Contributions to quadratic backward stochastic differential equations with jumps and applications

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CHAPTER 1

INTRODUCTION

1.1 Backward stochastic differential equation with jumps

The first part of this thesis is dedicated to the study of backward stochastic differential equation with jumps with exponential quadratic growth and unbounded terminal condition. First we recall the classical notion of BSDEs in the Brownian setting as well as the jump framework. We also review the major results on this subject.

1.1.1 Backward stochastic differential equation: The continuous setting

Backward stochastic differential equations were first introduced by Bismut in 1973 [18] and in the last two decades they became one of the main tools to solve different financial mathematics problems, for instance, pricing problem, stochastic differential games. In 1990, Pardoux and Peng [113] generalized this notion for bounded BSDEs with drivers that satisfy a general non-linear Lipschitz condition. Furthermore, Duffie and Epstein [36] introduced these equations independently in the context of recursive utility.

Given a complete filtered measurable probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{\{0\leq t\leq T\}}, \mathbb{P})$ generated by a d-dimensional Brownian motion B, a backward stochastic differential equation is presented as follows

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \ \mathbb{P}\text{-a.s.} \tag{1.1}$$

This equation is characterized by:

- Terminal condition ξ which is an $\mathcal{F}_T$-measurable random variable.
- Driver $f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ which is a progressively measurable process.

According to [113], solving this equation consists of finding a couple of progressively measurable processes $(Y_t, Z_t)_{0\leq t\leq T}$ taking values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ satisfying (1.1) such that

$$\mathbb{E}[\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T |Z_s|^2 ds] < +\infty. \tag{1.2}$$

The control process Z_t steers the process Y_t towards the terminal condition ξ and ensures the adaptedness of Y_t to the underlying filtration.

Before proceeding further, let us recall some classical existence and uniqueness results for BSDEs. We start by introducing a necessary assumption.

Assumption 1.1.1.

- *Integrability condition: $f_t(0,0)$ and ξ are square integrable.*
- *Lipschitz condition: The driver is Lipschitz in (y,z) and uniformly in (t,w) i.e. There exists a constant K such that, for all $(y,\bar{y},z,\bar{z})$ we have*

$$|f_t(y,z) - f_t(\bar{y},\bar{z})| \leq K(|y-\bar{y}| + |z-\bar{z}|).$$

Classically the solution (Y,Z) is defined in the following spaces

- $\mathbb{H}^2$ the set of all $\mathbb{R}^{k\times d}$-valued $\mathcal{F}_t$-progressively measurable processes Z such that

$$\mathbb{E}[\int_0^T |Z_s|^2 ds] < +\infty.$$

- $\mathcal{S}^2$ is the space of $\mathbb{R}^k$-valued $\mathcal{F}_t$-progressively measurable processes Y such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^2\right] < +\infty.$$

Theorem 1.1.2. *(Pardoux-Peng [113]) Under Assumption 1.1.1, the BSDE (1.1) has a unique solution $(Y,Z) \in \mathbb{H}^2(\mathbb{R}^k) \times \mathbb{H}^2(\mathbb{R}^{k\times d})$.*

The proof of this theorem is based on fixed point argument.
Another important result in the one-dimensional case is the comparison theorem given in the paper of El Karoui and al [43]. This result says that once we can compare the terminal conditions and the coefficients of two different BSDEs, the corresponding solutions can be ordered. More precisely, the comparison theorem given by EL Karoui and al. in [43] and relaxed in [39] is stated as follows

Theorem 1.1.3. *Let (Y,Z) and $(\bar{Y},\bar{Z})$ be two solutions of BSDEs associated respectively to (f,ξ) and $(\bar{f},\bar{\xi})$. Then, if whether f or $\bar{f}$ satisfies Assumption 1.1.1 and*

- $\xi \leq \bar{\xi}$, $\quad f(Y_t,Z_t) \leq \bar{f}(Y_t,Z_t), \quad \forall t \in [0,T]$, $\mathbb{P}$-*a.s.*

Then we have $Y_t \leq \bar{Y}_t, \ \forall t \in [0,T], \ \mathbb{P}$-a.s.

Several works proposed different approaches or techniques to prove that under weaker assumptions, either on the driver or on the terminal condition, we can guarante the existence of the solution. We can mention the case where

- f is locally Lipschitz and grows sub-linearally in z (see Hamadène [60]).
- f has linear growth (see Lepeltier and San Martin [90]).
- f is uniformly Lipschitz in z and satisfies a monotone condition (see Pardoux [112]).
- f is Lipschitz in z and has polynomial growth in y (see Carmona and Briand [21]).

In these papers, authors show existence of the solution using the monotone stability approach, that is based on approximation of the generator by a converging sequence. The problem is then reduced to showing the monotone convergence of the approximated BSDEs to the original one.

Quadratic BSDEs

Since their introduction, the main innovation about BSDEs is the study of quadratic BSDEs. This type of equations have been an object of intensive study during the last decades mainly due to the wide range of applications including financial problem (dynamic risk measure, portfolio maximization problem, etc).
As far as we know, BSDEs of this type were first considered by Schroder and Skiadas [125] in 1999 and followed by Kobylanski [84] who showed existence of the solution under the following assumption

Assumption 1.1.4. *Let $\alpha_0, \beta_0, b \in \mathbb{R}$ and c be an increasing continuous function. For all $(t, v, z) \in \mathbb{R}_+ \times \mathbb{R}^{1+d}$*

$$F(t, v, z) = a_0(t, v, z)v + F_0(t, v, z)$$

and

$$\beta_0 \le a_0(t, v, z) \le \alpha_0, \qquad |F_0(t, v, z)| \le b + c(|v|)|z|^2.$$

The existence and uniqueness result of Kobylanski [84] are stated as follow

Theorem 1.1.5. *Assume that $\xi \in \mathbb{L}^\infty$ and f satisfies Assumption 1.1.4, then the BSDE (1.1) has at least one solution $(Y_t, Z_t)_{0\le t\le T}$ such that Y is bounded a.s and $Z \in \mathbb{H}^2$. Moreover, there exists a minimal solution (resp.a maximal solution).*

The idea of the proof was inspired by the partial differential equation technics and is based on the following road mapp.
In the first step, we truncate the driver with respect to y and z to obtain a continuous monotone sequence of generators with linear growth for which solutions do uniquely exist. In the second

step, we show that the limit of these solutions exist and solves the original BSDE. Nonetheless, the technical difficulty in the so-called Kobylanski method is the strong convergence of the martingale part.
The uniqueness of the solution is also stated in this paper and based on the following comparison theorem.

Theorem 1.1.6. *Let (ξ^1, f^1) and (ξ^2, f^2) be two sets of parameters such that*

- $\xi^1 \le \xi^2$ *and* $f^1_t(Y_t, Z_t) \le f^2_t(Y_t, Z_t), \qquad \forall t \in [0,T], \ \mathbb{P}$*-a.s.*
- f^1 *or* f^2*satisfies the following assumption: For all* $M \in \mathbb{R}_+$ *there exist two constants* b *and* c *such that for all* $t \in [0,T]$ *,* $|y| \le M$ *and* $\forall z \in \mathbb{R}^d$*,*

$$|f(t,y,z)| \le b + c|z|^2 \text{ and } |\frac{\partial f}{\partial z}(t,y,z)| \le b + c|z|,$$
$$\frac{\partial f}{\partial y}(t,y,z) \le b + c|z|^2.$$

Then, if (Y^1, Z^1) and (Y^2, Z^2) are two solutions of the BSDE (1.1) associated respectively to (f^1, ξ^1) and (f^2, ξ^2), one has $Y^1_t \le Y^2_t, \ \forall t \in [0,T], \mathbb{P}$-a.s.

Another approach has been developed by Tevzadze [129] concerning well-possedness of this type of equations. More precisely, the author proved the existence and uniqueness of the solutions for BSDEs under locally Lipschitz-quadratic conditions. Although, this approach is limited to small bounded terminal condition, the strong convergence of martingale part is no longer needed.

Another important aspect of the theory of quadratic BSDEs is the well-posednes of these equation under weaker terminal condition. We would like to illustrate this through the following brief example. Consider a quadratic BSDE of the form

$$\begin{cases} dY_t = \frac{1}{2}|z|^2 dt - Z_t dB_t, & \forall t \in [0,T], \mathbb{P}\text{-a.s}, \\ Y_T = \xi. \end{cases}$$

Applying Itô's formula to $\exp(Y_t)$ and taking the conditional expectation, it is easy to see that $Y_t = \ln(\mathbb{E}[\exp(\xi)|\mathcal{F}_t])$ is a solution of the above BSDE. With this simple example we see that it is sufficient to consider a terminal condition with finite exponential moment.
This direction was first explored by Briand and Hu in [23, 24]. They showed existence and uniqueness of the solution under the following assumption

Assumption 1.1.7.

- $\forall t \in [0,T]$, $(y,z) \to f(t,y,z)$ *is continuous* $\mathbb{P}$*-a.s,.*

- $\forall (t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$, $\beta\geq 0$, $\gamma>0$, $\alpha\geq\frac{\beta}{\gamma}$, $|f(t,y,z)|\leq\alpha+\beta|y|+\frac{\gamma}{2}|z|^2$.
- $\exists\lambda>\gamma e^{\beta T}$, $\mathbb{E}[e^{\lambda\xi}]<+\infty$.

This result was followed by Delbaen et al [34] for a super quadratic growth driver under a specific Markovian assumption. The authors also argue that when the terminal condition is bounded, there are cases of BSDEs without any solution as well as infinitely many solutions.

A lot of research followed until Barrieu and EL Karoui [12] proposed in 2013 what is now known as "the forward approach". This is a new technique to deal with the well-posedness of quadratic BSDE allowing to find a solution in a direct way. The method is based essentially on a special class of semimartingale defined as follows

Definition 1.1.8. *A quadratic semimartingale Y is a continuous process such that $Y=Y_0-V+M$ with V a local finite variation process and M a local martingale with the following structure condition $\mathcal{Q}(\Lambda,C,\delta)$: For an increasing predictable processes C, Λ, and a constant δ*

$$-\frac{\delta}{2}d\langle M\rangle_t-d\Lambda_t-|Y_t|dC_t\ll dV_t\ll\frac{\delta}{2}d\langle M\rangle_t+d\Lambda_t+|Y_t|dC_t. \tag{1.3}$$

Note that the symbol "$\ll$" means that the difference is an increasing process.

The authors showed that a quadratic semimartingale Y who lives in the spaces $\mathcal{D}^{\exp}$ is in fact a quasi entropic submartingale .i.e for all stopping times σ, τ,

$$\ln\mathbb{E}[\exp(-Y_\tau)|\mathcal{F}_\sigma]\leq Y_\sigma\leq\ln\mathbb{E}[\exp(Y_\tau)|\mathcal{F}_\sigma]. \tag{1.4}$$

Via the exponential submartingale inequality (1.4), they derive the characterization of this semimartingale as well as integrability properties to investigate the behavior of this semimartingale as the limit of a sequence of quadratic semimartingale based on an stability result of Barlow and Protter [11]. Moreover, they prove that the quadratic semimartingale is stable in a suitable class and obtain different types of convergence of the martingale part.
The novelty of this method is that, it gives in a direct way, an existence result for quadratic BSDE. To the best of our knowledge, this is the only paper who gave a general existence result for quadratic BSDEs in the Brownian setting.

What about the uniqueness? The question of uniqueness is much more complicated then the existence of the solution. In the BSDEs literature the uniqueness result is only obtained under very limited conditions. More precisely, it requires additional assumptions on the driver and stands only for bounded terminal condition. We highlight that, Briand and Hu showed in [24] the uniqueness of quadratic BSDE's solution for unbounded terminal condition under stronger

hypothesis. More precisely, they proved comparison theorem where the driver f is convex with respect to the control process. Using this property and a localization procedure, they obtained a comparison result which leads to the uniqueness of the solution. This result was extended later by Mocha and Westray [107] to semimartingale BSDEs.
As explained before, BSDEs are of great theoretical interest due to their connection with various domains including finance, physics or biology. For an extensive overview of applications, we refer the reader to the relevant papers of El Karoui et al [120], Hu, Imkeller and Muller [73].

1.1.2 Backward stochastic differential equation : The jump setting

One of the directions who attracted many researchers is the case involving a discontinuity in the dynamics of the state solution. In this setting, the BSDE is no longer driven by Brownian motion but also by a random jump measure. To be more precisely, let a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which the filtration $\mathbb{F} = (\mathcal{F}_t)_{0<t<T}$ satisfies the usual conditions of completeness and right continuity. On this stochastic basis, let W a d-dimensional Brownian motion and $\mu(\omega, dt, de)$ an independent integer valued random measure defined on $([0,T] \times E, \mathcal{B}([0,T]) \otimes \mathcal{B}(E))$, with compensator $\nu(\omega, dt, de)$.
The BSDE with jumps were first introduced and studied by Tang and Li [128] and it has the following form

$$Y_t = \int_t^T f(s, Y_{s^-}, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{\mu}(dt, de), \ \forall t \in [0,T], \mathbb{P}\text{-a.s.} \tag{1.5}$$

This equation involves a second stochastic integral with respect to a compensated random measure. More precisely, the compensated random measure $\tilde{\mu}$ is given by

$$\tilde{\mu}(\omega, dt, de) = \mu(\omega, dt, de) - \nu(\omega, dt, de),$$

where the associated compensator ν is absolutely continuous with respect to $\lambda \otimes dt$ such that

$$\nu(\omega, dt, de) = \zeta(\omega, t, e)\lambda(de)dt, \qquad 0 \leq \zeta \leq C_\nu, \quad \text{for some constant } C_\nu.$$

for a σ-finite measure λ on $(E, \mathcal{B}(E))$ satisfying $\int_E 1 \wedge |e|^2 \lambda(de) < \infty$ and a bounded $\mathcal{P} \otimes \mathcal{B}(E)$-measurable non negative density function ζ. Here, $\mathcal{P}$ is the predictable σ-field on $\Omega \times [0,T]$.

In [128], Tang and Li studied existence and uniqueness of square integrable solutions under global Lipschitz assumption where the jump is of Poisson type. A solution of the BSDEJ associated to (f, ξ) consists of a triple $(Y, Z, U) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ of progressively measurable processes such that the equation (1.5) holds.

Here $\mathcal{S}^2$ is the set of real valued RCLL adapted process Y such that

$$\|Y\|^2_{\mathcal{S}^2} := \mathbb{E}[\sup_{0\le t\le T} |Y_t|^2] < +\infty.$$

The space associated to the jump component U is $\mathbb{H}^2_\nu$ which is the set of all $\tilde{\mathcal{P}}$-predictable processes U such that

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{E}} |U_s(e)|^2 \nu(de, dt)\right)\right]^{\frac{1}{2}} < +\infty,$$

where $\tilde{\mathcal{P}}$ is the respective σ-field on $\Omega \times [0, T] \times E$. We also need to define the following space. For $u, \bar{u} \in \mathbb{L}^0(\mathcal{B}(E), \nu)$, let

$$|u - \bar{u}|^2_t = \int_E |u(e) - \bar{u}(e)|^2 \zeta(t, e)\lambda(de).$$

Under the following standard assumption, Barles, Buckdahn and Pardoux [10], established an existence result in order to give a probabilistic interpretation of viscosity solution of semilinear integral Partial equations.

Assumption 1.1.9.

- *The terminal data* $\xi \in \mathbb{L}^2_T$.
- $\mathbb{E}[\int_0^T |f_s(0, 0, 0)|^2 ds] < +\infty$.
- *The generator* f *is Lipschitz continuous in the following sens: for all* $(t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(\mathcal{B}(E), \nu)$,

$$|f_t(y, z, u) - f_t(\bar{y}, \bar{z}, \bar{u})|^2 \le K[|y - \bar{y}|^2 + |z - \bar{z}|^2 + \int_E |u(e) - \bar{u}(e)|^2 \nu(dt, de)].$$

BSDEs with jumps (BSDEJs) and semi-linear partial differential equation (PDEs)

Let us focus now on how BSDEJs are linked to semi-linear PDEs. The starting point is the Markovian BSDEJs. From a diffusion process $(X_s)_{0\le s\le T}$ defined as the strong solution of the following standard stochastic differential equation (SDE):

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s + \int_0^t \int_E \beta(X_{s^-}, e)\tilde{\mu}(ds, de), \ \ \forall t \in [0, T], \mathbb{P}\text{-a.s}, \quad (1.6)$$

we define the Markovian BSDE as follows

$$Y_t = g(X_T) + \int_t^T f_s(X_s, Y_s, Z_s, V_s)ds - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e)\tilde{\mu}(de, dt), \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.} \quad (1.7)$$

Consider now the following PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \mathcal{K}u(t,x) + f(t,x,u(t,x),\nabla u(t,x)\sigma(t,x),Bu(t,x)) = 0, & \forall (t,x) \in [0,T]\times\mathbb{R}^d, \\ u(T,x) = g(x), \qquad \forall x \in \mathbb{R}^d, \end{cases} \tag{1.8}$$

where $\mathcal{K}$, B are defined by

$$\mathcal{K}u(t,x) := \mathcal{L}u(t,x) + \int_E [u(x+\beta(x,e)) - u(x) - \nabla u(x).\beta(x,e)]\lambda(de). \tag{1.9}$$

$$Bu(x) = \int_E [u(x+\beta(x,e)) - u(x)]\gamma(x,e)\lambda(de). \tag{1.10}$$

If we suppose that the PDE (1.8) has a smooth solution then, using Itô formula we can prove that the BSDEJ (1.7) admits $(Y_t, Z_t, V_t)_{0\le t\le T} := (u(t,X_t), \nabla u(t,X_t)\sigma(t,X_t))_{0\le t\le T}$ as a unique solution.
This result can be understood as the probabilistic representation of the solution of the semilinear PDE (1.8) via the solution of the BSDE (1.7). This probabilistic representation, also called Feynman-Kac representation is an interesting tool to solve PDE numerically and to prevent from the dimension problem. Various results have been obtained in this subject. Among then we quote Bouchard and Touzi [20], Gobet and al. [53].

As for the Brownian setting, the comparison theorem for BSDEJs is an important result since it implies the uniqueness of the solution. Although the comparison theorem holds true for Lipschitz BSDEs in the Brownian one dimensional case, it need further assumption to control the increment of the jump component in the discontinuous setting.

Royer [121] solves this problem and introduce necessary and sufficient condition to obtain a comparison result. This condition namely "A_γ-condition" is stated as follows.

Assumption 1.1.10. *There exits $-1 \le C_1 \le 0$ and $C_2 \ge 0$ such that $\forall y \in \mathbb{R}, \forall z \in \mathbb{R}^d$ and $\forall v, \bar{v} \in \mathbb{L}^0(\mathcal{B}(E),\nu)$.*

$$f_t(y,z,v) - f_t(y,z,\bar{v}) \le \int_E \gamma_t^{y,z,v,\bar{v}}(e)[v(e)-\bar{v}(e)]\nu(de), \qquad \forall t\in[0,T], \mathbb{P}\text{-a.s},$$

where $\gamma^{y,z,v,\bar{v}}$ is a $\mathcal{P}\otimes\mathcal{B}(R^{1+d})\otimes\mathcal{B}(E)$-measurable function satisfying $C_1(1\wedge|e|) \le \gamma_t(e) \le C_2(1\wedge|e|)$.

The comparison theorem is then stated jointly under the following assumption.

Assumption 1.1.11.

- *The terminal condition $\xi \in \mathbb{L}^2_T$.*

- $\mathbb{E}[\int_0^T |f_s(0,0,0)|^2 ds] < +\infty$.
- *The driver f is Lipschitz with respect to y, z.*

Theorem 1.1.12 (Royer [121]). *Assume that 1.1.11 and 1.1.10 are fulfilled. Consider (Y, Z, U) and $(\bar{Y}, \bar{Z}, \bar{U})$ the respective solutions of BSDEJs driven by (f, ξ) and $(\bar{f}, \bar{\xi})$. If $\xi \leq \bar{\xi}$ and $f_t(y, z, u) \leq f_t(\bar{y}, \bar{z}, \bar{u})$, then $Y_t \leq \bar{Y}_t, \quad \forall t \in [0, T], \mathbb{P}$-a.s.*

In 2006, Becherer et al. [15] studied the existence and uniqueness of bounded BSDE's solution when the generator is Lipschitz in (y, z) but not necessarily Lipschitz in the jump component. In this paper, the BSDE is driven by a Brownian motion and a random measure that is possibly inhomogeneous in time with finite jump activity. More precisely, the authors considered a driver with the following form

$$f_t(y, z, u) = \hat{f}_t(y, z, u) + \int_E g_t(u(e))\nu(de). \tag{1.11}$$

Assumption 1.1.13.

- $|\hat{f}_t(y, z, u) - \hat{f}_t(\bar{y}, \bar{z}, \bar{u})| \leq K[|y - \bar{y}| + |z - \bar{z}| + (\int_E |u(e) - \bar{u}(e)|^2 \nu(dt, de))^{\frac{1}{2}}]$.
- $\exists K^1, K^2 \in [0, +\infty)$ *such that* $|\hat{f}_t(y, z, u)| \leq K^1 + K^2|y|$.
- *g is locally Lipschitz in u, uniformly in (t, w) and*

$$\begin{cases} g(t, u) \leq |u|, & \text{if } u \leq 0, dt \otimes \mathbb{P}\text{-a.s} \\ g(t, u) \geq -|u|, & \text{if } u \geq 0, dt \otimes \mathbb{P}\text{-a.s.} \end{cases} \tag{1.12}$$

Under the above assumption, the existence result is stated as follows

Theorem 1.1.14. *(Becherer et al [15]) Let $\xi \in \mathbb{L}^\infty$. Assume that $\lambda(de)$ is finite and Assumption1.1.13 is fullfield. Then there exists a unique solution (Y, Z, U) in $S^\infty \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ to the BSDEJ* (1.5).

More recently, Becherer et al. continued in [14] their study by considering an extension to the infinite activity jumps. Moreover, the authors analyzed a jump market model in which they derive generalized good-deal valuation bounds for possibly path-dependent contingent claims using this class of BSDEJ. They also draw a close link between the BSDEs with infinite activity and the utility maximization problem in finance for power and exponential utility functions.

Quadratic BSDE with jumps

Many attempts have been suggested to relax the assumptions on the driver f as well as on the terminal condition in the discontinuous framework. In particular quadratic BSDEJ have found much attention. However, only few general existing results concerns the quadratic case. Most of the works are arising form utility maximization problem and hence deals with the wellposedness of BSDEJ with a specific form of f.

This gap has been successfully filled by El Karoui, Matoussi and Ngoupeyou [42] in the case of unbounded BSDEs with jumps, where the existence of the solution is provided by means of forward approach. In this paper the driver satisfies the following quadratic exponential structure condition

$$\underline{q}_t(y,z,u) = -\frac{1}{\delta}j_t(-\delta u) - \frac{\delta}{2}|z|^2 - l_t - c_t|y| \leq f_t(y,z,u) \leq \frac{1}{\delta}j_t(\delta u) + \frac{\delta}{2}|z|^2 + l_t + c_t|y| = \bar{q}_t(y,z,u),$$

with l, c are non negative processes and δ is a constant and

$$j_t(\delta u) := \int_E \frac{e^{\delta u_t(e)} - \delta u_t(e) + 1}{\delta}\nu(dt,de).$$

Let us point out that the choice of this quadratic exponential structure condition seems to be natural and realistic. In fact if we look the following canonical example of this type of BSDEJs

$$Y_t = \xi + \int_t^T \frac{\delta}{2}|Z_t|^2 + \frac{1}{\delta}j_t(\delta U)dt - \int_t^T Z_s dW_s - \int_t^T \int_E V_s(e)\tilde{\nu}(dt,de). \tag{1.13}$$

the solution is the process defined as $\rho_{\delta,t}(\eta_T) = \frac{1}{\delta}\ln \mathbb{E}\left[\exp(\delta\eta_T)\Big|\mathcal{F}_t\right]$. Under the simplest driver structure, we see that the related BSDEJ can characterize an entropic dynamic risk measure (see [42]) .
The approach exposed in [42] permits to relate these BSDEJs to a quadratic exponential semimartingale. In fact, this specific type of semimartingale yields to a suitable dominated inequalities in terms of entropic processes:

$$\rho_{-\delta,t}(\underline{U}_T) \leq Y_t \leq \rho_{\delta,t}(\overline{U}_T), \tag{1.14}$$

where $\underline{U}_T$ and $\overline{U}_T$ are two random variables depending only on l, c, δ and ξ.
From this dominated inequalities, the structure properties on the martingale part as well as the finite variation part are established. Under sufficient integrability condition, we can derive from the Doob decomposition of this class of semimartingale a general stability result. Then, existence of the solution of the BSDEJs follows from this stability theorem.
Nonetheless, we have to point out that the extension of the forward approach to the jumps framework is not straightforward or trivial. In fact, the presence of the jump induces many technical difficulties and requires some specific arguments.

Afterward, Kazi-Tani, Possamai and Zhou [117] established existence of a solution of Lipschitz quadratic BSDE with jumps by adopting the approach developed in [129] where the main tool is the fixed point argument. More precisely, under the following condition

- f is uniformly Lipschitz in y.
- There exits $\mu > 0$ and $\phi \in \mathbb{H}^2_{BMO}$ such that $\forall (t, y, z, z^{'}, u)$

$$|f_t(y, z, u) - f_t(y, z^{'}, u) - \phi_t(z - z^{'})| \leq \mu|z - z^{'}|(|z|^2 + |z|^2).$$

- There exit $\nu > 0$ and $\psi \in \mathbb{J}^2_{BMO}$ s.t $C_1(1 \wedge x) \leq \psi_t(x) \leq C_2(1 \wedge x)$.

$$|f_t(y, z, u) - f_t(y, z^{'}, u) - \langle \psi_t, z - z^{'} \rangle_t| \leq \mu|u - u^{'}|_t(|u|^2_t + |u|^2_t).$$

- $f_t(0, 0, 0)$ verify some integrability condition.

where $\|\phi\|_{\mathbb{J}_{BMO}} = \| \int_0^{\cdot} \int_E \phi_s(e)\tilde{\mu}(de, ds)\|_{BMO}$ for all predictable and $\mathcal{B}(E)$-measurable process U defined on $\Omega \times [0, T] \times E$ and $\|Z\|_{\mathbb{H}^2_{BMO}} = \| \int_0^{\cdot} Z_s dW_s\|_{BMO}$ for all $\mathbb{R}^d$-valued and $\mathcal{F}_t$-progressively measurable process Z. The authors show that if we assume that $\|\xi\|_\infty$ is small enough we can obtain existence of the solution. The construction of the solution is done via a fixed point argument. However, unlike the lipschitz case, the fixed point argument falls when f has local Lipschitz and quadratic growth. Regarding this and chosen a weighted norm of ξ , they prove that in sufficiently small ball the contraction can be obtained and hence the map has a fixed point. The authors also prove that under stronger assumption on f we can have the existence of the solution of the BSDEJ for any bounded terminal condition.

Recently, Fujii and Takahashi in [52] proved the existence and uniqueness of bounded solution under the exponential quadratic structure of [42]. They also derive sufficient conditions for the Malliavin differentiability of the solution.

In the context of exponential utility maximization problem, Morlais investigates in [108, 109] the solvability of bounded BSDEJs with the following specific driver

$$f^n_t(z, u) = \inf_{\pi \in C} [\frac{\alpha}{2}|\pi_t\sigma_t - (z + \frac{\theta}{\alpha})|^2 + \int_E j_\alpha(u - \pi_t\beta_t)\nu(de)] - \theta_t z - \frac{|\theta_t|^2}{2\alpha}.$$

The author shows the existence and uniqueness results of BSDEs governed by Wiener process and Poisson random measure using the monotone stability approach. In this paper, the particular case of admissible strategies valued in a compact set is considered in a model with bounded coefficients. More precisely, she characterized the associated value function as the unique solution of a quadratic BSDEJs using the well known dynamic programming principle.

A very recent result in this subject is the work of Laeven and Stadje [86], where they investigate the effect of ambiguity on a portfolio choice and indifference valuation problem. Using the well known dual approach, they express the value function in term of bounded BSDEJ's solution who grows at most quadratically. They proved a general existence result of BSDEJ with infinite activity jumps. However, the methodology adopted in this paper requires convexity assumption on the driver.

When the terminal condition is no longer bounded, the literature is less abounding. In fact the only existing reference in this subject is the paper of El Karoui, Matoussi, and Ngoupeyou [42]. As explained before, the approach adopted in this paper is the forward approach and the terminal condition is considered with finite exponential moment as follow

$$\forall\gamma > 0 \ \mathbb{E}\left[\exp\left(\gamma\left(e^{C_{t,T}}|\xi| + \int_t^T e^{C_{t,s}} d\Lambda_s\right)\right)\right] < +\infty.$$

The common feature in all these works except [42] is the use of BMO martingale. This notion plays a crucial role to prove the uniqueness of solutions in the continuous and discontinuous setting. In general, the uniqueness of BSDE's solution where the coefficient is no longer Lipschitz relies on two arguments:

- Classical linearization technic.
- Change of probability measure (Girsanov theorem).

Hence, to justify the use of Girsanov theorem, we are naturally lead to the BMO structure. More precisely, a square integrable càdlàg, $\mathbb{R}^d$-valued martingale M on a compact interval $[0,T]$ with $M_0 = 0$ is a BMO (bound mean oscillation) martingale if

$$\operatorname*{ess\,sup}_{\tau} \|\mathbb{E}^{\mathcal{F}_\tau}\left[[\langle M, M\rangle_T - \langle M, M\rangle_\tau]\right]\|_\infty \leq \infty, \tag{1.15}$$

where $\|X\|_\infty = \sup_{\omega\in\Omega}|X|$ for any bounded random variable and the supremum is taken over all stopping times τ valued in $[0,T]$. The first important property of this class is that if M is a càdlàg BMO martingale such that $\Delta M_t > -1, \forall t \in [0,T], \mathbb{P}$-a.s, then the stochastic exponential $\mathcal{E}(M)$ is strictly positive uniformly integrable martingale. A consequence of this result is that the probability measure $dQ = \mathcal{E}(M_T)/d\mathbb{P}$ is well defined. Moreover, $\tilde{M} = M + \langle M\rangle$ is a BMO(Q) martingale.
For further theoretical result about BMO martingales, we refer the interested reader to [81], [80].

Connection with financial and economics problem

When the jumps arise through a standard Poisson process or a Lévy process, several papers have been devoted to this type of equation. Among them, we quote Morlais [108], Yin and Mao [133], Kharoubi and Lim [82], Jiao and al [79], Ankirchner [4]. However, in a general setting that allows infinite activity jumps in the price process, only few papers studied this problem. Morlais [109] considered utility maximization problem in incomplete market and proved that the value process can be characterized as a solution of a specific quadratic BSDE with infinite activity jumps.
In [14], Becherer et al also analyzed a jump market model in which they derive generalized good-deal valuation bounds for possibly path-dependent contingent claims using this class of BSDEJ. They also draw a close link between the BSDE with infinite activity and the utility maximization problem in finance for power and exponential utility functions.

Robust utility maximization problem
In general the problem can be formulated as follows

$$V := \sup_{\pi \in \mathbb{A}} \inf_{Q \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}}[U(X_T^\pi - \xi)], \tag{1.16}$$

where $\mathcal{P}$ is the set of all possible probability measures and $\mathbb{A}$ is the set of admissible trading strategies. In contrast with standard utility maximization problem in which the "historical" probability $\mathbb{P}$ is well known, the investor has some uncertainty on the probability that describes the dynamics of the underlying asset. More precisely, instead of solving the utility maximization problem over one measure, a family of probability measures is considered.
In this context, the main goal of the investor is to find the optimal strategy under the worst scenario. In order to solve the problem, the information on the set $\mathcal{P}$ is fundamental where we can distinguish two cases, the dominated and non dominated case. So the set $\mathcal{P}$ is dominated means that every probability measure $\mathbb{P} \in \mathcal{P}$ is absolutely continuous with respect to some reference probability measure in $\mathcal{P}$.
In this case, several papers were devoted to this problem. However the starting point was by Gilboa and Schmeidler [124] and followed by many others such that of Anderson, Hansen and Sargent [3] and Hansen et al. [71], [58], [123] among others. In particular, JeanBlanc et al. [78] generalized the result of Bordigoni et al. [19] and proved that the solution of the robust problem is the solution of an exponential quadratic BSDE with jumps.

1.1.3 Summary of the main results

Motivation. In contrast to basic model with continuous paths, the jumps diffusion models, have been used to quantify and to capture adequately the risk of strong stock price movements. One reason for this move into, is that financial data in many cases cannot be accounting by a diffusion process and hence cannot provide a realistic observation of financial markets in a more accurate way.

Started with Merton, the jump diffusion models was developed in several setting. Classically, these model are build on a Brownian motion and a Poisson process to measure the variation of the normal asset prices and capture the large movement due to unexpected market information. We highlight that, recently, fiancial models allowing infinite activity jumps have been proposed.

Motivated by these models, our first contribution presented in this thesis is to study the solvability of quadratic BSDEs with unbounded terminal condition in a general jumps setting under the following assumption.

Assumption 1.1.15.

- *Structure condition.* $\forall (y,z,u) \in \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{L}^0(\mathcal{B}(E),\nu), \forall t \in [0,T]$

$$-\frac{1}{\delta}j_t(-\delta u) - \frac{\delta}{2}|z|^2 - l_t - c_t|y| \leq f_t(y,z,u) \leq \frac{1}{\delta}j_t(\delta u) + \frac{\delta}{2}|z|^2 + l_t + c_t|y|, \ \ \mathbb{P}\text{-a.s.}$$

- *Integrability condition.* $\forall \gamma > 0, \ \ \mathbb{E}\left[\exp\left(\gamma\left(e^{C_{t,T}}|\xi| + \int_t^T e^{C_{t,s}} d\Lambda_s\right)\right)\right] < +\infty,$

where l_t and C_t are two predictable positive processes.

Under this assumption, we prove the following existence result

Theorem 1.1.16. *Suppose that Assumption 1.1.15 and A_γ-condition are fulfilled. Then there exists a solution $(Y,Z,U) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ to the BSDEJ (1.5).*

We prove the existence of unbounded solution in a general framework using double truncation procedure and a stability theorem for a specific class of semimartingale. The idea of the construction is based on the following road map.

- Since we are dealing with BSDEs with infinite activity, we first rely on the truncation technics of Becherer and al [14]. More precisely, we truncate the infinite activity jump measure ν by an appropriate finite random measure.
- The second one is based on a double approximation of f by mean of the regularization

function of Moreau and Yosida to recover the case of driver with Lipschitz growth.
Combining the two steps we obtain a sequence of Lipschitz BSDEs with finite activity jumps.
In the other hand, since we rely on the forward approach, the solution of the truncated BSDEs will be seen as an exponential quadratic semimartingale with the following structure condition $\mathcal{Q}(\Lambda, C)$:

$$-\frac{\delta}{2}d\langle M^c\rangle_t - d\Lambda_t - |Y_t|dC_t - j_t(-\delta M_t^d) \ll dV_t \ll \frac{\delta}{2}d\langle M^c\rangle_t + d\Lambda_t + |Y_t|dC_t + j_t(\delta\Delta M_t^d).$$

From the entropy properties of this class of semimartingale, we apply a general stability result to obtain the existence of the solution.
As explained before, the uniqueness of the solution requires further hypothesis. For this reason, we state the uniqueness result under stronger assumption on the driver with respect to the control state process Z.
This work is concretized in the preprint [104].

1.2 Backward stochastic differential equation with random coefficients and jumps

1.2.1 Presentation of the problem and the link with doubly reflected BSDEJs

The second part of this thesis is devoted to the study of BSDE with jumps and unbounded terminal condition in more general framework. The first general setup concerns a class of BSDE called Generalized BSDE (GBSDEs in short). This class was introduced in the Brownian setting, by Pardoux and Zhang [115] in order to give a probabilistic representation for solutions of semilinear PDE with nonlinear Neumann boundary condition. This equation has the following form:

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s, U_s)ds + \int_t^T g_s(Y_s)dA_s - \int_t^T Z_s dB_s, \quad \forall t \in [0,T], \mathbb{P}\text{-a.s.} \tag{1.17}$$

Compared to the classical setting, we consider an additional integral with respect to an increasing continuous real valued process A that is independent of the Wiener process W.

In the paper [115], an existence and uniqueness result are established in the case where the drivers are monotone with respect to y and uniformly Lipschitz in z. This result was extended later by El otmani [44] to the Lévy jump setting. Several works have followed and developed in various subject. Among them we quote Ren and Xia [119], Ren and El Otmani [118] and Essaky and Hassani [49].

We should point out that, in all these works, the existence of the solution is based essentially on the fixed point argument. However the main problem in carrying out this methodology is that the construction of the contracting map (in an appropriate Banach spaces) requires a strong shape of integrability on the terminal condition of the type $\mathbb{E}\left[e^{\beta_T}|\xi|^2\right] < +\infty$ or $\mathbb{E}\left[e^{\beta\int_0^T \alpha_s^2 ds}|\xi|^2\right] < +\infty$. Unfortunately, since we are looking for unbounded solution, this approach may fail in our context.

We also consider more general assumptions on the driver. More precisely, we are looking for unbounded solution of Generalized BSDEJ where f has general stochastic growth.
Classically in the study of BSDEs, well-posedness results are obtained in the case where f satisfies some uniformly growth condition. However since BSDEs become an indispensable tool in various topics, these type of assumptions are too restrictive to be assumed and they are not verified in general.
If we look for the simplest example of a pricing problem in a Black and Scholes market we can see that the fair price of an European options is given by the solution of a linear BSDE associated to $(r(t)Y_t + \theta(t)Z_t, \xi)$, where θ and r are stochastic. Unfortunately, the interest rate and the risk premium are in general not bounded in the market.
In this sens, El karoui and Huang [40] introduced the following "Stochastic Lipschitz" hypothesis

Assumption 1.2.1. *[40]*

- *There are two predictable positive processes θ, r such that for all $(t, y, \bar{y}, z, \bar{z}) \in [0, T] \times \mathbb{R}^{2d} \times \mathbb{R}^{2n\times d}$, $|f_t(y, z) - f_t(\bar{y}, \bar{z})| \leq r_t|y - \bar{y}| + \theta_t|z - \bar{z}|$.*
- $\mathbb{E}\left[\int_0^T \exp(\beta\int_0^t \alpha_s^2 ds)\frac{|f(t,0,0)|^2}{\alpha_t^2}dt\right] < +\infty,$ *where* $\alpha_t^2 := r_t + \theta_t^2,\ \beta_t > 0.$
- $\mathbb{E}\left[\sup_{0\leq t\leq T} \exp\beta\int_0^T \alpha_s^2 ds|\xi|^2\right] < +\infty.$

Under this assumption, the authors established existence and uniqueness results for the following BSDE driven by a general càdlàg martingale and an increasing process

$$-dY_t = f(t, Y_t, Z_t)dC_t - Z_t^* dM_t - dN_t, \qquad \forall t \in [0, T],\ \mathbb{P}\text{-a.s.}$$

This result was extended in different direction

- Bender and Kohlmann [16] who studied BSDEs with random time under this condition.
- Bahlali et al. in [8] studied the case where the coefficient satisfies the Lipschitz Condition w.r.t z and some stochastic monotone condition in y.

- Essaky [50] extended this result to doubly reflected BSDEs with stochastic quadratic growth in z.
- Briand and Confortola [22] and Wang et al [130] among others.

To solve the unbounded generalized BSDE with jumps (1.17) under stochastic quadratic growth we were inspired by the work of Bahlali, Edhabi, Ouknine [7]. In this paper the authors gives a totally different approach to tackle the problem of existence and uniqueness of quadratic BSDE's solution in the continuous setting. This approach is based mainly on existing results of doubly reflected BSDEs.
In the following we explain the approach in more details. The existence of the solution of BSDE with a driver f satisfying the following structure condition

$$|f_s(y,z)| \leq a + b|y| + h(y)|z|^2 = g_s(y,z),$$

is based on the following road map. The first step is to prove that the BSDE associated respectively to g and $-g$ admits a maximal solution. More precisely, we denote respectively by (Y^g, Z^g) and (Y^{-g}, Z^{-g}) the associated solution of

$$Y_t^g = \xi + \int_t^T g_s(Y_s^g, Z_s^g)ds - \int_t^T Z_s^g dB_s, \qquad \forall t \in [0,T],\ \mathbb{P}\text{-a.s.}$$

$$Y_t^{-g} = \xi + \int_t^T -g_s(Y_s^{-g}, Z_s^{-g})ds - \int_t^T Z_s^{-g} dB_s, \qquad \forall t \in [0,T],\ \mathbb{P}\text{-a.s.}$$

In fact the solution process Y^{-g} and Y^g will be seen respectively as the obstacle L and U of the following doubly reflected BSDEs

$$\begin{cases} Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T^+ - K_T^-, \qquad \forall t \in [0,T],\ \mathbb{P}\text{-a.s.} \\ L_t = Y_t^{-g} \leq Y_t \leq U_t = Y_t^g, \qquad \forall t \in [0,T],\ \mathbb{P}\text{-a.s.} \\ \int_0^T (Y_s - L_s)dK_s^+ = \int_0^T (Y_s - U_s)dK_s^- = 0, \qquad \mathbb{P}\text{-a.s.} \end{cases}$$

Relying on the result of Essaky [50] related to doubly reflected BSDE, we can deduce the existence of the solution of the above system. Finally, if we show that the processes $dK^+ = dK^- = 0$, we can deduce existence result for non reflected case.
Our goal is to extend this result to a jump setting which leads naturally to the study of generalized doubly reflected BSDEs with infinite activity jumps and where the driver satisfies a general stochastic structure condition.
Unfortunately, as far as we know, there is no existing result on this subject. Hence, as a first step, we concern ourselves to the well-posedness of doubly reflected BSDEJ's where the driver has a general stochastic growth and unbounded terminal condition.

1.2.2 Doubly Reflected BSDEJs

As already mentioned, we fix our attention on BSDEJs with two reflecting barriers. These equations are commonly called doubly reflected BSDEJs (DRBSDE in short) and were first introduced and studied by Cvitanic and Karatzas [32]. However the notion of BSDEs with constrains started with the seminal paper of El Karoui, Kapudjian, Pardoux, Peng and Quenez [41] as the generalization of the work of Pardoux Peng [113].
Intuitively, the solution of these equation are constrained to be greater then a given process called obstacle. In this situation, the reflected BSDE takes the following form

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \qquad \forall t \in [0,T],\ \mathbb{P}\text{-a.s},$$

together with the following constraints

$$\begin{cases} L_t \leq Y_t, & \forall t \in [0,T],\ \mathbb{P}\text{-a.s.} \\ \int_0^T (Y_s - L_s)dK_s = 0,\ \mathbb{P}\text{-a.s.} \end{cases}$$

This system is also charcterized by an extra term K, which turns out to be crucial to the well posedness of those equation. This increasing process has to fulfilled the Skorohod condition, which means that K operate in a minimal way so that it only acts when the process Y try to over pass the obstacle L.
El Karoui et al. [41] proved in this paper, the existence of a unique $\mathbb{L}^2$-bounded solution when the generator satisfies uniform Lispchitz condition with respect to the processes (Y, Z). In addition, the obstacle L which is a continuous progressively measurable process satisfies $\mathbb{E}[\sup_{t\in[0,T]}(L_t^+)^2] < +\infty$.
The authors also draw a close link between reflected BSDE and optimal stopping control problem. More precisely, they show that the solution process Y_t characterize the value of an optimal stopping problem as follow

$$Y_t = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_t}\mathbb{E}\left[\int_t^\tau f_s(Y_s, Z_s)ds + L_\tau 1_{\tau<T} + \xi 1_{\tau=T}|\mathcal{F}_t\right], \tag{1.18}$$

where $\mathcal{T}$ is the set of all stopping times dominated by T and $\mathcal{T}_t := \{\tau \in \mathcal{T}; t \leq \tau \leq T\}$. In terms of applications, they proved that the reflected BSDEs are intrinsically linked to the hedging problem for American options.

Since then, this class of BSDE has been widely studied in the literature. In [103], Matoussi proved existence of the solution where the driver have linear growth with respect to (y, z).
Afterwards, Kobylanski et al [85] showed the existence of one dimensional BSDEs solution in the case when the terminal condition and the barrier are uniformly bounded. Also, the driver

has superlinear growth in y and quadratic growth in z as follow

$$|f(,y,z)| \leq l(y) + C|z|^2 \qquad \text{and} \int_0^{+\infty} \frac{1}{l(t)}dt = \int_{-\infty}^{0} \frac{1}{l(t)}dt = +\infty.$$

Several works has been obtained in this subject under weaker assumptions. We mention the work of Hamadène and Popier [69] or Klimsiak [83] who treated separately the case where f is continuous and monotone in y, Lipschitz continuous with respect to z and $\mathbb{L}^p$-integrable data. When f is Lipschitz and the barrier is RCLL, Lepeltier [92] and Hamadène [62] also investigated this problem. Lepeltier, Matoussi and Xu [91], considered the case of driver who grows arbitrary with respect to y. Xu [132] treated the case when the terminal condition is bounded and the driver is quadratic in z and monotone with arbitrary growth in y.

Later, Lepeltier and Xu [93] extended the result of [23] to the reflected setting and prove the existence of maximal solution in the case when the obstacle is bounded and the driver is linear increasing in y with quadratic growth in z as follow : there exits $\alpha, \beta \geq 0$, $\gamma > 0$ satisfying $\alpha \geq \frac{\beta}{\gamma}, \ |f(t,y,z)| \leq \alpha + \beta|y| + \frac{\gamma}{2}|z|^2$.
The extension to the unbounded obstacle case is obtained by Bayraktar and Yao [13].
More recently, Essaky et al [50] treat the case where the driver has general stochastic quadratic growth without assuming any integrability condition on the terminal data.

Building upon these works, RBSDE has been extended to jump setting as well as in more general settings(general martingale [9]). Among these works we mention the existence of the solution when

- f is uniformly Lipschtz, ξ is square integrable and the obstacle is RCLL with $\mathbb{E}[\sup_t (S_t)^2] < +\infty$ (see Essaky [48]),
- ξ is suqare integrable , f is uniformly Lipschitz and the barrier $(S_t)_{t\leq 1}$ is a càdlàg process with inaccessible jumps (see [68]).
- When the obstacle process has arbitrary jump structure (see [67]).

Doubly Reflected BSDEs

As we mentioned before, Cvitanic and Karatzas [32] introduce the notion of doubly reflected BSDE. Unlike reflected BSDE, the solution of the doubly reflected BSDE is maintained between two barriers L and U with a minimality condition that is the process K^+ and K^- only act when Y reaches the obstacle L and U.

In contrast with reflected BSDE we need to add additional assumption on the barrier to get

the existence result. In [32], the authors propose two different assumptions on the obstacle to get an existence result.
The first one is called *"the Mokobodski condition"* : This condition postulates that there exists two positives uniformly square integrable supermartingales h^1 and h^2 such that: $\forall t \in [0,T], L_t \leq h_t^1 - h_t^2 \leq U_t$.
The second one is that the two barriers are quasi-semimartinagles. They show that under either the first or the second condition and

$$\mathbb{E}\left[|\xi|^2 + \int_0^T |f(t,0,0)|^2 dt + \sup_{0\leq t\leq T} (|U_t|^2 + |L_t|^2)\right] < +\infty,$$

there exist a unique solution. The reason for preferring the last condition to the first one is simply that the Mokobodski's condition can be difficult to verify.

In general there is two possible approachs to solve the problem .
$-$ *Penalization procedure.* This approach has been widely studied in the literature. However the first work using this technique is of El karoui and al [41]. Roughly speaking, the reflected BSDE can be viewed as limit of penalized BSDEs. Consider (Y^n, Z^n) the solution of the following BSDE

$$Y_t^n = \xi + \int_t^T f_s(Y_s, Z_s)ds + n\int_t^T (Y_s^n - L_s)^- ds - \int_t^T Z_s^n dB_s \quad \forall t \in [0,T], \mathbb{P}\text{-a.s.} \tag{1.19}$$

In addition to the driver, the term $n(Y_s^n - L_s)^- dt$ measures how far the solution Y^n violate the obstacle L_t. When the penalty parameter n goes to infinity, it ward off Y^n from passing under L. Setting $K_t^n := n(Y_t^n - L_t)^- dt$ and showing that (Y^n, Z^n, K^n) converges in some sens to (Y, Z, K) in an appropriate spaces then we have that the limit of (Y^n, Z^n, K^n) solves the original reflected BSDE.
$-$ *Picard iterative method.* This approach is based essentially on the construction of an iterative sequence of processes (Y^n, Z^n, K^n). If one can prove that the sequence (Y^n, Z^n, K^n) converges in some sens then we can say that the limit is a solution to the reflected BSDE. As a final step, we prove that the solution (Y, Z, K) is the fixed point of a contracting mapping in an appropriate spaces.

DRBSDE in the Brownian seting as well as the jump setting have been considered by many authors. In the Brownian setting, we cite the work of Hamadène Lepeltier and Matoussi who investigated in [59] the existence of solution for DRBSDE when the driver has linear growth by assuming regularity condition on the one barrier and also the existence of a positive semimartingale between the barriers constraints. Later, Hamadène and Hassani [63] generalize the result of [59] and proved the existence of the solution when the barriers are completely separated .

