_R$ the ∞-category of derived k-algebras. The latter can be realized as the associated ∞-category to the usual model category of simplicial R-algebras.

We shall denote by $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}$ the ∞-category of derived adic algebras, introduced in [Lur16, §8.1]. Whenever R admits a non-trivial adic topology, we denote $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_R \coloneqq \left(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}\right)_{R/}$ the ∞-category of *derived adic R-algebras*, i.e. derived R-algebras equipped with an adic topology compatible with the adic topology on R together with continuous morphisms between these.

Let R be a field. We shall denote by $\mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_R$ the ∞-category of small augmented derived R-algebras. When $R = k$ we denote by $\mathrm{AnRing}^{\mathrm{sm}}_k$ the ∞-category of small augmented derived k-analytic rings over k, which is naturally equivalent to $\mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_k$, see [Por15a, §8.2].

Let R be a discrete ring. We denote by $\mathcal{C}\mathrm{Alg}^{\heartsuit}_R$ the 1-category of ordinary commutative rings over R. When R admits an adic topology we shall denote $\mathcal{C}\mathrm{Alg}^{\mathrm{ad},\heartsuit}_R \subseteq \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_R$ the full subcategory spanned by discrete derived adic R-algebras. Let R denote a derived ring. We denote Mod_R the derived ∞-category of R-modules and $\mathrm{Coh}^+(X) \subseteq \mathrm{Mod}_R$ the full subcategory spanned by those almost perfect R-modules.

We need sometimes to enlarge the starting Grothendieck universe, and we often do not make explicit such it procedure. Fortunately, this is innocuous for us. We will usually employ caligraphic letters $\mathcal{X}$, $\mathcal{Y}, \mathcal{Z}$ to denote ∞-topoi. The ∞-category of ∞-topoi together with geometric morphisms between these is denoted ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}$. Caligraphic letters such as $\mathcal{O}$, $\mathcal{A}, \mathcal{B}$ are often employed to denote *structures* on an ∞-topos. We will denote by $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ and $\mathcal{T}_{\mathrm{an}}(k)$ the adic and analytic pregeometries, respectively. Let $\mathcal{X} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ be an ∞-topos, we denote by $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \coloneqq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{ad}}(k^\circ)}(\mathcal{X})$ and $\mathrm{AnRing}_k(\mathcal{X}) \coloneqq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$.

We will denote by $\left(\mathrm{dAff}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\right)$ the algebraic geometric context and we denote by $\mathrm{dSt}\left(\mathrm{dAff}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\right)$ the ∞-category of derived geometric stacks with respect to $\left(\mathrm{dAff}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\right)$. Similary, whenever k denotes either the field $\mathbb{C}$ of complex numbers or a non-archimedean field we will denote by $\left(\mathrm{dAff}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\right)$ the analytic geometric context and correspondingly $\mathrm{dSt}\left(\mathrm{dAn}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\right)$ the ∞-category of derived geometric stacks with respect to the analytic geometric context.

Chapter 2

Brief overview of derived k-analytic geometry

For the purposes of clarity we introduce a small section reviewing the main foundational results in derived k-analytic geometry, proved in [PY16a, PY17a].

2.1 Derived rigid analytic geometry

Let k be a non-archimedean field with a non-trivial valuation. In [PY16a, PY17a] M. Porta and T. Yu Yue introduced the foundations of derived k-analytic geometry. Roughly speaking, a derived k-analytic space consists of a couple $(\mathcal{X}, \mathcal{O}^{\mathrm{alg}})$ where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}^{\mathrm{alg}}$ is a locally ringed sheaf on $\mathcal{X}$ having an additional k-analytic structure.

Definition 2.1.0.1. Let $\mathcal{T}_{\mathrm{an}}(k)$ denote the full subcategory of An_k spanned by smooth k-analytic spaces. We endow $\mathcal{T}_{\mathrm{an}}(k)$ with étale Grothendieck topology. We define a class of admissible morphisms on $\mathcal{T}_{\mathrm{an}}(k)$ as the class of étale morphisms on $\mathcal{T}_{\mathrm{an}}(k)$.

Definition 2.1.0.2. Let $\mathcal{X}$ be an ∞-topos. We say that a functor $\mathcal{O}\colon \mathcal{T}_{\mathrm{an}}(k) \to \mathcal{X}$ is a *local* $\mathcal{T}_{\mathrm{an}}(k)$*-structure* on $\mathcal{X}$ if the following conditions are satisfied:

(i) The functor $\mathcal{O}\colon \mathcal{T}_{\mathrm{an}}(k) \to \mathcal{X}$ commutes with finite produces in $\mathcal{T}_{\mathrm{an}}(k)$;

(ii) The functor $\mathcal{O}\colon \mathcal{T}_{\mathrm{an}}(k) \to \mathcal{X}$ commutes with pullbacks along admissible morphisms, i.e. given a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d, "g"] \\ U' \arrow[r] & V' \end{tikzcd} \tag{2.1.0.1}$$

in the category $\mathcal{T}_{\mathrm{an}}(k)$ such that $g\colon V \to V'$ is an admissible morphism then the induced commutative diagram

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{O}(V) \arrow[d, "g"] \\ \mathcal{O}(U') \arrow[r] & \mathcal{O}(V') \end{tikzcd}$$

in $\mathcal{X}$ is a pullback diagram.

(iii) Let $\coprod_i U_i \to U$ be an étale covering in $\mathcal{T}_{\mathrm{an}}(k)$, then the corresponding morphism $\coprod_i \mathcal{O}(U_i) \to \mathcal{O}(U)$ is an effective epimorphism in the ∞-topos $\mathcal{X}$. We say that a morphism

$$\alpha\colon \mathcal{O} \to \mathcal{O}'$$

between local $\mathcal{T}_{\mathrm{an}}(k)$-structures on $\mathcal{X}$ is local if the for admissible morphism $g\colon V \to U$ in $\mathcal{T}_{\mathrm{an}}(k)$ the induced commutative diagram

$$\begin{tikzcd} \mathcal{O}(V) \arrow[r] \arrow[d] & \mathcal{O}'(V) \arrow[d] \\ \mathcal{O}(U) \arrow[r] & \mathcal{O}'(U) \end{tikzcd}$$

is a pullback diagram in $\mathcal{X}$.

Notation 2.1.0.3. Let $\mathcal{X}$ be an ∞-topos. The subcategory of $\mathrm{Fun}\big(\mathcal{T}_{\mathrm{an}}(k), \mathcal{X}\big)$ spanned by local $\mathcal{T}_{\mathrm{an}}(k)$-structures and local morphisms between these is denoted by $\mathrm{AnRing}_k(\mathcal{X})$.

Remark 2.1.0.4. Let $\mathcal{O}\colon \mathcal{T}_{\mathrm{an}}(k) \to \mathcal{X}$ denote a local $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$. Condition (i) in Theorem 2.1.0.2 implies that evaluation on the affine line induces a sheaf $\mathcal{O}(\mathbf{A}^1_k) \in \mathcal{X}$ which can be promoted to a $\mathcal{C}\mathrm{Alg}_k(\mathcal{X})$-valued sheaf on $\mathcal{X}$. Similarly, the evaluation on the closed unit disk induces a $\mathcal{C}\mathrm{Alg}_k$-valued sheaf $\mathcal{O}(\mathbf{B}^1_k)$ on $\mathcal{X}$.

Definition 2.1.0.5. Let $\mathcal{O}\colon \mathcal{T}_{\mathrm{an}}(k) \to \mathcal{X}$ be a local $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$. We define its *underlying algebra* as the $\mathcal{C}\mathrm{Alg}_k$-valued sheaf $\mathcal{O}^{\mathrm{alg}} \coloneqq \mathcal{O}(\mathbf{A}^1_k)$ on $\mathcal{X}$. This association is functorial and the corresponding functor is denoted

$$(-)^{\mathrm{alg}}\colon \mathrm{AnRing}_k(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_k(\mathcal{X})$$

and referred to as the *underlying algebra functor*.

Remark 2.1.0.6. The object $\mathcal{O} \in \mathrm{AnRing}_k(\mathcal{X})$ does admit more structure than its algebraic counterpart $\mathcal{O}^{\mathrm{alg}} \in \mathcal{C}\mathrm{Alg}_k(\mathcal{X})$. For example, we have an induced morphism of derived rings on $\mathcal{X}$

$$\mathcal{O}(\mathbf{B}^1_k) \to \mathcal{O}^{\mathrm{alg}}$$

which one should interpret as the inclusion of radius-1 convergent holomorphic glocal sections on the sheaf of all global sections $\mathcal{O}^{\mathrm{alg}}$ on $\mathcal{X}$. Therefore, $\mathcal{O}^{\mathrm{alg}}$ admits a *k-analytic structure* which cannot be recovered solely by the algebraic structure on $\mathcal{O}^{\mathrm{alg}}$.

Definition 2.1.0.7. The ∞-category of $\mathcal{T}_{\mathrm{an}}(k)$-structured spaces, denoted ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$, is defined as the ∞-category of those couples $(\mathcal{X}, \mathcal{O})$ where $\mathcal{X} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ is an ∞-topos and $\mathcal{O}\colon \mathcal{T}_{\mathrm{an}}(k) \to \mathcal{X}$ is a local $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$, see [Lur11c, Definition 3.1.9] for a more rigourous construction of ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$.

Example 2.1.0.8. Let $X \in \mathrm{An}_k$ denote an ordinary k-analytic space. To X we can associate $\mathcal{X}_X \coloneqq \mathrm{Shv}(X_{\text{ét}})^\wedge$, the hypercompletion of the ∞-topos of sheaves on the (quasi-)étale site of X. We can define a $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$ as follows: given $V \in \mathcal{T}_{\mathrm{an}}(k)$ we associate it the sheaf $\mathcal{O}(V)$, on $\mathcal{X}$, defined on objects via the formula

$$U \in X_{\text{ét}} \mapsto \mathrm{Map}_{\mathrm{An}_k}(U, V) \in \mathcal{S}.$$

Notice that when $V = \mathbf{A}^1_k$, denotes the k-analytic affine line, the sheaf $\mathcal{O}(\mathbf{A}^1_k)$ coincides with the usual sheaf of sections on X.

Definition 2.1.0.9. A *derived k-analytic space* consists of a couple $X = (\mathcal{X}, \mathcal{O})$ where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}\colon \mathcal{T}_{\mathrm{an}}(k) \to \mathcal{X}$ is a $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$ satisfying the following conditions:

(i) The 0-truncation $\mathrm{t}_{\leq 0}X \coloneqq \big(\mathcal{X}, \pi_0(\mathcal{O})\big)$ is equivalent to an ordinary k-analytic space via Theorem 2.1.0.8.

(ii) For each $i > 0$, the homotopy sheaf $\pi_i\big(\mathcal{O}^{\mathrm{alg}}\big)$ is a coherent sheaf over $\big(\mathcal{X}, \pi_0(\mathcal{O})\big)$.

Notation 2.1.0.10. We will denote by dAn_k the ∞-category of derived k-analytic spaces.

The theory of derived k-analytic geometry is robust in the sense that in practice the main results of derived algebraic geometry do admit analogues in the k-analytic setting. We cite some of the most relevant results in derived k-analytic geometry:

Theorem 2.1.0.11 (Gluing along closed immersions, Theorem 6.5 [PY17a]). *Consider the following pushout diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "j"] & X' \arrow[d] \\ Y \arrow[r, "q"] & Y' \end{tikzcd}$$

in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. Suppose further that i and j are closed immersions and X, X', Y are derived k-analytic spaces. Then Y' is itself a derived k-analytic space.

Theorem 2.1.0.12 (Existence of an analytic cotangent complex). *Let $X := (\mathcal{X}, \mathcal{O}_X) \in \mathrm{dAn}_k$ be a derived k-analytic space and suppose we are given a morphism $f\colon X \to Y$. Consider the* relative analytic derivations functor $\mathrm{Der}^{\mathrm{an}}_{X/Y}(-)\colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathcal{S}$ *given on objects by the formula*

$$M \in \mathrm{Mod}_{\mathcal{O}_X} \mapsto \mathrm{Der}^{\mathrm{an}}_X(M) := \mathrm{Map}_{\mathrm{AnRing}_k(\mathcal{X})_{f^{-1}\mathcal{O}_Y//\mathcal{O}_X}}(\mathcal{O}_X, \mathcal{O}_X \oplus M) \in \mathcal{S},$$

where $\mathcal{O}_X \oplus M$ denotes the trivial square zero extension of $\mathcal{O}_X$ by M, see [PY17a, §5]. *Then $\mathrm{Der}^{\mathrm{an}}_{X/Y}$ is corepresentable. More precisely, there exists an object $\mathbb{L}^{\mathrm{an}}_{X/Y}$, which we refer to the* relative analytic cotangent complex *of $f\colon X \to Y$ such that for every $M \in \mathrm{Mod}_{\mathcal{O}_X}$ there exists a natural equivlence*

$$\mathrm{Map}_{\mathrm{AnRing}_k(\mathcal{X})_{f^{-1}\mathcal{O}_Y//\mathcal{O}_X}}(\mathcal{O}_X, \mathcal{O}_X \oplus M) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}_X}}\left(\mathbb{L}^{\mathrm{an}}_{X/Y}, M\right)$$

in the ∞-category of spaces $\mathcal{S}$. Whenever $f = \mathrm{Id}_X$ we refer to $\mathbb{L}^{\mathrm{an}}_X := \mathbb{L}^{\mathrm{an}}_{X/Y}$ as the absolute analytic cotangent complex *of X.*

Moreover, the analytic cotangent complex satisfies:

(i) Let $\mathsf{X} \in \mathrm{DM}_k$ denote a derived Deligne-Mumford stack over k. Then one has a natural equivalence

$$\left(\mathbb{L}_{\mathsf{X}}\right)^{\mathrm{an}} \simeq \mathbb{L}^{\mathrm{an}}_X$$

in the ∞-category $\mathrm{Coh}^+(X)$, where $\mathbb{L}_{\mathsf{X}}$ denotes the algebraic cotangent complex, introduced in [Lur12c, §7.3.5].

(ii) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms between derived k-analytic spaces. Then there exists a fiber sequence of relative cotangent complexes of the form

$$f^*\mathbb{L}^{\mathrm{an}}_{Y/Z} \to \mathbb{L}^{\mathrm{an}}_{X/Z} \to \mathbb{L}^{\mathrm{an}}_{X/Y}$$

in the ∞-category $\mathrm{Coh}^+(X)$.

(iii) Suppose we have a pullback square in the ∞-category dAn_k

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "g"] & Y' \arrow[d, "f"] \\ X \arrow[r] & Y. \end{tikzcd}$$

Then one has a natural equivalence

$$g^*\mathbb{L}^{\mathrm{an}}_{X'/Y'} \simeq \mathbb{L}^{\mathrm{an}}_{X/Y}$$

in the ∞-category $\mathrm{Coh}^+(X')$.

Theorem 2.1.0.13 (Compatibility with Postnikov towers, Corollary 5.44 [PY17a]). *Let $X := (\mathcal{X}, \mathcal{O}_X) \in \mathrm{dAn}_k$ be a derived k-analytic space. Then for every $n \geq 0$, the canonical map $\mathrm{t}_{\leq n}X \hookrightarrow \mathrm{t}_{\leq n+1}X$ is an analytic square zero extension. More precisely, we have a pushout diagram*

$$\begin{tikzcd} \mathrm{t}_{\leq n}X[\pi_{n+1}(\mathcal{O}_X)[n+2]] \arrow[r, "d_0"] \arrow[d, "d"] & \mathrm{t}_{\leq n}X \arrow[d] \\ \mathrm{t}_{\leq n}X \arrow[r] & \mathrm{t}_{\leq n+1}X \end{tikzcd}$$

in the ∞*-category* dAn_k*, where*

$$\mathrm{t}_{\leq n}X[\pi_{n+1}(\mathcal{O}_X)[n+2]] := \big(\mathcal{X}, \mathcal{O}_X \oplus \pi_{n+1}(\mathcal{O}_X)[n+2]\big) \in \mathrm{dAn}_k$$

denotes the trivial square extension of X *by* $\pi_{n+1}(\mathcal{O}_X)[n+2] \in \mathrm{Mod}_{\mathcal{O}_X}$*. Moreover,* d_0 *and* d *denote the trivial square-zero extension and a suitable analytic derivation*

$$d \colon \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_{\leq n}X} \to \pi_{n+1}(\mathcal{O}_X)[n+2]$$

in the ∞*-category* $\mathrm{Mod}_{\mathcal{O}_X}$*, respectively.*

Theorem 2.1.0.14 (Representability theorem, Theorem 7.1 [PY17a])**.** *Let* $F \in \mathrm{dSt}\big(\mathrm{dAfd}_k, \tau_{\text{ét}}\big)$ *denote a derived stack. Then the following assertions are equivalent:*

(i) F *is a geometric* n*-stack with respect to the geometric context* $\big(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$*.*

(ii) F *is compatible with Postnikov towers, has a global cotangent complex and its truncation* $\mathrm{t}_{\leq 0}F$ *is representable by an* n*-geometric stack with respect to the geometric context* $\big(\mathrm{Afd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$*.*

We refer the reader to [PY16c, §2] for the notions of geometric context and geometric stack with respect to a given geometric context.

The above results were proved by M. Porta and T. Yu Yue. They constitute an extensive review of derived methods in the context of k-analytic geometry. However, certain results of classical k-analytic geometry still did not have derived analogues prior to the current thesis. They constitute mainly the existence of formal models for k-analytic spaces and its applications in k-analytic geometry. Certain of these lacking results were desired in order to apply the techniques of derived k-analytic geometry to the study of certain problems coming from representation theory. We shall exemplify one such application, which is an ubiquitous theme in the current thesis.

Example 2.1.0.15. Let X be a smooth scheme over an algebraically closed field of positive characteristic $p > 0$. Let $\ell \neq p$ be a prime number and suppose we are given a continuous representation

$$\rho \colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n\big(\overline{\mathbb{Q}}_\ell\big).$$

As ρ is continuous we can suppose that ρ factors through a finite extension $k/\mathbb{Q}_\ell$. One would like to understand the space of continuous deformations of ρ. These should correspond to continuous group representations

$$\widetilde{\rho} \colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A)$$

where $A \in \mathrm{AnRing}_k^{\mathrm{sm}}$, as we are concerned with continuous deformations of ρ.

In such case, one would like to consider the formal moduli problem $F \colon \mathrm{AnRing}_k^{\mathrm{sm}} \to \mathcal{S}$ given on objects by the formula

$$\big(A \to k\big) \in \mathrm{AnRing}_k^{\mathrm{sm}} \mapsto \mathrm{Map}_{\mathrm{cont}}\left(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{BGL}_n(A)\right) \times_{\mathrm{Map}_{\mathrm{cont}}\left(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{BGL}_n(k)\right)} \{\rho\} \in \mathcal{S}. \qquad (2.1.0.2)$$

Unfortunately, we still do not have a precise definition of the right hand side of (2.1.0.2). We would like to define it as the space of *continuous group-like homomorphisms*

$$\widetilde{\rho} \colon \pi_1^{\text{ét}}(X) \to \mathrm{Aut}(A^n) \qquad (2.1.0.3)$$

such that its restriction along the morphism $A \to k$ in the ∞-category $\mathrm{AnRing}_k^{\mathrm{sm}}$ coincides with ρ, up to equivalence. However, we do not know what continuity means in this context. Indeed, $A \in \mathrm{AnRing}_k^{\mathrm{sm}}$ corresponds to a functor

$$A \colon \mathcal{T}_{\mathrm{an}}(k) \to \mathcal{S}$$

satisfying certain *admissibility conditions* captured in Theorem 2.1.0.2. Such k-analytic structure on A do not produce any sort of topological data. Therefore, we need to interpret A differently in order to be able to define continuous morphism of group-like objects as in (2.1.0.3). The avid reader might object by recalling the equivalence of ∞-categories

$$\mathrm{AnRing}_k^{\mathrm{sm}} \simeq \mathcal{C}\mathrm{Alg}_k^{\mathrm{sm}}$$

proved in [Por15a, §8.2]. However, if we regard $A \in \mathcal{C}\mathrm{Alg}_k^{\mathrm{sm}}$ as a plain $\mathbb{E}_\infty$-algebra over k we will restrict ourselves to plain group like morphisms

$$\widetilde{\rho} \colon \pi_1^{\text{ét}}(X) \to \mathrm{Aut}(A^n),$$

and we do not recover a good notion of continuity.

Remark 2.1.0.16. The content of Theorem 2.1.0.15 is in sharp contrast with the ordinary case. Indeed, if $A \in \mathrm{Afd}_k^{\mathrm{op}}$ is a k-affinoid algebra then we can equip A with a canonical topology induced by a choice of presentation

$$A \cong k\langle T_1, \dots, T_m \rangle / I.$$

Such topology on A does not depend on the choice of the presentation. Moreover, we can consider the A-points $\mathrm{GL}_n(A) \coloneqq \mathbf{GL}_n^{\mathrm{an}}(A)$ as a topological group whose topology is induced by the one on A. In this case it is reasonable to consider continuous group representations

$$\rho \colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A).$$

Moreover, when $A = k$ a continuous representation $\rho \colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(k)$ corresponds to the usual notion of ℓ-adic continuous representations of $\pi_1^{\text{ét}}(X)$.

Chapter 3

Derived k°-adic geometry and derived Raynaud localization theorem

Derived k°-adic geometry and derived Raynaud localization Theorem

Contents

3.1 Introduction

3.1.1 Background material

Let k be a non-archimedean field of discrete valuation, k° its ring of integers and let $t \in k^\circ$ be a fixed uniformizer for k. Denote fSch_{k° the category of *admissible* k°-adic formal schemes and An the category of k-analytic spaces. There exists a rigidification functor $(-)^{\mathrm{rig}} \colon \mathrm{fSch}_{k^\circ} \to \mathrm{An}$ such that every quasi-paracompact and quasi-separated k-analytic space X admits a formal model over $\mathrm{Spf}\, k^\circ$. That is to say, there exists $\mathfrak{X} \in \mathrm{fSch}_{k^\circ}$ such that

$$\mathfrak{X}^{\mathrm{rig}} \simeq X.$$

For this reason, one is able to understand the analytic structure on X through a formal model $\mathfrak{X}$ for X. The following is a classical result proved by Raynaud:

Theorem 1 (Raynaud, Theorem 8.4.3 [Bos05])**.** *The functor* $(-)^{\mathrm{rig}} \colon \mathrm{fSch}_{k^\circ} \to \mathrm{An}$ *is a localization functor. More specifically, the functor* $(-)^{\mathrm{rig}} \colon \mathrm{fSch}_{k^\circ} \to \mathrm{An}$ *factors through the localization of* fSch_{k° *at the class of admissible blow ups, S. Moreover such functor induces an equivalence of categories*

$$\mathrm{fSch}_{k^\circ}[S^{-1}] \to \mathrm{An}'$$

where $\mathrm{An}' \subseteq \mathrm{An}$ *denotes the full subcategory of quasi-paracompact quasi-separated k-analytic spaces.*

?? 1 it allows to use methods from algebraic geometry in order to establish certain results in the context of rigid analytic geometry. For instance, **??** 1 is useful to study flatness conditions for k-analytic spaces and base change theorems in the setting of k-analytic geometry. Raynaud's theory allows to bypass this problem the intrinsic analytic difficulties by reducing this problem to its analogue at the formal level. The latter situation can then be dealt using techniques from algebraic geometry.

3.1.2 Main results

The same situation occurs in the context of derived k-analytic geometry. Derived k-analytic geometry was developed by M. Porta and T. Yu Yue in [PY16a, PY17a]. In [Lur16, §8] the author introduces and studies at length derived and spectral formal geometry. Our main goal in this text is to prove an analogue of **??** 1 in the derived setting. However, in order to state a derived analogue of **??** 1 one needs another crucial ingredient, namely the existence of a derived rigidification functor. Inspired by the construction of the derived analytification functor [PY17a, §3], we will provide a construction of a derived rigidification functor. In order to so, we need to develop a structured spaces approach to derived formal geometry over $\mathrm{Spf}\, k^\circ$.

This is done in §2: we develop a theory of derived t-adic formal geometry by considering certain $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured spaces. Therefore, we will consider couples $(\mathcal{X}, \mathcal{O})$ where $\mathcal{X}$ is an ∞-topos and $\mathcal{O} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{X}$ is a *local* $\mathcal{T}_{\mathrm{ad}}(k^\circ)$*-structure*. To such a pair we can functorially associate a locally ringed ∞-topos $(\mathcal{X}, \mathcal{O}^{\mathrm{alg}})$. However, this construction loses information.

In general, the $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structure $\mathcal{O} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{X}$ encodes more information than its algebraic counterpart $\mathcal{O}^{\mathrm{alg}} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$. For example, one can show that $(-)^{\mathrm{alg}}$ factors through the canonical functor $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$, where $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ denotes the ∞-category of k°-adic algebra objects on the ∞-topos $\mathcal{X}$.

More specifically, $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ corresponds to the ∞-category whose objects are objects $\mathcal{O} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ together with an adic topology on $\pi_0(\mathcal{O})$ compatible with the adic topology on k° and continuous morphisms between these. Fortunately, we are able to fully understand the difference between the ∞-categories $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ and $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$.

Theorem 3.1.2.1 (Theorem 3.3.2.4)**.** *Let $\mathcal{X}$ be an ∞-topos and consider the underlying algebra functor*

$$(-)^{\mathrm{alg}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}),$$

given on objects by the formula

$$(\mathcal{X}, \mathcal{O}) \mapsto (\mathcal{X}, \mathcal{O}^{\mathrm{alg}}).$$

Then, this functor factors through the functor $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}(\mathcal{X})$ and the induced functor

$$(-)^{\mathrm{ad}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X}).$$

is fully faithful and moreover an equivalence of ∞-categories when restricted to those strictly Henselian objects topologically almost of finite presentation.

Theorem 3.1.2.1 implies that the ∞-category of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured spaces $(\mathcal{X}, \mathcal{O})$ whose $\mathcal{O}^{\mathrm{alg}} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ is topologically almost of finite presentation can be recovered as locally ringed ∞-topoi $(\mathcal{X}, \mathcal{O}^{\mathrm{alg}})$ such that $\pi_0(\mathcal{O}^{\mathrm{alg}})$ comes equipped with an adic topology compatible with the t-adic topology on k°. This can be regarded as a rectification type result for $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured spaces.

We will give a definition of derived formal k°-adic Deligne- Mumford stacks over k° in terms of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured spaces and show that this notion agrees with the notion introduced in [Lur16, §8]. We then proceed to study *k°-adic Postnikov tower decompositions* and the *k°-adic cotangent complex* with respect to maps between derived k°-adic Deligne-Mumford stacks, which, to the author's best knowledge, has never been addressed before in the literature.

In §3 we define a rigidification functor

$$(-)^{\mathrm{rig}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}\left(\mathcal{T}_{\mathrm{ad}}(k^\circ)\right) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}\left(\mathcal{T}_{\mathrm{an}}(k)\right)$$

which restricts to a functor $(-)^{\mathrm{rig}} \colon \mathrm{dfDM}_{k^\circ} \to \mathrm{dAn}$, where dfDM_{k° denotes the ∞-category of formal derived Deligne-Mumford stacks and dAn the ∞-category of derived k-anlytic spaces. We prove that the derived rigidification functor $(-)^{\mathrm{rig}}$ coincides with the usual rigidification functor when restricted to the category of ordinary formal schemes.

Whenever $Z \in \mathrm{dAn}$ is such that its 0-th truncation $\mathrm{t}_0(Z)$ is an (ordinary) quasi-separated and quasi-paracompact k-analytic space, we prove:

Theorem 3.1.2.2 (Theorem 6.2.3.15)**.** *Let $Z \in \mathrm{dAn}$ be a quasi-paracompact and quasi-separated derived k-analytic space. There exists $\mathsf{Z} \in \mathrm{dfDM}$ such that one has an equivalence $(\mathsf{Z})^{\mathrm{rig}} \simeq Z$ in the ∞-category dAn, in other words Z admits a formal model $\mathsf{Z} \in \mathrm{dfDM}_{k^\circ}$.*

Let dfSch_{k° denote the full subcategory of dfDM_{k° spanned by those $\mathsf{X} \in \mathrm{dfDM}_{k^\circ}$ such that $\mathrm{t}_0(\mathsf{X})$ is equivalent to an ordinary admissible quasi-paracompact formal scheme over k°. We say that a morphism $f \colon \mathsf{X} \to \mathsf{Y}$ in dfDM_{k° is *generically strong* if for each $i > 0$, the induced map

$$\pi_i\left(f^{-1}\mathcal{O}_{\mathsf{Y}}\right) \to \pi_i\left(\mathcal{O}_{\mathsf{X}}\right)$$

is an equivalence in $\mathrm{Coh}^+(\mathsf{X})$. Denote moreover $\mathrm{dAn}' \subseteq \mathrm{dAn}$ the full subcategory spanned by those $X \in \mathrm{dAn}$ such that its 0-th truncation $\mathrm{t}_{\leq 0}X$ is equivalent to a quasi-paracompact and quasi-separated ordinary k-analytic space. The following is a direct generalization of Raynaud's localization theorem in the derived setting:

Theorem 2 (Theorem 3.4.4.10)**.** *Let S denote the saturated class generated by those morphisms $f \colon \mathsf{X} \to \mathsf{Y}$ in dfSch_{k° such that $\mathrm{t}_0(f)$ is an admissible blow up and generically strong. Then the rigidification functor*

$$(-)^{\mathrm{rig}} \colon \mathrm{dfSch}_{k^\circ} \to \mathrm{dAn}'_k.$$

factors through the localization ∞-category $\mathrm{dfSch}_{k^\circ}[S^{-1}]$ and the induced functor

$$\mathrm{dfSch}_{k^\circ}[S^{-1}] \to \mathrm{dAn}'_k.$$

is an equivalence of ∞-categories.

Let us briefly sketch the proof of Theorem 3.4.4.10. In order to prove the statement it suffices to prove that given $X \in \mathrm{dAn}$ as in Theorem 3.4.4.10 the comma category $\mathcal{C}_X := \big(\mathrm{dfSch}\big)_{X/}$ is contractible. We will prove a slightly stronger result, namely $\mathcal{C}_X$ is a filtered ∞-category. In order to illustrate the main ideas behind the proof it suffices to deal with lifting a morphism $f \colon X \to Y$ in dAn to a morphism $f^+ \colon \mathsf{X} \to \mathsf{Y}$ in dfSch such that $(f^+)^{\mathrm{rig}} \simeq f$ as morphisms in dAn.

The lifting is done by induction on the Postnikov tower of X. Suppose that $X \simeq \mathrm{t}_{\leq 0}(X)$ in the ∞-category dAn. Notice that **??** 1 implies that we can lift $\mathrm{t}_{\leq 0}(f)$ to a morphism $f_0^+ \colon \mathsf{X}_0 \to \mathsf{Y}_0$ in the category fSch_{k°. As $X \to Y$ factors through the canonical morphism $\mathrm{t}_{\leq 0}Y \to Y$ in dAn, we conclude by Theorem 3.1.2.2 together with **??** 1 that we can find a formal model for $f \colon X \to Y$, up to an admissible blow up at the level of 0-th truncations.

Let $n \geq 0$ be an integer. Assume moreover that we are giving a morphism $(f_n^+) \colon \mathsf{X}_n \to \mathsf{Y}_n$ in dfSch such that $(f_n^+)^{\mathrm{rig}} \simeq \mathrm{t}_{\leq n} f \colon \mathrm{t}_{\leq n} X \to \mathrm{t}_{\leq n} Y$. Consider the $(n+1)$-st step of the Postnikov tower, namely the pushout diagram

$$\begin{tikzcd}
\mathrm{t}_{\leq n}X[\pi_{n+1}(\mathcal{O}_X)[n+2]] \arrow[r] \arrow[d] & \mathrm{t}_{\leq n}X \arrow[d] \\
\mathrm{t}_{\leq n}X \arrow[r] & \mathrm{t}_{\leq n+1}X
\end{tikzcd}$$

in the ∞-category dAn. In order to proceed, we will need to know that the adic cotangent complex is compatible with the analytic one via rigidification. Namely, we have the following proposition:

Proposition 3.1.2.3. *Let $\mathsf{X} \in \mathrm{dfSch}$ and denote $X := \mathsf{X}^{\mathrm{rig}} \in \mathrm{dAn}$. Then the rigidification functor induces a canonical equivalence*

$$\big(\mathbb{L}^{\mathrm{ad}}_{\mathsf{X}}\big)^{\mathrm{rig}} \simeq \mathbb{L}^{\mathrm{an}}_X$$

in the ∞-category $\mathrm{Coh}^+(X)$.

The induction hypothesis, together with the universal property of both the adic and analytic cotangent complexes plus refined results on the existence of formal models for almost perfect modules on X, proved in Appendix A, imply that we can extend the morphism $f_n^+ \colon \mathsf{X}_n \to \mathsf{Y}_n$ to a diagram

$$f_n^+ \leftarrow f_n^+[\pi_{n+1}(f)^+[n+2]] \to \mathrm{t}_{\leq n} f_n^+ \tag{3.1.2.1}$$

considered as an object in $\mathrm{Fun}\big(\Lambda^2_0, \mathrm{dfSch}^{\Delta^1}\big)$, where $\pi_{n+1}(f)^+ \in \mathrm{Coh}^+(\mathsf{X}_0)^{\Delta^1}$ in (3.1.2.1) denotes a formal model for $\pi_{n+1}(f)$. By taking pushouts along Λ^2_0 we obtain the desired lifting $f^+_{n+1} \colon \mathsf{X}_{n+1} \to \mathsf{Y}_{n+1}$ of $\mathrm{t}_{\leq n+1}(f)$.

The main technical difficulty of the proof comes from lifting higher coherences on diagrams of analytic derivations to suitable higher coherences of suitable diagrams of adic derivations. This is needed in order to extend (3.1.2.1) above in the case of more complex diagrams.

3.1.3 Related works

Let us give some examples of applications: it was proven in [Ant17a] that the moduli stack of continuous t-adic representations of a profinite group (topologically of finite generation) is representable by a geometric k-analytic stack. This object can be upgraded as a geometric derived k-analytic stack. This additional structure is crucial if one wants to obtain the correct cotangent complex and thus have a control of its obstruction theory. This additional structure led us to have a better understanding of the underlying geometry of such geometric derived k-analytic stack, in particular one is then able that it admits a shifted symplectic form.

However, the proof of the representability of such a derived k-analytic stack is not possible using only the techniques available from the structured spaces approach to derived k-analytic geometry, as in [PY16a]. The main drawback is that derived k-analytic spaces are defined as couples $(\mathcal{X}, \mathcal{O})$, where $\mathcal{X}$ denotes an (hypercomplete) ∞-topos and $\mathcal{O} \colon \mathcal{T}_{\mathrm{an}}(k) \to \mathcal{X}$ consists of a $\mathcal{T}_{\mathrm{an}}(k)$-local structure. This data should be interpreted as the given of a locally ringed space together with an additional structure, such structure consisting of the data of convergent t-adic holomorphic (derived) sections of the structure sheaf.

However such information does not provide directly any sort of topological structure on $\mathcal{O}$, in contrast with the classical setting in which $\mathcal{O}$ corresponds to a sheaf of Banach k-algebras. Since [Ant17a] studies continuous representations of a profinite group and, more generally, of pro-homotopy types, one needs to be able to recover back this topological data at the derived level. **??** 2 provides us with an answer to this matter and it plays a crucial role in the proof of representability of [Ant17a].

So far, Raynaud's viewpoint in the derived setting already encountered other applications: in a joint work with F. Petit and M. Porta one proves an HKR Theorem in the context of derived k-analytic geometry and the theory of formal models proves to be useful in the proof of such statement. Another such application is a joint work with M. Porta: we show the representability of the derived Hilbert stack as a derived k-analytic stack in which the theory of formal models plays a crucial role.

3.1.4 Notations and conventions

Throughout the text, unless otherwise stated, k denotes a non-archidemean field of discrete valuation and $k^\circ = \{x \in k : |x| \leq 1\}$ its ring of integers in k. We let t be a fixed uniformizer of k. Given an integer $n \geq 1$, we will denote by k°_n the reduction modulo (t^n) of k°. We denote fSch_{k° the (classical) category of formal schemes (topologically) of finite presentation over k°.

Let $n \geq 0$ be an integer, we define $k^\circ\langle T_1, \ldots, T_n\rangle$ as the sub-algebra of $k^\circ[[T_1, \ldots, T_n]]$ consisting of those formal power series which $f = \Sigma_I a_i T_I^{b_I}$, such that the coefficients $a_I \to 0$ in k°. Denote by $\mathfrak{A}^n_{k^\circ} := \operatorname{Spf} k^\circ\langle T_1, \ldots, T_n\rangle$, $\mathbf{B}^n_k := \operatorname{Sp}\langle T_1, \ldots, T_n\rangle$ the closed unit disk and $\mathbf{A}^n_k := \left(\mathbb{A}^n_k\right)^{\mathrm{an}}$ the k-analytic affine n-space.

We say that a morphism between two t-complete k°-algebras $A \to B$ is *formally étale* if, for each $n \geq 0$, its mod t^n reduction is an étale homomorphism of k°/t^n-algebras. We denote $\mathcal{S}$ the ∞-category of spaces and ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ the ∞-category of ∞-topoi together with geometric morphisms between these. Let R be a commutative simplicial ring, we denote $\mathcal{C}\mathrm{Alg}_R$ its ∞-category of derived R-algebras. Given an object $B \in \mathcal{C}\mathrm{Alg}_R$ we denote by $\pi_i\,(B)$ the i-th homotopy group of the underlying space associated to B. We will denote Mod_R the derived ∞-category of R-modules, it can be considered as an ∞-categorical upgrade of the usual (triangulated) derived category $\mathcal{D}(R)$ of R-complexes. Throughout the text we will employ homological convention, thus given $M \in \mathrm{Mod}_R$ we denote by $\pi_i(M) := H_i(M)$ its i-th homology group. Given an ∞-topos $\mathcal{X}$ we will denote $\mathcal{C}\mathrm{Alg}_R(\mathcal{X}) := \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{disc}(R)}}(\mathcal{X})$, $\mathcal{C}\mathrm{Alg}^{\mathrm{sh}}_R(\mathcal{X}) := \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\text{ét}(R)}}(\mathcal{X})$, $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) := \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{ad}(k^\circ)}}(\mathcal{X})$ and $\mathrm{AnRing}_k(\mathcal{X}) := \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}(k)}}(\mathcal{X})$. We will often denote a general pregeometry by the letter $\mathcal{T}$. Moreover, whenever we refer to an object $(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ we assume that $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ is a local $\mathcal{T}$-structure on $\mathcal{X}$.

3.1.5 Acknowledgments

The author is extremely thankful to Mauro Porta for sharing many ideas, remarks and advices about the contents of the paper. The author is also very thankful to his advisor B. Toën and M. Robalo for valuable discussions.

3.2 Review on derived algebraic and analytic geometry

3.2.1 Functor of points approach

Let $k \in \mathcal{C}\mathrm{Alg}$ denote a commutative ring or more generally a derived commutative ring.

Definition 3.2.1.1. The ∞-category of *derived affine schemes* over $\operatorname{Spec} k$ is defined as

$$\mathrm{dAff} := \left(\mathcal{C}\mathrm{Alg}_k\right)^{\mathrm{op}}.$$

Definition 3.2.1.2. Let $f \colon A \to B$ be a morphism of derived rings. We say that f is étale if $\pi_0(f) \colon \pi_0(A) \to \pi_0(B)$ is an étale morphism of ordinary commutative rings and for each $i > 0$ the induced morphism

$$\pi_i(f) \colon \pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$$

is an isomorphism of $\pi_0(B)$-modules.

Notation 3.2.1.3. One can equip the ∞-category dAff with the étale topology. We shall denote $(\mathrm{dAff}, \tau_{\text{ét}})$ the corresponding étale site.

Notation 3.2.1.4. Let $(\mathrm{dAff}, \tau_{\text{ét}})$ denote the étale Grothendieck site on the ∞-category dAff. Let $X = \operatorname{Spec} A \in \mathrm{dAff}$ we denote $\mathrm{Shv}_{\text{ét}}(X) := \mathrm{Shv}(\mathrm{dAff}_{/X}, \mathcal{T}_{\text{ét}})$ the ∞-topos of *étale sheaves* on X.

Remark 3.2.1.5. The ∞-category $\mathrm{Shv}_{\text{ét}}(X)$ can be realized as a presentable left localization of $\mathrm{PShv}(X) :=$ $\mathrm{Fun}\left(\mathrm{dAff}_{/X}, \mathcal{S}\right)$ given by localizing at the class of morphisms forcing étale descent on objects of $\mathrm{PShv}(X)$. In particular, the ∞-category $\mathrm{Shv}_{\text{ét}}(X)$ is presentable and indeed an ∞-topos.

Definition 3.2.1.6. We denote $\mathrm{dSt}\big(\mathrm{Aff}, \tau_{\text{ét}}\big) \subseteq \mathrm{Shv}_{\text{ét}}\big(\mathrm{Aff}\big)$ the ∞-category of *derived stacks* as the full subcategory spanned by those étale sheaves $X \in \mathrm{Shv}_{\text{ét}}(\mathrm{Aff})$ which are hypercomplete.

One can give a definition of the ∞-category of derived schemes in terms of (derived) locally ringed spaces:

Definition 3.2.1.7. Let X be a topological space and $\mathcal{O} \in \mathcal{C}\mathrm{Alg}(X)$ a $\mathcal{C}\mathrm{Alg}$-valued sheaf on X. We say that $\mathcal{O}$ is local if at every point $x \in X$, the stalk $\mathcal{O}_x \in \mathcal{C}\mathrm{Alg}$ is a *local derived ring*, i.e. $\pi_0(\mathcal{O}_x)$ is a local ordinary commutative ring.

Definition 3.2.1.8. Let $\mathrm{dLocRing}_k$ denote the ∞-category whose objects are pairs $(X, \mathcal{O})$ where X is a topological space and $\mathcal{O} \in \mathcal{C}\mathrm{Alg}(X)$ a local $\mathcal{C}\mathrm{Alg}(X)$-valued sheaf on X. We denote dSch_k the full subcategory of $\mathrm{dLocRing}_k$ spanned by those couples $(X, \mathcal{O})$ satisfying:

(i) Its 0-th truncation $\mathrm{t}_{\leq 0}(X, \mathcal{O}) := (X, \pi_0(\mathcal{O}))$ is isomorphic to an ordinary scheme over k;

(ii) For every $i \geq 0$, the higher homotopy sheaf $\pi_i(\mathcal{O})$ is a quasi-coherent sheaf on $\mathrm{t}_{\leq 0}(X, \mathcal{O})$.

Remark 3.2.1.9. One can think of a pair $(X, \mathcal{O}) \in \mathrm{dSch}_k$ as an infinitesimal deformation of the ordinary scheme $(X, \pi_0(\mathcal{O}))$ and the higher homotopy sheaves $\pi_i(\mathcal{O})$ encode the higher infinitesimal information.

Remark 3.2.1.10. One can realize the ∞-category as a full subcategory of $\mathrm{dLocRing}_k$. For this reason, we have a canonical functor $\mathrm{dLocRing}_k \to \mathrm{Fun}\big(\mathrm{Aff}, \mathcal{S}\big)$ which associates to every $(X, \mathcal{O}) \in \mathrm{dLocRing}_k$ the functor

$$\Big(\mathrm{Spec}\, A \mapsto \mathrm{Map}_{\mathrm{dLocRing}_k}\left(\mathrm{Spec}\, A, (X, \mathcal{O})\right)\Big) \in \mathrm{Fun}\big(\mathrm{Aff}, \mathcal{S}\big).$$

This provides a fully faithful embedding of the ∞-category of derived schemes in the ∞-category $\mathrm{dSt}\big(\mathrm{dAff}, \tau_{\text{ét}}\big)$.

Example 3.2.1.11. Let X be a usual scheme and Y, Z two full subschemes of X then we can define the *derived intersection* $Y \cap Z := Y \times^{\mathrm{R}}_X Z$ (in the ambient space X) as the derived scheme whose underlying topological space corresponds to the underlying topological space of the *ordinary pullback*, $Y \times_X Z$. Plus, the structure sheaf on $Y \cap Z$ coincides with the derived tensor product

$$\mathcal{O}_{Y \cap Z} := \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Z.$$

It turns out that the 0-th truncation of $Y \cap Z$ coincides with the ordinary fiber product of Y with Z over X. More specifically, one has $\pi_0(\mathcal{O}_{Y \cap Z}) \simeq \mathrm{Tor}^0_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)$ and isomorphisms of coherent sheaves on $Y \times_X Z$,

$$\pi_i(\mathcal{O}_{Y \cap Z}) \simeq \mathrm{Tor}^i_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z).$$

The Serre intersection formula implies that the Euler characteristic of the derived intersection $Y \cap Z$ agrees with the usual intersection number associated to the intersection of Y and Z inside of X.

3.2.2 Structured spaces approach

In [Lur11c], J. Lurie introduced the notion of a (spectral) scheme, and more generally (spectral) Deligne-Mumford stack via a structured spaces approach. Whenever k is a field of characteristic zero both approaches the functor of points and the structured spaces to derived algebraic geometry are equivalent. We review some of these notions which will be useful for our exposition. The reader is referred to [Lur11c] and [PY16a] for more details.

Definition 3.2.2.1. We refer $(\mathcal{X}, \mathcal{O})$ to a couple as a *ringed ∞-topos* whenever $\mathcal{X}$ is an ∞-topos and $\mathcal{O} \in \mathcal{C}\mathrm{Alg}(\mathcal{X})$ is a $\mathcal{C}\mathrm{Alg}$-valued sheaf on $\mathcal{X}$. We say that a ringed ∞-topos is a *locally ringed ∞-topos* if for each geometric point $x_* \colon \mathcal{X} \to \mathcal{S}$ the $\mathcal{C}\mathrm{Alg}$-valued sheaf $x^{-1}\mathcal{O}$ on $\mathcal{S}$ can be identified with a local derived k-ring.

Remark 3.2.2.2. Suppose given X a topological space. We can then form its associated ∞-topos $\mathcal{X} := \mathrm{Shv}(X)$ of $\mathcal{S}$-valued sheaves on X. The locally ringed pair $(X, \mathcal{O})$ induces naturally a locally ringed ∞-topos $(\mathcal{X}, \mathcal{O})$, as $\mathcal{O}$ can be by construction promoted to a local $\mathcal{C}\mathrm{Alg}$-valued sheaf on $\mathcal{X}$.

We now reformulate the notion of locally ringed ∞-topos in terms of pregeometries:

Definition 3.2.2.3. A *pregeometry* consists of an ∞-category $\mathcal{T}$ equipped with a class of *admissible morphisms* and a Grothendieck topology, which is generated by admissible morphisms satisfying the following conditions:

(i) $\mathcal{T}$ admits finite products;

(ii) Pullbacks along admissible morphisms exist and are again admissible;

(iii) If f and g are morphisms in $\mathcal{T}$ such that g and $g \circ f$ are admissible then so is f.

(iv) Retracts of admissible morphisms are again admissible.

We give a list of well known examples of pregeometries which will be useful later on.

Example 3.2.2.4. (i) Let $\mathcal{T}_{\mathrm{disc}}(k)$ denote the pregeometry whose underlying category consists of affine spaces $\mathbb{A}^n_k$ and morphisms between these. The family of admissible morphisms is the family of isomorphisms in $\mathcal{T}_{\mathrm{disc}}(k)$ and we equip it with the discrete Grothendieck topology.

(ii) Let $\mathcal{T}_{\mathrm{Zar}}(k)$ denote the pregeometry whose underlying category has objects those affine schemes which admit an open embedding in some n-th affine space, $\mathbb{A}^n_k$, whose admissible morphisms correspond to open immersions of schemes and the Grothendieck topology consists of usual Zariski topology.

(iii) Let $\mathcal{T}_{\text{ét}}$ be the pregeometry whose underlying category is the full subcategory of the category affine schemes spanned by affine schemes étale over $\mathbb{A}^n_k$, for some n. A morphism in $\mathcal{T}_{\text{ét}}$ is admissible if and only if it is an étale morphism of affine schemes.

Definition 3.2.2.5. Let $\mathcal{T}$ be a pregeometry and $\mathcal{X}$ an ∞-topos. A *$\mathcal{T}$-local structure on* $\mathcal{X}$ is defined as a functor between ∞-categories $\mathcal{O} : \mathcal{T} \to \mathcal{X}$ satisfying the following conditions:

(i) The functor $\mathcal{O}$ preserves finite products;

(ii) For a pullback square of the form

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ U \arrow[r] & X \end{tikzcd}$$

in $\mathcal{T}$ where f is admissible then the square

$$\begin{tikzcd} \mathcal{O}(U') \arrow[r] \arrow[d] & \mathcal{O}(X')^{\mathcal{O}(f)} \arrow[d] \\ \mathcal{O}(U) \arrow[r] & \mathcal{O}(X) \end{tikzcd}$$

is a pullback square in $\mathcal{X}$.

(iii) Given a covering $\{U_\alpha \to U\}$ in $\mathcal{T}$ consisting of admissible morphisms then the induced map

$$\coprod \mathcal{O}(U_\alpha) \to \mathcal{O}(U),$$

is an effective epimorphism in $\mathcal{X}$.

A morphism $\mathcal{O} \to \mathcal{O}'$ between $\mathcal{T}$-local structures is said to be *local* if it is a natural transformation satisfying the additional condition that for every admissible morphism $U \to X$ in $\mathcal{T}$, the resulting diagram

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{O}'(U) \arrow[d] \\ \mathcal{O}(X) \arrow[r] & \mathcal{O}'(X), \end{tikzcd}$$

is a pullback square in $\mathcal{X}$. We denote $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ the ∞-category of local $\mathcal{T}$-structures on $\mathcal{X}$ together with local morphisms between these.

Construction 3.2.2.6. (i) In virtue of [Lur11c, Example 3.1.6, Remark 4.1.2], we have an equivalence of ∞-categories $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{disc}}}(\mathcal{X}) \simeq \mathrm{Shv}_{\mathcal{C}\mathrm{Alg}}(\mathcal{X})$, where the latter denotes the ∞-category of $\mathcal{C}\mathrm{Alg}$-valued sheaves on $\mathcal{X}$. More explicitly, given a ringed ∞-topos $(\mathcal{X}, \mathcal{O})$, we can promoted it naturally to a $\mathcal{T}_{\mathrm{disc}}$-structured via the construction:

$$\mathbb{A}^n_k \in \mathcal{T}_{\mathrm{disc}} \mapsto (\mathcal{O}^n \in \mathrm{Shv}(\mathcal{X}) \simeq \mathcal{X}),$$

where we forget the additional ringed structure on $\mathcal{O}$.

(ii) Let $\mathcal{O} : \mathcal{T}_{\mathrm{Zar}}(k) \to \mathcal{X}$ be a $\mathcal{T}_{\mathrm{Zar}}(k)$-local structure on the ∞-topos. We can restrict it to a $\mathcal{T}_{\mathrm{disc}}(k)$-structure on $\mathcal{X}$ via the natural inclusion functor $\mathcal{T}_{\mathrm{disc}}(k) \to \mathcal{T}_{\mathrm{Zar}}$ which thus induces a $\mathcal{C}\mathrm{Alg}$-valued sheaf on $\mathcal{X}$, which we still denote by $\mathcal{O}$. Condition (iii) in Theorem 3.2.2.5 implies that $\mathcal{O}$ can be identified with a local $\mathcal{C}\mathrm{Alg}$-valued, see [Lur11c, Proposition 4.2.3].

(iii) Similarly a $\mathcal{T}_{\mathrm{\acute{e}t}}(k)$-local structure on $\mathcal{X}$ corresponds to a $\mathcal{C}\mathrm{Alg}_k$-valued sheaf on $\mathcal{X}$ whose stalks are strictly Henselian. We refer the reader to [Lur16, Lemma 1.4.3.9] for a detailed proof of this result.

Definition 3.2.2.7. A $\mathcal{T}$-structured space is a pair $X := (\mathcal{X}, \mathcal{O})$ where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}$ is $\mathcal{T}$-local structure on $\mathcal{X}$. We denote by ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ the ∞-category of $\mathcal{T}$-structured topoi, see [Lur11c, Definition 3.1.9].

Definition 3.2.2.8. A derived Deligne-Mumford stack is a couple $(\mathcal{X}, \mathcal{O})$, where $\mathcal{O}\colon \mathcal{T}_{\mathrm{\acute{e}t}} \to \mathcal{X}$ is a $\mathcal{T}_{\mathrm{\acute{e}t}}$-structure on $\mathcal{X}$ verifying the following conditions:

(i) The 0-truncation $\mathrm{t}_{\leq 0}(\mathcal{X}, \mathcal{O}) := \left(\mathcal{X}, \pi_0(\mathcal{O}^{\mathrm{alg}})\right)$ is equivalent to an (ordinary) Deligne-Mumford stack;

(ii) For each $i > 0$, the higher homotopy sheaf $\pi_i\left(\mathcal{O}^{\mathrm{alg}}\right)$ is a quasi-coherent sheaf on $(\mathcal{X}, \mathcal{O})$.

3.2.3 Derived k-analytic geometry

Let k denote a non-archimedean field of non-trivial valuation. Derived k-analytic geometry as introduced in [PY16a] is a vast generalization of the classical theory of rigid analytic geometry. It is far more complicated to introduce derived k-analytic geometry through a functor of points approach. The main drawback comes from the fact that there is no reasonable description of the ∞-category of (derived) affinoid spaces. For this reason, we prefer to adopt a structured spaces approach as in [PY16a]. We will review the basic definitions and we shall refer the reader to [PY16a, PY17a] for a detailed account of the foundational aspects of the theory.

Definition 3.2.3.1. Let $\mathcal{T}_{\mathrm{an}}(k)$ denote the pregeometry whose underlying category consists of those k-analytic spaces which are smooth and whose admissible morphisms correspond to étale maps between these. We equip $\mathcal{T}_{\mathrm{an}}(k)$ with the étale topology.

Construction 3.2.3.2. Let X be an ordinary k-analytc space and denote $X_{\mathrm{\acute{e}t}}$ the associated small étale site on X. Let $\mathcal{X} := \mathrm{Shv}_{\mathrm{\acute{e}t}}(X\mathrm{\acute{e}t})^{\wedge}$ denote the hypercompletion of the ∞-topos of étale sheaves on X. We can attach to X a $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$ as follows: given $U \in \mathcal{T}_{\mathrm{an}}(k)$, we define the sheaf $\mathcal{O}(U) \in \mathcal{X}$ by

$$X_{\mathrm{\acute{e}t}} \ni V \mapsto \mathrm{Hom}_{\mathrm{An}}(V, U) \in \mathcal{S}.$$

As in the algebraic case, we can canonically identify $\mathcal{O}(\mathbb{A}^1_k)$ with the usual sheaf of analytic functions on X.

Definition 3.2.3.3. We say that $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos $(\mathcal{X}, \mathcal{O})$ is a *derived k-analytic space* if the following conditions are satisfied:

(i) $\mathcal{X}$ is hypercomplete and there exists an effective epimorphism $\coprod_i U_i \to 1_{\mathcal{X}}$ on $\mathcal{X}$ verifying:

(ii) For each i, the couple $(\mathcal{X}_{|U_i}, \pi_0(\mathcal{O}^{\mathrm{alg}}|U_i))$ is equivalent in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ to an ordinary k-analytic space, by means of Theorem 3.2.3.2.

(iii) For each index i and $j \geq 1$, the $\pi_j(\mathcal{O}^{\mathrm{alg}}|U_i)$ is a coherent sheaf over $\pi_0(\mathcal{O}^{\mathrm{alg}}|U_i)$-modules on X_i.

3.3 Derived k°-adic geometry

In this section we will introduce the *k°-adic pregeometry*, $\mathcal{T}_{\mathrm{ad}}(k^\circ)$, and study the corresponding theory of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured spaces. Let ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ denote the ∞-category of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured spaces. Our first goal is to make precise the assertion that a $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured ∞-topos $(\mathcal{X}, \mathcal{O})$ can be realized as a locally ringed ∞-topos $(\mathcal{X}, \mathcal{O}^{\mathrm{alg}})$ together with an adic topology on $\pi_0\left(\mathcal{O}^{\mathrm{alg}}\right)$. We will prove such assertion in §2.1 in the case where $\mathcal{O}$ is almost of finite presentation. We will also extend the Spf-construction introduced in [Lur16, §8.2] to the context of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured spaces. We will then proceed to a formal study of the theory of modules and obstruction theory in this context.

We show that Postnikov towers for $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured spaces exist and they are controlled by the adic cotangent complex.

3.3.1 Derived k°-adic spaces

Definition 3.3.1.1. Denote $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ the full subcategory of the category of k°-formal schemes spanned by those formal affine schemes which are formally étale over some $\mathfrak{A}^n_{k^\circ}$. We consider $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ as a pregeometry by defining the class of admissible morphisms on $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ to be the class of étale morphisms. We equip $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ with the étale topology.

Notation 3.3.1.2. Denote by ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ the ∞-category of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured ∞-topoi. Given $\mathcal{X} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ an ∞-topos we define $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) := \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{ad}}(k^\circ)}(\mathcal{X})$ the ∞-category of local $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structures on $\mathcal{X}$.

Notation 3.3.1.3. We have canonical transformation of pregeometries, denoted

$$(-)^\wedge_t \colon \mathcal{T}_{\mathrm{disc}}(k^\circ) \to \mathcal{T}_{\mathrm{ad}}(k^\circ), \ \ (-)^\wedge_t \colon \mathcal{T}_{\acute{\mathrm{e}}\mathrm{t}}(k^\circ) \to \mathcal{T}_{\mathrm{ad}}(k^\circ)$$

obtained by performing completion along the t-locus. Precomposition along these transformations induce functors at the level of the ∞-categories of structured ∞-topoi:

$$(-)^{\mathrm{alg}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ)),$$
$$(-)^{\mathrm{sh}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\acute{\mathrm{e}}\mathrm{t}}(k^\circ))$$

which are determined by the association

$$(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \mapsto (\mathcal{X}, \mathcal{O}^{\mathrm{alg}}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ))$$
$$(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \mapsto (\mathcal{X}, \mathcal{O}^{\mathrm{sh}}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\acute{\mathrm{e}}\mathrm{t}}(k^\circ)).$$

Let $\mathcal{X} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ be an ∞-topos. Both functors $(-)^{\mathrm{alg}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ))$ and $(-)^{\mathrm{sh}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\acute{\mathrm{e}}\mathrm{t}}(k^\circ))$ induce well defined functors at the level of ∞-categories of structures on $\mathcal{X}$:

$$(-)^{\mathrm{alg}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}),$$
$$(-)^{\mathrm{sh}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}^{\mathrm{sh}}_{k^\circ}(\mathcal{X})$$

which we refer to as the *underlying algebra functor* and the *underlying $\mathcal{T}_{\acute{\mathrm{e}}\mathrm{t}}(k^\circ)$-structure functor*, respectively.

Remark 3.3.1.4. Let $\mathcal{X} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ be an ∞-topos. The underlying algebra functor $(-)^{\mathrm{alg}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ can be upgraded as a functor $(-)^{\mathrm{ad}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ as follows: for each integer $n \geq 1$ and for each $\mathcal{A} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$, consider the canonical morphism $\mathcal{A}^{\mathrm{alg}} \to \mathcal{A}^{\mathrm{alg}} \otimes_{k^\circ} k^\circ_n \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$. Denote by $I_n := \ker\left(\pi_0(\mathcal{A}^{\mathrm{alg}}) \to \pi_0(\mathcal{A}^{\mathrm{alg}} \otimes_{k^\circ} k^\circ_n)\right)$. The sequence of ideals $\{I_n\}_{n \geq 1}$ defines an adic structure on $\mathcal{A}^{\mathrm{alg}}$ which is moreover compatible with the t-adic topology on k°. Moreover, for every morphism $f \colon \mathcal{A} \to \mathcal{B}$ in $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ the forgetful $f^{\mathrm{alg}} \colon \mathcal{A}^{\mathrm{alg}} \to \mathcal{B}^{\mathrm{alg}}$ is compatible with the adic topologies on both $\mathcal{A}^{\mathrm{alg}}$ and $\mathcal{B}^{\mathrm{alg}}$: this can be checked at the level of π_0 in which case follows from the fact that every morphism $\mathcal{A}^{\mathrm{alg}} \to \mathcal{B}^{\mathrm{alg}} \to \mathcal{B}^{\mathrm{alg}} \otimes_{k^\circ} k^\circ_n$ induces a unique, up to contractible space of choices, morphism $\mathcal{A}^{\mathrm{alg}} \otimes_{k^\circ} k^\circ_n \to \mathcal{B}^{\mathrm{alg}} \otimes_{k^\circ} k^\circ_n$. Therefore, by the universal property of $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ as a pullback we conclude that the $(-)^{\mathrm{alg}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ can be upgraded to a functor

$$(-)^{\mathrm{ad}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X}),$$

as desired.

Proposition 3.3.1.5. *Both* $(-)^{\mathrm{alg}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ))$ *and* $(-)^{\mathrm{sh}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\acute{\mathrm{e}}\mathrm{t}}(k^\circ))$ *admit right adjoints*

$$L \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$$
$$L^{\mathrm{sh}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\acute{\mathrm{e}}\mathrm{t}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))).$$

Proof. This is an immediate consequence of [Lur11c, Theorem 2.1]. □

We now proceed to have a better understanding of the action of L at the level of $\mathcal{T}_{\mathrm{disc}}$-structures:

Construction 3.3.1.6. Let $(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ be a $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured ∞-topos. Consider the comma ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ which is a presentable ∞-category thanks to [Por15a, Corollary 9.4]. The underlying algebra functor induces a well defined functor at the level of comma ∞-categories:

$$(-)^{\mathrm{alg}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}.$$

Thanks to [Por15a, Corollary 9.5] the above functor commutes with limits and sifted colimits. Thanks to the Adjoint functor theorem it follows that $(-)^{\mathrm{alg}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}$ admits a left adjoint which we shall denote $\Psi_{\mathcal{X}} \colon \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$, or simply Ψ if the underlying ∞-topos $\mathcal{X}$ is made explicit.

We refer the reader to [Lur16, §7.3] for the notion of t-completeness of modules.

Construction 3.3.1.7. Let $\mathcal{A} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}$ be a $\mathcal{T}_{\mathrm{disc}}$-structure on $\mathcal{X}$. We define $\mathcal{A}_n$ as the pushout of the diagram

$$\begin{tikzcd} \mathcal{A}[u] \arrow[r, "u \mapsto t^n"] \arrow[d] & \mathcal{A} \arrow[d] \\ \mathcal{A} \arrow[r, "u \mapsto 0"] & \mathcal{A}_n \end{tikzcd} \tag{3.3.1.1}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}_n^{\mathrm{alg}}}$. Where $\mathcal{O}_n^{\mathrm{alg}}$ is defined in a similar way and $\mathcal{A}[u]$ denotes the free algebra on one generator in degree 0 over $\mathcal{A}$. As Ψ is a left adjoint we obtain a pushout square

$$\begin{tikzcd} \Psi(\mathcal{A}[u]) \arrow[r, "u \mapsto t^n"] \arrow[d] & \Psi(\mathcal{A}) \arrow[d] \\ \Psi(\mathcal{A}) \arrow[r, "u \mapsto 0"] & \Psi(\mathcal{A}_n) \end{tikzcd} \tag{3.3.1.2}$$

in the ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{\mathcal{O}_n}$. Moreover, as in an ∞-topos every epimorphism is effective, and Ψ preserves epimorphisms, see Theorem .3.0.1. We deduce that the top horizontal morphism displayed in (3.3.1.2) is an effective epimorphism in $\mathcal{X}$. As the transformation of pregeometries $\mathcal{T}_{\mathrm{disc}}(k^\circ) \to \mathcal{T}_{\mathrm{ad}}(k^\circ)$ is unramified, see Appendix B, we deduce thanks to [Lur11a, Proposition 10.3] that we have a pushout diagram

$$\begin{tikzcd} \Psi(\mathcal{A}[u])^{\mathrm{alg}} \arrow[r, "u \mapsto t^n"] \arrow[d] & \Psi(\mathcal{A})^{\mathrm{alg}} \arrow[d] \\ \Psi(\mathcal{A})^{\mathrm{alg}} \arrow[r, "u \mapsto 0"] & \Psi(\mathcal{A}_n)^{\mathrm{alg}} \end{tikzcd}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}_n^{\mathrm{alg}}}$. Therefore, for each integer $n \geq 1$ the unit of the adjunction $(\Psi, (-)^{\mathrm{alg}})$ induces morphisms

$$f_{\mathcal{A},n} \colon \mathcal{A}_n \to \Psi(\mathcal{A})_n^{\mathrm{alg}}$$

such that the ideal $I_n := \ker\left(\pi_0(\mathcal{A}) \to \pi_0(\mathcal{A}_n)\right)$ is sent to the ideal $J_n := \ker\left(\pi_0\left(\Psi(\mathcal{A})^{\mathrm{alg}}\right) \to \pi_0\left(\Psi(\mathcal{A})_n^{\mathrm{alg}}\right)\right)$. Therefore, the universal property of t-completion induces a canonical morphism

$$f_{\mathcal{A}} \colon \mathcal{A}_t^\wedge \to \Psi(\mathcal{A})^{\mathrm{alg}},$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$.

Remark 3.3.1.8. Let $\mathcal{A} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{\mathcal{O}^{\mathrm{alg}}}$ be as in Theorem 3.3.1.7. The natural morphism

$$f_{\mathcal{A}} \colon \mathcal{A}_t^\wedge \to \Psi(\mathcal{A})^{\mathrm{alg}},$$

is continuous if we equip both $\mathcal{A}$ and $\Psi(\mathcal{A})^{\mathrm{alg}}$ with the adic topologies determined by the ideals $\{I_n\}_n$ and $\{J_n\}_n$ as in Theorem 3.3.1.7, respectively. In this case we can upgrade the morphism $f_{\mathcal{A}}$ to a morphism in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ thanks to Theorem 3.3.1.4.

Definition 3.3.1.9. Let $(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}})$. Let $\mathcal{A} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$, we say that $\mathcal{A}$ is *strictly Henselian* if it belongs to the essential image of the functor $\mathcal{C}\mathrm{Alg}^{\mathrm{sh}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}$ given on objects by the formula

$$\mathcal{A} \in \mathcal{C}\mathrm{Alg}^{\mathrm{sh}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \mapsto \mathcal{A}^{\mathrm{alg}} := \mathcal{A} \circ \iota \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}$$

where $\iota \colon \mathcal{T}_{\mathrm{disc}}(k^\circ) \to \mathcal{T}_{\text{ét}}(k^\circ)$ is the canonical transformation of pregeometries.

Remark 3.3.1.10. Notice that the functor $\mathcal{C}\mathrm{Alg}^{\mathrm{sh}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ introduced in Theorem 3.3.1.9 is fully faithful. This follows from [Lur11c, Proposition 4.3.19, Remark 2.5.13] together with [Lur09b, Proposition 7.2.1.14] and the proof of [Por15a, Proposition 9.2]. Therefore, we will usually abusively consider $\mathcal{C}\mathrm{Alg}^{\mathrm{sh}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ as a full subcategory of $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$.

We can now understand explicitly the composite $(-)^{\mathrm{alg}} \circ \Psi$:

Proposition 3.3.1.11. *Let $(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ))$ such that the underlying ∞-topos $\mathcal{X}$ has enough geometric points and $\mathcal{O}$ is stricly Henselian. Let $\mathcal{A} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ be an almost of finite presentation $\mathcal{T}_{\mathrm{disc}}(k^\circ)$-structure on $\mathcal{X}$ which we assume further to be strictly Henselian. Then the canonical map*

$$f_{\mathcal{A}} \colon \mathcal{A}_t^\wedge \to \Psi\left(\mathcal{A}\right)^{\mathrm{alg}}$$

introduced in Theorem 3.3.1.7 is an equivalence in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})_{/\Psi(\mathcal{O})^{\mathrm{alg}}}$.

Proof. We wish to show that the natural map

$$f_{\mathcal{A}} \colon \mathcal{A}_t^\wedge \to \Psi(\mathcal{A})^{\mathrm{alg}}$$

constructed in Theorem 3.3.1.7 is an equivalence whenever $\mathcal{A} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}$ is almost of finite presentation.

By hypothesis $\mathcal{X}$ has enough geometric points. Thus in order to show that $f_{\mathcal{A}}$ is an equivalence it suffices to show that its inverse image under any geometric point $(x^{-1}, x_*) \colon \mathcal{X} \to \mathcal{S}$, $x^{-1} f_{\mathcal{A}}$, is an equivalence in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}$. Set $A := x^{-1}\mathcal{A}$. Thanks to [Por15c, Theorem 1.12] we deduce that $\Psi_{\mathcal{S}}(A)^{\mathrm{alg}} \simeq x^{-1}\Psi(\mathcal{A})^{\mathrm{alg}}$. We are thus reduced to the case where $\mathcal{X} = \mathcal{S}$.

The ∞-category $(\mathcal{C}\mathrm{Alg}_{k^\circ})_{/\mathcal{O}^{\mathrm{alg}}}$ is generated under sifted colimits by free objects of the form $\{k^\circ[T_1, \dots, T_m]\}_{m \geq 1}$. Thanks to Theorem .3.0.1 we conclude that $(\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ})_{/\mathcal{O}} := (\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ})_{/\mathcal{O}}(\mathcal{S})$ is generated under sifted colimits by the family $\{\Psi(k^\circ[T_1, \dots T_m])\}_m$. As $A \in (\mathcal{C}\mathrm{Alg}_{k^\circ})_{/x^{-1}\mathcal{O}^{\mathrm{alg}}}$ is almost of finite presentation we conclude that it can be written as a retract of a filtered colimit of a diagram of the form

$$A_0 \to A_1 \to A_2 \to \dots,$$

where A_0 is an ordinary commutative ring of finite presentation over k° and A_{i+1} can be obtained from A_i as the following pushout

$$\begin{tikzcd} k^\circ[S^n] \arrow[r] \arrow[d] & k^\circ[X] \arrow[d] \\ A_i \arrow[r] & A_{i+1}, \end{tikzcd} \tag{3.3.1.3}$$

where $k^\circ[S^n]$ is the free simplicial k°-algebra generated in degree n by a single generator. Notice that, since A is almost of finite presentation we can choose the above diagram in such a way that for $i > 0$ sufficiently large, we have surjections $\pi_0(A_i) \to \pi_0(A_{i+1})$. As Ψ is a left adjoint it commutes, in particular, with pushout diagrams. We conclude that the diagram

$$\begin{tikzcd} \Psi(k^\circ[S^n]) \arrow[r] \arrow[d] & \Psi(k^\circ[X]) \arrow[d] \\ \Psi(A_i) \arrow[r] & \Psi(A_{i+1}), \end{tikzcd} \tag{3.3.1.4}$$

is a pushout diagram in the ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ and the morphism $\Psi(A_i) \to \Psi(A_{i+1})$ is moreover an epimorphism on π_0. For each $n > 0$, the morphism $k^\circ[S^n] \to k^\circ[X]$ is an effective epimorphism. As Ψ is a left adjoint, the morphism $\Psi(k^\circ[S^n]) \to \Psi(k^\circ[X])$ is an epimorphism in the (hypercomplete) ∞-topos $\mathcal{X}$ and thus an effective epimorphism. Thanks to [PY16a, Proposition 3.14] it follows that the morphism

$$\Psi(k^\circ[S^n])^{\mathrm{alg}} \to \Psi(k^\circ[X])^{\mathrm{alg}}$$

is an effective epimorphism. Therefore, as the transformation of pregeometries $\theta\colon \mathcal{T}_{\text{ét}}(k^\circ) \to \mathcal{T}_{\mathrm{ad}}(k^\circ)$ is unramified, see Appendix B, [Lur11a, Propositon 10.3] implies that the diagram,

$$\begin{tikzcd} \Psi(k^\circ[S^n])^{\mathrm{alg}} \arrow[r] \arrow[d] & \Psi(k^\circ[X])^{\mathrm{alg}} \arrow[d] \\ \Psi(A_i)^{\mathrm{alg}} \arrow[r] & \Psi(A_{i+1})^{\mathrm{alg}}, \end{tikzcd} \tag{3.3.1.5}$$

is a pushout square in $\mathcal{C}\mathrm{Alg}_{k^\circ}$. By induction we might assume that $\Psi(A_i)^{\mathrm{alg}}$ is equivalent to $(A_i)^\wedge_t$.

The transformation of pregeometries $(-)^\wedge_t\colon \mathcal{T}_{\text{ét}}(k^\circ) \to \mathcal{T}_{\mathrm{ad}}(k^\circ)$ is given by t-completion along the (t)-locus. Therefore, one has a canonical equivalence

$$\Psi(k^\circ[X])^{\mathrm{alg}} \simeq k^\circ\langle X\rangle^{\mathrm{sh}},$$

where latter the t-completion of the strictly Henselianization of $k^\circ[X]$. We claim that the natural map

$$\Psi(k^\circ[S^n])^{\mathrm{alg}} \to \big(k^\circ[S^n]^{\mathrm{sh}}\big)^\wedge_t$$

is an equivalence: notice that $k[S^n]$ fits into a pushout diagram

$$\begin{tikzcd} k^\circ[S^{n-1}] \arrow[r] \arrow[d] & k^\circ[X] \arrow[d] \\ k^\circ[X] \arrow[r] & k^\circ[S^n], \end{tikzcd}$$

the result then follows by induction on $n \geq 0$ and the case $n = 0$ was already treated. Since

$$\Psi(A_i)^{\mathrm{alg}} \to \Psi(A_{i+1})^{\mathrm{alg}}$$

is surjective on π_0, it follows that $\pi_0(\Psi(A_{i+1})^{\mathrm{alg}})$ is t-complete. For each $i \geq 0$ the $\pi_0\big(\Psi(A_{i+1})^{\mathrm{alg}}\big)$-modules $\pi_n\big(\Psi(A_{i+1})^{\mathrm{alg}}\big)$ are of finite presentation, thus t-complete $\pi_0(\Psi(A_{i+1})^{\mathrm{alg}})$-modules. It follows that $\Psi(A_{i+1})^{\mathrm{alg}}$ is t-complete by [Lur16, Theorem 7.3.4.1].

Let $A_{i+1} \to B$ be a morphism in $\mathcal{C}\mathrm{Alg}_{k^\circ}$ whose target is strictly Henselian and t-complete. Thanks to (3.3.2) such morphism induces morphisms $A_i \to B$ and $k^\circ[T] \to B$ compatible with both $k^\circ[S^n] \to k^\circ[T]$ and $k^\circ[S^n] \to A_i$, in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}$. By induction the effect of $(-) \circ \Psi$ on A_i, $k^\circ[S^n]$ and $k^\circ[X]$ agrees with strictly henselianization followed by t-completion. As B is both strictly henselian and t-complete it follows that the map $A_{i+1} \to B$ induces a well defined morphism from the diagram displayed in (3.3.1.5) to B. It follows that $\Psi(A_{i+1})^{\mathrm{alg}}$ satisfies the universal property of t-completion for the derived k°-algebra A_{i+1}. As $\Psi\,(A_{i+1})^{\mathrm{alg}}$ is t-complete we conclude that the morphism

$$f_{A_{i+1}} : (A_{i+1}^{\mathrm{sh}})^\wedge_t \to \Psi(A_{i+1})^{\mathrm{alg}},$$

where A_{i+1}^{sh} denotes the strict henselianization of A_{i+1}, is necessarily an equivalence. Let $A := \mathrm{colim}_i\, A_i$ in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}$. Fix $i \geq 0$, then $\tau_{\leq i}\Psi(A)^{\mathrm{alg}} \simeq \tau_{\leq i}\Psi(A_j)^{\mathrm{alg}}$ for j sufficiently large. We conclude then that $\pi_i(\Psi(A)^{\mathrm{alg}})$ is t-complete for $i \geq 0$. [Lur16, Theorem 7.3.4.1] implies that $\Psi(A)^{\mathrm{alg}}$ is t-complete. Reasoning as before we conclude that it satisfies the universal property of t-completion with respect to A. It follows that

$$f_A : A^\wedge_t \to \Psi(A)^{\mathrm{alg}}$$

is an equivalence in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}$, the result now follows. $\square$

Warning 3.3.1.12. The functor $(-)^{\mathrm{alg}} \circ \Psi$ is not in general equivalent to the t-completion functor $(-)^{\wedge}_t$. In fact, both $(-)^{\mathrm{alg}}$ and Ψ commute with filtered colimits, thus also their composite $(-)^{\mathrm{alg}} \circ \Psi$. Therefore, the composite $(-)^{\mathrm{alg}} \circ \Psi$ commutes with filtered colimits which is not the case of the t-completion functor, in general.

We will need also the following ingredient:

Construction 3.3.1.13. Denote by k_n° the reduction of k° modulo (t^n). Reduction modulo (t^n) induces a transformation of pregeometries

$$p_n \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{T}_{\mathrm{disc}}(k_n^\circ)$$
$$\operatorname{Spf} R \mapsto \operatorname{Spec} R_n$$

where $R_n := R \otimes_{k^\circ} k_n^\circ$. Precomposition along p_n induces a morphism at the level of structured ∞-topoi

$$p_n^{-1} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$$

which is given on objects by the formula

$$(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ)) \mapsto (\mathcal{X}, \mathcal{O} \circ p_n) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)).$$

Given $\mathcal{X} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ we obtain an induced functor at the level of structures $p_n^{-1} \colon \mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X}) \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ given on objects by the formula

$$\mathcal{O} \in \mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X}) \mapsto p_n^{-1}\mathcal{O} := \mathcal{O} \circ p_n \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}).$$

Notice that, we have a commutative triangle of the transformations of pregeometries of the form

$$\begin{tikzcd} \mathcal{T}_{\mathrm{disc}}(k^\circ) \arrow[r, "(-)^{\wedge}_t"] \arrow[dr, "- \otimes_{k^\circ} k_n^\circ"'] & \mathcal{T}_{\mathrm{ad}}(k^\circ) \arrow[d, "p_n"] \\ & \mathcal{T}_{\mathrm{disc}}(k_n^\circ) \end{tikzcd}.$$

For this reason, for every $\mathcal{X} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$, it follows that the composite $(-)^{\mathrm{alg}} \circ p_n^{-1} \colon \mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ coincides with the usual forgetful functor $\mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ along the induced map of derived rings $k^\circ \to k_n^\circ$. Notice that the latter functor admits a left adjoint which is given by extension of scalars along $k^\circ \to k_n^\circ$, i.e. it is given on objects by the formula

$$\mathcal{O} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \mapsto \mathcal{O} \otimes_{k^\circ} k_n^\circ \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$$

Notation 3.3.1.14. We will denote by $(-)_n \colon \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X})$ the functor given by extension of scalars along the canonical morphism of derived rings $k^\circ \to k_n^\circ$:

$$\mathcal{O} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \mapsto \mathcal{O}_n := \mathcal{O} \otimes_{k^\circ} k_n^\circ \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$$

It follows by [Lur11c, Theorem 2.1] that p_n^{-1} admits a right adjoint $L_n \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ))$ which we can explicitly describe:

Proposition 3.3.1.15. *The functor $p_n^{-1} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ admits a right adjoint*

$$L_n \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ))$$

whose restriction to the full subcategory of ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ spanned by those couples $(\mathcal{X}, \mathcal{O})$ such that the underlying ∞-topos $\mathcal{X}$ has enough points is given on objects by the formula

$$(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \mapsto (\mathcal{X}, \mathcal{O}_n^{\mathrm{alg}}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ)).$$

Proof. The existence of a left adjoint $L_n\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ))$ follows directly from [Lur11c, Theorem 2.1]. Let $(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ))$ and $(\mathcal{Y}, \mathcal{O}') \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ be such that $\mathcal{X} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ has enough points. Given any geometric morphism $(f^{-1}, f_*)\colon \mathcal{X} \to \mathcal{Y}$ we have a morphism of fiber sequences of the form

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left(f^{-1}\mathcal{O}', p_n^{-1}\mathcal{O}\right) \arrow[r] \arrow[d, "q"] & \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))}\left((\mathcal{X}, p_n^{-1}\mathcal{O}), (\mathcal{Y}, \mathcal{O}')\right) \arrow[r] \arrow[d, "p"] & \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y}) \arrow[d, equal] \\
\mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X})}\left((f^{-1}\mathcal{O}')_n^{\mathrm{alg}}, \mathcal{O}\right) \arrow[r] & \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ))}\left((\mathcal{X}, \mathcal{O}), (\mathcal{Y}, (\mathcal{O}')_n^{\mathrm{alg}})\right) \arrow[r] & \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y})
\end{tikzcd} \tag{3.3.1.6}$$

where $q\colon \mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left(f^{-1}\mathcal{O}', p_n^{-1}, \mathcal{O}\right) \to \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X})}\left((f^{-1}\mathcal{O}')_n^{\mathrm{alg}}, \mathcal{O}\right)$ coincides with the composite

$$\mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left(f^{-1}\mathcal{O}', p_n^{-1}\mathcal{O}\right) \xrightarrow{(\text{-})^{\mathrm{alg}}} \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left((f^{-1}\mathcal{O}')^{\mathrm{alg}}, p_n^{-1}\mathcal{O}^{\mathrm{alg}}\right) \to \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X})}\left((f^{-1}\mathcal{O}')_n^{\mathrm{alg}}, \mathcal{O}\right).$$

In order to prove the assertion of the proposition it suffices to show that the morphism p displayed in (3.3.1.6) is an equivalence of mapping spaces. Thanks to the fact that the horizontal arrow diagrams in (3.3.1.6) form fiber sequences we are reduced to prove that q is an equivalence of mapping spaces. As $\mathcal{X}$ has enough points we reduce ourselves to prove the statement of the Theorem at the level of stalks. For this reason we can assume from the start that $\mathcal{X} = \mathcal{S}$. Both target and source of q commute with filtered colimits on the first argument, thus we are reduced, as in the proof of Theorem 3.3.1.11 to prove that q is an equivalence whenever $f^{-1}\mathcal{O}' \simeq \Psi\left(k^\circ[T_1, \ldots, T_n]\right)$. We have natural equivalences of mapping spaces

$$\begin{aligned}
\mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}}\left(\Psi(k^\circ[T_1, \ldots T_m]), p_n^{-1}\mathcal{O}\right) &\simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k^\circ}}\left(k^\circ[T_1, \ldots T_m], (p_n^{-1}\mathcal{O})^{\mathrm{alg}}\right) \\
&\simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k_n^\circ}}\left(k^\circ[T_1, \ldots T_m]_n, (p_n^{-1}\mathcal{O})^{\mathrm{alg}}\right) \\
&\simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k_n^\circ}}\left(k_n^\circ[T_1, \ldots T_m], \mathcal{O}\right).
\end{aligned}$$

The result now follows from the observation that $\Psi\left(k^\circ[T_1, \ldots T_m]\right)_n^{\mathrm{alg}} \simeq k_n^\circ[T_1, \ldots, T_m]$ in the ∞-category $\mathcal{C}\mathrm{Alg}_{k_n^\circ}$, which is a direct consequence Theorem 3.3.1.11. ☐

Corollary 3.3.1.16. *Let $\mathcal{X} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ be an ∞-topos. The functor $L_n\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ))$ introduced in Theorem 3.3.1.15 induces a well defined functor at the level of the corresponding ∞-categories of structures*

$$(-)_n^{\mathrm{ad}}\colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X}),$$

given on objects by the formula

$$\mathcal{O} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \mapsto \mathcal{O}_n^{\mathrm{alg}} \in \mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X}).$$

Moreover, the functor $(-)_n^{\mathrm{ad}}$ is a left adjoint to the forgetful $p_n^{-1}\colon \mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X}) \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$

Proof. The existence of $(-)_n^{\mathrm{ad}}$ is guaranteed by Theorem 3.3.1.15. The fact that $(-)_n^{\mathrm{ad}}$ is a left adjoint to $p_n^{-1}\colon \mathcal{C}\mathrm{Alg}_{k_n^\circ}(\mathcal{X}) \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ follows from the proof of Theorem 3.3.1.15 together with the fact that both $(-)_n^{\mathrm{ad}}$ and p_n^{-1} are defined at the level of ∞-categories of structures on the same underlying ∞-topos. ☐

Notation 3.3.1.17. Consider the forgetful functor ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ))$ given by restriction of scalars along the morphism $k^\circ \to k_n^\circ$. We will denote $- \times_{\mathrm{Spec}\, k^\circ} \mathrm{Spec}\, k_n^\circ\colon : {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ))$ its right adjoint.

Corollary 3.3.1.18. *For each $n \geq 1$, the composite $L_n \circ L\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ))$ coincides with the base change functor*

$$\begin{gathered}
- \times_{\mathrm{Spec}\, k^\circ} \mathrm{Spec}\, k_n^\circ\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ)), \\
(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ)) \mapsto (\mathcal{X}, \mathcal{O}) \times_{\mathrm{Spec}\, k^\circ} \mathrm{Spec}\, k_n^\circ \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k_n^\circ))
\end{gathered}$$

Proof. This is a direct consequence of the definitions together with the commutative triangle displayed in Theorem 3.3.1.13. ☐

3.3.2 Comparison with derived formal geometry

Our main goal now is to give comparison statements between $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured ∞-topoi and *locally adic ringed ∞-topoi*. The latter corresponding to couples $(\mathcal{X}, \mathcal{O})$ where $\mathcal{O}$ is a $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$-valued sheaf on the ∞-topos $\mathcal{X}$. We shall moreover fix a couple $(\mathcal{X}, \mathcal{O}) \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ throughout this whole §.

Definition 3.3.2.1. Let $\mathcal{X}$ be an ∞-topos and $\mathcal{A} \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ be a $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$-valued sheaf on $\mathcal{X}$. We say that $\mathcal{A}$ is *topologically almost of finite presentation* if $\mathcal{A}$ is t-complete, the sheaf $\pi_0(\mathcal{A})$ is topologically finitely generated and for each $i > 0$ the homotopy sheaf $\pi_i(\mathcal{A})$ is finitely generated as a $\pi_0(\mathcal{A})$-module.

Definition 3.3.2.2. Let $\mathcal{X}$ be an ∞-topos and consider the functor $(-)^{\mathrm{ad}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ introduced in Theorem 3.3.1.4. We say that $\mathcal{A} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ is *topologically almost of finite presentation* if the underlying sheaf of adic algebras $\mathcal{A}^{\mathrm{ad}}$ is topologically almost of finite presentation. We denote $\mathrm{f}\mathcal{C}\mathrm{Alg}^{\mathrm{taft}}_{k^\circ}(\mathcal{X})$ the ∞-category of topologically almost of finite presentation local $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structures on $\mathcal{X}$.

Construction 3.3.2.3. Consider the adjunction $\big(\Psi, (-)^{\mathrm{alg}}\big) \colon \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ of Theorem 6.2.3.5 and let

$$(-)^{\mathrm{disc}} \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$$

denote the canonical functor obtained by forgetting the adic structure. Then the couple

$$\big(\Psi \circ (-)^{\mathrm{disc}}, (-)^{\mathrm{ad}}\big) \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$$

forms an adjunction pair after restriction

$$\big(\Psi^{\mathrm{ad}}, (-)^{\mathrm{ad}}\big) := \big(\Psi \circ (-)^{\mathrm{disc}}, (-)^{\mathrm{ad}}\big) \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad,taft}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}^{\mathrm{taft}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}},$$

where $\mathcal{C}\mathrm{Alg}^{\mathrm{ad,taft}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}}$ denotes the full subcategory of $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}}$ spanned by those objects $\mathcal{A} \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ topologically almost of finite presentation.

In order to see this consider the unit $\mathrm{id} \to (-)^{\mathrm{alg}} \circ \Psi$ of the adjunction in Theorem 6.2.3.5. It follows by the construction of $(-)^{\mathrm{ad}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ that we have an equivalence

$$(-)^{\mathrm{alg}} \simeq (-)^{\mathrm{disc}} \circ (-)^{\mathrm{ad}}$$

in the ∞-category $\mathrm{Fun}\big(\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}, \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}\big)$. Therefore, for each $\mathcal{A} \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad,taft}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}}$ the unit of adjunction

$$\mathcal{A}^{\mathrm{disc}} \to \big(\Psi(\mathcal{A}^{\mathrm{disc}})\big)^{\mathrm{alg}}$$

induces a canonically defined, up to a contractible space of choices, morphism

$$\mathcal{A} \simeq \mathcal{A}^\wedge_t \to \big(\Psi^{\mathrm{ad}}(\mathcal{A})\big)^{\mathrm{ad}}.$$

This construction is functorial and thanks to our previous considerations it satisfies the universal property of a unit of adjunction. Therefore we obtain an adjunction $\big(\Psi^{\mathrm{ad}}, (-)^{\mathrm{ad}}\big) : \mathcal{C}\mathrm{Alg}^{\mathrm{ad,taft}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}^{\mathrm{taft}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$, as desired.

Theorem 3.3.2.4. *Let $\mathcal{X}$ be an ∞-topos with enough geometric points. Consider the functor*

$$(-)^{\mathrm{ad}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}}$$

introduced in Theorem 3.3.1.4. Then the induced restriction functor

$$(-)^{\mathrm{ad}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}^{\mathrm{taft}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}}$$

is fully faithful and its essential image coincides precisely with the full subcategory of $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}}$ spanned by those strictly henselian $\mathcal{A} \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}}$ topologically almost of finite presentation.

Proof. Consider the adjunction $\left(\Psi^{\mathrm{ad}}, (-)^{\mathrm{ad}}\right) : \mathcal{C}\mathrm{Alg}^{\mathrm{ad,taft}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}^{\mathrm{taft}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ constructed in Theorem 3.3.2.3. Thanks to Theorem 3.3.1.11 the composite $(-)^{\mathrm{ad}} \circ \Psi^{\mathrm{ad}}$ is an equivalence when restricted to the subcategory $\mathcal{C} \subseteq \mathcal{C}\mathrm{Alg}^{\mathrm{ad,taft}}_{k^\circ}(\mathcal{X})$ spanned by strictly Henselian objects. Therefore the left adjoint functor

$$\Psi^{\mathrm{ad}} \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad,taft}}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{ad}}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$$

is fully faithfully when restricted to the full subcategory $\mathcal{C}$. [PY16a, Lemma 3.13] implies that the right adjoint functor $(-)^{\mathrm{ad}}$ is conservative, the conclusion now follows. □

Remark 3.3.2.5. Theorem 3.3.2.4 can be interpreted as a rectification statement. Indeed, an element $\mathcal{A} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ corresponds to a functor $\mathcal{A} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{X}$ satisfying the axioms for a $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structure on $\mathcal{X}$. Morphisms in $\mathcal{A} \to \mathcal{B}$ in $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ correspond to local morphisms in $\mathrm{Fun}\left(\mathcal{T}_{\mathrm{ad}}(k^\circ), \mathcal{X}\right)$.

On the other hand, objects in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ correspond to a derived k°-algebra on $\mathcal{X}$, $\mathcal{A} \in \mathcal{C}\mathrm{Alg}(\mathcal{X})$, together with the given of an adic topology on the sheaf of ordinary k°-algebras, $\pi_0(\mathcal{A})$. Moreover, morphisms in $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$ correspond to morphisms on $\mathcal{C}\mathrm{Alg}(\mathcal{X})$ which are continuous adic morphisms at the level of π_0. Therefore, a priori, one could expect that specifying morphisms in the ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}^{\mathrm{taft}}_{k^\circ}(\mathcal{X})$ would require an increase amount of higher coherence data when compared to the adic case.

Construction 3.3.2.6 (The Spf-construction)**.** Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ be a derived adic k°-algebra. We can associate to A an object $\mathrm{Spf}\, A := (\mathcal{X}_A, \mathcal{O}_A) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ as follows: we let $\mathcal{X}_A := \mathcal{H}\mathrm{Shv}^{\mathrm{ad}}_A \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ denote the hypercompletion of the ∞-topos $\mathrm{Shv}^{\mathrm{ad}}_A$ introduced in [Lur16, Notation 8.1.1.8]. We define moreover $\mathcal{O}_A \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{X}_A$ as the $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structure on $\mathcal{X}_A$ determined by the formula

$$\mathrm{Spf}(R) \in \mathcal{T}_{\mathrm{ad}}(k^\circ) \mapsto \left(B \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad,\acute{e}t}}_A \mapsto \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}}(R, B)\right)$$

where $\mathcal{C}\mathrm{Alg}^{\mathrm{ad,\acute{e}t}}_A$ denotes the full subcategory of $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_A$ spanned by those derived A-algebras B étale over A. One checks directly that $\mathcal{O}_A \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{X}_A$ is indeed a $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structure on $\mathcal{X}_A$. Such association is functorial in $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ and we thus obtain a well defined functor (up to contractible space of choices)

$$\mathrm{Spf} \colon \left(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}\right)^{\mathrm{op}} \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)),$$

which we refer as the Spf-*construction functor*.

Remark 3.3.2.7. Given $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$, it follows immediately by the definitions that $\mathrm{Spf}(A)^{\mathrm{alg}} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ))$ agrees with the Spf-construction introduced in [Lur16, §8.1.1]

Remark 3.3.2.8. Let $n \geq 1$ and consider the right adjoint functor $L_n \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ_n))$ introduced in Theorem 3.3.1.14. Given $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$, it follows that $L_n\left(\mathrm{Spf}(A)\right) \simeq (\mathcal{X}_A, \mathcal{O}_{A,n})$ where $\mathcal{O}_{A,n} := \mathcal{O}^{\mathrm{alg}}_A \otimes_{k^\circ} k^\circ_n \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$.

Proposition 3.3.2.9. *The functor* $\mathrm{Spf} \colon \left(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}\right)^{\mathrm{op}} \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ *is fully faithful. Moreover, its essentially image corresponds precisely to the full subcategory of* ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ *spanned by those couples* $(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ *such that* $(\mathcal{X}, \mathcal{O}^{\mathrm{alg}}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ))$ *is equivalent to a formal spectrum.*

Proof. Let $A, B \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ and consider the corresponding formal spectrums $\mathrm{Spf}(A)$ and $\mathrm{Spf}(B) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$. The datum of a morphism of local $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structures $f \colon \mathrm{Spf}(A) \to \mathrm{Spf}(B)$ is equivalent to the datum of a geometric morphism of ∞-topoi $(f^{-1}, f_*) \colon \mathcal{X}_A \to \mathcal{Y}_B$ together with a natural transformation $\alpha \colon f^{-1}\mathcal{O}_B \to \mathcal{O}_A$. Applying the underlying algebra functor at the level of structures we obtain a morphism

$$\alpha^{\mathrm{alg}} \colon f^{-1}\left(\mathcal{O}^{\mathrm{ad}}_B\right)^{\mathrm{alg}} \to \left(\mathcal{O}^{\mathrm{ad}}_A\right)^{\mathrm{alg}}$$

in the ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_A)$. The unit of the adjunction (f^{-1}, f_*) produces a well defined morphism of derived k°-algebras $\phi \colon B \to A$, up to contractible indeterminacy.

By the construction of the underlying ∞-topoi of both $\mathrm{Spf}(A)$ and $\mathrm{Spf}(B)$ together with [Lur16, Remark 8.1.1.7] it follows that the morphism $\phi \colon B \to A$ is continuous with respect to the adic topologies for both A and B. We obtain thus a well defined morphism of mapping spaces

$$\Phi \colon \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))}(\mathrm{Spf}\, A, \mathrm{Spf}\, B) \to \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}}(B, A).$$

Let $\phi\colon B \to A$ be a continuous morphism of derived adic k°-algebras. In order to show that the functor

$$\mathrm{Spf}\colon \left(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}\right)^{\mathrm{op}} \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$$

is fully faithful it suffices to show that the fiber $Z_\phi := \mathrm{fib}_\phi(\Phi)$ is contractible for any such ϕ. To any continuous adic morphism, we can attach a well defined, up to contractible indeterminacy, morphism on the corresponding (formal) étale sites. We have thus a canonical morphism at the level of mapping spaces

$$\theta\colon \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}}(B, A) \to \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}}(\mathcal{X}_A, \mathcal{Y}_B).$$

Let $(f^{-1}, f_*)\colon \mathcal{X}_A \to \mathcal{Y}_B$ be a morphism of ∞-topoi such that it lies in the essential image of ϕ under θ. The fiber over (f^{-1}, f_*) induces a fiber sequence of mapping spaces:

$$\mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left(f^{-1}\mathcal{O}_B, \mathcal{O}_A\right) \longrightarrow \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))}(\mathrm{Spf}\, A, \mathrm{Spf}\, B) \xrightarrow{\theta} \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}}(\mathcal{X}_A, \mathcal{Y}_B).$$

Consider the commutative diagram in the ∞-category $\mathcal{S}$

$$\begin{tikzcd}
Z_\phi \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left(f^{-1}\mathcal{O}_B, \mathcal{O}_A\right) \arrow[r] \arrow[d] & \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))}(\mathrm{Spf}(A), \mathrm{Spf}(B)) \arrow[d] \\
\{\phi\} \arrow[r] & W \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}}(B, A) \arrow[d] \\
 & \{(f^{-1}, f_*)\} \arrow[r] & \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}}(\mathcal{X}_A, \mathcal{Y}_B)
\end{tikzcd} \tag{3.3.2.1}$$

where both the upper rectangle and and the bottom right square are pullback diagrams. It follows then that we can identify Z_ϕ with the pullback

$$Z_\phi \simeq \mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left(f^{-1}\mathcal{O}_B, \mathcal{O}_A\right) \times_W \{\phi\}.$$

Let $F\colon \mathrm{Spf}(A) \to \mathrm{Spf}(B)$ be a morphism of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured ∞-topoi such that $\Phi(F) \simeq \phi$. It follows by [Lur16, Remark 8.1.1.7] that the induce geometric morphism $(f^{-1}, f_*)\colon \mathcal{X}_A \to \mathcal{Y}_B$ can be identified with the restriction to closed subtopoi of the geometric morphism of ∞-topoi $\mathcal{X}_A \to \mathcal{Y}_B$. Thanks to the proof of [Lur16, Proposition 1.4.2.4] it follows that the latter is uniquely determined up to a contractible space of choices. For this reason (f^{-1}, f_*) is also uniquely determined by ϕ, up to a contractible space of choices. As a consequence we can identify Z_ϕ with the fiber product:

$$Z_\phi \simeq \mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left(f^{-1}\mathcal{O}_B, \mathcal{O}_A\right) \times_W \{\phi\}.$$

We have a sequence of equivalences of mapping spaces

$$\begin{aligned}
\mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left(f^{-1}\mathcal{O}_B, \mathcal{O}_A\right) \times_W \{\phi\} &\simeq \mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left(f^{-1}\mathcal{O}_B, \lim_{n\geq 1}(\mathcal{O}_{A,n})\right) \times_W \{\phi\} \\
&\simeq \left(\lim_{n\geq 1}\mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left(f^{-1}\mathcal{O}_B, \mathcal{O}_{A,n}\right)\right) \times_W \{\phi\}
\end{aligned}$$

We can further identify the last term with

$$\begin{gathered}
\left(\lim_{n\geq 1}\mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})}\left(f^{-1}\mathcal{O}_B, \mathcal{O}_{A,n}\right)\right) \times_W \{\phi\} \simeq \\
\left(\lim_{n\geq 1}\mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k^\circ_n}(\mathcal{X})}\left(f^{-1}\mathcal{O}_{B,n}, \mathcal{O}_{A,n}\right)\right) \times_W \{\phi\}
\end{gathered}$$

For each $n \geq 1$, denote ϕ_n the base change of ϕ to k°_n. Passing to the limit over $n \geq 1$ we can further identify the last term with

$$\lim_{n\geq 1}\left(\mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k^\circ_n}(\mathcal{X})}\left(f^{-1}\mathcal{O}_{B,n}, \mathcal{O}_{A,n}\right) \times_{W_n} \{\phi_n\}\right) \simeq \lim_{n\geq 1}\left(\mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{disc}}(k^\circ_n))}(A_n, B_n) \times_{W_n} \{\phi_n\}\right), \tag{3.3.2.2}$$

where W_n is defined as the fiber product of the corresponding diagram obtained as the reduction modulo t^n of the bottom right square, displayed in (3.3.2.1). Thanks to the proof of [Lur16, Corollary 1.2.3.5.] each term in displayed limit displayed in (3.3.2.2) can be identified with

$$\mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k_n^\circ}}(B_n, A_n) \times_{\mathcal{C}\mathrm{Alg}_{k_n^\circ}(B_n, A_n)} \{\phi_n\}$$

which is thus a contractible space. The result now follows by a simple analysis on the corresponding Milnor exact fiber sequence. $\square$

Definition 3.3.2.10. A derived k°-adic Deligne-Mumford stack is a couple $(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ such that $(\mathcal{X}, \mathcal{O}^{\mathrm{alg}})$ formal derived Deligne-Mumford stack as in [Lur16, Definition 8.1.3.1]. We say that a derived k°-adic Deligne-Mumford stack $(\mathcal{X}, \mathcal{O})$ is topologically almost of finite presentation if the underlying ∞-topos $\mathcal{X}$ is coherent (cf. [Lur11e, §3]) and the $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structure $\mathcal{O} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ is topologically almost of finite presentation.

Notation 3.3.2.11. We denote dfDM_{k° (resp., $\mathrm{dfDM}_{k^\circ}^{\mathrm{taft}}$) the full subcategory of ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ spanned by derived k°-adic Deligne-Mumford stacks (resp., topologically almost of finite presentation k°-adic Deligne-Mumford stacks).

Definition 3.3.2.12. We denote by dfSch the full subcategory of dfDM spanned by those objects $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ such that $(\mathcal{X}, \pi_0\mathcal{O}^{\mathrm{alg}})$ is equivalent to an ordinary derived formal scheme over k°. We refer to objects in dfSch as derived k°-adic formal schemes. We also define the ∞-category of topological almost of finite presentation derived k°-adic schemes as $\mathrm{dfSch}^{\mathrm{taft}} := \mathrm{dfDM}^{\mathrm{taft}} \cap \mathrm{dfSch}$.

Remark 3.3.2.13. The functor $\mathrm{Spf}\colon \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ factors through the fully faithful embedding $\mathrm{dfSch} \hookrightarrow {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$.

Remark 3.3.2.14 (Global spectrum construction)**.** Let fDM_{k° denote the category of ordinary Deligne-Mumford stacks and let $\mathsf{X} \in \mathrm{fDM}_{k^\circ}$ be a Deligne-Mumford stack. To X we can associate a $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured ∞-topos as follows: Let $\mathsf{X}_{\mathrm{fét}}$ denote the formally étale site on X. Denote by $\mathcal{X} := \mathrm{Shv}_{\mathrm{ét}}(\mathsf{X})^\wedge$ the hypercompletion of the ∞-topos $\mathrm{Shv}(\mathsf{X}_{\mathrm{fét}}, \mathcal{T}_{\mathrm{fét}})$. We define a $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structure on $\mathcal{X}$ by the formula

$$\mathrm{Spf}(R) \in \mathcal{T}_{\mathrm{ad}}(k^\circ) \mapsto \left(\mathsf{Y} \in \mathsf{X}_{\mathrm{fét}} \mapsto \mathrm{Map}_{\mathrm{fDM}_{k^\circ}}(\mathsf{Y}, \mathrm{Spf}(R)) \in \mathcal{S}\right).$$

In this case, $\mathcal{O}(\mathfrak{A}^1_{k^\circ}) \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ corresponds to the usual structure sheaf of continuous adic functions on X. This association is functorial and it provides us with a fully faithful embedding

$$\mathrm{fDM}_{k^\circ} \subseteq \mathrm{dfDM}_{k^\circ}$$

of ∞-categories.

3.3.3 Derived ∞-categories of modules for $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured spaces

Definition 3.3.3.1. Let $\mathsf{X} := (\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$. We define the ∞-category of modules on X as

$$\mathrm{Mod}_{\mathcal{O}} := \mathrm{Sp}\left(\mathrm{Ab}\left(\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}\right)\right),$$

where Ab denotes the abelianization functor, see [Lur12c, §1.4], and Sp the stabilization functor.

Remark 3.3.3.2. Let $(\mathcal{X}, \mathcal{O})$ be as above. The ∞-category $\mathrm{Mod}_{\mathcal{O}}$ is stable.

Construction 3.3.3.3. Given $(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ we can also consider the ∞-category of modules on its algebraization $(\mathcal{X}, \mathcal{O}^{\mathrm{alg}})$ defined as $\mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}} := \mathrm{Shv}_{\mathcal{D}(\mathrm{Ab})}(\mathcal{X})$, where $\mathcal{D}(\mathrm{Ab}) := \mathrm{Mod}_{\mathbb{Z}}$ denotes the derived ∞-category of $\mathbb{Z}$-modules. Thanks to [Lur12c, Theorem 7.3.4.13] one has a natural equivalence

$$\mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}} \simeq \mathrm{Sp}\left(\mathrm{Ab}\left(\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{\mathcal{O}^{\mathrm{alg}}}\right)\right),$$

in the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{st}}$. As the underlying algebra functor $(-)^{\mathrm{alg}}\colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}$ is a right adjoint it induces an exact functor at the level of derived ∞-categories of modules denoted

$$g^{\mathrm{alg}}\colon \mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}}.$$

We claim that the left adjoint $\Psi\colon \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ induces also a well defined functor

$$f^{\mathrm{ad}}\colon \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}} \to \mathrm{Mod}_{\mathcal{O}},$$

which is a left adjoint to g^{alg}. It suffices to prove that Ψ commutes with finite limits. We start by observing that as the composite $(-)^{\mathrm{alg}} \circ \Psi$ agrees with the t-completion functor on the full subcategory of almost of finite presentation objects $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})^{\mathrm{afp}}_{/\mathcal{O}}$ it commutes with small limits on $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})^{\mathrm{afp}}_{/\mathcal{O}^{\mathrm{alg}}}$. As $(-)^{\mathrm{alg}}$ is a conservative right adjoint, it follows that Ψ itself commutes with finite limits on $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})^{\mathrm{afp}}_{/\mathcal{O}^{\mathrm{alg}}}$. Let now $\mathcal{A} \in \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}$ be a general object. We can realize $\mathcal{A}$ as a filtered colimit of almost of finite presentation objects in $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}$. Let $\{\mathcal{A}_i\}_i$ be a diagram indexed by a finite ∞-category I, and for each $i \in I$ choose a presentation

$$\mathcal{A}_i \simeq \operatorname*{colim}_{m\in J} \mathcal{A}_{i,m},$$

where $\mathcal{A}_{i,m}$ is almost of finite presentation and J is a filtered ∞-category. We have thus a sequence of equivalences in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}$

$$\begin{aligned}
\Psi(\lim_i \mathcal{A}_i)^{\mathrm{alg}} &\simeq \operatorname*{colim}_m \Psi(\lim_i \mathcal{A}_{i,m})^{\mathrm{alg}} \\
&\simeq \operatorname*{colim}_m \lim_i (\mathcal{A}_{i,m})^\wedge_t \simeq \lim_i \operatorname*{colim}_m (\mathcal{A}_{i,m})^\wedge_t \\
&\simeq \lim_i \operatorname*{colim}_m \Psi(\mathcal{A}_{i,m})^{\mathrm{alg}} \simeq \lim_i \Psi(\mathcal{A}_{i,m})^{\mathrm{alg}}
\end{aligned}$$

and the conclusion now follows as in the preceding case.

Proposition 3.3.3.4. *Suppose $\mathcal{X}$ has enough geometric points and $\Psi(\mathcal{O}^{\mathrm{alg}})^{\mathrm{alg}} \simeq \mathcal{O}^{\mathrm{alg}}$. Then the functor*

$$g^{\mathrm{alg}}\colon \mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}}$$

is an equivalence of stable ∞-categories.

Proof. Let $f^{\mathrm{ad}}\colon \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}} \to \mathrm{Mod}_{\mathcal{O}}$ denote a left adjoint to g^{alg} which is induced by the functor Ψ introduced in §3.1. We want to show that f^{ad} is an inverse to g^{alg}, as functors. Notice that the functor g^{alg} is conservative as $(-)^{\mathrm{alg}}$ was already conservative. Therefore, we are reduced to show that f^{ad} is a fully faithfully functor. It suffices to show that the unit η of the adjunction $(f^{\mathrm{ad}}, g^{\mathrm{alg}})$ is an equivalence.

As $\mathcal{X}$ has enough geometric points we reduce ourselves to check the last assertion at the level of stalks. We are thus reduced to the case $\mathcal{X} = \mathcal{S}$. In this case, the ∞-category $\mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}}$ is compactly generated by $\mathcal{O}^{\mathrm{alg}} \in \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}}$. The $(-)^{\mathrm{alg}}$ commutes with filtered colimits (even sifted colimits) thus we deduce that also g^{alg} commutes with filtered colimits. As g^{alg} is an exact functor between stable ∞-categories we conclude that it commutes with all colimits. Therefore, the unit η commutes with colimits. We are thus reduced to check that η is an equivalence on the compact generator $\mathcal{O}^{\mathrm{alg}} \in \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}}$. By our assumption on $\mathcal{O}^{\mathrm{alg}}$ the result follows thanks to Theorem 3.3.1.11. □

Remark 3.3.3.5. The equivalence of stable ∞-categories provided in Theorem 3.3.3.4 allow us to define a t-structure on the ∞-category $\mathrm{Mod}_{\mathcal{O}}$ by means of the functor g^{alg}.

Definition 3.3.3.6. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$. We define the ∞-category of coherent $\mathcal{O}$-modules on $\mathcal{X}$, denoted $\mathrm{Coh}^+(\mathsf{X})$ as the full subcategory $\mathrm{Mod}_{\mathcal{O}}$ spanned by those $\mathcal{F}$ such that for each integer i the homotopy sheaves $\pi_i(\mathcal{F})$ are coherent $\pi_0(\mathcal{O}^{\mathrm{alg}})$-modules and vanish for sufficiently small i.

3.3.4 k°-adic cotangent complex

In this §we will introduce the notion of formal cotangent complex, which will prove to be of fundamental importance to us: we have a projection functor

$$\Omega^\infty_{\mathrm{ad}}\colon \mathrm{Mod}_{\mathcal{O}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}},$$

which is given by evaluation on the object $(S^0, *) \in \mathcal{S}^{\mathrm{fin}}_* \times \mathcal{T}_{\mathrm{Ab}}$. The functor $\Omega^\infty_{\mathrm{ad}}$ admits a left adjoint

$$\Sigma^\infty_{\mathrm{ad}}\colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}}.$$

We refer the reader to [PY17a, §5.1] and [Lur12c, §7.5] for more details about these constructions.

Definition 3.3.4.1. Let $M \in \mathrm{Mod}_{\mathcal{O}}$ we shall refer to $\mathcal{O} \oplus M := \Omega^\infty_{\mathrm{ad}}(M)$ as the *trivial adic square-zero extension of $\mathcal{O}$ by the module M.*

Definition 3.3.4.2. Let $\mathsf{X} := (\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ and let $\mathcal{A} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ be a $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ structure on $\mathcal{X}$. Given $M \in \mathrm{Mod}_{\mathcal{O}}$, we define the space of *$\mathcal{A}$-linear adic derivations* of $\mathcal{O}$ with values in M as

$$\mathrm{Der}^{\mathrm{ad}}_{\mathcal{A}}(\mathcal{O}, M) := \mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{\mathcal{A}//\mathcal{O}}}(\mathcal{O}, \mathcal{O} \oplus M) \in \mathcal{S}.$$

Proposition 3.3.4.3. *The functor*

$$\mathrm{Der}^{\mathrm{ad}}_{\mathcal{A}}(\mathcal{O}, -) : \mathrm{Mod}_{\mathcal{O}} \to \mathcal{S},$$

is corepresentable by an object

$$\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}/\mathcal{A}} \in \mathrm{Mod}_{\mathcal{O}}$$

which we refer to as the adic cotangent complex relative to $\mathcal{O} \to \mathcal{A}$.

Proof. The proof is a direct consequence of the existence of a left adjoint $\Sigma^\infty_{\mathrm{ad}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}}$. Set $\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}/\mathcal{A}} := \Sigma^\infty_{\mathrm{ad}}(\mathcal{O} \otimes_{\mathcal{A}} \mathcal{O})$. For every $M \in \mathrm{Mod}_{\mathcal{O}}$ we have a sequence of natural equivalences of mapping spaces of the form

$$\begin{aligned}
\mathrm{Der}^{\mathrm{ad}}_{\mathcal{A}}(\mathcal{O}, M) &\simeq \mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{\mathcal{A}//\mathcal{O}}}(\mathcal{O}, \mathcal{O} \oplus M) \\
&\simeq \mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{\mathcal{A}//\mathcal{O}}}(\mathcal{O}, \Omega^\infty_a d(M)) \\
&\simeq \mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{\mathcal{O}//\mathcal{O}}}(\mathcal{O} \otimes_{\mathcal{A}} \mathcal{O}, \mathcal{O} \oplus M) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}}(\Sigma^\infty_{\mathrm{ad}}(\mathcal{O} \otimes_{\mathcal{A}} \mathcal{O}), \mathcal{O}) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}}\left(\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}/\mathcal{A}}, M\right),
\end{aligned}$$

and the result follows. ☐

Proposition 3.3.4.4. *Let $\mathcal{A} \to \mathcal{B}$ be a morphism in $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ topologically almost of finite presentation. Then $\mathbb{L}^{\mathrm{ad}}_{\mathcal{B}/\mathcal{A}}$ is a compact object in the ∞-category $\mathrm{Mod}_{\mathcal{O}}$.*

Proof. The proof of [Lur16, Proposition 4.1.2.1] applies. ☐

Remark 3.3.4.5. Notice that we have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Mod}_{\mathcal{O}} \arrow[r, "g^{\mathrm{alg}}"] \arrow[d, "\Omega^\infty_{\mathrm{ad}}"] & \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}} \arrow[d, "\Omega^\infty"] \\
\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \arrow[r, "(-)^{\mathrm{alg}}"] & \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}},
\end{tikzcd}$$

therefore passing to left adjoints we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Mod}_{\mathcal{O}} & \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}} \arrow[l, "f^{\mathrm{ad}}"'] \\
\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \arrow[u, "\Sigma^\infty_{\mathrm{ad}}"] & \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}} \arrow[l, "\Psi"] \arrow[u, "\Sigma^\infty"']
\end{tikzcd} \tag{3.3.4.1}$$

in the ∞-category $\mathcal{C}\mathrm{at}_\infty$. The commutative of (3.3.4.1) provide us with a natural map

$$f^{\mathrm{ad}}\left(\mathbb{L}_{\mathcal{B}^{\mathrm{alg}}/\mathcal{A}^{\mathrm{alg}}}\right) \to \mathbb{L}^{\mathrm{ad}}_{\mathcal{B}/\mathcal{A}}$$

in the ∞-category $\mathrm{Mod}_{\mathcal{O}}$.

Proposition 3.3.4.6. *Let $\mathcal{A} \to \mathcal{B}$ be a morphism in $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ and consider the algebraic cotangent complex $\mathbb{L}_{\mathcal{B}^{\mathrm{alg}}/\mathcal{A}^{\mathrm{alg}}}$ associated to the morphism $\mathcal{A}^{\mathrm{alg}} \to \mathcal{B}^{\mathrm{alg}}$. Then the natural map introduced in Theorem 3.3.4.5*

$$f^{\mathrm{ad}}\left(\mathbb{L}_{\mathcal{B}^{\mathrm{alg}}/\mathcal{A}^{\mathrm{alg}}}\right) \simeq \mathbb{L}^{\mathrm{ad}}_{\mathcal{B}/\mathcal{A}}$$

is an equivalence in the ∞-category $\mathrm{Mod}_{\mathcal{O}}$.

Proof. The construction of the adic cotangent complex commutes with filtered colimits of local $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structures. Therefore we can suppose that the morphism $\mathcal{A} \to \mathcal{B}$ is topologically almost of finite presentation and $\mathcal{A}$ and $\mathcal{B}$ and $\mathcal{A}$ is itself topologically almost of finite presentation. Let $\mathcal{A} \to \mathcal{B}$ be a morphism in $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$ and consider $\mathbb{L}_{\mathcal{B}^{\mathrm{alg}}/\mathcal{A}^{\mathrm{alg}}} \in \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}}$ the algebraic cotangent complex associated to $\mathcal{A}^{\mathrm{alg}} \to \mathcal{B}^{\mathrm{alg}}$. By applying the functor

$$f^{\mathrm{ad}} \colon \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}} \to \mathrm{Mod}_{\mathcal{O}},$$

we obtain the following sequence of equivalences of mapping spaces

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}}\left(f^{\mathrm{ad}}(\mathbb{L}_{\mathcal{B}^{\mathrm{alg}}/\mathcal{A}^{\mathrm{alg}}}), M\right) &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}}}\left(\mathbb{L}_{\mathcal{B}^{\mathrm{alg}}/\mathcal{A}^{\mathrm{alg}}}, g^{\mathrm{alg}}(M)\right) \\
&\simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}}\left(\mathcal{A}^{\mathrm{alg}}, \mathcal{A}^{\mathrm{alg}} \oplus g^{\mathrm{alg}}(M)\right) \\
&\simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}}\left(\mathcal{A}^{\mathrm{alg}}, (\mathcal{A} \oplus M)^{\mathrm{alg}}\right) \\
&\simeq \mathrm{Map}_{\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}}\left(\mathcal{A}, \mathcal{A} \oplus M\right)
\end{aligned}$$

where the latter equivalence holds by fully faithfulness of the functor $(-)^{\mathrm{alg}}$, as $\mathcal{A}$ is topologically almost of finite presentation and t-complete and whenever M is coherent, which we can assume from the start thanks to Theorem 3.3.4.4. □

Proposition 3.3.4.7. *Let $f \colon \mathcal{A} \to \mathcal{B}$ and $g \colon \mathcal{B} \to \mathcal{C}$ be morphisms in the ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$. Then one has a fiber sequence*

$$\mathbb{L}^{\mathrm{ad}}_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{C} \to \mathbb{L}^{\mathrm{ad}}_{\mathcal{C}/\mathcal{A}} \to \mathbb{L}^{\mathrm{ad}}_{\mathcal{C}/\mathcal{B}}$$

in $\mathrm{Mod}_{\mathcal{O}}$.

Proof. This is a direct consequence of [PY17a, Proposition 5.10]. □

Proposition 3.3.4.8. *Suppose we are given a pushout diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

in the ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$. Then the natural morphism

$$\mathbb{L}^{\mathrm{ad}}_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}} \mathcal{D} \to \mathbb{L}^{\mathrm{ad}}_{\mathcal{D}/\mathcal{C}}$$

is an equivalence in the ∞-category $\mathrm{Mod}_{\mathcal{O}}$.

Proof. The assertion is a particular case of [PY17a, Proposition 5.12]. □

3.3.5 Postnikov towers of k°-adic spaces

Definition 3.3.5.1. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ and $M \in (\mathrm{Mod}_{\mathcal{O}})_{\geq 1}$ be an $\mathcal{O}$-module concentrated in homological degrees ≥ 1. A *k°-adic square zero extension of* X *by* M consists of a $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-adic structured ∞-topos $\mathsf{X}' = (\mathcal{X}, \mathcal{O}')$ equipped with a morphism $f \colon \mathsf{X} \to \mathsf{X}'$ satisfying:

(i) The underlying geometric morphism of f is equivalent to the identity of $\mathcal{X}$;

(ii) There exists an k°-adic derivation

$$d \colon \mathbb{L}^{\mathrm{ad}}_{\mathsf{X}} \to M[1] \in \mathrm{Mod}_{\mathcal{O}}$$

such that we have a pullback diagram in the ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$

$$\begin{tikzcd}
\mathcal{O}' \arrow[r] \arrow[d] & \mathcal{O} \arrow[d, "d"] \\
\mathcal{O} \arrow[r, "d_0"] & \mathcal{O} \oplus M[1]
\end{tikzcd}$$

where d_0 denotes the trivial k°-adic derivation.

Definition 3.3.5.2. Let $\mathcal{T}$ be a pregeometry and let $n \geq -1$ be an integer. We say that $\mathcal{T}$ is compatible with n-truncation if for every ∞-topos $\mathcal{X}$, every $\mathcal{T}$-structure $\mathcal{O} \colon \mathcal{T} \to \mathcal{X}$ and every admissible morphism $U \to V$ in $\mathcal{T}$, the induced square

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r] \arrow[d] & \tau_{\leq n}\mathcal{O}(U) \arrow[d] \\ \mathcal{O}(V) \arrow[r] & \tau_{\leq n}\mathcal{O}(U) \end{tikzcd}$$

is a pullback diagram in $\mathcal{X}$.

Remark 3.3.5.3. The above definition is equivalent to require that given a couple $(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$, the truncation $(\mathcal{X}, \tau_{\leq n}\mathcal{O})$, where $\tau_{\leq n} \colon \mathcal{X} \to \mathcal{X}$ denotes the n-truncation functor on $\mathcal{X}$, is again an object of the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$, or in other terms, $\tau_{\leq n}\mathcal{O} \colon \mathcal{T} \to \mathcal{X}$ is still a $\mathcal{T}$-structure on $\mathcal{X}$.

Notation 3.3.5.4. Let $\mathcal{T}$ be a preogeometry compatible with n-truncations. We will denote ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})^{\leq n} \subseteq {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ the full subcategory spanned by those couples $(\mathcal{X}, \mathcal{O})$ such that the $\mathcal{T}$-structure $\mathcal{O} \colon \mathcal{T} \to \mathcal{X}$ is n-truncated.

Remark 3.3.5.5. The inclusion functor ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})^{\leq n} \subseteq {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ admits a right adjoint $\mathrm{t}_{\leq n} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})^{\leq n}$ which is given on objects by the formula

$$(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}) \mapsto (\mathcal{X}, \tau_{\leq n}\mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})^{\leq n}.$$

Lemma 3.3.5.6. *The pregeometry $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ is compatible with n-truncations.*

Proof. We follow closely [Lur11c, Proposition 4.3.28]. Reasoning as in the proof of the cited reference or as in the proof of Theorem 3.3.1.11 it suffices to prove the following assertion: let $U \to V$ be an admissible morphism in $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ and $\mathcal{O} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{S})$ then the commutative square

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r] \arrow[d] & \tau_{\leq 0}\mathcal{O}(U) \arrow[d] \\ \mathcal{O}(U) \arrow[r] & \tau_{\leq 0}(V) \end{tikzcd} \tag{3.3.5.1}$$

is a pullback square in the ∞-topos $\mathcal{S}$. By the definition of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$, there are t-complete ordinary k°-algebras A and B such that $U \cong \operatorname{Spf} A$ and $V \cong \operatorname{Spf} B$. Moreover, by construction, B is étale over some ring of the form $k^\circ\langle T_1, \ldots T_m\rangle$. [dJ+, Tag A0R1, Lemma 8.0.10.3] implies that there exists an étale $k^\circ[T_1, \ldots T_m]$-algebra B' such that $B \cong (B')^\wedge_t$. The morphism $U \to V$ being admissible in $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ implies that the induced morphism $B \to A$ is formally étale. [dJ+, Tag A0R1, Lemma 7.9.10.3] implies that the morphism $B \to A$ can be realized as the t-completion of k°-algebras $B' \to A'$, where A' is an étale $k^\circ[T_1, \ldots, T_n]$ itself. Therefore, the morphism of spaces

$$\mathcal{O}(U) \to \mathcal{O}(V)$$

can be identified with a morphism

$$\mathcal{O}^{\mathrm{sh}}(\operatorname{Spec} A') \to \mathcal{O}^{\mathrm{sh}}(\operatorname{Spec} B')$$

and similaryly for the morphism $\tau_{\leq 0}\mathcal{O}(U) \to \mathcal{T}_{\leq 0}\mathcal{O}(V)$. Therefore we can identify the diagram (3.3.5.1) with

$$\begin{tikzcd} \mathcal{O}^{\mathrm{sh}}(\operatorname{Spec} A') \arrow[r] \arrow[d] & \tau_{\leq 0}\mathcal{O}^{\mathrm{sh}}(\operatorname{Spec} A') \arrow[d] \\ \mathcal{O}^{\mathrm{sh}}(\operatorname{Spec} B') \arrow[r] & \tau_{\leq 0}\mathcal{O}^{\mathrm{sh}}(\operatorname{Spec} B') \end{tikzcd}$$

in the ∞-category $\mathcal{S}$. The result now follows thanks to [Lur11c, Proposition 4.3.28]. $\square$

Definition 3.3.5.7. Let $\mathsf{X} = (\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$. We define its n-th truncation $\mathrm{t}_{\leq n}(\mathsf{X}) \coloneqq (\mathcal{X}, \tau_{\leq n}\mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$.

