Analyse probabiliste des règles de vote : méthodes et résultats

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Analyse probabiliste des règles de vote : méthodes et résultats.

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À la mémoire de mon père et de mon grand-père

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> S’il est une question centrale qui peut être envisagée comme la problématique principale motivant la théorie du choix social, c’est la suivante : comment est-il possible de parvenir à des jugements agrégés et incontestables au niveau de la société... – Amartya Sen [36]

Introduction

1. Vote et paradoxes

1.1 Vote majoritaire et paradoxe de Condorcet

L'agrégation des préférences individuelles en une préférence collective est une problématique importante dans la théorie économique moderne. Le choix social en a fait une question centrale. La sélection de la meilleure alternative globale parmi un ensemble d'alternatives a été étudiée depuis longtemps et sous diverses formes.

Dans un processus d'élection avec deux candidats, la majorité est la meilleure règle de vote. Le candidat élu est celui qu'une majorité d'électeurs préfère à l'autre. En 1952, May [28] en donne une caractérisation et une justification théorique : pour lui, dans un choix binaire avec deux options, la règle de la majorité est la seule règle de décision qui est à la fois neutre, anonyme, monotone et non manipulable. En d'autres mots, c'est une règle qui ne favorise ou désavantage aucun des deux candidats, où les électeurs sont égaux et sont incités à exprimer leurs vraies préférences.

Cependant, avec trois candidats ou plus, l'agrégation des préférences en un choix collectif pose quelques problèmes. En 1785, Condorcet[1] [9], dans son Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix, souligne les difficultés de l'agrégation de préférences et d'opinions individuelles en un choix collectif. Il a montré que le vote majoritaire peut conduire à une préférence collective non transitive et cyclique : il peut arriver lors d'une élection qu'un candidat A soit préféré à un autre candidat B par une majorité d'individus, qu'une autre majorité de votants préfère B à C, et qu'une autre encore préfère C à A. Donc les décisions prises peuvent ne pas être cohérentes avec celles que prendrait un individu rationnel, car le choix entre A et C ne serait pas le même selon que B est présent ou non. Ceci est historiquement le premier exemple de paradoxe de vote, on parle de paradoxe de Condorcet ou, suivant Guilbaud [19], de l'effet Condorcet. Par paradoxe, on désigne ici un résultat contre-intuitif auquel on peut aboutir à l'issu d'un scrutin, ou un phénomène allant à l'encontre de ce que dicterait la conformité sociale, plutôt qu'une contradiction purement

[1] Marie Jean Antoine Nicolas Caritat, marquis de Condorcet, est un mathématicien et homme politique français, représentant des Lumières, né le 17 septembre 1743 à Ribemont en Picardie et mort le 29 mars 1794 à Bourg-la-Reine

logique. L'exemple original présenté par Condorcet d'une situation de vote avec 60 électeurs et trois candidats est le suivant :

23	17	2	10	8
A	B	B	C	C
B	C	A	A	B
C	A	C	B	A

Table 1

Lorsqu'on effectue les comparaisons majoritaires par paires, ce système nous donne que :

A est préféré à B (33–27),

B est préféré à C (42–18),

C est préféré à A (35–25).

Ainsi, A $\succ$ B $\succ$ C $\succ$ A (le symbole $\succ$ désignant la préférence collective). On appelle cette situation un cycle de Condorcet. Chaque candidat est battu par au moins un autre ; il n'existe donc pas de *vainqueur de Condorcet*, c'est- à-dire de candidat capable de l'emporter à la majorité sur chacun de ses concurrents.

Le vainqueur de Condorcet peut donc ne pas exister. Notons cependant que, sous certaines conditions, on peut outrepasser ce paradoxe de Condor- cet. Par exemple lorsque les préférences sont unimodales[2], l'existence d'un vainqueur de Condorcet est garantie.

1.2 Borda et les règles positionnelles

Borda[3], qui était le contemporain de Condorcet, propose une approche alternative dite de classement par points. Dans la méthode qu'il propose, les électeurs construisent chacun une liste de n candidats par ordre de préférence. Le premier de la liste, reçoit n points, le second n-1 points, et ainsi de suite, le nième de la liste se voyant attribuer 1 point. Le score d'un candidat est alors la somme de tous les points qui lui ont été attribués. Le (ou les) candidat(s) dont le score est le plus élevé remporte(nt) l'élection. La règle de la pluralité (ou de la majorité simple),

[2] La condition d'unimodalité a été introduite par Black [5]. Elle revient à exclure certains ordres de préférences.

[3] Jean-Charles, chevalier de Borda, né le 4 mai 1733 à Dax et mort le 19 février 1799 à Paris, est un mathématicien, physicien, politologue et navigateur français.

qui consiste à donner 1 point pour une première place et 0 point pour toute autre position, constitue comme la règle de Borda une méthode de classement par points ; ces méthodes sont aussi appelées règles positionnelles simples. Cette famille de règles a la particularité de garantir la condition de consistance. C'est-à-dire que si un électorat est partagé en deux groupes, et si un candidat est le vainqueur dans chaque groupe, alors ce dernier va rester le vainqueur si les deux groupes sont réunis. Cette propriété de consistance constitue un argument fort en faveur de l'utilisation des règles de classement par points. Cependant, ces règles ne respectent pas toujours le critère dit de Condorcet (ou critère majoritaire) : le vainqueur peut ne pas être le vainqueur de Condorcet (lorsque celui-ci existe).

Ce qu'on appelle le paradoxe de Borda résulte d'une observation très intéressante concernant les conflits possibles entre le principe majoritaire et la règle de la pluralité pour déterminer le vainqueur d'une élection. L'exemple original de Borda [6] en ce qui concerne ce phénomène utilise la situation de vote de la table 2 pour 21 électeurs avec des préférences strictes et transitives lors d'une élection à trois candidats.

1	7	7	6
A	A	B	C
B	C	C	B
C	B	A	A

Table 2

Si on utilise la règle de la pluralité, on aura A ≻ B (8–7), A ≻ C (8–6) et B ≻ C (7–6), soit le classement A ≻ B ≻ C. Un résultat très différent est observé en utilisant la règle de la majorité. On a B ≻ A (13–8), C ≻ A (13–8) et C ≻ B (13–8), donc C ≻ B ≻ A (notons que le candidat A constitue ici ce qu'on appelle un *perdant de Condorcet*). Selon qu'on utilise la règle de la pluralité ou le principe majoritaire, le classement final sera inversé.

1.3 Le cas du vote majoritaire à deux tours

Parmi les méthodes de choix collectif, le vote majoritaire à deux tours est une règle très intéressante. Elle est utilisée dans de nombreux pays, et particulièrement en France pour l'élection présidentielle. Avec cette règle, on a un bulletin uninominal. Au premier tour, le candidat qui obtient plus de la moitié des voix est déclaré vainqueur, sinon les deux candidats ayant reçu le plus de voix vont au second tour. Au second tour, le candidat avec le plus de voix est élu. Un exemple pour illustrer cette méthode est présenté dans la table 3.

10	6	5
A	B	C
B	C	B
C	B	A

Table 3

Au premier tour, A et B obtiennent un score de 10 et 6 (respectivement), et C n'obtient qu'un score de 5. Comme A n'a pas la majorité absolue, A et B vont au second tour. Au second tour, B est élu avec un score de 11 contre 10 pour A.

Un des avantages du vote majoritaire à deux tours est qu'il n'élit jamais le perdant de Condorcet. Cependant, comme toute autre règle de vote, il présente un certain nombre d'inconvénients. Par exemple il ne respecte pas le critère de Condorcet : il arrive dans certaines élections que le vainqueur ne soit pas celui qui est préféré par la majorité. Au-delà du viol du critère de Condorcet, le vote majoritaire à deux tours présente l'inconvénient de ne pas être « monotone » [24], c'est-à-dire que ce système de vote peut réagir de manière contre-intuitive face à de légères modifications des préférences des votants.

6	5	4	2
A	C	B	B
B	A	C	A
C	B	A	C

Table 4

Dans la table 4, A et B vont au second tour et A est élu avec un score de 11 contre 6 pour B. Supposant maintenant que 2 électeurs changent leurs préférences au profit du candidat A et que l'on obtienne les préférences montrées dans la table 5 :

8	5	4
A	C	B
B	A	C
C	B	A

Table 5

Au premier tour, les scores de A, B et C sont respectivement 8, 4 et 5. Donc A et C vont au second tour et C est vainqueur avec un score de 9 contre 8 pour A. Par conséquent, ce n'est plus A mais C qui l'emporte : dans un vote majoritaire à deux tours, gagner des suffrages peut ainsi faire perdre l'élection. Ce qui est contraire au bon sens.

Un autre défaut du scrutin majoritaire à deux tours est qu'il est manipulable : certains électeurs sont incités à exprimer une préférence non sincère.

Exemple :

10	6	5
B	C	A
A	A	D
C	B	B
D	D	C

Table 6

Au premier tour, A est éliminé. B et C vont au second tour et B est vainqueur avec 15 voix contre 6 pour C.

Dans la table 6, 6 votants peuvent être incités à changer leur préférence, passant de (C préféré à A préféré à D préféré à B) à (A préféré à C préféré à D préféré à B). On obtient la table 7 :

10	6	5
B	A	A
A	C	D
C	D	B
D	B	C

Table 7

Au premier tour, c'est maintenant C qui est éliminé et au second tour, c'est A qui est élu. Donc certains votants, qui préfèrent A à B, sont gagnants en mentant sur leur préférence.

Au-delà de ces inconvénients, le vote majoritaire à deux tours peut présenter aussi d'autres problèmes (absence d'incitation à la participation, non consistance ou encore dictature de la majorité) ; nous renvoyons sur ce point à S. Konieczny [26].
Il existe bien sûr d'autres règles que le vote majoritaire à deux tours, comme le vote par approbation, la règle de Weber, le jugement majoritaire [3] ou encore le scrutin de Condorcet randomisé [22]... Toutes ces règles de vote comportent des avantages et des inconvénients et sont susceptibles d'exhiber des paradoxes. La littérature du choix social les a largement étudiés. Le lecteur pourra consulter par exemple Felsenthal [13], Nurmi [31] [30], Saari [33], Gehrlein et Lepelley [16] [17] pour une revue exhaustive de ces paradoxes.

On peut bien sûr se demander si ces paradoxes sont propres à certaines règles de vote, ou s'il existe une règle qui en serait dépourvue. La réponse à cette question est hélas négative : Arrow [1] en 1951, par son célèbre théorème d'impossibilité, affirme qu'il n'existe pas de processus de choix social indiscutable, qui permette d'exprimer, à partir de l'agrégation des préférences individuelles, une hiérarchie des préférences qui soit cohérente. De manière plus précise, dès que le nombre de choix possibles est supérieur ou égal à 3, il n'existe pas de fonction de choix social qui, simultanément, satisfait aux conditions d'universalité, de transitivité, d'unanimité, d'indépendance et de non dictature. Le théorème d'Arrow a engendré une très vaste littérature, dont on peut trouver un aperçu dans Kelly [25], Sen [37], Fishburn [14].

Il existe par ailleurs dans la littérature d'autre théorèmes d'impossibilités, tel que le théorème de Gibbard-Satterthwaite (Gibbard [18] , Satterthwaite [34]), qui établit que toutes les fonctions de choix social sont manipulables,ou encore le théorème de Sen [35] qui montre qu'on ne peut concilier le principe de Pareto et un libéralisme minimal. Tous ces théorème généralisent en quelque sorte la découverte de Condorcet en montrant qu'il n'y a pas de règles d'agrégation qui peuvent satisfaire un certain ensemble de propriétés ou de principes (Niemi et Riker [29]) qu'il paraît raisonnable de respecter (comme on l'a vu, avec le paradoxe de Condorcet, c'est la transitivité qui n'est pas respectée quand on applique le vote majoritaire).

2. La probabilité des paradoxes

Face à ces théorèmes d'impossibilité, il est naturel de se demander si les fréquences d'occurrences des différents paradoxes qu'ils décrivent sont réellement significatives, ou s'ils sont seulement une sorte de curiosité ma- thématique. Par fréquence d'occurrence, on fait référence au nombre total de situations de vote où l'événement est susceptible de se produire, divisé par le nombre total de situations de vote possibles. La littérature dans ce domaine est très riche ; Gehrlein et Lepelley [16] [17] en présentent un panorama complet.

Les travaux de calcul de probabilités en théorie du vote considèrent deux types d'événements élémentaires : d'abord les profils, qui sont des listes ordonnées de préférences individuelles, puis les situations de vote, qui sont des profils anonymes (seuls comptent les nombres d'électeurs ayant telle ou telle préférence ; l'identité de ces électeurs n'intervient pas, comme dans les exemples donnés ci-dessus). Deux modèles principaux sont alors utilisés :

- Le modèle de culture neutre (IC) (Impartial Culture) :

Guilbaud [19] l'utilise le premier. Sous l'hypothèse (IC), tous les profils sont supposés avoir la même probabilité d'apparition. Cette probabilité est donc de $(\frac{1}{m!})^{n}$ où m est le nombre d'options et n le nombre de votants.

- Le modèle de culture neutre et anonyme IAC (Impartial Anonymous culture) :

Introduit par Gehrlein et Fishburn [15] en 1976, il suppose que chaque situation de vote a la même probabilité de se produire, une situation de vote étant définie comme une distribution des électeurs sur les préférences possibles.

D'autres modèles existent : culture duale (DC), culture maximale (MC),... (Voir Gehrlein Lepelley [16]). Notre travail se situe explicitement dans le cadre du modèle IAC. Pour calculer la probabilité d'un événement (de vote) sous l'hypothèse IAC, on considère le cadre formel suivant :

- Un ensemble de n votants : 1, 2, ..., n
- Un ensemble de m candidats : A, B, C, ...
- La préférence d'un votant est décrite par un ordre linéaire sur l'ensemble des candidats (classement complet, anti symétrique et transitif)
- Pour m = 3, il y a 6 ordres de préférence possibles comme dans la figure 1. Les six ordres sont numéroté de R_1 à R_6. On désigne aussi par n_i le nombre d'électeurs ayant l'ordre préférence R_i.

A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A
R_1	R_2	R_3	R_4	R_5	R_6

Figure 1 – ordres de préférence

- Pour m = 4 on a 24 ordres, et 120 pour $m = 5$; plus généralement, pour m candidats on a m ! ordres ou préférences individuelles possibles.
- Un profil de préférences individuelles est une liste (ordonnée) de n ordres linéaires.

Dans ce cadre, si on considère un ensemble de m candidats et un nombre n de votants, on peut déduire que le nombre de situations de vote est donné par :

$$|V(n,m)| = \binom{n + m! - 1}{m! - 1}$$

En conséquence, pour calculer la probabilité d'un événement (de vote) sous l'hypothèse IAC, il suffit de calculer le nombre de situations correspondantes et de le diviser par le nombre total de situations$|V(n,m)|$. Formellement, pour m candidats et pour un événement de vote E, et (E, n, m) l'ensemble des éléments de $V(n,m)$ où E se réalise, la probabilité d'occurrence de E, en présence de n votants, est donnée par :

$$\Pr(E,n,m) = \frac{|(E,n,m)|}{|V(n,m)|}$$

Souvent le nombre de votant est grand, donc on s'intéresse aussi à la probabilité limite :

$$\Pr(E,\infty,m) = \lim_{n\to\infty} \Pr(E,n,m)$$

Un événement de vote E se présente généralement sous la forme d'un système de contraintes linéaires à coefficients rationnels sur les variables n_i et dépendant du paramètre n.