Under the Mokobodski condition, Bahlali et al. [9] treated the case where the driver has linear growth in y and quadratic in z. This result was followed by Hdhiri and Hamadène [65], they establish an existence result by assuming that the barriers are completely separated. Essaki and Hassani [50] proved existence and uniqueness of the solution of doubly reflected BSDE under more general assumption on the data by considering a driver with quadratic stochastic growth i.e $|f_s(y,z)| \leq \zeta_s(w) + \frac{C_s(w)}{2}|z|^2$ without imposing any integrability condition on the terminal condition. This result was generalized later by Essaky, Hassani and Ouknine to the case of RCLL obstacles.
More recently, Baadi and Ouknine [6] have studied doubly reflecting barrier BSDEs when the noise is driven by a Brownian motion. They showed existence and uniqueness of the solution when the reflecting barrier don't satisfy any regularity assumption.

Several authors have been regarding to extend these results to the discontinuous setting by adding a stochastic integral with respect to a jump measure. Among them, we cite the work of Crepey and Matoussi [31] who considered the case when the driver f is Lipschitz the Obstacles satisfy the Mokobodski's condition.

When the obstacles are completely separated, Hamadène and Hassani [63] showed that the DRBSDE driven by a Brownian motion and a Poisson noise has a unique solution. Let us mention that, in their study the proces Y has only inaccessible jumps. This result was generalized later by Hamadène and Wang [70].
Hamadène and Lepeltier [66] generalized the results of [32] in order to solve a non anticipative mixed zero-sum game. Doubly reflected BSDE is further developed to different framework, such work includes that of [30, 37, 38, 51, 64] among others.

1.2.3 Summary of the main results

The second contribution of this thesis is to study doubly reflected BSDEs with jumps in general setting. We prove the existence of a unique solution under a general structure condition. More precisely, we study the following system

$$(\mathcal{S})\begin{cases} Y_t = \xi + \int_t^T f_s(Y_{s^-}, Z_s, V_s)ds + \int_t^T dR_s + \int_t^T g_s(Y_s)dA_s - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e)\tilde{\mu}(ds, de) \\ \qquad + \int_t^T dK_s^+ - \int_t^T dK_s^-, \qquad\qquad 0 \le t \le T, \ \mathbb{P}\text{-a.s.} \\ L_t \le Y_t \le U_t \ \text{ and } \int_0^T (U_s - Y_{s^-})dK_s^+ = \int_0^T (Y_{s^-} - L_s)dK_s^- = 0, \quad 0 \le t \le T, \ \mathbb{P}\text{-a.s,} \end{cases}$$

where

- B is a d-dimensional Brownian motion.
- $\tilde{\mu} = \mu - \nu$ is a compensated random measure.
- $f : \Omega \times [0, T] \times \mathbb{R}^{2+d} \to \mathbb{R}$ is a $\mathcal{P}$-measurable continuous function with respect to (y, z, v).
- $g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is a $\mathcal{P}$-measurable continuous function with respect to y.
- R is a $\mathcal{P}$-measurable and continuous process such that there exist two continuous non decreasing process R^+ and R^- with $R_T^- < +\infty$ and $R_T^+ < +\infty$ satisfying $R = R^+ - R^-$.
- A is $\mathcal{P}$-measurable continuous non decreasing process such that $A_0 = 0$ and $A_T < +\infty$.

First, we define the solution of generalized doubly reflected BSDEJs associated to $(fds+gdA_s+dR_s, L, U)$ as the quintuple $(Y, Z, V, K^+, K^-) \in \mathcal{D} \times \mathcal{L}^{2,d} \times \mathcal{L}_\nu^2 \times \mathcal{K}^2$ such that the system $(\mathcal{S})$ is satisfied. More precisely, we look for solution in the following spaces
$\mathcal{L}^{2,d}$ is the spaces of $\mathbb{R}^d$ -valued and $\mathcal{P}$-measurable processes such that

$$\|Z\|_{\mathcal{L}^2}^2 := \int_0^T |Z_s|^2 ds < +\infty, \ \mathbb{P}\text{-a.s.}$$

$\mathcal{L}_\nu^{2,d}$ is the space of predictable processes such that

$$\|U\|_{\mathcal{L}_\nu^2}^2 := \int_0^T \int_E |U_s(e)|^2 \nu(de, ds) < +\infty, \ \mathbb{P}\text{-a.s.}$$

$\mathcal{K}$ the space of $\mathcal{P}$-measurable continuous non decreasing process such that $K_0 = 0$.
$\mathcal{D}$ (respectively $\mathcal{D}^c$) the space of $\mathbb{R}$-valued $\mathcal{P}$-measurable càdlàg processes resp. ($\Delta Y_t = 0$).
We ask the data $(fds + gdA_s + dR_s)$ to verify the following assumptions

Assumption 1.2.2 (Assumptions on the drivers)**.**

- *The first assumption characterize the growth of the driver f with a lower and an upper bound: For every $(y, z, v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(B(E), \nu)$, there exist $\delta > 0$ and two positives* *processes η and C respectively in $\mathbb{L}^1(\Omega,[0,T])$ and $\mathcal{D}^c$ such that*

$$\underline{q}_s(y,z,v) = -\eta_s(w) - \frac{C_s(w)}{2}|z|^2 - \frac{1}{\delta}J(v)$$
$$\leq f_s(w,y,z,v) \leq \bar{q}_s(y,z,v) = \eta_s(w) + \frac{C_s(w)}{2}|z|^2 + \frac{1}{\delta}J(v),$$

$dt \otimes d\mathbb{P}$*-a.s,* $(w,t) \in \Omega \times [0,T]$*, where*

$$J(v) = \int_E \left(e^{\delta v(e)} - \delta v(e) - 1\right)\nu(de).$$

- *The second assumption consists in specifying a lower and upper bound for g: For all $y \in [L_s(w), U_s(w)]$,*
$$|g_s(w,y)| \leq 1 \qquad A(dt)\otimes\mathbb{P}(dw)\text{-a.s.}$$

- *The last assumption known as the "A_γ-condition " deals with the increments of the driver f with respect to the jump component: For all $(y,z,u,\bar{u}) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{L}^0(B(E),\nu)$ there exists a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^{d+2})\otimes\mathcal{B}(E)$-measurable function γ where $C_1 \leq \gamma \leq C_2$ with $-1 < C_1 \leq 0$, $C_2 \geq 0$,*
$$f_t(y,z,v) - f_t(y,z,\bar{v}) \leq \int_E \gamma_t(e)[v(e)-\bar{v}(e)]\nu(de).$$

We will always assume that the upper and lower obstacle L and U verify the following properties

Assumption 1.2.3. *(Assumptions on the Obstacle)*
(i) There exists a semimartingale S with the following decomposition

$S = S_0 + V^+_\cdot - V^-_\cdot + \int_0^\cdot \alpha_s dB_s$*, where* $S_0 \in \mathbb{R}$*,* $V^+_\cdot, V^-_\cdot \in \mathcal{K}$ *and* $\alpha_s \in \mathcal{L}^{2,d}$*,* $\mathbb{P}$*-a.s.*

(ii) $L_t \leq S_t \leq U_t, \quad L_t \leq 0 \leq U_t, \quad \forall t \in [0,T], \mathbb{P}$*-a.s.*

(iii) For all $R \in \mathcal{K}, \quad dR_t \geq 0, \quad \forall t\in[0,T], \mathbb{P}$*-a.s.*

Let us point out that these assumptions are not restrictive and weaker than the existing assumption in the literature. Let us expose the example treated in [50].
We consider the case where the driver has general growth in y and quadratic growth in z. Let $\hat{\eta}, \tilde{\eta} \in \mathbb{L}^1([0,T])$ and a progressively measurable functions φ, Φ, ψ such that $\forall t \in [0,T]$, $\forall y \in [L_s, U_s]$, $\forall z \in \mathbb{R}^d$, we have

$$|f_s(w,y,z)| \leq \tilde{\eta}_s(w,z) + \Phi_s(w,y) + \psi(s,w,y)|z|^2, \qquad |g_s(w,y)| \leq \hat{\eta}_s(w) + \varphi_s(w,y).$$

We can see that the above structure condition holds in this case and thus, if we choose η and C as follow

$$\eta_t(w) = \tilde{\eta}_t(w) + \sup_{s\leq t} \sup_{\alpha\in[0,1]} |\Phi_s(w, \alpha L_s + (1-\alpha)U_s|$$
$$C_t(w) = 2\sup_{s\leq t} \sup_{\alpha\in[0,1]} |\psi_s(w, \alpha L_s + (1-\alpha)U_s|.$$

and

$$|g_t(w,y)| \leq \hat{\eta}_t + \sup_{s\leq t} \sup_{\alpha\in[0,1]} |\varphi_s(w, \alpha L_s + (1-\alpha)U_s]| = \bar{\eta}_t(w) \leq 1 + \bar{\eta}_t(w), \tag{1.20}$$

Put $g_t(w,y) = \frac{g_t(w,y)}{1+\bar{\eta}_t}$ and $dA_t = (1+\bar{\eta}_t)dA_t$ in the BSDE (1.17), we see that g also satisfies the above assumption. Our goal is then to find a solution to the Generalized DRBSDE with jumps $(\mathbb{S})$ under assumptions 1.2.2 and 1.2.3.

Theorem 1.2.4. *Assume that Assumptions 1.2.2 and 1.2.3 are fulfilled then the doubly reflected BSDE with jumps associated to $(\bar{f}ds + d\bar{R}_s + gdA_s, \xi, \bar{L}, \bar{U})$ has maximal solution.*

The construction's scheme of the solution is as follow. Following the classical terminology in the quadratic BSDE literature, we begin by an exponential change of variable. This transformation enables us to obtain another doubly reflected BSDE with more tractable coefficients. However, unlike the classical results on BSDE, this exponential transformation did not lead to a Lipschitz one. We are thus led naturally to a trancature procedure.
The second step will be the truncation procedure to obtain an approximation sequence of globally Lipschitz doubly reflected BSDEJs. More precisely, let us introduce the sequence of DRBSDE with jumps.

$$\begin{aligned} Y_t^{n,i} = \xi &+ \int_t^T f_s^n(Y_s^{n,i}, Z_s^{n,i}, U_s^{n,i})ds + \int_t^T g_s^n(Y_s^{n,i})dA_s^n + \int_t^T dR_s^i - \int_t^T Z_s^{n,i}dB_s \\ &- \int_t^T \int_E V_s^{n,i}(e)\tilde{\mu}(ds,de) + \int_t^T dK_s^{n,i+} - \int_0^T dK_s^{n,i-}, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \end{aligned}$$

with the constraint

$$L_t \leq Y_t^{n,i} \leq U_t, \quad 0 \leq t \leq T, \ \mathbb{P}\text{-a.s.}$$
$$\int_0^T (Y_{s^-}^{n,i} - L_s)dK_s^{n,i+} = \int_0^T (U_s^n - Y_{s^-}^{n,i})dK_s^{n,i-} = 0, \qquad 0 \leq t \leq T, \mathbb{P}\text{-a.s.}$$

To insure that the approximated coefficient conserves all the properties of f, we truncate the driver by means of sup-convolution .
Thanks to this truncation procedure, the coefficients f and g are respectively Lipschitz in (y,z,v) and y. However, since we work under weaker assumptions, we introduce a family stopping time

$\tau_i = \inf\left\{t \geq 0; A_t + R_t + C_t + \int_0^t \eta_s ds \geq i\right\} \wedge T$. We show that this approximated BSDE has a unique solution.
The solution is then constructed by proving a monotone limit theorem and achieving the convergence of the Lipschitz DRBSDEs constructed by the above trucation procedure. Then we show that the limit of those solution indeed solves the original DRBSDEJ. As a final step we comeback to our initial system and prove that admit a solution using a logarithmic transformation.
This work is concretized in the preprint [105].

1.3 Mean-field FBSDEs with jumps and application

The Last chapter of this thesis is dedicated to fully coupled Mean field FBSDEs with jumps and applications to the energy storage problem.

1.3.1 Forward Backward stochastic differential equation with jumps

An abundant literature has been dynamically devoted to investigate the solvability of forward backward stochastic differential equation (FBSDEs) as they naturally arise in the various field such as in finance (stochastic control problem), economics (contract theory) and Partial Differential equation (PDEs).

Forward-Backward SDEs were first introduced in the continuous framework by Antonelli in his Phd thesis [5]. He studied the well-posedness of these equations over a sufficiently small time duration. The author also argue through a counter example that, over a considerable time period, the solvability of the FBSDEs may fail. The same problem was solved by Pardoux and Tang [114] and proved continuous dependence of the solution on parameters. The authors also make a link with quasi parabolic PDE.

Forward Backward SDE driven by a Brownian motion have been studied by several authors using different approaches. Let us mention here the main approaches developed in the literature.

• *Four step method.* This method goes back to Ma, Protter and Young [100] and based essentially on PDE technics and probability method. In this paper, they proved existence and uniqueness of the solution of fully coupled FBSDEs when the coefficients are deterministic satisfy the non degeneracy condition (σ is non degenerate). They also showed that the solution is closely linked to some quasilinear PDE.

• *The continuation method.* Developed in [74] by Hu and Peng, it can be seen as a probabilistic approach. They proved existence and uniqueness under an arbitrary time horizon by relaxing the degeneracy condition with some monotonicity condition.
This result was followed by many others. Among them, we quote Peng and Wu [116], Yong [135,136]. In [116] the authors also showed some results about the links between FBSDEs and some linear quadratic stochastic optimal control problem using the maximum principle. We also refer to the paper of Yu and Ji [139] who studied the linear quadratic non zero sum games in term of FBSDE.

We also mentioned the work of Delarue [33] for decoupled FBSDEs. The existence and uniqueness are both proved in this paper using a fixed point argument and the so-called 'four step approach'. This result was extended later by Zhang [137].
• *Unified approach.* Developed in [101] by Ma et al. to study the solvability of a general type of FBSDE namely coupled FBSDE.

We can describe the difference between decoupled or coupled FBSDEs system as follows:
− The system of FBSDEs is said to be decoupled if the randomness of the terminal condition and the coefficients comes from a Forward SDE and does not depend on the BSDE's solution. Moreover, if the coefficients are deterministic, then the backward equation exhibits a Markovian property.
− Coupled FBSDEs: when the dependence of the coefficients of the Forward SDE is related to the Backward solution.

FBSDEs was extended naturally to the jumps setting. This subject started with the work of Zhen [138], where the author proved existence and uniqueness results for FBSDEs driven by a Brownian motion and a Poisson random measure. Later, Wu [131] generalized the result of [138] to fully coupled FBSDEs with jumps over a random interval. Recently, Situ [134] discussed the existence and uniqueness of the solution under the monotonicity condition using a probabilistic method.

Motivated by stochastic differential game and partial differential-integral equations (PDIE), Li and Wei [96], studied some $\mathbb{L}^p$-estimates as well as an existence and uniqueness result related to fully coupled FBSDE with jumps.

When it comes to application, a large trend of literature has been developed dynamically especially in connection with stochastic optimal control problem for random jumps and mathematical finance. We refer the interested reader to the work of Oksendal and Sulem [110, 111] , Shi [126, 127] and the reference therein.

1.3.2 Mean field theory

Mean-field game were first introduced around 2006 by Lasry, Lions [87–89, 99] and separately by Caines, Huang, Malhamé [75–77] in the engineering topics. The origin of mean field theory comes from statistical physics where they study the interaction of particles as well as the impact of the whole system on the behavior of every single partical. The mean-field game (MFGs in short) is the study of the asymptotic limits of a stochastic differential game of a large number of identical agents where their actions are influenced by the empirical distribution of the states of the ensemble players.

Let us describe informally the mean field setting. From a mathematical point of view MFGs, are captured typically by a system of forward Fokker-Plank equation and Backward Hamilton Jacobi Bellman (HJB):

$$\begin{cases} \partial_t u - \sigma \triangle u + H(x, \nabla(x,t)) = f(x, m(t)), \\ \partial_t m - \sigma \triangle m - div(m D_p H(x, \nabla u)) = 0, \\ m(0) = m_0, u(x,T) = g(x, m(T)), \end{cases} \tag{1.21}$$

where m is the density of probability measure. We consider that a representative player control, through a process α , his own dynamics $dx_t = \alpha_t dt + \sigma dW_t,\ x(0) = x_0$, in order to minimize the following cost function

$$\mathbb{E}[\int_0^T [L(x_t, \alpha_t) + f(x_t, m(t))]dt + g(x_T, m(T))]. \tag{1.22}$$

If the value function u of the problem (1.22) is smooth enough, then using standard control techniques the optimal control $\alpha^\star(x,t)$ is given by the first equation (Backward HJB equation) in (1.21). The second equation describes the aggregation of the action of the players. It presents the movement of the agents in time based on their initial distribution.

Since the seminal paper of Lasry and Lions, the literature on mean-field topic has experienced a significant increase in attention during this decades (see e.g Gomes [54–56], Cardialaguet [28] among others).
In particular, it is significantly used in divers areas such as in economics, finance, public wealth and power system. We can mentioned the work of Espinoza and Touzi [47] for investment problem, Elie, Mastrolia and Possamai [46], Aid et al. [1] for principal agent problem and Alasseur

et al. [2] for energie storage model.

For a more complete overview of the theory, we refer the reader to the books of Bensoussan et al. [17], Carmona, Delarue [29] or the notes of Cardaliaguet [27]. In particular, we refer to the paper of Lasry and Lions [57] for it application to Economics, Finance and game theory.

Motivated by theses works, Buckdahn, Djehiche and Li [25] introduced a new class of BSDEs namely Mean-field BSDEs. Following that study, Buckdahn, Li and Peng [26] have analysed existence and uniqueness of the solution and stated a comparison result for this type of BSDEs. The connection with some some type of PDE was also established. This result was generalized later by Min and Peng [106] to the case of fully coupled mean-field FBSDEs in the Brownian setting using a continuation method.

As for classical BSDEs, various applications have been considered in the mean field context, notably mean field control problem. For instance, Li et al. [98] investigated a linear quadratic optimal control problem and draw a correspondence with FBSDEs of mean-field type using a classic convex variation principle combined with variational method, Ma et al. studied the non convex control domain case and Tang and Meng generalized the result of [98] to the jump setting.

In this context, Min and Li [95] considered a new type of BSDEs namely mean-field BSDEs with jumps coupled with the value function. The authors proved that this type of BSDEs admits a unique solution, and established a comparison theorem as well as a dynamic programming principal. More recently, the authors extended their result [95] to the case of fully coupled mean-field BSDEs with poisson jump measure [97].

1.3.3 Summary of the main results

Existence and uniquess results for fully coupled mean-field FBSDEJs. Motivated by the above works on control problem, we investigate in this chapter fully coupled mean field FBSDEs driven jointly by a Brownian motion and a jump random measure and where the coefficients depend on Z and K. We prove existence and uniqueness under two different assumptions without imposing the non-degeneracy condition in the forward equation.
More precisely, we consider a system of fully coupled mean field FBSDEJs of the following form

$$
\textbf{(S)}\begin{cases} X_t = X_0 + \int_0^t b_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)})ds + \int_0^t \sigma_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)}))dW_s \\ \qquad + \int_0^t \int_E \beta(s, X_{s^-}, Y_{s^-}, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)})\tilde{\pi}(ds, de), \ \ 0 \le t \le T, \ \mathbb{P}\text{-a.s.} \\ Y_t = g(X_T, \mathbb{P}_{X_T}) + \int_t^T h_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)})ds - \int_t^T Z_s dW_s - \int_t^T \int_E K_s(e)\tilde{\pi}(ds, de), \end{cases}
$$

where $X_0 = x_0$ for deterministic $x_0 \in \mathbb{R}^d$ and terminal condition $Y_T = g(X_T, \mathbb{P}_{X_T})$, W is a m-dimensional Brownian motion and $\tilde{\pi}$ is compensated random jump measure.

For any random variable X on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we denote by $\mathbb{P}_X$ its probability law under $\mathbb{P}$. We denote by $\mathcal{M}_2(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$ with finite moments of order 2 equipped with the 2-Wassertein distance

$$
\begin{aligned}
\mathcal{W}_2(\mu, \mu') &:= \inf\{(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 F(dx, dy))^{\frac{1}{2}}, F \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d) \text{ with marginals } \mu, \mu^{'}\} \\
&:= \inf\{(\mathbb{E}|\xi - \xi'|^2)^{\frac{1}{2}} : \mu = \mathcal{L}(\xi),\ \mu^{'} = \mathcal{L}(\xi^{'})\},
\end{aligned}
$$

where $\mathcal{L}(\xi)$ and $\mathcal{L}(\xi^{'})$ are respectively the law of ξ and $\xi^{'}$ and the infimum is taken over $F \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d)$ with marginals μ and $\mu^{'}$.

Notice that if X^1 and X^2 are random variables of order 2 with values in $\mathbb{R}^d$, then we have the following inequality involving the Wasserstein metric between the laws of the square integrable random variables X^1 and X^2 and their L^2- distance:

$$
\mathcal{W}_2(\mathbb{P}_{X^1}, \mathbb{P}_{X^2}) \le \Big[\mathbb{E}|X^1 - X^2|^2\Big]^{\frac{1}{2}}. \tag{1.23}
$$

We make the following standing assumptions on the maps under consideration.

1- For $\phi \in \{f, h, g, \sigma\}$, ϕ is Lipschitz w.r.t x, y, z, k, ν with C_ϕ^x, C_ϕ^y, C_ϕ^k, C_ϕ^z and C_ϕ^ν as the Lipschitz constants.

2- The function $g : \Omega \times \mathbb{R}^d \times \mathbb{M}_2(\mathbb{R}^2) \to \mathbb{R}^d$ is Lipschitz in (x, μ) i.e. there exists $C > 0$ such that for all $x,\ x' \in \mathbb{R}^d$ and for all $\mu,\ \mu' \in \mathbb{M}_2(\mathbb{R}^d)$,

$$
|g(x, \mu) - g(x', \mu')| \le C(|x - x'| + d(\mu, \mu')), \mathbb{P}\text{-a.s.} \tag{1.24}
$$

We investigated the well-posedness of these equations under different assumptions.The first main result in this chapter is shown under the following hypothesis

$$\textbf{(H1)}\quad \begin{cases} (i) \text{ There exists } k>0, \text{s.t } \forall t\in[0,T], \nu\in\mathbb{M}_2(\mathbb{R}^d\times\mathbb{R}^d), u,u'\in\mathbb{R}^{2d+d\times d}\times\mathbb{L}^0(B(E),\nu) \\ \qquad\qquad \mathcal{A}(t,u,u',\nu)\leq -k|x-x'|^2, \mathbb{P}\text{-a.s.} \\ (ii) \text{ There exists } k'>0, \text{ s.t } \forall\nu\in\mathbb{M}_2(\mathbb{R}^d\times\mathbb{R}^d), x,x'\in\mathbb{R}^d \\ \qquad\qquad (g(x,\nu)-g(x',\nu)).(x-x')\geq k'|x-x'|^2, \mathbb{P}\text{-a.s.} \end{cases}$$

Theorem 1.3.1. *Under Assumption **(H1)**, there exists a unique solution $U=(X,Y,Z,K)$ of the mean-field FBSDE with jumps **(S)**.*

To do so, we use an approximation scheme based on a spike perturbation on the forward diffusion. More precisely, the idea behind is to consider a perturbed scheme of the following form:

Let $\delta\in]0,1]$ and define a sequence of processes (X^n,Y^n,Z^n,K^n) defined recursively by

• $(X^0,Y^0,Z^0,K^0)=(0,0,0,0)$, $\forall n\geq 1, U^n=(X^n,Y^n,Z^n,K^n)$, which satisfies

$$\begin{cases} X_t^{n+1}=X_0+\int_0^t[b_s(U_s^{n+1},\nu_s^n)-\delta(Y_s^{n+1}-Y_s^n)]ds+\int_0^t[\sigma_s(U_s^{n+1},\nu_s^n)-\delta(Z_s^{n+1}-Z_s^n)]dW_s \\ \qquad +\int_0^t\int_E\Big(\beta_s(U_s^{n+1},\nu_s^n)-\delta(K_s^{n+1}-K_s^n)\Big)\tilde{\pi}(ds,de), \\ Y_t^{n+1}=g(X_T^{n+1},\mu_T^n)-\int_t^T h_s(U_s^{n+1},\nu_s^n)ds-\int_t^T Z_s^{n+1}dW_s-\int_t^T\int_E K_s^{n+1}(e)\tilde{\pi}(ds,de). \end{cases} \tag{1.25}$$

We show that, in this case, the above explicit scheme converges, since we obtained that

$$\mathbb{E}[|\hat{X}_T^{n+1}|^2]+\mathbb{E}[\int_0^T\|\hat{U}_s^{n+1}\|^2ds]\leq\frac{\theta}{\gamma}\mathbb{E}[|\hat{X}_T^n|^2]+\mathbb{E}[\int_0^T\|\hat{U}_s^n\|^2ds],$$

where $\|U\|^2:=|x|^2+|y|^2+\|z\|^2+|k|_t^2$, for any $u=(x,y,z,k)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{d\times d}\times\mathbb{L}^0(\mathcal{B}(E),\eta)$

$$\begin{aligned} &\gamma:=min(k'-\frac{C_g^\nu\epsilon}{2},k-\frac{\tilde{\epsilon}C_h^\nu}{2},(\delta-\frac{\kappa\delta}{2}-\frac{\tilde{\epsilon}C_f^\nu}{2}),(\delta-\frac{\kappa\delta}{2}-\frac{\tilde{\epsilon}C_\sigma^\nu}{2}),(\delta-\frac{\kappa\delta}{2}-\frac{\tilde{\epsilon}C_\beta^\nu}{2})) \\ &\theta=max(\frac{C_g^\nu}{2\epsilon},-\frac{C_h^\nu+C_f^\nu+C_\sigma^\nu+C_\beta^\nu}{2\epsilon}+\frac{\delta}{2\kappa}). \end{aligned}$$

This inequality becomes a contraction and thus, $(\hat{X}^n)_{n\geq0},(\hat{Y}^n)_{n\geq0},(\hat{Z}^n)_{n\geq0}$ and $(\hat{K}^n)_{n\geq0}$ are Cauchy sequences, which lead to the existence of the solution of the system **(S)**.

In the other hand, we show that the system (**S**) admits a unique solution. This result is achieved by applying Itô formula combined with standard BSDEs argument as well as the following estimates.

Lemma 1.3.2. *The difference of the solutions $(Y^{'}, Z^{'}, K^{'})$ and (Y, Z, K) of the system **(S)** under (**H1**) , satisfies the following estimates*

$$\mathbb{E}[|\bar{Y}_s|^2] \leq \Theta^1 \mathbb{E}[|\bar{X}_T|^2] + \Theta^2 \int_0^T \mathbb{E}|\bar{X}_s|^2 ds.$$

$$\mathbb{E}[\int_0^T [|\bar{Z}_s|^2 + |\bar{K}|_s^2] ds \leq \bar{\Theta}^1 \mathbb{E}[|\bar{X}_T|^2] + \bar{\Theta}^2 \int_0^T \mathbb{E}|\bar{X}_s|^2 ds.$$

In the second part, we show that the system **(S)** admits a unique solution under the following monotonicity assumption

(H2)
$$\begin{cases} (i) \text{ There exists } k > 0, \text{ s.t } \forall t \in [0,T], \nu \in \mathbb{M}_2(\mathbb{R}^d \times \mathbb{R}^d), u, u' \in \mathbb{R}^{2d+d\times d} \times \mathbb{L}^0(B(E), \nu) \\ \qquad \mathcal{A}(t, u, u', \nu) \leq -k(|y - y'|^2 + ||z - z'||^2 + |k - k^{'}|_t), \mathbb{P}\text{-a.s.} \\ (ii) \text{ There exists } k' > 0, \text{ s.t } \forall \nu \in \mathbb{M}_2(\mathbb{R}^d \times \mathbb{R}^d), x, x' \in \mathbb{R}^d \\ \qquad (g(x, \nu) - g(x', \nu)).(x - x') \geq k'|x - x'|^2, \mathbb{P}\text{-a.s.} \end{cases}$$

Theorem 1.3.3. *Under Assumption **(H2)**, there exists a unique solution $U = (X, Y, Z, K)$ of the mean-field FBSDE with jumps **(S)**.*

The proof is almost the same as the previous theorem. Replacing assumption **(H1)** by the monotonicity hypothesis **(H2)**, we also construct an iterative scheme based on a spike perturbation not on the forward diffusion as in the previous result but rather on the Backward SDEJs as follows

$$\begin{cases} X_t^{n+1} = X_0 + \int_0^t b_s(U_s^{n+1}, \nu_s^n) ds + \int_0^t \sigma_s(U_s^{n+1}, \nu_s^n) dW_s + \int_0^t \int_E \beta_s(U_s^{n+1}, \nu_s^n) \tilde{\pi}(ds, de), \\ Y_t^{n+1} = g(X_T^{n+1}, \mu_T^n) + \delta(X_T^{n+1} - X_T^n) - \int_t^T \left[h_s(U_s^{n+1}, \nu_s^n) + \delta(X_s^{n+1} - X_s^n) \right] ds \\ \qquad - \int_t^T Z_s^{n+1} dW_s - \int_t^T \int_E K_s^{n+1}(e) \tilde{\pi}(ds, de). \end{cases} \tag{1.26}$$

Following the same steps as in the first existence result, we prove that for $\frac{\tilde{\theta}}{\tilde{\gamma}} < 1$ we have

$$\tilde{\gamma} \mathbb{E}[|\hat{X}_T^{n+1}|^2 + \int_0^T \|\hat{U}_s^{n+1}\|^2 ds] \leq \tilde{\theta} \mathbb{E}[|\hat{X}_T^n|^2 + \int_0^T \|\hat{U}_s^n\|^2 ds], \tag{1.27}$$

which provides the existence of the solution.
The uniqueness of the solution whether under **(H1)** or **(H2)** is achieved by applying Itô formula combined with standard BSDEs argument as well as the following estimate.

Lemma 1.3.4. *The difference of the solutions $(Y^{'}, Z^{'}, K^{'})$ and (Y, Z, K) of the system **(S)***

*under (**H2**), satisfies the following estimates*

$$\mathbb{E}[\int_0^T |\bar{X}_s|^2 ds] \leq \frac{\exp(T\Upsilon^1) - \Upsilon^1}{\Upsilon^1} \left[\Upsilon^2 E[\int_0^T |\bar{Y}_s|^2 ds + \Upsilon^3 E[\int_0^T |\bar{Z}_s|^2 ds + \Upsilon^4 E[\int_0^T |\bar{K}_s|_s^2 ds \right].$$

Application to storage energy problem Our second contribution in this chapter is devoted to storage problem in smart grids. We consider a macro grid system designed to analyses energy system of N nodes defined in a micro grid system. Each node is characterized by the following:

- The capacity of the battery S_t representing the total energy available in the storage device.
- The net power production of the energy (photvoltaic panels, diesel energy,..) that each nodes produces after all costs subtracted.

In contrast with the paper of Alasseur et al [2], we assume that the production of energy is unpredictable. This is due to its dependence on environmental conditions such as the sun, the speed of the wind etc. which are intermittent and irregular.
More precisely, including the jumps component is essential in our analyses. In fact when the production of energy is perturbed or sudden slowdown (winds) , the other region can compensate this variability by optimizing the balance between production and consumption and hence stone excess electricity from a nodes to another and therefore avoiding a later expensive production .
We formulate the problem via mean field type control. We show that it can be characterized though solving an associated FBSDEJS of mean field type.
In the particular case where the cost structure is quadratic and the pricing rule is linear, we show that the FBSDE which characterizes the solution of the EMFG can be solved explicitly. This provides a quite tractable and efficient setting to analyze numerically various questions arising in this power grid model.
This work is concretized in the preprint [102].

1.4 Work in preparation and future research perspectives

We present in this section some work in progress and future perspectives.

• As mentioned in section 2, our goal is to investigate the wellposedness of BSDE with infinite activity jumps in general setting, that is under a stochastic quadratic growth. The approach that we attempt to use is to one developed by Bahlali et al [7]. This method is purely based on doubly reflected BSDEJs. In our second work, we have proved that the doubly reflected BSDEs with jump admits a solution. Therefore, we are thinking of how far this approach can be applied in our setup.

• An other work in progress concerns the numerical aspects the influence of the environmental conditions on the energy production in the model exposed in the third chapter. More precisely, we assumed additionally that the battery has maximum fixed level S_{max}. However, it can be very useful to analyses the qualitative behavior of the battery. In fact there exist many type of storage: high capacity energy storage, intermediary storage and lower energy storage. The type of the battery is closely related to the number of nodes in the grid.
Moreover, we are also exploring the numerical result related to fully coupled mean-field FBSDEjs.

• We are also investigating a continuous time Principal-Agent problem under moral hazard in which the drift and the volatility are controlled by the agent. The Principal design a contract with the possibility to fire the agent. More precisely we study the impact of the retirement , quitting or replacement of the agent on the contract.

CHAPTER 2

QUADRATIC BSDEs WITH INFINITE ACTIVITY JUMPS

2.1 Overview of the content of this chapter

The aim of this chapter is to study the following backward stochastic differential equation with jumps

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U(s,e)\tilde{\mu}(ds, de), \ \forall t \in [0,T], \ \mathbb{P}\text{-a.s.}$$

where the generator f satisfies the following structure condition.

$$\frac{1}{\delta} j_t(-\delta u) - \frac{\delta}{2}|z|^2 - l_t - c_t|y| \leq f_t(y,z,u) \leq \frac{1}{\delta} j_t(\delta u) + \frac{\delta}{2}|z|^2 + l_t + c_t|y|.$$

where $\delta > 0$ and c, l are predictable non negative processes.

We will look more closely at the case where the jumps has infinite activity and a terminal condition with finite exponential moment. In that case, we prove that the above BSDEJs admits a solution via a forward approach.

The outline is as follows. In the first section, we recall briefly some notation and give our mathematical setting. We provide in Section 2 the precise definition of exponential quadratic BSDEJs with infinite activity jumps and give our main assumptions. In the next section, we present our main result which is the existence of the solution. Since we are adopting the forward approach, we start by given some properties of quadratic exponential semimatingales as well as their stability result. Next, we construct an auxiliary BSDEJs with bounded terminal condition and finite activity jumps. We prove that these intermediate sequence of BSDEJs is in fact an exponential quadratic semimartinagle with some suitable estimates. We are finally lead to prove by a stability theorem that these sequence converges in some specific sens to our initial BSDEJs. In the appendix, we provide a comparison principle for exponential quadratic BSDEJs under additional assumption on the driver f.

2.2 Framework

2.2.1 Notations and setting

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which the filtration $\mathbb{F} = (\mathcal{F}_t)_{0<t<T}$ satisfies the usual conditions of completeness and right continuity and that $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ be trivial. Due to these usual conditions, we can take all semimartingales to have right continuous paths with left limits.
On this stochastic basis, let W a d-dimensional Brownian motion and $\mu(\omega, dt, de)$ an independent integer valued random measure defined on $([0,T]\times E, \mathcal{B}([0,T])\otimes\mathcal{B}(E))$, with compensator $\nu(\omega, dt, de)$ that can be time-inhomogeneous and may allow for infinite activity of the jumps.
The predictable σ-field on $\Omega \times [0,T]$ is denoted by $\mathcal{P}$ and $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(E)$ is the respective σ-field on $\tilde{\Omega} = \Omega \times [0,T] \times E$. For a σ-finite measure λ on $(E, \mathcal{B}(E))$ satisfying $\int_E 1 \wedge |e|^2 \lambda(de) < \infty$ and a bounded $\tilde{\mathcal{P}}$-measurable non negative density function ζ, we will assume that the compensator ζ is absolutely continuous with respect to $\lambda \otimes dt$ such that

$$\nu(\omega, dt, de) = \zeta(\omega, t, e)\lambda(de)dt, \qquad 0 \le \zeta \le C_\eta, \quad \text{for some constant } C_\nu. \tag{2.1}$$

Finally, we will denote by $\tilde{\mu}$ the compensated measure of μ as

$$\tilde{\mu}(\omega, dt, de) = \mu(\omega, dt, de) - \nu(\omega, dt, de). \tag{2.2}$$

In this chapter , we specially pay attention to the case when the jumps have infinite activity meaning that $\lambda(E) = \infty$. As stated in [122], the infinite activity of the jumps is related to the behavior of the compensator ν near to 0. Since we have always a finite number of big jumps, the (in)finitness of the jumps is controlled by the number of small jumps and thus the behavior of the ν around the origin.
We also allowed to the compensator ν to be time-inhomogeneous and stochastic which permits richer dependence structure for $(W, \tilde{\mu})$.
We emphasize that, the case of finite activity is already considered in [42].
Let f be a $\mathcal{P} \otimes \mathcal{E}$-measurable function, the integral with respect to the random measure and the compensator are defined as follow

$$(f \star \mu)_t = \int_0^t \int_E f(s,e)\mu(ds, de) \quad , (f \star \mu)_t = \int_0^t \int_E f(s,e)\nu(ds, de).$$

The random measure $\tilde{\mu}$ is defined as the compensated measure of μ such that

$$\tilde{\mu}(dt, de) = \mu(dt, de) - \nu(dt, de).$$

In particular, the stochastic integral $U \star \tilde{\mu} = \int U_s(e)\tilde{\mu}(ds, de)$ is a local square integrable martingale, for any predictable locally integrable process U.
We will assume that W and $\tilde{\mu}$ satisfies the following weak representation property with respect to $(\mathcal{F}_t)_{0\leq t\leq T}$

$$M = M_0 + \int_0^{\cdot} Z_s.dW_s + \int_0^{\cdot}\int_E U_s(e)\tilde{\mu}(de, ds).$$

2.2.2 Spaces and norms

Now we introduce the following spaces of processes which will be often used in the sequel.
For any $p \geq 1$, $\mathcal{P}$ stands for the σ-field of all predictable sets of $[0, T] \times \Omega$.
• $\mathcal{G}_{loc}(\mu)$ the set of $\mathcal{P} \otimes \mathcal{E}$-measurable $\mathbb{R}$-valued functions H such that

$$|H|^2.\nu_t < +\infty.$$

• $\mathbb{H}^2$ the set of all $\mathbb{R}^d$-valued càdlàg and $\mathcal{F}_t$-progressively measurable processes Z such that

$$\mathbb{E}[\int_0^T |Z_s|^2 ds] < +\infty.$$

• $\mathcal{S}^2$ is the space of $\mathbb{R}$- valued càdlàg and $\mathcal{F}_t$-progressively measurable processes Y such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^2\right] < +\infty.$$

• For $u, \bar{u}$ in the space $\mathbb{L}^0(\mathcal{B}(E), \nu)$ of all $\mathcal{B}(E)$-measurable functions with the topology of convergence in measure, we define

$$|u - \bar{u}|_t = (\int_E |u - \bar{u}|^2 \zeta(t, e)\nu(de))^{\frac{1}{2}}.$$

• $\mathbb{H}^2_\nu$ the set of all predictable processes U such that

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{E}} |U_s(e)|^2 \nu(de, dt\right)\right] < +\infty.$$

• $\mathcal{D}^{\exp}$ is the space of progressively measurable processes $(X = (X)_t)_{t\leq T}$ with

$$\mathbb{E}\left[\exp\left(\gamma \sup_{0\leq t\leq T} |X_t|\right)\right] < +\infty.$$

• U_{exp} is the class of càdlàg martingales M such that $\mathcal{E}(M)$ is a uniformly integrable martingale.

2.2.3 Formulation of the Quadratic Exponential BSDEs with jumps

We are given the following objects:
• The terminal condition ξ is an $\mathcal{F}_T$-measurable random variable.
• $W = (W_t)_{t \le T}$ be a d-dimensional Brownian motion.
• μ a random measure with compensator ν and $\tilde{\mu}(ds, de) = \mu(ds, de) - \nu(ds, de)$.
• The generator $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(\mathcal{B}(E), \nu) \to \bar{\mathbb{R}}$ are always taken $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+1}) \otimes \mathcal{B}(\mathbb{L}^0(\mathcal{B}(E), \nu))$-measurable.
We consider a class of coefficient as follows

$$f_t(y, z, u) = \hat{f}_t(y, z) + \int_E g_t(u(e))\nu(de). \tag{2.3}$$

$\hat{f} : \Omega \times [0,T] \times \mathbb{R}^{1+d} \to \bar{\mathbb{R}}$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{1+d})$-measurable function and $g : \Omega \times [0,T] \times E \to \bar{\mathbb{R}}$ to be $\mathcal{P} \otimes B(E)$-measurable function.

Remark 2.2.1. *This family of generators was introduced by Becherer in [15] to prove the existence and uniqueness of solution of Lipschitz JBSDE when the terminal condition is bounded.*

We consider the following BSDEJ with data (f, ξ)

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U(t,e)\tilde{\mu}(dt, de), \quad \forall t \in [0,T], \ \mathbb{P}\text{-a.s.} \tag{2.4}$$

We shall now make these standing assumptions on the map under consideration.

Assumption 2.2.2.

- *Continuity condition.* $\forall t \in [0,T], (y, z, u) \to f_t(y, z, u)$ *is continuous,* $\mathbb{P}$*-a.s.*
- *Integrability condition.* $\forall \gamma > 0, \ \mathbb{E}[\exp(\gamma(e^{C_{t,T}}|\xi| + \int_t^T e^{C_{t,s}} d\Lambda_s))] < +\infty$, *where* Λ *and* C *are two positive continuous increasing processes.*
- *Structure condition. There exists a positive adapted processes* l, c *and a non negative bounded constant* δ *s.t* $\forall (y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(\mathcal{B}(E), \nu), \forall t \in [0,T]$,

  $\frac{1}{\delta} j_t(-\delta u) - \frac{\delta}{2}|z|^2 - l_t - c_t|y| \le f(t, y, z, u) \le \frac{1}{\delta} j_t(\delta u) + \frac{\delta}{2}|z|^2 + l_t + c_t|y|.$

  where

$$j_t(\delta u) = \int_{\mathbb{E}} \left(e^{\delta u_s(e)} - \delta u_s(e) - 1 \right) \nu(de).$$

Assumption 2.2.3. *A_γ-condition : there exists a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^{d+2})\otimes\mathcal{B}(E)$measurable function γ with $\gamma * \widetilde{\mu} \in \mathcal{U}_{\exp}$ and $\gamma > -1$ such that $\forall y, z \in \mathbb{R}\times\mathbb{R}^d, \forall u, \bar{u} \in \mathbb{L}^0(\mathcal{B}(E), \nu)$,*

$$f_t(y,z,u) - f_t(y,z,\bar{u}) \leq \int_E \gamma_t[u(e) - \bar{u}(e)]\xi(t,e)\lambda(de).$$

Remark 2.2.4. *The above assumptions are essential in our framework to get the existence of the BSDEJ's solution. Here we deal with a terminal condition having a finite exponential moment of order γ and a jump measure with infinite activity. It has been shown in [42], under (2.2.2), (2.2.3), the existence of solution of BSDEs with finite activity jumps.*
When the jumps have infinite activity, Becherer [14] proved the existence and the uniqueness of the solution of bounded JBSDE when the generator f is not necessary Lipschitz with respect the jump component. They give sufficient condition on the differential of the function g that is $\frac{\partial h}{\partial u}(Y,Z,U).\widetilde{\mu} \in \mathrm{BMO}$. The uniqueness of the solution were also proved using comparison result since the condition A_γ is satisfied.
In [109] motivated by the connection between the exponential utility maximization problem and BSDEJ, the author have proved the existence and uniqueness of solution of bounded BSDEJ when f have a specific form:

$$f^n(t,z,u) = \inf_{\pi\in C}[\frac{\alpha}{2}|\pi_t\sigma_t - (z+\frac{\theta}{\alpha})|^2 + \int_E j_\alpha(u - \pi_t\beta_t)\nu(dx)] - \theta_t z - \frac{|\theta_t|^2}{2\alpha}.$$

In the above π refers to the strategies investment valued in the closed constraint set C and θ, β and σ are related to the dynamic of the market.

2.3 Existence of solution with infinite activity

The question is how to prove existence of solution of JBSDE with infinite activity jumps and unbounded terminal condition without using the so called "Kobylanski approach".
The main tool is to consider an auxilary sequence of globally lipschitz BSDEs with finite random measure for which the existence and uniqueness of the solution are well known. To insure that the approximated coefficient conserves all the properties of f, we use a double approximation that is to consider a sequence of function bounded from below and above by linear quadratic function. More precisely, we split the coefficient of the BSDE into the sum of two positive and negative functions and approximate respectively each function by inf-convolution and sup-convolution. Furthermore, the BSDEJ have infinite activity jumps, we have to deal with some specific difficulties due to the infinite number of small jumps. The idea is then to introduce a truncated measure with $\lambda(A) < \infty$ in the auxiliary BSDEJ for which existence of the solution is guaranteed. Adopting the forward approach we prove that this sequence of BSDEJ is in fact

an exponential quadratic semimartingales. And then by a stability theorem we show that the limit of the auxiliary BSDE's solution solves the original one.

2.3.1 The forward approach: Exponential quadratic semimartingale and its properties

As explained above, our point of view is based on the forward approach which is essentially standing on semimartingale. Adopting this approach, we summarize in this section the essential properties of the quadratic exponential semimartingales as well as a stability result which we shall use for the construction of the BSDE's solution. We start by the definition of the quadratic exponential semimartingale and present some example for illustration .

Definition 2.3.1. *A Quadratic Exponential Special Semimartingale Y is a càdlàg process such that $Y = Y_0 - V + M$ with V a local finite variation process and M a local martingale part with the following structure condition $\mathcal{Q}(\Lambda, C)$: for an increasing predictable processes C, Λ and a constant δ*

$$-\frac{\delta}{2}d\langle M^c\rangle_t - d\Lambda_t - |Y_t|dC_t - j_t(-\delta\Delta M_t^d]) \ll dV_t \ll \frac{\delta}{2}d\langle M^c\rangle_t + d\Lambda_t + |Y_t|dC_t + j_t(\Delta M_t^d). \quad (2.5)$$

with

$$j_t(\delta u) = \int_E \frac{\exp \delta u(e) - \delta u(e) - 1}{\delta}\nu(de).$$

Note that the symbol "$\ll$" means that the difference is an increasing process.

Example .1. *the structure condition (2.5) holds in the cases below:*

- *A semimartingale Y where the finite variation process V is given by $V_t = \frac{1}{2}\langle M^c\rangle_t + j_t(\delta M_t^d)$ is a exponential quadratic semimartingale.*
- *If the finite variation part of a semimartingale Y satisfies*

  $$-\frac{1}{2}\langle M^c\rangle_t - j_t(-\delta M_t^d) \ll V_t \ll \frac{1}{2}\langle M^c\rangle_t + j_t(\delta M_t^d).$$

  then Y is a exponential quadratic semimartingale.

The following proposition gives us a characterization of a canonical class of quadratic exponential semimartingale. The canonical quadratic semimartingale is a semimartingale with $V = -\frac{1}{2}\langle \bar{M}^c\rangle_t - j(-\Delta M_t^d).\nu_t$ or $V = \frac{1}{2}\langle \bar{M}^c\rangle_t + j(\Delta M_t^d).\nu_t$. This characterization will be useful in the sequel.

Proposition 2.3.2 (Doléans dade martingale and canonical quadratic semimartingale)**.** *[42] Let $\bar{M} = \bar{M}^c + \bar{U}.\widetilde{\mu}$ and $\underline{M} = \underline{M}^c + \underline{U}.\widetilde{\mu}$ two càdlàg local martingales such that $\bar{M}^c + (e^{\bar{U}} - 1).\widetilde{\mu}$*

and $-\underline{M}^c+(e^{-\underline{U}}-1).\widetilde{\mu}$ *are still càdlàg local martingales. Let define the canonical local quadratic exponential semimartingale:*

$$r(\bar{M}_t) = r(\bar{M}_0) + \bar{M}_t - \frac{1}{2}\langle \bar{M}^c\rangle_t - (e^{\bar{U}} - \bar{U} - 1).\nu_t.$$
$$\underline{r}(\underline{M}_t) = \underline{r}(\underline{M}_0) + \underline{M}_t + \frac{1}{2}\langle \underline{M}^c\rangle_t + (e^{-\underline{U}} + \underline{U} - 1).\nu_t.$$

then the following processes:

$$\exp[r(\bar{M})-r(\bar{M}_0)] = \mathcal{E}\left(\bar{M}^c + (e^{\bar{U}} - 1).\widetilde{\mu}\right) \text{ and } \exp[-\underline{r}(\underline{M})+\underline{r}(\underline{M}_0)] = \mathcal{E}\left(-\underline{M}^c + (e^{-\underline{U}} - 1).\widetilde{\mu}\right)$$

are positive local martingales.