Proposition 3.3.5.8. *Let* $\mathsf{X} = (\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$*. Then for each integer* $n \geq 0$*, the* $n+1$*-th truncation* $\mathrm{t}_{\leq n+1}(\mathsf{X})$ *is a square zero extension of* $\mathrm{t}_{\leq n}(\mathsf{X})$*. In particular, when* X *is a derived* k°*-adic Deligne-Mumford stack then for each* $n \geq 0$ *the* n*-truncation* $\mathrm{t}_{\leq n}(\mathsf{X})$ *is again a derived* k°*-adic Deligne-Mumford stack.*

Proof. We have a canonical morphism $\mathrm{t}_{\leq n}(\mathsf{X}) \hookrightarrow \mathrm{t}_{\leq n+1}(\mathsf{X})$ induced by the identity functor on the underlying ∞-topos $\mathcal{X}$ and the natural map $\tau_{\leq n+1}\mathcal{O} \to \tau_{\leq n}\mathcal{O}$ at the level of structures. Let $\mathcal{B} := \tau_{\leq n+1}\mathcal{O}$ and $\mathcal{A} := \tau_{\leq n}\mathcal{O}$. Thanks to [Lur12c, Corollary 7.4.1.28] we deduce that the induced morphism at the level of underlying algebras

$$\mathcal{B}^{\mathrm{alg}} \to \mathcal{A}^{\mathrm{alg}}$$

is a square zero extension. Thus we can identify $\mathcal{B}^{\mathrm{alg}}$ with the pullback of the diagram

$$\begin{tikzcd} \mathcal{B}^{\mathrm{alg}} \arrow[r] \arrow[d] & \mathcal{A}^{\mathrm{alg}} \arrow[d, "d"] \\ \mathcal{A}^{\mathrm{alg}} \arrow[r, "d_0"] & \mathcal{A}^{\mathrm{alg}} \oplus \mathbb{L}_{\mathcal{B}^{\mathrm{alg}}/\mathcal{A}^{\mathrm{alg}}} \end{tikzcd} \tag{3.3.5.2}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\tau_{\leq n}\mathcal{O}}$. Consider the induced k°-adic derivation

$$f^{\mathrm{ad}}(d) \colon \mathbb{L}^{\mathrm{ad}}_{\mathcal{A}} \to \mathbb{L}^{\mathrm{ad}}_{\mathcal{B}/\mathcal{A}}$$

and form the pullback diagram

$$\begin{tikzcd} \mathcal{B}' \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\ \mathcal{A} \arrow[r, "d_0"] & \mathcal{A} \oplus \mathbb{L}^{\mathrm{ad}}_{\mathcal{B}/\mathcal{A}} \end{tikzcd} \tag{3.3.5.3}$$

in the ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{A}}$. In this way the canonical morphism $\mathcal{B}' \to \mathcal{A}$ is a k°-adic square zero extension and we have a canonical map $\mathcal{B} \to \mathcal{B}'$. As filtered colimits commute with finite limits we reduce ourselves to the case that $\mathcal{O}$, and therefore both $\mathcal{A}$ and $\mathcal{B}$, are topologically almost of finite presentation. Thanks to Theorem 3.3.1.11, the functor Ψ applied to the pullback diagram (6.3.2.12) is the identity. Thus by conservativity of $(-)^{\mathrm{alg}}$ it follows that the diagram

$$\begin{tikzcd} \Psi(\mathcal{B}^{\mathrm{alg}}) \arrow[r] \arrow[d] & \Psi(\mathcal{A}^{\mathrm{alg}}) \arrow[d, "d"] \\ \Psi(\mathcal{A}^{\mathrm{alg}}) \arrow[r, "d_0"] & \Psi(\mathcal{A}^{\mathrm{alg}} \oplus \mathbb{L}_{\mathcal{B}^{\mathrm{alg}}/\mathcal{A}^{\mathrm{alg}}}) \end{tikzcd} \tag{3.3.5.4}$$

is a pullback diagram in the ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{A}}$. Thanks to Theorem 3.3.2.4, one concludes that the diagram (3.3.5.4) is equivalent to the pullback diagram (3.3.5.3). Therefore, the canonical map $\mathcal{B}' \to \mathcal{B}$ is an equivalence in the ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}}$, as desired. □

3.4 Derived rigidification functor

3.4.1 Construction of the rigidification functor

Raynaud's generic fiber construction [Bos05, §8], induces a transformation of pregeometries

$$(-)^{\mathrm{rig}} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{T}_{\mathrm{an}}(k),$$

which is moreover a localization of categories with respect to those morphisms $\mathrm{Spf}(A) \to \mathrm{Spf}(B)$ such that there exists there exists a k°-adic complete algebra C together with continuous adic morphisms $C \to A$ and $C \to B$ such that they induce equivalence after inverting t,

$$A \otimes_{k^\circ} k \simeq C \otimes_{k^\circ} k \simeq B \otimes_{k^\circ} k.$$

Proposition 3.4.1.1. *Precomposition along the transformation of pregeometries* $(-)^{\mathrm{rig}} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{T}_{\mathrm{an}}(k)$ *induces a functor*

$$(-)^+ \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)),$$

which admits a right adjoint denoted

$$(-)^{\mathrm{rig}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)).$$

referred to as the (derived) rigidification functor.

Proof. It is a direct consequence of [Lur11c, Theorem 2.1]. □

Lemma 3.4.1.2. *For each integer* $n \geq 0$*, we have a commutative diagram*

$$\begin{tikzcd} {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \arrow[r, "(-)^+"] \arrow[d] & {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \arrow[d] \\ {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))^{\leq n} \arrow[r, "(-)^+"] & {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))^{\leq n} \end{tikzcd}$$

of ∞*-categories, where both* ∞*-categories* ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))^{\leq n}$ *and* ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))^{\leq n}$ *are as in Theorem 3.3.5.4.*

Proof. It follows immediately from the fact that both preogemetries $\mathcal{T}_{\mathrm{an}}(k)$ and $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ are compatible with n-truncations, see [PY16a, Theorem 3.23] and Theorem 3.3.5.6. □

Corollary 3.4.1.3. *Let* $n \geq -1$ *be an integer. The diagram*

$$\begin{tikzcd} {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \arrow[r, "(-)^{\mathrm{rig}}"] \arrow[d, "\mathrm{t}_{\leq n}"] & {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \arrow[d, "\mathrm{t}_{\leq n}"] \\ {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))^{\leq n} \arrow[r, "(-)^{\mathrm{rig}}"] & {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \end{tikzcd}$$

is commutative.

Proof. It follows by taking right adjoints in the diagram displayed in Theorem 3.4.1.2. □

These considerations imply the following useful result:

Corollary 3.4.1.4. *Let* $\mathsf{X} = (\mathcal{X}, \mathcal{O})$ *be a* $\mathcal{T}_{\mathrm{ad}}(k^\circ)$*-structured space which is equivalent to an ordinary* k°*-adic formal scheme topologically of finite presentation. Then* $\mathsf{X}^{\mathrm{rig}}$ *is equivalent to an ordinary* k*-analytic space which agrees with the usual generic fiber of* X*.*

Proof. The question is local on X. We can thus assume that $\mathsf{X} \simeq \mathrm{Spf}(A)$, where $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$, where A is a topologically of finite presentation ordinary k°-adic algebra. Therefore, choosing generators and relations for A we can find an (underived) pullback diagram of the form

$$\begin{tikzcd} \mathrm{Spf}(A) \arrow[r] \arrow[d] & \mathfrak{A}^m_{k^\circ} \arrow[d] \\ \mathrm{Spf}(k^\circ) \arrow[r] & \mathfrak{A}^n_{k^\circ} \end{tikzcd} \tag{3.4.1.1}$$

of ordinary k°-adic formal schemes. Let Z denote the (derived) pullback associated to (3.4.1.1) in the ∞-category dfSch, whose existence is guaranteed by [Lur16, Proposition 8.1.6.1]. It follows that $\mathrm{t}_{\leq 0}(Z) \simeq \mathrm{Spf}(A)$. As $\mathfrak{A}^m_{k^\circ}$, $\mathfrak{A}^n_{k^\circ}$ and $\mathrm{Spf}(k^\circ)$ are objects of the pregeometry $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ and $(-)^{\mathrm{rig}}$ is induced by the usual generic fiber functor if follows that

$$\mathrm{Spf}(k^\circ)^{\mathrm{rig}} \simeq \mathrm{Sp}\, k, \quad (\mathfrak{A}^m_{k^\circ})^{\mathrm{rig}} \simeq \mathbf{A}^m_k, \quad (\mathfrak{A}^n_{k^\circ})^{\mathrm{rig}} \simeq \mathbf{A}^n_k.$$

As $(-)^{\mathrm{rig}}$ is a right adjoint, it commutes with pullback diagrams. We thus have a pullback diagram in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$

$$\begin{tikzcd} Z^{\mathrm{rig}} \arrow[r] \arrow[d] & \mathbf{A}^m_k \arrow[d] \\ \mathrm{Sp}(k) \arrow[r] & \mathbf{A}^n_k \end{tikzcd}$$

Theorem 3.4.1.3 implies that $\mathrm{t}_{\leq 0}\left(Z^{\mathrm{rig}}\right) \simeq \mathrm{t}_{\leq 0}(Z)^{\mathrm{rig}}$. As $\mathrm{t}_{\leq 0}(Z) \simeq \mathrm{Spf}(A)$ we deduce that $(\mathrm{Spf}(A))^{\mathrm{rig}}$ is equivalent to the (underived) pullback diagram

$$\begin{tikzcd} \mathrm{Spf}(A)^{\mathrm{rig}} \arrow[r] \arrow[d] & \mathbf{A}^m_k \arrow[d] \\ \mathrm{Sp}(k) \arrow[r] & \mathbf{A}^n_k \end{tikzcd}$$

computed in the category of rigid k-analytic spaces. This is precisely the usual generic fiber construction applied to $\mathrm{Spf}\, A$. □

Lemma 3.4.1.5. *Let $f \colon \mathsf{Z} \to \mathsf{X}$ be a closed immersion of derived k°-adic Deligne-Mumford stacks topologically almost of finite presentation. Then f^{rig} is a closed immersion in the ∞-category dAn.*

Proof. It suffices to show that the truncation $\mathrm{t}_{\leq 0}(f^{\mathrm{rig}}) \colon \mathrm{t}_{\leq 0}(\mathsf{Z}^{\mathrm{rig}}) \to \mathrm{t}_{\leq 0}(\mathsf{X}^{\mathrm{rig}})$ is a closed immersion. Last assertion is a consequence of Theorem 3.4.1.4. □

Proposition 3.4.1.6. *Let X be a topological almost of finite presentation derived k°-derived Deligne-Mumford stack. Then $\mathsf{X}^{\mathrm{rig}}$ is a derived k-analytic space.*

Proof. Our proof is inspired on [PY17a, Proposition 3.7]. The question is étale local by [Lur11c, Lemma 2.1.3]. We can thus reduce ourselves to the case $\mathsf{X} = \mathrm{Spf}(A)$, where $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ is a t-complete topological of finite presentation derived k°-algebra. We wish to prove that $\mathrm{Spf}(A)^{\mathrm{rig}}$ is a derived k-affinoid space. Let $\mathcal{C}$ denote the full subcategory of dfDM_{k° spanned by those affine derived k°-adic formal Deligne-Mumford stacks $\mathrm{Spf}(A)$ such that $\mathrm{Spf}(A)^{\mathrm{rig}}$ is equivalent to a derived k-affinoid space. We have:

(i) The ∞-category $\mathcal{C}$ contains the essential image of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ thanks to [Lur11c, Proposition 2.3.18].

(ii) $\mathcal{C}$ is closed under pullbacks along closed immersions: Let

$$\begin{tikzcd} \mathsf{W} \arrow[r] \arrow[d] & \mathsf{Z} \arrow[d] \\ \mathsf{Y} \arrow[r, "f"] & \mathsf{X}, \end{tikzcd} \tag{3.4.1.2}$$

be a pullback diagram in the ∞-category dfDM_{k° such that X, Y and $\mathsf{Z} \in \mathcal{C}$ and that $f \colon \mathsf{Y} \to \mathsf{X}$ is a closed immersion. By unramifiedness of the pregeometry $\mathcal{T}_{\mathrm{ad}}(k^\circ)$, Theorem .2.0.3, the diagram (3.4.1.2) is also a pullback diagram in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$. As $(-)^{\mathrm{rig}}$ is a right adjoint the diagram

$$\begin{tikzcd} \mathsf{W}^{\mathrm{rig}} \arrow[r] \arrow[d] & \mathsf{Z}^{\mathrm{rig}} \arrow[d] \\ \mathsf{Y}^{\mathrm{rig}} \arrow[r] & \mathsf{X}^{\mathrm{rig}} \end{tikzcd} \tag{3.4.1.3}$$

is a pullback diagram in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. The ∞-category dAn is closed under pullbacks along closed immersions thanks to [PY16a, Proposition 6.2]. Theorem 3.4.1.5 then implies that the diagram (3.4.1.3) is a pullback square in the ∞-category dAn. Thus $\mathsf{W} \in \mathcal{C}$, as desired.

(iii) The ∞-category $\mathcal{C}$ is closed under finite limits. It suffices to prove that $\mathcal{C}$ is closed under finite products and pullbacks. [PY16a, Lemma 6.4] implies that $\mathcal{C}$ is closed under finite products. General pullback diagrams can be constructed as pullbacks along along closed immersions as in the proof of [PY16a, Theorem 6.5]. Thanks to Theorem 3.4.1.4, $(-)^{\mathrm{rig}}$ commutes with finite products of ordinary formal schemes and preserves closed immersions by Theorem 3.4.1.5, the assertion follows.

(iv) $\mathcal{C}$ is closed under retracts: let $X \in \mathcal{C}$ and let

$$\mathsf{Y} \xrightarrow{j} \mathsf{X} \xrightarrow{p} Y,$$

be a retract diagram in the ∞-category dfDM_{k°. Assume further that Y is affine. By assumption, $\mathsf{X}^{\mathrm{rig}} \in \mathrm{dAn}$ and $\mathrm{t}_{\leq 0}(\mathsf{Y})^{\mathrm{rig}} \in \mathrm{dAn}$ thanks to Theorem 3.4.1.4. It suffices to prove that for each $i > 0$, the homotopy sheaf $\pi_i\big(\mathcal{O}_{\mathsf{Y}}^{\mathrm{rig}}\big)$ is a coherent sheaf over $\pi_0\big(\mathcal{O}_{\mathsf{Y}}^{\mathrm{rig}}\big)$. The latter is a retract of $\pi_i\big(\mathcal{O}_{\mathsf{X}}^{\mathrm{rig}}\big)$, which is a coherent sheaf over $\pi_0\big(\mathcal{O}_{\mathsf{X}}^{\mathrm{rig}}\big)$. In this way, it follows that $\pi_i\big(\mathcal{O}_{\mathsf{Y}}^{\mathrm{rig}}\big)$ is coherent over $\pi_0\big(\mathcal{O}_{\mathsf{X}}^{\mathrm{rig}}\big)$. As $\pi_0\big(\mathcal{O}_{\mathsf{Y}}^{\mathrm{rig}}\big)$ is a retract of $\pi_0\big(\mathcal{O}_{\mathsf{X}}^{\mathrm{rig}}\big)$ we deduce that $\pi_i\big(\mathcal{O}_{\mathsf{Y}}^{\mathrm{rig}}\big)$ is coherent over $\pi_0\big(\mathcal{O}_{\mathsf{Y}}^{\mathrm{rig}}\big)$, as desired.

Let now $\mathsf{X} \in \mathrm{dfDM}_{k^\circ}^{\mathrm{taft}}$ be an affine object. Write $\mathsf{X} \simeq \mathrm{Spf}(A)$ for some adic derived k°-algebra $A \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}$ topologically almost of finite presentation. We wish to prove that $\mathsf{X} \in \mathcal{C}$. Theorem 3.4.1.4 guarantees that $\mathrm{t}_{\leq 0}\left(\mathsf{X}^{\mathrm{rig}}\right)$ is a k-analytic space. We are thus reduced to show that $\pi_i\big(\mathcal{O}_{\mathsf{X}}^{\mathrm{rig}}\big)$ is a coherent sheaf over $\pi_0\big(\mathcal{O}_{\mathsf{X}}^{\mathrm{rig}}\big)$. For every $n \geq 0$ the algebra $\tau_{\leq n}(A)$ is a compact object in the ∞-category $\left(\mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}\right)^{\leq n}$ of n-truncated derived adic k°-algebras. We can thus find a *finite* diagram of free simplicial k°-algebras

$$g \colon I \to \mathcal{C}\mathrm{Alg}_{k^\circ},$$

such that $\tau_{\leq n} A$ is a retract of $\tau_{\leq n}(B)$, where

$$B := \operatorname*{colim}_I \, (g)_t^\wedge \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}},$$

where $(g)_t^\wedge$ denotes the t-completion of the diagram $g \colon I \to \mathcal{C}\mathrm{Alg}_{k^\circ}$. As the t-completion functor commutes with finite colimits it follows that

$$B \simeq B_t^\wedge,$$

and in particular B is t-complete. As $\mathcal{C}$ is closed under finite limits and objects in the pregeometry $\mathcal{T}_{\mathrm{ad}}(k^\circ)$, we conclude that $\mathrm{Spf}(B) \in \mathcal{C}$. In particular $\mathrm{Spf}\left(\tau_{\leq n} B\right) \in \mathcal{C}$. As $\mathcal{C}$ is moreover closed under retracts, it follows that $\mathrm{Spf}\left(\tau_{\leq n} A\right) \in \mathcal{C}$ as well. It follows, that for each $0 \leq i \leq n$, $\pi_n\big(\mathcal{O}_{\mathsf{X}}^{\mathrm{rig}}\big)$ is coherent over $\pi_0\big(\mathcal{O}_{\mathsf{X}}^{\mathrm{rig}}\big)$. Repeating the argument for every $n \geq 0$ we conclude. $\square$

3.4.2 Rigidification of structures

Construction 3.4.2.1. Let $X = (\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ be a $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos. Suppose further that there exists $\mathsf{X} = (\mathcal{Z}, \mathcal{O}_0) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ such that we have an equivalence $\mathsf{X}^{\mathrm{rig}} \simeq X$ in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. Precomposition along the transformation of pregeometries

$$(-)^{\mathrm{rig}} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \to \mathcal{T}_{\mathrm{an}}(k)$$

induces a functor at the level of ∞-categories of structures

$$(-)^{+} \colon \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^+}$$

given on objects by the formula

$$\mathcal{A} \in \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}} \mapsto \mathcal{A}^+ := \mathcal{A} \circ (-)^{\mathrm{rig}} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^+}.$$

The functor of presentable ∞-categories $(-)^+ \colon \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^+}$ preserves limits and filtered colimits. Thanks to the Adjoint functor theorem it follows that there exists a left adjoint

$$(-)^{\mathrm{rig},\circ} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^+} \to \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}}. \tag{3.4.2.1}$$

The counit of the adjunction $\big((-)^+,\ (-)^{\mathrm{rig}}\big)\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ produces a well defined, up to contractible indeterminacy, morphism

$$f\colon X^+ = (\mathcal{X}, \mathcal{O}^+) \to (\mathcal{Z}, \mathcal{O}_0) = \mathsf{X} \tag{3.4.2.2}$$

in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$. Let $(f^{-1}, f_*) \to \mathcal{X} \to \mathcal{Z}$ denote the underlying geometric morphism associated to f. Then $f^{-1}\colon \mathcal{Z} \to \mathcal{X}$ induces a well defined functor

$$f^{-1}\colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_0} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/f^{-1}\mathcal{O}_0}. \tag{3.4.2.3}$$

Moreover, the morphism (3.4.2.2) induces a morphism at the level of structures

$$\theta\colon f^{-1}\mathcal{O}_0 \to \mathcal{O}^+,$$

which induces a well defined functor at the level of ∞-categories of structures

$$\theta\colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/f^{-1}\mathcal{O}_0} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^+} \tag{3.4.2.4}$$

given on objects by the formula

$$\big(\mathcal{A} \to f^{-1}\mathcal{O}_0\big) \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/f^{-1}\mathcal{O}_0} \mapsto \big(\mathcal{A} \to \mathcal{O}^+\big) \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^+}.$$

Therefore the composite $(-)^{\mathrm{rig}} := (-)^{\mathrm{rig},\circ} \circ \theta \circ f^{-1}$ induces a functor

$$(-)^{\mathrm{rig}}\colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_0} \to \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}}$$

which we refer to as *the rigidification functor at the level of structures*.

Remark 3.4.2.2. Notations as in Theorem 3.4.2.1 and suppose further that X is a derived k-analytic space and $\mathsf{X} \in \mathrm{dfDM}_{k^\circ}^{\mathrm{taft}}$. Thanks to Theorem 3.4.1.4 the geometric morphism underlying $f\colon X^+ \to \mathsf{X}$ corresponds to the classical specialization morphism at the level of ∞-topoi $\mathcal{X} \to \mathcal{Z}$.

Notation 3.4.2.3. We will denote the geometric morphism introduced in Theorem 3.4.2.1 $(f^{-1}, f_*)\colon \mathcal{X} \to \mathcal{Z}$ by $\mathrm{sp} = (\mathrm{sp}^{-1}, \mathrm{sp}_*)$.

Construction 3.4.2.4. Notations as in Theorem 3.4.2.1. Consider the following square of pregeometries

$$\begin{tikzcd} \mathcal{T}_{\mathrm{disc}}(k^\circ) \arrow[r, "-\otimes_{k^\circ} k"] \arrow[d, "(-)^\wedge_t"'] & \mathcal{T}_{\mathrm{disc}}(k) \arrow[d, "(-)^{\mathrm{an}}"] \\ \mathcal{T}_{\mathrm{ad}}(k^\circ) \arrow[r, "(-)^{\mathrm{rig}}"'] & \mathcal{T}_{\mathrm{an}}(k) \end{tikzcd} \tag{3.4.2.5}$$

Notice that (3.4.2.5) is not commutative. The lower composite sends

$$\mathbb{A}^1_{k^\circ} \in \mathcal{T}_{\mathrm{disc}} \mapsto \mathbf{A}^1_k \in \mathcal{T}_{\mathrm{an}}(k)$$

whereas the top composite sends

$$\mathbb{A}^1_{k^\circ} \in \mathcal{T}_{\mathrm{disc}} \mapsto \mathbf{B}^1_k \in \mathcal{T}_{\mathrm{an}}(k),$$

where $\mathbf{B}^1_k \in \mathcal{T}_{\mathrm{an}}(k)$ denotes the closed unit disk. Let $\mathcal{A} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_0}$, the counit of the adjunction $((-)^+, (-)^{\mathrm{rig}})$ induces a natural morphism at the level of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structures on $\mathcal{X}$

$$\theta_{\mathcal{A}}\colon \mathrm{sp}^{-1}\mathcal{A} \to \mathcal{A}^{\mathrm{rig},+} := \big(\mathcal{A}^{\mathrm{rig}}\big)^+.$$

Applying the underlying algebra functor $(-)^{\mathrm{alg}}\colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^+} \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{+,\mathrm{alg}}}$ to the morphism $\theta_{\mathcal{A}}$ we obtain a morphism

$$\theta_{\mathcal{A}}^{\mathrm{alg}}\colon (\mathrm{sp}^{-1}\mathcal{A})^{\mathrm{alg}} \to \mathcal{A}^{\mathrm{rig},+,\mathrm{alg}} \tag{3.4.2.6}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}^{+,\mathrm{alg}}}$. As $\mathcal{A}^{\mathrm{rig},+,\mathrm{alg}}$ lives naturally over the non-archimedean field k we obtain by adjunction a morphism

$$\overline{\theta}^{\mathrm{alg}}_{\mathcal{A}} \colon (\mathrm{sp}^{-1}\mathcal{A})^{\mathrm{alg}} \otimes_{k^\circ} k \to \mathcal{A}^{\mathrm{rig},+,\mathrm{alg}} := \left(\mathcal{A}^{\mathrm{rig},+}\right)^{\mathrm{alg}},$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_k(\mathcal{X})_{\mathcal{O}^{+,\mathrm{alg}}}$. We can identify $\mathcal{A}^{\mathrm{rig},+,\mathrm{alg}} \simeq \mathcal{A}(\mathbf{B}^1_k)$. There is a natural inclusion of k-analytic spaces $\mathbf{B}^1_k \to \mathbf{A}^1_k$. We obtain thus a canonical morphism

$$\mathcal{A}(\mathbf{B}^1_k) \to \mathcal{A}(\mathbf{A}^1_k), \tag{3.4.2.7}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_k(\mathcal{X})_{/\mathcal{O}(\mathbf{A}^1_k)}$. Composing both (3.4.2.6) with (3.4.2.7) we obtain a natural morphism

$$\overline{\theta}_{\mathcal{A}} \colon (\mathrm{sp}^{-1}\mathcal{A})^{\mathrm{alg}} \otimes_{k^\circ} k \to \mathcal{A}^{\mathrm{rig}}(\mathbf{A}^1_k) \tag{3.4.2.8}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}(\mathcal{X})_{/\mathcal{O}(\mathbf{A}^1_k)}$. We will take as (a probably confusing) convention to denote precomposition with $(-)^{\mathrm{an}}$ in (3.4.2.5) by

$$(-)^{\mathrm{alg}} \colon \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}} \to \mathcal{C}\mathrm{Alg}_k(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}.$$

In this case, we might as well write (3.4.2.8) as

$$\overline{\theta}_{\mathcal{A}} \colon (\mathrm{sp}^{-1}\mathcal{A})^{\mathrm{alg}} \otimes_{k^\circ} k \to \mathcal{A}^{\mathrm{rig},\mathrm{alg}} := \left(\mathcal{A}^{\mathrm{rig}}\right)^{\mathrm{alg}}.$$

Proposition 3.4.2.5. *Let $X = (\mathcal{X}, \mathcal{O}) \in \mathrm{dAn}$ and suppose there exists $\mathsf{X} = (\mathcal{Z}, \mathcal{O}_0) \in \mathrm{dfDM}_{k^\circ}$ such that $\mathsf{X}^{\mathrm{rig}} \simeq X$, in the ∞-category dAn. Then for every $\mathcal{A} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_0}$ the natural morphism*

$$\overline{\theta}_{\mathcal{A}} \colon (\mathrm{sp}^{-1}\mathcal{A})^{\mathrm{alg}} \otimes_{k^\circ} k \to \mathcal{A}^{\mathrm{rig},\mathrm{alg}}$$

introduced in Theorem 3.4.2.4 is an equivalence in the ∞-category $\mathcal{C}\mathrm{Alg}_k(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}$.

Proof. Both the underlying ∞-topoi of X and X have enough points, as these are hypercomplete and 1-localic. Therefore, thanks to [Por15c, Theorem 1.12], we are reduced to check the statement of the proposition on stalks, (notice that given a geometric point $x_* \colon \mathcal{S} \to \mathcal{X}$ the composite $\mathrm{sp}_* \circ x_* \colon \mathcal{S} \to \mathcal{Z}$ is also a geometric point).

By doing so, we might assume from the start that $\mathcal{X} = \mathcal{S} = \mathcal{Z}$. Both composites $(-)^{\mathrm{alg}} \circ (-)^{\mathrm{rig}}$ and $\left((-)^{\mathrm{alg}} \circ \mathrm{sp}^{-1}\right) \otimes_{k^\circ} k$ commute with sifted colimits. The proof of Theorem 3.3.1.11 implies that the ∞-category $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{S})_{/\mathcal{O}_0}$ is generated under sifted colimits by the family $\{\Psi\left(k^\circ[T_1, \dots, T_m]\right)\}_m$, where the T_i's sit in homological degree 0. It thus suffices to show that

$$\overline{\theta}_{\mathcal{A}} \colon (\mathrm{sp}^{-1}\mathcal{A})^{\mathrm{alg}} \otimes_{k^\circ} k \to \mathcal{A}^{\mathrm{rig},\mathrm{alg}}$$

is an equivalence whenever $\mathcal{A} \simeq \Psi\left(k^\circ[T_1, \dots, T_m]\right)$. But in this case, we have natural equivalences

$$(\mathrm{sp}^{-1}\Psi(k^\circ[T_1, \dots, T_m])^{\mathrm{alg}} \otimes_{k^\circ} k \simeq k\langle T_1, \dots, T_m\rangle,$$

and as $\Psi\left(k^\circ[T_1, \dots, T_m]\right)$ can be identified with (a germ) of $\mathfrak{A}^m_{k^\circ} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ it follows that

$$\Psi\left(k^\circ[T_1, \dots, T_m]\right)^{\mathrm{rig},\mathrm{alg}} \simeq k\langle T_1, \dots, T_m\rangle,$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_k(\mathcal{X})_{/\mathcal{O}^{\mathrm{alg}}}$. The result now follows. $\square$

3.4.3 Rigidification of modules

Definition 3.4.3.1. Let $X = (\mathcal{X}, \mathcal{O}) \in \mathrm{dAn}$ be a derived k-analytic space. Its ∞-category of modules is defined as

$$\mathrm{Mod}_{\mathcal{O}} := \mathrm{Sp}\left(\mathrm{Ab}\left(\mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}}\right)\right)$$

One has the following result:

Proposition 3.4.3.2. [PY17a, Theorem 4.5] *There exists a canonical equivalence of ∞-categories*

$$\mathrm{Mod}_{\mathcal{O}} \simeq \mathrm{Mod}_{\mathcal{O}^{\mathrm{alg}}}.$$

Lemma 3.4.3.3. *Let* $X = (\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ *and* $\mathsf{X} = (\mathcal{Z}, \mathcal{O}_0) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ *such that* $\mathsf{X}^{\mathrm{rig}} \simeq X$ *in the* ∞*-category* ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$*. The rigidification functor* $(-)^{\mathrm{rig}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_0} \to \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}}$ *induces a well defined functor, up to contractible indeterminacy,*

$$(-)^{\mathrm{rig}} \colon \mathrm{Mod}_{\mathcal{O}_0} \to \mathrm{Mod}_{\mathcal{O}}$$

which we shall refer to as the rigidification of modules functor.

Proof. It suffices to show that the functor $(-)^{\mathrm{rig}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_0} \to \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}}$ commutes with finite limits. Thanks to Theorem 3.4.2.5 the composite functor $(-)^{\mathrm{alg}} \circ (-)^{\mathrm{rig}}$ agrees with localization at t and therefore it commutes with finite limits. As $(-)^{\mathrm{alg}}$ is a conservative right adjoint it follows that $(-)^{\mathrm{rig}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_0} \to \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}}$ commutes with finite limits as well, and the proof is finished. ☐

We have a natural projection functor $\Omega^\infty_{\mathrm{an}} \colon \mathrm{Mod}_{\mathcal{O}} \to \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}}$. We shall denote $\mathcal{O} \oplus M \coloneqq \Omega^\infty_{\mathrm{an}}(M)$ and refer to it as the *analytic split square zero extension of* $\mathcal{O}$ *by* M. The functor $\Omega^\infty_{\mathrm{an}}$ admits a left adjoint $\Sigma^\infty_{\mathrm{an}} \colon \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}}$.

Suppose we are given $\mathcal{A} \in \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}}$ and consider the ∞-category $\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{O}}$. We can consider the *analytic derivations functor* $\mathrm{Der}^{\mathrm{an}}_{\mathcal{A}}(\mathcal{O}, -) \colon \mathrm{Mod}_{\mathcal{O}} \to \mathcal{S}$ given on objects by the formula

$$M \in \mathrm{Mod}_{\mathcal{O}} \mapsto \mathrm{Map}_{\mathrm{AnRing}_k(\mathcal{X})_{\mathcal{A}//\mathcal{O}}}(\mathcal{O}, \mathcal{O} \oplus M).$$

Such functor is corepresentable by the *analytic cotangent complex relative to* $\mathcal{A} \to \mathcal{O}$, which we denote by $\mathbb{L}^{\mathrm{an}}_{\mathcal{O}/\mathcal{A}}$. Explicitly, one has a natural equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}}\left(\mathbb{L}^{\mathrm{an}}_{\mathcal{O}/\mathcal{A}}, M\right) \simeq \mathrm{Der}^{\mathrm{an}}_{\mathcal{A}}(\mathcal{O}, M).$$

We can describe explicitly $\mathbb{L}^{\mathrm{an}}_{\mathcal{O}/\mathcal{A}} \simeq \Sigma^\infty_{\mathrm{an}}(\mathcal{O} \otimes_{\mathcal{A}} \mathcal{O}) \in \mathrm{Mod}_{\mathcal{O}}$.

Lemma 3.4.3.4. *Let* $X = (\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ *and* $\mathsf{X} = (\mathcal{Z}, \mathcal{O}_0) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$ *such that* $\mathsf{X}^{\mathrm{rig}} \simeq X$ *in the* ∞*-category* ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$*. Then the diagram*

$$\begin{tikzcd} \mathrm{Mod}_{\mathcal{O}_0} \arrow[r, "(-)^{\mathrm{rig}}"] & \mathrm{Mod}_{\mathcal{O}} \\ \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_0} \arrow[u, "\Sigma^\infty_{\mathrm{ad}}"] \arrow[r, "(-)^{\mathrm{rig}}"] & \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}} \arrow[u, "\Sigma^\infty_{\mathrm{an}}"'] \end{tikzcd}$$

is commutative up to coherent homotopy.

Proof. It suffices to prove that the corresponding diagram of right adjoints

$$\begin{tikzcd} \mathrm{Mod}_{\mathcal{O}_0} \arrow[d, "\Omega^\infty_{\mathrm{ad}}"] & \mathrm{Mod}_{\mathcal{O}} \arrow[l, "(-)^{+}"'] \arrow[d, "\Omega^\infty_{\mathrm{ad}}"] \\ \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_0} & \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}} \arrow[l, "(-)^{+}"'] \end{tikzcd}$$

is commutative. But this is immediate from the definitions and the result follows. ☐

Corollary 3.4.3.5. *We have a natural equivalence*

$$\left(\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}/\mathcal{A}}\right)^{\mathrm{rig}} \simeq \mathbb{L}^{\mathrm{an}}_{\mathcal{O}^{\mathrm{rig}}/\mathcal{A}^{\mathrm{rig}}}$$

in the ∞*-category* $\mathrm{Mod}_{\mathcal{O}}$*.*

Proof. It is an immediate consequence of Theorem 3.4.3.4 above. ☐

Definition 3.4.3.6. Let $M \in \mathrm{Coh}^+(X)$. We say that M admits a formal model if there exists an $\mathcal{O}_0$-module $M_0 \in \mathrm{Coh}^+(\mathcal{O}_0)$ such that

$$M_0^{\mathrm{rig}} \simeq M \in \mathrm{Mod}_{\mathcal{O}}.$$

Proposition 3.4.3.7. *Let* $\mathsf{X} = (\mathcal{Z}, \mathcal{O}_0) \in \mathrm{dfDM}_{k^\circ}$ *and let* $X = (\mathcal{X}, \mathcal{O}) \coloneqq \mathsf{X}^{\mathrm{rig}}$ *denote its rigidification. Then the functor* $(-)^{\mathrm{rig}} \colon \mathrm{Mod}_{\mathcal{O}_0} \to \mathrm{Mod}_{\mathcal{O}}$ *is t-exact.*

Proof. The statement follows readily from Theorem .1.1.4 and [HPV16a, Corollary 2.9]. ☐

3.4.4 Main results

In this §we our two main results. The first one concerns the existence of formal models for quasi-paracompact and quasi-separated derived k-analytic spaces. The second is a direct generalization of Raynaud's localization theorem.

Definition 3.4.4.1. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. We say that A is an *admissible adic derived k°-algebra* if A is topologically almost of finite presentation and t-complete and moreover, for every $i \geq 0$, the homotopy sheaf $\pi_i(A)$ is t-torsion free. We denote $\mathcal{C}\mathrm{Alg}^{\mathrm{adm}}_{k^\circ}$ the full subcategory of $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ spanned by admissible adic derived k°-algebras.

Definition 3.4.4.2. Let $\mathsf{X} \in \mathrm{dfDM}_{k^\circ}$ we say that X is a derived admissible k°-adic Deligne-Mumford stack (or derived admissible k°-adic scheme) if $\mathsf{X} \in \mathrm{dfDM}^{\mathrm{taft}}_{k^\circ}$ (resp., $\mathsf{X} \in \mathrm{dfSch}^{\mathrm{taft}}$) and we can find a covering

$$\coprod_i \mathrm{Spf}(A_i) \to \mathsf{X}$$

such that for each i, $A_i \in \mathcal{C}\mathrm{Alg}^{\mathrm{adm}}_{k^\circ}$. We denote by $\mathrm{dfDM}^{\mathrm{adm}}_{k^\circ}$ (resp., $\mathrm{dfSch}^{\mathrm{adm}}$) the ∞-category of derived admissible k°-adic Deligne-Mumford stacks (resp. derived admissible k°-adic schemes).

Definition 3.4.4.3. Let $X = (\mathcal{X}, \mathcal{O})$ be a derived k-analytic space. We say that X is quasi-paracompact and quasi-separated if the 0-th truncation $\mathrm{t}_{\leq 0}(X)$ is equivalent to a quasi-paracompact and quasi-separated ordinary k-analytic space.

Definition 3.4.4.4. Let $X = (\mathcal{X}, \mathcal{O})$ be a derived k-analytic space. We say that X admits a formal model if there exists $\mathsf{X} \in \mathrm{dfDM}_{k^\circ}$ such that

$$X \simeq \mathsf{X}^{\mathrm{rig}},$$

in the ∞-category dAn.

Thanks to [Bos05, Theorem 3, page 204] it follows that if $\mathrm{t}_{\leq 0}(X)$ is quasi-paracompact and quasi-separated then it admits a classical formal model. We generalize this result to the derived setting:

Theorem 3.4.4.5. *Let $X = (\mathcal{X}, \mathcal{O})$ be a quasi-paracompact and quasi-separated derived k-analytic space. Then X admits a derived formal model $\mathsf{X} = (\mathcal{Z}, \mathcal{O}_0) \in \mathrm{dfSch}$.*

Proof. Let $X_0 := \mathrm{t}_{\leq 0}(X)$ denote the 0-truncation of X. Thanks to [Bos05, Theorem 3, page 204] it follows that X_0 admits a formal model $\mathsf{X}_0 \in \mathrm{fSch}_{k^\circ}$ such that X_0 is admissible (i.e. it can be Zariski locally covered by affine formal spectrums of admissible k°-algebras). We inductively construct a sequence of derived admissible k°-adic schemes

$$\mathsf{X}_0 \to \mathsf{X}_1 \to \mathsf{X}_2 \to \dots,$$

such that we have equivalences

$$(\mathsf{X}_n)^{\mathrm{rig}} \simeq \mathrm{t}_{\leq n}(X),$$

for each $n \geq 0$.The case $n = 0$ being already dealt it suffices to treat the inductive step. Suppose $\mathsf{X}_n = (\mathcal{Z}, \mathcal{O}_{0,n}$ has already been constructed, for $n \geq 0$. As X is a derived k-analytic space, for each $n \geq 0$ the homotopy sheaf $\pi_n(\mathcal{O})$ is a coherent module over $\pi_0(\mathcal{O})$. Thanks to [PY17a, Corollary 5.42] there exists an analytic derivation $d \colon \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_{\leq n} X} \to \pi_{n+1}(\mathcal{O})[n+2]$ together with a pullback diagram

$$\begin{tikzcd} \tau_{\leq n+1}\mathcal{O} \arrow[r] \arrow[d] & \tau_{\leq n}\mathcal{O} \arrow[d] \\ \tau_{\leq n}\mathcal{O} \arrow[r, "d_0"] & \tau_{\leq n}\mathcal{O} \oplus \pi_{n+1}\mathcal{O})[n+2], \end{tikzcd} \tag{3.4.4.1}$$

in the ∞-category $\mathrm{AnRing}_k(\mathcal{X})_{/\tau_{\leq n}\mathcal{O}}$. Here d_0 denotes the trivial analytic derivation. Theorem .1.2.1 and its proof imply that we can find a formal model for d in the stable ∞-category $\mathrm{Coh}^+(\mathsf{X}_n)$

$$\delta \colon \mathbb{L}^{\mathrm{ad}}_{\mathsf{X}_n} \to M_{n+1}[n+2],$$

where $M_{n+1} \in \mathrm{Coh}^+(\mathsf{X}_n)^\heartsuit$ is of no t-torsion and we have that $M_{n+1}^{\mathrm{rig}} \simeq \pi_{n+1}(\mathcal{O})$ in the ∞-category $\mathrm{Coh}^+(\tau_{\leq n} X)$. We define $\mathcal{O}_{n+1} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_{0,n}}$ as the pullback of the diagram

$$\begin{tikzcd} \mathcal{O}_{0,n+1} \arrow[r] \arrow[d] & \mathcal{O}_{0,n} \arrow[d, "\delta"] \\ \mathcal{O}_{0,n} \arrow[r, "d_0"] & \mathcal{O}_{0,n} \oplus M_{n+1}[n+2] \end{tikzcd} \tag{3.4.4.2}$$

in $\mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_{0,n}}$. Define $\mathsf{X}_{n+1} \coloneqq (\mathcal{Z}, \mathcal{O}_{0,n+1})$. It is a derived k°-adic Deligne-Mumford stack and by construction it is admissible. Both $\mathsf{X}^{\mathrm{rig}}$ and $\mathrm{t}_{\leq n+1}(X)$ have equivalent underlying ∞-topoi. The rigidification functor $(-)^{\mathrm{rig}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{Z})_{/\mathcal{O}_0} \to \mathrm{AnRing}_k(\mathcal{X})_{/\mathcal{O}}$ commutes with finite limits. Thus the diagram (3.4.4.2) remains a pullback diagram after rigidification. For this reason, we obtain a canonical morphism

$$\alpha_{n+1} \colon (\mathcal{O}_{0,n+1})^{\mathrm{rig}} \to \tau_{\leq n+1}\mathcal{O}$$

in the ∞-category $\mathrm{AnRing}_k(\mathcal{X})_{/\tau_{\leq n+1}\mathcal{O}}$. We have thus a canonical morphism

$$\theta_{n+1} \colon \mathrm{t}_{\leq n+1}(X) \to \mathsf{X}_{n+1}^{\mathrm{rig}}.$$

We claim that θ_{n+1} is an equivalence in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. It suffices to show that α_{n+1} is an equivalence of structures. Thanks to Theorem 3.4.2.5 we have an equivalence

$$\left(\mathcal{O}_{0,n+1}^{\mathrm{rig}}\right)^{\mathrm{alg}} \simeq \left(\mathrm{sp}^{-1}\mathcal{O}_{0,n+1}\right)^{\mathrm{alg}} \otimes_{k^\circ} k.$$

By the inductive hypothesis together with the pullback diagrams (3.4.4.2) and it follows that

$$(\mathrm{sp}^{-1}\mathcal{O}_{0,n+1})^{\mathrm{alg}} \otimes_{k^\circ} k \simeq (\tau_{\leq n+1}\mathcal{O})^{\mathrm{alg}}$$

is an equivalence. By conservativity of $(-)^{\mathrm{alg}}$ it follows that

$$\alpha_{n+1} \colon \mathcal{O}_{0,n+1}^{\mathrm{rig}} \simeq \tau_{\leq n+1}\mathcal{O}$$

in the ∞-category $\mathrm{AnRing}_k(\mathcal{X})_{/\tau_{\leq n+1}\mathcal{O}}$. We conclude that

$$\theta_{n+1} \colon \mathsf{X}_{n+1}^{\mathrm{rig}} \simeq \mathrm{t}_{\leq n+1}X$$

is an equivalence in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. We define

$$\mathsf{X} \coloneqq \operatorname*{colim}_{n\geq 0} \mathsf{X}_n.$$

We claim that X is again an admissible derived k°-adic Deligne-Mumford stack: the question being local on X reduce ourselves to the case $\mathsf{X} = \mathrm{Spf}\, A$ and $\mathsf{X}_n \simeq \mathrm{Spf}\, A_n$, for $A,\ A_n \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{adm}}$, for each $n \geq 0$. By construction, $\mathcal{T}_{\leq n-1}A_n \simeq A_{n-1}$ for each $n \geq 1$. We have moreover an identification

$$\mathsf{X} \simeq \mathrm{Spf}\left(\lim_{n\geq 0} A_n\right).$$

As A_n is admissible we conclude that $\lim_{n\geq 0} A_n$ is also admissible. We have thus proved that X is an admissible derived k°-adic Deligne-Mumford stack.

We are finished if we prove that $\mathsf{X}^{\mathrm{rig}} \simeq X$. We have a sequence of equivalences

$$\mathrm{t}_{\leq n}(\mathsf{X}^{\mathrm{rig}}) \simeq (\mathrm{t}_{\leq n}\mathsf{X})^{\mathrm{rig}} \simeq \mathsf{X}_n^{\mathrm{rig}} \simeq \mathrm{t}_{\leq n}(X)$$

by convergence of derived k-analytic stacks, see [PY17a, §7]. Assembling these equivalences together produces a map

$$f \colon \mathsf{X}^{\mathrm{rig}} \to X$$

in the ∞-category dAn. The underlying morphism of ∞-topoi is an equivalence. The morphism f induces equivalences, for each $i \geq 0$,

$$\pi_i(\mathcal{O}_0^{\mathrm{rig}}) \simeq \pi_i(\mathcal{O}),$$

where $\mathcal{O}_0 \coloneqq \lim_{n \geq 0} \mathcal{O}_{0,n}$. By hypercompletion of the $\mathcal{X}$ it follows that

$$\mathcal{O}_0^{\mathrm{rig}} \simeq \mathcal{O}.$$

Thus proving that f is an equivalence, finishing the proof. $\square$

We now deal with our main result. We start with a useful lemma:

Lemma 3.4.4.6. *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Suppose that for any $D \in \mathcal{D}$ the following assertions are satisfied:*

(i) The ∞-category $\mathcal{C}_{/D} \coloneqq \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is contractible;

(ii) let $\mathcal{C}'_{/D}$ denote the full subcategory of $\mathcal{C}_{/D}$ spanned by those objects $(C, \psi \colon F(C) \to D)$ such that ψ is an equivalence in $\mathcal{D}$. Suppose further that $\mathcal{C}'_{/D}$ is non-empty and moreover the inclusion $\mathcal{C}'_{/D} \to \mathcal{C}_{/D}$ is cofinal.

Then $F \colon \mathcal{C} \to \mathcal{D}$ induces an equivalence of ∞-categories $\mathcal{C}[S^{-1}] \to \mathcal{D}$, where S denotes the class of morphisms $f \in \mathcal{C}^{\Delta^1}$ such that $F(f)$ is an equivalence.

Proof. Let $\mathcal{E}$ be an ∞-category. We have to prove that precomposition along F induces a fully faithful embedding of ∞-categories

$$F^* \colon \mathrm{Fun}\,(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}\,(\mathcal{C}, \mathcal{E})$$

whose essential image consists of those functors $G \colon \mathcal{C} \to \mathcal{E}$ which send morphisms in S to equivalences in $\mathcal{D}$. Given any functor $G \colon \mathcal{D} \to \mathcal{E}$, the composite $G \circ F \colon \mathcal{C} \to \mathcal{E}$ sends each morphism in S to an equivalence $\mathcal{E}$ as F does (in $\mathcal{D}$). Thanks to the colimit formula for left Kan extensions together with conditions (i) and (ii) in the statement of the Lemma, we conclude that given a functor $G \colon \mathcal{C} \to \mathcal{E}$ such that any morphism in S is sent to an equivalence, its left Kan extension $F_!(G) \in \mathrm{Fun}\,(\mathcal{D}, \mathcal{E})$ exists and we have natural. equivalence $F_! \circ F^* \simeq \mathrm{id}$ and $F^* \circ F_! \simeq \mathrm{id}$. The result now follows from the fact that $F_!$ is an inverse to F^* when restricted to the full subcategory of $\mathrm{Fun}\,(\mathcal{C}, \mathcal{E})$ spanned by those functors sending every morphism in S to an equivalence in $\mathcal{E}$. $\square$

Remark 3.4.4.7. Theorem 3.4.4.6 implies that the localization functor of classical Raynaud theorem is ∞-categorical, i.e. the usual category An' of quasi-paracompact and quasi-separated k-analytic spaces is the ∞-categorical localization of fSch_{k°. This is not a common phenomenon: if $\mathcal{C}$ is a 1-category and S a collection of morphisms in $\mathcal{C}$ then the ∞-categorical localization $\mathcal{C}[S^{-1}]$ is typically a genuine ∞-category.

Definition 3.4.4.8. Let $\mathrm{dAn}' \subseteq \mathrm{dAn}$ denote the full subcategory of dAn spanned by those quasi-paracompact and quasi-separated derived k-analytic spaces $X \in \mathrm{dAn}$.

Definition 3.4.4.9. Let $f \colon \mathsf{X} \to \mathsf{Y}$ be a morphism between derived k°-adic schemes. We say that f is generically strong if for each $i > 0$ the induced morphism

$$\pi_i \left(f^* \mathcal{O}_{\mathsf{Y}}\right)^{\mathrm{rig}} \to \pi_i \left(\mathcal{O}_{\mathsf{X}}\right)^{\mathrm{rig}}$$

is an equivalence in the ∞-category $\mathrm{Coh}^+(\mathsf{X}^{\mathrm{rig}})$.

Theorem 3.4.4.10 (Derived Raynaud Localization Theorem)**.** *Let S denote the saturated class of morphism of $\mathrm{dfSch}^{\mathrm{adm}}$ generated by those generically strong morphisms $f \colon \mathsf{X} \to \mathsf{Y}$ such that $\mathrm{t}_{\leq 0}(f)$ is an admissible blow-up of ordinary k°-adic schemes. Then the rigidification functor*

$$(-)^{\mathrm{rig}} \colon \mathrm{dfSch}^{\mathrm{adm}} \to \mathrm{dAn}'$$

induces an equivalence of ∞-categories

$$\mathrm{dfSch}^{\mathrm{adm}}[S^{-1}] \simeq \mathrm{dAn}'.$$

Theorem 3.4.4.10 is an immediate consequence of Theorem 3.4.4.6 together with the following Proposition:

Proposition 3.4.4.11. *The rigidification functor* $(-)^{\mathrm{rig}} \colon \mathrm{dfSch}^{\mathrm{adm}} \to \mathrm{dAn}'$ *satisfies the dual assumptions of the statement in Theorem 3.4.4.6.*

Proof. The verification of the assumptions of Theorem 3.4.4.6 are made simultaneously: Let $X \in \mathrm{dAn}'$, $\mathcal{C}_X := \left(\mathrm{dfSch}^{\mathrm{adm}}\right)_{X/}$ and $p_0 \colon \mathcal{C}_X \to \mathrm{dfSch}$, $p_1 \colon \mathcal{C}_X \to \mathrm{dAn}'$ denote the canonical projections. We will show that for every finite space K and every functor $f \colon K \to \mathcal{C}_X$ we can extend f to a (cone) functor $f^{\triangleleft} \colon K^{\triangleleft} \to \mathcal{C}_X$ in such a way that $f^{\triangleleft}(\infty)$ is a formal model for $X \in \mathrm{dAn}'$, where $\infty \in K^{\triangleleft}$ denotes the cone point. This will imply that $\mathcal{C}_X$ is a cofiltered ∞-category, hence of contractible homotopy type and moreover the inclusion of the full subcategory of formal models for X is final in $\mathcal{C}_X$.

Let us first sketch the rough idea of proof: By induction on Postnikov towers we allow ourselves to lift commutative diagrams of derived k-analytic spaces to the formal level. This is done, by reducing questions of lifting of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structures on certain ∞-topoi to lifting questions at the level of ∞-categories of coherent modules, using the universal property of the adic cotangent complex. The corresponding questions for coherent modules can be dealt using the refined results in Appendix A. The main technical difficulty is thus keeping track of higher coherences for commutative diagrams when passing from the analytic ∞-category to the k°-adic one.

We will construct a sequence $\{(\mathsf{X}_n, \mathrm{t}_{\leq n} X \to \mathsf{X}^{\mathrm{rig}}) \in \mathcal{C}_{\mathrm{t}_{\leq n} X}\}_{n \in \mathbb{N}}$ such that $\mathsf{X}_n := (\mathcal{X}_n, \mathcal{O}_{\mathsf{X}_n}) \in \mathrm{dfSch}^{\mathrm{adm}}$ satisfies the following conditions:

(i) For each $n \geq 0$, X_n is n-truncated.

(ii) For each $n \geq 0$, we have an equivalence

$$\left(\mathsf{X}_n\right)^{\mathrm{rig}} \simeq \mathrm{t}_{\leq n} \mathsf{X}.$$

(iii) For each $n \geq 0$, we have a canonical morphism

$$\mathsf{X}_n \to \mathrm{t}_{\leq n+1} \mathsf{X}_n$$

in the ∞-category $\mathrm{dfSch}^{\mathrm{adm}}$ which is moreover an equivalence. This implies, in particular, that the underlying ∞-topoi $\mathcal{X}_n \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ are all equivalent, for $n \geq 0$.

(iv) For each $n \geq 0$, there is a functor $f_n^{\triangleleft} \in \mathrm{Fun}\left(K^{\triangleleft}, \mathcal{C}_{\mathrm{t}_n X}\right)$ whose restriction $(f_n^{\triangleleft})_{|K}$ is naturally equivalent to $\mathrm{t}_{\leq n} f$ in the ∞-category $\mathrm{Fun}\left(K, \mathcal{C}_{\mathrm{t}_{\leq n} X}\right)$ and such that $p_0\left(f_n^{\triangleleft}(\infty)\right) \simeq \mathsf{X}_n$.

Assume that we have constructed such a sequence $\{(\mathsf{X}_n, \mathrm{t}_{\leq n} X \to \mathsf{X}^{\mathrm{rig}}) \in \mathcal{C}_{\mathrm{t}_{\leq n} X}\}_{n \in \mathbb{N}}$ satisfying conditions (i) through (iv). Define $\mathsf{X} := \mathrm{colim}_{n \geq 0} \mathsf{X}_n$ and notice that in such case the morphisms $\mathrm{t}_{\leq n} X \to \mathsf{X}_n^{\mathrm{rig}}$ assemble to induce a morphism

$$X \to \mathsf{X}^{\mathrm{rig}},$$

in the ∞-category dAn. Moreover, by the universal property of filtered limits the diagrams $f_n^{\triangleleft} \in \mathrm{Fun}\left(K^{\triangleleft}, \mathcal{C}_{\mathrm{t}_{\leq n} X}\right)$ assemble thus producing a well defined (up to contractible indeterminacy) extension $f^{\triangleleft} \in \mathrm{Fun}\left(K^{\triangleleft}, \mathcal{C}_X\right)$ of $f \colon K \to \mathcal{C}_X$. As the rigidification functor is compatible with n-truncations it follows that the functor $f^{\triangleleft}$ obtained in this way implies that $p_1\left(f^{\triangleleft}(\infty)\right) \in \left(\mathrm{dAn}'\right)_{X/}$ corresponding to the morphism

$$X \to \mathsf{X}^{\mathrm{rig}},$$

in the ∞-category dAn, is an equivalence. This finishes the proof of the claim. Therefore, we are reduced to prove the existence of a sequence $\{(\mathsf{X}_n, \mathrm{t}_{\leq n} X \to \mathsf{X}_n^{\mathrm{rig}}) \in \mathcal{C}_{\mathrm{t}_{\leq n} X}\}_{n \in \mathbb{N}}$ satisfying conditions (i) through (iv) above.

Step 1

(Case $n = 0$) Let $X_0 := \mathrm{t}_{\leq 0} X \in \mathrm{An}$ denote the underlying ordinary k-analytic space to X. By the universal property of n-truncation we can assume without loss of generality that for each vertex $x \in K$ the component $\left(\mathsf{Y}_x, \psi_x \colon X_0 \to \mathsf{Y}_x^{\mathrm{rig}}\right) := f(x) \in \mathcal{C}_{X_0}$ is actually discrete, i.e. Y_X is an ordinary k°-adic formal scheme. The result is now a direct consequence of [Bos05, Theorem 3, page 204].

Step 2

(Inductive assumptions) Suppose now, that for $n \geq 0$ we have constructed a diagram $f_n^{\triangleleft} \in \mathrm{Fun}\left(K^{\triangleleft}, \mathcal{C}_{\mathrm{t}_{\leq n} X}\right)$ satisfying conditions (i) through (iv) above. Denote by $\alpha_{n,x} \colon \mathsf{X}_n \to \mathsf{Y}_{n,x}$ the morphism associated to $\infty \to x$ in $K^{\triangleleft}$, where $\mathsf{Y}_{n,x} \coloneqq \mathrm{t}_{\leq n} \mathsf{Y}_x = \left(\mathcal{Y}_{n,x}, \mathcal{O}_{n,x}\right)$. The functor $f_n^{\triangleleft} \in \mathrm{Fun}\left(K^{\triangleleft}, \mathcal{C}_{\mathrm{t}_{\leq n} X}\right)$ corresponds to the following given:

(i) A diagram $f_{n,*}^{\triangleleft} \colon K^{\triangleleft} \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ such that $f(\infty) \simeq \mathcal{X}_n$ and for each $x \in K$ a morphism $\alpha_{n,x,*} \colon \mathcal{X}_n \to \mathcal{Y}_{n,x}$ in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}$. We remark that this data is constant for $0 \leq m \leq n$.

(ii) A diagram $f_n^{\triangleleft,-1} \colon K^{\triangleleft,\mathrm{op}} \to \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}}$ such that $f^{\triangleleft,-1}(\infty) \simeq \mathrm{id}_{\mathcal{O}_{\mathsf{X}_n}}$ and $f^{\triangleleft,-1}(x)$ corresponds to a morphism $h_{n,x} \colon \alpha_{n,x}^{-1} \mathcal{O}_{\mathsf{Y}_{n,x}} \to \mathcal{O}_{\mathsf{X}_n}$ in the ∞-category $\mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}}$.

A similar analysis for the diagram $\mathrm{t}_{\leq n+1} f \colon K \to \mathcal{C}_{\mathrm{t}_{\leq n+1} X}$ together with the Postnikov decomposition imply that we have a functor $f_{n+1}^{-1} \colon K^{\mathrm{op}} \times \left(\Delta_1\right)^2 \to \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}}$ such that for each $x \in K$ the induced morphism

$$f_{n+1,x}^{-1} \colon \left(\Delta_1\right)^2 \to \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}}$$

corresponds to a pullback diagram of the form

$$\begin{tikzcd}
\tau_{\leq n+1} \alpha_x^{-1} \mathcal{O}_{\mathsf{Y}_x} \arrow[r] \arrow[d] & \tau_{\leq n} \alpha_x^{-1} \mathcal{O}_{\mathsf{Y}_x} \arrow[d, "d_{n,x}"] \\
\tau_{\leq n} \alpha_x^{-1} \mathcal{O}_{\mathsf{Y}_x} \arrow[r, "d_{n,x}^0"] & \tau_{\leq n} \alpha_x^{-1} \mathcal{O}_{\mathsf{Y}_x} \oplus \alpha_x^{-1} \pi_{n+1}\left(\mathcal{O}_{\mathsf{Y}_x}\right)[n+2]
\end{tikzcd} \tag{3.4.4.3}$$

in the ∞-category $\mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}}$, where $d_{n,x}$ denotes a suitable k°-adic derivation and $d_{n,x}^0$ the trivial adic derivation.

Step 3

(Functoriality of the construction $\mathrm{fCAlg}_{k^\circ}(\mathcal{X})_{\mathcal{O}//\mathcal{O}}$) Consider the functor $I \colon \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}} \to \mathcal{C}\mathrm{at}_\infty$ given on objects by the formula

$$\left(\mathcal{O} \to \mathcal{O}_{\mathsf{X}_n}\right) \in \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}} \mapsto \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}//\mathcal{O}} \in \mathcal{C}\mathrm{at}_\infty$$

whose transition morphisms correspond to (suitable) base change functors. Let $\mathcal{D} \to \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}}$ denote the corresponding coCartesian fibration obtained via the unstraightening construction. Notice that pullback along $\mathcal{O} \to \mathcal{O}_{\mathsf{X}_n}$ induces a functor $g_{\mathcal{O}} \colon \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}} \to \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}//\mathcal{O}}$, which admits a left adjoint $f_{\mathcal{O}} \colon \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}//\mathcal{O}} \to \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}}$, obtained via base change along $\mathcal{O} \to \mathcal{O}_{\mathsf{X}_n}$. Therefore, applying the unstraightening construction, we obtain a well defined functor

$$G \colon \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}} \times \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}} \to \mathcal{D}$$

over $\mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}}$, whose fiber at $(\mathcal{O} \to \mathcal{O}_{\mathsf{X}_n}) \in \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}}$ coincides with $g_{\mathcal{O}}$ introduced above. Thanks to the (dual) discussion proceding [PY16c, Corollary 8.6] it follows that G admits a left adjoint $F \colon \mathcal{D} \to \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}} \times \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}}$.

Step 4

(Base change of (3.4.4.3) along the morphism $\tau_{\leq n} \alpha_x^{-1} \mathcal{O}_{\mathsf{Y}_x} \to \mathcal{O}_{\mathsf{X}_n}$) The zero derivations $d_{n,x}^0$ in (3.4.4.3) assemble to give a a well defined functor $d_n^0 \colon K^{\mathrm{op}} \to \mathcal{D}$ and similarly the $d_{n,x}$ induce a well defined functor $d_n \colon K^{\mathrm{op}} \to \mathcal{D}$. Denote $\Delta_0 \coloneqq F \circ d_n^0$ and $\Delta \coloneqq F \circ d_n \colon K^{\mathrm{op}} \to \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}}$, respectively. Notice that $\Delta_0 \colon K^{\mathrm{op}} \to \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}}$ is given on objects by the formula

$$x \in K^{\mathrm{op}} \mapsto \left(\mathcal{O}_{\mathsf{X}_n} \to \mathcal{O}_{\mathsf{X}_n} \oplus \alpha_x^* \pi_{n+1}\left(\mathcal{O}_{\mathsf{Y}_x}\right)[n+2] \xrightarrow{d_{n,x}^0} \mathcal{O}_{\mathsf{X}_n}\right) \in \mathrm{fCAlg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}}$$

and similarly for $\Delta \colon K^{\mathrm{op}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}}$ we have

$$x \in K^{\mathrm{op}} \mapsto \Big(\mathcal{O}_{\mathsf{X}_n} \to \mathcal{O}_{\mathsf{X}_n} \oplus \alpha_x^* \pi_{n+1}\big(\mathcal{O}_{\mathsf{Y}_x}\big)[n+2] \xrightarrow{d_{n,x}} \mathcal{O}_{\mathsf{X}_n} \Big) \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}}.$$

By construction, both functor Δ_0 and Δ factor through the full subcategory

$$\mathrm{f}\mathcal{C}\mathrm{Alg}^{\mathrm{der}}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}} \subseteq \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}}$$

spanned by those objects $\mathcal{O}_{\mathsf{X}_n} \to \mathcal{A} \to \mathcal{O}_{\mathsf{X}_n}$ which correspond to k°-adic derivations.

Step 5

(Reduction of the above diagrams to diagrams of modules) The universal property of the k°-adic cotangent complex implies that we have an equivalence of ∞-categories

$$(-)^{\mathrm{der}} \colon \mathrm{f}\mathcal{C}\mathrm{Alg}^{\mathrm{der}}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}} \simeq \big(\mathrm{Mod}_{\mathcal{O}_{\mathsf{X}_n}}\big)_{\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}}/}.$$

Therefore, the functors Δ_0 and Δ as above correspond, under the equivalence $(-)^{\mathrm{der}}$, to functors $\Delta_0,\ \Delta \colon K^{\mathrm{op}} \to \big(\mathrm{Mod}_{\mathcal{O}_{\mathsf{X}_n}}\big)_{\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}}}$ given on objects by the formulas

$$x \in K^{\mathrm{op}} \mapsto \big(d^0_{n,x} \colon \mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}} \to \alpha_x^* \pi_{n+1}\big(\mathcal{O}_{\mathsf{Y}_x}\big)[n+2]\big) \in \big(\mathrm{Mod}_{\mathcal{O}_{\mathsf{X}_n}}\big)_{\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}}/}$$

and

$$x \in K^{\mathrm{op}} \mapsto \big(d_{n,x} \colon \mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}} \to \alpha_x^* \pi_{n+1}\big(\mathcal{O}_{\mathsf{Y}_x}\big)[n+2]\big) \in \big(\mathrm{Mod}_{\mathcal{O}_{\mathsf{X}_n}}\big)_{\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}}/},$$

respectively. Thanks to the proofs of both [PY17a, Lemma 5.35 and Corollary 5.38] the k°-adic cotangent complex $\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}}$ is coherent and connective. Therefore the functors $\Delta_0,\ \Delta \colon K^{\mathrm{op}} \to \big(\mathrm{Mod}_{\mathcal{O}_{\mathsf{X}_n}}\big)_{\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}}/}$ factor through the full subcategory $\mathrm{Coh}^+(\mathcal{O}_{\mathsf{X}_n})_{\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}}/} \subseteq \big(\mathrm{Mod}_{\mathcal{O}_{\mathsf{X}_n}}\big)_{\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}}/}$.

Step 6

(Rigidification of the corresponding diagrams of modules) Consider now the composites

$$\Delta_0^{\mathrm{rig}} := (-)^{\mathrm{rig}} \circ \Delta_0, \quad \Delta^{\mathrm{rig}} := (-)^{\mathrm{rig}} \circ \Delta \colon K^{\mathrm{op}} \to \mathrm{Coh}^+(\mathcal{O}^{\mathrm{rig}}_{\mathsf{X}_n})_{\mathbb{L}^{\mathrm{an}}_{\mathsf{X}_n^{\mathrm{rig}}}/}.$$

The same reasoning as above applied to the rigidification of the diagram $\mathrm{t}_{\leq n+1} f \colon K \to \mathcal{C}_{\mathrm{t}_{\leq n+1} X}$ produces extensions

$$\widetilde{\Delta}_0^{\mathrm{rig}}, \quad \widetilde{\Delta}^{\mathrm{rig}} \colon K^{\triangleleft, \mathrm{op}} \to \mathrm{Coh}^+(\mathcal{O}^{\mathrm{rig}}_{\mathsf{X}_n})_{\mathbb{L}^{\mathrm{an}}_{\mathcal{O}^{\mathrm{rig}}_{\mathsf{X}_n}}}$$

of $(-)^{\mathrm{rig}} \circ \Delta_0$ and of $(-)^{\mathrm{rig}} \circ \Delta$, respectively, satisfying:

(i) We have equivalences

$$\big(\widetilde{\Delta_0}\big)_{|K^{\mathrm{op}}} \simeq (-)^{\mathrm{rig}} \circ \Delta_0, \quad \widetilde{\Delta}_{|K^{\mathrm{op}}} \simeq (-)^{\mathrm{rig}} \circ \Delta$$

in the ∞-category $\mathrm{Fun}\big(K^{\mathrm{op}}, \mathrm{Coh}^+(\mathcal{O}^{\mathrm{rig}}_{\mathsf{X}_n})_{\mathbb{L}^{\mathrm{an}}_{\mathsf{X}_n^{\mathrm{rig}}}/}\big)$.

(ii) We have moreover equivalences

$$\widetilde{\Delta_0}^{\mathrm{rig}}(\infty) \simeq \Big(d_0 \colon \mathbb{L}^{\mathrm{an}}_{\mathcal{O}^{\mathrm{rig}}_{\mathsf{X}_n}} \to \pi_{n+1}\big(\mathcal{O}_X\big)[n+2]\Big),$$

and

$$\widetilde{\Delta}^{\mathrm{rig}}(\infty) \simeq \Big(d \colon \mathbb{L}^{\mathrm{an}}_{\mathcal{O}^{\mathrm{rig}}_{\mathsf{X}_n}} \to \pi_{n+1}\big(\mathcal{O}_X\big)[n+2]\Big)$$

in the ∞-category $\mathrm{Coh}^+(\mathrm{t}_{\leq n}(X))_{\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_{\leq n} X}/}$.

Where the derivations d_0 and d considered above are induced by the pullback diagram

$$\begin{tikzcd} \tau_{\leq n+1}\mathcal{O}_X \arrow[r] \arrow[d] & \tau_{\leq n}\mathcal{O}_X \arrow[d, "d"] \\ \tau_{\leq n}\mathcal{O}_X \arrow[r, "d_0"] & \tau_{\leq n}\mathcal{O}_X \oplus \pi_{n+1}\big(\mathcal{O}_X\big)[n+2] \end{tikzcd} \tag{3.4.4.4}$$

in the ∞-category $\mathrm{AnRing}_k(X)_{/\mathcal{O}_X}$.

Step 7

(Lifting of $\widetilde{\Delta_0}^{\mathrm{rig}}$ and $\widetilde{\Delta}^{\mathrm{rig}}$ to diagrams in $\mathrm{Coh}^+(\mathsf{X}_n)$.) Thanks to Theorem .1.2.1 and its proof, we can lift both diagrams $\widetilde{\Delta}_0^{\mathrm{rig}}$ and $\widetilde{\Delta}^{\mathrm{rig}}$ to (formal model) diagrams $\underline{\Delta}_0,\ \underline{\Delta}\colon K^{\mathrm{op}} \to \mathrm{Coh}^+(\mathcal{O}_{\mathsf{X}_n})_{\mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}}/}$, respectively. We have moreover equivalences

$$\underline{\Delta}_{0|K^{\mathrm{op}}} \simeq \Delta_0, \quad \underline{\Delta}_{|K^{\mathrm{op}}} \simeq \Delta.$$

and

$$\underline{\Delta}_0(\infty) \simeq \Big(\delta_0 \colon \mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}} \to N[n+2]\Big) \tag{3.4.4.5}$$

$$\underline{\Delta}(\infty) \simeq \Big(\delta \colon \mathbb{L}^{\mathrm{ad}}_{\mathcal{O}_{\mathsf{X}_n}} \to N[n+2]\Big) \tag{3.4.4.6}$$

where $N \in \mathrm{Coh}^+(\mathcal{O}_{\mathsf{X}_n})$ denotes a t-torsion free formal model of $\pi_{n+1}(\mathcal{O}_{\mathsf{X}_n})$, concentrated in degree 0. The choice of such $N \in \mathrm{Coh}^+(\mathcal{O}_{\mathsf{X}_n})$ can be realized as follows: First choose a given formal model $N \in \mathrm{Coh}^+(\mathcal{O}_{\mathsf{X}_n})$ for $\pi_{n+1}(\mathcal{O}_X)$. As the rigidification functor $(-)^{\mathrm{rig}}$ is compatible with n-truncations, we can replace N with $\tau_{\leq n}N$ and thus suppose that N is truncated to begin with. We can kill the t-torsion on N by multiplying it by a sufficiently large power of t, i.e. consider $t^m N$ for $m > 0$ sufficiently large such that $t^m N$ is t-torsion free. The conclusion now follows thanks to the fact that the canonical map $t^m N \to N$ induces an equivalence $\big(t^m N\big)^{\mathrm{rig}} \simeq N^{\mathrm{rig}}$.

Step 8

(Recovering the extension of the extension of the original diagram f_{n+1}^{-1} by means of the right adjoint G above) Notice that the rigidication of both (3.4.4.5) and (3.4.4.6) concides with the derivations d_0 and d displayed in (3.4.4.4), respectively. We can also consider the diagrams $\underline{\Delta}_0$ and $\underline{\Delta}$ as morphisms $\Delta_0 \to \delta_0$ and $\Delta \to \delta$ in the ∞-category $\mathrm{Fun}\big(K^{\mathrm{op}}, \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}}\big)$. Thanks to [Lur09b, Proposition 3.3.3.2] we can lift both diagrams $\underline{\Delta}_0$ and $\underline{\Delta}$ as functors $K^{\triangleleft,\mathrm{op}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}//\mathcal{O}_{\mathsf{X}_n}} \times \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_n)_{/\mathcal{O}_{\mathsf{X}_n}}$ whose projection along the first component agrees with $\underline{\Delta}_0$ and $\underline{\Delta}$, respectively, and whose projection along the second component agrees with the composition $F \circ f^{\triangleleft,-1}$. By adjunction, we obtain thus diagrams $D_0,\ D\colon K^{\triangleleft,\mathrm{op}} \to \mathcal{D}$ inducing $D_0', D'\colon K^{\triangleleft,\mathrm{op}} \times \Delta^2 \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}/}$ given on vertices $x \in K$ by the formula

$$x \in K^{\mathrm{op}}$$
$$\mapsto \Big(\tau_{\leq n}\alpha_x^{-1}\mathcal{O}_{\mathsf{Y}_x} \xrightarrow{d_{0,n}} \tau_{\leq n}\alpha_x^{-1}\mathcal{O}_{\mathsf{Y}_x} \oplus \pi_{n+1}\big(\mathcal{O}_{\mathsf{Y}_x}\big)[n+2] \to \tau_{\leq n}\alpha_x^{-1}\mathcal{O}_{\mathsf{Y}_x}\Big) \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}/}$$
$$x \in K^{\mathrm{op}}$$
$$\mapsto \Big(\tau_{\leq n}\alpha_x^{-1}\mathcal{O}_{\mathsf{Y}_x} \xrightarrow{d_n} \tau_{\leq n}\alpha_x^{-1}\mathcal{O}_{\mathsf{Y}_x} \oplus \pi_{n+1}\big(\mathcal{O}_{\mathsf{Y}_x}\big)[n+2] \to \tau_{\leq n}\alpha_x^{-1}\mathcal{O}_{\mathsf{Y}_x}\Big) \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}/},$$

respectively. Moreover, their value at ∞ correspond to

$$\mathcal{O}_{\mathsf{X}_n} \xrightarrow{d_0} \mathcal{O}_{\mathsf{X}_n} \oplus N[n+2] \to \mathcal{O}_{\mathsf{X}_n}, \quad \mathcal{O}_{\mathsf{X}_n} \xrightarrow{d} \mathcal{O}_{\mathsf{X}_n} \oplus N[n+2] \to \mathcal{O}_{\mathsf{X}_n},$$

respectively.

Step 9

(Obtaining an extension $f^{\lhd}_{n+1}$ of the diagram $\mathrm{t}_{\leq n+1} f$) By taking fiber products along over each $\{x\} \times \Lambda^2_2$ we obtain thus a diagram $\mathrm{f}^{\lhd}_{n+1} \colon K^{\lhd, \mathrm{op}} \to \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_n)_{\mathcal{O}_{\mathsf{X}_n}}$ whose value on each $x \in K$ agrees with

$$f^{\lhd}_{n+1}(x) \simeq \tau_{\leq n+1} \alpha_x^{-1} \mathcal{O}_{\mathsf{Y}_x}.$$

More precisely, we have a canonical equivalence $\big(f^{\lhd}_{n+1}\big)_{|K} \simeq \tau_{\leq n+1} f^{-1}$. Moreover, for $f^{\lhd}_{n+1}(\infty) \simeq \mathcal{O}_{n+1} \in \mathrm{f}\mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X}_n)$ such that

$$\mathcal{O}^{\mathrm{rig}}_{n+1} \simeq \tau_{\leq n+1} \mathcal{O}_X,$$

in the ∞-category $\mathrm{AnRing}_k(X)$. Let $\mathsf{X}_{n+1} \coloneqq (\mathcal{X}_n, \mathcal{O}_{n+1})$. We obtain thus a well defined functor

$$f^{\lhd}_{n+1} \colon K^{\lhd} \to \mathrm{dfSch}^{\mathrm{adm}}$$

whose rigidification coincides with

$$\tau_{\leq n+1} f \colon K \to \big(\mathrm{dAn}'\big)_{\mathrm{t}_{\leq n+1} X /}.$$

Assembling these diagrams together we obtain a functor $f^{\lhd}_{n+1} \colon K^{\lhd} \to \mathcal{C}_X$ satisfying requirements (i) through (iv) above, which concludes the proof. $\square$

The proof of Theorem 6.2.3.15 also implies:

Corollary 3.4.4.12. *Let $f \colon X \to Y$ be a morphism between quasi-paracompact and quasi-separated derived k-analytic spaces. Then f admits a formal model, i.e. there exists a morphism $\mathsf{f} \colon \mathsf{X} \to \mathsf{Y}$ in $\mathrm{dfSch}^{\mathrm{adm}}$ such that $\mathsf{f}^{\mathrm{rig}} \simeq f$ in the ∞-category dAn.*

Corollary 3.4.4.13. *Let S be the saturated class generated by those morphisms $f \colon A \to B$ in $\mathcal{C}\mathrm{Alg}^{\mathrm{adm}}_{k^\circ}$ such that the induced map*

$$\big(\mathrm{Spf}\, f\big)^{\mathrm{rig}} \colon \mathrm{Spf}(B)^{\mathrm{rig}} \to \mathrm{Spf}(A)^{\mathrm{rig}}$$

is an equivalence in the ∞-category of derived k-affinoid spaces dAfd. Then the rigidification functor $(-)^{\mathrm{rig}} \colon \big(\mathcal{C}\mathrm{Alg}^{\mathrm{adm}}_{k^\circ}\big)^{\mathrm{op}} \to \mathrm{dAfd}$ factors as

$$\big(\mathcal{C}\mathrm{Alg}^{\mathrm{adm}}_{k^\circ}\big)^{\mathrm{op}}[S^{-1}] \to \mathrm{dAfd}$$

and the latter is an equivalence of ∞-categories.

Proof. The result is a direct application of the proof of Theorem 3.4.4.10 when $X \in \mathrm{dAfd}$. $\square$

Appendices

.1 Verdier quotients and Lemma on Coh^+

The results in this section were proved in a joint work with M. Porta on the representability of the derived Hilbert stack, many of the statements and proofs are due to him.

.1.1 Verdier Quotients

In this §we let X be a quasi-compact and quasi-separated scheme and Z denote the formal completion of X along the (t)-locus. Consider also $Z^{\mathrm{rig}} \in \mathrm{An}$ its rigidification. We have a rigidification functor at the level of the derived ∞-categories of almost perfect complexes

$$(-)^{\mathrm{rig}} \colon \mathrm{Coh}^+(Z) \to \mathrm{Coh}^+\big(Z^{\mathrm{rig}}\big).$$

Notation .1.1.1. Let $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ denote the ∞-category of small stable ∞-categories and exact functors between them.

Proposition .1.1.2. *[HPV16a, Theorem B.2] Let $\mathcal{C}$ be a stable ∞-category and $\mathcal{A} \hookrightarrow \mathcal{C}$ a full stable subcategory. Then the pushout diagram*

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ 0 \arrow[r] & \mathcal{D} \end{tikzcd}$$

exists in the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$.

Definition .1.1.3. Let $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{D}$ as in Theorem .1.1.2. We refer to $\mathcal{D}$ as the *Verdier quotient* of $\mathcal{C}$ by $\mathcal{A}$.

Proposition .1.1.4. *Let X be a quasi-compact quasi-separated derived scheme almost of finite type over k°. We denote Z its formal (t)-completion and $Z^{\mathrm{rig}} \in \mathrm{dAfd}$ its rigidification. Then there exists a cofiber sequence*

$$\mathcal{K}(Z) \hookrightarrow \mathrm{Coh}^+(Z) \to \mathrm{Coh}^+\big(Z^{\mathrm{rig}}\big) \tag{.1.1.1}$$

in the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$. Moreover, the functors in (.1.1.1) are t-exact. In particular, the rigidification functor

$$(-)^{\mathrm{rig}} \colon \mathrm{Coh}^+(Z) \to \mathrm{Coh}^+\big(Z^{\mathrm{rig}}\big)$$

exhibits $\mathrm{Coh}^+\big(Z^{\mathrm{rig}}\big)$ as a (t-exact) Verdier quotient of $\mathrm{Coh}^+(Z)$.

Proof. Let $\mathcal{K}(Z)$ denote the full subcategory of $\mathrm{Coh}^+(Z)$ spanned by t-torsion almost perfect modules on Z. Recall that $M \in \mathrm{Coh}^+(Z)$ is of t-torsion if $\pi_*(M)$ is of t-torsion. Consider the (quasi-compact) étale site $X_{\text{ét}}$ of X. We define a functor

$$\mathfrak{Coh}^+(-)/\mathcal{K}(-) \colon X_{\text{ét}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$$

given on objects by the formula

$$U \to X \text{ quasi-compact and étale} \mapsto \mathrm{Coh}^+(U_t^\wedge)/\mathcal{K}(U_t^\wedge) \in \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$$

where $U_t^\wedge$ denotes the formal completion of U along the (t)-locus. Thanks to [HPV16a, Theorem 7.3] this defines a uniquely, up to contractible indeterminacy, defined $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$-valued sheaf for the étale topology.

We will also need the following ingredient: define a functor

$$\mathfrak{Coh}^+_{\mathrm{rig}} \colon X_{\text{ét}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$$

given on objects by the formula

$$U \to X \text{ quasi-compact and étale} \mapsto \mathrm{Coh}^+\big((U_t^\wedge)^{\mathrm{rig}}\big) \in \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}.$$

We remark that $\mathrm{Coh}^+ \colon \mathrm{An} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ satisfies fpqc descent for k-analytic spaces which follows by the main theorem in [Con03a] together with the usual reasoning by induction on the Postnikov towers for almost perfect modules in order to reduce the statement for Coh^+ to a similar statement concerning the heart $\mathrm{Coh}^{+,\heartsuit}$. Moreover, the formal completion and rigidification functors are morphisms of sites. As a consequence we conclude that the assignment $\mathfrak{Coh}^+_r ig\colon X_{\text{ét}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ is a sheaf for the étale topology on X.

The universal property of pushout induces a canonical morphism of sheaves $\Psi\colon \mathfrak{Coh}^+(-)/\mathcal{K}(-) \to \mathfrak{Coh}^+_{\mathrm{rig}}$ in the ∞-category $\mathrm{Shv}_{\text{ét}}(X, \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}})$. We affirm that Ψ is an equivalence in $\mathrm{Shv}_{\text{ét}}(X, \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}})$. By descent, it suffices to prove the statement on affine objects of $X_{\text{ét}}$. In such case, the result follows readily from the observation that for a derived k°-algebra A_0 the ∞-category $\mathrm{Coh}^+\big(A_0 \otimes_{k^\circ} k\big) \in \mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ is obtained from $\mathrm{Coh}^+(A_0)$ by "modding out" the full subcategory spanned by t-torsion almost perfect modules. Moreover, thanks to [PY18a, Theorem 3.1] we have a canonical equivalence

$$\mathrm{Coh}^+\big((\mathrm{Spf}\, A_0)^{\mathrm{rig}}\big) \simeq \mathrm{Coh}^+\Big(\Gamma\big((\mathrm{Spf}\, A_0)^{\mathrm{rig}}\big)\Big)$$

in the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$, where $\Gamma\big((\mathrm{Spf}\, A_0)^{\mathrm{rig}}\big) \in \mathcal{C}\mathrm{Alg}_k$ denotes the derived global sections of $\mathrm{Spf}\, A_0^{\mathrm{rig}}$. On the other hand $\Gamma\big((\mathrm{Spf}\, A_0)^{\mathrm{rig}}\big) \simeq A_0 \otimes_{k^\circ} k$ and the result follows. □

.1.2 Existence of formal models for modules

In this §we prove some results concerning the existence of formal models with respect to the functor $(-)^{\mathrm{rig}}\colon \mathrm{Coh}^+(\mathsf{X}) \to \mathrm{Coh}^+(X)$ which prove to be fundamental in the proof of Theorem 3.4.4.10. I am thankful to Mauro Porta as the results in this §were proved in a joint work.

Proposition .1.2.1. *Let $X \in \mathrm{dAn}$ be a derived k-analytic stack admitting a formal model $\mathsf{X} \in \mathrm{dfSch}$, i.e. $(\mathsf{X})^{\mathrm{rig}} \simeq X$ in dAn. Let $\mathcal{F} \in \mathrm{Coh}^+(X)$ be concentrated in finitely many cohomological degrees. Then $\mathcal{F}$ admits a formal model, i.e. there exists $\mathcal{G} \in \mathrm{Coh}^+(\mathsf{X})$ such that $\mathcal{G}^{\mathrm{rig}} \simeq \mathcal{F}$ in $\mathrm{Coh}^+(X)$. Moreover, the ∞-category of those formal models for $\mathcal{F}$ is a filtered ∞-category.*

Proof. Let $\mathcal{F} \in \mathrm{Coh}^+(X)$, be as in the stament of the Theorem .1.2.1. Assume moreover that $\mathcal{F}$ is connective, i.e. its non-zero cohomology lives in non-positive degrees. Notice that, by definition of ind-completion, $\mathcal{F} \in \mathrm{Ind}\big(\mathrm{Coh}^+(X)\big)$ is a compact object.

Let $\Phi\colon \mathrm{Ind}(\mathrm{Coh}^+(X)) \to \mathrm{Ind}(\mathrm{Coh}^+(\mathsf{X}))$ denote a fully faithful right adjoint to $(-)^{\mathrm{rig}}$. It follows from the construction of Ind-completion that we have a canonical equivalence

$$\Phi(\mathcal{F}) \simeq \operatorname*{colim}_{\mathcal{G} \in \mathrm{Coh}^+(\mathsf{X})_{/\Phi(\mathcal{F})}} \mathcal{G}, \tag{.1.2.1}$$

in $\mathrm{Ind}\big(\mathrm{Coh}^+(X)\big)$, where, by construction, the limit indexing ∞-category appearing on the right hand side of (.1.2.1) is filtered. As Φ is a fully faithful functor, the counit of the adjunction $\big((-)^{\mathrm{rig}}, \Phi\big)$ is an equivalence. Our argument now follows by an inductive reasoning using the Postnikov tower as we now detail:

Suppose first that $\mathcal{F} \in \mathrm{Coh}^+(X)$ has cohomology concentrated in degree 0, then it is well known that $\mathcal{F}$ admits a formal model $\widetilde{\mathcal{F}} \in \mathrm{Coh}^{+,\heartsuit}(\mathsf{X})$, which we can moreover choose to be of no t-torsion. Moreover, we can choose $\widetilde{\mathcal{F}}$ in such a way that we have a monomorphism $\widetilde{\mathcal{F}} \hookrightarrow \mathcal{F}$ in the heart $\mathrm{Coh}^{+,\heartsuit}(X)$), whose rigidification becomes an equivalence, in the (heart of) $\mathrm{Ind}\big(\mathrm{Coh}^+(\mathsf{X})\big)$. We are then dealt with the base of our inductive reasoning.

Suppose now that $\mathcal{F}$ lives in cohomological degrees $[-n, 0]$, by the inductive hypothesis $\mathcal{T}_{\leq n-1}\mathcal{F} \in \mathrm{Coh}^+(X)$ admits a formal model $\widetilde{\mathcal{T}_{\leq n-1}\mathcal{F}} \in \mathrm{Coh}^+(\mathsf{X})$, which lives in cohomological degrees $[-n+1, 0]$ and is moreover of no t-torsion and we have a map $\widetilde{\mathcal{T}_{\leq n-1}\mathcal{F}} \to \mathcal{T}_{\leq n-1}\mathcal{F}$ in the ∞-category $\mathrm{Ind}\big(\mathrm{Coh}^+(\mathsf{X})\big)$, whose rigidification becomes an equivalence. We have a fiber sequence

$$\mathcal{F} \longrightarrow \mathcal{T}_{\leq n-1}\mathcal{F} \longrightarrow \pi_n(\mathcal{F})[n+1],$$

in the ∞-category $\mathrm{Coh}^+(X)$. By applying the exact functor Φ we also obtain a fiber sequence in the ∞-category $\mathrm{Coh}^+(\mathsf{X})$.

As $\widetilde{\mathcal{T}_{\leq n-1}\mathcal{F}} \in \mathrm{Ind}\big(\mathrm{Coh}^+(\mathsf{X})\big)$ is a compact object, the composite $\widetilde{\mathcal{T}_{\leq n-1}\mathcal{F}} \to \mathcal{T}_{\leq n-1}\mathcal{F} \to \pi_n(\mathcal{F})[n+1]$ factors through $\mathcal{G}[n+1]$, for an almost perfect complex $\mathcal{G} \in \mathrm{Coh}^+(\mathsf{X})^\heartsuit$, such that $\mathcal{G}^{\mathrm{rig}} \simeq \pi_n(\mathcal{F})$, which by the base step, we can choose to be of no p-torsion and admitting a monomorphism $\mathcal{G} \to \pi_n(\mathcal{F})$ in the heart of the ∞-category $\mathrm{Ind}\big(\mathrm{Coh}^+(\mathsf{X})\big)$.

Using the fact that Φ is a right adjoint and the counit is an equivalence, the rigidification of the constructed map $\widetilde{\mathcal{T}_{\leq n-1}\mathcal{F}} \to \mathcal{G}[n+1]$ is equivalent to $\mathcal{T}_{\leq n-1}\mathcal{F} \to \pi_n(\mathcal{F})[n+1]$.

Therefore $\widetilde{\mathcal{F}} := \mathrm{fib}\left(\widetilde{\mathcal{T}_{\leq n-1}\mathcal{F}} \to \mathcal{G}[n+1]\right)$ is a formal model for $\mathcal{F}$, which lives in cohomological degrees $[-n, 0]$, of no t-torsion and admitting a map $\widetilde{\mathcal{F}} \to \mathcal{F}$ in the ∞-category $\mathrm{Ind}\left(\mathrm{Coh}^+(\mathsf{X})\right)$, which become an equivalence after rigidification. The first part of Theorem .1.2.1 now follows.

We are now left to prove that the full subcategory $\mathcal{C}_{\mathcal{F}}$ of the filtered ∞-category $\mathrm{Coh}^+(\mathsf{X})_{/\mathcal{F}}$ spanned by those objects $\left(\widetilde{\mathcal{F}}, \psi\colon \widetilde{\mathcal{F}}^{\mathrm{rig}} \to \mathcal{F}\right)$ such that ψ is an equivalence, is also filtered.

By construction, the ∞-category $\mathrm{Coh}^+(\mathsf{X})_{/\mathcal{F}}$ is filtered. In order to prove that $\mathcal{C}_{\mathcal{F}}$ is filtered, it suffices to show that every $\left(\mathcal{G}, \phi\colon \mathcal{G}^{\mathrm{rig}} \to \mathcal{F}\right) \in \mathrm{Coh}^+(\mathsf{X})_{/\mathcal{F}}$ admits a morphism to an object in $\mathcal{C}_{\mathcal{F}}$.

We first treat the case where $\mathcal{F} \in \mathrm{Coh}^+(X)$ lies in the heart so then we can write $\mathcal{F} \simeq \mathrm{colim}_{i \in I}\, \mathcal{G}_i$ in $\mathrm{Ind}\left(\mathrm{Coh}^+(\mathsf{X})\right)^{\heartsuit}$, where I is filtered. Moreover, we can assume that the $\mathcal{G}_i \in \mathrm{Coh}^+(\mathsf{X})^{\heartsuit}$ are (of no t-torsion) and for each $i \in I$ they admit monomorphisms $\mathcal{G}_i \to \mathcal{F}$ such that after rigidification one has $\mathcal{G}_i^{\mathrm{rig}} \simeq \mathcal{F}$ in $\mathrm{Ind}\left(\mathrm{Coh}^+(X)\right)^{\heartsuit}$. The structural morphism $\psi\colon \mathcal{G}^{\mathrm{rig}} \to \mathcal{F}$ corresponds by adjunction to a morphism $\mathcal{G} \to \Phi(\mathcal{F}) \simeq \mathrm{colim}_{i \in I}\, \Phi(\mathcal{G}_i)$. By compactness of $\mathcal{G} \in \mathrm{Coh}^+(\mathsf{X})$ if follows that the later factors through one of the $\mathcal{G}_i$. To summarize, we have obtained a morphism $\mathcal{G} \to \mathcal{G}_i$ which induces a morphism in $\mathrm{Coh}^+(\mathsf{X})_{/\mathcal{F}}$ whose source corresponds to $\left(\mathcal{G}, \phi\colon \mathcal{G}^{\mathrm{rig}} \to \mathcal{F}\right)$ and the target is an object lying in $\mathcal{C}_{\mathcal{F}}$, as desired.

Suppose now that $\mathcal{F} \in \mathrm{Coh}^+(\mathsf{X})$ is connective whose non-zero cohomology lives in degress $[-n, 0]$. Given $\left(\mathcal{G}, \phi\colon \mathcal{G}^{\mathrm{rig}} \to \mathcal{F}\right) \in \mathrm{Coh}^+(\mathsf{X})_{/\mathcal{F}}$ we know by induction that $\left(\mathcal{G}, \mathcal{G}^{\mathrm{rig}} \to \mathcal{F} \to \mathcal{T}_{\leq n-1}\mathcal{F}\right) \in \mathrm{Coh}^+(\mathsf{X})_{/\mathcal{T}_{\leq n-1}\mathcal{F}}$ admits a factorization through one object

$$\left(\widetilde{\mathcal{T}_{\leq n-1}}\mathcal{F}, \widetilde{\mathcal{T}_{\leq n-1}}\mathcal{F}^{\mathrm{rig}} \to \mathcal{T}_{\leq n-1}\mathcal{F}\right) \in \mathrm{Coh}^+(\mathsf{X})_{/\mathcal{T}_{\leq n-1}\mathcal{F}},$$

as before. We have a commutative diagram

$$\begin{tikzcd}
\tau_{\leq n}\mathcal{G} \arrow[r] \arrow[d] & \mathcal{T}_{\leq n-1}\mathcal{G} \arrow[r] \arrow[d] & \pi_n(\mathcal{G})[n+1] \arrow[d] \\
\mathcal{F} \arrow[r] & \mathcal{T}_{\leq n-1}\mathcal{F} \arrow[r] & \pi_n(\mathcal{F})[n+1],
\end{tikzcd}$$

where the horizontal maps form fiber sequences in the ∞-category $\mathrm{IndCoh}^+(\mathsf{X})$. Moreover, there exists a sufficiently large formal model $\mathcal{H}_n \in \mathrm{Coh}^+(\mathsf{X})^{\heartsuit}$ for $\pi_n(\mathcal{F})$, without t-torsion together with a monomorphism $\mathcal{H}_n \to \pi_n(\mathcal{F})$ in $\mathrm{Ind}\left(\mathrm{Coh}^+(\mathsf{X})\right)^{\heartsuit}$ such that both the composites

$$\mathcal{T}_{\leq n-1}\mathcal{G} \to \widetilde{\mathcal{T}_{\leq n-1}}\mathcal{F} \to \mathcal{T}_{\leq n-1}\mathcal{F} \to \pi_n(\mathcal{F})[n+1]$$

and

$$\mathcal{T}_{\leq n-1}\mathcal{G} \to \pi_n(\mathcal{G})[n+1] \to \pi_n(\mathcal{F})[n+1]$$

factor through $\mathcal{H}_n[n+1]$. Thus we have a commutative diagram of fiber sequences in the ∞-category $\mathrm{Ind}\left(\mathrm{Coh}^+(\mathsf{X})\right)$

$$\begin{tikzcd}
\tau_{\leq n}\mathcal{G} \arrow[r] \arrow[d] & \mathcal{T}_{\leq n-1}\mathcal{G} \arrow[r] \arrow[d] & \pi_n(\mathcal{G})[n+1] \arrow[d] \\
\widetilde{\mathcal{F}} \arrow[r] \arrow[d] & \widetilde{\mathcal{T}_{\leq n-1}}\mathcal{F} \arrow[r] \arrow[d] & M_n[n+1] \arrow[d] \\
\mathcal{F} \arrow[r] & \mathcal{T}_{\leq n-1}\mathcal{F} \arrow[r] & \pi_n(\mathcal{F})[n+1]
\end{tikzcd}$$

which provides a factorization $\left(\mathcal{G}, \phi\colon \mathcal{G}^{\mathrm{rig}} \to \mathcal{F}\right) \to \left(\widetilde{\mathcal{F}}, \psi\colon \widetilde{\mathcal{F}}^{\mathrm{rig}} \to \mathcal{F}\right)$ in the ∞-category $\mathrm{Coh}^+(\mathsf{X})_{/\mathcal{F}}$ where $\left(\widetilde{\mathcal{F}}, \psi\colon \widetilde{\mathcal{F}}^{\mathrm{rig}} \to \mathcal{F}\right) \in \mathcal{C}_{\mathcal{F}}$, this concludes the proof. $\square$

Corollary .1.2.2. *Let $X \in \mathrm{dAn}$ and $f \colon \mathcal{F} \to \mathcal{G}$ be a morphism $\mathrm{Coh}^+(X)$, where $\mathcal{G}$ is of bounded cohomology, i.e. $\mathcal{G} \in \mathrm{Coh}^{\mathrm{b}}(X)$. Suppose we are given a formal model $\mathsf{X} \in \mathrm{dfSch}_{k^\circ}^{\mathrm{taft}}$ for X.*

Then we can find a morphism $\mathfrak{f} \colon \mathcal{F}' \to \mathcal{G}'$ in $\mathrm{Coh}^+(\mathsf{X})$ such that $\mathfrak{f}^{\mathrm{rig}}$ lies in the same connected component of f in the mapping space $\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathcal{F}, \mathcal{G})$.

Proof. We will actually prove more: Fix $\mathcal{F}' \in \mathrm{Coh}^+(\mathsf{X})$ a formal model for $\mathcal{F}$, whose existence is guaranteed by Theorem .1.2.1 then we can find a formal model $\mathcal{G}' \in \mathrm{Coh}^+(\mathsf{X})$ for $\mathcal{G}$ such that the morphism

$$f \colon \mathcal{F} \to \mathcal{G},$$

in the ∞-category $\mathrm{Coh}^+(X)$ lifts to a morphism,

$$\mathfrak{f} \colon \mathcal{F}' \to \mathcal{G}',$$

in the ∞-category $\mathrm{Coh}^+(\mathsf{X})$. Assume thus $\mathcal{F}' \in \mathrm{Coh}^+(\mathsf{X})$ fixed. Given a generic $\mathcal{G}' \in \mathrm{Coh}^+(\mathsf{X})$, denote by $\mathcal{H}\mathrm{om}\,(\mathcal{F}', \mathcal{G}') \in \mathrm{QCoh}((\mathsf{X}))$ the Hom-sheaf of (quasi-coherent) $\mathcal{O}_{\mathsf{X}}$-modules. Notice that if $\mathcal{G}' \in \mathrm{Coh}^{\mathrm{b}}\,(\mathsf{X})$ then the Hom-sheaf $\mathcal{H}\mathrm{om}\,(\mathcal{F}', \mathcal{G}')$ is still an object lying in the ∞-category $\mathrm{Coh}^+\,(\mathsf{X})$

By our assumption on $\mathcal{G} \in \mathrm{Coh}^+(X)$, we can find a cohomogically bounded formal model $\mathcal{G}' \in \mathrm{Coh}^{\mathrm{b}}\,(\mathsf{X})$ for $\mathcal{G}$, and thus $\mathcal{H}\mathrm{om}\,(\mathcal{F}', \mathcal{G}') \in \mathrm{Coh}^+\,(\mathsf{X})$. Consider the colimit,

$$\operatorname*{colim}_{\mathcal{G}' \in \mathcal{C}} \mathcal{H}\mathrm{om}\,(\mathcal{F}', \mathcal{G}') \simeq \mathcal{H}\mathrm{om}\,(\mathcal{F}', G\,(\mathcal{G})) \tag{.1.2.2}$$

$$\simeq \mathcal{H}\mathrm{om}\left((\mathcal{F}')^{\mathrm{rig}}, \mathcal{G}\right) \simeq G\,(\mathcal{H}\mathrm{om}\,(\mathcal{F}, \mathcal{G}))\,, \tag{.1.2.3}$$

where $\mathcal{C}$ denotes the ∞-category of (cohomological bounded) formal models for $\mathcal{G}$. The first equivalence in (.1.2.2) follows from the fact that $\mathcal{F}' \in \mathrm{Coh}^+(\mathsf{X})$ is a compact object in $\mathrm{Ind}(\mathrm{Coh}^+(\mathsf{X}))$, thus the Hom-sheaf, with source $\mathcal{F}'$, commutes with filtered colimits, and the second equivalence follows from adjunction. By applying the global sections functor on both sides of (.1.2.2) we obtain an equivalence of spaces (notice that Φ being a right adjoint respects global sections)

$$\operatorname*{colim}_{\mathcal{G}' \in \mathcal{C}} \mathrm{Map}(\mathcal{F}', \mathcal{G}') \simeq \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathcal{F}, \mathcal{G}).$$

We conclude thus that there exists $\mathcal{G}' \in \mathcal{C}$ and $\mathfrak{f} \colon \mathcal{F}' \to \mathcal{G}'$ such that $(\mathfrak{f})^{\mathrm{rig}}$ and f lie in the same connected component of $\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathcal{F}, \mathcal{G})$, as desired. □

Corollary .1.2.3. *Let $X \in \mathrm{dAn}$ and $f \colon \mathcal{F} \to \mathcal{G}$ be a morphism $\mathrm{Coh}^+(X)$, where $\mathcal{G}$ is of bounded cohomology, i.e. $\mathcal{G} \in \mathrm{Coh}^{\mathrm{b}}(X)$. Suppose we are given a formal model X for X together with formal models $\mathcal{F}'$, $\mathcal{G}' \in \mathrm{Coh}^+(\mathsf{X})$ for $\mathcal{F}$ and $\mathcal{G}$, respectively, where we assume moreover that $\mathcal{G}' \in \mathrm{Coh}^{\mathrm{b}}(\mathsf{X})$. Then given an arbitrary $f \colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{Coh}^+(X)$ we can find $\mathfrak{f} \colon \mathcal{F}' \to \mathcal{G}'$ in $\mathrm{Coh}^+(\mathsf{X})$ lifting $t^n f \colon \mathcal{F} \to \mathcal{G}$, for a sufficiently large $n > 0$.*

Proof. Consider the sequence of equivalences in (.1.2.2). Then by applying the same argument as in the proof of Theorem .1.2.1 we obtain that an equivalence,

$$(\mathcal{H}\mathrm{om}(\mathcal{F}', \mathcal{G}'))^{\mathrm{rig}} \simeq \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}),$$

in the ∞-category $\mathrm{Coh}^+(X)$. Therefore, by taking global sections we obtain,

$$\mathrm{Map}_{\mathrm{Coh}^+(\mathsf{X})}(\mathcal{F}', \mathcal{G}')[t^{-1}] \simeq \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathcal{F}, \mathcal{G}),$$

where the left hand side term denotes the colimit $\mathrm{colim}_{\text{mult by } t} \mathrm{Map}_{\mathrm{Coh}^+(\mathsf{X})}(\mathcal{F}', \mathcal{G}')$. Therefore, by multiplying $f \in \mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathcal{F}, \mathcal{G})$ by a sufficiently large power of t, say t^n, then $t^n f$ should lie in a connected component of $\mathrm{Map}_{\mathrm{Coh}^+(\mathsf{X})}(\mathcal{F}', \mathcal{G}')$, as desired. □

.2 Unramifiedness of $\mathcal{T}_{\mathrm{ad}}(k^\circ)$

In this §we prove that the k°-adic pregeometry $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ together with the transformation of pregeometries $(-)^\wedge_t \colon \mathcal{T}_{\text{ét}}(k^\circ) \to \mathcal{T}_{\mathrm{ad}}(k^\circ)$ are unramified.

Definition .2.0.1. Let $\mathcal{T}$ be a pregeometry. We say that $\mathcal{T}$ is unramified if for every morphism $f\colon X \to Y$ in $\mathcal{T}$ and every object $Z \in \mathcal{T}$, the diagram

$$\begin{tikzcd} X \times Z \arrow[r] \arrow[d] & X \times Y \times Z \arrow[d] \\ X \arrow[r] & X \times Y \end{tikzcd}$$

induces a pullback diagram

$$\begin{tikzcd} \mathcal{X}_{X \times Z} \arrow[r] \arrow[d] & \mathcal{X}_{X \times Y \times Z} \arrow[d] \\ \mathcal{X}_X \arrow[r] & \mathcal{X}_{X \times Y} \end{tikzcd}$$

in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}$, where $\mathcal{X}_{X\times Z}$, $\mathcal{X}_{X\times Y\times Z}$, $\mathcal{X}_X$ and $\mathcal{X}_{X\times Y}$ denote the underlying ∞-topoi associated to the absolute spectrum construction, introduced in [Lur11c, §2.2].

Remark .2.0.2. Both the pregeometries $\mathcal{T}_{\text{ét}}(k)$ and $\mathcal{T}_{\mathrm{an}}(k)$ are unramified, see [Lur11a, Proposition 4.1] and [PY16a, Corollary 3.11], respectively.

Proposition .2.0.3. *The pregeometry $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ is unramified.*

Proof. Let $Z \in \mathcal{T}_{\mathrm{ad}}(k^\circ)$ and denote $\mathcal{X}_Z$ denote the underlying ∞-topos of the corresponding absolute spectrum $\mathrm{Spec}^{\mathcal{T}_{\mathrm{ad}}(k^\circ)}(Z)$. The ∞-topos $\mathcal{X}_Z$ is equivalent to the hypercompletion of the étale ∞-topos on the special fiber of Z. As pullback diagrams are preserved by taking special fibers the result follows by unramifiedness of $\mathcal{T}_{\text{ét}}(k^\circ)$. ☐

There is also a notion of relative unramifiedness:

Definition .2.0.4. Let $\varphi\colon \mathcal{T} \to \mathcal{T}'$ be a transformation of pregeometries, and let $\Phi\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}') \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ the induced functor given on objects by the formula

$$(\mathcal{X}, \mathcal{O}) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}') \mapsto (\mathcal{X}, \mathcal{O} \circ \varphi) \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}).$$

We say that the transformation f is unramified if the following conditions are satisfied:

(i) Both $\mathcal{T}$ and $\mathcal{T}'$ are unramified;

(ii) For every morphism $f\colon X \in Y$ and every object $Z \in \mathcal{T}$, we have a pullback diagram

$$\begin{tikzcd} \Phi\big(\mathrm{Spec}^{\mathcal{T}'}(X \times Z)\big) \arrow[r] \arrow[d] & \Phi\big(\mathrm{Spec}^{\mathcal{T}'}(X \times Y \times Z)\big) \arrow[d] \\ \Phi\big(\mathrm{Spec}^{\mathcal{T}'}(Z)\big) \arrow[r] & \Phi\big(\mathrm{Spec}^{\mathcal{T}'}(X \times Y)\big) \end{tikzcd}$$

in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$.

Proposition .2.0.5. *The transformation of pregeometries $(-)^\wedge_t\colon \mathcal{T}_{\text{ét}}(k^\circ) \to \mathcal{T}_{\mathrm{ad}}(k^\circ)$ is unramified.*

Proof. It suffices to prove condition (ii) in Theorem .2.0.4. This follows from the fact that $\Phi\big(\mathrm{Spec}^{\mathcal{T}_{\mathrm{ad}}(k^\circ)}(-)\big)$ is an ind-étale spectrum, thus such construction commutes with finite limits. ☐

.3 Useful Lemma

In this § we will prove a formal statement that proved to be useful in the proof of Theorem 3.3.1.11:

Lemma .3.0.1. *Let $(F, G)\colon \mathcal{C} \to \mathcal{D}$ be an adjunction of presentable ∞-categories. Suppose further that:*

(i) Any epimorphism in $\mathcal{C}$ is effective;

(ii) G is conservative, preserves epimorphisms and sifted colimits;

Then epimorphisms in $\mathcal{D}$ are also effective, moreover, if $\{X_\alpha\}$ is a family of compact generators for $\mathcal{C}$ the family $\{F(X_\alpha)\}$ generates $\mathcal{D}$ under sifted colimits.

Proof. Let $g\colon V \to Y$ be an epimorphism in the ∞-category $\mathcal{D}$. We wan to show that it is effective, that is the canonical morphism $g'\colon Y' := |\check{\mathcal{C}}(g)|Y$, where Y' denotes the geometric realization of the Cech nerve of g, is an equivalence in $\mathcal{D}$. By assumption, $G(g)$ is an epimorphism. Since G is a right adjoint, we have a canonical equivalence

$$G\left(\check{\mathcal{C}}(g)\right) \simeq \check{\mathcal{C}}\left(G(g)\right).$$

As G commutes with sifted colimits, we see that $G(Y') \simeq |\check{\mathcal{C}}\left(G(g)\right)| \simeq G(Y)$. We thus conclude that $Y' \simeq Y$ using the conservativity of G. This finishes the proof of the first assertion.

Let $Y \in \mathcal{D}$. We can find a filtered category I and a diagram $T\colon I \to \mathcal{C}$ such that

$$\operatorname*{colim}_{\alpha \in I} T_\alpha \simeq G(Y) \in \mathcal{C}.$$

Consider the composition $F \circ T\colon I \to \mathcal{D}$. For every $\alpha \in I$, we obtain a natural map

$$\varphi_\alpha\colon F(T_\alpha) \to F(G(Y)) \to Y,$$

where the latter morphism is induced by the counit of the adjunction (F, G). These maps φ_α can be arranged into a cocone from $F \circ T$ to Y. For each α, we can form the Čech nerve $\check{\mathcal{C}}(\varphi_\alpha)$. This produces a functor

$$\widetilde{T}\colon I \times \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{D},$$

informally defined by

$$(\alpha, n) \mapsto \check{\mathcal{C}}(\varphi_\alpha)^n.$$

There is a natural cocone from $\widetilde{T}$ to Y, and we claim that the induced map

$$\psi\colon \operatorname*{colim}_{(\alpha,n) \in I \times \mathbf{\Delta}^{\mathrm{op}}} \widetilde{T}(\alpha, n) \to Y$$

is an equivalence. We remark that

$$\operatorname*{colim}_{(\alpha,n) \in I \times \mathbf{\Delta}^{\mathrm{op}}} \widetilde{T}(\alpha, n) \simeq \operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} \operatorname*{colim}_{\alpha \in I} \widetilde{T}(\alpha, n).$$

Since G is conservative, it is enough to check that $G(\psi)$ is an equivalence. Observe that, ,since G commutes with sifted colimits, we have

$$G\left(\operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} \operatorname*{colim}_{\alpha \in I} \widetilde{T}(\alpha, n)\right) \simeq \operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} \operatorname*{colim}_{\alpha \in I} G\left(\widetilde{T}(\alpha, n)\right).$$

Since I is a filtered category and G is a right adjoint, we obtain:

$$G\left(\operatorname*{colim}_{\alpha \in I} \check{\mathcal{C}}(\varphi_\alpha)^n\right) \simeq \check{\mathcal{C}}\left(\operatorname*{colim}_{\alpha \in I} G(F(T_\alpha)) \to G(Y)\right)^n.$$

The unit of the adjunction (F, G) provide us with maps $\eta_\alpha\colon T_\alpha \to G(F(T_\alpha))$ such that the induced composition

$$\operatorname*{colim}_{\alpha \in I} T_\alpha \to \operatorname*{colim}_{\alpha \in I} G(F(T_\alpha)) \to G(Y)$$

is an equivalence. In particular, the map

$$\operatorname*{colim}_{\alpha \in I} G(F(T_\alpha)) \to G(Y)$$

is an effective epimorphism. In particular,

$$\operatorname*{colim}_{\alpha \in I} G(F(T_\alpha)) \to G(Y)$$

is an effective epimorphism. Thus,,

$$\operatorname*{colim}_{(\alpha,n) \in \mathbf{\Delta}^{\mathrm{op}}} G(\widetilde{T}(\alpha, n)) \simeq |\check{\mathcal{C}}(\operatorname*{colim}_{\alpha \in I} G(F(T_\alpha)) \to G(Y)| \simeq G(Y).$$

Thus, $G(\psi)$ is an equivalence, and so we conclude that ψ was an equivalence to start with. $\square$

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Chapter 4

Derived non-archimedean analytic Hilbert space

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Contents

4.1 Introduction

Let k be a non-archimedean field equipped with a non-trivial valuation of rank 1. We let k° denote its ring of integers, $\mathfrak{m}$ an ideal of definition. We furthermore assume that $\mathfrak{m}$ is finitely generated. Given a separated k-analytic space X, we are concerned with the existence of the *derived* moduli space $\mathbf{R}\mathrm{Hilb}(X)$, which parametrizes flat families of closed subschemes of X. The truncation of $\mathbf{R}\mathrm{Hilb}(X)$ coincides with the classical Hilbert scheme functor, $\mathrm{Hilb}(X)$, which has been shown to be representable by a k-analytic space in [CG16]. On the other hand, in algebraic geometry the representability of the derived Hilbert scheme is an easy consequence of Artin-Lurie representability theorem. In this paper, we combine the analytic version of Lurie's representability obtained by T. Y. Yu and the second author in [PY17b] together with a theory of derived formal models developped by the first author in [Ant18a]. The only missing step is to establish the existence of the cotangent complex.

Indeed, the techniques introduced in [PY18b] allows to prove the existence of the cotangent complex at points $x\colon S \to \mathbf{R}\mathrm{Hilb}(X)$ corresponding to families of closed subschemes $j\colon Z \hookrightarrow S \times X$ which are of finite presentation in the derived sense. However, not every point of $\mathbf{R}\mathrm{Hilb}(X)$ satisfies this condition: typically, we are concerned with families which are *almost* of finite presentation. The difference between the two situations is governed by the relative analytic cotangent complex $\mathbb{L}\mathrm{an}_{Z/S\times X}$: Z is (almost) of finite presentation if $\mathbb{L}\mathrm{an}_{Z/S\times X}$ is (almost) perfect. We can explain the main difficulty as follows: if $p\colon Z \to S$ denotes the projection to S, then the cotangent complex of $\mathbf{R}\mathrm{Hilb}(X)$ at $x\colon S \to \mathbf{R}\mathrm{Hilb}(X)$ is computed by $p_+(\mathbb{L}\mathrm{an}_{Z/S\times X})$. Here, p_+ is a (partial) left adjoint for the functor p^*, which has been introduced in the k-analytic setting in [PY18b]. However, in loc. cit. the functor p_+ has only been defined on perfect complexes, rather than on almost perfect complexes. From this point of view, the main contribution of this paper is to provide an extension of the construction p_+ to almost perfect complexes. Our construction relies heavily on the existence results for formal models of derived k-analytic spaces obtained by the first author in [Ant18a]. Along the way, we establish three results that we deem to be of independent interest, and which we briefly summarize below.

Let $\mathfrak{X}$ be a derived formal k°-scheme topologically almost of finite presentation. One of the main construction of [Ant17b, Ant18a, ?] is the generic fiber $\mathfrak{X}^{\mathrm{rig}}$, which is a derived k-analytic space. The formalism introduced in loc. cit. provides as well an exact functor

$$(-)^{\mathrm{rig}}\colon \mathrm{Coh}^+(\mathfrak{X}) \longrightarrow \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}), \tag{4.1.0.1}$$

where Coh^+ denotes the stable ∞-category of almost perfect complexes on $\mathfrak{X}$ and on $\mathfrak{X}^{\mathrm{rig}}$. When $\mathfrak{X}$ is underived, this functor has been considered at length in [HPV16b], where in particular it has been shown to be essentially surjective, thereby extending the classical theory of formal models for coherent sheaves on k-analytic spaces. In this paper we extend this result to the case where $\mathfrak{X}$ is derived, which is a key technical step in our construction of the plus pushforward. In order to do so, we will establish the following descent statement, which is an extension of [HPV16b, Theorem 7.3]:

Theorem 3. *The functor* $\mathrm{Coh}^+_{\mathrm{loc}}\colon \mathrm{dAn}_k \to \mathrm{Cat}^{\mathrm{st}}_\infty$*, which associates to every derived formal derived scheme*

$$\mathfrak{X} \in \mathrm{dfDM} \mapsto \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}) \in \mathrm{Cat}^{\mathrm{st}}_\infty,$$

satisfies Zariski hyper-descent.

We refer the reader to Theorem 4.3.1.7 for the precise statement. S consequence of **??** 3 above is the following statement, concerning the properties of ∞-categories of formal models for almost perfect complexes on $X \in \mathrm{dAn}_k$:

Theorem 4 (Theorem 4.3.3.10)**.** *Let $X \in \mathrm{dAn}_k$ be a derived k-analytic space and let $\mathcal{F} \in \mathrm{Coh}^+(\mathcal{F})$ be a bounded below almost perfect complex on X. For any derived formal model $\mathfrak{X}$ of X, there exists $\mathcal{G} \in \mathrm{Coh}^+(\mathfrak{X})$ and an equivalence $\mathcal{G}^{\mathrm{rig}} \simeq \mathcal{F}$. Furthermore, the full subcategory of $\mathrm{Coh}^+(\mathfrak{X}) \times_{\mathrm{Coh}^+(X)} \mathrm{Coh}^+(X)_{/\mathcal{F}}$ spanned by formal models of $\mathcal{F}$ is filtered.*

?? 4 is another key technical ingredient in the proof of the existence of a plus pushforward construction. The third auxiliary result we need is a refinement of the existence theorem for formal models for morphisms of derived analytic spaces proven in [Ant18a]. It can be stated as follows:

Theorem 5 (Theorem 4.4.0.1)**.** *Let $f\colon X \to Y$ be a flat map between derived k-analytic spaces. Then there are formal models $\mathfrak{X}$ and $\mathfrak{Y}$ for X and Y respectively and a flat map $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ whose generic fiber is equivalent to f.*

The classical analogue of **??** 5 was proven by Bosch and Lutkëbohmert in [BLR95b]. The proof of this theorem is not entirely obvious: indeed the algorithm provided in [Ant18a] proceeds by induction on the Postnikov tower of both X and Y, and at each step uses [HPV16b, Theorem 7.3] to choose appropriately formal models for $\pi_i(\mathcal{O}_X\mathrm{alg})$ and $\pi_i(\mathcal{O}_Y\mathrm{alg})$. In the current situation, however, the flatness requirement on $\mathfrak{f}$ makes it impossible to freely choose a formal model for $\pi_i(\mathcal{O}_X\mathrm{alg})$. We circumvent the problem by proving a certain lifting property for morphisms of almost perfect complexes:

Theorem 6 (Theorem 4.3.3.11)**.** *Let $X \in \mathrm{dAn}_k$ be a derived k-analytic space and let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathrm{Coh}^+(X)$. Let $\mathfrak{X}$ denote a given formal model for X. Suppose, futhermore, that we are given formal models $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}} \in \mathrm{Coh}^+(X)$ for $\mathcal{F}$ and $\mathcal{G}$, respectively. Then, there exists a non-zero element $t \in \mathfrak{m}$ such that the map $t^n f$ admits a lift $\widetilde{f}\colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$, in the ∞-category $\mathrm{Coh}^+(\mathfrak{X})$.*

Finally, the techniques of the current text allow us to prove the following generalization of [PY18b, Theorem 8.6]:

Theorem 7 (Theorem 4.6.0.3)**.** *Let S be a rigid k-analytic space. Let X, Y be rigid k-analytic spaces over S. Assume that X is proper and flat over S and that Y is separated over S. Then the ∞-functor $\mathbf{Map}_S(X, Y)$ is representable by a derived k-analytic space separated over S.*

Notation and conventions In this paper we freely use the language of ∞-categories. Although the discussion is often independent of the chosen model for ∞-categories, whenever needed we identify them with quasi-categories and refer to [Lur09c] for the necessary foundational material.

The notations $\mathcal{S}$ and $\mathcal{C}\mathrm{at}_\infty$ are reserved to denote the ∞-categories of spaces and of ∞-categories, respectively. If $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ we denote by $\mathcal{C}^\simeq$ the maximal ∞-groupoid contained in $\mathcal{C}$. We let $\mathcal{C}\mathrm{at}_\infty^{\mathrm{st}}$ denote the ∞-category of stable ∞-categories with exact functors between them. We also let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category of presentable ∞-categories with left adjoints between them. Similarly, we let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$ denote the ∞-categories of stably presentable ∞-categories with left adjoints between them. Finally, we set

$$\mathcal{C}\mathrm{at}_\infty^{\mathrm{st},\otimes} \coloneqq \mathcal{C}\mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{st}}) \quad , \quad \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{st}} \coloneqq \mathcal{C}\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}).$$

Given an ∞-category $\mathcal{C}$ we denote by $\mathrm{PSh}(\mathcal{C})$ the ∞-category of $\mathcal{S}$-valued presheaves. We follow the conventions introduced in [PY16d, §2.4] for ∞-categories of sheaves on an ∞-site.

For a field k, we reserve the notation $\mathcal{C}\mathrm{Alg}_k$ for the ∞-category of simplicial commutative rings over k. We often refer to objects in $\mathcal{C}\mathrm{Alg}_k$ simply as *derived commutative rings*. We denote its opposite by dAff_k, and we refer to it as the ∞-category of *derived affine schemes*. We say that a derived ring $A \in \mathcal{C}\mathrm{Alg}_k$ is *almost of finite presentation* if $\pi_0(A)$ is of finite presentation over k and $\pi_i(A)$ is a finitely presented $\pi_0(A)$-module.[1] We denote by $\mathrm{dAff}_k^{\mathrm{afp}}$ the full subcategory of dAff_k spanned by derived affine schemes $\mathrm{Spec}(A)$ such that A is almost of finite presentation. When k is either a non-archimedean field equipped with a non-trivial valuation or is the field of complex numbers, we let An_k denote the category of analytic spaces over k. We denote by $\mathrm{Sp}(k)$ the analytic space associated to k.

[1]Equivalently, A is almost of finite presentation if $\pi_0(A)$ is of finite presentation and the cotangent complex $\mathbb{L}_{A/k}$ is an almost perfect complex over A.

Acknowledgments We are grateful to B. Hennion, M. Robalo, B. Toën, G. Vezzosi and T. Y. Yu for useful discussions related to the content of this paper.

We are especially grateful to B. Toën for inviting the second author to Toulouse during the fall 2017, when the bulk of this research has been carried out and to IRMA, Université de Strasbourg, for inviting the first author to Strasbourg during the spring 2018.

4.2 Preliminaries on derived formal and derived non-archimedean geometries

Let k denote a non-archimedean field equipped with a rank 1 valuation. We let $k^\circ = \{x \in k : |x| \leq 1\}$ denote its ring of integers. We assume that k° admits a finitely generated ideal of definition $\mathfrak{m}$.

Notation 4.2.0.1. (i) Let R be a discrete commutative ring. Let $\mathcal{T}_{\mathrm{disc}}(R)$ denote the full subcategory of R-schemes spanned by affine spaces $\mathbb{A}^n_R$. We say that a morphism in $\mathcal{T}_{\mathrm{disc}}(R)$ is *admissible* if it is an isomorphism. We endow $\mathcal{T}_{\mathrm{disc}}(R)$ with the trivial Grothendieck topology.

(ii) Let $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ denote the full subcategory of k°-schemes spanned by formally smooth formal schemes which are topologically finitely generated over k°. A morphism in $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ is said to be *admissible* if it is formally étale. We equip the category $\mathcal{T}_{\mathrm{ad}}(k^\circ)$ with the formally étale topology, τét.