Ainsi, l'événement E : « A est le vainqueur de Condorcet » est décrit par le système suivant :

$$\begin{cases} n_1 + \cdots + n_6 = n \\ n_i \geq 0, \;\; i = 1, \ldots, 6 \\ n_1 + n_2 + n_5 > n_3 + n_4 + n_6 \\ n_1 + n_2 + n_3 > n_4 + n_5 + n_6 \end{cases}$$

La probabilité recherchée est donc donnée par :

$$\Pr(E,3,m) = \frac{|(E,3,m)|}{|V(3,m)|}$$

Les techniques pour calculer la probabilité $\Pr(E,3,m)$ sont diverses ; elles se ramènent au dénombrement des solutions entières du système décrivant E.

La première méthode employée pour effectuer ce calcul est une méthode algébrique qui a été proposée par Gehrlein et Fishburn [1976]. Elle est basée sur l'utilisation des multi-sommes. Ils obtiennent :

$$\Pr(E,\infty,m) = \frac{15}{16}$$

Généralement, cette méthode de multi-somme est basée sur des calculs algébriques qui peuvent devenir fastidieux avec l'augmentation du nombre de candidats. Cependant, elle peut se révéler très judicieuses dans certaines situations : sa structure algébrique peut permettre des simplifications intéressantes dans le cas de généralisation à m candidats.

D'autres techniques ont été développées. Ainsi, Huang et Chua [23] (2000) ont montré que les solutions d'un système de contraintes linéaires à coefficients rationnels et dépendant d'un paramètre n se présentent sous forme d'un polynôme en n et à coefficients périodiques. Wilson et Pritchard [32] (2007) et Lepelley, Louichi et Smaoui [27] (2008) introduisent la théorie d'Ehrhart [11] (1967) et montrent que ce polynôme est associé à des polynômes d'Ehrhart, c'est-à-dire qu'un système linéaire à coefficients rationnels décrit un polytope, et que le paramètre n n'est que le coefficient de dilatation de ce dernier. Ainsi le résultat proposé par Huang et Chua n'est qu'un cas particulier d'une théorie plus vaste introduite il y a plus de 40 ans par le mathématicien français Eugène Ehrhart.

La connexion avec la théorie d'Ehrhart a ramené les calculs probabilistes sous IAC à un cadre mathématique bien connu : les polytopes (rationnels) et les quasi-polynômes. Ceci a permis l'introduction d'algorithmes et de techniques de calculs associés à la théorie et d'Ehrhart et aux polytopes dans la théorie du vote, tels que Clauss [8], Barvinok [38], LattE [10] (2004), ou encore Normaliz [7] (2012). L'algorithme le plus utilisé dans la littérature du choix social est celui qui est fondé sur la théorie de Barvinok [4] (1994).

3. Objet et plan de la thèse

L'objectif principal de cette thèse est d'optimiser les techniques de calcul des probabilités des paradoxes et de les appliquer à des questions non encore élucidées.

Au cours de ce travail, on a pu calculer un certain nombre de fréquences d'occurrence jusqu'ici inconnues. Pour y parvenir, nous proposons de nouvelles approches de calcul, qui ont pu bénéficier à la fois d'un progrès technique concernant les algorithmes de calcul et aussi du progrès des machines de calcul elles-mêmes, qui sont devenues plus puissantes qu'auparavant.

Notre thèse s'articule en quatre chapitres, de la manière suivante.

Dans le premier chapitre, nous présentons une description de l'évolution des différentes méthodes de calcul utilisées pour obtenir, sous l'hypothèse IAC, les probabilités d'occurrence des paradoxes du vote. Ce survol de la littérature récente a été écrit en collaboration avec Issofa Moyouwou et Hatem Smaoui. Le document associé est soumis pour publication en tant que chapitre d'ouvrage dans un livre à paraître chez Springer. On peut noter que cette contribution a été rédigée après les autres, mais il nous a semblé cohérent de la présenter avant les autres.

Le deuxième chapitre s'intéresse aux paradoxes du vote qui peuvent survenir dans des élections à quatre candidats. La probabilité d'occurrence de ces paradoxes (sous l'hypothèse IAC) a été obtenue par les travaux antérieurs en considérant, le plus souvent, le cadre d'élections à trois candidats. On va dans ce chapitre tenter d'étendre ces calculs au cadre d'élections à quatre candidats, avec par conséquent 24 ordres (linéaires) de préférence possibles. On retrouvera quelques résultats obtenus très récemment par d'autres chercheurs et on proposera de nombreux résultats nouveaux, concernant diverses règles de vote (incluant la règle de Borda et le vote majoritaire à deux tours) et divers paradoxes. Ce chapitre a été écrit en collaboration avec Dominique Lepelley et Hatem Smaoui. Il a été accepté pour publication dans Theory and Decision.

Le troisième chapitre est, techniquement, dans la continuité du deuxième. On s'y s'intéresse au vote par évaluation à trois valeurs (ou vote par note à trois niveaux). Cette méthode de vote a été introduite par Felsenthal (1989) [12] et Hillinger (2004, 2005) [20][21]. Elle fait partie de la large famille des méthodes de vote par évaluation. Selon Balinski et Laraki (2007) [2], et Hillinger (2004), ce genre de méthode cardinale d'agrégation, basée sur le principe d'évaluation, est préférable à l'approche ordinale des préférences individuelles, dans le sens où cette dernière est source de la plupart des paradoxes en théorie du vote. L'étude des paradoxes relatifs à cette règle présente ainsi un intérêt a la fois théorique et technique. Concrètement, le vote par évaluation à trois valeurs est une règle où chaque électeur évalue chaque candidat et lui attribue une note choisie dans l'ensemble 2,1,0. Le gagnant est le candidat qui obtient le plus grand nombre de points. Cette règle a beaucoup de bonnes propriétés, mais elle ne choisit pas systématiquement le vainqueur de Condorcet, c'est-à-dire le candidat (lorsqu'il existe) qui bat chacun des autres candidats dans des comparaisons majoritaires. En supposant que les préférences des électeurs sont de nature trichotomique, et en supposant qu'il n'y a que trois candidats, on se propose dans le chapitre 3 de mesurer et quantifier l'occurrence d'apparition de ce genre de situations, en considérant diverses conditions majoritaires (de type Condorcet) que ne vérifie pas le vote par évaluation. Ce chapitre a été co-écrit avec Dominique Lepelley et Hatem Smaoui. Il est en révision pour la revue Annals of Operations Research.

On conserve dans le chapitre 4 le cadre des préférences trichotomiques avec trois candidats, mais en considérant que les électeurs peuvent voter de manière stratégique. L'étude de la manipulabilité des règles de vote a fait l'objet d'une abondante littérature en théorie du vote. Cependant, il n'existe à ce jour aucune étude permettant de quantifier la manipulabilité théorique des règles de vote par évaluation. Le chapitre 4 se propose de mesurer la manipulabilité du vote par évaluation à trois niveaux, et de comparer cette manipulabilité à celles d'autres règles usuelles (Pluralité, Borda...) que l'on pourrait utiliser dans un contexte de préférences trichotomiques. On propose aussi une nouvelle technique de calcul, qui permet de simplifier considérablement la détermination des fréquences d'occurrence des événements de vote. La contribution associée à ce chapitre a bénéficié des conseils de Jérôme Serais, de l'université de Caen. Qu'il en soit ici remercié.

Nous concluons la thèse avec un résumé des différents résultats obtenus et en soulignant les limites des méthodes et techniques de calcul actuelles.

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Chapter 1 : IAC-Probability Calculations in Voting Theory: Progress Report

Abstract. *Over the past two decades, IAC probability calculations techniques have made substantial progress, particularly through methodological studies that have linked these calculations to their appropriate mathematical framework. We report on this progress by a brief description of the methods of calculation used in this field, and by reviewing some of the results that the application of these methods made possible to obtain.*

1. Introduction

In voting theory, probabilistic analysis aims to assess the frequency with which various electoral outcomes can be observed. The primary motivation is, in one hand, to quantify the potential impact of voting paradoxes on real-word elections, and on the other hand, to compare the alternative voting rules on the basis of their ability to meet certain normative criteria. These quantitative results can of course be obtained, in the form of estimates, by empirical and experimental methods (via actual election data and computer simulations).[4] However, the most significant part of the research on this topic makes use of analytical methods in order to obtain exact results describing the theoretical probabilities of the voting events under investigation. The book by Gehrlein (2006), entirely devoted to the famous Condorcet Paradox, and the two books by Gehrlein and Lepelley (2011, 2017), in addition to containing the most complete and essential literature reviews on the subject, constitute an excellent illustration of the richness and dynamism of this line of research. William Gehrlein and Dominique Lepelley are certainly the two most eminent and most prolific authors in this field, and one of the main objectives of this paper is also to pay tribute to their fundamental contribution to the probabilistic analysis of voting paradoxes and voting rules.

[4] A good summary of empirical and experimental studies can be found in Gehrlein (2006) and Gehrlein and Lepelley (2011). See also Regenwetter et al. (2006), Tideman and Plassmann (2012, 2014), Gehrlein et al. (2016, 2018) and Brandt et al. (2016, 2020).

The analytical approach uses theoretical models based on certain assumptions about the voters' preferences. In the literature, the most often used probabilistic models are the Impartial Culture condition (IC), introduced by Guilbaud (1952), and the Impartial Anonymous Culture condition (IAC), described initially by Kuga and Nagatani (1974) and formalized by Gehrlein and Fishburn (1976).[5] Over the past two decades, IAC probability calculation techniques have made substantial progress, particularly through methodological studies that have linked these calculations to their appropriate mathematical framework (Huang and Chua, 2000 ; Cervone et al., 2005 ; Pritchard and Wilson, 2007 ; Lepelley et al., 2008). In this paper, we wish to report on this progress, by a brief description of the methods of calculation used in this field, and by reviewing some of the results that the application of these methods made possible to obtain. For the sake of simplicity, we have chosen to restrict the themes of these representative results to four issues that are among the most often addressed by the probabilistic analysis of electoral outcomes: the election of the Condorcet winner, the election of the Condorcet loser, the (non)monotonicity of voting rules, and finally their manipulability.

In general, these specific voting events are described and studied in a very simple formal framework where individual preferences are represented by linear orderings on the set of candidates. For example, with three candidates (a, b, and c), there are six possible individual preference rankings: abc, acb, bac, bca, cab, and cba (the notation abc means that a is preferred to b, b is preferred to c and, by transitivity, a is preferred to c). With n voters and m candidates, a profile is an ordered list of n individual preferences chosen from the $m!$ possible rankings ; a voting situation is an anonymous profile. IC model assumes that each individual preference ranking (and so each profile) is equally likely to be observed. IAC assumes that each possible voting situation is equally likely to be observed. An (anonymous) voting rule is defined as a function that associates a winning candidate with each voting situation.

Most of the studies that will be presented in this brief report deal with voting rules that belong to the class of weighted scoring rules (WSRs) or to the class of scoring elimination rules (SERs). With m candidates, a WSR is defined by a scoring vector $(\lambda_1, \lambda_2, \cdots, \lambda_m)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $\lambda_1 > \lambda_m$, such that each candidate receives λ_k points each time he/she is ranked k^{th} by a voter. The candidate with the most total points wins. The most common WSRs are plurality rule, PR ($\lambda_1 = 1$ and $\lambda_k = 0$ for $k > 1$), negative plurality rule, NPR ($\lambda_m = 0$ and $\lambda_k = 1$ for $k < m$), and Borda rule, BR ($\lambda_k = (m-k)/(m-1)$). In three-candidate election, the scoring

[5] For a justification of research based on these assumptions, see Gehrlein and Lepelley (2004).

vector is of the form $(1, \lambda, 0)$, $0 \leq \lambda \leq 1$, and we have $\lambda = 0$ for PR, $\lambda = 1$ for NPR, and $\lambda = 1/2$ for BR. Scoring elimination rules use WSRs in a multi-stage process of sequential elimination : in each stage, the candidate with the lowest total points is eliminated. With three candidates, a WSR is used in a first round to eliminate the candidate with the lowest total points, and in a second round, the two remaining candidates are confronted and the one who obtains the majority of votes wins. Plurality elimination rule (PER), negative plurality elimination rule ($NPER$) and Borda elimination rule (BER) are the sequential versions of PR, NPR and BR, respectively.[6]

Given a voting situation, a Condorcet winner (CW) is a candidate who beats each other candidate in pairwise majority comparisons. In the same way, a Condorcet loser (CL) is a candidate who loses against every other candidate in pairwise majority contests. It is well known that the CW and the CL do not always exist. However, it is generally accepted that a "good" voting rule should select the CW when such a candidate exists (CW condition). Voting rules that satisfy this property are called Condorcet consistent.[7] In the same way, it seems reasonable to require the non-election of the CL, when such a candidate exists (CL condition). In this sense, the non-selection of the CW or the selection of the CL can be considered as voting paradoxes (the selection of the CL is known as (Strong) Borda Paradox). Failure of a given voting rule to meet the CW condition or the CL condition is viewed as a flaw of this rule. The other two imperfections that can affect the voting rules, and that we focus on in this paper, are monotonicity failure and vulnerability to strategic manipulation. A Monotonicity Paradox occurs when an increased support of a candidate who won an election makes him or her a loser (More is Less Paradox, MLP), or when a decreased support of a candidate who lost an election makes him or her a winner (Less is More Paradox, LMP).[8] A strategic manipulation of a voting rule occurs in an election when some voters express insincere preferences in order to obtain a final winner that they prefer to the candidate that would have been elected if they had voted in a sincere way.

[6] Note that the paper only deals with the classical form of elimination process. It is worth noting that other methods of elimination are studied in the literature (see for instance Kim and Roush, 1996).

[7] Black's Procedure, Copeland's Rule and Dodgson's Method are examples of methods belonging to this important class of voting rules (see, for example, Fishburn, 1977).

[8] Here MLP and LMP are defined for a fixed electorate. These two paradoxes can also be defined with a variable electorate (see Lepelley and Merlin, 2001).

In the remainder of this paper, we focus on the exact probabilistic results describing the theoretical frequency of these four voting events under each of the following six voting rules: PR, NPR, BR, PER, $NPER$ and BER (we also mention the results obtained for the entire class of WSRs and for that of SERs). Our main objective being to give a general idea on the evolution of the techniques of probabilities calculation in voting theory, we will not review here the results obtained by assuming IC hypothesis because computation methods used under this model are (almost) the same for twenty years. We therefore begin with a brief description of the general framework of probability calculations under IAC condition, and introduce some useful notations (Section 2). We then present the different methods used in these calculations, summing up the basic idea of each method and illustrating its scope by a short review of the results it has allowed to obtain (Sections 3-6). Finally, we conclude with a few remarks on the progress made so far and on the orientations to be considered to push even further the limits of the probabilistic analysis of voting rules.