Proposition 2.3.3. *Let* ψ_T *be an* $\mathcal{F}_T$*-random variable such that* $\exp(|\psi_T|) \in \mathbb{L}^1$ *and consider the two dynamic risk measures:*

$$\bar{\rho}_t(\psi_T) = \ln\left[\mathbb{E}\left(\exp(\psi_T)|\mathcal{F}_t\right)\right] \text{ and } \underline{\rho}_t(\psi_T) = -\ln\left[\mathbb{E}\left(\exp(-\psi_T)|\mathcal{F}_t\right)\right].$$

There exists local martingales $\bar{M} = \bar{M}^c + \bar{U}.\widetilde{\mu}$ *and* $\underline{M} = \underline{M}^c + \underline{U}.\widetilde{\mu}$ *such that:*

$$-\,d\bar{\rho}_t(\psi_T) = -d\bar{M}_t + \frac{1}{2}d\langle \bar{M}^c\rangle_t + \int_E (e^{\bar{U}(s,e)} - \bar{U}(s,e) - 1).\nu(dt,dx), \quad \bar{\rho}_T(\psi_T) = \psi_T.$$
$$-\,d\underline{\rho}_t(\psi_T) = -d\underline{M}_t - \frac{1}{2}d\langle \underline{M}^c\rangle_t - \int_E (e^{-\underline{U}(s,e)} + \underline{U}(s,e) - 1).\nu(dt,de), \quad \underline{\rho}_T(\psi_T) = \psi_T.$$

Moreover the local martingales $\bar{M}^c+(e^{\bar{U}}-1).\widetilde{\mu}$ *and* $-\underline{M}^c+(e^{-\underline{U}}-1).\widetilde{\mu}$ *belong to* $\mathcal{U}_{\exp}$*. The dynamic risk measures* $\bar{\rho}(\psi_T)$ *and* $\underline{\rho}(\psi_T)$ *are uniformly integrable canonical quadratic exponential semimartingales.*

Integrability of the $Q(\Lambda, C)$-semimartingale

In this part we want to investigate the integrability of this classs of semimartingale. This result will be extremely useful in the section (2.3.2). First, we recall the following transformation of a $Q(\Lambda, C, \delta)$-semimartingale Y

$$\bar{X}_t^{\Lambda,C}(|Y_t|) := e^{C_t}|Y_t| + \int_0^t e^{C_s}d\Lambda_s. \tag{2.6}$$

To explore the exponential integrability of this class, we proceed analogously to the proof of proposition (3.2) in [12]. Note that this decomposition appeared for the first time in the continuous setting in this paper.
For this propose, let us start by the definition of the Q-submartingale.

Definition 2.3.4. *A semimartingale $Y = Y_0 - V + M$ is a Q-submartingale if $-V + \frac{1}{2}\langle M^c\rangle + \frac{1}{2}j_t(\Delta M^d)$ is a predictable increasing process.*

Theorem 2.3.5. *let $\bar{X}$ be a càdlàg process given by (2.6)such that $\bar{X}_T^{\Lambda,C}(e^{|Y|}) \in \mathbb{L}^1$, the process $\bar{X}_T^{\Lambda,C}$ is an $\mathcal{Q}(\Lambda, C)$-semimartingale which belonging to $\mathcal{D}_{exp}$ if and only if for any stopping times $\sigma \leq \tau \leq T$:*

$$|Y_\sigma| \leq \rho_\sigma \left(e^{C_{\sigma,\tau}}|Y_\tau| + \int_\sigma^\tau e^{C_{\sigma,t}} d\Lambda_t\right). \tag{2.7}$$

Proof. First, we check that $\bar{X}_t^{\Lambda,C}(|Y|)$ is a Q-submartingale. Applying Itô-Tanaka formula we get for all $t \in [0,T]$

$$d|Y|_t = sign(Y_{t\_})dM_t - sign(Y_{t\_})dV_t + (|Y_{t\_} + U_t| - |Y_{t\_}|) \star \mu_t + L_t^Y. \tag{2.8}$$

where L^Y is a local time of Y in zero. We denote by $M^s = sign(Y).dM$ and $V^s = sign(Y).dV$. Hence,

$$\begin{aligned}
d\bar{X}_t^{\Lambda,C} &= e^{C_t}[|Y_t|dC_t - dV_t^s + dM_t^s + d(|Y_{t_} + U_t| - |Y_{t_}|) \star \mu + L_t^Y + d\Lambda_t] \\
&= e^{C_t}[|Y_t|dC_t + d\Lambda_t - dV_t^s + d(|Y_{t_} + U_t| - |Y_{t_}|)\nu_t + L_t^Y] + e^{C_t}[dM_t^s + d(|Y_{t_} + U_t| - |Y_{t_}|) \star \tilde{\mu}_t] \\
&= e^{C_t}[dA_t + \frac{1}{2}d\langle M_t^c\rangle - j_t(\Delta M_t^d) + d(|Y_{t_} + U_t| - |Y_{t_}|)\nu_t] \\
&+ e^{C_t}[dM_t^s - \frac{1}{2}d\langle M_t^c\rangle + d(|Y_{t_} + U_t| - |Y_{t_}|) \star \tilde{\mu}_t].
\end{aligned}$$

Thank's to structure condition of the semimartingale Y, the process $A = |Y_t|dC_t + d\Lambda_t$ is increasing. Notice that the martingale part of this last decomposition : $\bar{M} := e^{C_t}[dM_t^s + d(|Y_{t\_} + U_t| - |Y_{t\_}|) \star \tilde{\mu}_t]$ have the following quadratic variation $d\langle \bar{M}\rangle_t = e^{2C_t}d\langle M^{c,s}\rangle_t + e^{2C_t}d\langle |Y_{t\_} + U_t| - |Y_{t\_}|) \star \tilde{\mu}_t\rangle$.
Adding and subtracting respectively $j_t(e^{C_t}\Delta M_t^{s,d}), e^{2C_t}d\langle M^{s,c}\rangle$, to $\bar{X}$ yield to

$$\begin{aligned}
d\bar{X}_t^{\Lambda,C} &= d\tilde{A}_t - \frac{1}{2}d\langle e^{C_t}M_t^{s,c}\rangle - \frac{1}{2}j_t(e^{C_t}\Delta M_t^{s,d}) + e^{C_t}d(|Y_{t_} + U_t| - |Y_{t_}|)\nu_t \\
&+ e^{C_t}\left[dM_t^s + d(|Y_{t_} + U_t| - |Y_{t_}|) \star \tilde{\mu}_t\right].
\end{aligned}$$

The process $\tilde{A} = \tilde{A} + \frac{1}{2}\langle e^{C_t}M^{s,c}\rangle + \frac{1}{2}j(e^C\Delta M^{s,d})$ is increasing since $e^{C_t}j_t(\Delta M_t^{s,d}) - j_t(e^{C_t}\Delta M_t^{s,d}) \geq 0$. Furthermore, we have $e^{C_t}d\langle M^{s,c}\rangle_t - e^{2C_t}d\langle M^{s,c}\rangle_t \geq 0$. Once again we add and subtract $j_t(e^{C_t}\Delta M_t^{s,d})$

$$d\bar{X}_t^{\Lambda,C} = d\bar{A}_t - \frac{1}{2}d\langle e^{C_t}M_t^{s,c}\rangle - \frac{1}{2}j_t(e^{C_t}\Delta M_t^{s,d}) + e^{C_t}\left[dM_t^s + d(|Y_{t\_} + U_t| - |Y_{t\_}|) \star \tilde{\mu}_t\right].$$

The process $d\bar{A} = d\tilde{A} + d(j_t(e^{C_t}|Y_{t\_} + U_t| - |Y_{t\_}|)$ is increasing process. This decomposition shows that $\bar{X}^{\Lambda,C}$ is a Q-submartingale.

Since $\bar{X}^{\Lambda,C}$ is a Q-submartingale, it follows that $\exp(\bar{X}^{\Lambda,C})$ is a submartingale. Hence, for all stopping time σ, τ, we have

$$\begin{aligned}&\exp(\bar{X}^{\Lambda,C}_\sigma) \le \mathbb{E}[\exp(\bar{X}^{\Lambda,C}_\tau)|\mathcal{F}_\sigma]\\ &\exp(e^{C_\sigma}|Y_\sigma| + \int_0^\sigma e^{C_{\sigma,t}} d\Lambda_t) \le \mathbb{E}[\exp(e^{C_\tau}|Y_\tau| + \int_0^\tau e^{C_{\tau,t}} d\Lambda_t)|\mathcal{F}_\sigma].\end{aligned}$$

Taking $\int_0^\sigma e^{C_{\sigma,t}} d\Lambda_t$ in the right hand side we obtain

$$\exp(|Y_\sigma|) \le \mathbb{E}[\exp(e^{C_{\sigma,\tau}}|Y_\tau| + \int_\sigma^\tau e^{C_{\tau,t}} d\Lambda_t)].$$

Hence we can write

$$|Y_\sigma| \le \ln \mathbb{E}[\exp(e^{C_{\sigma,\tau}}|Y_\tau| + \int_\sigma^\tau e^{C_{\tau,t}} d\Lambda_t)]. \tag{2.9}$$

which end the proof.

Quadratic variation and Stability result

We introduce now the class of $\mathcal{S}_Q(|\xi|, \Lambda, C)$-semimartingale which will play an important role in the proof of the existence result.

Definition 2.3.6. *$\mathcal{S}_Q(|\xi|, \Lambda, C)$ is the class of all $\mathcal{Q}(\Lambda, C)$-semimartingales Y such that*

$$|Y_t| \le \bar{\rho}_t \left[e^{C_{t,T}}|\xi_T| + \int_t^T e^{C_{t,s}} d\Lambda_s \right], \quad a.s.$$

Theorem 2.3.7. *Let $(Y^n)_n$ a sequence of $\mathcal{S}_Q(|\xi|, \Lambda, C)$ special semimartingales which canonical decomposition $Y^n = X_0^n - V^n + M^n$ which converge in $\mathbb{H}^1$ to some process Y. Therefore the process Y is an adapted càdlàg process which belongs to $\mathcal{S}_Q(|\xi|, \Lambda, C)$ with the following canonical decomposition $Y = Y_0 - V + M$ such that:*

$$\lim_{n\to\infty} \mathbb{E}\left[(V^n - V)^*\right] = 0 \quad \textit{and} \quad \lim_{\kappa\to\infty} \|M^n - M\|_{\mathbb{H}^1} = 0.$$

Remark 2.3.8. *The proof is built on the stability theorem (2.5.2) based on uniform estimates of the quadratic variation part and the total variation of the semimartingale.*
In [12], the authors work in a continuous framework where they proved that the class $S_Q(|\xi|, \Lambda, C)$ is stable by a.s convergence. In the proposition above we prove that this class is also stable in the discontinuous setting.

Proof. From proposition (3.3) in [42], the following exponential transformation $(X^n)^{\Lambda,C}(Y) = Y^n + \Lambda + |Y^n| * C$ and $(X^n)^{\Lambda,C}(-Y)$ are $\mathcal{Q}$-local submartingale. Then by the Yoeurp-Meyer decomposition, there exists an increasing process A_t such that

$$\exp((X^n)_t^{\Lambda,C}(Y)) = \exp(Y_0)\mathcal{E}(\bar{M}_t + (e^{\bar{U}} - 1).\tilde{\mu})\exp(\bar{A}_t).$$
$$\exp((X^n)_t^{\Lambda,C}(-Y))) = \exp(-Y_0)\mathcal{E}((\bar{M}_t + (e^{\bar{U}} - 1).\tilde{\mu}))\exp(\underline{A}_t).$$

Therefore , for a stopping times $\sigma \leq T$, we obtain

$$\langle\bar{M}\rangle_\sigma = \int_\sigma^T \frac{d\langle\exp((X_t^n)^{\Lambda,C}(Y))\rangle}{\exp(2(X_t^n)^{\Lambda,C}(Y))}.$$

$$\langle\underline{M}\rangle_\sigma = \int_\sigma^T \frac{d\langle\exp((X_t^n)^{\Lambda,C}(-Y))\rangle}{\exp(2(X_t^n)^{\Lambda,C}(-Y))}.$$

Thank's to Garsia Lemma (2.5.3) and Itô formula we have for all $p \geq 1$

$$\mathbb{E}\left[\langle\exp(p(X_t^n)^{\Lambda,C}(Y))\rangle_T\right] \leq C_2 \quad \text{and} \quad \mathbb{E}\left[\langle\exp(p(X_t^n)^{\Lambda,C}(-Y))\rangle_T\right] \leq C_1. \tag{2.10}$$

Then, the estimates of $\langle\bar{M}\rangle$ and $\langle\underline{M}\rangle$ comes from Cauchy Schwartz inequality

$$\mathbb{E}\left[\langle\bar{M}_T^p\rangle\right] \leq C \quad \text{and } \mathbb{E}\left[\langle\underline{M}\rangle_T^p\right] \leq C.$$

In the other hand, applying Itô formula and using the fact that

$$2[(e^{\delta U} - \delta U - 1) + (e^{-\delta U} + \delta U - 1)] \leq |e^{\delta U} - 1|^2 + |e^{-\delta U} - 1|^2,$$

leads to

$$\mathbb{E}\left[\int_0^T |dV_s^n|\right] \leq \mathbb{E}\left[\int_0^T \frac{1}{2}d\langle(M^c)^n\rangle_s + \int_E [j(U_s^n(e)) + j(-U_s^n(e))]\nu(ds,de)\right] \leq C'.$$

Finally, from the Barlow-Protter stability theorem 2.5.2, we obtain that the limit process Y of Y^n is in fact a special semimartingale with the canonical decomposition $Y := Y_0 - V + M$ satisfying:

$$\mathbb{E}\Big[\int_0^T |dV_s|\Big] \leq C, \quad \text{and} \quad \mathbb{E}\left[(M)^*\right] \leq C. \tag{2.11}$$

and we have

$$\lim_{n\to\infty} \mathbb{E}\left[(V^n - V)^*\right] = 0 \quad \text{and} \quad \lim_{n\to\infty} \|M^n - M\|_{\mathbb{H}^1} = 0. \tag{2.12}$$

Moreover, the semimartingale Y is also a $S_Q(|\xi|, \Lambda, C)$ since a.s

$$|Y_\sigma^n| \leq \bar{\rho}_\sigma \left[e^{C_{\sigma,T}}|\xi| + \int_\sigma^T e^{C_{\sigma,s}} d\Lambda_s\right].$$

Passing to the limit when n goes to ∞. We finally obtain

$$|Y_\sigma| \leq \bar{\rho}_\sigma \left[e^{C_{\sigma,T}}|\xi| + \int_\sigma^T e^{C_{\sigma,s}} d\Lambda_s \right]. \tag{2.13}$$

2.3.2 The main result

We are now in position to give the main result of this paper.

Theorem 2.3.9. *Under Assumption 2.2.2 and Assumption 2.2.3, there exists a solution* $(Y, Z, U) \in \mathcal{S}_Q(|\xi|, \Lambda, C) \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ *of the BSDEJ (2.4)*

Proof. For the sake of clarity we split the proof into three main steps.

• The first step: we introduce an auxiliary generator $f^{n,m,\kappa}$ uniformly lipschitz (y, z) and locally lipschitz in u as follow

$$f^{n,m,\kappa}(y, z, u) := \bar{f}^{n,\kappa}(y, z, u) - \underline{f}^{m,\kappa}(y, z, u).$$

where $\bar{f} := f 1_{\{f>0\}}$ and $\underline{f} := f 1_{\{f\leq 0\}}$. From this and using a well known results, we justify the existence and the uniqueness of solution of BSDEJ associated to $(f^{n,m,\kappa}, |\xi|)$. The solution will be a triple of progressively measurable processes $(Y^{n,m,\kappa}, Z^{n,m,\kappa}, U^{n,m,\kappa})$.

• The second step: Since the truncated driver satisfies the structure condition of the quadratic exponential semimartingales, we prove that the solution $Y^{n,m,\kappa}$ of the BSDEJ associated to $(f^{n,m,\kappa}, |\xi|)$ is a $\mathcal{Q}(\Lambda, C)$-semimartingale.

• The last step will be the convergence of the approximated sequence of BSDEJ given by $(f^{n,m,\kappa}, |\xi|)$. Using the stability theorem 2.3.7, we prove that the limit of $(Y^{n,m,\kappa}, Z^{n,m,\kappa}, U^{n,m,\kappa})$ exists and solves the original BSDE.

Step 1. Construction of the truncated sequence of BSDEJs.

For $\kappa > 1$ we consider a random measure ν^κ as follows

$$\nu^\kappa(dt, de) := 1_{\{|e|\geq \frac{1}{k}\}} \nu(dt, de).$$

$\tilde{\mu}^\kappa(dt, de) = 1_{\{|e|\geq \frac{1}{k}\}} \tilde{\mu}(dt, de)$ and $f^\kappa(y, z, u)_t = f_t(y, z) + \int_E g_t(u(e))\nu^\kappa(de)$.

We emphasize that, the truncated random measure ν^κ introduced above is a finite random measure i.e for all borelian set A, $\nu^\kappa(A) < +\infty$.

Before proceeding with the proof, we will need the following proposition which provides essential properties of $(f^{n,m,\kappa})$ needed in the proof.

First, Let us introduce the regularization function $\bar{b}^n$ and $\underline{b}^m$ are the convex functions with linear growth defined by $\bar{b}_n(w, r, v) = n|w| + n|r| + n|v|_t$ and $\underline{b}_m(w, r, v) = -m|w| - m|r| - m|v|_t$.

Lemma 2.3.10. *Let us consider the generator $\bar{f} = f^+$ and $\underline{f} = f_-$. We define the sequence $\bar{f}^{n,\kappa}$,$\bar{q}^{n,\kappa}$, $\underline{f}^{m,\kappa}$, and $\underline{q}^{m,\kappa}$ respectively as the in-convolution and sup-convolution of $\bar{f}$, $\bar{q}$, $\underline{f}$ and $\underline{q}$ with the regularization function b_n. The regularized functions are defined as follows*

- $$\bar{f}^{n,\kappa}(y,z,u) := \bar{f}^{\kappa} \vee \bar{b}_n(y,z,u) = \bar{F}^n(y,z) + \bar{G}^{n,\kappa}(u) := \inf_{(r,w)\in\mathbb{Q}^{1+d}} \Big\{\hat{f}_t(r,w) + n|r-y| + n|w-z|\Big\} + \inf_{v\in\mathbb{L}^0(B(E),\nu)} \Big\{\int_E g_t(v(e))\nu^{\kappa}(de) + n|u-v|_t\Big\}.$$
- $$\underline{f}^{m,\kappa}(y,z,u) := \underline{f}^{\kappa} \wedge \underline{b}_m(y,z,u) = \underline{F}^m(y,z) + \underline{G}^m(u) := \sup_{(r,w)\in\mathbb{Q}^{d+1}} \Big\{\hat{f}_t(r,w) + m|r-y| + m|w-z|\Big\} + \sup_{v\in\mathbb{L}^0(B(E),\nu)} \Big\{\int_E g_t(v(e))\nu^{\kappa}(de) + m|u-v|_t\Big\}.$$
- $\bar{q}^{n,\kappa}(t,y,z,u) = \bar{q}^{\kappa} \wedge b_n(y,z,u)$ *and* $\underline{q}^{m,\kappa}(t,y,z,u) = \underline{q}^{\kappa} \vee b_m(y,z,u)$.

Under Assumption 2.2.2, we have the following essential properties

1. $(\bar{f}^{n,\kappa})$, $(\bar{q}^{n,\kappa})$,$(\underline{f}^{n,\kappa})$, $(\underline{q}^{n,\kappa})$ *are respectively increasing and decreasing sequences in* n, κ *and* m.
2. *The sequences* $(\bar{f}^{n,\kappa})$, $(\bar{q}^{n,\kappa})$,$(\underline{f}^{m,\kappa})$, $(\underline{q}^{m,\kappa})$ *are globally lipschitz continuous in* (y,z) *and locally lipschitz in* u *for each* n, m, κ.
3. *The sequences* $(\bar{f}^{n,\kappa})$, $(\bar{q}^{n,\kappa})$ *converge respectively to* $\bar{f}$ *and* $\bar{q}$ *as* n, κ *goes to* ∞ *i.e*
$$\bar{f}^{n,\kappa} \nearrow \bar{f} \quad \text{and} \quad \bar{q} \nearrow \bar{q}.$$
The convergence is also uniform.
4. *The sequences* $(\underline{f}^{m,\kappa})$, $(\underline{q}^{m,\kappa})$ *converge respectively to* $\underline{f}$ *and* $\underline{q}$ *as* m, κ *goes to* ∞ *i.e*
$$\underline{f}^{m,\kappa} \nearrow \underline{f} \quad \text{and} \quad \underline{q}^{m,\kappa} \nearrow \underline{q}.$$
5. $(f^{n,m,\kappa})_{n,m,\kappa}$ *satisfies the structure condition of assumption 2.2.2.*

The proof is relegated to the Appendix.

Remark 2.3.11. *We emphasize that the regularization technique in lemma 2.3.10 were inspired by the one developed in the paper [42]. Nonetheless, as explained before, the main difficulty in carrying out this construction is that the structure of this equations (2.4) are defined in infinite activity setting.*

Let as now introduce the BSDE associated to the truncated measure ν^κ

$$dY_t^\kappa = f_t^\kappa(Y_t^\kappa, Z_t^\kappa, U_t^\kappa)dt - Z_t^\kappa dW_t + \int_E U_t^\kappa(e)\mu(dt, de), \quad t \in [0,T], \ \mathbb{P}\text{-a.s,} \tag{2.14}$$

where $f_t(Y_t^\kappa, Z_t^\kappa, U_t^\kappa) = f_t(y,z) + \int_E g_t(u(e))\zeta(t,e)\lambda^\kappa(de)$. Let mention that this BSDE is driven jointly by the Brownian motion W and the truncated measure μ^κ. The associated filtration is denoted by $\mathcal{F}_t^\kappa \subset \mathcal{F}_t$.
Note that (2.14) is an exponential quadratic BSDE with finite activity jumps, the generator f^κ satisfies the same hypotheses as f.

After we construct the BSDE associated to the truncated random measure, we introduce the intermediate BSDE $(f^{n,m,\kappa}, |\xi|)$,

$$-dY_t^{n,m,\kappa} = f_t^{n,m,\kappa}(Y_t^{n,m,\kappa}, Z_t^{n,m,\kappa}, U_t^{n,m,\kappa})dt - Z_t^{n,m,\kappa}dW_t - \int_E U^{n,m,\kappa}(e)\tilde{\mu}(dt, de), \ \mathbb{P}\text{-a.s.} \tag{2.15}$$

First, we have to justify the existence of a solution to this BSDE. In fact this is a simple consequence of the existence results of [14]. Thanks to the above lemma our coefficient $(f^{n,m,\kappa})$ is Lipschitz with respect to (y,z,v). It remains to show that the A_γ-condition hold for $(f^{n,m,\kappa})$. let $u, \bar{u} \in \mathbb{L}^0(\mathcal{B}(E), \nu)$, $y, z \in \mathbb{R}, \mathbb{R}^d$ such that

$$\begin{aligned} f_t^{n,m,\kappa}(y,z,u) - f_t^{n,m,\kappa}(y,z,\bar{u}) &:= [G_t^{n,\kappa}(u) - G_t^{n,\kappa}(\bar{u})] + [G_t^{m,\kappa}(u) - G_t^{m,\kappa}(\bar{u})] \\ &\le \bar{q}_t^{n,\kappa}(y,z,u) - \bar{q}_t^{n,\kappa}(y,z,\bar{u}) + \underline{q}_t^{m,\kappa}(y,z,u) - \underline{q}_t^{m,\kappa}(y,z,\bar{u}). \end{aligned}$$

Following [42], we know that $\bar{q}^{n,\kappa}$ and $\underline{q}^{m,\kappa}$ satisfy respectively the A_γ-condition. Hence

$$\begin{aligned} \bar{q}_t^{n,\kappa}(y,z,u) - \bar{q}_t^{n,\kappa}(y,z,\bar{u}) + \underline{q}_t^{m,\kappa}(y,z,u) - \underline{q}_t^{m,\kappa}(y,z,\bar{u}) &\le \int_E \gamma^n(u(e), \bar{u}(e))(u(e) - \bar{u}(e))\nu^\kappa(de) \\ &+ \int_E \gamma^n(u(e), \bar{u}(e))(u(e) - \bar{u}(e))\nu^\kappa(de). \end{aligned}$$

where $-1 < \gamma^n < n$ and $-1 < \gamma^m < m$. Therefore

$$f_t^{n,m,\kappa}(y,z,u) - f_t^{n,m,\kappa}(y,z,\bar{u}) \le \int_E \gamma^{n,m}(u(e), \bar{u}(e))(u(e) - \bar{u}(e)\nu^\kappa(de).$$

Additionally, we have $-l_t \le f_t^{n,m,\kappa}(0,0,0) \le l_t$.
According to [14], which deals with the lipschitz BSDE with jumps, we know that the BSDE (3.22) has a unique solution $(Y^{n,m,\kappa}, Z^{n,m,\kappa}, U^{n,m,\kappa})$. Moreover since A_γ-condition holds for $(f^{n,m,\kappa})_{n,m,\kappa}$, we can also apply the comparison theorem, to obtain

$$Y_t^{n,m+1,\kappa} \le Y_t^{n,m,\kappa} \le Y_t^{n,m+1,\kappa+1}, \quad \forall t \in [0,T], \, \mathbb{P}\text{-a.s.} \tag{2.16}$$

Step 2: Construction of the sequence of $\mathcal{Q}(C,\Lambda)$-semimartingale

In view of lemma 2.3.10 and Assumption 2.2.2 we have ,

$$\underline{q}_t \leq \underline{q}_t^{\kappa,m} \leq f_t^{n,m,\kappa} \leq \bar{q}_t^{\kappa,n} \leq \bar{q}_t, \quad \forall t \in [0,T], \mathbb{P}\text{-a.s.}$$

Thus, we know that all the requirement of the definition 2.3.1 all full filled. It follow that $(Y^{n,m,\kappa})_{n,m,\kappa}$ defined as the unique solution of the BSDEJ (2.15) is a $\mathcal{Q}(C,\Lambda)$-semimartingale. Moreover from Lemma (2.3.10), we know that for all stopping times $\sigma \leq T$

$$Y_\sigma^{n,m,\kappa} \leq \bar{\rho}_\sigma \left[e^{C_{\sigma,T}}|\xi| + \int_\sigma^T e^{C_{\sigma,s}} d\Lambda_s \right], \qquad \mathbb{P}\text{-a.s.}$$

Step 3: The convergence of the semimartingale

The idea is now to prove that the limit in some sens of those sequence of $\mathcal{Q}(C,\Lambda)$-semimartingale is a solution of the BSDEJ (2.4).

• We know from the first step that the sequence $(Y^{n,m,\kappa})_\kappa$ is an increasing and bounded in κ then $(Y^{n,m,\kappa})_\kappa$ converge in $\mathbb{H}^2$ to $Y^{m,n}$ such that

$$Y^{n,m,\kappa} \nearrow Y^{n,m}, \quad \kappa \text{ goes to } \infty, \quad \mathbb{P}\text{-a.s.}$$

Thanks to the Dini's lemma [35] the convergence is uniform. Hence it follows from Theorem 2.3.7 that $Y^{n,m}$ is a $\mathcal{Q}(\Lambda,C)$-semimartingale and satisfies

$$\mathbb{E}\Big[\int_0^T |dV_s^{n,m}|\Big] \leq C, \quad \text{and} \quad \mathbb{E}\left[(M^{n,m})^\star\right] \leq C.$$

$$\lim_{\kappa\to\infty} \mathbb{E}\left[(V^{n,m,\kappa} - V^{n,m})^*\right] = 0 \quad \text{and} \quad \lim_{\kappa\to\infty} \|M^{n,m,\kappa} - M^{n,m}\|_{\mathcal{H}^1} = 0.$$

• We proceed exactly as [42] since $(Y^{n,m})_n$ and $(Y^{n,m})_m$ are monotone bounded uniformly sequence. Therefore they converge monotonically to some process Y i.e

$$\lim_{n,m} \searrow Y^{n,m} = Y, \quad \mathbb{P}\text{-a.s.}$$

Using the same arguments, we show that Y is a $\mathcal{Q}(\Lambda,C)$-semimartingale and the following estimates holds

$$\mathbb{E}\Big[\int_0^T |dV_s|\Big] \leq C, \quad \text{and} \quad \mathbb{E}\left[(M)^\star\right] \leq C.$$

$$\lim_{n,m\to\infty} \mathbb{E}\left[(V^{n,m} - V)^*\right] = 0 \quad \text{and} \quad \lim_{n,m\to\infty} \|M^{n,m} - M\|_{\mathbb{H}^1} = 0.$$

• It remains to show that the $\mathcal{Q}(\Lambda,C)$-semimartingale Y is the solution of the *exponential* *quadratic BSDEJ* (2.4) such that

$$f(Y,Z,U) = \lim_{n,m,\kappa} f^{n,m,\kappa}(Y^{n,m,\kappa}, Z^{n,m,\kappa}, U^{n,m,\kappa}), \qquad d\mathbb{P}\otimes d\nu \;\text{ a-s.}$$

$$(Z^{n,m,\kappa}.W + U^{n,m,\kappa}\tilde{\mu}) = \lim_{n,m,\kappa}(Z.W + U\tilde{\mu}).$$

We need to define a sequence of stopping times τ_l related to the class $\mathcal{Q}(\Lambda, C)$ such that τ_l goes to ∞ for a large l. Let us fix $l \in \mathbb{N}^*$ such that

$$\tau_l := \inf\{t \geq 0,\, \mathbb{E}\big[\exp(e^{C_T}|\xi| + \int_0^T e^{C_s} d\Lambda_s|\mathcal{F}_t] > l\}.$$

According to the first part of the proof, the monotone convergence of $(Y^{n,m,\kappa})_{.\wedge\tau_l} := ((M^{n,m,\kappa})^c + V^{n,m,\kappa} + U^{n,m,\kappa} \star \tilde{\mu})_{.\wedge\tau_l}$ is uniform. Therefore, we can extract a subsequence of $(M^{n,m,\kappa})_{.\wedge\tau_l}$ which converges strongly to $M_{.\wedge\tau_l}$

$$(M^{n,m,\kappa})_{t\wedge\tau_l} = Z_t^{n,m,\kappa}1_{\{t\leq\tau_l\}}.W + U^{n,m,\kappa}1_{\{t\leq\tau_l\}} \star \tilde{\mu}.$$

Once again, we can subtract a subsequence $Z_t^{n,m,\kappa}1_{\{t\leq\tau_l\}}$ and $U^{n,m,\kappa}1_{\{t\leq\tau_l\}}$ converge almost surely to Z and U in $\mathbb{H}^2 \times \mathbb{H}^2_\nu$

$$dV^{n,m,\kappa}_{.\wedge\tau_l} := \hat{f}_t^{n,m,\kappa}(Y^{n,m,\kappa}_{t\wedge\tau_l}, Z^{n,m,\kappa}_{t\wedge\tau_l})1_{\{t\leq\tau_l\}}dt + G^{n,m,\kappa}(U^{n,m,\kappa}_{t\wedge\tau_l})1_{\{t\leq\tau_l\}}.$$

As a last step, we will show that $f^{n,m,\kappa}(Y^{n,m,\kappa}, Z^{n,m,\kappa}, U^{n,m,\kappa})$ converge to $f(Y,Z,U)$ in $\mathbb{L}^1(d\mathbb{P}\otimes d\nu\otimes dt)$. In fact as $Z^{n,m,\kappa}$ and $U^{n,m,\kappa}$ are unbounded, one can decompose the expression above in 2 quantities: one in the region where $\{|Z^{n,m,\kappa}| + |U^{n,m,\kappa}| \leq C\}$ and the other in the region $\{|Z^{n,m,\kappa}| + |U^{n,m,\kappa}| > C\}$.

$$\begin{aligned}
&\mathbb{E}[\int_0^{\tau_l} |f_s^{n,m,\kappa}(Y_s^{n,m,\kappa}, Z_s^{n,m,\kappa}, U_s^{n,m,\kappa}) - f_s(Y_s, Z_s, U_s)|ds] \\
=&\mathbb{E}[\int_0^{\tau_l} |f_s^{n,m,\kappa}(Y_s^{n,m,\kappa}, Z_s^{n,m,\kappa}, U_s^{n,m,\kappa}) - f_s(Y_s, Z_s, U_s)|1_{\{|Z^{n,m,\kappa}|+|U^{n,m,\kappa}|\leq C\}}ds] =: A_1 \\
+&\mathbb{E}[\int_0^{\tau_l} |f_s^{n,m,\kappa}(Y_s^{n,m,\kappa}, Z_s^{n,m,\kappa}, U_s^{n,m,\kappa}) - f_s(Y_s, Z_s, U_s)|1_{\{|Z^{n,m,\kappa}|+|U^{n,m,\kappa}|> C\}}ds] =: A_2.
\end{aligned}$$

We start by studying the first term A_1.

Observe that in the region $\{|Z^{n,m,\kappa}| + |U^{n,m,\kappa}| \leq C\}$, since $Y^{n,m,\kappa}$ is bounded over $[0, \tau_l]$

$$
\begin{aligned}
A_1 &= \mathbb{E}[\int_0^{\tau_l} |f_s^{n,m,\kappa}(Y_s^{n,m,\kappa}, Z_s^{n,m,\kappa}, U_s^{n,m,\kappa}) - f_s(Y_s, Z_s, U_s)| 1_{\{|Z^{n,m,\kappa}|+|U^{n,m,\kappa}| \leq C\}} ds] \\
&\leq \phi_s + \mathbb{E}[\int_0^{\tau_l} |G_s^{n,\kappa}(u) - G_s(u) 1_{\{|Z^{n,m,\kappa}|+|U^{n,m,\kappa}| \leq C\}} ds] \\
&\leq \phi_s + E[\int_0^{\tau_l} j_s^{\kappa}(\delta U^{n,m,\kappa}(e)) - j_s(\delta U(e)) 1_{\{|U^{n,m,\kappa}| \leq C\}} ds].
\end{aligned}
$$

Therefore, to prove that $f^{n,m,\kappa}(Y^{n,m,\kappa}, Z^{n,m,\kappa}, U^{n,m,\kappa}) - f(Y, Z, U)$ is converges in $\mathbb{L}^1$, it is sufficient to show that $\mathbb{E}\left[\int_t^T j_s^{\kappa}(\delta U^{n,m,\kappa}(e)) - j_s(\delta U(e)) 1_{\{|U^{n,m,\kappa}| \leq C\}} ds\right]$ converge to zero as n, m, κ goes to ∞. By Dominated convergence theorem we get the desire result.
We start by following a technique similar to the one used in the proof of Theorem (4.13) of [14]. Observe that $j(\delta U(e)) = j(\delta U(e)) 1_{\{|e| \leq \kappa\}} + j(\delta U(e)) 1_{\{|e| \geq \kappa\}}$, hence one can write

$$
\begin{aligned}
&\mathbb{E}[\int_t^{\tau_l} j_s^{\kappa}(\delta U^{n,m,\kappa}(e)) - j_s(\delta U(e)) 1_{\{|U^{n,m,\kappa}| \leq C\}} ds] \\
\leq &\mathbb{E}[\int_t^{\tau_l} j_s^{\kappa}(\delta U^{n,m,\kappa}(e)) - j_s^{\kappa}(\delta U(e)) 1_{\{|U^{n,m,\kappa}| \leq C\}} 1_{\left\{|e| \geq \frac{1}{\kappa}\right\}} ds] \\
+ &\mathbb{E}[\int_t^{\tau_l} j_s(\delta U(e)) 1_{\{|U^{n,m,\kappa}| \leq C\}} 1_{\left\{|e| \leq \frac{1}{\kappa}\right\}} ds].
\end{aligned}
$$

The first term in the above inequality tend to zero since $j_t^{\kappa}(\delta U^{n,m,\kappa}) - j_t^{\kappa}(\delta U)$ is uniformly bounded in $\mathbb{L}^1$. For the last one, using the fact that $1_{\left\{|e| \leq \frac{1}{\kappa}\right\}}$ tend to zero as κ goes to infinity, we see that it also tend to zero.

• Finally let us study the term A_2. By Tchebychev, inequality we have

$$
\mathbb{E}\left[1_{\{|Z^{n,m,\kappa}|+|U^{n,m,\kappa}| \geq C\}}\right] \leq \frac{2}{C^2} \mathbb{E}\left[|Z^{n,m,\kappa}|^2 + |U^{n,m,\kappa}|^2\right].
$$

Hence, for $t \leq \tau_l$ from the dominated convergence theorem $f^{n,m,\kappa}(Y^{n,m,\kappa}, Z^{n,m,\kappa}, U^{n,m,\kappa})$ converge to $f(Y, Z, U)$ in $\mathbb{L}^1(d\mathbb{P} \otimes d\nu \otimes dt)$. Therefore, for $t \leq \theta_l$ $dV_t = f(t, Y, Z, U)dt$.

2.4 Uniqueness of solution of the BSDEJ

In this part, we shall prove the uniqueness of the solution of the BSDEJ (2.4). To do so, it is necessary to add further hypothesis on the coefficient f. This is one of the main reasons of the introduction of the BMO notion. In fact this will required to allows as to use the Girsanov theorem. We first provide some new assumption on the coefficient f to deal with the increments of the driver with respect to z.

Assumption 2.4.1. *There exists $C > 0$ such that for all $(t, y, z, z', u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times$*

$\mathbb{L}^0(\mathcal{B}(E),\nu)$ *such that*

$$|f_t(y,z,u)-f_t(y,z^{'},u)|\leq C(1+|z|+|z^{'}|)|z-z^{'}|.$$

Proposition 2.4.2. *Assume that $\xi\in\mathbb{L}^\infty$ and that the driver f satisfies Assumptions 2.2.2 and 2.4.1. Then the BSDE (4.11) associated to (f,ξ) has a unique solution (Y,Z,U).*

Proof. Let (Y^1,Z^1,U^1), (Y^2,Z^2,U^2) two solution of the BSDEJ (1.2). Applying Itô formula to $\Delta Y=(Y^1-Y^2)$, we obtain

$$\Delta Y_t=\Delta Y_T+\int_t^T f_s(Y_s^1,Z_s^1,U_s^1)-f_s(Y_s^2,Z_s^2,U_s^2)ds-\int_t^T\Delta Z_s dW_s-\int_t^T\int_E\Delta U_s(e)\tilde{\mu}(ds,de).$$

Let us introduce the following process $\lambda_t=\lambda(Y_t^1,Y_t^2)$ and $\beta_t=\beta(Z_t^1,Z_t^2)$ as follow

$$\lambda_t(Y_t^1,Y_t^2)=\begin{cases}\frac{f_t(Y_t^1,Z_t^1,U_t^1)-f_t(Y_t^2,Z_t^1,U_t^1)}{Y_t^1-Y_t^2} & \text{if } Y_t^1-Y_t^2\neq 0\\ 0 & \text{else.}\end{cases}$$
$$\beta_t(Z_t^1,Z_t^2)=\begin{cases}\frac{f_t(Y_t^2,Z_t^1,U_t^1)-f_t(Y_t^2,Z_t^2,U_t^1)}{\|Z_t^1-Z_t^2\|^2}(Z_t^1-Z_t^2) & \text{if } Z_t^1-Z_t^2\neq 0\\ 0 & \text{else.}\end{cases}$$

A standard linearization procedure of the increments of the coefficient f leads to

$$\begin{aligned}f_t(y^1,z^1,u^1)-f_t(y^2,z^2,u^2)&=f_t(y^1,z^1,u^1)-f(t,y^2,z^1,u^1)+f_t(y^2,z^1,u^1)-f(t,y^2,z^2,u^1)\\&\quad+f_t(y^2,z^2,u^1)-f(t,y^2,z^2,u^2)\\&\leq\lambda_t(y^1,y^2)\Delta y_t+\beta_t(z^1,z^2)\Delta Z_t.\end{aligned}$$

Applying Itô's formula to $P_t=e^{\lambda_t}\Delta Y_t$ we obtain

$$\begin{aligned}dP_t&=\lambda_t e^{\lambda_t}\Delta Y_{t\_}dt+e^{\lambda_t}d\Delta Y_{t\_}=e^{\lambda_t}\left[\lambda_t\Delta Y_{t\_}dt-\beta_t d\Delta Y_{t\_}\right]\\P_t&\leq P_T+\int_t^T\beta_s(Z_s^1,Z_s^2)\Delta Z_s ds+\int_t^T\int_E e^{\beta_s}\gamma_s(e)\Delta U_s(e)\nu(de,dt)\\&\quad-\int_t^T e^{\beta_s}\Delta Z_s dW_s-\int_t^T\int_E e^{\beta_s}\Delta U_s(e)\tilde{\mu}(ds,de).\end{aligned}$$

we denote by $M_t=M_t^1+M_t^2=\int_0^t e^{\beta_s}\Delta Z_s dW_s+\int_0^t\int_E e^{\beta_s}\Delta U_s(e)\tilde{\mu}(de,ds)$ and $N_t=\int_0^t\beta_s dW_s+\int_0^t\int_E\bar{\gamma}_s(e).\tilde{\mu}(de,ds)$ and we consider the probability measure Q such that $\frac{dQ}{d\mathbb{P}}=\mathcal{E}(N)_T$.
Since $-1+\delta_K\leq\gamma(U_s^1,U_s^2)\leq C_K$ and $|\beta_t(Z_1^s,Z_s^2)|\leq\bar{C}$ since Z^1 and Z^2 are BMO (W).Hence $\mathcal{E}(N)$ is a BMO-martingale. Using Girsanov theorem we obtain that $M-\langle M,N\rangle$ are locale Q-martingale.
Taking the conditional expectation between t and a sequence of stopping time $\tau^n\nearrow\infty$, we obtain

$$P_t\leq\mathbb{E}^Q\left[P_{\tau^n}|\mathcal{F}_t\right].$$

Sending n to infinity, we get $\Delta Y_t = Y_t^1 - Y_t^2 \leq 0$, $\forall t \in [0,T], \mathbb{P}$-a.s. Finally permitting the role of Y^1, Y^2 and using the same procedure we get the desired result.

2.5 Appendix

In order to prove existence of the solution, we need to prove the following intermediate result. As explained before, the principal significance of this lemma (2.3.10) is that it allows to approximate the driver functions by one of Lipschitz type.

Proof.[Proof of lemma (2.3.10)] (i) We can start to notice that, due to the properties of the inf-convolution of the sequence $(f^{\bar{n},\kappa})_n$ and $(\bar{q}^{n,\kappa})_n$ are increasing. Moreover $(\underline{f}^{m,\kappa})_m$ and $(\underline{q}^{m,\kappa})_m$ are decreasing. The monotonicity of the coefficient arises from the regularization function b. The main point for the monotonicity of the coefficients is to notice that $g(v(e))1_{\{|e|\geq \frac{1}{\kappa}\}}$ is smaller then $g(v(e))$.

(ii) To prove that $(\bar{f}^{n,\kappa})_{n,\kappa}$ is uniformly lipschitz in (y,z), we consider the function $\tilde{f}^{n,\kappa}$ such that $\forall \epsilon > 0$, y_1, y_2, z_1, z_2 and $y_\epsilon \in \mathbb{Q}$, $z_\epsilon \in \mathbb{Q}^d$ we have

$$f_t^{n,\kappa}(y^1, z^1, u) \geq \tilde{f}_t^{n,\kappa}(y_\epsilon, z_\epsilon, u) + n|y - y_\epsilon| + |z - z_\epsilon| - \epsilon.$$

where $\tilde{f}_t^{n,\kappa}(y_\epsilon, z_\epsilon, u) := \inf_{v \in \mathbb{L}^0(B(E),\nu)} \left\{ \int_E g_t(v(e))\nu^\kappa(de) + n|u - v|_t + \hat{f}(r,w) \right\}$.

$$\begin{aligned} f_t^{n,\kappa}(y^1, z^1, u) &\geq \tilde{f}(y_\epsilon, z_\epsilon, u) + n|y^2 - y_\epsilon| + n|z^2 - z_\epsilon| + n|y^1 - y_\epsilon| + n|z^1 - z_\epsilon| + \epsilon \\ &\geq \tilde{f}(y_\epsilon, z_\epsilon, u) - n|y^1 - y^2| - n|z^1 - z^2| + n|y^2 - y_\epsilon| + n|z^2 - z_\epsilon| + \epsilon. \end{aligned}$$

Hence,

$$f_t^{n,\kappa}(y^1, z^1, u) \geq f_t^{n,\kappa}(y^2, z^2, u) - n|y^1 - y^2| - n|z^1 - z^2| + \epsilon.$$

Indeed, by arbitrariness of ϵ and by interchanging respectively the roles of (y^1, z^1) and (y^2, z^2) we get the desire result. The same argument remains valid for $(\underline{f}^{m,\kappa})$ and $(\underline{q}^{m,\kappa})$.

Let $u, \bar{u} \in \mathbb{L}^0(\mathcal{B}(E), \nu)$ such that

$$\begin{aligned} f_t^{n,\kappa}(y,z,u) - f_t^{n,\kappa}(y,z,\bar{u}) &:= (\bar{F}_t^n(y,z) - G_t^{n,\kappa}(u)) - (\bar{F}_t^n(y,z) + G_t^{n,\kappa}(\bar{u})) \\ &= \inf_{v \in \mathbb{L}^0(B(E),\nu)} \left\{ \int_E g_t(v(e)) 1_{\{|e| \geq \frac{1}{\kappa}\}} \nu(de) + n|u - v|_t \right\} \\ &- \inf_{v \in \mathbb{L}^0(B(E),\nu)} \left\{ \int_E g_t(v(e)) 1_{\{|e| \geq \frac{1}{\kappa}\}} \nu(de) + n|\bar{u} - v|_t \right\}. \end{aligned}$$

Notice that

$$\inf_{v\in\mathbb{L}^0(B(E),\nu)}(f(v)) - \inf_{v\in\mathbb{L}^0(B(E),\nu)}(g(v)) \leq \sup_{v\in\mathbb{L}^0(B(E),\nu)}(f(v)-g(v)).$$

Thus we obtain

$$\begin{aligned} f_t^{n,\kappa}(y,z,u) - f_t^{n,\kappa}(y,z,\bar{u}) \leq & \sup_{v\in\mathbb{L}^0(B(E),\nu)} \{n|u-v|_t - n|\bar{u}-v|_t\} \\ \leq & \sup_{v\in\mathbb{L}^0(B(E),\nu)} \left\{\int_E n|u-v|^2 - n|\bar{u}-v|^2\nu(de)\right\} \\ \leq & n\int_E(|u|+|\bar{u}|)|u-\bar{u}|\nu(de). \end{aligned} \tag{2.17}$$

Then it follows from the Cauchy-Schwartz inequality that

$$f_t^{n,\kappa}(y,z,u) - f_t^{n,\kappa}(y,z,\bar{u}) \leq n\left\{|u|_t + n|\bar{u}|_t\right\}|u-\bar{u}|_t.$$

Hence the result.
(iii) The Convergence of the sequence is only obtained by the convergence of the second part in the expression of $(f^{n,\kappa})_\kappa$.
Hence

$$\begin{aligned} f_t^{n,\kappa}(y,z,u) - f_t^{n}(y,z,u) = & \inf_{v\in\mathbb{L}^0(B(E),\nu)} \left\{\int_{\left\{|e|\geq\frac{1}{\kappa}\right\}} g_t(v(e))\nu(de) + n|u-v|_t\right\} \\ & - \inf_{v\in\mathbb{L}^0(B(E),\nu)} \left\{\int_E g_t(v(e))\nu(de) + n|u-v|_t\right\} \\ \leq & \sup_{v\in\mathbb{L}^0(B(E),\nu)} \left\{\int_{\left\{|e|\geq\frac{1}{\kappa}\right\}} g_t(v(e))\nu(de) - \int_E g_t(v(e))\nu(de)\right\} \\ \leq & \sup_{v\in\mathbb{L}^0(B(E),\nu)} \left\{\int_E g_t(v(e))1_{[-\frac{1}{\kappa},\frac{1}{\kappa}]}\nu(de)\right\}. \end{aligned}$$

Since $\sup_{v\in\mathbb{L}^0(B(E),\nu)} g_t(v(e))1_{[-\frac{1}{\kappa},\frac{1}{\kappa}]}$ converge to zero, the result follows from the standard dominated convergence Theorem.
(v) Finally

$$\begin{cases} \bar{f}^{n,\kappa} \leq \bar{f}^n \leq q^{\bar{n},\kappa} \leq \bar{q} \\ \underline{f}^{m,\kappa} \leq \bar{f}^m \leq \underline{q}^{m,\kappa} \leq \underline{q} \end{cases} \Rightarrow \underline{q} \leq f^{n,m,\kappa} := \bar{f}^{n,\kappa} - \underline{f}^{m,\kappa} \leq \bar{q}.$$

In order to prove the existence of the solution we need the following lemma. Here we justify the existence of solution of the BSDE associated to $(\bar{q}^{n,\kappa},|\xi|)$ and $(\underline{q}^{m,\kappa},-|\xi|)$.

Lemma 2.5.1. *We consider the following approximation*

$$\bar{q}^{n,\kappa} =:= \bar{q} \wedge b(y,z,u) = \inf_{r,w,v} \{\bar{q}^{\kappa}(r,w,v) + n|y-r| + n|z-w| + n|u-v|_t\}. \tag{2.18}$$

$$\underline{q}^{m,\kappa} := \bar{q} \vee b(y,z,u) = \sup_{r,w,v} \left\{\underline{q}^{\kappa}(r,w,v) + m|y-r| + m|z-w| + m|u-v|_t\right\}. \tag{2.19}$$

where $\bar{q}^{\kappa}_t(r,w,v) = |l|_t + C_t|r| + \frac{1}{2}|w|^2 + \int_E (e^{v(e)} - v(e) - 1)\nu^{\kappa}(de)$ *and* $\underline{q}^{\kappa}_t(r,w,v) = -|l|_t - C_t|r| - \frac{1}{2}|w|^2 - \int_E (e^{-v(e)} + v(e) - 1)\nu^{\kappa}(de)$.
We have

• *The coefficient* $(\bar{q}^{n,\kappa})_{n,\kappa}$ *satisfy the structure condition in Assumption 2.2.2 converges monotonically to* $\bar{q}$ *in* (n,κ). *The same properties holds for* $\underline{q}^{n,\kappa}$.
Moreover, there exists respectively a unique solution $(\bar{\mathcal{Y}}^{n,\kappa}, \bar{\mathcal{Z}}^{n,\kappa}, \bar{\mathcal{U}}^{n,\kappa})$,$(\underline{\mathcal{Y}}^{m,\kappa}, \underline{\mathcal{Z}}^{m,\kappa}, \underline{\mathcal{U}}^{m,\kappa})$ *for the BSDEJ associated respectively to* $(\bar{q}^{n,\kappa}, \xi)$ *and* $(\underline{q}^{m,\kappa}, -\xi)$ *satisfying*

$$|\bar{\mathcal{Y}}^{n,\kappa}_t| \le \bar{\rho}_t \left[e^{C_{t,T}}|\xi| + \int_t^T e^{C_{t,s}} d\Lambda_s\right], \qquad \forall t \in [0,T], \mathbb{P}\text{-a.s.} \tag{2.20}$$

Proof. Due to the property of the inf-convolution, we have $\bar{q}^{n,\kappa} \le \bar{q}^{\kappa}$. However, we know that $\bar{q}^{\kappa}$ depend on κ through the function j. Hence, since for all $\kappa \in \mathbb{N}$ $j^{\kappa}(v) \le j(v)$ we get that $\bar{q}^{n,\kappa} \le \bar{q}$. Therefore , $\bar{q}^{n,\kappa}$ satisfies the structure condition . The monotonicity of the sequence follows from the function b . Hence we can conclude that $(\bar{q}^{n,\kappa})_{n,\kappa}$ converges to $\bar{q}$ and thank's to the Dini's lemma the convergence is uniform.