(iii) Denote $\mathcal{T}_{\mathrm{an}}(k)$ the category of smooth k-analytic spaces. A morphism in $\mathcal{T}_{\mathrm{an}}(k)$ is said to be *admissible* if it is étale. We endow $\mathcal{T}_{\mathrm{an}}(k)$ with the étale topology, τét.

In what follows, we will let $\mathcal{T}$ denote either one of the categories introduced above. We let τ denote the corresponding Grothendieck topology.

Definition 4.2.0.2. Let $\mathcal{X}$ be an ∞-topos. A $\mathcal{T}$-*structure* on $\mathcal{X}$ is a functor $\mathcal{O}\colon \mathcal{T} \to \mathcal{X}$ which commutes with finite products, pullbacks along admissible morphisms and takes τ-coverings in effective epimorphisms. We denote by $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ the full subcategory of $\mathrm{Fun}_{\mathcal{T}}(\mathcal{T}, \mathcal{X})$ spanned by $\mathcal{T}$-structures. A $\mathcal{T}$-*structured* ∞-*topos* is a pair $(\mathcal{X}, \mathcal{O})$, where $\mathcal{X}$ is an ∞-topos and $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$.

We can assemble $\mathcal{T}$-structured ∞-topoi into an ∞-category denoted ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$. We refer to [Lur11d, Definition 1.4.8] for the precise construction. The functor $\mathrm{Fun}(\mathcal{T}, -)\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ restricts to a functor

$$\mathrm{Fun}(\mathcal{T}, -)\colon \left({}^{\mathrm{R}}\mathcal{T}\mathrm{op}\right)\mathrm{op} \longrightarrow \mathcal{C}\mathrm{at}_\infty,$$

which sends a geometric morphism (f^{-1}, f_*) to the functor induced by composition with f^{-1}. Since the left adjoint of a geometric morphism preserves finite limits, it follows that it respects the full subcategories of $\mathcal{T}$-structures. In other words, we obtain a well defined functor

$$\mathrm{Str}_{\mathcal{T}}\colon \left({}^{\mathrm{R}}\mathcal{T}\mathrm{op}\right)\mathrm{op} \longrightarrow \mathcal{C}\mathrm{at}_\infty.$$

This defines a Cartesian fibration $p_{\mathrm{Str}}\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}\,\mathrm{op}$ and we can identify objects of ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ as pairs $(\mathcal{X}, \mathcal{O})$, where $\mathcal{X} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ and $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$. We say that an object of ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ is a $\mathcal{T}$-*structured* ∞-*topos*.

Definition 4.2.0.3. Let $\mathcal{X}$ be an ∞-topos. A morphism of $\mathcal{T}$-structures $\alpha\colon \mathcal{O} \to \mathcal{O}'$ is said to be *local* if for every admissible morphism $f\colon U \to V$ in $\mathcal{T}$ the diagram

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r, "\mathcal{O}(f)"] \arrow[d, "\alpha_U"] & \mathcal{O}(V) \arrow[d, "\alpha_V"] \\ \mathcal{O}'(U) \arrow[r, "\mathcal{O}'(f)"] & \mathcal{O}'(V) \end{tikzcd}$$

is a pullback square in $\mathcal{X}$. We denote by $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ the (non full) subcategory of $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ spanned by local structures and local morphisms between these.

Example 4.2.0.4. (i) Let R be a discrete commutative ring. A $\mathcal{T}_{\mathrm{disc}}(R)$-structure on an ∞-topos $\mathcal{X}$ is simply a product preserving functor $\mathcal{O}\colon \mathcal{T}_{\mathrm{disc}}(R) \to \mathcal{X}$. When $\mathcal{X} = \mathcal{S}$ is the ∞-topos of spaces, we can therefore use [Lur09c, Proposition 5.5.9.2] to identify the ∞-category $\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(R)}(\mathcal{X})$ with the underlying ∞-category $\mathcal{C}\mathrm{Alg}_R$ of the model category of simplicial commutative R-algebras. It follows that $\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(R)}(\mathcal{X})$ is canonically identified with the ∞-category of sheaves on $\mathcal{X}$ with values in $\mathcal{C}\mathrm{Alg}_R$. For this reason, we write $\mathcal{C}\mathrm{Alg}_R(\mathcal{X})$ rather than $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{disc}}(R)}(\mathcal{X})$.

(ii) Let $\mathfrak{X}$ denote a formal scheme over k° complete along $t \in k^\circ$. Denote by $\mathfrak{X}_{\mathrm{f\acute{e}t}}$ the small formal étale site on $\mathfrak{X}$ and denote $\mathcal{X} := \mathrm{Shv}(\mathfrak{X}_{\mathrm{ft}}, \tau_{\mathrm{\acute{e}t}})^\wedge$ denote the hypercompletion of the ∞-topos of formally étale sheaves on $\mathfrak{X}$. We define a $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structure on $\mathcal{X}$ as the functor which sends $U \in \mathfrak{X}_{\mathrm{f\acute{e}t}}$ to the sheaf $\mathcal{O}(U) \in \mathcal{X}$ defined by the association

$$V \in \mathfrak{X}_{\mathrm{f\acute{e}t}} \mapsto \mathrm{Hom}_{\mathrm{fSch}_{k^\circ}}(V, U) \in \mathcal{S}.$$

In this case, $\mathcal{O}(\mathbb{A}^1_{k^\circ})$ corresponds to the sheaf of functions on $\mathfrak{X}$ whose support is contained in the t-locus of $\mathfrak{X}$. To simplify the notation, we write $\mathrm{fCAlg}_{k^\circ}(\mathcal{X})$ rather than $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{ad}}(k^\circ)}(\mathcal{X})$.

(iii) Let X be a k-analytic space and denote Xét the associated small étale site on X. Let $\mathcal{X} := \mathrm{Shv}(X\text{ét}, \tau_{\mathrm{\acute{e}t}})^\wedge$ denote the hypercompletion of the ∞-topos of étale sheaves on X. We can attach to X a $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$ as follows: given $U \in \mathcal{T}_{\mathrm{an}}(k)$, we define the sheaf $\mathcal{O}(U) \in \mathcal{X}$ by

$$X\text{ét} \ni V \mapsto \mathrm{Hom}_{\mathrm{An}_k}(V, U) \in \mathcal{S}.$$

As in the previous case, we can canonically identify $\mathcal{O}(\mathbb{A}^1_k)$ with the usual sheaf of analytic functions on X. We write $\mathrm{AnRing}_k(\mathcal{X})$ rather than $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$.

Construction 4.2.0.5. Let $\mathcal{X}$ be an ∞-topos. We can relate the ∞-categories $\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k^\circ)}(\mathcal{X})$, $\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X})$, $\mathrm{Str}_{\mathcal{T}_{\mathrm{ad}}(k^\circ)}(\mathcal{X})$ and $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ as follows. Consider the following functors

(i) the functor

$$- \otimes_{k^\circ} k\colon \mathcal{T}_{\mathrm{disc}}(k^\circ) \longrightarrow \mathcal{T}_{\mathrm{disc}}(k).$$

induced by base change along the map $k^\circ \to k$.

(ii) The functor

$$(-)^\wedge_t\colon \mathcal{T}_{\mathrm{disc}}(k^\circ) \longrightarrow \mathcal{T}_{\mathrm{ad}}(k^\circ).$$

induced by the (t)-completion.

(iii) The functor

$$(-)\mathrm{an}\colon \mathcal{T}_{\mathrm{disc}}(k) \longrightarrow \mathcal{T}_{\mathrm{an}}(k),$$

induced by the analytification.

(iv) The functor

$$(-)^{\mathrm{rig}}\colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \longrightarrow \mathcal{T}_{\mathrm{an}}(k)$$

induced by Raynaud's generic fiber construction (cf. [Bos14, Theorem 8.4.3]).

These functors respect the classes of admissible morphisms and are continuous morphisms of sites. It follows that precomposition with them induce well defined functors

$$\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k^\circ)}(\mathcal{X}) \quad , \quad (-)\mathrm{alg}\colon \mathrm{Str}_{\mathcal{T}_{\mathrm{ad}}(k^\circ)}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k^\circ)}(\mathcal{X})$$
$$(-)^+\colon \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{ad}}(k^\circ)}(\mathcal{X}) \quad , \quad (-)\mathrm{alg}\colon \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}).$$

The first functor simply forgets the k-algebra structure to a k°-algebra one via the natural map $k^\circ \to k$. We refer to the second and fourth functors as the *underlying algebra functors*. The third functor is an analogue of taking the subring of power-bounded elements in rigid geometry.

Using the underlying algebra functors introduced in the above construction, we can at last introduce the definitions of derived formal scheme and derived k-analytic space. They are analogous to each other:

Definition 4.2.0.6. A $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured ∞-topos $\mathfrak{X} := (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ is said to be a *derived formal Deligne-Mumford stackk°-stack* if there exists a collection of objects $\{U_i\}_{i \in I}$ in $\mathcal{X}$ such that $\coprod_{i \in I} U_i \to \mathbf{1}_{\mathcal{X}}$ is an effective epimorphism and the following conditions are met:

(i) for every $i \in I$, the $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured ∞-topos $(\mathcal{X}_{/U_i}, \pi_0(\mathcal{O}_{\mathfrak{X}}|_{U_i}))$ is equivalent to the $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured ∞-topos arising from an affine formal k°-scheme via the construction given in Theorem 4.2.0.4.

(ii) For each $i \in I$ and each integer $n \geq 0$, the sheaf $\pi_n(\mathcal{O}_{\mathfrak{X}}\mathrm{alg}|_{U_i})$ is a quasi-coherent sheaf over $(\mathcal{X}_{/U_i}, \pi_0(\mathcal{O}_{\mathfrak{X}}|_{U_i}))$.

We say that $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ is a *formal derived k°-scheme* if it is a derived formal Deligne Mumford stack and furthermore its truncation $\mathrm{t}_0(\mathfrak{X}) := (\mathcal{X}, \pi_0(\mathcal{O}_{\mathfrak{X}}))$ is equivalent to the $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structured ∞-topos associated to a formal scheme via Theorem 4.2.0.4.

Definition 4.2.0.7. A $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos $X := (\mathcal{X}, \mathcal{O}_X)$ is said to be a *derived k-analytic space* if $\mathcal{X}$ is hypercomplete and there exists a collection of objects $\{U_i\}_{i \in I}$ in $\mathcal{X}$ such that $\coprod_{i \in I} U_i \to \mathbf{1}_{\mathcal{X}}$ is an effective epimorphism and the following conditions are met:

(i) for each $i \in I$, the $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos $(\mathcal{X}_{/U_i}, \pi_0(\mathcal{O}_X|_{U_i}))$ is equivalent to the $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos arising from an ordinary k-analytic space via the construction given in Theorem 4.2.0.4.

(ii) For each $i \in I$ and each integer $n \geq 0$, the sheaf $\pi_n(\mathcal{O}_X\mathrm{alg}|_{U_i})$ is a coherent sheaf on $(\mathcal{X}_{/U_i}, \mathcal{O}_X|_{U_i})$.

Theorem 4.2.0.8 (cf. [Ant18a,Lur11b,PY16b])**.** *Derived formal Deligne-Mumford stackk°-stacks and derived k-analytic spaces assemble into ∞-categories, denoted respectively* dfDM_{k° *and* dAn_k*, which enjoy the following properties:*

(i) fiber products exist in both dfDM_{k° *and* dAn_k*;*

(ii) The constructions given in Theorem 4.2.0.4 induce full faithful embeddings from the categories of ordinary formal Deligne-Mumford stackk°-stacks fDM_{k° *and of ordinary k-analytic spaces* An_k *in* dfDM_{k° *and* dAn_k*, respectively.*

Following [**?**, §8.1], we let $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}$ denote the ∞-category of simplicial commutative rings equipped with an adic topology on their 0-th truncation. Morphisms are morphisms of simplicial commutative rings that are furthermore continuous for the adic topologies on their 0-th truncations. We set

$$\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} := \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ/},$$

where we regard k° equipped with its $\mathfrak{m}$-adic topology. Thanks to [Ant18a, Remark 3.1.4], the underlying algebra functor $(-)\mathrm{alg} \colon \mathrm{fCAlg}_{k^\circ}(\mathcal{X}) \to \mathcal{C}\mathrm{Alg}_{k^\circ}(\mathcal{X})$ factors through $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$. We denote by $(-)^{\mathrm{ad}}$ the resulting functor:

$$(-)^{\mathrm{ad}} \colon \mathrm{fCAlg}_{k^\circ}(\mathcal{X}) \longrightarrow \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X}).$$

Definition 4.2.0.9. Let $A \in \mathrm{fCAlg}_{k^\circ}(\mathcal{X})$. We say that A is *topologically almost of finite type over k°* if the underlying sheaf of k°-adic algebras A^{ad} is t-complete, $\pi_0(A\mathrm{alg})$ is sheaf of topologically of finite type k°-adic algebras and for each $i > 0$, $\pi_i(A)$ is finitely generated as $\pi_0(A)$-module.

We say that a derived formal Deligne-Mumford stackstack $\mathfrak{X} := (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ if *topologically almost of finite type over k°* if its underlying ∞-topos is coherent (cf. [Lur11f, §3]) and $\mathcal{O}_{\mathfrak{X}} \in \mathrm{fCAlg}_{k^\circ}(\mathcal{X})$ is topologically almost of finite type over k°. We denote by $\mathrm{dfDM}^{\mathrm{taft}}$ (resp. $\mathrm{dfSch}^{\mathrm{taft}}$) the full subcategory of dfDM_{k° spanned by those derived formal Deligne-Mumford stackstacks $\mathfrak{X}$ that are topologically almost of finite type over k° (resp. and whose truncation $\mathrm{t}_0(\mathfrak{X})$ is equivalent to a formal k°-scheme).

The transformation of pregeometries

$$(-)^{\mathrm{rig}} \colon \mathcal{T}_{\mathrm{ad}}(k^\circ) \longrightarrow \mathcal{T}_{\mathrm{an}}(k)$$

induced by Raynaud's generic fiber functor induces ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ))$. [Lur11d, Theorem 2.1.1] provides a right adjoint to this last functor, which we still denote

$$(-)^{\mathrm{rig}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \longrightarrow {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)).$$

We refer to this functor as the *derived generic fiber functor* or as the *derived rigidification functor*.

Theorem 4.2.0.10 ([Ant18a, Corollary 4.1.4, Proposition 4.1.6]). *The functor* $(-)^{\mathrm{rig}}\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{ad}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ *enjoys the following properties:*

(i) it restricts to a functor

$$(-)^{\mathrm{rig}}\colon \mathrm{dfDM}^{\mathrm{taft}} \longrightarrow \mathrm{dAn}_k.$$

(ii) The restriction of $(-)^{\mathrm{rig}}\colon \mathrm{dfDM}^{\mathrm{taft}} \to \mathrm{dAn}_k$ *to the full subcategory* $\mathrm{fSch}^{\mathrm{taft}}_{k^\circ}$ *is canonically equivalent to Raynaud's generic fiber functor.*

(iii) Every derived analytic space $X \in \mathrm{dAn}_k$ *whose truncation is an ordinary* k*-analytic space*[2] *lies in the essential image of the functor* $(-)^{\mathrm{rig}}$.

Fix a derived formal Deligne-Mumford stackstack $\mathfrak{X} := (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ and a derived k-analytic space $Y := (\mathcal{Y}, \mathcal{O}_Y)$. We set

$$\mathcal{O}_{\mathfrak{X}}\text{-Mod} := \mathcal{O}_{\mathfrak{X}}\mathrm{alg}\text{-Mod} \quad , \quad \mathcal{O}_Y\text{-Mod} := \mathcal{O}_Y\mathrm{alg}\text{-Mod}.$$

We refer to $\mathcal{O}_{\mathfrak{X}}$-Mod as the *stable* ∞*-category of* $\mathcal{O}_{\mathfrak{X}}$*-modules*. Similarly, we refer to $\mathcal{O}_Y$-Mod as the *stable* ∞*-category of* $\mathcal{O}_Y$*-modules*. The derived generic fiber functor induces a functor

$$(-)^{\mathrm{rig}}\colon \mathcal{O}_{\mathfrak{X}}\text{-Mod} \longrightarrow \mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}}\text{-Mod}.$$

Definition 4.2.0.11. Let $\mathfrak{X} \in \mathrm{dfDM}_{k^\circ}$ be a derived k°-adic Deligne-Mumford stackstack and let $X \in \mathrm{dAn}_k$ be a derived k-analytic space. The ∞-category $\mathrm{Coh}^+(\mathfrak{X})$ (resp. $\mathrm{Coh}^+(X)$) of almost perfect complexes on $\mathfrak{X}$ (resp. on X) is the full subcategory of $\mathcal{O}_{\mathfrak{X}}$-Mod (resp. of $\mathcal{O}_X$-Mod) spanned by those $\mathcal{O}_{\mathfrak{X}}$-modules (resp. $\mathcal{O}_X$-modules) $\mathcal{F}$ such that $\pi_i(\mathcal{F})$ is a coherent sheaf on $\mathrm{t}_0(\mathfrak{X})$ (resp. on $\mathrm{t}_0(X)$) for every $i \in \mathbb{Z}$ and $\pi_i(\mathcal{F}) \simeq 0$ for $i \ll 0$.

For later use, let us record the following result:

Proposition 4.2.0.12 ([?] & [PY18b, Theorem 3.4]). *Let* $\mathfrak{X}$ *be a derived affine* k°*-adic scheme. Let* $A := \Gamma(\mathfrak{X}; \mathcal{O}_{\mathfrak{X}}\mathrm{alg})$*. Then the functor* $\Gamma(\mathfrak{X}; -)$ *restricts to*

$$\mathrm{Coh}^+(\mathfrak{X}) \longrightarrow \mathrm{Coh}^+(A)$$

and furthermore this is an equivalence. Similarly, if X *is a derived* k*-affinoid space,*[3] *and* $B := \Gamma(X; \mathcal{O}_X\mathrm{alg})$*, then* $\Gamma(X; -)$ *restricts to*

$$\mathrm{Coh}^+(X) \longrightarrow \mathrm{Coh}^+(B),$$

and furthermore this is an equivalence.

To complete this short review, we briefly discuss the notion of the k°-adic and k-analytic cotangent complexes. The two theories are parallel, and for sake of brevity we limit ourselves to the first one. We refer to the introduction of [PY17b] for a more thorough review of the k-analytic theory.

In [Ant18a, §3.4] it was constructed a functor

$$\Omega^\infty_{\mathrm{ad}}\colon \mathcal{O}_{\mathfrak{X}}\text{-Mod} \longrightarrow \mathrm{fCAlg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}_{\mathfrak{X}}},$$

which we refer to as the k°*-adic split square-zero extension functor*. Given $\mathcal{F} \in \mathcal{O}_{\mathfrak{X}}$-Mod, we often write $\mathcal{O}_{\mathfrak{X}} \oplus \mathcal{F}$ instead of $\Omega^\infty_{\mathrm{ad}}(\mathcal{F})$.

Remark 4.2.0.13. Although the ∞-category $\mathcal{O}_{\mathfrak{X}}$-Mod is *not* sensitive to the $\mathcal{T}_{\mathrm{ad}}(k^\circ)$-structure on $\mathcal{O}_{\mathfrak{X}}$, the functor $\Omega^\infty_{\mathrm{ad}}$ depends on it in an essential way.

Definition 4.2.0.14. The functor of k°*-adic derivations* is the functor

$$\mathrm{Der}^{\mathrm{ad}}_{k^\circ}(\mathfrak{X}; -)\colon \mathcal{O}_{\mathfrak{X}}\text{-Mod} \longrightarrow \mathcal{S}$$

defined by

$$\mathrm{Der}^{\mathrm{ad}}_{k^\circ}(\mathfrak{X}; \mathcal{F}) := \mathrm{Map}_{\mathrm{fCAlg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}_{\mathfrak{X}}}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}} \oplus \mathcal{F}).$$

[2]The ∞-category dAn_k also contains k-analytic Deligne-Mumford stackstacks.

[3]By definition, X is a derived k-affinoid space if $\mathrm{t}_0(X)$ is a k-affinoid space.

For formal reasons, the functor $\mathrm{Der}^{\mathrm{ad}}_{k^\circ}(\mathfrak{X}; -)$ is corepresentable by an object $\mathbb{L}^{\mathrm{ad}}_{\mathfrak{X}} \in \mathcal{O}_{\mathfrak{X}}\text{-}\mathrm{Mod}$. We refer to it as the *k°-adic cotangent complex of* $\mathfrak{X}$. The following theorem summarizes its main properties:

Theorem 4.2.0.15 ([Ant18a, Proposition 3.4.4, Corollary 4.3.5, Proposition 3.5.8]). *Let $\mathfrak{X} \coloneqq (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ be a derived k°-adic Deligne-Mumford stackstack. Let $\mathrm{t}_{\leq n}\mathfrak{X} \coloneqq (\mathcal{X}, \tau_{\leq n}\mathcal{O}_{\mathfrak{X}})$ be the n-th truncation of $\mathfrak{X}$. Then:*

(i) the k°-adic cotangent complex $\mathbb{L}^{\mathrm{ad}}_{\mathfrak{X}}$ belongs to $\mathrm{Coh}^+(\mathfrak{X})$;

(ii) in $\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$ there is a canonical equivalence

$$(\mathbb{L}^{\mathrm{ad}}_{\mathfrak{X}})^{\mathrm{rig}} \simeq \mathbb{L}\mathrm{an}_{\mathfrak{X}^{\mathrm{rig}}},$$

where $\mathbb{L}\mathrm{an}_{\mathfrak{X}^{\mathrm{rig}}}$ denotes the analytic cotangent complex of the derived k-analytic space $\mathfrak{X}^{\mathrm{rig}}$;

(iii) the algebraic derivation classifying canonical map $(\mathfrak{X}, \tau_{\leq n+1}\mathcal{O}_{\mathfrak{X}}) \to (\mathfrak{X}, \tau_{\leq n}\mathcal{O}_{\mathfrak{X}})$ can be canonically lifted to a k°-adic derivation

$$\mathbb{L}^{\mathrm{ad}}_{\mathrm{t}_{\leq n}\mathfrak{X}} \longrightarrow \pi_{n+1}(\mathcal{O}_{\mathfrak{X}})[n+2].$$

4.3 Formal models for almost perfect complexes

4.3.1 Formal descent statements

We assume that k° admits a finitely generated ideal of definition $\mathfrak{m}$. We also fix a set of generators $t_1, \ldots, t_n \in \mathfrak{m}$. We start by recalling the notion of $\mathfrak{m}$-nilpotent almost perfect complexes.

Definition 4.3.1.1. Let $\mathfrak{X}$ be a derived k°-adic Deligne-Mumford stackstack topologically almost of finite presentation. We let $\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$ denote the fiber of the generic fiber functor (4.1.0.1):

$$\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X}) \coloneqq \mathrm{fib}\left(\mathrm{Coh}^+(\mathfrak{X}) \xrightarrow{(-)^{\mathrm{rig}}} \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})\right).$$

We refer to $\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$ as the full subcategory of $\mathfrak{m}$-nilpotent almost perfect complexes on X.

A morphism $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathrm{dfDM}^{\mathrm{taft}}_{k^\circ}$ induces a commutative diagram

$$\begin{tikzcd} \mathrm{Coh}^+(\mathfrak{Y}) \arrow[r, "\mathfrak{f}^*"] \arrow[d, "(-)^{\mathrm{rig}}"] & \mathrm{Coh}^+(\mathfrak{X}) \arrow[d, "(-)^{\mathrm{rig}}"] \\ \mathrm{Coh}^+(\mathfrak{Y}^{\mathrm{rig}}) \arrow[r, "(\mathfrak{f}^{\mathrm{rig}})^*"] & \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}). \end{tikzcd} \tag{4.3.1.1}$$

In particular, we see that $\mathfrak{f}^*$ preserves the subcategory of $\mathfrak{m}$-nilpotent almost perfect complexes on X. Moreover, as both $\mathrm{Coh}^+(\mathfrak{X})$ and $\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$ satisfy étale descent, we conclude that $\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$ satisfies étale descent as well.

Lemma 4.3.1.2. *Let $\mathfrak{X}$ be a derived k°-adic Deligne-Mumford stackstack. Then an almost perfect sheaf $\mathcal{F} \in \mathrm{Coh}^+(X)$ is $\mathfrak{m}$-nilpotent if and only if for every $i \in \mathbb{Z}$ the coherent sheaf $\pi_i(\mathcal{F})$ is annihilated by some power of the ideal $\mathfrak{m}$.*

Proof. The question is étale local on $\mathfrak{X}$. In particular, we can assume that $\mathfrak{X}$ is a derived formal affine scheme topologically of finite presentation. Write

$$A \coloneqq \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}\mathrm{alg}).$$

Let $X \coloneqq \mathfrak{X}^{\mathrm{rig}}$. Then [Ant18a, Corollary 4.1.3] shows that

$$\mathrm{t}_0(\mathfrak{X}^{\mathrm{rig}}) \simeq (\mathrm{t}_0(\mathfrak{X}))^{\mathrm{rig}}.$$

In particular, we deduce that X is a derived k-affinoid space. Write

$$B \coloneqq \Gamma(X, \mathcal{O}_X\mathrm{alg}).$$

We can therefore use Theorem 4.2.0.12 to obtain canonical equivalences

$$\mathrm{Coh}^+(\mathfrak{X}) \simeq \mathrm{Coh}^+(A\mathrm{alg}) \quad , \quad \mathrm{Coh}^+(X) \simeq \mathrm{Coh}^+(B).$$

Under these identifications, the functor $(-)^{\mathrm{rig}}$ becomes equivalent to the base change functor

$$- \otimes_A B \colon \mathrm{Coh}^+(A) \longrightarrow \mathrm{Coh}^+(B).$$

Moreover, it follows from [Ant18a, Proposition A.1.4] that there is a canonical identification

$$B \simeq A \otimes_{k^\circ} k.$$

In particular, $(-)^{\mathrm{rig}} \colon \mathrm{Coh}^+(\mathfrak{X}) \to \mathrm{Coh}^+(X)$ is t-exact. The conclusion is now straightforward. □

Definition 4.3.1.3. Let $\mathfrak{X}$ be a derived k°-adic Deligne-Mumford stackstack. Let $\mathcal{F} \in \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$. We say that $\mathfrak{F} \in \mathrm{Coh}^+(\mathfrak{X})$ is a formal model for $\mathcal{F}$ if there exists an equivalence $\mathfrak{F}^{\mathrm{rig}} \simeq \mathcal{F}$ in $\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$. We let $\mathrm{FM}(\mathcal{F})$ denote the full subcategory of

$$\mathrm{Coh}^+(\mathfrak{X})_{/\mathcal{F}} \coloneqq \mathrm{Coh}^+(\mathfrak{X}) \times_{\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})} \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})_{/\mathcal{F}}$$

spanned by formal models of $\mathcal{F}$.

Our goal in this section is to study the structure of $\mathrm{FM}(\mathcal{F})$, and in particular to establish that it is non-empty and filtered when $\mathfrak{X}$ is a quasi-compact and quasi-separated derived k°-adic scheme. Notice that saying that $\mathrm{FM}(\mathcal{F})$ is non-empty for every choice of $\mathcal{F} \in \mathrm{Coh}^+(X)$ is equivalent to asserting that the functor (4.1.0.1)

$$(-)^{\mathrm{rig}} \colon \mathrm{Coh}^+(\mathfrak{X}) \longrightarrow \mathrm{Coh}^+(X)$$

is essentially surjective.

Lemma 4.3.1.4. *If $\mathfrak{X}$ is a derived k°-affine scheme topologically almost of finite presentation, then the functor* (4.1.0.1) *is essentially surjective.*

Proof. We let

$$A \coloneqq \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}\mathrm{alg}) \quad , \quad B \coloneqq \Gamma(\mathfrak{X}^{\mathrm{rig}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}}).$$

Then as in the proof of Theorem 4.3.1.2, we have identifications $\mathrm{Coh}^+(\mathfrak{X}) \simeq \mathrm{Coh}^+(A)$ and $\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}) \simeq \mathrm{Coh}^+(B)$, and under these identifications the functor $(-)^{\mathrm{rig}}$ becomes equivalent to

$$- \otimes_A B \colon \mathrm{Coh}^+(A) \longrightarrow \mathrm{Coh}^+(B).$$

As $B \simeq A \otimes_{k^\circ} k$, we see that $A \to B$ is a Zariski open immersion. The conclusion now follows from [HPV16b, Theorem 2.12]. □

To complete the proof of the non-emptiness of $\mathrm{FM}(\mathcal{F})$, it would be enough to know that the essential image of the functor $\mathrm{Coh}^+(\mathfrak{X}) \to \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$ satisfies descent. This is analogous to [HPV16b, Theorem 7.3].

Definition 4.3.1.5. Let $\mathfrak{X}$ be a derived k°-adic Deligne-Mumford stackstack locally topologically almost of finite presentation. We define the stable ∞-category $\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})$ of $\mathfrak{m}$-*local almost perfect complexes* as the cofiber

$$\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X}) \coloneqq \mathrm{cofib}\left(\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X}) \hookrightarrow \mathrm{Coh}^+(\mathfrak{X})\right).$$

We denote by $\mathrm{L} \colon \mathrm{Coh}^+(\mathfrak{X}) \to \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})$ the canonical functor. We refer to L as the *localization functor*.

We summarize below the formal properties of $\mathfrak{m}$-local almost perfect complexes:

Proposition 4.3.1.6. *Let $\mathfrak{X}$ be a derived k°-adic Deligne-Mumford stack-stack locally topologically almost of finite presentation. Then:*

(i) there exists a unique t-structure on the stable ∞-category $\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})$ having the property of making the localization functor

$$\mathrm{L} \colon \mathrm{Coh}^+(\mathfrak{X}) \longrightarrow \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})$$

t-exact.

(ii) The functor $(-)^{\mathrm{rig}} \colon \mathrm{Coh}^+(\mathfrak{X}) \to \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$ *factors as*

$$\Lambda \colon \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X}) \longrightarrow \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}).$$

Moreover, the essential images of $(-)^{\mathrm{rig}}$ *and* Λ *coincide.*

(iii) If $\mathfrak{X}$ *is affine, then the functor* Λ *is an equivalence.*

Proof. We start by proving (1). Using [HPV16b, Corollary 2.9] we have to check that the t-structure on $\mathrm{Coh}^+(\mathfrak{X})$ restricts to a t-structure on $\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$ and that the inclusion

$$i \colon \mathrm{Coh}^{\heartsuit}_{\mathrm{nil}}(\mathfrak{X}) \hookrightarrow \mathrm{Coh}^{\heartsuit}(\mathfrak{X})$$

admits a right adjoint R whose counit $i(R(X)) \to X$ is a monomorphism for every $X \in \mathrm{Coh}^{\heartsuit}(\mathfrak{X})$. For the first statement, we remark that it is enough to check that the functor $(-)^{\mathrm{rig}} \colon \mathrm{Coh}^+(\mathfrak{X}) \to \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$ is t-exact. As both $\mathrm{Coh}^+(\mathfrak{X})$ and $\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$ satisfy étale descent in $\mathfrak{X}$, we can test this locally on $\mathfrak{X}$. When $\mathfrak{X}$ is affine, the assertion follows directly from Theorem 4.2.0.12. As for the second statement, we first observe that

$$\mathrm{Coh}^{\heartsuit}(\mathfrak{X}) \simeq \mathrm{Coh}^{\heartsuit}(\mathrm{t}_0(\mathfrak{X})).$$

We can therefore assume that $\mathfrak{X}$ is underived. At this point, the functor R can be explicitly described as the functor sending $\mathfrak{F} \in \mathrm{Coh}^{\heartsuit}(\mathfrak{X})$ to the subsheaf of $\mathfrak{F}$ spanned by $\mathfrak{m}$-nilpotent sections. The proof of (1) is thus complete.

We now turn to the proof of (2). The existence of Λ and the factorization $(-)^{\mathrm{rig}} \simeq \Lambda \circ \mathrm{L}$ follow from the definitions. Moreover, $\mathrm{L} \colon \mathrm{Coh}^+(\mathfrak{X}) \to \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})$ is essentially surjective (cf. [HPV16b, Lemma 2.3]). It follows that the essential images of $(-)^{\mathrm{rig}}$ and of Λ coincide.

Finally, (3) follows directly from Theorem 4.2.0.12 and [HPV16b, Theorem 2.12]. □

The commutativity of (4.3.1.1) implies that a morphism $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathrm{dfDM}^{\mathrm{taft}}_{k^\circ}$ induces a well defined functor

$$\mathfrak{f}^{\circ *} \colon \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{Y}) \longrightarrow \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X}).$$

It is a simple exercise in ∞-categories to promote this construction to an actual functor

$$\mathrm{Coh}^+_{\mathrm{loc}} \colon \big(\mathrm{dfDM}^{\mathrm{taft}}_{k^\circ}\big)\mathrm{op} \longrightarrow \mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty.$$

Having Theorem 4.3.1.4 and Theorem 4.3.1.6 at our disposal, the question of the non-emptiness of $\mathrm{FM}(\mathcal{F})$ is essentially reduced to the the following:

Theorem 4.3.1.7. *Let* $\mathrm{dfSch}^{\mathrm{taft,qcqs}}_{k^\circ}$ *denote the* ∞*-category of derived* k°*-adic schemes which are quasi-compact, quasi separated and topologically almost of finite presentation. Then the functor*

$$\mathrm{Coh}^+_{\mathrm{loc}} \colon \big(\mathrm{dfSch}^{\mathrm{taft,qcqs}}_{k^\circ}\big)\mathrm{op} \longrightarrow \mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty$$

is a hypercomplete sheaf for the formal Zariski topology.

Proof. A standard descent argument reduces us to prove the following statement: let $\mathfrak{f}_\bullet \colon \mathfrak{U}_\bullet \to \mathfrak{X}$ be a derived affine k°-adic Zariski hypercovering. Then the canonical map

$$\mathfrak{f}^{\circ *}_\bullet \colon \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X}) \longrightarrow \lim_{[n] \in \mathbf{\Delta}} \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_\bullet) \tag{4.3.1.2}$$

is an equivalence. Using [HPV16b, Lemma 3.20] we can endow the right hand side with a canonical t-structure. It follows from the characterization of the t-structure on $\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})$ given in Theorem 4.3.1.6 that $\mathfrak{f}^{\circ *}_\bullet$ is t-exact.

We will prove in Theorem 4.3.2.4 that $\mathfrak{f}^{\circ *}_\bullet$ is fully faithful. Assuming this fact, we can complete the proof as follows. We only need to check that $\mathfrak{f}^{\circ *}_\bullet$ is essentially surjective. Let $\mathcal{C}$ be the essential image of $\mathfrak{f}^{\circ *}_\bullet$. We now make the following observations:

(i) the heart of $\lim_{\mathbf{\Delta}} \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_\bullet)$ is contained in $\mathcal{C}$. Indeed, Theorem 4.3.1.4 implies that

$$\Lambda_n \colon \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_n) \longrightarrow \mathrm{Coh}^+(\mathfrak{U}_n^{\mathrm{rig}})$$

is an equivalence. These equivalences induce a t-exact equivalence

$$\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}) \simeq \lim_{[n]\in\mathbf{\Delta}} \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_\bullet). \tag{4.3.1.3}$$

Passing to the heart and using the canonical equivalences

$$\mathrm{Coh}^\heartsuit_{\mathrm{loc}}(\mathfrak{X}) \simeq \mathrm{Coh}^\heartsuit_{\mathrm{loc}}(\mathrm{t}_0(\mathfrak{X})) \quad , \quad \mathrm{Coh}^\heartsuit(\mathfrak{X}^{\mathrm{rig}}) \simeq \mathrm{Coh}^\heartsuit(\mathrm{t}_0(\mathfrak{X}^{\mathrm{rig}})),$$

we can invoke the classical Rayanaud's theorem on formal models of coherent sheaves to deduce that the heart of the target of $\mathfrak{f}_\bullet^{\circ *}$ is contained in its essential image.

(ii) The subcategory $\mathcal{C}$ is stable. Indeed, let

$$\mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$$

be a fiber sequence in $\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}) \simeq \lim_{\mathbf{\Delta}} \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_\bullet)$ and suppose that two among $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$ belong to $\mathcal{C}$. Without loss of generality, we can assume that $\mathcal{F}$ and $\mathcal{F}''$ belong to $\mathcal{C}$. Then choose elements $\mathfrak{F}$ and $\mathfrak{F}''$ in $\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})$ representing $\mathcal{F}$ and $\mathcal{F}''$. Since $\mathfrak{f}_\bullet^{\circ *}$ is fully faithful, we can find a morphism $\widetilde{\psi} \colon \mathfrak{F} \to \mathfrak{F}''$ lifting ψ. Set

$$\mathfrak{F}' \coloneqq \mathrm{fib}(\widetilde{\psi} \colon \mathfrak{F} \to \mathfrak{F}'').$$

Then $\Lambda(\mathfrak{F}') \simeq \mathcal{F}'$, which means that under the equivalence (4.3.1.3) the object $\mathcal{F}'$ belongs to $\mathcal{C}$.

These two points together imply that $\mathfrak{f}_\bullet^{\circ *}$ is essentially surjective on cohomologically bounded elements. As both the t-structures on source and target of $\mathfrak{f}_\bullet^*$ are left t-complete, the conclusion follows. $\square$

Corollary 4.3.1.8. *Let $\mathfrak{X} \in \mathrm{dfSch}^{\mathrm{taft}}_{k^\circ}$ and assume moreover that $\mathfrak{X}$ is quasi-compact and quasi-separated. Then the canonical map*

$$\Lambda \colon \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X}) \longrightarrow \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$$

introduced in Theorem 4.3.1.6 is an equivalence.

Proof. Let $\mathfrak{f}_\bullet \colon \mathfrak{U}_\bullet \to \mathfrak{X}$ be a derived affine k°-adic Zariski hypercover. Consider the induced commutative diagram

$$\begin{tikzcd} \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X}) \arrow[r, "\mathfrak{f}_\bullet^*"] \arrow[d, "\Lambda"] & \lim_{[n]\in\mathbf{\Delta}} \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_n) \arrow[d, "\Lambda_\bullet"] \\ \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}) \arrow[r, "f_\bullet^*"] & \lim_{[n]\in\mathbf{\Delta}} \mathrm{Coh}^+(\mathfrak{U}_n^{\mathrm{rig}}), \end{tikzcd}$$

where we set $f_\bullet \coloneqq (\mathfrak{f}_\bullet)^{\mathrm{rig}}$. The right vertical map is an equivalence thanks to Theorem 4.3.1.6. On the other hand, $\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$ satisfies descent in $\mathfrak{X}$, and therefore the bottom horizontal map is also an equivalence. Finally, Theorem 4.3.1.7 implies that the top horizontal map is an equivalence as well. We thus conclude that $\Lambda \colon \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X}) \to \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$ is an equivalence. $\square$

Corollary 4.3.1.9. *Let $\mathfrak{X} \in \mathrm{dfSch}^{\mathrm{taft}}_{k^\circ}$ and assume moreover that it is quasi-compact and quasi-separated. For any $\mathcal{F} \in \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$, the ∞-category $\mathrm{FM}(\mathcal{F})$ is non-empty.*

Proof. The localization functor $\mathrm{L} \colon \mathrm{Coh}^+(\mathfrak{X}) \to \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})$ is essentially surjective by construction. Since $\mathfrak{X}$ is a quasi-compact and quasi-separted derived k°-adic scheme topologically of finite presentation, Theorem 4.3.1.8 implies that $\Lambda \colon \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X}) \to \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$ is an equivalence. The conclusion follows. $\square$

4.3.2 Proof of Theorem 4.3.1.7: fully faithfulness

The only missing step in the proof of Theorem 4.3.1.7 is the full faithfulness of the functor (4.3.1.2). We will address this question by passing to the ∞-categories of ind-objects. Let $\mathfrak{X}$ be a quasi-compact and quasi-separated derived k°-adic scheme locally topologically almost of finite presentation.

$$\mathfrak{f} \colon \mathfrak{U} \longrightarrow \mathfrak{X}$$

be a formally étale morphism. Then $\mathfrak{f}$ induces a commutative diagram

$$\begin{tikzcd} \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X})) \arrow[r, "\mathrm{L}_{\mathfrak{X}}"] \arrow[d, "\mathfrak{f}^*"] & \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})) \arrow[d, "\mathfrak{f}^{\circ *}"] \\ \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{U})) \arrow[r, "\mathrm{L}_{\mathfrak{U}}"] & \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U})). \end{tikzcd}$$

The functors $\mathfrak{f}^*$ and $\mathfrak{f}^{\circ *}$ commute with colimits, and therefore they admit right adjoints $\mathfrak{f}_*$ and $\mathfrak{f}^\circ_*$. In particular, we obtain a Beck-Chevalley transformation

$$\theta \colon \mathrm{L}_{\mathfrak{X}} \circ \mathfrak{f}_* \longrightarrow \mathfrak{f}^\circ_* \circ \mathrm{L}_{\mathfrak{U}}. \tag{4.3.2.1}$$

A key step in the proof of the full faithfulness of the functor (4.3.1.2) is to verify that θ is an equivalence when evaluated on objects in $\mathrm{Coh}^\heartsuit(\mathfrak{U})$. Let us start with the following variation on [HPV16b, Lemma 7.14]:

Lemma 4.3.2.1. *Let*

$$\begin{tikzcd} \mathcal{K}_{\mathcal{C}} \arrow[r, hook, "i_{\mathcal{C}}"] \arrow[d, "F_{\mathcal{K}}"] & \mathcal{C} \arrow[r, "L_{\mathcal{C}}"] \arrow[d, "F"] & \mathcal{Q}_{\mathcal{C}} \arrow[d, "F_{\mathcal{Q}}"] \\ \mathcal{K}_{\mathcal{D}} \arrow[r, hook, "i_{\mathcal{D}}"] & \mathcal{D} \arrow[r, "L_{\mathcal{D}}"] & \mathcal{Q}_{\mathcal{D}} \end{tikzcd} \tag{4.3.2.2}$$

be a diagram of stable ∞-categories and exact functors between them. Assume that:

(i) the functors $i_{\mathcal{C}}$ and $i_{\mathcal{D}}$ are fully faithful and admit right adjoints $R_{\mathcal{C}}$ and $R_{\mathcal{D}}$, respectively;

(ii) the functors $L_{\mathcal{C}}$ and $L_{\mathcal{D}}$ admit fully faithful right adjoints $j_{\mathcal{C}}$ and $j_{\mathcal{D}}$, respectively;

(iii) the rows are fiber and cofiber sequences in $\mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty$;

(iv) the functors F, $F_{\mathcal{K}}$ and $F_{\mathcal{Q}}$ admit right adjoints G, $G_{\mathcal{K}}$ and $G_{\mathcal{Q}}$, respectively.

Let $X \in \mathcal{D}$ be an object. Then the following statements are equivalent:

(i) the Beck-Chevalley transformation

$$q_X \colon L_{\mathcal{C}}(G(X)) \longrightarrow G_{\mathcal{Q}}(L_{\mathcal{D}}(X))$$

is an equivalence;

(ii) the Beck-Chevalley transformation

$$\kappa_{R_{\mathcal{D}}(X)} \colon i_{\mathcal{C}}(G_{\mathcal{K}}(R_{\mathcal{D}}(X))) \longrightarrow G(i_{\mathcal{D}}(R_{\mathcal{D}}(X)))$$

is an equivalence.

Proof. Since $j_{\mathcal{C}}$ and $i_{\mathcal{C}}$ are fully faithful, it is equivalent to check that

$$j_{\mathcal{C}}(L_{\mathcal{C}}(G(X))) \longrightarrow j_{\mathcal{C}}(G_{\mathcal{Q}}(L_{\mathcal{D}}(X)))$$

is an equivalence if and only if $\kappa_{R_{\mathcal{D}}(X)}$ is an equivalence. Using the natural equivalences

$$j_{\mathcal{C}} \circ G \simeq G j_{\mathcal{D}} \quad , \quad G_{\mathcal{K}} \circ R_{\mathcal{D}} \simeq R_{\mathcal{C}} \circ G$$

we obtain the following commutative diagram

$$\begin{array}{ccccc} i_{\mathcal{C}}(R_{\mathcal{C}}(G(X))) & \longrightarrow & G(X) & \longrightarrow & j_{\mathcal{C}}(L_{\mathcal{C}}(G(X))) \\ \downarrow & & \| & & \downarrow \\ G(i_{\mathcal{D}}(R_{\mathcal{D}}(X))) & \longrightarrow & G(X) & \longrightarrow & G(j_{\mathcal{D}}(L_{\mathcal{D}}(X))). \end{array}$$

Moreover, since the rows of the diagram (4.3.2.2) are Verdier quotients, we conclude that the rows in the above diagram are fiber sequences. Therefore, the leftmost vertical arrow is an equivalence if and only if the rightmost one is. ☐

Lemma 4.3.2.2. *The Beck-Chevalley transformation* (4.3.2.1) *is an equivalence whenever evaluated on objects in* $\mathrm{Coh}^{\heartsuit}(\mathfrak{U})$.

Proof. Using Theorem 4.3.2.1, we see that it is enough to prove that the Beck-Chevalley transformation associated to the square

$$\begin{array}{ccc} \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})) & \longrightarrow & \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X})) \\ \downarrow{\scriptstyle \mathfrak{f}^*} & & \downarrow{\scriptstyle \mathfrak{f}^*} \\ \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{U})) & \longrightarrow & \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{U})) \end{array}$$

is an equivalence when evaluated on objects of $\mathrm{Coh}^{\heartsuit}_{\mathrm{nil}}(\mathfrak{U})$. As the horizontal functors are fully faithful, it is enough to check that the functor

$$\mathfrak{f}_*\colon \mathrm{Ind}(\mathrm{Coh}^+(U)) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))$$

takes $\mathrm{Coh}^{\heartsuit}_{\mathrm{nil}}(\mathfrak{U})$ to $\mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X}))$. Let $\mathfrak{F} \in \mathrm{Coh}^{\heartsuit}_{\mathrm{nil}}(\mathfrak{U})$. We have to verify that $(\mathfrak{f}_*(\mathfrak{F}))^{\mathrm{rig}} \simeq 0$. Since $\mathfrak{F}$ is coherent and in the heart and since $\mathfrak{U}$ is quasi-compact we see that there exists an element $a \in \mathfrak{m}$ such that the map $\mu_a\colon \mathfrak{F} \to \mathfrak{F}$ given by multiplication by a is zero. Therefore $\mathfrak{f}_*(\mu_a)\colon \mathfrak{f}_*(\mathfrak{F}) \to \mathfrak{f}_*(\mathfrak{F})$ is homotopic to zero. Since $\mathfrak{f}_*(\mu_a)$ is equivalent to the endomorphism $\mathfrak{f}_*(\mathfrak{F})$ given by multiplication by a, we conclude that $(\mathfrak{f}_*(\mathfrak{F}))^{\mathrm{rig}} \simeq 0$. The conclusion follows. ☐

Having these adjointability statements at our disposal, we turn to the actual study of the full faithfulness of the functor (4.3.1.2). Let

$$\mathfrak{U}_\bullet\colon \mathbf{\Delta}\mathrm{op} \longrightarrow \mathrm{dfSch}^{\mathrm{taft}}_{k^\circ}$$

be an affine k°-adic Zariski hypercovering of $\mathfrak{X}$ and let $\mathfrak{f}_\bullet\colon \mathfrak{U}_\bullet \to \mathfrak{X}$ be the augmentation morphism. The morphism $\mathfrak{f}_\bullet$ induces functors

$$\mathfrak{f}^*_\bullet\colon \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X})) \longrightarrow \lim_{[n]\in\mathbf{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{U}_n))$$

and

$$\mathfrak{f}^{\circ*}_\bullet\colon \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})) \longrightarrow \lim_{[n]\in\mathbf{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_n)).$$

These functors commute by construction with filtered colimits, and therefore they admit right adjoints, that we denote respectively as

$$\mathfrak{f}_{\bullet*}\colon \lim_{[n]\in\mathbf{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{U}_n)) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))$$

and

$$\mathfrak{f}^{\circ}_{\bullet*}\colon \lim_{[n]\in\mathbf{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_n)) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})).$$

Moreover, the functors $\mathfrak{f}^*_\bullet$ and $\mathfrak{f}^{\circ*}_\bullet$ fit in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X})) & \xrightarrow{\mathfrak{f}^*_\bullet} & \lim_{[n]\in\mathbf{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{U}_\bullet)) \\ \downarrow{\scriptstyle \mathrm{L}} & & \downarrow{\scriptstyle \mathrm{L}_\bullet} \\ \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})) & \xrightarrow{\mathfrak{f}^{\circ*}_\bullet} & \lim_{[n]\in\mathbf{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_\bullet)). \end{array}$$

In particular, we have an associated Beck-Chevalley transformation

$$\theta\colon \mathrm{L} \circ \mathfrak{f}_{\bullet*} \longrightarrow \mathfrak{f}^{\circ}_{\bullet*} \circ \mathrm{L}_\bullet. \tag{4.3.2.3}$$

Proposition 4.3.2.3. *The Beck-Chevalley transformation* (4.3.2.3) *is an equivalence when restricted to the full subcategory* $\lim_{\boldsymbol{\Delta}} \mathrm{Coh}^{\heartsuit}(\mathfrak{U}_\bullet)$ *of* $\lim_{\boldsymbol{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{U}_\bullet))$.

Proof. The discussion right after [PY16d, Corollary 8.6] allows us to identify the functor

$$\mathfrak{f}_{\bullet *} \colon \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{U}_n)) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))$$

with the functor informally described by sending a descent datum $\mathfrak{F}_\bullet \in \lim_{\boldsymbol{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{U}_\bullet))$ to

$$\lim_{[n]\in\boldsymbol{\Delta}} \mathfrak{f}_{n*}\mathfrak{F}_n \in \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X})).$$

Similarly, the functor $\mathfrak{f}^\circ_{\bullet *}$ sends a descent datum $\mathcal{F}_\bullet \in \lim_{\boldsymbol{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_\bullet)$ to

$$\lim_{[n]\in\boldsymbol{\Delta}} \mathfrak{f}^\circ_{n*}\mathcal{F}_n \in \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})).$$

We therefore have to show that the Beck-Chevalley transformation

$$\theta \colon \mathrm{L}\Big(\lim_{[n]\in\boldsymbol{\Delta}} \mathfrak{f}_{n*}\mathfrak{F}_n\Big) \longrightarrow \lim_{[n]\in\boldsymbol{\Delta}} \mathfrak{f}^\circ_{n*}(\mathrm{L}_n\mathfrak{F}_n)$$

is an equivalence whenever each $\mathfrak{F}_n$ belongs to $\mathrm{Coh}^{\heartsuit}(\mathfrak{U}_n)$. First notice that the functors $\mathfrak{f}_{\bullet *}$ and $\mathfrak{f}^\circ_{\bullet *}$ are left t-exact. In particular, if $\mathfrak{F}_\bullet \in \lim_{\boldsymbol{\Delta}} \mathrm{Ind}(\mathrm{Coh}^{\heartsuit}(\mathfrak{U}_\bullet))$ then both $\mathrm{L}\mathfrak{f}_{\bullet *}(\mathfrak{F}_\bullet)$ and $\mathfrak{f}^\circ_{\bullet *}(\mathfrak{F}_\bullet)$ are coconnective. As the t-structures on $\lim_{\boldsymbol{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{U}_\bullet))$ and on $\lim_{\boldsymbol{\Delta}} \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_\bullet))$ are right t-complete, we conclude that it is enough to prove that $\pi_i(\theta)$ is an isomorphism for every $i \in \mathbb{Z}$. We now observe that for $m \geq i+2$ we have

$$\pi_i\Big(\lim_{[n]\in\boldsymbol{\Delta}} \mathfrak{f}^\circ_{n*}(\mathrm{L}_n\mathfrak{F}_n)\Big) \simeq \pi_i\Big(\lim_{[n]\in\boldsymbol{\Delta}_{\leq m}} \mathfrak{f}^\circ_{n*}(\mathrm{L}_n\mathfrak{F}_n)\Big),$$

and similarly

$$\pi_i\Big(\mathrm{L}\Big(\lim_{[n]\in\boldsymbol{\Delta}} \mathfrak{f}_{n*}\mathfrak{F}_n\Big)\Big) \simeq \mathrm{L}\Big(\pi_i\Big(\lim_{[n]\in\boldsymbol{\Delta}} \mathfrak{f}_{n*}\mathfrak{F}_n\Big)\Big) \simeq \mathrm{L}\Big(\pi_i\Big(\lim_{[n]\in\boldsymbol{\Delta}_{\leq m}} \mathfrak{f}_{n*}\mathfrak{F}_{n*}\Big)\Big).$$

It is therefore enough to prove that for every $m \geq 0$ the canonical map

$$\mathrm{L}\Big(\lim_{[n]\in\boldsymbol{\Delta}_{\leq m}} \mathfrak{f}_{n*}\mathfrak{F}_n\Big) \longrightarrow \lim_{[n]\in\boldsymbol{\Delta}_{\leq m}} \mathfrak{f}^\circ_{n*}(\mathrm{L}_n\mathfrak{F}_n)$$

is an equivalence. As L commutes with finite limits, we are reduced to show that the canonical map

$$\mathrm{L}(\mathfrak{f}_{n*}\mathfrak{F}_n) \longrightarrow \mathfrak{f}^\circ_{n*}(\mathrm{L}_n\mathfrak{F}_n)$$

is an equivalence whenever $\mathfrak{F}_n \in \mathrm{Coh}^{\heartsuit}(\mathfrak{U}_n)$, which follows from Theorem 4.3.2.2. □

Corollary 4.3.2.4. *Let* $\mathfrak{X}$ *and* $\mathfrak{f}_\bullet \colon \mathfrak{U}_\bullet \to \mathfrak{X}$ *be as in the above discussion. Then the functor*

$$\mathfrak{f}^{\circ *}_\bullet \colon \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X}) \longrightarrow \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_n)$$

is fully faithful.

Proof. As the functor $\mathfrak{f}^{\circ *}_\bullet$ is t-exact and the t-structure on both categories is left complete, we see that it is enough to reduce ourselves to prove that $\mathfrak{f}^*_\bullet$ is fully faithful when restricted to $\mathrm{Coh}^{\mathrm{b}}_{\mathrm{loc}}(\mathfrak{X})$. Consider the following commutative cube:

$$\begin{tikzcd}
\mathrm{Coh}^+(\mathfrak{X}) \arrow[rr, "\mathfrak{f}^*_\bullet"] \arrow[dr] \arrow[dd] & & \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Coh}^+(\mathfrak{U}_n) \arrow[dr] \arrow[dd] & \\
& \mathrm{Ind}\left(\mathrm{Coh}^+(\mathfrak{X})\right) \arrow[rr, "\mathfrak{f}^*_\bullet"] \arrow[dd, "\mathrm{L}_{\mathfrak{X}}"] & & \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Ind}\left(\mathrm{Coh}^+(\mathfrak{U}_n)\right) \arrow[dd, "\mathrm{L}_{\mathfrak{U}_\bullet}"] \\
\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X}) \arrow[rr, "\mathfrak{f}^{\circ *}_\bullet"] \arrow[dr] & & \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_n) \arrow[dr] & \\
& \mathrm{Ind}\left(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})\right) \arrow[rr, "\mathfrak{f}^{\circ *}_\bullet"] & & \lim_{[n]\in\boldsymbol{\Delta}} \mathrm{Ind}\left(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_n)\right).
\end{tikzcd}$$

(4.3.2.4)

First of all, we observe that the diagonal functors are all fully faithful. It is therefore enough to prove that the functor

$$\mathfrak{f}_\bullet^{\circ *}\colon \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}((\mathfrak{X})) \longrightarrow \lim_{[n]\in\boldsymbol{\Delta}}\mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{U}_n))$$

is fully faithful when restricted to $\mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})$. As this functor admits a right adjoint $\mathfrak{f}^\circ_{\bullet *}$, it is in turn enough to verify that for every $\mathcal{F} \in \mathrm{Coh}^{\mathrm{b}}_{\mathrm{loc}}(\mathcal{F})$ the unit transformation

$$\eta\colon \mathcal{F} \longrightarrow \mathfrak{f}^\circ_{\bullet *}\mathfrak{f}^{\circ *}_\bullet(\mathcal{F})$$

is an equivalence. Proceeding by induction on the number of nonvanishing homotopy groups of $\mathcal{F}$, we see that it is enough to deal with the case of $\mathcal{F} \in \mathrm{Coh}^\heartsuit_{\mathrm{loc}}(\mathcal{F})$.

As the functor $\mathrm{L}_{\mathfrak{X}}\colon \mathrm{Coh}^+(\mathfrak{X}) \to \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X})$ is essentially surjective and t-exact, we can choose $\mathfrak{F} \in \mathrm{Coh}^\heartsuit(\mathfrak{X})$ and an equivalence

$$\mathrm{L}_{\mathfrak{X}}(\mathfrak{F}) \simeq \mathcal{F}.$$

Moreover, the unit transformation

$$\mathfrak{F} \longrightarrow \mathfrak{f}_{\bullet *}\mathfrak{f}^*_\bullet\mathfrak{F}$$

is an equivalence. It is therefore enough to check that the Beck-Chevalley transformation associated to the front square is an equivalence when evaluated on objects in $\lim_{\boldsymbol{\Delta}}\mathrm{Coh}^\heartsuit(\mathfrak{U}_n)$. This is exactly the content of Theorem 4.3.2.3. □

4.3.3 Categories of formal models

Let $\mathfrak{X} \in \mathrm{dfSch}^{\mathrm{taft}}_{k^\circ}$ be a quasi-compact and quasi-separated derived k°-adic scheme topologically almost of finite presentation. We established in Theorem 4.3.1.9 that for any $\mathcal{F} \in \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$ the ∞-category of formal models $\mathrm{FM}(\mathcal{F})$ is non-empty. Actually, we can use Theorem 4.3.1.8 to be more precise about the structure of $\mathrm{FM}(\mathcal{F})$. We are in particular interested in showing that it is filtered. We start by recording the following immediate consequence of Theorem 4.3.1.8:

Lemma 4.3.3.1. *Let $\mathfrak{X} \in \mathrm{dfSch}^{\mathrm{taft}}_{k^\circ}$ be a quasi-compact and quasi-separated derived k°-adic scheme topologically almost of finite presentation. Then the functor*

$$(-)^{\mathrm{rig}}\colon \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X})) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}))$$

admits a right adjoint

$$j\colon \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X})),$$

which is furthermore fully faithful.

Proof. Theorem 4.3.1.8 implies that the functor $(-)^{\mathrm{rig}}$ induces the equivalence

$$\Lambda\colon \mathrm{Coh}^+_{\mathrm{loc}}(\mathfrak{X}) \xrightarrow{\sim} \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}).$$

In other words, we see that the diagram

$$\begin{tikzcd} \mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X}) \arrow[r] \arrow[d] & \mathrm{Coh}^+(\mathfrak{X}) \arrow[d, "(-)^{\mathrm{rig}}"] \\ 0 \arrow[r] & \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}) \end{tikzcd}$$

is a pushout diagram in $\mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty$. Passing to ind-completions, we deduce that $\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}))$ is a Verdier quotient of $\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))$. Applying [HPV16b, Lemma 2.5 and Remark 2.6] we conclude that $\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}))$ is an accessible localization of $\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))$. As these categories are presentable, we deduce that the localization functor $(-)^{\mathrm{rig}}$ admits a fully faithful right adjoint, as desired. □

Notation 4.3.3.2. Let $\mathfrak{X} \in \mathrm{dfDM}_{k^\circ}$. Given $\mathcal{F}, \mathcal{G} \in \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))$ we write $\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \in \mathrm{Mod}_{k^\circ}$ for the k°-enriched stable mapping space in $\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))$.

Lemma 4.3.3.3. *Let $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}^{\mathrm{taft}}$ be a quasi-compact and quasi-separated derived k°-adic scheme topologically almost of finite presentation. Let $\mathcal{F} \in \mathrm{Coh}^+(\mathfrak{X})$ and $\mathcal{G} \in \mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$. Then*

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \otimes_{k^\circ} k \simeq 0.$$

In other words, $\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G})$ *is* $\mathfrak{m}$*-nilpotent in* Mod_{k°.

Proof. Since $\mathfrak{X}$ is quasi-compact, we can find a finite formal Zariski cover $\mathfrak{U}_i = \mathrm{Spf}(A_i)$ by formal affine schemes. Let $\mathfrak{U}_\bullet$ be the Čech nerve. Since this is a formal Zariski cover, there exists $m \gg 0$ such that

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \simeq \lim_{[n] \in \mathbf{\Delta}_{\leq m}} \mathrm{Hom}_{\mathfrak{U}_n}(\mathcal{F}|_{\mathfrak{U}_n}, \mathcal{G}|_{\mathfrak{U}_n}).$$

Since the functor $- \otimes_{k^\circ} k \colon \mathrm{Mod}_{k^\circ} \to \mathrm{Mod}_k$ is exact, it commutes with finite limits. Therefore, we see that it is enough to prove that the conclusion holds after replacing $\mathfrak{X}$ by $\mathfrak{U}_m$. Since $\mathfrak{X}$ is quasi-compact and quasi-separated, we see that each $\mathfrak{U}_m$ is quasi-compact and separated. In other words, we can assume from the very beginning that $\mathfrak{X}$ is quasi-compact and separated. In this case, each $\mathfrak{U}_m$ will be formal affine, and therefore we can further reduce to the case where $\mathfrak{X}$ is formal affine itself.

Assume therefore $\mathfrak{X} = \mathrm{Spf}(A)$. In this case, $\mathrm{Coh}^+(\mathfrak{X}) \simeq \mathrm{Coh}^+(A)$ lives fully faithfully inside Mod_A. Notice that $A \to A \otimes_{k^\circ} k$ is a Zariski open immersion. Therefore,

$$\mathrm{Hom}_A(\mathcal{F}, \mathcal{G}) \otimes_{k^\circ} k \simeq \mathrm{Hom}_A(\mathcal{F}, \mathcal{G}) \otimes_A (A \otimes_{k^\circ} k) \simeq \mathrm{Hom}_A(\mathcal{F} \otimes_A k^\circ, \mathcal{G} \otimes_A k^\circ) \simeq 0.$$

Thus, the proof is complete. ◻

Corollary 4.3.3.4. *Let $\mathfrak{X}$ be as in the previous lemma. Given $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}^+(\mathfrak{X})$, the canonical map*

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \otimes_{k^\circ} k \longrightarrow \mathrm{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{F}^{\mathrm{rig}}, \mathcal{G}^{\mathrm{rig}})$$

is an equivalence.

Proof. Denote by $R \colon \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X})) \to \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X}))$ the right adjoint to the inclusion

$$i \colon \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})) \hookrightarrow \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X})).$$

Then for any $\mathcal{G} \in \mathrm{Coh}^+(\mathfrak{X})$ we have a fiber sequence

$$iR(\mathcal{G}) \longrightarrow \mathcal{G} \longrightarrow j(\mathcal{G}^{\mathrm{rig}}).$$

In particular, we obtain a fiber sequence

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, iR(\mathcal{G})) \longrightarrow \mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, j(\mathcal{G}^{\mathrm{rig}})).$$

Now observe that

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, j(\mathcal{G}^{\mathrm{rig}})) \simeq \mathrm{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{F}^{\mathrm{rig}}, \mathcal{G}^{\mathrm{rig}}).$$

Notice also that since $k^\circ \to k$ is an open Zariski immersion, $\mathrm{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{F}^{\mathrm{rig}}, \mathcal{G}^{\mathrm{rig}}) \otimes_{k^\circ} k \simeq \mathrm{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{F}^{\mathrm{rig}}, \mathcal{G}^{\mathrm{rig}})$. In particular, applying $- \otimes_{k^\circ} k \colon \mathrm{Mod}_{k^\circ} \to \mathrm{Mod}_k$ we find a fiber sequence

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, iR(\mathcal{G})) \otimes_{k^\circ} k \longrightarrow \mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \otimes_{k^\circ} k \longrightarrow \mathrm{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{F}^{\mathrm{rig}}, \mathcal{G}^{\mathrm{rig}}).$$

It is therefore enough to check that $\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, iR(\mathcal{G})) \otimes_{k^\circ} k \simeq 0$. Since i is a left adjoint, we can write

$$iR(\mathcal{G}) \simeq \operatorname*{colim}_{\alpha \in I} \mathcal{G}_\alpha,$$

where I is filtered and $\mathcal{G}_\alpha \in \mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$. As $\mathcal{F}$ is compact in $\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))$, we find

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, iR(\mathcal{G})) \otimes_{k^\circ} k \simeq \left(\operatorname*{colim}_{\alpha \in I} \mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}_\alpha) \right) \otimes_{k^\circ} k \simeq \operatorname*{colim}_{\alpha \in I} \mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}_\alpha) \otimes_{k^\circ} k.$$

Since each $\mathcal{G}_\alpha$ belongs to $\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$, Theorem 4.3.3.3 implies that $\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}_\alpha) \otimes_{k^\circ} k \simeq 0$. The conclusion follows. ◻

Remark 4.3.3.5. Notice that Theorem 4.3.3.4 holds without no bounded conditions on the cohomological amplitude on the considered almost perfect complexes. The key ingredient is the fact that the morphism $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k^\circ$ is an open immersion. Compare with [**?**, Lemma 6.5.3.7].

Construction 4.3.3.6. Choose generators $t_1, \dots, t_n$ for $\mathfrak{m}$. We consider $\mathbb{N}^n$ as a poset with order given by

$$(m_1, \dots, m_n) \leq (m_1', \dots, m_n') \iff m_1 \leq m_1', m_2 \leq m_2', \dots, m_n \leq m_n'$$

Introduce the functor

$$K \colon \mathbb{N}^n \longrightarrow \operatorname{Ind}(\operatorname{Coh}^\heartsuit(\operatorname{Spf}(k^\circ)))$$

defined as follows: K sends every object to k°, and it sends the morphism $\mathbf{m} \leq \mathbf{m}'$ to multiplication by $t^{\mathbf{m}'-\mathbf{m}}$. By abuse of notation, we still denote the composition of K with the inclusion $\operatorname{Ind}(\operatorname{Coh}^\heartsuit(k^\circ)) \to \operatorname{Ind}(\operatorname{Coh}^+(k^\circ))$ by K.

Let now $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}^{\mathrm{taft}}$ be a quasi-compact and quasi-separated derived k°-adic scheme topologically almost of finite presentation. Let $\mathcal{F} \in \operatorname{Coh}^+(\mathfrak{X})$. The natural morphism $q \colon \mathfrak{X} \to \operatorname{Spf}(k^\circ)$ induces a functor

$$q^* \colon \operatorname{Ind}(\operatorname{Coh}^+(\operatorname{Spf}(k^\circ))) \longrightarrow \operatorname{Ind}(\operatorname{Coh}^+(\mathfrak{X})).$$

We define the functor $K_{\mathcal{F}}$ as

$$K_{\mathcal{F}} \coloneqq q^*(K(-)) \otimes \mathcal{F} \colon \mathbb{N}^n \longrightarrow \operatorname{Ind}(\operatorname{Coh}^+(\mathfrak{X})).$$

We let $\mathcal{F}^{\mathrm{loc}}$ denote the colimit of the functor $K_{\mathcal{F}}$.

Let $\mathcal{G} \in \operatorname{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$ and let $\alpha \colon \mathcal{F}^{\mathrm{rig}} \to \mathcal{G}$ be a given map. Notice that the natural map

$$\mathcal{F}^{\mathrm{rig}} \longrightarrow \operatorname*{colim}_{\mathbb{N}^n} (K_{\mathcal{F}}(-))^{\mathrm{rig}}$$

is an equivalence. Therefore α induces a cone

$$(K_{\mathcal{F}}(-))^{\mathrm{rig}} \longrightarrow \mathcal{G},$$

which is equivalent to the given of a cone

$$K_{\mathcal{F}}(-) \longrightarrow j(\mathcal{G}).$$

Specializing this construction for $\alpha = \mathrm{id}_{\mathcal{F}^{\mathrm{rig}}}$, we obtain a canonical map

$$\gamma_{\mathcal{F}} \colon \mathcal{F}^{\mathrm{loc}} \longrightarrow j(\mathcal{F}^{\mathrm{rig}}).$$

Lemma 4.3.3.7. *Let $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}^{\mathrm{taft}}$ be a quasi-compact and quasi-separated derived k°-adic scheme topologically almost of finite presentation. Let $\mathcal{F} \in \operatorname{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$. Then $\mathcal{F}^{\mathrm{loc}} \simeq 0$.*

Proof. For any $\mathcal{G} \in \operatorname{Coh}^+(\mathfrak{X})$, we write $\operatorname{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F}) \in \operatorname{Mod}_{k^\circ}$ for the k°-enriched mapping space. As $\mathcal{G}$ is compact in $\operatorname{Ind}(\operatorname{Coh}^+(\mathfrak{X}))$, we have

$$\operatorname{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F}^{\mathrm{loc}}) \simeq \operatorname*{colim}_{\mathbb{N}^n} \operatorname{Hom}_{\mathfrak{X}}(\mathcal{G}, K_{\mathcal{F}}(-)) \simeq \operatorname{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F}) \otimes_{k^\circ} k.$$

Theorem 4.3.3.4 implies that

$$\operatorname{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F}) \otimes_{k^\circ} k \simeq \operatorname{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{G}^{\mathrm{rig}}, \mathcal{F}^{\mathrm{rig}}) \simeq 0.$$

It follows that $\mathcal{F}^{\mathrm{loc}} \simeq 0$. $\square$

Lemma 4.3.3.8. *Let $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}^{\mathrm{taft}}$ be a quasi-compact and quasi-separated derived k°-adic scheme topologically almost of finite presentation. Let $\mathcal{F} \in \operatorname{Coh}^+(\mathfrak{X})$. Then for any $\mathcal{G} \in \operatorname{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$, one has*

$$\operatorname{Map}_{\operatorname{Ind}(\operatorname{Coh}^+(\mathfrak{X}))}(\mathcal{G}, \mathcal{F}^{\mathrm{loc}}) \simeq 0.$$

Proof. It is enough to prove that for every $i \geq 0$ we have

$$\pi_i \mathrm{Map}_{\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))}(\mathcal{G}, \mathcal{F}^{\mathrm{loc}}) \simeq 0.$$

Up to replacing $\mathcal{F}$ by $\mathcal{F}[i]$, we see that it is enough to deal with the case $i = 0$. Let therefore $\alpha \colon \mathcal{G} \to \mathcal{F}^{\mathrm{loc}}$ be a representative for an element in $\pi_0 \mathrm{Map}_{\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))}(\mathcal{G}, \mathcal{F}^{\mathrm{loc}})$. As $\mathcal{G}$ is compact in $\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))$, the map α factors as $\alpha' \colon \mathcal{G} \to \mathcal{F}$, and therefore it induces a map $\widetilde{\alpha} \colon \mathcal{G}^{\mathrm{loc}} \to \mathcal{F}^{\mathrm{loc}}$ making the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\alpha'"] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{G}^{\mathrm{loc}} \arrow[r, "\widetilde{\alpha}"] & \mathcal{F}^{\mathrm{loc}} \end{tikzcd}$$

commutative, where both compositions are equivalent to α. Now, Theorem 4.3.3.7 implies that $\mathcal{G}^{\mathrm{loc}} \simeq 0$, and therefore α is nullhomotopic, completing the proof. □

Lemma 4.3.3.9. *Let $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}^{\mathrm{taft}}$ be a quasi-compact and quasi-separated derived k°-adic scheme topologically almost of finite presentation. Let $\mathcal{F} \in \mathrm{Coh}^+(\mathfrak{X})$. Then the canonical map*

$$\gamma_{\mathcal{F}} \colon \mathcal{F}^{\mathrm{loc}} \longrightarrow j(\mathcal{F}^{\mathrm{rig}})$$

is an equivalence.

Proof. Let $\mathcal{G} \in \mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$. Then

$$\mathrm{Map}_{\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))}(\mathcal{G}, j(\mathcal{F}^{\mathrm{rig}})) \simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}}))}(\mathcal{G}^{\mathrm{rig}}, \mathcal{F}) \simeq 0.$$

Theorem 4.3.3.8 implies that the same holds true replacing $j(\mathcal{F}^{\mathrm{rig}})$ with $\mathcal{F}^{\mathrm{loc}}$. As $\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$ is a stable full subcategory of $\mathrm{Coh}^+(\mathfrak{X})$, it follows that

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{G}, j(\mathcal{F})) \simeq \mathrm{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F}^{\mathrm{loc}}) \simeq 0.$$

Let $\mathcal{H} \coloneqq \mathrm{fib}(\gamma_{\mathcal{F}})$. Then for any $\mathcal{G} \in \mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$, one has

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{H}) \simeq 0.$$

On the other hand,

$$\mathcal{H}^{\mathrm{rig}} \simeq \mathrm{fib}(\gamma_{\mathcal{F}}^{\mathrm{rig}}) \simeq 0.$$

It follows that $\mathcal{H} \in \mathrm{Ind}(\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X}))$, and hence that $\mathcal{H} \simeq 0$. Thus, $\gamma_{\mathcal{F}}$ is an equivalence. □

Theorem 4.3.3.10. *Let $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}$ be a quasi-compact and quasi-separated derived k°-adic scheme. Let $\mathcal{F} \in \mathrm{Coh}^+(\mathfrak{X}^{\mathrm{rig}})$. Then the ∞-category $\mathrm{FM}(\mathcal{F})$ of formal models for $\mathcal{F}$ is non-empty and filtered.*

Proof. We know that $\mathrm{FM}(\mathcal{F})$ is non-empty thanks to Theorem 4.3.1.9. Pick one formal model $\mathcal{F} \in \mathrm{FM}(\mathcal{F})$. Then Theorem 4.3.3.9 implies that the canonical map

$$\gamma_{\mathcal{F}} \colon \mathcal{F}^{\mathrm{loc}} \longrightarrow j(\mathcal{F}^{\mathrm{rig}}) \simeq j(\mathcal{F})$$

is an equivalence. We now observe that $\mathrm{FM}(\mathcal{F})$ is by definition a full subcategory of

$$\mathrm{Coh}^+(\mathfrak{X})_{/\mathcal{F}} \coloneqq \mathrm{Coh}^+(\mathfrak{X}) \times_{\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))} \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))_{/j(\mathcal{F})}.$$

As this ∞-category is filtered, it is enough to prove that every object $\mathcal{G} \in \mathrm{Coh}^+(\mathfrak{X})_{/\mathcal{F}}$ admits a map to an object in $\mathrm{FM}(\mathcal{F})$. Let $\alpha \colon \mathcal{G} \to j(\mathcal{F})$ be the structural map. Using the equivalence $\gamma_{\mathcal{F}}$ and the fact that $\mathcal{G}$ is compact in $\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))$, we see that α factors as $\mathcal{G} \to \mathcal{F}$, which belongs to $\mathrm{FM}(\mathcal{F})$ by construction. □

Corollary 4.3.3.11. *Let $X \in \mathrm{dAn}_k$ and $f \colon \mathcal{F} \to \mathcal{G}$ be a morphism $\mathrm{Coh}^+(X)$. Suppose we are given a formal model $\mathfrak{X}$ for X together with formal models $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}^+(\mathfrak{X})$ for $\mathcal{F}$ and $\mathcal{G}$, respectively. Then there exists a morphism $\mathfrak{f} \colon \mathcal{F}' \to \mathcal{G}'$ in the ∞-category $\mathrm{Coh}^+(\mathfrak{X})$ lifting*

$$t_1^{m_1} \dots t_n^{m_n} f \colon \mathcal{F} \to \mathcal{G}, \quad \textit{in } \mathrm{Coh}^+(X)$$

for suitable non-negative integers $m_1, \dots, m_n \geq 0$.

Proof. Any map $\mathcal{F} \to \mathcal{G}$ induces a map $\mathfrak{F} \to j(\mathcal{F}) \to j(\mathcal{G})$. Using the equivalence $j(\mathcal{G}) \simeq \mathcal{G}^{\mathrm{loc}}$ and the fact that $\mathcal{F}$ is compact in $\mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X}))$, we see that the map $\mathfrak{F} \to j(\mathcal{G})$ factors as $\mathfrak{F} \to \mathcal{G}$. Unraveling the definition of the functor $K_{\mathcal{G}}(-)$, we see that the conclusion follows. $\square$

For later use, let us record the following consequence of Theorem 4.3.3.9:

Corollary 4.3.3.12. *Let $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}^{\mathrm{taft}}$ be a quasi-compact and quasi-separated derived k°-adic scheme topologically almost of finite presentation. Let $\mathcal{F} \in \mathrm{Coh}^+(\mathfrak{X})$. Then $\mathcal{F}$ is $\mathfrak{m}$-nilpotent if and only if $\mathcal{F}^{\mathrm{loc}} \simeq 0$.*

Proof. If $\mathcal{F}$ is $\mathfrak{m}$-nilpotent, the conclusion follows from Theorem 4.3.3.7. Suppose vice-versa that $\mathcal{F}^{\mathrm{loc}} \simeq 0$. Then Theorem 4.3.3.9 implies that

$$j(\mathcal{F}^{\mathrm{rig}}) \simeq \mathcal{F}^{\mathrm{loc}} \simeq 0.$$

Now, Theorem 4.3.3.1 shows that j is fully faithful. In particular it is conservative and therefore $\mathcal{F}^{\mathrm{rig}} \simeq 0$. In other words, $\mathcal{F}$ belongs to $\mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X})$. $\square$

4.4 Flat models for morphisms of derived analytic spaces

Using the study of formal models for almost perfect complexes carried out in the previous section, we can prove the following derived version of [BL93b, Theorem 5.2]:

Theorem 4.4.0.1. *Let $f \colon X \to Y$ be a proper map of quasi-paracompact derived k-analytic spaces. Assume that:*

(i) the truncations of X and Y are k-analytic spaces.[4]

(ii) The map f is flat.

Then there exists a proper flat formal model $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathrm{dfSch}_{k^\circ}^{\mathrm{taft}}$ for f.

Proof. We construct, by induction on n, the following data:

(i) derived k°-adic schemes $\mathfrak{X}_n$ and $\mathfrak{Y}_n$ equipped with equivalences

$$\mathfrak{X}_n^{\mathrm{rig}} \simeq \mathrm{t}_{\leq n}(X), \qquad \mathfrak{Y}_n^{\mathrm{rig}} \simeq \mathrm{t}_{\leq n}(Y).$$

(ii) Morphisms $\mathfrak{X}_n \to \mathfrak{X}_{n-1}$ and $\mathfrak{Y}_n \to \mathfrak{Y}_{n-1}$ exhibiting $\mathfrak{X}_{n-1}$ and $\mathfrak{Y}_{n-1}$ as $(n-1)$-truncations of $\mathfrak{X}_n$ and $\mathfrak{Y}_n$, respectively.

(iii) A proper flat morphism $\mathfrak{f}_n \colon \mathfrak{X}_n \to \mathfrak{Y}_n$ and homotopies making the cube

$$\begin{tikzcd}
& \mathfrak{X}_n^{\mathrm{rig}} \arrow[rr, "\mathfrak{f}_n^{\mathrm{rig}}"] \arrow[dl] \arrow[dd] & & \mathfrak{Y}_n^{\mathrm{rig}} \arrow[dl] \arrow[dd] \\
\mathfrak{X}_{n-1}^{\mathrm{rig}} \arrow[rr, "\mathfrak{f}_{n-1}^{\mathrm{rig}}" near end] \arrow[dd] & & \mathfrak{Y}_{n-1}^{\mathrm{rig}} \arrow[dd] & \\
& \mathrm{t}_{\leq n}(X) \arrow[rr] \arrow[dl] & & \mathrm{t}_{\leq n}(Y) \arrow[dl] \\
\mathrm{t}_{\leq n-1}(X) \arrow[rr] & & \mathrm{t}_{\leq n-1}(Y) &
\end{tikzcd}$$

commutative.

[4] As opposed to k-analytic Deligne-Mumford stackstacks.

Having these data at our disposal, we set

$$\mathfrak{X} \coloneqq \operatorname*{colim}_n \mathfrak{X}_n, \qquad \mathfrak{Y} \coloneqq \operatorname*{colim}_n \mathfrak{Y}_n,$$

and we let $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ be map induced by the morphisms $\mathfrak{f}_n$. The properties listed above imply that $\mathfrak{f}$ is proper and flat and that its generic fiber is equivalent to f.

We are therefore left to construct the data listed above. When $n = 0$, we can apply the flattening technique of Raynaud-Gruson (see [BL93b, Theorem 5.2]) to produce a proper flat formal model $\mathfrak{f}_0 \colon \mathfrak{X}_0 \to \mathfrak{Y}_0$ for $\mathrm{t}_0(f) \colon \mathrm{t}_0(X) \to \mathrm{t}_0(Y)$. Assume now that we constructed the above data up to n and let us construct it for $n+1$. Set $\mathcal{F} \coloneqq \pi_{n+1}(\mathcal{O}_X)[n+2]$ and $\mathcal{G} \coloneqq \pi_{n+1}(\mathcal{O}_Y)[n+2]$. Using [PY17b, Corollary 5.44], we can find analytic derivations $d_\alpha \colon (\mathrm{t}_{\le n}X)[\mathcal{F}] \to \mathrm{t}_{\le n}X$ and $d_\beta \colon (\mathrm{t}_{\le n}Y)[\mathcal{G}] \to \mathrm{t}_{\le n}Y$ making the following cube

$$\begin{tikzcd}
& (\mathrm{t}_{\le n}X)[\mathcal{F}] \arrow[rr, "d_0"] \arrow[dl, "d_\alpha"'] \arrow[dd] & & \mathrm{t}_{\le n}X \arrow[dl] \arrow[dd, "f_n"] \\
\mathrm{t}_{\le n}X \arrow[rr] \arrow[dd] & & \mathrm{t}_{\le n+1}X \arrow[dd, "f_{n+1}" near start] & \\
& (\mathrm{t}_{\le n}Y)[\mathcal{G}] \arrow[rr, "d_0" near start] \arrow[dl, "d_\beta"'] & & \mathrm{t}_{\le n}Y \arrow[dl] \\
\mathrm{t}_{\le n}Y \arrow[rr] & & \mathrm{t}_{\le n+1}Y &
\end{tikzcd} \tag{4.4.0.1}$$

commutative. Here d_0 denote the zero derivation and we set $f_n \coloneqq \mathrm{t}_{\le n}(f)$, $f_{n+1} \coloneqq \mathrm{t}_{\le n+1}(f)$. The derivations d_α and d_β correspond to morphisms $\alpha \colon \mathbb{L}\mathrm{an}_{\mathrm{t}_{\le n}X} \to \mathcal{F}$ and $\beta \colon \mathbb{L}\mathrm{an}_{\mathrm{t}_{\le n}Y} \to \mathcal{G}$, respectively. Moreover, the commutativity of the left side square in (4.4.0.1) is equivalent to the commutativity of

$$\begin{tikzcd}
f_n^* \mathbb{L}\mathrm{an}_{\mathrm{t}_{\le n}Y} \arrow[r, "f_n^*\beta"] \arrow[d] & f_n^* \mathcal{G} \arrow[d] \\
\mathbb{L}\mathrm{an}_{\mathrm{t}_{\le n}X} \arrow[r, "\alpha"] & \mathcal{F}
\end{tikzcd}$$

in $\mathrm{Coh}^+(\mathrm{t}_{\le n}X)$. Notice that, since f is flat, the morphism $f_n^*\mathcal{F} \to \mathcal{G}$ is an equivalence. Using Theorem 4.2.0.15 and the induction hypothesis, we know that $\mathbb{L}^{\mathrm{ad}}_{\mathfrak{Y}_n}$ is a canonical formal model for $\mathbb{L}\mathrm{an}_{\mathrm{t}_{\le n}X}$. Using Theorem 4.3.3.10, we can therefore find a formal model $\overline{\beta} \colon \mathbb{L}^{\mathrm{ad}}_{\mathfrak{Y}_n} \to \mathfrak{G}$ for the map β. We now set

$$\mathfrak{F} \coloneqq \mathfrak{f}_n^* \mathfrak{G}.$$

Using Theorem 4.3.3.11, we can find $\mathbf{m} \in \mathbb{N}^n$ and a formal model $\widetilde{\alpha} \colon \mathbb{L}^{\mathrm{ad}}_{\mathfrak{X}_n} \to \mathfrak{F}$ for $t^{\mathbf{m}}\alpha$ together with a homotopy making the diagram

$$\begin{tikzcd}
\mathfrak{f}_n^* \mathbb{L}^{\mathrm{ad}}_{\mathfrak{Y}_n} \arrow[r, "t^{\mathbf{m}} \mathfrak{f}_n^* \overline{\beta}"] \arrow[d] & \mathfrak{f}_n^* \mathfrak{G} \arrow[d, equal] \\
\mathbb{L}^{\mathrm{ad}}_{\mathfrak{X}_n} \arrow[r, "\tilde{\alpha}"] & \mathfrak{F}
\end{tikzcd}$$

commutative. Set $\widetilde{\beta} \coloneqq t^{\mathbf{m}}\overline{\beta} \colon \mathbb{L}^{\mathrm{ad}}_{\mathfrak{Y}_n} \to \mathfrak{G}$. Then $\widetilde{\alpha}$ and $\widetilde{\beta}$ induce a commutative square

$$\begin{tikzcd}
\mathfrak{X}_n[\mathfrak{F}] \arrow[r, "d_{\tilde{\alpha}}"] \arrow[d] & \mathfrak{X}_n \arrow[d, "\mathfrak{f}_n"] \\
\mathfrak{Y}_n[\mathfrak{G}] \arrow[r, "d_{\tilde{\beta}}"] & \mathfrak{Y}_n .
\end{tikzcd} \tag{4.4.0.2}$$

We now define $\mathfrak{X}_{n+1}$ and $\mathfrak{Y}_{n+1}$ as the square-zero extensions associated to $\widetilde{\alpha}$ and $\widetilde{\beta}$. In other words, they are defined by the following pushout diagrams:

$$\begin{tikzcd} \mathfrak{X}_n[\mathfrak{F}] \arrow[r, "d_0"] \arrow[d, "d_{\tilde{\alpha}}"] & \mathfrak{X}_n \arrow[d] \\ \mathfrak{X}_n \arrow[r] & \mathfrak{X}_{n+1} \end{tikzcd} , \qquad \begin{tikzcd} \mathfrak{Y}_n[\mathfrak{G}] \arrow[r, "d_0"] \arrow[d, "d_{\tilde{\beta}}"] & \mathfrak{Y}_n \arrow[d] \\ \mathfrak{Y}_n \arrow[r] & \mathfrak{Y}_{n+1}. \end{tikzcd}$$

The commutativity of (4.4.0.2) provides a canonical map $\mathfrak{f}_{n+1} \colon \mathfrak{X}_{n+1} \to \mathfrak{Y}_{n+1}$, which is readily verified to be proper and flat. We are therefore left to verify that $\mathfrak{f}_{n+1}$ is a formal model for f_{n+1}. Unraveling the definitions, we see that it is enough to produce equivalences $a \colon (\mathrm{t}_{\le n} X)[\mathcal{F}] \xrightarrow{\sim} (\mathrm{t}_{\le n} X)[\mathcal{F}]$ and $b \colon (\mathrm{t}_{\le n} Y)[\mathcal{G}] \xrightarrow{\sim} (\mathrm{t}_{\le n} Y)[\mathcal{G}]$ making the following diagrams

$$\begin{tikzcd} (\mathrm{t}_{\le n} X)[\mathcal{F}] \arrow[r, "d_{t^{\mathbf{m}}\alpha}"] \arrow[d, "a"] & \mathrm{t}_{\le n} X \arrow[d, equal] \\ (\mathrm{t}_{\le n} X)[\mathcal{F}] \arrow[r, "d_\alpha"] & \mathrm{t}_{\le n} X \end{tikzcd} , \qquad \begin{tikzcd} (\mathrm{t}_{\le n} Y)[\mathcal{G}] \arrow[r, "d_{t^{\mathbf{m}}\beta}"] \arrow[d, "b"] & \mathrm{t}_{\le n} Y \arrow[d, equal] \\ (\mathrm{t}_{\le n} Y)[\mathcal{G}] \arrow[r, "d_\beta"] & \mathrm{t}_{\le n} Y \end{tikzcd} \tag{4.4.0.3}$$

commutative. The situation is symmetric, so it is enough to deal with $\mathrm{t}_{\le n} X$. Consider the morphism

$$t^{-\mathbf{m}} \colon \mathcal{F} \longrightarrow \mathcal{F},$$

which exists because all the elements $t_i \in \mathfrak{m}$ are invertible in k. For the same reason it is an equivalence, with inverse given by multiplication by $t^{\mathbf{m}}$. This morphism induces a map

$$a \colon (\mathrm{t}_{\le n} X)[\mathcal{F}] \longrightarrow (\mathrm{t}_{\le n} X)[\mathcal{F}],$$

which by functoriality is an equivalence. We now observe that the commutativity of (4.4.0.3) is equivalent to the commutativity of

$$\begin{tikzcd} \mathbb{L}\mathrm{an}_{\mathrm{t}_{\le n} X} \arrow[r, "t^{\mathbf{m}}\alpha"] \arrow[d, equal] & \mathcal{F} \arrow[d, "t^{-\mathbf{m}}"] \\ \mathbb{L}\mathrm{an}_{\mathrm{t}_{\le n} X} \arrow[r, "\alpha"] & \mathcal{F}, \end{tikzcd}$$

which is immediate. The proof is therefore achieved. $\square$

4.5 The plus pushforward for almost perfect sheaves

Let $f \colon X \to Y$ be a proper map between derived k-analytic spaces of finite tor-amplitude. In [PY18b, Definition 7.9] it is introduced a functor

$$f_+ \colon \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(Y),$$

and it is shown in Proposition 7.11 in loc. cit. that for every $\mathcal{G} \in \mathrm{Coh}^+(Y)$ there is a natural equivalence

$$\mathrm{Map}_{\mathrm{Coh}^+(X)}(\mathcal{F}, f^*\mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Coh}^+(Y)}(f_+(\mathcal{F}), \mathcal{G}).$$

In this section we extend the definition of f_+ to the entire $\mathrm{Coh}^+(X)$, at least under the stronger assumption of f being flat.

Remark 4.5.0.1. In algebraic geometry, the extension of f_+ to $\mathrm{Coh}^+(X)$ passes through the extension to $\mathrm{QCoh}(X) \simeq \mathrm{Ind}(\mathrm{Perf}(X))$. This is ultimately requires being able to describe every element in $\mathrm{Coh}^+(X)$ as a filtered colimit of elements in $\mathrm{Perf}(X)$, which in analytic geometry is possible only locally.

Therefore, this technique cannot be applied in analytic geometry. When dealing with non-archimedean analytic geometry, formal models can be used to circumvent this problem.

Proposition 4.5.0.2. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper map between derived k°-adic schemes. Assume that f has finite tor amplitude. Then the functor*

$$f^*\colon \mathrm{Coh}^+(\mathfrak{Y}) \to \mathrm{Coh}^+(\mathfrak{X})$$

admits a left adjoint

$$f_+\colon \mathrm{Coh}^+(\mathfrak{X}) \to \mathrm{Coh}^+(\mathfrak{Y}).$$

Proof. Let $\mathfrak{X}_n \coloneqq \mathfrak{X} \times_{\mathrm{Spf}(k^\circ)} \mathrm{Spec}(k^\circ/\mathfrak{m}^n)$ and define similarly $\mathfrak{Y}_n$. Let $f_n\colon \mathfrak{X}_n \to \mathfrak{Y}_n$ be the induced morphism. Then by definition of k°-adic schemes,

$$\mathfrak{X} \simeq \operatorname*{colim}_{n\in\mathbb{N}} \mathfrak{X}_n, \qquad \mathfrak{Y} \simeq \operatorname*{colim}_{n\in\mathbb{N}} \mathfrak{Y}_n,$$

and therefore

$$\mathrm{Coh}^+(\mathfrak{X}) \simeq \lim_{n\in\mathbb{N}} \mathrm{Coh}^+(\mathfrak{X}_n), \qquad \mathrm{Coh}^+(\mathfrak{Y}) \simeq \lim_{n\in\mathbb{N}} \mathrm{Coh}^+(\mathfrak{Y}_n).$$

Combining [**?**, Remark 6.4.5.2(b) & Proposition 6.4.5.4(1)], we see that each functor

$$f_n^*\colon \mathrm{Coh}^+(\mathfrak{Y}_n) \longrightarrow \mathrm{Coh}^+(\mathfrak{X}_n)$$

admits a left adjoint f_{n+}. Moreover, Proposition 6.4.5.4(2) in loc. cit. implies that these functors f_{n+} can be assembled into a natural transformation, and that therefore they induce a well defined functor

$$f_+\colon \mathrm{Coh}^+(\mathfrak{X}) \longrightarrow \mathrm{Coh}^+(\mathfrak{Y}).$$

Now let $\mathcal{F} \in \mathrm{Coh}^+(\mathfrak{X})$ and $\mathcal{G} \in \mathrm{Coh}^+(\mathfrak{Y})$. Let $\mathcal{F}_n$ and $\mathcal{G}_n$ be the pullbacks of $\mathcal{F}$ and $\mathcal{G}$ to $\mathcal{X}_n$ and $\mathcal{Y}_n$, respectively. Then

$$\begin{aligned} \mathrm{Map}_{\mathrm{Coh}^+(\mathfrak{X})}(\mathcal{F}, f^*(\mathcal{G})) &\simeq \lim_{n\in\mathbb{N}} \mathrm{Map}_{\mathrm{Coh}^+(\mathfrak{X}_n)}(\mathcal{F}_n, f_n^*(\mathcal{G}_n)) \\ &\simeq \lim_{n\in\mathbb{N}} \mathrm{Map}_{\mathrm{Coh}^+(\mathfrak{Y}_n)}(f_{n+}(\mathcal{F}_n), \mathcal{G}_n) \\ &\simeq \mathrm{Map}_{\mathrm{Coh}^+(\mathfrak{Y})}(f_+(\mathcal{F}), \mathcal{G}), \end{aligned}$$

which completes the proof. $\square$

Corollary 4.5.0.3. *Let $f\colon X \to Y$ be a proper map between derived analytic spaces. Assume that f is flat. Then the functor*

$$f^*\colon \mathrm{Coh}^+(Y) \to \mathrm{Coh}^+(X)$$

admits a left adjoint

$$f_+\colon \mathrm{Coh}^+(X) \to \mathrm{Coh}^+(Y).$$

Proof. Using Theorem 4.4.0.1, we can choose a proper flat formal model $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ for f. Thanks to Theorem 4.5.0.2, we have a well defined functor

$$\mathfrak{f}_+\colon \mathrm{Coh}^+(\mathfrak{X}) \longrightarrow \mathrm{Coh}^+(\mathfrak{Y}).$$

We claim that it restricts to a functor

$$\mathfrak{f}_+\colon \mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{X}) \longrightarrow \mathrm{Coh}^+_{\mathrm{nil}}(\mathfrak{Y}).$$

Using Theorem 4.3.3.12, it is enough to prove that

$$\mathfrak{f}_+(\mathcal{F})^{\mathrm{loc}} \simeq 0.$$

Extending $\mathfrak{f}_+$ to a functor $\mathfrak{f}_+\colon \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{X})) \to \mathrm{Ind}(\mathrm{Coh}^+(\mathfrak{Y}))$, we see that

$$\mathfrak{f}_+(\mathcal{F})^{\mathrm{loc}} \simeq \mathfrak{f}_+(\mathcal{F}^{\mathrm{loc}}) \simeq 0.$$

Using Theorem 4.3.1.8, we get a well defined functor

$$f_+\colon \mathrm{Coh}^+(X) \longrightarrow \mathrm{Coh}^+(Y).$$

We only have to prove that it is left adjoint to f^*. Let $\mathcal{F} \in \mathrm{Coh}^+(X)$ and $\mathcal{G} \in \mathrm{Coh}^+(Y)$. Choose a formal model $\mathfrak{F} \in \mathrm{Coh}^+(\mathfrak{X})$. Then unraveling the construction of f_+, we find a canonical equivalence

$$f_+(\mathcal{F}) \simeq \mathfrak{f}_+(\mathfrak{F})^{\mathrm{rig}}.$$

We now have the following sequence of natural equivalences:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Coh}^+(Y)}(f_+(\mathcal{F}), \mathcal{G}) &\simeq \mathrm{Map}_{\mathrm{Coh}^+(Y)}((\mathfrak{f}_+(\mathfrak{F}))^{\mathrm{rig}}, \mathfrak{G}^{\mathrm{rig}}) \\
&\simeq \mathrm{Map}_{\mathrm{Coh}^+(\mathfrak{X})}(\mathfrak{f}_+(\mathfrak{F}), \mathfrak{G}) \otimes_{k^\circ} k \qquad \text{by Theorem 4.3.3.4} \\
&\simeq \mathrm{Map}_{\mathrm{Coh}^+(\mathfrak{X})}(\mathfrak{F}, \mathfrak{f}^*\mathfrak{G}) \otimes_{k^\circ} k \\
&\simeq \mathrm{Map}_{\mathrm{Coh}^+(\mathfrak{X})}(\mathfrak{F}^{\mathrm{rig}}, (\mathfrak{f}^*\mathfrak{G})^{\mathrm{rig}}) \qquad \text{by Theorem 4.3.3.4} \\
&\simeq \mathrm{Map}_{\mathrm{Coh}^+(\mathfrak{X})}(\mathcal{F}, f^*\mathcal{G}).
\end{aligned}$$

The proof is therefore complete. □

Corollary 4.5.0.4. *Let $f\colon X \to Y$ be a proper and flat map between derived analytic spaces. Let $p\colon Z \to Y$ be any other map and consider the pullback square*

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Z \arrow[r, "p"] & Y. \end{tikzcd}$$

Then for any $\mathcal{F} \in \mathrm{Coh}^+(X)$ the canonical map

$$g_+(q^*(\mathcal{F})) \longrightarrow p^*(f_+(\mathcal{F}))$$

is an equivalence.

Proof. Using Theorem 4.4.0.1, we find a flat formal model $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$. Choose a formal model $\mathfrak{p}\colon \mathfrak{Z} \to \mathfrak{Y}$ for $p\colon Z \to Y$, and form the pullback square

$$\begin{tikzcd} \mathfrak{W} \arrow[r, "\mathfrak{q}"] \arrow[d, "\mathfrak{g}"] & \mathfrak{X} \arrow[d, "\mathfrak{f}"] \\ \mathfrak{Z} \arrow[r, "\mathfrak{p}"] & \mathfrak{Y}. \end{tikzcd}$$

Choose also a formal model $\mathfrak{F} \in \mathrm{Coh}^+(\mathfrak{X})$ for $\mathcal{F}$. It is then enough to prove that the canonical map

$$\mathfrak{g}_+(\mathfrak{q}^*(\mathfrak{F})) \longrightarrow \mathfrak{p}^*(\mathfrak{f}_+(\mathfrak{F}))$$

is an equivalence. This follows at once by [**?**, Proposition 6.4.5.4(2)]. □

4.6 Representability of $\mathrm{RHilb}(X)$

Let $p\colon X \to S$ be a proper and flat morphism of underived k-analytic spaces. We define the functor

$$\mathrm{RHilb}(X/S)\colon \mathrm{dAfd}_S\mathrm{op} \longrightarrow \mathcal{S}$$

by sending $T \to S$ to the space of diagrams

$$\begin{tikzcd} Y \arrow[rr, hook, "i"] \arrow[dr, "q_T"'] & & T \times_S X \arrow[dl, "p_T"] \\ & T & \end{tikzcd} \tag{4.6.0.1}$$

where i is a closed immersion of derived k-analytic spaces, and q_T is flat.

Proposition 4.6.0.1. *Keeping the above notation and assumptions,* $\mathrm{RHilb}(X/S)$ *admits a global analytic cotangent complex.*

Proof. Let $x\colon T \to \mathrm{RHilb}(X/S)$ be a morphism from a derived k-affinoid space $T \in \mathrm{dAfd}_S$. It classifies a diagram of the form (4.6.0.1). Unraveling the definitions, we see that the functor

$$\mathrm{Der}^{\mathrm{an}}_{\mathrm{RHilb}(X/S),x}(T;-)\colon \mathrm{Coh}^+(T) \longrightarrow \mathrm{RHilb}(X/S)$$

can be explicitly written as

$$\mathrm{Der}^{\mathrm{an}}_{\mathrm{RHilb}(X/S),x}(T;\mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Coh}^+(Y)}(\mathbb{L}\mathrm{an}_{Y/T\times_S X}, q_T^*(\mathcal{F})).$$

Since $q_T\colon Y \to T$ is proper and flat, Theorem 4.5.0.3 implies the existence of a left adjoint $q_{T+}\colon \mathrm{Coh}^+(Y) \to \mathrm{Coh}^+(T)$ for q_T^*. Moreover, [PY17b, Corollary 5.40] implies that $\mathbb{L}\mathrm{an}_{Y/T\times_S X} \in \mathrm{Coh}^+{\geq}\,0(Y)$. Therefore, we find

$$\mathrm{Der}^{\mathrm{an}}_{\mathrm{RHilb}(X/S),x}(T;\mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Coh}^+(T)}(q_{T+}(\mathbb{L}\mathrm{an}_{Y/T\times_S X}), \mathcal{F}),$$

and therefore $\mathrm{RHilb}(X/S)$ admits an analytic cotangent complex at x. Using Theorem 4.5.0.4, we see that it admits as well a global analytic cotangent complex. □

Theorem 4.6.0.2. *Let X be a k-analytic space. Then* $\mathbf{R}\mathrm{Hilb}(X)$ *is a derived analytic space.*

Proof. We only need to check the hypotheses of [PY17b, Theorem 7.1]. The representability of the truncation is guaranteed by [CG16, Proposition 5.3.3]. The existence of the global analytic cotangent complex has been dealt with in Theorem 4.6.0.1. Convergence and infinitesimal cohesiveness are straightforward checks. The theorem follows. □

As a second concluding applications, let us mention that the theory of the plus pushforward developed in this paper allows to remove the lci assumption in [PY18b, Theorem 8.6]:

Theorem 4.6.0.3. *Let S be a rigid k-analytic space. Let X, Y be rigid k-analytic spaces over S. Assume that X is proper and flat over S and that Y is separated over S. Then the ∞-functor $\mathbf{Map}_S(X,Y)$ is representable by a derived k-analytic space separated over S.*

Proof. The same proof of [PY18b, Theorem 8.6] applies. It is enough to observe that Corollaries 4.5.0.3 and 4.5.0.4 allow to prove Lemma 8.4 in loc. cit. by removing the assumption of $Y \to S$ being locally of finite presentation. □

4.7 Coherent dualizing sheaves

It should be possible to apply the formalism of this paper to get a reasonable construction for the dualizing sheaf of a morphism of derived k-analytic schemes.

Definition 4.7.0.1. Let $f\colon X \to Y$ be a morphism of derived k-analytic schemes. Choose a formal model $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ and let $\omega_{\mathfrak{X}/\mathfrak{Y}}$ be a dualizing sheaf. We set

$$\omega_{X/Y} \coloneqq (\omega_{\mathfrak{X}/\mathfrak{Y}})^{\mathrm{rig}}.$$

Proposition 4.7.0.2. *Suppose $f\colon X \to Y$ is proper and flat. Then:*

(i) We have

$$f_+(\mathcal{F}) = f_*(\mathcal{F} \otimes \omega_{X/Y}).$$

(ii) the functor

$$\mathcal{F} \mapsto f^!(\mathcal{F}) \coloneqq f^*(\mathcal{F} \otimes \omega_{X/Y})$$

is a right adjoint for the functor f_*.

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Chapter 5

Moduli of continuous p-adic representations of a profinite group

Moduli of p-adic representations of a profinite group

Contents

5.1 Introduction

5.1.1 Main results

Let X be a proper and smooth scheme over an algebraically closed field. The goal of the present text is to show the existence of the moduli of rank n étale p-adic lisse sheaves on X, study its geometry and its corresponding deformation theory. More precisely, let k denote a non-archimedean field extension of $\mathbb{Q}_p$. We will construct a functor

$$\mathrm{LocSys}_{\ell,n}(X)\colon \mathrm{Afd}_k^{\mathrm{op}} \to \mathcal{S},$$

where Afd_k denotes the category of k-affinoid spaces and $\mathcal{S}$ the ∞-category of ∞-groupoids, given on objects by the formula

$$A \in \mathrm{Afd}_k^{\mathrm{op}} \mapsto \mathrm{LocSys}_{\ell,n}(X)(A) \in \mathcal{S}$$

where $\mathrm{LocSys}_{\ell,n}(X)(A)$ denotes the groupoid of conjugation classes of continuous morphisms

$$\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A)$$

where we endow $\mathrm{GL}_n(A)$ with the topology induced by the non-archimedean topology on $A \in \mathrm{Afd}_k^{\mathrm{op}}$. Our first main result is the following:

Theorem 5.1.1.1. *The moduli functor*

$$\mathrm{LocSys}_{\ell,n}(X)\colon \mathrm{Afd}_k^{\mathrm{op}} \to \mathcal{S}$$

is representable by a k-analytic stack. More precisely, there exists a k-analytic space $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \in \mathrm{An}_k$ together with a canonical smooth map

$$q\colon \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \to \mathrm{LocSys}_{\ell,n}(X)$$

which exhibits $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ as a smooth atlas of $\mathrm{LocSys}_{\ell,n}(X)$. Moreover, $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ admits a canonical action of the k-analytic group $\mathbf{GL}_n^{\mathrm{an}}$ and $\mathrm{LocSys}_{\ell,n}(X)$ can be realized as the stack quotient of $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ by the $\mathbf{GL}_n^{\mathrm{an}}$-action.

We can construct $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ explicitly via its functor of points. Explicitly, $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ represents the functor $\mathrm{Afd}_k^{\mathrm{op}} \to \mathrm{Set}$ given on objects by the formula

$$A \in \mathrm{Afd}_k^{\mathrm{op}} \mapsto \mathrm{Hom}_{\mathrm{cont}}\big(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(A)\big) \in \mathrm{Set}. \tag{5.1.1.1}$$

Showing that the functor given by formula (5.1.1.1) is representably by a k-analytic space $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \in \mathrm{An}_k$ will occupy most of §2. Our proof follows the scheme of proof of the analogous result for smooth and proper schemes over the field of complex numbers $\mathbb{C}$. However, our argument is considerably more involved as in general the topologies on $\pi_1^{\text{ét}}(X)$ and $\mathrm{GL}_n(A)$ are of different natures. More precisely, the former admits a profinite topology whereas the latter group admits a ind-pro-topology, where the pro-structure comes from the choice of a formal model A_0 for A and the ind-structure by the existence of an isomorphism

$$A_0 \otimes_{k^\circ} k \cong A$$

of topological algebras. Moreover, thanks to the formula (5.1.1.1) it is clear that $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ admits a canonical action of the k-analytic general linear group $\mathbf{GL}_n^{\mathrm{an}}$, given by conjugation. The rest of §2 is devoted to present the theory of k-analytic stacks and to show that $\mathrm{LocSys}_{\ell,n}(X)$ can be identified with the k-analytic stack obtained by "quotientening" $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)$ by $\mathbf{GL}_n^{\mathrm{an}}$. We observe that k-analytic stacks are the k-analytic analogues of Artin stacks in algebraic geometry.

We show moreover that our definition of $\mathrm{LocSys}_{\ell,n}(X)$ is correct. More precisely we show:

Proposition 5.1.1.2. *Let $A \in \mathrm{Afd}_k^{\mathrm{op}}$. Then the groupoid*

$$\mathrm{LocSys}_{\ell,n}(X)(A) \in \mathcal{S}$$

can be identified with the groupoid of rank n pro-étale A-local systems on X.

We then proceed to study the deformation theory of $\mathrm{LocSys}_{\ell,n}(X)$. We prove more precisely that $\mathrm{LocSys}_{\ell,n}(X)$ admits a canonical derived enhancement. By derived enhancement we mean that there exists a derived k-analytic stack, following Porta and Yu Yue approach to derived k-analytic geometry [PY17a], whose 0-truncation is naturally equivalent to $\mathrm{LocSys}_{\ell,n}(X)$.

In order to construct such derived structure on $\mathrm{LocSys}_{\ell,n}(X)$ we need to first extend its definition to derived coefficients, i.e. we need to extend $\mathrm{LocSys}_{\ell,n}(X)$ to a functor defined on the ∞-category of derived k-analytic spaces dAfd_k, such that when restricted to the full subcategory of discrete objects

$$\mathrm{Afd}_k \hookrightarrow \mathrm{dAfd}_k$$

we recover the k-analytic stack $\mathrm{LocSys}_{\ell,n}(X)$. In order to provide a correct definition of a derived enhancement of $\mathrm{LocSys}_{\ell,n}(X)$ we employ the language of enriched ∞-categories. Namely, given $Z \in \mathrm{dAfd}_k$ a derived k-affinoid a continuous representation

$$\rho\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n\big(\Gamma(Z)\big),$$

where $\mathrm{Sh}^{\text{ét}}(X)$ denotes the étale homotopy type of X and $\Gamma\colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{C}\mathrm{Alg}_k$ the derived global sections functor, corresponds to an object in the ∞-category of functors

$$\mathrm{Fun}_{\mathcal{C}\mathrm{at}_\infty(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(\Gamma(Z))\big), \tag{5.1.1.2}$$

where we interpret the ∞-category of perfect complexes $\mathrm{Perf}(\Gamma(Z))$ as enriched over $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$ in a suitable sense. We will explore these constructions in both §4 and §5. More precisely, in §4 we treat the case of continuous representations $\rho\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{Perf}(A)$ where A is a derived k°-adic algebra. Studying *derived k°-adic continuous representations* of $\mathrm{Sh}^{\text{ét}}(X)$ will prove useful in order to show that the ∞-category

$$\mathrm{Fun}_{\mathcal{C}\mathrm{at}_\infty(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(\Gamma(Z))\big) \in \mathcal{C}\mathrm{at}_\infty$$

satisfies many pleasant conditions. We deal with this in §5, where we prove new results concerning the lifting of continuous representations

$$\rho\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{B}\mathcal{E}\mathrm{nd}(\Gamma(Z))$$

to a continuous representation

$$\rho\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{B}\mathcal{E}\mathrm{nd}(A)$$

where A is a derived k°-adic algebra such that $\mathrm{Spf}(A)$ is a formal model for $Z \in \mathrm{dAfd}_k$. This is possible thanks to results concerning the existence of formal models for derived k-analytic spaces, proved in [Ant18b].

We will then show that when we restrict ourselves to the full subcategories of (5.1.1.2) spanned by rank n free modules we get the desired derived enhancement of $\mathrm{LocSys}_{\ell,n}(X)$. With this knowledge at our disposal we are able to show the following important result:

Theorem 5.1.1.3. *The k-analytic stack*

$$\mathrm{LocSys}_{\ell,n}(X)\colon \mathrm{Afd}_k^{\mathrm{op}} \to \mathcal{S}$$

admits a derived enhancement, which we denote $\mathrm{RLocSys}_{\ell,n}(X)$. Moreover, the derived moduli stack $\mathrm{RLocSys}_{\ell,n}(X)$ admits a global analytic cotangent complex. Given a $Z \in \mathrm{dAfd}_k$-point of $\mathrm{RLocSys}_{\ell,n}(X)$

$$\rho\colon Z \to \mathrm{RLocSys}_{\ell,n}(X)$$

the analytic cotangent complex of $\mathrm{RLocSys}_{\ell,n}(X)$ *is canonically equivalent to*

$$\mathbb{L}^{\mathrm{an}}_{\mathrm{RLocSys}_{\ell,n}(X),\rho} \simeq C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)^{\vee}[-1] \in \mathrm{Mod}_{\Gamma(Z)},$$

where $C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)^{\vee}$ *denotes the complex of étale cochains on the étale site of* X *with values in the derived local system*

$$\mathrm{Ad}(\rho) \coloneqq \rho \otimes \rho^{\vee}.$$

Using the main theorem [PY17a, Theorem 7.1] we are thus able to show the following second main result:

Theorem 5.1.1.4. *The functor*

$$\mathrm{RLocSys}_{\ell,n}(X) \colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S}$$

is representable by a derived k*-analytic stack whose* 0*-truncation agrees canonically with* $\mathrm{LocSys}_{\ell,n}(X)$.

5.1.2 Notations and Conventions

We shall denote k a non-archimedean field equipped with a non-trivial valuation, k° its ring of integers and sometimes we will use the letter $t \in k^\circ$ to denote a uniformizer for k. We denote An_k the category of strict k-analytic spaces and Afd_k the full subcategory spanned by strict k-affinoid spaces and we adopt the convention that whenever we mention k-affinoid or k-analytic space we mean strict k-affinoid and strict k-analytic space, respectively. We denote fSch_{k° the category of quasi-separated formal schemes over the formal spectrum $\mathrm{Spf}(k^\circ)$, where we consider k° equipped with its canonical topology induced by the valuation on k. In order to make clear that we consider formal schemes over $\mathrm{Spf}(k^\circ)$, we shall often employ the terminology k°-adic scheme to refer to formal scheme over $\mathrm{Spf}\, k^\circ$.

Let $n \geq 1$, we shall make use of the following notations:

$$\mathbb{A}^n_k \coloneqq \mathrm{Spec}\, k[T_1, \dots, T_m], \quad \mathfrak{A}^n_{k^\circ} \coloneqq \mathrm{Spf}(k^\circ\langle T_1, \dots, T_m\rangle)$$

and

$$\mathbf{A}^n_k \coloneqq \big(\mathbb{A}^n_k\big)^{\mathrm{an}}, \quad \mathbf{B}^n_k \coloneqq \mathrm{Sp}(k\langle T_1, \dots T_m\rangle),$$

where $(-)^{\mathrm{an}}$ denotes the usual analytification functor $(-)^{\mathrm{an}} \colon \mathrm{Sch}_k \to \mathrm{An}_k$, see [Ber93a]. We denote by $\mathbf{GL}^{\mathrm{an}}_n$ the analytification of the usual general linear group scheme over k, which associates to every k-affinoid algebra $A \in \mathrm{Afd}_k$ the general linear group $\mathrm{GL}_n(A)$ with A-coefficients.

In this thesis we extensively use the language of ∞-categories. Most of the times, we reason model independently, however whenever needed we prove ∞-categorical results using the theory of quasi-categories and we follows closely the notations in [Lur09b]. We use caligraphic letters $\mathcal{C}$, $\mathcal{D}$ to denote ∞-categories. We denote $\mathcal{C}\mathrm{at}_\infty$ the ∞-category of (small) ∞-categories. We will denote by $\mathcal{S}$ the ∞-category of spaces, $\mathcal{S}^{\mathrm{fc}}$ the ∞-category of finite constructible space, see [Lur09a, §3.1]. Let $\mathcal{C}$ be an ∞-category, we denote by $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ the corresponding ∞-categories of ind-objects and pro-objects on $\mathcal{C}$, respectively. When $\mathcal{C} = \mathcal{S}^{\mathrm{fc}}$, the ∞-category $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ is referred as the ∞-category of *profinite spaces*.

Let R be a derived commutative ring. We will denote by $\mathcal{C}\mathrm{Alg}_R$ the ∞-category of derived k-algebras. The latter can be realized as the associated ∞-category to the usual model category of simplicial R-algebras.

We shall denote by $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}$ the ∞-category of derived adic algebras, introduced in [Lur16, §8.1]. Whenever R admits a non-trivial adic topology, we denote $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_R \coloneqq \big(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}\big)_{R/}$ the ∞-category of *derived adic* R*-algebras*, i.e. derived R-algebras equipped with an adic topology compatible with the adic topology on R together with continuous morphisms between these.

Let R be a field. We shall denote by $\mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_R$ the ∞-category of small augmented derived R-algebras. When $R = k$ we denote by $\mathrm{AnRing}^{\mathrm{sm}}_k$ the ∞-category of small augmented derived k-analytic rings over k, which is naturally equivalent to $\mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_k$, see [Por15a, §8.2].

Let R be a discrete ring. We denote by $\mathcal{C}\mathrm{Alg}^{\heartsuit}_R$ the 1-category of ordinary commutative rings over R. When R admits an adic topology we shall denote $\mathcal{C}\mathrm{Alg}^{\mathrm{ad},\heartsuit}_R \subseteq \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_R$ the full subcategory spanned by discrete derived adic R-algebras. Let R denote a derived ring. We denote Mod_R the derived ∞-category of R-modules and $\mathrm{Coh}^+(X) \subseteq \mathrm{Mod}_R$ the full subcategory spanned by those almost perfect R-modules.

We need sometimes to enlarge the starting Grothendieck universe, and we often do not make explicit such it procedure. Fortunately, this is innocuous for us. Given $Z \in \mathrm{dAfd}_k$ a derived k-affinoid space and $M \in \mathrm{Coh}^+(Z)$

an almost perfect sheaf on Z its mapping space of endomorphisms $\mathrm{End}(M) \in \mathcal{S}$ admits a natural enrichment over the ∞-category $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$. We shall denote such enrichement by $\mathcal{E}\mathrm{nd}(M)$. We will employ the same notation whenever $M \in \mathrm{Coh}^+(A)$ where $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. Namely, for such $M \in \mathrm{Coh}^+(A)$ we denote $\mathcal{E}\mathrm{nd}(M)$ the $\mathbb{E}_1$-monoid like object on the ∞-category $\mathrm{Pro}(\mathcal{S})$.

We will denote by $\big(\mathrm{dAff}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$ the algebraic geometric context and we denote by $\mathrm{dSt}\big(\mathrm{dAff}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$ the ∞-category of derived geometric stacks with respect to $\big(\mathrm{dAff}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$. Similary, whenever k denotes either the field $\mathbb{C}$ of complex numbers or a non-archimedean field we will denote by $\big(\mathrm{dAff}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$ the analytic geometric context and correspondingly $\mathrm{dSt}\big(\mathrm{dAn}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$ the ∞-category of derived geometric stacks with respect to the analytic geometric context.

5.1.3 Acknowledgments

I would like to express my deep gratitude to my advisor B. Toen for all his advice and by sharing many ideas and suggestions concerning the contents of this text. I am also in great debt to M. Porta from which I learned derived non- archimedean geometry. I am thankful to Mauro Porta also for so many useful mathematical suggestions along these last years. I would also like to thank to M. Robalo, A. Vezzani, B. Hennion, D. Gapner, M. d'Addezio, V. Daruvar, C. Simpson, V. Melani, P. Scholze, B. Bhatt, B. Conrad, B. Stroh, and J. Tapia for useful commentaries and suggestions during the elaboration of the text. I would also like to express my gratitude to the establishment Université Paris 13 and Université de Strasbourg, where I could discuss and share many of the ideas presented in this text.

5.2 Representability of the space of morphisms

Let G be a profinite group topologically of finite generation. One can consider the functor

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G)\colon \mathrm{Afd}_k^{\mathrm{op}} \to \mathrm{Set}$$

given on objects by the formula

$$A \in \mathrm{Afd}_k \mapsto \mathrm{Hom}_{\mathrm{cont}}\big(G, \mathrm{GL}_n(A)\big) \in \mathrm{Set},$$

where $\mathrm{Hom}_{\mathrm{cont}}\big(G, \mathrm{GL}_n(A)\big)$ denotes the set of continuous group homomorphisms

$$G \to \mathrm{GL}_n(A),$$

where we consider $\mathrm{GL}_n(A)$ as a topological group via the induced topology on $A \in \mathrm{Afd}_k$. We will prove that $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G)$ is representable by a k-analytic space, i.e.

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(X) \in \mathrm{An}_k.$$

The proof of representability is established first when G is a free profinite group. This is the main result of the section. The case where G is a more general topologically finitely generated profinite group follows directly from the case of topologially free profinite groups.

Our main motivation to study $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G)$ follows from the fact that it forms a smooth atlas of the moduli of continuous representations of G, which we shall designate the latter by $\mathrm{LocSys}_{\ell,n}(G)$. One can show that $\mathrm{LocSys}_{\ell,n}(G)$ is equivalent to the "stack-quotient" of $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G)$ by its natural action of the k-analytic general linear group $\mathbf{GL}_n^{\mathrm{an}}$ under conjugation.

Furthermore, the representability of $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G)$ entails the representability of $\mathrm{LocSys}_{\ell,n}(G)$ as a geometric stack with respect to the k-analytic context. We shall prove this latter assertion in §2.3 and review the main basic notions concerning k-analytic stacks.

5.2.1 Preliminaries

This § reviews the basic notions of k-analytic geometry that we will use more often.

Definition 5.2.1.1. Let $n \geq 1$ be an integer. The *Tate k-algebra on n generators with radius* $(r_1, \dots, r_n)$ is defined as

$$k\langle r_1^{-1}T_1, \dots, r_n^{-1}T_n \rangle := \{ \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} T_1^{i_1} \dots T_n^{i_n} \in k[[T_1, \dots T_n]] | \, a_{i_1,\dots,i_n} r_1^{i_1} \dots r_n^{i_n} \to 0 \},$$

whose multiplicative structure is induced by the multiplicative structure on the formal power series ring $k[[T_1, \dots T_n]]$.

Definition 5.2.1.2. A k-affinoid algebra is a quotient of a Tate algebra $k\langle r_1^{-1}T_1, \dots, r_n^{-1}T_n \rangle$ by a finitely generated ideal I.

Definition 5.2.1.3. Let A be a k-affinoid algebra we say that A is strict k-affinoid if we can choose such a presentation for A with the $r_i = 1$, for each i. We denote by $\mathrm{Afd}_k^{\mathrm{op}}$ the category of strict k-affinoid algebras together with continuous k-algebra homomorphisms between them.

Remark 5.2.1.4. The k-algebra $k\langle r_1^{-1}T_1, \dots, r_n^{-1}T_n \rangle$ admits a canonical k-Banach structure induced by the usual Gauss norm. Moreover, any finitely generated ideal $I \subset k\langle r_1^{-1}T_1, \dots, r_n^{-1}T_n \rangle$ is closed which implies that any k-affinoid algebra A admits a k-Banach structure, depending on the choice of a presentation of A. Nonetheless it is possible to show that any two such k-Banach structures on A are equivalent and therefore the latter inherits a canonical topology, induced from the one on $k\langle r_1^{-1}T_1, \dots, r_n^{-1}T_n \rangle$ given by the Gauss norm.

Strict k-affinoid algebras correspond to the affine objects in (rigid) k-analytic geometry. Therefore, we define the category of k-affinoid spaces as

$$\mathrm{Afd}_k := (\mathrm{Afd}_k^{\mathrm{op}})^{\mathrm{op}}.$$

Remark 5.2.1.5. Let $A \in \mathrm{Afd}_k^{\mathrm{op}}$ denote a k-affinoid algebra. The given of a presentation of A of the form

$$A \cong k\langle T_1, \dots, T_m \rangle / I$$

determines a formal model for A, i.e a p-complete k°-adic algebra of topological finite presentation A_0 such that

$$A \simeq A_0 \otimes_{k^\circ} k.$$

in the category of k-algebras. One can simply take A_0 to be

$$A_0 := k^\circ \langle T_1, \dots T_m \rangle / I \cap k^\circ \langle T_1, \dots T_m \rangle.$$

Definition 5.2.1.6. Given a k-affinoid alge bra A we denote by $\mathrm{M}(A)$ the set of semi-multiplicative seminorms on A. Given $x \in \mathrm{M}(A)$ we can associate it a (closed) prime ideal of A. Namely, it corresponds to the kernel of $x \colon A \to \mathbb{R}$, $\ker(x) \subseteq A$. The fact that it defines a prime ideal of A follows from multiplicativity of $x \in \mathrm{M}(A)$.

Notation 5.2.1.7. We denote by $\mathrm{H}(x)$ the completion of the residue field $\mathrm{Frac}(A/\mathfrak{p})$, where $\mathfrak{p} := \ker(x)$. The field $\mathrm{H}(x)$ possesses a canonical valuation, denoted $| \bullet |_x$, induced by the one on A and given $a \in A$ we denote by $|a|_x \in \mathbb{R}$ the evaluation of $| \bullet |_x$ on the image of a in $\mathrm{H}(x)$.