2. Probabilities calculations under the IAC condition

In an election with n voters and m candidates, we denote by R_1, …, $R_{m!}$ the $m!$ possible individual preference rankings. A voting situation is then represented by an $m!$-tuple of integers, n_i, that sums to n, where n_i denotes the number of voters having the individual preference R_i. For $m = 3$, voting situations are 6-tuples, $(n_1, \dots, n_6)$, with the six possible individual rankings labeled as follows: abc (R_1), acb (R_2), bac (R_3), cab (R_4), bca (R_5), and cba (R_6). Note that voting situations are 24-tuples for $m = 4$, 120-tuples for $m = 5$, etc. We denote by $V(n, m)$ the set of all possible voting situations with n voters and m candidates. Under the IAC assumption, the elementary events are the voting situations. Thus, for a voting event E, and for a fixed m, if we denote by $E(n, m)$ the set of elements of $V(n, m)$ in which E occurs, the probability of E is a function of n that is given by:

$$Pr(E, n, m) = |E(n, m)|/|V(n, m)| \qquad (1)$$

In this identity, $|V(n, m)|$ and $|E(n, m)|$ denote the cardinalities of sets $V(n, m)$ and $E(n, m)$, respectively. The expression of $|V(n, m)|$ is well known and is given by:

$$|V(n, m)| = \binom{n + m! - 1}{m! - 1} \qquad (2)$$

In general, $E(n, m)$ is described by a parametric system, $S(n)$, of linear (in)equalities with integer (or rational) coefficients on the variables n_i and on the parameter n. Therefore, the computation of $Pr(E, n, m)$ is reduced to the enumeration of all the integer solutions of $S(n)$. Note that this is a combinatorial problem that is not always easy to solve, and that the transition from three options ($m = 3$) to four options ($m = 4$) is actually a move from a calculation with 6 variables to a calculation with 24 variables (with $m = 5$, we move to much more complex computations, involving 120 variables). Often, especially when the probability of an event is difficult to obtain as a function of n, or when this expression is too cumbersome, one is satisfied to calculate the limiting probability of E, defined by:

$$Pr(E, \infty, m) = \lim_{n\to\infty} Pr(E, n, m) \qquad (3)$$

One of the very first events examined by probabilistic studies is the event CW: “there exists a Condorcet Winner”. Consider the event CW^a: “candidate a is the Condorcet winner”. By formula (1), and using the symmetry of IAC with respect to the m candidates, we have:

$$Pr(CW, n, m) = m|CW^a(n, m)|/|V(n, m)| \qquad (4)$$

With three candidates ($m = 3$), the set $CW^a(n, 3)$ is characterized by the following parametric linear system:

$$S^a(n): \begin{cases} n_1, n_2, n_3, n_4, n_5, n_6 \geq 0 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_1 + n_2 + n_4 > n/2 \\ n_1 + n_2 + n_3 > n/2 \end{cases}$$

The two first conditions (the six sign inequalities and the equality) characterize the set of all voting situations with three candidates and n voters, $V(n, 3)$. The two last conditions describe the fact that a is the Condorcet winner (a beats b by a majority of votes and a beats c by a majority of votes).

Knowing the probability that a CW exists, one can consider calculating the Condorcet Efficiency of a voting rule F, denoted by $CE(F, n, m)$, and defined as the conditional probability that F elects the CW, given that such a candidate exists. Some studies have also investigated the probability of electing the CL, when such a candidate exists, $Pr(CL - F, n, m)$. The other notations that will be useful later in this paper are the following. For a Monotonicity Paradox M (MLP or LMP), we denote by $Pr(M - F, n, m)$ the vulnerability of F to M (i.e., the probability that M occurs when using F). The global vulnerability of F to Monotonicity Paradoxes (i.e., the probability that a voting situation gives rise to MLP or LMP under F) is denoted by $Pr(GMP - F, n, m)$. Finally, the vulnerability of F to coalitional manipulability is denoted by $VM(F, n, m)$.

We close this section with three brief remarks on the IC and IAC models and on the relevance of the theoretical results obtained under these conditions:

- The two models are based on a hypothesis of equiprobability. In both cases, this hypothesis can be justified by the absence of information *a priori* on the voter preferences. Note that with IC, individual preferences are completely independent. By contrast, IAC implicitly introduces a certain degree of interaction between individuals, which induces less heterogeneous preferences than with IC.
- The probabilities obtained under IAC are in general (slightly) lower than those obtained with IC. As pointed by Berg and Lepelley (1992), this can be explained intuitively by the fact that the homogeneity introduced by IAC makes the occurrence of voting paradoxes less likely (for a paradox to occur, a certain antagonism of individual preferences is required).
- In general, IC and IAC represent scenarios that exaggerate the probability of voting events. Thus, the probabilities computed with these two models should be perceived, not as estimates of the likelihood of these events in real situations, but rather as upper bounds. In particular, when the theoretical probability of a voting event is very small, this event is assuredly very unlikely to be observed in reality (Gehrlein and Lepelley, 2004).

3. The algebraic Approach

The first method for probabilities computation under IAC was developed by Gehrlein and Fishburn (1976). This simple algebraic counting technique, which was used until the early 2000s (and beyond in some cases), is based on the use of multiple summations and their reduction by the formulas on sums of powers of integers (Selby, 1965). Let's go back to the example of event CW^a: "a is the Condorcet winner", with three candidates ($m = 3$). If we assume that n is odd, then the last two inequalities in the parametric system $S^a(n)$ can be written as $n_3 + n_5 + n_6 \leq (n-1)/2$ and $n_4 + n_5 + n_6 \leq (n-1)/2$, respectively. As the variables n_i are inter-related, the first step in Gehrlein-Fishburn procedure is to transform $S^a(n)$ into a form that will facilitate the enumeration. From the equality in the second condition in $S^a(n)$, we can replace n_1 by $n - (n_2 + n_3 + n_4 + n_5 + n_6)$. It is then easy to show that the number of integer solutions of $S^a(n)$ is equal to the number of 5-tuples of integers, $(n_2, n_3, n_4, n_5, n_6)$, that meet the following five restrictions:

$$0 \leq n_2 \leq n - n_6 - n_5 - n_4 - n_3, \quad 0 \leq n_3 \leq \tfrac{n-1}{2} - n_6 - n_5,$$
$$0 \leq n_4 \leq \tfrac{n-1}{2} - n_6 - n_5, \quad 0 \leq n_5 \leq \tfrac{n-1}{2} - n_6, \quad 0 \leq n_6 \leq \tfrac{n-1}{2}$$

With this rearrangement of the conditions on the n_i's, the cardinality of the set $CW^a(n, 3)$ can be computed as:

$$|CW^a(n,3)| = \sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} 1 \qquad (5)$$

The second step is to algebraically reduce this multiple summation by sequentially using known relations for sums of powers of integers. The process starts by the evaluation of the last summation, $\sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} 1$, which can be obviously replaced by $(n - n_6 - n_5 - n_4 - n_3 + 1)$. Then, the n_3 summation in (5) becomes $\sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} [(n - n_6 - n_5 - n_4 + 1) - n_3]$, and can be easily calculated (using the formula $\sum_{t=0}^{k} t = k(k+1)/2$). Continuing this way, it can be showed that $|CW^a(n,3)| = (n+1)(n+3)^3(n+5)/384$. Using formulas (1), (2) and (4) for $m = 3$, the analytic representation of the probability of the event $CW(n,3)$, for odd n, is obtained as (Gehrlein and Fishburn, 1976):

$$Pr(CW,n,3) = \frac{15(n+1)^2}{16(n+2)(n+4)} \qquad (6)$$

The representation for even n, calculated by Lepelley (1989), is given by:

$$Pr(CW,n,3) = \frac{15n(n+2)(n+3)}{16(n+1)(n+3)(n+5)} \qquad (7)$$

As we can see, the two formulas (for odd n and for even n) give the same limit as $n \to \infty$; thus the value of the limiting probability is given by $Pr(CW,\infty,3) = 15/16$.

Despite of its apparent simplicity, this algebraic method has made it possible to produce a very large number of results that have significantly contributed to advance the probabilistic analysis of voting rules. It is not an exaggeration to say that, until the early 2000s, (almost) all the analytical representations of the likelihood of voting events, under IAC, were achieved by using this method. With a few exceptions, all these results deal with the case of three-candidate elections.[9] We limit ourselves here to mentioning only a small part of this abundant bibliography, the one that gives the first results on the probabilities of the four voting events and the six voting rules under consideration.

Representations for the Condorcet Efficiency, as a function of n, were obtained by Gehrlein (1982) and Gehrlein and Lepelley (2001), for PR, NPR, BR, PER and $NPER$ (BER is known to always select the CW when such a candidate exists). The limiting probabilities are given by $CE(PR,\infty,3) = 88.15\%$, $CE(NPR,\infty,3) = 62.96\%$, $CE(BR,\infty,3) = 91.11\%$,

[9] A description of the few studies dealing with the case of four (and more) candidates can be found in (Gehrlein, 2006), 107-152.

$CE(PER, \infty, 3) = 95.85\%$, and $CE(NPER, \infty, 3) = 97.04\%$. The probability of electing the Condorcet loser (when such a candidate exists), with n voters, was calculated by Lepelley (1993) for PR and NPR (the other four voting rules never select the CL). The limiting probabilities are respectively given by $Pr(CL - PR, \infty, 3) = 2.96\%$ and $Pr(CL - NPR, \infty, 3) = 3.15\%$. The first analytical results on the frequency of Monotonicity Paradoxes were obtained by Lepelley et al. (1996), for PER and $NPER$. The associated limiting values are $Pr(MLP - PER, \infty, 3) = 4.51\%$, $Pr(MLP - NPER, \infty, 3) = 5.56\%$, $Pr(LMP - PER, \infty, 3) = 1.97\%$ and $Pr(LMP - NPER, \infty, 3) = 6.48\%$. Other results, dealing with variable electorate versions of MLP and LMP, were proposed by Lepelley and Merlin (2001). Finally, the exact formulas (as a function of n), describing the vulnerability of PR, NPR and PER to strategic manipulation (by a coalition of voters) were provided by Lepelley and Mbih (1987, 1994). For a large number of voters, the obtained formulas give: $VM(PR, \infty, 3) = 29.16\%$, $VM(NPR, \infty, 3) = 51.85\%$, $VM(PER, \infty, 3) = 11.11\%$. For $NPER$, only the limiting probability was possible to compute in Lepelley and Mbih (1994): $VM(NPER, \infty, 3) = 43.05\%$.

This sample of results shows the importance of Gehrlein-Fishburn procedure in probability calculations under the IAC hypothesis. However, the implementation of this method often faces a number of difficulties. First, we must start by rearranging the inequalities in $S(n)$ in a way that allows the use of multiple summations, which is not always easy to do, especially because there is no mechanical procedure to accomplish this operation. Second, it often happens that the Min and Max functions appear in certain lower and upper summation bounds, which leads to partitioning the set $E(n, m)$ into several sub-spaces and to further complicate the calculations (see, for example, Gehrlein and Lepelley, 2001). Finally, in the expression of these bounds, when some of the n_i's coefficients are not integer, the Gehrlein-Fishburn procedure may fail to produce the desired result. The scope of this method is therefore rather limited to simple voting events in three-candidate elections. For example, it does not allow to compute the analytical representations of $Pr(MLP - BR, n, 3)$, $Pr(LMP - BR, n, 3)$, $VM(NPER, n, 3)$ and $VM(BR, n, 3)$. As for the results for $m = 4$ and the results depending on a second parameter (other than n) with $m = 3$, these representations seem to be (in general) inaccessible until now with this approach.

4. The geometric approach for limiting probabilities

Saari (1994) is the first author to introduce tools of geometric analysis in the study of voting rules. His extensive work has significantly contributed to a more complete and deeper understanding of most voting paradoxes and impossibility theorems. His geometric approach has also made it possible to develop a probability calculation technique in the limit case where the number of voters tends to infinity (Saari and Tataru, 1994, 1999). This method, based on volume calculations and on results from Schläfli (1950), was later used, in Merlin and Tataru (1997), Saari and Valognes (1999), and Merlin et al. (2000, 2002), among other studies, to obtain a number of interesting asymptotic results under the IC hypothesis.

With the assumption of IAC, Cervone et al. (2005) developed a very similar method that allows to reduce the problem of computing limiting probabilities, in three-candidate elections, to a problem of pure geometry. They start by transforming each voting situation $(n_1, \ldots, n_6)$ into a normalized (anonymous) profile $(x_1, \ldots, x_6)$ where $x_i = n_i/n$ represents the fraction of voters who favor the preference ranking R_i. Since $x_i \geq 0$, for each i, and $\sum x_i = 1$, the normalized profiles correspond to points (with rational coordinates) in Δ^5, the 5-simplex in $\mathbb{R}^6$. In the same way, for a voting event, E, described by a linear system $S(n)$, we can associate the convex region R described by the linear system $S(1)$ (where the n_i's are replaced by the x_i's and n is replaced by 1). If we respectively denote by $\mathrm{Vol}(\Delta^5)$ and $\mathrm{Vol}(R)$ the 5-volume of Δ^5 and the 5-volume of R, then, the limiting probability of E is obtained as:

$$Pr(E, 3, \infty) = \mathrm{Vol}(R)/\mathrm{Vol}(\Delta^5) \qquad (8)$$

$\mathrm{Vol}(\Delta^5)$ is easy to obtain and is known to be equal to $\sqrt{6}/120$. To compute $\mathrm{Vol}(R)$, the authors apply a procedure based on the general formula giving the volume of a pyramid and on a recursive technique of triangulation. The volume of a pyramid in dimension d is equal to Vh/d, where V is the $(d-1)$-dimensional volume of the base, and h is the height of the apex above the base. The method begins by determining all the vertices of R and then uses one of them to decompose R into a collection of pyramids having the chosen vertex as their apex and the various faces of R as their bases. The faces of R are 4-dimensional convex regions and are in turn broken in pyramids, and so forth. This generates a recursive procedure for computing the volume of the region R; the base case for the recursion is the 2-dimensional case where the "pyramid" is simply a triangle.

With this method, the calculation of the limit probabilities, $Pr(E, \infty, m)$, is reduced to the calculation of the volumes of convex regions (in general, of dimension 5 for $m = 3$). It is no longer necessary to obtain the exact expressions of $Pr(E, n, m)$ as a function of n and then calculate their limit when n tends to infinity (this is a significant simplification when $Pr(E, n, m)$ is difficult to obtain). For example, to compute $Pr(CW, \infty, 3)$, it suffices to introduce the convex region R^a associated with $CW^a(n, 3)$ when $n \to \infty$; then (using formulas (4) and (8)) we have $Pr(CW, \infty, 3) = 3\mathrm{Vol}(R^a)/\mathrm{Vol}(\Delta^5)$. We know that $\mathrm{Vol}(\Delta^5) = \sqrt{6}/120$ and it can be showed, applying the technique we have just outlined , that $\mathrm{Vol}(R^a) = \sqrt{6}/384$ (Cervone et al., 2005). We thus recover the result of Gehrlein and Fishburn (1976) and Lepelley (1989), $\mathrm{Pr}(CW, \infty, 3) = 15/16$.

Thanks to their geometric approach, Cervone et al. (2005) have been able to provide a complete answer to a much more complex problem. They obtained the exact analytical representation of the limiting Condorcet efficiency, $CE(\lambda, \infty, 3)$, of all weighted scoring rules $WSR(\lambda), \lambda \in [0,1]$. In particular, they showed that the Borda rule ($\lambda = 0.5$) does not maximize $CE(\lambda, \infty, 3)$ (the maximum is reached for $\lambda = 0.37228$). Other asymptotic results in the form of general formulas (in λ) for all WSRs were obtained by applying the technique of Cervone et al. (2005). For example, Diss and Gehrlein (2012) developed limiting representations for the probability that a Borda Paradox will be observed under each WSR. The main conclusion of this paper is that, in realistic voting scenarios, it is very unlikely that a Strict Borda Paradox[10] would ever be observed for any WSR, and that occurrences of a strong Borda paradox (electing the Condorcet loser) should be relatively rare, but not impossible to observe.