• Let us prove the existence of solution of the BSDEJ $(\bar{q}^{n,\kappa}, |\xi|)$. First, notice that the coefficient $\bar{q}^{n,\kappa}$ is Lipschitz continuous in (y,z,u) and since we are dealing with truncated measure having a finite activity, by lemma (2.3.10) $\bar{q}^{n,\kappa}$ satisfies A_γ condition . Hence by theorem (4.13) in [14], there exits a unique solution $(\bar{\mathcal{Y}}^{n,\kappa}, \bar{\mathcal{Z}}^{n,\kappa}, \bar{\mathcal{U}}^{n,\kappa})$ to the BSDEJ associated to $(\bar{q}^{n,\kappa}, |\xi|)$.
To conclude we know that $\bar{q}^{n,\kappa}$ satisfy the structure condition (2.5), then the process $\bar{\mathcal{Y}}^{n,\kappa}$ is an exponential quadratic semimartingale. Hence by theorem 13 we get that
$|\bar{\mathcal{Y}}^{n,\kappa}_t| \le \bar{\rho}_t \left[e^{C_{t,T}}|\xi|_T + \int_t^T e^{C_{t,s}} d\Lambda_s\right], \quad \forall t \in [0,T], \ \mathbb{P}$-a.s.

For the reader convenient, let us recall the following theorem and lemma, for more details see [11].

Theorem 2.5.2. *(Barlow-Protter [11])*
If $Y^n = Y_0 + M^n - V^n$ *is a sequence of semimartingales in* $\mathbb{H}^1$ *converging uniformly in* $\mathbb{L}^1$ *to an adapted process* X. *such that, for some positive constant* C,

$$\mathbb{E}\Big[\int_0^T |dV^n_s|\Big] \le C, \quad \textit{and} \quad \mathbb{E}\left[(M^n)^*\right] \le C. \tag{2.21}$$

Then X is a semimartingale in $\mathbb{H}^1$ with canonical decomposition: $X = X_0 + M - V$ satisfying:

$$\mathbb{E}\Big[\int_0^T |dV_s|\Big] \leq C, \quad \textit{and} \quad \mathbb{E}\left[(M)^*\right] \leq C. \tag{2.22}$$

and we have

$$\lim_{n\to\infty} \mathbb{E}\left[(V^n - V)^*\right] = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|M^n - M\|_{\mathbb{H}^1} = 0. \tag{2.23}$$

Lemma 2.5.3. *(Garsia-Neveu) Let A be a predictable càdlàg increasing process and Φ a random variable, positive integrable. For any stopping times $\sigma \leq T$, we have $\mathbb{E}\left[A_T - A_\sigma|\mathcal{F}_\sigma\right] \leq \mathbb{E}\left[U.1_{\sigma<T}|\mathcal{F}_\sigma\right]$.*
Then for all $p \geq 1$,

$$\mathbb{E}[A_T^p] \leq p^p \mathbb{E}[U^p].$$

CHAPTER 3

EXPONENTIAL QUADRATIC DOUBLY REFLECTED BSDE WITH JUMPS

3.1 Overview of the content of this chapter

The motivation behind this chapter is to go beyond the results of Chapter 2 to provide an existence and uniqueness result for backward stochastic differential equation with jumps whose generator has stochastic quadratic growth. The approach that we aim to adopt relies on a suitable class of quadratic BSDEJs with two reflecting barriers. To do so, we concern ourselves to the well-posedness of generalized doubly reflected BSDEJ's with unbounded terminal condition.

The chapter is organized as follows, we first recall briefly some notation and give a precise definition of the solution of generalized doubly reflected BSDEJs and show by an exponential transformation how they are connected to another BSDEJ with more tractable coefficients. Indeed, we prove that this auxiliary BSDEJs admits a unique solution using monotonic approximation techniques. Finally, in the appendix we prove a comparison theorem as well as an existence result for generalized doubly reflected BSDEJs under stronger assumptions which plays a crucial role in our proofs.

3.2 Framework

The set up and notations are the same as the ones introduced in Chapter 2, we will therefore limit ourselves here to introduce the specific spaces corresponding to our framework.

• $\mathcal{L}^{2,d}$ is the spaces of $\mathbb{R}^d$-valued and $\mathcal{P}$-measurable processes such that

$$\|Z\|^2_{\mathcal{L}^2} := \int_0^T |Z_s|^2 ds < +\infty, \ \mathbb{P}\text{-a.s.}$$

• $\mathcal{L}^{2,d}_\nu$ is the space of $\tilde{\mathcal{P}}$-measurable processes such that

$$\|U\|^2_{\mathcal{L}^2_\nu} := \int_0^T \int_{\mathbb{E}} |U_s(e)|^2 \nu(de, ds) < +\infty, \ \mathbb{P}\text{-a.s.}$$

• $\mathcal{K}$ the space of $\mathcal{P}$-measurable continuous non decreasing process such that $K_0 = 0$.
• $\mathcal{D}$ (respectively $\mathcal{D}^c$) the space of $\mathbb{R}$-valued $\mathcal{P}$-measurable càdlàg processes resp. ($\Delta Y_t = 0$).

We also introduce the following classical spaces
• $\mathbb{H}^2$ the set of all $\mathcal{P}$-measurable processes Z such that $\mathbb{E}[\int_0^T |Z_s|^2 ds] < +\infty, \quad \mathbb{P}$-a.s.
• For $u, \bar{u}$ in the space $\mathbb{L}^0(\mathcal{B}(E), \nu)$ of all $\mathcal{B}(E)$-measurable functions with the topology of convergence in measure, we define

$$|u - \bar{u}|_t = (\int_E |u - \bar{u}|^2 \zeta(t, e)\lambda(de))^{\frac{1}{2}}.$$

• $\mathbb{H}^2_\nu$ the set of all $\tilde{\mathcal{P}}$-measurable processes Z such that

$$\mathbb{E}[\int_0^T \int_E |V_s(e)|^2 \nu(de, ds)] < +\infty, \quad \mathbb{P}\text{-a.s.}$$

3.3 Generalized Doubly reflected BSDE with jumps

3.3.1 Formulation

In this section, we aim to prove existence of solution of generalized doubly reflected BSDE with jumps given by $(f.ds + gdA_s + dR_s, \xi, L, U)$ under weaker assumption. Let us first introduce the following definition of *generalized doubly reflected backward stochastic differential equation with jumps* .

We are given the following objects:

- ξ an $\mathcal{F}_T$-measurable real valued random variable.
- A function $f : \Omega \times [0, T] \times \mathbb{R}^{1+d} \times \mathbb{L}^0(B(E), \nu) \to \mathbb{R}$ such that f is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{1+d}) \otimes \mathcal{B}(\mathbb{L}^0(\mathcal{B}(E), \nu))$-measurable.
- A function $g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ such that g is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable .
- Two continuous $\mathbb{R}$-valued processes L_t and U_t such that $L_t \leq U_t$ satisfying $L_t \leq \xi \leq U_t$.
- A positive random measure dR and a non-decreasing continuous process A.

Definition 3.3.1. *We say that a quintuple (Y, Z, V, K^+, K^-) is a solution to the generalized* *doubly reflected BSDE with jumps associated to* $(f.ds + gdA_s + dR_s, \xi, L, U)$*, if*

$$
(\mathcal{E}) \begin{cases} Y \in \mathcal{D}, \quad Z \in \mathcal{L}^{2,d}, \quad V \in \mathcal{L}^{2,d}_{\nu}, \quad K^{\pm} \in \mathcal{K}. \\ Y_t = \xi + \int_t^T f_s(Y_s, Z_s, V_s)ds + \int_t^T dR_s + \int_t^T g_s(Y_s)dA_s - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e)\tilde{\mu}(ds, de) \\ \qquad + \int_t^T dK_s^+ - \int_t^T dK_s^-, \qquad 0 \le t \le T, \ \mathbb{P}\text{-a.s.} \\ L_t \le Y_t \le U_t \ \text{ and } \int_0^T (Y_{s^-} - U_s)dK_s^+ = \int_0^T (Y_{s^-} - L_s)dK_s^- = 0, \quad 0 \le t \le T, \ \mathbb{P}\text{-a.s.} \end{cases}
$$

The last condition is called the Skorohod condition. It requires that the processes K^+ and K^- are minimal in the sense that they only act when Y reaches the obstacles L and U. This condition is crucial to obtain the wellposedness of generalized doubly reflected BSDEs with jumps. Note that when there is no barrier, the system becomes an ordinary BSDE with jumps.

Definition 3.3.2. *We say that* dM_t^1 *and* dM_t^2 *are two singular measure and we denote* $dM_s^1 \perp dM_s^2$ *if*

$$
\int_0^T 1_{\{A_s(w)\}} dM_t^1 = \int_0^T 1_{\{A_s^c(w)\}} dM_s^2 = 0, \qquad \forall A \in \mathcal{P}. \tag{3.1}
$$

We shall make the following standing assumptions on the maps under consideration.

Assumption 3.3.3 (Assumptions on the drivers)**.**

- *The first assumption characterize the growth of the driver* f *with a lower and an upper bound: For every* $(y, z, v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(B(E), \nu)$*, there exist two positives processes* η *and* C *respectively in* $\mathbb{L}^1(\Omega, [0, T])$ *and* $\mathcal{D}^c$ *such that*

$$
\begin{aligned} \underline{q}_s(y, z, v) &= -\eta_s(w) - \frac{C_s(w)}{2}|z|^2 - \frac{1}{\delta}J(v) \\ &\le f_s(w, y, z, v) \le \bar{q}_s(y, z, v) = \eta_s(w) + \frac{C_s(w)}{2}|z|^2 + \frac{1}{\delta}J(v), \end{aligned}
$$

  $dt \otimes d\mathbb{P}$*-a.s,* $(w, t) \in \Omega \times [0, T]$*, where*

$$
J(v) = \int_E \left(e^{\delta v(e)} - \delta v(e) - 1\right) \nu(de).
$$

- *The second assumption consists in specifying a lower and upper bound for* g*: For all* $y \in [L_s(w), U_s(w)]$*,*

$$
|g_s(w, y)| \le 1 \qquad A(ds) \otimes \mathbb{P}(dw)\text{-a.s.}
$$

- *The last assumption known as the "*A_γ*-condition " deals with the increments of the driver* f *with respect to the jump component: For all* $(y, z, u, \bar{u}) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(B(E), \nu)$ *there* *exists a* $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^{d+2})\otimes\mathcal{B}(E)$*-measurable function* γ *where* $C_1 \leq \gamma \leq C_2$ *with* $-1 < C_1 \leq 0,\ C_2 \geq 0,$

$$f_t(y,z,v) - f_t(y,z,\bar{v}) \leq \int_E \gamma_t(e)[v(e)-\bar{v}(e)]\nu(de), \ \ \forall t\in[0,T], \mathbb{P}\text{-a.s}$$

Remark 3.3.4. *We emphasize that, usually the above structure condition is uniform that is the constants in front of* z *and* u *are constants. However in our context, we look for solution of generalized doubly reflected BSDEs where the driver has stochastic growth i.e.* η *and* C *are no longer constants but predictable processes.*

To conclude this part we introduce the following requirement on the obstacle processes.

Assumption 3.3.5 (Assumptions on the Obstacle)**.**
(i) There exists a semimartingale S with the following decomposition

$S = S_0 + V^+_{\cdot} - V^-_{\cdot} + \int_0^{\cdot} \alpha_s dB_s$, *where* $S_0 \in \mathbb{R}$, $V^+_{\cdot}, V^-_{\cdot} \in \mathcal{K}$ *and* $\alpha_s \in \mathcal{L}^{2,d}$*,* $\mathbb{P}$*-a.s.*

(ii) $L_t \leq S_t \leq U_t, \quad L_t \leq 0 \leq U_t, \ \ t\in[0,T], \mathbb{P}$*-a.s.*

(iii) For all $R \in \mathcal{K}, \quad dR_t \geq 0, \ \ \mathbb{P}$*-a.s.*

3.3.2 Exponential transformation and estimates

To achieve our main result, that is the existence of solutions of generalized doubly reflected BSDEJs, we first introduce an auxiliary BSDEJ which is explicitly given in terms of exponential transformation of the original one. We then establish a correspondence between solutions of the auxilary one and those of BSDEJs given by $(fds + gdA_s + dR_s, \xi, L, U)$. To do so, we consider the following $\mathcal{F}_t$-adapted continuous increasing process

$$m_t = 2\sup_{0\leq s\leq t}|C_s| + \sup_{0\leq s\leq t}|U_s| + |R_t| + A_t + \frac{1}{\delta} + 1. \tag{3.2}$$

Then we have the following result.

Proposition 3.3.6. *There exists a solution* $(Y,Z,V,K^+,K^-) \in \mathcal{D}\times\mathcal{L}^{2,d}\times\mathcal{L}^{2,d}_{\nu}\times\mathcal{K}^2$ *to the system* $(\mathcal{E})$ *if and only if* $(\bar{Y},\bar{Z},\bar{V},\bar{K}^+,\bar{K}^-)$ *is solution of* $(\mathcal{E})$ *with data* $(\bar{f},\bar{\xi},\bar{L},\bar{U})$
where

$$\bar{Y}_t = e^{m_t(Y_t-m_t)}, \quad \bar{Z}_t = m_t\bar{Y}_tZ_t, \quad \bar{V}_t = \bar{Y}_t[e^{m_tV_t(e)}-1], \quad \bar{K}^{\pm}_s = m_s\bar{Y}_sK^{\pm}_s.$$
$$\bar{f} = \tilde{f}_s((\bar{y}\wedge\bar{L}_s)\vee\bar{U}_s,\bar{z},\bar{v}) - \eta_s m_s \quad \text{and} \quad \bar{g} = \frac{1}{8m_s}\tilde{g}_s((\bar{y}\wedge\bar{L}_s)\vee\bar{U}_s) - \frac{1}{2}$$
$$\bar{U}_t = e^{m_t(U_t-m_t)}, \quad \bar{L}_t = e^{m_t(L_t-m_t)}, d\bar{R}_s = \frac{1}{2}d\bar{A}_s + \eta_s m_s ds, \ d\bar{A}_s = 8m_s dm_s. \tag{3.3}$$

with

$$\begin{cases} \tilde{g}_s(\bar{y}) = \bar{y}[m_s g_s(\frac{\ln(\bar{y})}{m_s} + m_s)\frac{dA_s}{dm_s} + m_s\frac{dR_s}{dm_s} + m_s - \frac{ln(\bar{y})}{m_s}], \\ \tilde{f}_s(\bar{y},\bar{z},\bar{v}) = m_s\bar{y}[f_s(\frac{\ln(\bar{y})}{m_s} + m_s, \frac{\bar{z}}{m_s\bar{y}}, \frac{1}{m_s}ln(\frac{\bar{v}}{\bar{y}}+1)). \\ \qquad +m_s\bar{y} - \frac{|\bar{z}|^2}{2m_s\bar{y}^2} - \frac{1}{m_s}\int_E(e^{ln(1+\frac{\bar{v}}{\bar{y}})} - ln(1+\frac{\bar{v}}{\bar{y}}) - 1)\nu(de)]. \end{cases}$$

Proof. Applying Itô's formula to $\bar{Y}_t = e^{m_t(Y_t - m_t)}$, we obtain for all $t \in [0,T]$, $\mathbb{P}$-a.s

$$\begin{aligned} \bar{Y}_t &= \bar{Y}_T + \int_t^T \bar{Y}_{s^-}[m_s f_s(Y_s,Z_s,V_s) - \frac{1}{2}|m_s Z_s|^2 - \int_E[e^{m_s V_s(e)} - m_s V_s(e) - 1]\nu(de)]ds \\ &+ \int_t^T m_s\bar{Y}_{s^-}dK_s^+ - \int_t^T m_s\bar{Y}_{s^-}dK_s^- - \int_t^T m_s\bar{Y}_{s^-}Z_s dB_s - \int_t^T\int_E \bar{Y}_{s^-}[e^{m_sV_s(e)} - 1]\tilde{\mu}(de,ds) \\ &+ \int_t^T \bar{Y}_{s^-}[2m_s - Y_s]dm_s + \int_t^T [\bar{Y}_{s^-}m_s g_s(Y_s)\frac{dA_s}{dm_s}]dm_s + \int_t^T [\bar{Y}_{s^-}m_s\frac{dR_s}{dm_s}]dm_s. \end{aligned}$$

Thus, taking $\bar{f}$, $\bar{g}$, $\bar{A}$ and $d\bar{R}$ as in (3.3) yield to

$$\begin{aligned} \bar{Y}_t &= \bar{Y}_T + \int_t^T \bar{f}_s(\bar{Y}_s,\bar{Z}_s,\bar{V}_s)ds + \int_t^T \bar{g}_s(\bar{Y}_s)d\bar{A}_s + \int_t^T d\bar{R}_s - \int_t^T \bar{Z}_s dB_s - \int_t^T\int_E \bar{V}_s(e)\tilde{\mu}(ds,de) \\ &+ \int_t^T d\bar{K}_s^+ - \int_t^T d\bar{K}_s^-. \end{aligned}$$

We then deduce that if (Y,Z,V,K^+,K^-) is a solution of $(\mathcal{E})$ then $(\bar{Y},\bar{Z},\bar{V},\bar{K}^+,\bar{K}^-)$ is a solution of the generalized doubly reflected BSDEJs associated to $(\bar{f}ds + \bar{g}d\bar{A}_s + d\bar{R}_s, \bar{\xi}, \bar{L}, \bar{U})$.
Conversely, let $(\bar{Y},\bar{Z},\bar{V},\bar{K}^+,\bar{K}^-)$ be the solution of $(\mathcal{E})$ with data $(\bar{f},\bar{\xi},\bar{L},\bar{U})$. Applying Itô's formula to $Y_t = \frac{ln(\bar{Y}_t)}{m_t} + m_t$, we obtain that (Y,Z,V,K^+,K^-) is a solution of $(\mathcal{E})$.
The following lemma summarizes the properties satisfied by the transformed data.

Lemma 3.3.7. *The data $(\bar{f}ds + d\bar{R}_s + \bar{g}d\bar{A}_s, \bar{\xi}, \bar{L}, \bar{U})$ obtained by the above exponential transformation satisfy the following properties*

(i) $0 \le \bar{L}_t \le e^{-m_t^2} \le \bar{U}_t \le e^{-1} < 1$ *and* $\bar{L}_T \le \bar{\xi} \le \bar{U}_T, \ \forall t \in [0,T], \ \mathbb{P}$*-a.s.*

(ii) $\bar{f}$ *is a* $\mathcal{P}$*-measurable function such that for all* $(s,y,z,v) \in [0,T] \times [\bar{L}_s,\bar{U}_s] \times \mathbb{R}^{1+d} \times \mathbb{L}^0(B(E),\nu)$:

$$-2m_s\eta_s - \frac{|z|^2}{L_s} - \frac{1}{\delta}J(ln(\frac{\bar{v}}{L_s}+1)) \le \bar{f}_s(w,y,z,v) \le 0, \quad \mathbb{P}\text{-a.s.}$$

(iii) $-1 \le g_s(w,y) \le 0, \quad \forall y \in [L_s,U_s].$

(iv) $d\bar{R}$ *is a random positive measure .*

Proof. (i) Since $U_t \leq m_t - 1$, it follows directly that $0 \leq \bar{L}_t \leq e^{-m_t^2} \leq \bar{U}_t \leq e^{-1} < 1$.
Let us now prove that $\bar{f}$ satisfies the property(ii). Recalling the expression of $\tilde{f}$ and using Assumption (3.3.3), we aim to determine an upper bound to $\tilde{f}$

$$\begin{aligned}
&\tilde{f}_s(w,\bar{y},\bar{z},\bar{v})\\
&\leq m_s\bar{y}[\eta_s + \frac{C_s}{2}\frac{|\bar{z}|^2}{m_s^2\bar{y}^2} + \frac{1}{\delta}J(\frac{1}{m_s}ln(1+\frac{\bar{v}}{\bar{y}})) - \frac{|\bar{z}|^2}{2m_s\bar{y}^2} - \frac{1}{m_s}\int_E (e^{ln(1+\frac{\bar{v}}{\bar{y}})} - ln(1+\frac{\bar{v}}{\bar{y}}) - 1)\nu(de)]\\
&\leq \bar{y}m_s.\eta_s + \left(\frac{C_s}{2m_s} - \frac{1}{2}\right)\frac{|z|^2}{\bar{y}} + \frac{m_s}{\delta}\bar{y}J(\frac{1}{m_s}ln(1+\frac{\bar{v}}{\bar{y}})) - \int_E (e^{ln(1+\frac{\bar{v}}{\bar{y}})} - ln(1+\frac{\bar{v}}{\bar{y}}) - 1)\nu(de).
\end{aligned}$$

Using the following inequality $J(ku) \geq kJ(u)$, $\forall k \geq 1$ and $\bar{y} \leq \bar{U} \leq e^{-1}$, we obtain

$$\tilde{f}_s(w,\bar{y},\bar{z},\bar{v}) \leq e^{-1}m_s\eta_s + \left(\frac{C_s}{2m_s} - \frac{1}{2}\right)\frac{|z|^2}{\bar{y}}.$$

Since $C_s \leq m_s$ yield to $\frac{C_s}{2m_s} - \frac{1}{2} \leq 0$, we see that $\tilde{f}_s(w,\bar{y},\bar{z},\bar{v}) \leq \eta_s m_s$.

Now we aim to find a lower bound of $\tilde{f}$. In fact

$$\begin{aligned}
\tilde{f}_s(w,\bar{y},\bar{z},\bar{v}) &\geq m_s\bar{y}\Big[\Big(-\eta_s - \frac{C_s}{2}\frac{|\bar{z}|^2}{m_s^2\bar{y}^2} - \frac{1}{\delta}J_s(\frac{1}{m_s}ln(1+\frac{\bar{v}}{\bar{y}}))\Big)\\
&\quad - \frac{|\bar{z}|^2}{2m_s\bar{y}^2} - \frac{1}{m_s}\int_E (e^{ln(1+\frac{\bar{v}}{\bar{y}})} - ln(1+\frac{\bar{v}}{\bar{y}}) - 1)\nu(de)\Big]\\
&\geq -e^{-1}m_s\eta_s - \left(\frac{C_s}{2m_s} + \frac{1}{2}\right)\frac{|z|^2}{\bar{y}} - \int_E (e^{ln(1+\frac{\bar{v}}{\bar{y}})} - ln(1+\frac{\bar{v}}{\bar{y}}) - 1)\nu(de).
\end{aligned}$$

Using once again $J(ku) \geq kJ(u)$ together with $\bar{L}_s \leq \bar{y} \leq \bar{U}_s$ and $\frac{C_s}{2m_s} + \frac{1}{2} \leq 1$, we finally get

$$\begin{aligned}
\tilde{f}_s(w,\bar{y},\bar{z},\bar{v}) &\geq -e^{-1}m_s\eta_s - \left(\frac{C_s}{2m_s} + \frac{1}{2}\right)\frac{|z|^2}{\bar{y}} - \frac{1}{\delta}J_s(ln(1+\frac{\bar{v}}{\bar{y}}))\\
&\geq m_s\eta_s - \frac{|z|^2}{\bar{L}_s} - \frac{1}{\delta}J_s(ln(1+\frac{\bar{v}}{\bar{L}_s})).
\end{aligned}$$

Similar arguments can be used to prove that g satisfy $-1 \leq g_s(w,y) \leq 0$.
The property (iv) follows from Assumption 3.3.5 and the definition 3.2.

3.3.3 Auxiliary generalized doubly reflected BSDE with jumps: Existence and uniqueness results

Our problem is then reduced to find a maximal solution for generalized doubly reflected BSDEJs under the following weaker assumptions.

Assumption 3.3.8.

1. *There exist two positives processes* $\eta \in \mathbb{L}^1(\Omega, [0,T]), C \in \mathcal{D}^c$ *such that*

$$\underline{q} := -\eta_t(w) - \frac{C_t(w)}{2}|z|^2 - \frac{1}{\delta}j(v) \leq f_t(w,y,z,v) \leq 0, \quad \forall t \in [0,T].$$

2. *For all* $y \in \mathbb{R}, \qquad -1 \leq g_t(w,y) \leq 0, \quad 0 \leq t \leq T.$

3. *There exists a continuous non decreasing process* $S = S_0 - V_\cdot$, *where* $S_0 \in \mathbb{R}, V \in \mathcal{K}$ *such that* $L_t \leq S_t \leq U_t, \quad \mathbb{P}$*-a.s.*

4. $\forall t \in [0,T], \forall R \in \mathcal{K}, \quad dR_t \geq 0$ *and* $0 \leq L_t \leq U_t < 1, \quad \mathbb{P}$*-a.s.*

Theorem 3.3.9. *Assume that Assumption 3.3.8 and* A_γ*-condition are fulfilled then the generalized DRBSDE with jumps* $(\mathcal{E})$ *associated to* $(fds + dR_s + gdA_s, \xi, L, U)$ *has a maximal solution.*

Classically, when we want to manage a quadratic BSDE, it seems natural to start by an exponential change of variable to obtain a Lipschitz BSDE. However in general, this method may fail as it can be seen in lemma (3.3.7). A possible way to do so is to approximate the BSDE by mean of sup-convolution. This technique was introduced by Lepeltier and San Martin [90] in the backward theory.
To be a little bit more precise, the scheme of our proof is the following.
• The first step consists on introducing an auxiliary generator $(f^n)_n$ globally Lipschitz with respect to (y,z,u) as follows

$$f^n_t(y,z,u) := \sup_{(p,q,r)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{L}^0(\mathcal{B}(E),\nu)} \{f_t(p,q,r) + n|y-p| + n|z-q| + n|v-r|_t\}$$

Moreover, since the integrability conditions on the data are weaker, we will introduce a family of stopping times $(\tau_i)_{i\geq 0}$. Hence, using the existence results of Appendix, we justify the existence of a unique processes $(Y^{n,i}, Z^{n,i}, U^{n,i}, K^{-,n,i}, K^{+,n,i})$ solution for the truncated generalized doubly reflected BSDEJs.
In the last step, we prove a stability result for the approximating sequence of this type of BSDEJs and hence we deduce from it that the limit exists and solves the original one.
Before proceeding in the proof, we will need the following lemma which provides essential properties of the truncated drivers. Define

$$f^n(t,y,z,v) = \sup_{(p,q,r)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{L}^0(\mathcal{B}(E),\nu)} \{f(t,p,q,r) - n|p-y| - n|q-z| + n|r-v|_t\}.$$

$$\underline{q}^n(t,y,z,v) = \sup_{(p,q,r)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{L}^0(\mathcal{B}(E),\nu)} \left\{\underline{q}(t,p,q,r) - n|p-y| - n|q-z| + n|r-v|_t\right\}.$$

$$g^n(t,y) = \sup_{p\in\mathbb{R}} \{g(t,p) - n|p-y|\}.$$

Lemma 3.3.10. *[104] Under Assumption 3.3.8 and A_γ-condition, we have*

- *The sequences $(f^n)_n$, $(g^n)_n$ are Lipschitz with respect to (y, z, u).*
- *For all $(t, \omega, y, z, u) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^0(B(E), \nu)$, $\forall n \in \mathbb{N}$*

$$\underline{q}_t(y, z, u) \leq \underline{q}_t^n(y, z, u) \leq f_t^n(y, z, u) \leq 0.$$

- *For all $(t, \omega, y) \in [0, T] \times \Omega \times \mathbb{R}$, $\forall n \in \mathbb{N}$, $-1 \leq g_t^n(w, y) \leq 0$.*
- *The sequences $(f^n)_n$ and $(g^n)_n$ are increasing and converges uniformly in every compact set respectively to f and g $\mathbb{P}$-a.s.*

Proof.[proof of Theorem 3.3.9] The proof falls naturally into four steps
First step: Construction of the sequence of generalized doubly reflected BSDE with jumps:

Let $j, i, p \in \mathbb{N}$ such that $j \leq i \leq p$ and $t \in [0, \tau_j]$ where τ_j is a stationary family of stopping times defined as follow

$$\tau_j = \inf\left\{t \geq 0; A_t + R_t + C_t + \int_0^t \eta_s ds \geq j\right\} \wedge T.$$

Let us now introduce the doubly reflected BSDEJ associated to the truncated driver$(f^n)_n$

$$(\mathcal{E}^1)\begin{cases} (i)\ Y_t^{n,i} = \xi + \int_t^T f_s^n(Y_s^{n,i}, Z_s^{n,i}, V_s^{n,i})ds + \int_t^T dR_s^i + \int_t^T g_s^n(Y_s^{n,i})dA_s^n + \int_t^T dK_s^{n,i,+} \\ \quad - \int_0^T dK_s^{n,i,-} - \int_t^T Z_s^{n,i}dB_s - \int_t^T \int_E V_s^{n,i}(e)\tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s}, \\ (ii)\ L_t^{n,i} \leq Y_t^{n,i} \leq U_t^{n,i}, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s}, \\ (iii)\ \int_0^T (Y_{s^-}^{n,i} - L_s^{n,i})dK_s^{n,i,+} = \int_0^T (U_s^{n,i} - Y_{s^-}^{n,i})dK_s^{n,i,-} = 0, \ \mathbb{P}\text{-a.s.} \\ (iv)\ dK_s^{n,i,+} \perp dK_s^{n,i,-}. \end{cases} .$$

where $dR^i = 1_{\{s \leq \tau_i\}} dR_s$ and $dA^n = 1_{\{s \leq \tau_n\}} dA_s$.
First, we have to justify the existence of the solution for the system $(\mathcal{E}^1)$. Using both Lemma (3.3.10) and the associate A_γ-condition, it follows from Theorem 3.4.2, the existence of a unique solution $(Y^{n,i}, Z^{n,i}, V^{n,i}, K^{n,i,-}, K^{n,i,+})$. Moreover, we have the following estimate: $\forall n, i \in \mathbb{N}$

$$\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t^{n,i}|^2 + \int_0^T |Z_t^{n,i}|^2 dt + \int_0^T \int_E |V_t^{n,i}(e)|^2 \nu(de, dt) + (K_T^{n,i,+})^2 + (K_T^{n,i,-})^2] < +\infty. \quad (3.4)$$

In the other hand, since $(\tau_i)_{i \geq 0}$ is an increasing family of stopping times, we deduce from the comparison Theorem 3.4.1 and (3.4), that the solution satisfy the following properties

$$\text{For all } i,n\in\mathbb{N},\ dR^i\le dR^{i+1},\ L_t\le Y_t^{n+1,i}\le Y_t^{n,i}\le Y_t^{n,i+1}\le U_t,\qquad \forall t\in[0,T],\ \mathbb{P}\text{-a.s} \quad (3.5)$$

and

$$dK_t^{n,i+1,+}\le dK_t^{n,i,+}\le dK_t^{n+1,i,+},\ dK_t^{n+1,i,-}\le dK_t^{n,i,-}\le dK_t^{n,i+1,-},\forall t\in[0,T],\ \mathbb{P}\text{-a.s.} \quad (3.6)$$

For a fixed n, since $(Y^{n,i})_i$ is increasing we can define Y^n as follows

$$Y_t^n=\lim_{i\to+\infty}\nearrow Y_t^{n,i},\qquad \forall t\in[0,T],\quad \mathbb{P}\text{-a.s.}$$

From (3.4), the sequences $(Z^{n,i})$ and $(V^{n,i})$ are bounded which entails the weak convergence. We denote respectively (Z^n) and (V^n) their weak limits .
In the next step we prove a stability result for the approximating sequence of generalized doubly reflected BSDEJs. We define an order for the convergence: first we will send i to infinity and then in the third step we let n goes to infinity.

Step2: The convergence of the approximating generalized doubly reflected BSDEJ.

In this part we shall freeze $n\in\mathbb{N}$ and let i goes to ∞. For simplicity, we shall make the following notations $\delta Y=Y^{n,i}-Y^{n,p}$, $\delta Z=Z^{n,i}-Z^{n,p}$ and $\delta V=V^{n,i}-V^{n,p}$.
Consider $e_t^n:=e^{2nA_t^n+(2n^2+2n)t}$, we obtain by applying Itô's formula to $e_s^n(Y_s^{n,i}-Y_s^{n,p})^2$ between t and τ_j, the following

$$\begin{aligned} e_t^n(\delta Y)_t^2 &= e_{\tau_j}^n(\delta Y)_{\tau_j}^2+\int_t^{\tau_j}2e_s^n\delta Y_s\left[f_s^n(Y_s^{n,i},Z_s^{n,i},V_s^{n,i})-f_s^n(Y_s^{n,p},Z_s^{n,p},V_s^{n,p})\right]ds-\int_t^{\tau_j}dR_s^p\\ &+2\int_t^{\tau_j}e_s^n\delta Y_s d\delta K_s^+-2\int_t^{\tau_j}e_s^n\delta Y_s d\delta K_s^--2\int_t^{\tau_j}e_s^n\delta Y_s\left[g_s^n(Y_s^{n,i})-g_s^n(Y_s^{n,p})\right]dA_s^n\\ &-2\int_t^{\tau_j}e_s^n\delta Y_s\delta Z_s dB_s-\int_t^{\tau_j}e_s^n|\delta Z_s|^2ds-2\int_t^{\tau_j}\int_E e_s^n\delta Y_s\delta V_s(e)\tilde{\mu}(de,ds)+\int_t^{\tau_j}dR_s^i\\ &-\int_t^{\tau_j}\int_E e_s^n|\delta V_s(e)|^2\nu(de,ds)-\int_t^{\tau_j}e_s^n\delta Y_s^2\left(2nd A_s^n+(2n+n^2)ds\right). \end{aligned} \quad (3.7)$$

Before going any further, we need to estimate the following difference

$$\int_t^{\tau_j}2e_s^n\delta Y_s\left[f_s^n(Y_s^{n,i},Z_s^{n,i},V_s^{n,i})-f_s^n(Y_s^{n,p},Z_s^{n,p},V_s^{n,p})\right]ds.$$

We first rely on the classical inequality: $\forall\epsilon>0,\ a.b\le\epsilon a^2+\frac{1}{\epsilon}b^2$ and the fact that both $(f^n)_n$ and

$(g^n)_n$ are uniformly Lipschitz in (y, z, v). In this way we obtain what follows

$$\begin{aligned}&\int_t^{\tau_j} 2e_s^n \delta Y_s \left[f_s^n(Y_s^{n,i}, Z_s^{n,i}, V_s^{n,i}) - f_s^n(Y_s^{n,p}, Z_s^{n,p}, V_s^{n,p}) \right] ds \\ &\leq \int_t^{\tau_j} 2.n.e_s^n \delta Y_s \left[|\delta Y_s| + |\delta Z_s| + |\delta V_s|_s \right] ds. \\ &\leq \epsilon \int_t^{\tau_j} n e_s^n |\delta Y_s|^2 ds + \frac{1}{\epsilon} \int_t^{\tau_j} n e_s^n \left[|\delta Y_s|^2 + |\delta Z_s|^2 + |\delta V_s|_s^2 \right] ds. \end{aligned} \tag{3.8}$$

Besides, we have

$$\int_t^{\tau_j} 2e_s^n \delta Y_s \left(g_s^n(Y_s^{n,i}) - g_s^n(Y_s^{n,p}) \right) dA_s^n \leq \int_t^{\tau_j} 2.n.e_s^n |\delta Y_s|^2 dA_s^n. \tag{3.9}$$

On the other hand, it follow from (3.6)

$$\int_t^{\tau_j} e_s^n \delta Y_s d\delta K_s^+ \leq 0 \qquad \text{and} \qquad \int_t^{\tau_j} e_s^n \delta Y_s d\delta K_s^- \geq 0. \tag{3.10}$$

Besides, Notice that if we use the standard localization procedure we can prove that the local martingale $\int_t^{\tau_j} e_s^n \delta Y_s \delta Z_s dB_s - \int_t^{\tau_j} \int_E e_s^n \delta Y_s \delta V_s(e) \tilde{\mu}(de, ds)$ is in fact a true $(\mathcal{F}, \mathbb{P})$-martingale. Now notice that since the family of stopping time τ_j is increasing then,

$$\begin{aligned}\int_t^{\tau_j} dR_s^i - \int_t^{\tau_i} dR_s^p &= \int_t^{\tau_j \wedge \tau_i} dR_s - \int_t^{\tau_j \wedge \tau_p} dR_s \\ &= \int_t^{\tau_j} dR_s - \int_t^{\tau_j} dR_s = 0 \end{aligned} \tag{3.11}$$

Combining (3.8), (3.9), (3.10) and (3.11) and putting all terms containing δZ and δV in the left-hand side, we can rewrite (3.7) as follow

$$\mathbb{E}[e_t^n |\delta Y_t|^2 + \int_t^{\tau_j} e_s^n [|\delta Z_s|^2 + |\delta V_s|_s^2] ds] \leq \mathbb{E}[e_{\tau_j}^n |\delta Y_{\tau_j}|^2].$$

Now in order to justify the passage to the limit in the right hand side as i goes to $+\infty$, we apply Lebesgue's Dominated convergence theorem for a fixed n, since we know that the process $Y^{n,i}$ is bounded $\mathbb{E}[|\delta Y_{\tau_j}|^2]$ goes to 0 as i goes to infinity.
Hence there exits $Z^n \in \mathcal{L}^{2,d}$ and $V^n \in \mathcal{L}_\nu^{2,d}$ such that for any $n \in \mathbb{N}$ we have

$$\lim_{i \to +\infty} \mathbb{E}[\int_0^{\tau_n} (|Z_s^{n,i} - Z_s^n|^2 + \int_E |V_s^{n,i}(e) - V_s^n(e)|^2 \nu(de)) ds] = 0.$$

Now apply Itô formula to $|Y_t^{n,i} - Y_t^{n,p}|^2$ and taking first the supremum over $t \in [0, \tau_j]$ then the conditional expectation we get using the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\big[\sup_{t\leq\tau_j}|Y_t^{n,i}-Y_t^{n,p}|^2\big]\leq\mathbb{E}[|Y_{\tau_j}^{n,i}-Y_{\tau_j}^{n,p}|^2]+2c\mathbb{E}\big[\int_0^{\tau_j}|Y_s^{n,i}-Y_s^{n,p}|^2|Z_s^{n,i}-Z_s^{n,p}|^2ds\big]^{\frac{1}{2}}$$
$$+2c\mathbb{E}\big[\int_0^{\tau_j}\int_E|Y_s^{n,i}-Y_s^{n,p}|^2|V_s^{n,i}(e)-Z_s^{n,p}(e)|^2\nu(de,ds)\big]^{\frac{1}{2}}.$$

Letting n goes to infinity in the above inequality, we can deduce from the monotone convergence theorem and dominated convergence

$$\lim_{i\to+\infty}\mathbb{E}\big[\sup_{t\leq\tau_j}|Y_t^{n,i}-Y_t^{n,p}|^2\big]=0.$$

To conclude that the process Y^n is càdlàg and hence belongs to $\mathcal{D}$, the idea is to define $Y^{\tilde{p}}$ the projection of Y as the unique predictable process such that $Y_{\tilde{\tau}}=\mathbb{E}_{\tilde{\tau}-}[Y_\tau]$ on $\{\tilde{\tau}<\infty\}$ for all predictable time $\tilde{\tau}$ and then $(Y^{n,i})^{\tilde{p}}=Y_-^{n,i}$.
Putting $Y^n=\lim\limits_{i\to\infty}Y^{n,i}$ together with the fact that $Y^{n,i}$ is càdlàg from [11] we deduce by the weak convergence of $Z^{n,i}$ and $V^{n,i}$ that $Y_-^{n,i}=(Y^{n,i})^{\tilde{p}}\uparrow(Y^n)^{\tilde{p}}$ as $i\to\infty$.
Similar to the arguments used in [48] we can prove that the processes K^+ and K^- belongs to $\mathcal{K}$. From (3.6), we deduce that $K^{n,i,+}$ converges weakly to the continuous increasing process $K^{n,+}$. Furthermore, using Fatou lemma :for fixed $n\in\mathbb{N}$ and for all $i\in\mathbb{N}$

$$\mathbb{E}\left[(K^{n,+})^2\right]\leq\mathbb{E}\left[(K^{n,i,+})^2\right]\leq\mathbb{E}\left[(K^{n,0,+})^2\right]<+\infty.$$

Hence, we get that $\mathbb{E}\left[(K^{n,+})^2\right]<\infty$ which proves that $K^{n,+}$ belongs to $\mathcal{K}$.
Now since τ_j is a stationary family of stopping times , it follows from the system $(\mathcal{E}^1)$ with monotone convergence theorem that for a fixed $n\in\mathbb{N}$, $\quad K_T^{n,-}<+\infty,\quad\mathbb{P}$-a.s.

Finally, letting i goes to infinity for fixed $n\in\mathbb{N}$ in the system $(\mathcal{E}^1)$ we get that the quintuple $(Y^n,Z^n,V^n,K^{n,+},K^{n,-})$ solves the following system

$$Y_t^n=\xi+\int_t^T f_s^n(Y_s^n,Z_s^n,U_s^n)ds+dR_s^n+g_s^n(Y_s^n)dA_s^n-\int_t^T Z_s^n dB_s \tag{3.12}$$
$$+\int_t^T dK_s^{n,+}-\int_t^T dK_s^{n,-}-\int_t^T Z_s^n dB_s-\int_t^T\int_E V_s^n(e)\tilde{\mu}(ds,de),$$
$$\text{with}\quad L_t^n\leq Y_t^n\leq U_t^n,\quad 0\leq t\leq T,\mathbb{P}\text{-a.s.} \tag{3.13}$$

Note that Y^n satisfies the above system for all $t\in[0,\tau_j]$. However , since the family of stopping time τ_j satisfies $\mathbb{P}\left[\cup_{i=1}^\infty(\tau_j=T)\right]=1$, we have immediately that Y^n satisfies the system (3.12) for all t in $[0,T]$.
To complete this step, it remains to prove the Skorohod condition of Y^n. Since $0\leq U_t-Y_t^n\leq$

$U_t - Y_t^{n,i}$, we clearly have

$$\int_0^T (U_t - Y_{t^-}^n) dK_t^{n,i,-} = 0, \quad \mathbb{P}\text{-a.s.}$$

Therefore, since the process $K^{n,i,-}$ converges to the continuous increasing process K^n, we obtain the weak convergence of the measure $dK^{n,i}$. Hence it follows that

$$\int_0^T (U_t - Y_{t^-}^n) dK_t^{n,-} = 0, \quad \mathbb{P}\text{-a.s.}$$

The proof of $\int_0^T (Y_{t^-}^n - L_t) dK_t^{n,+} = 0$ is in the same spirit, we only have to notice that we have the weak convergence of the measure $dK^{n,i}$ to obtain that

$$0 \leq \int_0^T (Y_{t^-}^{n,i} - L_t) dK_t^{n,i,+} \leq \int_0^T (Y_{t^-}^{n,i} - L_t) dK_t^{n,+} = 0, \qquad \mathbb{P}\text{-a.s.}$$

With the help of Fatou lemma we get the desired result.
The 5-uplet $(Y^n, Z^n, V^n, K^{n,+}, K^{n,-})$ is solution of the following generalized doubly reflected BSDEs with jumps.

$$(\mathcal{E}^2) \begin{cases} (i)\ Y_t^n = \xi + \int_t^T f_s^n(Y_s^n, Z_s^n, V_s^n) ds + \int_t^T dR_s^n + \int_t^T g_s^n(Y_s^n) dA_s^n - \int_t^T Z_s^n dB_s \\ \qquad + \int_t^T dK_s^{n,+} - \int_0^T dK_s^{n,-} - \int_t^T Z_s^n dB_s - \int_t^T \int_E V_s^n(e) \tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \ \mathbb{P}\text{-a.s.} \\ (ii)\ L_t^n \leq Y_t^n \leq U_t^n, \quad 0 \leq t \leq T, \ \mathbb{P}\text{-a.s.} \\ (iii)\ \int_0^T (Y_{s^-}^n - L_s^n) dK_s^{n,+} = \int_0^T (Y_{s^-}^n - U_s^n) dK_s^{n,-} = 0, \quad \mathbb{P}\text{-a.s.} \end{cases}.$$

Moreover the processes $Z^n, V^n, K^{n,+}$ and $K^{n,-}$ inherits what follows : for all $n \in \mathbb{N}$

$$\mathbb{E}\Big[\int_0^{\tau_j} |Z_s^n|^2 ds + \int_0^{\tau_j} \int_E |V_s^n(e)|^2 \nu(de, ds) + (K_T^{n,+})^2 + (K_T^{n,-})^2\Big] < +\infty.$$

To conclude this step, since $dK^{n,+} = \inf_i dK^{n,i,+}$ and $dK^{n,-} = \sup_i dK^{n,i}$ we have that $dK^{n,+}$ and $dK^{n,-}$ are singular.