In Berkovich's non-archimedean geometry it is possible to define the notion of relative interior, which is very useful in practice. Let

$$\phi : A \to A'$$

denote a bounded morphism of k-affinoid algebras. The *relative interior* of ϕ, denote $\mathrm{Int}(\mathrm{M}(A')/\mathrm{M}(A))$ is by definition the set of points,

$$\mathrm{Int}\big(\mathrm{M}(A')/\mathrm{M}(A)\big) := \{x' \in \mathrm{M}(A') | A' \to \mathrm{H}(x') \text{ is inner with respect to } A\},$$

where inner with respect to A means that there exist a continuous surjective map

$$A\langle r_1^{-1}T_1, \dots, r_n^{-1}T_n \rangle \to A'$$

of k-affinoid algebras which induces a norm on A' equivalent to its original one and such that, for each i, we have $|T_i|_{x'} < r_i$.

Definition 5.2.1.8. One can glue k-affinoid spaces, as in algebraic geometry. A *k-analytic space* is defined as a locally ringed space which locally is equivalent to a k-affinoid space. We denote by An_k the category of k-analytic spaces and morphisms between these.

One then is able to globalize most of the previous notions, in particular it is possible to give a global definition of the relative interior of a morphism between k-analytic spaces. We refer the reader to [Ber93a], [Con08a] and [Bos05] for a more detailed exposition on rigid geometry, from different points of view.

5.2.2 Hom spaces

Let G denote a profinite group of topological finite type which we fix throughout this §. Consider the functor

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}} \colon \mathrm{Afd}_k^{\mathrm{op}} \to \mathrm{Set}$$

given informally on objects by the formula

$$A \mapsto \mathrm{Hom}_{\mathrm{cont}}\big(G, \mathrm{GL}_n(A)\big),$$

where $\mathrm{Hom}_{\mathrm{cont}}$ denotes the set of morphisms in the category of continuous groups, and we consider $\mathrm{GL}_n(A)$ with the topology induced by the topology of A viewed as a k-affinoid algebra.

Our goal in this section is to show that

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(G) \in \mathrm{Fun}\big(\mathrm{Afd}_k^{\mathrm{op}}, \mathrm{Set}\big)$$

is representable by a k-analytic space. Let $A \in \mathrm{Afd}_k^{\mathrm{op}}$ be a (strictly) k-affinoid algebra.

Notation 5.2.2.1. We will typically denote by A_0 a formal model for A, i.e., a (p-adic complete) k°-algebra of topological finite presentation such that we have an isomorphism

$$A_0 \otimes_{k^\circ} k \simeq A.$$

Remark 5.2.2.2. By choice of A_0, we conclude that A_0 can be identified with an open subring of A. For this reason, the topology of A can be thought as an ind-pro topology, in which the pro-structure comes from the fact that formal models are p-adic complete and the ind-structure arises after localizing at p.

Remark 5.2.2.3. Fix a formal model A_0 for A, as above. The topology on A_0 admits the family $\{\pi^n A_0\}_{n \geq 1}$ as a fundamental family of open neighborhoods around $0 \in A_0$. Consequently, for $k \geq 0$, we have a fundamental family of normal open subgroups

$$\mathrm{Id} + p^{k+1} \cdot \mathrm{M}_n(A_0) \trianglelefteq \mathrm{GL}_n(A_0).$$

These form a basis of normal open subgroups for the topology on $\mathrm{GL}_n(A_0)$ induced by A_0. We have moreover canonical isomorphisms

$$\mathrm{GL}_n(A_0)/\big(\mathrm{Id} + p^k M_n(A_0)\big) \simeq \mathrm{GL}_n(A_0/p^k A_0).$$

We have thus a canonical isomorphism

$$\mathrm{GL}_n(A_0) \cong \lim_{k \geq 1} \big(\mathrm{GL}_n(A_0)/\big(\mathrm{Id} + p^k M_n(A_0)\big)\big).$$

Thus it is p-adically complete. The same reasoning holds for the topological group $\mathrm{Id} + p^k \cdot \mathrm{M}_n(A_0)$, for each $k \geq 1$. More concretely, we have isomorphisms

$$\mathrm{Id} + p^k \cdot \mathrm{M}_n(A_0) \cong \lim_{m \geq 1} \big(\mathrm{Id} + p^k \cdot \mathrm{M}_n(A_0)/(\mathrm{Id} + p^{k+m} \cdot \mathrm{M}_n(A_0)\big).$$

Notation 5.2.2.4. We denote by $\widehat{\mathrm{F}}_r$ a fixed free profinite group of rank r. It can be explicitly realized as the profinite completion of a free group on r generators, F_r. The latter can be realized as a dense full subgroup of $\widehat{\mathrm{F}}_r$. We will thus fix throughout the text a continuous dense group inclusion homomorphism $\mathrm{F}_r \to \widehat{\mathrm{F}}_r$ and a set of generators $e_1, \dots, e_r \in \mathrm{F}_r$ which become topological generators of the profinite group $\widehat{\mathrm{F}}_r$.

Remark 5.2.2.5. Let FinGrp denote the category of finite groups. The category of profinite group corresponds to its pro-completion, $\mathrm{Pro}(\mathrm{FinGrp})$. For each $r \geq 1$, the groups $\widehat{\mathrm{F}}_r \in \mathrm{Pro}(\mathrm{FinGrp})$ satisfy the universal property given by the formula

$$\mathrm{Hom}_{\mathrm{Pro}(\mathrm{FinGrp})}(\widehat{\mathrm{F}}_r, G) \cong G^r, \quad \text{for any } G \in \mathrm{Pro}(\mathrm{FinGrp}).$$

Notation 5.2.2.6. Let us fix $\mathcal{J}_r$ a final family of normal open subgroups of finite index in $\widehat{\mathrm{F}}_r$, i.e., such that we have a continuous group isomorphism,

$$\lim_{U \in \mathcal{J}_r} \widehat{\mathrm{F}}_r / U \simeq \widehat{\mathrm{F}}_r.$$

Remark 5.2.2.7. Given $U \in \mathfrak{J}_r$, the quotient group

$$\widehat{\mathrm{F}}_r / U \simeq \Gamma$$

is finite and therefore of finite presentation. It follows that U admits a finite family of generators $\sigma_1, \ldots, \sigma_l$. Moreover, thanks to the Nielsen-Schreier theorem the group U is topologically finitely generated free profinite. Consider furthermore the dense group inclusion homomorphism

$$\mathrm{F}_r \to \widehat{\mathrm{F}}_r,$$

then $U \cap \mathrm{F}_r \to U$ is a discrete subgroup of U which is again dense in U. Therefore, we can assume without loss of generality that $\sigma_1, \ldots, \sigma_l \in U \cap \mathrm{F}_r$.

Notation 5.2.2.8. Let $\sigma = \prod_i e_i^{n_{j_i}} \in \widehat{\mathrm{F}}_r$ be a general element of the profinite group $\widehat{\mathrm{F}}_r$. Suppose furthermore we are given a group morphism

$$\rho \colon \widehat{\mathrm{F}}_r \to \mathrm{GL}_n(A)$$

such that $M_1 \coloneqq \rho(e_1), \ldots, M_r \coloneqq \rho(e_r)$. We denote by

$$\sigma\big(M_1, \ldots M_r\big) \coloneqq \prod_i M_i^{n_{j_i}} \in \mathrm{GL}_n(A),$$

whenever the right hand side is well defined, (which is always the case when the product on the left hand side is indexed by a finite set).

Definition 5.2.2.9. Let $U \in \mathfrak{J}_r$ and fix $\sigma_1, \ldots, \sigma_l \in U \cap \widehat{\mathrm{F}}_r$ a finite number of topological generators for U. We define the functor

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(U, \sigma_1, \ldots, \sigma_l) \colon \mathrm{Afd}_k \to \mathrm{Set},$$

given on objects by the formula,

$$\begin{aligned} A \mapsto & \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(U, \sigma_1, \ldots, \sigma_l)(A) \\ & \coloneqq \{(M_1, \ldots, M_r) \in \mathrm{GL}_n(A)^r \colon \text{for each } i \in [1, l],\ |\sigma_i(M_1, \ldots, M_r) - \mathrm{Id}| \leq |p|\}. \end{aligned}$$

When A_0 is a formal model for A, we denote by $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(U, \sigma_1, \ldots, \sigma_l)(A_0)$ the set of those $(M_1, \ldots, M_r) \in \mathrm{GL}_n(A_0)^r$ such that the mod p reduction

$$\sigma(M_1, \ldots, M_r) = \mathrm{Id}, \quad \mathrm{mod}\, p$$

Remark 5.2.2.10. Let $U \in \mathfrak{J}_r$, $A \in \mathrm{Afd}_k^{\mathrm{op}}$ and A_0 be a formal model for A. Then the set

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(U, \sigma_1, \ldots, \sigma_l)(A_0) \in \mathrm{Set}$$

does not depend on the choice of the topological generators for U. More precisely, if $\tau_1, \ldots \tau_s \in U \cap F_r$ denote a different choice of topological generators for U, we have a natural bijection of sets

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(U, \sigma_1, \ldots, \sigma_l)(A_0) = \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(U, \tau_1, \ldots, \tau_s)(A_0).$$

In order to see this, it suffices to note that, for each $n \geq 1$, the mod p^n reduction of

$$(M_1, \ldots, M_r) \in \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(U, \sigma_1, \ldots, \sigma_l)(A_0/\pi^n A_0)$$

corresponds to a group homomorphism

$$\Gamma \coloneqq \widehat{\mathrm{F}}_r / U \to \mathrm{GL}_n(A_0/\pi A_0).$$

As group homomorphisms are independent of the choice of presentation for Γ the set

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(U, \sigma_1, \ldots, \sigma_l)(A_0/\pi^n) \in \mathrm{Set}$$

does not depend on such choices, either. As A_0 is π-adic complete,, passing to inverse limits we deduce that

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(U, \sigma_1, \ldots, \sigma_l)(A_0)$$

is independent of the choice of topological generators for U, as desired.

Notation 5.2.2.11. Following Theorem 5.2.2.10 we will denote the set $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(U, \sigma_1, \ldots, \sigma_l)$ simply by

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(U)(A_0) \in \mathrm{Set}.$$

Lemma 5.2.2.12. *Let $A \in \mathrm{Afd}_k^{\mathrm{op}}$ be an k-affinoid algebra and A_0 an k°-formal model for A. Then there is a bijection*

$$\mathrm{Hom}_{\mathrm{cont}}\big(\widehat{\mathrm{F}}_r, \mathrm{GL}_n(A_0)\big) \cong \operatorname*{colim}_{U \in \mathfrak{J}_r} \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U\big)(A_0),$$

of sets, for each $r \geq 1$.

Proof. Since

$$\mathrm{Hom}_{\mathrm{cont}}\big(\widehat{\mathrm{F}}_r, \mathrm{GL}_n(A_0))\big)$$

denotes the set of continuous group homomorphisms in the category of pro-discrete groups we have a bijection

$$\mathrm{Hom}_{\mathrm{cont}}\big(\widehat{\mathrm{F}}_r, \mathrm{GL}_n(A_0)\big) \cong \lim_k \operatorname*{colim}_{\Gamma \in \mathscr{J}_r} \mathrm{Hom}_{\mathrm{Grp}}\big(\Gamma, \mathrm{GL}_n(A_0/p^{k+1}A_0)\big).$$

of sets. It therefore suffices to show that we have a bijection,

$$\lim_k \operatorname*{colim}_{U \in \mathfrak{J}_r} \mathrm{Hom}_{\mathrm{Grp}}\big(\Gamma_U, \mathrm{GL}_n(A_0/p^{k+1}A_0)\big) \cong \operatorname*{colim}_{U \in \mathfrak{J}_r} \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U\big)(A_0),$$

where Γ_U denotes the finite group $\widehat{\mathrm{F}}_r/U$. We assert that there exists a canonical morphism,

$$\phi : \lim_k \operatorname*{colim}_{U \in \mathfrak{J}_r} \mathrm{Hom}_{\mathrm{Grp}}\big(\Gamma_U, \mathrm{GL}_n(A_0/p^{k+1}A_0)\big) \to \operatorname*{colim}_{U \in \mathfrak{J}_r} \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U\big)(A_0).$$

In order to prove this assertion we observe that a group morphism

$$\rho_k : \Gamma_{U_k} \to \mathrm{GL}_n(A_0/p^{k+1}A_0),$$

with $U_k \in \mathfrak{J}_r$, is determined by the image of the r generators of Γ_{U_k} which correspond to r matrices in $\mathrm{GL}_n(A_0/p^{k+1}A_0)$. Therefore given such a system of compatible group homomorphisms $\{\rho_k\}_k$ one can associate an r-vector $(M_1, \ldots, M_r) \in \mathrm{GL}_n(A_0)^r$ such that its mod p reduction satisfies

$$\sigma_i(M_1, \ldots, M_r) = \mathrm{Id},$$

where $\sigma_1, \ldots, \sigma_l \in U \cap F_r$ denotes a choice of a finite set of topological generators for U_1. Thus

$$(M_1, \ldots, M_r) \in \mathscr{X}_{U_1}(A_0).$$

This shows the existence of the desired map. We now construct maps

$$\psi_U : \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U\big)(A_0) \to \lim_k \operatorname*{colim}_{U' \in \mathfrak{J}_r} \mathrm{Hom}_{\mathrm{Grp}}\big(\Gamma, \mathrm{GL}_n(A_0/p^{k+1}A_0)\big),$$

for each $U \in \mathfrak{J}_r$, such that when we assemble these together we obtain the desired inverse for ϕ. In order to construct ψ_U, we start by fixing topological generators

$$\sigma_1, \ldots, \sigma_l \in U \cap F_r$$

for U. Let $(M_1, \ldots, M_r) \in \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U\big)(A_0)$. As we have seen these matrices define a continuous group homomorphism

$$\widehat{\mathrm{F}}_r \to \mathrm{GL}_n(A_0/pA_0).$$

Thanks to Theorem 5.2.2.13 below the matrices

$$\sigma_1(M_1, \ldots, M_r), \ldots, \sigma_l(M_1, \ldots, M_r) \in \mathrm{Id} + p\mathrm{M}_n(A_0)$$

determine a continuous group homomorphism

$$\rho_1 : U \to \mathrm{Id} + p\mathrm{M}_n(A_0).$$

Then the inverse image

$$U_2' := \rho_1^{-1}\big(\mathrm{Id} + p^2 \mathrm{M}_n(A_0)\big)$$

is an open normal subgroup of U of finite index. As U itself is an open subgroup of $\widehat{\mathrm{F}}_r$ of finite index we conclude that U_2' is also a finite index subgroup of $\widehat{\mathrm{F}}_r$. As open normal subgroup of finite index in $\widehat{\mathrm{F}}_r$ define a final family for $\widehat{\mathrm{F}}_r$ we conclude that there exists $U_2 \in \mathfrak{J}_r$ such that $\rho_1(U_2)$ is a subgroup of $\mathrm{Id} + p^2\mathrm{M}_n(A_0)$. Consequently, the matrices $(M_1, \dots, M_r) \in \mathscr{X}_U(A_0)$ define a group homomorphism

$$\rho_2 \colon \widehat{\mathrm{F}}_r / U_2 \to \mathrm{GL}_n(A_0/p^2 A_0).$$

By iterating the process we obtain a sequence of continuous group homomorphisms

$$\{\rho_i : \widehat{\mathrm{F}}_r/U_i \to \mathrm{GL}_n(A_0/p^i A_0)\}_i \in \lim_i \operatorname*{colim}_{U \in \mathfrak{J}_r} \mathrm{Hom}_{\mathrm{grp}}(\Gamma_U, \mathrm{GL}_n(A_0/p^i A_0)).$$

Assembling these together we obtain a continuous group homomorphism $\rho \in \mathrm{Hom}_{\mathrm{cont}}(\widehat{\mathrm{F}}_r, \mathrm{GL}_n(A_0)$. It follows easily by our construction that,

$$\operatorname*{colim}_{U \in \mathfrak{J}_r}(\psi_U) \colon \operatorname*{colim}_{U \in \mathfrak{J}_r} \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U\big)(A_0) \to \mathrm{Hom}_{\mathrm{cont}}\big(\widehat{\mathrm{F}}_r, \mathrm{GL}_n(A_0)\big),$$

is the inverse map of ϕ, as desired. □

Lemma 5.2.2.13 (Burnside problem for topologically nilpotent p-groups)**.** *Let $A \in \mathrm{Afd}_k^{\mathrm{op}}$ and A_0 be a formal model for A. For each $k \geq 1$, we have a natural bijection*

$$\mathrm{Hom}_{\mathrm{cont}}\big(\widehat{\mathrm{F}}_r, \mathrm{Id} + p^{k+1}\mathrm{M}_n(A_0)\big) = \mathrm{Id} + p^{k+1}\mathrm{M}_n(A_0).$$

Proof. Noticing that the quotient groups,

$$\big(\mathrm{Id} + p^k \mathrm{M}_n(A_0)/(\mathrm{Id} + p^{k+m+1}\mathrm{M}_n(A_0)\big),$$

are torsion, i.e., every element has finite order and we conclude that

$$\mathrm{Hom}_{\mathrm{cont}}\big(\widehat{\mathbb{Z}}, \mathrm{Id} + p^{k+1}\mathrm{M}_n(A_0)\big) = \mathrm{Id} + p^{k+1}\mathrm{M}_n(A_0),$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$. This finishes the proof when $r = 1$. The same holds for general $\widehat{\mathrm{F}}_r$, i.e., we have a canonical equivalence,

$$\mathrm{Hom}_{\mathrm{cont}}\big(\widehat{\mathrm{F}}_r, \mathrm{Id} + p^{k+1}\mathrm{M}_n(A_0)\big) = \big(\mathrm{Id} + p^{k+1}\mathrm{M}_n(A_0)\big)^r.$$

In order to prove this last assertion it suffices to show that any finitely generated subgroup of the quotient

$$\big(\mathrm{Id} + p^k \mathrm{M}_n(A0)/(\mathrm{Id} + p^{k+m+1}\mathrm{M}_n(A_0)\big),$$

for some positive integer $m \geq 1$, is finite (i.e. the Burnside problem admits an affirmative answer in this particular case). In order to justify the given assertion we fix G a finitely generated subgroup of

$$\big(\mathrm{Id} + p^k \mathrm{M}_n(A_0)/(\mathrm{Id} + p^{k+l+1}\mathrm{M}_n(A_0)\big).$$

By assumption it is generated by matrices of the form $\mathrm{Id} + p^{k+1}N_1, \dots, \mathrm{Id} + p^{k+1}N_s$, for some $s \geq 1$. Therefore a general element of G can be written as,

$$\mathrm{Id} + p^{k+1}(n_{1,1}N_1 + \dots + n_{1,s}N_s) + \dots + p^{k+l-1}(n_{a-1,1}N_{a-1,1} + \dots + n_{a-1,s^{l-1}}N_{a-1,s^{l-1}}),$$

where the $N_{i,j}$, for $i, j \in [1, a-1] \times [1, s^{a-1}]$, denote products of the N_i having $a-1$ multiplicative terms, where a denotes the least integer such that $k \times (a+1) \geq l$. By the Pigeonhole principle there are only finite number of such choices for the integers $n_{i,j}$ for $(i, j) \in [1, l-1] \times [1, s^{l-1}]$ and the result follows. □

Proposition 5.2.2.14. *Let $A \in \mathrm{Afd}_k^{\mathrm{op}}$ be an k-affinoid algebra then we have a natural bijection,*

$$\mathrm{Hom}_{\mathrm{cont}}\big(\widehat{\mathrm{F}}_r, \mathrm{GL}_n(A)\big) \simeq \underset{U \in \mathcal{J}_r, \sigma_1, \ldots, \sigma_l \text{ generators}}{\mathrm{colim}} \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U, \sigma_1, \ldots, \sigma_l\big)(A).$$

Proof. Let $\rho : \widehat{\mathrm{F}}_r \to \mathrm{GL}_n(A)$ be a continuous homomorphism of topological groups and let $e_1, \ldots, e_r$ be the fixed topological generators of $\widehat{\mathrm{F}}_r$. Let

$$M_i := \rho(e_i) \in \mathrm{GL}_n(A)$$

for each $1 \leq i \leq r$. The group $\mathrm{Id} + p \cdot \mathrm{M}_n(A_0)$ is open in $\mathrm{GL}_n(A_0)$ and the latter open in $\mathrm{GL}_n(A)$. We thus deduce that the inverse image

$$U := \rho^{-1}\big(\mathrm{Id} + p\mathrm{M}_n(A_0)\big)$$

is an open subgroup of $\widehat{\mathrm{F}}_r$ and it has thus finite index in $\widehat{\mathrm{F}}_r$, moreover as $\mathcal{J}_r$ is a final family for $\widehat{\mathrm{F}}_r$ one can suppose without loss of generality, up to shrinking U, that $U \in \mathcal{J}_r$ and thus normal in $\widehat{\mathrm{F}}_r$. Choosing a finite set of topological generators for $\rho^{-1}\big(\mathrm{Id} + \pi \cdot \mathrm{M}_n(A_0)\big)$ we deduce that the $(M_1, \ldots, M_r)$ satisfy the inequalities, in $\mathrm{GL}_n(A)$, associated to such generators, therefore

$$(M_1, \ldots, M_r) \in \underset{U \in \mathscr{J}_r, \text{generators } \sigma_1, \ldots, \sigma_l}{\mathrm{colim}} \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U, \sigma_1, \ldots, \sigma_l\big)(A)$$

which proves the direct inclusion. We conclude that the association

$$\rho \in \mathrm{Hom}_{\mathrm{cont}}(\widehat{\mathrm{F}}_r, \mathrm{GL}_n(A)) \mapsto (\rho(e_1), \ldots, \rho(e_r)) \in \underset{U \in \mathcal{J}_r, \sigma_1, \ldots, \sigma_l \text{ generators}}{\mathrm{colim}} \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U, \sigma_1, \ldots, \sigma_l\big)(A),$$

defines a well defined map of sets. Let us prove that we have a well defined inverse map. We consider

$$(M_1, \ldots, M_r) \in \mathrm{GL}_n(A)^r$$

such that $\sigma_i(M_1, \ldots, M_r) \in \mathrm{Id} + \pi \cdot M_n(A_0)$ for a finite family $\{\sigma_i\}_{i \in [1,l]}$, all lying in the dense subgroup $F_r \subset \widehat{\mathrm{F}}_r$, of topological generators of a finite index normal open subgroup of $\widehat{\mathrm{F}}_r$, which we shall denote by U.

We remark that U is a free profinite by the version of Nielsen-Schreier theorem for open subgroups of free profinite groups, see [RS08, Theorem 3.3.1]. By Theorem 5.2.2.13 we conclude that

$$(\sigma_1(M_1, \ldots, M_r), \ldots, \sigma_l(M_1, \ldots, M_r)) \in \mathrm{GL}_n(A)^r$$

defines a continuous group homomorphism

$$\bar{\rho} : U \simeq \hat{F}_l^{\mathrm{pf}} \to \mathrm{Id} + p\mathrm{M}_n(A_0).$$

Therefore, we have the following diagram in the category of topological groups,

$$\begin{tikzcd}
F_r \arrow[r] \arrow[d, "{(M_1,\ldots,M_r)}"] & \widehat{\mathrm{F}}_r & U \arrow[l] \arrow[d] \\
\mathrm{GL}_n(A) \arrow[r, "="] & \mathrm{GL}_n(A) & \mathrm{Id} + p\mathrm{M}_n(A_0). \arrow[l]
\end{tikzcd}$$

We want to show that we can fill the above diagram with a continuous morphisms $\widehat{\mathrm{F}}_r \to \mathrm{GL}_n(A)$ making the whole diagram commutative. Since U is of finite index in $\widehat{\mathrm{F}}_r$ we can choose elements

$$g_1, \ldots g_m \in F_r \subset \widehat{\mathrm{F}}_r$$

such that these form a (faithful) system of representatives for the finite group $\widehat{\mathrm{F}}_r/U$. For $i \in [1, m]$ write

$$g_i := \prod_{j_i} e_{j_i}^{n_{j_i}},$$

where this product is finite and unique by the assumption that the $g_i \in F_r$. Every element of $h \in \widehat{\mathrm{F}}_r$ can be written as $h = g_i\sigma$, for some g_i as above and $\sigma \in U$. Let us then define

$$\rho(h) := \big(\prod_{j_i} M_{j_i}^{n_{j_i}}\big)\bar{\rho}(\sigma) \in \mathrm{GL}_n(A).$$

We are left to verify that the association

$$h \in \widehat{\mathrm{F}}_r \mapsto \bar{\rho}(h) \in \mathrm{GL}_n(A)$$

gives a well defined continuous group homomorphism. Let

$$g := \prod_s e_s^{n_s} \in F_r \subset \widehat{\mathrm{F}}_r$$

and $\sigma' \in U$ such that $g\sigma' = h = g_i\sigma$. We first prove that

$$\bar{\rho}(h) = \big(\prod_s M_s^{n_s}\big)\bar{\rho}(\sigma').$$

Suppose that $\sigma, \sigma' \in U \cap F_r$, then it follows that $h \in F_r$. The result then follows in this case, since we have fixed a group homomorphism

$$(M_1, \dots, M_r) \colon F_r \to \mathrm{GL}_n(A),$$

which is necessarily continuous. Suppose then that it is not the case that

$$\sigma, \sigma' \in U \cap F_r.$$

Let $(\sigma_n)_n$ and $(\sigma_{n'})_{n'}$ be sequences of elements in $U \cap F_r$ converging to σ and σ', respectively. We observe that this is possible since F_r is dense in $\widehat{\mathrm{F}}_r$ and $U \cap F_r$ is a free (discrete) group whose profinite completion is canonically equivalent to U, thus dense in U. For this reason, we obtain that

$$g^{-1}g_i\sigma = \sigma'$$

and we get moreover that $g^{-1}g_i\sigma_n$ converges to σ'. Thus the elements

$$(\prod_s M_{s^{-1}}^{-n_{s^{-1}}})(\prod_{j_i} M_{j_i}^{n_{j_i}})\rho(\sigma_m) \in \mathrm{GL}_n(A)\ ,$$

where our notations are clear from the context, converge to $\rho(\sigma')$ by continuity of ρ. They also converge to the element

$$(\prod_s M_{s^{-1}}^{-n_{s^{-1}}})(\prod_{j_i} M_{j_i}^{n_{j_i}}) \in \mathrm{GL}_n(A)$$

by continuity of the group multiplication on $\mathrm{GL}_n(A)$. Since the topology on A comes from a norm on A, making the latter a Banach k-algebra we conclude that A is Hausdorff and so it is $\mathrm{GL}_n(A)$. This implies that converging sequences in $\mathrm{GL}_n(A)$ admit a unique limit. We conclude therefore that,

$$\rho(\sigma') = (\prod_s M_{s^{-1}}^{-n_{s^{-1}}})(\prod_{j_i} M_{j_i}^{n_{j_i}})\rho(\sigma),$$

thus giving the desired equality,

$$(\prod_s M_s^{n_s})\rho(\sigma') = (\prod_{j_i} M_{j_i}^{n_{j_i}})\rho(\sigma),$$

proving that $\overline{\rho} \colon \widehat{\mathrm{F}}_r \to \mathrm{GL}_n(A)$ is a well defined map. We wish to show that it is a continuous group homomorphism. Our definitions make clear that to check multiplicativity of $\overline{\rho}$ it suffices to show that for every $g \in F_r$ and $sigma \in U$ we have,

$$\overline{\rho}(g\sigma g^{-1}) = \overline{\rho}(g)\rho(\sigma)\overline{\rho}(g^{-1}).$$

Pick again a converging sequence $(\sigma_n)_n$, in $F_r \cap U$, such that σ_n converges to σ. Then for each n we have,

$$\overline{\rho}(g\sigma_n g^{-1}) = \overline{\rho}(g)\rho(\sigma_n)\overline{\rho}(g^{-1}),$$

and by passing to the limit, we obtain the desired equality. We are reduced to show that $\overline{\rho}$ is continuous. Let V be an open subset of $\mathrm{GL}_n(A)$. The intersection $V \cap (\mathrm{Id} + pM_n(A_0)$ is open in $\mathrm{GL}_n(A)$. Thus,

$$\overline{\rho}^{-1}(V \cap (\mathrm{Id} + pM_n(A_0) = \rho^{-1}(V \cap (\mathrm{Id} + pM_n(A_0))$$

is open in U. Therefore, the quotient $U/(V \cap (\mathrm{Id} + pM_n(A_0))$ is discrete, since U is of finite index in $\widehat{\mathrm{F}}_r$. We conclude that,

$$U/\rho^{-1}(V \cap (\mathrm{Id} + pM_n(A_0)) \to \widehat{\mathrm{F}}_r/\overline{\rho}^{-1}(V)$$

exhibits the quotient

$$U/\rho^{-1}(V \cap (\mathrm{Id} + pM_n(A_0))$$

as a subgroup of finite index in $\widehat{\mathrm{F}}_r/\overline{\rho}^{-1}(V)$. Thus the latter is necessarily discrete. The result now follows, since we have that $\overline{\rho}^{-1}(V)$ is an open subset in $\widehat{\mathrm{F}}_r$. □

Notation 5.2.2.15. We will denote

$$\begin{aligned}\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)(A) &:= \operatorname*{colim}_{U \in \mathcal{J}_r, \sigma_1, \dots, \sigma_l} \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U, \sigma_1, \dots, \sigma_l\big)(A) \\ &\cong \mathrm{Hom}_{\mathrm{cont}}\big(\widehat{\mathrm{F}}_r, \mathrm{GL}_n(A)\big).\end{aligned}$$

Remark 5.2.2.16. Let $\mathbf{GL}_n^{\mathrm{an}}$ denote the analytification of the general linear group scheme GL_n over $\mathrm{Spec} k$. Proposition 5.2.2.14 allows us to write $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)$ as a union of subfunctors, $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(U, \sigma_1, \dots, \sigma_l)$. Where, for each $U \in \mathcal{J}_r$ and $\sigma_1, \dots, \sigma_l$, $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(U, \sigma_1, \dots, \sigma_l)$ is given on the objects of Afd_k by the formula,

$$\begin{aligned}&A \mapsto \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U, \sigma_1, \dots, \sigma_l\big)(A) \\ &\quad := \{(M_1, \dots, M_r) \in \mathrm{GL}_n(A)^r : \text{ such that for each } i \in [1, l], |\sigma_i(M_1, \dots, M_r) - \mathrm{Id}| \leq |\pi|\}.\end{aligned}$$

Lemma 5.2.2.17. *The functor* $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(U, \sigma_1, \dots, \sigma_l)$ *as above, is representable by a (strict) k-analytic space.*

Proof. Let $\mathrm{GL}_n^0 = \mathrm{Sp}_{\mathrm{B}}\big(k\langle T_{ij}\rangle[\frac{1}{\det}]\big)$ denote the closed unit disk of $\mathbf{GL}_n^{\mathrm{an}}$ and

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U, \sigma_1, \dots, \sigma_l\big)^0 \in \mathrm{An}_k$$

denote the pullback of $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U, \sigma_1, \dots, \sigma_l\big)$ along the inclusion morphism $\mathrm{GL}_n^0 \hookrightarrow \mathbf{GL}_n^{\mathrm{an}}$ computed in the category An_k. Consider the following cartesian diagram

$$\begin{tikzcd} \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U, \sigma_1, \dots, \sigma_l, B\big)^0 \arrow[r] \arrow[d] & \mathrm{Sp}_{\mathrm{B}} B \arrow[d] \\ \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U, \sigma_1, \dots, \sigma_l\big)^0 \arrow[r] & (\mathrm{GL}_n^0)^r \end{tikzcd}$$

where $B \in \mathrm{Afd}_k^{\mathrm{op}}$ is a k-affinoid algebra and

$$(M_1, \dots, M_r) \in \mathrm{GL}_n^0(\mathrm{B})^r = \mathrm{GL}_n(\mathrm{B}^0)^r$$

corresponds to a given morphism of k-analytic spaces $\mathrm{Sp}_{\mathrm{B}} B \to (\mathrm{GL}_n^0)^r$. It follows that

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U, \sigma_1, \dots, \sigma_l, B\big)^0 \in \mathrm{Fun}\big(\mathrm{Afd}_k^{\mathrm{op}}, \mathrm{Set}\big)$$

corresponds to the subfunctor of $\mathrm{Sp}_{\mathrm{B}} B$ whose value on k-affinoid algebra A consists of the set

$$\mathscr{X}^0_{U,\sigma_1,\dots,\sigma_l,B}(A) := \{f : B \to A : \text{for each } i, |\sigma_i(M_1,\dots,M_r) - \mathrm{Id}| \leq |p|, \text{ in } A\}.$$

Therefore, at the level of the points, the functor

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l,B\big)^0$$

parametrizes those points $x \in \mathrm{Sp}_B \mathrm{B}$ such that, for each i,

$$|\sigma(M_1,\dots,M_r) - \mathrm{Id}|(x) \leq |p|(x)$$

It is clear from our description, that this latter functor is representable by a Weierstrass subdomain of $\mathrm{Sp}_{\mathrm{B}} B$. As $(\mathrm{GL}^0_n)^r$ is a (strict) k-affinoid space, it follows that $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)$ is representable in the category Afd_k, (consider in the above diagram with $\mathrm{Sp}_B \mathrm{B} = \mathrm{GL}^0_n$ and $(M_1,\dots,M_r)$ the r-vector whose matrix components correspond to identity morphism of GL^0_n).

Let $c_i \in |k^\times|$ be a decreasing sequence of real numbers converging to 0, there is a natural isomorphism

$$(\mathrm{GL}^0_n)^r \simeq \operatorname*{colim}_i (\mathrm{GL}^0_n)^r_{c_i},$$

where $(\mathrm{GL}^0_n)^r_{c_i}$ denotes a copy of $(\mathrm{GL}^0_n)^r$ indexed by c_i, and the inclusion morphisms in the corresponding diagram sends $(\mathrm{GL}^0_n)^r_{c_i}$ to the closed disk of radius c_i^{-1} inside of $(\mathrm{GL}^0_n)^r_{c_{i+1}}$. Henceforth we have canonical isomorphisms

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big) \cong \operatorname*{colim}_i \left(\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)^0\right)_{c_i},$$

which is a union of k-affinoid subdomains where the image of an element in the filtered diagram lies in the interior, in Berkovich's sense, of the successive one. We thus conclude that $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)$ is itself representable by an k-analytic space. □

Theorem 5.2.2.18. *For each $r \geq 1$, the functor*

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r) : \mathrm{Afd}^{\mathrm{op}}_k \to \mathrm{Set},$$

given on objects by the formula,

$$A \in \mathrm{Afd}_k \mapsto \{(M_1,\dots,M_r) \in \mathrm{GL}_n(A) : \textit{there exists } i,\ |\sigma_i(M_1,\dots M_r) - \mathrm{Id}| \leq |p|\} \in \mathrm{Set},$$

is representable by a (strict) k-analytic space.

Proof. We start by remarking that if $U' \subset U$ is an inclusion of subgroups lying in the family $\mathfrak{J}_r$ then they induce an inclusion

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big) \hookrightarrow \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U',\tau_1,\dots,\tau_s\big).$$

We employ the notation

$$\sigma_i(M_1,\dots,M_r) \in \mathrm{Id} + p\mathrm{M}_n(A_0)$$

where the σ_i denote a choice of generators for U, lying in the dense subgroup $U \cap F_r$. It follows that we have necessarily

$$\tau'_j(M_1,\dots,M_r) \in \mathrm{Id} + p\mathrm{M}_n(A_0)$$

for a choice of generators for U', lying in $U' \cap F_r$, denoted $\tau_1,\dots \tau_s$. By the proof of our previous result it follows that the functor

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big) \colon \mathrm{Afd}^{\mathrm{op}}_k \to \mathrm{Set}$$

is representable by a k-analytic subdomain of $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U',\tau_1,\dots,\tau_s\big)$. Since we are interested in the representability of the space

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(\widehat{\mathrm{F}}_r\big) \cong \operatorname*{colim}_{U \in \mathscr{J}_r, \text{generators} \sigma_1,\dots\sigma_l} \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big),$$

we need to check that the inclusions

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U,\sigma_1,\dots,\sigma_l\big)\hookrightarrow \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U',\tau_1,\dots,\tau_s\big)$$

are nice enough whenever U denotes a sufficiently large (finite) index subgroup of $\widehat{\mathrm{F}}_r$. Fix $U\in\mathfrak{J}_r$ and let $\sigma_1,\dots,\sigma_l$ be a finite set of generators for U. Given a k-affinoid algebra A and

$$(M_1,\dots,M_r)\in \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U,\sigma_1,\dots,\sigma_l\big)(A)$$

we can write

$$\sigma_i(M_1,\dots,M_r)=\mathrm{Id}+p\cdot N_i,$$

for suitable matrices $N_i\in M_n(A0)$, for each i. Moreover the r-tuple $(M_1,\dots,M_r)$ define a continuous group homomorphism

$$\rho\colon \widehat{\mathrm{F}}_r\to \mathrm{GL}_n(A).$$

By Theorem 5.2.2.13, quotients of the pro-p-group $\mathrm{Id}+p\mathrm{M}_n(A_0)$ are of p-torsion. Let $U'\in\mathfrak{J}_r$ such that

$$U'\subset\rho^{-1}(\mathrm{Id}+p^2\mathrm{M}_n(A_0)).$$

Given $\tau_1,\dots,\tau_s$ generators for U' as above we have

$$|\tau_i(M_1,\dots,M_r)-\mathrm{Id}|\le|p^2|<|p|,$$

for each $i\in[1,s]$. This implies that

$$(M_1,\dots,M_r)\in \mathrm{Int}\big(\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U',\tau_1,\dots,\tau_s\big)/(\mathbf{GL}_n^{\mathrm{an}})^r\big),$$

For each $U\in\mathfrak{J}_r$, $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U,\sigma_1,\dots,\sigma_l\big)$ is a k-analytic subdomain of $(\mathbf{GL}_n^{\mathrm{an}})^r$, and therefore

$$\mathrm{Int}\big(\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U,\sigma_1,\dots,\sigma_l\big)/(\mathbf{GL}_n^{\mathrm{an}})^r\big)\hookrightarrow(\mathbf{GL}_n^{\mathrm{an}})^r$$

is an open subset of $(\mathbf{GL}_n^{\mathrm{an}})^r$, [Con08a, Exercise 4.5.3]. Moreover, the functor

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(\widehat{\mathrm{F}}_r)\hookrightarrow\colon \mathrm{Afd}_k^{\mathrm{op}}\to\mathrm{Set}$$

is a subfunctor of $(\mathbf{GL}_n^{\mathrm{an}})^r$, which follows readily from the definitions. We can therefore (canonically) associate to $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(\widehat{\mathrm{F}}_r)$ a topological subspace

$$\begin{aligned}\mathscr{X}&:=\bigcup_{\Gamma\in\mathscr{J}_r}\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U,\sigma_1,\dots,\sigma_l\big)\\&\cong\bigcup_{\Gamma\in\mathscr{J}_r}\mathrm{Int}\big(\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(U,\sigma_1,\dots,\sigma_l)/(\mathbf{GL}_n^{\mathrm{an}})^r\big),\end{aligned}$$

Therefore, the topological space $\mathscr{X}$ corresponds to an open subspace of the underlying topological space of $(\mathbf{GL}_n^{\mathrm{an}})^r$. Consequently, the former is necessarily an Hausdorff space. We will construct a canonical k-analytic structure on it and show that such k-analytic space represents the functor $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(\widehat{\mathrm{F}}_r)$. As each

$$\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U,\sigma_1,\dots,\sigma_l\big)\in\mathrm{An}_k$$

is a k-analytic space we can take the maximal atlas and quasi-net on it consisting of k-affinoid subdomains of $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U,\sigma_1,\dots,\sigma_l\big)$, which we denote by $\mathcal{T}_{U,\sigma_1,\dots,\sigma_l}$. As $\mathscr{X}$ can be realized as a filtered union of the $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U,\sigma_1,\dots,\sigma_l\big)$ we conclude that the union of the quasi-nets $\mathcal{T}_{U,\sigma_1,\dots,\sigma_l}$ induces a quasi-net $\mathcal{T}$ on $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(\widehat{\mathrm{F}}_r)$. In order to prove this we shall show that given a point $x\in\mathscr{X}$ we need to be able to find a finite collection $V_1,\dots,V_n$ of compact Hausdorff subsets of $\mathscr{X}$ such that $x\in\bigcap_i V_i$ and moreover $V_1\cup\dots\cup V_n$ is an open neighborhood of x inside $\mathscr{X}$. In order to show such condition on $\mathcal{T}$, we notice first that that we can choose $U\in\mathfrak{J}_r$ of sufficiently large finite index in $\widehat{\mathrm{F}}_r$ such that

$$x\in\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}\big(U,\sigma_1,\dots,\sigma_l\big)$$

lies in its relative interior $\mathrm{Int}\big(\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)/(\mathbf{GL}_n^{\mathrm{an}})^r\big)$. By the k-analytic structure on

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)\in \mathrm{An}_k$$

we conclude that we can take $V_1,\dots,V_n$ k- affinoid subdomains of $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)$ satisfying the above condition.

We are reduced to show that the union $V_1\cup\dots\cup V_n$ is open in $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)$. By shrinking the V_i of larger index in $\widehat{\mathrm{F}}_r$, if necessary, we can assume that the union $V_1\cup\dots\cup V_n$ lies inside the relative interior $\mathrm{Int}\big(\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)/(\mathbf{GL}_n^{\mathrm{an}})^r\big)$ and is open in $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)$. As a consequence, the latter is open in

$$\mathrm{Int}\big(\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)/(\mathbf{GL}_n^{\mathrm{an}})^r\big).$$

The latter is also open in $(\mathbf{GL}_n^{\mathrm{an}})^r$, consequently also the union $V_1\cup\dots\cup V_n$ is open in $(\mathrm{GL}_n)^r$. We conclude that there exists an open subset W of $(\mathbf{GL}_n^{\mathrm{an}})^r$ such that

$$V_1\cup\dots\cup V_n=\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)\cap U=\mathrm{Int}\big(\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)/(\mathbf{GL}_n^{\mathrm{an}})^r\big)\cap W$$

and therefore $V_1\cup\dots\cup V_n$ is itself open in $(\mathbf{GL}_n^{\mathrm{an}})^r$. We conclude thus that $V_1\cup\dots\cup V_n$ is also open in $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)$, as desired. Clearly, $\mathcal{T}$ induces quasi-nets on the intersections

$$W\cap W'$$

for any $W,\ W'\in\mathcal{T}$ as we can always choose a sufficiently large finite quotient Γ of $\widehat{\mathrm{F}}_r$ such that $W,\ W'\subset \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)$. The fact that the union of the maximal atlas on each $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)$ gives an atlas on $\mathscr{X}$, with respect to $\mathcal{T}$, is also clear from the definitions. We conclude that the topological space $\mathscr{X}$ is endowed with a natural structure of k-analytic space (in fact, a k-analytic subdomain of $(\mathbf{GL}_n^{\mathrm{an}})^r$). We shall show that the k-analytic space $\mathscr{X}$ represents the functor

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)\colon \mathrm{Afd}_k^{\mathrm{op}}\to\mathrm{Set}.$$

As k-affinoid spaces are compact we conclude that any map

$$\mathrm{Sp}_{\mathrm{B}}A\to\mathscr{X}$$

factor through some $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)$ as their union equals the union of the respective relative interiors. As $\mathrm{Sp}_B A$ is quasi-compact, it follows from Theorem 5.2.2.14, that the functor of points, thanks to associated to $\mathscr{X}$ is canonically equivalent to $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)$ and the result follows. □

Corollary 5.2.2.19. *Let G be a profinite group topologically of finite type then the functor*

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G):\mathrm{Afd}_k^{\mathrm{op}}\to\mathrm{Set},$$

given on objects by the formula

$$A\in\mathrm{Afd}_k^{\mathrm{op}}\to\mathrm{Hom}_{\mathrm{cont}}\big(G,\mathrm{GL}_n(A)\big)\in\mathrm{Set},$$

is representable by a k-analytic space.

Proof. Let us fix a continuous surjection of profinite groups

$$q\colon\widehat{\mathrm{F}}_r\to G,$$

for some integer $r\le 1$. Let H denote the kernel of q. Thanks to Theorem 5.2.2.18 we know that $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)$ is representable by a k-analytic stack. We have an inclusion at the level of functor of points

$$q_*\colon\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G)\to\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)$$

induced by precomposing continuous homomorphisms $\rho\colon G\to\mathrm{GL}_n(A)$ with q. We show that the morphism q_* is representable and a closed immersion. Let $\mathrm{Sp}_{\mathrm{B}}\,A$ be a k-affinoid space and suppose given a morphism of k-analytic spaces,

$$\rho\colon\ \mathrm{Sp}_{\mathrm{B}}\,A\to\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r),$$

which corresponds to a continuous representation $\rho\colon G \to \mathrm{GL}_n(A)$. We want to compute the fiber product

$$\mathrm{Sp_B}\, A \times_{\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)} \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G).$$

Since $\mathrm{Sp_B}\, A$ is quasi-compact and we have an isomorphism at the underlying topological spaces,

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r) \cong \operatorname*{colim}_{U \in \mathfrak{J}_r, \text{generators } \sigma_1,\dots,\sigma_l} \mathrm{Int}\big(\mathscr{X}_{U,\sigma_1,\dots,\sigma_l}/(\mathrm{GL}_n)^r\big),$$

we conclude that

$$\rho\colon\ \mathrm{Sp}\, A \to \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)$$

factors through a k-analytic subspace of the form $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U, \sigma_1, \dots, \sigma_l\big)$, for suitable such $U \in \mathfrak{J}_r$ and $\sigma_1, \dots, \sigma_l$. By applying again the same reasoning we can assume further that

$$\rho\colon\ \mathrm{Sp_B}\, A \to \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)$$

factors through some $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U, \sigma_1, \dots, \sigma_l\big)^0$ as in the proof of Theorem 5.2.2.17. The latter is k-affinoid, say

$$\mathscr{X}^0_{U,\sigma_1,\dots,\sigma_l} \cong \mathrm{Sp_B}\, B$$

in the category Afd_k, for some k-affinoid algebra B. Let $\mathscr{X}^0_{G,U,\sigma_1,\dots,\sigma_l}$ denote the fiber product,

$$\begin{tikzcd}
\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(G, U, \sigma_1, \dots, \sigma_l\big)^0 \arrow[r] \arrow[d] & \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G) \arrow[d] \\
\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U, \sigma_1, \dots, \sigma_l\big)^0 \arrow[r] & \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)
\end{tikzcd}.$$

By construction, the set

$$\mathscr{X}^0_{G,U,\sigma_1,\dots,\sigma_l}(A) \in \mathrm{Set}$$

corresponds to those $(M_1, \dots, M_r) \in \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U, \sigma_1, \dots, \sigma_l\big)^0(A)$ such that

$$h(M_1, \dots, M_r) = \mathrm{Id},$$

for every $h \in H \cap F_r \subset H$. Then we have an equivalence of fiber products,

$$\begin{aligned}
Z &:= \mathrm{Sp}\, A \times_{\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\widehat{\mathrm{F}}_r)} \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G) \\
&\cong \mathrm{Sp}\, A \times_{\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(U,\sigma_1,\dots,\sigma_l\big)^0} \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}\big(G, U, \sigma_1, \dots, \sigma_l\big)^0
\end{aligned}$$

As every k-affinoid algebra is Noetherian, [Con08a, Theorem 1.1.5], we conclude that Z parametrizes points which determined by finitely many equations with coefficients in $A \in \mathrm{Afd}^{\mathrm{op}}_k$, induced from the relations defining H inside $\widehat{\mathrm{F}}_r$, (after choosing topological generators for $\widehat{\mathrm{F}}_r$). We conclude that Z is a closed subspace of $\mathrm{Sp}\, A$ and thus representable. The result now follows. $\square$

Remark 5.2.2.20. Given G a profinite group as above there is a canonical action of the k-analytic group GL_n on $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G)$ via conjugation. Furthermore, continuous representations of a group correspond precisely to the conjugacy classes of elements in $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G)$ under the action of $\mathbf{GL}^{\mathrm{an}}_n$.

5.2.3 Geometric contexts and geometric stacks

Our next goal is to give an overview of the general framework that allow us to define the notion of a geometric stack. Our motivation comes from the need to define the moduli stack of continuous representations of a profinite group G (of topological finite presentation) as a non-archimedean geometric stack. This latter object should be obtained by taking the quotient of $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G)$ by the conjugation action of $\mathbf{GL}^{\mathrm{an}}_n$ on $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(G)$. We will review these notions and show that such we are able to construct such a quotient via a formal procedure.

Definition 5.2.3.1. A geometric context $(\mathcal{C}, \tau, \mathbf{P})$ consists of an ∞-site $(\mathcal{C}, \tau)$, see [Lur09b, Definition 6.2.2.1], and a class $\mathbf{P}$ of morphisms in $\mathcal{C}$ verifying:

(i) Every representable sheaf is a hypercomplete sheaf on $(\mathcal{C}, \tau)$.

(ii) The class $\mathbf{P}$ is closed under equivalences, compositions and pullbacks.

(iii) Every τ-covering consists of moprhisms in $\mathbf{P}$.

(iv) For any morphism $f : X \to Y$ in $\mathcal{C}$, if there exists a τ-covering $\{U_i \to X\}$ such that each composition $U_i \to Y$ belongs to $\mathbf{P}$ then f belongs to $\mathbf{P}$.

Notation 5.2.3.2. Let $(\mathcal{C}, \tau)$ denote an ∞-site. We denote by $\mathrm{Shv}(\mathcal{C}, \tau)$ the ∞-category of sheaves on $(\mathcal{C}, \tau)$. It can be realized as a presentable left localization of the ∞-category of presheaves on $\mathcal{C}$, $\mathrm{PSh}(\mathcal{C}) \coloneqq \mathrm{Fun}\big(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\big)$.

Given a geometric context $(\mathcal{C}, \tau, \mathbf{P})$ it is possible to form an ∞-category of geometric stacks $\mathrm{Geom}(\mathcal{C}, \tau, \mathbf{P})$ via an inductive definition as follows:

Definition 5.2.3.3. A morphism in $F \to G$ in $\mathrm{Shv}(\mathcal{C}, \tau)$ is (-1)-representable if for every map $X \to G$, where X is a representable object of $\mathrm{Shv}(\mathcal{C}, \tau)$, the base change $F \times_G X$ is also representable. Let $n \geq 0$, we say that $F \in \mathrm{Shv}(\mathcal{C}, \tau)$ is n-geometric if it satisfies the following two conditions:

(i) It admits an n-atlas, i.e. a morphism $p : U \to F$ from a representable object U such that p is $(n-1)$-representable and it lies in $\mathbf{P}$.

(ii) The diagonal map $F \to F \times F$ is $(n-1)$-representable.

Definition 5.2.3.4. We say that $F \in \mathrm{Shv}(\mathcal{C}, \tau)$ is locally geometric if F can be written as an union of n-geometric stacks $F = \bigcup_i G_i$, for possible varying n, such that each G_i is open in F, i.e., after base change by representable objects the corresponding inclusion morphisms are open immersions.

An important feature that one desires to be satisfied in a geometric context $(\mathcal{C}, \tau, \mathbf{P})$ is the notion of closedness under τ-descent.

Definition 5.2.3.5. Let $(\mathcal{C}, \tau)$ be an ∞-site. The ∞-category $\mathcal{C}$ is closed under τ-descent if for any morphism $F \to Y$, where $F,\ Y \in \mathrm{Shv}(\mathcal{C}, \tau)$ and Y is required to be representable and for any τ-covering $\{Y_i \to Y\}$ the pullback $F \times_Y Y_i$ is representable then so is F.

Remark 5.2.3.6. When the geometric context is closed under τ-descent the definition of a geometric stack becomes simpler since it turns out to be ambiguous to require the representability of the diagonal map.

Example 5.2.3.7. Many examples of geometric contexts can be given but our main object of study will be the geometric context $(\mathrm{Afd}_k, \tau_{\text{ét}}, \mathbf{P}_{\mathrm{sm}})$, where $\tau_{\text{ét}}$ denotes the quasi-étale topology on Afd_k, and $\mathbf{P}_{\mathrm{sm}}$ denotes the collection of quasi-smooth morphisms, see [Ber94a] chapter 3 for the definitions of quasi-étale and quasi-smooth morphisms of k-analytic spaces. Such geometric context is closed under $\tau_{\text{ét}}$-descent and we will call the corresponding geometric stacks as k-analytic stacks.

Let G be a smooth group object in the ∞-category $\mathrm{Shv}(\mathcal{C}, \tau)$. Suppose that G acts on a representable object X. We can form its quotient stack via the (homotopy) colimit of the diagram,

$$\cdots \quad G^2 \times X \quad G \times X \quad X$$

We denote such (homotopy) colimit by $[X/G]$ and refer it as the stacky quotient of X by G.

Lemma 5.2.3.8. *Let $(\mathcal{C}, \tau, \boldsymbol{P})$ be a geometric context satisfying τ-descent. Let G be a smooth group object in the ∞-category $\mathrm{Shv}(\mathcal{C}, \tau)$ acting on a representable object X. Then the stacky quotient $[X/G]$ is a geometric stack.*

Proof. It suffices to verify condition (1) of Definition 2.12. By definitoin of $[X/G]$ we have a canonical morphism $X \to [X/G]$ which is easily seen to be (-1)-representable and smooth. Therefore, $[X/G]$ is a 0-geometric stack. □

Definition 5.2.3.9. Let G be a profinite group of topological finite presentation. We define the k-analytic stack of continuous representations of G

$$\mathrm{LocSys}_{\ell,n}(G) \coloneqq [\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(G)/\mathbf{GL}_n^{\mathrm{an}}] \in \mathrm{St}\big(\mathrm{Afd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big).$$

Thanks to Theorem 5.2.2.19 we obtain the following important result:

Theorem 5.2.3.10. *Let G be a profinite group of topological finite presentation. Then the groupoid-valued functor*

$$\mathrm{LocSys}_{\ell,n}(G)\colon \mathrm{Afd}_k \to \mathcal{S}$$

is representable by a geometric stack.

Proof. The result is a direct consequence of Theorem 5.2.3.8 together with Theorem 5.2.2.19. □

Corollary 5.2.3.11. *Let X be a smooth and proper scheme over an algebraically closed field. Then the k-analytic stack parametrizing continuous representations of $\pi_1^{\text{ét}}(X)$ is representable by a geometric stack.*

Proof. It follows immediately by Theorem 5.2.3.10 together with the fact that under such assumptions on X its étale fundamental group $\pi_1^{\text{ét}}(X)$ is topologically of finite generation. □

Remark 5.2.3.12. As $\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(G)$ is a representable object in the ∞-category $\mathrm{Shv}(\mathrm{Afd}_k^{\mathrm{op}}, \tau_{\text{ét}})$, $\mathbf{GL}_n^{\mathrm{an}}$ is a smooth group object in $\mathrm{Shv}(\mathrm{Afd}_k^{\mathrm{op}}, \tau_{\text{ét}})$ and the corresponding geometric context satisfies descent we conclude by Theorem 5.2.3.8 that the quotient $[\mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(G)/\mathbf{GL}_n^{\mathrm{an}}]$ is representable by a geometric stack.

Remark 5.2.3.13. The geometric stack $\mathrm{LocSys}_{\ell,n}(G)$ is not, in general, a mapping stack. However the reader should think of it as a continuous version of the latter. It would thus be desirable to say that $\mathrm{LocSys}_{\ell,n}(G)$ is equivalent to

$$\mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\big(\mathrm{B}G, \mathrm{B}\mathbf{GL}_n^{\mathrm{an}}\big),$$

where the latter consists of the stack of morphisms between $\mathrm{B}G$ and $\mathrm{B}\mathbf{GL}_n^{\mathrm{an}}$, considered as ind-pro-stacks. This is not really the case, but it a reasonable conceptual approximation. We will explore this idea in detail using the language of $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$-enriched ∞-categories.

5.3 Moduli of k-lisse sheaves on the étale site of a proper normal scheme

Let X be a proper normal scheme over an algebraically closed field K. Let

$$\bar{x} : \mathrm{Spec}K \to X$$

be a fixed geometric point of X. Thanks to [GR, Theorem 2.9, exposé 10] the étale fundamental group $\pi_1^{\text{ét}}(X)$ is topologically of finite presentation. As a consequence, the results proved in the previous §hold true for the profinite group $G = \pi_1^{\text{ét}}(X)$. In particular,

$$\mathrm{LocSys}_{\ell,n}(X) \coloneqq \mathrm{LocSys}_{\ell,n}(\pi_1^{\text{ét}}(X))$$

is representable by a k-analytic stack. In this §, we will show that the moduli $\mathrm{LocSys}_{\ell,n}(X)$ parametrizes pro-étale local systems of rank n on X. This is a consequence of the fact that the étale fundamental group of X parametrizes étale local systems on X with finite coefficients. As we are interested in the local systems valued in k-affinoid algebras, the pro-étale topology is thus more suited for us. This is dealt with in §3.2.

In §3.1 we prove some results concerning perfecteness of étale cohomology chains with derived coefficients. These results are known to experts but hard to locate in the literature so we prefer to give a full account of these as they will be important for us in order to show the existence of the cotangent complex of $\mathrm{LocSys}_{\ell,n}(X)$.

5.3.1 Étale cohomology of perfect local systems

Let

$$\pi : X \to \mathrm{Spec} K$$

denote the structural morphism. For each integer $n \geq 1$, we have a canonical equivalence of ∞-categories

$$\mathrm{Shv}(\mathrm{Spec} K, \mathbb{Z}/\ell^n\mathbb{Z}) \simeq \mathrm{Mod}_{\mathbb{Z}/\ell^n\mathbb{Z}}.$$

We have a pullback functor

$$p^* : \mathrm{Shv}_{\text{ét}}(\mathrm{Spec} K, \mathbb{Z}/\ell^n\mathbb{Z}) \to \mathrm{Shv}_{\text{ét}}(X, \mathbb{Z}/\ell^n\mathbb{Z}),$$

which associates to each $\mathbb{Z}/\ell^n\mathbb{Z}$-module M the étale constant sheaf on X with values in M.

Proposition 5.3.1.1. *Let X be a proper normal scheme over an algebraically closed field K. Then $\mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$ is a perfect complex of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules.*

Proof. This is a direct consequence of the more general result [GL14, Proposition 4.2.15]. □

Definition 5.3.1.2. Let A be a derived ring. We say that A is Noetherian if it satisfies the following conditions:

(i) $\pi_0(A)$ is a Noetherian ring;

(ii) For each $i \geq 0$, $\pi_i(A)$ is an $\pi_0(A)$-module of finite type.

Definition 5.3.1.3. Let A be a derived ring and $M \in \mathrm{Mod}_A$ and A-module. We say that M has tor-amplitude $\leq n$ if, for every discrete A-module N, (which can be automatically seen as a $\pi_0(A)$), the homotopy groups

$$\pi_i(M \otimes_A N) \in \mathrm{Mod}_A$$

vanish for every integer $i > n$.

Lemma 5.3.1.4. *Let A be a Noetherian simplicial ring and $M \in \mathrm{Mod}_A$ be an A-module such that $\pi_i(M) \simeq 0$ for sufficiently small $i \leq 0$. Then M is a perfect A-module if and only if the following two conditions are satisfied:*

(i) For each i, $\pi_i(M)$ is of finite type over $\pi_0(A)$;

(ii) M is of finite Tor-dimension.

Proof. It is part of [Lur12c, Proposition 7.2.4.23]. □

Remark 5.3.1.5. Let A be a derived $\mathbb{Z}/\ell^n\mathbb{Z}$-algebra and let $N \in \mathrm{Shv}(X_{\text{ét}}, A)$ be a local system of perfect A-modules on $X_{\text{ét}}$. Thanks to [GL14, Proposition 4.2.2] it follows that N can be written as a (finite sequence) of retracts of

$$(f_V)_!(A) \in \mathrm{Shv}(X_{\text{ét}}, A),$$

where

$$(f_V)_! \colon \mathrm{Shv}(V_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to \mathrm{Shv}(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}),$$

denotes the exceptional direct image functor associated to an étale map $f_V \colon V \to X$.

Lemma 5.3.1.6 (Projection Formula)**.** *Let X be a scheme over an algebraically closed field K. Let A be a simplicial ring and let $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(X, A)$. Then, for any $M \in \mathrm{Mod}_A$ we have a natural equivalence,*

$$\pi_*(\mathcal{F} \otimes_A \pi^*(M)) \simeq \pi_*(\mathcal{F}) \otimes_A M,$$

in the derived ∞-category Mod_A, where π denotes the structural morphism $\pi \colon X \to \mathrm{Spec}\, K$.

Proof. Let $\mathcal{C} \subset \mathrm{Mod}_A$ be the full subcategory spanned by those A-modules M such that there exists a canonical equivalence $\mathrm{R}\Gamma(X_{\text{ét}}, M) \simeq \mathrm{R}\Gamma(X_{\text{ét}}, A) \otimes_A M$. It is clear that $A \in \mathcal{C}$ and $\mathcal{C}$ is closed under small colimits as both tensor product and the direct image functor π_* commute with small colimits. Consequently, by the fact that the ∞-category $\mathrm{Mod}_{\mathbb{Z}/\ell^n\mathbb{Z}}$ is compactly generated (under small colimits) by the object A, the result follows. □

Proposition 5.3.1.7. *Let $A \in \mathcal{C}\mathrm{Alg}_{\mathbb{Z}/\ell^n\mathbb{Z}}$ be a Noetherian simplicial $\mathbb{Z}/\ell^n\mathbb{Z}$-algebra. Let N be a local system of A-modules on $X_{\text{ét}}$ with values in the ∞-category of perfect A-modules, $\mathrm{Perf}(A)$. Then the étale cohomology of the local system N, denoted $\mathrm{R}\Gamma(X_{\text{ét}}, N)$, is a perfect A-module.*

Remark 5.3.1.8. Note that the statement of Theorem 5.3.1.7 concerns the chain level and not the étale cohomology of the complex $\mathrm{R}\Gamma(X_{\text{ét}}, N)$, thus it is a stronger statement than just requiring finiteness of the corresponding étale cohomology groups.

Proof of Theorem 5.3.1.7. Let N be a local system on $X_{\text{ét}}$ of perfect A-modules, i.e., there exists an étale covering $U \to X$, such that $N_{|U} \simeq f^*(P)$, where

$$f : U \to \mathrm{Spec}K$$

denotes the structural map and $P \in \mathrm{Perf}(A)$ is a perfect A-module. Our goal is to show that

$$\mathrm{R}\Gamma(X_{\text{ét}}, N) \in \mathrm{Perf}(A).$$

By Theorem 5.3.1.4 it suffices to show that for each $i \in \mathbb{Z}$, the cohomology groups

$$H^{-i}(X_{\text{ét}}, N) := \pi_i(\mathrm{R}\Gamma(X_{\text{ét}}, N))$$

is of finite type over $\pi_0(A)$ and moreover

$$\mathrm{R}\Gamma(X_{\text{ét}}, N) \in \mathrm{Mod}_A$$

is of finite Tor-dimension over A. Without loss of generality we can assume that N is a connective perfect A-module on the étale site $X_{\text{ét}}$, i.e., the discrete $\pi_0(A)$-étale sheaves on X, $\pi_i(N)$ vanish for $i < 0$.

Thanks to [GL14, Proposition 4.2.10] and its proof we deduce that $\pi_i(N)$ is an étale local system of finitely presented (discrete) $\pi_0(A)$-modules on $X_{\text{ét}}$, for each $i \geq 0$. For a fixed integer $i \geq 0$ the homotopy sheaf $\pi_i(N)$ is a local system of finitely presented (discrete) $\pi_0(A)$-modules. It thus follows that there exists an étale covering

$$V \to X$$

such that

$$\pi_i(N)_{|V} \simeq g^* E,$$

where $g : V \to \mathrm{Spec}K$ denotes the structural map and E denotes a suitable $\pi_0(A)$-module of finite presentation. As X is a normal scheme we can assume without loss of generality that the étale map $V \to X$ is a Galois covering (in particular it is finite étale). It follows by Galois descent that

$$\mathrm{R}\Gamma(X_{\text{ét}}, \pi_i(N)) \simeq \mathrm{R}\Gamma(G, \mathrm{R}\Gamma(X_{\text{ét}}, g^* E)),$$

where G is the finite group of automorphisms of the Galois covering $V \to X$. Assume first that $\mathrm{R}\Gamma(X_{\text{ét}}, g^* E)$ is an A- module whose homotopy groups are finitely generated over $\pi_0(A)$. Since the group G is finite, the group cohomology of G with $\mathbb{Z}$-coefficients is finitely generated and of torsion, we thus conclude by the corresponding Grothendieck spectral sequence that the homotopy groups of the complex

$$\mathrm{R}\Gamma(G, \mathrm{R}\Gamma(X_{\text{ét}}, g^* E)) \in \mathrm{Mod}_{\pi_0(A)}$$

are finitely generated over $\pi_0(A)$. We are thus reduced to the case where $\pi_i(N)$ is itself a constant $\pi_0(A)$-module on $X_{\text{ét}}$. By the projection formula we can reduce to the case where $\pi_i(N)$ is $\pi_0(A)$ itself. Again by the projection formula we can reduce to the case where $\pi_0(A) \simeq \mathbb{Z}/\ell^n\mathbb{Z}$ in which case the result follows readily by Theorem 5.3.1.1.

By induction on the Postnikov tower associated to N we conclude that given $n \geq 0$ we have a fiber sequence of étale A-modules,

$$\tau_{\leq n+1} N \to \tau_{\leq n} N \to \pi_{n+1}(N)[n+2],$$

such that, by our inductive hypothesis both complexes

$$\mathrm{R}\Gamma(X_{\text{ét}}, \tau_{\leq n} N) \text{ and } \mathrm{R}\Gamma(X_{\text{ét}}, \pi_{n+1}(N))[n+2] \in \mathrm{Mod}_A$$

have homotopy groups which are finitely generated $\pi_0(A)$-modules. Therefore, as $\mathrm{R}\Gamma(X_{\text{ét}}, -)$ is an exact functor the assertion also follows for $\tau_{\leq n+1}N$. We are thus dealt with condition (i) in Theorem 5.3.1.4 We have another fiber sequence of the form,

$$\tau_{>n}N \to N \to \tau_{\leq n}N.$$

As X is of finite cohomological dimension it follows by [GL14, Lemma 2.2.4.1] that for any given integer i there is a sufficiently large integer n such that

$$\pi_i(\mathrm{R}\Gamma(X_{\text{ét}}, \tau_{>n}N)) \in \mathrm{Mod}_{\pi_0(A)}$$

vanishes. By exactness of the global sections functor $\mathrm{R}\Gamma$ it implies that $\pi_i(\mathrm{R}\Gamma(X_{\text{ét}}, N))$ and $\pi_i(\mathrm{R}\Gamma(X_{\text{ét}}, \tau_{\leq n}N))$ agree for sufficiently large n.

As X is of finite cohomological dimension we conclude that given M a discrete $\pi_0(A)$-module, the A-module $\pi_*\pi^*(M)$ has non-zero homotopy groups lying in a finite set of indices. Using the projection formula we conclude once more that

$$\mathrm{R}\Gamma(X_{\text{ét}}, N) \otimes_A M \simeq \mathrm{R}\Gamma(X_{\text{ét}}, N \otimes_A M)$$

can be obtained by a finite sequence of retracts of the A-module $\pi_*\pi^*(M)$. Consequently, under our hypothesis on M, it follows that

$$\pi_i(\mathrm{R}\Gamma(X_{\text{ét}}, N) \otimes_{\pi_0(A)} M) \simeq 0$$

for large enough i. Thus we conclude that $\mathrm{R}\Gamma(X_{\text{ét}}, N)$ is of finite Tor-dimension as an A-module and thus a perfect A-module. □

5.3.2 Pro-etale lisse sheaves on $X_{\text{ét}}$

It follows by our hypothesis on X and [BS13, Lemma 7.4.10] that the pro-étale and étale fundamental groups of X agree henceforth it suffices to consider representations of the étale fundamental group of X, $\pi_1^{\text{ét}}(X)$.

Definition 5.3.2.1 (Noohi group)**.** Let G be a topological group and consider the category of G-sets, denoted G-Set. Consider the forgetful functor

$$F_G \colon G\text{-Set} \to \text{Set}.$$

We say that G is a Noohi group if there is a canonical equivalence $G \simeq \mathrm{Aut}(F_G)$, where $\mathrm{Aut}(F_G)$ is topologized with the compact-open topology on Aut(S) for each $S \in$ Set.

Lemma 5.3.2.2. *Let G be a topological group which admits an open Noohi subgroup U, then G is itself a Noohi group.*

Proof. This is [BS13, Lemma 7.1.8]. □

Lemma 5.3.2.3. *Let A be an k-affinoid algebra, then $\mathrm{GL}_n(A)$ is a Noohi group.*

Proof. Let A_0 be a formal model for A, it is a p-adically complete ring and we have the equivalence,

$$\mathrm{GL}_n(A_0) \simeq \lim_k \mathrm{GL}_n(A_0/p^k A_0) \simeq \lim_k \mathrm{GL}_n(A_0)/\left(\mathrm{Id} + p^k \mathrm{M}_n(A_0)\right),$$

which induces its structure of topological group, in particular it is a pro-discrete group as in [Noo04, Definition 2.1]. Moreover, the system $\{\mathrm{GL}_n(A_0)\} \cup \{\mathrm{Id} + p^k M_n(A_0)\}$ is a basis of open normal subgroups for the topology on $\mathrm{GL}_n(A_0)$ and thus by [Noo04, Proposition 2.14] we conclude that $\mathrm{GL}_n(A_0)$ is a Noohi group. As A_0 is an open subgroup of A the same holds for $\mathrm{GL}_n(A_0) \subset \mathrm{GL}_n(A)$ and by [BS13, Lemma 7.1.8] we conclude that $\mathrm{GL}_n(A)$ is a Noohi group. □

The following Proposition is a generalization of [BS13, Lemma 7.4.7] and its proof is just an adaption of that one. We give it here for the sake of completeness.

Proposition 5.3.2.4. *Let A be a k-affinoid algebra. Then there is an equivalence of groupoids,*

$$\mathrm{LocSys}_{\ell,n}(X) \simeq \mathrm{Loc}_{X,n}(A),$$

and $\mathrm{Loc}_{X,n}(A)$ the groupoid of (pro-)étale local systems of rank n A-free modules on X.

Proof. Let A_0 be a formal model for A. Note that A_0 is an open subring of A which is p-adically complete and therefore is a pro-discrete ring implying that the group $\mathrm{GL}_n(A_0)$ is a pro-discrete group as in [Noo04, Definition 2.1], thanks to [BS13, Lemma 7.4.6] the result follows if we replace A by A_0 in the statement of the Lemma. Let,

$$\rho : \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A),$$

be a continuous representation and

$$U = \rho^{-1}(\mathrm{GL}_n(A_0)),$$

note that U is an open subgroup of $\pi_1^{\text{ét}}(X)$, therefore it defines a pointed covering $X_U \to X$ with $\pi_1^{\text{ét}}(X_U) = U$. The induced representation,

$$\pi_1^{\text{ét}}(X_U) \to \mathrm{GL}_n(A_0),$$

defines thus an element $M \in \mathrm{Loc}_{X_U}(A_0)$ and hence, by inverting p,it produces a local system $M' \in \mathrm{Loc}_{X_U}(A)$. Such element M' comes equipped with descent data for $X_U \to X$ and therefore comes from a unique $N(\rho)$ in $\mathrm{Loc}_X(A)$. Conversely, fix some $N \in \mathrm{Loc}_X(A)$ which, for suitable n, we can see it as a $\mathcal{F}_{\mathrm{GL}_n(A)}$-torsor, which is a sheaf for the pro-étale topology on X via [BS13, Lemma 4.2.12], here $\mathcal{F}_{\mathrm{GL}_n(A)}$ denotes the sheaf on $X_{\text{proét}}$ defined informally via,

$$T \in X_{\text{proét}} \mapsto \mathrm{Map}_{\mathrm{cont}}(T, \mathrm{GL}_n(A)).$$

Let $S \in \mathrm{GL}_n(A)$-Set then we have an induced representation,

$$\rho_S : \mathcal{F}_{\mathrm{GL}_n(A)} \to \mathcal{F}_{\mathrm{Aut}(S)},$$

of pro-étale local sheaves. The pushout of N along ρ_S defines an element $N_S \in \mathrm{Loc}_X$ with stalk S, which is functorial in S and therefore it defines a functor $\mathrm{GL}_n(A)\text{-Set} \to \mathrm{Loc}_{X,n}(A)$ compatible with the fiber functor. By Theorem 5.3.2.3, $\mathrm{GL}_n(A)$ is Noohi and therefore it is possible to associated it a continuous homomorphism $\rho_N : \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A)$, which gives an inverse for the previous construction. This establishes the equivalence of the statement, as desired. $\square$

Corollary 5.3.2.5. *The non-archimedean stack* $\mathrm{LocSys}_{\ell,n}(X)$ *represents the functor* $\mathrm{Afd}_k^{\mathrm{op}} \to \mathcal{S}$ *given on objects by the formula,*

$$A \mapsto \mathrm{Loc}_{X,n}(A),$$

where $\mathrm{Loc}_{X,n}(A)$ *denotes the groupoid of local systems of projective* A*-modules locally of rank* n *on the pro-étale topology of* X*.*

Proof. It follows by the construction of quotient stack and Theorem 5.3.2.4. $\square$

5.4 Moduli of continuous k°-adic representations

In this section we prove several results concerning the ∞-category of derived continuous k°-adic representations and the associated derived moduli stack. Even though such results are somewhat secondary to our main goal they will prove useful in proving the representability of derived moduli stack of rank n continuous k-adic representations.

5.4.1 Preliminaries

Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ be a profinite space which we suppose fixed throughout this §. Assume further that X is connected, i.e.

$$\pi_0 \mathrm{Mat}(X) \simeq *$$

where $\mathrm{Mat}(X) \coloneqq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})}(*, X) \in \mathcal{S}$.

Definition 5.4.1.1. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ and $n \geq 1$ an integer. We define $A_n \in \mathcal{C}\mathrm{Alg}_{k^\circ_n}$ as the derived k°_n-algebra defined as the pushout of the diagram

$$\begin{tikzcd} A[u] \arrow[r, "u \mapsto p^n"] \arrow[d, "u \mapsto 0"] & A \arrow[d] \\ A \arrow[r] & A_n \end{tikzcd}$$

computed in $\mathcal{C}\mathrm{Alg}_{k^\circ_n}$, where $A[t]$ denotes the derived A-algebra obtained from A by freely adding a variable t in degree 0.

Remark 5.4.1.2. Suppose $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ is p-complete. Thanks to [Lur16, Remark 8.1.2.4] we have an equivalence of derived k°-algebras

$$A \simeq \lim_{n \geq 1} A_n.$$

Moreover, perfect A-modules are necessarily p-complete and we have an equivalence

$$M \simeq \lim_{n \geq 1} (M \otimes_A A_n)$$

in the ∞-category $\mathrm{Perf}(A)$. Thanks to [Lur16, Proposition 8.1.2.3] it follows that one has an equivalence of ∞-categories

$$\mathrm{Perf}(A) \to \lim_{n \geq 1} \mathrm{Perf}(A_n).$$

Therefore, we can (functorially) associate to $\mathrm{Perf}(A)$ a pro-object $\{\mathrm{Perf}(A_n)\}_n \in \mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)$.

Construction 5.4.1.3. Let $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ be an ∞-category. The ∞-category of pro-objects on $\mathcal{C}$, denoted $\mathrm{Pro}(\mathcal{C})$, is defined by means of the following universal property: the ∞-category $\mathrm{Pro}(\mathcal{C})$ admits small cofiltered colimits and there exists a fully faithful Yoneda embedding

$$j \colon \mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$$

such that for any ∞-category $\mathcal{D}$ admitting small cofiltered colimits we have that pre-composition with j induces an equivalence of ∞-categories

$$\mathrm{Fun}'\big(\mathrm{Pro}(\mathcal{C}), \mathcal{D}\big) \to \mathrm{Fun}\big(\mathcal{C}, \mathcal{D}\big)$$

where the left hand side denotes the full subcategory of $\mathrm{Fun}\big(\mathrm{Pro}(\mathcal{C}), \mathcal{D}\big)$ spanned by those functors which preserve small cofiltered limits. Moreover, if $\mathcal{C}$ is an accessible ∞-category which admits finite limits one can give a more explicit of $\mathrm{Pro}(\mathcal{C})$ as the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ spanned by those left exact accessible functors

$$f \colon \mathcal{C} \to \mathcal{S}.$$

The existence of $\mathrm{Pro}(\mathcal{C}$ is general (e.g. when $\mathcal{C}$ is not necessarily accessible) is guaranteed by [Lur09b, Proposition 5.3.6.2]. We observe that, up to enlarge Grothendieck universes, one can consider the ∞-category $\mathrm{Pro}(\widehat{\mathcal{C}\mathrm{at}_\infty})$ of pro-objects in the ∞-category of (not necessarily small) ∞-categories, denoted $\widehat{\mathcal{C}\mathrm{at}_\infty}$.

Remark 5.4.1.4. Since the ∞-category $\mathcal{S}$ is presentable, we can identify $X \in \mathrm{Pro}(\mathcal{S})$ with a functor

$$f \colon \mathcal{S} \to \mathcal{S}.$$

Such functor induces a unique, up to contractible indeterminacy, left fibration

$$F \colon \mathcal{C} \to \mathcal{S}$$

obtained as a pullback of the diagram

$$\begin{tikzcd} \mathcal{S}_{/f} \arrow[r] \arrow[d] & \mathcal{S}_{/*} \arrow[d] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{S} \end{tikzcd}$$

in the ∞-category $\mathcal{C}\mathrm{at}_\infty$. Thanks to [Lur09b, Propositino 5.3.2.5], the ∞-category $\mathcal{S}_{/f}$ is cofiltered. Therefore, the association

$$X \mapsto \mathcal{S}_{/f}$$

allow us to interpret X as a pro-system $\{T_i\}_i$, where $T_i \in \mathcal{S}$. Moreover, given $X \in \mathcal{S}$, we have an equivalence

$$f(X) \simeq \mathrm{Map}_{T_i \in \mathcal{S}_{/f}}\big(uT_i, X\big),$$

where $u \colon \mathcal{S}_{/f} \to \mathcal{S}$ is the forgetful functor.

Remark 5.4.1.5. The ∞-categories $\mathcal{S}$ and $\mathcal{C}\mathrm{at}_\infty$ are presentable. It follows by [Lur09a, Remark 3.1.7] that one has a fully faithful embedding

$$\mathrm{Pro}(\mathcal{S}) \to \mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty).$$

The analogous statement the larger versions $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{C}\mathrm{at}_\infty}$ holds by the same reasoning.

Definition 5.4.1.6. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$, we define $\mathrm{Perf}^{\mathrm{ad}}(X)(A)$ as the functor category

$$\mathrm{Fun}_{\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)}\big(X, \mathrm{Perf}(A)\big) \in \mathcal{C}\mathrm{at}_\infty,$$

where we consider $\mathrm{Perf}(A) \in$
$pro(\mathcal{C}\mathrm{at}_\infty)$. We define also the ∞-categories

$$\begin{aligned}
\mathrm{Mod}^{\mathrm{ad}}(X)(A) &\coloneqq \mathrm{Ind}\big(\mathrm{Perf}^{\mathrm{ad}}(X)(A)\big) \in \mathcal{C}\mathrm{at}_\infty,\\
\mathrm{Coh}^+(X)^{\mathrm{ad}}(A) &\coloneqq \mathrm{Fun}_{\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)}\big(X, \mathrm{Coh}^+(A)\big) \in \mathcal{C}\mathrm{at}_\infty\\
\mathrm{Vect}(X)^{\mathrm{ad}}(A) &\coloneqq \mathrm{Fun}_{\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)}\big(X, \mathrm{Vect}(A)\big) \in \mathcal{C}\mathrm{at}_\infty,
\end{aligned}$$

where $\mathrm{Vect}(A) \subseteq \mathrm{Perf}(A)$ denotes the full subcategory spanned by free A-modules.

Remark 5.4.1.7. The ∞-category $\mathrm{Perf}^{\mathrm{ad}}(X)(A)$ can be identified with

$$\begin{aligned}
\mathrm{Perf}^{\mathrm{ad}}(X)(A) &\simeq \mathrm{Fun}_{\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)}\big(X, \mathrm{Perf}(A)\big)\\
&\simeq \lim_{n \geq 1} \mathrm{Fun}_{\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)}\big(X, \mathrm{Perf}(A_n)\big)\\
&\simeq \lim_{n \geq 1} \operatorname*{colim}_{X_i \in \mathcal{S}_{/f}} \big(X_i, \mathrm{Perf}(A_n)\big)
\end{aligned}$$

in the ∞-category $\mathcal{C}\mathrm{at}_\infty$.

Remark 5.4.1.8. By construction, the ∞-category $\mathrm{Mod}^{\mathrm{ad}}(X)(A)$ is compactly generated and the compact objects span the full subcategory $\mathrm{Perf}^{\mathrm{ad}}(X)(A) \subseteq \mathrm{Mod}^{\mathrm{ad}}(X)(A)$.

Definition 5.4.1.9. Let $\mathcal{C}$ be an additive symmetric monoidal ∞-category. Let $R \in \mathcal{C}\mathrm{Alg}$ be a commutative derived ring and consider its derived ∞-category of modules $\mathrm{Mod}_R \in \mathcal{C}\mathrm{at}^{\otimes}_\infty$. We say that $\mathcal{C}$ is *equipped with an A-linear action* if there exists a finite direct sum preserving symmetric monoidal functor

$$F \colon \mathrm{Mod}^{\mathrm{ff}}_R \to \mathcal{C}$$

see [Lur16, Definition D.1.1.1] for a definition. If $\mathcal{C}$ is presentable, then the datum of a linear R-action is equivalent to the existence of a colimit preserving symmetric monoidal functor

$$F \colon \mathrm{Mod}^{\mathrm{cn}}_R \to \mathcal{C}$$

see [Lur16, Remark D.1.1.5]. If moreover, $\mathcal{C}$ is presentable and stable, then the datum of a linear R-action on $\mathcal{C}$ is equivalent to give a colimit preserving monoidal functor

$$F \colon \mathrm{Mod}_R \to \mathcal{C}.$$

Proposition 5.4.1.10. *Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. The ∞-category $\mathrm{Mod}^{\mathrm{ad}}(X)(A)$ is a symmetric monoidal presentable A-linear stable ∞-category. The ∞-categories $\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)$ and $\mathrm{Perf}^{\mathrm{ad}}(X)(A)$ are both symmetric monoidal A-linear idempotent complete stable ∞-categories and the former admits a canonical t-structure. The ∞-category $\mathrm{Vect}^{\mathrm{ad}}(X)(A)$ is symmetric monoidal, admits an A-linear action and it is moreover additive.*

Proof. Let $X_i \in \mathcal{S}_{/f}$ and $n \geq 1$ an integer. Each transition functor

$$\mathrm{Fun}\big(X_i, \mathrm{Perf}(A_n)\big) \to \mathrm{Fun}\big(X_j, \mathrm{Perf}(A_n)\big)$$

is an exact functor between stable ∞-categories, see [Lur12c, Proposition 1.1.4.6]. Thanks to [Lur12c, Theorem 5.5.3.18] the colimit

$$\operatorname*{colim}_{T_i \in \mathcal{S}_{/f}} \mathrm{Fun}\big(X_i, , \mathrm{Perf}(A_n)\big) \tag{5.4.1.1}$$

is again a stable ∞-category, as the transition maps are exact. Furthermore, each ∞-category

$$\mathrm{Fun}\big(X_i, \mathrm{Perf}(A_n)\big)$$

admits a symmetric monoidal structure which is induced by the one on $\mathrm{Perf}(A_n)$ objectwise. Since the transition maps above are symmetric monoidal functors, one concludes that the ∞-category in **??** is naturally endowed with a symmetric monoidal structure and by construction it is A_n-linear. Each of the transition functors

$$\operatorname*{colim}_{X_i \in \mathcal{S}_{/f}} \mathrm{Fun}\big(X_i, \mathrm{Perf}(A_n)\big) \to \operatorname*{colim}_{X_i \in \mathcal{S}_{/f}} \mathrm{Fun}\big(X_i, \mathrm{Perf}(A_m)\big)$$

are colimit preserving and exact. The fact that $\mathrm{Perf}^{\mathrm{ad}}(X)(A)$ is idempotent complete follows by stability of idempotent completion under filtered colimits [Lur09b, Propositin 4.4.5.21] and limits of ∞-categories. Therefore, thanks to [Lur12c, Proposition 1.1.4.4] one deduces that the limit

$$\lim_n \operatorname*{colim}_{X_i \in \mathcal{S}_{/f}} \mathrm{Fun}\big(X_i, \mathrm{Perf}(A_n)\big) \tag{5.4.1.2}$$

is stable, as desired. The fact the ∞-category displayed in (5.4.1.2) is symmetric monoidal follows from our previous considerations together with [Lur09b, Proposition 3.3.3.2]. By taking ind-completion one deduces that $\mathrm{Mod}^{\mathrm{ad}}(X)(A)$ is presentable. The statements for $\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)$ and $\mathrm{Vect}(X)^{\mathrm{ad}}(A)$ are similar but easier. $\square$

Remark 5.4.1.11. The ∞-category $\mathrm{Perf}^{\mathrm{ad}}(X)(A)$ is rigid, i.e. every object in $\mathrm{Perf}^{\mathrm{ad}}(X)(A)$ is dualizable, as the tensor product is computed objectwise. We conclude that the conditions in [Lur16, Definition D.7.4.1] are verified, thus $\mathrm{Mod}^{\mathrm{ad}}(X)(A)$ is a locally rigid ∞-category.

Construction 5.4.1.12. We have a functor

$$\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \times \mathbb{N}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$$

given on objects by the formula

$$(A, n) \mapsto \mathrm{Perf}(A_n) \in \mathcal{C}\mathrm{at}_\infty.$$

Thanks to [GHN15a, Lemma 6.2 and Example 6.3] the association

$$(\mathcal{C}, \mathcal{D}) \in \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty \mapsto \mathrm{Fun}\big(\mathcal{C}, \mathcal{D}\big) \in \mathcal{C}\mathrm{at}_\infty$$

is functorial. Thus we can consider the composite

$$F \colon \mathcal{S}^{\mathrm{op}} \times \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \times \mathbb{N}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$$

given on objects by the formula

$$(X, A, n) \mapsto \mathrm{Fun}\big(X, \mathrm{Perf}(A_n)\big) \in \mathcal{C}\mathrm{at}_\infty.$$

Via straightening we obtain a coCartesian fibration

$$\mathcal{D} \to \mathcal{S} \times \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \times \mathbb{N}^{\mathrm{op}}.$$

Given a cofiltered diagram $h \colon I \to \mathcal{S}$ we can consider the pullback diagram

$$\begin{tikzcd}
\mathcal{D}_I \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
I^{\mathrm{op}} \times \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathbb{N}^{\mathrm{op}} \arrow[r] & \mathcal{S}^{\mathrm{op}} \times \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \times \mathbb{N}^{\mathrm{op}}.
\end{tikzcd}$$

Thus we obtain a coCartesian fibration

$$\mathcal{D}_I \to I^{\mathrm{op}} \times \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \times \mathbb{N}^{\mathrm{op}}$$

whose fiber at $(i, A, n) \in I^{\mathrm{op}} \times \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \times \mathbb{N}^{\mathrm{op}}$ can be identified with the ∞-category

$$\mathrm{Fun}\big(h(i), \mathrm{Perf}(A_n)\big) \in \mathcal{C}\mathrm{at}_\infty.$$

Furthermore, the composition

$$\mathcal{D}_I \to \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \to \mathbb{N}^{\mathrm{op}},$$

under the natural projection $I^{\mathrm{op}} \times \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \to \mathbb{N}^{\mathrm{op}} \to \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \to \mathbb{N}^{\mathrm{op}}$, is a coCartesian fibration whose fiber at (A, n) is naturally equivalent to

$$\operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathrm{Fun}\big(h(i), \mathrm{Perf}(A_n)\big).$$

Unstraightening produces a functor

$$F \colon \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \times \mathbb{N}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$$

given on objects by the formula

$$(A, n) \mapsto \operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathrm{Fun}\big(h(i), \mathrm{Perf}(A_n)\big) \in \mathcal{C}\mathrm{at}_\infty.$$

Composing it with the projection functor $\mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \times \mathbb{N}^{\mathrm{op}} \to \mathbb{N}^{\mathrm{op}}$ produces a coCartesian fibration

$$\mathcal{D}_I \to \mathbb{N}^{\mathrm{op}}.$$

Consider the ∞-category of coCartesian sections

$$\mathrm{Map}^\flat\big(\mathbb{N}^{\mathrm{op}}, \mathcal{D}_I\big) \in \mathcal{C}\mathrm{at}_\infty.$$

We have a canonical functor

$$\begin{aligned} g \colon \mathrm{Map}^\flat\big(\mathbb{N}^{\mathrm{op}}, \mathcal{D}_I\big) &\to \mathrm{Map}^\flat\big(\mathbb{N}^{\mathrm{op}}, \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \times \mathbb{N}^{\mathrm{op}}\big) \\ &\simeq \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}. \end{aligned}$$

The coCartesian fibration g produces a well defined functor, up to contractible indeterminacy,

$$\mathrm{Perf}^{\mathrm{ad}}(X) \colon \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \to \mathcal{C}\mathrm{at}_\infty$$

given on objects by the formula

$$A \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \mapsto \mathrm{Perf}^{\mathrm{ad}}(X)(A) \in \mathcal{C}\mathrm{at}_\infty.$$

Similarly, we can define functors

$$\mathrm{Mod}^{\mathrm{ad}}(X), \quad \mathrm{Coh}^+(X)^{\mathrm{ad}}, \quad \mathrm{Vect}^{\mathrm{ad}}(X) \colon \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \to \mathcal{C}\mathrm{at}_\infty$$

given on objects by the formulas

$$\begin{aligned} A \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} &\mapsto \mathrm{Mod}^{\mathrm{ad}}(X)(A) \in \mathcal{C}\mathrm{at}_\infty, \\ A \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} &\mapsto \mathrm{Coh}^+(X)^{\mathrm{ad}}(A) \in \mathcal{C}\mathrm{at}_\infty, \\ A \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} &\mapsto \mathrm{Vect}^{\mathrm{ad}}(X)(A) \in \mathcal{C}\mathrm{at}_\infty, \end{aligned}$$

respectively.

5.4.2 Geometric properties of $\mathrm{Perf}^{\mathrm{ad}}(X)$

In this §we prove that $\mathrm{Perf}^{\mathrm{ad}}(X)$ has a rich geometrical information, namely it satisfies hyper-descent, it is nilcomplete and cohesive and it admits a global k°-adic cotangent complex.

Definition 5.4.2.1. We equip the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ with the étale topology. Denote by $(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}, \tau_{\text{ét}})$ the corresponding étale ∞-site. Let $\mathrm{P_{sm}}$ denote the class of smooth morphisms in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. The triple $(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}, \tau_{\text{ét}}, \mathrm{P_{sm}})$ forms a geometric context, which we refer to as the k°-adic geometric context. The ∞-category of geometric stacks on $(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}, \tau_{\text{ét}}, \mathrm{P_{sm}})$ is denoted as $\mathrm{dSt}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}, \tau_{\text{ét}}, \mathrm{P_{sm}})$.

Lemma 5.4.2.2. *The pre-sheaf* $\mathrm{Perf}^{\mathrm{ad}}(X)\colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathcal{C}\mathrm{at}_\infty$ *satisfies étale hyper-descent.*

Proof. Let $A^\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ denote an hyper-covering of a given derived k°-adic algebra $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. We have thus an equivalence

$$A \simeq \lim_{[n]\in \mathbf{\Delta}^{\mathrm{op}}} A^{[n]}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. Let $m \geq 1$ be an integer. Modding out by p^m produces an étale hyper-covering

$$A^\bullet_m\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}\mathrm{Alg}_{k^\circ_m}$$

in the ∞-site $(\mathcal{C}\mathrm{Alg}_{k^\circ_n}, \tau_{\text{ét}})$. Therefore, we have an equivalence

$$A_m \simeq \lim_{[n]\in \mathbf{\Delta}^{\mathrm{op}}} A^{[n]}_m$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ_m}$. Therefore, we have a chain of equivalences of the form

$$\begin{aligned}
\mathrm{Perf}^{\mathrm{ad}}(X)(\lim_{[n]\in \mathbf{\Delta}^{\mathrm{op}}} A^{[n]}) &\simeq \lim_{[n]\in \mathbf{\Delta}^{\mathrm{op}}} \lim_{k\geq 1} \mathrm{Fun}_{\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)}\big(X, \mathrm{Perf}(A^{[n]}_k)\big) \\
&\simeq \lim_{m\geq 1} \lim_{[n]\in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}_{\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)}\big(X, \mathrm{Perf}(A^{[n]}_k)\big) \\
&\simeq \lim_{[n]\in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{Perf}^{\mathrm{ad}}(X)(A)
\end{aligned}$$

where we used in a crucial way the fact that $\mathrm{Perf}\colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathcal{C}\mathrm{at}_\infty$ satisfies étale hyper-descent. □

Proposition 5.4.2.3. *The stack* $\mathrm{Perf}^{\mathrm{ad}}(X)\colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathcal{C}\mathrm{at}_\infty$ *is cohesive and nilcomplete.*

Before proving Theorem 5.4.2.3 we prove first some preliminary results:

Lemma 5.4.2.4. *Let* $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ *be* p*-complete and* $M \in \mathrm{Coh}^+(A)$*, then we have a canonical equivalence*

$$\Omega^\infty_{\mathrm{ad}}(M) \simeq \lim_{n\geq 1} \Omega^\infty_{\mathrm{ad}}(M \otimes_A A_n)$$

where $\Omega^\infty_{\mathrm{ad}}\colon \mathrm{Coh}^+(A) \to (\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ})_{A/}$ *is the projection functor introduced in [Ant18b, §3.3].*

Proof. By definition $\Omega^\infty_{\mathrm{ad}}(M) \simeq A \oplus M$. As M is eventually connective it follows that we have an equivalence

$$M \simeq \lim_{n\geq 1} M \otimes_A A_n.$$

The functor $\Omega^\infty_{\mathrm{ad}}$ being a right adjoint commutes with limits. As a consequence, we have a chain of equivalences of the form

$$\begin{aligned}
A \oplus M &\simeq \Omega^\infty_{\mathrm{ad}}(M) \\
&\simeq \Omega^\infty_{\mathrm{ad}}(\lim_{n\geq 1} M \otimes_A A_n) \\
&\simeq \lim_{n\geq 1} \Omega^\infty_{\mathrm{ad}}(M \otimes_A A_n)
\end{aligned}$$

in the ∞-category $\mathrm{Coh}^+(A)$. Moreover, the morphism $A \to A_n$ induces a canonical equivalence of functors

$$(- \otimes_A A_n) \circ \Omega^\infty_{\mathrm{ad}} \simeq \Omega^\infty_{\mathrm{ad}} \circ (- \otimes_A A_n).$$

The result now follows by the fact that $A \oplus M$ is p-complete together with a standard cofinality argument. □

Proof of Theorem 5.4.2.3. We first treat cohesiveness of $\mathrm{Perf}^{\mathrm{ad}}(X)$. The pre-sheaf of perfect complexes

$$\mathrm{Perf}\colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathcal{C}\mathrm{at}_\infty,$$

is cohesive thanks to [Lur12a, Proposition 3.4.10] together with Theorem 5.4.2.4. As both filtered colimits and limits commute with fiber products, we deduce that the same is true for $\mathrm{Perf}^{\mathrm{ad}}(X)$.

We now prove that $\mathrm{Perf}^{\mathrm{ad}}(X)$ is nilcomplete. Fix an integer $n \geq 1$, the pre-sheaf of perfect modules

$$\mathrm{Perf}\colon \mathcal{C}\mathrm{Alg}_{k^\circ_n} \to \mathcal{C}\mathrm{at}_\infty$$

is nilcomplete, thanks to [Lur12a, Proposition 3.4.10]. This implies that given $A_n \in \mathcal{C}\mathrm{Alg}_{k^\circ_n}$ we have natural equivalences

$$\mathrm{Perf}(A_n) \simeq \lim_{m\geq 0} \mathrm{Perf}(\tau_{\leq m} A_n).$$

Thanks to the $\mathcal{C}\mathrm{at}_\infty$-enriched version of [Lur09b, Proposition 5.3.5.3] the inclusion

$$\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty) \hookrightarrow \mathcal{P}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{op}})^{\mathrm{op}}$$

preserves cofiltered limits, where the latter denotes the ∞-category of pre-sheaves on $\mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}$. Therefore, cofiltered limits in $\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)$ can be computed objectwise. Thus given $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ it follows that we have natural equivalences

$$\begin{aligned}
\mathrm{Perf}^{\mathrm{ad}}(X)(A) &\simeq \mathrm{Fun}_{\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)}\big(X, \mathrm{Perf}(A)\big)\\
&\mathrm{Fun}_{\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)}\big(X, \lim_{n\geq 0} \mathrm{Perf}(\tau_{\leq n} A)\big)\\
&\lim_{\lim_{n\geq 0}} \mathrm{Fun}_{\mathrm{Pro}(\mathcal{C}\mathrm{at}_\infty)}\big(X, \mathrm{Perf}(\tau_{\leq m} A)\big)\\
&\lim_{\lim_{n\geq 0}} \mathrm{Perf}^{\mathrm{ad}}(X)(\tau_{\leq m}),
\end{aligned}$$

as the functor $\tau_{\leq m}\colon \mathcal{C}\mathrm{Alg}_{k^\circ} \to \mathcal{C}\mathrm{Alg}_{k^\circ}$ is a left adjoint and therefore commutes with pushouts, thus

$$\tau_{\leq m} A_n \simeq (\tau_{\leq m} A)_n.$$

□

Remark 5.4.2.5. The above result holds true with an analogous proof for the functors $\mathrm{Vect}^{\mathrm{ad}}(X)$ and $\mathrm{Coh}^+(X)^{\mathrm{ad}}$. However, the result does not hold for $\mathrm{Mod}^{\mathrm{ad}}(X)$ as it is not true already in the discrete case.

We now devote ourselves to the computation of a cotangent complex for $\mathrm{Perf}^{\mathrm{ad}}(X)$. We will need a few preliminary results first.

Proposition 5.4.2.6. *Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ and $M \in \mathrm{Coh}^+(A)$ which we assume furthermore to be p-torsion free. Let $\rho \in \mathrm{Perf}^{\mathrm{ad}}(X)(A)$. Given $\rho' \in \mathrm{Perf}^{\mathrm{ad}}(X)(A \oplus M)$ together with a morphism*

$$\theta\colon \rho' \to \rho \otimes_A (A \oplus M)$$

in the ∞-category $\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)$, which we assume to be an equivalence after base change along the canonical morphism

$$A \to A \oplus M,$$

in the ∞-category $\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)$. Then θ is an equivalence in the ∞-category $\mathrm{Coh}^+(X)^{\mathrm{ad}}(A \oplus M)$.

Proof. It suffices to prove the result in the case where

$$\alpha\colon \rho' \to \rho \otimes_A (A \oplus M)$$

in the ∞-category $\mathrm{Perf}^{\mathrm{ad}}(X)(A \oplus M)$ coincides with the identify morphism

$$\mathrm{Id}\colon \rho \to \rho$$

in the ∞-category $\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)$. Consider the cofiber $\mathrm{cofib}(\alpha) \in \mathrm{Coh}^+(X)^{\mathrm{ad}}(A \oplus M)$, it is a dualizable object in the ∞-category $\mathrm{Coh}^+(X)^{\mathrm{ad}}(A \oplus M)$ and its image in $\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)$ is equivalent to the zero object. We wish to prove that the coevaluation morphism

$$\mathrm{coev} \colon A \oplus M \to \mathrm{cofib}(\alpha) \otimes \mathrm{cofib}(\alpha)^\vee,$$

is the zero map in the ∞-category $\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)$. Consider the inclusion morphism

$$A \to A \oplus M$$

in $\mathrm{Mod}(A)$. By naturality of taking tensor products we obtain that tensoring the above morphism with the coevaluation morphism induces a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "\mathrm{coev} \otimes A"] \arrow[d] & (\mathrm{cofib}(\alpha) \otimes \mathrm{cofib}(\alpha)^\vee) \otimes A \arrow[d] \\ A \oplus M \arrow[r, "\mathrm{coev}"] & \mathrm{cofib}(\alpha) \otimes \mathrm{cofib}(\alpha)^\vee. \end{tikzcd}$$

Since $\mathrm{coev} \otimes A$ corresponds to the coevaluation morphism of the dualizable object

$$\mathrm{cofib}(\alpha) \otimes_{A \oplus M} A \in \mathrm{Coh}^+(X)^{\mathrm{ad}}(A)$$

it coincides with the identity morphism

$$\mathrm{cofib}(\alpha) \otimes A \to \mathrm{cofib}(\alpha) \otimes A,$$

which is the zero morphism, by our assumption on α. It follows that the A-linear morphism

$$A \to \mathrm{cofib}(\alpha) \otimes \mathrm{cofib}(\alpha)^\vee$$

is the zero morphism, and thus by adjunction (with respect to the extension and restriction of scalars along $A \to A \oplus M$), the coevaluation map

$$\mathrm{coev} \colon A \oplus M \to \mathrm{cofib}(\alpha) \otimes \mathrm{cofib}(\alpha)^\vee$$

is the zero morphism, as desired. Thus $\mathrm{cofib}(\alpha) \simeq 0$ in the ∞-category $\mathrm{Coh}^+(X)^{\mathrm{ad}}(A \oplus M)$, as desired. $\square$

Proposition 5.4.2.7. *Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ and $M \in \mathrm{Coh}^+(A)$ a p-torsion free almost perfect A-module. Given $\rho \in \mathrm{Perf}^{\mathrm{ad}}(X)(A)$ the fiber at ρ of the natural functor*

$$(p_{A,\rho})^\simeq \colon \mathrm{Perf}^{\mathrm{ad}}(X)(A \oplus M)^\simeq \to \mathrm{Perf}^{\mathrm{ad}}(X)(A)^\simeq$$

can be identified canonically with

$$\mathrm{fib}_\rho(p_{A,\rho}) \simeq \mathrm{Map}_{\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)}\big(\rho \otimes \rho^\vee, M[1]\big).$$

Proof. The canonical functor $p_{A,M}$ exhibits $\mathrm{Perf}^{\mathrm{ad}}(X)(A \oplus M)$ as an object in the ∞-category $\mathcal{S}_{/\mathrm{Perf}^{\mathrm{ad}}(X)(A)^{\mathrm{ad}}}$. This is a right fibration of spaces and thus it induces a functor

$$p_M \colon \mathrm{Perf}^{\mathrm{ad}}(X)(A)^\simeq \to \mathcal{S}.$$

Given $\rho \in \mathrm{Perf}^{\mathrm{ad}}(X)(A)$ we have an equivalence of spaces

$$p_M(\rho) \simeq \mathrm{fib}_\rho(p_{A,M}).$$

Thanks to Theorem 5.4.2.6 and its proof the loop space based at the identify of the object

$$\rho \otimes_A (A \oplus M) \in \mathrm{Coh}^+(X)^{\mathrm{ad}}(A \oplus M),$$

which we denote simply by $\Omega(p_m(\rho))$, can be identified with

$$\Omega(p_M(\rho)) \simeq$$
$$\simeq \mathrm{fib}_{\mathrm{Id}_\rho}\big(\mathrm{Map}_{\mathrm{Coh}^+(X)^{\mathrm{ad}}(A\oplus M)}\big(\rho \otimes (A \oplus M), \rho \otimes (A \oplus M)\big) \to \mathrm{Map}_{\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)}\big(\rho, \rho)\big).$$

We denote the latter object simply by $\mathrm{Map}_{/\rho}\big(\rho \otimes_A (A \oplus M), \rho \otimes_A (A \oplus M)\big)$. Since the underlying A-module of $\rho \otimes_A (A \oplus M)$ can be identified with $\rho \oplus \rho \otimes_A M$, we have a chain of natural equivalences of mapping spaces of the form

$$\begin{aligned}\mathrm{Map}_{/\rho}\big(\rho \otimes_A (A \oplus M), \rho \otimes_A (A \oplus M)\big) &\simeq \\ &\simeq \mathrm{Map}_{\mathrm{Coh}^+(X)(A)_{/\rho}}\big(\rho \otimes_A (A \oplus M), \rho \otimes_A (A \oplus M)\big) \\ &\simeq \mathrm{Map}_{\mathrm{Coh}^+(X)(A)_{/\rho}}\big(\rho, \rho \otimes_A (A \oplus M)\big)\end{aligned}$$

where the latter mapping space is pointed at the zero morphism. Since $\rho \in \mathrm{Perf}^{\mathrm{ad}}(X)(A)$ is a dualizable object we have an identification of mapping spaces

$$\mathrm{Map}_{\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)}\big(\rho, \rho \otimes_A M\big) \simeq \mathrm{Map}_{\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)}\big(\rho \otimes \rho^\vee, M\big).$$

Consider teh pullback diagram of derived k°-adic extensions

$$\begin{tikzcd} A \oplus M \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & A \oplus M[1] \end{tikzcd}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. As $\mathrm{Perf}^{\mathrm{ad}}(X)$ is cohesive and the right adjoint $(-)^\simeq \colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ commutes with limits we obtain a pullback diagram of the form

$$\begin{tikzcd} \mathrm{Perf}^{\mathrm{ad}}(X)(A \oplus M) \arrow[r] \arrow[d] & \mathrm{Perf}^{\mathrm{ad}}(X)(A) \arrow[d] \\ \mathrm{Perf}^{\mathrm{ad}}(X)(A) \arrow[r] & \mathrm{Perf}(A \oplus M[1]) \end{tikzcd}$$

in the ∞-category $\mathcal{S}$. By taking fibers at $\rho \in \mathrm{Perf}^{\mathrm{ad}}(X)(A)$ we have a pullback diagram of spaces

$$\begin{tikzcd} p_M(\rho) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & p_{M[1]}(\rho) \end{tikzcd}$$

in the ∞-category $\mathcal{S}$. By our previous computations, replacing M with the shift $M[1]$ produces the chain of equivalences

$$\begin{aligned}p_M(\rho) &\simeq \\ &\simeq \Omega(p_{M[1]}(\rho)) \\ &\simeq \mathrm{Map}_{\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)}\big(\rho \otimes \rho^\vee, M[1]\big),\end{aligned}$$

as desired. $\square$

Definition 5.4.2.8. Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ be a connected profinite space. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ and $\rho \in \mathrm{Perf}^{\mathrm{ad}}(X)(A)$. We say that X is *locally p-cohomologically perfect at* ρ if the object

$$\mathrm{Map}_{\mathrm{Perf}^{\mathrm{ad}}(X)(A)}(1, \rho) \in \mathrm{Sp}$$

where $1 \in \mathrm{Perf}^{\mathrm{ad}}(X)(A)$ denotes the unit for the symmetric monoidal strucutre, equipped with its canonical A-linear action is equivalent to a perfect A-module. We say that X is cohomologically perfect if it is locally cohomologically perfect for every $\rho \in \mathrm{Perf}^{\mathrm{ad}}(X)(A)$ for every admissible derived k°-adic algebra $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$.

Proposition 5.4.2.9. *Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ be a p-cohomologically perfect profinite space. Then for every $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ and every $\rho \in \mathrm{Perf}^{\mathrm{ad}}(X)(A)$ the functor*

$$F \colon \mathrm{Coh}^+(A) \to \mathcal{S}$$

given informally via the association

$$M \in \mathrm{Coh}^+(A) \mapsto \mathrm{fib}_\rho\big(\mathrm{Perf}^{\mathrm{ad}}(X)(A \oplus M) \to \mathrm{Perf}^{\mathrm{ad}}(X)(A)\big) \in \mathcal{S}$$

is corepresentable by the A-module

$$\mathrm{Map}_{\mathrm{Perf}^{\mathrm{ad}}(X)(A)}\big(1, \rho \otimes \rho^\vee[1]\big)^\vee \in \mathrm{Mod}_A$$

Proof. We first prove the following assertion: let $\mathcal{C}$ be an A-linear stable presentable ∞-category, and $C \in \mathcal{C}$ denote a compact object of $\mathcal{C}$. Then for every object $M \in \mathrm{Mod}_A$ we have an equivalence

$$\mathrm{Map}_{\mathcal{C}}\big(C, M\big) \simeq \mathrm{Map}_{\mathcal{C}}\big(C, 1_{\mathcal{C}}\big) \otimes_A M,$$

in the ∞-category Mod_A, where $1_{\mathcal{C}}$ denotes the unit for symmetric monoidal structure on $\mathcal{C}$. Let $\mathcal{D} \subseteq \mathrm{Mod}_A$ denote the full subcategory spanned by those A-modules M such that the assertion holds true. Clearly $A \in \mathcal{D}$. Since $A \in \mathrm{Mod}_A$ generates the ∞-category Mod_A under small colimits, it suffices to show that $\mathcal{D}$ is closed under small colimits. Suppose that

$$M \simeq \operatorname*{colim}_{i \in I} M_i,$$

such that $M_i \in \mathcal{D}$ and I is a filtered ∞-category. Then by our compactness assumption it follows that we have a chain of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathcal{C}}(C, M) &\simeq \mathrm{Map}_{\mathcal{C}}\big(C, \operatorname*{colim}_i M_i\big) \\
&\simeq \operatorname*{colim}_i \mathrm{Map}_{\mathcal{C}}\big(C, M_i\big) \\
&\simeq \operatorname*{colim}_i \mathrm{Map}_{\mathcal{C}}\big(C, 1\big) \otimes_A M_i \\
&\simeq \mathrm{Map}_{\mathcal{C}}\big(C, 1\big) \otimes_A (\operatorname*{colim}_i M_i) \\
&\simeq \mathrm{Map}_{\mathcal{C}}\big(C, 1_{\mathcal{C}}\big) \otimes_A M.
\end{aligned}$$

Thus $\mathcal{D}$ is closed under filtered colimits. It suffices to show to show then that $\mathcal{D}$ is closed under finite colimits. Since Mod_A is a stable ∞-category it suffices to show that $\mathcal{D}$ is closed under finite coproducts and cofibers. Let

$$f \colon C \to D$$

be a morphism in $\mathcal{D}$, we wish to show that $\mathrm{cofib}(f)\mathcal{D}$. Thanks to [Lur12c, Theorem 1.1.2.14] we have an equivalence

$$\mathrm{cofib}(f) \simeq \mathrm{fib}(f)[1].$$

As a consequence, we can write

$$\begin{aligned}
\mathrm{Map}_{\mathcal{C}}\big(C, \mathrm{cofib}(f)\big) &\simeq \mathrm{Map}_{\mathcal{C}}\big(C, \mathrm{fib}(f)[1]\big) \\
&\simeq \mathrm{fib}\big(\mathrm{Map}_{\mathcal{C}}\big(C, f\big)\big)[1] \\
&\simeq \mathrm{cofib}\big(\mathrm{Map}_{\mathcal{C}}\big(C, f\big)\big) \\
&\simeq \mathrm{Map}_{\mathcal{C}}\big(C, 1\big) \otimes_A \mathrm{cofib}(f).
\end{aligned}$$

The case of coproducts follows along the same lines and it is easier. From this we conclude that

$$\mathcal{D} \simeq \mathrm{Mod}_A,$$

as desired. In our case, let $\mathcal{C} := \mathrm{Mod}^{\mathrm{ad}}(X)(A)$ and observe that the assertion implies that for every $M \in \mathrm{Mod}_A$ we have a chain of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Mod}^{\mathrm{ad}}(X)(A)}\big(\rho \otimes \rho^\vee, M[1]\big) &\simeq \mathrm{Map}_{\mathrm{Mod}^{\mathrm{ad}}(X)(A)}\big(\Omega(\rho \otimes \rho^\vee), 1\big) \otimes_A M \\
&\mathrm{Map}_{\mathrm{Mod}^{\mathrm{ad}}(X)(A)}\big(1, \rho \otimes \rho^\vee[1]\big) \otimes_A M.
\end{aligned}$$

Furthermore, it follows by our hypothesis on X that

$$C := \mathrm{Map}_{\mathrm{Mod}^{\mathrm{ad}}(X)(A)}\big(1, \rho \otimes \rho^\vee[1]\big) \in \mathrm{Mod}_A$$

is a perfect A-module. Thus we have a chain of equivalences

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mod}_A}\big(C^\vee, M\big) &\simeq \mathrm{Map}_{\mathrm{Mod}_A}\big(A, C \otimes_A M\big)\\ &\mathrm{Map}_{\mathrm{Perf}^{\mathrm{ad}}(X)(A)}\big(1, (\rho \otimes \rho)^\vee \otimes_A M\big)\\ &\mathrm{Map}_{\mathrm{Perf}^{\mathrm{ad}}(X)(A)}\big(\rho \otimes \rho^\vee, M[1]\big)\end{aligned}$$

and the result now follows by Theorem 5.4.2.9. □

5.5 Enriched ∞-categories

In this §we will state and prove the results in the theory of enriched ∞-categories that will prove to be more useful for us in the body of the present text. We will follow mainly the expositions presented in [GH15] and [Lur12c, §4.2].

5.5.1 Preliminaries on $\mathrm{Pro}(\mathcal{S})$ and $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$-enriched ∞-categories

We will moreover be more interested in the case where the enrichments are over the ∞-categories $\mathrm{Pro}(\mathcal{S})$ and $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$ of pro-objects and ind-pro-objects in $\mathcal{S}$, respectively. The following remark states that both definitions [Lur12c, Definition 4.2.25] and [GH15, Definition 5.4.3] are equivalent in the Cartesian symmetric monoidal case:

Remark 5.5.1.1. Recall that in [Lur12c, Definition 4.2.5.25] the notion of a enrichment is defined as a *pseud-enrichment* $p\colon \mathcal{C}^\otimes \to \mathcal{LM}^\otimes$ together with the requirement that for any two objects $M,\ N \in \mathcal{C}_{\mathfrak{m}}$ we have a functorial *morphism space object* $\mathrm{Mor}\big(M, N\big) \in \mathcal{C}_{\mathfrak{a}}$. Following a private communication with David Gepner, whenever $\mathcal{C}_{\mathfrak{a}}$ is a Cartesian symmetric monoidal ∞-category, both definitions [GH15, Definition 5.4.3] and [GH15, Definition 7.2.14] are equivalent. Thanks to this fact, we can consider the ∞-category of $\mathcal{C}_{\mathfrak{a}}$-enriched ∞-categories, $\mathcal{C}\mathrm{at}_\infty\big(\mathrm{Pro}(\mathcal{S})\big)$ as the full subcategory of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{C}_{\mathfrak{a}})$ spanned by those complete objects, in the sense of [GH15, Definition 4.3.1].

We start with a general Lemma which will be helpful for us:

Lemma 5.5.1.2. *Let $\mathcal{V}^\otimes$ be a presentably symmetric monoidal ∞-category. Suppose we are given a small diagram $F\colon I \to \mathcal{C}\mathrm{at}_\infty(\mathcal{V}^\otimes)$. Then the limit $\mathcal{C} := \lim_I F$ exists in the ∞-category $\mathcal{C}\mathrm{at}_\infty(\mathcal{V}^\otimes)$. Furthermore, given any two objects $x,\ y \in \mathcal{C}$ we have an equivalence of mapping objects*

$$\mathcal{C}(x, y) \simeq \lim_{i \in I} \mathcal{C}_i(x_i, y_i) \in \mathcal{V}^\otimes$$

where $\mathcal{C}_i := F(i)$, for each $i \in I$ and $x_i,\ y_i$ denote the images of both x and y under the projection functor $\mathcal{C} \to \mathcal{C}_i$, respectively.

Proof. We use the notations of [GH15]. In this case, we have a chain of equivalences in $\mathcal{V}^\otimes$

$$\begin{aligned}\mathcal{C}(x, y) &\simeq\\ &\simeq \underline{\mathrm{Map}}_{* \coprod */\mathcal{C}\mathrm{at}_\infty(\mathcal{V}^\otimes)}\big(E^1, \mathcal{C}\big)\\ &\simeq \lim_{i \in I} \underline{\mathrm{Map}}_{* \coprod */\mathcal{C}\mathrm{at}_\infty(\mathcal{V}^\otimes)}\big(E^1, \mathcal{C}_i\big)\\ &\simeq \lim_{i \in I} \mathcal{C}_i(x_i, y_i)\end{aligned}$$

where $\underline{\mathrm{Map}}$ denotes the internal mapping object in $\mathcal{V}^\otimes$. This finishes the proof of the statement. □

Construction 5.5.1.3. Let $\mathrm{Mat}\colon \mathrm{Pro}(\mathcal{S}) \to \mathcal{S}$ denote the materialization functor given on objects by the formula

$$X \in \mathrm{Pro}(\mathcal{S}) \mapsto \mathrm{Map}_{\mathrm{Pro}(\mathcal{S})}\big(*, X\big)$$

The functor Mat preserves limits and it thus lifts to a symmetric monoidal functor

$$\mathrm{Mat}^{\times}\colon \mathrm{Pro}(\mathcal{S})^{\times} \to \mathcal{S}^{\times},$$

where we consider the corresponding Cartesian symmetric monoidal structures on both ∞-categories. Furthermore, thanks to [GH15, Corollary 5.7.6] we have a realization functor

$$\mathrm{Mat}_{\mathrm{cat}}\colon \mathcal{C}\mathrm{at}_{\infty}\big(\mathrm{Pro}(\mathcal{S})\big) \to \mathcal{C}\mathrm{at}_{\infty}(\mathcal{S}).$$

We have an equivalence of ∞-categories $\mathcal{C}\mathrm{at}_{\infty}(\mathcal{S}) \simeq \mathcal{C}\mathrm{at}_{\infty}$, thanks to [GH15, Theorem 5.7.6]. Therefore, we obtain an induced functor

$$\mathrm{Mat}_{\mathrm{cat}}\colon \mathcal{C}\mathrm{at}_{\infty}\big(\mathrm{Pro}(\mathcal{S})\big) \to \mathcal{C}\mathrm{at}_{\infty}.$$

Similarly, we have a materialization functor $\mathrm{Mat}\colon \mathrm{Ind}(\mathrm{Pro}(\mathcal{S})) \to \mathcal{S}$ given on objects by the formula

$$X \in \mathrm{Ind}\big(\mathrm{Pro}(\mathcal{S})\big) \to \mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\big(*, X\big) \in \mathcal{S}.$$

By construction, this functor commutes with finite limits. Therefore, we are given a well defined functor

$$\mathcal{E}\mathcal{C}\mathrm{at}_{\infty} \to \mathcal{C}\mathrm{at}_{\infty}.$$

Construction 5.5.1.4. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ be a derived k°-adic algebra. Denote by $A_n \in \mathcal{C}\mathrm{Alg}_{k^\circ_n}$ the pushout of the diagram

$$\begin{tikzcd} A[u] \arrow[r, "u\mapsto 0"] \arrow[d, "u\mapsto t^n"] & A \arrow[d] \\ A \arrow[r] & A_n \end{tikzcd}$$

computed in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}$. For this reason, base change along the morphism of derived algebras

$$A \to A_n, \qquad \text{for each } n$$

induces a natural morphism

$$\mathrm{Perf}(A) \to \lim_{n\geq 1}\mathrm{Perf}(A_n),$$

which is an equivalence, thanks to [Lur16, Lemma 8.1.2.1]. For this reason, we can consider the stable ∞-category $\mathrm{Perf}(A)$ enriched over $\mathrm{Pro}(\mathcal{S})$, i.e. $\mathrm{Perf}(A) \in \mathcal{C}\mathrm{at}_{\infty}\big(\mathrm{Pro}(\mathcal{S})\big)$. Given $M \in \mathrm{Perf}(A)$ we can enhance $\mathrm{End}(M) \in \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S})$ with a pro-structure via the equivalence

$$\mathrm{End}(M) \simeq \lim_{n\geq 1}\mathrm{End}(M \otimes_A A_n).$$

Therefore, we can consider $\mathrm{End}(M)$ naturally as an object in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}\big(\mathrm{Pro}(\mathcal{S})\big)$.

Remark 5.5.1.5. We have a canonical functor

$$\mathrm{Pro}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_{\infty}\big(\mathrm{Pro}(\mathcal{S})\big)$$

induced by the universal property of the pro-construction together with the canonical inclusion functor $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_{\infty}$. For this reason, given $X \in \mathrm{Pro}(\mathcal{S})$ we can consider the ∞-category of continuous k°-adic representations of X defined as the functor ∞-category

$$\mathrm{Fun}_{\mathcal{C}\mathrm{at}_{\infty}(\mathrm{Pro}(\mathcal{S}))}\big(X, \mathrm{Perf}(A)\big) \in \mathcal{C}\mathrm{at}_{\infty}.$$

Suppose $X \in \mathrm{Pro}(\mathcal{S})$ is connected, i.e.

$$\pi_0\mathrm{Mat}(X) \simeq \pi_0\mathrm{Map}_{\mathrm{Pro}(\mathcal{S})}(*, X) \simeq *.$$

Then there exists a canonical functor $\pi_A\colon \mathrm{Fun}_{\mathcal{C}\mathrm{at}_{\infty}(\mathrm{Pro}(\mathcal{S}))}\big(X, \mathrm{Perf}(A)\big) \to \mathrm{Perf}(A)$ which sends a continuous A-adic representation

$$\rho\colon X \to \mathrm{Perf}(A)$$

to the underlying perfect A-module $M \coloneqq \rho(*) \in \mathrm{Perf}(A)$.