Moyouwou (2012) made it more systematic to obtain this type of results (general exact formulas for the whole class of WSRs or SERs), by using the triangulation algorithm of Cohen and Hickey (1979) and by introducing routines using MAPLE codes to undertake the operations involved in this algorithm.[11] This method of calculation was used in a series of articles, including Gehrlein et al. (2013, 2015), Moyouwou and Tchantcho (2017) and Lepelley et al. (2018). In the last article, the authors offer some new exact results describing the vulnerability to Monotonicity Paradoxes (MLP, LMP, GMP) of the whole class of SERs. In particular, they show that when three-candidate elections are close, the risk of monotonicity failure is high for

[10] A Strict Borda Paradox occurs when a voting rule completely reverses the rankings (on the set of candidates) that are obtained by the pairwise majority comparisons (in particular, the CW becomes the loser and the CL becomes the winner under the considered voting rule).
[11] See also Moyouwou and Tchantcho (2017).

PER, $NPER$, and BER (this is especially true under PER, for which the probability of GMP is higher than 32%). In Gehrlein et al. (2013), analytical formulas in term of λ are provided for $VM(\lambda, \infty, 3)$, the asymptotic vulnerability of $SER(\lambda)$ to coalitional manipulation. These analysis was extended by Moyouwou and Tchantcho (2017) who notably showed that the plurality rule minimizes $VM(\lambda, \infty, 3)$ (when the size of the manipulating coalition is unrestricted).

It appears from the studies just cited that the geometric approach developed by Cervone et al. (2005) is very useful when analyzing an entire class of voting rules (typically the WSRs and the SERs) and comparing the rules belonging to this class on the basis of their asymptotic probabilities to meet certain normative criteria. However, this type of simultaneous analysis remains limited to the case of three candidates, and it seems difficult, for the moment, to envisage similar investigations for four-candidate elections with this calculation technique. In fact, the scope of this method essentially depends on the efficiency of the procedure used to find the vertices and perform the triangulations; it is therefore not excluded that future improvements in triangulation algorithms will make it possible to deal with the case of four candidates.

5. Huang-Chua method and EUPIA procedure

In the results obtained in the literature applying the Gehrlein-Fishburn procedure, it has been observed that the analytical representations of the probabilities of the voting events always appear in the form of a quotient of two polynomials in n, and that these representations are periodic in n (with, in general, a period equal to 2, 6, 9 or 12). Huang and Chua (2000) transformed this observation into a general result that for any voting event E such that $E(n, m)$ is described by a system of linear constraints $S(n)$, the number $|E(n, m)|$, for fixed m, can be described by a periodic polynomial (i.e., a polynomial, $f(n)$, with coefficients depending on a certain period q). Consider for example the event CW^a: "a is the Condorcet winner", characterized by the system $S^a(n)$. We have seen in Section 3 that $|CW^a(n, 3)| = (n+1)(n+3)^3(n+5)/384$, for odd n. And, using formulas (1), (2), (4) and (7), we can also see that for even n, we have $|CW^a(n, 3)| = (n+2)^3(n+3)(n+4)/384$. Thus, $|CW^a(n, 3)|$ is described by a five degree periodic polynomial with periodicity $q = 2$.

The Huang-Chua result leads to fundamental simplification in probabilistic calculations under the IAC condition, avoiding in particular to go through the cumbersome (manual) partitioning of the set $E(n, m)$, as it is frequently the case with the Gehrlein-Fishburn procedure. Indeed, knowing the degree of the periodic polynomial expression $f(n)$, it is enough to find the period q and to process by interpolation to determine the periodic coefficients of $f(n)$. Huang and Chua (2000) suggest a simple algorithm that allows to simultaneously identify these unknown values. This algorithm is based on the interpolation technique and an iterative process of computer enumeration of the elements of $E(n, m)$ for initial values of the parameter n. Gehrlein (2002) has improved this approach by developing EUPIA procedure,[12] which applies to both IAC and MC models,[13] and overcomes a number of technical difficulties that may be encountered when using the Huang-Chua algorithm. An extension of this procedure (EUPIA 2), proposed by Gehrlein (2005), allows to obtain representations for the conditional probability that voting outcomes are observed, given that voting situations are constrained to have some specified values of a measurable parameter (describing, in general, the degree of homogeneity of individual preferences).

The use of these new computational tools has generated a number of results that would have been difficult to obtain with the algebraic approach of Gehrlein and Fishburn. The Huang and Chua algorithm was mainly used in the study of the manipulability of voting rules. Huang and Chua (2000) completed the results of Lepelley and Mbih (1987, 1994), especially by providing the exact expression of $VM(NPER, n, 3)$ as a function of n. Favardin et al. (2002) obtained representations for the vulnerability of the Borda rule to individual manipulation. Favardin et al. (2006) consider various electoral environments in which strategic manipulation can occur and derive some analytical representations for the manipulability of a large number of voting rules. EUPIA was used by Gehrlein (2002) to develop probability representations for a number of different voting outcomes, which are considered to be intractable to obtain with the use of standard algebraic techniques (as for example the probability that all weighted scoring rules on three candidates give the same winner). This procedure was also applied by Gehrlein and Lepelley (2003) to compare the median voting rule with other voting rules, notably on the basis of their manipulability and their Condorcet Efficiency. Finally, the two-parameter algorithm EUPIA 2 has been very useful in a number of studies on the impact that different degrees of mutual coherence of individual preferences may have on the probability of certain voting electoral outcomes, such as the existence of the Condorcet winner (Gehrlein, 2005), the election

[12] EUPIA : Effectively Unlimited Precision Integer Arithmetic.

[13] Under the Maximal Culture assumption (MC), all voting situations with at most n voters are assumed to be equally likely to be observed.

of the Condorcet winner (Gehrlein and Lepelley, 2009 ; Gehrlein et al., 2011) and the occurrence of Borda Paradox (Gehrlein and Lepelley, 2010).

As we have already pointed out, the result of Huang and Chua (2000) corresponds to a crucial change in the methods of calculating probabilities of voting events under IAC. As a consequence of this result, the technical efforts focused on the development of a procedure for the systematic computation of the period and the coefficients of the periodic polynomial representing $|E(n,m)|$. This goal has been partially achieved with the Huang-Chua, EUPIA, and EUPIA 2 algorithms that have been successfully applied to solve problems that lead to calculations involving small periodicities. Unfortunately, the execution time of the interpolation procedure increases exponentially depending on the periodicity, and these algorithms become inoperative when the (unknown) periods are too large. This is the problem encountered, for example, by Favardin et al. (2006) who could not obtain the exact expression of $VM(BR,n,3)$ and the exact value of $VM(BR,\infty,3)$. This difficulty severely reduces the efficiency of the Huang-Chua algorithm, the Eupia procedure, and all methods based on an interpolation technique, and prevents them from being used to analyze a large number of voting events with three candidates and (almost) all voting events with four candidates.

6. Ehrhart theory based methods

We have seen in Section 4 that the calculation of the limiting probability of a voting event E can be formulated as a geometric problem. This is also true for the calculation of the probability of E as a function of n (the number of voters). Indeed, when m (the number of candidates) is fixed, the parametric linear system $S(n)$, describing the set $E(n,m)$, defines a (rational) parametric polytope P_n (with a single parameter, n).[14] Computing $Pr(E,n,m)$, i.e., counting the number of integer solutions of $S(n)$, is equivalent to the geometric problem of counting the number of integer points belonging to P_n. Wilson and Pritchard (2007) and Lepelley et al. (2008) drew the attention of voting theorists to the existence of a well-established mathematical approach for performing such a calculation, based on Ehrhart's theory (Ehrhart, 1962) and efficient counting algorithms. The basic result of this theory concerns a particular type of parametric polytopes, that of the dilatation of a rational polytope P by a positive integer factor

[14] A rational polytope P of dimension d is a bounded subset of $\mathbb{R}^d$, defined by a system of integer linear inequalities. P is said to be semi-open when some of these inequalities are strict. A parametric polytope of dimension d (with a single parameter n) is a d-dimensional rational polytope P_n of the form $P_n = \{x \in \mathbb{R}^d : Mx \geq bn + c\}$, where M is a $t \times d$ integer matrix, b and c are two integer vectors with t components.

n, denoted by nP. In this case, the number of integer points of nP is a quasi-polynomial on n (i.e., a polynomial on n with periodic coefficients), of degree equal to the dimension of P. For exemple, in the event CW^a: "a is the Condorcet winner", with $m = 3$, the system $S^a(n)$ defines the dilatation nP where P is the (semi-open) rational polytope, of dimension 5, defined by the system $S^a(1)$ (obtained when n is replaced by 1). We can therefore deduce from Ehrhart's theorem that $|CW^a(n, 3)|$ is a quasi-polynomial on n of degree 5.

As it can be seen, the theoretical result proposed by Huang and Chua (2000) corresponds to the algebraic version of the basic result of Ehrhart theory. However, this theory is more general and more advanced, and continues to be enriched by numerous studies in mathematics and computer science. For instance, Ehrhart's theorem has been extended to the general class of parametric polytopes, with one or more parameters (Clauss and Loechner, 1998), and algorithms have been proposed to compute the coefficients of the quasi-polynomial describing the number of integer points in parametric polytopes.

The first algorithms, based on Ehrhart theory, that have been introduced in probability calculations under IAC condition are Clauss's method (Clauss, 1996), Barvinok's algorithm (Barvinok, 1994; [Barvinok]) and LattE (De Loera et al., 2004; [LattE]). The use of these powerful tools greatly facilitates the derivation of probability representations for voting outcomes. In particular, for the four voting events that interest us here, and in the case of three candidates, they made it possible to compute all the probabilities (as a function of n, and when $n \to \infty$) that the previous methods failed to obtain. The limiting value of the vulnerability of Borda rule to coalitional manipulation was obtained (separately) by Wilson and Pritchard (2007) and Lepelley et al. (2008): $VM(BR, \infty, 3) = 132953/264600$. It is worth noticing that this exact result (50.247%) is very close to the approximation given in Favardin et al. (2006), 50.25%. Note also that the quasi-polynomial involved in the expression of $VM(BR, n, 3)$ is of period 210, which explains why this result was not possible to obtain with the Huang-Chua algorithm. For Monotonicity Paradoxes, the five missing analytical representations were provided by Smaoui et al. (2016) : $Pr(MLP - BER, \infty, 3) = 1.12\%$, $Pr(LMP - BER, \infty, 3) = 0.28\%$, $Pr(GMP - PER, \infty, 3) = 1.05\%$, $Pr(GMP - NPER, \infty, 3) = 6.02\%$, and $Pr(GMP - BER, \infty, 3) = 1.40\%$.

Among the three algorithms cited above, Barvinok algorithm is the most used in IAC calculations, in the case of three candidates. Since 2008, the use of the program [Barvinok] has led to many analytical results describing the frequency of various voting events, and should allow us to solve most of the probabilistic problems that we could consider in voting theory for three-candidate elections. Recall that in this case ($m = 3$), there are only 6 variables and the quasi-polynomials describing $|E(n, 3)|$ are generally of degree 5. With four candidates ($m = 4$), there are 24 variables, and the quasi-polynomials are of degree 23. In this case, [Barvinok], as well as the other two programs cited, fails to produce the desired quasi-polynomials (the maximum number of variables that they can deal with seems to be about 20). Consequently, it is not possible to analyze four-candidate elections with these three programs. However, we know that for $m = 4$, the periods of the quasi-polynomials can be very large and that the exact formulas for $Pr(E, n, 4)$ can be far too heavy for meaningful analysis. Therefore, the probabilistic calculations, for $m = 4$, must focus on obtaining the limiting probabilities, $Pr(E, \infty, 4)$. This amounts, as we have already seen, to the computation of the volume of the polytope associated with the system $S(n)$ describing the set $E(n, m)$.

The volume of a rational polytope P can be obtained either by a direct use of a volume computation algorithm, or as the leading coefficient of the quasi-polynomial associated with the dilated polytope nP (it is well known that this coefficient is equal to the (normalized) volume of P). Until 2015, computational algorithms (for volumes or quasi-polynomials) could not handle the case of 23-dimentional polytopes. Nevertheless, exact probabilistic results dealing with the case $m = 4$ could be obtained from 2013. Schürmann (2013) proposed a method that enables to reduce the number of variables involved in the volume calculation, by exploiting the possible symmetries in the linear systems describing voting events. Applying this method, and using a new version of LattE (lattE integral, De Loera et al., 2013), he was able to obtain the first exact results (after Gehrlein, 2001) giving the exact limiting probability of voting events with four candidates. The Condorcet efficiency of plurality rule is one of three limiting values computed in Schürmann (2013):

$$CE(PR, \infty, 4) = \frac{10658098255011916449318509}{14352135440302080000000000} \qquad (74.26\%)$$

This value was recovered by Bruns and Söger (2015) and El Ouafdi et al. (2019). It should be noted that Bruns and Söger performed their calculations by an improved version of Normaliz (Bruns and Söger, 2015; [Normaliz]) which became the first algorithm to be able to calculate (most) volumes and quasi-polynomials in dimension 23. For their part, El Ouafdi et al. (2019) have developed a method that combines the use of LattE and Lrs, a program for computing the coordinates of the vertices of a rational polytope (see [Lrs]). The limiting values for the Condorcet efficiency of PER and for the probability of electing the CL for PR and NPR were calculated by Bruns et al. (2019): $CE(PER, \infty, 4) = 91.16\%$, $Pr(CL - PR, \infty, 4) = 2.27\%$ and $Pr(CL - NPR, \infty, 4) = 2.38\%$.[15] These values were also found independently by El Ouafdi et al. (2019), who also calculated the Condorcet efficiency for NPR, BR, $NPER$ and BER: $CE(NPR, \infty, 4) = 55.16\%$, $CE(BR, \infty, 4) = 87.06\%$, $CE(NPER, \infty, 4) = 84.50\%$, and $CE(BER, \infty, 4) = 99.66\%$. By using the LattE-Lrs method (and the latest version of Normaliz, based on a new computation technique called "Descent", see Bruns and Ichim, 2018), the last authors were able to obtain the first results on the vulnerability to coalitional manipulation in four-candidate elections : $VM(PR, \infty, 4) = 87.28\%$ and $VM(PER, \infty, 4) = 38.63\%$. The number of digits in the fraction giving the exact value of $VM(PER, \infty, 4)$ gives an indication on the complexity of calculations. We show it here as a comparison with one of the first probabilities calculated under IAC ($Pr(CW, \infty, 3) = 15/16$):

$$\frac{27894075660803530530375814597857426621349385364922065051212334152466919 31}{722117096321071170617881346289051953079972591461990400000000000000000000}$$

We have limited ourselves here to the results concerning the four election outcomes and the six voting procedures that interest us. To our knowledge, the only other studies offering analytical representations for probabilities in four-candidate elections are Brandt et al. (2016, 2020) which deal with certain Condorcet extensions, and Diss and Doghmi (2016), Bubboloni et al. (2018), Diss and Mahajne (2019) and Diss et al. (2019) which analyze multi-winner voting rules in committee elections. It should be mentioned that all analytical findings in Brandt et al. (2016, 2020) were obtained by applying Normaliz, and that (almost) all volume computations in Diss and Mahajne (2019) and Diss et al. (2019) were performed by Convex, the second software, after Normaliz, capable of processing 23-dimensional polytopes (see Franz, 2016, [Convex]).

It is clear from the studies presented in this section, and more generally all the recent literature, that the methods based on Ehrhart theory and on volume computation techniques are today the natural tools for probabilities calculation under IAC condition. The connection with these mathematical themes also made it possible to use specialized software to analyze voting events previously considered as very difficult to tackle. The first results obtained in the case of four

[15] It is important to mention that, in Bruns et al. (2019) as well as in El Ouafdi et al. (2019), all results concerning PER, $NPER$ and BER deal with a truncated version of these three iterative procedures, in which in a first step, the two candidates obtaining the lowest scores are eliminated and the second (and final) step is a majority contest between the two remaining candidates (in this case, PER coincides with the so-called Plurality Runoff rule, often used in political elections).

candidates are a good illustration of the power and the efficiency of these new tools. They seem to us to be able to answer most of the problems that we may consider in the case $m = 4$, to begin with that of determining the limiting probabilities, not yet calculated, for the four voting events and the six voting rules considered in this paper.