Step 3 : In this part, we will derive a stability result of the system $(\mathcal{E}^2)$. We proceed exactly as in the second step. To this end, since we know that the sequence $(Y^n)_n$ is decreasing and uniformly bounded, we only need to prove that there exits a Z and V in $\mathcal{L}^{2,d}$ and $\mathcal{L}_\nu^{2,d}$ such that

$$\lim_{n \to \infty} \mathbb{E}\Big[\int_0^T |Z_s^n - Z_s|^2 ds + \int_0^T \int_E |V_s^n(e) - V_s(e)|^2 \nu(de, ds)\Big] = 0. \tag{3.14}$$

As in [50, 85], we first consider the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\phi(x) = \frac{1}{4j}(e^{4jx} - 4jx - 1)$ with the following properties

$$\begin{cases} 0 \le x \le 1, \qquad \phi(0) = 0,\ \phi^{'}(0) = 0, \qquad \phi^{'}(x), \ \ \phi^{''}(x) \ge 0, \\ \phi^{'}(x) = e^{4jx} - 1, \qquad \phi^{''}(x) = 4je^{4jx}, \\ \phi^{''}(x) = 4j\phi^{'}(x) + 4j. \end{cases} \tag{3.15}$$

For the sake of clarity we define the processes $\bar{Y}$,$\bar{Z}$ and $\bar{V}$

$$\forall n \le m,\ \bar{Y} := Y^n - Y^m, \qquad \bar{Z} := Z^n - Z^m, \qquad \bar{V} := V^n - V^m.$$

Since ϕ is $\mathcal{C}^2$, by Itô's formula we get

$$\begin{aligned} \phi(\bar{Y}_t) &= \phi(\bar{Y}_{\tau_j}) - \int_t^{\tau_j} \phi^{'}(\bar{Y}_s) d\bar{K}_s^- + \int_t^{\tau_j} \phi^{'}(\bar{Y}_s) d\bar{K}_s^+ - \int_t^{\tau_j} \phi^{'}(\bar{Y}_s)\bar{Z}_s dB_s \\ &- \frac{1}{2}\int_t^{\tau_j} \phi^{''}(\bar{Y}_s)|\bar{Z}_s|^2 ds - \int_t^{\tau_j}\int_E \phi^{'}(\bar{Y}_s)\bar{V}_s(e)\tilde{\mu}(de, ds) + \int_t^{\tau_j} \phi^{'}(\bar{Y}_s) g_s^n(Y_s^n) dA_s^n \\ &- \int_t^{\tau_j} \phi^{'}(\bar{Y}_s) g_s^m(Y_s^m) dA_s^m - \int_t^T \int_E (\phi(\bar{Y}_s + \bar{V}_s) - \phi(\bar{Y}_s) - \phi^{'}(\bar{Y}_s)\bar{V}_s)\mu(de, ds) \\ &+ \int_t^{\tau_j} \phi^{'}(\bar{Y}_s)\left[(f_s^n(Y_s^n, Z_s^n, V_s^n) - f_s^m(Y_s^m, Z_s^m, V_s^m)\right] ds \end{aligned} \tag{3.16}$$

Clearly $f^n \le f$. By the growth assumption on f we get for all $n \in \mathbb{N}$

$$-\eta_s - \frac{C_s}{2}|Z_s^n|^2 - \frac{1}{\delta}J_s(V_s^n) \le f_s^n(Y_s^n, Z_s^n, V_s^n) \le 0.$$

Then we have

$$0 \le -f_s^m(Y_s^m, Z_s^m, V_s^m) \le \eta_s + \frac{C_s}{2}|Z_s^m|^2 + \frac{1}{\delta}J(V_s^m).$$

Now the elementary inequality $|Z^m|^2 = |Z^m - Z^n + Z^n|^2 \le 2|\bar{Z}|^2 + 2|Z^n|^2$ yield to an upper bound of $f^n - f^m$.

$$\begin{aligned} -\eta_s - \frac{C_s}{2}|Z_s^n|^2 - \frac{1}{\delta}J_s(V_s^n) &\le f_s^n(Y_s^n, Z_s^n, V_s^n) - f_s^m(Y_s^m, Z_s^m, V_s^m) \\ &\le \eta_s + C_s(w)\left(|\bar{Z}_s|^2 + |Z_s^n|^2\right) + \frac{1}{\delta}J_s(V^n) \\ &\le \eta_s + C_s(w)\left(|\bar{Z}_s|^2 + |Z_s^n|^2\right) + \epsilon|V_s^m|_s^2 \\ &\le \eta_s + C_s(w)\left(|\bar{Z}_s|^2 + |Z_s^n|^2\right) + 2\epsilon|V_s^n|_s^2 + 2\epsilon|\bar{V}_s|_s^2. \end{aligned} \tag{3.17}$$

In the other hand

$$\int_t^{\tau_j} g_s^n(Y_s^n)dA_s^n - \int_t^{\tau_j} g_s^m(Y_s^m)dA_s^m = \int_t^{\tau_j} g_s^n(Y_s^n)1_{\{s\leq\tau_n\}}dA_s - \int_t^{\tau_j} g_s^m(Y_s^m)1_{\{s\leq\tau_m\}}dA_s$$
$$= \int_t^{\tau_j} (g_s^n(Y_s^n) - g_s^m(Y_s^m))1_{\{s\leq\tau_n\}}dA_s - \int_t^{\tau_j} \phi'(\bar{Y}_s)g_s^m(Y_s^m)1_{\{\tau_n\leq s\leq\tau_m\}}dA_s.$$

By lemma (3.3.10), we have

$$\begin{aligned}\int_t^{\tau_j} \phi'(\bar{Y}_s)(g_s^n(Y_s^n) - g_s^m(Y_s^m))1_{\{s\leq\tau_n\}}dA_s &\leq -\int_t^{\tau_j} \phi'(\bar{Y}_s)g_s^m(Y_s^m)1_{\{s\leq\tau_n\}}dA_s \\ &\leq \int_t^{\tau_j} \phi'(\bar{Y}_s)1_{\{s\leq\tau_n\}}dA_s \\ &\leq \int_t^{\tau_j} 4je^{4j}\bar{Y}_s dA_s. \end{aligned} \tag{3.18}$$

Hence, reporting (3.17) and (3.18) in (3.16) yield to

$$\begin{aligned}&\phi(\bar{Y}_t) + \int_t^T \int_E (\phi(\bar{Y}_s + \bar{V}_s) - \phi(\bar{Y}_s) - \phi'(\bar{Y}_s)\bar{V}_s)\mu(de, ds) \\ &\leq \phi(\bar{Y}_{\tau_j}) - \int_t^{\tau_j}\int_E \bar{V}_s(e)\phi'(\bar{Y}_s)\tilde{\mu}(de, ds) - \int_t^{\tau_j} \phi'(\bar{Y}_s)\bar{Z}_s dB_s \\ &- \frac{1}{2}\int_t^{\tau_j} \phi''(\bar{Y}_s)|\bar{Z}_s|^2 ds - \int_t^{\tau_j} \phi'(\bar{Y}_s)d\bar{K}_s^- + \int_t^{\tau_j} \phi'(\bar{Y}_s)d\bar{K}_s^+ \\ &+ \int_t^{\tau_j} \phi'(\bar{Y}_s)\left[\eta_s + C_s(w)\left(|\bar{Z}_s|^2 + |Z_s^n|^2\right) + 2\epsilon|V_s^n|_s^2 + 2\epsilon|\bar{V}_s|_s^2\right] ds \\ &+ 4j\int_t^{\tau_j} e^{4j}\bar{Y}_s dA_s + 4j\int_t^{\tau_j} e^{4j}\bar{Y}_s 1_{\{\tau_n\leq s\leq\tau_m\}}dA_s. \end{aligned} \tag{3.19}$$

Now, since the process $C_s(w)$ is bounded by j for all $s\in[0,\tau_j]$ it follows

$$\begin{aligned}&\int_t^{\tau_j} \phi'(\bar{Y}_s)C_s(w)|\bar{Z}_s|^2 ds - \frac{1}{2}\int_t^{\tau_j} \phi''(\bar{Y}_s)|\bar{Z}_s|^2 ds \\ &\leq \int_t^{\tau_j} j\phi'(\bar{Y}_s)|\bar{Z}_s|^2 ds - \frac{1}{2}\int_t^{\tau_j}\left(4j\phi'(\bar{Y}_s) + 4j\right)|\bar{Z}_s|^2 ds \\ &\leq j\int_t^{\tau_j} \phi'(\bar{Y}_s)|\bar{Z}_s|^2 ds - 2j\int_t^{\tau_j} \phi'(\bar{Y}_s)|\bar{Z}_s|^2 ds - 2j\int_t^{\tau_j} |\bar{Z}_s|^2 ds \\ &\leq -j\int_t^{\tau_j} |\bar{Z}_s|^2 ds - 2j\int_t^{\tau_j} \phi'(\bar{Y}_s)|\bar{Z}_s|^2 ds. \end{aligned} \tag{3.20}$$

Using the same argument as in [52], we obtain

$$\tilde{C}|\bar{V}_s(e)|^2 \leq \phi(\bar{Y}_{s^-} + \bar{V}_s(e)) - \phi(\bar{Y}_{s^-}) - \phi'(\bar{Y}_{s^-})\bar{V}_s(e), \quad d\nu\text{a.e.} \tag{3.21}$$

Plugging (3.20) and (3.21) in the previous Itô equation (3.19) yields

$$
\begin{aligned}
&\phi(\bar{Y}_t)+\tilde{C}\int_t^T\int_E|\bar{V}_s(e)|^2\nu(de,ds)\leq\\
&\phi(\bar{Y}_{\tau_j})-2\epsilon\int_t^{\tau_j}\phi'(\bar{V}_s)|\bar{V}_s|_s^2ds-2\epsilon\int_t^{\tau_j}\phi'(\bar{V}_s)|V_s^n|_s^2ds-2j\int_t^{\tau_j}|\bar{Z}_s|^2ds-j\int_t^{\tau_j}\phi'(\bar{Y}_s)|\bar{Z}_s|^2ds\\
&-j\int_t^{\tau_j}\phi'(\bar{Y}_s)|Z_s^n|^2ds+\int_t^{\tau_j}(e^{j\bar{Y}_s}-1)dA_s+\int_t^{\tau_j}\left(e^{j\bar{Y}_s}-1\right)d\bar{K}_s^-+\int_t^{\tau_j}\left(e^{j\bar{Y}_s}-1\right)d\bar{K}_s^+\\
&-\int_t^{\tau_j}(e^{j\bar{Y}_s}-1)\bar{Z}_sdB_s+dR_s^{n,m}+4j\int_t^{\tau_j}\eta_se^{4j}\bar{Y}_sds-\int_t^{\tau_j}\int_E\bar{V}_s(e)\phi'(\bar{Y}_s)\tilde{\mu}(de,ds). \qquad (3.22)
\end{aligned}
$$

Let us underline that the process $\bar{K}$ acts only when $\bar{Y}$ reaches the obstacles L and U, it is easy to see that $K^{m,+}$ only increases when $Y_{t^-}^n=S_t$ and $\bar{K}^{n,-}$ only increases when $Y_{t^-}^m=L_t$. Therefore, we have

$$
\begin{aligned}
&\phi(\bar{Y}_t)+\tilde{C}\int_t^T\int_E|\bar{V}_s(e)|^2\nu(de,ds)\leq\phi(\bar{Y}_{\tau_j})-2\epsilon\int_t^{\tau_j}\phi'(\bar{V}_s)|\bar{V}_s|_s^2ds-2\epsilon\int_t^{\tau_j}\phi'(\bar{V}_s)|V_s^n|_s^2ds\\
&-2j\int_t^{\tau_j}[\frac{1}{2}+\phi'(\bar{Y}_s)]|\bar{Z}_s|^2ds-j\int_t^{\tau_j}\phi'(\bar{Y}_s)|Z_s^n|^2ds+\int_t^{\tau_j}4je^{4j}\left(\bar{Y}_s+1_{\{\tau_n\leq s\leq\tau_m\}}\right)dA_s\\
&+\int_t^{\tau_j}j\left(Y_{s^-}^n-L_s\right)dK_s^{m,+}+dR_s^{n,m}+\int_t^{\tau_j}j\left(-Y_{s^-}^m+U_s\right)dK_s^{n,-}-\int_t^{\tau_j}\int_E\bar{V}_s(e)\phi'(\bar{Y}_s)\tilde{\mu}(de,ds)\\
&-\int_t^{\tau_j}(e^{j\bar{Y}_s}-1)\bar{Z}_sdB_s+4j\int_t^{\tau_j}\eta_se^{4j}\bar{Y}_sds.
\end{aligned}
$$

where

$$
d\tilde{R}_s^{n,m}=dR_s^{n,m}+\left(e^{4j\left(Y_{s^-}^n-L_s\right)}-4j\left(Y_{s^-}^n-L_s\right)\right)dK_s^{m,+}-\left(e^{4j\left(-Y_{s^-}^m+U_s\right)}-4j\left(-Y_{s^-}^m+U_s\right)\right)dK_s^{n,-}.
$$

Taking in the left sides all the terms containing either $\bar{Z}$ or $\bar{V}$, we get

$$
\begin{aligned}
&\phi(\bar{Y}_t)+\tilde{C}\int_t^T\int_E|\bar{V}_s(e)|^2\nu(de,ds)+2\epsilon\int_t^{\tau_j}\phi'(\bar{V}_s)|\bar{V}_s|_s^2ds+2j\int_t^{\tau_j}[1+\phi'(\bar{Y}_s)]|\bar{Z}_s|^2ds\\
&+4j\int_t^{\tau_j}j.\left(Y_{s^-}^n-L_s\right)dK_s^{m,+}+4j\int_t^{\tau_j}j.\left(-Y_{s^-}^m+U_s\right)dK_s^{n,-}+\int_t^{\tau_j}d\tilde{R}_s^{n,m}.\\
&\leq\phi(\bar{Y}_{\tau_j})+j\int_t^{\tau_j}\phi'(\bar{Y}_s)|Z_s^n|^2ds-2\epsilon\int_t^{\tau_j}\phi'(\bar{V}_s)|V_s^n|_s^2ds+\int_t^{\tau_j}4je^{4j}\left(\bar{Y}_s+1_{\{\tau_n\leq s\leq\tau_m\}}\right)dA_s\\
&+4j\int_t^{\tau_j}\eta_se^{4j}\bar{Y}_sds-\int_t^{\tau_j}\phi'(\bar{Y}_s)\bar{Z}_sdB_s-\int_t^{\tau_j}\int_E\bar{V}_s(e)\phi'(\bar{Y}_s)\tilde{\mu}(de,ds). \qquad (3.23)
\end{aligned}
$$

Then using the fact that $Y^n - Y^m \leq 1$ we obtain for $t = 0$,

$$
\begin{aligned}
\mathbb{E}[2j(1+\phi^{'}(1))\int_0^{\tau_j}|Z_s^n - Z_s^m|^2 ds + (2\epsilon\phi^{'}(1)+\tilde{C})\int_0^{\tau_j}\int_E |V_s^n - V_s^m|^2\nu(de,ds)] \\
\leq \phi(1) + \mathbb{E}[4je^{4j}\int_t^{\tau_j}\eta_s ds + \phi^{'}(1)\int_0^{\tau_j} dA_s + \int_0^{\tau_j} d\tilde{R}_s^{n,m}] \\
+ \mathbb{E}[\int_0^{\tau_j} j.\phi^{'}(\bar{Y})|Z_s^n|^2 - 2\epsilon\phi^{'}(1)\mathbb{E}\int_t^{\tau_j}|V_s^n|_s^2 ds].
\end{aligned}
\tag{3.24}
$$

Let us underline that since $\phi(Y^{n,m})$ is uniformly bounded process, the conditional expectation of the martingale part of equation (3.23) vanishes. Now, taking $n = 0$ we get from the apriori estimates of $(Y^n, Z^n, K^{n,+}, K^{n,-})$ that

$$
\mathbb{E}[\int_0^{\tau_j}|Z_s^0 - Z_s^m|^2 ds + \int_0^{\tau_j}\int_E |V_s^0 - V_s^m(e)|^2\nu(de,ds)] \leq C_j^{'}. \tag{3.25}
$$

and

$$
\sup_{m\in\mathbb{N}}\mathbb{E}[\int_0^{\tau_j}|Z_s^m|^2 ds + \int_0^{\tau_j}\int_E |V_s^m(e)|\nu(ds,de)] < +\infty. \tag{3.26}
$$

For all $t \in [0,\tau_j]$, we can then extract a subsequence $(m_k^j)_{k\in\mathbb{N}}$ such that $Z_t^{m_k^j}$ and $V_t^{m_k^j}$ converges weakly respectively to an $\mathcal{F}_t$-adapted processes $\hat{Z}_t$ and $\hat{V}_t$ in $\mathbb{H}^2$ and $\mathbb{H}^2_\nu$. It is obvious that $(\phi^{'}(Y_s^n - Y_s^{m_k^j}))^{\frac{1}{2}}1_{[0,\tau_j]}(Z_s^n - Z_s^{m_k^j})$ respectively $(\phi^{'}(Y_s^n - Y_s^{m_k^j}))^{\frac{1}{2}}1_{[0,\tau_j]}(V_s^n - V_s^{m_k^j})$ converge weakly in $\mathbb{H}^2$ and in $\mathbb{H}^2_\nu$ to $(\phi^{'}(Y_s^n - Y_s^{m_k^j}))^{\frac{1}{2}}1_{[0,\tau_j]}(Z_s^n - Z_s)$ and $(\phi^{'}(Y_s^n - Y_s^{m_k^j}))^{\frac{1}{2}}1_{[0,\tau_j]}(V_s^n - V_s)$. Hence, recalling the Itô equation (3.23) we obtain

$$
\begin{aligned}
&\mathbb{E}[\int_0^{\tau_j} 2j(\frac{1}{2}+\phi^{'}(\bar{Y}_s))|Z_s^n - Z_s^{m_k^j}|^2 ds + \int_0^{\tau_j}\int_E(2\epsilon\phi^{'}(\bar{Y}_s)+\tilde{C})|V_s^n - V_s^{m_k^j}(e)|^2\nu(de,ds)] \\
&\leq \mathbb{E}[\phi(\bar{Y}_{\tau_j}) + 4je^{4j}\int_0^{\tau_j}\eta_s ds + \int_0^{\tau_j} 4je^{4j}\left(\bar{Y}_s + 1_{\{\tau_n\leq s\leq\tau_m\}}\right)dA_s + \int_0^{\tau_j} j.\phi^{'}(\bar{Y}_s)|Z_s^n|^2 ds] \\
&- 2\epsilon\mathbb{E}\Big[\int_0^{\tau_j}\phi^{'}(\bar{Y}_s)|V_s^n|_s ds\Big].
\end{aligned}
$$

Notice that

$$
j\mathbb{E}[\int_0^{\tau_j}\phi^{'}(\bar{Y}_s)|Z_s^n|^2 ds] \leq 2j\mathbb{E}[\int_0^{\tau_j}\phi^{'}(\bar{Y}_s)|Z_s^n - \hat{Z}_s|^2 ds + \int_0^{\tau_j}\phi^{'}(\bar{Y}_s)|\hat{Z}_s|^2 ds],
$$

and

$$
\begin{aligned}
2\epsilon\mathbb{E}[\int_t^{\tau_j}\phi^{'}(Y_s^n - \hat{Y}_s)|V_s^n|_s^2 ds] \leq 2\tilde{\epsilon}\mathbb{E}[\int_t^{\tau_j}\phi^{'}(Y_s^n - \hat{Y}_s)|V_s^n - \hat{V}_s|_s^2 ds] \\
+ 2\tilde{\epsilon}\mathbb{E}[\int_t^{\tau_j}\phi^{'}(Y_s^n - \hat{Y}_s)|\hat{V}_s|_s^2 ds].
\end{aligned}
$$

Now since we have

$$\mathbb{E}[2j(1+\phi^{'}(Y^n_s-\hat{Y}_s))\int_0^{\tau_j}|Z^n_s-\hat{Z}_s|^2ds+\int_0^{\tau_j}\int_E(2\epsilon\phi^{'}(Y^n_s-\hat{Y}_s)+\tilde{C})|V^n_s(e)-\hat{V}_s(e)|^2\nu(de,ds)]$$
$$\leq \liminf_k \mathbb{E}[2j(1+\phi^{'}(Y^n_s-Y^{m^j_k}_s))\int_0^{\tau_j}|Z^n_s-Z^{m^j_k}_s|^2ds]$$
$$+\liminf_k \mathbb{E}[(2\epsilon\phi^{'}(Y^n_s-Y^{m^k_j}_s)+\tilde{C})\int_0^{\tau_j}\int_E|V^n_s(e)-V^{m^j_k}_s(e)|^2\nu(de,ds)],$$

letting k goes to infinity allows us to obtain

$$\begin{aligned}
&\mathbb{E}[2j\int_0^{\tau_j}|Z^n_s-\hat{Z}_s|^2ds+\tilde{C}\int_0^{\tau_j}\int_E|V^n_s(e)-\hat{V}_s(e)|^2\nu(de,ds)]\\
&\quad\leq \liminf_k\Big[\mathbb{E}[2j(1+\phi^{'}(Y^n_s-Y^{m^j_k}))\int_0^{\tau_j}|Z^n_s-Z^{m^j_k}_s|^2ds\\
&\qquad+(2\epsilon\phi^{'}(Y^n_s-Y^{m^k_j}_s)+\tilde{C})\int_0^{\tau_j}\int_E|V^n_s(e)-V^{m^j_k}_s(e)|^2\nu(de,ds)]\Big].\\
&\quad\leq \mathbb{E}[\phi(\bar{Y}_{\tau_j})+4je^{4j}\int_t^{\tau_j}\eta_sds+\int_0^{\tau_j}\phi^{'}(\bar{Y}_s)dA_s+2j\int_0^{\tau_j}\phi^{'}(\bar{Y}_s)|\hat{Z}_s|^2ds\Big]\\
&\qquad+2\tilde{\epsilon}\mathbb{E}[\int_t^{\tau_j}\phi^{'}(Y^n_s-\hat{Y}_s)|V^n_s|^2_sds].
\end{aligned}$$

Since $\phi^{'}(Y^n-\hat{Y}_s)$ goes to zero as k goes to infinity, we conclude by the dominated convergence theorem, that

$$\mathbb{E}[\int_0^{\tau_j}|Z^n_s-\hat{Z}^j_s|^2ds+\int_0^{\tau_j}\int_E|V^n_s(e)-\hat{V}^j_s(e)|^2\nu(de,ds)]\longrightarrow_{n\to\infty}0. \tag{3.27}$$

Therefore by the uniqueness of the limit we have , $\hat{Z}^j_s(w)=\hat{Z}^{j+1}_s(w)$ and $\hat{V}^j_s(w)=\hat{V}^{j+1}_s(w)$, $\mathbb{P}$-a.s.

We then denote by Z and V their respective limits when j goes to infinity. Finally since $\tau_j=T$, we obtain

$$\lim_{n\to\infty}\mathbb{E}[\int_0^T|Z^n_s-Z_s|^2ds+\int_0^T\int_E|V^n_s(e)-V_s(e)|^2\nu(de,ds)]=0. \tag{3.28}$$

Hence, it remains to prove that the limit process Y is càdlàg.

$$\begin{aligned}
|Y_s-Y^n_s|\leq&\int_t^{\tau_j}|(f_s(Y_s,Z_s,V_s)-f^n_s(Y^n_s,Z^n_s,V^n_s)|ds+\int_t^{\tau_j}|g_s(Y_s)-g^n_s(Y^n_s)|dA^n_s\\
&+\int_t^{\tau_j}|g_s(Y_s)1_{\{s\geq\tau_n\}}|dA_s+|\int_t^{\tau_j}(Z_s-Z^m_s)dB_s|+|\int_t^{\tau_j}\int_E(V_s(e)-V^m_s(e))\tilde{\mu}(de,ds)|.
\end{aligned}$$

Taking the supremum over t and the conditional expectation yields to

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le\tau_j}|Y_s-Y_s^n|\Big] &\le \mathbb{E}\Big[\int_0^{\tau_j}|(f_s(Y_s,Z_s,V_s)-f_s^n(Y_s^n,Z_s^n,V_s^n)|ds\Big] \\ &+\mathbb{E}\Big[\int_0^{\tau_j}|g_s(Y_s)1_{\{s\ge\tau_n\}}|dA_s\Big]+\mathbb{E}\big[\int_0^{\tau_j}|g_s(Y_s)-g_s^n(Y_s^n)|dA_s^n\big] \\ &+\mathbb{E}\big[\sup_{0\le t\le\tau_j}|\int_t^{\tau_j}(Z_s-Z_s^m)dB_s|+\sup_{0\le t\le\tau_j}|\int_t^{\tau_j}\int_E(V_s(e)-V_s^m(e))\tilde{\mu}(de,ds)|\big].\end{aligned} \tag{3.29}$$

Furthermore, the Burkholder-Davis-Gundy inequality allows us to deduce that

$$\mathbb{E}\big[\sup_{0\le t\le\tau_j}\int_0^{\tau_j}(Z_s-Z_s^m)dB_s\big]\le 2\mathbb{E}\big[\int_0^{\tau_j}|Z_s-Z_s^n|^2ds\big]^{\frac{1}{2}}. \tag{3.30}$$

and

$$\mathbb{E}\big[\sup_{0\le t\le\tau_j}\int_0^{\tau_j}\int_E(V_s(e)-V_s^n(e))\tilde{\mu}(de,ds)\big]\le 2\mathbb{E}\big[\int_0^{\tau_j}\int_E|V_s(e)-V_s^n(e)|^2\nu(de,ds)\big]^{\frac{1}{2}}. \tag{3.31}$$

Reporting (3.30), (3.31) in the previous inequality yields to

$$\mathbb{E}\big[\sup_{0\le t\le\tau_j}|Y_t-Y_t^n|\big]\to_n 0.$$

Step 4: Identification of the limit

In this part we aim to prove that quintuple (Y,Z,V,K^-,K^+) is the solution of the generalized doubly reflected BSDEJ $(\mathcal{E})$. To do so we first prove that

$$\lim_{n\to\infty}\mathbb{E}\big[\int_0^{\tau_j}|f_t^n(Y_t^n,Z_t^n,U_t^n)-f_t(Y_t,Z_t,U_t)|dt\big]=0 \text{ and } \lim_{n\to\infty}\mathbb{E}\big[\int_0^{\tau_j}|g_t^n(Y_t^n)-g_t(Y_t)|dt\big]=0.$$

By Lemma 3.3.10, $f_s^n(y,z,v)$ converges to $f_s(y,z,v)$, $\forall s\in[0,T]$, $\mathbb{P}$-a.s. Moreover, from the càdlàg version of Dini's theorem, $f_s^n(y,z,v)$ converges uniformly on every compact set. Thus, $f_s^n(y,z,v)1_{[0,\tau_j]}$ converges uniformly to $f_s(y,z,v)1_{[0,\tau_j]}$ as n goes to infinity, $dt\otimes d\mathbb{P}\otimes d\nu$ a.s.
In fact as $Z^{n_k^j}$ and $U^{n_k^j}$ are unbounded, one can decompose the expression above in the following way

$$\begin{aligned}\mathbb{E}\big[\int_0^{\tau_j}|f_t^n(Y_t^n,Z_t^n,U_t^n)-f_t(Y_t,Z_t,U_t)|dt\big] &= \mathbb{E}\big[\int_0^{\tau_j}|f_t^n(Y_t^n,Z_t^n,U_t^n)-f_t(Y_t,Z_t,U_t)|\mathbf{1}_{\{|Z_s^{n_k^j}|+|U_s^{n_k^j}|\le C\}}dt\big] \\ &+\mathbb{E}\big[\int_0^{\tau_j}|f_t^n(Y_t^n,Z_t^n,U_t^n)-f_t(Y_t,Z_t,U_t)|\mathbf{1}_{\{|Z_s^{n_k^j}|+|U_s^{n_k^j}|\ge C\}}dt\big]\end{aligned}$$

The first term in the right-hand side goes to zero as k goes to infinity since $Y^{n_k^j}$ is bounded over $[0,\tau_j]$ and

$$|f_t^{n_k^j}(Y_t^{n_k^j},Z_t^{n_k^j},V_t^{n_k^j})-f_t(Y_t,Z_t,V_t)|\leq \eta_t+\frac{j}{2}|Z_t^{n_k^j}|^2+\frac{1}{\delta}J(V_t^{n_k^j}).$$

For the last one, using Markov inequality we have

$$\mathbb{E}[\mathbf{1}_{\{|Z^{n_k^j}|+|V^{n_k^j}|\geq C\ \}}]\leq \frac{2}{C^2}\mathbb{E}[|Z^{n_k^j}|^2+|V^{n_k^j}|_t^2].$$

Hence, using the dominated convergence theorem we obtain that $f_t^{n_k^j}(Y_t^{n_k^j},Z_t^{n_k^j},U_t^{n_k^j})$ converge to $f_t(Y_t,Z_t,U_t)$ in $\mathbb{L}^1(dt\otimes d\mathbb{P}\otimes d\nu)$ for all $t\leq\tau_j$.
Using the same argument we can prove that $\mathbb{E}\left[\int_0^{\tau_j}|g_t^n(Y_t^n)-g_t(Y_t)|dt\right]$ goes to zero as n goes to infinity.
Now, notice from the system $(\mathcal{E}^2)$ that $\sup_n\mathbb{E}[K_{\tau_j}^{n,+}]<\infty$. By Fatou lemma we deduce that $\mathbb{E}[K_{\tau_j}^{+}]<\infty$. Thus, we obtain that $K_T^+<\infty,\ \mathbb{P}$-a.s. We can show similarly that $K_T^-<\infty$.
In order to finish the proof we need to show that the limit process Y satisfies the following minimality condition

$$\int_0^T(Y_{s^-}-L_s)dK_s^+=\int_0^T(U_s-Y_{s^-})dK_s^-=0,\quad 0\leq t\leq T,\ \mathbb{P}\text{-a.s.}$$

This is deduced from the following facts :
• From the system (3.4.1) we have $\int_0^T(Y_s^n-L_s^n)dK_s^{n,+}=\int_0^T(U_s^n-Y_s^n)dK_s^{n,-}=0$, $\mathbb{P}$-a.s.
• $K^{n,+}$ respectively $K^{n,-}$ converges uniformly to K^+, K^-.
The only point remaining concerns the singularity of the measures dK^+ and dK^-. The result follows from the singularity of $dK^{n,-}$ and $dK^{n,+}$ with $dK^-=\inf_n dK^{n,-}$, $dK^+=\sup_n dK^{n,+}$.
We can hence conclude that the 5-uplet (Y,Z,V,K^+,K^-) is a solution of the system $(\mathcal{E}^1)$.

Theorem 3.3.11. *Under Assumptions 3.3.3 and 3.3.5, there exists a maximal solution* (Y,Z,V,K^+,K^-) *for the doubly reflected BSDE with jumps associated to* (f,ξ,L,U) *satisfying the system* $(\mathcal{E})$.

Proof. As already explained, the existence is obtained directly from a logarithmic change of variable. Let $Y_t=\frac{ln(\bar{Y}_t)}{m_t}+m_t,\ Z_t=\frac{\bar{Z}_t}{m_s\bar{Y}_t}$ and $V_t=\frac{1}{m_t}ln(\frac{\bar{V}_t}{\bar{Y}_t}+1)$. Then applying Itô's formula, we obtain

$$\begin{aligned}Y_t=\xi+\int_t^T f_s(Y_s,Z_s,V_s)ds+\int_t^T dR_s+\int_t^T g_s(Y_s)dA_s+\int_t^T dK_s^+-\int_t^T dK_s^-\\-\int_t^T Z_sdB_s-\int_t^T\int_E V_s(e)\tilde{\mu}(ds,de),\quad 0\leq t\leq T,\mathbb{P}\text{-a.s,}.\end{aligned}$$

In addition, since $\int_0^T(\bar{Y}_{s^-}-\bar{L}_s)d\bar{K}_s^+=\int_0^T(\bar{Y}_{s^-}-\bar{U}_s)d\bar{K}_s^-=0$. we have $\int_0^T(Y_{s^-}-L_s)dK_s^+=\int_0^T(Y_{s^-}-U_s)dK_s^-=0,\ \mathbb{P}$-a.s.

3.4 Appendix

3.4.1 Existence and uniqueness result: The Lipschitz case.

In this section, we extend some of the results of Pardoux Zhang [115] concerning generalized BSDEs with no reflection to the case of doubly reflected BSDEJs. Let us note that the majority of the following proofs follows straightforwardly from the original proofs of [115], [68] [118] and [45] with some minor modifications due to jumps and reflection. However, we still provide the proof of existence since it will be needed in the construction of stochastic quadratic BSDEJ's solution. To the best of our knowledge, they do not appear anywhere else in the literature.
We look for the solution of the following generalized doubly reflected BSDE with jumps,

$$(S)\begin{cases} Y_t = \xi + \int_t^T f_s(Y_s, Z_s, V_s)ds + \int_t^T g_s(Y_s)dA_s - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e)\tilde{\mu}(ds, de) \\ \quad + \int_t^T dK_s^+ - \int_t^T dK_s^- \quad 0 \le t \le T, \mathbb{P}\text{-a.s.} \\ L_t \le Y_t \le U_t \text{ and } \int_0^T (Y_{s^-} - U_s)dK_s^+ = \int_0^T (Y_{s^-} - L_s)dK_s^- = 0, \quad 0 \le t \le T, \mathbb{P}\text{-a.s.} \end{cases}$$

under the following assumption

$$\textbf{(H1)} \begin{cases} (i) \text{ There exists a positive constant } L_f \text{ such that } \forall y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d, u, u' \in \mathbb{L}^0(\mathcal{B}(E), \nu), \\ \qquad |f_t(y, z, u) - f_t(y', z', u')| \le L_f(|y - y'| + |z - z'| + |k - k'|_t), \ \forall t \in [0, T], \mathbb{P}\text{-a.s.} \\ (ii) \text{ There exists a positive constant } L_g \text{ such that } \forall y, y' \in \mathbb{R}, \\ \qquad |g_t(y) - g_t(y')| \le L_g|y - y'|, \text{ and } -1 \le g_t(y) \le 0, \quad \forall t \in [0, T]. \\ (iii) \forall R \in \mathcal{K}, \quad dR_t \ge 0 \text{ and } 0 \le L_t \le U_t < 1, \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.} \end{cases}$$

Comparison result

Theorem 3.4.1. *Let $(Y^1, Z^1, V^1, K^{1,-}, K^{1,+})$ and $(Y^2, Z^2, V^2, K^{2,-}, K^{2,+})$ be to two solutions of $(\mathcal{S})$ associated to $(f^i, \xi^i, L^i, U^i)_{i=1,2}$, such that, for $(i = 1, 2)$ Assumption **(H1)** is satisfied. Assume moreover that*

$$\begin{cases} \bullet \quad \xi^1 \le \xi^2, \ \mathbb{P}\text{-a.s.} \\ \bullet \quad L_t^1 \le L_t^2 \text{ and } S_t^1 \le S_t^2, \ \forall t \in [0, T], \mathbb{P}\text{-a.s.} \\ \bullet \quad f_s^1(Y_s^2, Z_s^2, V_s^2) \le f_s^2(Y_s^2, Z_s^2, V_s^2), \quad \forall t \in [0, T], \mathbb{P}\text{-a.s.} \\ \bullet \quad g_s^1(Y_s^2) \le g_s^2(Y_s^2), \quad \forall t \in [0, T], \mathbb{P}\text{-a.s,} \end{cases}$$

then we have $Y_t^1 \le Y_t^2, \ \forall t \in [0, T], \mathbb{P}$-a.s.
Furthermore, if $U_t^1 = U_t^2$, $L_t^1 = L_t^2$, $L_t \le U_t, \ \forall t \in [0, T]$, $\mathbb{P}$-a.s, then $K_t^{-,1} \le K_t^{-,2}$ and $K_t^{+,2} \le K_t^{+,1}$, $\forall t \in [0, T], \mathbb{P}$-a.s.

Proof. The proof follows the lines of the proof of the theorem (1.3) in [63] in the continuous setting . For simplicity, we shall make the following notations.

$$(\delta Y_t, \delta Z_t, \delta V_t) = (Y_t^1 - Y_t^2, Z_t^1 - Z_t^2, V_t^1 - V_t^2), \qquad \delta\xi = \xi^1 - \xi^2$$

$$\delta f_t = f_t^1(Y_t^2, Z_t^2, V_t^2) - f_t^2(Y_t^2, Z_t^2, V_t^2) \quad \delta g_t = g_t^1(Y_t^2) - g_t^2(Y_t^2)$$

Let us define the following bounded processes

$$\alpha_s = \frac{f_s^2(Y_s^1, Z_s^1, V_s^1) - f_s^2(Y_s^2, Z_s^1, V_s^1)}{Y_s^1 - Y_s^2} 1_{\{Y_s^1 \neq Y_s^2\}}, \quad \tilde{\alpha}_s = \frac{g_s^2(Y_s^1) - g_s^2(Y_s^2)}{Y_s^1 - Y_s^2} 1_{\{Y_s^1 \neq Y_s^2\}}.$$

$$\beta_s = \frac{f_s^2(Y_s^2, Z_s^1, V_s^1) - f_s^2(Y_s^2, Z_s^1, V_s^1)}{\|Z_s^1 - Z_s^2\|^2}(Z_s^1 - Z_s^2) 1_{\{Z_s^1 \neq Z_s^2\}}.$$

Consider the following stopping times

$$\tau_k = \inf\left\{t \geq 0, \int_0^t \left(|Z_s^1|^2 + |Z_s^2|^2\right) ds + \int_0^t \int_E \left(|V_s^1(e)|^2 + |V_s^2(e)|^2\right) \nu(de, ds) \geq k\right\} \wedge T.$$

We start by applying Itô formula to $R_t \delta Y_t = e^{\alpha_t}(Y_t^1 - Y_t^2)^+$ where $R_t = e^{\alpha_t}$.

$$\begin{aligned}
&R_{t\wedge\tau_k}(\delta Y_{t\wedge\tau_k})^+ = R_{\tau_k}(\delta Y_{\tau_k})^+ + \int_{t\wedge\tau_k}^{\tau_k} 1_{\{Y_s^1 \geq Y_s^2\}} R_s \delta Y_s \left[f_s^1(Y_s^1, Z_s^1, V_s^1) - f_s^2(Y_s^2, Z_s^2, V_s^2)\right] ds \\
&+ \int_{t\wedge\tau_k}^{\tau_k} R_s \delta Y_s [g_s(Y_s^1) - g_s(Y_s^2)] dA_s - \int_{t\wedge\tau_k}^{\tau_k} 1_{\{Y_s^1 \geq Y_s^2\}} \delta Y_s R_s \delta Z_s dB_s \\
&- \int_{t\wedge\tau_k}^{\tau_k} \int_E R_s \delta Y_s \delta V_s(e) \tilde{\mu}(de, ds) - \int_{t\wedge\tau_k}^{\tau_k} \alpha_s R_s \delta Y_s ds \\
&+ \int_{t\wedge\tau_k}^{\tau_k} 1_{\{Y_s^1 \geq Y_s^2\}} R_s (dK_s^{+,1} - dK_s^{-,1}) - \int_{t\wedge\tau_k}^{\tau_k} 1_{\{Y_s^1 \geq Y_s^2\}} R_s (dK_s^{+,2} - dK_s^{-,2}).
\end{aligned}$$

Notice that when $Y^1 \geq Y^2$ we have $U_t^2 \geq Y_t^2$ and $Y_t^1 \geq L_t^1$, we obtain

$$\int_{t\wedge\tau_k}^{\tau_k} 1_{\left\{Y_{s-}^1 \geq Y_{s-}^2\right\}} \delta Y_s (dK_s^{+,1} - dK_s^{+,2}) - \int_{t\wedge\tau_k}^{\tau_k} 1_{\left\{Y_{s-}^1 \geq Y_{s-}^2\right\}} \delta Y_s (dK_s^{-,1} - dK_s^{-,2}) \leq 0.$$

Hence, using Assumption (H1), we get

$$\begin{aligned}
R_{t\wedge\tau_j}(\delta Y_{t\wedge\tau_k})^+ &\leq R_{\tau_k}(\delta Y_{\tau_k})^+ + \int_{t\wedge\tau_k}^{\tau_k} 1_{\{Y_s^1\geq Y_s^2\}}\alpha_s R_s|\delta Y_s|ds \\
&\underbrace{- \int_{t\wedge\tau_k}^{\tau_k} 1_{\{Y_s^1\geq Y_s^2\}} R_s\delta Z_s dB_s - \int_{t\wedge\tau_k}^{\tau_k}\int_E 1_{\{Y_s^1\geq Y_s^2\}} R_s\delta V_s(e))\tilde{\mu}(de)ds + \int_{t\wedge\tau_k}^{\tau_k} R_s\tilde{\alpha}_s 1_{\{Y_s^1\geq Y_s^2\}}|\delta Y_s|dA_s}_{M} \\
&\underbrace{+ \int_{t\wedge\tau_k}^{\tau_k} 1_{\{Y_s^1\geq Y_s^2\}} R_s|\delta Z_s|\beta_s ds + \int_{t\wedge\tau_k}^{\tau_k}\int_E 1_{\{Y_s^1\geq Y_s^2\}} R_s\gamma_s(e)\delta V_s(e)\nu_s(de,ds).}_{M-<M,N>}
\end{aligned} \tag{3.32}$$

Now we define the probability measure $\tilde{\mathbb{P}}$ such that $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}(N)_T$. Since $-1+\delta_K \leq \gamma(U_s^1, U_s^2) \leq C_K$ and $|\beta_t(Z_s^1, Z_s^2)| \leq \bar{C}$ since Z^1 and Z^2 are of BMO types.
Hence $\mathcal{E}(M)$ is a BMO-martingale with $M = \int_0^t R_s\tilde{\alpha}_s\delta Y_s dA_s + \int_0^t R_s(Z_s^1 - Z_s^2)dB_s + \int_. R_s(V_s^1 - V_s^2)\tilde{\mu}(de)$ and $N = \int_0^t \beta_s dB_s + \int_0^t\int_E \gamma(e)\tilde{\mu}(de,ds)$. Using Girsanov theorem, we obtain that $M - \langle M, N\rangle$ are locale $\tilde{P}$-martingale.
Hence, taking the conditional expectation in (3.4.1) between t and τ_n when τ_k converges to T as long as k goes to infinity yield

$$R_t(Y_t^1 - Y_t^2)^+ \leq \mathbb{E}^{\tilde{\mathbb{P}}}\left[R_{\tau_k}(Y_{\tau_k}^1 - Y_{\tau_k}^2)^+|\mathcal{F}_t\right].$$

Sending k to ∞ we get $\delta Y_t = Y_t^1 - Y_t^2 = 0, \forall t\in[0,T]$, $\mathbb{P}$-a.s.
• Let as now prove that

$$K_t^{-,1} \leq K_t^{-,2}, \quad K_t^{+,2} \leq K_t^{+,1}, \ \forall t\in[0,T],\ \mathbb{P}\text{-a.s.}$$

Exactly as (Theorem (1.3), [68]), we define the following family of stopping times τ

$$\tau = \inf\left\{t\geq 0, \quad K_t^{-,1} \geq K_t^{-,2}\right\}\wedge T.$$

Suppose that $\mathbb{P}(\tau < T) > 0$. Hence, $K_\tau^{-,1} = K_\tau^{-,2}$ on $\{\tau < T\}$. Moreover

$$Y_\tau^1 = Y_\tau^2 = U_\tau \qquad \text{on } \{\tau < T\}.$$

if $Y_{\tau(w)}^1(w) \neq U_{\tau(w)}(w)$ then $Y_t(w) < U_t(w)$ for all $t \in]\tau(w) - p(w), \tau + p(w)[$ where $p(w)$ is a positive real number.
Thus, it follows that for all $t \in]\tau(w) - p(w), \tau + p(w)[$, $K_{t(w)}^{-,1}(w) = K_{t(w)}^{-,2}(w) = Y_{t(w)}^2(w)$ which contradicts the definition of the stopping family $\tau(w)$. Hence

$$Y_{t(w)}^1(w) = U_{t(w)}(w) = Y_{t(w)}^2(w).$$

In the other hand, we consider the family of stopping times $\delta = \inf\{t \geq \tau; Y_t^1 = L_t\} \wedge T$ such that $\{\tau < T\} \subset \{\delta < T\}$. Notice that
- if $\tau_w < T$ then $Y^1_{\tau(w)} = U_{\tau(w)}(w)$.
- if $\delta(w) = \tau(w)$ then $Y_{\delta(w)}(w) = L_{\delta(w)} = U_{t(w)} = L_{t(w)}(w)$ which contradicts the fact that the process U_t remain above the process L_t . We can deduce that $\mathbb{P}[\delta < \tau] > 0$.
This implies that $K_\delta^{+,1} = K_t^{+,1}$ and $K_\delta^{+,2} = K_t^{+,2}$, $\forall t \in [\tau, \delta]$, $\mathbb{P}$-a.s, since $Y^1 \leq Y^2$ and K^+ (resp. K^-) moves when Y^1 (resp. Y^2 reaches the lower obstacles L. Henceforth, we have

$$Y_t^1 = Y_\delta^1 + \int_t^\delta f_s^1(Y_s^1, Z_s^1, V_s^1)ds + \int_t^T g_s^1(Y_s^1)dA_s - K_\delta^{1,-} + K_t^{1,+} - \int_t^\delta Z_s^1 dB_s - \int_t^\delta \int_E V_s^1(e)\tilde{\mu}(de, ds).$$
$$Y_t^2 = Y_\delta^2 + \int_t^\delta f_s^2(Y_s^2, Z_s^2, V_s^2)ds + \int_t^T g_s^2(Y_s^2)dA_s - K_\delta^{2,-} + K_t^{2,+} - \int_t^\delta Z_s^2 dB_s - \int_t^\delta \int_E V_s^2(e)\tilde{\mu}(de, ds).$$

Now in order to conclude, we define $(\bar{Y}_t^1, \bar{Z}_t^1, \bar{V}_t^1, \bar{K}_t^1)_{t\leq\delta}$ (resp. $(\bar{Y}_t^2, \bar{Z}_t^2, \bar{V}_t^2, \bar{K}_t^2)_{t\leq\delta}$) solution of the reflected BSDE with jumps in the upper obstacle L associated to $(\bar{f}^1 ds + \bar{g}^1 dA_s, Y_\delta^1)$ (resp. $(f^2 ds + \bar{g}^2 dA_s, Y_\delta^2)$). Then by the comparison theorem given in [118], we have

$$\bar{Y}_t^1 \leq \bar{Y}_t^2, \quad \text{and} \quad \bar{K}_t^{1,-} - \bar{K}_s^{1,-} \geq \bar{K}_t^{2,+} - \bar{K}_s^{2,+}, \ \forall t \in [s, \delta], \quad \mathbb{P}\text{-a.s.}$$

Using the growth property of $\bar{f}^1$ (resp. $\bar{g}^1$) and $\bar{f}^2$(resp. $\bar{g}^1$), we obtain that $\bar{Y}_t^1 = Y_t^1$, $\bar{Y}_t^2 = Y_t^2$, $\bar{Z}_t^1 = Z_t^1$, $\bar{Z}_t^2 = Z_t^2$, $\bar{V}_t^1 = V_t^1$ and $\bar{V}_t^2 = V_t^2$, $\forall t \in [\tau, \delta]$, $\mathbb{P}$-a.s. Hence, we immediately get

$$\bar{K}_\delta^1 - \bar{K}_t^1 = K_\delta^1 - K_t^1, \ \bar{K}_\delta^2 - \bar{K}_t^2 = K_\delta^2 - K_t^2, \forall t \in [\tau, \delta], \ \mathbb{P}\text{-a.s},$$

which contradicts the definition of the stopping time τ. Therefore, $\mathbb{P}[\tau < T]$ which implies that $K^{-,1} \leq K^{-,2}$, $\mathbb{P}$-a.s. To conclude the proof, we can show similarly that $K^{+,2} \leq K^{+,1}$, $\mathbb{P}$-a.s..