Lemma 5.5.1.6. *Let $X \in \operatorname{Pro}(\mathcal{S}^{\mathrm{fc}})$ be a connected profinite space. Given $A \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}$ and $M \in \operatorname{Perf}(A)$ the fiber of*

$$\pi_A \colon : \operatorname{Fun}_{\mathcal{C}\mathrm{at}_\infty(\operatorname{Pro}(\mathcal{S}))}\big(X, \operatorname{Perf}(A)\big) \to \operatorname{Perf}(A)$$

over $M \in \operatorname{Perf}(A)$ is canonical equivalent to the space

$$\operatorname{Map}_{\operatorname{Mon}_{\mathbb{E}_1}(\operatorname{Pro}(\mathcal{S}))}\big(\Omega X, \operatorname{End}(M)\big) \in \mathcal{S}.$$

Proof. Let $M \in \operatorname{Perf}(A)$. The fiber of π_A over A is equivalent to

$$\operatorname{Map}_{\operatorname{Pro}(\mathcal{S})}\big(X, \mathrm{B}\mathcal{E}\mathrm{nd}(M)\big) \in \mathcal{S}$$

and applying May's Theorem together with the limit colimit formula for mapping spaces in $\operatorname{Pro}(\mathcal{S})$ we obtain the canonical equivalence

$$\operatorname{Map}_{\operatorname{Pro}(\mathcal{S})}\big(X, \mathrm{B}\mathcal{E}\mathrm{nd}(M)\big) \simeq \operatorname{Map}_{\operatorname{Mon}_{\mathbb{E}_1}(\operatorname{Pro}(\mathcal{S}))}\big(\Omega X, \mathcal{E}\mathrm{nd}(M)\big)$$

of mapping spaces. The result now follows. □

Construction 5.5.1.7. The fully faithful embedding

$$\mathcal{S} \hookrightarrow \operatorname{Pro}(\mathcal{S})$$

induces a fully faithful functor

$$\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty\big(\operatorname{Pro}(\mathcal{S})\big)$$

which can be extended, via the universal property of pro-completion, by cofiltered limits to a functor

$$F \colon \operatorname{Pro}(\mathcal{C}\mathrm{at}_\infty) \to \mathcal{C}\mathrm{at}_\infty\big(\operatorname{Pro}(\mathcal{S})\big).$$

Moreover, given $A \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}$ the essential image of $\operatorname{Perf}(A) \in \operatorname{Pro}(\mathcal{C}\mathrm{at}_\infty)$ by F can be identified with $\operatorname{Perf}(A) \in \mathcal{C}\mathrm{at}_\infty(\operatorname{Pro}(\mathcal{S}))$ as in Theorem 5.5.1.4 as the latter can be identified with

$$\operatorname{Perf}(A) \simeq \lim_{n \geq 1} \operatorname{Perf}(A_n) \in \mathcal{C}\mathrm{at}_\infty(\operatorname{Pro}(\mathcal{S}))$$

and by construction F preserves cofiltered limits. Similarly, we have a commutative diagram of the form

$$\begin{tikzcd}
\operatorname{Pro}(\mathcal{S}) \arrow[rr] \arrow[dr] & & \operatorname{Pro}(\mathcal{C}\mathrm{at}_\infty) \arrow[dl, "F"] \\
& \mathcal{C}\mathrm{at}_\infty\big(\operatorname{Pro}(\mathcal{S})\big) &
\end{tikzcd}$$

and for this reason $X \in \operatorname{Pro}(\mathcal{S}) \subseteq \operatorname{Pro}(\mathcal{C}\mathrm{at}_\infty)$ is sent via F to $X \in \mathcal{C}\mathrm{at}_\infty(\operatorname{Pro}(\mathcal{S}))$. As a consequence the functor $F \colon \operatorname{Pro}(\mathcal{C}\mathrm{at}_\infty) \to \mathcal{C}\mathrm{at}_\infty(\operatorname{Pro}(\mathcal{S}))$ induces a well defined functor

$$\beta_A \colon \operatorname{Fun}_{\operatorname{Pro}(\mathcal{C}\mathrm{at}_\infty)}\big(X, \operatorname{Perf}(A)\big) \to \operatorname{Fun}_{\mathcal{C}\mathrm{at}_\infty(\operatorname{Pro}(\mathcal{S}))}\big(X, \operatorname{Perf}(A)\big).$$

Moreover, β_A is functorial in A and thus produces a well defined morphism

$$\beta \colon \operatorname{Perf}^{\mathrm{ad}}(X) \to \operatorname{Fun}_{\mathcal{C}\mathrm{at}_\infty(\operatorname{Pro}(\mathcal{S}))}\big(X, \operatorname{Perf}(-)\big)$$

in the ∞-category $\operatorname{Fun}\big(\mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}, \mathcal{C}\mathrm{at}_\infty\big)$.

We have now two potential definitions for the moduli stack $\operatorname{Perf}^{\mathrm{ad}}(X)$, namely the one provided in the previous §and the second one by defining it via the $\operatorname{Pro}(\mathcal{S})$-enriched ∞-categories approach. The following result implies that there is no ambiguity involved in chosen one of these:

Proposition 5.5.1.8. *Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ be a connected profinite space. Then the functor*

$$\beta \colon \mathrm{Perf}^{\mathrm{ad}}(X) \to \mathrm{Fun}_{\mathcal{C}\mathrm{at}_\infty(\mathrm{Pro}(\mathcal{S}))}\big(X, \mathrm{Perf}(-)\big)$$

is an equivalence in the ∞-category $\mathrm{Fun}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}, \mathcal{C}\mathrm{at}_\infty)$.

Before giving a proof of Theorem 5.5.1.8 we need a preliminary lemma:

Lemma 5.5.1.9. *Let $X \in \mathrm{Pro}(\mathcal{S})$ be a connected pro-space. Then $X \in \mathrm{Pro}(\mathcal{S}^{\leq 1})$, where the latter denotes the ∞-category of pro-objects in the ∞-category of connected spaces, $\mathcal{S}^{\leq 1}$.*

Proof. Let $X \in \mathrm{Pro}(cS)$ be indexed by a cofiltered ∞-category I. More explicitly, we can identify X with

$$X \simeq \lim_{i \in I} X_i$$

for suitable $X_i \in \mathcal{S}$. By construction, for each $i \in I$, we have an induced morphism

$$\lambda_i \colon X \to X_i.$$

By our hypothesis on X,, we conclude that each λ_i should factor through a connected component $X_i^\circ \subseteq X_i$. We can thus form the pro-system $\{X_i^\circ\}_{i \in I} \in \mathrm{Pro}(\mathcal{S})$, which lies in the essential image of the inclusion functor

$$\mathrm{Pro}(\mathcal{S}^{\leq 1}) \hookrightarrow \mathrm{Pro}(\mathcal{S}).$$

Our goal is to show that the induced maps $X \to X_i$ induce an equivalence

$$X \simeq \{X_i^\circ\}_{i \in I}$$

in the ∞-category $\mathrm{Pro}(\mathcal{S})$. It suffices to show that we have an equivalence of mapping spaces

$$\theta \colon \lim_{i \in I} \mathrm{Map}_{\mathcal{S}}\big(Y, X_i^\circ\big) \simeq \lim_{i \in I} \mathrm{Map}_{\mathcal{S}}\big(Y, X_i\big)$$

for every connected space $Y \in \mathcal{S}^{\leq 1}$. Notice that θ is the cofiltered limit of monomorphisms in the ∞-category $\mathcal{S}$. Thus it is itself a monomorphism. It suffices to show that it is also an effective epimorphism in $\mathcal{S}$.

Let $* \to Y$ be the unique, up to contractible indeterminacy, morphism in the ∞-category $\mathcal{S}$. For each $i \in I$, consider also the canonical map

$$\lim_{i \in I} \mathrm{Map}_{\mathcal{S}}\big(Y, X_i\big) \to X_i$$

induced by λ_i. Such morphism must necessarily factor through $X_i^\circ \subseteq X_i$, by our choice of X_i°. It now follows that $\pi_0(\theta)$ is surjective. Consequently, the morphism θ is an effective epimorphism. The result now follows. □

Proof of Theorem 5.5.1.8. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. Both the ∞-categories

$$\mathrm{Perf}^{\mathrm{ad}}(X)(A), \quad \mathrm{Fun}_{\mathcal{C}\mathrm{at}_\infty(\mathrm{Pro}(\mathcal{S}))}\big(X, \mathrm{Perf}(A)\big) \in \mathcal{C}\mathrm{at}_\infty$$

are fibered over $\mathrm{Perf}(A)$ and we have a commutative diagram of the form

$$\begin{tikzcd}
\mathrm{Perf}^{\mathrm{ad}}(X)(A) \arrow[rr, "\beta_A"] \arrow[dr] & & \mathrm{Fun}_{\mathcal{C}\mathrm{at}_\infty(\mathrm{Pro}(\mathcal{S}))}\big(X, \mathrm{Perf}(A)\big) \arrow[dl] \\
& \mathrm{Perf}(A) &
\end{tikzcd} \tag{5.5.1.1}$$

of coCartesian fibrations over $\mathrm{Perf}(A)$. Therefore, it suffices to show that for each $M \in \mathrm{Perf}(A)$ the fiber products over M of both vertical functors displayed in (5.5.1.1) are equivalent via β_A. Thanks to Theorem 5.5.1.9 together with limit-colimit formula for mapping spaces on ∞-categories of pro-objects it follows that we can identify the fiber of the left hand side with

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S})}\big(X, \mathrm{B}\mathcal{E}\mathrm{nd}(M)\big).$$

The result now follows due to Theorem 5.5.1.6. □

5.5.2 Enriched ∞-categories and p-adic continuous representations of homotopy types

Definition 5.5.2.1. Let S denote the sphere spectrum. It has a natural $\mathbb{E}_\infty$-ring structure, being the unit object of the stable ∞-category of spectra, Sp. For each $n \geq 1$, denote by S/p^n the pushout of the diagram

$$\begin{tikzcd} S[u] \arrow[r, "u \mapsto p^n"] \arrow[d, "u \mapsto 0"] & S \arrow[d] \\ S \arrow[r] & S/p^n \end{tikzcd}$$

computed in the ∞-category $\mathcal{C}\mathrm{Alg}(\mathrm{Sp})$. We can also define the p-completion

$$S_p^\wedge \simeq \lim_{n \geq 1} S/p^n \in \mathrm{Sp}\,.$$

The derived ∞-category $\mathrm{Mod}_{S_p^\wedge}$ admits an induced symmetric monoidal structure from Sp such that $S_p^\wedge$ is the unit object. Moreover, we can consider the full subcategory

$$\big(\mathrm{Mod}_{S_p^\wedge}\big)_{\mathrm{nil}} \subseteq \mathrm{Mod}_{S_p^\wedge}$$

spanned by p-nilpotent modules, see [Lur16, §7.1]. Moreover, the full subcategory $\big(\mathrm{Mod}_{S_p^\wedge}\big)_{\mathrm{nil}}$ is naturally symmetric monoidal. We denote by $\mathrm{Sp}_{\mathrm{pro}}(p)$ the ∞-category $\mathrm{Pro}\big((\mathrm{Mod}_{S_p^\wedge})_{\mathrm{nil}}\big)$.

Notation 5.5.2.2. We shall denote the symmetric monoidal ∞-category associated to $\mathrm{Sp}_{\mathrm{pro}}(p)$ by $\mathrm{Sp}_{\mathrm{pro}}(p)^\otimes$.

Remark 5.5.2.3. For each $n \geq 1$, the ∞-category Mod_{S/p^n} admits a symmetric monoidal structure induced from the smash product of spectra. Moreover, we have a natural lax symmetric monoidal functor

$$\mathrm{Mod}^\otimes_{S/p^n} \to \big(\mathrm{Mod}_{S_p^\wedge}\big)_{\mathrm{nil}}$$

which factors the usual restriction functor

$$\mathrm{Mod}^\otimes_{S/p^n} \to \mathrm{Mod}^\otimes_{S_p^\wedge}$$

along the canonical morphism $S_p^\wedge \to S/p^n$ in the ∞-category Sp. This implies that we have a canonical symmetric monoidal functor

$$\mathrm{Mod}^\otimes_{S/p^n} \to \mathrm{Sp}_{\mathrm{pro}}(p)^\otimes \tag{5.5.2.1}$$

which commutes with cofiltered limits, see [Lur12c, Proposition 6.3.1.13] and its proof. Moreover, given $M := \lim_\alpha M_\alpha$ and $M' := \lim_\beta M_\beta$ objects in $\mathrm{Sp}_{\mathrm{pro}}(p)$ we have an equivalence

$$M \otimes M' \simeq \lim_{\alpha,\,\beta}\big(M_\alpha \otimes M_\beta\big)$$

in the ∞-category $\mathrm{Sp}_{\mathrm{pro}}(p)$.

Remark 5.5.2.4. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ be a derived k°-adic algebra. For each $n \geq 1$, the derived k_n°-algebra A_n admits a natural action of S/p^n. Therefore, the stable ∞-category $\mathrm{Perf}(A_n)$ is not only enriched over spectra but actually enriched over the derived ∞-category Mod_{S/p^n}. The existence of the symmetric monoidal functor displayed in (5.5.2.1) implies the existence of an induced action of the ∞-category $\mathrm{Sp}_{\mathrm{pro}}(p)$ on $\mathrm{Perf}(A_n)$. Passing to the limit, we deduce that the ∞-category $\mathrm{Perf}(A)$ can be upgraded naturally to an object in the ∞-category $\mathcal{C}\mathrm{at}_\infty\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big)$.

Thanks to Theorem 5.5.2.4 we have two natural enriched structures on $\mathrm{Perf}(A)$, for $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. Namely, a $\mathrm{Pro}(\mathcal{S})$-enriched structure on $\mathrm{Perf}(A)$ and an $\mathrm{Sp}_{\mathrm{pro}}(p)$-enriched structure. We will show that these are compatible in a sense which we will precise hereafter.

Remark 5.5.2.5. Consider the usual connective cover functor

$$\Omega^\infty\colon\ \mathrm{Sp} \to \mathcal{S}.$$

It can be upgraded to a lax symmetric monoidal functor $\Omega^{\infty,\otimes}\colon \mathrm{Sp}^{\otimes} \to \mathcal{S}^{\times}$, where $\mathcal{S}$ is considered with its Cartesian symmetric monoidal structure. Therefore, Ω^{∞} induces a lax symmetric monoidal functor

$$\Omega^{\infty}_{\mathrm{pro}}(p)\colon \mathrm{Sp}_{\mathrm{pro}}(p)^{\otimes} \to \mathrm{Pro}(\mathcal{S})^{\times}.$$

For this reason, there exists a natural functor $\Omega^{\infty}_{\mathrm{pro}}(p)\colon \mathcal{C}\mathrm{at}_{\infty}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big) \to \mathcal{C}\mathrm{at}_{\infty}\big(\mathrm{Pro}(\mathcal{S})\big)$. Moreover, the image of $\mathrm{Perf}(A) \in \mathcal{C}\mathrm{at}_{\infty}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big)$ under $\Omega^{\infty}_{\mathrm{pro}}(p)$ is naturally equivalent to $\mathrm{Perf}(A) \in \mathcal{C}\mathrm{at}_{\infty}\big(\mathrm{Pro}(\mathcal{S})\big)$ whose $\mathrm{Pro}(\mathcal{S})$-structure is the one introduced in Theorem 5.5.1.4.

Construction 5.5.2.6. Consider now the ∞-category $\mathrm{Ind}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big)$ of ind-objects on the presentable stable ∞-category $\mathrm{Sp}_{\mathrm{pro}}(p)$. For each $M \in \mathrm{Sp}_{\mathrm{pro}}(p)$, the multiplication map

$$p\colon M \to M$$

induces a functor

$$(-)[p^{-1}]\colon \mathrm{Ind}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big) \to \mathrm{Ind}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big)$$

given informally on objects by the formula

$$M \mapsto M \otimes_{k^\circ} k \coloneqq \operatorname*{colim}_{p\colon M \to M} M.$$

Notation 5.5.2.7. Denote by $\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$ the essential image of the full subcategory $\mathrm{Sp}_{\mathrm{pro}}(p) \subseteq \mathrm{Ind}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big)$ under the functor

$$(-)[p^{-1}]\colon \mathrm{Ind}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big) \to \mathrm{Ind}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big).$$

Moreover, we will denote by

$$\mathrm{L}_p\colon \mathrm{Sp}_{\mathrm{pro}}(p) \to \mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$$

the induced functor from $(-)[p^{-1}]$ restricted to $\mathrm{Sp}_{\mathrm{pro}}(p)$.

Lemma 5.5.2.8. *The ∞-category $\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$ admits a natural symmetric monoidal structure induced from the one on $\mathrm{Ind}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big)$. Moreover, the functor*

$$\mathrm{L}_p\colon \mathrm{Sp}_{\mathrm{pro}}(p) \to \mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$$

admits an essentially unique natural extension to a symmetric monoidal functor $\mathrm{L}_p^{\otimes}\colon \mathrm{Sp}_{\mathrm{pro}}(p) \to \mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$.

Proof. The symmetric monoidal structure on $\mathrm{Ind}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big)$ is induced by the symmetric monoidal structure on $\mathrm{Sp}_{\mathrm{pro}}(p)$, by extending via filtered colimits. For this reason, given $M,\ M' \in \mathrm{Sp}_{\mathrm{pro}}(p)$ we have natural equivalences

$$M \otimes_{k^\circ} k \otimes M'[p^{-1}] \simeq \big(M \otimes M'\big)[p^{-1}] \in \mathrm{Sp}_{\mathrm{pro}}(p),$$

and the result follows. ☐

Lemma 5.5.2.9. *Let $M,\ M' \in \mathrm{Sp}_{\mathrm{pro}}(p)$. Then we have an equivalence of mapping spaces*

$$\mathrm{Map}_{\mathrm{Sp}_{\mathrm{pro}}(p)}\big(M, M'\big)[p^{-1}] \to \mathrm{Map}_{\mathrm{Sp}_{\mathrm{pro}}(p)}\big(M \otimes_{k^\circ} k, M' \otimes_{k^\circ} k\big)$$

where

$$\mathrm{Map}_{\mathrm{Sp}_{\mathrm{pro}}(p)}\big(M, M'\big)[p^{-1}] \coloneqq \operatorname*{colim}_{\text{mult by } p} \mathrm{Map}_{\mathrm{Sp}_{\mathrm{pro}}(p)}\big(M, M'\big).$$

Proof. It is a direct consequence of the characterization of mapping spaces in ∞-categories of ind-objects. ☐

Proposition 5.5.2.10. *Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ be a derived k°-adic algebra. Denote by $\mathrm{Coh}^+(A \otimes_{k^\circ} k)$ denote the ∞-category of almost connectibe coherent $A \otimes_{k^\circ} k$-modules. Then the ∞-category $\mathrm{Coh}^+(A \otimes_{k^\circ} k)$ is naturally enriched over the ∞-category $\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$.*

Proof. We will actually prove a stronger statement, namely we will show that $\mathrm{L}_p(\mathrm{Coh}^+(A))$ is a canonical upgrade of the ∞-category $\mathrm{Coh}^+(A \otimes_{k^\circ} k)$ as an $\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$-enriched ∞-category. There exists a materialization functor $\mathrm{Mat}\colon \mathrm{Sp}_{\mathrm{pro}}(p) \to \mathcal{S}$, given on objects by the formula

$$M \in \mathrm{Sp}_{\mathrm{pro}}(p) \mapsto \mathrm{Map}_{\mathrm{Sp}_{\mathrm{pro}}(p)}(*, M) \in \mathcal{S}.$$

Moreover, the functor $\mathrm{Mat}_{\mathrm{cat}}$ is lax symmetric monoidal. Thus it induces a well defined functor at the level of enriched ∞-categories

$$\mathrm{Mat}_{\mathrm{cat}}\colon \mathcal{C}\mathrm{at}_\infty\big(\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}\big) \to \mathcal{C}\mathrm{at}_\infty.$$

Thanks to [GH15, Corollary 5.7.6] the functor $\mathrm{L}_p^\otimes\colon \mathrm{Sp}_{\mathrm{pro}}(p)^\otimes \to \mathrm{Sp}_{\mathrm{pro}}(p)^\otimes$ induces a well defined functor

$$\mathrm{L}_p\colon \mathcal{C}\mathrm{at}_\infty\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big) \to \mathcal{C}\mathrm{at}_\infty\big(\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}\big).$$

Therefore, given $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ the ∞-category $\mathrm{L}_p\big(\mathrm{Coh}^+(A)\big)$ is naturally enriched over $\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$. Furthermore, given $M,\ N \in \mathrm{Coh}^+(A)$ we have an equivalence of mapping objects

$$\mathrm{Map}_{\mathrm{L}_p\left(\mathrm{Coh}^+(A)\right)}\big(M, N\big) \simeq \mathrm{Map}_{\mathrm{Coh}^+(A)}\big(M, N\big)[p^{-1}] \in \mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}},$$

whose essential image under $\mathrm{Mat}_{\mathrm{cat}}$ coincides with the mapping space

$$\operatorname*{colim}_{\text{mult by } p} \mathrm{Map}_{\mathrm{Coh}^+(A)}\big(M, N\big) \simeq \mathrm{Map}_{\mathrm{Coh}^+(A \otimes_{k^\circ} k)}\big(M \otimes_{k^\circ} k, N \otimes_{k^\circ} k\big).$$

[Ant18b, Proposition A.1.5] imlies that every object in the ∞-category $\mathrm{Coh}^+(A \otimes_{k^\circ} k)$ admits a formal model living in the ∞-category $\mathrm{Coh}^+(A)$. Thus, the underlying space of the enriched ∞-category $\mathrm{L}_p\big(\mathrm{Coh}^+(A)\big)$ is equivalent to $\mathrm{Coh}^+(A \otimes_{k^\circ} k)^\simeq$. It then follows that we have a natural equivalence

$$\mathrm{Mat}_{\mathrm{cat}}\big(\mathrm{L}_p(\mathrm{Coh}^+(A)\big) \simeq \mathrm{Coh}^+(A \otimes_{k^\circ} k)$$

in the ∞-category $\mathcal{C}\mathrm{at}_\infty$. □

Definition 5.5.2.11. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ be a derived k°-algebra. Denote by $\underline{\mathrm{Perf}}(A \otimes_{k^\circ} k)$ the $\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$-enriched subcategory of $\mathrm{L}_p\big(\mathrm{Coh}^+(A)\big)$ spanned by those perfect $A \otimes_{k^\circ} k$.

Remark 5.5.2.12. We can express Theorem 5.5.2.11 more concretely as follows: the ∞-category $\mathrm{L}_p\big(\mathrm{Coh}^+(A)\big)$ is equivalent to the given of a functor

$$\mathcal{C}_{A \otimes_{k^\circ} k}\colon \mathrm{L}_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{Coh}^+(A)^\simeq} \to \mathrm{Sp}_{\mathrm{pro}}(p)^\otimes_{p^{-1}},$$

which is an $\mathrm{L}_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{Coh}^+(A)^\simeq}$-algebra object in $\mathrm{Sp}_{\mathrm{pro}}(p)^\otimes_{p^{-1}}$. Let $X \subseteq \mathrm{Coh}^+(A)^\simeq$ denote the subspace spanned by those A-modules whose extension of scalars along the morphism $k^\circ \to k$ is a perfect $A \otimes_{k^\circ} k$-module. We can consider the restriction of the functor $\mathcal{C}_{A \otimes_{k^\circ} k}$ to $\mathrm{L}_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_X$ and thus obtain an $\mathbf{\Delta}^{\mathrm{op}}_X$-algebra on $\mathrm{Sp}_{\mathrm{pro}}(p)^\otimes_{p^{-1}}$, which is complete in the sense of [GH15].

Remark 5.5.2.13. By construction, one has an equivalence $\mathrm{Mat}_{\mathrm{cat}}\big(\underline{\mathrm{Perf}}(A \otimes_{k^\circ} k)\big) \simeq \mathrm{Perf}(A \otimes_{k^\circ} k)$ as stable ∞-categories.

Warning 5.5.2.14. The enriched mapping objects in $\underline{\mathrm{Perf}}(A \otimes_{k^\circ} k)$ depend of the given choice of a lifting, i.e. of an A-formal model. However, whenever A is n-truncated such choice is canonical up to a contractible space of choices

Remark 5.5.2.15. The association $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \mapsto \underline{\mathrm{Perf}}(A) \in \mathcal{C}\mathrm{at}_\infty\big(\mathrm{Sp}_{\mathrm{Pro}}(p)_{p^{-1}}\big)$ is functorial. Therefore, the usual functor

$$\mathrm{Perf}\colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathcal{C}\mathrm{at}^{\mathrm{st}}_\infty$$

can be upgraded naturally to a functor $\underline{\mathrm{Perf}}\colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathcal{C}\mathrm{at}_\infty(\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}})$.

Proposition 5.5.2.16. *The functor* $\underline{\mathrm{Perf}}\colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathcal{C}\mathrm{at}_\infty$ *is infinitesimally cartesian.*

Proof. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. Thanks to [GH15, Proposition 5.5.3 and Corollary 5.5.4] it suffices to show that given an k°-adic derivation

$$d \colon \mathbb{L}^{\mathrm{ad}}_A \to M$$

in the ∞-category $\mathrm{Coh}^+(A)$, the induced functor

$$F_{A,d} \colon \underline{\mathrm{Perf}}\big(A_d[M]\big) \to \underline{\mathrm{Perf}}(A) \times_{\underline{\mathrm{Perf}}(A \oplus M)} \underline{\mathrm{Perf}}(A) \tag{5.5.2.2}$$

is both essentially surjective and fully faithful. By construction and [GH15, Definition 5.3.3] essential surjectiveness can be checked after applying the materialization functor

$$\mathrm{Mat}_{\mathrm{cat}} \colon \mathcal{C}\mathrm{at}_\infty\big(\mathrm{Sp}_{\mathrm{Pro}}(p)_{p^{-1}}\big) \to \mathcal{C}\mathrm{at}_\infty.$$

Furthermore, after applying $\mathrm{Mat}_{\mathrm{cat}}$ the functor displayed in (5.5.2.2) is equivalent to the canonical functor

$$F_{A,d} \colon \mathrm{Perf}(A \otimes_{k^\circ} k) \to \mathrm{Perf}(A \otimes_{k^\circ} k) \times_{\mathrm{Perf}(A_d[M] \otimes_{k^\circ} k} \mathrm{Perf}(A \otimes_{k^\circ} k)$$

which is essential surjective thanks to [Lur12a, Proposition 3.4.10]. Thus, we are reduced to show that the functor $F_{A,d}$ is fully faithful on mapping objects. This is a consequence of Theorem 5.5.1.2 together with the fact that the analogous statement holds for the $\mathrm{Pro}(\mathcal{S})$-enriched version of $\mathrm{Coh}^+(A_d[M])$, objects in $\underline{\mathrm{Perf}}(A_d[M])$ admit formal models in $\mathrm{Coh}^+(A_d[M])$ and when computing mapping objects in $\underline{\mathrm{Perf}}(A_d[M])$ we are taking the filtered colimit under multiplication by p of the corresponding mapping objects in $\mathrm{Coh}^+(A_d[M])$ and such colimits commute with finite limits. $\square$

The functor $\underline{\mathrm{Perf}} \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathcal{C}\mathrm{at}_\infty\big(\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}\big)$ does not depend on the choice of formal model as the following result illustrates:

Proposition 5.5.2.17. *Let $A,\ A' \in \big(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}\big)^{<\infty}$ be truncated derived k°-adic algebras. Suppose that there exists an equivalence*

$$A \otimes_{k^\circ} k \simeq A' \otimes_{k^\circ} k$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_k$. Then we have an equivalence

$$\underline{\mathrm{Perf}}(A) \simeq \underline{\mathrm{Perf}}(A')$$

in the ∞-category $\mathcal{C}\mathrm{at}_\infty\big(\mathrm{Sp}_{\mathrm{Pro}}(p)_{p^{-1}}\big)$.

Proof. Let $A,\ A' \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ as in the statement of the proposition. Thanks to [Ant18b, Theorem 4.4.10] we can suppose that there exists a morphism $f \colon A \to A'$ in the ∞-category $(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ})^{<\infty}$ such that the rigidification of

$$\mathrm{Spf}(f) \colon \mathrm{Spf}(A) \to \mathrm{Spf}(A')$$

is an equivalence in the ∞-category dAfd_k. The induced functor

$$\underline{\mathrm{Perf}}(A) \to \underline{\mathrm{Perf}}(A') \tag{5.5.2.3}$$

is essentially surjective, as this condition can be checked after applying the materialization functor $\mathrm{Mat}_{\mathrm{cat}}$. Thanks to [GH15, Proposition 5.5.3 and Corollary 5.5.4] it suffices to show that the functor displayed in (5.5.2.3) is fully faithful on mapping objects.

Notice that, fully faithfulness can be checked already at the level of $\mathcal{C}\mathrm{at}_\infty\big(\mathrm{Sp}_{\mathrm{Pro}}(p)_{p^{-1}}\big)$. Let $M,\ N \in \mathrm{Coh}^+(A)$ such that their images in $\mathrm{Coh}^+(A \otimes_{k^\circ} k)$ are perfect. We need to show that the functor displayed in (5.5.2.3) induces an equivalence

$$\theta \colon \lim_{n \geq 1} \mathrm{Map}_{\mathrm{Perf}(A_n)}\big(M \otimes_A A_n, N \otimes_A A_n\big)[p^{-1}] \to \lim_{n \geq 1} \mathrm{Map}_{\mathrm{Perf}(A'_n)}\big(M \otimes_A A_n, N \otimes_A A_n\big)[p^{-1}] \tag{5.5.2.4}$$

in the ∞-category $\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$, where

$$M' \coloneqq M \otimes_A A', \quad N' \coloneqq N \otimes_A A' \in \mathrm{Mod}_{A'}.$$

The functor θ can be realized as the ind-localization at p of the map

$$\theta' \colon \lim_{n\geq 1}\mathrm{Map}_{\mathrm{Perf}(A_n)}\big(M\otimes_A A_n, N\otimes_A A_n\big)\to \lim_{n\geq 1}\mathrm{Map}_{\mathrm{Perf}(A_n)}\big(M'\otimes_{A'} A'_n, N'\otimes_{A'} A'_n\big)$$

in the ∞-category $\mathrm{Sp}_{\mathrm{pro}}(p)$. It thus suffices to show that θ' is an equivalence in the ∞-category $\mathrm{Sp}_{\mathrm{pro}}(p)$ after multiplying by a sufficiently large power of p. The ∞-category $\mathrm{Sp}_{\mathrm{pro}}(p)$ is a stable ∞-category, since it consists of pro-objects in the ∞-category $\big(\mathrm{Mod}_{S_p^\wedge}\big)_{\mathrm{nil}}$. We are thus reduced to prove that $\mathrm{cofib}(\theta')$ is equivalent to the zero morphism in $\mathrm{Sp}_{\mathrm{pro}}(p)$ after multiplication by a sufficiently large power of p.

Furthermore, we have an equivalence $\mathrm{cofib}(\theta')\simeq \lim_{n\geq 1}\mathrm{cofib}(\theta'_n)$ where

$$\theta' \colon \mathrm{Map}_{\mathrm{Perf}(A_n)}\big(M\otimes_A A_n, N\otimes_A A_n\big)\to \mathrm{Map}_{\mathrm{Perf}(A'_n)}\big(M'\otimes_{A'} A'_n, N'\otimes_{A'} A'_n\big) \tag{5.5.2.5}$$

is the canonical morphism at the level of mapping spaces. The previous statement is a consequence of the dual statement concerning the commutation of filtered colimits with finite limits, see the proof of [Lur12c, Proposition 1.1.3.6].

By assumption, both A and A' are m-truncated for a sufficiently large integer $m>0$. It follows then that both $M,\ N\in \mathrm{Mod}_A$ and $M',\ N'\in\mathrm{Mod}_{A'}$ have non-zero homotopy groups concentrated in a finite number of degrees. We thus conclude that each mapping spectrum displayed in **??** has non-trivial homotopy groups living in a finite number of degrees, which do not depend on the integer $n\geq 0$, and only at the special fiber at $n=1$. Therefore, there exists a sufficiently large $k>0$ such that each cofiber $\mathrm{cofib}(\theta'_n)$ is killed by p^k, and k does not depend on the chosen n. Denote $X_n \coloneqq \mathrm{cofib}(\theta'_n)$ the cofiber of θ'_n. Let $Z\coloneqq \underset{i}{Z_i}\in\mathrm{Sp}_{\mathrm{pro}}(p)$. We have a chain of equivalences

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}_{\mathrm{pro}}(p)}\big(\lim_i Z_i, \mathrm{cofib}(\theta')\big) &\simeq \mathrm{Map}_{\mathrm{Sp}_{\mathrm{pro}}(p)}\big(\lim_i Z_i, \lim_n X_n\big)\\ \lim_n \mathrm{Map}_{\mathrm{Sp}_{\mathrm{pro}}(p)}\big(\lim_i Z_i, X_n\big) &\simeq \lim_n \operatorname*{colim}_i \mathrm{Map}_{(\mathrm{Mod}_{S_p^\wedge})_{\mathrm{nil}}}\big(Z_i, X_n\big)\in \big(\mathrm{Mod}_{S_p^\wedge}\big)_{\mathrm{nil}}.\end{aligned}$$

Each X_n is killed by a certain power of p, we deduce that the same holds for $\mathrm{Map}_{(\mathrm{Mod}_{S_p^\wedge})_{\mathrm{nil}}}\big(Z_i, X_n\big)\in\mathcal{S}$. Such property is closed under filtered colimits, thus we conclude that $\operatorname*{colim}_i \mathrm{Map}_{(\mathrm{Mod}_{S_p^\wedge})_{\mathrm{nil}}}\big(Z_i, X_n\big)$ is of p-torsion. Therefore, since we have assumed A and A' to be truncated derived k°-adic algebras, also the projective limit

$$\lim_n \operatorname*{colim}_i \mathrm{Map}_{(\mathrm{Mod}_{S_p^\wedge})_{\mathrm{nil}}}\big(Z_i, X_n\big)$$

is of p-torsion. We conclude thus that for any compact object $Z\in\mathrm{Ind}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big)$ the mapping space

$$\mathrm{Map}_{\mathrm{Ind}(\mathrm{Sp}_{\mathrm{pro}}(p))}\big(Z, \lim_n X_n[p^{-1}]\big)\simeq \mathrm{Map}_{\mathrm{Sp}_{\mathrm{pro}}(p)}\big(Z,\lim_n X_n\big)[p^{-1}]$$

is trivial. As a consequence we obtain that

$$\mathrm{cofib}(\theta')[p^{-1}]\simeq \big(\lim_n X_n\big)[p^{-1}]\simeq 0$$

in the ∞-category $\mathrm{Ind}\big(\mathrm{Sp}_{\mathrm{pro}}(p)\big)$. This implies that θ is an equivalence, thus the functor displayed in (5.5.2.3) is fully faithful and thus an equivalence of ∞-categories. □

Corollary 5.5.2.18. *Let $A\in\big(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}\big)^{<\infty}$. Suppose we are given $M,\ N\in\mathrm{Coh}^+(A)$ such that*

$$M\otimes_{k^\circ} k\in\mathrm{Perf}(A\otimes_{k^\circ} k),\quad N\otimes_{k^\circ} k\in\mathrm{Perf}(A\otimes_{k^\circ} k).$$

Then the mapping object

$$\mathrm{Map}_{\underline{\mathrm{Perf}(A)}}\big(M\otimes_{k^\circ} k, N\otimes_{k^\circ} k\big)\in\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$$

does not depend on the choice of M and $N\in\mathrm{Coh}^+(A)$.

Proof. This is a direct consequence of Theorem 5.5.2.17. □

Construction 5.5.2.19. There exists a lax symmetric monoidal functor

$$\Omega^\infty_{\mathrm{pro}}(p)_{p^{-1}} \colon \mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}} \to \mathrm{Ind}\big(\mathrm{Pro}(\mathcal{S})\big)$$

induced by $\Omega^\infty_{\mathrm{pro}}(p) \colon \mathrm{Sp}_{\mathrm{pro}}(p) \to \mathrm{Pro}(\mathcal{S})$. Therefore, we have a canonical, up to contractible indeterminacy, functor

$$\Omega^\infty_{\mathrm{pro}}(p)_{p^{-1}} \colon \mathcal{C}\mathrm{at}_\infty\big(\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}\big) \to \mathcal{E}\mathcal{C}\mathrm{at}_\infty.$$

Thus, given $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ we can consider the ∞-category $\mathrm{Perf}(A \otimes_{k^\circ} k)$ as naturally enriched over the symmetric monoidal ∞-category $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$.

Notation 5.5.2.20. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. By abuse of notation we will denote by $\underline{\mathrm{Perf}}(A)$ either the $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$-enriched or the $\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$-enriched versions of the stable ∞-category $\mathrm{Perf}(A \otimes_{k^\circ} k)$.

Corollary 5.5.2.21. *The functor* $\underline{\mathrm{Perf}} \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathcal{E}\mathcal{C}\mathrm{at}_\infty$ *is infinitesimally cartesian. Moreover, its restriction to the full subcategory* $(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ})^{<\infty} \subseteq \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ *is invariant under generically equivalences.*

Proof. The statement of the corollary is a direct consequence of Theorem 5.5.2.17 and Theorem 5.5.2.16. $\square$

Remark 5.5.2.22. Suppose, $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ is truncated derived k°-algebra. Let $M \in \mathrm{Perf}(A \otimes_{k^\circ} k)$ be such that there exists a perfect A-module $M' \in \mathrm{Perf}(A)$ such that

$$M' \otimes_{k^\circ} k \simeq M.$$

Then we can regard $\mathrm{End}(M') \in \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S})$ with an enhanced pro-structure seen as the pro-object

$$\mathcal{E}\mathrm{nd}(M') \coloneqq \{\mathrm{End}(M' \otimes_A A_n)\}_n \in \mathrm{Pro}(\mathcal{S}). \tag{5.5.2.6}$$

Moreover, the transition maps appearing in the diagram displayed in (5.5.2.6) preserve the monoid structures. Thus, we can consider $\mathcal{E}\mathrm{nd}(M')$ naturally as an object in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}))$. Similary, we can consider $\mathrm{End}(M) \in \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S})$ with an enhanced ind-pro-structure via the diagram

$$\mathcal{E}\mathrm{nd}(M) \coloneqq \operatorname*{colim}_{\text{mult by } p} \mathcal{E}\mathrm{nd}(M') \in \mathrm{Ind}(\mathrm{Pro}(\mathcal{S})).$$

Thanks to Theorem 5.5.2.18 it follows that the above definition does not depend on the choice of the perfect formal model $M' \in \mathrm{Perf}(A)$. Moreover, as filtered colimits commute with finite limits, when taking ind-completions, it follows that the monoid structure on $\mathcal{E}\mathrm{nd}(M') \in \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}))$ induces a monoid structure on $\mathcal{E}\mathrm{nd}(M) \in \mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$. Thus we can consider $\mathcal{E}\mathrm{nd}(M)$ naturally as an object in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$.

Remark 5.5.2.23. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ and $M \in \mathrm{Perf}(A)$. Then we can also consider $\mathcal{E}\mathrm{nd}(M) \in \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Sp}_{\mathrm{pro}}(p))$. We can also naturally consider $\mathcal{E}\mathrm{nd}(M \otimes_{k^\circ} k) \in \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}})$. These considerations will prove to be useful for us as they allow us to use methods from stable homotopy theory the study of continuous representations of profinite spaces.

Notation 5.5.2.24. Consider the ∞-category dAfd_k of derived k-affinoid spaces. We denote $\mathrm{dAfd}_k^{<\infty}$ the full subcategory spanned by those truncated derived k-affinoid spaces.

Remark 5.5.2.25. The rigidification functor $(-)^{\mathrm{rig}} \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathrm{dAfd}_k^{\mathrm{op}}$ introduced in [Ant18b, §4] induces, by restriction, a well defined functor

$$(-)^{\mathrm{rig}} \colon \big(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}\big)^{<\infty} \to \big(\mathrm{dAfd}_k^{<\infty}\big)^{\mathrm{op}}.$$

Moreover, the derived Raynaud localization theorem [Ant18b, Theorem 4.4.10] and its proof imply that $\mathrm{dAfd}_k^{<\infty}$ is a localization of the ∞-category $(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ})^{<\infty}$ at the saturated class of generically strong morphisms.

The functor $\underline{\mathrm{Perf}} \colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathcal{E}\mathcal{C}\mathrm{at}_\infty$ actually descends to a well defined functor $\underline{\mathrm{Perf}} \colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{E}\mathcal{C}\mathrm{at}_\infty$, which is the content of the following proposition:

Proposition 5.5.2.26. *Let $S^{<\infty}$ denote the saturated class of generically strong morphisms in $(\mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}})^{<\infty}$. Then the functor*

$$\underline{\mathrm{Perf}} \colon \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}} \to \mathcal{E}\mathcal{C}\mathrm{at}_\infty$$

sends morphisms in $S^{<\infty}$ to equivalences of ∞-categories in $\mathcal{E}\mathcal{C}\mathrm{at}_\infty$. In particular, one has a canonical induced functor

$$\underline{\mathrm{Perf}} \colon \big(\mathrm{dAfd}_k^{<\infty}\big)^{\mathrm{op}} \to \mathcal{E}\mathcal{C}\mathrm{at}_\infty$$

which associates to every $Z \in \mathrm{dAfd}_k^{<\infty}$ the $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$-enriched ∞-category $\underline{\mathrm{Perf}}(\Gamma(Z)') \in \mathcal{E}\mathcal{C}\mathrm{at}_\infty$, where $\Gamma(Z)' \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}$ is a given formal model for $\Gamma(Z) \in \mathcal{C}\mathrm{Alg}_k$.

Proof. The first part of the statement follows from Theorem 5.5.2.18. The second part of the statement follows from the derived Raynaud localization theorem, [Ant18b, Theorem 4.4.10]. □

5.6 Moduli of derived continuous p-adic representations

5.6.1 Construction of the functor

We start with the following important construction:

Construction 5.6.1.1. The ∞-category $\mathcal{C}\mathrm{at}_\infty\big(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))\big)$ can be naturally upgraded to an $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$-$(\infty, 2)$-$\infty$-category, thanks to [GH15, Remark 7.4.11]. Therefore, given $\mathcal{C}, \mathcal{D} \in \mathcal{C}\mathrm{at}_\infty\big(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))\big)$ we have a natural functor ∞-category

$$\underline{\mathrm{Fun}}\big(\mathcal{C}, \mathcal{D}\big) \in \mathcal{C}\mathrm{at}_\infty\big(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))\big).$$

Applying the materialization functor $\mathrm{Mat}_{\mathrm{cat}} \colon \mathcal{C}\mathrm{at}_\infty\big(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))\big) \to \mathcal{C}\mathrm{at}_\infty$ we produce an ∞-category

$$\mathcal{C}\mathrm{ontFun}\big(\mathcal{C}, \mathcal{D}\big) \in \mathcal{C}\mathrm{at}_\infty$$

which we designate by the ∞-category of *continuous* functors between $\mathcal{C}$ and $\mathcal{D}$.

We have a canonical inclusion functor $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})) \hookrightarrow \mathcal{C}\mathrm{at}_\infty\big(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))\big)$. Given $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ we can consider it as an ind-pro-space via the composite

$$\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}) \hookrightarrow \mathrm{Pro}(\mathcal{S}) \hookrightarrow \mathrm{Ind}(\mathrm{Pro}(\mathcal{S})).$$

Given $A \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}$, we can thus consider the ∞-category of continuous functors

$$\mathrm{Perf}_\ell(X)(A) := \mathcal{C}\mathrm{ontFun}\big(X, \mathrm{Perf}(A)\big) \in \mathcal{C}\mathrm{at}_\infty.$$

Definition 5.6.1.2. Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ and $A \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}$. We designate the ∞-category $\mathrm{Perf}_\ell(X)(A)$ as the ∞-category of $A \otimes_{k^\circ} k$*-adic continuous representations of* X.

Lemma 5.6.1.3. *Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ be a profinite space. Suppose further that $X \simeq *$, then we have a canonical equivalence $\mathrm{Perf}_\ell(X)(A) \simeq \mathrm{Perf}(A \otimes_{k^\circ} k) \in \mathcal{C}\mathrm{at}_\infty$.*

Proof. Whenever $X \simeq *$ in $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ the ∞-category $\mathrm{Perf}_\ell(X)(A)$ coincides with the materialization of $\underline{\mathrm{Perf}}(A) \simeq \mathrm{Perf}(A \otimes_{k^\circ} k) \in \mathcal{C}\mathrm{at}_\infty$, as desired. □

Let $X \in \mathrm{Pro}(\mathcal{X}^{\mathrm{fc}})$ be connected profinite space. We have thus an unique, up to contractible indeterminacy, morphism

$$f \colon * \to X$$

in the ∞-category $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$. Precomposition along $f \colon * \to X$ induces a canonical map

$$\mathrm{ev}(*) := \mathrm{Perf}_\ell(X)(A) \to \mathrm{Perf}(A \otimes_{k^\circ} k)$$

in the ∞-category $\mathcal{C}\mathrm{at}_\infty$.

Proposition 5.6.1.4. *Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ be a connected profinite space. Let $A \in \big(\mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}\big)^{<\infty}$. Consider a perfect module $M \in \mathrm{Perf}(A \otimes_{k^\circ} k)$ admitting a formal model $M_0 \in \mathrm{Coh}^+(A)$. The fiber of the functor*

$$\mathrm{ev}(*) := \mathrm{Perf}_\ell(X)(A) \to \mathrm{Perf}(A \otimes_{k^\circ} k)$$

at $M \in \mathrm{Perf}(A \otimes_{k^\circ} k)$ is naturally equivalent to the mapping space

$$\mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))}\big(\Omega X, \mathcal{E}\mathrm{nd}(M)\big) \in \mathcal{S},$$

where $\mathcal{E}\mathrm{nd}(M) \in \mathrm{Mon}_{\mathbb{E}_1}\big(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))\big)$ denotes the enriched mapping object of M equipped with is multiplicative monoid structure.

Proof. By hypothesis $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ is connected. Therefore, the fiber of $\mathrm{ev}(*)$ at M is naturally equivalent to the mapping space

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_*)}\big(X, \mathrm{B}\mathcal{E}\mathrm{nd}(M)\big) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_*)}\big(X, \mathrm{B}\mathcal{E}\mathrm{nd}(M_0)\big)[p^{-1}]. \tag{5.6.1.1}$$

Consider the usual loop functor $\Omega \colon \mathcal{S}_* \to \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S})$. It induces a canonical functor

$$\Omega \colon \mathrm{Pro}(\mathcal{S}_*) \to \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S})).$$

Notice that every transition morphism in the pro-system $\mathcal{E}\mathrm{nd}(M_0) \in\in \mathrm{Pro}(\mathcal{S})$ is actually a morphism of monoid objects, i.e. it admits a natural lifting in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S})$. Using the limit-colimit formula for mapping spaces in ∞-categories of pro-objects together with the Bar-Cobar equivalence (B, Ω) we obtain a natural equivalence

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_*)}\big(X, \mathrm{B}\mathcal{E}\mathrm{nd}(M_0)\big)[p^{-1}] \simeq \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}))}\big(\Omega X, \mathcal{E}\mathrm{nd}(M_0)\big)[p^{-1}].$$

The universal property of localization at p induces a canonical map

$$\theta \colon \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}))}\big(\Omega X, \mathcal{E}\mathrm{nd}(M_0)\big)[p^{-1}] \to \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))}\big(\Omega X, \mathcal{E}\mathrm{nd}(M)\big),$$

in the ∞-category of spaces $\mathcal{S}$. The result follows if we are able to prove that θ is an equivalence. We notice that we cannot apply May delooping theorem component-wise as multiplication by $p \colon \mathcal{E}\mathrm{nd}(M_0) \to \mathcal{E}\mathrm{nd}(M_0)$ is not a morphism of monoid-objects. However, the map θ is induced by a morphism of the form

$$\theta' \colon \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}))}\big(\Omega X, \mathcal{E}\mathrm{nd}(M_0)\big) \to \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))}\big(\Omega X, \mathcal{E}\mathrm{nd}(M)\big).$$

Furthermore, the fiber of the morphism $\mathcal{E}\mathrm{nd}(M_0) \to \mathcal{E}\mathrm{nd}(M)$ in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}})$ coincides with the colimit

$$\operatorname*{colim}_n \mathcal{E}\mathrm{nd}(M_0)/p^n[-1], \tag{5.6.1.2}$$

which is of p-torsion. Therefore, passing to the filtered colimit along multiplication by p the term in (5.6.1.2) becomes the zero object in the stable ∞-category $\mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$. As a consequence, it follows that $\theta \simeq \theta'[p^{-1}]$ has contractible fiber and therefore it is an equivalence. ☐

Remark 5.6.1.5. Theorem 5.6.1.4 implies that the functor $\mathrm{Perf}_p(X)(A) \to \mathrm{Perf}(A \otimes_{k^\circ} k$ is a coCartesian fibration which corresponds to a functor

$$F \colon \mathrm{Perf}(A) \to \mathcal{C}\mathrm{at}_\infty$$

given on objects by the formula

$$M \in \mathrm{Perf}(A \otimes_{k^\circ} k) \mapsto \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))}\big(\Omega X, \mathcal{E}\mathrm{nd}(M)\big) \in \mathcal{S}.$$

Therefore, we can regard the ∞-category $\mathrm{Perf}_p(X)(A)$ as the ∞-category of continuous representations of X with values in perfect $A \otimes_{k^\circ} k$-modules.

Definition 5.6.1.6. We define the $\mathcal{C}\mathrm{at}_\infty$-valued functor of p-adic perfect modules on X as the functor

$$\mathrm{Perf}_p(X) \colon \big(\mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}\big)^{<\infty} \to \mathcal{C}\mathrm{at}_\infty,$$

given on objects by the formula

$$A \in \big(\mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}\big)^{<\infty} \mapsto \mathrm{Perf}_p(X)(A) \in \mathcal{C}\mathrm{at}_\infty.$$

An important consequence of Theorem 5.5.2.26 is the following result:

Proposition 5.6.1.7. *The functor* $\mathrm{Perf}_p(X)\colon \big(\mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}\big)^{<\infty} \to \mathcal{C}\mathrm{at}_\infty$ *descends to a well defined functor*

$$\mathrm{Perf}_p(X)\colon \big(\mathrm{dAfd}_k^{<\infty}\big)^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$$

which is given on objecs by the formula

$$Z \in \mathrm{dAfd}_k^{<\infty} \mapsto \mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\big(X, \underline{\mathrm{Perf}}(Z)\big) \in \mathcal{C}\mathrm{at}_\infty.$$

Proof. The result is a direct consequence of the equivalent statement for $\underline{\mathrm{Perf}}$ which is the content of Theorem 5.5.2.26. $\square$

5.6.2 Lifting results for continuous p-adic representations of profinite spaces

The following definition is crucial for our purposes:

Definition 5.6.2.1. Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ be a connected profinite space. We say that X is p-cohomologically compact if for any p-torsion $\mathbb{Z}_p^\wedge$-module $N \in \mathrm{Mod}_{\mathbb{Z}_p^\wedge}$ with

$$N \simeq \operatorname*{colim}_\alpha N_\alpha,$$

where $N_\alpha \in \mathrm{Mod}_{\mathbb{Z}}^{\heartsuit}$ for each α, we have an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}))}\big(\Omega X, N\big) \simeq \operatorname*{colim}_\alpha \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}))}\big(\Omega X, N_\alpha\big),$$

i.e. taking continuous cohomology of X iwth torsion coefficients commutes with filtered colimits.

Remark 5.6.2.2. The above definition makes sense when we consider $X \in \mathcal{S}$. In this case, it is equivalent to ask for a cellular decomposition of X with finitely man cells in each dimension. However X itself might have infinitely many non-zero (finite) homotopy groups.

Example 5.6.2.3.
(i) Suppose $Y \to X$ is a finite morphism in $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$, i.e. its fiber is a finite constructible space $Z \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$. If we assume further that X is p-cohomologically compact, then so it is Y. More generically, the notion of cohomologically compactness is stable under fiber sequences.

(ii) Suppose $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ is the étale homotopy type of a smooth variety over an algebraically closed field. Then X is cohomologically almost of finite type, see

Theorem 5.6.2.4. *Let* $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ *be a connected profinite space which we assume further to be* p*-cohomologically compact. Let* $A \in (\mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}})^{<\infty}$ *and suppose we are given* $\rho \in \mathrm{Perf}_p(X)(A)$ *such that* $M := \mathrm{ev}(*)(\rho) \in \mathrm{Perf}(A \otimes_{k^\circ} k)$ *admits a perfect formal model* $M' \in \mathrm{Perf}(A)$*. Then there exists* $Y \in \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}))$ *together with a morphism*

$$f\colon Y \to \Omega X$$

in the ∞*-category* $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}))$ *such that* $\mathrm{fib}(f)$ *is finite constructible and such that we have a factorization of the form*

$$\begin{array}{ccc} Y & \xrightarrow{\rho'} & \mathcal{E}\mathrm{nd}(M') \\ \downarrow & & \downarrow \\ \Omega X & \xrightarrow{\rho} & \mathcal{E}\mathrm{nd}(M) \end{array}$$

in the ∞*-category* $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$.

Proof. As $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ is assumed to be connected, an object $\rho \in \mathrm{Perf}_p(X)(A)$ corresponds to a morphism

$$\rho\colon \Omega X \to \mathcal{E}\mathrm{nd}(M)$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$, where M denotes $\mathrm{ev}(*)(\rho) \in \mathrm{Perf}(A \otimes_{k^\circ} k)$. We first remark assumption on the existence of a perfect formal model for M, $M' \in \mathrm{Perf}(A)$ can be dropped, since by [TT90] the trivial square zero extension $M \otimes M[-1] \in \mathrm{Perf}(A \otimes_{k^\circ} k)$ admits a formal model and M is a retract of the latter. Therefore, we can replace M with $M \oplus M[-1]$ and assume from the start that M admits a formal model $M' \in \mathrm{Perf}(A)$.

Let $n \geq 0$ be an integer and consider the truncation functor

$$\tau_{\leq n} \colon \mathrm{Ind}(\mathrm{Pro}(\mathcal{S})) \to \mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$$

induced by the usual truncation functor $\tau_{\leq n} \colon \mathcal{S} \to \mathcal{S}$ by means of the universal property of both ind and pro completions. More explicitly,

$$\tau_{\leq n}(\operatorname*{colim}_i \lim_j X_{i,}j) \simeq \operatorname*{colim}_i \lim_j \tau_{\leq n}(X_{i,j}) \in \mathrm{Ind}(\mathrm{Pro}(\mathcal{S})).$$

As $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ is assumed to be truncated and $M' \in \mathrm{Perf}(A)$ is perfect over A it follows that also M' is a truncated A-module. The same conclusion holds for the couple $(A \otimes_{k^\circ} k, M)$. Therefore, there exists a sufficiently large $m > 0$ such that

$$\tau_{\leq m}\mathcal{E}\mathrm{nd}(M) \simeq \mathcal{E}\mathrm{nd}(M). \tag{5.6.2.1}$$

We now proceed to construct such a profinite $Y \in \mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}))$ satisfying the conditions of the statement. Our construction is by means of an inductive reasoning via the relative Postnikov tower of the canonical morphism

$$g \colon \mathcal{E}\mathrm{nd}(M') \to \mathcal{E}\mathrm{nd}(M)$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$. We start by observing that the cofiber of g is equivalent to the filtered colimit

$$\mathrm{cofib}(g) \simeq \operatorname*{colim}_k \mathcal{E}\mathrm{nd}(M')/p^k \in \mathrm{Ind}(\mathrm{Pro}(\mathcal{S})).$$

For this reason, we can identify the fiber of the morphism g with

$$\mathrm{fib}(g) \simeq \operatorname*{colim}_k \mathcal{E}\mathrm{nd}(M')/p^k[-1].$$

Suppose then $n = 0$. Consider the pullback diagram

$$\begin{tikzcd}
\mathcal{E}\mathrm{nd}(M')_{\leq 0} \arrow[r] \arrow[d, "q_0"] & \pi_0(\mathcal{E}\mathrm{nd}(M')) \arrow[d] \\
\mathcal{E}\mathrm{nd}(M) \arrow[r] & \pi_0(\mathcal{E}\mathrm{nd}(M))
\end{tikzcd}$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$. By construction, we can identify the fiber of the morphism $q \colon \mathcal{E}\mathrm{nd}(M')_{\leq 0} \to \mathcal{E}\mathrm{nd}(M)$ with

$$\mathrm{fib}(q) \simeq \operatorname*{colim}_k \pi_0(\mathcal{E}\mathrm{nd}(M')/p^k)[-1]).$$

By the universal of the 0-truncation functor in (5.6.2.1) the composite

$$\Omega X \xrightarrow{\rho} \mathcal{E}\mathrm{nd}(M) \to \pi_0(\mathcal{E}\mathrm{nd}(M))$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$ factors through a continuous group homomorphism

$$\rho_0 \colon \pi_1(X) \to \pi_0(\mathcal{E}\mathrm{nd}(M)),$$

where the topology on the left hand side group is the profinite one induced from X and the topology on the right hand side group is the topology induced by the ind-pro structure on $\mathcal{E}\mathrm{nd}(M)$. Since $\pi_0(\mathcal{E}\mathrm{nd}(M'))$ is an open subgroup of $\pi_0(\mathcal{E}\mathrm{nd}(M))$ and $\pi_1(X)$ is profinite, it follows that the inverse image

$$\rho_0^{-1}(\pi_0(\mathcal{E}\mathrm{nd}(M'))) \leq \pi_1(X)$$

is of finite index in $\pi_1(X)$. Let $U \lhd X$ be an open normal subgroup such that

$$\rho_0(U) \subseteq \pi_0(\mathcal{E}\mathrm{nd}(M')) \subseteq \pi_0(\mathcal{E}\mathrm{nd}(M))$$

and such that $\pi_1(X)/U \cong G$, where G is a finite group. Consider the pullback diagram

$$\begin{tikzcd} Y_{\leq 0} \arrow[r] \arrow[d, "h_0"] & U \arrow[d] \\ \Omega X \arrow[r] & \pi_1(X) \end{tikzcd}$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}))$. By construction, the morphism $Y_{\leq 0} \to \Omega X$ admits a finite constructible fiber, namely G. Furthermore, we have an equivalence

$$X \simeq \mathrm{B}Y_{\leq 0}/G$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$. The base step of our inductive reasoning is thus finished. Suppose now that for a given integer $n \geq 0$ we have constructed a commutative diagram

$$\begin{tikzcd} Y_{\leq n} \arrow[r] \arrow[d, "h_n"] & \mathcal{E}\mathrm{nd}(M')_{\leq n} \arrow[d] \\ \Omega X \arrow[r, "\rho"] & \mathcal{E}\mathrm{nd}(M) \end{tikzcd}$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$, where $h_n \colon Y_{\leq n} \to \Omega X$ admits a finite constructible fiber. Let

$$g_{n+1} \colon \mathcal{E}\mathrm{nd}(M')_{\leq n+1} \to \mathcal{E}\mathrm{nd}(M)$$

denote the relative $(n+1)$-st truncation of the canonical morphism $g \colon \mathcal{E}\mathrm{nd}(M') \to \mathcal{E}\mathrm{nd}(M)$. We have thus a commutative diagram of the form

$$\begin{tikzcd} \mathcal{E}\mathrm{nd}(M')_{\leq n+1} \arrow[r, "j_n"] \arrow[drrr] & \mathcal{E}\mathrm{nd}(M')_{\leq n} \arrow[r, "j_{n-1}"] \arrow[drr] & \dots \arrow[r, "j_0"] & \mathcal{E}\mathrm{nd}(M')_{\leq 0} \arrow[d, "g_0"] \\ & & & \mathcal{E}\mathrm{nd}(M) \end{tikzcd}$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$ such that

$$\mathrm{fib}(j_n) \simeq \operatorname*{colim}_k \pi_{n+1}(\mathcal{E}\mathrm{nd}(M')/p^k)[n+2].$$

Consider the following pullback diagram

$$\begin{tikzcd} \widetilde{Y}_{\leq n+1} \arrow[r] \arrow[d, "\pi_n"] & \mathcal{E}\mathrm{nd}(M')_{\leq n+1} \arrow[d] \\ Y_{\leq n} \arrow[r] & \mathcal{E}\mathrm{nd}(M')_{\leq n} \end{tikzcd}$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$. The fiber of the morphism $\pi_n \colon \widetilde{Y}_{\leq n+1} \to Y_{\leq n}$ is equivalent

$$\mathrm{fib}(\pi_n) \simeq \operatorname*{colim}_k \pi_{n+2}(\mathcal{E}\mathrm{nd}(M')/p^k)[n+2].$$

The fiber sequence

$$\mathrm{fib}(\pi_n) \to \widetilde{Y}_{\leq n+1} \to Y_{\leq n}$$

is classified by a morphism

$$\varphi_n \colon Y_{\leq n} \to \operatorname*{colim}_k \pi_{n+2}(\mathcal{E}\mathrm{nd}(M')/p^k)[n+3]$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$. Notice that, $\pi_n + 2(\mathcal{E}\mathrm{nd}(M')/p^k)$ is a discrete group and the monoid structure on $\pi_{n+2}(\mathcal{E}\mathrm{nd}(M')/p^k)[n+3]$ is abelian, as $n+3 \geq 2$. Therefore, the transition maps in the ind-filtered colimit

$$\operatorname*{colim}_k \pi_{n+2}(\mathcal{E}\mathrm{nd}(M')/p^k)[n+3]$$

do preserve the monoid structure on each term. Thus we find that φ_n is actually a morphism in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$. As $Y_{\leq n} \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}) \hookrightarrow \mathrm{Pro}(\mathcal{S})$ it follows that φ_n factors through $\mathcal{E}\mathrm{nd}(M')/p^k$ for sufficiently large $k \geq 1$. This induces a fiber sequence of the form

$$\pi_{n+2}(\mathcal{E}\mathrm{nd}(M')/p^k)[n+2] \to \overline{Y}_{\leq n+1} \to Y_{\leq n},$$

such that we have an induced map

$$\overline{Y}_{\leq n+1} \to \mathcal{E}\mathrm{nd}(M')_{\leq n+1}$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$ which coincides with the composite

$$\overline{Y}_{\leq n+1} \to \widetilde{Y}_{\leq n+1} \to \mathcal{E}\mathrm{nd}(M')_{\leq n+1}.$$

As X is p-cohomologically compact and the morphism $\mathrm{B}Y_{\leq n} \to X$ admits a finite constructible fiber, by assumption, it follows that $\mathrm{B}Y_{\leq n}$ is also p-cohomologically compact. As $\pi_{n+2}(\mathcal{E}\mathrm{nd}(M')/p^k)[n+2]$ is p-torsion over $\mathbb{Z}_p^\wedge$ it follows, by Lazard's theorem, that we have an equivalence

$$\pi_{n+2}(\mathcal{E}\mathrm{nd}(M')/p^k)[n+2] \operatorname*{colim}_\alpha N_\alpha[n+2],$$

in the derived ∞-category $\mathrm{Mod}_{\mathbb{Z}_{p^k}}$, where each N_α is a finite discrete $\mathbb{Z}/p^k\mathbb{Z}$-module. We obtain thus, by p-cohomological compactness, an equivalence of mapping spaces

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}))}\big(Y_{\leq n}, \pi_{n+2}(\mathcal{E}\mathrm{nd}(M')/p^k\big)[n+3] \simeq \\ \simeq \operatorname*{colim}_\alpha \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}))}\big(Y_{\leq n}, N_\alpha[n+3]\big).\end{aligned}$$

Therefore, the map φ_n above factors through a morphism

$$\varphi_{\beta,n} \colon Y_{\leq n} \to N_\beta[n+3]$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathcal{S}^{\mathrm{fc}})$ and such factorization produces an extension

$$N_\beta[n+2] \to Y_{\leq n+1} \xrightarrow{j_{n+1}} Y_{\leq n},$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}))$. Moreover, by construction, it follows that the composite

$$Y_{\leq n+1} \to Y_{\leq n} \to \cdots \to \Omega X \to \mathcal{E}\mathrm{nd}(M)$$

factors through the canonical morphism $\mathcal{E}\mathrm{nd}(M')_{\leq n+1} \to \mathcal{E}\mathrm{nd}(M)$. The inductive step is thus completed.

In order to finish the proof of the statement it suffices now to observe that there exists a sufficiently large integer $n \geq 1$ such that both

$$\mathcal{E}\mathrm{nd}(M'), \quad \mathcal{E}\mathrm{nd}(M)$$

are n-truncated objects in the ∞-category $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$. Thus for such $n \geq 1$ we have an equivalence

$$\mathcal{E}\mathrm{nd}(M')_{\leq n+1} \simeq \mathcal{E}\mathrm{nd}(M').$$

We have thus produced a finite morphism $Y \to \Omega$ with $Y := Y_{\leq n+1}$ and a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{E}\mathrm{nd}(M') \\ \big\downarrow & & \big\downarrow \\ \Omega X & \longrightarrow & \mathcal{E}\mathrm{nd}(M) \end{array}$$

in the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))$ and the claim is proved. $\square$

Remark 5.6.2.5. In the above proof there exists no need to assume $M \in \mathrm{Perf}(A \otimes_{k^\circ} k)$ admits a perfect module $M' \in \mathrm{Perf}(A)$. Actually, it suffices to assume that there exists a formal model $M' \in \mathrm{Coh}^+(A)$ for M which is truncated. This assumption is always verified. Indeed, there exists a formal model $M' \in \mathrm{Coh}^+(A)$ for $M \in \mathrm{Perf}(A \otimes_{k^\circ} k)$ thanks to [Ant18b, Proposition A.2.1]. Even if M' is not truncated, its rigidification $M \otimes_{k^\circ} k$ is, as A and thus $A \otimes_{k^\circ} k$ are assumed from the start to be truncated. If we pick $m > 0$ sufficiently large such that M is m-truncated then $\tau_{\leq m} M' \in \mathrm{Coh}^+(A)$ is still a formal model for $M \in \mathrm{Perf}(A \otimes_{k^\circ} k)$.

Construction 5.6.2.6. Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ be a truncated derived k°-adic algebra. The functor $\mathrm{L}_p \colon \mathrm{Sp}_{\mathrm{pro}}(p) \to \mathrm{Sp}_{\mathrm{pro}}(p)_{p^{-1}}$ induces a base change functor $f_A^* \colon \mathrm{Perf}^{\mathrm{ad}}(X)(A) \to \mathrm{Perf}_p(X)(A)$ of stable ∞-categories.

Definition 5.6.2.7. We say that $\rho \in \mathrm{Perf}_\ell(X)(A)$ is liftable if ρ lies in the essential image of the base change functor f_A^* as above.

As a consequence of Theorem 5.6.2.4 we have the following Corollary:

Corollary 5.6.2.8. *Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ be a connected and p-cohomologically compact profinite space. Consider a truncated derived k°-adic algebra $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$. Then every object $\rho \in \mathrm{Perf}_p(X)(A)$ is a retract of a liftable $\rho' \in \mathrm{Perf}_p(X)(A)$. Moreover, $\mathrm{Perf}_p(X)(A)$ is canonically equivalent to the idempotent completion of the base change*

$$\mathrm{Perf}^{\mathrm{ad}}(X)(A) \otimes_{k^\circ} k$$

in the ∞-category of stable and idempotent complete small ∞-categories, $\mathcal{C}\mathrm{at}_\infty^{\mathrm{st,idem}}$.

Proof. Let $\rho \in \mathrm{Perf}_p(X)(A)$, $M := \mathrm{ev}(*)(\rho) \in \mathrm{Perf}(A \otimes_{k^\circ} k)$ and let $\pi \colon Y \to X$ be a finite morphism in $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ with Y connected such that the composite representation

$$\Omega Y \to \Omega X \to \mathcal{E}\mathrm{nd}(M)$$

factors through a morphism $\mathcal{E}\mathrm{nd}(M') \to \mathcal{E}\mathrm{nd}(M)$ where we can suppose that $M' \in \mathrm{Perf}(A)$. The proof of Theorem 5.6.2.4 implies that we have a commutative diagram of the form

$$\begin{tikzcd} Y \simeq Y_{\leq n+1} \arrow[r, "g_n"] \arrow[drrr] & Y_{\leq n} \arrow[r, "g_{n-1}"] \arrow[drr] & \dots \arrow[r, "j_0"] & Y_{\leq 0} \arrow[d, "g_0"] \\ & & & X \end{tikzcd}$$

in the ∞-category $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$. Furthermore, we have by construction a canonical equivalence $X \simeq Y_{\leq 0}/\Gamma$, where Γ is a suitable finite group (not necessarily abelian). In particular, we have an equivalence of ∞-categories

$$\mathrm{Perf}_p(X)(A) \simeq \mathrm{Perf}_p(Y_{\leq 0})(A)^\Gamma$$

of $A \otimes_{k^\circ} k$-linear stable ∞-categories. Moreover, the proof of Theorem 5.6.2.4 implies that for each integer $0 \leq i \leq n-1$ we can choose the morphism

$$g_i \colon Y_{\leq i+1} \to Y_{\leq i}$$

such that it is a $M_i[n+2]$-torsor for a given finite abelian group M_i. As $A \otimes_{k^\circ} k$ lives over a field of characteristic zero, namely k it follows that we have an equivalence of ∞-categories

$$\mathrm{Perf}_p(Y_{\leq 0})(A) \simeq \mathrm{Perf}_p(Y_{\leq i})(A)$$

for each integer $0 \leq i \leq n$. As a consequence, we deduce that one has an equivalence of ∞-categories

$$\mathrm{Perf}_p(X)(A) \simeq \mathrm{Perf}_p(Y)(A)^\Gamma.$$

Thus we have an adjunction of the form

$$\pi^* \colon \mathrm{Perf}_p(X)(A) \rightleftarrows \mathrm{Perf}_p(Y)(A) \colon \pi_*$$

where π_* denotes the restriction functor along $\pi\colon Y \to X$. Given $\rho \in \mathrm{Perf}_p(Y)(A)$ we have an equivalence

$$\pi^* \rho \simeq \rho \otimes_{k^\circ} k[\Gamma],$$

where $k[\Gamma]$ denotes the free k-algebra on the finite group Γ. The representation ρ is a retract of $\pi_* \pi^* \rho$, given by the trivial morphism of groups

$$\{1\} \to \Gamma.$$

Observe further that the representation $\pi_* \pi^* \rho$ is liftable by the choice of Y, as $\pi^* \rho$ is so. We are thus reduced to prove the second part of the statementn, namely that $\mathrm{Perf}_p(X)(A)$ is equivalent to the idempotent completion of the $A \otimes_{k^\circ} k$-linear stable ∞-category $\mathrm{Perf}^{\mathrm{ad}}(X)(A) \otimes_{k^\circ} k$. It suffices, in fact, to prove that $\mathrm{Perf}_p(X)(A)$ is idempotent complete. Thanks to Theorem 5.6.2.4 we can assume from the start that ρ is liftable from the start. In this case, it suffices to show that for every idempotent

$$f \colon \rho \to \rho,$$

in the ∞-category $\mathrm{Perf}_p(X)(A)$ admits a fiber and cofiber in the ∞-category $\mathrm{Perf}_p(X)(A)$. Let $M \coloneqq \mathrm{ev}(*)(\rho) \in \mathrm{Perf}(A \otimes_{k^\circ} k)$. Under the evaluation functor $\mathrm{ev}(*)$ f corresponds to an idempotent morphism

$$\widetilde{f} \colon M \to M.$$

We might not be able to lift $\widetilde{f}$ but thanks to [Ant18b, Corollary A.2.3] and [Lur09b, Proposition 4.4.5.20] there exists a formal model $M' \in \mathrm{Coh}^+(A)$ for M for which we can lift $\widetilde{f} \colon M \to M$ and higher homotopy coherences associated to the diagram $\widetilde{f} \colon \mathrm{Idem} \to \mathrm{Perf}(A \otimes_{k^\circ} k)$ to a diagram $\overline{f} \colon \mathrm{Idem} \to \mathrm{Perf}(A)$ such that its rigidification coincides with $\widetilde{f}$.

As in Theorem 5.6.2.5 we can suppose that M' has non-trivial homotopy groups concentrated in a finite number of degrees. by Theorem 5.6.2.5 the proof of Theorem 5.6.2.4 applies in this case. In this case, it suffices to show that $\overline{f}$ admits fiber and cofiber sequences in the ∞-category $\mathrm{Coh}^+(X)^{\mathrm{ad}}(A)$, which follows as the latter ∞-category is idempotent complete. $\square$

Corollary 5.6.2.9. *Let $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}$ be a truncated derived k°-adic algebra. Then the ∞-category $\mathrm{Perf}_p(X)(A)$ is stable and admits a natural symmetric monoidal structure.*

Proof. Thanks to Theorem 5.6.2.8 and the formula for mapping spaces it follows that $\mathrm{Perf}_p(X)(A)$ is an $A \otimes_{k^\circ} k$-linear ∞-category which is equivalent to the tensor product

$$\mathrm{Perf}^{\mathrm{ad}}(X)(A) \otimes_{k^\circ} k$$

in the ∞-category $\mathcal{C}\mathrm{at}^{\mathrm{st,idem}}_\infty$. This implies that $\mathrm{Perf}_p(X)(A)$ is stable and the symmetric monoidal structure on $\mathrm{Perf}_p(X)(A)$ is induced from the one on $\mathrm{Perf}^{\mathrm{ad}}(X)(A)$. $\square$

Definition 5.6.2.10. Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ and $A \in (\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ})^{<\infty}$. We define

$$\mathrm{Mod}_p(X)(A) \coloneqq \mathrm{Ind}(\mathrm{Perf}_p(X)(A)).$$

Corollary 5.6.2.11. *The ∞-category $\mathrm{Mod}_p(X)(A)$ is a presentable stable ∞-category which is moreover locally rigid and we have an equivalence of presentable ∞-categories*

$$\mathrm{Mod}^{\mathrm{ad}}(X)(A) \otimes_{k^\circ} k \simeq \mathrm{Mod}_p(X)(A) \in \mathcal{P}\mathrm{r}^{\mathrm{st}}_{\mathrm{L}}.$$

Proof. Presentability of $\mathrm{Mod}_p(X)(A)$ follows from Theorem 5.6.2.8. Moreover, we have a chain of equivalences

$$\begin{aligned}\mathrm{Mod}_p(X)(A) &\simeq \\ &\simeq \mathrm{Mod}^{\mathrm{ad}}(X)(A) \otimes_{k^\circ} k \\ &\simeq \mathrm{Mod}_{A \otimes_{k^\circ} k}\big(\mathrm{Mod}^{\mathrm{ad}}(X)(A)\big)\end{aligned}$$

where the latter equivalence follows from [Lur12c, Proposition 6.3.4.6]. As $\mathrm{Mod}^{\mathrm{ad}}(X)(A)$ is locally rigid it follows also that $\mathrm{Mod}_p(X)(A)$ is locally rigid, thanks to [Lur16, Lemma D.7.7.2]. The result now follows. $\square$

5.6.3 Moduli of derived continuous p-adic representations

In this §we define the moduli of derived continuous p-adic representations of a profinite space X and we show that it admits a derived structure under certain mild assumptions on the profinite space $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$.

Definition 5.6.3.1. Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$. The *moduli of derived continuous p-adic representations* of X is defined as the right Kan extension along the canonical inclusion functor

$$j\colon \mathrm{dAfd}_k^{<\infty} \to \mathrm{dAfd}_k$$

of the moduli functor

$$\mathrm{PerfSys}_\ell(X) := (-)^{\simeq} \circ \mathrm{Perf}_p(X)\colon \left(\mathrm{dAfd}_k^{<\infty}\right)^{\mathrm{op}} \to \mathcal{S}$$

which is given on objects by the formula

$$Z \in \mathrm{dAfd}_k^{<\infty} \mapsto \mathrm{Perf}_p(X)(\Gamma(Z))^{\simeq} \in \mathcal{S}.$$

The following result is a reality check:

Lemma 5.6.3.2. *Let $Z \in \mathrm{dAfd}_k$. Then we have a natural equivalence*

$$\mathrm{PerfSys}_\ell(X)(Z) \simeq \lim_n \mathrm{PerfSys}_\ell(X)(\mathrm{t}_{\leq n} Z)$$

in the ∞-category $\mathcal{S}$. In particular, the functor $\mathrm{PerfSys}_\ell(X)$ is nilcomplete.

Proof. This statement was stated without proof in [GR14, p. 10]. Let $\mathcal{T}_X := (\mathrm{dAfd}_k^{<\infty})^{op}_{Z/}$ and denote $\mathcal{T}'_Z$ the full subcategory of $\mathcal{T}_Z$ spanned by those objects of the form $\mathrm{t}_{\leq n} \to Z$, for each $n \geq 0$. By the end formula for right Kan extensions it suffices to show that the inclusion functor

$$\mathcal{T}'_Z \to \mathcal{T}_Z$$

is a final functor. Thanks to the dual statement of [Lur09b, Theorem 4.1.3.1] it suffices to show that for every $(Y \to Z)^{\mathrm{op}}$ in $\mathcal{T}_Z$, the ∞-category

$$(\mathcal{T}'_Z)_{/Y}$$

has weakly contractible enveloping groupoid. We can identify the ∞-category $(\mathcal{T}'_Z)_{/Y}$ with the ∞-category of factorizations of the morphism $(Y \to Z)^{\mathrm{op}}$. Thanks to the universal property of n-th truncations and the fact that Y is a truncated derived k-affinoid space it follows that there exists a sufficiently large integer m such that $(Y \to X)^{\mathrm{op}}$ factors uniquely (up to contractible indeterminacy) as

$$(Y \to \mathrm{t}_{\leq m} X \to X)^{\mathrm{op}}.$$

Therefore the ∞-category $(\mathcal{T}'_X)_{/Y}$ is cofiltered and thus weakly contractible, as desired. $\square$

Proposition 5.6.3.3. *The functor $\mathrm{PerfSys}_\ell(X)\colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S}$ satisfies étale hyper-descent.*

Proof. $\square$

Proposition 5.6.3.4. *The functor $\mathrm{PerfSys}_\ell(X)\colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S}$ is cohesive.*

Proof. The right adjoint $(-)^{\simeq}\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ commutes with small limits and in particular with finite limits. Moreover, $\mathrm{PerfSys}_\ell(X)$ is nilcomplete, thus we can restrict ourselves to prove the assertion when restricted to truncated objects. As a consequence, it suffices to show that the functor

$$\mathrm{Perf}_p(X)\colon \mathrm{dAfd}_k^{<\infty} \to \mathcal{C}\mathrm{at}_\infty$$

is infinitesimally cartesian. Let $Z \in \mathrm{dAfd}_k^{<\infty}$ and let $d\colon \mathbb{L}_Z^{\mathrm{an}} \to M$ be a k-analytic derivation of Z, with $M \in \mathrm{Coh}^+(A)$. Thanks to [Ant18b, Theorem A.2.1] we can lift d to a formal derivation

$$d'\colon \mathbb{L}_A^{\mathrm{ad}} \to M'$$

in the ∞-category $\mathrm{Coh}^+(A)$ where $A \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}$ is a formal model for $Z \in \mathrm{dAfd}_k$ which we can assume to be truncated. In this case, the canonical functor

$$\mathrm{Perf}_p(X)(A_{d'}[M']) \to \mathrm{Perf}_p(X)(A) \times_{\mathrm{Perf}_p(X)(A \otimes M')} \mathrm{Perf}_p(X)(A)$$

is an equivalence, which follows immediately from Theorem 5.5.2.16. $\square$

In order to show the existence of a cotangent complex for $\mathrm{PerfSys}_\ell(X)$ we will need the following technical result:

Proposition 5.6.3.5. *Let $F \in \mathrm{St}(\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$ and denote by $F^{\mathrm{rig}} \in \mathrm{St}(\mathrm{dAfd}_k, , \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$ its rigidification. Then if F admits an adic cotangent complex at a point*

$$x \colon \mathrm{Spf}(A) \to F,$$

denoted $\mathbb{L}^{\mathrm{ad}}_{F,x}$, then F^{rig} admits a cotangent complex at the rigidification

$$x^{\mathrm{rig}} \colon \mathrm{Spf}(A)^{\mathrm{rig}} \to F^{\mathrm{rig}}$$

which we denote by $\mathbb{L}^{\mathrm{an}}_{F^{\mathrm{rig}}, x^{\mathrm{rig}}}$. Moreover, we have a canonical equivalence

$$(\mathbb{L}^{\mathrm{ad}}_{F,x})^{\mathrm{rig}} \simeq \mathbb{L}^{\mathrm{an}}_{F^{\mathrm{rig}}, x^{\mathrm{rig}}},$$

in the ∞-category $\mathrm{Coh}^+(Z)$, where $Z := \mathrm{Spf}(A)^{\mathrm{rig}}$.

Proof. The existence of $\mathbb{L}^{\mathrm{ad}}_{F,x}$ implies that for every $M \in \mathrm{Coh}^+(A)$ we have functorial equivalences

$$\mathrm{Map}_{\mathrm{Coh}^+(A)}\big(\mathbb{L}^{\mathrm{ad}}_{F,x}, M\big) \simeq \mathrm{fib}_x\big(F(A \oplus M) \to F(A)\big).$$

Thanks to [Ant18b, Proposition A.1.4] the ∞-category $\mathrm{Coh}^+(X)$ is a Verdier quotient of $\mathrm{Coh}^+(A)$ with respect to the full subcategory of torsion objects in the ∞-category $\mathrm{Coh}^+(A)$. Furthermore, it follows from [Ant18b, Proposition A.2.1] and its proof that the we have an adjunction

$$(-)^{\mathrm{rig}} \colon \mathrm{Ind}\big(\mathrm{Coh}^+(A)\big) \rightleftarrows \mathrm{Ind}\big(\mathrm{Coh}^+(X)\big) \colon (-)^+$$

between presentable ∞-categories where $(-)^{\mathrm{rig}}$ is an accessible localization functor and $(-)^+$ is consequently fully faithful. We have an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{Coh}^+(X)}\big(\mathbb{L}^{\mathrm{rig}}_{F,x}, M\big) \simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Coh}^+(A))}\big(\mathbb{L}_{F,x}, (M^{\mathrm{rig}})^+\big).$$

Since $(M^{\mathrm{rig}})^+ \in \mathrm{Ind}(\mathrm{Coh}^+(A))$ we can write it as a filtered colimit

$$\begin{aligned}(M^{\mathrm{rig}})^+ &\simeq \operatorname*{colim}_i M_i \\ &\simeq \operatorname*{colim}_{M \in \mathrm{Coh}^+(A)_{/M^{\mathrm{rig}}}} M,\end{aligned}$$

where the last equivalence follows by the adjunction. Therefore, we can write

$$\mathrm{Map}_{\mathrm{Ind}(\mathrm{Coh}^+(A))}\big(\mathbb{L}_{F,x}, (M^{\mathrm{rig}})^+\big) \simeq \operatorname*{colim}_i \mathrm{Map}_{\mathrm{Coh}^+(A)}\big(\mathbb{L}_{F,x}, M_i\big)$$

since $\mathbb{L}_{F,x}$ is a compact object in the ∞-category $\mathrm{Ind}(\mathrm{Coh}^+(A))$. Let $N \in \mathrm{Coh}^+(X)$, where $X := \mathrm{Spf}(A)^{\mathrm{rig}}$. We have a chain of equivalences

$$\begin{aligned}\operatorname*{colim}_{A' \in (\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ})_{/X}} \operatorname*{colim}_{M_i \in \mathrm{Coh}^+(A')_{/N}} \mathrm{fib}_{x'}\big(F(A' \oplus M_i) \to F(A')\big) \simeq& \\ \simeq \operatorname*{colim}_{A' \in (\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ})_{/X}} \operatorname*{colim}_{M_i \in \mathrm{Coh}^+(A')_{/N}} \mathrm{Map}_{\mathrm{Coh}^+(A')}\big(\mathbb{L}_{F,x}, M_i\big)& \\ \simeq \operatorname*{colim}_{A' \in (\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ})_{/X}} \mathrm{Map}_{\mathrm{Ind}(\mathrm{Coh}^+(A'))}\big(\mathbb{L}_{F,x}, N^+\big)& \\ \simeq \operatorname*{colim}_{A' \in (\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ})_{/X}} \mathrm{Map}_{\mathrm{Coh}^+(X)}\big(\mathbb{L}^{\mathrm{rig}}_{F,x}, N\big)& \\ \simeq \mathrm{Map}_{\mathrm{Coh}^+(X)}\big(\mathbb{L}^{\mathrm{rig}}_{F,x}, N\big).&\end{aligned}$$

where both colimit indexing ∞-categories are filtered and x' denotes the composite

$$x\colon \operatorname{Spf} A' \to \operatorname{Spf} A \to F.$$

This is justified as in the above colimit diagrams it suffices to consider only colimits indexed by the full subcategories of formal models for X and lying under A. Furthermore, we have an equivalence

$$\operatorname*{colim}_{A' \in (\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ})_{/X}} \operatorname*{colim}_{M_i \in \mathrm{Coh}^+(A')_{/N}} \mathrm{fib}_{x'}\big(F(A' \oplus M_i) \to F(A')\big) \simeq \operatorname*{colim}_{\mathcal{C}} \mathrm{fib}_{x'}\big(F(A' \oplus M') \to F(A')\big),$$

where $\mathcal{C}$ denotes the ∞-category of admissible formal models for $X[N]$. This last assertion follows from the observation that a formal model for $X[N]$ consists of the given of an admissible formal model for X together with a formal model for N. Observe that filtered colimits commute with finite limits in the ∞-category of spaces. Thus

$$\begin{aligned}\operatorname*{colim}_{\mathcal{C}} \mathrm{fib}_{x'}\big(F(A' \oplus M') \to F(A')\big) \simeq \\ &\simeq \mathrm{fib}_{x^{\mathrm{rig}}} \operatorname*{colim}_{\mathcal{C}} \big(F(A' \oplus M') \to F(A')\big) \\ &\simeq \mathrm{fib}_{x^{\mathrm{rig}}}\big(F^{\mathrm{rig}}(X[N]) \to F^{\mathrm{rig}}(X)\big).\end{aligned}$$

We have thus an equivalence of the form

$$\mathrm{fib}_{x^{\mathrm{rig}}}\big(F^{\mathrm{rig}}(X[N]) \to F^{\mathrm{rig}}(X)\big) \simeq \mathrm{Map}_{\mathrm{Coh}^+(X)}\big(\mathbb{L}_{F^{\mathrm{rig}}, x^{\mathrm{rig}}}, N\big) \tag{5.6.3.1}$$

as desired. The result now follows from the observation that the right hand side of (5.6.3.1) is an invariant under hyper-descent. $\square$

Theorem 5.6.3.6. *Let $Z \in \mathrm{dAfd}_k$ and $M \in \mathrm{Coh}^+(Z)$. Suppose we are given furthermore a morphism*

$$\rho\colon Z \to \mathrm{PerfSys}_\ell(X)$$

then we have a canonical identification

$$\mathrm{fib}_\rho\big(\mathrm{PerfSys}_\ell(X)(Z[M]) \to \mathrm{Perf}(X)(Z)\big) \simeq \mathrm{Map}_{\mathrm{Mod}_p(X)(Z)}\big(\rho \otimes \rho^\vee, M[1]\big)$$

in the ∞-category $\mathrm{Coh}^+(Z)$, where $Mod_p(X)(Z) \coloneqq \mathrm{Ind}\big(\mathrm{Perf}_p(X)(Z)$.

Proof. We first observe that the derived Tate acyclicity theorem implies that we have a canonical equivalence of ∞-categories

$$\mathrm{Coh}^+(Z) \simeq \mathrm{Coh}^+(\Gamma(Z)).$$

As in the proof of Theorem 5.4.2.7 we consider the right fibration of spaces

$$\lambda_M\colon \mathrm{PerfSys}_\ell(X)(Z[M]) \to \mathrm{PerfSys}_\ell(X)(Z)$$

which classifies a functor

$$\lambda_M\colon \mathrm{PerfSys}_\ell(X)(Z) \to \mathcal{S},$$

whose value at $\rho \in \mathrm{PerfSys}_\ell(X)(Z)$ coincides with the fiber

$$\mathrm{fib}_\rho\big(\lambda_M\big) \in \mathcal{S}.$$

As the rigidification functor $(-)^{\mathrm{rig}}\colon \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{k^\circ} \to \mathrm{dAn}_k$ preserves small extensions it follows that the statement of Theorem 5.4.2.6 still holds in this case. Therefore, reasoning as in the proof of Theorem 5.4.2.7 we obtain a chain of equivalences of $\Gamma(Z)$-modules

$$\begin{aligned}\Omega(\lambda_M(\rho)) \simeq \\ &\simeq \mathrm{Map}_{/\rho}\big(p_M^*(\rho), p_M^*(\rho)\big) \\ &\simeq \mathrm{Map}_{/\rho}\big(\rho, \rho \oplus (\rho \otimes M)\big) \\ &\simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Perf}_p(X)(Z))}\big(\rho, \rho \otimes M\big) \\ &\simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Perf}_p(X)(Z))}\big(\rho \otimes \rho^\vee, M\big)\end{aligned}$$

where $p_M^*\colon \mathrm{Perf}_p(X)(Z) \to \mathrm{Perf}_p(X)(Z[M])$ denotes the usual base change functor along the canonical functor

$$p_M\colon Z[M] \to Z, \quad \text{in } \mathrm{dAfd}_k.$$

As $\mathrm{PerfSys}_\ell(X)$ is cohesive, it follows that we have a pullback diagram of the form

$$\begin{tikzcd} \lambda_M(\rho) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \lambda_{M[1]}(\rho). \end{tikzcd}$$

The conclusion now follows as in Theorem 5.4.2.7. □

Corollary 5.6.3.7. *Let $X \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ be a connected p-cohomologically compact and p-cohomologically perfect profinite space. Then for every p-complete $Z \in \mathrm{dAfd}_k$ and every $\rho \in \mathrm{PerfSys}_\ell(X)(Z)$ the functor*

$$F\colon \mathrm{Coh}^+(\Gamma(Z)) \to \mathcal{S}$$

given on objects by the formula

$$M \in \mathrm{Coh}^+(\Gamma(Z)) \mapsto \mathrm{fib}_\rho\big(\mathrm{PerfSys}_\ell(X)(Z[M]) \to \mathrm{PerfSys}_\ell(Z)\big) \in \mathcal{S}$$

is corepresentable by the $\Gamma(Z)$-module

$$\mathrm{Map}_{\mathrm{Perf}_p(X)(Z)}\big(1, \rho \otimes \rho^\vee[1]\big)^\vee \quad \in \mathrm{Coh}^+(\Gamma(Z)).$$

Proof. The result is a direct consequence of Theorem 5.4.2.9 together with Theorem 5.6.3.5 whenever

$$\rho \in \mathrm{PerfSys}_\ell(X)(Z)$$

is liftable. For a general $\rho \in \mathrm{PerfSys}_\ell(X)(Z)$ the result follows thanks to Theorem 5.6.2.8 as it implies that ρ is a retract of a liftable object.

□

5.7 Main results

5.7.1 Representability theorem

As we shall see, the moduli stack $\mathrm{LocSys}_{\ell,n}(X)\colon \mathrm{Afd}_k^{\mathrm{op}} \to \mathcal{S}$ admits a natural derived extension which it is representable with respect to the derived k-analytic context. Nonetheless, the moduli $\mathrm{LocSys}_{\ell,n}(X)$ cannot be realized as a usual k-analytic space, instead it corresponds to a k-analytic stack. Therefore, one must show that the derived enhancement of $\mathrm{LocSys}_{\ell,n}(X)$ is representable not by a derived k-analytic space but instead by a derived k-analytic stack. It would be thus desirable to have a representability type statement in the context of derived k-analytic geometry. Fortunately, such a result has been proved by M. Porta and T. Yu Yue in [PY17a]. As it will be of fundamental importance we shall motivate such result.

Definition 5.7.1.1. Denote by $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$ the *derived k-analytic geometric context* where $\tau_{\text{ét}}$ denotes the étale topology on dAfd_k and P_{sm} denotes the class of smooth morphisms on dAfd_k.

Definition 5.7.1.2. Let $F \in \mathrm{dSt}\big(\mathrm{dAfd}_k, \tau_{\text{ét}}\big)$ be a stack. We say that F is a *derived k-analytic stack* if it is representable by a geometric stack with respect to $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$.

Theorem 5.7.1.3. [PY17a, Theorem 7.1] *Let $F \in \mathrm{dSt}(\mathrm{dAfd}_k, \tau_{\text{ét}})$. The following assertions are equivalent:*

(i) F is a geometric stack;

(ii) The truncation $\mathrm{t}_{\leq 0}(F) \in \mathrm{St}(\mathrm{Afd}_k, \tau_{\text{ét}})$ is geometric, F admits furthermore a cotangent complex and it is cohesive and nilcomplete.

We shall review the main definitions:

Definition 5.7.1.4. Let $F \in \mathrm{dSt}\big(\mathrm{dAfd}_k, \tau_{\text{ét}}\big)$. We say that F admits a global analytic cotangent complex if the following two conditions are verified:

(i) Given $Z \in \mathrm{dAfd}_k$ and $z \colon Z \to F$ a morphism, the functor

$$\mathrm{Der}^{\mathrm{an}}_F(Z, -) \colon \mathrm{Coh}^+(Z) \to \mathcal{S}$$

given on objects by the formula

$$M \mapsto \mathrm{fib}_z\big(F(Z[M]) \to F(Z)\big),$$

is corepresented by an eventually connective object $\mathbb{L}^{\mathrm{an}}_{F,z} \in \mathrm{Coh}^+(Z)$.

(ii) For any morphism $f \colon Z \to Z'$ in the ∞-category dAfd_k and any morphism $z' \colon Z \to F$ we have a canonical equivalence,

$$f^* \mathbb{L}^{\mathrm{an}}_{F,z'} \simeq \mathbb{L}^{\mathrm{an}}_{F,z}$$

where $z := z' \circ f$.

Definition 5.7.1.5. Let $F \in \mathrm{dSt}(\mathrm{dAfd}_k, \tau_{\text{ét}})$. We say that F is cohesive if for every $Z \in \mathrm{dAfd}_k$ and every coherent sheaf $\mathcal{F} \in \mathrm{Coh}^{\geq 1}(Z)$ together with a derivation

$$d \colon \mathbb{L}^{\mathrm{an}}_X \to \mathcal{F}$$

the natural map

$$F\big(Z_d[\mathcal{F}[-1]]\big) \to F(Z) \times_{F(Z[\mathcal{F}])} F(Z)$$

is an equivalence in the ∞-category $\mathcal{S}$.

Definition 5.7.1.6. We say that $F \in \mathrm{dSt}\big(\mathrm{dAfd}_k, \tau_{\text{ét}}\big)$. We say that F is convergent if for every derived k-affinoid space Z the canonical morphism,

$$F(Z) \to \lim_{n \geq 0} F(\mathrm{t}_{\leq n} Z),$$

is an equivalence in the ∞-category $\mathcal{S}$.

5.7.2 Main results

Let X be a proper and smooth scheme over an algebraically closed field. To such X we can associate it a profinite space, namely its étale homotopy type

$$\mathrm{Sh}^{\text{ét}}(X) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}).$$

see [Lur09a, §3.6]. By construction, $\mathrm{Sh}^{\text{ét}}(X) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ classifies étale local systems on X. Moreover, we have a canonical identification

$$\pi_1\big(\mathrm{Sh}^{\text{ét}}(X)\big) \simeq \pi_1^{\text{ét}}(X)$$

as profinite groups. Therefore, it is natural to consider the moduli stack

$$\mathrm{PerfSys}_\ell\big(\mathrm{Sh}^{\text{ét}}(X)\big) \in \mathrm{dSt}\big(\mathrm{dAfd}_k, \tau_{\text{ét}}\big)$$

as a derived extension of the moduli $\mathrm{LocSys}_{\ell,n}(X) \in \mathrm{St}\big(\mathrm{Afd}_k, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}}\big)$.

Definition 5.7.2.1. Let $\mathrm{RLocSys}_{\ell,n}(X) \subseteq \mathrm{PerfSys}_\ell\big(\mathrm{Sh}^{\text{ét}}(X)\big)$ denote the substack spanned by continuous p-adic representations of $\mathrm{Sh}^{\text{ét}}(X) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ with values in rank n free modules.

Proposition 5.7.2.2. *We have a canonical equivalence of stacks*

$$\mathrm{t}_{\leq 0}(\mathrm{RLocSys}_{\ell,n}(X)) \simeq \mathrm{LocSys}_{\ell,n}(X)$$

in the ∞-category $\mathrm{St}(\mathrm{Afd}_k, \tau_{\text{ét}})$.

Proof. Let $A \in \mathrm{Afd}_k$, then $\mathrm{RLocSys}_{\ell,n}(X)(\mathrm{Sp}(A))$ can be identified with the space

$$\mathrm{RLocSys}_{\ell,n}(X)(\mathrm{Sp}(A)) \simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{B}\mathcal{E}\mathrm{nd}(A)\big),$$

of non-pointed continuous morphisms

$$\mathrm{Sh}^{\text{ét}}(X) \to \mathrm{B}\mathcal{E}\mathrm{nd}(A).$$

As the $A \in \mathrm{Afd}_k$ is discrete, it follows that $\mathrm{BGL}_n(A)$ is a $K(\mathrm{GL}_n(A), 1)$-space. Therefore, by the universal property of 1-truncation we have a chain of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{B}\mathcal{E}\mathrm{nd}(A)\big) \simeq&\\
&\simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\big(\tau_{\leq 1}(\mathrm{Sh}^{\text{ét}}(X)), \mathrm{B}\mathcal{E}\mathrm{nd}(A)\big)\\
&\simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\big(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{B}\mathcal{E}\mathrm{nd}(A)\big)\\
&\simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\big(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{BGL}_n(A)\big)
\end{aligned}$$

where the last equivalence follows from the fact that $\pi_1^{\text{ét}}(X)$ is a group, therefore every morphism $\pi_1^{\text{ét}}(X) \to \mathcal{E}\mathrm{nd}(A)$ should factor through the group of units of $\mathcal{E}\mathrm{nd}(A)$ which coincides with $\mathrm{GL}_n(A)$ with its k-analytic induced topology. The result now follows, by the fact that we can realize

$$\mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\big(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{BGL}_n(A)\big)$$

with the geometric realization of the diagram

$$\ldots \ \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \ \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))}}\big(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(A)\big)^{\times 2} \ \substack{\longrightarrow\\\longrightarrow} \ \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))}}\big(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(A)\big) \longrightarrow *$$

and the fact that

$$\mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1(\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})))}}\big(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(A)\big) \in \mathcal{S}$$

can be identified with the set $\mathrm{Hom}_{\mathrm{cont}}\big(\pi_1^{\text{ét}}(X), \mathrm{GL}_n(A)\big)$. $\square$

Notation 5.7.2.3. As $\mathrm{t}_{\leq 0}\mathrm{RLocSys}_{\ell,n}(X) \simeq \mathrm{LocSys}_{\ell,n}(X)$ we will denote $\mathrm{RLocSys}_{\ell,n}(X)$ simply by $\mathrm{LocSys}_{\ell,n}(X)$ from now on.

Theorem 5.7.2.4. *The moduli stack $\mathrm{LocSys}_{\ell,n}(X) \in \mathrm{St}(\mathrm{dAfd}_k, \tau_{\text{ét}})$ admits a cotangent complex. Given $\rho \in \mathrm{LocSys}_{\ell,n}(X)(Z)$ where $Z \in \mathrm{dAfd}_k$ is a derived k-affinoid space, we have an equivalence*

$$\mathbb{L}^{\mathrm{an}}_{\mathrm{LocSys}_{\ell,n}(X),\rho} \simeq C^*_{\text{ét}}(X, \mathrm{Ad}(\rho),)^\vee[-1]$$

where $C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)$ denotes the étale cohomology of X with coefficients in

$$\mathrm{Ad}(\rho) \coloneqq \rho \otimes \rho^\vee.$$

Proof. Since X is smooth and proper it follows that $\mathrm{Sh}^{\text{ét}}(X)$ is p-cohomologically compact and p-cohomologically perfect. Therefore, $\mathrm{PerfSys}_\ell(X)$ admits a cotangent complex and by restriction so does $\mathrm{LocSys}_{\ell,n}(X)$. Moreover, the tangent complex of $\mathrm{LocSys}_{\ell,n}(X)$ at the morphism

$$\rho \colon Z \to \mathrm{LocSys}_{\ell,n}(X)$$

can be identified with the mapping space

$$\mathbb{T}^{\mathrm{an}}_{\mathrm{LocSys}_{\ell,n}(X),\rho} \simeq \mathrm{Map}_{\mathrm{Perf}_p(X)(Z)}\big(1, \rho \otimes \rho\big)[1].$$

We are thus reduced to prove that

$$\mathrm{Map}_{\mathrm{Perf}_p(X)(Z)}\big(1, \rho \otimes \rho\big)[1] \simeq C^*_{\text{ét}}(X, \mathrm{Ad}(\rho))[1].$$

But this follows by the universal property of $\mathrm{Sh}^{\text{ét}}(X)$ together with the fact that global sections of local systems with torsion coefficients on $\mathrm{Sh}^{\text{ét}}(X)$ classify étale cohomology on X with torsion coefficients. The result follows now for liftable such ρ and for general ρ by Theorem 5.6.2.4. $\square$

Proposition 5.7.2.5. *The moduli stack* $\mathrm{LocSys}_{\ell,n}(X)$ *is cohesive and nilcomplete.*

Proof. This is a direct consequence of the analogous statement for $\mathrm{PerfSys}_{\ell}(X)$. $\square$

As a consequence we obtain our main result:

Theorem 5.7.2.6. *The moduli stack* $\mathrm{LocSys}_{\ell,n}(X) \in \mathrm{dSt}(\mathrm{dAfd}_k, \tau_{\text{ét}})$ *is representable by a derived* k*-analytic stack.*

Proof. The proof follows by the Representability theorem together with Theorem 5.7.2.2, Theorem 5.2.2.19, Theorem 5.7.2.4 and Theorem 5.7.2.5. $\square$

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Chapter 6

Moduli of ℓ-adic continuous representations of étale fundamental groups of non-proper varieties

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Moduli of ℓ-adic representations (Continuation)
June 23, 2019

Contents

6.1 Introduction

6.1.1 The goal of this paper

Let X be a smooth scheme over an algebraically closed field k of positive characteristic $p > 0$. Without the properness assumption the étale homotopy group $\pi_1^{\text{ét}}(X)$ fits in a short exact sequence of profinite groups

$$1 \to \pi_1^w(X) \to \pi_1^{\text{ét}}(X) \to \pi_1^{\mathrm{tame}}(X) \to 1, \tag{6.1.1.1}$$

where $\pi_1^w(X)$ and $\pi_1^{\mathrm{tame}}(X)$ denote the *wild* and *tame* fundamental groups of X, respectively. One can prove that the profinite group $\pi_1^{\mathrm{tame}}(X)$ is topologically of profinite type. However, the profinite group $\pi_1^{\text{ét}}(X)$ is, in general, a profinite pro-p group satisfying no finiteness condition or whatsoever. Needless to say, the étale fundamental group $\pi_1^{\text{ét}}(X)$ will in general not admit a finite number of topological generators. Consider $X = \mathbb{A}_k^1$, the affine line. Its étale and wild fundamental groups agree, but they are not topologically of finite type.

For this reason, the main results of Theorem 6.2.3.4 do not apply for a general smooth scheme X. In particular, one cannot expect that the moduli of ℓ-adic continuous representations of X, $\mathrm{LocSys}_{\ell,n}(X)$, is representable by a $\mathbb{Q}_\ell$-analytic stack. The purpose of the current text, is to study certain moduli substacks of $\mathrm{LocSys}_{\ell,n}$ parametrizing continuous representations

$$\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A), \quad A \in \mathrm{Afd}_{\mathbb{Q}_\ell}$$

such that the restriction $\rho_{|\pi_1^w(X)}$ factors through a finite quotient $p_\Gamma\colon \pi_1^w(X) \to \Gamma$. Denote $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$ such stack. Our main result is the following:

Theorem 6.1.1.1. *The moduli stack* $\mathrm{LocSys}_{\ell,n,\Gamma}(X)\colon \mathrm{Afd}_{\mathbb{Q}_\ell} \to \mathcal{S}$ *can be promoted naturally to a derived moduli stack*

$$\mathrm{RLocSys}_{\ell,n,\Gamma}(X)\colon \mathrm{dAfd}_{\mathbb{Q}_\ell} \to \mathcal{S}$$

which is representable by a derived $\mathbb{Q}_\ell$-analytic stack. Given $\rho \in \mathrm{RLocSys}_{\ell,n,\Gamma}(X)$, the analytic cotangent complex $\mathbb{L}_{\mathrm{RLocSys}_{\ell,n,\Gamma},\rho} \in \mathrm{Mod}_{\mathbb{Q}_\ell}$ is naturally equivalent to

$$\mathbb{L}^{\mathrm{an}}_{\mathrm{RLocSys}_{\ell,n,\Gamma}(X),\rho} \simeq \mathrm{C}^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)^\vee[-1]$$

in the derived ∞-category $\mathrm{Mod}_{\mathbb{Q}_\ell}$.

In particular, Theorem 6.1.1.1 implies that the inclusion morphism of stacks

$$j_\Gamma\colon \mathrm{RLocSys}_{\ell,n,\Gamma}(X) \hookrightarrow \mathrm{RLocSys}_{\ell,n}(X)$$

induces an equivalence on contangent complexes, in particular it is an étale morphism. We can thus regard $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$ as an admissible substack of $\mathrm{RLocSys}_{\ell,n}$, in the sense of $\mathbb{Q}_\ell$-analytic geometry.

The knowledge of the analytic cotangent complex allow us to have a better understanding of the local geometry of $\mathrm{RLocSys}_{\ell,n}$. In particular, given a continuous representation

$$\overline{\rho}\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$$

one might ask how $\overline{\rho}$ can be deformed into a continuous representation $\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\overline{Q}_\ell)$. This amounts to understand the formal moduli problem $\mathrm{Def}_{\overline{\rho}}\colon \mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_{\mathbb{F}_\ell} \to \mathcal{S}$ given on objects by the formula

$$A \in \mathcal{C}\mathrm{Alg}^{\mathrm{sm}} \mapsto \mathrm{Map}_{\mathrm{cont}}\left(\mathrm{Sh}_{\text{ét}}(X), \mathrm{BGL}_n(A)\right) \times_{\mathrm{Map}_{\mathrm{cont}}(\mathrm{Sh}_{\text{ét}}(X), \mathrm{BGL}_n(\mathbb{F}_\ell))} \{\rho\} \in \mathcal{S},$$

where $\mathrm{Sh}_{\text{ét}}(X) \in \mathrm{Pro}\big(\mathcal{S}^{\mathrm{fc}}\big)$ denotes the étale homotopy type of X. Given $\overline{\rho}$ as above, the functor $\mathrm{Def}_{\overline{\rho}}$ was first considered by Mazur in [24], for Galois representations, in the discrete case. More recently, Galatius and Venkatesh studied its derived structure in detail, see [10] for more details.

One can prove, using similar methods to those in [1] that the tangent complex of $\mathrm{Def}_{\overline{\rho}}$ is naturally equivalent to

$$\mathbb{T}_{\mathrm{Def}_{\overline{\rho}}} \simeq \mathrm{C}^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)[1],$$

in the derived ∞-category $\mathrm{Mod}_{\overline{\mathbb{F}}_\ell}$. We can consider $\mathrm{Def}_{\overline{\rho}}$ as a derived $W(\mathbb{F}_\ell)$-adic scheme which is locally admissible, in the sense of [2]. Therefore, one can consider its rigidification

$$\mathrm{Def}_{\overline{\rho}}^{\mathrm{rig}} \in \mathrm{dAn}_{\mathbb{Q}_\ell}.$$

By construction, we have a canonical inclusion functor

$$j_{\overline{\rho}}\colon \mathrm{Def}_{\overline{\rho}}^{\mathrm{rig}} \to \mathrm{LocSys}_{\ell,n}(X).$$

By comparing both analytic cotangent complexes, one arrives at the following result:

Proposition 6.1.1.2. *The morphism of derived stacks*

$$j_{\overline{\rho}}\colon \mathrm{Def}_{\overline{\rho}}^{\mathrm{rig}} \to \mathrm{LocSys}_{\ell,n}(X)$$

exhibits $\mathrm{Def}_{\overline{\rho}}^{\mathrm{rig}}$ *as an admissible open substack of* $\mathrm{LocSys}_{\ell,n}(X)$.

Theorem 6.1.1.2 implies, in particular, that $\mathrm{LocSys}_{\ell,n}(X)$ admits as an admissible analytic substack the disjoing union $\coprod_{\overline{\rho}} \mathrm{Def}_{\overline{\rho}}^{\mathrm{rig}}$, indexed by the set of contininuous representations $\overline{\rho}\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. Nonetheless, the moduli $\mathrm{LocSys}_{\ell,n}(X)$ admits more (analytic) points in general than those contained in the disjoint union $\coprod_{\overline{\rho}} \mathrm{Def}_{\overline{\rho}}^{\mathrm{rig}}$. This situation renders difficult the study of trace formulas on $\mathrm{LocSys}_{\ell,n}(X)$ which was the first motivation for the study of such moduli. Ideally, one would like to "glue" the connected components of $\mathrm{LocSys}_{\ell,n}(X)$ in order to have a better behaved global geometry. More specifically, one would like to exhibit a moduli algebras or analytic stack $\mathcal{M}_{\ell,n}(X)$ of finite type over $\mathbb{Q}_\ell$ such that the space closed points $\mathcal{M}_{\ell,n}(X)(\overline{\mathbb{Q}}_\ell) \in \mathcal{S}$ would correspond to continuous ℓ-adic representations of $\pi_1^{\text{ét}}(X)$. Moreover, one should expect such moduli stack to have a natural derived structure which would provided an understanding of deformations of ℓ-continuous representations ρ.

Such principle has been largely successful for instance in the context of continuous p-adic representations of a Galois group of a local field of mixed characteristic $(0,p)$. Via p-adic Hodge structure and a scheme-image construction provided in [16], the authors consider the moduli of Kisin modules which they prove to be an ind-algebraic stack admitting strata given by algebraic stacks of Kisin modules of a fixed height. Unfortunately, the methods used in [16], namely the scheme-image construction, do not directly generalize to the derived setting. Recent unpublished work of M. Porta and V. Melani regarding formal loop stacks might provide an effective answer to this problem, which we pretend to explore in the near future. However, to the best of the author's knowledge, there is no other successful attempts outside the scope of p-adic Hodge theory.

We will also study the existence of a $2-2d$-shifted symplectic form on $\mathrm{LocSys}_{\ell,n}(X)$, where $d = \dim X$. Even though $\mathrm{LocSys}_{\ell,n}(X)$ is not an instance of an analytic mapping stack it behaves as such. We need to introduce the moduli stack $\mathrm{PerfSys}_\ell(X)$ which corresponds to the moduli of objects associated to the $\mathcal{C}\mathrm{at}_\infty^{\mathrm{st},\omega,\otimes}$-valued moduli stack given on objects by the formula

$$Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell} \mapsto \mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\left(|X|_{\text{ét}}, \mathrm{Perf}\big(\Gamma(Z)\big)\right)$$

where $\mathcal{E}\mathcal{C}\mathrm{at}_\infty$ denotes the ∞-category of (small) $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$-enriched ∞-categories. We are then able to prove:

Theorem 6.1.1.3. *The derived moduli stack* $\mathrm{PerfSys}_\ell(X)$ *admits a natural shifted symplectic form* ω*. Explicitly, given* $\rho \in \mathrm{PerfSys}_\ell(X)$ ω *induces a non-degenerated pairing*

$$C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)[1] \otimes C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)[1] \to \mathbb{Q}_\ell[2-2d],$$

which agrees with Poincaré duality.

By transport of strucure, the substack $\mathrm{LocSys}_{\ell,n}(X) \hookrightarrow \mathrm{PerfSys}_\ell(X)$ can be equipped with a natural shifted sympletic structure. By restricting further, we equip the $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$ with a shifted symplectic form ω_Γ.

6.1.2 Summary

Let us give a brief review of the contents of each section of the text. Both §2.1 and §2.2 are devoted to review the main aspects of ramification theory for local fields and smooth varities in positive characteristic. Our exposition is classical and we do not pretend to prove anything new in this context. In §2.3 we construct the (ordinary) *moduli stack of continuous* ℓ*-adic representations*. Our construction follows directly the methods applied in [1]. Given $q\colon \pi_1^w(X) \to \Gamma$ a continuous group homomorphism whose target is finite we construct the moduli stack $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$ parametrizing ℓ-adic continuous representations of $\pi_1^{\text{ét}}(X)$ such that $\rho_{|\pi_1^w(X)}$ factors through Γ. We then show that $\mathrm{LocSys}_{\ell,n,\Gamma}$ is representable by a $\mathbb{Q}_\ell$-analytic stack (the analogue of an Artin stack in the context of $\mathbb{Q}_\ell$ analytic geometry).

In §3, we show that both the $\mathbb{Q}_\ell$-analytic stacks $\mathrm{LocSys}_{\ell,n}(X)$ and $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$ can be given natural derived structures and we compute their corresponding cotangent complexes. It follows then by [29, Theorem 7.1] that $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$ is representable by a derived $\mathbb{Q}_\ell$-analytic stack.

§4 is devoted to state and prove certain comparison results. We prove Theorem 6.1.1.2 and relate this result to the moduli of pseudo-representations introduced in [6].

Lastly, in §5 we study the existence of a shifted symplectic form on $\mathrm{LocSys}_{\ell,n}(X)$. We state and prove Theorem 6.1.1.3 and analysize some of its applications.

6.1.3 Convention and Notations

Throught the text we will employ the following notations:

(i) $\mathrm{Afd}_{\mathbb{Q}_\ell}$ and $\mathrm{dAfd}_{\mathbb{Q}_\ell}$ denote the ∞-categories of ordinary $\mathbb{Q}_\ell$-affinoid spaces and derived $\mathbb{Q}_\ell$-affinoid spaces, respectively.;

(ii) $\mathrm{An}_{\mathbb{Q}_\ell}$ and $\mathrm{dAn}_{\mathbb{Q}_\ell}$ denote the ∞-categories of analytic $\mathbb{Q}_\ell$-spaces and derived $\mathbb{Q}_\ell$-analytic spaces, respectively;

(iii) We shall denote $\mathcal{S}$ the ∞-category of spaces and $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S})) := \mathrm{Ind}\big(\mathrm{Pro}(\mathcal{S})\big)$ the ∞-category of ind-pro-objects on $\mathcal{S}$.

(iv) $\mathcal{C}\mathrm{at}_\infty$ denotes the ∞-category of small ∞-categories and $\mathcal{E}\mathcal{C}\mathrm{at}_\infty$ the ∞-category of $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$-enriched ∞-categories.

(v) Given a continuous representation ρ, we shall denote $\mathrm{Ad}(\rho) := \rho \otimes \rho^\vee$ the corresponding adjoint representation;

(vi) Given $Z \in \mathrm{Afd}_{\mathbb{Q}_\ell}$ we sometimes denote $\Gamma(Z) := \Gamma(Z)$ the derived $\mathbb{Q}_\ell$-algebra of global sections of Z.

6.1.4 Acknowledgements

I am grateful to Jean-Baptiste Tessier and Bertrand Toën for many useful discussions and suggestions on the contents of the present text. I would also like to acknowledge le *Séminaire Groupes Réductifs et formes automorphes* for the invitation to expose many of my ideas about the subject.

6.2 Setting the stage

6.2.1 Recall on the monodromy of (local) inertia

In this subsection we recall some well known facts on the monodromy of the local inertia, our exposition follows closely [9, §1.3].

Let K be a local field, $\mathcal{O}_K$ its ring of integers and k the residue field which we assume to be of characteristic $p > 0$ different from ℓ. Fix $\overline{K}$ an algebraic closure of K and denote by $G_K := \mathrm{Gal}\left(\overline{K}/K\right)$ its absolute Galois group.

Definition 6.2.1.1. Given a finite Galois extension L/K with Galois group $\mathrm{Gal}\,(L/K)$ we define its *inertia group*, denoted $I_{L/K}$, as the subgroup of $\mathrm{Gal}\,(L/K)$ spanned by those elements of $\mathrm{Gal}\,(L/K)$ which act trivially on $\mathfrak{l} := \mathcal{O}_L/\mathfrak{m}_L$, where $_L$ denotes the ring of integers of L and $\mathfrak{m}_L$ the corresponding maximal ideal.

Remark 6.2.1.2. We can identify the inertia subgroup $I_{L/K}$ of $\mathrm{Gal}(L/K)$ with the kernel of the surjective continuous group homomorphism $q\colon \mathrm{Gal}(L/K) \to \mathrm{Gal}(l/k)$. We have thus a short exact sequence of profinite groups

$$1 \to I_{L/K} \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(l/k) \to 1. \tag{6.2.1.1}$$

In particular, we deduce that the inertia subgroup $I_{L/K}$ can be identified with a normal subgroup of $\mathrm{Gal}(L/K)$.

Remark 6.2.1.3. Letting the field extension L/K vary, we can assemble together the short exact sequences displayed in (6.2.1.1) thus obtaining a short exact sequence of profinite groups

$$1 \to I_K \to G_K \to G_k \to 1, \tag{6.2.1.2}$$

where $G_k := \mathrm{Gal}(\overline{k}/k)$ where $\overline{k}$ denotes the algebraic closure of k determined by $\overline{K}$.

Definition 6.2.1.4 (Absolute inertia)**.** Define the *(absolute) inertia group of* K as the inverse limit

$$I_K := \lim_{L/K \text{ finite}} I_{L/K},$$

which we canonically identify with a subgroup of G_K.

Definition 6.2.1.5 (Wild inertia)**.** Let L/K be a field extension as above. We let $P_{L/K}$ denote the subgroup of $I_{L/K}$ which consists of those elements of $I_{L/K}$ acting trivially on $\mathcal{O}_L/\mathfrak{m}_L^2$. We refer to $P_{L/K}$ as the *wild inertia group* associated to L/K.

Definition 6.2.1.6 (Absolute wild inertia)**.** We define the absolute wild inertia group of K as:

$$P_K := \lim_{L \text{ finite}} P_{L/K}.$$

Remark 6.2.1.7. We can identify the absolute wild inertia group P_K with a normal subgroup of I_K.

Consider the exact sequence

$$1 \to P_K \to I_K \to I_K/P_K \to 1. \tag{6.2.1.3}$$

Thanks to [35, Lemma 53.13.6] it follows that the wild inertia group P_K is a *pro-p* group. When $K = \mathbb{Q}_p$ a theorem of Iwasawa implies that P_K is not topologically of finite generation, even though G_K is so. Nonetheless, the quotient I_K/P_K is much more amenable:

Proposition 6.2.1.8. *[4, Corollary 13] Let $p := \mathrm{char}(k)$ denote the residual characteristic of K. The quotient I_K/P_K is canonically isomorphic to $\widehat{\mathbb{Z}}'(1)$, where the latter denotes the profinite group $\prod_{q\neq p} \mathbb{Z}_q(1)$. In particular, the quotient profinite group I_K/P_K is topologically of finite generation.*

Define $P_{K,\ell}$ to be the inverse image of $\prod_{q\neq \ell,p} \mathbb{Z}_q$ in I_K. We have then a short exact sequence of profinite groups

$$1 \to P_K \to P_{K,\ell} \to \prod_{q\neq \ell,p} \mathbb{Z}_q \to 1.$$

Define similarly $G_{K,\ell} := G_K/P_{K,\ell}$ the quotient of G_K by $P_{K,\ell}$. We have a short exact sequence of profinite groups

$$1 \to P_{K,\ell} \to G_K \to G_{K,\ell} \to 1. \tag{6.2.1.4}$$

Assembling together (6.2.1.3) and Theorem 6.2.1.8 we obtain a short exact sequence

$$1 \to \mathbb{Z}_\ell(1) \to G_{K,\ell} \to G_k \to 1. \tag{6.2.1.5}$$

Remark 6.2.1.9. As a consequence of both (6.2.1.4) and (6.2.1.5) the quotient $G_{K,\ell}$ is topologically of finite type.

Suppose we are now given a continuous representation

$$\rho\colon G_K \to \mathrm{GL}_n(E_\ell),$$

where E_ℓ denotes a finite field extension of $\mathbb{Q}_\ell$. Up to conjugation, we might assume that ρ preserves a lattice of E_ℓ. More explicitly, up to conjugation we have a commutative diagram of the form

$$\begin{tikzcd} G_K \arrow[r, "\widetilde{\rho}"] \arrow[dr, "\rho"'] & \mathrm{GL}_n(\mathbb{Z}_\ell) \arrow[d] \\ & \mathrm{GL}_n(\mathbb{Q}_\ell) \end{tikzcd}.$$

Therefore $\widetilde{\rho}\,(G_K)$ is a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_\ell)$. Consider the short exact sequence

$$1 \to N_1 \to \mathrm{GL}_n(\mathbb{Z}_\ell) \to \mathrm{GL}_n(\mathbb{F}_\ell) \to 1,$$

where N_1 denotes the group of $\mathrm{GL}_n(\mathbb{Z}_p)$ formed by congruent to Id mod ℓ matrices. In particular, N_1 is a profinite pro-ℓ group. By construction, every finite quotient of $P_{K,\ell}$ is of order prime to ℓ. One then has necessarily

$$\rho\,(P_{K,\ell}) \cap N_1 = \{1\}.$$

As a consequence, the group $\rho(P_{K,\ell})$ injects into the finite group $\mathrm{GL}_n(\mathbb{F}_\ell)$ under ρ. Which in turn implies that the (absolute) wild inertia group P_K itself acts on $\mathrm{GL}_n(\mathbb{Q}_\ell)$ via a finite quotient.

6.2.2 Geometric étale fundamental groups

Let X be a geometrically connected smooth scheme over an algebraically closed field k of positive characteristic. Fix once and for all a geometric point $\iota_x\colon \overline{x} \to X$ and consider the corresponding étale fundamental group $\pi_1^{\text{ét}}(X) := \pi_1^{\text{ét}}(X, \overline{x})$, a profinite group. If we assume that X is moreover proper one has the following classical result:

Theorem 6.2.2.1. *[12, Exposé 10, Thm 2.9] Let X be a smooth and proper scheme over an algebraically closed field. Then its étale fundamental group $\pi_1^{\text{ét}}\left(\overline{X}\right)$ is topologically of finite type.*

Unfortunately, the statement of Theorem 6.2.2.1 does not hold in the non-proper case as the following proposition illustrates:

Proposition 6.2.2.2. *Let k be an algebraically closed field of positive characteristic. Then the étale fundamental group of the affine line $\pi_1^{\text{ét}}(\mathbb{A}^1_k)$ is not topologically finitely generated.*

Proof. For each integer $n \geq 1$, one can exhibit Galois covers of $\mathbb{A}^1_k$ whose corresponding automorphism group is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$. This statement readily implies that $\pi_1^{\text{ét}}(\mathbb{A}^1_k)$ does not admit a finite number of topological generators. In order to construct such coverings, we consider the following endomorphism of the affine line

$$\phi_n\colon \mathbb{A}^1_k \to \mathbb{A}^1_k,$$

defined via the formula

$$\phi_n\colon x \mapsto x^{p^n} - x.$$

The endormophism ϕ_n respects the additive group structure on $\mathbb{A}^1_k$. Moreover, the differential of ϕ_n equals -1. For this reason, ϕ_n induces an isomorphism on cotangent spaces and, in particular, it is an étale morphism. As k is algebraically closed, ϕ_n is surjective and it is finite, thus a finite étale covering. The automorphism group of ϕ_n is naturally identified with its kernel, which is isomorphic to $\mathbb{F}_{p^n}$. The statement of the proposition now follows. $\square$

Definition 6.2.2.3. Let G be a profinite group and p a prime numbger, we say that G is *quasi-p* if G equals the subgroup generated by all p-Sylow subgroups of G.

Examples of quasi-2 finite groups include the symmetric groups S_n, for $n \geq 2$. Moreover, for each prime p, the group $\mathrm{SL}_n(\mathbb{F}_p)$ is quasi-p. Let $X = \mathbb{A}^1_k$ be the affine line over an algebraically closed field k of characteristic $p > 0$. We have the following result proved by Raynaud which was originally a conjecture of Abhyankar:

Theorem 6.2.2.4. *[7, Conjecture 10] Every finite quasi-p group can be realized as a quotient of $\pi_1^{\text{ét}}(X)$.*

Remark 6.2.2.5. In the example of the affine line the infinite nature of $\pi_1(\mathbb{A}^1_k)$ arises as a phenomenon of the existence of étale coverings whose ramification at infinity can be as large as we desire. This phenomenon is special to the positive characteristic setting. Neverthless, we can prove that $\pi_1^{\text{ét}}(X)$ admits a topologically finitely generated quotient which corresponds to the group of automorphisms of tamely ramified coverings. On the other hand, in the proper case every finite étale covering of X is everywhere unramified.

Definition 6.2.2.6. Let $X \hookrightarrow \overline{X}$ be a normal compactification of X, whose existence is guaranteed by [26]. Let $f\colon Y \to X$ be a finite étale cover with connected source. We say that f is *tamely ramified along* the divisor $D := \overline{X}\backslash X$ if every codimension-1 point $x \in D$ is tamely ramified in the corresponding extension field extension $k(Y)/k(X)$.