7. Concluding remarks

We can now consider that computing IAC probabilities for three-candidate elections with linear preferences (implying calculations with 6 variables) has become easy: we can obtain not only a wide variety of probability representations depending on the number of voters (and, of course, the corresponding limiting probabilities), but also some limiting representations depending on other parameters such as the degree of homogeneity of preferences or the value of λ in an election using a scoring rule. A lot of results have been obtained but we believe that some further studies remain to be conducted: for example, analyzing the impact of group coherence on the manipulability of various voting rules would be of great interest. The most recent software also allow to consider three-candidate elections with preferences that are not necessarily linear, with calculations implying more than 6 variables. For instance, some studies exist that consider dichotomous or trichotomous preferences (implying calculations with 12 or 24 variables).

The case of four-candidate elections can now be addressed (24 variables when preferences are supposed to be linear). About ten papers have already studied this case and we think that some other papers analyzing four-candidate elections will be published in the next few years. Note however that representations as a function of n, the number of voters, although possible, are too complicated to be useful in this framework. But limiting representations depending on some other parameter can certainly be obtained.

If we except some easy problems where symmetries exist (e.g., the probability of having a Condorcet winner), the move from four to five candidate elections (120 variables) seems to be out of reach with the current techniques. Progress in software (and in mathematics) has to be made if we want to deal with five-alternative elections. Observe however that in this case, for most of the probabilities of interest, the exact fractions associated to the probabilities could be too large to be exhibited! Finally, if we want to know what happens when the number m of candidates increases, a way of doing (in addition to simulations studies) could be to investigate analytically the IAC probabilities of various electoral outcomes as a function on m for a given number of voters.

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Chapter 2 : Probabilities of electoral outcomes : From three candidate to four candidate elections

Abstract. *The main purpose of this paper is to compute the theoretical likelihood of some electoral outcomes under the Impartial Anonymous Culture (IAC) in four-candidate elections by using the last versions of software like LattE or Normaliz. By comparison with the three-candidate case, our results allow to analyze the impact of the number of candidates on the occurrence of these voting outcomes.*

JEL: D7 - Analysis of Collective Decision Making
Keywords: voting rules, voting paradoxes, Condorcet efficiency, Condorcet loser, manipulability

1. Introduction

A significant part of voting theory is concerned with the computation of the likelihood of various electoral outcomes, including voting paradoxes. The basic motivation for these studies is of course to determine whether these possible paradoxical events might actually pose real threats to election; a good illustration of this line of research is the book by Gehrlein (2006), entirely devoted to the famous Condorcet's paradox. Another possible motivation is to measure and compare the ability of alternative voting rules to meet some normative criteria, often based upon majority principle (see, *e.g.*, Gehrlein and Lepelley, 2011, 2017).

In the literature, the most often used probabilistic model for computing the likelihood of these events is the IAC model, introduced by Gehrlein and Fishburn (1976), with IAC standing for Impartial Anonymous Culture. IAC condition assumes that every voting situation is equally likely to occur, a voting situation being defined as a distribution of the voters on the possible preferences. The IAC computations have recently made substantial progress by using the connection between IAC, on the one hand, and Ehrhart's theory on the other hand (see Huang and Chua, 2000; Wilson and Pritchard, 2007; Lepelley *et al.*, 2008). However, with some notable exceptions (Gehrlein, 2001; Schürmann, 2013; Brandt *et al.*, 2016; Diss and Doghmi, 2016, Bubboloni *et al.*, 2018; and very recently, Bruns *et al.*, 2019; Brandt *et al.*, 2019; Diss and Mahajne, 2019; Diss *et al*., 2019), the results available in the literature only deal with three-candidate elections, not because it is the most interesting case but due to the difficulties arising when considering more than three candidates. The first goal of this paper is to present some further illustrations of the following observation, first suggested by Schürmann, 2013, and Bruns *et al.*, 2019: an appropriate use of the last versions of software like LattE (De Loera *et al*., 2004, 2013) or Normaliz (Bruns and Söger, 2015; Bruns *et al.*, 2019) now allows to obtain exact results for four-candidate elections. Our second (and correlated) objective is to study the impact of the number of candidates on the occurrence of various electoral outcomes by comparing the results obtained with four candidates with the ones previously derived for the three-candidate case.

We first provide a series of results on the likelihood of majority condition violations by some usual voting rules in four-candidate elections. Interestingly, some of these results have been obtained (independently) by Bruns *et al.* (2019), who use a method different from ours. As emphasized by these authors, it is a good test of the correctness of the algorithms involved.

The other results we derive deal with the manipulability of two widely used voting procedures (plurality rule and plurality runoff), on the one hand, and with the concordance of scoring rules to determine the winner on the other.

The remainder of the paper is organized as follows. The basic notions used in our study are introduced in Section 2. As our technical approach is partly original, Section 3 is devoted to methodological considerations. Sections 4, 5 and 6 offer our results and Section 7 concludes.

2. Voting rules and electoral outcomes

We consider an election with four candidates (a, b, c and d) and n voters ($n \geq 2$). The 24 possible complete preferences that a voter could have on the four candidates are numbered as follows.

$abcd$	(R_1)	$bacd$	(R_7)	$cabd$	(R_{13})	$dabc$	(R_{19})
$abdc$	(R_2)	$badc$	(R_8)	$cadb$	(R_{14})	$dacb$	(R_{20})
$acbd$	(R_3)	$bcad$	(R_9)	$cbad$	(R_{15})	$dbac$	(R_{21})
$acdb$	(R_4)	$bcda$	(R_{10})	$cbda$	(R_{16})	$dbca$	(R_{22})
$adbc$	(R_5)	$bdac$	(R_{11})	$cdab$	(R_{17})	$dcab$	(R_{23})
$adcb$	(R_6)	$bdca$	(R_{12})	$cdba$	(R_{18})	$dcba$	(R_{24})

Figure 1. The possible complete preference rankings on four candidates

We suppose that voters' preferences are anonymous and we denote by n_i $(1 \leq i \leq 24)$ the number of voters with preference R_i, so that n_1 voters rank a first, b second, c third and d fourth. A *voting situation* (of size n) reports the value of each n_i and can be represented by a 24-tuple $(n_1, \dots, n_{24})$ such that:

$$\sum_{i=1}^{24} n_i = n \qquad (1)$$

and

$$n_i \geq 0 \; (1 \leq i \leq 24) \qquad (2)$$

We denote by $V(n)$ the set of all voting situations with n voters and by V the set of all voting situations. As mentioned above, the IAC model assumes that all possible voting situations are equally likely to occur. For a voting situation x in $V(n)$ and two different candidates w, w', we will denote by $P_x(w, w')$ the number of voters that prefer w to w'. For example, the numbers involved in the binary comparisons between a and b are:

$$P_x(a,b) = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_{13} + n_{14} + n_{17} + n_{19} + n_{20} + n_{23}$$
$$P_x(b,a) = n_7 + n_8 + n_9 + n_{10} + n_{11} + n_{12} + n_{15} + n_{16} + n_{18} + n_{21} + n_{22} + n_{24}$$

A *Condorcet Winner* (CW) is a candidate who beats each other candidate in pairwise majority comparisons. In the same way, a *Condorcet Loser* (CL) is a candidate who loses against every other candidate in pairwise majority contests. To illustrate, candidate a is the CW if and only if the following inequalities are satisfied:

$$P_x(a,b) - P_x(b,a) > 0,\; P_x(a,c) - P_x(c,a) > 0 \text{ and } P_x(a,d) - P_x(d,a) > 0 \qquad (3)$$

We will also make use of the notions of *Absolute Condorcet Winner* (ACW) and *Absolute Condorcet Loser* (ACL); a ACW is a candidate who is top ranked by more than half of the voters, and, similarly, a ACL is a candidate who is bottom ranked by more than half of the voters.

A voting rule is a mapping F associating with every voting situation x in V a (winning) candidate $F(x)$ in W. A "good" voting rule should select the CW when such a candidate exists (*CW condition)* and should not select the CL when such a candidate exists (*CL condition*). If a voting rule does not select the CW, it should at least select the ACW when such a candidate exists (*ACW condition*); similarly, a voting rule should not select the ACL when such a candidate exists (*ACL condition*). In this sense, the non-selection of the CW (ACW) or the selection of the CL (ACL) can be considered as voting paradoxes. All voting rules studied in this paper belong to the class of (simple) *scoring rules* or to the class of *elimination scoring rules*. We evaluate the conditional probability of electing the CW or the ACW (given that such candidates exist) and the conditional probability to select the CL or the ACL (given that such candidates exist) for the following voting rules:

- Plurality Rule (PR) : the widely used Plurality Rule selects the candidate with a majority of first preferences.
- Negative Plurality Rule (NPR): it selects the candidate who obtains the minimum of last place votes.
- Borda Rule (BR) : in a four-candidate election, each candidate gets 0 points for each last place vote received, 1 point for each third place vote, 2 points for each second place vote and 3 points for each first place vote. The candidate with the largest total point wins the election.
- Plurality Elimination Rule (PER): it is an iterative procedure, in which, at each step, the candidate who obtained the minimum number of first place votes is eliminated. The last candidate non eliminated is the winner.
- Negative Plurality Elimination Rule ($NPER$): at each step of this iterative procedure, the candidate with the maximum number of last place votes is eliminated.
- Borda Elimination Rule (BER): at each step of this iterative procedure, the candidate with the minimum Borda score is eliminated.

For the sake of simplicity, we will consider a truncated version of the three iterative procedures, in which in a first step, the two candidates obtaining the lowest scores are eliminated and the second (and final) step is a majority contest between the two remaining candidates (in this case, PER coincides with the so-called *Plurality Runoff rule*, often used in political elections). The particular versions of these elimination rules will be denoted by PRR (PR Runoff), $NPRR$ (NPR Runoff) and BRR (BR Runoff). It is worth noticing that BRR is susceptible to elect a candidate different from the CW when such a candidate exists, in contrast to BER (the non-truncated version), which always selects the CW; BRR can even choose a candidate different from the ACW, as shown in the following example: consider an election with 4 candidates and 15 voters: 4 voters have preference R_1, 4 voters have preference R_2 and 7 voters have preference R_{22} (see Figure 1); candidate a is ranked first by an absolute majority of voters and the Borda scores of a, b, c, d are (respectively) 24, 30, 11 and 25; thus c and a (the ACW) are eliminated at the first step of the procedure.

Hence, the six voting rules we consider here violate the CW condition. And, among these six rules:

(i) BR, NPR, BRR and $NPRR$ (and only these rules) violate the ACW condition;

(ii) PR and NPR are the only rules violating the CL condition;

(iii) and PR is the only rule violating the ACL condition (see, *e.g.*, Lepelley, 1989).

The results on the frequency of violation of each of the Condorcet (or majority) conditions we have introduced will be presented in Section 4. In Section 5, we will tackle a completely different problem: we will compute the vulnerability of PR and PRR to strategic misrepresentation of preferences by coalitions of voters in four-alternative elections. Finally, in Section 6, we will evaluate the probability that all the scoring rules select the same winner when the number of candidates is equal to four.

In the remainder of this study, we will need to compute the scores of the candidates under each of the three scoring rules PR, NPR and BR. For a scoring rule F, a candidate w and a voting situation x, we will denote by $S_F(w,x)$ the score of w under F. We only write the scores of candidate a under each of the three rules (the other scores are easily obtained in the same way):

$$S_{PR}(a,x) = (n_1 + n_2 + n_3 + n_4 + n_5 + n_6),$$

$$S_{NPR}(a,x) = (n_1 + n_2 + n_3 + n_4 + n_5 + n_6) + (n_7 + n_8 + n_{13} + n_{14} + n_{19} + n_{20})$$
$$+(n_9 + n_{11} + n_{15} + n_{17} + n_{21} + n_{23}),$$

$$S_{BR}(a,x) = 3(n_1 + n_2 + n_3 + n_4 + n_5 + n_6) + 2(n_7 + n_8 + n_{13} + n_{14} + n_{19} + n_{20})$$
$$+(n_9 + n_{11} + n_{15} + n_{17} + n_{21} + n_{23}).$$

Note that the score of candidate a under the Negative Plurality rule can be obtained more simply as $S_{NPR}(a,x) = n - (n_{10} + n_{12} + n_{16} + n_{18} + n_{22} + n_{24})$. We conclude this section by emphasizing that, as we only consider large electorates, the problem of tied elections can be disregarded: under each of the voting rules we study, the probability of *ex aequo* tends to 0 when n tends to infinity (see, *e.g.*, Lepelley, 1989).

3. Methodology

Under the IAC assumption, the voting events are often described by a parametric system of linear constraints with integer (or rational) coefficients on the variables n_i and the parameter n. For example, with n voters, the event "a is the CW" is characterized by the system formed by equality (1), the 24 sign inequalities in (2) and the three strict inequalities in (3). Thus, the frequency of a voting event E can be evaluated by computing the number of integer solutions of the parametric linear system describing E. It is now well known in voting theory, since

Wilson and Pritchard (2007) and Lepelley *et al*. (2008), that the use of polytopes and quasi-polynomials is the most appropriate mathematical tool for such computations.

3.1. Integral points in parametric polytopes. A rational polytope P of dimension d is a bounded subset of $\mathbb{R}^d$ defined by a system of integer linear inequalities. P is said to be semi-open when some of these inequalities are strict. A parametric polytope of dimension d (with a single parameter n) is a sequence of d-dimensional rational polytopes P_n ($n \in \mathbb{N}$) of the form $P_n = \{x \in \mathbb{R}^d : Mx \geq bn + c\}$, where M is an $t \times d$ integer matrix and b and c two integer vectors with t components. When the constant term c is equal to the zero vector, P_n is denoted nP and corresponds to the dilatation, by the positive integer factor n, of the rational polytope P defined by $P = \{x \in \mathbb{R}^d : Mx \geq b\}$. In this case,t Ehrhart's theorem (1962) tells us that the number of integer points (lattice points) in nP is a quasi- polynomial in n of degree d, *i.e.* a polynomial expression $f(n)$ in the parameter n where the coefficients are not constants, but periodic functions of n with integral period. Each coefficient can have its own period, but we can always write $f(n)$ in a form where the coefficients have a common period called the period of the quasi-polynomial (or the denominator of P) and defined as the least common multiple (lcm) of the periods of all coefficients. The leading coefficient of the quasi polynomial f is the same for all congruence classes and is equal to the (relative) volume of P.

Clauss *et al*. (1997) extended Ehrhart's result to the general class of parametric polytopes P_n, showing that the number of lattice points in P_n can be described by a finite set of quasi-polynomials, each valid on a different subset of $\mathbb{N}$. Note that this implies that for n large enough, the number of lattice points in P_n is given by a single quasi-polynomial. Note also that this generalization makes possible to count the number of lattice points inside the dilatation of a semi-open polytope P. It suffices to use the rule " $\forall x \in \mathbb{Z}, x > 0 \Leftrightarrow x \geq 1$ " to transform each strict inequality in the system describing nP into a non-strict inequality, and thus obtain a parametric polytope having the same number of lattice points than nP.