Theorem 3.4.2. *Under Assumption **(H1)**, there exists a unique solution (Y, Z, V, K^+, K^-) to the generalized doubly reflected backward stochastic differential equation with jumps associated to $(fds + gdA_s, \xi)$. Moreover, it satisfies*

$$\mathbb{E}[\sup_{t\leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt + \int_0^T \int_E |V_t(e)|^2 \nu(de)dt + |K_T^+|^2 + |K_T^-|^2] < +\infty. \tag{3.33}$$

Proof. The uniqueness is a simple consequence of the above comparison theorem. Let us prove the existence of the solution. We consider the following penalized generalized BSDEJ: for any $n, m \in \mathbb{N}^*$

$$\begin{aligned} Y_t^{n,m} &= \xi + \int_t^T f_s(Y_s^{n,m}, Z_s^{n,m}, V_s^{n,m})ds + \int_t^T g_s(Y_s^{n,m})dA_s + m\int_t^T (Y_s^{n,m} - L_s)^- ds \\ &\quad - n\int_t ({}^T U_s - Y_s^{n,m})^- ds - \int_t^T Z_s^{n,m} dW_s - \int_t^T \int_E V_s^{n,m} \tilde{\mu}(de, ds), \ \forall t \in [0, T], \mathbb{P}\text{-a.s}, \end{aligned} \tag{3.34}$$

were $f^{n,m}(s,y,z,v) = f(s,y,z,v) + m(y-L_s)^- - n(U_s-y)^-$. Referring to the results of [72], we obtain existence and uniqueness for a solution $(Y^{n,m}, Z^{n,m}, V^{n,m})$ to the BSDEs given by $(f^{n,m},\xi)$. We set $K_t^{n,m+} = m\int_0^t (Y_{s^-}^{n,m} - L_s)^- ds$ and $K_t^{n,m-} = n\int_0^t (U_s - Y_{s^-}^{n,m})^- ds$.
Step1. we aim to prove the following estimate. There exists a constant C such that

$$\sup_{n,m\in\mathbb{N}^*} \mathbb{E}\big[\sup_{t\le T}|Y_t^{n,m}|^2 + \int_0^T |Z_t^{n,m}|^2 dt + \int_0^T\int_E |V_t^{n,m}(e)|^2\nu(de,dt) + |K_T^{n,m,+}|^2 + |K_T^{n,m,-}|^2\big] < +\infty. \tag{3.35}$$

As usual we start by applying Itô formula to $e^{\lambda A_t}|Y_t^{n,m}|^2$, relying on the Lipschitz property of f and g and using Young's inequality we obtain

$$\begin{aligned}
e^{\lambda A_t}|Y_t^{n,m}|^2 &\le e^{\lambda A_T}|Y_T^{n,m}|^2 - \int_t^T \lambda e^{\lambda A_s}|Y_s^{n,m}|^2 dA_s + \int_t^T e^{\lambda A_s}[g_s(0)dA_s + f_s(0,0,0)ds] \\
&- \frac{1}{2}\int_t^T e^{\lambda A_s}[|Z_s^{n,m}|^2 + |V_s^{n,m}|_s^2]ds + (1+6L_f^2)\int_t^T e^{\lambda A_s}|Y_s^{n,m}|^2 ds \\
&+ (1+2L_g)\int_t^T e^{\lambda A_s}|Y_s^{n,m}|^2 dA_s + 2\int_t^T e^{\lambda A_s} Y_s^{n,m} dK_s^{n,m+} - 2\int_t^T e^{\lambda A_s} Y_s^{n,m} dK_s^{n,m-} \\
&- 2\int_t^T e^{\lambda A_s} Y_s^{n,m}[Z_s^{n,m} dB_s + \int_E V_s^{n,m}(e)\tilde{\mu}(de,ds)].
\end{aligned}$$

From the Skorokhod condition and Young inequality we have

$$2\int_t^T Y_s dK_s^- = \int_t^T L_s dK_s^- \le \frac{1}{\epsilon}\sup_{0\le s\le T}|L_s|^2 + \epsilon|K_T^- - K_0^-|^2. \tag{3.36}$$

$$2\int_t^T Y_s dK_s^+ = \int_t^T U_s dK_s^+ \le \frac{1}{\tilde{\epsilon}}\sup_{0\le s\le T}|U_s|^2 + \tilde{\epsilon}|K_T^+ - K_0^+|^2. \tag{3.37}$$

Therefore taking the conditional expectation we get

$$\begin{aligned}
&\mathbb{E}[e^{\lambda A_t}|Y_t^{n,m}|^2 + \frac{1}{2}\int_t^T e^{\lambda A_s}[|Z_s^{n,m}|^2 + |V_s^{n,m}|_s^2]ds] \\
&\le \mathbb{E}[e^{\lambda A_T}|Y_T^{n,m}|^2] + \int_t^T e^{\lambda A_s}[g_s(0)dA_s + f_s(0,0,0)ds] + (-\lambda+1+2L_g)\mathbb{E}[\int_t^T e^{\lambda A_s}|Y_s^{n,m}|^2 dA_s] \\
&+ (1+6L_f^2)\mathbb{E}[\int_t^T e^{\lambda A_s}|Y_s^{n,m}|^2 ds] + \frac{1}{\epsilon}(\mathbb{E}\sup_{0\le s\le T}|L_s|^2 + \mathbb{E}[\sup_{0\le s\le T}|U_s|^2]) + \epsilon(\mathbb{E}[|K_T^-|^2 + \mathbb{E}|K_T^+|^2]) \\
&- 2\mathbb{E}[\int_t^T e^{\lambda A_s} Y_s^{n,m}[Z_s^{n,m} dB_s + \int_E V_s^{n,m}(e)\tilde{\mu}(de,ds)].
\end{aligned} \tag{3.38}$$

In the other hand, we consider a sequence of stopping time τ_n such that

$$\begin{aligned}
\tau_{n+1} &= \inf\{t \ge \tau_n, \quad Y_t^{n,m} \le L_t\} \wedge T \\
\tau_{n+2} &= \inf\{t \ge \tau_{n+1}, \quad Y_t^{n,m} \ge U_t\} \wedge T.
\end{aligned}$$

Using the same argument as in [63] or [70] we can rewrite the penalized generalized BSDEJ as follow

$$
\begin{aligned}
e^{\lambda A_{\tau_n}} Y^{n,m}_{\tau_n} &= e^{\lambda A_{\tau_{n+1}}} Y^{n,m}_{\tau_{n+1}} + \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} [f_s(Y^{n,m}_s, Z^{n,m}_s, V^{n,m}_s) ds + g_s(Y^{n,m}_s) dA_s] \\
&\quad - n \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} (U_s - Y^{n,m}_s)^- ds - \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} Z^{n,m}_s dW_s - \int_{\tau_n}^{\tau_{n+1}} \int_E e^{\lambda A_s} V^{n,m}_s \tilde{\mu}(de, ds).
\end{aligned} \tag{3.39}
$$

Relying on assumption **(H1)** we get

$$
\begin{aligned}
&Y^{n,m}_{\tau_n} \geq S_{\tau_n} \text{ on } \{\tau_n < T\}; \quad Y^{n,m}_{\tau_n} = S_{\tau_n} \text{ on } \{\tau_n = T\}. \\
&Y^{n,m}_{\tau_{n+1}} \leq S_{\tau_{n+1}} \text{ on } \{\tau_{n+1} < T\}; \quad Y^{n,m}_{\tau_{n+1}} = S_{\tau_{n+1}} \text{ on } \{\tau_{n+1} = T\}.
\end{aligned}
$$

Hence from (3.39) with the help of the Lipschitz assumption of f and g we derive the following estimate

$$
\begin{aligned}
n \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} (U_s - Y^{n,m}_s)^- ds \leq & - \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} (\alpha_s - Z^{n,m}_s) dW_s - \int_{\tau_n}^{\tau_{n+1}} \int_E e^{\lambda A_s} V^{n,m}_s \tilde{\mu}(de, ds) \\
& + \int_{\tau_n}^{\tau_{n+1}} e^{\lambda A_s} [f_s(0,0,0) + L_f(|Y^{n,m}_s| + |Z^{n,m}_s| + |V^{n,m}_s|_s)] ds \\
& + \int_{\tau_n}^{\tau_{n+1}} (e^{\lambda A_s} [g_s(0) + L_g |Y^{n,m}_s| dA_s] + e^{\lambda A_s} d\tilde{V}^+_s - e^{\lambda A_s} d\tilde{V}^-_s).
\end{aligned}
$$

Hence, since τ_n is stationary sequence [63] we obtain from assumption **(H1)**

$$
\begin{aligned}
\mathbb{E}[n \int_t^T e^{\lambda A_s} (U_s - Y^{n,m}_s)^- ds]^2 \leq C\{1 &+ \mathbb{E}[\int_t^T [e^{\lambda A_s} |Y^{n,m}_s|^2 ds] + \mathbb{E}[\int_t^T e^{\lambda A_s} |Z^{n,m}_s|^2 ds \\
&+ \mathbb{E}[\int_t^T e^{\lambda A_s} |V^{n,m}_s|^2_s] ds] + \mathbb{E}[\int_t^T e^{\lambda A_s} |Y^{n,m}_s|^2 dA_s]\}.
\end{aligned} \tag{3.40}
$$

Similarly we obtain

$$
\begin{aligned}
\mathbb{E}[m \int_t^T (Y^{n,m}_s - L_s)^- ds]^2 \leq C\{1 &+ \mathbb{E}[\int_t^T [e^{\lambda A_s} |Y^{n,m}_s|^2 ds] + \mathbb{E}[\int_t^T e^{\lambda A_s} |Z^{n,m}_s|^2 ds \\
&+ \mathbb{E}[\int_t^T e^{\lambda A_s} |V^{n,m}_s|^2_s] ds] + \mathbb{E}[\int_t^T e^{\lambda A_s} |Y^{n,m}_s|^2 dA_s]\}.
\end{aligned} \tag{3.41}
$$

where C is a generic constant. Plugging (3.40) and (3.41) in (3.38) we obtain

$$
\begin{aligned}
&\mathbb{E}[e^{\lambda A_t}|Y_t^{n,m}|^2 + \frac{1}{2}\int_t^T e^{\lambda A_s}[|Z_s^{n,m}|^2 + \int_E |V_s^{n,m}(e)|^2\nu(de)]ds] \qquad (3.42)\\
&\leq \mathbb{E}[e^{\lambda A_T}|Y_T^{n,m}|^2 + \int_t^T e^{\lambda A_s}[g_s(0)dA_s + f_s(0,0,0)ds]] + \frac{1}{\epsilon}(\mathbb{E}[\sup_{0\leq s\leq T}|L_s|^2] + \mathbb{E}[\sup_{0\leq s\leq T}|U_s|^2])\\
&+ \mathbb{E}[\int_t^T(-\lambda + 1 + 2L_g)e^{\lambda A_s}|Y_s^{n,m}|^2 dA_s] + (1+6L_f^2)\mathbb{E}[\int_t^T e^{\lambda A_s}|Y_s^{n,m}|^2 ds]\\
&+ 2\epsilon C\Big[1 + \mathbb{E}[\int_t^T [e^{\lambda A_s}|Y_s^{n,m}|^2 ds] + \mathbb{E}[\int_t^T e^{\lambda A_s}(|Z_s^{n,m}|^2 + |V_s^{n,m}|_s^2)ds] + \mathbb{E}[\int_t^T e^{\lambda A_s}|Y_s^{n,m}|^2 dA_s]\Big]
\end{aligned}
$$

Hence, using Gronwall inequality we get

$$
\sup_{0\leq t\leq T}\mathbb{E}|Y_t^{n,m}|^2 + \mathbb{E}\int_0^T |Z_s|^2 ds + \mathbb{E}\int_0^T\int_E |V_s(e)|^2\nu(de,ds) \leq C,
$$

which implies the desired result.

Step 2: There exists a constant C such that for any $n \in \mathbb{N}^*$ we have $\mathbb{E}\left[\sup_t |Y_t^n|^2\right] \leq C$ and there exists an $\mathcal{F}_t$-adapted process $(Y_t)_t$ such that $\lim_{n\to+\infty}\mathbb{E}[\int_0^T |Y_t^n - Y_t|^2 dt] = 0$.

We know that all the requirements of existence result of [118] are fullfilled. Thus we know that for fixed $m \in \mathbb{N}$ $Y^{n,m}$, $Z^{n,m}$ and $V^{n,m}$ converge respectively to Y^n , Z^n and V^n as m goes to infinity. Moreover the limit process (Y^n, Z^n, V^n) is the unique solution of the generalized reflected BSDEJs associated to (ξ, f^n, g, K^{n+}) where $f^n(y,z,v) = f(y,z,v) - n(U_s - y)^-$. Moreover the limit process inherits the property (3.35).

Notice that for all $n \in \mathbb{N}$ and $\forall(s,y,z,v)$

$$
f_s^n(y,z,v) \leq f_s^{n+1}(y,z,v).
$$

Therefore by comparison Theorem [68] we have, $Y_t^n \leq Y_t^{n+1}, \quad \forall t \in [0,T]$, $\mathbb{P}$-a.s.

Hence, $Y_t^n \nearrow Y_t$ $\mathbb{P}$-a.s. Now from the property (3.35) and Fatou's lemma we deduce that $\mathbb{E}\left[|Y_t|^2\right] \leq C$ and then by Dominated convergence theorem we obtain that

$$
\lim_n \mathbb{E}[\int_0^T |Y_t^n - Y_t|^2] = 0.
$$

Step3 In this step, we aim to prove $\lim_{n\to+\infty}[\sup_{0\le t\le T}|(Y_t^n-U_t)^+|^2]=0.$
Let $(\bar{Y}_t^n,\bar{Z}_t^n,\bar{V}_t^n,\bar{K}_t^n)$ be the solution of the following reflected generalized BSDE with jumps

$$\begin{cases} \bar{Y}_t^n=\xi+\int_t^T[f_s(\bar{Y}_s^n,\bar{Z}_s^n,\bar{V}_s^n)+n(U_s-Y_s^n)]ds+\int_t^T g_s(\bar{Y}_s^n)dA_s+\bar{K}_T^{n+}-\bar{K}_t^{n-}\\ \qquad -\int_t^T\bar{Z}_s^n dW_s-\int_t^T\int_E\bar{V}_s^n(e)\tilde{\mu}(de,ds),\qquad \forall t\in[0,T],\mathbb{P}\text{-a.s,} \\ \bar{Y}_t^n\ge L_t,\qquad \int_0^T(\bar{Y}_t^n-L_t)d\bar{K}_t^{n+}=0,\ \mathbb{P}\text{-a.s.}\end{cases} \tag{3.43}$$

In addition, we consider $(\bar{Y}_t^{n,m},\bar{Z}_t^{n,m},\bar{V}_t^{n,m})$ the solution of the penalized generalized BSDEJ associated to the system (3.43). By comparison, since

$$f_t^{n,m}(y,z,v)\le f_t(y,z,v)+n(U_t-y)+m(y-L_t)^-.$$

We have, for any $n,m\in\mathbb{N}^*$, $Y_t^{n,m}\le\bar{Y}_t^{n,m}$, $\mathbb{P}$-a.s. Letting m goes to infinity we obtain that $Y_t^n\le\bar{Y}_t^n$. Now, applying Itô's formula to $\bar{Y}_t^n e^{-nt}$, yield to

$$\begin{aligned}\bar{Y}_\tau^n=\operatorname{ess\,sup}_\tau\mathbb{E}_\tau\,[e^{-n(T-t)}\xi 1_{\{\tau=T\}}+e^{-n(\tau-t)}L_\tau 1_{\{\tau<T\}}\\ +\int_\tau^T e^{-n(s-t)}[f_s(\bar{Y}_s^n,\bar{Z}_s^n,\bar{V}_s^n)ds+g_s(\bar{Y}_s^n)dA_s]+n\int_\tau^T e^{-n(s-t)}U_s ds],\end{aligned} \tag{3.44}$$

where τ is an $\mathcal{F}_t$-family of stopping times $\tau\le T$. Now since U is continuous then

$$e^{-n(T-t)}\xi 1_{\{\tau=T\}}+n\int_\tau^T e^{-n(s-t)}U_s ds\overset{n\to\infty}{\longrightarrow}\xi 1_{\{\tau=T\}}+U_\tau 1_{\{\tau\le T\}}\quad \mathbb{P}\text{-a.s and in }\mathbb{H}^2.$$

In addition, we have

$$\mathbb{E}_\tau[\int_\tau^T e^{-n(s-\tau}|f_s(\bar{Y}_s^n,\bar{Z}_s^n,\bar{V}_s^n)|ds]\le\frac{1}{\sqrt{2n}}\mathbb{E}[\int_0^T|f_s(\bar{Y}_s^n,\bar{Z}_s^n,\bar{V}_s^n)|^2ds]^{\frac12}\longrightarrow 0. \tag{3.45}$$

$$\mathbb{E}_\tau[\int_\tau^T e^{-n(s-\tau}|g_s(\bar{Y}_s^n)|dA_s]\le\frac{1}{\sqrt{2n}}\mathbb{E}[\int_0^T|g_s(\bar{Y}_s^n)|^2dA_s]^{\frac12}\longrightarrow 0. \tag{3.46}$$

Besides, since we have that $L_t\le S_t\le U_t$ and

$$\mathbb{E}_\tau[\int_\tau^\nu e^{-n(s-\tau}(dV_s^+-dV_s^-)]\le\frac{1}{\sqrt{2n}}\mathbb{E}[|V_T^+|^2+|V_T^+|^2]\longrightarrow 0. \tag{3.47}$$

where ν is a stopping time with $\tau\le\nu\le T$. Combining (3.45), (3.46) and (3.47), we finally obtain

$$\bar{Y}_\tau\le\bar{Y}_\tau^n\overset{n}{\longrightarrow}\xi 1_{\{\tau=T\}}+U_{\{\tau\le T\}}\text{ in }\mathbb{H}^2. \tag{3.48}$$

Henceforth, we deduce from (Theorem 86, [35]) that $\forall t \in [0,T]$ $Y_t \leq U_t$ $\mathbb{P}$-a.s. Consequently, $(Y_t^n - U_t)^- \searrow 0, \quad \forall t \in [0,T], \quad \mathbb{P}$-a.s. Then, from Dini's theorem [35], we deduce that $\sup_t (Y_t^n - U_t)^- \searrow 0$.

Therefore, since for any $n \in \mathbb{N}$, $(Y_t^n - U_t) \leq Y_t^0 - S_t$ and $(Y^n - U_t)^+ \leq |Y^0 + |U_t|$, by dominated convergence theorem, we obtain that

$$\lim_{n\to\infty} \mathbb{E}[\sup_{0\leq t\leq T} |(Y_t^n - U_t)^+|^2] = 0, \text{ a.s.}$$

Step 4 In this step, we aim to prove that $\lim_{n\to\infty} \mathbb{E}[\sup_{0\leq t\leq T} |Y_t^n - Y_t|^2] = 0$ and there exists Z , V K^+ and K^- such that

$$\lim_n \mathbb{E}[\int_0^T |Z_t|^2 dt + \int_0^T \int_E |V_t(e)|^2 \nu(de,dt) + |K_T^+|^2 + |K_T^-|^2] = 0 \tag{3.49}$$

Let $n, p \in \mathbb{N}$ and consider the following processes $\delta Y_t = Y_t^n - Y_t^p$, $\delta Z_t = Z_t^n - Z_t^p$ and $\delta V_t = V_t^n - V_t^p$. Applying Itô's formula to $|\Delta Y|^2$ between t and T

$$\begin{aligned}
|\delta Y_t|^2 + \int_t^T |\delta Z_s|^2 ds + \int_t^T \int_E |\delta V_s(e)|^2 \nu(de,ds) &= |\delta Y_T|^2 + 2\int_t^T \delta Y_s [g_s(Y_s^n) - g_s(Y_s^p)] dA_s \\
&+ 2\int_t^T \int_E \delta Y_s \left[f_s(Y_s^n, Z_s^n, V_s^n) - f_s(Y_s^p, Z_s^p, V_s^p)\right] ds - 2\int_t^T \int_E \delta Y_s \delta V_s(e) \tilde{\mu}(de,ds) \\
&- 2\int_t^T \delta Y_s \delta Z_s dB_s + 2\int_t^T \int_E \delta Y_s (dK_s^{+,n} - dK_s^{+,p}) - 2\int_t^T \delta Y_s (dK_s^{-,n} - dK_s^{-,p}).
\end{aligned} \tag{3.50}$$

Since $n \leq p$ we have $2\int_t^T \int_E \delta Y_t (dK_s^{+,n} - dK_s^{+,p}) \leq 0$ and

$$\begin{aligned}
2\int_t^T \int_E \delta Y_s (dK_s^{-,n} - dK_s^{-,p}) &= 2\int_t^T \int_E \delta Y_s dK_s^{-,n} - 2\int_t^T \int_E \delta Y_t dK_s^{-,p} \\
&\leq 2\sup_{0\leq t\leq T} |Y_t^p - U_t| K_T^{-,n} + 2\sup_{0\leq t\leq T} |U_t - Y_t^n| K_T^{-,p}.
\end{aligned}$$

In the other hand, from the growth assumption of g we have

$$2\int_t^T \delta Y_s [g_s(Y_s^n) - g_s(Y_s^p)] dA_s \leq 2C\int_t^T |\delta Y_s|^2 dA_s.$$

Using step 2 and 3, we can deduce that

$$\begin{aligned}
\mathbb{E}[|\delta Y_t|^2] - 2L_g \mathbb{E}[\int_t^T |\delta Y_s|^2 dA_s] + 2\mathbb{E}[\int_t^T |\delta Z_s|^2 ds + \int_t^T \int_E |\delta V_s(e)|^2 \nu(de,ds)] \\
\leq C\mathbb{E}[\int_0^T |\delta Y_s|^2 ds + 2\sup_{0\leq t\leq T} |Y_t^p - U_t| K_T^{-,n} + 2\sup_{0\leq t\leq T} |U_t - Y_t^n| K_T^{-,p}] \xrightarrow{n,p} 0.
\end{aligned}$$

Therefore (Z^n) and (V^n) are a Cauchy sequences in a complete spaces. Consequently there exits two processes Z and V such that (Z^n) and (V^n) converge respectively to Z and V.
Since $\mathbb{E}[\int_0^T |\delta Y_s|^2 dA_s] \leq 2L_g\sqrt{T}(\mathbb{E}[\int_0^T |\delta Y_s|^2 ds])^{\frac{1}{2}}$, using BDG inequality we can rewrite (3.50) as follows,

$$\mathbb{E}[\sup_{0\leq t\leq T}|\delta Y_t|^2] \leq 2L_g\sqrt{T}(\mathbb{E}[\int_0^T |\delta Y_s|^2 ds)^{\frac{1}{2}} \\ + C\mathbb{E}[\int_0^T |\delta Y_s|^2 ds + 2\sup_{0\leq t\leq T}|Y_t^p - U_t|K_T^{-,n} + 2\sup_{0\leq t\leq T}|U_t - Y_t^n|K_T^{-,p}].$$

Therefore, $\lim_{p\to+\infty}\mathbb{E}[\sup_{0\leq t\leq T}|Y_t^n - Y_t^p|^2] = 0$ and then $\lim_{n\to+\infty}\mathbb{E}[\sup_{0\leq t\leq T}|Y_t^n - Y_t^p|^2] = 0$.
Moreover, since K^{n+} is an increasing sequence then $K^{n,+}$ converges to the process K^+. In addition, $\mathbb{E}[K_T^{n,+}|^2] \leq C$ we deduce that $\mathbb{E}[|K_T^+|^2] < +\infty$.
Besides, for any $n \in \mathbb{N}$ we have $L_t \leq Y_t^n \leq Y_t \leq U_t$ and $\lim_n \mathbb{E}[\sup_{0\leq t\leq T}(Y_t^n - U_t)^+]^2 = 0$.
Then $L_t \leq Y_t \leq U_t$. In the other hand, $\int_0^T (Y_t^n - L_t)dK_t^{n,+} = 0$ and $\lim_n \int_0^T (Y_t^n - L_t)dK_t^{n,+} = \int_0^T (Y_t - L_t)dK_t^+$. Thus the Skorohod condition is satisfied and the proof of the Theorem (3.4.2) is complete.

CHAPTER 4

FORWARD-BACKWARD SDE WITH JUMPS OF MCKEAN-VLASOV TYPE AND STORAGE PROBLEM IN SMART GRIDS

4.1 Overview of the content of this chapter

In this chapter, we investigate the wellposedness of the following fully coupled forward backward stochastic differential equation with jumps of Mckean-Vlasov type

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t b_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)})ds + \int_0^t \sigma_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)}))dW_s \\ \qquad + \displaystyle\int_0^t \int_E \beta(s, X_{s-}, Y_{s-}, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)})\tilde{\pi}(ds, de), \quad t \in [0,T], \ \mathbb{P}\text{-a.s.} \\ Y_t = g(X_T, \mathbb{P}_{X_T}) + \displaystyle\int_t^T h_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)})ds - \int_t^T Z_s dW_s - \int_t^T \int_E K_s(e)\tilde{\pi}(ds, de), \end{cases}$$

where W is d-Brownian motion and π is an integer valued random measure defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, (X, Y, Z, K) is an $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{L}^0(B(E), \eta)$-valued adapted process and $\mathbb{P}(X_t, Y_t)$ is the marginal distribution of (X_t, Y_t) .
We prove existence and uniqueness of the solution under two different assumptions using an approximation scheme based on a spike perturbation either on the forward or the backward equation. We highlight that we do not impose any degeneracy restriction on the forward coefficient. Our second contribution is dedicated to energy storage problem in smart grids. We formulate the problem via mean field type control. We show that it can be characterized though solving an associated FBSDEJS of mean field type.
In the particular case where the cost structure is quadratic and the pricing rule is linear, we show that the FBSDE which characterizes the solution of the EMFG can be solved explicitly. This provides a quite tractable and efficient setting to analyze numerically various questions arising in this power grid model. The rest of this chapter is organised as follows. In section 4.2, we recall briefly the mathematical setting and some basic definition. In section 4.3, we prove existence and uniqueness of the fully coupled FBSDEJs under assumption $(H1)$ and $(H2)$. In

Section 4.4 we consider an energy storage problem and we study the connection with some mean-field FBSDEJ's solution. In the appendix, we extend some of the results concerning FB-SDE's in the Brownian setting to the case of jumps.

4.2 Framework: Notations and setting

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which the filtration $\mathbb{F} = (\mathcal{F}_t)_{0<t<T}$ satisfies the usual conditions of completeness and right continuity. On this stochastic basis, let W a d-dimensional Brownian motion and $\pi(\omega, dt, de)$ an independent integer valued random measure defined on $([0,T] \times E, \mathcal{B}([0,T]) \otimes \mathcal{B}(E))$, with compensator $\eta(\omega, dt, de)$.
The predictable σ-field on $\Omega \times [0,T]$ is denoted by $\mathcal{P}$ and $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(E)$ is the respective σ-field on $\tilde{\Omega} = \Omega \times [0,T] \times E$. For a σ-finite measure λ on $(E, \mathcal{B}(E))$ satisfying $\int_E 1 \wedge |e|^2 \lambda(de) < \infty$ and a bounded $\tilde{\mathcal{P}}$-measurable non negative density function ζ, we will assume that the compensator η is absolutely continuous with respect to $\lambda \otimes dt$ such that

$$\eta(\omega, dt, de) = \zeta(\omega, t, e)\lambda(de)dt, \qquad 0 \leq \zeta \leq C_\eta, \quad \text{for some constant } C_\eta.$$

Finally, we will denote by $\tilde{\pi}$ the compensated measure of π as

$$\tilde{\pi}(\omega, dt, de) = \pi(\omega, dt, de) - \eta(\omega, dt, de).$$

For any random variable X on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we denote by $\mathbb{P}_X$ its probability law under $\mathbb{P}$. We denote by $\mathcal{M}_2(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$ with finite moments of order 2 equipped with the 2-Wassertein distance

$$\begin{aligned}\mathcal{W}_2(\mu, \mu') &:= \inf\{(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 F(dx, dy))^{\frac{1}{2}}, F \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d) \text{ with marginals } \mu, \mu'\} \\ &:= \inf\{(\mathbb{E}|\xi - \xi'|^2)^{\frac{1}{2}} : \mu = \mathcal{L}(\xi),\ \mu' = \mathcal{L}(\xi')\},\end{aligned}$$

where $\mathcal{L}(\xi)$ and $\mathcal{L}(\xi')$ are respectively the law of ξ and ξ' and the infimum is taken over $F \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d)$ with marginals μ and μ'.
Notice that if X^1 and X^2 are random variables of order 2 with values in $\mathbb{R}^d$, then we have the following inequality involving the Wasserstein metric between the laws of the square integrable random variables X^1 and X^2 and their L^2- distance:

$$\mathcal{W}_2(\mathbb{P}_{X^1}, \mathbb{P}_{X^2}) \leq \left[\mathbb{E}|X^1 - X^2|^2\right]^{\frac{1}{2}}. \tag{4.1}$$

Now, we define the spaces of processes which will be used in the present work.

- $\mathbb{H}^2$ is the space of all $\mathbb{R}^d$-valued and $\mathcal{F}_t$-progressively measurable process such that

$$\|Z\|^2_{\mathbb{H}^2} := \mathbb{E}\int_0^T |Z_s|^2 ds < +\infty, \ \mathbb{P}\text{-a.s.}$$

- $\mathbb{H}^2_\eta$ is the space of all predictable processes such that

$$\|K\|^2_\eta := \mathbb{E}\int_0^T \int_{\mathbb{E}} |K_s(e)|^2 \eta(de, ds) < +\infty, \ \mathbb{P}\text{-a.s.}$$

- For $k, \bar{k}$ in the space $\mathbb{L}^0(\mathcal{B}(E), \eta)$ of all $\mathcal{B}(E)$-measurable functions with the topology of convergence in measure, we define

$$|k - \bar{k}|^2_t = \int_E |k(e) - \bar{k}(e)|^2 \zeta(t, e)\lambda(de).$$

- For $u = (x, y, z, k) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d\times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta)$, we set $\|u\|^2 := |x|^2 + |y|^2 + \|z\|^2 + |k|^2_t$

Finally, we will assume W and η jointly have the weak predictable representation property with respect to the filtration $(\mathcal{F}_t)_{0\le t\le T}$. This means that every square integrable martingale $\mathcal{M}$ has a the following representation,

$$\mathcal{M} = \mathcal{M}_0 + \int Z dB + K \star \tilde{\pi},$$

where $Z \in \mathbb{H}^2$ and $K \in \mathbb{H}^2_\eta$.
Finally, we set $\mathbb{R}^{m+m+m\times m} = \mathbb{R}^m \times \mathbb{R}^m \times L(\mathbb{R}^m, \mathbb{R}^m)$. For $x, y \in \mathbb{R}^d$, $x.y$ denotes the scalar product and for $x, y \in L(\mathbb{R}^m, \mathbb{R}^m)$, $[x, y] = \sum_1^m x^j y^j$ where x^j (resp. y^j) refers to the j-th columns of x (resp. y). However, we suppressed the bracket for notational simplicity.

4.3 The system of forward-backward SDE with jumps of Mckean-Vlasov type

In this section, we study the solvability of the following fully coupled mean-field forward backward SDE with jumps driven by a Brownian motion B and an integer valued independent random jump measure $\tilde{\pi}$

$$\textbf{(S)}\begin{cases} X_t = X_0 + \displaystyle\int_0^t b_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)})ds + \int_0^t \sigma_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)}))dW_s \\ \qquad + \displaystyle\int_0^t \int_E \beta(s, X_{s-}, Y_{s^-}, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)})\tilde{\pi}(ds, de), \quad t \in [0, T], \ \mathbb{P}\text{-a.s.} \\ Y_t = g(X_T, \mathbb{P}_{X_T}) + \displaystyle\int_t^T h_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)})ds - \int_t^T Z_s dW_s - \int_t^T \int_E K_s(e)\tilde{\pi}(ds, de). \end{cases}$$

where (X, Y, Z, K) is an $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta)$-valued adapted processes and $\mathbb{P}_{(X_t,Y_t)}$ is the marginal distribution of (X_t, Y_t).
We require that the coefficients of the system **(S)** satisfy the following assumptions.

Assumption 4.3.1. ***Lipschitz Assumptions***

1- The functions b, h, σ and β are Lipschitz in (x, y, z, k, ν) i.e. there exists a constant $C > 0$ such that for all $t \in [0, T]$, $u = (x, y, z, k)$, $u' = (x', y', z', k') \in \mathbb{R}^{d+d+d\times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta)$ and $\nu, \nu' \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\begin{aligned}&|b(t,u,\nu) - b(t,u',\nu')| + |h(t,u,\nu) - h(t,u',\nu')| + |\sigma(t,u,\nu) - \sigma(t,u',\nu')|\\&+ |\beta(t,u,\nu) - \beta(t,u',\nu')| \leq C\Big[|x-x'| + |y-y'| + \|z-z'\| + |k-k'|_t + \mathcal{W}_2(\nu,\nu')\Big].\end{aligned}$$

2- The function $g : \Omega \times \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d) \to \mathbb{R}^d$ is Lipschitz in (x, μ) i.e. there exists $C > 0$ such that for all $x,\ x' \in \mathbb{R}^d$ and for all $\mu,\ \mu' \in \mathcal{M}_2(\mathbb{R}^d)$,

$$|g(x,\mu) - g(x',\mu')| \leq C(|x-x'| + \mathcal{W}_2(\mu,\mu')),\ \mathbb{P}\text{-a.s.} \tag{4.2}$$

3- For $\phi \in \{b, h, g, \sigma, \beta\}$, ϕ is Lipschitz with respect to x, y, z, k and ν with C^x_ϕ, C^y_ϕ, C^z_ϕ, C^k_ϕ and C^ν_ϕ as the Lipschitz constants.

For $u = (x, y, z, k)$ and $u' = (x', y', z', k') \in \mathbb{R}^{d+d+d\times d} \times \mathbb{L}^0(\mathcal{B}(E), \eta)$, $\nu \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d)$ we define the operator $\mathcal{A}$ in the following way

$$\begin{aligned}\mathcal{A}(t,u,u',\nu) &= (b(s,u,\nu) - b(s,u',\nu)).(y-y') + (h(s,u,\nu) - h(s,u',\nu)).(x-x')\\&+ [(\sigma(s,u,\nu) - \sigma(s,u',\nu)), (z-z')]\\&+ \int_E (\beta(s,u,\nu) - \beta(s,u',\nu))(k-k')(e)\eta(ds,de).\end{aligned}$$

4.3.1 Existence and uniqueness under (H1)

In this part, we shall prove the existence and uniqueness of the solution of the system **(S)** under the following assumption:

(H1)
$$\begin{cases}(i) \text{ There exists } k > 0, \text{ s.t } \forall t \in [0,T], \nu \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d), u, u' \in \mathbb{R}^{d+d+d\times d} \times \mathbb{L}^0(\mathcal{B}(E),\eta),\\ \qquad \mathcal{A}(t,u,u',\nu) \leq -k|x-x'|^2, \mathbb{P}\text{-a.s.}\\(ii) \text{ There exists } k' > 0, \text{ s.t } \forall \nu \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d), x, x' \in \mathbb{R}^d\\ \qquad (g(x,\nu) - g(x',\nu)).(x-x') \geq k'|x-x'|^2, \mathbb{P}\text{-a.s.}\end{cases}$$

We start by giving a key estimate for the difference of two solutions of the mean-field fully coupled FBSDEs with jumps **(S)** satisfying **(H1).**

Lemma 4.3.2. *Let $(Y^{'}, Z^{'}, K^{'})$ another solution of the system* ***(S)****. Then, under $(\textbf{\textit{H1}})$, we have the following estimates*

$$\mathbb{E}[|Y_s - Y_s^{'}|^2] \leq \Theta^1 \mathbb{E}[|X_T - X_T^{'}|^2] + \Theta^2 \int_0^T \mathbb{E}|X_s - X_s^{'}|^2 ds, \tag{4.3}$$

$$\mathbb{E}[\int_0^T [|Z_s - Z_s^{'}|^2 + |K_s - K_s^{'}|_s^2] ds \leq \bar{\Theta}^1 \mathbb{E}[|X_T - X_T^{'}|^2] + \bar{\Theta}^2 \int_0^T \mathbb{E}|X_s - X_s^{'}|^2 ds, \tag{4.4}$$

where

$$\begin{cases} \bar{\Theta}^1 = 2[(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)]\Theta^1 + 2(C_g^x + C_g^\nu)^2 \\ \bar{\Theta}^2 = 2[(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)]\Theta^2 + 2(C_h^x + C_h^\nu) \\ \Theta^1 = e^{(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)T}(C_g^x + C_g^\nu)^2 \\ \Theta^2 = e^{(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)T}(C_h^x + C_h^\nu). \end{cases}$$

Proof. For simplicity, we shall make the following notations that will be used all along this chapter: $\Delta X = X' - X,\ \Delta Y = Y' - Y,\ \Delta Z = Z' - Z,\ \Delta K = K' - K$.

• We start by proving the first estimate. Let us consider the following processes

$$\begin{aligned} \zeta_s^1 &= \frac{h(X_s', Y_s', Z_s', K_s', \mathbb{P}_{(X_s', Y_s')}) - h(X_s, Y_s', Z_s', K_s', \mathbb{P}_{(X_s', Y_s')})}{X_s' - X_s} 1_{\{X_s \neq X_s'\}} \\ \zeta_s^2 &= \frac{h(X_s, Y_s', Z_s', K_s', \mathbb{P}_{(X_s', Y_s')}) - h(X_s, Y_s, Z_s', K_s', \mathbb{P}_{(X_s', Y_s')})}{Y_s' - Y_s} 1_{\{Y_s \neq Y_s'\}} \\ \zeta_s^3 &= \frac{h(X_s, Y_s, Z_s', K_s', \mathbb{P}_{(X_s', Y_s')}) - h(X_s, Y_s, Z_s, K_s', \mathbb{P}_{(X_s', Y_s')})}{\|Z_s' - Z_s\|^2} (Z_s^{'} - Z_s) 1_{\{Z_s \neq Z_s'\}}, \end{aligned}$$

which are respectively bounded by C_h^x, C_h^y, C_h^z due to the Lipschitz assumption on h. We apply Itô's formula to the process $|\Delta Y|^2$ and we obtain

$$\begin{aligned} &\mathbb{E}[|\Delta Y_t|^2] = \mathbb{E}[g(X_T', \mathbb{P}_{X_T'}) - g(X_T, \mathbb{P}_{X_T})]^2 + 2\mathbb{E}\int_t^T \Delta Y_s[\zeta_s^1 \Delta X_s + \zeta_s^2 \Delta Y_s + \zeta_s^3 \Delta Z_s] ds \\ &- \mathbb{E}[\int_t^T \int_E \Big(|\Delta Y_{s^-} + \Delta K_s(e)|^2 - |\Delta Y_{s^-}|^2 \Big) \tilde{\pi}(de, ds) - \int_t^T \|\Delta Z_s\|^2 ds] \\ &- \mathbb{E}[\int_t^T \int_E \Big(|\Delta Y_{s^-} + \Delta K_s(e)|^2 - |\Delta Y_{s^-}|^2 - 2|\Delta Y_{s^-} \Delta K_s(e)| \Big) \eta(de, ds) - \int_t^T 2\Delta Y_s \Delta Z_s dW_s] \\ &+ 2\mathbb{E}[\int_t^T \Delta Y_s \big[h(U_s^{'}, \mathbb{P}_{(X_s', Y_s')}) - h(U_s^{'}, \mathbb{P}_{(X_s, Y_s)}) \big] ds \\ &+ 2\mathbb{E}[\int_t^T \Delta Y_s \big[h(X_s, Y_s, Z_s, K_s', \mathbb{P}_{(X_s', Y_s')}) - h(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s', Y_s')}) \big] ds. \end{aligned}$$

Since the stochastic integrals are true martingales, we conclude that

$$\begin{aligned}
\mathbb{E}[|\Delta Y_t|^2]+&\mathbb{E}[\int_t^T \|\Delta Z_s\|^2 ds] + \mathbb{E}[\int_t^T \int_E |\Delta U_s(e)|^2 \eta(de, ds)] \\
&= \mathbb{E}[|g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T})|^2] + 2\mathbb{E}\int_t^T \Delta Y_s[\zeta_s^1 \Delta X_s + \zeta_s^2 \Delta Y_s + \zeta_s^3 \Delta Z_s] ds \\
&+ \mathbb{E}[\int_t^T \Delta Y_s[h(X_s, Y_s, Z_s, K'_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X'_s, Y'_s)})] ds] \\
&+ 2\mathbb{E}[\int_t^T \Delta Y_s[(U_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(U'_s, \mathbb{P}_{(X_s, Y_s)})] ds]. \qquad (4.5)
\end{aligned}$$

Using the Lipschitz property of h, we obtain that

$$\begin{aligned}
\mathbb{E}[|\Delta Y_t|^2 + \int_t^T \|\Delta Z_s\|^2 ds + \int_t^T \int_E |\Delta K_s(e)|^2 \eta(de, ds)] &\leq \mathbb{E}[|g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T})|^2] \\
&+ 2\mathbb{E}\int_t^T \Delta Y_s[C_h^x|\Delta X_s| + C_h^y|\Delta Y_s| + C_h^z|\Delta Z_s| + C_h^k|\Delta K_s|_s] ds \\
&+ 2\mathbb{E}[\int_t^T \Delta Y_s[h(U'_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(U'_s, \mathbb{P}_{(X_s, Y_s)})] ds. \qquad (4.6)
\end{aligned}$$

Notice that, in one hand, we have

$$2\Delta Y_s[h(U'_s, \mathbb{P}_{(X'_s, Y'_s)}) - h(U'_s, \mathbb{P}_{(X_s, Y_s)})] \leq 2C_h^\nu|\Delta Y_s|(\sqrt{\mathbb{E}[|\Delta X_s|^2]} + \sqrt{\mathbb{E}[|\Delta Y_s|^2]}), \qquad (4.7)$$

and in the other hand, we have

$$\begin{aligned}
&2\Delta Y_s[C_h^x|\Delta X_s| + C_h^y|\Delta Y_s| + C_h^z|\Delta Z_s| + C_h^k|\Delta K_s|_s] ds \qquad (4.8)\\
&\leq C_h^x|\Delta Y_s|^2 + C_h^x|\Delta X_s|^2 + (C_h^z)^2|\Delta Y_s|^2 + |\Delta Z_s|^2 + (C_h^k)^2|\Delta Y_s|^2 + |\Delta K|_s^2 + 2C_h^y|\Delta Y_s|^2.
\end{aligned}$$

Moreover, by Young inequality and the Lispchitz property on g we obtain

$$\begin{aligned}
|g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T})|^2 &= |g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X'_T}) + g(X_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T})|^2 \\
&\leq |C_g^x|X'_T - X_T| + C_g^\nu \mathcal{W}_2(\mu', \mu)|^2 \\
&\leq (C_g^x)^2|\Delta X_T|^2 + (C_g^\nu)^2|\Delta X_T|^2 + 2C_g^x C_g^\nu|\Delta X_T|\mathcal{W}_2(\mu', \mu) \\
&\leq (C_g^x)^2|\Delta X_T|^2 + (C_g^\nu)^2|\Delta X_T|^2 + C_g^x C_g^\nu|\Delta X_T|^2 + C_g^x C_g^\nu \mathcal{W}_2^2(\mu', \mu) \\
&\leq (C_g^x)^2|\Delta X_T|^2 + (C_g^\nu)^2|\Delta X_T|^2 + 2C_g^x C_g^\nu|\Delta X_T|^2 \\
&\leq (C_g^x + C_g^\nu)^2|\Delta X_T|^2. \qquad (4.9)
\end{aligned}$$

Now, plugging (4.9), (4.8) and (4.7) in (4.6) yields

$$\mathbb{E}[|\Delta Y_t|^2] \leq (C_g^x + C_g^\nu)^2\mathbb{E}[|\Delta X_T|^2] + \int_t^T (C_h^x + C_h^\nu)\mathbb{E}|\Delta X_s|^2 ds$$
$$+ \mathbb{E}[\int_t^T (C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)|\Delta Y_s|^2 ds].$$

Finally, Gronwall's lemma implies

$$\mathbb{E}[|\Delta Y_s|^2] \leq e^{[(C_h^x+(C_h^z)^2+(C_h^k)^2+2C_h^\nu+2C_h^y)]T}\Big[(C_g^x + C_g^\nu)^2\mathbb{E}[|\Delta X_T|^2] + (C_h^x + C_h^\nu)\int_0^T \mathbb{E}|\Delta X_s|^2 ds\Big],$$

and we obtain the following inequality

$$\mathbb{E}[|\Delta Y_s|^2] \leq \Theta^1\mathbb{E}[|\Delta X_T|^2] + \Theta^2\int_0^T \mathbb{E}|\Delta X_s|^2 ds. \tag{4.10}$$

• Let us now prove the second estimate. Recalling (4.6) and noting that

$$\begin{cases} 2C_h^z|\Delta Y_s||\Delta Z_s| \leq 2(C_h^z)^2|\Delta Y_s|^2 + \frac{1}{2}|\Delta Z_s|^2 \\ 2C_h^k|\Delta Y_s||\Delta K_s|_s \leq 2(C_h^k)^2|\Delta Y_s|^2 + \frac{1}{2}|\Delta K_s|_s^2, \end{cases}$$

we obtain

$$\frac{1}{2}\mathbb{E}[\int_t^T [|\Delta Z_s|^2 + |\Delta K_s|_s^2]ds] \leq (C_g^x + C_g^\nu)^2\mathbb{E}[|\Delta X_T|^2] + \int_t^T (C_h^x + C_h^\nu)\mathbb{E}|\Delta X_s|^2 ds$$
$$+ \mathbb{E}[\int_t^T (C_h^x + 2(C_h^z)^2 + 2(C_h^k + C_h^\nu)^2 + 2C_h^y)|\Delta Y_s|^2 ds].$$

Henceforth, making the following notations

$$\bar\Theta^1 = 2[(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)]\Theta^1 + 2(C_g^x + C_g^\nu)^2$$
$$\bar\Theta^2 = 2[(C_h^x + (C_h^z)^2 + (C_h^k)^2 + 2C_h^\nu + 2C_h^y)]\Theta^2 + 2(C_h^x + C_h^\nu),$$

we obtain the desired result

$$\mathbb{E}[\int_t^T (|\Delta Z_s|^2 + |\Delta K|_s^2)ds \leq \bar\Theta^1\mathbb{E}[|\Delta X_T|^2] + \bar\Theta^2\int_0^T \mathbb{E}|\Delta X_s|^2 ds. \tag{4.11}$$

These previous estimates allow us to prove the following uniqueness result of the solution of the mean-field FBSDE with jumps **(S)**.

Proposition 4.3.3. *Under **(H1)**, there exists a unique solution $U = (X, Y, Z, K)$ of the mean field FBSDE with jumps **(S)**.*

Proof. Suppose that **(S)** has another solution $U' = (X', Y', Z', K')$. Applying Itô's formula to the product $\Delta X_t \Delta Y_t$ gives

$$d(\Delta X_t \Delta Y_t) = \Delta X_t d(\Delta Y_t) + \Delta Y_t d(\Delta X_t) + d\langle \Delta X_t, \Delta Y_t \rangle_t$$

Taking the conditional expectation, we obtain

$$\begin{aligned}
\Gamma_T = \mathbb{E}[\Delta X_T \Delta Y_T] &= \mathbb{E}\Big[\int_0^T \{(b(s, U_s, \nu_s) - b(s, U'_s, \nu'_s))\Delta Y_s \\
&+ (h(s, U_s, \nu_s) - h(s, U'_s, \nu'_s)\Delta X_s + (\sigma(s, U_s, \nu_s) - \sigma(s, U'_s, \nu'_s))\Delta Z_s\} ds \\
&+ \int_0^T \int_E (\beta(s, U_s, \nu_s) - \beta(s, U'_s, \nu'_s))\Delta K_s \eta(ds, de)\Big] + \mathbb{E}[\int_0^T \Delta X_s \Delta Z_s dW_s] \\
&+ \int_0^T \int_E \Delta X_s \Delta K_s \tilde{\pi}(ds, de)] + \mathbb{E}[\int_0^T \Delta Y_s(\sigma(s, U_s, \nu) - \sigma(s, U'_s, \nu'_s)dW_s] \\
&+ \mathbb{E}[\int_0^T \int_E \Delta Y_s(\beta(s, U_s, \nu_s) - \beta(s, U'_s, \nu'_s))\tilde{\pi}(de, ds)].
\end{aligned}$$

Let us observe that the local martingale $\int_0^t \Delta X_s \Delta Z_s dW_s + \int_0^T \Delta X_s \Delta K_s \tilde{\pi}(ds, de)$ is a true $(\mathbb{P}, \mathcal{F})$ martingale. Indeed, using the BDG inequality with the help of the square integrability of ΔY, ΔZ and ΔK, we get

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T} |\int_0^t \Delta X_s \Delta Z_s dW_s|\Big] &\le C\mathbb{E}[\sup_{0\le t\le T} |\Delta X_t|^2 \int_0^T |\Delta Z_s|^2 ds]^{\frac{1}{2}} \\
&\le C(\mathbb{E}[\sup_{0\le t\le T} |\Delta X_t|^2] + \mathbb{E}[\int_0^T |\Delta Z_s|^2 ds]) < +\infty,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\sup_{0\le t\le T} |\int_0^t \int_E \Delta X_s \Delta K_s \tilde{\pi}(de, ds)] &\le C\mathbb{E}[\int_0^T \int_E |\Delta X_s \Delta K_s|^2 \eta(de, ds)]^{\frac{1}{2}} \\
&\le C(\mathbb{E}[\sup_{0\le t\le T} |\Delta X_t|^2] + \mathbb{E}[\int_0^T \int_E |\Delta K_s|^2 \eta(de, ds)]) < +\infty.
\end{aligned}$$

In the same way, we can prove that

$$\int_0^T \Delta Y_s(\sigma(s, U_s, \nu) - \sigma(s, U'_s, \nu'_s)dW_s + \int_0^T \int_E \Delta Y_s(\beta(s, U_s, \nu_s) - \beta(s, U'_s, \nu'_s)\tilde{\pi}(de, ds),$$

is a $(\mathbb{P}, \mathcal{F})$-martingale. Afterwards, we study each term separately. Let us start by the term

$\Delta X_T \Delta Y_T$: In one hand, using **(H1)**, we make the following computation

$$\begin{aligned}\Gamma_T &= \mathbb{E}[(\Delta X_T)(g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T}))] \\ &\geq \mathbb{E}[k'|\Delta X_T|^2 - C_g^\nu(|\Delta X_T|.\mathcal{W}_2(\mathbb{P}_{X'_T}, \mathbb{P}_{X_T}))] \\ &\geq k'\mathbb{E}[|\Delta X_T|^2] - C_g^\nu \mathbb{E}[|\Delta X_T|]\mathbb{E}[|\Delta X_T|^2]^{\frac{1}{2}} \\ &\geq (k' - C_g^\nu)\mathbb{E}[|\Delta X_T|^2]. \end{aligned} \tag{4.12}$$

On the other hand, we have

$$\begin{aligned}\Gamma_T \leq \mathbb{E}\Big[\int_0^T &\{\mathcal{A}(s, U_s, U'_s, \nu_s) + ((b(s, U'_s, \nu_s) - b(s, U'_s, \nu'_s)).\Delta Y_s \\ &+ (h(s, U'_s, \nu_s) - h(s, U'_s, \nu'_s)).\Delta X_s + (\sigma(s, U'_s, \nu_s) - \sigma(s, U'_s, \nu'_s))\Delta Z_s\}ds \\ &+ \int_0^T \int_E (\beta(s, U'_s, \nu_s) - \beta(s, U'_s, \nu'_s))\Delta K_s \eta(ds, de)\Big].\end{aligned}$$

The Lipschitz assumption together with Young inequality ($ab \leq \frac{1}{2}(a^2 + b^2)$) imply that

$$\begin{aligned}\Gamma_T \leq\; &\mathbb{E}\Big[\int_0^T [\mathcal{A}(s, U_s, U_s^{'}, \nu) + (C_h^\nu|\Delta X_s| + C_f^\nu|\Delta Y_s| + C_\sigma^\nu|\Delta Z_s| + C_\beta^\nu|\Delta K_s|_s)\mathcal{W}_2(\nu_s, \nu_s^{'})]ds\Big] \\ \leq\; &\mathbb{E}[\int_0^T -k|\Delta X_s|^2 ds] + \frac{1}{2}\mathbb{E}[\int_0^T (C_h^\nu|\Delta X_s|^2 + C_h^\nu \mathcal{W}_2^2(\nu_s^{'}, \nu_s) + C_b^\nu|\Delta Y_s|^2 + C_b^\nu \mathcal{W}_2^2(\nu_s^{'}, \nu_s))ds] \\ &+ \frac{1}{2}\mathbb{E}[\int_0^T (C_\sigma^\nu\|\Delta Z_s\|^2 + C_\sigma^\nu \mathcal{W}_2^2(\nu_s^{'}, \nu_s))ds + \frac{1}{2}\mathbb{E}[\int_0^T (C_\beta^\nu|\Delta K_s|_s^2 + C_\beta^\nu \mathcal{W}_2^2(\nu_s^{'}, \nu_s))ds].\end{aligned}$$

Using the following inequality

$$\mathcal{W}_2^2(\nu'_s, \nu_s) \leq \mathbb{E}[|\Delta X_s|^2] + E[|\Delta Y_s|^2], \tag{4.13}$$

we obtain that

$$\begin{aligned}\Gamma_T \leq\; &\mathbb{E}[\int_0^T -k|\Delta X_s|^2 ds\Big] + \mathbb{E}[\int_0^T [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)]|\Delta X_s|^2 ds] \\ &+ \mathbb{E}[\int_0^T [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)]|\Delta Y_s|^2 ds] \\ &+ \frac{1}{2}C_\sigma^\nu \mathbb{E}[\int_0^T \|\Delta Z_s\|^2 ds] + \frac{1}{2}C_\beta^\nu \mathbb{E}[\int_0^T |\Delta K_s|_s^2 ds].\end{aligned}$$

Now, using the estimates in Lemma 4.3.5, we get

$$
\begin{aligned}
\Gamma_T \leq \mathbb{E}\Big[& \int_0^T (-k + [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)]|\Delta X_s|^2 ds] \\
& + [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)]\Big(\Theta^1 \mathbb{E}[|\Delta X_T|^2] + \Theta^2 \int_0^T \mathbb{E}|\Delta X_s|^2 ds\Big) \\
& + \frac{1}{2}(C_\sigma^\nu \vee C_\beta^\nu)(\bar{\Theta}^1 \mathbb{E}[|\Delta X_T|^2] + \bar{\Theta}^2 \int_0^T \mathbb{E}|\Delta X_s|^2 ds)\Big].
\end{aligned}
$$

Hence, (4.12) gives that

$$
\begin{aligned}
&(k' - C_g^\nu)\mathbb{E}[|\Delta X_T|^2] \leq \Big(\Theta^1 [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)] + \frac{1}{2}(C_\sigma^\nu \vee C_\beta^\nu)\bar{\Theta}^1\Big)\mathbb{E}[|\Delta X_T|^2] \\
&+ \Big(-k + [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)] + [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)]\Theta^2 + \frac{1}{2}(C_\sigma^\nu \vee C_\beta^\nu)\bar{\Theta}^2\Big)\mathbb{E}[\int_0^T |\Delta X_s|^2 ds].
\end{aligned} \tag{4.14}
$$

Henceforth, taking

$$
\begin{aligned}
&(k' - C_g^\nu) \geq \Theta^1 [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)] + \frac{1}{2}(C_\sigma^\nu \vee C_\beta^\nu)\bar{\Theta}^1, \\
&k \geq [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)] + [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)]\Theta^2 + \frac{1}{2}(C_\sigma^\nu \vee C_\beta^\nu)\bar{\Theta}^2,
\end{aligned}
$$

we obtain that $X_T = X'_T$, and $\forall t \in [0,T], X'_t = X_t$ $\mathbb{P}$-a.s. Hence, (Y,Z,K) and (Y,Z,K) are two solutions of

$$
Y_t = g(X_T, \mathbb{P}_{X_T}) + \int_t^T h_s(X_s, Y_s, Z_s, K_s, \mathbb{P}_{(X_s,Y_s)})ds - \int_t^T Z_s dW_s - \int_t^T \int_E K_s(e)\tilde{\pi}(ds,de).
$$

However, this mean field BSDEs with jumps admits a unique solution (see [94]). Therefore, the system **(S1)** admits a unique solution.