Proposition 6.2.2.7. *Tamely ramified extensions along $D := \overline{X}\backslash X$ of X are classified by a quotient $\pi_1^{\text{ét}}(X) \to \pi_1^{\text{t}}(X, D)$, referred to as the* tame fundamental group of X along D.

Remark 6.2.2.8. Let $\overline{X}$ denote a smooth compactification of X and $D := \overline{X}\backslash X$. We denote by $\pi_1^w(X, D)$, the *wild fundamental group of X along D*, the kernel of the continuous morphism $\pi_1^{\text{ét}}(X) \to \pi_1^{\text{t}}(X)$.

Definition 6.2.2.9. Assume X is a normal connected scheme over k.

(i) Let $f\colon Y \to X$ be an étale covering. We say that f is *divisor-tame* if for every normal compactification $X \hookrightarrow \overline{X}$, f is tamely ramified along $D = \overline{X}\backslash X$.

(ii) Let $f\colon Y \to X$ be an étale covering. We shall refer to f as *curve-tame* if for every smooth curve C over k and morphism $g\colon C \to X$, the base change $Y \times_X C \to C$ is a tame covering of the curve C.

Remark 6.2.2.10. In Theorem 6.2.2.9 X is assumed to be a normal connected scheme over a field of positive characteristic. Currently, we lack a resolution of singularities theorem in this setting. Therefore, a priori, one cannot expect that both divisor-tame and curve-tame notions agree in general. Indeed, one can expect many regular normal crossing compactifications of X to exist, or none.

Neverthless, one has the following result:

Proposition 6.2.2.11. *[17, Theorem 1.1] Let X be a smooth scheme over k and let $f\colon Y \to X$ be an étale covering. Then f is divisor-tame if and only if it is curve-tame.*

Definition 6.2.2.12. The *tame fundamental group* $\pi_1^{\text{t}}(X)$ is defined as the quotient of $\pi_1^{\text{ét}}(X)$ by the normal closure of opens subgroup of $\pi_1^{\text{ét}}(X)$ generated by the wild fundamental groups $\pi_1^w(X, D)$ along D, for each normal compactification $X \hookrightarrow \overline{X}$.

Remark 6.2.2.13. The notion of tameness is stable under arbitrary base changes between smooth schemes. In particular, given a morphism $f\colon Y \to X$ between smooth schemes over k, one has a functorial well defined morphism $\pi_1^{\text{t}}(Y) \to \pi_1^{\text{t}}(X)$ fitting in a commutative diagram of profinite groups

$$\begin{tikzcd}
\pi_1^{\text{ét}}(Y) \arrow[r] \arrow[d] & \pi_1^{\text{ét}}(X) \arrow[d] \\
\pi_1^{\text{t}}(Y) \arrow[r] & \pi_1^{\text{t}}(X).
\end{tikzcd}$$

Moreover, the profinite group $\pi_1^{\text{t}}(X)$ classifies tamely ramified étale coverings of X.

Remark 6.2.2.14. The tame fundamental group $\pi_1^{\text{t}}(X)$ classifies finite étale coverings $f\colon X \to Y$ which are tamely ramified along any divisor at infinity.

Definition 6.2.2.15. We define the *wild fundamental group* of X, denoted $\pi_1^w(X)$, as the kernel of the surjection $\pi_1^{\text{ét}}(X) \to \pi_1^{\text{t}}(X)$. It is an open normal subgroup of $\pi_1^{\text{ét}}(X)$.

Proposition 6.2.2.16. *[7] Let C be a geometrically connected smooth curve over k. Then the wild fundamental group $\pi_1^w(C)$ is a pro-p-group.*

Theorem 6.2.2.17. *[5, Appendix 1, Theorem 1] Let X be a smooth and geometrically connected scheme over k. There exists a smooth, geometrically connected curve C/k together with a morphism $f \colon C \to X$ of varieties such that the corresponding morphism at the level of fundamental groups $\pi_1^{\text{ét}}(C) \to \pi_1^{\text{ét}}(X) \to \pi_1^{\text{t}}(X)$ is surjective and it factors by a well defined morphism $\pi_1^{\text{t}}(C) \to \pi_1^{\text{t}}(X)$. In particular, $\pi_1^{\text{t}}(X)$ is topologically finitely generated.*

Remark 6.2.2.18. Theorem 6.2.2.17 implies that $\pi_1^{\text{t}}(\mathbb{A}_k^1)$ admits a finite number of topological generators. In fact, the group $\pi_1^{\text{t}}(\mathbb{A}_k^1)$ is trivial.

6.2.3 Moduli of continuous ℓ-adic representations

In this §, X denotes a smooth scheme over an algebraically closed field of positive characteristic $p > 0$. Nevertheless, our arguments apply when X is the spectrum of a local field of mixed characteristic.

Remark 6.2.3.1. Let $A \in \mathrm{Afd}$ be $\mathbb{Q}_\ell$-affinoid algebra $A \in \mathrm{Afd}$. It admits a natural topology induced from a choice of a norm on A, compatible with the usual ℓ-adic valuation on $\mathbb{Q}_\ell$. Given $\mathbf{G}$ an analytic $\mathbb{Q}_\ell$-group space we can consider the corresponding group of A-points on $\mathbf{G}$, $\mathbf{G}(A)$. The group $\mathbf{G}(A)$ admits a natural topology induced from the non-archimedean topology on A. In the current text we will be interested in studying the moduli functor parametrizing continuous representations

$$\rho \colon \pi_1^{\text{ét}}(X) \to \mathbf{GL}_n^{\text{an}}(A).$$

Nevertheless, our arguments can be directly applied when we instead consider the moduli of continuous representations

$$\pi_1^{\text{ét}}(X) \to \mathbf{G}^{\text{an}}(A),$$

where $\mathbf{G}$ denotes a reductive group scheme.

Definition 6.2.3.2. Let G be a profinite group. Denote by

$$\mathrm{LocSys}_{\ell,n}^{\text{framed}}(G) \colon \mathrm{Afd}_{\mathbb{Q}_\ell} \to \mathrm{Set},$$

the *functor of rank n continuous ℓ-adic group homomorphisms of G*. It is given on objects by the formula

$$A \in \mathrm{Afd}^{\text{op}} \mapsto \mathrm{Hom}_{\text{cont}}\left(G, \mathrm{GL}_n(A)\right) \in \mathrm{Set}, \tag{6.2.3.1}$$

where the right hand side of (6.2.3.1) denotes the set of continuous group homomorphisms $G_K \to \mathrm{GL}_n(A)$.

Notation 6.2.3.3. Whenever $G = \pi_1^{\text{ét}}(X)$ we denote $\mathrm{LocSys}_{\ell,n}(X) := \mathrm{LocSys}_{\ell,n}(\pi_1^{\text{ét}}(X))$.

Proposition 6.2.3.4. *[1, Corollary 2.2.16] Suppose G is a topologically finitely generated profinite group. Then the functor $\mathrm{LocSys}_{\ell,n}^{\text{framed}}(G)$ is representable by a $\mathbb{Q}_\ell$-analytic space.*

By the results of the previous §, the étale fundamental group $\pi_1^{\text{ét}}(X)$ is almost never topologically finitely generated in the non-proper case. For this reason, we cannot expect the functor $\mathrm{LocSys}_{\ell,n}^{\text{framed}}(G_X)$ to be representable by an object in the category $\mathrm{An}_{\mathbb{Q}_\ell}$ of $\mathbb{Q}_\ell$-analytic spaces. Nevertheless, we can prove an analogue of Theorem 6.2.3.4 if we consider instead certain subfunctors of $\mathrm{LocSys}_{\ell,n}^{\text{framed}}$. More specifically, given a finite quotient $q \colon \pi_1^w(X) \to \Gamma$ we can consider the moduli parametrizing continuous ℓ-adic representations of $\pi_1^{\text{ét}}(X)$ whose restriction to $\pi_1^w(X)$ factors through Γ:

Construction 6.2.3.5. Let $q \colon \pi_1^{\text{w}}(X) \to \Gamma$ denote a surjective continuous group homomorphism, whose target is a finite group (equipped with the discrete topology). We define the functor of *continuous group homomorphisms $\pi_1^{ét}(X)$ to $\mathrm{GL}_n(-)$ with Γ-bounded ramification at infinity*, as the fiber product

$$\mathrm{LocSys}_{\ell,n,\Gamma}^{\text{framed}}(\pi_1^{\text{ét}}(X)) := \mathrm{LocSys}_{\ell,n}^{\text{framed}}(\pi_1^{\text{ét}}(X)) \times_{\mathrm{LocSys}_{\ell,n}^{\text{framed}}(\pi_1^{\text{w}}(X))} \mathrm{LocSys}_{\ell,n}^{\text{frame}}(\Gamma), \tag{6.2.3.2}$$

computed in the category $\mathrm{Fun}\left(\mathrm{Afd}^{\text{op}}, \mathrm{Set}\right)$.

Remark 6.2.3.6. The moduli functor $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n,\Gamma}(X)$ introduced in Theorem 6.2.3.5 depends on the choice of the continuous surjective homomorphism $q\colon \mathrm{P}_X \to \Gamma$. However, for notational convenience we drop the subscript q.

We have the following result:

Theorem 6.2.3.7. *The functor* $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n,\Gamma}(X)$ *is representable by a* $\mathbb{Q}_\ell$*-analytic stack.*

Proof. Let r be a positive integer and denote $\mathrm{F}^{[r]}$ a free profinite group on r topological generators. The finite group Γ and the quotient G_X/P_X are topologically of finite generation. Therefore, it is possible to choose a continuous group homomorphism

$$p\colon \mathrm{F}^{[r]} \to \pi_1^{\text{ét}}(X),$$

such that the images $p(e_i)$, for $i = 1, \dots, r$, form a set of generators for Γ, seen as a quotient of $\pi_1^w(X)$, and for $\pi_1^{\mathrm{t}}(X) \cong \pi_1^{\text{ét}}(X)/\pi_1^w(X)$. Restriction under φ induces a closed immersion of functors

$$\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n,\Gamma}(G_X) \hookrightarrow \mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(\mathrm{F}^{[r]}).$$

Thanks to [1, Theorem 2.2.15.], the latter is representable by a rigid $\mathbb{Q}_\ell$-analytic space, denoted $X^{[r]}$. It follows that $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n,\Gamma}(G_X)$ is representable by a closed subspace of $X^{[r]}$, which proves the statement. $\square$

Definition 6.2.3.8. Let $\mathrm{PShv}\,(\mathrm{Afd}_{\mathbb{Q}_\ell}) := \mathrm{Fun}\big(\mathrm{Afd}^{\mathrm{op}}_{\mathbb{Q}_\ell}, \mathcal{S}\big)$ denote the ∞-category of $\mathcal{S}$-valued preasheaves on $\mathrm{Afd}_{\mathbb{Q}_\ell}$. Consider the étale site $(\mathrm{Afd}, \tau_{\text{ét}})$. We define the ∞-category of *higher stacks* on $(\mathrm{Afd}, \tau_{\text{ét}})$, $\mathrm{St}\,(\mathrm{Afd}, \tau_{\text{ét}})$, as the full subcategory of $\mathrm{PShv}\,(\mathrm{Afd})$ spanned by those pre-sheaves which satisfying étale hyper-descent, [19, §7].

Remark 6.2.3.9. The inclusion functor $\mathrm{St}\,(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}) \subseteq \mathrm{PShv}\,(\mathrm{Afd})$ admits a left adjoint, which is a left localization functor. For this reason, the ∞-category $\mathrm{St}\,(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}})$ is a presentable ∞-category. One can actually prove that $\mathrm{St}\big(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}\big)$ is the hypercompletion of the ∞-topos of étale sheaves on $\mathrm{Afd}_{\mathbb{Q}_\ell}$, $\mathrm{Shv}_{\text{ét}}\big(\mathrm{Afd}\big)$.

Definition 6.2.3.10. Consider the geometric context $(\mathrm{dAfd}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$, [1, Definition 2.3.1]. Let $\mathrm{St}\,(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$ denote the full subcategory of $\mathrm{St}(\mathrm{Afd}, \tau_{\text{ét}})$ spanned by geometric stacks, [1, Definition 2.3.2]. We will refer to an object $\mathcal{F} \in \mathrm{St}\,(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$ as the a $\mathbb{Q}_\ell$-analytic stack and we refer to $\mathrm{St}\big(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}\big)$ as the ∞-category of $\mathbb{Q}_\ell$*-analytic stacks*.

Example 6.2.3.11. Let $\mathbf{G}$ be a group object in the ∞-category $\mathrm{St}\,(\mathrm{Afd}, \tau_{\text{ét}}, \mathrm{P}_{\mathrm{sm}})$. Given a $\mathbf{G}$-equivariant object $\mathcal{F} \in \mathrm{St}\,(\mathrm{Afd}, \tau_{\text{ét}}, P_{\mathrm{sm}})^{\mathbf{G}}$ we denote $[\mathcal{F}/\mathbf{G}]$ the geometric realization of the simplicial object

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \mathbf{G}^2 \times \mathcal{F} \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \mathbf{G} \times \mathcal{F} \;\rightrightarrows\; \mathcal{F}$$

computed in the ∞-category $\mathrm{St}\big(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}\big)$. We refer to $[\mathcal{F}/\mathbf{G}]$ as the *quotient stack objec*t of $\mathcal{F}$ by $\mathbf{G}$.

Lemma 6.2.3.12. *[1, §2.3]. Suppose* $\mathbf{G} \in \mathrm{St}\,(\mathrm{Afd}, \tau_{\acute{e}t}, P_{\mathrm{sm}})$ *is a smooth group object and* $\mathcal{F}$ *is representable by a* $\mathbb{Q}_\ell$*-analytic space. Then the quotient stack object* $[\mathcal{F}/\mathbf{G}]$ *is representable by a geometric stack.*

Remark 6.2.3.13. The smooth group $\mathbf{GL}_n^{\mathrm{an}} \in \mathrm{An}_{\mathbb{Q}_\ell}$ acts by conjugation on the moduli functor $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}$.

Definition 6.2.3.14. Let $\mathrm{LocSys}_{\ell,n}(X) := [\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n}(X)/\mathrm{GL}_{\mathbf{n}}{}^{\mathrm{an}}]$ denote the *moduli stack of rank* n ℓ*-adic pro-étale local systems on* X. Given a continuous surjective group homomorphism $q\colon \pi_1^{\mathrm{w}}(X) \to \Gamma$ whose target is a finite group we define the substack of $\mathrm{LocSys}_{\ell,n}(X)$ spanned by rank n ℓ-adic pro-étale local systems on X *ramified at infinity by level* Γ as the fiber product

$$\mathrm{LocSys}_{\ell,n,\Gamma} := \mathrm{LocSys}_{\ell,n}(X) \times_{\mathrm{LocSys}_{\ell,n}(\pi_1^w(X)} \mathrm{LocSys}_{\ell,n}(\Gamma)$$

Theorem 6.2.3.15. *The moduli stack* $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$ *is representable by a* $\mathbb{Q}_\ell$*-analytic stack.*

Proof. We have a canonical map $\mathrm{LocSys}^{\mathrm{framed}}_{\ell,n,\Gamma}(G_X) \to \mathrm{LocSys}_{\ell,n,\Gamma}(X)$, which exhibits the former as a smooth atlas of the latter. The result now follows formally, as explained in [1, §2.3]. $\square$

One can prove that there is an equivalence between the space of continuous representations

$$\rho\colon \pi_1^{\text{ét}}(X) \to \mathbf{GL}_n^{\text{an}}(A), \quad A \in \mathrm{Afd}_{\mathbb{Q}_\ell}$$

and the space of rank n pro-étale A-local systems on X. We thus have the following statement:

Proposition 6.2.3.16. *[1, Corollary 3.2.5] The functor* $\mathrm{LocSys}_{\ell,n}(X)$ *parametrizes pro-étale local systems of rank* n *on* X.

Proof. The same proof of [1, Corollary 3.2.5] applies. □

6.3 Derived structure

Let X be a smooth scheme over an algebraically closed field k and fix a finite quotient $q\colon \pi_1^w(X) \to \Gamma$. In this §we will study at full the deformation theory of both the $\mathbb{Q}_\ell$-analytic moduli stacks $\mathrm{LocSys}_{\ell,n}(X)$ and $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$. Our goal is to show that $\mathrm{LocSys}_{\ell,n}(X)$ and $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$ can be naturally promoted to *derived* $\mathbb{Q}_\ell$*-stacks*, denoted $\mathrm{RLocSys}_{\ell,n}(X)$ and $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$, respectively. Therefore the corresponding 0-truncations $\mathrm{t}_{\leq 0}\mathrm{RLocSys}_{\ell,n}(X)$ and $\mathrm{t}_{\leq 0}\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$ are equivalent to $\mathrm{LocSys}_{\ell,n}(X)$ and $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$, respectively. We will prove moreover that both $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$ and $\mathrm{LocSys}_{\ell,n}(X)$ admit tangent complexes and give a precise formula for these. Moreover, we show that the substack $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$ is geometric with respect to the geometric context $\left(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}, \mathrm{P}_{\text{sm}}\right)$. In particular, $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$ admits a cotangent complex which we can understand at full.

We compute the corresponding cotangent complexes and analyze some consequences of the existence of derived structures on theses objects. We will use extensively the language of derived $\mathbb{Q}_\ell$-analytic geometry as developed in [28, 29].

6.3.1 Derived enhancement of $\mathrm{LocSys}_{\ell,n}(X)$

Recall the ∞-category of derived $\mathbb{Q}_\ell$-affinoid spaces $\mathrm{dAfd}_{\mathbb{Q}_\ell}$ introduced in [28]. Given a derived $\mathbb{Q}_\ell$-affinoid space $Z := (\mathcal{Z}, \mathcal{O}_Z) \in \mathrm{dAfd}_{\mathbb{Q}_\ell}$, we denote

$$\Gamma(Z) := \Gamma\left(\mathcal{O}_Z^{\text{alg}}\right) \in \mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}$$

the corresponding derived ring of *global sections on* Z, see [27, Theorem 3.1] for more details. [2, Theorem 4.4.10] implies that $\Gamma(Z)$ always admits a formal model, i.e., a ℓ-complete derived $\mathbb{Z}_\ell$-algebra $A_0 \in \mathcal{C}\mathrm{Alg}_{\mathbb{Z}_\ell}$ such that $(\mathrm{Spf}\, A_0)^{\text{rig}} \simeq X$. Here $(-)^{\text{rig}}$ denotes the rigidification functor from derived formal $\mathbb{Z}_\ell$-schemes to derived $\mathbb{Q}_\ell$-analytic spaces, introduced in [2, §4]. This allow us to prove:

Proposition 6.3.1.1. *[1, Proposition 4.3.6] The* ∞*-category of perfect complexes on* A, $\mathrm{Perf}(A)$, *admits a natural structure of* $\mathrm{Ind}\big(\mathrm{Pro}(\mathcal{S})\big)$*-enriched* ∞*-category, i.e., it can be naturally upgraded to an object in the* ∞*-category* $\mathcal{E}\mathcal{C}\mathrm{at}_\infty$.

Definition 6.3.1.2. Let $Y \in \mathrm{Ind}\big(\mathrm{Pro}(\mathcal{S})\big)$. We define its *materialization* by the formula

$$\mathrm{Mat}\,(\mathcal{X}) := \mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\,(*, \mathcal{X}) \in \mathcal{S},$$

where $* \in \mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$ denotes the terminal object. This formula is functorial. For this reason, we have a well defined, up to contractible indeterminacy functor, *materialization functor* $\mathrm{Mat}\colon \mathrm{Ind}(\mathrm{Pro}(\mathcal{S})) \to \mathcal{S}$.

As a consequence of Theorem 6.3.1.1, there exists an object $\mathrm{B}\mathcal{E}\mathrm{nd}(Z) \in \mathrm{Ind}\big(\mathrm{Pro}(\mathcal{S})\big)$, functorial in $Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}$, such that its *materalization* is equivalent to

$$\mathrm{Mat}\,(\mathcal{E}\mathrm{nd}(Z)) \simeq \mathrm{BEnd}(\Gamma(Z)^n) \in \mathcal{S}. \tag{6.3.1.1}$$

The right hand side of (6.3.1.1) denotes the usual Bar-construction applied to $\mathbb{E}_1$-monoid object $\mathrm{End}(\Gamma(Z)) \in \mathcal{S}$. Moreover, given $Y \in \mathrm{Ind}\big(\mathrm{Pro}(\mathcal{S})\big)$ every continuous morphism

$$Y \to \mathrm{B}\mathcal{E}\mathrm{nd}(Z), \text{in } \mathrm{Ind}\big(\mathrm{Pro}(\mathcal{S})\big)$$

is such that its materialization factors as

$$\mathrm{Mat}\,(Y) \to \mathrm{BGL}_n(\Gamma(Z)) \hookrightarrow \mathrm{BEnd}(\Gamma(Z))$$

in the ∞-category $\mathcal{S}$. See [1, §4.3 and §4.4] for more details.

Definition 6.3.1.3. [22, Notation 3.6.1] We shall denote $\mathrm{Sh}^{\text{ét}}(X)$ the *étale shape of* X defined as the fundamental groupoid associated to the ∞-topos $\mathrm{Shv}_{\text{ét}}\,(X)^{\wedge}$, of hyper-complete étale sheaves on X.

Definition 6.3.1.4. Let X be as above. We define the *derived moduli stack of ℓ-adic pro-étale local systems of rank n on X* as the functor

$$\mathrm{RLocSys}_{\ell,n}(X) \colon \mathrm{dAfd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \to \mathcal{S},$$

given informally on objects by the formula

$$Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \mapsto \lim_{n \geq 0} \mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_1}(\mathcal{C})} \left(\mathrm{Sh}^{\text{ét}}(X), \mathrm{B}\mathcal{E}\mathrm{nd}\left(\mathrm{t}_{\leq n}(Z)\right)\right),$$

where $\mathrm{t}_{\leq n}(Z)$ denotes the n-th truncation functor on derived $\mathbb{Q}_\ell$-affinoid spaces.

Notation 6.3.1.5. Given $Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}$ we sometimes prefer to employ the notation

$$\mathrm{RLocSys}_{\ell,n}(X)(\Gamma(Z)) := \mathrm{RLocSys}_{\ell,n}(X)(Z).$$

Let $\rho \in \mathrm{RLocSys}_{\ell,n}(X)(\Gamma(Z))$, we refer to it as a *continuous representation of* $\mathrm{Sh}^{\text{ét}}(X)$ *with coefficients in* $\Gamma(Z)$.

Definition 6.3.1.6. Let $Y := \lim_m Y_m \in \mathrm{Pro}\,(\mathcal{S})$. Given an integer $n \geq 0$, we define the *n-truncation of Y* as

$$\tau_{\leq n}\,(Y) := \lim_m \tau_{\leq n}(Y_m) \in \mathrm{Pro}(\mathcal{S}_{\leq n}),$$

i.e. we apply pointwise the truncation functor $\tau_{\leq n} \colon \mathcal{S} \to \mathcal{S}$ to the diagram defining $Y = \lim_m Y_m \in \mathrm{Pro}(\mathcal{S})$.

$\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$

Notation 6.3.1.7. Let $\iota \colon \mathrm{Afd} \to \mathrm{dAfd}_{\mathbb{Q}_\ell}$ denote the canonical inclusion functor. Denote by

$$\mathrm{t}_{\leq 0}\left(\mathrm{RLocSys}_{\ell,n}(X)\right) := \mathrm{RLocSys}_{\ell,n}(X) \circ \iota,$$

the restriction of $\mathrm{RLocSys}_{\ell,n}(X)$ to $\mathrm{Afd}_{\mathbb{Q}_\ell}$.

Given $Z \in \mathrm{Afd}^{\mathrm{op}}$, the object $\mathrm{B}\mathcal{E}\mathrm{nd}(Z) \in \mathrm{Ind}\big(\mathrm{Pro}(\mathcal{S})\big)$ is 1-truncated. As a consequence, we have an equivalence of mapping spaces:

$$\mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{B}\mathcal{E}\mathrm{nd}(Z)\big) \simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\left(\tau_{\leq 1}\mathrm{Sh}^{\text{ét}}(X), \mathrm{B}\mathcal{E}\mathrm{nd}(Z)\right).$$

We have moreover an equivalence of profinite spaces $\tau_{\leq 1}\mathrm{Sh}^{\text{ét}}(X) \simeq \mathrm{B}\pi_1^{\text{ét}}(X)$. Given a continuous group homomorphism $\rho \colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A)$ we can associate, via the cobar construction performed in the ∞-category $\mathcal{T}\mathrm{op}_{\mathrm{na}}$, a well defined morphism

$$\mathrm{B}\rho \colon \mathrm{B}\pi_1^{\text{ét}}(X) \to \mathrm{B}\mathcal{E}\mathrm{nd}(A),$$

in the ∞-category $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$. This construction provide us with a well defined, up to contractible indeterminacy,

$$p_A \colon \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X)(A) \to \mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\left(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{B}\mathcal{E}\mathrm{nd}(Z)\right).$$

On the other hand, the morphisms p_A assemble to provide a morphism of stacks

$$p \colon \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \to \mathrm{t}_{\leq 0}\mathrm{RLocSys}_{\ell,n}(X).$$

Proposition 6.3.1.8. *The canonical morphism*

$$p \colon \mathrm{LocSys}_{\ell,n}^{\mathrm{framed}}(X) \to \mathrm{t}_{\leq 0}\mathrm{RLocSys}_{\ell,n}(X),$$

in the ∞-category $\mathrm{St}\big(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}\big)$ which induces an equivalence of stacks

$$\mathrm{LocSys}_{\ell,n}(X) \simeq \mathrm{t}_{\leq 0}\mathrm{RLocSys}_{\ell,n}(X).$$

Proof. The proof of [1, Theorem 4.5.8] applies. □

Notation 6.3.1.9. Let $Z := (\mathcal{Z}, \mathcal{O}_Z) \in \mathrm{dAn}$ denote a derived $\mathbb{Q}_\ell$-analytic space and $M \in \mathrm{Mod}_{\mathcal{O}_Z}$. In [29, §5] it was introduced the analytic square zero extension of Z by M as the derived $\mathbb{Q}_\ell$-analytic space $Z[M] := (\mathcal{Z}, \mathcal{O}_Z \oplus M) \in \mathrm{dAn}$, where $\mathcal{O}_Z \oplus M := \Omega^\infty_{\mathrm{an}} \in \mathrm{AnRing}_k(\mathcal{Z})_{/\mathcal{O}_Z}$ denotes the trivial square zero extension of $\mathcal{O}_Z$ by M. In this case, we have a natural composite

$$\mathcal{O}_Z \to \mathcal{O}_Z \oplus M \to \mathcal{O}_Z \tag{6.3.1.2}$$

in the ∞-category $\mathrm{AnRing}_k(\mathcal{Z})_{/\mathcal{O}_Z}$ which is naturally equivalent to the identity on $\mathcal{O}_Z$. We denote $p_{Z,M} \colon \mathcal{O}_Z \oplus M \to \mathcal{O}_Z$ the natural projection displayed in (6.3.1.2)

Definition 6.3.1.10. Let $Z \in \mathrm{dAfd}^{\mathrm{op}}_{\mathbb{Q}_\ell}$ be a derived $\mathbb{Q}_\ell$-affinoid space. Let $\rho \in \mathrm{RLocSys}_{\ell,n}(X)(\mathcal{O}_Z)$ be a continuous representation with values in $\mathcal{O}_Z$. The *tangent complex* of $\mathrm{RLocSys}_{\ell,n}(X)$ at ρ is defined as the fiber

$$\mathbb{T}_{\mathrm{RLocSys}_{\ell,n}(X),\rho} := \mathrm{fib}_\rho\left(p_{\mathcal{O}_Z}\right)$$

where

$$p_{\mathcal{O}_Z} \colon \mathrm{RLocSys}_{\ell,n}(X)(\mathcal{O}_Z \oplus^{\mathrm{an}} \mathcal{O}_Z) \to \mathrm{RLocSys}_{\ell,n}(\mathcal{O}_Z),$$

is the morphism of stacks induced from the canonical projection map $p_{\mathcal{O}_Z,\mathcal{O}_Z} \colon \mathcal{O}_Z \oplus \mathcal{O}_Z \to \mathcal{O}_Z$.

The derived stack $\mathrm{RLocSys}_{\ell,n}$ is not, in general, representable as derived $\mathbb{Q}_\ell$-analytic stack, as this would entail the representability of its 0-truncation. Nevertheless we can compute its tangent complex explicitly:

Lemma 6.3.1.11. *[1, Proposition 4.4.9.] Let $\rho \in \mathrm{RLocSys}_{\ell,n}(X)(\mathcal{O}_Z)$. We have a natural morphism*

$$\mathbb{T}_{\mathrm{RLocSys}_{\ell,n}(X),\rho} \to C^*_{\text{ét}}\left(X, \mathrm{Ad}\left(\rho\right)\right)[1],$$

which is an equivalence in the derived ∞-category $\mathrm{Mod}_{\Gamma(Z)}$.

Proof. The proof of [1, Proposition 4.4.9] applies. □

6.3.2 The bounded ramification case

In this §we are going to define a natural derived enhancement of $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$ and prove its representability by a derived $\mathbb{Q}_\ell$-analytic stack. Let X be a smooth scheme over an algebraically closed field k of positive characteristic $p \neq \ell$.

Definition 6.3.2.1. Consider the sub-site $X^{\mathrm{tame}}_{\text{ét}}$ of the small étale site $X_{\text{ét}}$ spanned by those étale coverings $Y \to X$ satisfying condition (2) in Theorem 6.2.2.9. We can form the ∞-topos $\mathrm{Shv}^{\mathrm{tame}}(X) := \mathrm{Shv}\left(X^{\mathrm{tame}}_{\text{ét}}\right)$ of *tamely ramified* étale sheaves on the Grothendieck site $X^{\mathrm{tame}}_{\text{ét}}$.

Consider the inclusion of sites $\iota \colon X^{\mathrm{tame}}_{\text{ét}} \hookrightarrow X_{\text{ét}}$, it induces a geometric morphism of ∞-topoi

$$g_* \colon \mathrm{Shv}_{\text{ét}}(X) \to \mathrm{Shv}^{\mathrm{tame}}_{\text{ét}}(X) \tag{6.3.2.1}$$

which is a right adjoint functor to the functor induced by precomposition with ι.

Lemma 6.3.2.2. *The geometric morphism of ∞-topoi $g_* \colon \mathrm{Shv}^{\mathrm{tame}}_{\text{ét}}(X) \to \mathrm{Shv}_{\text{ét}}(X)$ introduced in* (6.3.2.1) *is fully faithful.*

Proof. As the Grothendieck topology on $X^{\mathrm{tame}}_{\text{ét}}$ is induced by the inclusion functor $\iota \colon X^{\mathrm{tame}}_{\text{ét}} \to X_{\text{ét}}$, it suffices to prove the corresponding statement for the ∞-categories of presheaves. More specifically, the statement of the lemma is a consequence of the assertion that the left adjoint

$$\iota^* \colon \mathrm{PShv}\left(X_{\text{ét}}\right) \to \mathrm{PShv}\left(X^{\mathrm{tame}}_{\text{ét}}\right),$$

given by precomposition along ι, admits a fully faithful right adjoint. The existence of a right adjoint for ι^*, denoted ι_*, follows by the Adjoint functor theorem. The required right adjoint is moreover computed by means of a right Kan extension along ι. Let $Y \in X^{\mathrm{tame}}_{\text{ét}}$, we can consider $Y \in X_{\text{ét}}$ by means of the inclusion functor

$\iota\colon X^{\text{ét}}_{\text{tame}} \to X_{\text{ét}}$. The comma ∞-category $\left(X^{\text{tame}}_{\text{ét}}\right)_{Y/}$ admits an initial object, namely Y itself. Let $\mathcal{C}_Y := \left(X^{\text{tame}}_{\text{ét}}\right)_{Y/}$. Given $\mathcal{F} \in \text{PShv}\left(X^{\text{tame}}_{\text{ét}}\right)$ one can compute

$$\begin{aligned}\iota^*\iota_*\mathcal{F}(Y) \simeq &\\ &\simeq \iota_*\mathcal{F}(Y)\\ &\simeq \iota^*\lim_{V\in\mathcal{C}_Y}\mathcal{F}(V)\\ &\simeq \mathcal{F}(Y)\end{aligned}$$

In particular, the counit of the adjunction $\theta\colon \iota^* \circ \iota_* \to \text{Id}$ is an equivalence. Reasoning formally we deduce that ι_* is fully faithful and therefore so it is g_*. □

Definition 6.3.2.3. Let $\text{Sh}^{\text{tame}}(X) \in \text{Pro}\,(\mathcal{S})$ denote the fundamental ∞-groupoid associated to the ∞-topos $\text{Shv}(X^{\text{tame}}_{\text{ét}})$, which we refer to as the *tame étale homotopy type of* X.

Remark 6.3.2.4. The fact that the geometric morphism $g_*\colon \text{Shv}(X^{\text{tame}}_{\text{ét}}) \to \text{Shv}(X_{\text{ét}})$ is fully faithful implies that the canonical morphism

$$\text{Sh}^{\text{tame}}(X) \to \text{Sh}^{\text{ét}}(X)$$

induces an equivalence of profinite abelian groups $\pi_i\left(\text{Sh}^{\text{tame}}(X)\right) \simeq \pi_i\left(\text{Sh}^{\text{ét}}(X)\right)$ for each $i > 1$. As a consequence one has a fiber sequence

$$\text{B}\pi_1^w(X) \to \text{Sh}^{\text{ét}}(X) \to \text{Sh}^{\text{tame}}(X),$$

in the ∞-category $\text{Pro}(\mathcal{S}^{\text{fc}})$ of profinite spaces.

Definition 6.3.2.5. The derived moduli stack of *wild (pro)-étale rank n ℓ-local systems on X* is defined as the functor $\text{RLocSys}^w_{\ell,n}(X)\colon \text{dAfd}^{\text{op}} \to \mathcal{S}$ given informally by the association

$$Z \in \text{dAfd}^{\text{op}}_{\mathbb{Q}_\ell} \mapsto \lim_{n\geq 0}\text{Map}_{\text{Ind}(\text{Pro}(\mathcal{S}))}\left(\text{B}\pi_1^w(X), \text{BGL}_{\mathbf{n}}\left(\tau_{\leq n}\left(\Gamma(Z)\right)\right)\right) \in \mathcal{S}.$$

Remark 6.3.2.6. The functor $\text{RLocSys}^w_{\ell,n}(X)$ satisfies descent with respect to the étale site $(\text{dAfd}, \tau_{\text{ét}})$, thus we can naturally consider $\text{RLocSys}^w_{\ell,n}(X)$ as an object of the ∞-category of *derived stacks* $\text{dSt}\,(\text{dAfd}, \tau_{\text{ét}},)$.

Suppose now we have a surjective continuous group homomorphism $q\colon \pi_1^w(X) \to \Gamma$, where Γ is a finite group. Such morphism induces a well defined morphism (up to contractible indeterminacy)

$$\text{B}q\colon \text{B}\pi_1^w(X) \to \text{B}\Gamma.$$

Precomposition along $\text{B}q$ induces a morphism of derived moduli stacks $\text{B}q^*\colon \text{RLocSys}_{\ell,n}(\Gamma) \to \text{RLocSys}^w_{\ell,n}(X)$. Where $\text{RLocSys}_{\ell,n}(\Gamma)\colon \text{dAfd}_{\mathbb{Q}_\ell} \to \mathcal{S}$ is the functor informally defined by the association

$$Z \in \text{dAfd}_{\mathbb{Q}_\ell} \mapsto \text{Map}_{\text{Ind}(\text{Pro}(\mathcal{S}))}\left(\text{B}\Gamma, \text{B}\mathcal{E}\text{nd}(Z)\right).$$

Remark 6.3.2.7. As $\text{B}\Gamma \in \mathcal{S}^{\text{fc}} \subseteq \text{Pro}\left(\mathcal{S}^{\text{fc}}\right)$ it follows that, for each $Z \in d\text{Afd}_{\mathbb{Q}_\ell}$, one has a natural equivalence of mapping spaces

$$\text{Map}_{\text{Ind}(\text{Pro}(\mathcal{S}))}\left(\text{B}\Gamma, \text{B}\mathcal{E}\text{nd}(Z)\right) \simeq \text{Map}_{\mathcal{S}}\left(\text{B}\Gamma, \text{BGL}_n(\mathcal{O}_Z)\right).$$

Therefore the moduli stack $\text{RLocSys}_{\ell,n}\,(\text{B}\Gamma)$ is always representable by a derived $\mathbb{Q}_\ell$-analytic stack which is moreover equivalent to the analytification of the usual (algebraic) *mapping stack* $\underline{\text{Map}}\,(\text{B}\Gamma, \text{BGL}_n(-))$. The latter is representable by an Artin stack, see [23, Proposition 19.2.3.3.].

We can now give a reasonable definition of the moduli of local systems with bounded ramification at infinity:

Definition 6.3.2.8. The derived moduli stack of derived étale local systems on X wtih Γ-*bounded ramification at infinity* is defined as the fiber product

$$\text{RLocSys}_{\ell,n,\Gamma}(X) := \text{RLocSys}_{\ell,n}(X) \times_{\text{RLocSys}^w_{\ell,n}(X)} \text{RLocSys}_{\ell,n}(\text{B}\Gamma)$$

Proposition 6.3.2.9. *Let $q\colon \pi_1^w(X) \to \Gamma$ be a surjective continuous group homomorphism whose target is finite. Then the 0-truncation of $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$ is naturally equivalent to $\mathrm{LocSys}_{\ell,n,\Gamma}(X)$. In particular, the former is representable by a $\mathbb{Q}_\ell$-analytic stack.*

Proof. It suffices to prove the statement for the corresponding moduli associated to $\mathrm{Sh}^{\text{ét}}(X)$, $\mathrm{B}\pi_1^w(X)$ and $\mathrm{B}\Gamma$. Each of these three cases can be dealt as in Theorem 6.3.1.8. □

Similarly to the derived moduli stack $\mathrm{RLocSys}_{\ell,n}(X)$ we can compute the tangent complex of $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$ explicitly. In order to do so, we will first need some preparations:

Construction 6.3.2.10. Let $Y \in \mathrm{Pro}\left(\mathcal{S}^{\mathrm{fc}}_{\geq 1}\right)$ be a 1-*connective* profinite space. Fix moreover a morphism

$$c\colon * \to \mathcal{X},$$

in the ∞-category $\mathrm{Pro}\left(\mathcal{S}^{\mathrm{fc}}\right)$. Notice that such choice is canonical up to contractible indeterminacy due to connectedness of X.

Let $\mathrm{Perf}\left(\mathbb{Q}_\ell\right)$ the ∞-category of perfect $\mathbb{Q}_\ell$-modules. One can canonically enhance $\mathrm{Perf}(\mathbb{Q}_\ell)$ to an object in the ∞-category $\mathcal{E}\mathcal{C}\mathrm{at}_\infty$ of $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$-enriched ∞-categories. Consider the full subcategory

$$\mathrm{Perf}_\ell\left(Y\right) \coloneqq \mathrm{Fun}_{\mathrm{cont}}\left(Y, \mathrm{Perf}(\mathbb{Q}_\ell)\right)$$

of $\mathrm{Fun}\left(\mathrm{Mat}\left(Y\right), \mathrm{Perf}(\mathbb{Q}_\ell)\right)$ spanned by those functors $F\colon Y \to \mathrm{Perf}(\mathbb{Q}_\ell)$ with $M \coloneqq F(*)$ such that the induced morphism

$$\Omega \mathrm{Mat}\left(\mathcal{X}\right) \to \mathrm{End}\left(M\right) \tag{6.3.2.2}$$

is equivalent to the materialization of a continuous morphism

$$\Omega \mathcal{X} \to \mathcal{E}\mathrm{nd}\left(M\right)$$

in the ∞-category $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$. Thanks to [1, Corollary 4.3.23] the ∞-category $\mathrm{Perf}_\ell(\mathcal{X})$ is an idempotent complete stable $\mathbb{Q}_\ell$-linear ∞-category which admits a symmetric monoidal structure given by point-wise tensor product.

Consider the *ind-completion* $\mathrm{Mod}_{\mathbb{Q}_\ell}(\mathcal{X}) \coloneqq \mathrm{Ind}\left(\mathrm{Perf}_\ell(\mathcal{X})\right)$, which is a presentable stable symmetric monoidal $\mathbb{Q}_\ell$-linear ∞-category, [1, Corollary 4.3.25]. We have a canonical functor $p_\ell(\mathcal{X})\colon \mathrm{Mod}_{\mathbb{Q}_\ell}(\mathcal{X}) \to \mathrm{Mod}_{\mathbb{Q}_\ell}$ given informally by the formula

$$\operatorname*{colim}_i F_{\in} \mathrm{Mod}_{\mathbb{Q}_\ell}\left(Y\right) \mapsto \operatorname*{colim}_i \left(F_i(*)\right) \in \mathrm{Mod}_{\mathbb{Q}_\ell}.$$

Given $Z \coloneqq (\mathcal{Z}, \mathcal{O}_Z) \in \mathrm{dAfd}_{\mathbb{Q}_\ell}$ a derived $\mathbb{Q}_\ell$-affinoid space, we denote $\Gamma(Z) \coloneqq \Gamma\left(Z\right)$ the corresponding derived ring of global sections. Consider the extension of scalars ∞-category

$$\mathrm{Mod}_{\Gamma(Z)}\left(Y\right) \coloneqq \mathrm{Mod}_{\mathbb{Q}_\ell}\left(Y\right) \otimes_{\mathbb{Q}_\ell} \Gamma(Z),$$

which is a presentable stable symmetric monoidal $\Gamma(Z)$-linear ∞-category, [1, Corollary 4.3.25]. We can base change $p_\ell(Y)$ to a well defined (up to contractible indeterminacy) functor $p_{\Gamma(Z)}\left(Y\right) : \mathrm{Mod}_{\Gamma(Z)}\left(Y\right) \to \mathrm{Mod}_{\Gamma(Z)}$ given informally by the association

$$\left(\operatorname*{colim}_i F_i\right) \otimes_{\mathbb{Q}_\ell} \Gamma(Z) \in \mathrm{Mod}_{\Gamma(Z)}\left(\mathcal{X}\right) \mapsto \operatorname*{colim}_i \left(F_i(*) \otimes_{\mathbb{Q}_\ell} \Gamma(Z)\right) \in \mathrm{Mod}_{\Gamma(Z)}.$$

Proposition 6.3.2.11. *Let $Z \in \mathrm{dAfd}$ be a derived $\mathbb{Q}_\ell$-affinoid space and $\rho \in \mathrm{RLocSys}_{\ell,n,\Gamma}(X)(\mathcal{O}_Z)$. The inclusion morphism of stacks*

$$\mathrm{RLocSys}_{\ell,n,\Gamma}(X) \hookrightarrow \mathrm{RLocSys}_{\ell,n}(X)$$

induces a natural morphism at the corresponding tangent complexes at ρ

$$\mathbb{T}_{\mathrm{RLocSys}_{\ell,n,\Gamma},\rho} \to \mathbb{T}_{\mathrm{RLocSys}_{\ell,n},\rho}$$

is an equivalence in the ∞-category $\mathrm{Mod}_{\Gamma(Z)}$. In particular, we have an equivalence of $\Gamma(Z)$-modules

$$\mathbb{T}_{\mathrm{RLocSys}_{\ell,n,\Gamma},\,\rho} \simeq C^*_{\text{ét}}\left(X, \mathrm{Ad}\left(\rho\right)\right)[1] \in \mathrm{Mod}_{\Gamma(Z)}.$$

Proof. Let $\Pi \coloneqq \mathrm{B}q\colon \mathrm{B}\pi_1^w(X) \to \mathrm{B}\Gamma$ denote the morphism of profinite homotopy types induced from a continuous surjective group homomorphism $q\colon \pi_1^w(X) \to \Gamma$ whose target is finite. We can form a fiber sequence

$$\mathcal{Y} \to \mathrm{B}\pi_1^w(X) \to \mathrm{B}\Gamma \tag{6.3.2.3}$$

in the ∞-category $\mathrm{Pro}\left(\mathcal{S}^{\mathrm{fc}}_{\geq 1}\right)_{*/}$ of pointed 1-connective profinite spaces. Let $A \coloneqq \Gamma(Z)$ and consider the ∞-categories $\mathrm{Mod}_A\left(\mathrm{Sh}^w(X)\right)$ and $\mathrm{Mod}_A\left(\mathrm{B}\Gamma\right)$ introduced in Theorem 6.3.2.10. Let $\mathcal{C}_{A,n}\left(\mathrm{B}\pi_1^w(X)\right)$ and $\mathcal{C}_{A,n}\left(\mathrm{B}\Gamma\right)$ denote the full subcategories of $\mathrm{Mod}_A\left(\mathrm{B}\pi_1^w(X)\right)$ and $\mathrm{Mod}_A\left(\mathrm{B}\Gamma\right)$, respectively, spanned by modules rank n free A-modules. It is a direct consequence of the definitions that one has an equivalence of spaces

$$\mathrm{RLocSys}_{\ell,n}\left(\mathrm{B}\pi_1^w(X)\right) \simeq \mathcal{C}_{A,n}\left(\mathrm{B}\pi_1^w(X)\right)^{\simeq} \text{ and } \mathrm{RLocSys}_{\ell,n}\left(\mathrm{B}\Gamma\right) \simeq \mathcal{C}_{A,n}\left(\mathrm{B}\Gamma\right)^{\simeq}$$

where $(-)^{\simeq}$ denotes the underlying ∞-groupoid functor. The fiber sequence displayed in (6.3.2.3) induces an equivalence of ∞-categories

$$\mathrm{Mod}_A\left(\mathrm{B}\Gamma\right) \simeq \mathrm{Mod}_A\left(\mathrm{B}\pi_1^w(X)\right)^{\mathcal{Y}} \tag{6.3.2.4}$$

where the right hand side of (6.3.2.4) denotes the ∞-category of $\mathcal{Y}$-equivariant continuous representations of $\mathrm{B}\pi_1^w(X)$ with A-coefficients. Thanks to [1, Proposition 4.4.9.] we have an equivalence of A-modules

$$\mathbb{T}_{\mathrm{RLocSys}_{\ell,n}\left(\mathrm{B}\pi_1^w(X)\right),\ \rho_{|\mathrm{B}\pi_1^w(X)}} \simeq \mathrm{Map}_{\mathrm{Mod}_{\Gamma(Z)}\left(\mathrm{B}\pi_1^w(X)\right)}\left(1, \rho_{|\mathrm{B}\pi_1^w(X)} \otimes \rho^{\vee}_{|\mathrm{B}\pi_1^w(X)}\right)[1] \tag{6.3.2.5}$$

and similarly,

$$\mathbb{T}_{\mathrm{RLocSys}_{\ell,n}(\mathrm{B}\Gamma),\ \rho_\Gamma} \simeq \mathrm{Map}_{\mathrm{Mod}_{\Gamma(Z)}(\mathrm{B}\Gamma)}\left(1, \rho_\Gamma \otimes \rho_\Gamma^{\vee}\right)[1] \tag{6.3.2.6}$$

By definition of ρ, we have an equivalence $\rho^{\mathcal{Y}} \simeq \rho$, where $(-)^{\mathcal{Y}}$ denotes (homotopy) fixed points with respect to the morphism $\mathcal{Y} \to \mathrm{B}\pi_1^w(X)$. Thus we obtain a natural equivalence of A-modules:

$$\mathrm{Map}_{\mathrm{Mod}_{\Gamma(Z)}\left(\mathrm{B}\pi_1^w(X)\right)}\left(1, \rho \otimes \rho^{\vee}\right)[1] \simeq \mathrm{Map}_{\mathrm{Mod}_{\Gamma(Z)}\left(\mathrm{B}\pi_1^w(X)\right)}\left(1, (\rho_\Gamma \otimes \rho_\Gamma^{\vee})^{\mathcal{Y}}\right)[1]. \tag{6.3.2.7}$$

Homotopy $\mathcal{Y}$-fixed points are computed by $\mathcal{Y}$-indexed limits. As the $\mathcal{Y}$-indexed limit computing the right hand side of (6.3.2.7) has identity transition morphisms we conclude that the right hand side of (6.3.2.7) is naturally equivalent to the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_A\left(\mathrm{B}\pi_1^w(X)\right)}\left(1, (\rho \otimes \rho^{\vee})^{\mathcal{Y}}\right)[1] \simeq \mathrm{Map}_{\mathrm{Mod}_A(\mathrm{B}\Gamma)}\left(1, \Pi_*(\rho \otimes \rho^{\vee})\right)[1] \tag{6.3.2.8}$$

where $\Pi_*\colon \mathrm{Mod}_A\left(\mathrm{B}\pi_1^w(X)\right) \to \mathrm{Mod}_A\left(\mathrm{B}\Gamma\right)$ denotes a right adjoint to the forgetful $\Pi^*\colon \mathrm{Mod}_A\left(\mathrm{B}\Gamma\right) \to \mathrm{Mod}_A\left(\mathrm{B}\pi_1^w(X)\right)$. As a consequence we have an equivalence

$$\mathrm{Map}_{\mathrm{Mod}_A\left(\mathrm{B}\pi_1^w(X)\right)}\left(1, \rho \otimes \rho^{\vee}\right)[1] \simeq \mathrm{Map}_{\mathrm{Mod}_A(\mathrm{B}\Gamma)}\left(1, \Pi_*(\rho \otimes \rho^{\vee})\right)[1] \tag{6.3.2.9}$$

in the ∞-category $\mathcal{S}$. Notice that, by construction

$$\rho_\Gamma \otimes \rho_\Gamma^{\vee} \simeq \left(\rho \otimes \rho^{\vee}\right)_\Gamma \tag{6.3.2.10}$$

in the ∞-category $\mathrm{Mod}_A\left(\mathrm{B}\Gamma\right)$. One has moreover equivalences

$$\Pi_*\left(\rho \otimes \rho^{\vee}\right) \simeq \left(\rho \otimes \rho^{\vee}\right)_\Gamma, \tag{6.3.2.11}$$

as the restriction of $\rho \otimes \rho^{\vee}$ to $\mathcal{Y}$ is trivial. Thanks to (6.3.2.5) through (6.3.2.11) we conclude that the canonical morphism $\mathrm{LocSys}_{\ell,n}\left(\mathrm{B}\Gamma\right) \to \mathrm{LocSys}_{\ell,n}\left(\mathrm{B}\pi_1^w(X)\right)$ induces an equivalence on tangent spaces, as desired. $\square$

Construction 6.3.2.12. Fix a continuous surjective group homomorphism $q\colon \pi_1^w(X) \to \Gamma$, whose target is finite. Denote by H the kernel of q. The profinite group H is an open subgroup of $\pi_1^w(X)$. For this reason, there exists an open subgroup $U \leq \pi_1^{\text{ét}}(X)$ such that $U \cap \pi_1^w(X) = H$. In particular, the subgroup U has finite index in $\pi_1^{\text{ét}}(X)$. As finite étale coverings of X are completely determined by finite continuous representations of $\pi_1^{\text{ét}}(X)$, there exists a finite étale covering

$$f_U\colon Y_U \to X$$

such that $\pi_1^{\text{ét}}(X)$ acts on it canonically. Moreover, one has an isomorphism of profinite groups

$$\pi_1^{\text{ét}}(Y) \cong U$$

As a consequence, it follows that $\pi_1^w(Y_U) \cong H$. Given $Z \in \mathrm{Afd}_{\mathbb{Q}_\ell}$ and $\rho \in \mathrm{RLocSys}_{\ell,n,\Gamma}(X)(\mathcal{O}_Z)$ it follows by the construction of $f_U \colon Y_U \to X$ that the restriction

$$\rho_{|\mathrm{Sh}^{\text{ét}}(Y)}$$

factors through $\mathrm{Sh}^{\mathrm{tame}}(Y)$. The morphism $f_U \colon Y_U \to X$ induces a morphism of profinite spaces

$$\mathrm{Sh}^{\text{ét}}(Y) \to \mathrm{Sh}^{\text{ét}}(X),$$

which on the other hand induces a morphism of stacks $\mathrm{RLocSys}_{\ell,n}(X) \to \mathrm{RLocSys}_{\ell,n}(Y_U)$. Moreover, by the above considerations the composite

$$\mathrm{RLocSys}_{\ell,n,\Gamma}(X) \to \mathrm{RLocSys}_{\ell,n}(X) \to \mathrm{RLocSys}_{\ell,n}(Y),$$

factors through the substack of tamely ramified local systems $\mathrm{RLocSys}_{\ell,n}\big(\mathrm{Sh}^{\mathrm{tame}}(Y_U)\big) \hookrightarrow \mathrm{RLocSys}_{\ell,n}(Y_U)$.

Lemma 6.3.2.13. *The canonical restriction morphism of Theorem 6.3.2.12*

$$\mathrm{RLocSys}_{\ell,n,\Gamma}(X) \to \mathbf{RLocSys}_{\ell,n}(Y_U)$$

induces an equivalence

$$\mathrm{RLocSys}_{\ell,n,\Gamma}(X) \simeq \mathbf{RLocSys}_{\ell,n}\big(\mathrm{Sh}^{\text{ét}}(Y_U)\big)^{\mathrm{B}\Gamma'}$$

of stacks.

Proof. By Galois descent, the restriction morphism along $f_U \colon Y_U \to X$ induces an equivalence of stacks

$$\mathrm{RLocSys}_{\ell,n}(X) \simeq \mathrm{RLocSys}_{\ell,n}(Y_U)^{\mathrm{B}\Gamma'}.$$

Moreover, the considerations of Theorem 6.3.2.12 imply that we have a pullback square

$$\begin{tikzcd} \mathrm{RLocSys}_{\ell,n,\Gamma}(X) \arrow[r] \arrow[d] & \mathrm{RLocSys}_{\ell,n}(X) \arrow[d] \\ \mathrm{RLocSys}_{\ell,n}\big(\mathrm{Sh}^{\mathrm{tame}}(Y_U)\big) \arrow[r] & \mathrm{RLocSys}_{\ell,n}(Y_U) \end{tikzcd} \tag{6.3.2.12}$$

in the ∞-category $\mathrm{dSt}\big(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}\big)$. The result now follows since we can identify (6.3.2.12) with

$$\begin{tikzcd} \mathrm{RLocSys}_{\ell,n}\big(\mathrm{Sh}^{\mathrm{tame}}(Y_U)\big)^{\mathrm{B}\Gamma'} \arrow[r] \arrow[d] & \mathrm{RLocSys}_{\ell,n}(Y_U)^{\mathrm{B}\Gamma'} \arrow[d] \\ \mathrm{RLocSys}_{\ell,n}\big(\mathrm{Sh}^{\mathrm{tame}}(Y_U)\big) \arrow[r] & \mathrm{RLocSys}_{\ell,n}(Y_U) \end{tikzcd}$$

in the ∞-category $\mathrm{dSt}\big(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}\big)$. ◻

Theorem 6.3.2.14. *The (derived) moduli stack $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$ is representable by a derived $\mathbb{Q}_\ell$-analytic stack.*

Proof. Thanks to [29, Theorem 7.1] we need to check that the functor $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$ has representable 0-truncation, it admits a (global) cotangent complex and it is compatible with Postnikov towers. The representability of $t_0(\mathrm{RLocSys}_{\ell,n,\Gamma}(X)) \simeq \mathrm{LocSys}_{\ell,n,\Gamma}(X)$ follows from Theorem 6.2.3.15. Theorem 6.3.2.11 implies that $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$ admits a global tangent complex. Moreover, by finiteness of ℓ-adic cohomology for smooth varieties in characteristic $p \neq \ell$, [25, Theorem 19.1] together with [1, Proposition 3.1.7] for each $\rho \in \mathrm{RLocSys}_{\ell,n,\Gamma}(X)(Z)$, the tangent complex at ρ

$$\mathbb{T}_{\mathrm{RLocSys}_{\ell,n,\Gamma}(X),\rho} \simeq C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)[1] \in \mathrm{Mod}_{\Gamma(Z)}$$

is a dualizable object of the derived ∞-category $\mathrm{Mod}_{\Gamma(Z)}$. Thanks to Theorem 6.3.2.13 we deduce that the existence of a cotangent complex is equivalent to the existence of a global cotangent complex for the derived moduli stack

$$\mathrm{RLocSys}_{\ell,n}\big(\mathrm{Sh}^{\mathrm{tame}}(Y_U)\big) \in \mathrm{dSt}\big(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}\big).$$

We are thus reduced to show that $\mathrm{Sh}^{\mathrm{tame}}(Y) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ is cohomologically perfect and cohomologically compact, see [1, Definition 4.2.7] and [1, Definition 4.3.17] for the definitions of these notions. As Y_U is a smooth scheme over a field of characteristic $p \neq \ell$, cohomologically perfectness of $\mathrm{Sh}^{\mathrm{tame}}(Y_U)$ follows by finiteness of étale cohomology with ℓ-adic coefficients, [25, Theorem 19.1] together with [1, Proposition 3.1.7]. To show that $\mathrm{Sh}^{\mathrm{tame}}(Y)$ is cohomologically compact we pick a torsion $\mathbb{Z}_\ell$-module N which can be written as a filtered colimit $N \simeq \mathrm{colim}_\alpha N_\alpha$ of perfect $\mathbb{Z}_\ell$-modules. As the tame fundamental group is topologically of finite type and for each $i > 0$, the stable homotopy groups $\pi_i\big(\mathrm{Sh}^{\mathrm{tame}}(Y_U)\big)^{\mathrm{st}}$ are finitely presented the result follows. For these reasons, the derived moduli stack $\mathrm{RLocSys}_{\ell,n}\big(\mathrm{Sh}^{\mathrm{tame}}(Y_U)\big)$ admits a glocal cotangent complex. Theorem 6.3.2.13 implies now that the same is true for $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$. Compatibility with Postnikov towers of $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$ follows from the fact that the latter moduli is defined as a pullback of stacks compatible with Postnikov towers. $\square$

6.4 Comparison statements

6.4.1 Comparison with Mazur's deformation functor

Let L be a finite extension of $\mathbb{Q}_\ell$, $\mathcal{O}_L$ its ring of integers and $\mathfrak{l} := {}_L/\mathfrak{m}_L$ its residue field. We denote $\mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_{/\mathfrak{l}}$ the ∞-category of *derived small k-algebras* augmented over $\mathfrak{l}$.

Let G be a profinite group and $\rho \colon G \to \mathrm{GL}_n(L)$ a continuous ℓ-adic representation of G. Up to conjugation, ρ factors through $\mathrm{GL}_n({}_L) \subseteq \mathrm{GL}_n(L)$ and we can consider its corresponding residual continuous $\mathfrak{l}$-representation

$$\overline{\rho} \colon G \to \mathrm{GL}_n(\mathfrak{l}).$$

The representation ρ can the be obtained as the inverse limit of $\{\overline{\rho}_n \colon G \to \mathrm{GL}_n(\mathcal{O}_L/\mathfrak{m}_L^{n+1})\}_n$, where each $\overline{\rho}_n \simeq \rho \bmod \mathfrak{m}^{n+1}$. For each $n \geq 0$, $\overline{\rho}_n$ is a deformation of the residual representation $\overline{\rho}$ to the ring $\mathcal{O}_L/\mathfrak{m}_L^{n+1}$. Therefore, in order to understand continuous representations $\rho \colon G \to \mathrm{GL}_n(L)$ one might hope to understand residual representations $\overline{\rho} \colon G \to \mathrm{GL}_n(\mathfrak{l})$ together with their corresponding deformation theory. For this reason, it is reasonable to consider the corresponding *derived formal moduli problem*, see [23, Definition 12.1.3.1], associated to $\overline{\rho}$:

$$\mathrm{Def}_{\overline{\rho}} \colon \mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_{/\mathfrak{l}} \to \mathcal{S},$$

given informally via the formula

$$A \in \mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_{/\mathfrak{l}} \mapsto \mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\left(\mathrm{B}G, \mathrm{B}\mathcal{E}\mathrm{nd}(A)\right) \times_{\mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}(\mathrm{B}G, \mathrm{B}\mathcal{E}\mathrm{nd}(A))} \{\overline{\rho}\} \in \mathcal{S}. \tag{6.4.1.1}$$

Construction 6.4.1.1. [1, Proposition 4.2.6] and its proof imply that one has an equivalence between the tangent complex of $\mathrm{Def}_{\overline{\rho}}$ and the complex of continuous cochains of $\mathrm{Ad}(\overline{\rho})$

$$\mathbb{T}_{\mathrm{Def}_{\overline{\rho}}} \simeq C^*_{\mathrm{cont}}\left(G, \mathrm{Ad}(\rho)\right)[1] \tag{6.4.1.2}$$

in the ∞-category $\mathrm{Mod}_{\mathfrak{l}}$. Replacing $\mathrm{B}G$ in (6.4.1.1) by étale homotopy type of X, $\mathrm{Sh}^{\text{ét}}(X)$, and C^*_{cont} by $C^*_{\text{ét}}$ in (6.4.1.2) it follows by [25, Theorem 19.1] together with [21, Theorem 6.2.5] that $\mathrm{Def}_{\overline{\rho}}$ is *pro-representable* by a local Noetherian derived ring $A_{\overline{\rho}} \in \mathcal{C}\mathrm{Alg}_{/\mathfrak{l}}$ whose residue field is equivalent to $\mathfrak{l}$. Moreover, $A_{\overline{\rho}}$ is complete with respect to the augmentation ideal $\mathfrak{m}_{A_{\overline{\rho}}}$ (defined as the kernel of the homomorphism $\pi_0\left(A_{\overline{\rho}}\right) \to k$ of ordinary rings). It follows that $A_{\overline{\rho}}$ admits a natural structure of a derived $W(\mathfrak{l})$-algebra, where $W(\mathfrak{l})$ denotes the ring of Witt vector of $\mathfrak{l}$. As $\overline{\rho}$ admits deformations to $\mathcal{O}_L$, for e.g. ρ itself, we have that $\ell \neq 0$ in $\pi_0(A_{\overline{\rho}})$.

Notation 6.4.1.2. Denote by $L^{\mathrm{unr}} := \mathrm{Frac}\left(W(\mathfrak{l})\right)$ the field of fractions of $W(\mathfrak{l})$. It corresponds to the maximal unramified extension of $\mathbb{Q}_\ell$ contained in L.

Proposition 6.4.1.3. *Let* $\mathrm{t}_{\leq 0}\left(\mathrm{Def}_{\overline{\rho}}\right)$ *denote the* 0*-truncation of the derived formal moduli problem* $\mathrm{Def}_{\overline{\rho}}$*, i.e. the restriction of* $\mathrm{Def}_{\overline{\rho}}$ *to the full subcategory of ordinary Artinian rings augmented over* $\mathfrak{l}$, $\mathcal{C}\mathrm{Alg}^{\mathrm{sm},\heartsuit}_{/\mathfrak{l}} \subseteq \mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_{/\mathfrak{l}}$. *Then* $\mathrm{t}_{\leq 0}\left(\mathrm{Def}_{\overline{\rho}}\right)$ *is equivalent to Mazur's deformation functor introduced in [24, Section 1.2] and* $\pi_0(A_{\overline{\rho}})$ *is equivalent to Mazur's universal deformation ring.*

Proof. Given $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{sm},\heartsuit}_{/\mathfrak{l}} \subseteq \mathcal{C}\mathrm{Alg}^{\mathrm{sm}}_{/\mathfrak{l}}$ an ordinary (Artinian) local $\mathfrak{l}$-algebra, the object $\mathrm{B}\mathcal{E}\mathrm{nd}(R) \in \mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$ is 1-*truncated*. Therefore one has a natural equivalence of spaces

$$\mathrm{t}_0\left(\mathrm{Def}_{\overline{\rho}}\right)(R) \simeq \mathrm{Map}_{\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))}\left(\mathrm{B}\pi_1^{\text{ét}}(X), \mathrm{B}\mathcal{E}\mathrm{nd}(A)\right) \times_{\mathrm{Def}_{\overline{\rho}}(k)} \{\overline{\rho}\}. \tag{6.4.1.3}$$

By construction, the ordinary $W(\mathfrak{l})$-algebra $\pi_0(A_{\overline{\rho}})$ pro-represents the functor $\mathrm{t}_0\left(\mathrm{Def}_{\overline{\rho}}\right) : \mathcal{C}\mathrm{Alg}^{\mathrm{sm},\heartsuit}_{/\mathfrak{l}} \to \mathcal{S}$. As a consequence, the mapping space on the right hand side of (6.4.1.3) is 0-truncated and the set of R-points corresponds to deformations of $\overline{\rho}$ valued in R. This is precisely Mazur's deformation functor, as introduced in [24, Section 1.2], concluding the proof. ☐

6.4.2 Comparison with S. Galatius, A. Venkatesh derived deformation ring

In the case where X corresponds to the spectrum of a maximal unramified extension, outside a finite set S of primes, of a number field L and $\rho\colon G_X \to \mathrm{GL}_n(K)$ is a continuous representation, the corresponding derived $W(k)$-algebra was first introduced and extensively studied in [10].

6.4.3 Comparison with G. Chenevier moduli of pseudo-representations

In this section we will compare our derived moduli stack $\mathrm{RLocSys}_{\ell,n}(X)$ with the construction of the moduli of *pseudo-representations* introduced in [6]. We prove that $\mathrm{RLocSys}_{\ell,n}(X)$ admits an admissible analytic substack which is a disjoint union of the various $\mathrm{Def}_{\overline{\rho}}$. Such disjoint union of deformation functors admits a canonical map to the moduli of pseudo-representations of introduced in [6]. Such morphism of derived stacks is obtained as the composite of the 0-truncation functor followed by the morphism which associates to a continuous representation ρ its corresponding pseudo-representation, see [6, Definition 1.5]. Nevertheless, the derived moduli stack $\mathrm{RLocSys}_{\ell,n}(X)$ has more points in general, and we will provide a typical example in order to illustrate this phenomena.

Proposition 6.4.3.1. *Let* $\overline{\rho}\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ *be a continuous residual* ℓ*-adic representation. To* $\overline{\rho}$ *we can attach a derived* $\mathbb{Q}_\ell$*-analytic space* $\mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}} \in \mathrm{dAn}_{\mathbb{Q}_\ell}$ *for which every closed point* $\rho\colon \mathrm{Sp}\, L \to \mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}}$ *is equivalent to a continuous deformation of* $\overline{\rho}$ *over* L.

Proof. Denote by $\mathrm{dfSch}_{W(\mathfrak{l})}$ the ∞-category of *derived formal schemes* over $W(\mathfrak{l})$, introduced in [23, section 2.8]. The local Noetherian derived $W(\mathfrak{l})$-algebra $A_{\overline{\rho}}$ is complete with respect to its maximal ideal $\mathfrak{m}_{A_{\overline{\rho}}}$. For this reason, we can consider its associated derived formal scheme $\mathrm{Spf}\, A_{\overline{\rho}} \in \mathrm{dfSch}_{W(\mathfrak{l})}$.

Let $A \in \mathcal{C}\mathrm{Alg}_{W(\mathfrak{l})}$ denote an admissible derived $W(\mathfrak{l})$-algebra, see [2, Definition 3.1.1]. We have an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{dfSch}_{W(\mathfrak{l})}}\left(\mathrm{Spf}\, A, \mathrm{Spf}\, A_{\overline{\rho}}\right) \simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{W(\mathfrak{l})}}\left(A_{\overline{\rho}}, A\right).$$

Notice that as A is a ℓ-complete topological almost of finite type over $W(k)$, the image of each $t \in \mathfrak{m}_{A_{\overline{\rho}}}$ is necessarily a topological nilpotent element of the ordinary commutative ring $\pi_0(A)$. Let $\mathfrak{m} \subseteq \pi_0(A)$ denote a maximal ideal of $\pi_0(A)$ and let $(A)^\wedge_{\mathfrak{m}}$ denote the $\mathfrak{m}$-completion of A. There exists a faithfully flat morphism of derived adic $W(k)$-algebra

$$A \to A' := \prod_{\mathfrak{m} \subseteq \pi_0(A)} (A)^\wedge_{\mathfrak{m}}$$

where the product is labeled by the set of maximal ideals of $\pi_0(A)$. By fppf descent we have an equivalence of mapping spaces

$$\mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{W(k)}}\left(A_{\overline{\rho}}, A\right) \simeq \lim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{W(k)}}\left(A_{\overline{\rho}}, A'_{[n]}\right) \tag{6.4.3.1}$$

where $A'_{[n]} := A' \widehat{\otimes}_A \dots \widehat{\otimes}_A A'$ denotes the $n+1$-tensor fold of A' with itself over A computed in the ∞-category of derived adic $W(k)$-algebras $\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{W(k)}$. For a fixed $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ we an equivalence of spaces

$$\mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{W(k)}}\left(A_{\overline{\rho}}, A'_{[n]}\right) \simeq \mathrm{Def}_{\overline{\rho}}\left(A'_{[n]}\right).$$

For each $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ we obtain thus a natural inclusion morphism $\theta_{[n]}\colon \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{W(k)}}\left(A_{\overline{\rho}}, A'_{[n]}\right) \to \mathrm{RLocSys}_{\ell,n}(X)(A'_{[n]})$. The $\theta_{[n]}$ assemble together and by fppf descent induce a morphism $\theta\colon \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{W(k)}}(A_{\overline{\rho}}, A) \to \mathrm{RLocSys}_{\ell,n}(X)(A)$. By construction, θ induces a natural map of mapping spaces

$$\mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{W(k)}}(A_{\overline{\rho}}, A) \to \prod_{\mathfrak{m} \subseteq \pi_0(A)} \left(\mathrm{RLocSys}_{\ell,n}(X)(A) \times_{\mathrm{Def}_{\overline{\rho}}(A^\wedge_{\mathfrak{m}})} \mathrm{RLocSys}_{\ell,n}(X)(A^\wedge_{\mathfrak{m}})\right)$$

which is equivalence of spaces. In order words $\mathrm{Spf}\ A_{\overline{\rho}}$ represents the moduli functor which assigns to each affine derived formal scheme $\mathrm{Spf}\ A$, over $W(\mathfrak{l})$, the space of continuous representations $\rho\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n(A)$ such that for each maximal ideal $\mathfrak{m} \subseteq \pi_0(A)$ the induced representation

$$(\rho)^\wedge_{\mathfrak{m}} : \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n\left((A^\wedge_{\mathfrak{m}})\right)$$

is a deformation of $\overline{\rho}\colon \mathrm{Sh}^{\text{ét}}(X) \to \mathrm{BGL}_n(k)$. The formal spectrum $\mathrm{Spf}\ A_{\overline{\rho}}$ is locally admissible, see [2, Definition 3.1.1]. We can thus consider its rigidificiation introduced in [2, Proposition 3.1.2] which we denote by $\mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}} := (\mathrm{Spf}\ A_{\overline{\rho}})^{\mathrm{rig}} \in \mathrm{dAn}_{\mathbb{Q}_\ell}$. Notice that $\mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}}$ is not necessarily derived affinoid.

Let $Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}$, [2, Corollary 4.4.13] implies that any given morphism $f\colon Z \to (\mathrm{Spf}\ A_{\overline{\rho}})^{\mathrm{rig}}$ in $\mathrm{dAn}_{\mathbb{Q}_\ell}$ admits necessarily a formal model, i.e., it is equivalent to the rigidification of a morphism

$$\mathfrak{f}\colon\ \mathrm{Spf}\ A \to \mathrm{Spf}\ A_{\overline{\rho}},$$

where $A \in \mathcal{C}\mathrm{Alg}^{\mathrm{ad}}_{W(k)}$ is a suitable admissible derived $W(\mathfrak{l})$-algebra. The proof now follows from our previous discussion. $\square$

The proof of Theorem 6.4.3.1 provides us with a canonical morphism of derived moduli stacks $\mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}} \to \mathrm{LocSys}_{\ell,n}(X)$. Therefore, passing to the colimit over all continuous representations

$$\overline{\rho}\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\mathbb{F}_\ell)$$

provides us with a morphism

$$\theta\colon \coprod_{\overline{\rho}} \mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}} \to \mathrm{RLocSys}_{\ell,n}(X) \tag{6.4.3.2}$$

in the ∞-category $\mathrm{dSt}(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}})$.

Proposition 6.4.3.2. *The morphism of derived $\mathbb{Q}_\ell$-analytic stacks*

$$\theta\colon \coprod_{\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)} \mathrm{Def}^{\mathrm{rig}}_{\rho} \to \mathrm{LocSys}_{\ell,n}(G)$$

displayed in (6.4.3.2) *exhibits the left hand side as an analytic subdomain of the right hand side.*

Proof. Let $\overline{\rho}\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(\mathbb{F}_\ell)$ be a continuous representation. The induced morphism

$$\theta_{\overline{\rho}}\colon \mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}} \to \mathrm{RLocSys}_{\ell,n}(X)$$

is an étale morphism of derived stacks, which follows by noticing that $\theta_{\overline{\rho}}$ induces an equivalence at the level of tangent complexes. Moreover, Theorem 6.4.3.1 implies that $\theta_{\overline{\rho}}\colon \mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}} \to \mathrm{RLocSys}_{\ell,n}(X)$ exhibits the former as a substack of the latter. It then follows that the morphism is locally an admissible subdomain inclusion. The result now follows. $\square$

Theorem 6.4.3.2 implies that $\mathrm{RLocSys}_{\ell,n}(X)$ admits as an analytic subdomain the disjoint union of those derived $\mathbb{Q}_\ell$-analytic spaces $\mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}}$. One could then ask if θ is itself an epimorphism of stacks and thus an equivalence of such. However, this is not the case in general as the following example illustrates:

Example 6.4.3.3. Let $G = \mathbb{Z}_\ell$ with its additive structure and let $A = \mathbb{Q}_\ell\langle T\rangle$ be the (classical) Tate $\mathbb{Q}_\ell$-algebra on one generator. Consider the following continuous representation

$$\rho\colon G \to \mathrm{GL}_2(\mathbb{Q}_\ell\langle T\rangle),$$

given by

$$1 \mapsto \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}.$$

It follows that ρ is a $\mathbb{Q}_\ell\langle T\rangle$-point of $\mathrm{LocSys}_{\ell,n}(\mathbb{Z}_\ell)$ but it does not belongs to the image of the disjoint union $\mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}}$ as ρ cannot be factored as a point belonging to the interior of the closed unit disk $\mathrm{Sp}\left(\mathbb{Q}_\ell\langle T\rangle\right)$.

Remark 6.4.3.4. As Theorem 6.4.3.3 suggests, when $n = 2$ the derived moduli stack $\mathrm{RLocSys}_{\ell,n}(X)$ does admit more points than those that come from deformations of its closed points. However, we do not know if $\mathrm{RLocSys}_{\ell,n}$ can be written as a disjoint union of the closures of $\mathrm{Def}^{\mathrm{rig}}_{\overline{\rho}}$ in $\mathrm{LocSys}_{\ell,n}(X)$. However, when $n = 1$ the analytic subdomain morphism θ is an equivalence in the ∞-category $\mathrm{dSt}\big(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}\big)$.

6.5 Shifted symplectic structure on $\mathrm{RLocSys}_{\ell,n}(X)$

Let X be a smooth and proper scheme over an algebraically closed field of positive characteristic $p > 0$. Poincaré duality provide us with a canonical map

$$\varphi\colon C^*_{\text{ét}}\left(X, \mathbb{Q}_\ell\right) \otimes_{\mathbb{Q}_\ell} C^*_{\text{ét}}\left(X, \mathbb{Q}_\ell\right) \to \mathbb{Q}_\ell[-2d]$$

in the derived ∞-category $\mathrm{Mod}_{\mathbb{Q}_\ell}$ is non-degenerate, i.e., it induces an equivalence of *derived* $\mathbb{Q}_\ell$-modules

$$C^*_{\text{ét}}\left(X, \mathbb{Q}_\ell\right) \to C^*_{\text{ét}}\left(X, \mathbb{Q}_\ell\right)^\vee[-2d], \tag{6.5.0.1}$$

in $\mathrm{Mod}_{\mathbb{Q}_\ell}$. As we have seen in the previous section, we can identify the left hand side of (6.5) with a (shit) of the tangent space of $\mathrm{RLocSys}_{\ell,n}(X)$ at the trivial representation. Moreover, the equivalence holds if we consider étale (co)chains with more general coefficients. The case that interest us is taking étale cohomology with $\mathrm{Ad}(\rho)$-coefficients for a continuous representation $\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_n(A)$, with $A \in \mathrm{Afd}_{\mathbb{Q}_\ell}$. Let $\rho \in \mathrm{RLocSys}_{\ell,n}(X)(Z)$, we can regard ρ as a dualizable object of the symmetric monoidal ∞-category $\mathrm{Perf}^{\mathrm{ad}}_\ell(X) \coloneqq \mathrm{Fun}_{\mathcal{E}\mathrm{cat}_\infty}\big(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}(A)\big)$. Let $\rho^\vee$ denote a dual for ρ. By definition of dualizable objects, we have a canonical trace map

$$\mathrm{tr}_\rho\colon \rho \otimes \rho^\vee \to 1_{\mathrm{Perf}^{\mathrm{ad}}_\ell(X)}$$

in the ∞-category $\mathrm{Perf}^{\mathrm{ad}}_\ell(X)$ and $1_{\mathrm{Perf}^{\mathrm{ad}}_\ell(X)}$ denotes the unit object of the latter ∞-category. Therefore, passing to mapping spaces, we obtain a natural composite

$$\mathrm{Map}_{\mathrm{Perf}^{\mathrm{ad}}_\ell(X)}\left(1, \mathrm{Ad}(\rho)\right) \otimes \mathrm{Map}_{\mathrm{Perf}^{\mathrm{ad}}_\ell(X)}\left(1, \mathrm{Ad}(\rho)\right) \xrightarrow{\mathrm{mult}} \mathrm{Map}_{\mathrm{Perf}^{\mathrm{ad}}_\ell(X)}\left(1, \mathrm{Ad}(\rho)\right) \tag{6.5.0.2}$$

$$\xrightarrow{\mathrm{tr}_\rho} \mathrm{Map}_{\mathrm{Perf}^{\mathrm{ad}}_\ell(X)}\left(1, 1\right) \tag{6.5.0.3}$$

in the ∞-category $\mathrm{Mod}_{\Gamma(Z)}$. By identifying the above with étale cohomology coefficients with coefficients we obtain a non-degenerate bilinear form

$$C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)[1] \otimes C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)[1] \to C^*_{\text{ét}}\big(X, \mathrm{Ad}(\rho)\big)[2] \xrightarrow{\mathrm{tr}_\rho} C^*_{\text{ét}}\big(X, \Gamma(Z)\big)[2-2d] \tag{6.5.0.4}$$

in the ∞-category $\mathrm{Mod}_{\Gamma(Z)}$. Moreover, this non-degenerate bilinear form can be interpreted as a Poincaré duality statement with $\mathrm{Ad}(\rho)$-coefficients.

Our goal in this §is to construct a shifted symplectic form ω on $\mathrm{RLocSys}_{\ell,n}(X)$ in such a way that its underlying bilinear form coincides precisely with the composite (6.5.0.4). We will also analyze some of its consequences. Before continuing our treatment we will state a $\mathbb{Q}_\ell$-analytic version of the derived HKR theorem, first proved in the context of derived algebraica geometry in [31].

Theorem 6.5.0.1 (Analytic HKR Theorem)**.** *Let k denote either the field of complex numbers or a non-archimedean field of characteristic 0 with a non-trivial valuation. Let $X \in \mathrm{dAn}_k$ be a derived k-analytic space. Then there is an equivalence of derived analytic spaces*

$$X \times_{X \times X} X \simeq \mathrm{T}X[-1],$$

compatible with the projection to X.

The proof of Theorem 6.5.0.1 is a work in progress together with F. Petit and M. Porta, which the author hopes to include in his PhD thesis.

6.5.1 Shifted symplectic structures

In this §we fix X a smooth scheme over an algebraically closed field k of positive characteristic p.

In [32] the author proved the existence of shifted symplectic structures on certain derived algebraic stacks which cannot be presented as certain mapping stacks. As $\mathrm{RLocSys}_{\ell,n}(X)$ cannot be presented as usual analytic mapping stack, we will need to apply the results of [32] to construct the desired shifted sympletic structure on $\mathrm{RLocSys}_{\ell,n}(X)$.

Definition 6.5.1.1. Consider the canonical inclusion functor $\iota\colon \mathrm{dSt}\left(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}, P_{\text{sm}}\right) \subseteq \mathrm{Fun}\left(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \mathcal{S}\right)$. The functor ι admits a left adjoint which we refer to as *the stackification functor* $(-)^{\text{st}}\colon \mathrm{Fun}\left(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \mathcal{S}\right) \to \mathrm{dSt}\left(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}, P_{\text{sm}}\right)$.

Definition 6.5.1.2. Consider the functor $\mathrm{PerfSys}_\ell^f\colon \mathrm{dAfd}_{\mathbb{Q}_\ell} \to \mathcal{S}$ which is defined via the assignment

$$Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}^{\text{op}} \mapsto \mathrm{Map}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\left(\mathrm{Sh}^{\text{ét}}(X), \mathrm{Perf}\big(\Gamma(Z)\big)\right) \in \mathcal{S}$$

where we designate $\mathrm{Perf}\big(\Gamma(Z)\big)$ to be the $\mathrm{Ind}(\mathrm{Pro}(\mathcal{S}))$-enriched ∞-category of perfect $\Gamma(Z)$-modules, which is equivalent to the subcategory of dualizable objects in the ∞-category of Tate modules on $\Gamma(Z)$, $\mathrm{Mod}_{\Gamma(Z)}^{\text{Tate}}$, [**?**]. We define the moduli stack $\mathrm{PerfSys}_\ell \in \mathrm{dSt}\left(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}, \right)$ as the stackyfication of $\mathrm{PerfSys}_\ell^f$.

Remark 6.5.1.3. This is an example of a moduli stack which cannot be presented as a usual mapping stack, instead one should think of it as an example of a *continuous mapping stack*.

Notation 6.5.1.4. We will denote $\mathcal{C}\mathrm{at}_\infty^\otimes$ the ∞-category of (small) symmetric monoidal ∞-categories.

Definition 6.5.1.5. Let $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty^\otimes$ be a symmetric monoidal ∞-category. We say that $\mathcal{C}$ is a rigid symmetric monoidal ∞-category if every object $C \in \mathcal{C}$ is dualizable.

Notation 6.5.1.6. We denote by $\mathcal{C}\mathrm{at}_\infty^{\text{st},\omega,\otimes}$ the ∞-category of small rigid symmetric monoidal ∞-categories.

Consider the usual inclusion of ∞-categories $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$, it admits a right adjoint, denoted

$$(-)^{\simeq}\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$$

which we refer as the *underlying ∞-groupoid functor*. Given $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ its underlying ∞-groupoid $\mathcal{C}^{\simeq} \in \mathcal{S}$ consists of the maximal subgroupoid of $\mathcal{C}$, i.e., the subcategory spanned by equivalences in $\mathcal{C}$.

Lemma 6.5.1.7. *There exists a valued $\mathcal{C}\mathrm{at}_\infty^{\text{st},\omega,\otimes}$-valued pre-sheaf*

$$\mathrm{Perf}_\ell^{\text{ad}}(X)\colon \mathrm{dAfd}_{\mathbb{Q}_\ell} \to \mathcal{C}\mathrm{at}_\infty$$

given on objects by the formula

$$Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell} \mapsto \mathrm{Fun}_{\mathcal{E}\mathcal{C}\mathrm{at}_\infty}\left(X, \mathrm{Perf}\big(\Gamma(Z)\big)\right).$$

Moreover, the underlying derived stack $(-)^{\simeq} \circ \mathrm{Perf}_\ell^{\text{ad}}(X) \in \mathrm{dSt}\big(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\acute{e}t}\big)$ is naturally equivalent to derived stack $\mathrm{PerfSys}_\ell \in \mathrm{dSt}\left(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}\right)$.

Proof. The construction of $\mathrm{Perf}_\ell^{\text{ad}}(X)$ is already provided in [1, Definition 4.3.11]. Moreover, it follows directly from the definitions that $\big(\mathrm{Perf}_\ell^{\text{ad}}(X)\big)^{\simeq} \simeq \mathrm{PerfSys}_\ell(X)$. □

Theorem 6.5.1.7 is useful because it place us in the situation of [32, §3]. Therefore, we can run the main argument presented in [32, §3]. Before doing so, we will need to introduce some more ingredients:

Definition 6.5.1.8. Let $H\left(\mathrm{Perf}_\ell^{\mathrm{ad}}(X)\right)\colon \mathrm{dAfd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \to \mathcal{S}$ denotes the sheaf defined on objects via the formula

$$Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \mapsto \mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)\left(\Gamma(Z)\right)}(1,1) \in \mathcal{S},$$

where $\mathbf{1} \in \mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z))$ denotes the unit of the corresponding symmetric monoidal structure on $\mathrm{Perf}_\ell^{\mathrm{ad}}(\Gamma(Z))$.

Definition 6.5.1.9. Let $\mathcal{O}\colon \mathrm{dAfd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \to \mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}$ denote the sheaf on $(\mathrm{Afd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}})$ given on objects by the formula

$$Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}^{\mathrm{op}} \mapsto \Gamma(Z) \in \mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}.$$

Construction 6.5.1.10. One is able to define a *pre-orientation*, in the sense of [32, Definition 3.3], on the $\mathcal{C}\mathrm{at}_\infty^{\mathrm{st},\omega,\otimes}$-value stack $\mathrm{Perf}_\ell^{\mathrm{ad}}(X)$

$$\theta\colon H\left(\mathrm{Perf}_\ell^{\mathrm{ad}}(X)\right) \to \mathcal{O}[-2d],$$

as follows: let $Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}$ be a derived $\mathbb{Q}_\ell$-affinoid space. We have a canonical equivalence in the ∞-category $\mathrm{Mod}_{\Gamma(Z)}$

$$\beta_{\Gamma(Z)}\colon \mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z))}(1,1) \simeq C^*_{\text{ét}}(X, \Gamma(Z)), \tag{6.5.1.1}$$

by the very construction of $\mathrm{Perf}_\ell^{\mathrm{ad}}\big(\Gamma(Z)\big)$. Moreover, the projection formula for étale cohomology produces a canonical equivalence

$$C^*_{\mathrm{et}}(X, \Gamma(Z)) \simeq C^*_{\mathrm{et}}(X, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \Gamma(Z)$$

in the ∞-category $\mathrm{Mod}_{\mathbb{Q}_\ell}$. As X is a connected smooth scheme of dimension d over an algebraically closed field we have a canonical map on cohomology groups

$$\alpha\colon \mathbb{Q}_\ell \simeq H^0(X_{\text{ét}}, \mathbb{Q}_\ell) \otimes H^{2d}(X_{\text{ét}}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell$$

which is induced by Poincaré duality. Consequently, the morphism α induces, up to contractible indeterminacy, a canonical morphism

$$C^*_{\text{ét}}(X, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell[-2d]. \tag{6.5.1.2}$$

in the ∞-category $\mathrm{Mod}_{\mathbb{Q}_\ell}$. (6.5.1.1) together with base change of (6.5.1.2) along the morphism $\mathbb{Q}_\ell \to \Gamma(Z)$ provides us with a natural morphism

$$\mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z))}(1,1) \to \Gamma(Z)[-2d].$$

By naturality of the previous constructions, we obtain a morphism pre-orientation

$$\theta\colon H\left(\mathrm{Perf}_\ell^{\mathrm{ad}}(X)\right) \to \mathcal{O}[-2d],$$

which corresponds to the desired orientation.

Given $Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}$, the ∞-category $\mathrm{Perf}_\ell^{\mathrm{ad}}\big(\Gamma(Z)\big)$ is rigid. Thus for a given object $\rho \in \mathrm{Perf}_\ell^{\mathrm{ad}}\big(\Gamma(Z)\big)$ we have a canonical trace map

$$\mathrm{tr}_\rho\colon \mathrm{Ad}(\rho) \to 1.$$

which together with the symmetric monoidal structure provide us with a composite of the form

$$\mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z))}(1, \mathrm{Ad}(\rho)) \otimes \mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z))}(1, \mathrm{Ad}(\rho)) \to \mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z))}(1, \mathrm{Ad}(\rho) \otimes \mathrm{Ad}(\rho)) \tag{6.5.1.3}$$

$$\to \mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z))}(1, \mathrm{Ad}(\rho)) \to \mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z)}(1,1) \to \Gamma(Z)[-2d] \tag{6.5.1.4}$$

which we can right equivalently as a morphism

$$C^*_{\text{ét}}(X, \mathrm{Ad}(\rho)) \otimes C^*_{\text{ét}}(X, \mathrm{Ad}(\rho)) \to \Gamma(Z)[2-2d],$$

which by our construction coincides with the base change along $\mathbb{Q}_\ell \to \Gamma(Z)$ of the usual *pairing* given by *Poincaré Duality*.

Lemma 6.5.1.11. *Let $Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}$ be a derived $\mathbb{Q}_\ell$-affinoid space. The pairing of Theorem 6.5.1.10*

$$\mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z)}\left(1, \mathrm{Ad}(\rho)\right) \otimes \mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z))}\left(1, \mathrm{Ad}(\rho)\right) \to \Gamma(Z)[-2d]$$

is non-degenerate. In particular, the pre-orientation $\theta \colon H\left(\mathrm{Perf}_\ell^{\mathrm{ad}}(X)\right) \to \mathcal{O}[-2d]$ is an orientation, see [32, Definition 3.4] for the latter notion.

Proof. Let $\rho \in \mathrm{PerfSys}_\ell(X)(\mathcal{O}_Z)$ be an arbitrary continuous representation with $\mathcal{O}_Z$-coefficients. We wish to prove that the natural mapping

$$\mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z)}\left(1, \mathrm{Ad}(\rho)\right) \otimes \mathrm{Map}_{\mathrm{Perf}_\ell^{\mathrm{ad}}(X)(\Gamma(Z))}\left(1, \mathrm{Ad}(\rho)\right) \to \Gamma(Z)[-2d]$$

is non-degenerate. As Z lives over $\mathbb{Q}_\ell$ and $p \neq \ell$ it follows that $\rho \in \mathbf{PerfSys}_{\ell,\Gamma}(X)$ for a sufficiently large finite quotient $q \colon \pi_1^w(X) \to \Gamma$. It then follows by [1, Proposition 4.3.19] together with Theorem 6.3.2.13 that ρ can be realized as the $\mathrm{B}\Gamma$-fixed points of a given $\widetilde{\rho} \colon \mathrm{Sh}^{\mathrm{tame}}(Y) \to \mathrm{BGL}_n(A_0)$, where $Y \to X$ is a suitable étale covering and $A_0 \in \mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}$ is an admissible derived $\mathbb{Z}_\ell$-algebra such that

$$\left(\mathrm{Spf}\, A_0\right)^{\mathrm{rig}} \simeq Z,$$

in the ∞-category $\mathrm{dAfd}_{\mathbb{Q}_\ell}$. We notice that it suffices then to show the statement for the residual representation $\rho_0 \colon \mathrm{Sh}^{\mathrm{tame}}(Y) \to \mathrm{BGL}_n(A_0/\ell)$, where A_0/ℓ denotes the pushout

$$\begin{tikzcd} A_0[t] \arrow[r, "t \mapsto \ell"] \arrow[d, "t \mapsto 0"] & A_0 \arrow[d] \\ A_0 \arrow[r] & A_0/\ell \end{tikzcd}$$

computed in the ∞-category $\mathcal{C}\mathrm{Alg}_{k^\circ}^{\mathrm{ad}}$. We can write A_0/ℓ as a filtered colimit of free $\mathbb{F}_\ell$-algebras $\mathbb{F}_\ell[T_0, \dots, T_m]$, where the T_i sit in homological degree 0. As $\mathrm{Sh}^{\mathrm{tame}}(Y)$ is cohomological compact we reduce ourselves to prove the statement by replacing ρ_0 with a continuous representation with values in some polynomial algebra $\mathbb{F}_\ell[T_0, \dots, T_m]$. The latter is a flat module over $\mathbb{F}_\ell$. Therefore, thanks to Lazard's theorem [18, Theorem 8.2.2.15] we can further reduce ourselves to the case where ρ_0 is valued in a finite $\mathbb{F}_\ell$-module. The result now follows by the Theorem 6.5.1.10 together with the projection formula for étale cohomology and Poincaré duality for étale cohomology. ☐

As a corollary of [32, Theorem 3.7] one obtains the following important result:

Theorem 6.5.1.12. *The derived moduli stack $\mathrm{PerfSys}_\ell(X) \in \mathrm{dSt}\left(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\acute{e}t}\right)$ admits a canonical shifted symplectic structure $\omega \in \mathrm{HC}\big(\mathrm{PerfSys}_\ell(X)\big)$, where the latter denotes cyclic homology of the derived moduli stack $\mathrm{PerfSys}_\ell(X)$. Moreover, given $Z \in \mathrm{dAfd}_{\mathbb{Q}_\ell}$ and $\rho \in \mathrm{PerfSys}_\ell\big(\Gamma(Z)\big)$, the shifted symplectic structure ω on $\mathrm{PerfSys}_\ell(X)$ is induced by* étale Poincaré duality

$$C^*_{\text{ét}}\left(X, \mathrm{Ad}(\rho)\right)[1] \otimes C^*_{\text{ét}}\left(X, \mathrm{Ad}(\rho)\right)[1] \to \Gamma(Z)[2-2d].$$

Proof. This is a direct consequence of our previous discussion together with the argument used in [32, Theorem 3.7]. ☐

6.5.2 Applications

Consider the canonical inclusion $\iota \colon \mathrm{RLocSys}_{\ell,n}(X) \hookrightarrow \mathrm{PerfSys}_\ell(X)$. Pullback along the morphism ι on cyclic homology induces a well defined, up to contractible indeterminacy, morphism

$$\iota^* \colon \mathrm{HC}\big(\mathrm{PerfSys}_\ell(X)\big) \to \mathrm{HC}\big(\mathrm{RLocSys}_{\ell,n}(X)\big).$$

We then obtain a canonical closed form $\iota^*(\omega) \in \mathrm{HC}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$. Moreover, as ι induces an equivalence on tangent complexes, the closed form $\iota^*(\omega) \in \mathrm{HC}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$ is non-degenerate, thus a $2-2d$-shifted symplectic form. Similarly, given a finite quotient $q \colon \pi_1^w(X) \to \Gamma$, we obtain a $2-2d$-shifted symplectic form on the derived $\mathbb{Q}_\ell$-analytic stack $\mathrm{RLocSys}_{\ell,n,\Gamma}(X)$. The existence of the sifted symplectic form entails the following interesting result:

Definition 6.5.2.1. Let $\mathbb{L}_{\mathrm{RLocSys}_{\ell,n}(X)}$ denote the cotangent complex of the derived moduli stack $\mathrm{RLocSys}_{\ell,n}(X)$. We will denote by

$$C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big) := \mathrm{Sym}^*\big(\mathbb{L}_{\mathrm{RLocSys}_{\ell,n}(X)}\big) \in \mathrm{Coh}^+\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$$

Remark 6.5.2.2. Notice that $C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$ admits, by construction, a natural mixed algebra structure. However, we will be mainly interested in the corresponding "plain module" and $\mathbb{E}_\infty$-algebra structures underlying the given mixed algebra structure on $C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$.

Proposition 6.5.2.3. *Let X be a proper and smooth scheme over an algebraically closed field of positive characteristic $p > 0$. We then have a well defined canonical morphism*

$$C^*_{\mathrm{dR}}\big(\mathrm{B}\mathbf{GL}_n^{\mathrm{an}}\big) \otimes C^*_{\text{ét}}\big(X, \mathbb{Q}_\ell\big)^\vee \to C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$$

Proof. Let $\rho \in \mathrm{PerfSys}_\ell(X)$ be a continuous representation. We have a canonical morphism

$$\mathrm{BEnd}(\rho) \to \mathrm{BEnd}\big(\rho(*)\big)$$

in the ∞-category $\mathcal{S}$, where $\rho(*)$ denotes the module underlying ρ. This association induces a well defined, up to contractible indeterminacy, morphism

$$\mathrm{PerfSys}_\ell(X) \to \mathrm{Perf}^{\mathrm{an}},$$

where $\mathrm{Perf}^{\mathrm{an}} \in \mathrm{dSt}\big(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}\big)$ denotes the analytification of the algebraic stack of perfect complexes, Perf. Therefore, we obtain a canonical morphism

$$f^* \colon \mathrm{HC}\big(\mathrm{Perf}^{\mathrm{an}}\big) \otimes \mathrm{H}\big(\mathrm{Perf}^{\mathrm{an}}\big) \to \mathrm{HC}\big(\mathrm{PerfSys}_\ell(X)\big) \otimes \mathrm{H}\big(\mathrm{PerfSys}_\ell(X)\big) \tag{6.5.2.1}$$

in the ∞-category $\mathrm{Mod}_{\mathbb{Q}_\ell}$, where $\mathrm{H}\big(\mathrm{Perf}^{\mathrm{an}}\big) := \mathrm{Map}_{\mathrm{Perf}(\mathbb{Q}_\ell)}\big(\mathbb{Q}_\ell, \mathbb{Q}_\ell\big) \simeq \mathbb{Q}_\ell$ and $\mathrm{H}\big(\mathrm{PerfSys}_\ell(X)\big) \simeq C^*_{\text{ét}}(X, \mathbb{Q}_\ell)$. Thus we can rewrite (6.5.2.1) simply as

$$f^* \colon \mathrm{HC}\big(\mathrm{Perf}^{\mathrm{an}}\big) \to \mathrm{HC}\big(\mathrm{PerfSys}_\ell(X)\big) \otimes C^*_{\text{ét}}(X, \mathbb{Q}_\ell). \tag{6.5.2.2}$$

As étale cohomology $C^*_{\text{ét}}(X, \mathbb{Q}_\ell) \in \mathrm{Mod}_{\mathbb{Q}_\ell}$ is a perfect module we can dualize (6.5.2.2) to obtain a canonical morphism

$$f^* \colon \mathrm{HC}\big(\mathrm{Perf}^{\mathrm{an}}\big) \otimes C^*_{\text{ét}}(X, \mathbb{Q}_\ell) \to \mathrm{HC}\big(\mathrm{PerfSys}_\ell(X)\big).$$

in the ∞-category $\mathrm{Mod}_{\mathbb{Q}_\ell}$. Consider now the commutative diagram

$$\begin{tikzcd} \mathrm{RLocSys}_{\ell,n}(X) \arrow[r] \arrow[d, "j"] & \mathrm{B}\mathbf{GL}_n^{\mathrm{an}} \arrow[d] \\ \mathrm{PerfSys}_\ell(X) \arrow[r] & \mathrm{Perf}^{\mathrm{an}} \end{tikzcd}$$

in the ∞-category $\mathrm{dSt}\big(\mathrm{dAfd}_{\mathbb{Q}_\ell}, \tau_{\text{ét}}\big)$. Then we have a commutative diagram at the level of loop stacks

$$\begin{tikzcd} \mathrm{Map}\left(S^1, \mathrm{RLocSys}_{\ell,n}(X)\right) \arrow[r, "i"] \arrow[d, "j"] & \mathrm{Map}\left(S^1, \mathrm{B}\mathbf{GL}_n^{\mathrm{an}}\right) \arrow[d] \\ \mathrm{Map}\left(S^1, \mathrm{PerfSys}_\ell(X)\right) \arrow[r] & \mathrm{Map}\left(S^1, \mathrm{Perf}^{\mathrm{an}}\right). \end{tikzcd}$$

By taking global sections in the above diagram we conclude that the composite

$$f^* \circ i_! \mathcal{HH}\big(\mathcal{O}_{\mathrm{B}\mathbf{GL}_n^{\mathrm{an}}}\big) \simeq f^* \circ i_! \mathcal{O}_{\mathrm{Map}(S^1, \mathrm{B}\mathbf{GL}_n^{\mathrm{an}})}$$

has support in $\mathrm{Map}\left(S^1, \mathrm{RLocSys}_{\ell,n}(X)\right) \hookrightarrow \mathrm{Map}\left(S^1, \mathrm{PerfSys}_\ell(X)\right)$. Therefore, we can factor the composite

$$\mathrm{HH}\big(\mathrm{B}\mathbf{GL}_n^{\mathrm{an}}\big) \otimes C^*_{\text{ét}}(X, \mathbb{Q}_\ell)^\vee \to \mathrm{HH}\big(\mathrm{Perf}^{\mathrm{an}}\big) \otimes C^*_{\text{ét}}(X, \mathbb{Q}_\ell)^\vee \to \mathrm{HH}\big(\mathrm{PerfSys}_\ell(X)\big)$$

as a morphism

$$\mathrm{HH}\big(\mathrm{B}\mathbf{GL}_n^{\mathrm{an}}\big) \otimes C^*_{\text{ét}}(X, \mathbb{Q}_\ell)^\vee \to \mathrm{HH}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$$

in the ∞-category $\mathrm{Mod}_{\mathbb{Q}_\ell}$. The analytic HKR theorem then provide us with the desired morphism

$$C^*_{\mathrm{dR}}\big(\mathrm{B}\mathbf{GL}_n^{\mathrm{an}}\big) \otimes C^*_{\text{ét}}\big(X, \mathbb{Q}_\ell\big)^\vee \to C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$$

in the ∞-category $\mathrm{Mod}_{\mathbb{Q}_\ell}$. □

Remark 6.5.2.4. A type GAGA theorem for reductive groups together with a theorem of B. Totaro, see [33, Theorem 10.2], that the de Rham cohomology of the classifying stack $\mathbf{GL}_n^{\mathrm{an}}$ coincides with ℓ-adic cohomology

$$C^*_{\mathrm{dR}}\big(\mathrm{B}\mathbf{GL}_n^{\mathrm{an}}\big) \simeq C^*_{\mathrm{dR}}\big(\mathrm{BGL}_n^{\mathrm{top}}\big)$$

in the ∞-category $\mathrm{Mod}_{\mathbb{Q}_\ell}$, where $\mathrm{BGL}_n^{\mathrm{top}}$ denotes the topological classifying stack associated to the general linear group GL_n. In particular, we obtain a morphism

$$C^*_{\text{ét}}\big(\mathrm{BGL}_n, \mathbb{Q}_\ell\big) \otimes C^*_{\text{ét}}\big(X, \mathbb{Q}_\ell\big) \to C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big).$$

in the ∞-category $\mathrm{Mod}_{\mathbb{Q}_\ell}$. As $C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$ admits a natural $\mathbb{E}_\infty$-algebra structure we obtain, by the universal property of the Sym construction, a well defined morphism

$$\mathrm{Sym}\,\Big(C^*_{\text{ét}}\big(\mathrm{BGL}_n, \mathbb{Q}_\ell\big) \otimes C^*_{\text{ét}}\big(X, \mathbb{Q}_\ell\big)\Big) \to C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big). \tag{6.5.2.3}$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}$. Assuming further that X is a proper and smooth curve over an algebraically closed field, an ℓ-adic version of Atiyah-Bott theorem proved in [15] implies that we can identify the left hand side of (6.5.2.3) with a morphism

$$C^*_{\text{ét}}\left(\mathrm{Bun}_{\mathrm{GL}_n}(X), \mathbb{Q}_\ell\right) \to C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}$.

As a corollary we obtain:

Corollary 6.5.2.5. *Let X be a smooth scheme over an algebraically closed field of positive characteristic $p > 0$. We have a canonical morphism*

$$\varphi\colon C^*_{\acute{e}t}\big(\mathrm{BGL}_n, \mathbb{Q}_\ell\big) \otimes C^*_{\acute{e}t}\big(X, \mathbb{Q}_\ell\big)^\vee \to C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}$. Moreover, assuming further that X is also a proper curve we obtain a canonical morphism

$$C^*_{\text{ét}}\left(\mathrm{Bun}_{\mathrm{GL}_n}(X), \mathbb{Q}_\ell\right) \to C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$$

in the ∞-category $\mathcal{C}\mathrm{Alg}_{\mathbb{Q}_\ell}$.

Remark 6.5.2.6. By forgetting the mixed k-algebra structure on $C^*_{\mathrm{dR}}\big(\mathrm{RLocSys}_{\ell,n}(X)\big)$ one can prove that the moprhism φ sends the product of the canonical classes on $C^*_{\text{ét}}\big(\mathrm{BGL}_n, \mathbb{Q}_\ell\big) \otimes C^*_{\text{ét}}(X, \mathbb{Q}_\ell)^\vee$ to the underlying cohomology class of the shifted symplectic form ω on $\mathrm{RLocSys}_{\ell,n}(X)$.

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Chapter 7

Analytic HKR theorems

Analytic HKR theorems

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June 23, 2019

Contents

7.1 Introduction

Main results

The goal of this paper is to prove a highly structured version of the Hochschild-Kostant-Rosenberg theorem in the setting of analytic geometry. We do not wish to make any smoothness assumption, and this leads us to work with derived analytic geometry, as developed by J. Lurie, T. Y. Yu and the last author. In first approximation, the theorem we wish to prove is the following:

Theorem 7.1.0.1. *Let k denote either the field $\mathbb{C}$ of complex numbers or a non-archimedean field of characteristic 0 with a non-trivial valuation. Let X be a k-analytic analytic space. Then there is an equivalence of derived analytic spaces*

$$X \times_{X \times X} X \simeq \mathrm{T}X[-1],$$

compatible with the projection to X.

Suppose that X is Stein (when $k = \mathbb{C}$) or affinoid. Let $A := \Gamma(X; \mathcal{O}_X)$. Then the above theorem implies that there is the following equivalence of simplicial algebras:

$$A \widehat{\otimes}_{A \widehat{\otimes}_k A} A \simeq \mathrm{Sym}_A^{\mathrm{an}}(\mathbb{L}\mathrm{an}_A[1]),$$

where $\mathrm{Sym}_A^{\mathrm{an}}$ denotes the analytification relative to A of the algebraic Sym_A. From this point of view, we see that on the right hand side one has an extra structure that we ignored so far, namely the de Rham differential. Taking this extra structure into account leads to the following more precise version of the HKR theorem:

Theorem 7.1.0.2. *There are ∞-categories ε-AnRing_k of mixed analytic rings and S^1-AnRing_k of S^1-equivariant mixed analytic rings. These categories are equivalent compatibly with their forgetful functors to AnRing_k.*

Strategy of the proof

One novelty of this paper is the strategy itself that we use. Our method is new even in the algebraic case and provides an alternative proof of the main result of [TV11]. In order to explain our main ideas, we provide an axiomatic treatment of the HKR theorem.

Warning 7.1.0.3. In this axiomatic presentation, we formulate stronger hypotheses than what is actually needed. This is done in order to obtain a neater exposition. These extra assumptions will be satisfied in the algebraic and $\mathbb{C}$-analytic setting, but not in the k-analytic one.

We start with an ∞-category $\mathcal{A}$, that plays the role of either $\mathcal{C}\mathrm{Alg}_k$ or AnRing_k.

Assumption 7.1.0.4. *The ∞-category $\mathcal{A}$ is presentable.*

In particular, $\mathcal{A}$ has pushouts. We denote the pushout of the diagram $A' \leftarrow A \to B$ by $A' \otimes_A^{\mathcal{A}} B$.
Given an object $A \in \mathcal{A}$ we let

$$A\text{-}\mathrm{Mod} := \mathrm{Sp}(\mathcal{A}_{/A}).$$

Given $M \in A$-Mod, we let

$$A \oplus M := \Omega^\infty(M).$$

The functor

$$\mathrm{Der}_{\mathcal{A}}(A;-)\colon A\text{-Mod}\longrightarrow \mathcal{S}$$

given by sending M to $\mathrm{Map}_{\mathcal{A}_{/A}}(A, A\oplus M)$ commutes with limits and κ-filtered colimits for κ a big enough regular cardinal. Therefore, it is representable by an object in A-Mod that we denote $\mathbb{L}_A$.

Assumption 7.1.0.5. *For every $A\in\mathcal{A}$ there is a conservative functor*

$$U_A\colon \mathcal{A}_{A/}\longrightarrow A\text{-Mod}.$$

Furthermore, this functor admits a left adjoint, denoted $\mathrm{Sym}_A^{\mathcal{A}}(-)$.

Step 1: Construction of the categories of mixed and S^1-equivariant objects. It is easy to construct the ∞-category of S^1-equivariant objects in $\mathcal{A}$. Indeed, we set

$$S^1\text{-}\mathcal{A}\coloneqq \mathrm{Fun}(\mathrm{B}(S^1),\mathcal{A}).$$

This category is equipped with a forgetful functor

$$U_{S^1}\colon S^1\text{-}\mathcal{A}\longrightarrow\mathcal{A},$$

which is conservative and admits both a left and a right adjoint. In particular, it is monadic. We denote by T_{S^1} the associated monad. Notice that we can identify the endofunctor of $\mathcal{A}$ underlying the monad with the functor

$$S^1\otimes -\colon \mathcal{A}\longrightarrow\mathcal{A}$$

sending A to $A\otimes_{A\otimes A} A$.

It is less trivial to construct the ∞-category of mixed objects in $\mathcal{A}$. In this general setting, we need an assumption:

Assumption 7.1.0.6. *There is an ∞-category ε-$\mathcal{A}$ equipped with a functor*

$$U_\varepsilon\colon \varepsilon\text{-}\mathcal{A}\longrightarrow\mathcal{A}$$

satisfying the following properties:

(i) the functor U_ε is conservative, commutes with sifted colimits and it admits a left adjoint

$$\mathrm{DR}\colon \mathcal{A}\longrightarrow\varepsilon\text{-}\mathcal{A}.$$

(ii) For every $A\in\mathcal{A}$, there is a canonical equivalence

$$U_\varepsilon(\mathrm{DR}(A))\simeq \mathrm{Sym}_A^{\mathcal{A}}(\mathbb{L}_A[1]).$$

In particular, U_ε exhibits ε-$\mathcal{A}$ as monadic over $\mathcal{A}$. We let T_ε denote the associated monad.

At this point, we can distinguish two versions of the HKR theorem:

(i) The *plain HKR*: this is the statement that the underlying endofunctors of T_{S^1} and T_ε are equivalent. It implies the familiar algebraic formulation of the HKR theorem, i.e. the existence of an equivalence

$$\mathrm{Sym}_A^{\mathcal{A}}(\mathbb{L}_A[1])\simeq A\otimes_{A\otimes A} A$$

for every $A\in\mathcal{A}$.

(ii) The *structured HKR*: this is the statement that T_{S^1} and T_ε are equivalent as monads. It implies the existence of an equivalence of ∞-categories

$$\varepsilon\text{-}\mathcal{A}\simeq S^1\text{-}\mathcal{A}$$

compatible with the forgetful functors U_{S^1} and U_ε to $\mathcal{A}$.

Notice that the structured HKR implies the plain HKR, but the vice-versa is obviously not true.

Step 2: Passage to nonconnective objects We can describe the most basic idea behind our strategy as follows: instead of proving directly the equivalence between the monads T_{S^1} and T_ε, we look at the comonads. Indeed, the functor $U_{S^1}: S^1\text{-}\mathcal{A} \to \mathcal{A}$ also admits a right adjoint, given by sending $A \in \mathcal{A}$ to

$$A^{S^1} := A \times_{A\times A} A.$$

In the cases of our interest, it turns out that the forgetful functor $U_\varepsilon : \varepsilon\text{-}\mathcal{A} \to \mathcal{A}$ also admits a right adjoint, given by sending $A \in \mathcal{A}$ to the split square-zero extension $A \oplus \tau_{\geq 0}(A[-1])$. This implies that both S^1-$\mathcal{A}$ and ε-$\mathcal{A}$ are comonadic over $\mathcal{A}$. Let C_{S^1} and C_ε denote the respective comonads. It is then enough to prove that C_{S^1} and C_ε are equivalent comonads, but unfortunately this comparison is as difficult as the original problemma.

In order to decrease the complexity of the problemma, we would like to prove that our comonads to be induced by coalgebra objects in $\mathcal{A}$, via the correspondence provided by the following lemmama:

Lemma 7.1.0.7 (cf. **??**). *Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category. The functor*

$$\mathcal{C} \longrightarrow \mathrm{End}(\mathcal{C})$$

informally given by $X \mapsto X \otimes -$ induces a well defined functor

$$\mathrm{CoAlg}(\mathcal{C}^\otimes) \longrightarrow \mathrm{CoMonads}(\mathcal{C}).$$

We refer to the comonads in the essential image of this functor as representable comonads.

Indeed, if could construct two coalgebras A_{S^1} and A_ε in $\mathcal{A}$ whose associated comonads are C_{S^1} and C_ε, it would then be enough to prove that A_{S^1} and A_ε are equivalent in the ∞-category $\mathrm{CoAlg}(\mathcal{A})$. Unfortunately, this does not happen in our case. Nevertheless, we can make it true after suitably enlarging the ∞-category $\mathcal{A}$.

Example 7.1.0.8. In order to get a feeling for what is the obstruction to exhibit C_{S^1} as the comonad associated to a coalgebra A_{S^1}, it is useful to look at the algebraic situation. In this case, the ∞-category $\mathcal{A}$ coincides with the ∞-category of simplicial commutative algebras $\mathcal{C}\mathrm{Alg}_k$. The comonad C_{S^1} sends an object $A \in \mathcal{C}\mathrm{Alg}_k$ to $A \times_{A\times A} A$. On the other hand, if $R \in \mathrm{CoAlg}(\mathcal{C}\mathrm{Alg}_k)$, then its associated comonad sends A to $A \otimes R$. If we assume that $C_{S^1}(A)$ can be written as $A \otimes R$ for every choice of A, then we would obtain

$$R \simeq k \otimes R \simeq C_{S^1}(k) \simeq k \times_{k\times k} k.$$

But in $\mathcal{C}\mathrm{Alg}_k$ one has $k \times_{k\times k} k \simeq k$, and the comonad associated to k is simply the identity. On the other hand, when A is not discrete, then the underlying module of $A \times_{A\times A} A$ is not equivalent to A itself, but rather to $A \oplus \tau_{\geq 0}(A[-1])$.

This is suggesting that working with *connective* commutative algebras is too much restrictive for this problemma. Using the Dold-Kan equivalence we can identify $\mathcal{C}\mathrm{Alg}_k$ with the underlying ∞-category $\mathbf{cdga}_k^{\geq 0}$ of connective cdgas (we use homological convention). The inclusion

$$\mathbf{cdga}_k^{\geq 0} \hookrightarrow \mathbf{cdga}_k$$

does not commute with limits, and in $\mathbf{cdga}_k$ one has $k \times_{k\times k} k \simeq k \oplus k[-1]$. Notice that all the functors introduced so far (U_{S^1}, $S^1 \otimes -$, DR, ...) extend to the unbounded setting. Furthermore, in the unbounded setting we always have $C_{S^1}(A) \simeq A \otimes (k \times_{k\times k} k)$.

In the light of the above example, our actual strategy can be summarized as follows:

(i) Construct a "nonconnective enlargement" $\mathcal{A}^{\mathrm{nc}}$ of the ∞-category $\mathcal{A}$.

(ii) Prove that the comonads C_{S^1} and C_ε extend to comonads $C_{S^1}^{\mathrm{nc}}$ and $C_\varepsilon^{\mathrm{nc}}$ in $\mathcal{A}^{\mathrm{nc}}$.

(iii) Prove that the extended comonads $C_{S^1}^{\mathrm{nc}}$ and $C_\varepsilon^{\mathrm{nc}}$ are representable by objects A_{S^1} and A_ε in $\mathrm{CoAlg}(\mathcal{A}^{\mathrm{nc}})$.

(iv) Finally prove that A_{S^1} and A_ε are equivalent in $\mathrm{CoAlg}(\mathcal{A}^{\mathrm{nc}})$.

The construction of the nonconnective enlargement $\mathcal{A}^{\mathrm{nc}}$ in the analytic setting constitutes the technical heart of this paper. The ideas introduced are a natural extension of the ones that appear in [Lur11d], and they allow to construct nonconnective enlargements in many situations. We believe that the ∞-categories of nonconnective structures that we construct here as an auxiliary tool to prove the HKR theorem are interesting on their own and will prove useful in a variety of different situations.

Let us now give more details about the important features that a "nonconnective enlargement" $\mathcal{A}^{\mathrm{nc}}$ should satisfy in order to be useful to our problemma.

Assumption 7.1.0.9. *There exists a presentable ∞-category $\mathcal{A}^{\mathrm{nc}}$ equipped with a* connective cover functor

$$\tau_{\geq 0} \colon \mathcal{A}^{\mathrm{nc}} \longrightarrow \mathcal{A}.$$

Furthermore, this functor admits a fully faithful left adjoint

$$i \colon \mathcal{A} \hookrightarrow \mathcal{A}^{\mathrm{nc}}.$$

This assumption together with the chain rule for Goodwillie's derivative implies that for every $A \in \mathcal{A}$ there is a fully faithful functor

$$\partial(i) \colon A\text{-}\mathbf{Mod} \longrightarrow \mathrm{Sp}(\mathcal{A}^{\mathrm{nc}}_{/i(A)}).$$

In turn, this allows to prove that for $A \in \mathcal{A}$ the cotangent complex $\mathbb{L}_A$ coincides with the cotangent complex $\mathbb{L}_{i(A)}$ of $i(A) \in \mathcal{A}^{\mathrm{nc}}$. It also allows to define nonconnective split square-zero extensions: if $A \in \mathcal{A}^{\mathrm{nc}}$ and $M \in \mathrm{Sp}(\mathcal{A}^{\mathrm{nc}}_{/A})$, we set

$$A \oplus^{\mathrm{nc}} M := \Omega^\infty(M),$$

where Ω^∞ is the natural functor $\Omega^\infty \colon \mathrm{Sp}(\mathcal{A}^{\mathrm{nc}}_{/A}) \to \mathcal{A}^{\mathrm{nc}}_{/A}$.

Assumption 7.1.0.10. *There exist coalgebras $A_\varepsilon, A_{S^1} \in \mathrm{CoAlg}(\mathcal{A}^{\mathrm{nc}})$ such that for every $A \in \mathcal{A}$ one has a canonical equivalences*

$$i(A) \otimes A_\varepsilon \simeq i(A) \oplus^{\mathrm{nc}} \partial(i)(A[-1])$$

and

$$i(A) \otimes A_{S^1} \simeq i(A) \times_{i(A) \times i(A)} i(A).$$

This assumption is easily verified when $\mathcal{A}^{\mathrm{nc}} = \mathbf{cdga}_k$. In the analytic setting however its verification is one of the most delicate points of the paper. It requires a relative version of the Van Est theorem, which we will discuss in the next step. However, once proven it implies that ε-$\mathcal{A}$ and S^1-$\mathcal{A}$ can be identified with the full subcategories of $\mathrm{CoMod}_{A_\varepsilon}(\mathcal{A}^{\mathrm{nc}})$ and $\mathrm{CoMod}_{A_{S^1}}(\mathcal{A}^{\mathrm{nc}})$ spanned by connective objects. In other words, we have the following pullback diagrams

$$\begin{tikzcd} \varepsilon\text{-}\mathcal{A} \arrow[r] \arrow[d, "U_\varepsilon"] & \mathrm{CoMod}_{A_\varepsilon}(\mathcal{A}^{\mathrm{nc}}) \arrow[d] \\ \mathcal{A} \arrow[r, hook, "i"] & \mathcal{A}^{\mathrm{nc}} \end{tikzcd} \quad , \quad \begin{tikzcd} S^1\text{-}\mathcal{A} \arrow[r] \arrow[d, "U_{S^1}"] & \mathrm{CoMod}_{A_{S^1}}(\mathcal{A}^{\mathrm{nc}}) \arrow[d] \\ \mathcal{A} \arrow[r, hook, "i"] & \mathcal{A}^{\mathrm{nc}}. \end{tikzcd}$$

In this way, we are reduced to prove that there is an equivalence in $\mathrm{CoAlg}(\mathcal{A}^{\mathrm{nc}})$ between A_{S^1} and A_ε. In order to do this, we need one final structural property of the ∞-category $\mathcal{A}^{\mathrm{nc}}$:

Assumption 7.1.0.11. *There exists a conservative functor*

$$U \colon \mathcal{A}^{\mathrm{nc}} \longrightarrow \mathrm{Mod}_k,$$

where Mod_k denotes the ∞-category of (unbounded) k-modules. Furthermore, this functor admits a left adjoint, denoted $\mathrm{Sym}^{\mathrm{nc}}_k$.

Step 3: Use of a theorem of type Van Est Using the conservativity of the functor $U\colon \mathcal{A}^{\mathrm{nc}} \to \mathrm{Mod}_k$, we see that it would be enough to construct a morphism $A_{S^1} \to A_\varepsilon$ in $\mathrm{CoAlg}(\mathcal{A}^{\mathrm{nc}})$ that becomes an equivalence in Mod_k. Unfortunately, it is unreasonable to be able to construct such a morphism directly. This can be seen by attempting to work over the sphere spectrum $k = \mathbb{S}$ instead of over a field of characteristic 0.

Example 7.1.0.12. Let $k = \mathbb{S}$ be the sphere spectrum. Then Böckstedt's computation shows that

$$\pi_* \mathrm{THH}(\mathrm{H}\mathbb{F}_p) = \mathbb{F}_p[\sigma],$$

with $|\sigma| = 2$. On the other hand, $\pi_2(\mathbb{L}_{\mathrm{H}\mathbb{F}_p/\mathbb{S}}) = \mathbb{F}_p$, which implies that $\pi_3(\mathrm{Sym}_{\mathrm{H}\mathbb{F}_p}(\mathbb{L}_{\mathrm{H}\mathbb{F}_p/\mathbb{S}}[1])) = \mathbb{F}_p$. Therefore $\mathrm{THH}(\mathrm{H}\mathbb{F}_p) = \mathrm{H}\mathbb{F}_p \otimes_{\mathrm{H}\mathbb{F}_p \otimes_{\mathbb{S}} \mathrm{H}\mathbb{F}_p} \mathrm{H}\mathbb{F}_p$ cannot be equivalent to $\mathrm{Sym}_{\mathrm{H}\mathbb{F}_p}(\mathbb{L}_{\mathrm{H}\mathbb{F}_p/\mathbb{S}}[1])$, and so even the plain HKR theorem fails over the sphere spectrum. On the other hand, A_{S^1} is $\mathbb{S} \times_{\mathbb{S} \times \mathbb{S}} \mathbb{S}$ and A_ε is the split square-zero extension $\mathbb{S} \oplus \mathbb{S}[-1]$. In particular, the underlying spectra of A_{S^1} and A_ε are equivalent. This means that the equivalence of the underlying modules cannot even be lifted to an equivalence at the $\mathbb{E}_\infty$ level, let alone at the bialgebra level.

This example suggests that it is more reasonable to look for a correspondence between A_{S^1} and A_ε. When working over a field of characteristic zero, both morphisms in this correspondence will be equivalences, but in general they will not. Notice that $U(A_{S^1})$ is forced to be $k \times_{k \times k} k \simeq k \oplus k[-1]$.

Assumption 7.1.0.13. *One has $U(A_\varepsilon) = k \oplus k[-1]$.*

This assumption is easy to verify in all cases of interests. When $\mathcal{A}^{\mathrm{nc}} = \mathbf{cdga}$ it is a consequence of [Lur12c, 7.3.4.15]. In the analytic case, it is easy to reduce oneself to the algebraic situation.

This provides us with the following canonical correspondence in $\mathcal{A}^{\mathrm{nc}}$:

$$\begin{tikzcd}
 & \mathrm{Sym}_k^{\mathrm{nc}}(k[-1]) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
A_{S^1} & & A_\varepsilon.
\end{tikzcd}$$

Assumption 7.1.0.14. *Both $U(p)$ and $U(q)$ are equivalences in* Mod_k.