3.2. Limiting probabilities of voting events. Consider an election with n voters and m candidates. Let E be a voting event for which we want to calculate the probability under the IAC hypothesis. Let $V(n)$ be the set of all possible voting situations of size n and (E,n) the set of all elements of $V(n)$ in which E occurs. The probability of (E,n) is a function of n and is given by:

$$Pr(E,n) = |(E,n)|/|V(n)| \quad (4)$$

The expression of $|V(n)|$ is well known: with m candidates, it is given by $|V(n)| = \binom{n + m! - 1}{m! - 1}$. Hence $|V(n)|$ is a polynomial of degree $m! - 1$ and the coefficient of the leading term is equal to $1/(m! - 1)$. In general, (E, n) is described by a parametric linear system $S(n)$ that defines a dilatation of a semi-open rational polytope P of dimension $m! - 1$. Thus, $|(E, n)|$ is equal to the number of lattice points inside nP and is given by the quasi-polynomial describing this number.

To compute $|(E, n)|$, we usually resort to (parametrized) Barvinok's algorithm (Barvinok, 1994). The software [Barvinok] (see Verdoolaege and Bruynooghe, 2008) applies to any parametric polytope and can therefore deals with the case of interest for us, that of a dilated semi open polytope. [Barvinok] performs very well for $m = 3$ and, since 2008, the use of this program has yielded many results giving the exact analytical representation for the frequency of various voting events. Note that in the case $m = 3$, there are only 6 variables and the quasi polynomials describing $|(E, n)|$ are generally of degree 5. Unfortunately, with $m = 4$, there are 24 variables, the quasi-polynomials are of degree 23 and [Barvinok] does not allow to obtain the desired results. Other software packages such as LattE with its new version Latte integrale (see [latte]) and Normaliz (see [Normaliz]) allow, in some cases, to calculate quasi-polynomials corresponding to polytopes of dimension 23. However, we know that for $m = 4$, the periods of the quasi-polynomials can be very large and that the exact formulas for $Pr(E, n)$ can be far too heavy for meaningful analysis. Therefore, in what follows, attention will be focused on the limiting case where the number of voters, n, tends to infinity.

We set the number of candidates to $m = 4$ and we denote by $Pr(E, \infty)$ the limit of $Pr(E, n)$ when n tends to infinity. From the above, $Pr(E, n)$ is the quotient of the quasi-polynomial $|(E, n)|$ by the polynomial $|V(n)|$. For $|V(n)|$, the coefficient of the leading term is equal to $1/23!$. For $|(E, n)|$, this coefficient is independent of n and is equal to the volume of the semi open polytope P obtained by taking $n = 1$ in the linear system $S(n)$. Going to the limit in (4), we get**:**

$$Pr(E, \infty) = 23! \operatorname{vol}(P) \qquad (5)$$

It is obvious that the same reasoning can be applied for conditional probabilities. In this case, $Pr(E, n)$ is of the form $Pr(E, n) = |(E_1, n)|/|(E_2, n)|$ where (E_1, n) and (E_2, n) are two voting events characterized by some linear systems $S(n)$ and $T(n)$ that define two dilated semi-open polytopes, nP and nQ. If P and Q are of the same dimension, we can write:

$$Pr(E, \infty) = \mathrm{vol}(P)/\mathrm{vol}(Q) \qquad (6)$$

In general, algorithms that compute the volume of polytopes are not always efficient when, as in this paper, the number of variables is equal to 24. However, recent improvements in algorithms such as LattE, Normaliz or Convex (see [Convex]) have made it possible to obtain some results describing the probability of voting events with four candidates, requiring the calculation of the volumes of certain polytopes of dimension 23 (see Schürmann, 2013; Bruns and Söger, 2015; Brandt *et al.*, 2016; Bruns *et al.*, 2019). To compute the volumes involved in the calculations developed in the remainder of this paper, we will not use any algorithm of direct volume computation (with, however, some exceptions[16]). Instead, we will apply a (new) method based on Ehrhart theory and on the combined use of two software, LattE integrale and lrs (see [lrs]). The command (count-ehrhart-polynomial) in the first program allows to calculate in a reasonable time (from a few seconds to a few hours) the quasi-polynomial associated with a dilated polytope nP. With LattE integrale, this computation is possible only when P is an integral polytope ($i.e.$ when all its vertices have integer coordinates). In this case, the quasi-polynomial has period equal to 1 and hence is simply a polynomial. The second program, lrs, allows to obtain (usually within seconds) the coordinates of all vertices of a rational polytope. Since in our calculations, P is in general a non-integral polytope, we proceed as follows to calculate $\mathrm{vol}(P)$. We start by dilating P by a positive integer factor k such that the obtained polytope kP is integral; for this, k must be a multiple of the period of P. Now, we know by Ehrhart theorem that the period of P is a divisor of the lcm of the denominators of the vertices of P. It suffices then to take k equal to this number that we can easily obtain by applying the lrs program. After this step, we apply LattE integrale to the integral polytope kP and we obtain the polynomial associated with the dilated polytope nkP. It is obvious that if A is the coefficient of the leading term of this polynomial, then: $A = \mathrm{vol}(kP) = k^{24}\,\mathrm{vol}(P)$. Finally, we have: $\mathrm{vol}(P) = A/k^{24}$.

4. Results on Condorcet conditions

For a total number of voters equal to n, let $(X - F, n)$ be the event "X is elected under F, given that X exists", with X in $\{CW, CL, ACW, ACL\}$ and F a voting rule in $\{PR, NPR, BR, PRR, NPRR, BRR\}$. We denote by $Pr(X - F, n)$ and $Pr(X - F,\ \infty)$ the IAC probability of $(X - F, n)$ and the limit of this probability when n tends to infinity. We know

[16]See Section 6 and the Appendix where we make use of the last version of Normaliz to deal with some particularly complicated computations.

that $Pr(CL - F, n) = 0$ (and thus $Pr(CL - F, \infty) = 0$) for F in $\{BR, PRR, NPRR, BRR\}$, $Pr(ACW - F, n) = 1$ for F in $\{PR, PRR\}$ and $Pr(ACL - F, n) = 0$ for F in $\{BR, NPR, PRR, NPRR, BRR\}$. We derive the other probabilities in the following subsections.

4.1. Condorcet Winner election. We assume without loss of generality that a is the CW. We denote by (CW^a, n) the event "a is the CW" and by (CW_F^a, n) the event "a is the CW and a is selected under F". It is easy to see that under IAC:

$$Pr(CW - F, n) = \frac{|(CW_F^a, n)|}{|(CW^a, n)|} \qquad (7)$$

The voting situations x associated with the event (CW^a, n) are characterized by the following parametric linear system:

$$\begin{cases} n_1 + \cdots + n_{24} = n \\ n_i \geq 0, \;\; i = 1, \ldots, 24 \\ P_x(a,b) - P_x(b,a) > 0 \\ P_x(a,c) - P_x(c,a) > 0 \\ P_x(a,d) - P_x(d,a) > 0 \end{cases} \qquad T(n)$$

Let Q_1 be the (semi-open) polytope defined by the system $T(1)$. Applying the method described in Section 3, we obtain:

$$\mathrm{Vol}(Q_1) = \frac{101 \times 23!}{124576306544085722726400000}$$

4.1.1. *Voting rules PR, NPR and BR*. For a voting rule F in $\{PR, NPR, BR\}$, the voting situations x associated with the event (CW_F^a, n) are characterized by the parametric linear system, $S^F(n)$, consisting of the constraints in $T(n)$ and the following three inequalities:

$$S_F(a,x) - S_F(b,x) > 0, \; S_F(a,x) - S_F(c,x) > 0, \; S_F(a,x) - S_F(d,x) > 0$$

Let P_1^F be the (semi-open) polytope defined by the system $S^F(1)$. Taking the limit in (7) and using formula (6), we obtain the limit of the probability $Pr(CW - F, n)$ as:

$$Pr(CW - F, \infty) = \frac{\mathrm{Vol}(P_1^F)}{\mathrm{Vol}(Q_1)}$$

We have already calculated $\mathrm{Vol}(Q_1)$. To calculate $\mathrm{Vol}(P_1^F)$ for PR, NPR and BR, we replace successively, in system $S^F(1)$, the voting rule F by PR, NPR and BR (by referring to the scores defined in section 2) and then we use the calculation method based on the LattE and lrs algorithms. Finally we obtain:

$$Pr(CW - PR, \infty) = \frac{106580982550119164493185 09}{143521354403020800000000000} \approx 74.26\ \%$$

$$Pr(CW - NPR, \infty) = \frac{2431999845589783615}{4408976007260798976} \approx 55.16\ \%$$

$$Pr(CW - BR, \infty) = \frac{828894710496058365982223276647}{952076453898607919942860800000} \approx 87.06\ \%$$

Our result for $Pr(CW - PR, \infty)$ is in accordance with the value obtained by Schürmann (2013) and (more recently) by Bruns et al. (2019).

4.1.2. *Runoff voting rules*. Let F be a voting rule in $\{PR, NPR, BR\}$ and FR the runoff voting rule using F, so that FR belongs to $\{PRR, NPRR, BRR\}$. Let $(CW1_F^a, n)$ and $(CW2_F^a, n)$ be the events defined respectively by "a is the CW and is ranked first under F" and "a is the CW and obtains the second score under F". As the CW always wins the second round, these two events describe the two possible configurations for the occurrence of the event (CW_{FR}^a, n) and we can then write: $(CW_{FR}^a, n) = (CW1_F^a, n) \cup (CW2_F^a, n)$. The voting situations associated with $(CW1_F^a, n)$ are the same as those associated with (CW_F^a, n), and are therefore characterized by the system $S^F(n)$. To characterize $(CW2_F^a, n)$, we must distinguish three cases according to the identity of the candidate ranked first under F (b, c or d). Since these three cases are symmetrical, we have $|(CW2_F^a, n)| = 3|(E, n)|$, where (E, n) is the set of voting situations belonging to $(CW2_F^a, n)$ and satisfying the additional condition that b is ranked first by the scoring rule F. This set is characterized by the parametric linear system, $Z^F(n)$, formed by the five constraints in $T(n)$ and the three additional inequalities $S_F(b, x) - S_F(a, x) > 0$, $S_F(a, x) - S_F(c, x) > 0$ and $S_F(a, x) - S_F(d, x) > 0$. Let K_1^F be the (semi-open) polytope defined by the system $Z^F(1)$. Using formula (6), we obtain the limit of the probability $Pr(CW - FR, n)$ as:

$$Pr(CW - FR, \infty) = \frac{\mathrm{Vol}(P_1^F) + 3\,\mathrm{Vol}(K_1^F)}{\mathrm{Vol}(Q_1)}$$

We substitute successively, in system $Z^F(1)$, the voting rule F by PR, NPR and BR and we use the calculation method based on LattE and lrs algorithms. We obtain:

$$Pr(CW - PRR, \infty) = \frac{19627224002877404784030049}{21528203160453120000000000} \approx 91.16\ \%$$

$$Pr(CW - NPRR, \infty) = \frac{18192354603646054002780049}{21528203160453120000000000} \approx 84.50\ \%$$

$$Pr(CW - BRR, \infty) = \frac{55789461223667462820836026969}{56004497288153407055462400000} \approx 99.66\ \%$$

Note that our result for PRR is in accordance with Bruns *et al.*(2019), who have obtained this probability (91.16%) by using Normaliz.

4.2. Condorcet Loser election. As already mentioned, among the six rules studied, only PR and NPR are succeptible to elect the CL, when such a candidate exists. We assume without loss of generality that candidate a is the CL and we denote by (CL^a, n) the event "a is the CL" and by (CL^a_F, n), for F in $\{PR, NPR\}$, the event "a is the CL and a is selected under F". It is easy to show that:

$$Pr(CL - F, n) = \frac{|(CL^a_F, n)|}{|(CL^a, n)|}$$

The voting situations x associated with the event (CL^a, n) are characterized by the following parametric linear system:

$$\begin{cases} n_1 + \cdots + n_{24} = n \\ n_i \geq 0, \ \ i = 1, \dots, 24 \\ P_x(b, a) - P_x(a, b) > 0 \\ P_x(c, a) - P_x(a, c) > 0 \\ P_x(d, a) - P_x(a, d) > 0 \end{cases} \qquad L(n)$$

The voting situations x associated with the event (CL^a_F, n) are characterized by the parametric linear system, $M^F(n)$, consisting of the constraints in $L(n)$ and the following three inequalities:

$$S_F(a, x) - S_F(b, x) > 0, \ S_F(a, x) - S_F(c, x) > 0, \ S_F(a, x) - S_F(d, x) > 0$$

To get the limit values of $Pr(CL - F, n)$, it is enough to compute the volume of the semi-open polytope defined by $L(1)$ and the volume of the semi-open polytope defined by $M^F(1)$ for F in $\{PR, NPR\}$. After calculation, we obtain:

$$Pr(CL - PR, \infty) = \frac{32545167483582855068149 1}{14352135440302080000000000} \approx 2.27\ \%$$

$$Pr(CL - NPR, \infty) = \frac{104898234852130241}{4408976007260798976} \approx 2.38\ \%$$

The same results have been obtained by Bruns *et al*. (2019) *via* Normaliz.

4.3. Absolute Condorcet Winner election and Absolute Condorcet Loser election. We assume without loss of generality that candidate a is the ACW and we denote by (ACW^a, n) the event "a is the ACW". The voting situations associated with this event are characterized by the parametric linear system obtained from $T(n)$ when the three inequalities in (3) are replaced by the following single condition: $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 > n/2$. Let Q'_1 be the polytope associated with this new system. Computing $\mathrm{Vol}(Q'_1)$ and applying (5), we get:

$$Pr((ACW^a, \infty) = \frac{5569}{1048576}$$

This implies that the probability of having a ACW is equal to $4 \times \ Pr((ACW^a, \infty) = \frac{5569}{262144} \approx$ 2.12 %. We know from Lepelley (1989) that the corresponding probability for the three-candidate case is $\frac{9}{16} \approx 56.25\ \%$: consequently, moving from three to four candidates dramatically decreases the percentage of voting situations with a ACW.

Suppose, however, that such a candidate exists. What is the probability for this candidate to be selected? Proceeding as in subsection 4.1, but replacing, everywhere in the calculations concerning NPR, BR, $NPRR$ and BRR, the inequalities describing the event (CW^a, n) with the one describing the event (ACW^a, n), we obtain:

$$Pr(ACW - NPR, \infty) = \frac{6712690981925}{10775556292608} \approx 62.30\ \%$$
$$Pr(ACW - BR, \infty) = \frac{36216780125610009500388529}{36278317087318348922880000} \approx 99.83\ \%$$
$$Pr(ACW - NPRR, \infty) = \frac{396415547534699}{436410029850624} \approx 90.84\%$$
$$Pr(ACW - BRR, \infty) = \frac{18139154487212563566077 6587}{18139158543659174461440 0000} \approx 99.99\ \%$$

Consider now the election of the ACL. Let (ACL^a, n) be the event "a is the ACL". The voting situations associated with this event are characterized by the system obtained from $L(n)$ when the last three inequalities are replaced with the single condition $n_{10} + n_{12} + n_{16} + n_{18} + n_{22} + n_{24} > n/2$.

By a symmetry argument, the volume associated with this new system is equal to $\mathrm{Vol}(Q_1')$, so we have $Pr(ACL^a, \infty) = Pr((ACW^a, \infty)$. When an *Absolute Condorcet Loser* exists, the only voting rule (among the six rules we consider) susceptible to elect such a candidate is the Plurality Rule. Proceeding as in subsection 4.2, but replacing everywhere in the calculations concerning PR the inequalities describing (CL^a, n) with the one describing (ACL^a, n), we obtain:

$$Pr(ACL - PR, \infty) = \frac{3950740911499}{872820059701248} \approx 0.45\%$$

4.4. Summary of the results on Condorcet conditions. The following Table summarizes our four-candidate results on the ability of various voting rules to fulfill Condorcet conditions and compares these results to known results obtained in the literature for the three-candidate case (see Lepelley, 1989; Gehrlein and Lepelley, 2011 and Diss *et al.*, 2018). The four-candidate results with an asterix* have been independently obtained by Schürmann (2013) and Bruns *et al*. (2019).