Theorem 4.3.4. *Under Assumption **(H1)**, there exists a solution $U = (X,Y,Z,K)$ of the mean field FBSDE with jumps system **(S)**.*

Proof. In order to prove the existence of the solution, we use an approximation scheme based on perturbations of the forward equation. Let $\delta \in]0,1]$ and consider a sequence (X^n, Y^n, Z^n, K^n) of processes defined recursively by $(X^0, Y^0, Z^0, K^0) = (0,0,0,0)$ and for $n \geq 1, U^n = (X^n, Y^n, Z^n, K^n)$ satisfies

$$
\begin{cases}
X_t^{n+1} = X_0 + \int_0^t [b_s(U_s^{n+1}, \nu_s^n) - \delta(Y_s^{n+1} - Y_s^n)]ds + \int_0^t [\sigma_s(U_s^{n+1}, \nu_s^n) - \delta(Z_s^{n+1} - Z_s^n)]dW_s \\
\qquad + \int_0^t \int_E \Big(\beta_s(U_s^{n+1}, \nu_s^n) - \delta(K_s^{n+1} - K_s^n)\Big)\tilde{\pi}(ds,de), \\
Y_t^{n+1} = g(X_T^{n+1}, \mu_T^n) - \int_t^T h_s(U_s^{n+1}, \nu_s^n)ds - \int_t^T Z_s^{n+1} dW_s - \int_t^T \int_E K_s^{n+1}(e)\tilde{\pi}(ds,de).
\end{cases} \tag{4.15}
$$

Hereafter, we shall use the following simplified notations: For $n \geq 1$, $t \in [0,T]$, we set

$$\hat{X}_t^{n+1} := X_t^{n+1} - X_t^n, \quad \hat{Y}_t^{n+1} := Y_t^{n+1} - Y_t^n, \quad \hat{Z}_t^{n+1} := Z_t^{n+1} - Z_t^n, \quad \hat{K}_t^{n+1} := K_t^{n+1} - K_t^n$$

and for a function $\phi = \{b, h, \sigma, \beta\}$, we set

$$\hat{\phi}_t^{n+1} := \phi(t, U_t^{n+1}, \nu_t^n) - \phi(t, U_t^n, \nu_t^{n-1}), \quad \tilde{\phi}_t^n := \phi(t, U_t^n, \nu_t^n) - \phi(t, U_t^n, \nu_t^{n-1}).$$

We first apply Itô's formula to the product $\hat{X}^{n+1}\hat{Y}^{n+1}$

$$\begin{aligned}
&\mathbb{E}[\hat{X}_T^{n+1}\hat{Y}_T^{n+1}] = \mathbb{E}[\int_0^T \hat{Y}_s^{n+1}[\hat{b}_s^{n+1} - \delta(\hat{Y}_s^{n+1} - \hat{Y}_s^n)]ds + \mathbb{E}[\int_0^T \hat{Y}_s^{n+1}[\hat{\sigma}_s^{n+1} - \delta(\hat{Z}_s^{n+1} - \hat{Z}_s^n)]dW_s] \\
&\quad + \mathbb{E}[\int_0^T \int_E \hat{Y}_s^{n+1}[\hat{\beta}_s^{n+1} - \delta(\hat{K}_s^{n+1}(e) - \hat{K}_s^n(e)]\tilde{\pi}(de, ds) + \mathbb{E}[\int_0^T \hat{X}_s^{n+1}\hat{h}_s^{n+1}ds] \\
&\quad - \mathbb{E}[\int_0^T \hat{X}_s^{n+1}\hat{Z}_s^{n+1}dW_s] - \mathbb{E}[\int_0^T \int_E \hat{X}_s^{n+1}\hat{K}_s^{n+1}(e)\tilde{\pi}(de, ds)] \\
&\quad + \mathbb{E}[\int_0^T (\hat{\sigma}_s^{n+1} - \delta(\hat{Z}_s^{n+1} - \hat{Z}_s^n), \hat{Z}_s^{n+1})ds] + \mathbb{E}[\int_0^T \int_E \hat{K}_s^{n+1}(\hat{\beta}_s^{n+1} - \delta(\hat{K}_s^{n+1} - \hat{K}_s^n))\eta(de, ds)].
\end{aligned}$$

Using the BDG inequality, we can easily see that the stochastic integrals in the above expression are a true martingale. Hence we obtain

$$\begin{aligned}
&\mathbb{E}[\hat{X}_T^{n+1}\hat{Y}_T^{n+1}] = \mathbb{E}[\int_0^T \hat{Y}_s^{n+1}[\hat{b}_s^{n+1} - \delta(\hat{Y}_s^{n+1} - \hat{Y}_s^n)]ds + \mathbb{E}[\int_0^T \hat{X}_s^{n+1}\hat{h}_s^{n+1}ds] \\
&\quad + \mathbb{E}[\int_0^T (\hat{\sigma}_s^{n+1} - \delta(\hat{Z}_s^{n+1} - \hat{Z}_s^n), \hat{Z}_s^{n+1})ds] + \mathbb{E}[\int_0^T \int_E \hat{K}_s^{n+1}(\hat{\beta}_s^{n+1} - \delta(\hat{K}_s^{n+1} - \hat{K}_s^n))\eta(de, ds)].
\end{aligned}$$

Rearranging terms, we get

$$\begin{aligned}
&\delta\mathbb{E}[\int_0^T \hat{Y}_s^{n+1}\hat{Y}_s^n ds + \int_0^T \hat{Z}_s^{n+1}\hat{Z}_s^n ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{K}_s^n \eta(de, ds)] = \mathbb{E}[\hat{X}_T^{n+1}\hat{Y}_T^{n+1}] \\
&- \mathbb{E}[\int_0^T \hat{X}_s^{n+1}\hat{h}_s^{n+1} + \hat{Y}_s^{n+1}\hat{b}_s^{n+1} + \hat{Z}_s^{n+1}\hat{\sigma}_s^{n+1}]ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds)]. \\
&+ \delta\mathbb{E}[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds].
\end{aligned} \tag{4.16}$$

Since $\hat{X}_T^{n+1}\hat{Y}_T^{n+1} = \hat{X}_T^{n+1}[g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^{n-1})]$, we have from **(H1)**

$$\begin{aligned}
\hat{X}_T^{n+1}\hat{Y}_T^{n+1} &= \hat{X}_T^{n+1}[g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^{n-1})] \\
&= \hat{X}_T^{n+1}[g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^n)] + \hat{X}_T^{n+1}[g(X_T^n, \mu_T^n) - g(X_T^n, \mu_T^{n-1})] \\
&\geq k'|\hat{X}_T^n|^2 - C_g^\nu|\hat{X}_T^n|\mathcal{W}_2(\mu_T^n, \mu_T^{n-1}).
\end{aligned}$$

Using (4.1) and the elementary inequality : $\forall \epsilon > 0,\ 2ab \leq \epsilon^{-1}a^2 + \epsilon b^2$, we obtain

$$\mathbb{E}[\hat{X}_T^{n+1}\hat{Y}_T^{n+1}] \geq (k^{'} - \frac{C_g^{\nu}\epsilon}{2})\mathbb{E}[|\hat{X}_T^{n+1}|^2] - \frac{C_g^{\nu}}{2\epsilon}\mathbb{E}[|\hat{X}_T^{n}|^2]. \tag{4.17}$$

In the other hand, using once again **(H1)**

$$\begin{aligned}
&\int_0^T [\hat{X}_s^{n+1}\hat{h}_s^{n+1} + \hat{Y}_s^{n+1}\hat{b}_s^{n+1} + \hat{Z}_s^{n+1}\hat{\sigma}_s^{n+1}]ds + \int_0^T\int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de,ds) \\
&= \int_0^T [\mathcal{A}(s, U_s^{n+1}, U_s^{n}, \nu^n) + \hat{Y}_s^{n+1}\bar{b}_s^{n} + \hat{X}_s^{n+1}\bar{h}_s^{n} + \hat{Z}_s^{n+1}\bar{\sigma}_s^{n}]ds + \int_0^T\int_E \hat{K}_s^{n+1}\bar{\beta}_s^{n}\eta(de,ds). \\
&\leq -k\int_0^T |\hat{X}_s^{n+1}|^2 ds + \int_0^T [C_h^{\nu}|\hat{X}_s^{n+1}| + C_b^{\nu}|\hat{Y}_s^{n+1}| + C_\sigma^{\nu}\|\hat{Z}_s^{n+1}\| + C_\beta^{\nu}|\hat{K}_s^{n+1}|_s]\mathcal{W}_2(\nu^n, \nu^{n-1})ds.
\end{aligned}$$

Using Young inequality : $\forall \tilde{\epsilon} > 0,\ 2ab \leq \tilde{\epsilon}^{-1}a^2 + \tilde{\epsilon}b^2$, we obtain

$$\begin{aligned}
&\int_0^T [\hat{X}_s^{n+1}\hat{h}_s^{n+1} + \hat{Y}_s^{n+1}\hat{b}_s^{n+1} + \hat{Z}_s^{n+1}\hat{\sigma}_s^{n+1}]ds + \int_0^T\int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de,ds) \\
&\leq (\frac{\tilde{\epsilon}C_h^{\nu}}{2} - k)\int_0^T |\hat{X}_s^{n+1}|^2 ds + \frac{\tilde{\epsilon}}{2}\int_0^T (C_b^{\nu}|\hat{Y}_s^{n+1}|^2 + C_\sigma^{\nu}\|\hat{Z}_s^{n+1}\|^2 + C_\beta^{\nu}|\hat{K}_s^{n+1}|_s^2)ds \\
&+ \frac{C_h^{\nu} + C_b^{\nu} + C_\sigma^{\nu} + C_\beta^{\nu}}{2\tilde{\epsilon}}\mathcal{W}_2^2(\nu_s^n, \nu_s^{n-1}).
\end{aligned}$$

Notice that $\mathcal{W}_2^2(\nu_s^n, \nu_s^{n-1}) \leq \mathbb{E}[|\hat{X}_s^n|^2 + |\hat{Y}_s^n|^2]$. Hence, taking the conditional expectation in the expression above, we obtain

$$\begin{aligned}
&\mathbb{E}[\int_0^T [\hat{X}_s^{n+1}\hat{h}^{n+1}(s) + \hat{Y}_s^{n+1}\hat{b}^{n+1}(s) + \hat{Z}_s^{n+1}\hat{\sigma}^{n+1}(s)]ds + \int_0^T\int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de,ds)] \\
&\leq (\frac{\tilde{\epsilon}C_h^{\nu}}{2} - k)\mathbb{E}[\int_0^T |\hat{X}_s^{n+1}|^2 ds] + \frac{\tilde{\epsilon}}{2}\mathbb{E}[\int_0^T (C_b^{\nu}|\hat{Y}_s^{n+1}|^2 + C_\sigma^{\nu}\|\hat{Z}_s^{n+1}\|^2 + C_\beta^{\nu}|\hat{K}_s^{n+1}|_s^2)ds] \\
&+ \frac{C_h^{\nu} + C_b^{\nu} + C_\sigma^{\nu} + C_\beta^{\nu}}{2\tilde{\epsilon}}\mathbb{E}[\int_0^T (|\hat{X}_s^n|^2 + |\hat{Y}_s^n|^2)ds].
\end{aligned} \tag{4.18}$$

In addition,

$$\begin{aligned}
&\mathbb{E}[\int_0^T (\hat{Y}_s^{n+1}\hat{Y}_s^{n} + \hat{Z}_s^{n+1}\hat{Z}_s^{n})ds + \int_0^T\int_E \hat{K}_s^{n+1}\hat{K}_s^{n}\,\eta(de,ds)] \\
&\leq \frac{\kappa}{2}\mathbb{E}[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds] + \frac{1}{2\kappa}\mathbb{E}[\int_0^T |\hat{Y}_s^{n}|^2 + \|\hat{Z}_s^{n}\|^2 + |\hat{K}_s^{n}|_s^2 ds].
\end{aligned} \tag{4.19}$$

Plugging (4.17), (4.18) and (4.19) in (4.16) we obtain

$$
\begin{aligned}
&(k^{'}-\frac{C_g^{\nu}\epsilon}{2})\mathbb{E}[|\hat{X}_T^{n+1}|^2]-\frac{C_g^{\nu}}{2\epsilon}\mathbb{E}[|\hat{X}_T^{n}|^2]+\delta\mathbb{E}[\int_0^T|\hat{Y}_s^{n+1}|^2+\|\hat{Z}_s^{n+1}\|^2+|\hat{K}_s^{n+1}|_s^2ds]\\
&+(-\frac{\tilde{\epsilon}C_h^{\nu}}{2}+k)\mathbb{E}[\int_0^T|\hat{X}_s^{n+1}|^2ds]-\frac{\tilde{\epsilon}}{2}\mathbb{E}[\int_0^T C_b^{\nu}|\hat{Y}_s^{n+1}|^2+C_\sigma^{\nu}\|\hat{Z}_s^{n+1}\|^2+C_\beta^{\nu}|\hat{K}_s^{n+1}|_s^2ds]\\
&-\frac{C_h^{\nu}+C_b^{\nu}+C_\sigma^{\nu}+C_\beta^{\nu}}{2\epsilon}\mathbb{E}[\int_0^T|\hat{X}_s^{n}|^2+|\hat{Y}_s^{n}|^2ds]\\
&\leq\frac{\delta\kappa}{2}\mathbb{E}[\int_0^T|\hat{Y}_s^{n+1}|^2+\|\hat{Z}_s^{n+1}\|^2+|\hat{K}_s^{n+1}|_s^2ds]+\frac{\delta}{2\kappa}\mathbb{E}[\int_0^T|\hat{Y}_s^{n+1}|^2+\|\hat{Z}_s^{n+1}\|^2+|\hat{K}_s^{n+1}|_s^2ds].
\end{aligned}
$$

Rearranging terms we get

$$
\begin{aligned}
(k^{'}-\frac{C_g^{\nu}\epsilon}{2})\mathbb{E}[|\hat{X}_T^{n+1}|^2]&+(k-\frac{\tilde{\epsilon}C_h^{\nu}}{2})\mathbb{E}[\int_0^T|\hat{X}_s^{n+1}|^2ds]+(\delta-\frac{\kappa\delta}{2}-\frac{\tilde{\epsilon}C_f^{\nu}}{2})\mathbb{E}[\int_0^T|\hat{Y}_s^{n+1}|^2ds]\\
&+(\delta-\frac{\kappa\delta}{2}-\frac{\tilde{\epsilon}C_\sigma^{\nu}}{2})\mathbb{E}[\int_0^T\|\hat{Z}_s^{n+1}\|^2ds]+(\delta-\frac{\kappa\delta}{2}-\frac{\tilde{\epsilon}C_\beta^{\nu}}{2})\mathbb{E}[\int_0^T|\hat{K}_s^{n+1}|_s^2ds]\\
&\leq\frac{C_g^{\nu}}{2\epsilon}\mathbb{E}[|\hat{X}_T^{n}|^2]+\frac{C_h^{\nu}+C_b^{\nu}+C_\sigma^{\nu}+C_\beta^{\nu}}{2\epsilon}\mathbb{E}[\int_0^T|\hat{X}_s^{n}|^2ds]\\
&+\frac{\delta}{2\kappa}E[\int_0^T\|\hat{Z}_s^{n}\|^2+|\hat{K}_s^{n}|_s^2ds]+(\frac{C_h^{\nu}+C_b^{\nu}+C_\sigma^{\nu}+C_\beta^{\nu}}{2\epsilon}+\frac{\delta}{2\kappa})\mathbb{E}[\int_0^T|\hat{Y}_s^{n}|^2ds].
\end{aligned}
$$

Setting

$$
\begin{cases}
\gamma:=min(k^{'}-\frac{C_g^{\nu}\epsilon}{2},k-\frac{\tilde{\epsilon}C_h^{\nu}}{2},(\delta-\frac{\kappa\delta}{2}-\frac{\tilde{\epsilon}C_b^{\nu}}{2}),(\delta-\frac{\kappa\delta}{2}-\frac{\tilde{\epsilon}C_\sigma^{\nu}}{2}),(\delta-\frac{\kappa\delta}{2}-\frac{\tilde{\epsilon}C_\beta^{\nu}}{2}))\\
\theta=max(\frac{C_g^{\nu}}{2\epsilon},-\frac{C_h^{\nu}+C_b^{\nu}+C_\sigma^{\nu}+C_\beta^{\nu}}{2\epsilon}+\frac{\delta}{2\kappa}),
\end{cases}
\tag{4.20}
$$

we obtain that

$$
\mathbb{E}[|\hat{X}_T^{n+1}|^2]+\mathbb{E}[\int_0^T\|\hat{U}_s^{n+1}\|^2ds]\leq\frac{\theta}{\gamma}(\mathbb{E}[|\hat{X}_T^{n}|^2]+\mathbb{E}[\int_0^T\|\hat{U}_s^{n}\|^2ds]).
\tag{4.21}
$$

Choosing $\tilde{\epsilon}$ and ϵ so that $\theta<\gamma$, the inequality becomes a contraction. Thus, $(\hat{X}_T^n)_{n\geq0}$ is a Cauchy sequence in $\mathbb{H}^2(\Omega,\mathbb{P})$ and $(\hat{X}^n)_{n\geq0},(\hat{Y}^n)_{n\geq0},(\hat{Z}^n)_{n\geq0}$ and $(\hat{K}^n)_{n\geq0}$ are Cauchy sequences respectively in $\mathbb{H}^2([0,T],\Omega,dt\otimes d\mathbb{P})$and $\mathbb{H}^2_\eta([0,T],\Omega,dt\otimes d\eta)$. Hence, if X,Y, Z and K are the respective limits of these sequences, passing to the limit in (4.15), we see that (X,Y,Z,K) is a solution of (4.3).

4.3.2 Existence and uniqueness under (H2)

Our second main result is an extension to the case where the datas satisfy a weaker monotonicity assumptions. We adopt here a common strategy which is the Picard approach: we construct a schema based on small perturbation. This helps us to construct the contracting maps and therefore deduce the existence of a unique solution of the system **(S).** Consider the following assumption

$$\textbf{(H2)}\quad \begin{cases} (i) \text{ There exists } k>0, \text{ s.t } \forall t\in[0,T], \nu\in\mathcal{M}_2(\mathbb{R}^d\times\mathbb{R}^d), u,u'\in\mathbb{R}^{d+d+d\times d}\times\mathbb{L}^0(\mathcal{B}(E),\eta),\\ \qquad\qquad \mathcal{A}(t,u,u',\nu)\le -k(|y-y'|^2+||z-z'||^2+|k-k^{'}|_s), \mathbb{P}\text{-a.s.}\\ (ii) \text{ There exists } k'>0, \text{ s.t } \forall\nu\in\mathcal{M}_2(\mathbb{R}^d\times\mathbb{R}^d), x,x'\in\mathbb{R}^d\\ \qquad\qquad (g(x,\nu)-g(x',\nu)).(x-x')\ge k'|x-x'|^2, \mathbb{P}\text{-a.s.} \end{cases}$$

As in the previous section we will give a useful a priori estimate.

Lemma 4.3.5. *Let $(Y^{'},Z^{'},K^{'})$ another solution of the the system **(S)**. Then, under (**H2**) we have the following estimates*

$$\mathbb{E}[\int_0^T|\Delta X_s|^2ds]\le\frac{\exp(T.\Upsilon^1)-\Upsilon^1}{\Upsilon^1}[\Upsilon^2\mathbb{E}[\int_0^T|\Delta Y_s|^2ds+\Upsilon^3\mathbb{E}[\int_0^T|\Delta Z_s|^2ds+\Upsilon^4E[\int_0^T|\Delta K_s|_s^2ds],$$

where

$$\begin{cases} \Upsilon^1:=3+2C_b^x+5(C_\sigma^x)^2+5(C_\beta^x)^2+2C_b^\nu+5(C_\sigma^\nu)^2+5(C_\beta^\nu)^2\\ \Upsilon^2:=(C_b^y)^2+5(C_\sigma^y)^2+5(C_\beta^y)^2+C_b^\nu+5(C_\sigma^\nu)^2+5(C_\beta^\nu)^2\\ \Upsilon^3:=(C_b^z)^2+5(C_\sigma^z)^2+5(C_\beta^x)^2\\ \Upsilon^4:=(C_b^k)^2+5(C_\sigma^k)^2+5(C_\beta^k)^2, \end{cases}$$

Proof. Applying Itô formula to $|\Delta X|^2$, we compute using the Lipschitz assumption

$$\mathbb{E}[|\Delta X_t|^2]\le 2\mathbb{E}[\int_0^t|\Delta X_s|(C_b^x|\Delta X_s|+C_b^y|\Delta Y_s|+C_b^z\|\Delta Z_s\|+C_b^k|\Delta K_s|_{\mathcal{L}^2(\eta)}+C_b^\nu\mathcal{W}_2(\nu_s',\nu_s))ds$$
$$+5\mathbb{E}[\int_0^t[(C_\sigma^x)^2|\Delta X_s|^2+(C_\sigma^y)^2|\Delta Y_s|^2+(C_\sigma^z)^2\|\Delta Z_s\|^2+(C_\sigma^k)^2|\Delta K_s|^2_{\mathcal{L}^2(\eta)}+(C_\sigma^\nu)^2\mathcal{W}_2^2(\nu_s',\nu_s)]ds$$
$$+5\mathbb{E}[\int_0^t\Big[(C_\beta^x)^2|\Delta X_s|^2+(C_\beta^y)^2|\Delta Y_s|^2+(C_\beta^z)^2\|\Delta Z_s\|^2+(C_\beta^k)^2|\Delta K_s|^2_{\mathcal{L}^2(\eta)}+(C_\beta^\nu)^2\mathcal{W}_2(\nu_s',\nu_s)^2\Big]$$

Then, we apply Young inequality and we obtain

$$\mathbb{E}[|\Delta X_t|^2]\le\Big(3+2C_b^x+5(C_\sigma^x)^2+5(C_\beta^x)^2+2C_b^\nu+5(C_\sigma^\nu)^2+5(C_\beta^\nu)^2\Big)\mathbb{E}[\int_0^t|\Delta X_s|^2ds$$
$$+((C_b^y)^2+5(C_\sigma^y)^2+5(C_\beta^y)^2+C_b^\nu+5(C_\sigma^\nu)^2+5(C_\beta^\nu)^2E[\int_0^t|\Delta Y_s|^2ds$$
$$+((C_b^z)^2+5(C_\sigma^z)^2+5(C_\beta^x)^2)E[\int_0^t|\Delta Z_s|^2ds]+[(C_b^k)^2+5(C_\sigma^k)^2+5(C_\beta^k)^2]E[\int_0^t|\Delta K_s|_s^2ds].$$

Thus, taking Υ^1, Υ^2, Υ^3, and Υ^4 as in (4.22) and applying Gronwall lemma imply

$$\mathbb{E}[|\Delta X_t|^2 \leq \frac{\exp(t.\Upsilon^1) - \Upsilon^1}{\Upsilon^1}[\Upsilon^2 E[\int_0^t |\Delta Y_s|^2 ds + \Upsilon^3 E[\int_0^t |\Delta Z_s|^2 ds + \Upsilon^4 E[\int_0^t |\Delta K_s|_s^2 ds]],$$

which gives the desired result.

Proposition 4.3.6. *Under Assumption **(H2)**, there exists a unique solution* (X, Y, Z, K) *of the FBSDE with jumps system* (4.3).

Proof. Let $U = (X, Y, Z, K)$ and $U' = (X', Y', Z', K')$ be two solutions of the mean-field FBSDE with jumps system **(S)**. Using the same notation as in Proposition 4.3.3, We have as proved earlier in (4.12)

$$\Gamma_T \geq (k' - C_g^\nu)\mathbb{E}[|\Delta X_T|^2]. \tag{4.22}$$

On the other hand, using **(H2)** and the Lipschitz assumption, we compute

$$\begin{aligned}\Gamma_T \leq \mathbb{E}[-k\int_0^T (|\Delta Y_s|^2 + |\Delta Z_s|^2 + |\Delta K_s|^2)ds + \int_0^T [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)]|\Delta X_s|^2 ds] \\ + \mathbb{E}[\int_0^T [C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)]|\Delta Y_s|^2 ds + C_\sigma^\nu \int_0^T \|\Delta Z_s\|^2 ds + C_\beta^\nu \int_0^T |\Delta K_s|_s^2 ds].\end{aligned}$$

Combining (4.3.5) and (4.22) we obtain

$$\begin{aligned}&(k' - C_g^\nu)\mathbb{E}[|\Delta X_T|^2] + k\mathbb{E}[\int_0^T |\Delta Y_s|^2 + |\Delta Z_s|^2 + |\Delta K_s|_s^2 ds] \\ &\leq [\Upsilon^\nu \frac{\exp(t.\Upsilon^1) - \Upsilon^1}{\Upsilon^1}\Upsilon^2 + (C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)] E[\int_0^t |\Delta Y_s|^2] ds \\ &+ [\Upsilon^\nu \frac{\exp(t.\Upsilon^1) - \Upsilon^1}{\Upsilon^1}\Upsilon^3 + \frac{C_\sigma^\nu}{2}] E[\int_0^t |\Delta Z_s|^2 ds] \\ &+ [\Upsilon^\nu \frac{\exp(t.\Upsilon^1) - \Upsilon^1}{\Upsilon^1}\Upsilon^4 + \frac{C_\beta^\nu}{2})] E[\int_0^t |\Delta K_s|_s^2 ds],\end{aligned}$$

where $\Upsilon^\nu := [C_h^\nu + \frac{1}{2}(C_b^\nu + C_\sigma^\nu + C_\beta^\nu)]$. Choosing the Lipschitz constants small enough to obtain

$$\begin{aligned}&k > [\Upsilon^\nu \frac{\exp(t.\Upsilon^1) - \Upsilon^1}{\Upsilon^1}\Upsilon^2 + (C_b^\nu + \frac{1}{2}(C_h^\nu + C_\sigma^\nu + C_\beta^\nu)] \\ &k > [\Upsilon^\nu \frac{\exp(t.\Upsilon^1) - \Upsilon^1}{\Upsilon^1}\Upsilon^3 + \frac{C_\sigma^\nu}{2}], \quad k > [\Upsilon^\nu \frac{\exp(t.\Upsilon^1) - \Upsilon^1}{\Upsilon^1}\Upsilon^4 + \frac{C_\beta^\nu}{2})],\end{aligned}$$

and $k' - C_g^\nu > 0$. Thus, we have

$$(k' - C_g^\nu)\mathbb{E}[|\Delta X_T|^2] + k\int_0^T (|Y_s' - Y_s|^2 + \|Z_s' - Z_s\|^2 + |K_s' - K_s|_s^2)ds] \leq 0.$$

This implies that $X'_T = X_T$ and for all $t \in [0,T]$, $X'_t = X_t$, $Y'_t = Y_t$, $Z'_t = Z_t$ and $K'_t = K_t$,$\mathbb{P}$-a.s which gives the desired result.

Theorem 4.3.7. *Under Assumption **(H2)**, there exists a solution (X,Y,Z,K) of the FBSDE with jumps (4.3).*

Proof. Following the same approach as in Proposition 4.3.4, we use an approximation scheme based on perturbation. However, perturbations here are made in the backward SDE with jumps. Let $\delta > 0$ and consider a sequence (X^n, Y^n, Z^n, K^n) of processes defined recursively by : $(X^0, Y^0, Z^0, K^0) = (0,0,0,0)$ and for $n \geq 1$, $U^n = (X^n, Y^n, Z^n, K^n)$ satisfies

$$\begin{cases} X_t^{n+1} = X_0 + \int_0^t b_s(U_s^{n+1}, \nu_s^n)ds + \int_0^t \sigma_s(U_s^{n+1}, \nu_s^n)dW_s + \int_0^t \int_E \beta_s(U_s^{n+1}, \nu_s^n)\tilde{\pi}(ds, de), \\ Y_t^{n+1} = g(X_T^{n+1}, \mu_T^n) + \delta(X_T^{n+1} - X_T^n) - \int_t^T [h_s(U_s^{n+1}, \nu_s^n) + \delta(X_s^{n+1} - X_s^n)]ds \\ \qquad - \int_t^T Z_s^{n+1}dW_s - \int_t^T \int_E K_s^{n+1}(e)\tilde{\pi}(ds, de). \end{cases} \tag{4.23}$$

with $\mu_T^n = \mathbb{P}_{X_T^n}$, $\nu_t^n = \mathbb{P}_{(X_t^n, Y_t^n)}$.
We keep the same notation as in Theorem 4.3.4 and we apply Itô formula to the product $\hat{X}_s^{n+1}\hat{Y}_s^{n+1}$.

$$\begin{aligned} \hat{X}_T^{n+1}\hat{Y}_T^{n+1} - \hat{X}_0^{n+1}\hat{Y}_0^{n+1} &= \int_0^T \hat{Y}_s^{n+1}\hat{b}_s^{n+1}ds + \int_0^T \hat{Y}_s^{n+1}\hat{\sigma}_s^{n+1}dW_s \\ + \int_0^T \hat{X}_s^{n+1}[\hat{h}_s^{n+1} - \delta(\hat{X}_s^{n+1} - \hat{X}_s^n)]ds &- \int_0^T \hat{X}_s^{n+1}\hat{Z}_s^{n+1}dW_s - \int_0^T \int_E \hat{X}_s^{n+1}\hat{K}_s^{n+1}(e)\tilde{\pi}(de, ds) \\ + \int_0^T (\hat{\sigma}_s^{n+1}, \hat{Z}_s^{n+1})ds &+ \int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds) + \int_0^T \int_E \hat{Y}_s^{n+1}\hat{\beta}_s^{n+1}\tilde{\pi}(de, ds). \end{aligned} \tag{4.24}$$

Notice that, since the terminal condition is given by

$$\hat{Y}_T^{n+1} = [g(X^{n+1}, \mu_T^n) - g(X^n, \mu_T^{n-1})] + \delta(X_T^{n+1} - X_T^n),$$

we rewrite the above equation as follows

$$\begin{aligned} &\mathbb{E}[\hat{X}_s^{n+1}(g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^{n-1})) + \delta[|\hat{X}_T^{n+1}|^2 - \hat{X}_T^{n+1}\hat{X}_T^n]] \\ &= \mathbb{E}[\int_0^T \hat{Y}_s^{n+1}\hat{b}_s^{n+1}ds + \int_0^T \hat{X}_s^{n+1}\hat{h}_s^{n+1}ds + \int_0^T (\hat{\sigma}_s^{n+1}, \hat{Z}_s^{n+1})ds] \\ &+ \mathbb{E}[\int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds) - \delta(\int_0^T |\hat{X}_s^{n+1}|^2 ds - \int_0^T \hat{X}_s^{n+1}\hat{X}_s^n ds)]. \end{aligned}$$

Using Assumption **(H2)** and Young's inequality, we obtain

$$\begin{aligned}
\mathbb{E}\big[\hat{X}_T^{n+1}(g(X_T^{n+1},\mu_T^n)-g(X_T^n,\mu_T^{n-1}))\big] &= \mathbb{E}\Big[\hat{X}_T^{n+1}(g(X_T^{n+1},\mu_T^n)-g(X_T^n,\mu_T^n))\Big] \\
&+ \mathbb{E}\big[\hat{X}_T^{n+1}(g(X_T^n,\mu_T^n)-g(X_T^n,\mu_T^{n-1}))\big] \\
&\geq -C_g^\nu \mathbb{E}[|\hat{X}_T^{n+1}|]\mathcal{W}_2(\mu_T^n,\mu_T^{n-1}) + k'\mathbb{E}|\hat{X}_T^{n+1}|^2] \\
&\geq (k'-\frac{C_g^\nu\epsilon}{2})\mathbb{E}[]|\hat{X}_T^{n+1}|^2] - \frac{2C_g^\nu}{\epsilon}\mathcal{W}_2^2(\mu_T^n,\mu_T^{n-1}) \\
&\geq (k'-\frac{C_g^\nu\epsilon}{2})\mathbb{E}[|\hat{X}_T^{n+1}|^2] - \frac{2C_g^\nu}{\epsilon}\mathbb{E}[|\hat{X}_T^n|^2]. \qquad (4.25)
\end{aligned}$$

Besides, classical linearization technics imply that

$$\begin{aligned}
&\mathbb{E}[\int_0^T[\hat{Y}_s^{n+1}\hat{b}_s^{n+1}+\hat{X}_s^{n+1}\hat{h}_s^{n+1}(\hat{\sigma}_s^{n+1},\hat{Z}_s^{n+1})]ds + \int_0^T\int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de,ds)] \\
=&\mathbb{E}[\int_0^T[\mathcal{A}(s,U_s^{n+1},U_s^n,\nu_s^n)+\hat{Y}_s^{n+1}\bar{b}_s^n+\hat{X}_s^{n+1}\bar{h}_s^n+\hat{Z}_s^{n+1}\bar{\sigma}_s^n]ds \\
+&\mathbb{E}[\int_0^T\int_E \hat{K}_s^{n+1}\bar{\beta}_s^n\eta(de,ds)].
\end{aligned}$$

Once again, Assumption **(H2)** and Young inequality give

$$\begin{aligned}
&\mathbb{E}[\int_0^T[\hat{Y}_s^{n+1}\hat{b}_s^{n+1}+\hat{X}_s^{n+1}\hat{h}_s^{n+1}+(\hat{\sigma}_s^{n+1},\hat{Z}_s^{n+1})]ds + \int_0^T\int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de,ds)] \\
&\leq -k\mathbb{E}[\int_0^T|\hat{Y}_s^{n+1}|^2+\|\hat{Z}_s^{n+1}\|^2+|\hat{K}_s^{n+1}|_s^2ds] + (\frac{C_b^\nu\alpha}{2}-k)\mathbb{E}[\int_0^T|\hat{Y}_s^{n+1}|^2ds] \\
&+\mathbb{E}[\int_0^T[C_h^\nu|\hat{X}_s^{n+1}|+C_b^\nu|\hat{Y}_s^{n+1}|+C_\sigma^\nu\|\hat{Z}_s^{n+1}\|+C_\beta^\nu|\hat{K}_s^{n+1}|_{\mathcal{L}^2(\eta)}]\mathcal{W}_2(\nu_s^n,\nu_s^{n-1})ds] \\
&\leq \frac{C_h^\nu\alpha}{2}\mathbb{E}[\int_0^T|\hat{X}_s^{n+1}|^2ds] + \frac{C_h^\nu+C_b^\nu+C_\sigma^\nu+C_\beta^\nu}{2\alpha}\mathbb{E}[(|\hat{X}_s^n|^2+|\hat{Y}_s^n|^2)ds] \\
&+(\frac{C_\sigma^\nu\alpha}{2}-k)\mathbb{E}[\int_0^T\|\hat{Z}_s^{n+1}\|^2ds] + (\frac{C_\beta^\nu\alpha}{2}-k)\mathbb{E}[\int_0^T|\hat{K}_s^{n+1}|_s^2ds]. \qquad (4.26)
\end{aligned}$$

Therefore, we obtain from (4.25) and (4.26)

$$\begin{aligned}
&\mathbb{E}[(k'-\frac{c_g^\nu\epsilon}{2}+\frac{\delta}{2})|\hat{X}_T^{n+1}|^2 + \int_0^T(-\frac{C_h^\nu\alpha}{2}+\delta-\frac{\delta\rho}{2})|\hat{X}_s^{n+1}|^2ds + \int_0^T(\frac{C_b^\nu\alpha}{2}-k)|\hat{Y}_s^{n+1}|^2ds] \\
&+\mathbb{E}[\int_0^T(\frac{C_\sigma^\nu\alpha}{2}-k)|\hat{Z}_s^{n+1}|^2ds] + \mathbb{E}[\int_0^T(\frac{C_\beta^\nu\alpha}{2}-k)|\hat{K}_s^{n+1}|_s^2ds] \\
&\leq \mathbb{E}[(\frac{2C_g^\nu}{\epsilon}+\frac{\delta}{2})|\hat{X}_T^n|^2] + \int_0^T(\frac{C_h^\nu+C_b^\nu+C_\sigma^\nu+C_\beta^\nu}{2\alpha}+\frac{\delta}{2\rho})|\hat{X}_s^n|^2ds] \\
&+\mathbb{E}[\int_0^T\frac{C_h^\nu+C_b^\nu+C_\sigma^\nu+C_\beta^\nu}{2\alpha}|\hat{Y}_s^n|^2ds].
\end{aligned}$$

Henceforth,

$$\tilde{\gamma}\mathbb{E}[|\hat{X}_T^{n+1}|^2 + \int_0^T \|\hat{U}_s^{n+1}\|^2 ds] \leq \tilde{\theta}\mathbb{E}[|\hat{X}_T^n|^2 + \int_0^T \|\hat{U}_s^n\|^2 ds]. \tag{4.27}$$

Choosing $\tilde{\epsilon}$, α and ϵ so that $\frac{\tilde{\theta}}{\tilde{\gamma}} < 1$, the inequality (4.27) becomes a contraction. Thus, $(\hat{X}_T^n)_{n\geq 0}$ is a Cauchy sequence in $\mathbb{H}^2(\Omega, \mathbb{P})$ and $(\hat{X}^n)_{n\geq 0}, (\hat{Y}^n)_{n\geq 0}, (\hat{Z}^n)_{n\geq 0}$ and $(\hat{K}^n)_{n\geq 0}$ are Cauchy sequences respectively in $\mathbb{H}^2([0,T], \Omega, dt \otimes d\mathbb{P})$ and $\mathbb{H}^2_\eta([0,T], \Omega, dt \otimes d\eta)$. Hence, if X, Y, Z and K are the respective limits of these sequences, passing to the limit in (4.15), we see that (X, Y, Z, K) is a solution of (4.23).

4.4 Application: Storage problem

4.4.1 Description of the model

We consider a stylized model for a power grid with distributed local energy generation and storage. The grid connects N nodes indexed by $i = 1, \cdots, N$. Each node is characterized by two state variables:

- The storage level S_t representing the total energy available in the storage device.
- The net power production of the energy (photvoltaic panels, diesel energy,..) that each nodes produces after all costs subtracted Q_t.

We assume that the nodes forming this grid can be partitioned in Γ different groups: the nodes in the same group γ share the same characteristics of local net power production and storage, yet these characteristics vary from one group to the other.

We denote by N_γ the number of nodes in group γ so that $N = \sum_{\gamma=1}^{\Gamma} N_\gamma$ and we define $\pi^\gamma = N_\gamma / N$ as the ratio of the population size of region γ to the whole population. We shall abusively write $i \in \gamma$ to signify that the node i is in region γ.

In order to model the dynamics of the state variables, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying $N+1$ Brownian motions $B^0, B^1, \cdots, B^N$ and a Poisson process defined on $[0,T] \times \Omega \times \mathbb{R}^*$ to which is associated a counting measure $\hat{N}(de, dt) = n(de)dt$. We suppose that the predictable measure $n(de)$ is positive, finite and satisfies the following integrability condition

$$\int_{\mathbb{R}^*} (1 \wedge |e|)^2 n(de) < \infty. \tag{4.28}$$

We also consider $N+1$ independant Poisson measures $(N^0, N^1, \cdots, N^N)$ a N independant identically distributed random variables $x_0^i = (s_0^i, q_0^i)$ which are independant from B^0, N^0 the B^i

and the N^i. We denote by $\mathcal{F} = \{\mathcal{F}_t\}$ the filtration defined by

$$\mathcal{F}_t = \sigma\{(s_0^i, q_0^0, q_0^i), B_s^0, B_s^i, N^0, N^i \text{ where } i = 1, \cdots N,\ s \leq t\},$$

and the filtration $\mathcal{F}^0 = \{\mathcal{F}_t^0\}$ generated by B^0 and N^0. We also denote by $\mathcal{A}$ the set of $\mathcal{F}$-adapted real-valued processes $a = \{a_t\}$ such that $\mathbb{E}[\int_0^T |a_u|^2] < \infty$. Let us now define the dynamics of the state variables.

- The power production of the energy Q_t^i of each node $i \in \{1, \cdots, N\}$ in the region γ at time t is modeled in the following way:

$$\begin{cases} dQ_t^i = \mu^\gamma(t, Q_t^i)dt + dM_t^i + dM_t^0 \\ Q_0^r = q_0^r, \end{cases} \tag{4.29}$$

where

$$\begin{cases} dM_t^i = \sigma^\gamma(t, Q_t^i)dB_t^i + \int_E \beta^\gamma(t, e, Q_{t-}^i)\tilde{N}^i(dt, de), \\ dM_t^0 = \sigma^{\gamma^0}(t, Q_t^0)dB_t^0 + \int_E \beta^{\gamma^0}(t, e, Q_{t-}^0)\tilde{N}^0(dt, de). \end{cases}$$

- The battery level S_t^i of the node i in the region γ is controlled through a storage action $\alpha^{\gamma,i} \in \mathcal{A}$ according to

$$\begin{cases} S_t^i = S_0^i + \int_0^t \alpha_s^i ds, \\ 0 \leq S_t^i \leq S_{\max}. \end{cases} \tag{4.30}$$

The quantity $Q_t^i - \alpha_t^i$ is the net injection of the node. It can be either positive or negative:

- If $Q_t^i - \alpha_t^i$ is positive: It corresponds to electricity being sold from the node i to the grid.
- If $Q_t^i - \alpha_t^i$ is negative: It corresponds to electricity being bought by the node i from the grid.

Remark 4.4.1. *In our framework, in contrast with the paper of Alasseur et al [2], we assume that the production of energy is unpredictable. This is due to its dependence on environmental conditions such as the sun, the speed of the wind which are intermittent and irregular which is traduced by including the jump component in our analyses. We will also assume that the storage level will be enforced by a constraint. In other words, we assume that there is a maximal level for which the battery can support. S_{max} is the battery's maximum instantaneous power output.*

As in [2], we include a micro grid system indexed by 0 called the "rest of the world", which is characterized by one state variable, its *local net power production* Q_t^0, and which does not possess any storage. The net production of the rest of the world is given by

$$\begin{cases} dQ_t^0 = \mu^0(t, Q_t^0)dt + \sigma^0(t, Q_t^0)dB_t^0 + \int_E \beta^0(t, e, Q_{t^-}^0)\tilde{N}^0(dt, de), \\ Q_0^0 = q_0^0. \end{cases} \tag{4.31}$$

In our model B_0 and $\tilde{N}^0$ represent a common signal which affects the energy demand of the whole grid. Then for each i, $\sigma^\gamma : \mathbb{R} \to \mathbb{R}$ and β^γ are given functions which allow to model how the node i of region γ is affected by the common signal B_t^0 and N_t^0 . We assume that the rest of the world is only affected by this common signal B_t^0 and N_t^0.

Electricity spot price

We make the assumption that the electricity price per Watt-hour depends on the instantaneous demand. When the strategy $\alpha = (\alpha^1, \cdots, \alpha^N) \in \mathcal{A}^N$ is implemented, the spot price is given by

$$P_t^{N,\alpha} = p\left(-Q_t^0 - \sum_{i=1}^N \eta(Q_t^i - \alpha_t^i)\right), \tag{4.32}$$

where $p(\cdot)$ is the exogenous inverse demand function for electricity and η is a scaling parameter which weights the contribution of each individual node i to the whole system.

We assume that p is a strictly increasing function. Since the energy model here is given through a macro grid system connecting a large number of small nodes i, we shall consider the limit when $N \to \infty$ and $\eta \to 0$. Here we assume that $\eta = \frac{1}{N}$. So, the spot price can be written as follows

$$P_t^{N,\alpha} = p\left(-Q_t^0 - \sum_{i=1}^N \frac{1}{N}(Q_t^i - \alpha_t^i)\right), \tag{4.33}$$

where $\frac{1}{N}\sum_{i=1}^N (Q_t^i - \alpha_t^i)$ is the averaged net injections.

The control problem

We consider a finite time horizon $T > 0$. When the control action $\alpha = (\alpha^1, \cdots, \alpha^N)$ is implemented, the cost incurred at the node i in the region γ is given by

$$J^{i,\gamma,N}(\alpha) = \mathbb{E}[\int_0^T P_t^{N,\alpha}.\left(\alpha_t^i - Q_t^i\right) + L_T^\gamma(Q_t^i, \alpha_t^i) + L_S(S_t^{i,\alpha^i}, \alpha_t^i)dt + g(S_T^{i,\alpha^i})], \tag{4.34}$$

where $L_T^\gamma, L_s : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions.

- $P_t^{N,\alpha}(\alpha_t^i - Q_t^i)$ represents the current volumetric cost (resp. profit) of electricity consumed (resp. produced) at the spot price $P_t^{N,\alpha}$.

- $L_S(S^{i,\alpha^i}, \alpha_t^i$ represent the current and it is assumed to be identical in all the regions γ.

- $L^{\gamma}_T((Q^i_t, \alpha^i_t$ is the volumetric charge. This electricity cost is closely related to the power that the system requires in peak hours and hence produce enough power to satisfy the highest level of peak demand.

The rest of the world incurs only energy and transmission costs

$$J^{0,N}(\alpha) = \mathbb{E}\left[\int_0^T -P_t^{N,\alpha}.Q_t^0 + L_T^0(Q_t^0, 0)dt\right]. \tag{4.35}$$

Central Planner control problem

The central planner aims to dictate a storage rule: $\alpha = (\alpha^1, \cdots, \alpha^N)$ in order to minimize the *egalitarian* cost function between the nodes and the rest of the world

$$J^{\mathrm{C,N}}(\alpha) = J^{0,N}(\alpha) + \sum_{i=1}^{N} \frac{1}{N} J^{i,\gamma,N}(\alpha).$$

where $1/N$ is the scaling parameter which weights the contribution of each individual node to the system. The cost function $J^{C,N}(\alpha)$ can also be written as

$$J^{\mathrm{C,N}}(\alpha) = J^{0,N}(\alpha) + \sum_{\gamma=1}^{\Gamma} \pi^{\gamma} \sum_{i=1}^{N_\gamma} \frac{1}{N_\gamma} J^{i,\gamma,N}(\alpha).$$

Definition 4.4.2 (Optimal coordinated plan)**.** *We say that $\hat{\alpha} = (\hat{\alpha}^1, \cdots, \hat{\alpha}^N) \in \mathcal{A}^N$ is an optimal coordinated plan if: $\hat{\alpha} = \mathrm{argmin}_{\alpha \in \mathcal{A}^N} \ J^{C,N,\eta}(\alpha)$.*

Assumption 4.4.3.

- *The current cost $(s, q, \alpha) \mapsto L^{\gamma}_T(q, \alpha) + L_S(s, \alpha)$ is strictly convex with respect to (s, α). The terminal cost $s \mapsto g(s)$ is strictly convex with respect to s.*

- *There exists some constant $C > 0$ such that*

$$|L^{\gamma}_T(q, a)| + |L_S(s, a)| + |g(s)| \quad \leq C \left(|q|^2 + |s|^2 + |a|^2\right)$$

- *The functions L^{γ}_T, L_S and g are continuously differentiable and their derivatives are a Lipschitz continuous functions.*

- *The coefficients μ^0 and σ^0 (respectively μ^{γ} and σ^{γ}) are Lipschitz continuous functions and with linear growth in the state variable.*

4.4.2 Reformulation: Mean field type control problem

In this section we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ carrying Γ standard Brownian motions $\{B^\gamma, \gamma = 1, \cdots, \Gamma\}$ and an independant Poisson random measure $\{N^\gamma, \gamma = 1, \cdots, \Gamma\}$ which are mutually independent and independent from the filtration $\mathbb{F}^0$. We shall use the following notation. If $\xi = \{\xi_t\}$ is an $\mathbb{F}$-adapted process, then $\bar{\xi} = \{\bar{\xi}_t\}$ denotes the process defined by : $\bar{\xi}_t := \mathbb{E}[\xi_t|\mathcal{F}_t^0]$.