We warn the reader that the above assumption is really strong and it is not always satisfied. For instance, when k has characteristic $p > 0$ one has

$$\pi_{-1}(U(\mathrm{Sym}_k^{\mathrm{nc}}(k[-1]))) = \bigoplus_{\mathbb{N}} k.$$

In practice, in order to verify this assumption we need to really unravel the construction of $\mathcal{A}^{\mathrm{nc}}$ and of the functor $\mathrm{Sym}_k^{\mathrm{nc}}$. The reason we are able to go through this computation is that $\mathcal{A}^{\mathrm{nc}}$ is constructed in a fairly geometric way, and $U(\mathrm{Sym}_k^{\mathrm{nc}}(k[-1]))$ can be identified with the cohomology complex of the classifying stack $\mathrm{B}(\mathrm{B}G_a)$, where $\mathrm{B}G_a$ denotes the analytic 1-dimensional additive group. To actually compute this cohomology, we resort to a theorem due to Van Est, that identifies the group cohomology of $\mathrm{B}G_a$ with the Lie algebra cohomology of its Lie algebra.

Warning 7.1.0.15. The above discussion is simplistic. It is only accurate when $k = \mathbb{C}$ or it is a non-archimedean field of equicharacteristic 0. In the mixed characteristic case one needs to replace the middle comparison term with the nonconnective analytic algebra of global sections of the classifying stack $\mathrm{B}(\mathbb{D}^1_k(r))$, where $\mathbb{D}^1_k(r)$ denotes the non-archimedean closed disk of radius r, and r is supposed to be less or equal the converging radius of the exponential. In this case, the theorem of Van Est is replaced by its non-archimedean analogue, which is due to Lazard.

Step 4: Contractibility of the space of coalgebra structures The final step required for the completion of the proof is to prove that the morphisms $p\colon \mathrm{Sym}_k^{\mathrm{nc}}(k[-1]) \to A_{S^1}$ and $q\colon \mathrm{Sym}_k^{\mathrm{nc}}(k[-1]) \to A_\varepsilon$ can be promoted to morphisms in $\mathrm{CoAlg}(\mathcal{A}^{\mathrm{nc}})$. Even in the algebraic setting, this verification has never been done explicitly. The main theorem of [TV11] implies that A_{S^1} and A_ε are equivalent as bialgebras. However, in loc. cit. the authors do not give a direct argument, and therefore they are taking quite a long detour. One possible way of expressing the difficulty is the lack of a rectification theorem for bialgebras. With our approach it is possible to show directly that the morphism p respects the coalgebra structure. However, to verify that q also respects the coalgebra structure is a nontrivial task. Our method consists in verifying the following stronger statement:

Assumption 7.1.0.16. *The space of coalgebra structures on* $\mathrm{Sym}_k^{\mathrm{nc}}(k[-1])$, *formally defined as the pullback*

$$\begin{tikzcd} \mathrm{CoAlg}(\mathrm{Sym}_k^{\mathrm{nc}}(k[-1])) \arrow[r] \arrow[d] & \mathrm{CoAlg}(\mathcal{A}^{\mathrm{nc}}) \arrow[d] \\ \{*\} \arrow[r, "\mathrm{Sym}_k^{\mathrm{nc}}(k[-1])"] & \mathcal{A}^{\mathrm{nc}}, \end{tikzcd}$$

is contractible.

Pairing this assumption with Theorem 7.1.0.14 finally completes the proof of our main theorem. Let us be more specific about the way of checking this last assumption in practice. Once again, we need to unravel the actual construction of $\mathcal{A}^{\mathrm{nc}}$ and relate $\mathrm{Sym}_k^{\mathrm{nc}}(k[-1])$ with the cohomology of a classifying stack like $\mathrm{B}(\mathrm{B}G_a)$. This allows to canonically identify the space of coalgebra structures on $\mathrm{Sym}_k^{\mathrm{nc}}(k[-1])$ with the space of group structures on $\mathrm{B}(\mathrm{B}G_a)$. In the cases of interest, we see $\mathrm{B}(\mathrm{B}G_a)$ as an object in the ∞-topos of derived (analytic) stacks. In particular, we are entitled to use the ∞-categorical version of May's delooping theorem. This reduces the computation of the group structures on $\mathrm{B}(\mathrm{B}G_a)$ to the computation of the space of $\mathbb{E}_1$-structures on $\mathrm{B}G_a$ that are compatible with its additive structure. At this point, the Eckmann-Hilton argument implies that this latter space is contractible. See Theorem 7.2.3.2.

Structure of the paper

In Section 7.2 we briefly revisit the main theorems of [TV11] providing shortened proofs following our general strategy. We notice that in this case we have a natural candidate for the category $\mathcal{A}^{\mathrm{nc}}$, namely the category of unbounded cdgas. Furthermore, Van Est theorem can easily be bypassed by means of a direct computation of $\mathrm{Sym}_k(k[-1])$. On the other hand the contractibility of the space of coalgebra structures on $\mathrm{Sym}_k(k[-1])$ (cf. Theorem 7.2.3.2) is a new result that was missing from both [BZN12] and [TV11].

In **??** we provide a general framework to produce the category $\mathcal{A}^{\mathrm{nc}}$. In the algebraic setting, the category we obtain is bigger than the category of unbounded cdgas. It is nevertheless possible to canonically recover the category of unbounded cdgas out of our $\mathcal{A}^{\mathrm{nc}}$. From the point of view of the HKR theorem, the distinction between the two categories is not relevant because Axioms (1) through (4) are satisfied in both cases.

In **??** we apply the machinery previously introduced to construct the ∞-categories $\varepsilon\text{-}\mathrm{AnRing}_k$, $S^1\text{-}\mathrm{AnRing}_k$ and the nonconnective variations. We conclude by proving the main theorem.

7.2 Revisiting the algebraic case

We start the paper by reviewing the algebraic setting for the HKR theorem. In this case, the machinery of nonconnective structures is not needed, as we have a natural candidate, namely the ∞-category of unbounded cdgas. Nevertheless, we take the opportunity to collect a few basic facts about mixed algebras and S^1-objects that are needed in what follows. In Section 7.2.1 we recast the theory of mixed algebras as developed in [TV11, **?**, **?**] in purely ∞-categorical terms. We give two different description of this ∞-category. The first one is equivalent in a more or less tautological way to the classical one introduced in the aforementioned papers, and it is "monadic" in nature. The second one is "comonadic" in nature and it has two main advantages: first of all, it makes the underyling derivation associated to the mixed algebra appear naturally. Secondly, it generalizes to the analytic setting.

In Section 7.2.2 we turn our attention to S^1-algebras. As before, we provide two equivalent description of this ∞-category, one that is naturally monadic and one that is comonadic. The equivalence between these two description builds on the presentation as Segal objects of modules for a monoid in a Cartesian ∞-category. This comparison allows us to prove the equivalence between mixed algebras and S^1-algebras by checking that the comonads are equivalent.

Finally, in Section 7.2.3 we realize the general strategy described in the introduction by proving the HKR statement in the algebraic setting. As main auxiliary step, we prove the contractibility of the space of coalgebra structures on $\mathrm{Sym}_k(k[-1])$.

7.2.1 Mixed algebras

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. We set

$$\mathcal{C}\mathrm{Alg}(\mathcal{C}) \coloneqq \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}^{\otimes}).$$

Given $A \in \mathcal{C}\mathrm{Alg}(\mathcal{C})$, we can consider the ∞-category of A-modules

$$A\text{-}\mathrm{Mod}(\mathcal{C}) \coloneqq \mathrm{LMod}_A(\mathcal{C}),$$

formally defined as the fiber product $\{A\} \times_{\mathcal{C}\mathrm{Alg}(\mathcal{C})} \mathrm{LMod}(\mathcal{C})$.

Similarly, we define the ∞-category of *coalgebras in* $\mathcal{C}$ as

$$\mathrm{CoAlg}(\mathcal{C}) \coloneqq \mathcal{C}\mathrm{Alg}(\mathcal{C}\mathrm{op})\mathrm{op}.$$

Given $A \in \mathrm{CoAlg}(\mathcal{C})$ we set

$$A\text{-}\mathrm{CoMod}(\mathcal{C}) \coloneqq (A\text{-}\mathrm{Modop})\mathrm{op}.$$

A monadic presentation for mixed algebras

Fix a field k of characteristic 0. We let Mod_k denote the stable ∞-category of (unbounded) k-modules and Perf_k be the full stable subcategory spanned by perfect complexes. We endow Mod_k with its canonical symmetric monoidal structure. We set

$$\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \coloneqq \mathcal{C}\mathrm{Alg}(\mathrm{Mod}_k).$$

It can be identified with the ∞-category of unbounded cdgas. Given $A \in \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ we have a canonical equivalence

$$A\text{-}\mathrm{Mod} \simeq \mathrm{Sp}(\mathcal{C}\mathrm{Alg}_{k/A}^{\mathrm{nc}}).$$

For $M \in A\text{-}\mathrm{Mod}$ we set

$$A \oplus M \coloneqq \Omega^\infty(M).$$

In particular, we pose

$$k[\varepsilon] \coloneqq k \oplus k[1].$$

As an algebra, it coincides with the split square-zero extension of k by $k[1]$. Since $k[\varepsilon]$ and all its finite tensor powers are formal, we can define a coalgebra structure on $k[\varepsilon]$ simply by setting

$$\Delta(\varepsilon) \coloneqq 1 \otimes \varepsilon + \varepsilon \otimes 1,$$

where $\Delta\colon k[\varepsilon] \to k[\varepsilon] \otimes_k k[\varepsilon]$ denotes the comultiplication. It is easily checked that $k[\varepsilon]$ inherits in this way the structure of a bialgebra.

Lemma 7.2.1.1. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Let A be a bialgebra in $\mathcal{C}$. Then the ∞-category A-Mod admits a symmetric monoidal structure such that the forgetful functor*

$$A\text{-}\mathrm{Mod}(\mathcal{C}) \longrightarrow \mathcal{C}$$

is strong monoidal.

Proof. We observe that the ∞-functor

$$\mathcal{C}\mathrm{Alg}(\mathcal{C})\mathrm{op} \longrightarrow \mathcal{C}\mathrm{at}_\infty$$

sending A to A-$\mathrm{Mod}(\mathcal{C})$ is lax monoidal. In particular, it brings an object $A \in \mathrm{CoAlg}(\mathcal{C}\mathrm{Alg}(\mathcal{C}))$ to a symmetric monoidal ∞-category A-$\mathrm{Mod}(\mathcal{C})$. Unraveling the definition, we see that the induced forgetful functor is strong monoidal. □

Consider the ∞-category ε-Mod_k of $k[\varepsilon]$-modules. We denote by $\otimes_k$ the monoidal structure on ε-Mod_k provided by the previous lemmama.

Definition 7.2.1.2. The *∞-category of (nonconnective) mixed algebras* is

$$\varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \coloneqq \mathcal{C}\mathrm{Alg}(\varepsilon\text{-}\mathrm{Mod}_k^{\otimes_k}).$$

By definition, ε-$\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ comes equipped with a forgetful functor

$$U_\varepsilon \colon \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \longrightarrow \varepsilon\text{-}\mathrm{Mod}_k,$$

which is monadic. On the other hand the forgetful functor

$$u_\varepsilon \colon \varepsilon\text{-}\mathrm{Mod}_k \longrightarrow \mathrm{Mod}_k$$

is strong monoidal. In particular, it induces a functor

$$v_\varepsilon \colon \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \longrightarrow \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}.$$

Proposition 7.2.1.3. *The commutative diagram*

$$\begin{tikzcd} \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[r, "v_\varepsilon"] \arrow[d, "U_\varepsilon"] & \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[d, "U"] \\ \varepsilon\text{-}\mathrm{Mod}_k \arrow[r, "u_\varepsilon"] & \mathrm{Mod}_k \end{tikzcd}$$

is vertically left adjointable. Furthermore, the functor v_ε commutes with all limits and colimits, and in particular it is both monadic and comonadic.

Proof. We first observe that since U, U_ε and u_ε are conservative, the same goes for v_ε. As both the ∞-categories ε-$\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ and $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ are presentable, the fact that v_ε is monadic and comonadic follows at once if we prove that it commutes with both limits and colimits.

Observe that the functors U, U_ε and u_ε commute with limits and sifted colimits. As U is conservative, it follows that v_ε commutes with limits and sifted colimits as well. In order to prove that v_ε commutes with arbitrary colimits, it is therefore enough to prove that it commutes with arbitrary coproducts of free objects. This is a direct consequence of the vertical left adjointability of the diagram.

We are thus left to prove that the diagram is vertically left adjointable. The functors U_ε and U admit left adjoints L_ε and L. We have to prove that the Beck-Chevalley transformation

$$\alpha \colon L \circ u_\varepsilon \longrightarrow v_\varepsilon \circ L_\varepsilon$$

is an equivalence. As U is conservative, it is enough to prove that

$$U(\alpha) \colon U \circ L \circ u_\varepsilon \longrightarrow U \circ v_\varepsilon \circ L_\varepsilon$$

is an equivalence. Since $U \circ v_\varepsilon \simeq u_\varepsilon \circ U_\varepsilon$, we are reduced to check that the natural transformation

$$U \circ L \circ u_\varepsilon \longrightarrow u_\varepsilon \circ U_\varepsilon \circ L_\varepsilon$$

is an equivalence. We now recall from [Lur12c, 3.1.3.13] that there are canonical equivalences

$$U(L((M)) \simeq \bigoplus_{n \geq 0} M^{\otimes_k n}/\Sigma_n \quad , \quad U_\varepsilon(L_\varepsilon(N)) \simeq \bigoplus_{n \geq 0} N^{\otimes_k n}/\Sigma_n.$$

The conclusion now follows from the fact that u_ε is strong monoidal and commutes with arbitrary colimits. □

A comonadic presentation for mixed algebras

We now provide a second construction for the ∞-category of mixed algebras. We start by observing that the complex underlying $k[\varepsilon]$ is perfect. As Perf_k is a rigid symmetric monoidal ∞-category, we deduce that the k-linear dual

$$k[\eta] \coloneqq \mathrm{Hom}(k[\varepsilon], k)$$

also acquires the structure of a bialgebra. In particular, $k[\eta]$ is a coalgebra in Mod_k. We let

$$\eta\text{-}\mathrm{Mod}_k \coloneqq k[\eta]\text{-}\mathrm{CoMod}(\mathrm{Mod}_k)$$

denote the ∞-category of $k[\eta]$-comodules. Write

$$u_\eta \colon \eta\text{-}\mathrm{Mod}_k \longrightarrow \mathrm{Mod}_k$$

for the canonical forgetful functor.

Since $k[\eta]$ is a bialgebra, we can also consider the ∞-category

$$\eta\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \coloneqq k[\eta]\text{-}\mathrm{CoMod}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}).$$

We let

$$v_\eta \colon \eta\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \longrightarrow \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$$

denote the natural forgetful functor. By construction, we obtain a commutative diagram

$$\begin{tikzcd} \eta\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[r, "v_\eta"] \arrow[d, "U_\eta"] & \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[d, "U"] \\ \eta\text{-}\mathrm{Mod}_k \arrow[r, "u_\eta"] & \mathrm{Mod}_k. \end{tikzcd}$$

Moreover, the functors u_η and v_η are comonadic. Using **??** we can identify the respective comonads with the ones induced by the coalgebra structure on $k[\eta]$ (considered as an elemmaent in Mod_k, resp. in $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$).

In the next section we will prove that there is a canonical equivalence $\varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k \simeq \eta\text{-}\mathcal{C}\mathrm{Alg}_k$. Before arguing about this, however, let us explore some of the basic features of the ∞-category $\eta\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$. We start by observing that if $A \in \eta\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ in particular we have a commutative triangle

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[dr, equal] & A \otimes_k k[\eta] \arrow[d, "1_A \otimes c"] \\ & A, \end{tikzcd}$$

where γ is the coaction of $k[\eta]$ on A and $c \colon k[\eta] \to k$ is the counit of k. We now remark that the algebra structure on $k[\eta]$ is the one induced by the coalgebra structure of $k[\varepsilon]$. Unraveling the definitions, we see that $\eta^2 = 0$ and therefore we can identify $k[\eta]$ with $\Omega^\infty(k[-1])$, the (nonconnective) split square-zero extension of k by $k[-1]$. We claim that $A \otimes_k k[\eta]$ can be canonically identified with the split square-zero extension $A \oplus A[-1]$.

Notation 7.2.1.4. Let $f \colon A \to B$ be a morphism in $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$. The operations of pullback and pushout along f induce an adjunction

$$f^* \colon \mathcal{C}\mathrm{Alg}_{A//A}^{\mathrm{nc}} \leftrightarrows \mathcal{C}\mathrm{Alg}_{B//B}^{\mathrm{nc}} \colon f_*.$$

From an informal point of view, f^* sends an augmented A-algebra $A \to R \to A$ to $B \to B \otimes_A R \to B$. Similarly, f_* sends an augmented B-algebra $B \to R \to B$ to $A \to A \times_B R \to A$.

Lemma 7.2.1.5. *Let $f \colon A \to B$ be a morphism in $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$. The diagram*

$$\begin{tikzcd} B\text{-}\mathrm{Mod} \arrow[r, "f_*"] \arrow[d, "\Omega^\infty"] & A\text{-}\mathrm{Mod} \arrow[d, "\Omega^\infty"] \\ \mathcal{C}\mathrm{Alg}_{B//B}^{\mathrm{nc}} \arrow[r, "f_*"] & \mathcal{C}\mathrm{Alg}_{A//A}^{\mathrm{nc}} \end{tikzcd}$$

is commutative and horizontally left adjointable.

Proof. The functor $f_* \colon \mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_{A//A} \to \mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_{B//B}$ commutes with all limits. Taking its first Goodwillie derivative we obtain the above commutative diagram. In particular, for every $M \in A\text{-}\mathrm{Mod}$ we have a Beck-Chevalley transformation

$$\alpha \colon \Omega^\infty(M) \otimes_A B \longrightarrow \Omega^\infty(M \otimes_A B),$$

and we have to prove that it is an equivalence. We now observe that the underlying module of $\Omega^\infty(M \otimes_A B)$ is $B \oplus f^*(M)$. On the other hand, the underlying module of $\Omega^\infty(M)$ is $A \oplus M$. Since the pushout in $\mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_k$ is computed by the ordinary tensor product, we see that

$$\Omega^\infty(M) \otimes_A B \simeq B \oplus f^*(M).$$

As the forgetful functor $B\text{-Mod} \to \mathrm{Mod}_k$ is conservative, we conclude that α is an equivalence. $\square$

In particular, the coaction $\gamma \colon A \to A \otimes_k k[\eta]$ can be canonically identified with a derivation d_γ of A with values in $A[-1]$. We refer to d_γ as the *derivation underlying the η-algebra A*. At this point, we can prove the following important result:

Proposition 7.2.1.6. *The functor*

$$- \otimes_k k[\eta] \colon \mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_k \longrightarrow \mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_k$$

admits a left adjoint DR*, that informally sends A to*

$$\mathrm{DR}(A) \coloneqq \mathrm{Sym}_A(\mathbb{L}_A[1]).$$

Proof. For any $A, B \in \mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_k$ composition with the canonical map $A \to \mathrm{DR}(A)$ induces a morphism

$$\mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_k}(\mathrm{DR}(A), B) \longrightarrow \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_k}(A, B).$$

Fix a morphism $f \colon A \to B$. Then we have a fiber sequence

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_A}(\mathrm{DR}(A), B) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_k}(\mathrm{DR}(A), B) \arrow[d] \\
\{f\} \arrow[r] & \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_k}(A, B).
\end{tikzcd}$$

Recall that

$$\mathrm{DR}(A) \simeq \mathrm{Sym}_A(\mathbb{L}_{A/k}[1]).$$

Using the universal property of the free algebra, we therefore obtain

$$\mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_A}(\mathrm{DR}(A), B) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(\mathbb{L}_{A/k}[1], f_*(B)).$$

On the other hand, composing with the canonical projection

$$1_B \otimes c \colon B \otimes_k k[\eta] \longrightarrow B,$$

we obtain a natural map

$$\mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_k}(A, B \otimes_k k[\eta]) \longrightarrow \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_k}(A, B).$$

Using Theorem 7.2.1.5 we can canonically identify the fiber at $f \colon A \to B$ with

$$\mathrm{Der}_k(A, f_*(B)) \coloneqq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}^{\mathrm{nc}}_k/B}(A, B \oplus B[-1]) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(\mathbb{L}_{A/k}[1], f_*(B)).$$

Finally, we observe that there is a canonical map

$$A \to \mathrm{DR}(A) \otimes_k k[\eta] \tag{7.2.1.1}$$

which is determined by the condition that the underlying map $A \to \mathrm{DR}(A)$ coincides with the canonical inclusion, and the derivation $\mathbb{L}_{A/k}[1] \to \mathrm{DR}(A)$ corresponds to the inclusion of $\mathbb{L}_{A/k}[1]$ in $\mathrm{Sym}_k(\mathbb{L}_{A/k}[1])$. This transformation induces a morphism of fiber sequences

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_A^{\mathrm{nc}}}(\mathrm{DR}(A), B) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}}(\mathrm{DR}(A), B) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}}(A, B) \arrow[d] \\ \mathrm{Der}_k(A, f_*(B)) \arrow[r] & \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}}(A, B \otimes_k k[\eta]) \arrow[r] & \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}}(A, B). \end{tikzcd}$$

As the outer vertical morphisms are equivalences and this holds for every choice of $f\colon A \to B$, we conclude that the morphism (7.2.1.1) induces a functorial equivalence

$$\mathrm{Map}_{\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}}(\mathrm{DR}(A), B) \simeq \mathrm{Map}_{\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}}(A, B \otimes_k k[\eta]),$$

thus completing the proof of the claim. □

Equivalence of the ∞-categories of mixed algebras

In the previous sections we introduced two ∞-categories, ε-$\mathcal{C}\mathrm{Alg}_k$ and η-$\mathcal{C}\mathrm{Alg}_k$. We now prove that they are equivalent:

Theorem 7.2.1.7. *There is an equivalence $\phi\colon \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k \simeq \eta\text{-}\mathcal{C}\mathrm{Alg}_k$ making the diagram*

$$\begin{tikzcd} \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[rr, "\phi", "\sim"'] \arrow[dr, "v_\varepsilon"'] & & \eta\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[dl, "v_\eta"] \\ & \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} & \end{tikzcd}$$

commutative.

Proof. We observe that v_η is comonadic by construction. **??** implies that the comonad associated to v_η coincides with $\Phi(k[\eta])$, which is the comonad associated to the coalgebra $k[\eta] \in \mathrm{CoAlg}(\mathcal{C}\mathrm{Alg}_k)$. On the other hand, Theorem 7.2.1.3 implies that the functor

$$v_\varepsilon\colon \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \longrightarrow \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$$

is both monadic and comonadic. It is therefore enough to prove that the comonad associated to v_ε can be identified with $\Phi(k[\eta])$. Let us temporarily denote by C_ε the comonad associated to v_ε. Recall from **??** that we have a fully faithful functor

$$\Phi'\colon \mathrm{CoAlg}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}) \longrightarrow \mathrm{CoMonads}^{\mathrm{lax}}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}).$$

This functor has a right adjoint Ψ', that is informally given by evaluation on k. Notice that C_ε can be naturally promoted to an elemmaent in $\mathrm{CoMonads}^{\mathrm{lax}}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}})$. In particular, it is enough to construct an equivalence

$$\alpha\colon \Phi'(k[\eta]) \longrightarrow C_\varepsilon$$

in $\mathrm{CoMonads}^{\mathrm{lax}}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}})$. As the functor $\mathrm{CoMonads}^{\mathrm{lax}}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}) \to \mathrm{End}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}})$ is conservative, it is enough to construct the morphism α and to check afterwards that it induces an equivalence on the underlying endofunctors.

Since Ψ' is right adjoint to Φ', to produce a morphism $\Phi'(k[\eta]) \to C_\varepsilon$ is equivalent to produce a morphism $k[\eta] \to \Psi'(C_\varepsilon)$ in $\mathrm{CoAlg}(\mathcal{C}\mathrm{Alg}_k)$. We are therefore lead to identify $\Psi'(C_\varepsilon)$. Consider the commutative diagram

$$\begin{tikzcd} \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[r, "v_\varepsilon"] \arrow[d, "U_\varepsilon"] & \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[d, "U"] \\ \varepsilon\text{-}\mathrm{Mod}_k \arrow[r, "u_\varepsilon"] & \mathrm{Mod}_k. \end{tikzcd} \tag{7.2.1.2}$$

Theorem 7.2.1.3 guarantees that this diagram is vertically left adjointable. Denoting by L (resp. L_ε) the left adjoint to U (resp. U_ε), we obtain in particular the commutativity of the diagram

$$\begin{tikzcd} \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[r, "v_\varepsilon"] & \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \\ \varepsilon\text{-}\mathrm{Mod}_k \arrow[r, "u_\varepsilon"] \arrow[u, "L_\varepsilon"] & \mathrm{Mod}_k. \arrow[u, "L"] \end{tikzcd}$$

Passing to right adjoints, we obtain the commutativity of the diagram

$$\begin{tikzcd} \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[d, "U_\varepsilon"] & \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[l, "\sigma_\varepsilon"'] \arrow[d, "U"] \\ \varepsilon\text{-}\mathrm{Mod}_k & \mathrm{Mod}_k, \arrow[l, "\rho_\varepsilon"'] \end{tikzcd}$$

where we denoted by ρ_ε and σ_ε the right adjoints to u_ε and v_ε, respectively. In other words, the diagram (7.2.1.2) is horizontally right adjointable. This provides us with the following alternative description for σ_ε: as u_ε is strong monoidal, the functor ρ_ε is lax monoidal. In particular, it induces a well defined functor $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \to \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$, which coincides with σ_ε. Observe on the other hand that the coalgebra structure on $k[\eta]$ corresponds to the algebra structure on $k[\varepsilon]$. Coupling this with the fact that $k[\eta]$ is dualizable as k-module, we obtain a natural equivalence

$$\varepsilon\text{-}\mathrm{Mod}_k \simeq \eta\text{-}\mathrm{Mod}_k,$$

compatible with the forgetful functors to Mod_k. This implies that the endofunctor $u_\varepsilon \circ \rho_\varepsilon$ is canonically identified with $- \otimes_k k[\eta]$. As a consequence, we see that can identify $\Psi'(C_\varepsilon)$ with $C_\varepsilon(k) \simeq k[\eta]$ with its canonical bialgebra structure. In other words, $k[\eta] \simeq \Psi'(C_\varepsilon)$.

This provides us with the natural transformation $\alpha \colon \Phi'(k[\eta]) \to C_\varepsilon$ we were looking for. To complete the proof, it is enough to observe that the previous discussion also showed that the endofunctor underlying C_ε coincides with $- \otimes_k k[\eta]$. Therefore α is an equivalence, and the proof is complete. □

The equivalence provided by the previous theorem has the following two non-trivial consequences:

Corollary 7.2.1.8. *The forgetful functor $v_\eta \colon \eta\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \to \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ is monadic.*

Proof. The functor $v_\varepsilon \colon \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \to \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ is obviously monadic, hence the conclusion follows from Theorem 7.2.1.7. □

Corollary 7.2.1.9. *The endofunctor underlying the monadic functor $u_\varepsilon \colon \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \to \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ coincides with the functor*

$$\mathrm{DR} \colon \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \longrightarrow \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$$

informally sending A to $\mathrm{Sym}_A(\mathbb{L}_A[1])$.

Proof. We start with a simple consideration. Let

$$U \colon \mathcal{C} \longrightarrow \mathcal{D}$$

be a functor between ∞-categories. Suppose that U is both monadic and comonadic and let L (resp. R) denote its left (resp. right) adjoint. Then $U \circ L$ is left adjoint to $U \circ R$.

Applying this remark to the forgetful functor $u_\varepsilon \colon \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \to \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$, we can characterize the endofunctor underlying the associated monad with the left adjoint to the endofunctor underlying the associated comonad. Using the equivalence provided by Theorem 7.2.1.7, we identify the latter with $- \otimes_k k[\eta]$. At this point, the conclusion follows from Theorem 7.2.1.6. □

7.2.2 S^1-algebras

We now introduce the second major character of the HKR equivalence, namely the ∞-category of S^1-algebras. As for mixed algebras, we have at our disposal two different description for this ∞-category, one that is naturally monadic and another one which is naturally comonadic. Again, as for mixed algebras, the one we are truly interested in is the monadic one, because it encodes Hochschild homology. However, the comonadic one is easier to study and manipulate. For this reason, we devote this section to the study of the equivalence between the two presentations.

We start with some general considerations. We consider $\mathbb{Z}$ as a discrete, grouplike $\mathbb{E}_\infty$-monoid in $\mathcal{S}$. The ∞-categorical version of May's theorem (see [Lur12c, 5.2.6.15]) provides an equivalence

$$\mathrm{Bar}^{(1)} \colon \mathcal{S}_*^{\geq 1} \leftrightarrows \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}_*) \colon \Omega,$$

where $\mathcal{S}_*$ is the ∞-category of pointed spaces and $\mathcal{S}_*^{\geq 1}$ denotes the full subcategory spanned by the spaces that are 1-connective. Using the additivity theorem of Dunn-Lurie of $\mathbb{E}_n$-operads (see [Lur12c, 5.1.2.2]) and the equivalence

$$\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S}_*) \simeq \varprojlim_n \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_n}(\mathcal{S}_*)$$

(provided by [Lur12c, 5.1.1.5]), we obtain an ∞-functor

$$\mathrm{Bar}^{(1)} \colon \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S}_*) \to \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S}_*).$$

We denote by $U \colon \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S}_*) \to \mathcal{S}$ the forgetful functor and we set

$$\mathrm{B} \coloneqq U \circ \mathrm{Bar}^{(1)}.$$

We therefore define

$$S^1 \coloneqq \mathrm{B}(\mathbb{Z}) \in \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathcal{S}_*).$$

Notice that the underlying homotopy type of S^1 coincides with the one coming from its standard model, i.e. we have an equivalence

$$S^1 \simeq * \amalg_{* \amalg *} *$$

in $\mathcal{S}$. Defining $S^1 \coloneqq \mathrm{B}(\mathbb{Z})$ has merely the advantage of explicitly fixing the $\mathbb{E}_\infty$-structure on S^1. Observe now that since S^1 is again a group-like $\mathbb{E}_\infty$-monoid, we are entitled to consider $\mathrm{B}(S^1)$, which inherits the same kind of structure. When needed, we will therefore consider $\mathrm{B}(S^1)$ as a group-like $\mathbb{E}_\infty$-monoid.

Definition 7.2.2.1. Let $\mathcal{C}$ be an ∞-category. The ∞-category of *S^1-representations with values in* $\mathcal{C}$ is the ∞-category

$$S^1\text{-}\mathcal{C} \coloneqq \mathrm{Fun}(\mathrm{B}(S^1), \mathcal{C}).$$

Notice that there is a canonical map

$$u \colon * \to \mathrm{B}(S^1),$$

corresponding to the unit in $\mathrm{B}(S^1)$. Precomposition with u provides us with a forgetful functor

$$u_{S^1} \colon S^1\text{-}\mathcal{C} \longrightarrow \mathcal{C}.$$

Lemma 7.2.2.2. *Suppose that $\mathcal{C}$ is complete and cocomplete. Then the functor $u_{S^1} \colon S^1\text{-}\mathcal{C} \to \mathcal{C}$ is conservative and admits both a left and a right adjoint. In particular, u_{S^1} is both monadic and comonadic.*

Proof. The left (resp. right) adjoint to u_{S^1} is given by left (resp. right) Kan extension along $u \colon * \to \mathrm{B}(S^1)$. We are therefore left to check that u_{S^1} is conservative. This follows at once because u is an effective epimorphism. $\square$

Our next goal is to identify the monad and the comonad associated to $u_{S^1} \colon S^1\text{-}\mathcal{C} \to \mathcal{C}$. Observe that the situation is dual: switching from $\mathcal{C}$ to $\mathcal{C}$op interchanges the monad and the comonad. It is therefore enough to focus on the description of the monad.

A rectification result for S^1-object in spaces

We start our investigation in the simplest case possible, namely when $\mathcal{C}$ coincides with the ∞-category of spaces $\mathcal{S}$. In this situation, S^1 is an internal group object in $\mathcal{C}$. We are therefore allowed to form the ∞-category $\mathrm{LMod}_{S^1}(\mathcal{S})$. This category is equipped with a forgetful functor

$$v_{S^1}: \mathrm{LMod}_{S^1}(\mathcal{S}) \longrightarrow \mathcal{S},$$

which is obviously monadic. Both the categories S^1-$\mathcal{S}$ and $\mathrm{LMod}_{S^1}(\mathcal{S})$ encode the idea of spaces with the action of S^1. It is therefore reasonable to expect them to be equivalent.

Remark 7.2.2.3. Let us denote by T_{S^1} (resp. R_{S^1}) the monad associated to u_{S^1} (resp. v_{S^1}). It is easy to verify that these two monads have the same underlying endofunctor. To see this, start by observing that $\mathrm{B}(S^1)_{/*} \simeq * \times_{\mathrm{B}(S^1)} * \simeq S^1$. Therefore, the formula for the left Kan extension yields for every $X \in \mathcal{S}$:

$$T_{S^1}(X) \simeq \operatorname*{colim}_{\mathrm{B}(S^1)/*} X \simeq \operatorname*{colim}_{S^1} X \simeq X \amalg_{X \amalg X} X.$$

On the other hand, **??** implies that

$$R_{S^1}(X) \simeq S^1 \times X.$$

As $\mathcal{S}$ is an ∞-topos, we have a canonical equivalence

$$S^1 \times X \simeq X \amalg_{X \amalg X} X.$$

This implies that the two endofunctor are equivalent.

In virtue of the above remark, all we have to do is to verify that the equivalence between the endofunctors can be lifted to an equivalence between the monads. This is however less obvious than one might expect. Our proof passes through the simplicial description of the ∞-category $\mathrm{LMod}_{S^1}(\mathcal{S})$ given in [Lur12c, 4.2.2.11].

Proposition 7.2.2.4. *There exists a canonical ∞-functor $f\colon S^1$-$\mathcal{S} \to \mathrm{LMod}_{S^1}(\mathcal{S})$ making the diagram*

$$\begin{tikzcd} S^1\text{-}\mathcal{S} \arrow[rr, "f"] \arrow[dr, "u_{S^1}"'] & & \mathrm{LMod}_{S^1}(\mathcal{S}) \arrow[dl, "v_{S^1}"] \\ & \mathcal{S} & \end{tikzcd} \tag{7.2.2.1}$$

commutative. Moreover, f is an equivalence.

Proof. Applying [Lur12c, 4.2.2.11] to the ($\mathbb{E}_\infty$ and hence) $\mathbb{E}_1$ monoid S^1, we obtain a functor

$$G\colon \mathbf{\Delta}\mathrm{op} \to \mathcal{S},$$

which can be informally described by $G([n]) \simeq (S^1)^{\times n}$. Using again [Lur12c, 4.2.2.11], we can identify $\mathrm{LMod}_{S^1}(\mathcal{S})$ with the full subcategory

$$^{\Delta}\mathrm{LMod}_{S^1}(\mathcal{S}) \subset \{G\} \times_{\mathrm{Fun}(\mathbf{\Delta}\mathrm{op},\mathcal{S})} \mathrm{Fun}(\mathbf{\Delta}\mathrm{op} \times \Delta^1, \mathcal{S})$$

spanned by the functors $F\colon \mathbf{\Delta}\mathrm{op} \times \Delta^1 \to \mathcal{S}$ for which the arrows $F([n],0) \to F([n],1)$ and $F([n],0) \to F(\{n\},0) \simeq F([0],0)$ induces an equivalence

$$F([n],0) \simeq F([n],1) \times_{F([0],1)} F([0],0).$$

Consider now the trivial Cartesian fibration

$$p\colon \mathrm{Fun}(\Delta^1,\mathcal{S}) \times \mathbf{\Delta}\mathrm{op} \to \mathbf{\Delta}\mathrm{op}.$$

The functor G determines a map

$$g\colon \mathbf{\Delta}\mathrm{op} \to \mathcal{S} \times \mathbf{\Delta}\mathrm{op}$$

compatible with the projection over $\mathbf{\Delta}\mathrm{op}$. We let $\mathcal{E}$ be the ∞-category fitting in the following pullback diagram:

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "q"] & \mathrm{Fun}(\Delta^1, \mathcal{S}) \times \mathbf{\Delta}\mathrm{op} \arrow[d, "\mathrm{ev}_1 \times \mathrm{id}_{\mathbf{\Delta}\mathrm{op}}"] \\ \mathbf{\Delta}\mathrm{op} \arrow[r, "g"] & \mathcal{S} \times \mathbf{\Delta}\mathrm{op}. \end{tikzcd} \tag{7.2.2.2}$$

We claim that $q\colon \mathcal{E} \to \mathbf{\Delta}\mathrm{op}$ is a Cartesian fibration. Indeed, [Lur09c, 2.4.7.12] shows that $\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathcal{S}) \to \mathcal{S}$ is a Cartesian fibration. Therefore, the stability of Cartesian fibrations under base change (see [Lur09c, 2.4.2.3]) implies first that $\mathrm{ev}_1 \times \mathrm{id}_{\mathbf{\Delta}\mathrm{op}}$ is a Cartesian fibration, and then that the same goes for q.

Inspection reveals that the fiber of q at $[n] \in \mathbf{\Delta}\mathrm{op}$ is equivalent to the ∞-category $\mathcal{S}_{/(S^1)^{\times n}}$. Unraveling the definitions, we see that a morphism

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ (S^1)^{\times n} \arrow[r] & (S^1)^{\times m} \end{tikzcd}$$

in $\mathcal{E}$ is q-Cartesian if and only if it is a pullback square and the morphism $(S^1)^{\times n} \to (S^1)^{\times m}$ is equivalent to $G(s)$ for some $s\colon [n] \to [m]$ in $\mathbf{\Delta}\mathrm{op}$. In turn, this implies that the associated unstraightened functor

$$\mathrm{Un}(q) \colon \mathbf{\Delta} \to \mathcal{C}\mathrm{at}_\infty$$

can be informally described as the functor

$$[n] \mapsto \mathcal{S}_{/(S^1)^{\times n}}.$$

Applying the functor $\mathrm{Fun}_{/\mathbf{\Delta}\mathrm{op}}(\mathbf{\Delta}\mathrm{op}, -)$ to (7.2.2.2), we obtain the following pullback diagram:

$$\begin{tikzcd} \mathrm{Fun}_{/\mathbf{\Delta}\mathrm{op}}(\mathbf{\Delta}\mathrm{op}, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Fun}_{/\mathbf{\Delta}\mathrm{op}}(\mathbf{\Delta}\mathrm{op}, \mathrm{Fun}(\Delta^1, \mathcal{S}) \times \mathbf{\Delta}\mathrm{op}) \arrow[d, "\mathrm{ev}_1"] \\ \{*\} \arrow[r, "G"] & \mathrm{Fun}_{/\mathbf{\Delta}\mathrm{op}}(\mathbf{\Delta}\mathrm{op}, \mathcal{S} \times \mathbf{\Delta}\mathrm{op}). \end{tikzcd}$$

Under the natural identifications

$$\mathrm{Fun}_{/\mathbf{\Delta}\mathrm{op}}(\mathbf{\Delta}\mathrm{op}, \mathrm{Fun}(\Delta^1, \mathcal{S}) \times \mathbf{\Delta}\mathrm{op}) \simeq \mathrm{Fun}(\mathbf{\Delta}\mathrm{op} \times \Delta^1, \mathcal{S}), \qquad \mathrm{Fun}_{/\mathbf{\Delta}\mathrm{op}}(\mathbf{\Delta}\mathrm{op}, \mathcal{S} \times \mathbf{\Delta}\mathrm{op}) \simeq \mathrm{Fun}(\mathbf{\Delta}\mathrm{op}, \mathcal{S}),$$

we obtain an equivalence

$$\mathrm{Fun}_{/\mathbf{\Delta}\mathrm{op}}(\mathbf{\Delta}\mathrm{op}, \mathcal{E}) \simeq \{G\} \times_{\mathrm{Fun}(\mathbf{\Delta}\mathrm{op}, \mathcal{S})} \mathrm{Fun}(\mathbf{\Delta}\mathrm{op} \times \Delta^1, \mathcal{S}).$$

It follows from the description we gave above of the q-Cartesian edges that we can identify ${}^{\Delta}\mathrm{LMod}_{S^1}(\mathcal{S})$ with the full subcategory of the left hand side spanned by Cartesian sections. Therefore, [Lur09c, 3.3.3.2] provides us with the following chain of equivalences:

$${}^{\Delta}\mathrm{LMod}_{S^1}(\mathcal{S}) \simeq \mathrm{Fun}_{/\mathbf{\Delta}\mathrm{op}}(\mathbf{\Delta}\mathrm{op}, \mathcal{E}) \simeq \lim_{\mathbf{\Delta}} \mathrm{Un}(q).$$

On the other hand, we observe that the inclusion $\mathcal{S} \subseteq \mathcal{C}\mathrm{at}_\infty$ has both a left and a right adjoint. In particular, it commutes with colimits. As a consequence,

$$\mathrm{Fun}(\mathrm{B}(S^1), \mathcal{S}) \simeq \lim_{\mathbf{\Delta}} \mathrm{Fun}(G, \mathcal{S}).$$

Using [Lur09c, 2.2.1.2], we see that for every $[n] \in \mathbf{\Delta}$, we have a natural identification

$$\mathrm{Fun}(G([n]), \mathcal{S}) \simeq \mathrm{Fun}((S^1)^{\times n}, \mathcal{S}) \simeq \mathcal{S}_{/(S^1)^{\times n}}.$$

In other words, we can identify $\mathrm{Fun}(G, \mathcal{S}) \colon \mathbf{\Delta}\mathrm{op} \to \mathcal{C}\mathrm{at}_\infty$ with the functor $\mathrm{Un}(q)$ introduced above. This completes the proof. $\square$

Building on Theorem 7.2.2.4 it is easy to bootstrap and extend to the case of presheaves. Let us introduce the necessary terminology. Let $\mathcal{C}$ be an ∞-category. Precomposition with the canonical functor

$$\pi \colon \mathcal{C} \to *$$

gives rise to a functor

$$\pi^p \colon \mathcal{S} \to \mathrm{PSh}(\mathcal{C})$$

that sends a space K to the constant presheaf $\underline{K}$ on $\mathcal{C}$ associated to K. As π^p commutes with limits, it can be promoted to a symmetric monoidal functor

$$\pi^p \colon \mathcal{S}^\times \longrightarrow \mathrm{PSh}(\mathcal{C})^\times.$$

The $\mathbb{E}_\infty$-structure on S^1 induces therefore a canonical $\mathbb{E}_\infty$-structure on $\underline{S}^1$, which is easily checked to be grouplike. This allows us to consider once again the ∞-category $\mathrm{LMod}_{\underline{S}^1}(\mathrm{PSh}(\mathcal{C}))$. Having fixed these notations, Theorem 7.2.2.4 has the following immediate corollary:

Corollary 7.2.2.5. *There exists a canonical ∞-functor $f \colon S^1\text{-}\mathrm{PSh}(\mathcal{C}) \to \mathrm{LMod}_{\underline{S}^1}(\mathrm{PSh}(\mathcal{C}))$ making the diagram*

$$\begin{tikzcd} S^1\text{-}\mathrm{PSh}(\mathcal{C}) \arrow[rr, "f"] \arrow[dr, "\mathrm{forget}"'] & & \mathrm{LMod}_{\underline{S}^1}(\mathrm{PSh}(\mathcal{C})) \arrow[dl, "\mathrm{forget}"] \\ & \mathrm{PSh}(\mathcal{C}) & \end{tikzcd} \tag{7.2.2.3}$$

commutative. Moreover, f is an equivalence.

Proof. We have canonical equivalences

$$\mathrm{LMod}_{\underline{S}^1}(\mathrm{PSh}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{C}\mathrm{op}, \mathrm{LMod}_{S^1}(\mathcal{S}))$$

and

$$S^1\text{-}\mathrm{PSh}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}\mathrm{op}, S^1\text{-}\mathcal{S}).$$

The result is therefore a direct consequence of Theorem 7.2.2.4. □

A rectification result in the general case

Theorem 7.2.2.5 works for any ∞-category $\mathcal{C}$. Notice however that when applied to $\mathcal{C} = \mathcal{S}$ it does not recover the statement of Theorem 7.2.2.4. Our goal is to formulate an analogue of Theorem 7.2.2.4 for a more general ∞-category $\mathcal{C}$.[1] For this, we will need to make certain assumptions on $\mathcal{C}$ itself.

To set the stage, suppose that $\mathcal{C}$ is a presentable ∞-category. In this case $\mathcal{C}$ is canonically enriched with tensor and cotensor over $\mathcal{S}^{\mathrm{fin}}$. In particular, we have a functor

$$\otimes \colon \mathcal{S}^{\mathrm{fin}} \times \mathcal{C} \longrightarrow \mathcal{C}.$$

For any $X \in \mathcal{C}$ we obtain an adjunction

$$- \otimes X \colon \mathcal{S} \leftrightarrows \mathcal{C} \colon \mathrm{Map}_{\mathcal{C}}(X, -).$$

The right adjoint commutes with products and therefore it can be promoted to a symmetric monoidal functor (with respect to the cartesian monoidal structures on both sides). In particular, it can be canonically lifted to a functor

$$M_X \colon \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{C}^\times) \longrightarrow \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}^\times)$$

[1] In the applications, $\mathcal{C}$ will be the opposite of the ∞-category of nonconnective structures.

fitting in the commutative diagram

$$\begin{tikzcd} \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}^{\times}) \arrow[d] & \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{C}^{\times}) \arrow[l, "M_X"'] \arrow[d] \\ \mathcal{S} & \mathcal{C}, \arrow[l, "\mathrm{Map}_{\mathcal{C}}(X,-)"'] \end{tikzcd} \tag{7.2.2.4}$$

where the vertical arrows are the forgetful functors. Since they are conservative and commute with limits and sifted colimits, we deduce that M_X admits a left adjoint, that we denote L_X. When $X = 1_{\mathcal{C}}$, we write L instead of $L_{1_{\mathcal{C}}}$. We let

$$S^1_{\mathcal{C}} \coloneqq L(S^1) \in \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{C}^{\times}).$$

As in general the functor $-\otimes 1_{\mathcal{C}}$ is only oplax monoidal, the diagram (7.2.2.4) is not horizontally left adjointable in general.

Definition 7.2.2.6. Let $\mathcal{C}$ be a presentable ∞-category. We say that $\mathcal{C}$ *satisfies the condition* (M) if the oplax monoidal functor

$$-\otimes 1_{\mathcal{C}} \colon \mathcal{S}^{\mathrm{fin}} \longrightarrow \mathcal{C}$$

is strong monoidal.

When $\mathcal{C}$ satisfies the condition (M) we have the following improved situation:

Lemma 7.2.2.7. *Suppose that $\mathcal{C}$ satisfies the condition* (M)*. Then the diagram*

$$\begin{tikzcd} \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}^{\mathrm{fin},\times}) \arrow[r, "L"] \arrow[d] & \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{C}^{\times}) \arrow[d] \\ \mathcal{S}^{\mathrm{fin}} \arrow[r, "-\otimes 1_{\mathcal{C}}"] & \mathcal{C} \end{tikzcd}$$

is commutative. In particular, there is an equivalence $S^1_{\mathcal{C}} \simeq S^1 \otimes 1_{\mathcal{C}}$ as objects of $\mathcal{C}$.

Proof. As the functor $-\otimes 1_{\mathcal{C}} \colon \mathcal{S}^{\mathrm{fin}} \to \mathcal{C}$ is strong monoidal, it induces a commutative diagram

$$\begin{tikzcd} \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}^{\mathrm{fin},\times}) \arrow[r, "L'"] \arrow[d] & \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{C}^{\times}) \arrow[d] \\ \mathcal{S}^{\mathrm{fin}} \arrow[r, "-\otimes 1_{\mathcal{C}}"] & \mathcal{C}, \end{tikzcd}$$

and therefore we have to produce a natural isomorphism between L and L'. In order to do this, it is enough to prove that for every $X \in \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}^{\mathrm{fin},\times})$ there is a morphism

$$\eta \colon X \longrightarrow M_{1_{\mathcal{C}}}(L'(X))$$

in $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}^{\mathrm{fin},\times})$ inducing an equivalence

$$\mathrm{Map}_{\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}^{\mathrm{fin},\times})}(X, M_{1_{\mathcal{C}}}(Y)) \simeq \mathrm{Map}_{\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{C}^{\times})}(L'(X), Y). \tag{7.2.2.5}$$

Represent X as a Segal object

$$F_X \colon \mathbf{\Delta}\mathrm{op} \longrightarrow \mathcal{S}^{\mathrm{fin}}.$$

Since $\mathcal{C}$ satisfies the condition (M), we see that the functor

$$\mathrm{Map}_{\mathcal{C}}(1_{\mathcal{C}}, F_X(-) \otimes 1_{\mathcal{C}}) \colon \mathbf{\Delta}\mathrm{op} \longrightarrow \mathcal{S}^{\mathrm{fin}}$$

still satisfies the Segal condition, and it corresponds precisely to $M_{1_{\mathcal{C}}}(L'(X))$. The unit of the adjunction $-\otimes 1_{\mathcal{C}} \colon \mathcal{S} \rightleftarrows \mathcal{C} \colon \mathrm{Map}_{\mathcal{C}}(1_{\mathcal{C}}, -)$ induces therefore a natural transformation from F_X to $M_{1_{\mathcal{C}}}(L'(X))$, which is easily checked to induce an equivalence (7.2.2.5). □

Since the Yoneda embedding $y\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ commutes with products, we see that $y(S^1_{\mathcal{C}})$ inherits the structure of a grouplike $\mathbb{E}_1$-monoid in $\mathrm{PSh}(\mathcal{C})$. Recall that the canonical functor $\pi\colon \mathcal{C} \to *$ induces an adjunction

$$\pi^p\colon \mathcal{S} \leftrightarrows \mathrm{PSh}(\mathcal{C})\colon {}_p\pi,$$

and that furthermore π^p commutes with all limits. In particular, this adjunction lifts to another adjunction

$$\pi^p\colon \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}) \leftrightarrows \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathrm{PSh}(\mathcal{C}))\colon {}_p\pi.$$

Observe that

$${}_p\pi(y(S^1_{\mathcal{C}})) \simeq \mathrm{Map}(1_{\mathcal{C}}, S^1_{\mathcal{C}}).$$

In particular, the unit of the adjunction $L \dashv M_{1_{\mathcal{C}}}$ induces a morphism in $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S})$

$$S^1 \longrightarrow {}_p\pi(y(S^1_{\mathcal{C}})).$$

In turn, this corresponds to a morphism in $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathrm{PSh}(\mathcal{C}))$

$$\rho\colon \underline{S}^1 = \pi^p(S^1) \longrightarrow y(S^1_{\mathcal{C}}),$$

which induces a forgetful functor

$$\rho_*\colon \mathrm{LMod}_{y(S^1_{\mathcal{C}})}(\mathrm{PSh}(\mathcal{C})) \longrightarrow \mathrm{LMod}_{\underline{S}^1}(\mathrm{PSh}(\mathcal{C})).$$

We would like to say that ρ_* is an equivalence of ∞-categories, but this will not be true in general. Therefore, we need to formulate some stronger assumption on $\mathcal{C}$:

Definition 7.2.2.8. Let $\mathcal{C}$ be a presentable ∞-category. Let $i\colon \mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory closed under products. We say that $\mathcal{C}_0$ *satisfies the condition* (UM) (*relative to* $\mathcal{C}$) if for every $X \in \mathcal{C}_0$ and every $K \in \mathcal{S}^{\mathrm{fin}}$ the natural morphism

$$K \otimes X \longrightarrow (K \otimes 1_{\mathcal{C}}) \times X$$

is an equivalence.

Remark 7.2.2.9. Notice that since $\mathcal{C}_0$ is closed under products in $\mathcal{C}$, the final object $1_{\mathcal{C}}$ belongs to $\mathcal{C}_0$. In particular, if $\mathcal{C}_0$ satisfies the condition (UM) and for any $H \in \mathcal{S}^{\mathrm{fin}}$ the object $H \otimes 1_{\mathcal{C}}$ still belongs to $\mathcal{C}_0$, then the natural morphism

$$K \otimes (H \otimes 1_{\mathcal{C}}) \longrightarrow (K \otimes 1_{\mathcal{C}}) \times (H \otimes 1_{\mathcal{C}})$$

is an equivalence. On the other hand, $K \otimes (H \otimes 1_{\mathcal{C}}) \simeq (K \times H) \otimes 1_{\mathcal{C}}$.

$$\begin{aligned} \mathrm{Map}(K \otimes (H \otimes 1_{\mathcal{C}}), Y) &\simeq \mathrm{Map}(K, \mathrm{Map}(H \otimes 1_{\mathcal{C}}, Y)) \\ &\simeq \mathrm{Map}(K, \mathrm{Map}(H, \mathrm{Map}(1_{\mathcal{C}}, Y))) \\ &\simeq \mathrm{Map}(K \times H, \mathrm{Map}(1_{\mathcal{C}}, Y)) \end{aligned}$$

and therefore the Yoneda lemmama implies that $K \otimes (H \otimes 1_{\mathcal{C}}) \simeq (K \times H) \otimes 1_{\mathcal{C}}$. Therefore, we conclude that in this case $\mathcal{C}$ satisfies the condition (M). In particular, if $\mathcal{C}_0 = \mathcal{C}$ satisfies (UM) then $\mathcal{C}$ satisfies (M) as well.

However, in our applications $\mathcal{C}$ will be the opposite of the ∞-category of nonconnective structures, and $\mathcal{C}_0$ will be the full subcategory spanned by the connective ones. In this case, $\mathcal{C}$ satisfies the condition (M) and $\mathcal{C}_0$ satisfies the condition (UM), but $\mathcal{C}_0$ is not closed in $\mathcal{C}$ under tensor with finite spaces.

At this point, we are ready to state the main result of this section:

Theorem 7.2.2.10. *Let $\mathcal{C}$ be a presentable ∞-category and let $\mathcal{C}_0$ be a full subcategory of $\mathcal{C}$ closed under products. Suppose that $\mathcal{C}_0$ satisfies the condition* (UM). *Then the functor $\rho_*\colon \mathrm{LMod}_{y(S^1_{\mathcal{C}})}(\mathrm{PSh}(\mathcal{C})) \to \mathrm{LMod}_{\underline{S}^1}(\mathrm{PSh}(\mathcal{C}))$ restricts to an equivalence of ∞-categories*

$$\mathrm{LMod}_{y(S^1_{\mathcal{C}})}(\mathrm{PSh}(\mathcal{C})) \times_{\mathrm{PSh}(\mathcal{C})} \mathcal{C}_0 \simeq \mathrm{LMod}_{\underline{S}^1}(\mathrm{PSh}(\mathcal{C})) \times_{\mathrm{PSh}(\mathcal{C})} \mathcal{C}_0.$$

Proof. To simplify the notations, write

$$\mathrm{LMod}_{y(S^1_{\mathcal{C}})}(\mathcal{C}_0) \coloneqq \mathrm{LMod}_{y(S^1_{\mathcal{C}})}(\mathrm{PSh}(\mathcal{C})) \times_{\mathrm{PSh}(\mathcal{C})} \mathcal{C}_0$$

and similarly

$$\mathrm{Mod}_{\underline{S}^1}(\mathcal{C}_0) \coloneqq \mathrm{LMod}_{\underline{S}^1}(\mathrm{PSh}(\mathcal{C})) \times_{\mathrm{PSh}(\mathcal{C})} \mathcal{C}_0.$$

We start by observing that if $X, Y \in \mathcal{C}_0$ then the natural morphism

$$\mathrm{Map}_{\mathcal{C}}((S^1_{\mathcal{C}})^{\times n} \times X, Y) \longrightarrow \mathrm{Map}_{\mathrm{PSh}(\mathcal{C})}((\underline{S}^1)^{\times n} \times y(X), y(Y)) \tag{7.2.2.6}$$

is an equivalence. Indeed, we can identify the right hand side with $\mathrm{Map}_{\mathcal{S}}((S^1)^{\times n}, \mathrm{Map}_{\mathcal{C}}(X,Y))$. On the other hand, since $\mathcal{C}_0$ satisfies the condition (UM) we see that

$$(S^1_{\mathcal{C}})^{\times n} \times X \simeq (S^1)^{\times n} \otimes X$$

and therefore

$$\mathrm{Map}_{\mathcal{C}}((S^1_{\mathcal{C}})^{\times n} \times X, Y) \simeq \mathrm{Map}_{\mathcal{S}}((S^1)^{\times n}, \mathrm{Map}_{\mathcal{C}}(X,Y)).$$

Let now $X, Y \in \mathrm{LMod}_{y(S^1_{\mathcal{C}})}(\mathcal{C}_0)$. Using [Lur12c, 4.2.2.11] we can represent X and Y as functors

$$F_X, F_Y \colon \mathbf{\Delta}\mathrm{op} \times \Delta^1 \longrightarrow \mathcal{C}$$

satisfying the conditions already described at the beginning of the proof of Theorem 7.2.2.4. Furthermore we can describe $\rho_*(X)$ and $\rho_*(Y)$ as the functors

$$\widetilde{F}_X, \widetilde{F}_Y \colon \mathbf{\Delta}\mathrm{op} \times \Delta^1 \longrightarrow \mathcal{C}$$

whose restriction to $\mathbf{\Delta}\mathrm{op} \times \{1\}$ coincides with the simplicial presentation of $\underline{S}^1$ and such that the diagrams

$$\begin{tikzcd} \widetilde{F}_X([n],0) \arrow[r] \arrow[d] & F_X([n],0) \arrow[d] \\ (\underline{S}^1)^{\times n} \arrow[r] & y(S^1_{\mathcal{C}})^{\times n} \end{tikzcd} , \quad \begin{tikzcd} \widetilde{F}_Y([n],0) \arrow[r] \arrow[d] & F_Y([n],0) \arrow[d] \\ (\underline{S}^1)^{\times n} \arrow[r] & y(S^1_{\mathcal{C}})^{\times n} \end{tikzcd}$$

are pullback squares. As already remarked in the proof of Theorem 7.2.2.4, we can describe morphisms in $\mathrm{LMod}_{\underline{S}^1}(\mathcal{C}_0)$ as fiber products

$$\mathrm{Map}_{\underline{S}^1}(X,Y) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}\mathrm{op}\times\Delta^1,\mathcal{C})}(F_X,F_Y) \times_{\mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}\mathrm{op},\mathcal{C})}(F_X|_{\mathbf{\Delta}\mathrm{op}\times\{1\}}, F_Y|_{\mathbf{\Delta}\mathrm{op}\times\{1\}})} \{\mathrm{id}_{\underline{S}^1}\}$$

Using the end formula to compute natural transformations, we can describe $\mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}\mathrm{op}\times\Delta^1,\mathcal{C})}(F_X,F_Y)$ as

$$\int_{([n],i)\in\mathbf{\Delta}\mathrm{op}\times\Delta^1} \mathrm{Map}_{\mathcal{C}}(F_X([n],i), F_Y([n],i)) \simeq \int_{i\in\Delta^1}\int_{[n]\in\mathbf{\Delta}\mathrm{op}} \mathrm{Map}_{\mathcal{C}}(F_X([n],i), F_Y([n],i)).$$

Bringing the fiber product inside the end, we can rewrite

$$\mathrm{Map}_{\underline{S}^1}(X,Y) \simeq \int_{i\in\Delta^1}\int_{[n]\in\mathbf{\Delta}\mathrm{op}} \mathrm{Map}_{\mathcal{C}}(F_X([n],i), F_Y([n],i)) \times_{\mathrm{Map}_{\mathcal{C}}(F_X([n],i),F_Y([n],1))} \{p_n\}$$

where p_n is the identity when $i=1$ and the natural projection

$$p_n \colon F_X([n],0) \to F_X([n],1) \simeq F_Y([n],1)$$

when $i=0$. A similar description holds for $\mathrm{Map}_{y(S^1_{\mathcal{C}})}(\rho_*(X), \rho_*(Y))$. We now observe that the fiber product

$$\mathrm{Map}_{\mathcal{C}}(F_X([n],i), F_Y([n],i)) \times_{\mathrm{Map}_{\mathcal{C}}(F_X([n],i),F_Y([n],1))} \{p_n\}$$

is contractible when $i = 1$ and coincides with

$$\mathrm{Map}_{\mathcal{C}}((\underline{S}^1)^{\times n} \times X, Y)$$

when $i = 0$. Similarly, the fiber product

$$\mathrm{Map}_{\mathcal{C}}(\widetilde{F}_X([n], i), \widetilde{F}_Y([n, i]) \times_{\mathrm{Map}_{\mathcal{C}}(\widetilde{F}_X([n],i), \widetilde{F}_Y([n],1)} \{\widetilde{p}_n\}$$

is contractible when $i = 1$ and coincides with

$$\mathrm{Map}_{\mathcal{C}}(y(S^1_{\mathcal{C}})^{\times n} \times X, Y)$$

when $i = 0$. Since we saw that the morphism (7.2.2.6) is an equivalence, we finally conclude that ρ_* is fully faithful.

As for essential surjectivity, we observe that giving an object in $\mathrm{LMod}_{\underline{S}^1}(\mathcal{C}_0)$ is equivalent to provide a morphism of $\mathbb{E}_1$-monoids in $\mathrm{PSh}(\mathcal{C})$

$$\underline{S}^1 \longrightarrow \mathbf{Map}_{\mathrm{PSh}(\mathcal{C})}(y(X), y(X)),$$

where $X \in \mathcal{C}_0$. Using the monoidal adjunction $\pi^p \dashv {}_p\pi$, this is equivalent to give a morphism of $\mathbb{E}_1$-monoids

$$S^1 \longrightarrow \mathrm{Map}_{\mathcal{C}}(X, X)$$

in $\mathcal{S}$. Finally, the condition (UM) again implies that such a datum is equivalent to the datum of an action of $S^1_{\mathcal{C}}$ on X. We therefore obtain an object in $\mathrm{LMod}_{S^1_{\mathcal{C}}}(\mathcal{C}_0)$ that induces the $\underline{S}^1$-object in $\mathcal{C}_0$ we started with. Therefore the functor

$$\mathrm{LMod}_{\underline{S}^1}(\mathcal{C}_0) \longrightarrow \mathrm{LMod}_{y(S^1)}(\mathcal{C}_0)$$

is essentially surjective. Thus, the proof is complete. $\square$

7.2.3 Algebraic HKR theorem

We now put in fruition the technology developed so far to obtain a proof of the HKR theorem in the algebraic setting. We put in motion the general strategy outlined in the introduction.

We denote by $\mathcal{C}\mathrm{Alg}_k$ the ∞-category of connective cdgas over k and we let $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ denote the ∞-category of nonconnective ones. In Section 7.2.1 we introduced and studied at length the ∞-categories ε-$\mathcal{C}\mathrm{Alg}_k$ and ε-$\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$. In particular, in Theorem 7.2.1.7 we provided a comonadic description of this category over $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$, and in Theorem 7.2.1.6 we identified the left adjoint to the forgetful functor with the de Rham algebra functor. The comonad of ε-$\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ over $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ is given by tensor product with the bialgebra $k[\eta]$. Recall that the algebra structure on $k[\eta]$ is, by definition, the one coming from the split square-zero extension. Let us denote this bialgebra by A_ε.

On the other hand, in Section 7.2.2 we studied S^1-$\mathcal{C}\mathrm{Alg}_k$ and S^1-$\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$. Set $\mathcal{C} \coloneqq (\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}})\mathrm{op}$. Since the tensor product of nonconnective cdgas commutes with finite limits, we see that the condition (M) (and in fact the stronger condition (UM)) is satisfied. In particular, Theorem 7.2.2.7 guarantees that

$$S^1_{\mathcal{C}} \simeq k \times_{k \times k} k.$$

Let us denote this bialgebra by A_{S^1}.

Instead of comparing ε-$\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ and S^1-$\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ directly we will compare the two ∞-categories

$$A_\varepsilon\text{-}\mathrm{CoMod}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}) \quad \text{and} \quad A_{S^1}\text{-}\mathrm{CoMod}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}).$$

This is enough for our purposes: in fact, Theorem 7.2.1.7 provides us with an equivalence

$$\varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \simeq \mathrm{LMod}_{A_\varepsilon}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}).$$

On the other hand, since $\mathcal{C} = (\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}})\mathrm{op}$ satisfies the condition (UM), we see that Theorem 7.2.2.10 provides an equivalence

$$S^1\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \simeq \mathrm{LMod}_{A_{S^1}}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}).$$

Warning 7.2.3.1. In the analytic setting the latter equivalence will not be satisfied. However the weaker equivalence

$$S^1\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \times_{\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}} \mathcal{C}\mathrm{Alg}_k \simeq \mathrm{LMod}_{A_{S^1}}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}) \times_{\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}} \mathcal{C}\mathrm{Alg}_k$$

will still hold.

As outlined in the introduction, we are then reduced to compare the bialgebras A_ε and A_{S^1}. For this, we introduce a middle comparison term. From a purely algebraic point of view, we simply take $\mathrm{Sym}_k(k[-1])$. The universal property of the symmetric algebra provides us with a canonical zig-zag in $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$

$$\begin{tikzcd} & \mathrm{Sym}_k(k[-1]) \arrow[dl] \arrow[dr] & \\ A_\varepsilon & & A_{S^1}. \end{tikzcd} \tag{7.2.3.1}$$

Since we are in characteristic 0, we see that the chain complex underlying $\mathrm{Sym}_k(k[-1])$ coincides with $k \oplus k[-1]$.[2] Therefore the two morphisms in the above zig-zag are equivalences. We are left to check that these two morphisms can be promoted to morphisms of bialgebras. This is not entirely tautological, and to prove it we need to resort to a more geometrical description of $\mathrm{Sym}_k(k[-1])$. Indeed, let dSt_k denote the ∞-category of derived stacks over k. This ∞-category comes equipped with a global section functor

$$\Gamma \colon \mathrm{dSt}_k \longrightarrow \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}},$$

that can be obtained as left Kan extension of the global section functor on affine derived schemes. Then we have a canonical equivalence

$$\mathrm{Sym}_k(k[-1]) \simeq \Gamma(\mathrm{B}(\mathbb{G}_a)).$$

The functor Γ admits a left adjoint, denoted Spec, which can be described as a restricted Yoneda embedding. This functor is not fully faithful, but it becomes so when restricted to the full subcategory of $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$ spanned by coconnective algebras [Toë06a]. In particular, this allows to identify the space of coalgebra structures on $\mathrm{Sym}_k(k[-1])$ (compatible with the given algebra structure) with the space of $\mathbb{E}_\infty$-group structures on $\mathrm{B}(\mathbb{G}_a)$. We have:

Proposition 7.2.3.2. *The space of $\mathbb{E}_\infty$-group structures on $\mathrm{B}(\mathbb{G}_a)$ is contractible.*

Proof. The space X of $\mathbb{E}_\infty$-group structures on $\mathrm{B}(\mathbb{G}_a)$ is defined as the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Mon}_{\mathbb{E}_\infty}^{\mathrm{gp}}(\mathrm{dSt}_k) \arrow[d, "\mathrm{forget}"] \\ \{*\} \arrow[r, "\mathrm{B}(\mathbb{G}_a)"] & \mathrm{dSt}_k. \end{tikzcd} \tag{7.2.3.2}$$

Notice that $\pi_0(\mathrm{B}(\mathbb{G}_a)) \simeq \mathrm{Spec}(k)$, which is the final object in the ∞-topos dSt_k. In other words, $\mathrm{B}(\mathbb{G}_a) \in \mathrm{dSt}_k^{\geq 1}$. Notice furthermore that the inclusion $\mathrm{dSt}_k^{\geq 1} \hookrightarrow \mathrm{dSt}_k$ commutes with products. It follows that we can split the square (7.2.3.2) into the following ladder of pullbacks:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Mon}_{\mathbb{E}_\infty}^{\mathrm{gp}}(\mathrm{dSt}_k^{\geq 1}) \arrow[r] \arrow[d] & \mathrm{Mon}_{\mathbb{E}_\infty}^{\mathrm{gp}}(\mathrm{dSt}_k) \arrow[d] \\ \{*\} \arrow[r, "\mathrm{B}(\mathbb{G}_a)"] & \mathrm{dSt}_k^{\geq 1} \arrow[r] & \mathrm{dSt}_k. \end{tikzcd}$$

It is therefore enough to compute the fiber product on the left. Consider the commutative rectangle

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Mon}_{\mathbb{E}_\infty}^{\mathrm{gp}}(\mathrm{dSt}_k^{\geq 1}) \arrow[r, "\Omega"] \arrow[d] & \mathrm{Mon}_{\mathbb{E}_\infty}^{\mathrm{gp}}(\mathrm{Mon}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathrm{dSt}_k)) \arrow[d] \\ \{*\} \arrow[r, "\mathrm{B}(\mathbb{G}_a)"] & \mathrm{dSt}_k^{\geq 1} \arrow[r, "\Omega"] & \mathrm{Mon}_{\mathbb{E}_1}^{\mathrm{gp}}(\mathrm{dSt}_k) \end{tikzcd}$$

[2] The assumption on the characteristic is truly necessary. When $k = \mathbb{F}_p$, one can show that $\mathrm{H}^1(\mathrm{Sym}_k(k[-1]))$ is a countable direct sum of copies of $\mathbb{F}_p$.

May's delooping theorem implies that the horizontal morphism in the square on the right are equivalences. In particular, the square in question is a pullback. As a consequence, it is enough to compute the outer pullback.

Observe now that $\Omega(\mathrm{B}(\mathbb{G}_a)) \simeq \mathbb{G}_a$ and that this is a discrete object in dSt_k. Furthermore, the induced $\mathbb{E}_1$-group structure on $\mathbb{G}_a$ coincides with the additive one. We now observe that, since $\mathbb{G}_a$ is discrete and since the $\mathbb{E}_1$-structure is fixed, being $\mathbb{E}_\infty$ is now a property rather than a structure. In other words, we see that X is either empty or contractible. As the additive group structure on $\mathbb{G}_a$ is commutative, we see that it is indeed the latter case. $\square$

This proposition implies therefore that the space of bialgebra structure on $\mathrm{Sym}_k(k[-1])$ extending the given algebra structure is contractible. In particular, both morphisms $\mathrm{Sym}_k(k[-1]) \to A_\varepsilon$ and $\mathrm{Sym}_k(k[-1]) \to A_{S^1}$ can be promoted to equivalences of bialgebras. Thus, we obtain an equivalence $A_\varepsilon \simeq A_{S^1}$ as bialgebras. This implies immediately the structure HKR:

Theorem 7.2.3.3. *Let k be a field of characteristic zero. There is a natural equivalence $\phi\colon S^1\text{-}\mathcal{C}\mathrm{Alg}_k \simeq \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k$ making the diagram*

$$\begin{tikzcd} S^1\text{-}\mathcal{C}\mathrm{Alg}_k \arrow[rr, "\sim"] \arrow[dr, "v_{S^1}"'] & & \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k \arrow[dl, "v_\varepsilon"] \\ & \mathcal{C}\mathrm{Alg}_k & \end{tikzcd}$$

commutative.

Proof. Since k has characteristic zero, both maps in the zig-zag (7.2.3.1) are equivalences. It follows from Theorem 7.2.3.2 that both maps $\mathrm{Sym}_k(k[-1]) \to A_{S^1}$ and $\mathrm{Sym}_k(k[-1]) \to A_\varepsilon$ can be promoted to equivalences of bialgebras. This provides us with an equivalence $A_{S^1} \simeq A_\varepsilon$ of nonconnective bialgebras. Therefore, we obtain an equivalence

$$A_{S^1}\text{-}\mathrm{CoMod}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}) \simeq A_\varepsilon\text{-}\mathrm{CoMod}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}})$$

compatible with the forgetful functors to $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$. Notice that $A_\varepsilon\text{-}\mathrm{CoMod}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}})$ coincides by definition with the ∞-category we previously denoted $\eta\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$. We can therefore invoke Theorem 7.2.1.7 to deduce that

$$\varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k \simeq A_\varepsilon\text{-}\mathrm{CoMod}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}).$$

Furthermore, this equivalence is compatible with the forgetful functors to $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$. On the other hand, Theorem 7.2.2.10 provides us with an equivalence

$$A_{S^1}\text{-}\mathrm{CoMod}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}) \simeq S^1\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}},$$

also compatible with the forgetful functors to $\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}$. Putting all the information together, we obtain the following commutative diagram:

$$\begin{tikzcd} \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[r, "\sim"] \arrow[drr, "v_\varepsilon"'] & A_\varepsilon\text{-}\mathrm{CoMod}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}) \arrow[r, "\sim"] \arrow[dr] & A_{S^1}\text{-}\mathrm{CoMod}(\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}) \arrow[r, "\sim"] \arrow[d] & S^1\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \arrow[dl, "v_{S^1}"] \\ & & \mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}. & \end{tikzcd}$$

We now remark that there are canonical equivalences

$$\varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k \simeq \varepsilon\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \times_{\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}} \mathcal{C}\mathrm{Alg}_k$$

and

$$S^1\text{-}\mathcal{C}\mathrm{Alg}_k \simeq S^1\text{-}\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}} \times_{\mathcal{C}\mathrm{Alg}_k^{\mathrm{nc}}} \mathcal{C}\mathrm{Alg}_k.$$

The conclusion follows. $\square$

7.3 Nonconnective contexts and structures

As we saw in the algebraic case, non-connective algebras play a fundamental role in the proof of the HKR theorem. However, one limitation of derived analytic geometry (as introduced in [Lur11b, PY16b]) is that it only allows to work within the connective framework. Indeed, if $X = (\mathcal{X}, \mathcal{O}_X)$ is a derived analytic space, then the underlying algebra $\mathcal{O}_X^{\mathrm{alg}}$ is always a *simplicial* commutative ring. The goal of this section is to explain how this problemma can be solved, by introducing a suitable notion of non-connective analytic ring.

7.3.1 Definitions

We work in the general context of pregeometries. This will allow us to obtain several versions of HKR theorems. We refer the reader to [Lur11d, Definition 3.1.1] for the notion of pregeometry.

Definition 7.3.1.1. Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry. A $\mathcal{T}$-*geometric context* is the data of:

(i) a full subcategory $\mathcal{C} \subset {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ containing at least all the objects of the form $\mathrm{Spec}^{\mathcal{T}}(X)$ for $X \in \mathcal{T}$;

(ii) a choice of mophisms $\mathbb{P}$ in $\mathcal{C}$.

Furthermore, we impose the following conditions:

(i) the τ-topology on ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ restricts to a Grothendieck topology on $\mathcal{C}$, which we still denote τ;

(ii) for every $X \in \mathcal{T}$ the morphism $X \to *$ is in $\mathbb{P}$;

(iii) $\mathcal{C}$ has finite limits;

(iv) the inclusion $\mathcal{T} \subset \mathcal{C}$ commutes with products;

(v) the triple $(\mathcal{C}, \tau, \mathbb{P})$ forms a geometric context in the sense of [PY16d, §2.2].

The following are the fundamental examples considered in this paper:

Example 7.3.1.2. Let k be a classical commutative ring (of any characteristic). Let $\mathcal{T} = \mathcal{T}_{\text{ét}}(k)$ be the étale pregeometry (see [Lur11d, Definition 4.3.1]). Then we take $\mathcal{C} := \mathrm{dAff}_k$, the ∞-category of derived affine k-schemes. The topology τ coincides with the étale topology, and we take $\mathbb{P}$ to be the collection of smooth morphisms in dAff_k. Notice that in this example already the inclusion $\mathcal{C} \subset {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ does *not* commute with finite limits.

Example 7.3.1.3. Let $\mathcal{T}_{\mathrm{an}}(\mathbb{C})$ be the complex analytic pregeometry (see [Lur11b, Definition 11.1]). Then we take $\mathcal{C} := \mathrm{dStn}_{\mathbb{C}}$, the ∞-category of derived Stein spaces. This is the full subcategory of $\mathrm{dAn}_{\mathbb{C}}$ spanned by those derived complex analytic spaces X whose truncation $\mathrm{t}_0(X)$ is Stein (cf. [Por15b, Definition 3.2]). The topology τ coincides with the analytic topology, and we take $\mathbb{P}$ to be the collection of smooth morphisms in $\mathrm{dStn}_{\mathbb{C}}$.

Example 7.3.1.4. Let k be a non-archimedean field equipped with a non-trivial valuation. Let $\mathcal{T} = \mathcal{T}_{\mathrm{an}}(k)$ be the k-analytic pregeometry (see [PY16b, Construction 2.2]). Then we take $\mathcal{C} := \mathrm{dAn}_k$, the ∞-category of derived k-affinoid spaces. As in the previous example, this is the full subcategory of dAn_k spanned by those derived k-analytic spaces X whose truncation $\mathrm{t}_0(X)$ is k-affinoid (cf. [PY16b, Definition 7.3]).

Remark 7.3.1.5. In the case of a generic pregeometry $(\mathcal{T}, \mathrm{adm}, \tau)$ there is always a canonical choice for the category $\mathcal{C}$. Indeed, if $\mathcal{G}$ denotes a geometric envelope for $\mathcal{T}$ in the sense of [Lur11d, Definition 3.4.1], then one can take $\mathcal{C}$ to be the ∞-category of $\mathcal{G}$-schemes. However, at this level of generality there is no good choice of the collection of morphisms $\mathbb{P}$. One could take the collection of étale morphisms, but this choice would lead to a rather degenerate situation in what follows.

For the following definition, we recall that whenever $(\mathcal{C}, \tau, \mathbb{P})$ is a geometric context, one can define an associated ∞-category of *geometric stacks*. We denote this ∞-category by $\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})$ and we refer to [PY16d, §2.2] for the definition. We also recall that if $\mathcal{E}$ is an ∞-category with finite limits then

$$\mathrm{Sp}(\mathcal{E}) \simeq \mathrm{Sp}(\mathcal{E}_*).$$

In particular, any spectrum object $E \in \mathrm{Sp}(\mathcal{E})$ receives a canonical map $* \to \Omega^{\infty - n}(E)$, where $*$ denotes the final object of $\mathcal{E}$.

Definition 7.3.1.6. Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry. A *pre-$\mathcal{T}$-nonconnective context* is the given of:

(i) a $\mathcal{T}$-geometric context $(\mathcal{C}, \tau, \mathbb{P})$;

(ii) a spectrum object $E \in \mathrm{Sp}(\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}))$.

Furthermore, we impose the following two conditions on E:

(i) the canonical map $\Omega^\infty(E) \to *$ is in $\mathbb{P}$;

(ii) for every $n > 0$, the canonical morphisms $* \to \Omega^{\infty - n}(E)$ is a $\mathbb{P}$-atlas;

We say that a pre-$\mathcal{T}$-nonconnective context is a *$\mathcal{T}$-nonconnective context* if the following additional condition is satisfied:

(3) for every $X \in \mathcal{C}$ and every $n \geq$, one has $\pi_0 \mathrm{Map}_{\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})}(X, \Omega^{\infty-n}(E)) \simeq *$.

Notation 7.3.1.7. Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry and let $(\mathcal{C}, \mathbb{P}, E)$ be a $\mathcal{T}$-nonconnective context. We set

$$E^n \coloneqq \Omega^{\infty - n}(E) \in \mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}).$$

In particular, we have the relation $E^n \simeq \Omega(E^{n+1})$.

Example 7.3.1.8. The reader should keep in mind the following fundamental examples:

(i) when $\mathcal{T} = \mathcal{T}_{\text{ét}}(k)$, we take $E_{\mathrm{alg}} \coloneqq \{\mathrm{B}^n(\mathbb{G}_a)\}$, the spectrum associated via May's delooping theorem to the commutative k-group scheme $\mathbb{G}_a$. As the relations

$$\mathrm{B}^n(\mathbb{G}_{a,k}) \simeq \mathrm{Spec}(k) \times_{\mathrm{B}^{n+1}(\mathbb{G}_{a,k})} \mathrm{Spec}(k)$$

are satisfied, we see that $\{\mathrm{B}^n(\mathbb{G}_{a,k})\}_{n \geq 0}$ form indeed a spectrum object. Furthermore, the morphisms $\mathrm{Spec}(k) \to \mathrm{B}^n(\mathbb{G}_{a,k})$ are smooth atlases.

(ii) When $\mathcal{T} = \mathcal{T}_{\mathrm{an}}(\mathbb{C})$ or $\mathcal{T} = \mathcal{T}_{\mathrm{an}}(k)$, we take $E \coloneqq \{\mathrm{B}^n(\mathbf{G}_a)\}_{n \geq 0}$. Here $\mathbf{G}_a$ denotes the analytic affine line $\mathbf{A}^1_k \simeq (\mathbb{A}^1_k)^{\mathrm{an}}$, seen as an analytic commutative group. As in the algebraic setting, this is indeed a spectrum object, and each $\mathrm{B}^n(\mathbf{G}_a)$ is a geometric stack with smooth atlas given by $\mathrm{Spec}(k) \to \mathrm{B}^n(\mathbf{G}_a)$.

(iii) When $\mathcal{T} = \mathcal{T}_{\mathrm{an}}(k)$ for a nonarchimedean field k equipped with a non-trivial valuation we have many natural choices for E. Indeed, for any $r \in \mathbb{R}_{>0}$, we can consider the disk $\mathbf{D}^1_k(0, r)$. Since the k is nonarchimedean, we see that $\mathbf{D}^1_k(0, r)$ is an abelian group object in $\mathcal{T}_{\mathrm{an}}(k)$ and therefore we can consider its delooping stacks $\mathrm{B}^n(\mathbf{D}^1_k(0, r))$. We denote the spectrum $\{\mathrm{B}^n(\mathbf{D}^1_k(0, r))\}_{n \geq 0}$ by $E(r)$, with the understanding that when $r = \infty$ we get back the spectrum of the previous example. We will see later that the extra freedom in choosing the spectrum E is one of the keys to the nonarchimedean HKR in many interesting situations (such as when $k = \mathbb{Q}_p$).

Whenever a $\mathcal{T}$-nonconnective context is fixed, we can define $\mathcal{T}$-nonconnective structures.

Notation 7.3.1.9. Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry and let $(\mathcal{C}, \mathbb{P}, E)$ be a pre-$\mathcal{T}$-nonconnective context. We let $\mathcal{T}^{\mathrm{nc}}$ be the smallest full subcategory of $\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})$ closed under finite products and containing the objects of the form $\mathrm{Spec}^{\mathcal{T}}(X)$ and the geometric stacks E^n for $n \geq 0$.

Definition 7.3.1.10. A *nonconnective pregeometry* is the given of a pregeometry $(\mathcal{T}, \mathrm{adm}, \tau)$ and of a $\mathcal{T}$-nonconnective context $(\mathcal{C}, \mathbb{P}, E)$. Committing an abuse of notation, we usually denote a nonconnective pregeometry simply by the symbol $\mathcal{T}^{\mathrm{nc}}$.

Definition 7.3.1.11. Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry and let $(\mathcal{C}, \mathbb{P}, E)$ be a pre-$\mathcal{T}$-nonconnective context. Let $\mathcal{X}$ be an ∞-topos. A *nonconnective $\mathcal{T}$-structure on $\mathcal{X}$* (or a *$\mathcal{T}^{\mathrm{nc}}$-structure on $\mathcal{X}$*) is a product preserving functor

$$\mathcal{O} \colon \mathcal{T}^{\mathrm{nc}} \to \mathcal{X}$$

such that

(i) the restriction $\mathcal{O}|_{\mathcal{T}}$ is a $\mathcal{T}$-structure;

(ii) $\mathcal{O}$ preserves the pullbacks of the form

$$\begin{tikzcd} \Omega^{\infty-n}(E) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Omega^{\infty-n-1}(E). \end{tikzcd} \tag{7.3.1.1}$$

If $\mathcal{O}$, $\mathcal{O}'$ are $\mathcal{T}^{\mathrm{nc}}$-structures on $\mathcal{X}$, a natural transformation $\mathcal{O} \to \mathcal{O}'$ is said to be *local* if its restriction $\mathcal{O}|_{\mathcal{T}} \to \mathcal{O}'|_{\mathcal{T}}$ is a local transformation of $\mathcal{T}$-structures.

We denote the ∞-category of $\mathcal{T}^{\mathrm{nc}}$-structures on $\mathcal{X}$ and local transformations between them by $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$.

Remark 7.3.1.12. In what follows we will mainly restrict our attention to nonconnective structures for a nonconnective pregeometry $\mathcal{T}^{\mathrm{nc}}$. In other words, we almost always work with $\mathcal{T}$-nonconnective contexts rather than pre-$\mathcal{T}$-nonconnective contexts. There is only one exception: in Section 7.3.8 it is important to allow pre-$\mathcal{T}$-nonconnective contexts. This is the reason we formulated the above definition in this more general setting.

The above definition is justified by Theorem 7.3.4.1, that proves in particular that if k is a field of characteristic zero, then the ∞-category of $\mathcal{T}^{\mathrm{nc}}_{\mathrm{\acute{e}t}}(k)$-structures on $\mathcal{X}$ coincides with the ∞-category of sheaves with values in the ∞-category of cdgas.

However, before stating and proving this result, we need to study some general features of the ∞-category of nonconnective structures.

7.3.2 Underlying spectrum object

Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry and let $(\mathcal{C}, \mathbb{P}, E)$ be a $\mathcal{T}$-nonconnective context. Using [Lur09c, 5.3.6.2] we find an ∞-category $\mathcal{G}^{\mathrm{nc}}$ equipped with an ∞-functor $j \colon \mathcal{T}^{\mathrm{nc}} \to \mathcal{G}^{\mathrm{nc}}$ enjoying the following properties:

(i) the ∞-category $\mathcal{G}^{\mathrm{nc}}$ is idempotent complete and admits finite limits;

(ii) the functor j commutes with products, admissible pullbacks in $\mathcal{T}$ and take diagrams of the form (7.3.1.1) to pullbacks;

(iii) for every other idempotent complete ∞-category with finite limits $\mathcal{E}$, composition with j induces an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{E}) \longrightarrow \mathrm{Fun}'(\mathcal{T}^{\mathrm{nc}}, \mathcal{E}),$$

where $\mathrm{Fun}'(\mathcal{T}^{\mathrm{nc}}, \mathcal{E})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{T}^{\mathrm{nc}}, \mathcal{E})$ spanned by those functors commuting with products, admissible pullbacks in $\mathcal{T}$ and taking diagrams of the form (7.3.1.1) to pullbacks;

(iv) j is fully faithful.

Since the inclusion $\mathcal{T} \subseteq \mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})$ commutes with products, admissible pullbacks and takes the diagrams (7.3.1.1) to pullbacks, we obtain a canonical left exact functor

$$i \colon \mathcal{G}^{\mathrm{nc}} \longrightarrow \mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}).$$

Let $p \colon \mathfrak{G} \to \mathbb{N}\mathrm{op}$ be the Cartesian fibration associated to the diagram

$$\cdots \xrightarrow{\Omega} \mathcal{G}^{\mathrm{nc}} \xrightarrow{\Omega} \mathcal{G}^{\mathrm{nc}} \xrightarrow{\Omega} \mathcal{G}^{\mathrm{nc}},$$

and let $q \colon \mathfrak{C} \to \mathbb{N}\mathrm{op}$ be the Cartesian fibration associated to

$$\cdots \xrightarrow{\Omega} \mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}) \xrightarrow{\Omega} \mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}) \xrightarrow{\Omega} \mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}).$$

Since $i \colon \mathcal{G}^{\mathrm{nc}} \to \mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})$ is left exact, it induces a morphism of Cartesian fibrations $f \colon \mathfrak{G} \to \mathfrak{C}$. Let us represent an object in $\mathfrak{G}$ (resp. in $\mathfrak{C}$) by a pair (X, n) where $n \in \mathbb{N}$ and $X \in \mathcal{G}^{\mathrm{nc}}$ (resp. $X \in \mathrm{Geom}(\mathcal{T}, \tau, \mathbb{P})$). We let $\mathfrak{G}_E$ be the full subcategory of $\mathfrak{G}$ spanned by the family of objects $\{(E^n, n)\}_{n\in\mathbb{N}}$. Since $\Omega(E^n) \simeq E^{n-1}$, we see that the inclusion $\mathfrak{G}_E \subset \mathfrak{G}$ preserves Cartesian edges. Furthermore, the composition

$$f_E \colon \mathfrak{G}_E \hookrightarrow \mathfrak{G} \xrightarrow{f} \mathfrak{C}$$

is fully faithful.

Using [Lur09c, 3.3.3.2] we can identify $\mathrm{Sp}(\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}))$ with ∞-category of Cartesian sections of $q \colon \mathfrak{C} \to \mathbb{N}\mathrm{op}$. Let $s_E \colon \mathbb{N}\mathrm{op} \to \mathfrak{C}$ be the section determined by E. Notice that $s_E(n) \simeq E^n$. In particular, s_E factors through the essential image of f_E. Since f_E is fully faithful, we can therefore a Cartesian section $s'_E \colon \mathbb{N}\mathrm{op} \to \mathfrak{G}_E$ such that $f_E \circ s'_E \simeq s_E$. Composing with the inclusion $\mathfrak{G}_E \subset \mathfrak{G}$, we can review s'_E as a Cartesian section of $\mathfrak{G}$. In other words, we can associate to E a spectrum object in $\mathcal{G}^{\mathrm{nc}}$. In what follows, we denote this spectrum object again by E. This abuse of notation is justified by the full faithfulness of f_E.

Let us now fix an ∞-topos $\mathcal{X}$. The universal property of $\mathcal{G}^{\mathrm{nc}}$ produces an equivalence of ∞-categories

$$\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{nc}}, \mathcal{X}).$$

Since $\mathcal{G}^{\mathrm{nc}}$ has finite limits, we can identify $\mathrm{Sp}(\mathcal{G}^{\mathrm{nc}})$ with the ∞-category $\mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{G}^{\mathrm{nc}})$ spanned by strongly excisive functors. In particular, we obtain an evaluation map

$$\mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{G}^{\mathrm{nc}}) \times \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{nc}}, \mathcal{X}) \longrightarrow \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{X}).$$

Evaluation at $E \in \mathrm{Sp}(\mathcal{G}^{\mathrm{nc}})$ provides us with a functor

$$U \colon \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Sp}(\mathcal{X}).$$

Definition 7.3.2.1. Let $\mathcal{O}$ be a $\mathcal{T}^{\mathrm{nc}}$-structure on an ∞-topos $\mathcal{X}$. Then we refer to $U(\mathcal{O})$ as the *underlying spectrum object of* $\mathcal{O}$.

Remark 7.3.2.2. Loosely speaking, we can identify $U(\mathcal{O})$ with the collection of objects $\{\mathcal{O}(E^n)\}_{n\geq 0}$. Notice that the assumption on $\mathcal{O}$ guarantees that

$$\mathcal{O}(E^n) \simeq \Omega(\mathcal{O}(E^{n+1})).$$

We are therefore authorized to think to the sequence $\{\mathcal{O}(E^n)\}_{n\geq 0}$ as an Ω-spectrum in $\mathcal{X}$. The above construction, is a formalization of this rough idea.

Example 7.3.2.3. Consider the case $\mathcal{T} = \mathcal{T}_{\text{ét}}(k)$ and let $\mathcal{O}$ be a nonconnective $\mathcal{T}$-structure on $\mathcal{S}$.

The reader should however observe that the spaces $\Omega^{\infty - n}(U(\mathcal{O}))$ do *not* have a ring structure. In particular, one cannot interpret $U(\mathcal{O})$ as a spectrum object in the category of simplicial commutative rings. This is due to the fact that there is no *multiplication* map

$$\mathrm{B}(\mathbb{G}_a) \times \mathrm{B}(\mathbb{G}_a) \longrightarrow \mathrm{B}(\mathbb{G}_a),$$

although there are of course multiplication maps

$$\mathrm{B}^n(\mathbb{G}_a) \times \mathrm{B}^m(\mathbb{G}_a) \longrightarrow \mathrm{B}^{n+m}(\mathbb{G}_a),$$

corresponding to the cup product in cohomology.

Remark 7.3.2.4. In the general case, we can roughly think of a nonconnective $\mathcal{T}$-structure in $\mathcal{S}$ as the given of:

(i) a spectrum $A \in \mathrm{Sp}$;

(ii) a $\mathcal{T}$-structure on $\Omega^\infty(A)$;

(iii) an additional structure on A exhibiting A as an algebra over the ring of cohomology operations associated to E.

However, as the previous example shows, there is no natural $\mathcal{T}$-structure on the spaces $\Omega^{\infty-n}(A)$. Furthermore, the additional structure coming from E might consist of significantly less operations than the ones provided by $\mathcal{T}$. For example, we will see later in the paper that a $\mathcal{T}_{\mathrm{an}}(k)$-structure essentially consists of an unbounded cdga A equipped with an analytic structure on $\tau_{\geq 0}(A)$.

Proposition 7.3.2.5. *The forgetful functor* $U\colon \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \to \mathrm{Sp}(\mathcal{X})$ *commutes with limits and filtered colimits. Furthermore, suppose that* $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ *is closed under sifted colimits in* $\mathrm{Fun}(\mathcal{T}, \mathcal{X})$*. Then* U *commutes with sifted colimits.*

Proof. Let us first observe that since limits and filtered colimits commute with finite limits, the inclusion $\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}}, \mathcal{X})$ preserves limits and filtered colimits. Since limits and colimits in a category of functors are computed objectwise and U is given by evaluation, the statement follows immediately.

Suppose now that $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ is closed under sifted colimits in $\mathrm{Fun}(\mathcal{T}, \mathcal{X})$. It is enough to prove that if

$$\mathcal{O}^\bullet\colon \mathbf{\Delta} \longrightarrow \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$$

is a simplicial diagram, then its geometric realization $\mathcal{O} := |\mathcal{O}^\bullet|$ computed in $\mathrm{Fun}(\mathcal{T}^{\mathrm{nc}}, \mathcal{X})$ is again a $\mathcal{T}^{\mathrm{nc}}$-structure. Observe that $\mathcal{O}$ commutes with finite products because sifted colimits commute with finite products. Furthermore, the assumption implies that the restriction of $\mathcal{O}$ to $\mathcal{T}$ is a $\mathcal{T}$-structure. We are therefore left to check that

$$\mathcal{O}(E^n) \simeq \Omega(\mathcal{O}(E^{n+1})).$$

Notice that using the ∞-categorical version of May's delooping theorem we can factor evaluation at E^n through $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}^n}(\mathcal{X})$. In this way, the looping functor Ω gets identified with the forgetful functor

$$\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}^n}(\mathcal{X}) \longrightarrow \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}^{n-1}}(\mathcal{X}).$$

As this forgetful functor commutes with sifted colimits, the conclusion follows. $\square$

7.3.3 Connective covers

Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry and let $(\mathcal{C}, \mathbb{P}, E)$ be a $\mathcal{T}$-nonconnective context. Let $\mathcal{X}$ be an ∞-topos. Precomposition with the natural inclusion $j\colon \mathcal{T} \hookrightarrow \mathcal{T}^{\mathrm{nc}}$ induces a well defined functor

$$\tau_{\geq 0}\colon \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}}(\mathcal{X}).$$

We refer to this functor as the *connective cover functor*.

Since both $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ and $\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$ are presentable and $\tau_{\geq 0}$ commutes with limits and filtered colimits, the adjoint functor theorem implies the existence of a left adjoint i to $\tau_{\geq 0}$. The goal of this section is to study the properties of the functor i. In particular, we will prove that in many cases i is fully faithful, and we will provide a characterization of its essential image.

We start by providing a sufficient criterion to check that a functor $\mathcal{O}\colon \mathcal{T}^{\mathrm{nc}} \to \mathcal{X}$ is a $\mathcal{T}^{\mathrm{nc}}$-structure. Let us begin by fixing some notation. Let $\mathcal{D}$ be a Cartesian symmetric monoidal ∞-category. Using [Lur12c, 4.1.2.11] we can identify the ∞-category $\mathrm{Mon}_{\mathbb{E}_1}(\mathcal{D})$ of $\mathbb{E}_1$-monoid objects in $\mathcal{D}$ with the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}\mathrm{op}, \mathcal{D})$ spanned by those simplicial objects satisfying the Segal condition (cf. [Lur12c, 4.1.2.5]). We denote by ${}^{\Delta}\mathrm{B}$ the corresponding functor:

$${}^{\Delta}\mathrm{B}\colon \mathrm{Mon}_{\mathbb{E}_1}(\mathcal{D}) \longrightarrow \mathrm{Fun}(\mathbf{\Delta}\mathrm{op}, \mathcal{D}).$$

When $\mathcal{D}$ admits geometric realization of simplicial objects, then we have a natural transformation of functors $\mathrm{Mon}_{\mathbb{E}_1}(\mathcal{D}_*) \to \mathcal{D}_*$

$$\left|{}^{\Delta}\mathrm{B}(-)\right| \longrightarrow \mathrm{Bar}^{(1)}(-)$$

that is in fact an equivalence.

Suppose now given a product-preserving functor $\mathcal{O}\colon \mathcal{T}^{\mathrm{nc}} \to \mathcal{X}$. As E is a spectrum object in $\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})$, we see that each E^n acquires the structure of a grouplike ($\mathbb{E}_\infty$ and hence) $\mathbb{E}_1$-monoid in $\mathcal{T}^{\mathrm{nc}}$. Since $\mathcal{O}$ respects the Cartesian structures, we see that the canonical morphism in $\mathrm{Fun}(\mathbf{\Delta}\mathrm{op}, \mathcal{X}_*)$

$$\mathcal{O}\left({}^{\Delta}\mathrm{B}(E^n)\right) \longrightarrow {}^{\Delta}\mathrm{B}(\mathcal{O}(E^n))$$

is an equivalence. On the other hand, we can identify ${}^{\triangle}\mathrm{B}(E^n)$ with the Čech nerve of the $\mathbb{P}$-atlas $* \to E^{n+1}$. We therefore obtain a canonical morphism

$$\phi^n_{\mathcal{O}} \colon \left| \mathcal{O}\left({}^{\triangle}\mathrm{B}(\mathcal{O}(E^n))\right) \right| \longrightarrow \mathcal{O}(E^{n+1}).$$

With these notations, we can now prove the following result:

Lemma 7.3.3.1. *Let $\mathcal{O} \in \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}}, \mathcal{X})$ be a functor. Suppose that:*

(i) the restriction $\mathcal{O}|_{\mathcal{T}}$ is a (local) $\mathcal{T}$-structure;

(ii) the functor $\mathcal{O}$ commutes with products;

(iii) for every $n \geq 1$, the canonical morphism $\phi^n_{\mathcal{O}} \colon \left| \mathcal{O}\left({}^{\triangle}\mathrm{B}(\mathcal{O}(E^n))\right) \right| \to \mathcal{O}(E^{n+1})$ is an equivalence $\mathcal{X}$;

Then $\mathcal{O}$ is a (local) $\mathcal{T}^{\mathrm{nc}}$-structure.

Proof. We only need to check that the canonical morphism

$$\mathcal{O}(E^n) \longrightarrow \Omega\mathcal{O}(E^{n+1}) \tag{7.3.3.1}$$

is an equivalence for every $n \geq 0$. Since $\mathcal{O}$ commutes with products, the canonical morphism

$$\mathcal{O}\left({}^{\triangle}\mathrm{B}(E^n)\right) \longrightarrow {}^{\triangle}\mathrm{B}(\mathcal{O}(E^n))$$

is an equivalence. Coupling this observation with the hypothesis on $\phi^n_{\mathcal{O}}$, we deduce that the canonical morphism

$$\mathrm{Bar}^{(1)}\left(\mathcal{O}(E^n)\right) \longrightarrow \mathcal{O}(E^{n+1})$$

is an equivalence in $\mathcal{X}_*^{\geq 1}$. In virtue of May's delooping theorem [Lur12c, 5.2.6.15], we see that the canonical morphism

$$\mathcal{O}(E^n) \longrightarrow \mathrm{Cobar}^{(1)}(\mathcal{O}(E^{n+1}))$$

is an equivalence in $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{X}_*)$. Applying the forgetful functor to $\mathcal{X}_*$ and [Lur12c, 5.2.6.12], we finally obtain that the morphism (7.3.3.1) is an equivalence, thus completing the proof. □

Later in this section we will characterize $\mathcal{T}$-structures exactly as those $\mathcal{T}^{\mathrm{nc}}$-structures satisfying condition (3) in the above lemmama. The following example illustrates why it is reasonable to expect a similar characterization:

Example 7.3.3.2. Let $X \in \mathcal{T}$ be any object. Then the functor $A^{\mathrm{nc}}_X \colon \mathcal{T}^{\mathrm{nc}} \to \mathcal{S}$ given by $A_X(Y) \coloneqq \mathrm{Map}_{\mathcal{T}^{\mathrm{nc}}}(X, Y)$ is a $\mathcal{T}^{\mathrm{nc}}$-structure. Furthermore, assumption (3) in Theorem 7.3.1.6 implies that for every $n \geq 1$ one has:

$$\pi_0(A^{\mathrm{nc}}_X(E^n)) \simeq \pi_0\mathrm{Map}_{\mathcal{T}^{\mathrm{nc}}}(X, E^n) \simeq \{*\}.$$

In other words, the morphism $A^{\mathrm{nc}}_X(*) \to A^{\mathrm{nc}}_X(E^n)$ is an effective epimorphism. Observe now that A^{nc}_X commutes with all limits. In particular, it commutes with the Čech nerve $\check{\mathcal{C}}(p)$ of the $\mathbb{P}$-atlas $p \colon * \to E^n$. This implies that we can identify the simplicial object

$${}^{\triangle}\mathrm{B}(\mathcal{O}(E^n)) \simeq \mathcal{O}\left({}^{\triangle}\mathrm{B}(E^n)\right) \simeq \mathcal{O}\big(\check{\mathcal{C}}(p)\big)$$

with the Čech nerve of $A^{\mathrm{nc}}_X(p) \colon A^{\mathrm{nc}}_X(*) \to A^{\mathrm{nc}}_X(E^n)$. As we already argued that this is an effective epimorphism, we finally conclude that $\phi^n_{A^{\mathrm{nc}}_X}$ is an equivalence.

Combining Theorem 7.3.3.1 and the above example we can produce many $\mathcal{T}^{\mathrm{nc}}$-structures out of $\mathcal{T}$-structures. Before stating the result, let us introduce some notations. We let

$$\iota_{\mathcal{X}} \colon \mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{T}, \mathcal{X}) \quad , \quad \iota^{\mathrm{nc}}_{\mathcal{X}} \colon \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}}, \mathcal{X})$$

denote the canonical inclusions and we denote by $L_{\mathcal{X}}$ and $L^{\mathrm{nc}}_{\mathcal{X}}$ their left adjoints, respectively. When $\mathcal{X} = \mathcal{S}$ is the ∞-topos of spaces, we omit the $\mathcal{X}$ in the subscript.

Lemma 7.3.3.3. *Assume that the pregeometry $\mathcal{T}$ is discrete. Let I be a sifted category and let $F\colon I \to \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ a diagram. Suppose that for every $\alpha \in I$, the $\mathcal{T}$-structure $\mathcal{O}_\alpha := F(\alpha)$ is of the form $A_{X_\alpha} := \mathrm{Map}_{\mathcal{T}}(X_\alpha, -)$ for some $X_\alpha \in \mathcal{T}$. Let $\mathcal{O}$ be a colimit for F. Then $\mathrm{Lan}_j(\iota \circ \mathcal{O}) \in \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}}, \mathcal{S})$ is a $\mathcal{T}^{\mathrm{nc}}$-structure.*

Proof. To simplify the notations, we simply write $\mathrm{Lan}_j(\mathcal{O})$ instead of $\mathrm{Lan}_j(\iota \circ \mathcal{O})$. Notice that the functor $\iota\colon \mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \to \mathrm{Fun}(\mathcal{T}, \mathcal{X})$ commutes with sifted colimits. As $\mathrm{Lan}_j\colon \mathrm{Fun}(\mathcal{T}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}}, \mathcal{S})$ is a left adjoint, we conclude that

$$\mathrm{Lan}_j(\mathcal{O}) \simeq \operatorname*{colim}_{\alpha \in I} \mathrm{Lan}_j(\mathcal{O}_\alpha).$$

As $\mathcal{O}_\alpha \simeq A_{X_\alpha}$, we have

$$\mathrm{Lan}_j(\mathcal{O}_\alpha) \simeq \mathrm{Map}_{\mathcal{T}^{\mathrm{nc}}}(X_\alpha, -) = A^{\mathrm{nc}}_{X_\alpha}.$$

In particular, each $\mathrm{Lan}_j(\mathcal{O}_\alpha)$ is a $\mathcal{T}^{\mathrm{nc}}$-structure. As I is sifted, we easily conclude that $\mathrm{Lan}_j(\mathcal{O})$ commutes with finite products. Furthermore, for each $\alpha \in I$, Theorem 7.3.3.2 implies that the natural morphism

$$\phi^n_{A^{\mathrm{nc}}_{X_\alpha}} : \left|{}^{\Delta}\mathrm{B}(A^{\mathrm{nc}}_{X_\alpha}(E^n)\right| \longrightarrow A^{\mathrm{nc}}_{X_\alpha}(E^{n+1})$$

is an equivalence for every $n \geq 1$. Since I is sifted, we see that

$$\operatorname*{colim}_{\alpha \in I} {}^{\Delta}\mathrm{B}(A^{\mathrm{nc}}_{X_\alpha}(E^n)) \simeq {}^{\Delta}\mathrm{B}(\mathrm{Lan}_j(\mathcal{O})(E^n)).$$

Since colimits commute with colimits, we conclude that $\phi^n_{\mathrm{Lan}_j(\mathcal{O})}$ is an equivalence as well. Therefore, Theorem 7.3.3.1 implies that $\mathrm{Lan}_j(\mathcal{O})$ is a $\mathcal{T}^{\mathrm{nc}}$-structure. $\square$

Theorem 7.3.3.4. *Suppose that the pregeometry $\mathcal{T}$ is discrete. Then the functor*

$$i\colon \mathrm{Str}_{\mathcal{T}}(\mathcal{S}) \longrightarrow \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{S})$$

is fully faithful and its essential image consists of those $\mathcal{T}^{\mathrm{nc}}$-structures $\mathcal{O}$ satisfying condition (3) in Theorem 7.3.3.1.

Proof. If $\mathcal{T}$ is discrete then $\mathrm{Str}_{\mathcal{T}}(\mathcal{S}) \simeq \mathrm{Fun}^{\times}(\mathcal{T}, \mathcal{S})$, i.e. $\mathrm{Str}_{\mathcal{T}}(\mathcal{S})$ coincides with the sifted completion $\mathcal{P}_\Sigma(\mathcal{T})$ of $\mathcal{T}$. In this case every $\mathcal{T}$-structure can be written as sifted colimit of $\mathcal{T}$-structures of the form A_X for $X \in \mathcal{T}$. In this case, Theorem 7.3.3.3 implies that the composition

$$\mathrm{Str}_{\mathcal{T}}(\mathcal{S}) \xhookrightarrow{\iota} \mathrm{Fun}(\mathcal{T}, \mathcal{S}) \xrightarrow{\mathrm{Lan}_j} \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}}, \mathcal{S})$$

factors through $\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{S})$. As both Lan_j and ι^{nc} are fully faithful, this immediately implies the full faithfulness of i.

This argument implies furthermore that for every $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}}(\mathcal{S})$, the $\mathcal{T}^{\mathrm{nc}}$-structure $i(\mathcal{O})$ satisfies condition (3) of Theorem 7.3.3.1. Suppose vice-versa that $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{S})$ is such that the morphism

$$\phi^n_{\mathcal{O}}\colon \left|{}^{\Delta}\mathrm{B}(\mathcal{O}(E^n))\right| \simeq \mathcal{O}(E^{n+1})$$

is an equivalence for every $n \geq 1$. As in an ∞-topos every groupoid object is effective, we conclude that ${}^{\Delta}\mathrm{B}(\mathcal{O}(E^n))$ can be identified with the Čech nerve of the map $\mathcal{O}(p_{n+1})\colon \mathcal{O}(*) \to \mathcal{O}(E^{n+1})$. In other words, we have

$$\check{\mathcal{C}}(\mathcal{O}(p_{n+1})) \simeq {}^{\Delta}\mathrm{B}(\mathcal{O}(E^n)) \simeq \mathcal{O}\left({}^{\Delta}\mathrm{B}(\mathcal{O}(E^n))\right) \simeq \mathcal{O}(\check{\mathcal{C}}(p_{n+1})).$$

Observe now that the adjunction $(i, \tau_{\geq 0})$ provides a counit map $\varepsilon_{\mathcal{O}}\colon i(\tau_{\geq 0}\mathcal{O}) \to \mathcal{O}$. As i is fully faithful, we see that this map induces an equivalence on the connective covers. As both functors commute with products, we see that it is enough to check that $\varepsilon_{\mathcal{O}}$ is an equivalence when evaluated on E^n for every $n \geq 1$. This is easily checked by induction on n, starting with $n = 0$ and using the fact that both $\mathcal{O}$ and $i(\tau_{\geq 0}\mathcal{O})$ commute with the geometric realization of the Čech nerves of the maps $p_{n+1}\colon * \to E^{n+1}$. $\square$

Corollary 7.3.3.5. *Suppose that the pregeometry $\mathcal{T}$ is discrete. Then for every ∞-topos $\mathcal{X}$ the functor $i\colon \mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \to \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$ is fully faithful.*

Proof. It is enough to prove that the composition

$$\mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \xrightarrow{\iota_{\mathcal{X}}} \mathrm{Fun}(\mathcal{T},\mathcal{X}) \xrightarrow{\mathrm{Lan}_j} \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}},\mathcal{X})$$

factors through $\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$.

Suppose first that $\mathcal{X} = \mathrm{PSh}(\mathcal{D})$ is the ∞-topos of presheaves on an ∞-category $\mathcal{D}$. For every $d \in \mathcal{D}$ let $\mathrm{ev}_d \colon \mathrm{PSh}(\mathcal{D}) \to \mathcal{S}$ be the evaluation functor. Since the family $\{\mathrm{ev}_d\}_{d\in\mathcal{D}}$ is conservative and each ev_d commutes with (finite) limits, we see that a functor $\mathcal{O} \in \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}}, \mathrm{PSh}(\mathcal{D}))$ is a $\mathcal{T}^{\mathrm{nc}}$-structure if and only if the composition $\mathrm{ev}_d \circ \mathcal{O}$ is a $\mathcal{T}^{\mathrm{nc}}$-structure in $\mathcal{S}$ for every $d \in \mathcal{D}$. Observe now that for every $d \in \mathcal{D}$ the diagram

$$\begin{tikzcd}
\mathrm{Str}_{\mathcal{T}}(\mathrm{PSh}(\mathcal{D})) \arrow[r, "\iota_{\mathrm{PSh}(\mathcal{D})}"] \arrow[d, "\mathrm{ev}_d"] & \mathrm{Fun}(\mathcal{T},\mathrm{PSh}(\mathcal{D})) \arrow[r, "\mathrm{Lan}_j"] \arrow[d, "\mathrm{ev}_d"] & \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}},\mathrm{PSh}(\mathcal{D})) \arrow[d, "\mathrm{ev}_d"] \\
\mathrm{Str}_{\mathcal{T}}(\mathcal{S}) \arrow[r, "\iota"] & \mathrm{Fun}(\mathcal{T},\mathcal{S}) \arrow[r, "\mathrm{Lan}_j"] & \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}},\mathcal{S})
\end{tikzcd}$$

commutes. Indeed, this is because the functors ev_d commute with arbitrary colimits (in particular, the ones computing the left Kan extension along j). As we saw in the proof of Theorem 7.3.3.4, the bottom row factors through $\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{S})$.

We now deal with the general case. Choose a presentation of $\mathcal{X}$ as left exact accessible localization of an ∞-category of presheaves, $y \colon \mathcal{X} \rightleftarrows \mathrm{PSh}(\mathcal{D}) \colon \lambda$. Let us write $\mathrm{Lan}_j^{\mathcal{X}}$ and $\mathrm{Lan}_j^{\mathcal{D}}$ to denote the following left Kan extension functors

$$\mathrm{Lan}_j^{\mathcal{X}} \colon \mathrm{Fun}(\mathcal{T},\mathcal{X}) \longrightarrow \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}},\mathcal{X}) \quad , \quad \mathrm{Lan}_j^{\mathcal{D}} \colon \mathrm{Fun}(\mathcal{T},\mathrm{PSh}(\mathcal{D})) \longrightarrow \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}},\mathrm{PSh}(\mathcal{D})).$$

Similarly, let us write λ_* and λ_*^{nc} to denote the following functors, given by composition with λ:

$$\lambda_* \colon \mathrm{Fun}(\mathcal{T},\mathrm{PSh}(\mathcal{D})) \longrightarrow \mathrm{Fun}(\mathcal{T},\mathcal{X}) \quad , \quad \lambda_*^{\mathrm{nc}} \colon \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}},\mathrm{PSh}(\mathcal{D})) \longrightarrow \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}},\mathcal{X}).$$

With these notations, the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{T},\mathrm{PSh}(\mathcal{D})) \arrow[r, "\mathrm{Lan}_j^{\mathcal{D}}"] \arrow[d, "\lambda_*"] & \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}},\mathrm{PSh}(\mathcal{D})) \arrow[d, "\lambda_*^{\mathrm{nc}}"] \\
\mathrm{Fun}(\mathcal{T},\mathcal{X}) \arrow[r, "\mathrm{Lan}_j^{\mathcal{X}}"] & \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}},\mathcal{X})
\end{tikzcd}$$

commutes. Let $y_* \colon \mathrm{Fun}(\mathcal{T},\mathcal{X}) \to \mathrm{Fun}(\mathcal{T},\mathrm{PSh}(\mathcal{D}))$ denote the functor given by composition with y. Then the commutativity of the previous diagram and the full faithfulness of y_* imply that

$$\mathrm{Lan}_j^{\mathcal{X}} \simeq \lambda_*^{\mathrm{nc}} \circ \mathrm{Lan}_j \circ y_*.$$

As the functor λ_*^{nc} commute with finite limits, it preserves $\mathcal{T}^{\mathrm{nc}}$-structures. It is therefore sufficient to prove that the composition

$$\mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \xrightarrow{\iota_{\mathcal{X}}} \mathrm{Fun}(\mathcal{T},\mathcal{X}) \xrightarrow{y_*} \mathrm{Fun}(\mathcal{T},\mathrm{PSh}(\mathcal{D})) \xrightarrow{\mathrm{Lan}_j^{\mathcal{D}}} \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}},\mathrm{PSh}(\mathcal{D}))$$

factors through $\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathrm{PSh}(\mathcal{D}))$. As the diagram

$$\begin{tikzcd}
\mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \arrow[r, "\iota_{\mathcal{X}}"] \arrow[d, "y_*"] & \mathrm{Fun}(\mathcal{T},\mathcal{X}) \arrow[d, "y_*"] \\
\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathrm{PSh}(\mathcal{D})) \arrow[r, "\iota_{\mathrm{PSh}(\mathcal{D})}"] & \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}},\mathrm{PSh}(\mathcal{D}))
\end{tikzcd}$$

commutes, we are reduced to the case where $\mathcal{X} = \mathrm{PSh}(\mathcal{D})$, that has already been dealt with previously. $\square$

In both Theorem 7.3.3.4 and Theorem 7.3.3.5 we used in an essential way the fact that the pregeometry $\mathcal{T}$ is discrete. Although this case is sufficient to deal with the algebraic setting and the $\mathbb{C}$-analytic setting, it is unfortunately too restrictive to deal with the rigid analytic setting, at least at the current state of development of derived rigid analytic geometry. The following theorem is a variation of Theorem 7.3.3.5 that removes the discreteness assumption on $\mathcal{T}$ at the cost of working only with local $\mathcal{T}$-structures. The proof is conceptually similar to the one of Theorem 7.3.3.4, but we replace the equivalence $\mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \simeq \mathcal{P}_\Sigma(\mathcal{T})$ with the existence of sifted resolution by elemmaentary structures, see [Lur11b, Proposition 2.11].

Corollary 7.3.3.6. *Let $\mathcal{X}$ be an ∞-topos with enough points. The connective cover functor*

$$\tau_{\geq 0} \colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$$

admits a fully faithful left adjoint.

Proof. Consider the functor

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{T}, \mathcal{X}) \xrightarrow{\mathrm{Lan}_j} \mathrm{Fun}(\mathcal{T}^{\mathrm{nc}}, \mathcal{X}).$$

It is enough to prove that this functor factors through $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$. This will provide a functor $i \colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$ and prove that it is fully faithful. Let therefore $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Since Lan_j is fully faithful, we already know that $\mathrm{Lan}_j(\mathcal{O})|_{\mathcal{T}}$ is a local $\mathcal{T}$-structure. It is therefore enough to prove that $\mathrm{Lan}_j(\mathcal{O})$ commutes with finite products and that for every $n \geq 1$ the canonical morphism

$$\mathrm{Lan}_j(\mathcal{O})(E^n) \to \Omega \mathrm{Lan}_j(\mathcal{O}(E^{n+1}))$$

is an equivalence. Using Theorem 7.3.3.1, we rather prove that for every $n \geq 1$ the morphism

$$\mathrm{Lan}_j(\mathcal{O})(*) \to \mathrm{Lan}_j(\mathcal{O})(E^n)$$

is an effective epimorphism.

Using [Lur11b, Proposition 2.11] we choose a Cartesian fibration $q \colon \mathcal{D} \to \mathcal{X}$ and a diagram $Q \colon \mathcal{D}^{\triangleright} \to \underline{\mathrm{Str}}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ such that:

(i) the fibers of q are essentially small sifted ∞-categories;

(ii) Q is a colimit diagram relative to $\mathcal{X}$;

(iii) for each $C \in \mathcal{D}$ the object $Q(C)$ is an elemmaentary $\mathcal{T}$-structure (see [Lur11b, Definition 2.6]);

(iv) the image of the cone point via Q is equivalent to $\mathcal{O}$.

Recall from [Lur11d, Proposition 3.3.1] that the inclusion $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \to \mathrm{Fun}(\mathcal{T}, \mathcal{X})$ commutes with sifted colimits. Therefore, reasoning as in Theorem 7.3.3.3, we see that it is enough to prove that if $\mathcal{O}$ is an elemmaentary $\mathcal{T}$-structure, then $\phi^n_{\mathrm{Lan}_j(\mathcal{O})}$ is an equivalence for every $n \geq 1$.

Let therefore $Y \in \mathcal{T}$ and let $f \colon (\mathcal{X}, \mathcal{O}_X) \to \mathrm{Spec}^{\mathcal{T}}(Y)$ be a fixed morphism. We have to prove that

$$\left| {}^{\Delta}\mathrm{B}\left(\mathrm{Lan}_j(f^{-1}(\mathcal{O}_Y))(E^n)\right) \right| \longrightarrow \mathrm{Lan}_j(f^{-1}(\mathcal{O}_Y))(E^{n+1})$$

is an equivalence. This can be checked on stalks. After passing at the stalk at a geometric point $p_* \colon \mathcal{S} \rightleftarrows \mathcal{X} \colon p^{-1}$, we are reduced to the situation where $\mathcal{X} = \mathcal{S}$. In this case, $f^{-1}\mathcal{O}_Y$ can be written as a sifted colimit of $\mathcal{T}$-structures of the form A_{Y_α}. As the functor Lan_j commutes with colimits, the conclusion follows from Theorem 7.3.3.2. $\square$

Corollary 7.3.3.7. *For any ∞-topos $\mathcal{X}$ with enough points, the commutative diagram*

$$\begin{tikzcd} \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \arrow[r, "\tau_{\geq 0}"] & \mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \\ \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \arrow[u, hook] \arrow[r, "\tau_{\geq 0}"] & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \arrow[u, hook] \end{tikzcd}$$

is left adjointable horizontally.

Proof. The functor $\tau_{\geq 0} \colon \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \to \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ commutes with limits and filtered colimits, and both categories are presentable. It follows that $\tau_{\geq 0}$ admits a left adjoint, that we denote i. Recall that [Lur11d, Proposition 3.3.1] proves that $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ is closed under sifted colimits in $\mathrm{Fun}(\mathcal{T}, \mathcal{X})$, and hence in $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$. Therefore, the same argument of Theorem 7.3.3.6 implies that the composition

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \hookrightarrow \mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \xrightarrow{i} \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$$

is fully faithful. In particular, it factors through $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$ and the resulting functor coincides with the left adjoint to $\tau_{\geq 0} \colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ we constructed in Theorem 7.3.3.6. The conclusion follows. $\square$

Proposition 7.3.3.8. *For any ∞-topos $\mathcal{X}$, the forgetful functor*

$$(\tau_{\geq 0}, U) \colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X}) \times \mathrm{Sp}(\mathcal{X})$$

is conservative. Furthermore, if $\mathcal{T}$ is discrete then the forgetful functor

$$(\tau_{\geq 0}, U) \colon \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \to \mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \times \mathrm{Sp}(\mathcal{X})$$

is conservative as well.

Proof. Let $\varphi \colon \mathcal{O} \to \mathcal{O}'$ be a morphism of nonconnective $\mathcal{T}$-structures. If $U(\varphi)$ is an equivalence, then unraveling the definition of the underlying spectrum we deduce that

$$\varphi_{E^n} \colon \mathcal{O}(E^n) \longrightarrow \mathcal{O}'(E^n)$$

is an equivalence. On the other hand, whenever $X \in \mathcal{T}$, Corollaries 7.3.3.5 and 7.3.3.6 imply that

$$\mathcal{O}(X) \simeq (\tau_{\geq 0}\mathcal{O})(X) \qquad \text{and} \qquad \mathcal{O}'(X) \simeq (\tau_{\geq 0}\mathcal{O}')(X).$$

Therefore $\varphi_X \colon \mathcal{O}(X) \to \mathcal{O}'(X)$ is an equivalence as well. The conclusion now follows from the fact that both $\mathcal{O}$ and $\mathcal{O}'$ commute with finite products. $\square$

7.3.4 Nonconnective structures in the algebraic case

In this section we focus on the special case where $\mathcal{T} = \mathcal{T}_{\mathrm{disc}}(k)$ and the nonconnective context is the one of Theorem 7.3.1.8(1). We can summarize the main results as follow:

(i) when k is a ring containing $\mathbb{Q}$, we provide a canonical equivalence between the ∞-category of nonconnective $\mathcal{T}_{\mathrm{disc}}(k)$-structures and the ∞-category of cdga's;

(ii) when k is a field of positive characteristic, we provide a fully faithful embedding of cohomologically connected cosimplicial algebras in the ∞-category of nonconnective $\mathcal{T}_{\text{ét}}(k)$-structures.

Let us start by assuming that k contains $\mathbb{Q}$. In this case there is a canonical model structure on cdga_k. We denote the underlying ∞-category by $\mathbf{cdga}_k$. We know that

$$\mathbf{cdga}_k^{\geq 0} \simeq \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{S}) \simeq \mathcal{P}_\Sigma(\mathcal{T}_{\mathrm{disc}}(k)),$$

and this equivalence is realized by sending a $\mathcal{T}_{\mathrm{disc}}(k)$-structure to its evaluation on $\mathbb{A}^1_k \in \mathcal{T}_{\mathrm{disc}}(k)$. In particular, we see that the inclusion

$$\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{S}) \hookrightarrow \mathrm{Fun}(\mathcal{T}_{\mathrm{disc}}(k), \mathcal{S})$$

commutes with sifted colimits. As a consequence, we can invoke Theorem 7.3.2.5 to deduce that the underlying spectrum

$$U \colon \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)^{\mathrm{nc}}}(\mathcal{S}) \to \mathrm{Sp}$$

commutes with sifted colimits. Furthermore, since the equivalence above is realized by evaluation on $\mathbb{A}^1_k \simeq E^0$, we deduce from Theorem 7.3.3.8 that the underlying spectrum functor U is also conservative. It follows that U is monadic.