Events	3 candidates	4 candidates
$CW - PR$	88.15%	74.26%*
$CW - NPR$	62.96%	55.16%
$CW - BR$	91.11%	87.06%
$CW - PRR$	96.85%	91.16%*
$CW - NPRR$	97.04%	84.50%
$CW - BRR$	100%	99.61%
$CL - PR$	2.96%	2.27%*
$CL - NPR$	3.15%	2.38%*
$ACW - NPR$	60.76%	62.30%
$ACW - BR$	96.32%	99.83%
$ACW - NPRR$	97.53%	90.84%
$ACW - BRR$	100%	99.99%
$ACL - PR$	2.47%	0.45%

Table 1: CW election, CL election, ACW election and ACL election

Some interesting conclusions emerge from this comparison. First, it turns out that the probability of electing the CW, given that such a candidate exists, decreases when the number of candidates moves from three to four for each of the voting rules we have considered. Note however that the decreasing rate is lower for BR (4.4%) than for PR (16%), NPR (12.4%), PRR (5.9%) and $NPRR$ (12.9%). The ability to electing the CW (or *Condorcet Efficiency*) of BR is now closer to the two-stage PRR value, and it is higher than the $NPRR$ value. These results reinforce the conclusion recently obtained by Gehrlein *et al.* (2018) that the expected benefit that would be gained from using two-stage voting rules like PRR or $NPRR$ instead of BR is quite small.

Second, we find that the probability of electing the CL (the so-called *Strong Borda Paradox*) decreases from three to four candidates for PR and NPR as well, thus (slightly) increasing the ability of these two voting rules to fulfill the CL condition.

Third, our results show that the impact of the number of candidates on the *Absolute Condorcet Winner* election depends on the voting rule under consideration: when moving from three to four candidates, the probability of electing the ACW increases for NPR and BR but decreases for $NPRR$ (and, of course, for BRR, which satisfies the ACW condition in the three-candidate case). It is worth noticing that, in the four candidate case, the BR probability is close to 100%: in this case, the possible non- election of the ACW should not be considered as a significant flaw of the Borda rule. In addition, it turns out that the truncated version of the Borda

Elimination Rule we consider here has only a very marginal impact on the ability of this rule to elect the ACW.

Finally, we obtain that the likelihood of the ACL election under PR is divided by 5.5 when we move from three to four candidates: such an event becomes very unlikely when four candidates are in contention.

5. Results on coalitional manipulability

5.1. Coalitional manipulability of plurality rule. A strategic manipulation of a voting rule occurs in an election when some voters express insincere preferences in order to obtain a final winner that they prefer to the candidate that would have been elected if they had voted in a sincere way. To illustrate, consider the following voting situation (with 30 voters and 4 candidates), supposed to correspond to the sincere preferences: $n_1 = 12,\ n_7 = 10,\ n_{15} = 8,\ n_i = 0$ for all $i \notin \{1, 7, 15\}$ (the numbering of the preferences is the one given in Figure 1). Under PR and sincere voting, a is the winner (with 12 votes for a, 10 votes for b, 8 votes for c and 0 votes for d. If (at least) three of the eight electors who rank c in first position vote for their second choice (b), then b is elected and the voters who vote in an insincere way are better off since they prefer b to a (the "sincere" winner). Such a voting situation is said to be *instable*: a coalition of voters, by misrepresenting their preferences, may secure an outcome that they all prefer to the result of sincere voting.

It makes sense to evaluate the *coalitional manipulability* of a voting rule by calculating the proportion of instable voting situations when the voting rule under consideration is used. We consider first the plurality rule. Let x be a voting situation where candidate a is elected under PR:

$$S_{PR}(a,x) - S_{PR}(b,x) > 0,\ S_{PR}(a,x) - S_{PR}(c,x) > 0,\ S_{PR}(a,x) - S_{PR}(d,x) > 0 \qquad (8)$$

According to Lepelley and Mbih (1987), PR is *not* vulnerable to strategic manipulation by a coalition of voters at this voting situation if, in addition, $S_{PR}(a,x)$ is higher than the number of voters preferring b to a, the number of voters preferring c to a and the number of voters preferring d to a, *i.e.*:

$$S_{PR}(a,x) - P_x(b,a) > 0,\ S_{PR}(a,x) - P_x(c,a) > 0 \text{ and } S_{PR}(a,x) - P_x(d,a) > 0 \qquad (9)$$

Let P_n be the (semi-open) parametric polytope defined by the system formed by equality (1), the 24 sign inequalities in (2), the three inequalities in (8) and the three inequalities in (9). Applying formula (5) and multiplying by 4 (the number of candidates), we obtain that the probability for PR to be vulnerable to misrepresentation of preferences by coalitions of voters, denoted by $Pr(Manip - PR, \infty)$, is given as: $Pr(Manip - PR, \infty) = 1 - 4 \times 23!\,\mathrm{Vol}(P_1)$. Evaluating $\mathrm{Vol}(P_1)$ by the method described in section 3, we obtain for the four-candidate case:

$$Pr(Manip - PR, \infty) = 1 - \frac{1938509031230593}{15116544000000000} \approx 87.28\%$$

Lepelley and Mbih (1987) have shown that, in the three-candidate case, the vulnerability of PR to strategic manipulation by coalitions of voters for large electorates is equal to $7/24$, *i.e.* 29.17%. We conclude that moving from three candidates to four candidates very significantly increases the PR vulnerability to strategic manipulation

5.2. Coalitional manipulability of plurality rule with runoff. Do we obtain a similar conclusion for PRR? We know from Lepelley (1989) that the vulnerability of PRR to strategic manipulation for large electorate in three-candidate elections is equal to $1/9$, *i.e.* 11.11%. The aim of the current subsection is to investigate what happens when a further candidate is added.

Our computations will be based on the two following propositions.

Proposition 1 (Lepelley, 1989). If a voting situation is such that either there is no CW or a CW exists and is not the PRR winner, then this voting situation is instable for PRR.

This first proposition is valid regardless of the number of candidates. The second one only deals with four-candidate elections and needs some additional notation: $F_x(a)$ is the number of voters in x who rank a in first position (this is simply $S_{PR}(a, x)$), $F_x^{ab}(c)$ is the number of voters in x who rank c in first position and prefer a to b, $F_x^{ab}(d)$ is the number of voters in x who rank d in first position and prefer a to b, and y will denote the voting situation obtained from x after manipulation by a coalition of voters.

Proposition 2. Consider a four-candidate election and a voting situation x in $V(n)$ in which candidate a is both the CW and the PRR winner. Then x is instable under PRR if and only if there are two candidates, say b and c, different from a, such that

(i) $P_x(b, a) > F_x(a)$, (ii) $P_x(b, a) > F_x^{ab}(d)$,

(iii) $P_x(b, a) + F_x^{ab}(c) > 2F_x(a)$, (iv) $P_x(b, a) + F_x^{ab}(c) > 2F_x^{ab}(d)$,

(v) $P_x(b, c) > n/2$.

Proof.[17]

Necessity. Suppose that x is instable for PRR. It means that there exists a candidate different from a, say b, and a voting situation y derived from x, such that $PRR(y) = b$, and in which the manipulating voters belong to the set of voters preferring b to a in x. This implies that the scores of a and b in y are such that: (α) $S_{PR}(a,y) = F_x(a)$ and (β) $S_{PR}(b,y) \leq P_x(b,a)$. As a is the Condorcet Winner in x, (β) implies that b cannot win in the first stage in y. Thus, there exists a candidate different from a and b, say c, who goes to the second stage with b in y and is beaten by b in this second stage. The only possible strategies for the manipulating voters being to rank b or c in first position, it follows that: (γ) $S_{PR}(b,y) + S_{PRR}(c,y) \leq P_x(b,a) + F_x^{ab}(c)$ and (δ) $S_{PR}(d,y) \geq F_x^{ab}(d)$.

Condition (i) is necessary because, if it does not hold, by (α) and (β), we would have $S_{PR}(b,y) < S_{PR}(a,y)$ and this implies that b is either eliminated in the first stage or confronted to a in the second stage; in both cases, it contradicts the fact that b and c are together in the second stage in y. Similarly, (ii) has to hold: if not, by (β) and (δ), we would have $S_{PR}(b,y) < S_{PR}(d,y)$, which would imply that b is either eliminated in the first round or confronted to d in the second stage, contradicting the presence of b and c in the second stage in y.

Condition (iii) is also necessary: if not, using (β) and (γ), we would have $S_{PR}(b,y) < S_{PR}(a,y)$ or $S_{PR}(c,y) < S_{PRR}(a,y)$. This would imply that either b or c (or both of them) would be eliminated in the first stage in y (contradicting the presence of b et c in the second stage). A similar argument using (γ) and (δ) instead of (β) and (γ) shows that (iv) is necessary as well.

Finally, condition (v) is necessary because, to be the winner in y, b has to beat c in the second stage by a majority of votes.

Sufficiency. Assume there exist two candidates, say b and c, different from a, such that conditions (i)-(v) hold. Let $r = \max\{F_x(a), F_x^{ab}(d)\} + 1$ and $s = P_x(b,a) - r$; by (iii) and (iv), we have $s \geq 0$. Let y be the voting situation resulting from x where voters preferring b to a (all or part of them) strategically vote in order to have b ranked first exactly r times and c ranked first exactly s times (it is possible by (i) and (ii)). Thus we have: $S_{PR}(a,y) = F_x(a)$, $S_{PR}(b,y) = r$, $S_{PR}(c,y) = s + F_x^{ab}(c)$, and $S_{PR}(d,y) = F_x^{ab}(d)$. It is then easy to see that $S_{PR}(b,y) > S_{PR}(a,y)$ and $S_{PR}(b,y) > S_{PR}(d,y)$ (by definition of r), and $S_{PR}(c,y) > S_{PR}(a,y)$ and $S_{PR}(c,y) > S_{PR}(d,y)$ (by definition of r and s, and by (iii) and (iv)).

[17] Recall that we only consider large electorates; consequently, we ignore here the cases where two candidates obtain the same score: for instance, if the score of a is not strictly higher than the score of b, it means that the score of a is strictly lower than the score of b.

Consequently, b and c are selected for the second stage in y and b beats c in the second stage, by (v). Hence $PRR(y) = b$, showing that x is instable for PRR.

Let E_1 denote the event "there is no CW", E_2 the event "a is the CW and is not selected under PRR" and E_3 the event " a is the CW, is selected under PRR and the voting situation is instable for PRR". It follows from Proposition 1 that the probability for PRR to be vulnerable to misrepresentation of preferences by coalitions of voters can be written as (we assume large electorates):

$$Pr(Manip - PRR, \infty) = Pr(E_1, \infty) + 4(Pr(E_2, \infty) + Pr(E_3, \infty)) \qquad (10)$$

We know from Gehrlein (2001) that, in four-candidate elections:

$$Pr(E_1, \infty) = \frac{331}{2048}$$

We easily deduce from the above-computed Condorcet Efficiency of PRR for four candidates (see Subsection 4.1.2) that:

$$Pr(E_2, \infty) = \frac{1717}{8192}\left(1 - \frac{19627224002877404784030049}{21528203160453120000000000}\right)$$
$$= \frac{19009791575757152159699951}{102713477163909120000000000}$$

and we have used Proposition 2 to obtain the following fraction for $Pr(E_3, \infty)$:

$$\frac{10877280648064963377199686333074553289294052519565566601468366152466919 31}{288846838528428468247152538515620781231989036584796160000000000000000000000}$$

The computations are tedious and are detailed in Appendix. Using (10), we finally obtain the following result for $Pr(Manip - PRR, \infty)$:

$$\frac{2789407566080353053037581459785742662134938536492206505121233415246691931}{7221170963210711706178813462890519530799725914619904000000000000000000000}$$

i.e. $Pr(Manip - PRR, \infty) \approx$ 38.63%. Consequently, the vulnerability of PRR to coalitional manipulation is multiplied by a factor higher than 3.4 when a fourth candidate is introduced! The manipulability of PRR remains however significantly lower than the one of PR.

6. Concordance of all scoring rules.

In four-candidate elections, a scoring rule can be defined by a 4-tuple $(1, \lambda, \mu, 0)$, with $1 \geq \lambda \geq \mu \geq 0$. Candidates get 1 point for each first position in voters' rankings, λ points for each second position, μ points for each third position and 0 points for each last position. We obtain PR by taking $\lambda = \mu = 0$, NPR by taking $\lambda = \mu = 1$ and BR by taking $\lambda = 2/3$ and $\mu = 1/3$. We wish to compute the probability that all the scoring rules agree, *i.e.* select the same winner, in four-candidate elections. This calculation is of interest since it allows to know, *a contrario*, the proportion of voting situations for which the choice of a specific scoring rule is susceptible to impact the determination of the winner. In three-candidate elections, the result is known: Gehrlein (2002) shows that, in this case, the probability that all scoring rules give the same winner is equal to $113/216 = 0.5231$; thus, the proportion of voting situations where the choice of a particular voting rule really matters is about 48%. We would like to know how these figures are modified when we consider four-candidate elections. We know from Moulin (1988) that, in four-candidate elections, all the scoring rules will select the same winner if and only if the three "elementary" scoring rules $(1,0,0,0), (1,1,0,0)$ and $(1,1,1,0)$ lead to the choice of the same winner. The first and the third elementary scoring rules are simply PR and NPR; we denote the second elementary rule by IR (the "intermediate" rule). The voting situations x (of size n) at which the event "All the scoring rules select candidate a" occurs are characterized by the system formed by (1), (2) and the following nine inequalities:

$$S_{PR}(a,x) - S_{PR}(b,x) > 0,\ S_{PR}(a,x) - S_{PR}(c,x) > 0,\ S_{PR}(a,x) - S_{PR}(d,x) > 0$$
$$S_{IR}(a,x) - S_{IR}(b,x) > 0,\ S_{IR}(a,x) - S_{IR}(c,x) > 0,\ S_{IR}(a,x) - S_{IR}(d,x) > 0$$
$$S_{NPR}(a,x) - S_{NPR}(b,x) > 0,\ S_{NPR}(a,x) - S_{NPR}(c,x) > 0,\ S_{NPR}(a,x) - S_{NPR}(d,x) > 0$$

Let P_n be the (semi-open) parametric polytope defined by this system. Our method failed to compute the volume of P_1; but we have been able to obtain the desired result by using the latest version of Normaliz, based on a new computation technique called "Descent" (see Bruns and Ichim, 2018). The numerator and the denominator of the fraction we obtain are very high:

$$\frac{93491394011276905335667964185570257949502235924011178804737669535180034916049 67}{3686714233060324324252649132772264674310716738771957491852880834872585173144239303735377920000000000000}$$

Using formula (5), multiplying by 4 (the number of candidates), and evaluating this fraction, we obtain the following probability for the event SW: "All the scoring rules give the same winner":

$$Pr(SW, \infty) = 0.2622325388$$

We conclude that the probability that the choice of the voting rule impacts the winner determination increases from 48% to about 74% when the number of candidates moves from three to four. We note also that our result is consistent with the probability obtained by Bruns and Ichim (2018) for the concordance of the following four voting rules: PR, NPR, BR and MR (Majority Rule): they found that the probability that these voting rules select the same winner is about 31%.