Let $x_0 = (s_0, q_0) = (x_0^\gamma = (s_0^\gamma, q_0^\gamma))_{1\leq\gamma\leq\Gamma}$ be a random vector which is independent from $\mathbb{F}^0$. Let Q^0 and Q^γ be the production processes defined by

$$
\begin{aligned}
Q^\gamma &= q_0^\gamma + \int_0^t \mu^\gamma(u, Q_u^\gamma)du + \int_0^t \sigma^\gamma(u, Q_u^\gamma)dB_u^\gamma + \int_0^t \sigma^{\gamma,0}(u, Q_u^\gamma)dB_u^0 \\
&\quad + \int_0^t \int_E \beta^\gamma(u, e, Q_{u^-}^\gamma)\tilde{N}^\gamma(du, de) + \int_0^t \int_E \beta^{\gamma^0}(u, e, Q_{u^-}^0)\tilde{N}^0(du, de). \\
Q_t^0 &= q_0^0 + \int_0^t \mu^r(u, Q_u^0)du + \int_0^t \sigma^0(u, Q_u^0)dB_u^0 + \int_0^t \int_E \beta^\gamma(u, e, Q_{u^-}^0)\tilde{N}^0(du, de). \qquad (4.36)
\end{aligned}
$$

If $\bar{\nu} = (\bar{\nu}^1, \cdots, \bar{\nu}^\Gamma)$ is an $\mathbb{F}^0$-adapted $\mathbb{R}^\Gamma$-valued process, we denote

$$P_t^{\bar{\nu}} = p\left(-Q_t^0 - \sum_{\gamma\in\Gamma} \pi^\gamma \left(\mathbb{E}[Q_t^\gamma|\mathcal{F}_t^0] - \bar{\nu}_t^\gamma\right)\right). \qquad (4.37)$$

Now, for $\bar{\nu}^0 = (\bar{\nu}^{1,0}, \cdots, \bar{\nu}^{\Gamma,0})$, $\alpha = (\alpha^1, \cdots, \alpha^\Gamma)$ and for each $\gamma = 1, \cdots, \Gamma$, we consider the two following cost functions

$$J_{x_0}^\gamma(\alpha^\gamma, \bar{\nu}) = \mathbb{E}\int_0^T \left[P_t^{\bar{\nu}}(\alpha_t^\gamma - Q_t^\gamma) + L_T^\gamma(Q_t^\gamma, \alpha_t^\gamma) + L_S(S_t^\gamma, \alpha_t^\gamma)\right]dt + \mathbb{E}\left[g(S_t^\gamma)\right]. \qquad (4.38)$$

$$J_{x_0}^C(\alpha) = \mathbb{E}\int_0^T [-P_t^{\bar{\alpha}}Q_t^0 + L_T^0(Q_t^0, 0)]dt + \sum_{\gamma=1}^\Gamma \pi^\gamma J_{x_0}^\gamma(\alpha^\gamma, \bar{\alpha}_t). \qquad (4.39)$$

where $S_t^\gamma = s_0^\gamma + \int_0^t \alpha_u^\gamma du$.

Definition 4.4.4 (Mean field Nash equilibrium). *Let $x_0 = (s_0, q_0)$ be a random vector independent from $\mathcal{F}^0$. We say that $\alpha^\star = \{\alpha^{\gamma,\star}, 1 \leq \gamma \leq \Gamma\}$ is a mean field Nash equilibrium if, for each γ, $\alpha^{\gamma,\star}$ minimizes the function $\alpha^\gamma \mapsto J_{x_0}^\gamma(\alpha^\gamma, \{\mathbb{E}[\alpha_t^\star|\mathcal{F}_t^0]\})$.*

Definition 4.4.5 (Mean field optimal control). *Let $x_0 = (s_0, q_0)$ be a random vector independent from $\mathbb{F}^0$. We say that $\hat{\alpha} = \{\hat{\alpha}^\gamma, 1 \leq \gamma \leq \Gamma\}$ is a mean field optimal control if, $\hat{\alpha}$ minimizes the function $\alpha \mapsto J_{x_0}^C(\alpha)$.*

Characterization of mean field Nash equilibrium

Proposition 4.4.6. *Let $\bar{\nu}$ be a given $\mathbb{F}^0$-adapted $\mathbb{R}^\Gamma$-valued process. Then there exists a unique control $(\alpha^{1,\star}, \dots, \alpha^{\Gamma,\star}) = \alpha^\star(\bar{\nu}, x_0)$ such that*

- *For each $\gamma \in 1, \dots, \Gamma$, $\alpha^{\gamma,\star}$ minimizes the function $\alpha^\gamma \mapsto J^\gamma_{x_0}(\alpha^\gamma, \bar{\nu})$.*
- *If $(S^{\gamma,\star}, Q^\gamma)$ is the state process corresponding to the initial data condition x^γ_0, to the control $\alpha^{\gamma,\star}$, and to the dynamic above, then there exists a unique adapted solution $(Y^{\gamma,\star}, Z^{0,\gamma,\star}, Z^{\gamma,\star}, V^{\gamma,\star}, V^{0,\gamma,\star})$ of the BDSE with jumps*

$$\begin{aligned} Y^{\gamma,\star}_t &= \partial_s g(S^{\gamma,\star}_T) + \int_0^T \partial_s L_S(S^{\gamma,\star}_t, \alpha^{\gamma,\star}_t)dt + \int_0^T Z^{0,\gamma,\star}_t dB^0_t + Z^{\gamma,\star}_t dB^\gamma_t \\ &\quad + \int_0^T \int_E V^{\gamma,\star}_t(e)\tilde{N}^\gamma(dt, de) + \int_0^T \int_E V^{0,\gamma,\star}_t(e)\tilde{N}^0(dt, de), \end{aligned} \tag{4.40}$$

satisfying the coupling condition

$$Y^{\gamma,\star}_t + P^{\bar{\nu}}_t + \partial_\alpha L^\gamma_T(Q^\gamma_t, \alpha^{\gamma,\star}_t) + \partial_\alpha L_S(S^{\gamma,\star}_t, \alpha^{\gamma,\star}_t) = 0. \tag{4.41}$$

Conversely, assume that there exists $(\alpha^{\gamma,\star}, S^{\gamma,\star}, Y^{\gamma,\star}, Z^{0,\gamma,\star}, Z^{\gamma,\star}, V^{\gamma,\star}, V^{0,\gamma,\star})$ which satisfy the coupling condition (4.41) as well as the FBSDEJ, then $\alpha^{\gamma,\star}$ is the optimal control minimizing $J^\gamma_{x_0}(\alpha^\gamma, \bar{\nu})$ and $S^{\gamma,\star}$ is the optimal trajectory.

- *If in addition: $\forall \gamma = 1, \cdots, \Gamma \ \mathbb{E}\left[\alpha^{\gamma,\star}_t | \mathcal{F}^0_t\right] = \bar{\nu}^{\gamma,0}_t$, then $\alpha^\star$ is a mean field Nash equilibrium.*

Proof. • Since the dynamic programming principal does not work in this context, our proof consists on the classical Pontryagin's maximum principle where the characterization of the Mean field Nash equilibrium is given by the associated McKean-Vlasov FBSDEs. Uing the fact that $J^\gamma_{x_0}$ is a strictly convex coercive function and Gateaux-differentiable (see Assumption (4.4.3)), we have

$$d_\beta J^\gamma_{x_0}(., \bar{\nu}) := 0. \tag{4.42}$$

We start by computing the functional directional derivative of $J^\gamma_{x_0}(., \bar{\nu})$

$$\begin{aligned} d_\beta J^\gamma_{x_0}(., \bar{\nu}) &= \mathbb{E}\left[\int_0^T [P^{\bar{\nu}}_u + \partial_\alpha L^\gamma_T(Q^\gamma_u, \alpha^\gamma_u) + \partial_\alpha L_S(S^\gamma_u, \alpha^\gamma_u) + \partial_u L_S(S^\gamma_u, \alpha^\gamma_u)]\beta_u du\right] \\ &\quad + \mathbb{E}\left[\tilde{S}^\beta_T \partial_s g(S^\gamma_T)\right]. \end{aligned} \tag{4.43}$$

Hence, there exists a unique optimal control $\alpha^{\gamma,\star} = \alpha^{\gamma,\star}(\bar{\nu}, x_0)$ satisfying the following Euler optimality condition

$$\mathbb{E}\left[\int_0^T [P^{\bar{\nu}}_u + \partial_\alpha L^\gamma_T(Q^\gamma_u, \alpha^\gamma_u) + \partial_\alpha L_S(S^\gamma_u, \alpha^\gamma_u) + \partial_s L_S(S^\gamma_u, \alpha^\gamma_u)]\beta_u du + \tilde{S}^\beta_T \partial_s g(S^\gamma_T)\right] = 0. \tag{4.44}$$

We denote by $S^{\gamma,\star}$ the optimal trajectory associated to $\alpha^{\gamma,\star}$. Applying Itô Tanaka formula to

$S_t^\beta Y_t^{\gamma,\star}$ we get

$$\tilde{S}_t^\beta Y_t^{\gamma,\star} = \tilde{S}_T^\beta Y_T^{\gamma,\star} + \int_t^T Y_s^{\gamma,\star}\beta_s ds - \int_t^T \tilde{S}_s^\beta \partial_s L_S(S_s^{\gamma,\star}, \alpha_s^{\gamma,\star})ds + \int_0^T \tilde{S}_s^\beta Z_s^{0,\gamma,\star} dB_s^0$$
$$+ \int_t^T \tilde{S}_s^\beta Z_s^{\gamma,\star} dB_s^\gamma + \int_0^T \int_E \tilde{S}_s^\beta V_s^{\gamma,\star}(e)\tilde{N}^\gamma(ds,de) + \int_0^T \int_E \tilde{S}_s^\beta V_s^{0,\gamma,\star}(e)\tilde{N}^{0,\gamma}(ds,de)$$

Taking the conditional expectation in the equation above, we obtain

$$\mathbb{E}\left[\tilde{S}_T^\beta Y_T^{\gamma,\star}\right] = \mathbb{E}[\int_t^T Y_s^{\gamma,\star}\beta_s ds - \int_t^T \tilde{S}_s^\beta \partial_s L_S(S_s^{\gamma,\star}, \alpha_s^{\gamma,\star})ds]. \tag{4.45}$$

Taking into account the terminal the terminal condition $Y_T^\star = \partial_s g(S_T^{\gamma,\star})$ and the Euler optimality condition (4.44), the previous equation leads to

$$\mathbb{E}\left[\int_0^T \left(Y_s^{\gamma,\star} + P_s^{\bar{\nu}} + \partial_\alpha L_T^\gamma(Q_s^\gamma, \alpha_s^{\gamma,\star}) + \partial_\alpha L_S(S_s^{\gamma,\star}, \alpha_s^{\gamma,\star})\right)\beta_s ds\right] = 0. \tag{4.46}$$

• Suppose that $(\alpha^{\gamma,\star}, S^{\gamma,\star}, Y^{\gamma,\star}, Z^{0,\gamma,\star}, Z^{\gamma,\star}, V^{\gamma,\star}, V^{0,\gamma,\star})$ is a solution of the following coupled Forward-Backward SDE with jumps.

$$\begin{cases} Y_T^{\gamma,\star} = \partial_s g(S_T^{\gamma,\star}) + \int_0^T \partial_s L_S(S_s^{\gamma,\star}, \alpha_s^{\gamma,\star})ds + \int_0^T Z_s^{0,\gamma,\star} dB_s^0 + Z_s^{\gamma,\star} dB_s^\gamma \\ \qquad + \int_0^T \int_E V_s^{\gamma,\star}(e)\tilde{N}(ds,de) + \int_0^T \int_E V_s^{0,\gamma,\star}(e)\tilde{N}^0(ds,de). \\ Q_T^\gamma = q_0^\gamma + \int_0^T \mu^\gamma(s, Q_s^\gamma)ds + \int_0^T \sigma^\gamma(s, Q_s^\gamma)dB_s^\gamma + \int_0^T \sigma^{\gamma,0}(s, Q_s^\gamma)dB_s^0 \\ \qquad + \int_0^T \int_E \beta^\gamma(s,e,Q_{s^-}^\gamma)\tilde{N}^\gamma(ds,de) + \int_0^T \int_E \beta^{\gamma,0}(s,e,Q_{s^-}^{\gamma,0})\tilde{N}^0(ds,de) \\ S_t^\gamma = s_0^\gamma + \int_0^t \alpha_s^\gamma ds. \end{cases} \tag{4.47}$$

with

$$Y_s^{\gamma,\star} + P_s^{\bar{\nu}} + \partial_\alpha L_T^\gamma(Q_s^\gamma, \alpha_s^{\gamma,\star}) + \partial_s L_S(S_s^{\gamma,\star}, \alpha_s^{\gamma,\star}) = 0,$$

The idea is then to compute the the gâteau derivative of $J_{x_0}^{\gamma,\mathrm{MFG}}(\cdot,\bar{\nu})$ at $\alpha^{\gamma,\star}$ to obtain zero and then from the strict convexity of $J_{x_0}^{\gamma,\mathrm{MFG}}(\cdot,\bar{\nu})$ we obtain the desire result.

Characterization of mean field optimal controls

Proposition 4.4.7. *Assume that $\hat{\alpha} = (\hat{\alpha}^1, \cdots, \hat{\alpha}^\Gamma)$ minimizes the functional $J_{x_0}^C(\alpha)$, and denote by $\hat{S} = (\hat{S}^1, \cdots, \hat{S}^\Gamma)$ is the corresponding controlled trajectory. Then there exists a unique adapted solution $(\hat{Y} = (\hat{Y}^1, \cdots \hat{Y}_t^\Gamma)$, $\hat{Z} = (\hat{Z}^1, \cdots, \hat{Z}^\Gamma)$, $\hat{Z}^0 = (\hat{Z}^{0,1}, \cdots, \hat{Z}^{0,\Gamma})$, $\hat{V} = (\hat{V}^1, \cdots, \hat{V}^\Gamma)$, $\hat{V}^0 =$*

$(\hat{V}^{0,1}, \cdots, \hat{V}^{0,\Gamma}))$ *of the BDSE*

$$\begin{cases} d\hat{Y}_t^\gamma = -\partial_s L_S(\hat{S}_t^\gamma, \hat{\alpha}_t^\gamma)dt + \hat{Z}_t^{0,\gamma} dB_t^0 + \hat{Z}_t^\gamma dB_t^\gamma + \int_E V_t^\gamma(e)\tilde{N}(dt,de) + \int_E V_t^{0,\gamma}(e)\tilde{N}^0(dt,de) \\ \hat{Y}_T^\gamma = \partial_s g(\hat{S}_T^\gamma). \end{cases} \tag{4.48}$$

satisfying the coupling condition: for all $\gamma = 1, \ldots, \Gamma$

$$\begin{aligned} 0 = \hat{Y}_t^\gamma + \partial_\alpha L_T^\gamma(Q_t^\gamma, \hat{\alpha}_t^\gamma) + \partial_\alpha L_S(\hat{S}_t, \hat{\alpha}_t^\gamma) + P_t^{\bar{\hat{\alpha}}} \\ - p'\left(-Q_t^0 - \Pi_\Gamma \cdot \left(\bar{Q}_t - \bar{\hat{\alpha}}_t\right)\right)\left(-Q_t^0 - \Pi_\Gamma \cdot \left(\bar{Q}_t - \bar{\hat{\alpha}}_t\right)\right), \end{aligned} \tag{4.49}$$

with $\bar{\hat{\alpha}}_t = \mathbb{E}[\hat{\alpha}_t|\mathcal{F}_t^0]$ *and* $\Pi_\Gamma = (\pi_1, \ldots, \pi_\Gamma)^T$*. Conversely, suppose that* $(\hat{S}, \hat{\alpha}, \hat{Y}, \hat{Z}^0, \hat{Z})$ *is an adapted solution to the forward-backward system (4.30)-(4.48) with the coupling condition (4.49), then* $\hat{\alpha}$ *is the optimal control minimizing* $J_{x_0}^{\mathrm{MFC}}(\alpha)$ *and* $\hat{S}$ *is the optimal trajectory.*

Proof. Exactly as [2], we only prove the necessary condition of Pontryagin's maximum principle for optimality. The sufficient condition could be proven exactly as it is done in Proposition 4.4.6. Thanks to Assumption 4.4.3 insures that the cost function $\alpha \in \mathcal{A} \mapsto J_{x_0}^{\mathrm{C}}(\alpha)$ is Gâteaux differentiable with Gâteaux derivative given by

$$\begin{aligned} d_\beta J_{x_0}^C(\alpha) &= \sum_\gamma \pi^\gamma \mathbb{E}\left[\partial_s g(S_T^\gamma)\tilde{S}_T^{\beta^\gamma} + \int_0^T \partial_s L_S(S_u^\gamma, \alpha_u^\gamma) S_u^{\beta^\gamma} du\right] \\ &+ \sum_\gamma \pi^\gamma \mathbb{E}\left[\int_0^T \left\{P_u^{\bar{\alpha}} + \partial_\alpha L_T^\gamma(Q^\gamma, \alpha_t^\gamma) + \partial_\alpha L_S(S^\gamma, \alpha^\gamma)\right\} \beta_u^\gamma du\right] \\ &- \sum_\gamma \pi^\gamma \mathbb{E}\left[\int_0^T p'\left(-Q_u^0 - \Pi_\Gamma \cdot \left(\bar{Q}_u - \bar{\alpha}_u\right)\right)\left(-Q_u^0 - \Pi_\Gamma \cdot \left\{\bar{Q}_u - \bar{\alpha}_u\right)\right\} \beta_u^\gamma du\right], \end{aligned}$$

where $\tilde{S}_u^{\beta^\gamma}$ is the process defined by

$$d\tilde{S}_u^{\beta^\gamma} = \beta_u^\gamma du, \quad \tilde{S}_0^{\beta^\gamma} = 0\,.$$

Hence the optimal control $\hat{\alpha}$ satisfies the Euler optimality condition: for all $\beta = (\beta^1, \cdots, \beta^\Gamma)$

$$\begin{aligned} 0 &= \sum_\gamma \pi^\gamma \mathbb{E}\left[\partial_s g(S_T^\gamma)\tilde{S}_T^{\beta^\gamma} + \int_0^T \partial_s L_S(S_u^\gamma, \alpha_u^\gamma) S_u^{\beta^\gamma} du\right] \\ &+ \sum_\gamma \pi^\gamma \mathbb{E}\left[\int_0^T \left\{P_u^{\bar{\alpha}} + \partial_\alpha L_T^\gamma(Q_u^\gamma, \alpha_u^\gamma) + \partial_\alpha L_S(S_u^\gamma, \alpha_u^\gamma)\right\} \beta_u^\gamma du\right] \\ &- \sum_\gamma \pi^\gamma \mathbb{E}\left[\int_0^T \left\{p'\left(-Q_u^0 - \Pi_\Gamma \cdot \left(\bar{Q}_u^0 - \bar{\alpha}_u^0\right)\right)\left(-Q_u^0 - \Pi_\Gamma \cdot (\bar{Q}_u - \bar{\alpha}_u)\right\} \beta_u^\gamma du\right], \end{aligned}$$

• Now, let $(\hat{Y}, \hat{Z}, \hat{Z}^0, \hat{V}^\gamma, \hat{V}^{0,\gamma})$ be the unique solution to the BSDE with jump (4.48), and let $\hat{S}$ be the state process associated to the optimal control $\hat{\alpha}$, applying Itô formula, we obtain

$$\sum_\gamma \pi^\gamma \mathbb{E}\left[\hat{Y}_T^\gamma \tilde{S}_T^{\beta\gamma}\right] = \sum_\gamma \pi^\gamma \mathbb{E}\left[\int_0^T \left\{-\partial_s L_S(\hat{S}_u, \hat{\alpha}_u) + \beta_u^\gamma \hat{Y}_u^\gamma\right\} du\right].$$

Taking into account the terminal condition $\hat{Y}_T^\gamma = \partial_s g(\hat{S}_T^\gamma)$ and the Euler Optimality condition for $\hat{\alpha}$ we get: for all $\beta = (\beta^1, \cdots, \beta^\Gamma) \in \mathcal{A}^\Gamma$:

$$\begin{aligned} 0 = \sum_\gamma \pi^\gamma \mathbb{E}\Bigg[\int_0^T &\left\{\hat{Y}_u^\gamma + P_u^{\bar{\hat{\alpha}}} + \partial_\alpha L_T^\gamma(Q_u^\gamma, \hat{\alpha}_u + \partial_\alpha L_S(\hat{S}_u, \hat{\alpha}_u)\right. \\ &\left. -p'\left(-Q_u^0 - \Pi_\Gamma \cdot \left(\bar{Q}_u - \bar{\hat{\alpha}}_u\right)\right)\left(-Q_u^0 - \Pi_\Gamma \cdot \left(\bar{Q}_u - \bar{\hat{\alpha}}_u\right)\right)\right\} \beta_u^\gamma du\Bigg]. \end{aligned}$$

We deduce the coupling condition (4.49).

Proposition 4.4.8. *Assume that $\hat{\alpha}$ is a mean field optimal control for the problem with a pricing rule p. Then $\hat{\alpha}$ is a mean field Nash equilibrium for the MFG problem with pricing rule*

$$p^{\mathrm{MFG}}(x) = p(x) + xp'(x)\,. \tag{4.50}$$

Proof. This result follows straightforward from comparing the two coupling condition (4.49) and (4.41) since the two McKean-Vlasov BSDEs (4.40) and (4.48) are of the same form.

4.4.3 Explicit solution of the MFC with 1 region

In this section, we provide an example where an explicit solution of the MFC problem is obtained. We consider a linear pricing rule of the following form

$$p(x) = p_0 + p_1 x. \tag{4.51}$$

The storage cost L_S is defined by: For $A_1 < 0, A_2 > 0, C < 0$,

$$L_S(s, \alpha) = A_1 s + \frac{A_2}{2}s^2 + \frac{C}{2}\alpha^2.$$

For some given positive constant $\{K^\gamma\}_{\gamma=1}^\Gamma$, the transmission cost L_T^γ is defined by

$$L_T^\gamma(q, \alpha) = \frac{K^\gamma}{2}(q - \alpha)^2.$$

For some constants B_1 and $B_2 > 0$, the terminal cost

$$g(s) = \frac{B_2}{2}\left(s - \frac{B_1}{B_2}\right)^2.$$

Now, we will consider the simple case of one region, i.e. when $\pi = 1$. we aim to find an explicit solution to the MFC problem associated to the linear quadratic case.

Step 1 In this first step, we use the forward-backward system (4.48)-(4.30) and the coupling condition (4.49) in order to get the optimal control $\bar{\alpha}$ and the optimal trajectory $\bar{S}$ associated to one node in this region. We have

$$\begin{cases} d\bar{S}_t = \bar{\alpha}_t dt, \quad \bar{S}_0 = 0, \\ d\bar{Y}_t = -(A_2\bar{S}_t + A_1)dt + \bar{Z}^0_t dB^0_t + \int_E \bar{V}^{\gamma,\star}_t(e)\tilde{N}(dt,de), \quad \bar{Y}_T = B_2\bar{S}_T - B_1. \end{cases} \tag{4.52}$$

To find the optimal control $\bar{\alpha}$, we use firstly the coupling condition (4.49) to obtain

$$\bar{Y}_t - K(\bar{Q}_t - \bar{\alpha}_t) + C\bar{\alpha}_t + P^{\bar{\alpha}}_t - p'(-Q^0_t - \bar{Q}_t + \bar{\alpha}_t)(-Q^0_t - \bar{Q}_t + \bar{\alpha}_t) = 0, \tag{4.53}$$

where Q^0 and Q are defined by (4.36). Now, Proposition 3.3 in [2] and the linear form of p in (4.51) give

$$P^{\bar{\alpha}}_t = p_0 + 2p_1(-Q^0_t - \bar{Q}_t + \bar{\alpha}_t), \tag{4.54}$$

and we obtain the following expression of the optimal control $\bar{\alpha}$:

$$\begin{aligned} \bar{\alpha}_t &= -\frac{1}{K + C + p_1}\Big(\bar{Y}_t + p_0 - p_1 Q^0_t - (p_1 + K)\bar{Q}_t\Big) \\ &= -\Delta(\bar{Y}_t + b_t), \end{aligned}$$

where $\Delta = \frac{1}{K+C+p_1}$ and $b_t = p_0 - p_1 Q^0_t - (p_1 + K)\bar{Q}_t$.
We expect the solution of the FBSDE (4.52) to be affine. It has the following form:

$$\bar{Y}_t = \bar{\phi}_t \bar{S}_t + \bar{\psi}_t, \tag{4.55}$$

where ϕ and ψ are deterministic functions. Computing $d\bar{Y}_t$ from this expression, we obtain

$$d\bar{Y}_t = \bar{S}_t(-\Delta\bar{\phi}^2_t + \dot{\bar{\phi}}_t)dt - \Delta\bar{\phi}_t(\bar{\psi}_t dt + b_t)dt + \dot{\bar{\psi}}_t. \tag{4.56}$$

Identifying the two expressions of $d\bar{Y}_t$ we get, in one hand, that

$$\dot{\bar{\phi}}_t - \Delta\bar{\phi}^2_t + A_2 = 0, \quad \bar{\phi}(T) = B_2 \tag{4.57}$$

which is a Riccati equation. In the other hand, we obtain that ψ is the unique solution of the BSDE

$$d\bar{\psi}_t = \Delta\bar{\phi}_t(\bar{\psi}_t + b_t)dt - A_1dt + \bar{Z}_t^0 dB_t^0 + \int_E \bar{V}_t^{\gamma,\star}(e)\tilde{N}(dt,de). \tag{4.58}$$

Consequently, substituting b_t, we get the following BSDE

$$d\bar{\psi}_t = \Delta\bar{\phi}_t(\bar{\psi}_t + \bar{P}_t)dt + \bar{Z}_t^0 dB_t^0 + \int_E \bar{V}_t^{\gamma,\star}(e)\tilde{N}(dt,de), \quad \psi_T = -B_1. \tag{4.59}$$

This allows us to find the expression of the approximated electricity price. In fact, in one hand we have

$$\begin{aligned}
\bar{\psi}_t &= \Delta\bar{\phi}_t(\bar{\psi}_t + \bar{P}_t)dt + \bar{Z}_t^0 dB_t^0 + \int_E \bar{V}_t^{\gamma,\star}(e)\tilde{N}(dt,de) \\
&= \Delta\bar{\phi}_t(\bar{Y}_t - \bar{\phi}_t\bar{S}_t + \bar{P}_t)dt + \bar{Z}_t^0 dB_t^0 + \int_E \bar{V}_t^{\gamma,\star}(e)\tilde{N}(dt,de) \\
&= \Delta\bar{\phi}_t\bar{Y}_t dt - \Delta(\bar{\phi}_t)^2\bar{S}_t dt + \int_E \bar{V}_t^{\gamma,\star}(e)\tilde{N}(dt,de).
\end{aligned}$$

In the other hand, we have that $d\bar{\psi}_t = d\bar{Y}_t - \dot{\bar{\phi}}_t S_t dt - \bar{\phi}_t dS_t$. So, we obtain

$$-(A_2\bar{S}_t dt + A_1)dt - \Delta\bar{\phi}_t\bar{P}_t dt = \dot{\bar{\phi}}_t\bar{S}_t dt + \bar{\phi}_t d\bar{S}_t + \Delta\bar{\phi}_t\bar{Y}_t - \Delta(\bar{\phi}_t)^2\bar{S}_t dt.$$

Finally, using the Riccati equation (4.57) in the equation above, we obtain directly the following price expression

$$\bar{P}_t = -\frac{A_1}{\Delta\bar{\phi}_t} + b_t.$$

As it can be seen, $\bar{\Psi}$ is the solution of linear BSDE with jumps. So it has the following expression

$$\bar{\Psi}_t = \mathbb{E}[-\Gamma_{t,T}B_1 + \int_t^T \Gamma_{t,u}\Delta\bar{\phi}_u\bar{P}_u du|\mathcal{F}_t^0],$$

where $\Gamma_{t,T}$ is the adjoint process and in this case, it is the solution of

$$d\Gamma_{t,s} = \Gamma_{t,s}\Delta\bar{\phi}_s ds,$$

which is $\Gamma_{t,s} = \exp(\int_t^s \Delta\bar{\phi}_s ds)$. Consequently, Ψ_t is given by

$$\bar{\Psi}_t = -B_1\exp\left\{-\int_t^T \Delta\bar{\phi}(u)du\right\} - \mathbb{E}\left[\int_t^T \Delta\bar{\phi}(u)\exp\left\{-\int_t^u \Delta\bar{\phi}(s)ds\right\}\bar{P}_u du|\mathcal{F}_t^0\right].$$

The function $\bar{\phi}$ is given by

$$\bar{\phi}(t) \quad = \quad -\frac{\rho}{\Delta}\frac{e^{-\rho(T-t)}(-B_2\Delta+\rho)-e^{\rho(T-t)}(B_2\Delta+\rho)}{e^{-\rho(T-t)}(-B_2\Delta+\rho)+e^{\rho(T-t)}(B_2\Delta+\rho)} \quad \text{with} \quad \rho := \sqrt{A_2\Delta},$$

Now, to find $\bar{S}_t$, it suffices to solve the following simple EDO

$$d\bar{S}_t = \big[-\Delta\bar{\phi}_t\bar{S}_t - \Delta(\Psi_t + \bar{P}_t + \frac{A_1}{\Delta\bar{\phi}_t})\big]dt, \tag{4.60}$$

for which the solution is given BY

$$\bar{S}_t = \bar{S}_0\exp(\int_0^t -\Delta\bar{\phi}_s ds) - \Delta\int_0^t \exp(\int_u^t -\Delta\bar{\phi}_s ds)(\Psi_u + \bar{P}_u + \frac{A_1}{\Delta\bar{\phi}_u})du. \tag{4.61}$$

As $\bar{S}_0 = 0$, the solution is then

$$\bar{S}_t = -\Delta\int_0^t \exp\left\{-\int_u^t \Delta\bar{\phi}(s)ds\right\}\left(\bar{P}_u + \bar{\Psi}_u + \frac{A_1}{\Delta\bar{\phi}(u)}\right)du.$$

Step 2 Once we obtain all the optimal elements of one node in the first step, we use the FBSDE (4.40) and the coupling condition (4.41) to find the optimal objects associated to one region containing a number of identical nodes. Using the FBSDE (4.40), we have

$$\begin{aligned} dS_t &= -\delta\left(Y_t + P_t + \frac{A_1}{\delta\phi(t)}\right)dt, \quad S_0 = s_0, \\ dY_t &= -(A_2S_t + A_1)dt + Z_t^0 dB_t^0 + Z_t dB_t + \int_E V_t(e)\tilde{N}(dt,de), \quad Y_T = B_2S_T - B_1. \end{aligned}$$

Again, we look at a solution of the form

$$Y_t \quad = \quad \varphi(t)S_t + \psi_t,$$

and using the coupling condition (4.41), we obtain the expression the optimal control α. In fact,

$$Y_t - K(Q_t - \alpha_t) + C\alpha_t + P_t^{\bar{\alpha}} - p_1(-Q_t^0 - Q_t - \alpha_t) = 0$$

where $P_t^{\bar{\alpha}} = p_0 + 2p_1(-Q_t^0 - \bar{Q}_t + \bar{\alpha}_t)$. Then

$$\begin{aligned} \alpha_t &= -\delta(Y_t + p_0 - KQ_t - 2p_1(Q_t^0 + \bar{Q}_t - \bar{\alpha}_t) + \frac{A_1}{\delta\phi_t}) \\ &= -\delta(Y_t + P_t + \frac{A_1}{\delta\phi_t}), \end{aligned}$$

where $\delta = \frac{-1}{K+C}$, $P_t = p_0 - KQ_t - 2p_1(Q_t^0 + \bar{Q}_t - \bar{\alpha}_t) - \frac{A_1}{\delta\phi_t}$. Once again, we aim to find a solution to the FBSDE above which has the following form

$$Y_t = \varphi(t)S_t + \psi_t,$$

where φ and ψ can be explicitly calculated in the same way as before. In fact, ψ is the solution of the following equation

$$d\psi_t = -\delta\varphi_t(\psi_t dt + P_t) + Z_t^0 dB_t^0 + Z_t dB_t + \int_E V_t(e)\tilde{N}(de, dt) \tag{4.62}$$

and φ_t satisfies the following Riccati equation

$$\delta\varphi_t^2 - \dot{\varphi}_t - A_2 = 0 \tag{4.63}$$

whose solution is given by

$$\varphi(t) = -\frac{\rho}{\delta}\,\frac{e^{-\rho(T-t)}(-B_2\delta + \rho) - e^{\rho(T-t)}(B_2\delta + \rho)}{e^{-\rho(T-t)}(-B_2\delta + \rho) + e^{\rho(T-t)}(B_2\delta + \rho)} \quad \text{with} \quad \rho := \sqrt{A_2\delta},$$

As it can be seen, ψ_t is the solution of the following BSDE driven by a 2-dimensional Brownian motion

$$d\psi_t = -\delta\varphi_t(\psi_t + P_t)dt + \tilde{Z}_t d\tilde{B}_t + \int_E V_t(e)\tilde{N}(de, ds), \tag{4.64}$$

where $\tilde{Z}_t = (Z_t^0, Z_t)$ and $\tilde{B}_t = (B_t^0, B_t)$.

Finally

$$S_t = s_0 \exp\left\{-\int_0^t \delta\varphi(u)du\right\} - \delta\int_0^t \exp\left\{-\int_u^t \delta\varphi(s)ds\right\}\left(P_u + \psi_u + \frac{A_1}{\delta\phi(u)}\right)du.$$

Remark 4.4.9. *Notice that in the example that we treated above, we only assume the presence of a common noise B^0 (no common jump $\tilde{N}^0$) in order to simplify computations. However, we emphasize that the presence of the common jump $\tilde{N}^0$ make just little changes in the proof.*

4.5 Appendix

In this section, we extend some of the results of Hamadène [61] concerning FBSDEs in the Brownian setting to the case of jumps. Let us note that arguments of proof are close to the one used by Hamadǹe in [61] with some minor modifications due to jumps setting. However, we still provide the proof of existence.

We look for the solution of the following fully coupled forward-backward SDE with jumps

$$
(\mathbf{S})\begin{cases} X_t = X_0 + \displaystyle\int_0^t b_s(X_s,Y_s,Z_s,K_s)ds + \int_0^t \sigma_s(X_s,Y_s,Z_s,K_s(e))dW_s \\ \qquad + \displaystyle\int_0^t\int_E \beta_s(X_{s^-},Y_{s^-},Z_s,K_s(e))\tilde{\pi}(ds,de), \;\; 0\le t\le T, \; \mathbb{P}\text{-a.s.} \\ Y_t = g(X_T,\mathbb{P}_{X_T}) - \displaystyle\int_t^T h_s(X_s,Y_s,Z_s,K_s)ds - \int_t^T Z_s dW_s - \int_t^T\int_E K_s(e)\tilde{\pi}(ds,de), \end{cases} \tag{4.65}
$$

We assume the following assumptions.

• The functions f,h,σ,β defined on $\mathbb{R}^{d+d+d\times d}\times\mathbb{L}^0(\mathcal{B}(E),\nu)$ are Lipschitz in (x,y,z,k) and uniformly in $\omega\in\Omega$.

• The function g is defined on $\Omega\times\mathbb{R}^d$ and valued in $\mathbb{R}^d$ such that for any $x\in\mathbb{R}^d$, g is $\mathcal{F}_T$-measurable and square integrable. Moreover, g is Lipschitz in x and uniformly in $\omega\in\Omega$.

Finally, for $u=(x,y,z,k)$ and $u'=(x',y',z',k')\in\mathbb{R}^{d+d+d\times d}$, we define the function $\mathcal{A}$ as follows

$$
\begin{aligned}
\bar{\mathcal{A}}(t,u,u') &= [b_t(x,y,z,k)-b_t(x',y',z',k')](y-y') + [h_t(x,y,z,k)-h_t(x',y',z',k')](x-x') \\
&+ [\sigma_t(x,y,z,k)-\sigma_t(x',y',z',k')](z-z') + \int_E(\beta_t(x,y,z,k)-\beta_t(x',y',z',k'))(k-k')(e)\eta(dt,de).
\end{aligned}
$$

We make the following assumption

Assumption 4.5.1.

$$
(\bar{H}1)\begin{cases} (i) \text{ *There exists* } k>0, \text{ *s.t* } \forall t\in[0,T],\nu\in\mathcal{M}_2(\mathbb{R}^d\times\mathbb{R}^d), u,u'\in\mathbb{R}^{d+d+d\times d}\times\mathbb{L}^0(\mathcal{B}(E),\eta), \\ \qquad \bar{\mathcal{A}}(t,u,u')\le -k|x-x'|^2, \mathbb{P}\text{-a.s.} \\ (ii) \text{ *There exists* } k'>0, \text{ *s.t* } \forall\nu\in\mathcal{M}_2(\mathbb{R}^d\times\mathbb{R}^d), x,x'\in\mathbb{R}^d \\ \qquad (g(x,\nu)-g(x',\nu)).(x-x')\ge k'|x-x'|^2, \mathbb{P}\text{-a.s.} \\ (iii)\; min\{k,k'\}\ge C(\tilde{C}+1). \end{cases}
$$

Proposition 4.5.2. *Under Assumption* $(\bar{H}1)$*, there exists a unique solution* $U=(X,Y,Z,K)$ *of the FBSDE with jumps* (4.65).

Proof. In the following poof, we will use the notation $\tilde{C}$ to denote a generic constant that may change from line to line and that depends in an implicit way on T and the Lipschitz constants. The key point of the proof is to consider a sequence $U^n=(X^n,Y^n,Z^n,K^n)$ of processes defined recursively by : $(X^0,Y^0,Z^0,K^0)=(0,0,0,0)$ and for $n\ge 1$, $U^n=(X^n,Y^n,Z^n,K^n)$

satisfies, for all $t \in [0,T]$ and $\delta \in]0,1]$, the following system

$$\begin{cases} X_t^{n+1} = x + \int_0^t \Big(f_s(X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1}, K_s^{n+1}) - \delta(Y_s^{n+1} - Y_s^n) \Big) ds \\ \qquad + \int_0^t \Big(\sigma_s(X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1}, K_s^{n+1}) - \delta(Y_s^{n+1} - Y_s^n) \Big) dW_s \\ \qquad + \int_0^t \int_E \Big(\beta_s(X_{s^-}^{n+1}, Y_{s^-}^{n+1}, Z_s^{n+1}, K_s^{n+1}) - \delta(K_s^{n+1} - K_s^n) \Big) \tilde{\pi}(ds, de), \\ Y_t^{n+1} = g(X_T^{n+1}) - \int_t^T h_s(X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1}, K_s^{n+1}) ds - \int_t^T Z_s^{n+1} dW_s - \int_t^T \int_E K_s^{n+1}(e) \tilde{\pi}(ds, de)). \end{cases}$$

For $n \geq 1$, $t \in [0,T]$, we consider the following processes

$$\hat{X}_t^{n+1} := X_t^{n+1} - X_t^n, \quad \hat{Y}_t^{n+1} := Y_t^{n+1} - Y_t^n, \quad \hat{Z}_t^{n+1} := Z_t^{n+1} - Z_t^n, \qquad \hat{K}_t^{n+1} := K_t^{n+1} - K_t^n.$$

and for a function $\phi = \{f, h, \sigma, \beta\}$, we set

$$\hat{\phi}_t^{n+1} := \phi(t, U_t^{n+1}, \nu_t^n) - \phi(t, U_t^n, \nu_t^{n-1}), \quad \tilde{\phi}_t^n := \phi(t, U_t^n, \nu_t^n) - \phi(t, U_t^n, \nu_t^{n-1}).$$

In order to prove the existence of the solution, we will show that $(X^n, Y^n, Z^n, K^n)_{n\geq 0}$ is a Cauchy sequence. First, we apply Itô's formula to $\hat{X}^{n+1}\hat{Y}^{n+1}$ and take the expectation

$$\mathbb{E}[\hat{X}_T^{n+1}\hat{Y}_T^{n+1}] = \mathbb{E}[\int_0^T \hat{Y}_s^{n+1}[\hat{f}_s^{n+1} - \delta(\hat{Y}_s^{n+1} - \hat{Y}_s^n)]ds + \mathbb{E}[\int_0^T \hat{X}_s^{n+1}\hat{h}_s^{n+1} ds]$$
$$+ \mathbb{E}[\int_0^T (\hat{\sigma}_s^{n+1} - \delta(\hat{Z}_s^{n+1} - \hat{Z}_s^n), \hat{Z}_s^{n+1}) ds + \int_0^T \int_E \hat{K}_s^{n+1}(\hat{\beta}_s^{n+1} - \delta(\hat{K}_s^{n+1} - \hat{K}_s^n))\eta(de, ds)].$$

Rearranging terms, we get

$$\begin{aligned} &\mathbb{E}[\hat{X}_T^{n+1}(g(X_T^{n+1}) - g(X_T^n))] + \delta\mathbb{E}[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds] \\ &-\mathbb{E}[\int_0^T \hat{X}_s^{n+1}\hat{h}_s^{n+1} + \hat{Y}_s^{n+1}\hat{f}_s^{n+1} + \hat{\sigma}^{n+1}(s)\hat{Z}_s^{n+1} ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{\beta}_s^{n+1}\eta(de, ds)] \\ &= \delta\mathbb{E}[\int_0^T \hat{Y}_s^{n+1}\hat{Y}_s^n + \hat{Z}_s^{n+1}\hat{Z}_s^n ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{K}_s^n \eta(de, ds)]. \end{aligned} \tag{4.66}$$

Using Assumption 4.5.1, we get

$$\begin{aligned} &\mathbb{E}[k'|\hat{X}_T^{n+1}|^2 + \delta \int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds + k\int_0^T |\hat{X}_s^{n+1}|^2 ds] \\ &\leq \delta\mathbb{E}[\int_0^T \hat{Y}_s^{n+1}\hat{Y}_s^n + \hat{Z}_s^{n+1}\hat{Z}_s^n ds + \int_0^T \int_E \hat{K}_s^{n+1}\hat{K}_s^n \eta(de, ds)]. \end{aligned} \tag{4.67}$$

In addition, the elementary inequality $ab \leq a^2/2 + b^2/2$ yields to

$$\begin{aligned}
&\mathbb{E}[\int_0^T \hat{Y}_s^{n+1}\hat{Y}_s^n + \hat{Z}_s^{n+1}\hat{Z}_s^n ds] + \int_0^T \int_E \hat{K}_s^{n+1}\hat{K}_s^n \, \eta(de, ds) \\
&\leq \frac{1}{2}\mathbb{E}[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds] \\
&+ \frac{1}{2}\mathbb{E}[\int_0^T |\hat{Y}_s^n|^2 + \|\hat{Z}_s^n\|^2 + |\hat{K}_s^n|_s^2 ds].
\end{aligned} \tag{4.68}$$

Pugging (4.68) in (4.67), we obtain

$$k^{'}\mathbb{E}[|\hat{X}_T^{n+1}|^2] + k\mathbb{E}[\int_0^T |\hat{X}_s^{n+1}|^2 ds] + \frac{\delta}{2}\mathbb{E}[\int_0^T |\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{K}_s^{n+1}|_s^2 ds] \tag{4.69}$$

$$\leq \frac{\delta}{2}\left[\mathbb{E}[\int_0^T |\hat{Y}_s^n|^2 + \|\hat{Z}_s^n\|^2 + |\hat{K}_s^n|_s^2 ds]\right]. \tag{4.70}$$

Now, using Itô's formula to $|\hat{Y}^n|^2$, we obtain classically that

$$\forall n \geq 1, \exists C^1 > 0,\ \mathbb{E}\left[\int_0^T |\hat{Y}_s^n|^2 + \|\hat{Z}_s^n\|^2 + |\hat{K}^n|_s^2 ds\right] \leq C^1\Big(\mathbb{E}[|\hat{X}_T^n|^2 + \int_0^T |\hat{X}_s^n|^2 ds]\Big).$$

Plugging this estimates in (4.69), we obtain by choosing $\delta := \min(k, k^{'})/\tilde{C}$,

$$\mathbb{E}[|\hat{X}_T^{n+1}|^2] + \mathbb{E}[\int_0^T |\hat{X}_s^{n+1}|^2] \leq \frac{1}{2^n}[\mathbb{E}|\hat{X}_T^1|^2 + \mathbb{E}\int_0^T |\hat{X}_s^1|^2 ds].$$

Besides, we also have for all $n \geq 1$

$$\mathbb{E}\left[\int_0^T |\hat{Y}_s^n|^2 + \|\hat{Z}_s^n\|^2 + |\hat{K}^n|_s^2 ds\right] \leq \frac{C^1}{2^{n-1}}\Big(\mathbb{E}\left[|\hat{X}_T^1|^2 + \int_0^T |\hat{X}_s^1|^2 ds\right]\Big).$$

Thus, $(\hat{X}_T^n)_{n\geq 0}$ is a Cauchy sequence in $\mathbb{H}^2(\Omega, \mathbb{P})$ and $(\hat{X}^n)_{n\geq 0}, (\hat{Y}^n)_{n\geq 0}, (\hat{Z}^n)_{n\geq 0}$ and $(\hat{K}^n)_{n\geq 0}$ are Cauchy sequences respectively in $\mathbb{H}^2([0,T], \Omega, dt \otimes d\mathbb{P})$and $\mathbb{H}^2_\eta([0,T], \Omega, dt \otimes d\eta)$. Hence, if X, Y, Z and K are the respective limits of these sequences, passing to the limit in (4.5), gives us the desired result.

In order to prove that the system (4.65) has a unique solution, we suppose that $(\bar{X}, \bar{Y}, \bar{Z}, \bar{K})$ is also solution of (4.65). Let $\bar{X} = X^1 - X^2$, $\bar{Y} = Y^1 - Y^2$, $\bar{X} = Z^1 - Z^2$, $\bar{K} = K^1 - K^2$.

Apply Itô's formula to the product $\bar{X}\bar{Y}$, it follows straightforwardly from Assumption $(\bar{H}1)$ that

$$k^{'}\mathbb{E}[|\bar{X}_T|^2] + k\mathbb{E}[\int_0^T |\bar{X}_s|^2 ds] \leq 0,$$

and conclude that $\bar{X}_T = 0$, $\bar{X}_t = 0,\ \forall t \in [0,T],\ \mathbb{P}$-a.s. Thus, in order to finish the proof we need to show that $(\bar{Y}, \bar{Z}, \bar{K}) = (0, 0, 0)$. This is actually a simple consequence of [138].

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UNIVERSITE BRETAGNE LOIRE / MATHSTIC

Le Mans Université

Titre : Contributions aux équations différentielles stochastiques rétrogrades quadratiques avec sauts et applications.

Mots clés : EDSR avec sauts, EDSR doublement réfléchies avec sauts, croissance quadratique (stochastique), activité infinie, EDSR progressive-rétrograde de type champ moyen.

Résumé : Cette thèse porte sur l'étude des équations différentielles stochastiques rétrogrades (EDSR) avec sauts et leurs applications.
Dans le chapitre 1, nous étudions une classe d'EDSR lorsque le bruit provient d'un mouvement Brownien et d'une mesure aléatoire de saut indépendante à activité infinie. Plus précisément, nous traitons le cas où le générateur est à croissance quadratique et la condition terminale est non bornée. L'existence et l'unicité de la solution sont prouvées en combinant à la fois la procédure d'approximation monotone et une approche progressive. Cette méthode permet de résoudre le cas où la condition terminale est non bornée.
Le chapitre 2 est consacré aux EDSR avec sauts généralisées doublement réfléchies sous des hypothèses d'integrabilités faibles. Plus précisément, on montre l'existence d'une solution pour un générateur à croissance quadratique stochastique et une condition terminale non bornée.
Dans le chapitre 3, nous considérons une classe générale d'EDSR progressive-rétrograde couplée avec sauts de type Mackean Vlasov sous une condition faible de monotonicité. Les résultats d'existence et d'unicité sont établis sous deux classes d'hypothèses en se basant sur des schémas de perturbations soit de l'equation differentielle stochastique progressive, soit de l'equation differentielle stochastique retrograde.
On conclut le chapitre par un problème de stockage optimal d'énergie dans un parc électrique de type champs moyen.

Title: Contributions to quadratic backward stochastic differential equation with jump and applications.

Keywords : BSDEs with jumps, doubly reflected BSDEs, stochastic quadratic growth, infinite activity jump, Mean-field Forward Backward SDEs with jumps.

Abstract :
This thesis focuses on backward stochastic differential equation with jumps and their applications.
In the first chapter, we study a backward stochastic differential equation (BSDE for short) driven jointly by a Brownian motion and an integer valued random measure that may have infinite activity with compensator being possibly time inhomogeneous. In particular, we are concerned with the case where the driver has quadratic growth and unbounded terminal condition. The existence and uniqueness of the solution are proven by combining a monotone approximation technics and a forward approach.
Chapter 2 is devoted to the well-posedness of generalized doubly reflected BSDEs (GDRBSDE for short) with jumps under weaker assumptions on the data. In particular, we study the existence of a solution for a one-dimensional GDRBSDE with jumps when the terminal condition is only measurable with respect to the related filtration and when the coefficient has general stochastic quadratic growth.
In chapter 3, we investigate a general class of fully coupled mean field forward-backward under weak monotonicity conditions without assuming any non-degeneracy assumption on the forward equation. We derive existence and uniqueness results under two different sets of conditions based on proximation schema weither on the forward or the backward equation. Later, we give an application for storage in smart grids.