We will now prove that $\mathbf{cdga}_k$ and $\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)^{\mathrm{nc}}}(\mathcal{S})$ are equivalent using the Barr-Beck-Lurie criterion for equivalences [Lur12c, 4.7.3.16]. We start by constructing a functor between these two categories. Given an unbounded cdga $A \in \mathbf{cdga}_k$, we define a functor

$$\phi(A) \colon \mathcal{T}_{\mathrm{disc}}(k)^{\mathrm{nc}} \to \mathcal{S}$$

by setting

$$\phi(A)(\mathrm{B}^n(\mathbb{G}_a)) \coloneqq \mathrm{Map}(\mathrm{Sym}_k(k[-n]), A).$$

Notice that $\phi(A)(\mathrm{B}^n(\mathbb{G}_a))$ is equivalent, as chain complex, to $\tau_{\geq 0}(A[n])$. This implies that we can identify $U(\phi(A))$ with the underlying spectrum of A. In other words, the diagram

$$\begin{tikzcd} \mathbf{cdga}_k \arrow[rr, "\phi"] \arrow[dr, "V"'] & & \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)^{\mathrm{nc}}}(\mathcal{S}) \arrow[dl, "U"] \\ & \mathbf{Mod}_k & \end{tikzcd} \tag{7.3.4.1}$$

commutes. Here V denotes the forgetful functor.

Theorem 7.3.4.1. *The functor ϕ is an equivalence.*

Proof. Recall from Theorem 7.3.2.5 that U commutes with limits and filtered colimits. As both $\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)^{\mathrm{nc}}}(\mathcal{S})$ and $\mathbf{Mod}_k$ are presentable, it follows that U admits a left adjoint, that we denote F.

In virtue of [Lur12c, 4.7.3.16] it is enough to prove that for every $M \in \mathbf{Mod}_k$ the canonical map

$$\phi(\mathrm{Sym}_k(M)) \longrightarrow F(M)$$

is an equivalence. Notice that the commutativity of (7.3.4.1) coupled with the fact that both U and V are conservative and commute with sifted colimits implies that ϕ commutes with sifted colimits as well. In particular, it is enough to prove the statement when $M = k[-n]$ for $n \geq 0$. In this case, we observe that $\mathrm{Sym}_k(k[-n])$ can be identified with the global sections of $\mathrm{B}^n(\mathbb{G}_a)$, while the Yoneda lemmama allows to identify $F(k[-n])$ with $\mathrm{Map}_{\mathcal{T}^{\mathrm{nc}}}(\mathrm{B}^n(\mathbb{G}_a), -)$. Now, the adjunction $\mathcal{O} \dashv \mathrm{Spec}$ introduced in [Toë06a] implies that

$$\phi(\mathrm{Sym}_k(k[-n])) \simeq \mathrm{Map}_{\mathcal{T}^{\mathrm{nc}}}(\mathrm{B}^n(\mathbb{G}_a), -),$$

thus completing the proof. □

7.3.5 Nonconnective cotangent complex

In this section we prove that the adjunction $(i, \tau_{\geq 0})$ of Section 7.3.3 induces an equivalence after stabilization. In particular, this allows to introduce nonconnective split square-zero extensions.

Fix a pregeometry $(\mathcal{T}, \mathrm{adm}, \tau)$ and a $\mathcal{T}$-nonconnective context $(\mathcal{C}, \mathbb{P}, E)$. Fix also an ∞-topos $\mathcal{X}$ with enough points. Then the connective cover functor

$$\tau_{\geq 0} \colon \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$$

commutes with limits and filtered colimits, and therefore it has a left adjoint, denoted i. Notice that Corollaries 7.3.3.6 and 7.3.3.7 imply that the functor i is fully faithful when restricted to $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Fix a local $\mathcal{T}$-structure $A \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Then $\tau_{\geq 0}(i(A)) \simeq A$ and we therefore obtain a well defined ∞-functor

$$\tau_{\geq 0} \colon \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/i(A)} \longrightarrow \mathrm{Str}_{\mathcal{T}}(\mathcal{X})_{/A}.$$

This functor still commutes with limits and filtered colimits, and therefore it admits a left adjoint, that we still denote i. Now, recall from [Por15b, Corollary 9.4] that $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/A}$ is a presentable ∞-category and that the functor

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/A} \longrightarrow \mathrm{Str}_{\mathcal{T}}(\mathcal{X})_{/A}$$

is fully faithful and admits a left adjoint. Furthermore, after passing to the stabilization we obtain an equivalence

$$\mathrm{Sp}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/A}) \simeq \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}}(\mathcal{X})_{/A}). \tag{7.3.5.1}$$

Notice that Corollaries 7.3.3.6 and 7.3.3.7 imply that the composition

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/A} \longrightarrow \mathrm{Str}_{\mathcal{T}}(\mathcal{X})_{/A} \longrightarrow \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/i(A)}$$

is fully faithful. Therefore, the chain rule for Goodwillie derivatives implies that the induced functor

$$\mathrm{Sp}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/A}) \longrightarrow \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/i(A)})$$

is fully faithful. Pairing this with the equivalence (7.3.5.1), we deduce that the functor

$$\partial(i) \colon \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}}(\mathcal{X})_{/A}) \longrightarrow \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/i(A)})$$

is fully faithful. We summarize this in the following:

Proposition 7.3.5.1. *Let $\mathcal{X}$ be an ∞-topos with enough points and let $A \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ be a connective local $\mathcal{T}$-structure. Then the functor*

$$\partial(i) \colon \mathrm{Sp}(\mathrm{Sp}(\mathrm{Str}_{\mathcal{T}}(\mathcal{X})_{/A})) \longrightarrow \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/i(A)})$$

is fully faithful. Furthermore, it has a right adjoint given by $\partial(\tau_{\geq 0})$.

Proof. The fact that $(\partial(i), \partial(\tau_{\geq 0})$ can be promoted to an adjoint pair is a standard consequence of Goodwillie's calculus. The full faithfulness of $\partial(i)$ follows from the above discussion. ☐

Remark 7.3.5.2. It seems likely that the above adjunction is actually an equivalence. However, for our purposes, full faithfulness is largely sufficient.

We now use Theorem 7.3.5.1 to develop the theory of nonconnective cotangent complex.

Definition 7.3.5.3. Let $\mathcal{X}$ be an ∞-topos and let $A \in \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$. We refer to the functor

$$\Omega^\infty \colon \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/A}) \longrightarrow \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/A}$$

as the $\mathcal{T}$-*theoretic nonconnective split square-zero extension functor* (or simply as the nonconnective split square-zero extension functor when $\mathcal{T}$ is clear from the context). Given $M \in \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/A})$ we set

$$A \oplus_{\mathcal{T}^{\mathrm{nc}}} M \coloneqq \Omega^\infty(M).$$

When $\mathcal{T}^{\mathrm{nc}}$ is clear from the context, we simply write $A \oplus M$ to denote this nonconnective $\mathcal{T}$-structure.

Definition 7.3.5.4. Let $\mathcal{X}$ be an ∞-topos and let $A \in \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$. The *functor of nonconnective $\mathcal{T}$-derivations* is by definition the functor

$$\mathrm{Der}_{\mathcal{T}^{\mathrm{nc}}}(A; -) \colon \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/A}) \longrightarrow \mathcal{S}$$

defined by

$$M \mapsto \mathrm{Map}_{/A}(A, A \oplus M),$$

the mapping space being computed in $\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/A}$.

Notice that the the functor $\mathrm{Der}_{\mathcal{T}^{\mathrm{nc}}}(A; -)$ commutes with limits and with κ-filtered colimits for κ a sufficiently big regular cardinal. The adjoint functor theorem implies therefore that it is corepresentable.

Definition 7.3.5.5. Let $\mathcal{X}$ be an ∞-topos and let $A \in \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})$. We denote the corepresentative of the functor $\mathrm{Der}_{\mathcal{T}^{\mathrm{nc}}}(A; -)$ by $\mathbb{L}^{\mathcal{T}^{\mathrm{nc}}}_A$ and we refer to it as the $\mathcal{T}^{\mathrm{nc}}$-*theoretic cotangent complex* (or as the nonconnective $\mathcal{T}$-theoretic cotangent complex).

When A is a $\mathcal{T}$-structure we therefore have at our disposal two cotangent complexes, $\mathbb{L}^{\mathcal{T}}_A$ (cf. [PY17b, Definition 5.4]) and $\mathbb{L}^{\mathcal{T}^{\mathrm{nc}}}_A$. Our next goal is to prove that these two objects are canonically equivalent.

Lemma 7.3.5.6. *Let $\mathcal{X}$ be an ∞-topos and let A be a $\mathcal{T}^{\mathrm{nc}}$-structure on $\mathcal{X}$. The connective cover functor $\tau_{\geq 0} \colon \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}) \to \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ induces a commutative diagram*

$$\begin{tikzcd} \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/A}) \arrow[r, "\partial(\tau_{\geq 0})"] \arrow[d, "\Omega^\infty"] & \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}}(\mathcal{X})_{/\tau_{\geq 0}A}) \arrow[d, "\Omega^\infty"] \\ \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/A} \arrow[r, "\tau_{\geq 0}"] & \mathrm{Str}_{\mathcal{T}}(\mathcal{X})_{/\tau_{\geq 0}A}. \end{tikzcd}$$

In other words,

$$\tau_{\geq 0}\left(A \oplus_{\mathcal{T}^{\mathrm{nc}}} M\right) \simeq \tau_{\geq 0}(A) \oplus_{\mathcal{T}} \partial\tau_{\geq 0}(M).$$

Proof. This simply follows from the fact that $\tau_{\geq 0}$, being a right adjoint, commutes with limits. ☐

Proposition 7.3.5.7. *Let $\mathcal{X}$ be an ∞-topos with enough points. Let $A \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$. Then there is a canonical equivalence*

$$\partial(i)(\mathbb{L}^{\mathcal{T}}_A) \simeq \mathbb{L}^{\mathcal{T}^{\mathrm{nc}}}_{i(A)}$$

in the ∞-category $\mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/i(A)})$. In particular, $\mathbb{L}^{\mathcal{T}^{\mathrm{nc}}}_{i(A)}$ belongs to the full subcategory $\mathrm{Sp}(\mathrm{Str}_{\mathcal{T}}(\mathcal{X})_{/A})$.

Proof. Let $M \in \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/i(A)})$. Then, using Theorem 7.3.5.6, we obtain

$$\begin{aligned} \mathrm{Map}_{\mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/i(A)})}(\partial(i)(\mathbb{L}^{\mathcal{T}}_A), M) &\simeq \mathrm{Map}_{\mathrm{Sp}(\mathrm{Str}_{\mathcal{T}}(\mathcal{X})_{/A})}(\mathbb{L}^{\mathcal{T}}_A, \partial(\tau_{\geq 0})(M)) \\ &\simeq \mathrm{Map}_{\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/A}}(A, A \oplus^{\mathcal{T}} \partial(\tau_{\geq 0})(M)) \\ &\simeq \mathrm{Map}_{\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/\tau_{\geq 0}(i(A))}}(A, \tau_{\geq 0}(i(A) \oplus^{\mathcal{T}^{\mathrm{nc}}} M)) \\ &\simeq \mathrm{Map}_{\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/A}}(i(A), i(A) \oplus^{\mathcal{T}^{\mathrm{nc}}} M) \\ &\simeq \mathrm{Map}_{\mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X})_{/i(A)})}(\mathbb{L}^{\mathcal{T}^{\mathrm{nc}}}_A, M). \end{aligned}$$

We can therefore deduce from the Yoneda lemmama that $\partial(i)(\mathbb{L}^{\mathcal{T}}_A) \simeq \mathbb{L}^{\mathcal{T}^{\mathrm{nc}}}_A$. The second statement follows from the full faithfulness of $\partial(i)$. ☐

Remark 7.3.5.8. When $\mathcal{T} = \mathcal{T}_{\mathrm{disc}}(k)$ and k contains $\mathbb{Q}$, then we can identify $\mathrm{Sp}(\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{S})_{/A})$ with the category of unbounded A-modules. If $M \in A\text{-}\mathrm{Mod}$, then the underlying module of $A \oplus_{\mathcal{T}_{\mathrm{disc}}(k)^{\mathrm{nc}}} M$ really coincides with $A \oplus M$. On the other hand, Theorem 7.3.5.6 implies that the underlying module of $A \oplus_{\mathcal{T}_{\mathrm{disc}}(k)} M$ coincides with $A \oplus \tau_{\geq 0}(M)$.

7.3.6 Change of spectrum

The leitmotiv of this paper is to provide an axiomatic context where to formulate and prove the HKR theorem. In later sections we will provide four different contexts where our formalism applies. Given the abundance of such HKR theorems, a very natural question is to compare them whenever the question makes sense. In order to carry out such a task, we need to introduce a suitable notion of transformation of nonconnective contexts. Furthermore, the ideas introduced in this context will prove fundamental to prove certain cases of the analytic HKR (namely, when the residue field has positive characteristic).

We start by analyzing a simple situation, where we keep the pregeometry $\mathcal{T}$ and the $\mathcal{T}$-geometric context $\mathcal{C}$ fixed, but we change the spectrum E. Consider therefore two spectra $E_0, E_1 \in \mathrm{Sp}(\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}))$ satisfying the assumptions of Theorem 7.3.1.6. We denote by $\mathcal{T}^{\mathrm{nc}}_i$ the nonconnective pregeometry generated by E_i for $i = 0, 1$. Define $\mathcal{T}^{\mathrm{nc}}_{01}$ to be the smallest full subcategory of $\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})$ closed under products and containing E^n_i for $i = 0, 1$ and $n \geq 0$. Notice that we have fully faithful embeddings $j_i \colon \mathcal{T}^{\mathrm{nc}}_i \to \mathcal{T}^{\mathrm{nc}}_{01}$ for $i = 0, 1$. We define a $\mathcal{T}^{\mathrm{nc}}_{01}$-structure in an ∞-topos $\mathcal{X}$ to be a functor

$$\mathcal{O} \colon \mathcal{T}^{\mathrm{nc}}_{01} \to \mathcal{X}$$

which commutes with products and whose restrictions to $\mathcal{T}^{\mathrm{nc}}_i$ are $\mathcal{T}^{\mathrm{nc}}_i$-structures in the sense of Theorem 7.3.1.11.

By definition, we obtain forgetful functors

$$\rho_i \colon \mathrm{Str}_{\mathcal{T}_{01}^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_i^{\mathrm{nc}}}(\mathcal{X}),$$

for $i = 0, 1$. These functors commute with limits and filtered colimits and therefore the adjoint functor theorem guarantees the existence of left adjoints

$$\lambda_i \colon \mathrm{Str}_{\mathcal{T}_i^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_{01}^{\mathrm{nc}}}(\mathcal{X}).$$

As a consequence, we obtain two functors

$$\gamma_{01} \coloneqq \rho_1 \circ \lambda_0 \colon \mathrm{Str}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X}), \qquad \gamma_{10} \coloneqq \rho_0 \circ \lambda_1 \colon \mathrm{Str}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X}),$$

that allow to change spectrum.

Definition 7.3.6.1. Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry and let $(\mathcal{C}, \tau, \mathbb{P})$ be a $\mathbb{T}$-geometric context. Let $E_0, E_1 \in \mathrm{Sp}(\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}))$ be such that $(\mathcal{C}, \mathbb{P}, E_i)$ is a $\mathcal{T}$-nonconnective context, for $i = 0, 1$. The *change of spectrum functors* are the pair of functors γ_{01} and γ_{10} introduced above.

As in the case of a single spectrum, the restriction to $\mathcal{T}$ inside $\mathcal{T}_{01}^{\mathrm{nc}}$ allows to associate to each $\mathcal{T}_{01}^{\mathrm{nc}}$-structure a $\mathcal{T}$-structure. We denote this functor once again by

$$\tau_{\geq 0} \colon \mathrm{Str}_{\mathcal{T}_{01}^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}}(\mathcal{X}).$$

This functor admits a left adjoint, given by left Kan extension along $\mathcal{T} \hookrightarrow \mathcal{T}_{01}^{\mathrm{nc}}$. This follows from the same argument given in **??**.

Proposition 7.3.6.2. *For $i = 0, 1$ the diagram*

$$\begin{tikzcd} \mathrm{Str}_{\mathcal{T}_{01}^{\mathrm{nc}}}(\mathcal{X}) \arrow[rr, "\rho_i"] \arrow[dr, "\tau_{\geq 0}"'] & & \mathrm{Str}_{\mathcal{T}_i^{\mathrm{nc}}}(\mathcal{X}) \arrow[dl, "\tau_{\geq 0}"] \\ & \mathrm{Str}_{\mathcal{T}}(\mathcal{X}) & \end{tikzcd}$$

commutes and it is left adjointable.

Proof. The commutativity of the diagram follows just from unraveling the definitions. For the left adjointability, we remark that for $X \in \mathcal{T}_i^{\mathrm{nc}}$ one has an equivalence of comma categories

$$\mathcal{T}^{\triangleright} \times_{\mathcal{T}_{01}^{\mathrm{nc}}} \{X\} \simeq \mathcal{T}^{\triangleright} \times_{\mathcal{T}_i^{\mathrm{nc}}} \{X\}$$

induced by full faithfulness of both functors in the composition $\mathcal{T} \hookrightarrow \mathcal{T}_i^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{01}^{\mathrm{nc}}$. The conclusion now follows because the adjoint to $\tau_{\geq 0}$ is computed in both cases by a plain left Kan extension. ☐

It is a more subtle question to understand whether the same properties hold for the functors λ_i.

Proposition 7.3.6.3. *For $i = 0, 1$ the diagram*

$$\begin{tikzcd} \mathrm{Str}_{\mathcal{T}_i^{\mathrm{nc}}}(\mathcal{X}) \arrow[rr, "\lambda_i"] & & \mathrm{Str}_{\mathcal{T}_{01}^{\mathrm{nc}}}(\mathcal{X}) \\ & \mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \arrow[ul, "i"] \arrow[ur, "i"'] & \end{tikzcd}$$

commutes. Suppose furthermore that the left Kan extension functor along $\mathcal{T}_i^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{01}^{\mathrm{nc}}$ takes $\mathcal{T}_i^{\mathrm{nc}}$-structures to $\mathcal{T}_{01}^{\mathrm{nc}}$-structures. Then the above diagram is right adjointable.

Remark 7.3.6.4. The condition on $\mathcal{T}_i^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{01}^{\mathrm{nc}}$ is typically asymmetric. This means that it will often be satisfied for only one of the two inclusions $\mathcal{T}_0^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{01}^{\mathrm{nc}}$, $\mathcal{T}_1^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{01}^{\mathrm{nc}}$.

Proof. The first statement follows immediately from Theorem 7.3.6.2 by passing to left adjoints. The second statement follows from the fact that under the assumption on $\mathcal{T}_i^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{01}^{\mathrm{nc}}$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Str}_{\mathcal{T}_i^{\mathrm{nc}}}(\mathcal{X}) \arrow[r, "\lambda_i"] \arrow[d, hook] & \mathrm{Str}_{\mathcal{T}_{01}^{\mathrm{nc}}}(\mathcal{X}) \arrow[d, hook] \\ \mathrm{Fun}(\mathcal{T}_i^{\mathrm{nc}}, \mathcal{X}) \arrow[r, "\mathrm{Lan}"] & \mathrm{Fun}(\mathcal{T}_{01}^{\mathrm{nc}}, \mathcal{X}). \end{tikzcd}$$

In other words, we can identify the functor λ_i with the left Kan extension. As a consequence, we have

$$\tau_{\geq 0} \circ \lambda_i \simeq \tau_{\geq 0} \circ \rho_i \circ \lambda_i \simeq \tau_{\geq 0},$$

because the left Kan extension is fully faithful. $\square$

Corollary 7.3.6.5. *Suppose that left Kan extension along $\mathcal{T}_0^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{01}^{\mathrm{nc}}$ takes $\mathcal{T}_0^{\mathrm{nc}}$-structures to $\mathcal{T}_{01}^{\mathrm{nc}}$-structures. Then the diagram*

$$\begin{tikzcd} \mathrm{Str}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X}) \arrow[rr, "\gamma_{01}"] \arrow[dr, "\tau_{\geq 0}"'] & & \mathrm{Str}_{\mathcal{T}_1^{\mathrm{nc}}} \arrow[dl, "\tau_{\geq 0}"] \\ & \mathrm{Str}_{\mathcal{T}}(\mathcal{X}) & \end{tikzcd}$$

is commutative.

Proof. As $\gamma_{01} = \rho_1 \circ \lambda_0$, this is a direct consequence of Propositions 7.3.6.2 and 7.3.6.3. $\square$

In virtue of the above results, it is useful to have a more geometrical condition only involving the spectra E_0 and E_1 implying that left Kan extension along $\mathcal{T}_i^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{01}^{\mathrm{nc}}$ takes $\mathcal{T}_i^{\mathrm{nc}}$-structures to $\mathcal{T}_{01}^{\mathrm{nc}}$-structures.

Definition 7.3.6.6. Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry and let $(\mathcal{C}, \tau, \mathbb{P})$ be a $\mathcal{T}$-geometric context. We say that $(\mathcal{C}, \tau, \mathbb{P})$ is *ordinary* if the topology τ is quasi-compact and there exists a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ satisfying the following conditions:

(i) $\mathcal{C}_0$ is a 1-category with finite limits;

(ii) the topology τ and the class of morphisms $\mathbb{P}$ restrict to $\mathcal{C}_0$ and make $(\mathcal{C}_0, \tau, \mathbb{P})$ into a geometric context;

(iii) $\mathcal{T}$ is contained in $\mathcal{C}_0$.

We say that a nonconnective pregeometry $\mathcal{T}^{\mathrm{nc}}$ is *ordinary* if the underlying $\mathcal{T}$-geometric context is ordinary.

Example 7.3.6.7. All the contexts introduced so far are ordinary. The choice of $\mathcal{C}_0$ is in each case canonical and it corresponds to the full subcategory spanned by discrete objects. For instance, when $\mathcal{T} = \mathcal{T}_{\text{ét}}(k)$, $\mathcal{C}_0$ is the category of classical affine schemes. When $\mathcal{T} = \mathcal{T}_{\mathrm{an}}(k)$, $\mathcal{C}_0$ is the category of classical k-analytic spaces.

In virtue of the above example, whenever $(\mathcal{C}, \tau, \mathbb{P})$ is an ordinary $\mathcal{T}$-geometric context, we refer to objects in $\mathcal{C}_0$ as the *discrete objects*. Furthermore, we refer to the stacks in $\mathrm{Geom}(\mathcal{C}_0, \tau, \mathbb{P})$ as the *geometric stacks*.

When the $\mathcal{T}$-nonconnective context is ordinary we have an extra amount of control on the spectrum E, as the following couple of result shows.

Lemma 7.3.6.8. *Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry and let $(\mathcal{C}, \tau, \mathbb{P})$ be an ordinary $\mathcal{T}$-geometric context. Then the restriction $E^n|_{\mathcal{C}_0}$ is n-truncated.*

Proof. The same proof of [TV08b, lemmama 2.1.1.2] applies. $\square$

Lemma 7.3.6.9. *Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry and let $(\mathcal{C}, \tau, \mathbb{P})$ be an ordinary $\mathcal{T}$-geometric context. Let $F \colon I \to \mathrm{Geom}(\mathcal{C}_0, \tau, \mathbb{P})$ be a filtered diagram of discrete geometric stacks. Let*

$$F_\infty := \operatorname*{colim}_I F \in \mathrm{PSh}(\mathcal{C}_0)$$

be the colimit of F computed in $\mathrm{PSh}(\mathcal{C}_0)$. Suppose that there exists an $n \geq 0$ such that for each $\alpha \in I$, the geometric stack $F_\alpha := F(\alpha)$ is n-truncated. Then F_∞ satisfies τ-hyperdescent.

Proof. First we remark that, since colimits in $\mathrm{PSh}(\mathcal{C}_0)$ are computed objectwise and since n-truncated objects are stable under filtered colimits, the presheaf F_∞ is n-truncated.

Let now $X \in \mathcal{C}_0$ be an object and let $U^\bullet$ be a τ-hypercover of X. Since the topology τ is quasi-compact, we can suppose that for every $[m] \in \mathbf{\Delta}$, U^m is disjoint union of finitely many objects in $\mathcal{C}_0$. We have to prove that the canonical map

$$F_\infty(X) \longrightarrow \varprojlim_{\mathbf{\Delta}} F_\infty(U^\bullet)$$

is an equivalence. Since $F(X)$ and $F_\infty(U^m)$ are n-truncated for every $[m] \in \mathbf{\Delta}$ and since $\mathcal{S}^{\leq m}$ is closed under limits in $\mathcal{S}$, we see that we can compute the above limit in $\mathcal{S}^{\leq n}$. Since $\mathcal{S}^{\leq n}$ is an n-category and $\mathbf{\Delta}_{\leq n+2} \hookrightarrow \mathbf{\Delta}$ is n-cofinal, we see that there is a canonical equivalence

$$\varprojlim_{\mathbf{\Delta}} F_\infty(U^\bullet) \simeq \varprojlim_{\mathbf{\Delta}_{\leq n+2}} F_\infty(U^\bullet).$$

It is therefore enough to prove that the canonical map from $F(X)$ to the right hand side is an equivalence. Notice that, since each U^m is a finite disjoint union of objects in $\mathcal{C}_0$, the limit on the right is a finite. Since filtered colimits commute with finite limits, we have a canonical equivalence

$$\varprojlim_{\mathbf{\Delta}_{\leq n+2}} F_\infty(U^\bullet) \simeq \operatorname*{colim}_{\alpha \in I} \varprojlim_{\mathbf{\Delta}_{\leq n+2}} F_\alpha(U^\bullet).$$

Since each F_α is n-truncated, we can use once more the n-cofinality of the inclusion $\mathbf{\Delta}_{\leq n+2} \hookrightarrow \mathbf{\Delta}$ to deduce that the canonical map

$$F_\alpha(X) \longrightarrow \varprojlim_{\mathbf{\Delta}_{\leq n+2}} F_\alpha(U^\bullet)$$

is an equivalence. The conclusion follows. $\square$

This is provided by our next result:

Proposition 7.3.6.10. *Let $(\mathcal{T}, \mathrm{adm}, \tau)$ be a pregeometry and let $(\mathcal{C}, \tau, \mathbb{P})$ be an ordinary $\mathcal{T}$-geometric context. Let $E_0, E_1 \in \mathrm{Sp}(\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}))$ be such that $(\mathcal{C}, \mathbb{P}, E_i)$ is a $\mathcal{T}$-nonconnective context for $i = 0, 1$.*

Suppose furthermore that there is an endomorphism

$$a \colon E_0^0 \longrightarrow E_0^0$$

in $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}))$ and an equivalence

$$E_1^0 \simeq \operatorname*{colim}_{\mathbb{N}} \left(E_0^0 \xrightarrow{a} E_0^0 \xrightarrow{a} E_0^0 \xrightarrow{a} \cdots \right)$$

in $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}))$. Then the left Kan extension along $\mathcal{T}_0^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{01}^{\mathrm{nc}}$ takes $\mathcal{T}_0^{\mathrm{nc}}$-structures to $\mathcal{T}_{01}^{\mathrm{nc}}$-structures.

Proof. Let us simply write Lan for the functor

$$\mathrm{Fun}(\mathcal{T}_0^{\mathrm{nc}}, \mathcal{X}) \longrightarrow \mathrm{Fun}(\mathcal{T}_{01}^{\mathrm{nc}}, \mathcal{X})$$

given by left Kan extension along $\mathcal{T}_0^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{01}^{\mathrm{nc}}$.

Consider the following claim:

Claim. For every $n \geq 0$ the canonical map

$$\mathrm{Lan}(\mathcal{O})(E_1^n) \longrightarrow \operatorname*{colim}_{\mathbb{N}} \left(\mathcal{O}(E_0^n) \xrightarrow{\mathcal{O}(a)} \mathcal{O}(E_0^n) \xrightarrow{\mathcal{O}(a)} \mathcal{O}(E_0^n) \xrightarrow{\mathcal{O}(a)} \cdots \right)$$

is an equivalence in $\mathcal{X}$.

Assume this claim. Then, as filtered colimits commute with finite limits, we deduce immediately that

$$\mathrm{Lan}(\mathcal{O})(E_1^n) \simeq \Omega \mathrm{Lan}(\mathcal{O})(E_1^{n+1}).$$

We are therefore left to check that $\operatorname{Lan}(\mathcal{O})$ commutes with finite products. This follows from an analysis case by case as in the proof of **??**.

It is therefore enough to prove the claim. We first observe that since a is a morphism of $\mathbb{E}_\infty$-groups, the ∞-categorical version of May's theorem implies that it induces an endomorphism

$$a_n \colon E_0^n \longrightarrow E_0^n$$

for every $n \geq 0$ (when $n = 0$, $a_0 = a$). Moreover, since the delooping functor

$$\mathrm{B}^n \colon \mathrm{Mon}_{\mathbb{E}_\infty}^{\mathrm{gp}}(\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})) \longrightarrow \mathrm{Mon}_{\mathbb{E}_\infty}^{\mathrm{gp}}(\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})^{\geq n})$$

is a left adjoint, it commutes with filtered colimits. As a consequence, we obtain canonical equivalences in $\mathrm{Mon}_{\mathbb{E}_\infty}^{\mathrm{gp}}(\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}))$

$$E_1^n \simeq \operatorname*{colim}_{\mathbb{N}} \left(E_0^n \xrightarrow{a_n} E_0^n \xrightarrow{a_n} E_0^n \xrightarrow{a_n} \cdots \right)$$

for every $n \geq 0$. Let now $X \in \mathcal{T}_0^{\mathrm{nc}}$ be any object. We claim that the canonical map

$$\mathrm{Map}_{\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})}(X, E_1^n) \longrightarrow \operatorname*{colim}_{\mathbb{N}} \mathrm{Map}_{\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})}(X, E_0^n)$$

is an equivalence. When X belongs to $\mathcal{T}$, this is a direct consequence of Theorem 7.3.6.9. In the general case, we can find an hypercover $U^\bullet$ of X such that each $U^\bullet$ is a disjoint union of finitely many objects in $\mathcal{T}$. In particular, we have equivalences

$$\mathrm{Map}_{\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})}(X, E_i^n) \simeq \lim_{\mathbf{\Delta}} \mathrm{Map}_{\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})}(U^\bullet, E_i^n),$$

for $i = 0, 1$. Since E_i^n is n-truncated in virtue of Theorem 7.3.6.8, we see that the above is a limit in $\mathcal{S}^{\leq n}$. As $\mathcal{S}^{\leq n}$ is an n-category and $\mathbf{\Delta}_{\leq n+2} \hookrightarrow \mathbf{\Delta}$ is n-cofinal, we conclude that we also have an equivalence

$$\mathrm{Map}_{\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})}(X, E_i^n) \simeq \lim_{\mathbf{\Delta}_{\leq n+2}} \mathrm{Map}_{\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P})}(U^\bullet, E_i^n).$$

The conclusion now follows from the fact that each U^m is a finite disjoint union of finitely many objects in $\mathcal{T}$, from Theorem 7.3.6.9 and from the fact that filtered colimits commute with finite limits.

We now consider the comma category $(\mathcal{T}_0^{\mathrm{nc}})_{/E_1^n}$. The above argument implies that the diagram

$$\mathbb{N} \longrightarrow (\mathcal{T}_0^{\mathrm{nc}})_{/E_1^n}$$

corresponding to the iteration of the morphism $a_n \colon E_0^n \to E_0^n$ is cofinal. Therefore, the claim follows from the explicit formula for left Kan extensions. □

Corollary 7.3.6.11. *Under the same assumptions of Theorem 7.3.6.10 and for any $\mathcal{T}_0^{\mathrm{nc}}$-structure A, the diagram*

$$\begin{tikzcd}
\mathrm{Sp}(\mathrm{Str}_{\mathcal{T}_{01}^{\mathrm{nc}}}(\mathcal{X})_{/\lambda_i(A)}) \arrow[r, "\partial(\rho_i)"] \arrow[d, "\Omega^\infty"] & \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}_i^{\mathrm{nc}}}(\mathcal{X})_{/A}) \arrow[d, "\Omega^\infty"] \\
\mathrm{Str}_{\mathcal{T}_{01}^{\mathrm{nc}}}(\mathcal{X})_{/\lambda_i(A)} \arrow[r, "\rho_i"] & \mathrm{Str}_{\mathcal{T}_i^{\mathrm{nc}}}(\mathcal{X})_{/A}
\end{tikzcd}$$

commutes and it is left adjointable. In particular, the functor $\gamma_{01} \colon \mathrm{Str}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X}) \to \mathrm{Str}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X})$ commutes with the formation of split square-zero extensions.

Proof. The commutativity of the diagram simply follows from the fact that the functor ρ_i commutes with limits. For the left adjointability, we observe that the proof of Theorem 7.3.6.3 implies that the functor Lan commutes with finite limits, because it can be computed by filtered colimits. The conclusion follows. □

7.3.7 Change of context

We now turn to the general case, where we do not keep underlying pregeometry fixed.

Definition 7.3.7.1. Let $\mathcal{T}_0^{\mathrm{nc}}$ and $\mathcal{T}_1^{\mathrm{nc}}$ be two nonconnective pregeometries. A *transformation of nonconnective pregeometries from* $\mathcal{T}_0^{\mathrm{nc}}$ *to* $\mathcal{T}_1^{\mathrm{nc}}$ is the given of a morphism of geometric contexts $\varphi\colon (\mathcal{C}_0, \tau_0, \mathbb{P}_0) \to (\mathcal{C}_1, \tau_1, \mathbb{P}_1)$ satisfying the following two conditions:

(i) the morphism of geometric contexts φ restricts to a transformation of pregeometries $\mathcal{T}_0 \to \mathcal{T}_1$;

(ii) let $\mathcal{T}_2$ be the $\mathcal{T}_1$-nonconnective context determined by $(\mathcal{C}_1, \tau_1, \mathbb{P}_1)$ and the spectrum object $\varphi_s(E_0)$. Then the left Kan extension along $\mathcal{T}_2^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{12}^{\mathrm{nc}}$ takes $\mathcal{T}_2^{\mathrm{nc}}$-structures to $\mathcal{T}_{12}^{\mathrm{nc}}$-structures.

We denote a transformation of pregeometries by $\varphi\colon (\mathcal{C}_0, \mathbb{P}_0, E_0) \to (\mathcal{C}_1, \mathbb{P}_1, E_1)$.

Definition 7.3.7.2. A transformation of nonconnective pregeometries $\varphi\colon (\mathcal{C}_0, \mathbb{P}_0, E_0) \to (\mathcal{C}_1, \mathbb{P}_1, E_1)$ is said to be a *change of spectrum* if the underlying transformation of geometric contexts is an equivalence.

Definition 7.3.7.3. A transformation of nonconnective pregeometries $\varphi\colon (\mathcal{C}_0, \mathbb{P}_0, E_0) \to (\mathcal{C}_1, \mathbb{P}_1, E_1)$ is said to be *strong* if there is an equivalence

$$\varphi_s(E_0) \simeq E_1$$

of spectra in $\mathrm{Geom}(\mathcal{C}_1, \tau_1, \mathbb{P}_1)$.

Notice that if φ is a strong transformation of nonconnective pregeometries it induces a well defined functor $\varphi\colon \mathcal{T}_0^{\mathrm{nc}} \to \mathcal{T}_1^{\mathrm{nc}}$. In this paper we abusively identify the strong transformation with the functor $\varphi\colon \mathcal{T}_0^{\mathrm{nc}} \to \mathcal{T}_1^{\mathrm{nc}}$.

It follows from the definitions that we can always factor a transformation of nonconnective pregeometries as a strong transformation followed by a change of spectrum. We already performed an in-depth analysis of the change of spectrum situation in the previous section. Therefore, we focus now on strong transformations.

Let $\varphi\colon \mathcal{T}_0^{\mathrm{nc}} \to \mathcal{T}_1^{\mathrm{nc}}$ be a strong transformation of pregeometries. For any ∞-topos $\mathcal{X}$, precomposition with φ induces a well defined functor

$$\varphi_*\colon \mathrm{Str}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X}).$$

This functor commutes with limits and filtered colimits. In particular, it admits a left adjoint that we denote φ^*.

Proposition 7.3.7.4. *Let $\varphi\colon \mathcal{T}_0^{\mathrm{nc}} \to \mathcal{T}_1^{\mathrm{nc}}$ be a strong transformation of pregeometries. For any ∞-topos $\mathcal{X}$ and any $\mathcal{T}_1^{\mathrm{nc}}$-structure $A \in \mathrm{Str}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X})$ the diagram*

$$\begin{tikzcd}
\mathrm{Sp}(\mathrm{Str}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X})_{/A}) \arrow[r, "\partial(\varphi_*)"] \arrow[d, "\Omega_1^\infty"] & \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X})_{/\varphi_*(A)}) \arrow[d, "\Omega_0^\infty"] \\
\mathrm{Str}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X})_{/A} \arrow[r, "\varphi_*"] & \mathrm{Str}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X})_{/\varphi_*(A)}
\end{tikzcd}$$

commutes.

Proof. This is an immediate consequence of the fact that φ_* commutes with (not necessarily) finite limits. ☐

Corollary 7.3.7.5. *Let $\varphi\colon \mathcal{T}_0^{\mathrm{nc}} \to \mathcal{T}_1^{\mathrm{nc}}$ be a strong transformation of pregeometries. For any ∞-topos $\mathcal{X}$, any $\mathcal{T}_1^{\mathrm{nc}}$-structure $A \in \mathrm{Str}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X})$ and any $M \in \mathrm{Sp}(\mathrm{Str}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X})_{/\varphi_*(A)})$ there is a canonical morphism of $\mathcal{T}_1^{\mathrm{nc}}$-structures on $\mathcal{X}$*

$$\varphi^*(\Omega_0^\infty(M)) \longrightarrow \Omega_1^\infty(\partial(\varphi^*)(M)).$$

Here $\partial(\varphi^*)$ denotes the Goodwillie derivative of φ^*.

Proof. This follows immediately from the fact that $\partial(\varphi^*)$ is left adjoint to $\partial(\varphi_*)$ and from the commutativity of the diagram asserted in the previous proposition. Indeed, the canonical map is simply the Beck-Chevalley transformation. ☐

7.3.8 Morita equivalences

Fix a pregeometry $(\mathcal{T}, \mathrm{adm}, \tau)$ and an ∞-topos $\mathcal{X}$. We are often more interested in manipulating the ∞-category $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ than the ∞-category $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$. The reason $\mathcal{T}$-structures are only a tool needed to set up the theory of $\mathcal{T}$-structured spaces and, ultimately of $\mathcal{T}$-schemes.

Unfortunately, the ∞-category $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ is typically not presentable. This can be fixed as follows: for any $A \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$, the ∞-category $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})_{/A}$ becomes presentable (see [Por15b, Corollary 9.4]). This fact has been used in an extensive way to obtain several of the main results of [PY17b].

On the other hand, $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ is much more flexible of $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$. Indeed there are typically several different pregeometries giving rise to the same ∞-categories of local structures. This has been studied in [Lur11d, §3.2].

The same picture applies when dealing with nonconnective pregeometries. In the nonconnective setting the Morita equivalence plays an even greater role. To understand the reason consider the following example:

Example 7.3.8.1. Let k be a nonarchimedean field equipped with a non-trivial valuation. Let $\mathcal{T}_{\mathrm{an}}(k)$ be the pregeometry introduced in [PY16b]. We recall that the objects of $\mathcal{T}_{\mathrm{an}}(k)$ are smooth k-analytic spaces. Choose $\mathcal{C} := \mathrm{dAn}_k$, the ∞-category of derived k-analytic spaces. The results of [PY17b, §5.6] imply that $(\mathcal{C}, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})$ is a $\mathcal{T}_{\mathrm{an}}(k)$-geometric context. The sequence $\{\mathrm{B}^n(\mathrm{B}G_a)\}_{n\geq 0}$ defines a spectrum object E in $\mathrm{Geom}(\mathcal{C}, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})$, and $(\mathcal{C}, \mathbb{P}_{\mathrm{sm}}, E)$ becomes a pre-$\mathcal{T}_{\mathrm{an}}(k)$-nonconnective context. Nevertheless, this is not a $\mathcal{T}_{\mathrm{an}}(k)$-nonconnective context, because it is not true that for every $X \in \mathcal{T}_{\mathrm{an}}(k)$ one has $\mathrm{H}^1(\mathcal{O}_X) = 0$ (take for example $X = \mathbb{P}^1_k$). In this situation, **??** is no longer valid.[3]

In order to fix this result one can change the starting pregeometry $\mathcal{T}_{\mathrm{an}}(k)$ as follows: we define $\mathcal{T}_{\mathrm{an}}(k)'$ to be the category of smooth k-*affinoid* spaces. The same choices of admissible morphisms and of the topology for $\mathcal{T}_{\mathrm{an}}(k)$ endow $\mathcal{T}_{\mathrm{an}}(k)'$ with a the structure of a pregeometry. Furthermore, the inclusion $\mathcal{T}_{\mathrm{an}}(k)' \to \mathcal{T}_{\mathrm{an}}(k)$ is a Morita equivalence, and $(\mathrm{dAn}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})$ is a $\mathcal{T}_{\mathrm{an}}(k)'$-geometric context. Therefore, $(\mathrm{dAn}_k, \mathbb{P}_{\mathrm{sm}}, E)$ is a pre-$\mathcal{T}_{\mathrm{an}}(k)'$-nonconnective context. The difference with before is that now Tate's acyclicity theorem implies that $(\mathrm{dAn}_k, \mathbb{P}_{\mathrm{sm}}, E)$ is actually a $\mathcal{T}_{\mathrm{an}}(k)'$-nonconnective context.

The conclusion is that although the results we developed in the previous sections might not apply to $\mathcal{T}_{\mathrm{an}}(k)$-structures, they will apply for $\mathcal{T}_{\mathrm{an}}(k)'$-structures, and thanks to Morita equivalence, they will also apply to $\mathcal{T}_{\mathrm{an}}(k)$-*local* structures.[4]

Motivated by the previous example, we introduce the notion of Morita equivalence.

Definition 7.3.8.2. Let $(\mathcal{T}_0, \mathrm{adm}_0, \tau_0)$ and $(\mathcal{T}_1, \mathrm{adm}_1, \tau_1)$ be two pregeometries. For $i = 0, 1$, let $(\mathcal{C}_i, \mathbb{P}_i, E_i)$ be a pre-$\mathcal{T}_i$-nonconnective context. A *Morita equivalence of pre-nonconnective contexts from* $(\mathcal{C}_0, \mathbb{P}_0, E_0)$ *to* $(\mathcal{C}_1, \mathbb{P}_1, E_1)$ is the given of a Morita equivalence of geometric contexts

$$\varphi\colon (\mathcal{C}_0, \tau_0, \mathbb{P}_0) \to (\mathcal{C}_1, \tau_1, \mathbb{P}_1)$$

with the following properties:

(i) it restricts to a Morita equivalence of pregeometries $\varphi\colon \mathcal{T}_0 \to \mathcal{T}_1$;

(ii) under the equivalence $\mathrm{Geom}(\mathcal{C}_0, \tau_0, \mathbb{P}_0) \simeq \mathrm{Geom}(\mathcal{C}_1, \tau_1, \mathbb{P}_1)$ the spectrum E_0 is equivalent to E_1.

Let $\varphi\colon (\mathcal{C}_0, \mathbb{P}_0, E_0) \to (\mathcal{C}_1, \mathbb{P}_1, E_1)$ be a transformation of pre-nonconnective contexts. Then φ induces a functor

$$\varphi\colon \mathcal{T}_0^{\mathrm{nc}} \longrightarrow \mathcal{T}_1^{\mathrm{nc}}.$$

Precomposition with this functor provides, for every ∞-topos $\mathcal{X}$, restriction morphisms

$$\varphi_*\colon \mathrm{Str}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X})$$

and

$$\varphi_*\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X}).$$

The following is the analogue of [Lur11d, Proposition 3.28] in the nonconnective setting:

[3] Notice that the adjoint functor theorem implies the existence of a left adjoint to $\tau_{\geq 0}$. What is no longer clear, however, is that this left adjoint is fully faithful.

[4] The reader might wonder why bother working with $\mathcal{T}_{\mathrm{an}}(k)$-structures rather than with $\mathcal{T}_{\mathrm{an}}(k)'$-structures, since the results for the latter are nicer. The reason is that the analytification functor only defines a transformation of pregeometries $\mathcal{T}_{\text{ét}}(k) \to \mathcal{T}_{\mathrm{an}}(k)$, and this transformation of pregeometries is crucial in derived k-analytic geometry.

Theorem 7.3.8.3. *Let $(\mathcal{T}_0, \mathrm{adm}_0, \tau_0)$ and $(\mathcal{T}_1, \mathrm{adm}_1, \tau_1)$ be two pregeometries. For $i = 0, 1$, let $(\mathcal{C}_i, \mathbb{P}_i, E_i)$ be a pre-$\mathcal{T}_i$-nonconnective context. Let $\varphi \colon (\mathcal{C}_0, \mathbb{P}_0, E_0) \to (\mathcal{C}_1, \mathbb{P}_1, E_1)$ be a Morita equivalence. Suppose furthermore that the restriction $\varphi \colon \mathcal{T}_0 \to \mathcal{T}_1$ satisfies the following properties:*

(i) $\varphi \colon \mathcal{T}_0 \to \mathcal{T}_1$ is fully faithful;

(ii) a morphism $f \colon U \to X$ in $\mathcal{T}_0$ is admissible if and only if $\varphi(f) \colon \varphi(U) \to \varphi(X)$ is admissible;

(iii) a collection of admissible morphisms $\{f_\alpha \colon U_\alpha \to X\}$ generates a covering sieve on X in $\mathcal{T}_0$ if and only if the collection $\{\varphi(f_\alpha) \colon \varphi(U_\alpha) \to \varphi(X)\}$ generates a covering sieve on $\varphi(X)$ in $\mathcal{T}_1$;

(iv) for every $X \in \mathcal{T}_1$ there exists a collection of objects $\{U_\alpha\}$ in $\mathcal{T}_0$ and a collection of morphisms $\{f_\alpha \colon \varphi(U_\alpha) \to X\}$ generating a covering sieve.

If in addition $(\mathcal{C}_0, \mathbb{P}_0, E_0)$ is a $\mathcal{T}_0$-nonconnective context, then for any ∞-topos $\mathcal{X}$ the restriction functor

$$\varphi_* \colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X})$$

is an equivalence of ∞-categories.

Proof. The transformation φ induces a fully faithful functor $\mathcal{T}_0^{\mathrm{nc}} \to \mathcal{T}_1^{\mathrm{nc}}$ that we still denote by φ. Consider the left Kan extension functor

$$\mathrm{Lan}_\varphi \colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X}) \to \mathrm{Fun}(\mathcal{T}_1^{\mathrm{nc}}, \mathcal{X}).$$

Notice, since $(\mathcal{C}_0, \mathbb{P}_0, E_0)$ is a $\mathcal{T}_0$-nonconnective context, for every $X \in \mathcal{T}_1^{\mathrm{nc}}$ the functor

$$(\mathcal{T}_1^{\mathrm{nc}})_{X/} \times_{\mathcal{T}_0^{\mathrm{nc}}} \mathcal{T}_0 \longrightarrow (\mathcal{T}_1^{\mathrm{nc}})_{X/} \times_{\mathcal{T}_0^{\mathrm{nc}}} \mathcal{T}_0^{\mathrm{nc}}$$

is cofinal. It follows that the diagram

$$\begin{tikzcd}
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X}) \arrow[r, "\mathrm{Lan}_\varphi"] \arrow[d, "\tau_{\geq 0}"] & \mathrm{Fun}(\mathcal{T}_1^{\mathrm{nc}}, \mathcal{X}) \arrow[d, "\tau_{\geq 0}"] \\
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_0}(\mathcal{X}) \arrow[r, "\mathrm{Lan}_\varphi"] & \mathrm{Fun}(\mathcal{T}_1, \mathcal{X})
\end{tikzcd}$$

commutes. Thus [Lur11d, Proposition 3.28] implies that

$$\mathrm{Lan}_\varphi \colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_0}(\mathcal{X}) \to \mathrm{Fun}(\mathcal{T}_1, \mathcal{X})$$

factors through the (non full) subcategory $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_1}(\mathcal{X})$ and that such factorization is an equivalence. It is now enough to remark that since $\varphi \colon (\mathcal{C}_0, \tau_0, \mathbb{P}_0) \to (\mathcal{C}_1, \tau_1, \mathbb{P}_1)$ is a Morita equivalence and $\varphi_s(E_0) \simeq E_1$, for every $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X})$ we have

$$\mathrm{Lan}_\varphi(\mathcal{O})(E_1^n) \simeq \mathcal{O}(E_0^n).$$

This proves at the same time that

$$\mathrm{Lan}_\varphi \colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_0^{\mathrm{nc}}}(\mathcal{X}) \longrightarrow \mathrm{Fun}(\mathcal{T}_1^{\mathrm{nc}}, \mathcal{X})$$

factors through the (non full) subcategory $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_1^{\mathrm{nc}}}(\mathcal{X})$ and that such factorization is an equivalence. □

Remark 7.3.8.4. It is possible to push the theory of nonconnective pregeometries much farther than what we did so far. For example, it would be possible to develop a theory of nonconnective schemes, which would provide an analogue of the theory of spectral affine schemes introduced in [Lur11f]. Having a notion of nonconnective scheme for a suitably general pregeometry might be useful: for example, the paper [AHR10] suggests the existence of a complex analytic version of tmf. In order to make rigorous the considerations done in loc. cit. it is necessary to consider nonconnective derived analytic spaces. This can easily be achieved by the formalism of nonconnective pregeometries. Another possible application of nonconnective structures would be to develop analogues of the results in [Toë06a] in the complex analytic and in the non-archimedean analytic setting.

7.4 The analytic case

From this point on we specialize to the analytic setting. Our first task is to introduce the categories of mixed and S^1-equivariant analytic algebras.

In this section we let k denote either the field of complex numbers or a non-archimedean field equipped with a non-trivial valuation and of characteristic zero. In the latter case, we let $|-|$ denote the associated absolute value. Following the use of [PY16d, PY17b], we write $\mathbb{C}$-*analytic* to mean complex analytic and k-*analytic* to mean non-archimedean analytic over the non-archimedean field k. When statements apply to both settings, we simply write *analytic*.

7.4.1 The analytic nonconnective contexts

Let $\mathcal{T}_{\mathrm{an}}(k)$ denote the analytic pregeometry. See Theorem 7.3.1.3 for the $\mathbb{C}$-analytic case and Theorem 7.3.1.4 for the k-analytic case. Let dAn_k denote the ∞-category of derived analytic spaces, as defined in [Lur11b, Definition 12.3] and in [PY16b, Definition 2.5]. We endow dAn_k with the étale topology $\tau_{\text{ét}}$. This is indeed the restriction of the $\tau_{\text{ét}}$-topology on ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))$: see [Por15b, Lemma 3.4] for the $\mathbb{C}$-analytic case and [PY16b, Theorem 5.4] for the k-analytic case. The notion of smooth morphism between derived analytic spaces has been introduced and studied in [PY17b, §5.6]. Let us denote it by $\mathbb{P}_{\mathrm{sm}}$. Then $(\mathrm{dAn}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})$ is a geometric context in the sense of [PY16d, §2]. The ∞-category dAn_k has finite limits in virtue of [Lur11b, Proposition 12.12] in the $\mathbb{C}$-analytic case and of [PY16b, Proposition 6.2] in the k-analytic case. Finally, the inclusion $\mathcal{T}_{\mathrm{an}}(k) \to \mathrm{dAn}_k$ preserves products, as it is shown in [Lur11b, Lemma 12.14(5)] and in [PY16b, Proposition 6.2(v)]. We can summarize these considerations in the following result:

Proposition 7.4.1.1. *The choice of the étale topology and the collection of smooth morphisms make* $(\mathrm{dAn}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})$ *into a* $\mathcal{T}_{\mathrm{an}}(k)$*-geometric context.*

We can also define a variant $\mathcal{T}^{\mathrm{afd}}_{\mathrm{an}}(k)$ of $\mathcal{T}_{\mathrm{an}}(k)$ as follows: the object of $\mathcal{T}^{\mathrm{afd}}_{\mathrm{an}}(k)$ are smooth Stein spaces (resp. smooth k-affinoid spaces), while the notion of admissible morphism and the Grothendieck topology are left unchanged. It follows from [Lur11d, Proposition 3.2.8] that the inclusion $\mathcal{T}^{\mathrm{afd}}_{\mathrm{an}}(k) \hookrightarrow \mathcal{T}_{\mathrm{an}}(k)$ is a Morita equivalence of pregeometries.

Following the convention introduced in [PY17b], we let Afd_k denote the full subcatgory of ${}^{\mathrm{An}}_{k}$ spanned by Stein spaces (in the $\mathbb{C}$-analytic case) or by k-affinoid spaces (in the k-analytic case). Furthermore, we let dAfd_k denote the full subcategory of dAn_k spanned by those derived analytic spaces whose truncation belongs to Afd_k.

Corollary 7.4.1.2. *The étale topology and the collection of smooth morphisms make* $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})$ *into a* $\mathcal{T}^{\mathrm{afd}}_{\mathrm{an}}(k)$*-geometric context.*

Proof. It is enough to prove that dAfd_k is closed under fiber products in dAn_k. As the truncation functor commutes with fiber products, it is enough to prove that Afd_k is closed under fiber products in ${}^{\mathrm{An}}_{k}$. In the $\mathbb{C}$-analytic case, this follows from [GR84, §1.4.4]. In the k-analytic case, this is a consequence of [Con08b, Exercise 2.2.3(1)]. □

Notice that the inclusion $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}}) \hookrightarrow (\mathrm{dAn}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})$ induces an equivalence of the ∞-categories of geometric stacks thanks to [PY16d, Corollary 2.26].

Let now $\mathrm{B}G_a$ denote the analytic affine line equipped with its additive group structure. Since $\mathrm{B}G_a$ is commutative, May's delooping theorem for ∞-topoi [Lur12c, 5.2.6.15] provides us with a spectrum object

$$E \in \mathrm{Sp}(\mathrm{Geom}(\mathrm{dAn}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}}))$$

such that

$$E^n \simeq \mathrm{B}^n(\mathrm{B}G_a).$$

In this way, $(\mathrm{dAn}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}}, E)$ becomes a pre-$\mathcal{T}_{\mathrm{an}}(k)$-nonconnective context, while $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}}, E)$ becomes a $\mathcal{T}^{\mathrm{afd}}_{\mathrm{an}}(k)$-nonconnective context. The latter assertion is a consequence of the derived version of Tate's acyclicity theorem (see [PY18a, Theorem 3.4]). In particular, $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}}, E)$ defines a nonconnective pregeometry, that accordingly to our convention we should denote $\mathcal{T}^{\mathrm{afd}}_{\mathrm{an}}(k)^{\mathrm{nc}}$. However, when no confusion is possible we simplify the notation and denote it instead by $\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)$.

In the rigid analytic situation, there are other variations that are important for us. For every $r \in |k^\times|$ let $\mathbb{D}^1_k(r)$ denote the closed disk of radius r centered at the origin. Since the valuation is non-archimedean, $\mathbb{D}^1_k(r)$ acquires the structure of a commutative group object in dAn_k. Applying once again May's delooping theorem for ∞-topoi, we obtain a spectrum object

$$E(r) \in \mathrm{Sp}(\mathrm{Geom}(\mathrm{dAn}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}}))$$

such that

$$E(r)^n \simeq \mathrm{B}^n(\mathbb{D}^1_k(r)).$$

Similarly to the case discussed above, $(\mathrm{dAn}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}}, E(r))$ is then a pre-$\mathcal{T}_{\mathrm{an}}(k)$-nonconnective context and also similary $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}}, E(r))$ is a pre-$\mathcal{T}_{\mathrm{an}}(k)$-nonconnective context, moreover we have the following:

Lemma 7.4.1.3. *The pre-$\mathcal{T}_{\mathrm{an}}(k)$-nonconenective context $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}}, E(r))$ is a $\mathcal{T}^{\mathrm{afd}}_{\mathrm{an}}(k)$-nonconnective context.*

Proof. We are left to check condition (3) in Theorem 7.3.1.6. Let $X \in \mathrm{dAfd}_k$ be a derived k-affinoid space, we want to show that the mapping space

$$\mathrm{Map}_{\mathrm{Geom}(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})}\left(X, \mathrm{B}^n(\mathbb{D}^1_k(r))\right)$$

is connected, whenever $n \geq 1$ and $r \in]0, \infty]$ is a given radius.

Let us treat first the case where $r = \infty$, in this case $E^n(r) \simeq \mathrm{B}^n(\mathrm{B}G_a)$. By the universal property of the k-analytic group $\mathrm{B}G_a$ together with the universal property of the delooping functor we obtain an equivalence between spaces

$$\mathrm{Map}_{\mathrm{Geom}(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})}\left(X, \mathrm{B}^n(\mathrm{B}G_a)\right) \simeq \Gamma\left(X, \mathcal{O}_X\right)[-n].$$

By definition of derived k-affinoid space we know that, for each $j \geq 0$, $\pi_j\left(\mathcal{O}_X\right)$ is a coherent sheaf over the 0-truncated k-affinoid space $t_{\leq 0}X = (\mathcal{X}, \tau_{\leq 0}\mathcal{O}_X)$. Moreover, thanks to (the dual of) [Lur12c, 1.2.2.14] we have a spectral sequence of the form

$$E_2^{i,j} := \pi_i\left(\Gamma\left(t_{\leq 0}X, \pi_j\left(\mathcal{O}_X\right)\right)\right) \Rightarrow \pi_{i+j}\left(\Gamma\left(t_{\leq 0}X, \pi_0\left(\mathcal{O}_X\right)\right)\right)$$

and by Tate's acyclicity theorem it follows that we have an equivalence of spaces

$$\pi_0\left(\mathrm{Map}_{\mathrm{Geom}(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})}\left(X, \mathrm{B}^n(\mathrm{B}G_a)\right)\right) \simeq \pi_0\left(\Gamma\left(X, \pi_0\left(\mathcal{O}_X\right)\right)[-n]\right) \simeq H^n\left(t_{\leq 0}X, \pi_0\mathcal{O}_X\right) \simeq *,$$

whenever $n > 0$. Suppose now that $r < \infty$, by May's delooping theorem we reduce ourselves to show that the mapping space

$$\mathrm{Map}_{\mathrm{Geom}(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})}\left(X, \mathbb{D}^1_k(r)\right) \tag{7.4.1.1}$$

is a discrete space, i.e., the only non-trivial homotopy groups live in degree 0. We have a monomorphism $\mathbb{D}^1_k(r) \to \mathrm{B}G_a$ in the ∞-category dAfd_k and therefore for each $n \geq 1$ we have monomorphisms $\mathrm{B}^n(\mathbb{D}^1_k(r)) \to \mathrm{B}^n\mathrm{B}G_a$ and therefore we have a monomorphism of mapping spaces

$$\mathrm{Map}_{\mathrm{Geom}(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})}\left(X, \mathbb{D}^1_k(r)\right) \to \mathrm{Map}_{\mathrm{Geom}(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbb{P}_{\mathrm{sm}})}\left(X, \mathrm{B}G_a\right) \simeq \Gamma\left(t_{\leq 0}X, \pi_0\left(\mathcal{O}_X\right)\right). \tag{7.4.1.2}$$

and we can identify the left hand side of (7.4.1.2) with global sections of the subsheaf $\pi_0\left(\mathcal{O}_X\right)(r)$ of $\pi_0\left(\mathcal{O}_X\right)$ on $t_{\leq 0}X$ spanned by those sections which (locally) are uniformly bounded in norm by $r > 0$.Therefore, covering $t_{\leq 0}X$ by a finite number of open affinoid spaces $X = \bigcup_i X_i$, with (classical) intersections $X_{i,j} = X_i \cap X_j$ we have a commutative diagram of the form

$$\begin{tikzcd} \Pi_{i,j}A_{i,j}(r) \arrow[r] \arrow[d] & \Pi_i A_i(r) \arrow[r] \arrow[d] & A(r) \arrow[d] \\ \Pi_{i,j}A_{i,j} \arrow[r] & \Pi_i A_i \arrow[r] & A \end{tikzcd} \tag{7.4.1.3}$$

where the vertical maps are monomorphisms and $A_{i,j}(r) := \pi_0\left(\mathcal{O}_X\right)(r)(X_i, j)$, $A_{i,j} := \pi_0\left(\mathcal{O}_X\right)(X_i, j)$, $A_i(r) := \pi_0\left(\mathcal{O}_X\right)(r)(X_i)$, $A_i := \pi_0\left(\mathcal{O}_X\right)(r)(X_i)$, $A(r) := \pi_0\left(\mathcal{O}_X\right)(r)(X)$ and $A := \pi_0\left(\mathcal{O}_X\right)(X)$. Computing cohomology of the associated complex to the vertical sequence in (7.4.1.3) we conclude that cycles in such a complex are (co)homologous if and only if the boundary element corresponds to a section uniformly bounded in norm by r on the intersections and thus on all X. This implies that we have a monomorphism of graded modules

$$\pi_*\left(\Gamma\left(t_{\leq 0}X, \pi_0\mathcal{O}_X(r)\right)\right) \to \pi_*\left(\Gamma\left(t_{\leq 0}X, \pi_0\mathcal{O}_X\right)\right)$$

and therefore, we conclude that the mapping space in (7.4.1.1) is discrete, as desired. $\square$

Therefore, the latter defines a pregeometry, that we denote by $\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k;r)$.

7.4.2 Nonconnective analytification

In [Por15b, PY17b] the authors studied and exploited the derived analytification functor. This is a functor that associates to every derived scheme locally almost of finite presentation over k a derived analytic space, which is characterized by a certain universal property. At the level of algebras, the analytification functor can simply be understood as the left adjoint to the underlying algebra functor. In our approach to the analytic HKR theorem, the analytification functor plays a major role, for instance in the definition of mixed analytic algebras.

The classical analytification functor induces a transformation of pregeometries

$$(-)^{\mathrm{an}} \colon \mathcal{T}_{\mathrm{\acute{e}t}}(k) \longrightarrow \mathcal{T}_{\mathrm{an}}(k).$$

The analysis carried over in [Por15b, §4] and in [PY17b, §3] shows that this functor extends to a transformation of geometric contexts

$$(-)^{\mathrm{an}} \colon (\mathrm{dAff}_k^{\mathrm{afp}}, \tau_{\mathrm{\acute{e}t}}, \mathbb{P}_{\mathrm{sm}}) \longrightarrow (\mathrm{dAn}_k, \tau_{\mathrm{\acute{e}t}}, \mathbb{P}_{\mathrm{sm}}).$$

In particular, [PY16d, Proposition 2.25] provides us with a well defined functor

$$(-)^{\mathrm{an}} \colon \mathrm{Geom}(\mathrm{dAff}_k^{\mathrm{afp}}, \tau_{\mathrm{\acute{e}t}}, \mathbb{P}_{\mathrm{sm}}) \longrightarrow \mathrm{Geom}(\mathrm{dAn}_k, \tau_{\mathrm{\acute{e}t}}, \mathbb{P}_{\mathrm{sm}}).$$

Notice that this functor commutes with colimits by construction, and therefore that it brings $\mathrm{B}^n(\mathbb{G}_a)$ to $\mathrm{B}^n(\mathrm{B}G_a)$. As a consequence, we obtain a strong transformation of nonconnective pre-contexts

$$(-)^{\mathrm{an}} \colon \mathcal{T}_{\mathrm{\acute{e}t}}^{\mathrm{nc}}(k) \longrightarrow \mathcal{T}_{\mathrm{an}}(k)^{\mathrm{nc}}.$$

Remark 7.4.2.1. In the $\mathbb{C}$-analytic setting, $\mathcal{T}_{\mathrm{an}}(k)^{\mathrm{nc}}$ is a nonconnective context. On the other hand, in the rigid analytic setting, $\mathcal{T}_{\mathrm{an}}(k)^{\mathrm{nc}}$ is only a nonconnective pre-context. However, the inclusion $\mathcal{T}_{\mathrm{an}}^{\mathrm{afd}}(k)^{\mathrm{nc}} \hookrightarrow \mathcal{T}_{\mathrm{an}}(k)^{\mathrm{nc}}$ satisfies the assumptions of **??**, therefore providing for every ∞-topos $\mathcal{X}$ an equivalence

$$\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{afd}}(k)^{\mathrm{nc}}}(\mathcal{X}) \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)^{\mathrm{nc}}}(\mathcal{X}).$$

We denote this ∞-category simply by $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X})$.

The analytification functor $(-)^{\mathrm{an}} \colon \mathcal{T}_{\mathrm{\acute{e}t}}^{\mathrm{nc}}(k) \to \mathcal{T}_{\mathrm{an}}(k)^{\mathrm{nc}}$ fits into the following commutative square

$$\begin{tikzcd}
\mathcal{T}_{\mathrm{\acute{e}t}}(k) \arrow[r, "(-)^{\mathrm{an}}"] \arrow[d, hook] & \mathcal{T}_{\mathrm{an}}(k) \arrow[d, hook] \\
\mathcal{T}_{\mathrm{\acute{e}t}}^{\mathrm{nc}}(k) \arrow[r, "(-)^{\mathrm{an}}"] & \mathcal{T}_{\mathrm{an}}(k)^{\mathrm{nc}}.
\end{tikzcd}$$

Therefore, for every ∞-topos $\mathcal{X}$, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}^{\mathrm{nc}}(k)}(\mathcal{X}) \arrow[d, "\tau_{\geq 0}"] & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X}) \arrow[l, "(-)^{\mathrm{alg}}"'] \arrow[d, "\tau_{\geq 0}"] \\
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X}) & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}). \arrow[l, "(-)^{\mathrm{alg}}"']
\end{tikzcd}$$

Consider the left adjoints

$$i\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X}) \longrightarrow \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X}) \quad , \quad i\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}) \longrightarrow \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(\mathcal{X})$$

to the connective cover functors $\tau_{\geq 0}$. The commutativity of the above diagram induces a Beck-Chevalley transformation

$$\beta\colon i \circ (-)^{\mathrm{alg}} \longrightarrow (-)^{\mathrm{alg}} \circ i.$$

Proposition 7.4.2.2. *Let $\mathcal{X}$ be an ∞-topos with enough points. For every $A \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ the natural transformation $\beta_A\colon i(A^{\mathrm{alg}}) \to i(A)^{\mathrm{alg}}$ is an equivalence.*

Proof. Notice that [Lur11d, Proposition 3.3.1] implies that the functor $(-)^{\mathrm{alg}}\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}) \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})$ commutes with sifted colimits. On the other hand, Theorem 7.3.3.6 shows that if $F\colon I \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ is a sifted diagram, then the colimit of $i \circ F$ can be computed objectwise. It follows that the composition

$$(-)^{\mathrm{alg}} \circ i\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}) \longrightarrow \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})$$

commutes with sifted colimits. Using [Lur11b, Proposition 2.11], we reduce ourselves to the case where A is an elementary $\mathcal{T}_{\mathrm{an}}(k)$-structure. In this case, the formula follows from direct inspection. $\square$

Let us fix $A \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$. Since $\tau_{\geq 0}(i(A)) \simeq A$ and $\tau_{\geq 0}(i(A^{\mathrm{alg}})) \simeq A^{\mathrm{alg}}$, we obtain the following commutative square:

$$\begin{tikzcd}
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})_{/i(A^{\mathrm{alg}})} \arrow[d, "\tau_{\geq 0}"] & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(\mathcal{X})_{/i(A)} \arrow[l, "(-)^{\mathrm{alg}}"'] \arrow[d, "\tau_{\geq 0}"] \\
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})_{/A^{\mathrm{alg}}} & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})_{/A}. \arrow[l, "(-)^{\mathrm{alg}}"']
\end{tikzcd}$$

Using [Por15b, Corollary 9.4] we see that all the categories appearing in this diagram are presentable. As the functors $(-)^{\mathrm{alg}}$ commute with limits and filtered colimits, we deduce that they admit left adjoints, that we still denote by

$$(-)^{\mathrm{an}}\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})_{/A^{\mathrm{alg}}} \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})_{/A} \quad , \quad (-)^{\mathrm{an}}\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})_{/i(A^{\mathrm{alg}})} \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(\mathcal{X})_{/i(A)}.$$

We refer to the functor on the right as the nonconnective (derived) analytification functor.

These functors induce the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})_{/i(A^{\mathrm{alg}})} \arrow[r, "(-)^{\mathrm{an}}"] & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(\mathcal{X})_{/i(A)} \\
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})_{/A^{\mathrm{alg}}} \arrow[u, "i"] \arrow[r, "(-)^{\mathrm{an}}"] & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})_{/A}. \arrow[u, "i"']
\end{tikzcd}$$

In particular, we see that the nonconnective analytification functor respects connective structures.

Remark 7.4.2.3. The functor $(-)^{\mathrm{an}}\colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})_{/A^{\mathrm{alg}}} \to \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})_{/A}$ does not depend on the choice of A in the following sense. If $f\colon A \to B$ is any morphism in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$, then the diagram

$$\begin{tikzcd}
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})_{/A^{\mathrm{alg}}} \arrow[r, "{}_A(-)^{\mathrm{an}}"] \arrow[d, "f_*^{\mathrm{alg}}"] & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})_{/A} \arrow[d, "f_*"] \\
\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})_{/B^{\mathrm{alg}}} \arrow[r, "{}_B(-)^{\mathrm{an}}"] & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})_{/B}
\end{tikzcd}$$

commutes. Here we denoted by ${}_A(-)^{\mathrm{an}}$ and ${}_B(-)^{\mathrm{an}}$ the analytification functors constructed using A (resp. B) as auxiliary choice, and f_* and f_*^{alg} denote the functors induced by composition with f (resp. f_*^{alg}). For this reason we suppress the dependence on A in the notation for the analytification functor.

A key feature of the underlying algebra functor on local structures is the fact that it is conservative. Let us record this property explicitly for later use:

Proposition 7.4.2.4. *Let $\mathcal{X}$ be an ∞-topos and let $A \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$. The underlying algebra functors*

$$(-)^{\mathrm{alg}} \colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})_{/A} \longrightarrow \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{\acute{e}t}}(k)}(\mathcal{X})_{/A^{\mathrm{alg}}}$$
$$(-)^{\mathrm{alg}} \colon \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(\mathcal{X})_{/i(A)} \longrightarrow \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(\mathcal{X})_{/i(A^{\mathrm{alg}})}$$

are conservative.

Proof. In the connective case, this has already been proven in [Lur11b, Proposition 11.9] (in the $\mathbb{C}$-analytic setting) and in [PY16b, Lemma 3.13] (in the rigid analytic setting). In the nonconnective situation, the statement follows at once from the connective one and from Theorem 7.3.3.8. $\square$

For the rest of this section, we restrict ourselves to the ∞-topos of spaces $\mathcal{S}$. In this case, the Yoneda lemma allows to produce several important examples of nonconnective analytic structures. Indeed, for every derived analytic stack $X \in \mathrm{dAnSt}_k$, we can define a functor

$$\Gamma^{\mathrm{an}}_k(X) \colon \mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k) \longrightarrow \mathcal{S}$$

by

$$\Gamma^{\mathrm{an}}_k(X)(U) \coloneqq \mathrm{Map}_{\mathrm{dAnSt}_k}(X, U).$$

This is a $\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)$-structure, but in general it is not local. The following proposition isolates a special class of derived analytic stack X for which $\Gamma^{\mathrm{an}}_k(X)$ is local.

Proposition 7.4.2.5. *Suppose that X is underived and $\pi_0(X) \simeq \mathrm{Sp}(k)$. Then $\Gamma^{\mathrm{an}}_k(X)$ is a local $\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)$-structure.*

Proof. We only need to check that $\tau_{\geq 0}(\Gamma^{\mathrm{an}}_k(X)))$ is a local $\mathcal{T}_{\mathrm{an}}(k)$-structure. Let $U \in \mathcal{T}_{\mathrm{an}}(k)$. Since X is underived, we have

$$\mathrm{Map}_{\mathrm{dAnSt}_k}(X, U) \simeq \mathrm{Map}_{\mathrm{AnSt}_k}(X, U).$$

Since U is discrete, we have

$$\mathrm{Map}_{\mathrm{AnSt}_k}(X, U) \simeq \mathrm{Map}_{\mathrm{AnSt}_k}(\pi_0(X), U) \simeq \mathrm{Map}_{\mathrm{AnSt}_k}(\mathrm{Sp}(k), U).$$

This functor clearly sends $\tau_{\mathrm{\acute{e}t}}$-coverings to effective epimorphism. $\square$

In particular, the objects

$$\Gamma^{\mathrm{an}}_k(E^n), \Gamma^{\mathrm{an}}_k(E(r)^n) \in \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(\mathcal{S}) \quad (r \in |k|^\times)$$

are local structures. We denote them by $S^{\mathrm{an}}(n)$ and $S^{\mathrm{an}}_k(n, r)$ respectively and we refer to them as the *free nonconnective analytic algebra of rank* 1*, degree n (and radius r).*

Notation 7.4.2.6. We denote by $\mathrm{AnRing}^{\mathrm{nc}}_k$ the ∞-category $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(\mathcal{S})_{/k}$. We refer to it as the ∞-category of local nonconnective analytic rings. Notice that it is a presentable ∞-category. In particular, it admits pushouts. Given maps $A \to A'$ and $A \to A''$ we denote their pushout by $A' \mathbin{\widehat{\otimes}}_A A''$.

Notice that in the algebraic case, the identification of $\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{disc}}(k)}(\mathcal{S})$ with $\mathbf{cdga}_k$ the local nonconnective algebraic structure

$$\Gamma(E^n_{\mathrm{alg}}) = \Gamma(\mathrm{B}^n(\mathbb{G}_a)) \in \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{disc}}(k)}(\mathcal{S})$$

corresponds to $\mathrm{Sym}_k(k[-n])$. We denote it by $S_k(n)$.

Lemma 7.4.2.7. *There is a natural equivalence*

$$S_k(n)^{\mathrm{an}} \simeq S^{\mathrm{an}}_k(n).$$

Proof. This simply follows from the fact that $(\mathrm{B}^n(\mathbb{G}_a))^{\mathrm{an}} \simeq \mathrm{B}^n(\mathrm{B}G_a)$. $\square$

It is less trivial to identify $(S_k^{\mathrm{an}}(n,r))^{\mathrm{alg}}$. We start by observing that for every $r \in |k^\times|$ there is a natural morphism

$$\mathrm{B}^n(\mathbb{D}^1(r)) \hookrightarrow \mathrm{B}^n(\mathrm{B}G_a),$$

that induces a (local) morphism of nonconnective analytic rings

$$S_k^{\mathrm{an}}(n) \longrightarrow S_k^{\mathrm{an}}(n,r).$$

On the other hand, the unit of the adjunction $(-)^{\mathrm{an}} \dashv (-)^{\mathrm{alg}}$ provides us with a map

$$\eta_{n,r} \colon S_k(n) \longrightarrow (S_k^{\mathrm{an}}(n))^{\mathrm{alg}} \longrightarrow (S_k^{\mathrm{an}}(n,r))^{\mathrm{alg}}$$

The following result is the key to the analytic HKR:

Theorem 7.4.2.8 (Van Est, Lazard)**.** *Let $\rho \in |k^\times| \cup \{\infty\}$ denote the converging radius of the exponential function. Then if $r \leq \rho$ the morphism $\eta_{n,r}$ is an equivalence. Moreover, if $r, r' \in |k^\times| \cup \{\infty\}$ satisfy $r < r' \leq \rho$ the natural morphism*

$$S_k^{\mathrm{an}}(n,r') \longrightarrow S_k^{\mathrm{an}}(n,r)$$

is an equivalence in $\mathrm{AnRing}_k^{\mathrm{nc}}$.

Proof. Since the underlying algebra functor $(-)^{\mathrm{alg}} \colon \mathrm{AnRing}_k^{\mathrm{nc}} \to \mathbf{cdga}_k$ is conservative, the second statement follows at once from the first one.

Since the forgetful functor

$$\mathbf{cdga}_k \longrightarrow \mathbf{Mod}_k$$

is conservative, it is enough to check that the image of $\eta_{n,r}$ in $\mathbf{Mod}_k$ is an equivalence. It follows from the definitions that this is the same morphism obtained by applying the underlying spectrum functor to $\eta_{n,r}$. Unraveling the definitions, we see that the object $U(S_k^{\mathrm{an}}(n,r)) \in \mathbf{Mod}_k$ is computed as the totalization of the following cosimplicial object:

$$k \rightrightarrows k\langle r^{-1}T\rangle \overset{\rightarrow}{\rightrightarrows} k\langle r^{-1}T_1, r^{-1}T_2\rangle \overset{\rightrightarrows}{\rightrightarrows} \cdots,$$

which in degree n has $k\langle r^{-1}T_1, \ldots, r^{-1}T_n\rangle$, and the ith morphism

$$k\langle r^{-1}T_1, \ldots, r^{-1}T_n\rangle \longrightarrow k\langle r^{-1}T_1, \ldots, r^{-1}T_{n+1}\rangle$$

is determined by the rule

$$T_j \mapsto \begin{cases} T_{j+1} & \text{if } j < i \\ T_j + T_{j+1} & \text{if } j = i \\ T_j & \text{if } j > i. \end{cases}$$

When $r = \infty$, we denote by $k\langle r^{-1}T_1, \ldots, r^{-1}T_n\rangle$ the algebra of analytic functions on $\mathbb{A}_k^n$.

Let us first deal with the complex analytic case. In this case $r = \rho = \infty$. After applying the cosimplicial Dold-Kan, we can identify the above cosimplicial object with the cochain complex computing holomorphic cohomology of $\mathrm{B}G_a$ with coefficients in its trivial representation of rank 1. The Van Est theorem (originally formulated for continuous cochains and extended to holomorphic cochains by Hochschild and Mostow) implies that we can identify the above complex with the complex computing the cohomology of the Lie algebra $\mathfrak{g}_a$ of $\mathrm{B}G_a$ with coefficients in its trivial representation of rank 1. Inspection reveals that the latter is quasi-isomorphic to $k \oplus k[-1]$. The conclusion now follows from **??**.

In the non-archimedean analytic case the same thing happens, with the difference that the cochain complex is no longer acyclic in degrees ≤ -2 unless $r \leq \rho$. This has originally been proven by Lazard. See [**?**] for a modern treatment. ☐

We also need a relative version of Van Est theorem. Let $A \in \mathrm{AnRing}_k^{\mathrm{nc}}$ and set

$$S_A(n) \coloneqq A^{\mathrm{alg}} \otimes_k S_k(n), \quad S_A^{\mathrm{an}}(n) \coloneqq A \widehat{\otimes}_k S_k^{\mathrm{an}}(n), \quad S_A^{\mathrm{an}}(n,r) \coloneqq A \widehat{\otimes}_k S_k^{\mathrm{an}}(n,r).$$

The map $\eta_{n,r}$ induces a well defined map $S_A(n) \to (S_A^{\mathrm{an}}(n,r))^{\mathrm{alg}}$, that we still denote $\eta_{n,r}$.

Corollary 7.4.2.9. *Let $\rho \in |k|^\times \cup \{\infty\}$ denote the converging radius of the exponential. Then if $r \leq \rho$ and $A \in \mathrm{AnRing}_k$, the map $\eta_{n,r} \colon S_A(n) \to (S_A^{\mathrm{an}}(n,r))^{\mathrm{alg}}$ is an equivalence.*

Proof. We start by observing that the formation of the map $\eta_{n,r} \colon S_A(n) \to (S_A^{\mathrm{an}}(n,r))^{\mathrm{alg}}$ commutes with filtered colimits in A. Using **??**, we reduce ourselves to the case where $A \simeq x^{-1}\mathcal{O}_X$, for $X = (\mathcal{X}, \mathcal{O}_X) \in \mathrm{dAn}_k$ and $x_* \colon \mathcal{S} \leftrightarrows \mathcal{X} \colon x^{-1}$ a geometric point.

Notice that $x^{-1}\mathcal{O}_X \widehat{\otimes}_k S_k^{\mathrm{an}}(n,r) \simeq x^{-1}(\mathcal{O}_X \widehat{\otimes} S_{\mathcal{X}}^{\mathrm{an}}(n,r))$.

we can reduce ourselves to the case where A is the germ of analytic functions at a point in the interior of the disk $\mathbb{D}_k^m$. As the formation of the map $\eta_{n,r}$ commutes with filtered colimits, we see that it is enough to prove that it is an equivalence replacing A with the analytic ring of overconvergent analytic functions on $\mathbb{D}_k^m$. In this case the Yoneda lemma allows to identify the completed tensor product $A \widehat{\otimes}_k S_A^{\mathrm{an}}(n,r)$ with the analytic ring associated to the stack $(\mathbb{D}_k^m)^\dagger \times \mathrm{B}^n(\mathbb{D}_k^1(r))$, and hence with the inverse limit

$$\varprojlim \left(A \rightrightarrows A \widehat{\otimes}_k S_k^{\mathrm{an}}(n-1,r) \to\mathrel{\mkern-14mu}\rightrightarrows A \widehat{\otimes}_k (S_k^{\mathrm{an}}(n-1,r))^{\widehat{\otimes}\, 2} \rightrightarrows\mathrel{\mkern-14mu}\rightrightarrows \cdots \right).$$

As the underlying algebra functor commutes with inverse limits, we can reason by induction on n, and we are immediately reduced to the case $n = 1$. In this case

$$(S_k^{\mathrm{an}}(1,r))^{\widehat{\otimes}\, l} \simeq k\langle r^{-1}T_1, \ldots, r^{-1}T_l \rangle,$$

and the Yoneda lemma allows to identify the categorical completed tensor product $A \widehat{\otimes}_k (S_k^{\mathrm{an}}(1,r))^{\widehat{\otimes}\, l}$ with the usual completed tensor product $A \widehat{\otimes}_k k\langle r^{-1}T_1, \ldots, r^{-1}T_l\rangle$ of functional analysis. At this point, the conclusion follows because A is Fréchet nuclear: see [Dem, Example 5.12] in the $\mathbb{C}$-analytic case and [BBB15] in the k-analytic case. In particular, the completed tensor product with A is acyclic. Thus, the conclusion follows directly from Theorem 7.4.2.8. □

7.4.3 A nonconnective base change

Before starting to discuss S^1 and mixed analytic algebras, we need a couple of preliminaries on the general features of nonconnective analytic structures. We start with a discussion of a very particular base change property.

Notice that we have a functor

$$\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(-) \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op} \longrightarrow \mathcal{P}\mathrm{r}^{\mathrm{R}}$$

defined informally by sending an ∞-topos $\mathcal{X}$ to the presentable ∞-category $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X})$, and a geometric morphism $f_* \colon \mathcal{X} \leftrightarrows \mathcal{Y} \colon f^{-1}$ to the functor

$$f_* \colon \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X})$$

obtained by composing with f_*. The resulting functor

$$\mathrm{dAn}_k \longrightarrow {}^{\mathrm{R}}\mathcal{T}\mathrm{op} \xrightarrow{\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(-)} \mathcal{P}\mathrm{r}^{\mathrm{R}}$$

takes hypercoverings to limit diagrams. This is indeed a consequence of the descent theory for ∞-topoi [Lur09c, 6.1.3.9] and [Por15b, Lemma 3.4] (in the $\mathbb{C}$-analytic case) or [PY16b, Theorem 5.4] (in the k-analytic case). In particular, it extends to a functor

$$\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(-) \colon \mathrm{dAnSt}_k \longrightarrow \mathcal{P}\mathrm{r}^{\mathrm{R}}.$$

When $X = (\mathcal{X}, \mathcal{O}_X)$ is a derived k-analytic space, we therefore abusively write $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(X)$ instead of $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X})$.

Proposition 7.4.3.1. *Let $X \in \mathrm{dAnSt}_k$ and fix $r \in |k^\times| \cup \{\infty\}$. Consider the pullback square*

$$\begin{tikzcd} X \times \mathrm{B}(\mathbb{D}_k^m(r)) \arrow[r, "q'"] \arrow[d, "p'"] & \mathrm{B}(\mathbb{D}_k^m(r)) \arrow[d, "p"] \\ X \arrow[r, "q"] & \mathrm{Sp}(k). \end{tikzcd}$$

If $r \leq \rho$, where ρ is the converging radius of the exponential, then the Beck-Chevalley transformation

$$\beta \colon q^* p_*(\mathcal{O}_{\mathrm{B}(\mathbb{D}^m_k(r))}) \longrightarrow q'_* p'^*(\mathcal{O}_{\mathrm{B}(\mathbb{D}^m_k(r))})$$

is an equivalence in $\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(X)$.

Proof. By definition, we have

$$p_*(\mathcal{O}_{\mathrm{B}(\mathbb{D}^m_k(r))}) \simeq S^{\mathrm{an}}_k(m,r),$$

and therefore

$$q^* p_*(\mathcal{O}_{\mathrm{B}(\mathbb{D}^m_k(r))}) = \mathcal{O}_X \widehat{\otimes}_k q^{-1}(S^{\mathrm{an}}_k(m,r)).$$

Since $q'^*(\mathcal{O}_{\mathrm{B}(\mathbb{D}^m_k(r))}) \simeq \mathcal{O}_{X \times \mathrm{B}(\mathbb{D}^m_k(r))}$, we see that we have to prove that the canonical morphism

$$\mathcal{O}_X \widehat{\otimes}_k q^{-1}(S^{\mathrm{an}}_k(m,r)) \longrightarrow p'_* \mathcal{O}_{X \times \mathrm{B}(\mathbb{D}^m_k(r))}$$

is an equivalence in $\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(X)$ In order to check this, we can reason locally on X. In particular, we can assume X to be a derived analytic space. Let us write $X = (\mathcal{X}, \mathcal{O}_X)$, so that $\mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(X) \simeq \mathrm{Str}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(\mathcal{X})$. As $\mathcal{O}_X \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$, we can use [Lur11b, Proposition 2.11] to further reduce ourselves to the case where $X \in \mathcal{T}_{\mathrm{an}}(k)$. At this point, the conclusion follows from the Yoneda lemma and the fact that $\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)$-structures commute with products by assumption. ☐

Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived analytic space. Let

$$q_* \colon \mathcal{X} \leftrightarrows \mathcal{S} \colon q^{-1}$$

be the canonical geometric morphism. We set

$$S^{\mathrm{an}}_{\mathcal{X}}(m,r) \coloneqq q^{-1}(S^{\mathrm{an}}_k(m,r)).$$

As observed in the previous proof, we have a canonical identification $q^* p_*(\mathcal{O}_{\mathrm{B}(\mathbb{D}^m_k(r))}) \simeq \mathcal{O}_X \widehat{\otimes}_k S^{\mathrm{an}}_{\mathcal{X}}(m,r)$. We first address the question of whether this is a local structure or not.

Proposition 7.4.3.2. *Let $X \in \mathrm{dAn}_k$ be a derived analytic space. Then the nonconnective $\mathcal{T}_{\mathrm{an}}(k)$-structure $\mathcal{O}_X \widehat{\otimes}_k S^{\mathrm{an}}_{\mathcal{X}}(m,r)$ is local, and the natural transformation $\mathcal{O}_X \widehat{\otimes}_k S^{\mathrm{an}}_{\mathcal{X}}(m,r) \to \mathcal{O}_X$ is local as well.*

Proof. As $\mathcal{O}_X$ is a connective local structure, it is enough to prove the second statement. Using Theorem 7.4.3.1, we can rewrite

$$\mathcal{O}_X \widehat{\otimes}_k S^{\mathrm{an}}_{\mathcal{X}}(m,r) \simeq q'_*(\mathcal{O}_{X \times \mathrm{B}(\mathbb{D}^m_k(r))}).$$

We observe now that we can explicitly represent $\tau_{\geq 0}(q'_*(\mathcal{O}_{X \times \mathrm{B}(\mathbb{D}^m_k(r))}))$ as the functor $\mathcal{T}_{\mathrm{an}}(k) \to \mathcal{X}$ sending $Y \in \mathcal{T}_{\mathrm{an}}(k)$ to the sheaf on $\mathcal{X}$ defined by sending an étale morphism $U \to X$ from a derived Stein (resp. k-affinoid) U to

$$U \mapsto \mathrm{Map}_{\mathrm{dAnSt}_k}(U \times \mathrm{B}(\mathbb{D}^m_k(r)), Y).$$

Let $V \to Y$ be an étale map. Then we have to check that the square

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{dAnSt}_k}(U \times \mathrm{B}(\mathbb{D}^m_k(r)), V) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dAnSt}_k}(U \times \mathrm{B}(\mathbb{D}^m_k(r)), Y) \arrow[d] \\
\mathrm{Map}_{\mathrm{dAnSt}_k}(U, V) \arrow[r] & \mathrm{Map}_{\mathrm{dAnSt}_k}(U, Y)
\end{tikzcd}$$

is a pullback diagram.

We start by proving an auxiliary statement. The canonical morphism $\mathrm{t}_0(U) \hookrightarrow U$ induces a commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{dAnSt}_k}(U \times \mathrm{B}(\mathbb{D}^m_k(r)), V) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dAnSt}_k}(U \times \mathrm{B}(\mathbb{D}^m_k(r)), Y) \arrow[d] \\
\mathrm{Map}_{\mathrm{dAnSt}_k}(\mathrm{t}_0(U) \times \mathrm{B}(\mathbb{D}^m_k(r)), V) \arrow[r] & \mathrm{Map}_{\mathrm{dAnSt}_k}(\mathrm{t}_0(U) \times \mathrm{B}(\mathbb{D}^m_k(r)), Y).
\end{tikzcd}$$

We claim that this square is a pullback. Observe that

$$\mathrm{Map}_{\mathrm{dAnSt}_k}(U \times \mathrm{B}(\mathbb{D}_k^m(r)), V) \simeq \varprojlim_{[n]\in\mathbf{\Delta}} (U \times (\mathbb{D}_k^m(r))^{\times n}, V),$$

and we can give similar description of the other mapping spaces appearing in the above square. As limits commute with limits, in order to prove that we have a pullback, it is enough to show that for every $n \geq 0$ the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{dAnSt}_k}(U \times (\mathbb{D}_k^m(r))^{\times n}, V) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{dAnSt}_k}(U \times (\mathbb{D}_k^m(r))^{\times n}, Y) \arrow[d] \\ \mathrm{Map}_{\mathrm{dAnSt}_k}(\mathrm{t}_0(U) \times (\mathbb{D}_k^m(r))^{\times n}, V) \arrow[r] & \mathrm{Map}_{\mathrm{dAnSt}_k}(\mathrm{t}_0(U) \times (\mathbb{D}_k^m(r))^{\times n}, Y). \end{tikzcd}$$

This follows at once from the fact that the étale topoi of $U \times (\mathbb{D}_k^m(r))^{\times n}$ and $\mathrm{t}_0(U) \times (\mathbb{D}_k^m(r))^{\times n}$ coincide and from the universal property of étale morphisms proved in [Lur11d, Remark 2.3.4].

At this point, consider the commutative cube

$$\begin{tikzcd}[column sep=small] & \mathrm{Map}(U \times \mathrm{B}(\mathbb{D}_k^m(r))) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Map}(U \times \mathrm{B}(\mathbb{D}_k^m(r)), Y) \arrow[dl] \arrow[dd] \\ \mathrm{Map}(U, V) \arrow[rr, crossing over] \arrow[dd] & & \mathrm{Map}(U, Y) & \\ & \mathrm{Map}(\mathrm{t}_0(U) \times \mathrm{B}(\mathbb{D}_k^m(r)), V) \arrow[rr] \arrow[dl] & & \mathrm{Map}(\mathrm{t}_0(U) \times \mathrm{B}(\mathbb{D}_k^m(r)), Y) \arrow[dl] \\ \mathrm{Map}(\mathrm{t}_0(U), V) \arrow[rr] & & \mathrm{Map}(\mathrm{t}_0(U), Y). \arrow[from=uu, crossing over] & \end{tikzcd}$$

We just argued that the back square is a pullback. A similar (but easier) use of [Lur11d, Remark 2.3.4] shows that the front square is also a pullback. Therefore, in order to prove that the top square is a pullback, it is enough to prove that the bottom square is a pullback. We now observe that both Y and $\mathrm{t}_0(U) \times \mathrm{B}(\mathbb{D}_k^m(r))$ are underived. Furthermore, the functor of points of Y takes values in sets. It follows that

$$\mathrm{Map}(\mathrm{t}_0(U) \times \mathrm{B}(\mathbb{D}_k^m(r)), Y) \simeq \mathrm{Map}(\pi_0(\mathrm{t}_0(U) \times \mathrm{B}(\mathbb{D}_k^m(r)), Y) \simeq \mathrm{Map}(\mathrm{t}_0(U), Y).$$

This proves that the vertical morphisms in the diagram

$$\begin{tikzcd} \mathrm{Map}(\mathrm{t}_0(U) \times \mathrm{B}(\mathbb{D}_k^m(r)), V) \arrow[r] \arrow[d] & \mathrm{Map}(\mathrm{t}_0(U) \times \mathrm{B}(\mathbb{D}_k^m(r)), Y) \arrow[d] \\ \mathrm{Map}(\mathrm{t}_0(U), V) \arrow[r] & \mathrm{Map}(\mathrm{t}_0(U), Y) \end{tikzcd}$$

are equivalences, and hence that this diagram is a pullback. This completes the proof. ☐

We now use the results of **??** to extend the above result to a more general situation.

Proposition 7.4.3.3. *Let $\mathcal{X}$ be an ∞-topos with enough points. Let $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ be a connective and local $\mathcal{T}_{\mathrm{an}}(k)$-structure. Then the nonconnective structure $\mathcal{O}\widehat{\otimes}_k S^{\mathrm{an}}_{\mathcal{X}}(m,r)$ is local, and the natural transformation $\mathcal{O}\widehat{\otimes}_k S^{\mathrm{an}}_{\mathcal{X}}(m,r) \to \mathcal{O}$ is local as well.*

Proof. Observe that since $\mathcal{X}$ has enough points, in order to check that $\tau_{\geq 0}(\mathcal{O}\widehat{\otimes}_k S^{\mathrm{an}}_{\mathcal{X}}(m,r))$ is local, it is enough to check that for every geometric morphism $x_* \colon \mathcal{S} \leftrightarrows \mathcal{X} \colon x^{-1}$, the stalk $x^{-1}(\tau_{\geq 0}(\mathcal{O}\widehat{\otimes}_k S^{\mathrm{an}}_{\mathcal{X}}(m,r)))$ is local. Now,

$$x^{-1}(\tau_{\geq 0}(\mathcal{O}\widehat{\otimes}_k S^{\mathrm{an}}_{\mathcal{X}}(m,r))) \simeq \tau_{\geq 0}(x^{-1}\mathcal{O}\widehat{\otimes}_k x^{-1}(S^{\mathrm{an}}_{\mathcal{X}}(m,r))) \simeq \tau_{\geq 0}(x^{-1}\mathcal{O}\widehat{\otimes}_k S^{\mathrm{an}}_k(m,r)).$$

Using **??**, we can write

$$x^{-1}\mathcal{O} \simeq \operatorname*{colim}_{\mathcal{O}' \in \mathcal{E}} \mathcal{O}',$$

where $\mathcal{E} \subset \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{S})_{/x^{-1}\mathcal{O}}$ is the full subcategory spanned by germs of $\mathcal{T}_{\mathrm{an}}(k)$-structures (see **??**). As a consequence, we obtain

$$x^{-1}\mathcal{O}\widehat{\otimes}_k S^{\mathrm{an}}_k(m,r) \simeq \operatorname*{colim}_{\mathcal{O}'\in\mathcal{E}}(\mathcal{O}'\widehat{\otimes}_k S^{\mathrm{an}}_k(m,r)).$$

As $\mathcal{O}'$ is a germ of a $\mathcal{T}$-structure, we can apply Theorem 7.4.3.2 to deduce that $\tau_{\geq 0}(\mathcal{O}'\widehat{\otimes}_k S^{\mathrm{an}}_k(m,r))$ is a local structure. Using **??** again, we see that the ∞-category $\mathcal{E}$ is filtered. As $\tau_{\geq 0}$ commutes with filtered colimits, we finally conclude that

$$\tau_{\geq 0}(x^{-1}\mathcal{O}\widehat{\otimes}_k S^{\mathrm{an}}_k(m,r)) \simeq \operatorname*{colim}_{\mathcal{O}'\in\mathcal{E}}\tau_{\geq 0}(\mathcal{O}'\widehat{\otimes}_k S^{\mathrm{an}}_k(m,r)).$$

As local structures are closed under filtered colimits, the conclusion follows. □

7.4.4 Relative Van Est

Let $\mathcal{X}$ be an ∞-topos with enough points and let $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ be a local $\mathcal{T}_{\mathrm{an}}(k)$-structure. Let $p_* \colon \mathcal{X} \rightleftarrows \mathcal{S} \colon p^{-1}$ be the canonical geometric morphism. We set

$$S^{\mathrm{an}}_{\mathcal{X}}(n,r) \coloneqq p^{-1}S^{\mathrm{an}}_k(n,r).$$

Recall that $(S^{\mathrm{an}}_k(n,r))^{\mathrm{alg}} \simeq S_k(n)$. There is a canonical morphism

$$\eta_{\mathcal{O},n,r} \colon \mathcal{O}^{\mathrm{alg}} \otimes_k S_k(n) \longrightarrow (\mathcal{O}\widehat{\otimes}_k S_k(n,r))^{\mathrm{alg}}.$$

The main result of this section is the following:

Theorem 7.4.4.1. *For every $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$, the canonical morphism $\eta_{\mathcal{O},n,r}$ is an equivalence.*

We prove this theorem by several reduction steps. Consider the full subcategory $\mathcal{C}_{\mathcal{X}}$ of $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ spanned by those $\mathcal{O} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ such that $\eta_{\mathcal{O},n,r}$ is an equivalence. Notice that $\mathcal{O} \in \mathcal{C}_{\mathcal{X}}$ if and only if for every geometric point $x_* \colon \mathcal{S} \rightleftarrows \mathcal{X} \colon x^{-1}$, one has $x \in \mathcal{O} \in \mathcal{C}_{\mathcal{S}}$. We can therefore suppose that $\mathcal{X} = \mathcal{S}$. Recall now that both the functor $-\widehat{\otimes}_k S_k(n,r)$ and $(-)^{\mathrm{alg}}$ commute with filtered colimits. Using **??**, we are therefore reduced to prove the theorem in the case where $\mathcal{O} \simeq x^{-1}\mathcal{O}_X$ for $X \in \mathrm{dAn}_k$ and $x_* \colon \mathcal{S} \rightleftarrows \mathcal{X} \colon x^{-1}$ a given geometric point. For this, it is enough to prove that

$$\eta_{\mathcal{O}_X,n,r} \colon \mathcal{O}^{\mathrm{alg}}_X \otimes_k S_k(n) \longrightarrow (\mathcal{O}\widehat{\otimes}_k S_k(n,r))^{\mathrm{alg}}$$

is an equivalence in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}^{\mathrm{nc}}_{\mathrm{an}}(k)}(\mathcal{X}_X)$. We proceed by induction on the Postnikov tower of X. When X is underived, the result follows from Theorem 7.4.2.9. Suppose therefore that the statement has been proven for $X_n \coloneqq \mathrm{t}_{\leq n}(X)$. We have a pullback diagram in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$

$$\begin{tikzcd} \tau_{\leq n+1}\mathcal{O}_X \arrow[r] \arrow[d] & \tau_{\leq n}\mathcal{O}_X \arrow[d, "d"] \\ \tau_{\leq n}\mathcal{O}_X \arrow[r, "d_0"] & \tau_{\leq n}\mathcal{O}_X \oplus M, \end{tikzcd}$$

where $M \coloneqq \pi_{n+1}(\mathcal{O}_X)[n+2]$ and where d_0 corresponds to the zero derivation. As both functors $(-)^{\mathrm{alg}}$ and $-\otimes_k S_k(n)$ commute with limits, we conclude that the diagram

$$\begin{tikzcd} \tau_{\leq n+1}\mathcal{O}^{\mathrm{alg}}_X \otimes_k S_k(n) \arrow[r] \arrow[d] & \tau_{\leq n}\mathcal{O}^{\mathrm{alg}}_X \otimes_k S_k(n) \arrow[d] \\ \tau_{\leq n}\mathcal{O}^{\mathrm{alg}}_X \otimes_k S_k(n) \arrow[r] & (\tau_{\leq n}\mathcal{O}^{\mathrm{alg}}_X \oplus M) \otimes_k S_k(n) \end{tikzcd}$$

is again a pullback.

We now claim that the diagram

$$\begin{array}{ccc} \tau_{\le n+1}\mathcal{O}_X\widehat{\otimes}_k S_k^{\mathrm{an}}(n,r) & \longrightarrow & \tau_{\le n}\mathcal{O}_X\widehat{\otimes}_k S_k^{\mathrm{an}}(n,r) \\ \downarrow & & \downarrow \\ \tau_{\le n}\mathcal{O}_X\widehat{\otimes}_k S_k^{\mathrm{an}}(n,r) & \longrightarrow & (\tau_{\le n}\mathcal{O}_X \oplus M)\widehat{\otimes}_k S_k^{\mathrm{an}}(n,r) \end{array} \tag{7.4.4.1}$$

is a pullback square in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X}_X)$. For this, we observe that Theorem 7.4.3.1 allows to rewrite $\tau_{\le n}\mathcal{O}_X\widehat{\otimes}_k S_k^{\mathrm{an}}(n,r)$ as the pushforward on $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X}_X)$ of the structure sheaf of $\mathrm{t}_{\le n}(X)\times \mathrm{B}(\mathbf{D}^n_k(r))$, and similarly for $\tau_{\le n+1}\mathcal{O}_X\widehat{\otimes}_k S_k^{\mathrm{an}}(n,r)$ and $(\tau_{\le n}\mathcal{O}_X \oplus M)\widehat{\otimes}_k S_k^{\mathrm{an}}(n,r)$. We now observe that in the ∞-category of derived analytic geometric stack we have a pushout

$$\begin{array}{ccc} \mathrm{t}_{\le n}X[M] \times \mathrm{B}(\mathbf{D}^n_k(r)) & \longrightarrow & \mathrm{t}_{\le n}X \times \mathrm{B}(\mathbf{D}^n_k(r)) \\ \downarrow & & \downarrow \\ \mathrm{t}_{\le n}X \times \mathrm{B}(\mathbf{D}^n_k(r)) & \longrightarrow & \mathrm{t}_{\le n+1}X \times \mathrm{B}(\mathbf{D}^n_k(r)), \end{array}$$

which implies that (7.4.4.1) is a pullback. Using again the fact that $(-)^{\mathrm{alg}}$ commutes with limits and invoking the induction hypothesis, we see that we are reduced to prove that the canonical map

$$(\tau_{\le n}\mathcal{O}_X^{\mathrm{alg}} \oplus M)\otimes S_k(n) \longrightarrow ((\tau_{\le n}\mathcal{O}_X \oplus M)\widehat{\otimes}_k S_k^{\mathrm{an}}(n,r))^{\mathrm{alg}}$$

is an equivalence. Let $q\colon \mathrm{t}_{\le n}X \times \mathrm{B}(\mathbb{D}^n_k(r)) \to \mathrm{t}_{\le n}X$ be the natural projection. Then

$$\mathrm{t}_{\le n}X[M] \times \mathrm{B}(\mathbb{D}^n_k(r)) \simeq (\mathrm{t}_{\le n}X \times \mathrm{B}(\mathbb{D}^n_k(r)))[q^*M].$$

Therefore

$$\begin{aligned} ((\tau_{\le n}\mathcal{O}_X \oplus M)\widehat{\otimes}_k S_k^{\mathrm{an}}(n,r))^{\mathrm{alg}} &\simeq \varprojlim_{m\in\mathbf{\Delta}} \left(\tau_{\le n}\mathcal{O}_X\widehat{\otimes}_k S_k^{\mathrm{an}}(mn,r) \oplus M\otimes_k S_k^{\mathrm{an}}(mn,r)\right)^{\mathrm{alg}} \\ &\simeq \varprojlim_{m\in\mathbf{\Delta}} (\tau_{\le n}\mathcal{O}_X\widehat{\otimes}_k S_k^{\mathrm{an}}(mn,r))^{\mathrm{alg}} \oplus \varprojlim_{m\in\mathbf{\Delta}} M\otimes_k S_k(mn), \end{aligned}$$

whence the conclusion.

7.4.5 Analytic S^1-algebras

Recall that we denote by $\underline{S}^1$ the (derived) algebraic stack

$$\underline{S}^1\colon (\mathrm{dAff}_k^{\mathrm{afp}})\mathrm{op} \longrightarrow \mathcal{S}$$

obtained as sheafification of the constant presheaf associated to the space S^1. We denote by $(\underline{S}^1)^{\mathrm{an}}$ the (derived) analytic stack

$$(\underline{S}^1)^{\mathrm{an}}\colon \mathrm{dAfd}_k\mathrm{op} \longrightarrow \mathcal{S}$$

obtained as the analytification of $\underline{S}^1$. Equivalently, $(\underline{S}^1)^{\mathrm{an}}$ is the derived analytic stack obtained as sheafification of the constant presheaf associated to the space S^1. Notice that $(\underline{S}^1)^{\mathrm{an}}$ is in fact underived and that

$$\pi_0((\underline{S}^1)^{\mathrm{an}}) \simeq \mathrm{Sp}(k).$$

In particular, Theorem 7.4.2.5 implies that $\Gamma_k^{\mathrm{an}}((\underline{S}^1)^{\mathrm{an}})$ is a local nonconnective analytic ring.

Notation 7.4.5.1. We denote the local k-analytic ring $\Gamma_k^{\mathrm{an}}((\underline{S}^1)^{\mathrm{an}})$ simply by $\Gamma_k^{\mathrm{an}}(S^1)$. Let $\mathcal{X}$ be any ∞-topos. Let $p_*\colon \mathcal{X} \leftrightarrows \mathcal{S}\colon p^{-1}$ be the canonical geometric morphism. We set

$$\Gamma_{\mathcal{X}}^{\mathrm{an}}(S^1) \coloneqq p^{-1}\Gamma_k^{\mathrm{an}}(S^1) \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X}).$$

There is a third alternative way to describe the derived analytic stack $(\underline{S}^1)^{\mathrm{an}}$. We can in fact identify it with the delooping of the constant stack $(\underline{\mathbb{Z}})^{\mathrm{an}}$ associated to $\mathbb{Z}$, seen as a discrete topological space. As $\mathbb{Z}$ is initial among discrete groups, we see that there is a canonical morphism of group stacks

$$(\underline{\mathbb{Z}})^{\mathrm{an}} \longrightarrow \mathbb{D}^1(r).$$

In particular, applying the delooping functor provides us with a morphism

$$(\underline{S}^1)^{\mathrm{an}} \longrightarrow \mathrm{B}(\mathbb{D}^1(r)).$$

Passing to global sections, we obtain a morphism of local nonconnective analytic rings

$$\pi_r \colon S_k^{\mathrm{an}}(n,r) \longrightarrow \Gamma_k^{\mathrm{an}}(S^1).$$

Proposition 7.4.5.2. *Let $\rho \in |k^\times| \cup \{\infty\}$ be the converging radius of the exponential function. If $r \leq \rho$, then π_r is an equivalence.*

Proof. We know from Theorem 7.4.2.4 that the underlying algebra functor $(-)^{\mathrm{alg}}$ is conservative. It is therefore enough to prove that π_r^{alg} is an equivalence. Thanks to Theorem 7.4.2.8 and **??**, we see that it suffices to prove that the canonical morphism

$$\Gamma_k(S^1) \longrightarrow \Gamma_k^{\mathrm{an}}(S^1)^{\mathrm{alg}}$$

is an equivalence. In order to see this, we only need to prove that for every $n \geq 0$, one has

$$\mathrm{Map}_{\mathrm{dAnSt}_k}((\underline{S}^1)^{\mathrm{an}}, \mathrm{B}^n(\mathrm{B}G_a)) \simeq \mathrm{Map}_{\mathrm{dSt}_k}(\underline{S}^1, \mathrm{B}^n(\mathbb{G}_a)).$$

As $S^1 \simeq * \amalg_{* \amalg *} *$, we are reduced to check that

$$\mathrm{Map}_{\mathrm{dAnSt}_k}(\mathrm{Spec}(k)^{\mathrm{an}}, \mathrm{B}^n(\mathrm{B}G_a)) \simeq \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(k), \mathrm{B}^n(\mathbb{G}_a)),$$

which is tautological. □

Fix an ∞-topos $\mathcal{X}$. We are interested in studying the ∞-category of S^1-objects

$$S^1\text{-}\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X}) \coloneqq \mathrm{Fun}\left(\mathrm{B}(S^1), \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X})\right).$$

As in **??**, we are looking for a reformulation in terms of analytic comodules. In order to do this, we need first to discuss the structure of coalgebra of $\Gamma_k^{\mathrm{an}}(S^1)$.

Proposition 7.4.5.3. *Let $X \in \mathrm{dAnSt}_k$ be a derived analytic stack. Consider the cartesian diagram*

$$\begin{tikzcd} X \times (\underline{S}^1)^{\mathrm{an}} \arrow[r, "q'"] \arrow[d, "p'"] & (\underline{S}^1)^{\mathrm{an}} \arrow[d, "p"] \\ X \arrow[r, "q"] & \mathrm{Sp}(k) \end{tikzcd}$$

in dAnSt_k. Then the induced Beck-Chevalley transformation

$$q^* p_* \mathcal{O}_{(\underline{S}^1)^{\mathrm{an}}} \longrightarrow p'_* q'^* \mathcal{O}_{(\underline{S}^1)^{\mathrm{an}}}$$

in an equivalence in $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(X)$.

Proof. Recall that $S^1 \simeq * \amalg_{* \amalg *} *$. Hence

$$(\underline{S}^1)^{\mathrm{an}} \simeq \mathrm{Sp}(k) \amalg_{\mathrm{Sp}(k) \amalg \mathrm{Sp}(k)} \mathrm{Sp}(k),$$

and therefore

$$p_* \mathcal{O}_{S^1} \simeq k \times_{k \times k} k,$$

the product being taken in $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{S})$. Similarly, since colimits in dAnSt_k are universal, we have $X \times (\underline{S}^1)^{\mathrm{an}} \simeq X \amalg_{X \amalg X} X$ and therefore

$$p'_* q'^* \mathcal{O}_{S^1} \simeq p'_* \mathcal{O}_{X \times (\underline{S}^1)^{\mathrm{an}}} \simeq \mathcal{O}_X \times_{\mathcal{O}_X \times \mathcal{O}_X} \mathcal{O}_X,$$

the product being taken in $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X})$.

We are therefore reduced to prove that the canonical morphism

$$\mathcal{O}_X \widehat{\otimes}_k q^{-1}(k \times_{k \times k} k) \longrightarrow \mathcal{O}_X \times_{\mathcal{O}_X \times \mathcal{O}_X} \mathcal{O}_X$$

is an equivalence. We now invoke Theorem 7.4.5.2 to obtain an equivalence of $\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)$-structures

$$k \times_{k \times k} k \simeq S_k^{\mathrm{an}}(1, r),$$

where r is less than the converging radius of the exponential. Using Theorem 7.4.3.1 we now identify $\mathcal{O}_X \widehat{\otimes}_k q^{-1}(k \times_{k \times k} k)$ with the pushforward along $X \times \mathrm{B}(\mathbb{D}_k^1(r)) \to X$ of the structure sheaf of $X \times \mathrm{B}(\mathbb{D}_k^1(r))$, i.e. $S_{\mathcal{O}_X}^{\mathrm{an}}(1, r)$. Using Theorem 7.4.2.9, we can further identify the underlying algebra of $S_{\mathcal{O}_X}^{\mathrm{an}}(1, r)$ with $\mathrm{Sym}_{\mathcal{O}_X^{\mathrm{alg}}}(\mathcal{O}_X^{\mathrm{alg}}[-1])$. As the underlying algebra functor commutes with limits, we are therefore reduced to check that the canonical morphism

$$\mathrm{Sym}_{\mathcal{O}_X^{\mathrm{alg}}}(\mathcal{O}_X^{\mathrm{alg}}[-1]) \longrightarrow \mathcal{O}_X^{\mathrm{alg}} \times_{\mathcal{O}_X^{\mathrm{alg}} \times \mathcal{O}_X^{\mathrm{alg}}} \mathcal{O}_X^{\mathrm{alg}}$$

is an equivalence. Using [Lur11b, Proposition 2.11], we can further reduce to the case where $X \in \mathcal{T}_{\mathrm{an}}(k)$. In this case we can identify the underlying module of both sides with $\mathcal{O}_X^{\mathrm{alg}} \oplus \mathcal{O}_X^{\mathrm{alg}}[-1]$, whence the conclusion. □

Remark 7.4.5.4. Observe that $p'_* q'^* \mathcal{O}_{S^1}$ (and hence $q^* p_* \mathcal{O}_{S^1}$) belongs to $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X})$.

Corollary 7.4.5.5. *Let $\mathcal{D}$ be the smallest full subcategory of dAnSt_k closed under products and containing $\mathcal{T}_{\mathrm{an}}(k)$ and $(\underline{S}^1)^{\mathrm{an}}$. Then the restriction*

$$\Gamma_k^{\mathrm{an}} \colon \mathcal{D}\mathrm{op} \longrightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{S})$$

is strongly monoidal.

Proof. We first observe that the functor Γ_k^{an} is lax monoidal. Indeed, it is right adjoint to the functor

$$\mathrm{AnSpec} \colon \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{S}) \longrightarrow \mathrm{dAnSt}_k\mathrm{op}$$

that sends $A \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{S})$ to the derived analytic stack sending $U \in \mathrm{dAfd}_k$ to

$$\mathrm{AnSpec}(A)(U) \coloneqq \mathrm{Map}_{\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}}(A, \Gamma_k^{\mathrm{an}}(U)).$$

The Yoneda lemma implies immediately that AnSpec is strong monoidal. It follows that its right adjoint is lax monoidal.

Let now $X, Y \in \mathcal{D}$. Suppose at first that $X, Y \in \mathcal{T}_{\mathrm{an}}(k)$. Then for any $A \in \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{S})$, we have

$$A(X \times Y) \simeq A(X) \times A(Y).$$

Applying the Yoneda lemma, we therefore obtain

$$\mathrm{Map}_{\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}}(\Gamma_k^{\mathrm{an}}(X \times Y), A) \simeq \mathrm{Map}_{\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}}(\Gamma_k^{\mathrm{an}}(X), A) \times \mathrm{Map}_{\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}}(\Gamma_k^{\mathrm{an}}(Y), A),$$

and hence

$$\Gamma_k^{\mathrm{an}}(X) \widehat{\otimes}_k \Gamma_k^{\mathrm{an}}(Y) \simeq \Gamma_k^{\mathrm{an}}(X \times Y).$$

In general, we can write $X \simeq X' \times ((\underline{S}^1)^{\mathrm{an}})^{\times n}$ and $Y \simeq Y' \times ((\underline{S}^1)^{\mathrm{an}})^{\times m}$, where $X', Y' \in \mathcal{T}_{\mathrm{an}}(k)$. In this case, the conclusion follows by induction on n and m, using Theorem 7.4.5.3 to deal with the induction step. □

As a consequence of this corollary, we can promote $\Gamma_k^{\mathrm{an}}(S^1)$ to an object in $\mathrm{Comon}_{\mathbb{E}_\infty}(\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{S}))$. Observe furthermore that $\Gamma_k^{\mathrm{an}}((S^1)^{\times n})$ belongs to $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{S})$. We can therefore review $\Gamma_k^{\mathrm{an}}(S^1)$ as an object in $\mathrm{Comon}_{\mathbb{E}_\infty}(\mathrm{AnRing}_k^{\mathrm{nc}})$. If $\mathcal{X}$ is an ∞-topos and $p_* \colon \mathcal{X} \leftrightarrows \mathcal{S} \colon p^{-1}$ is the canonical geometric morphism, the functor

$$p^{-1} \colon \mathrm{AnRing}_k^{\mathrm{nc}} \simeq \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{S}) \longrightarrow \mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X})$$

is a left adjoint. This allows to identify $\Gamma_{\mathcal{X}}^{\mathrm{an}}(S^1)$ with an object in $\mathrm{Comon}_{\mathbb{E}_\infty}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X}))$.

Proposition 7.4.5.6. *Let $\mathcal{X}$ be an ∞-topos. There is a canonical equivalence*

$$S^1\text{-}\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X}) \simeq \mathrm{Comod}_{\Gamma_{\mathcal{X}}^{\mathrm{an}}(S^1)}(\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}^{\mathrm{nc}}(k)}(\mathcal{X})).$$

Proof. This is just a consequence of the comonadic version of Barr-Beck-Lurie's theorem. □

7.4.6 Nonconnective analytic square-zero extensions

We now turn our attention to the second main character in the HKR theorem: the analytic split square-zero extension.

Using **??** we obtain an equivalence

$$\mathrm{Mod}_k \simeq \mathrm{Sp}(\mathrm{AnRing}_k^{\mathrm{nc}}).$$

We simply denote the forgetful functor

$$\Omega^\infty\colon \mathrm{Sp}(\mathrm{AnRing}_k^{\mathrm{nc}}) \to \mathrm{AnRing}_k^{\mathrm{nc}}$$

by

$$M \mapsto k \oplus^{\mathrm{an}} M,$$

and we refer to it as the analytic split square-zero extension. Notice that Mod_k has an exotic monoidal structure given by

$$(M, N) \mapsto M \oplus N \oplus M \otimes_k N.$$

With respect to this monoidal structure, the functor Ω^∞ is lax monoidal.

Using Theorem 7.3.7.5, we obtain a canonical map

$$(k \oplus k[-1])^{\mathrm{an}} \longrightarrow k \oplus^{\mathrm{an}} k[-1],$$

which in general is not an equivalence.

On the other hand, using the formal nonconnective context, the generic fiber transformation and Theorem 7.3.6.10, we obtain a canonical map

$$S_k^{\mathrm{an}}(1, r) \longrightarrow k \oplus^{\mathrm{an}} k[-1].$$

Proposition 7.4.6.1. *If $r \leq \rho$, the map $S_k^{\mathrm{an}}(1, r) \to k \oplus^{\mathrm{an}} k[-1]$ is an equivalence.*

Proof. It is enough to apply $(-)^{\mathrm{alg}}$ and use Theorem 7.4.2.8. □

Using **??**, we can reprove (with exactly the same proof) the contractibility of the space of Hopf structures on $S_k^{\mathrm{an}}(1, r)$:

Proposition 7.4.6.2. *For every $r \in |k^\times| \cup \{\infty\}$, the space of Hopf structures on $S_k^{\mathrm{an}}(1, r)$ is contractible.*

Corollary 7.4.6.3. *The canonical map $S_k^{\mathrm{an}}(1, r) \to k \oplus^{\mathrm{an}} k[-1]$ can be promoted to an equivalence of analytic Hopf algebras.*

(i) The $\Omega^\infty_{\mathrm{an}}$ is oplax monoidal;

(ii) construction of the Beck-Chevalley transformation;

(iii) promotion of Beck-Chevalley to a transformation of Hopf algebras;

(iv) comparison between $\mathrm{B}(\mathbb{G}_a)$ and the split square-extension.

7.4.7 Mixed analytic rings

Definition 7.4.7.1. We define the ∞-category of mixed nonconnective analytic rings as

$$\varepsilon\text{-}\mathrm{AnRing}_k^{\mathrm{nc}} \coloneqq \mathrm{Comod}_{k \oplus^{\mathrm{an}} k[-1]}(\mathrm{AnRing}_k^{\mathrm{nc}}).$$

Proposition 7.4.7.2. *The forgetful functor*

$$\varepsilon\text{-}\mathrm{AnRing}_k^{\mathrm{nc}} \longrightarrow \mathrm{AnRing}_k^{\mathrm{nc}}$$

is monadic. The endofunctor underlying the monad is canonically equivalent to

$$A \mapsto \mathrm{AnDR}(A) \coloneqq \mathrm{Sym}_A^{\mathrm{an}}(^{\mathbb{L}}\mathrm{an}_A[1]).$$

We now provide an alternative construction for the category $\varepsilon\text{-AnRing}_k^{\mathrm{nc}}$.

Lemma 7.4.7.3. *The canonical morphism*

$$A^{\mathrm{alg}} \otimes (k \oplus k[-1]) \longrightarrow (A \widehat{\otimes} (k \oplus^{\mathrm{an}} k[-1]))^{\mathrm{alg}}$$

is an equivalence.

Proof. Need relative Van Est. □

7.4.8 Analytic HKR

Prove S^1-equivariant HKR (define mixed analytic rings as comodules over the analytic split square-zero extension).

Our goal is to prove the following theorem:

Theorem 7.4.8.1. *The forgetful functor $v_\eta^{\mathrm{an}} \colon k[\eta]^{\mathrm{an}}\text{-Comod}(\mathrm{AnRing}_k) \to \mathrm{AnRing}_k$ admits a left adjoint L. Furthermore, the composition $L \circ v_\eta^{\mathrm{an}}$ can be canonically identified with the assignment*

$$A \mapsto \mathrm{Sym}_A^{\mathrm{an}}(\mathbb{L}\mathrm{an}_A[1]).$$

Proof. The existence of L is a consequence of the adjoint functor theorem and the fact that the comonad $k[\eta]^{\mathrm{an}} \widehat{\otimes}_k -$ acts on A^{alg} by

$$A \mapsto A \oplus A[-1].$$

We now remark that for every $f \colon A \to B$ in AnRing_k, we have a fiber sequence

$$\begin{array}{ccc} \mathrm{Map}_{A^{\mathrm{alg}}\mathrm{Mod}}(\mathbb{L}\mathrm{an}_A[-1], f_* B^{\mathrm{alg}}) & \longrightarrow & \mathrm{Map}_{\mathrm{AnRing}_k}(\mathrm{Sym}_A^{\mathrm{an}}(\mathbb{L}\mathrm{an}_A[1]), B) \\ \downarrow & & \downarrow \\ \{f\} & \longrightarrow & \mathrm{Map}_{\mathrm{AnRing}_k}(A, B). \end{array}$$

Similarly, we have a fiber sequence

$$\begin{array}{ccc} \mathrm{Map}_{A^{\mathrm{alg}}\mathrm{Mod}}(\mathbb{L}\mathrm{an}_A[-1], f_* B^{\mathrm{alg}}) & \longrightarrow & \mathrm{Map}_{\mathrm{AnRing}_k^{\mathrm{nc}}}(A, k[\eta]^{\mathrm{an}} \widehat{\otimes}_k B) \\ \downarrow & & \downarrow \\ \{f\} & \longrightarrow & \mathrm{Map}_{\mathrm{AnRing}_k}(A, B). \end{array}$$

Finally, we observe that there is a canonical map

$$A \to k[\eta]^{\mathrm{an}} \widehat{\otimes}_k \mathrm{Sym}_A^{\mathrm{an}}(\mathbb{L}\mathrm{an}_A[1])$$

induced by $A \to \mathrm{Sym}_A^{\mathrm{an}}(\mathbb{L}\mathrm{an}_A[1])$. This map induces a morphism

$$\mathrm{Map}_{\mathrm{AnRing}_k}(\mathrm{Sym}_A^{\mathrm{an}}(\mathbb{L}\mathrm{an}_A[1]), B) \longrightarrow \mathrm{Map}_{\mathrm{AnRing}_k^{\mathrm{nc}}}(A, k[\eta]^{\mathrm{an}} \widehat{\otimes}_k B).$$

The above fiber sequences imply that this morphism is an equivalence.

Hence the functor $A \mapsto \mathrm{Sym}_A^{\mathrm{an}}(\mathbb{L}\mathrm{an}_A[1])$ is left adjoint to $A \mapsto k[\eta]^{\mathrm{an}} \widehat{\otimes}_k A$. This completes the proof. □

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