Another interesting result given in Gehrlein (2002) for three-candidate elections is the probability that all the scoring rules select the CW. Gehrlein obtains $3437/6912 = 0.4973$. Let $SW = CW$ denote this event. Adding (3) to (1), (2) and the above inequalities, we obtain *via* Normaliz and using (5) that the probability of having candidate a as both the CW and the winner of all the scoring rules is given as:

$$\frac{56805533835478620517477392716788353889762986166521058744514015680894892856328332595075 3}{92191183923235569884368281442342607854309693855410582305557180205391195144192000000000 00}$$

Multiplying by 4, we have:

$$Pr(SW = CW, \infty) = 0.2464683993$$

Hence, as in the case of three-candidate elections, the addition of the restriction that the common winner of the scoring rules is also a CW has little impact on the probability that all the scoring rules select the same winner.

7. Conclusion

We have derived in this paper some exact results for the likelihood of various electoral outcomes and voting paradoxes under the IAC assumption in four-candidate elections. These computations have made possible a first investigation (based on exact results rather than on estimates obtained from simulations) of the impact of the number of candidates on the occurrence of these voting outcomes. Among other results, we showed that the non-election of the Absolute Condorcet Winner under the Borda rule and the election of the Absolute Condorcet Loser under the plurality rule are not a big concern when the number of candidates is equal to four. By contrast, the introduction of a fourth candidate significantly increases 1) the manipulability of the plurality and plurality with runoff rules, and 2) the significance of voting rule selection.

From a technical point of view, the major part of our calculations have been done thanks to an original method, based on a combination of the software packages LattE and lrs. It seems however that the latest version of Normaliz is, at the present time, the most efficient software tool to obtain the IAC probabilities of electoral outcomes when more than three alternatives are in contention, as suggested by the recent paper of Bruns and Ichim (2018) and illustrated by the computations we have conducted in Section 6 and in Appendix.

Appendix: Computation of $\boldsymbol{Pr(E_3,\infty)}$

Let (E_3,n) be the set of all voting situations, of size n, in which E_3 occurs. Since a must be first or second in the first stage of the sincere vote, by symmetry we can write:

$$|(E_3,n)| = 3(|(G,n)| + |(H,n)|) \qquad \text{(A1)}$$

where (G,n) is the set of the voting situations in (E_3,n) for which a is first and b is second in the first stage, and (H,n) the set of the voting situations in (E_3,n) for which a is second and b is first in the first stage. Considering all possibilities for the choice of the candidate who wins after manipulation and the candidate who goes with him to the second stage, we obtain:

$$(G,n) = (G^{bc},n) \cup (G^{cb},n) \cup (G^{bd},n) \cup (G^{db},n) \cup (G^{cd},n) \cup (G^{dc},n) \qquad \text{(A2)}$$

$$(H,n) = (H^{bc},n) \cup (H^{cb},n) \cup (H^{bd},n) \cup (H^{db},n) \cup (H^{cd},n) \cup (H^{dc},n) \qquad \text{(A3)}$$

Here, for α, β in $\{b,c,d\}$ and $\alpha \neq \beta$, the notation $(G^{\alpha\beta},n)$ (resp. $(H^{\alpha\beta},n)$) denotes the subset of (G,n) (resp. (H,n)) of voting situations where α and β go to the second stage after manipulation (in favor of α) and α beats β by a majority of votes. For simplicity, in what follows, these subsets will be denoted by $G^{\alpha\beta}$ (resp. $H^{\alpha\beta}$).

Using Proposition 2 and deleting the redundant inequalities, it follows that the voting situations x in G^{bc}, G^{cb} and G^{cd} are characterized by the following parametric linear systems:

$$(S_n^{bc})\begin{cases} n_1+\dots+n_{24}=n \\ n_i \geq 0,\ \ i=1,\dots,24 \\ P_x(a,b)-P_x(b,a)>0 \\ P_x(a,c)-P_x(c,a)>0 \\ P_x(a,d)-P_x(d,a)>0 \\ F_x(a)-F_x(b)>0 \\ F_x(b)-F_x(c)>0 \\ F_x(b)-F_x(d)>0 \\ P_x(b,a)-F_x(a)>0 \\ P_x(b,a)+F_x^{ab}(c)-2F_x(a)>0 \\ P_x(b,c)-P_x(c,b)>0 \end{cases} \quad (S_n^{cb})\begin{cases} n_1+\dots+n_{24}=n \\ n_i \geq 0,\ \ i=1,\dots,24 \\ P_x(a,b)-P_x(b,a)>0 \\ P_x(a,c)-P_x(c,a)>0 \\ P_x(a,d)-P_x(d,a)>0 \\ F_x(a)-F_x(b)>0 \\ F_x(b)-F_x(c)>0 \\ F_x(b)-F_x(d)>0 \\ P_x(c,a)-F_x(a)>0 \\ P_x(c,a)+F_x^{ac}(b)-2F_x(a)>0 \\ P_x(c,b)-P_x(b,\mathrm{c})>0 \end{cases} \quad (S_n^{cd})\begin{cases} n_1+\dots+n_{24}=n \\ n_i \geq 0,\ \ i=1,\dots,24 \\ P_x(a,b)-P_x(b,a)>0 \\ P_x(a,c)-P_x(c,a)>0 \\ P_x(a,d)-P_x(d,a)>0 \\ F_x(a)-F_x(b)>0 \\ F_x(b)-F_x(c)>0 \\ F_x(b)-F_x(d)>0 \\ P_x(c,a)-F_x(a)>0 \\ P_x(c,a)+F_x^{ac}(d)-2F_x(a)>0 \\ P_x(c,d)-P_x(d,\mathrm{c})>0 \end{cases}$$

By symmetry between candidates c and d, the systems characterizing G^{bd}, G^{db} and G^{dc} are obtained by permuting c and d in S_n^{bc}, S_n^{cb} and S_n^{cd} respectively; so we have $|G^{bd}| = |G^{bc}|$, $|G^{db}| = |G^{cb}|$ and $|G^{dc}| = |G^{cd}|$.

Now, we use (A2) and we apply the inclusion-exclusion principle to calculate $|(G,n)|$. For the 15 pairwise intersections, it is obvious that $G^{bc} \cap G^{cb}$, $G^{bd} \cap G^{db}$ and $G^{cd} \cap G^{dc}$ are empty, and that by symmetry we have $|G^{cb} \cap G^{bd}| = |G^{bc} \cap G^{db}|$, $|G^{bd} \cap G^{dc}| = |G^{bc} \cap G^{cd}|$, $|G^{bd} \cap G^{cd}| = |G^{bc} \cap G^{dc}|$, $|G^{db} \cap G^{dc}| = |G^{cb} \cap G^{cd}|$, and $|G^{db} \cap G^{cd}| = |G^{cb} \cap G^{dc}|$. Of the 20 triple intersections, the only ones that are (possibly) non-empty are the 8 that are obtained by choosing one and only one element in each of the three sets $\{G^{bc},G^{cb}\}$, $\{G^{bd},G^{db}\}$ and $\{G^{cd},G^{dc}\}$; and by symmetry we have $|G^{bc} \cap G^{bd} \cap G^{cd}| = |G^{bc} \cap G^{bd} \cap G^{dc}|$, $|G^{bc} \cap G^{db} \cap G^{cd}| = |G^{cb} \cap G^{bd} \cap G^{dc}|$, $|G^{cb} \cap G^{bd} \cap G^{cd}| = |G^{bc} \cap G^{db} \cap G^{dc}|$ and $|G^{cb} \cap G^{db} \cap G^{cd}| = |G^{cb} \cap G^{db} \cap G^{dc}|$. Finally, all intersections of 4, 5 or 6 subsets $G^{\alpha\beta}$ (α, β in $\{b,c,d\}$ and $\alpha \neq \beta$) are empty, because each of them is included in (at least) one of the three empty intersections, $G^{bc} \cap G^{cb}$, $G^{bd} \cap G^{db}$ and $G^{cd} \cap G^{dc}$ (to form an intersection of 4, 5 or 6 subsets

$G^{\alpha\beta}$, it is necessary to choose the two elements of at least one of the sets $\{G^{bc}, G^{cb}\}$, $\{G^{bd}, G^{db}\}$ and $\{G^{cd}, G^{dc}\}$).
We can now write the formula giving the cardinality of (G, n):

$$
\begin{aligned}
|(G,n)| &= 2(|G^{bc}| + |G^{cb}| + |G^{cd}|) \\
&-(|G^{bc} \cap G^{bd}| + |G^{cb} \cap G^{db}|) + 2|G^{bc} \cap G^{db}| + 2|G^{bc} \cap G^{cd}| + 2|G^{bc} \cap G^{dc}| \\
&+2|G^{cb} \cap G^{cd}| + 2|G^{cb} \cap G^{dc}|) \\
&+2(|G^{bc} \cap G^{bd} \cap G^{cd}| + |G^{bc} \cap G^{db} \cap G^{dc}| + |G^{cb} \cap G^{db} \cap G^{dc}| + |G^{cb} \cap G^{bd} \cap G^{dc}|)
\end{aligned}
\tag{A4}
$$

To obtain $\Pr(G, \infty)$, we replace each cardinality that appears in the second member of (A4) by the volume of the associated polytope (for example, the polytope associated with G^{bc} is the one described by the system S_1^{bc}), and then we divide by the volume associated with the total number of voting situations (*i.e.* by $1/23!$). Using the method based on LattE and Lrs (and Normaliz for the triple intersections), we get the following results:

	volume of the associated polytope
G^{bc}	215799651022148336618223418954725782642961476613018207489968576368915345171956001 $\vert$ 16966274854513042168017066937694599890902256969427606313893971886080000000000000000
G^{cb}	557328725615816454482227910067415486390657165825493267957282636323213591 7 $\vert$ 538163337264415891621828147363607120928617085478067328201523200000000000000000
G^{cd}	690763588675926208892200201275635242424317217116771810711456400097168422586192409 $\vert$ 108584159068883469875309228401245439301774444604336680408921420070912000000000000000
$G^{bc} \cap G^{bd}$	437495870930736649855127528255082994516834134015800580312416390677735588662159241840 7 $\vert$ 15783821593121799418221188750549477369596125747406423643972960256000000000000000000000000
$G^{cb} \cap G^{db}$	3716683076047559220040529128094170263114916523334203603814335905161079123205584086818332149078766150973 08537557287 $\vert$ 14986578623275516071460371349256714458889547296595518795179705199098832012821943562641233597584900096000000000000000000
$G^{bc} \cap G^{db}$	7372879509769191060803267580297964950417758211260999955915536110690105995008349978266426944382880049365434496288618834769 3 $\vert$ 31976950682702112545122688614706446500385377035566310493107178621284727313905373632621843167367746301827481600000000000000000000
$G^{bc} \cap G^{cd}$	1128678697638588406020837480381418976300873421867855410386970684729753714450967582951899081133195953970779591091254147647624909 $\vert$ 573026956234021856808598579975539521286905956477348284036480640893422313465184295496583429559230013728748470272000000000000000000000
$G^{bc} \cap G^{dc}$	179120884374630246903781735660307774495056889927342505569670205616577977805908161982115664774078815947037152735182335 7 $\vert$ 71965550548969028175152703219130742831587606118251681254452944366072591325570972987803203735602690260992000000000000000000
$G^{cb} \cap G^{cd}$	156090914075338229952814649806280933859900502452413621807273601730707932075993177383819 $\vert$ 104173222514603876160259845753626550639334429932882396050221537689600000000000000000000000000
$G^{cb} \cap G^{dc}$	169871965226875995373638321448221960851557358072284253875160535563392943747406634199804234059979635043769392424873208758859 87 $\vert$ 1169442767824534401650201183623550043442665217300710783747919675292698598908539378564455978692306150466833612800000000000000000000
$G^{bc} \cap G^{bd} \cap G^{cd}$	674324033423332239489570724345486219235182962634052751321547924919297782590615439333218137279830648518311866993975500318502409742149903 $\vert$ 800337852832421947317129795594046017972207282304842286498696861526387919674178698272333760160140951092350184062976000000000000000000000000
$G^{bc} \cap G^{db} \cap G^{dc}$	6622408494773299194809505762540975520979526374814683196354563322261796216124815329855721189005229633888195819632671107118857573386123 7 $\vert$ 10260741702979768555347817892248776946118214464493387546793201104521010153428434228554274051487360193708141261619200000000000000000000000000

<table>
<tr><td>$G^{cb} \cap G^{db} \cap G^{dc}$</td><td>172499040886591309105147664280974842140465399609634373368829431585429271796091241844107326681316968683271609632632217239661825101449 | 3374674704134010572259781563482056846842725283481549285924564370689149907098236927927278529330469282801631887360000000000000000000000000</td></tr>
<tr><td>$G^{cb} \cap G^{bd} \cap G^{dc}$</td><td>6433925339577046922056349489616669597071348937361564870673006196214723723 | 14770885442042769376198830238432027684092123485420931514368000000000000000000</td></tr>
</table>

After calculation, we obtain

$$\Pr(G,\infty) = \frac{52683297709949532142119507583496364663740732115091118336072352908066132417}{134874482602274422408582167690411633024678174847653642240000000000000000000000}$$

To compute $\Pr(H,\infty)$, we use (A3) and we proceed in exactly the same way as for $\Pr(G,\infty)$. Using Proposition 2 and deleting the redundant inequalities, we obtain the systems characterizing the voting situations x in H^{bc}, H^{cb} and H^{cd}:

$$(T_n^{bc})\begin{cases} n_1 + \cdots + n_{24} = n \\ n_i \geq 0,\ \ i = 1, \dots, 24 \\ P_x(a,b) - P_x(b,a) > 0 \\ P_x(a,c) - P_x(c,a) > 0 \\ P_x(a,d) - P_x(d,a) > 0 \\ F_x(b) - F_x(a) > 0 \\ F_x(a) - F_x(c) > 0 \\ F_x(a) - F_x(d) > 0 \\ P_x(b,a) - F_x(a) > 0 \\ P_x(b,a) + F_x^{ab}(c) - 2F_x(a) > 0 \\ P_x(b,c) - P_x(c,b) > 0 \end{cases} \quad (T_n^{cb})\begin{cases} n_1 + \cdots + n_{24} = n \\ n_i \geq 0,\ \ i = 1, \dots, 24 \\ P_x(a,b) - P_x(b,a) > 0 \\ P_x(a,c) - P_x(c,a) > 0 \\ P_x(a,d) - P_x(d,a) > 0 \\ F_x(b) - F_x(a) > 0 \\ F_x(a) - F_x(c) > 0 \\ F_x(a) - F_x(d) > 0 \\ P_x(c,a) - F_x(a) > 0 \\ P_x(c,a) + F_x^{ac}(b) - 2F_x(a) > 0 \\ P_x(c,b) - P_x(b,c) > 0 \end{cases} \quad (T_n^{cd})\begin{cases} n_1 + \cdots + n_{24} = n \\ n_i \geq 0,\ \ i = 1, \dots, 24 \\ P_x(a,b) - P_x(b,a) > 0 \\ P_x(a,c) - P_x(c,a) > 0 \\ P_x(a,d) - P_x(d,a) > 0 \\ F_x(b) - F_x(a) > 0 \\ F_x(a) - F_x(c) > 0 \\ F_x(a) - F_x(d) > 0 \\ P_x(c,a) - F_x(a) > 0 \\ P_x(c,a) - F_x(b) > 0 \\ P_x(c,a) + F_x^{ac}(d) - 2F_x(a) > 0 \\ P_x(c,a) + F_x^{ac}(d) - 2F_x(b) > 0 \\ P_x(c,d) - P_x(d,c) > 0 \end{cases}$$

Here again, c and d being symmetrical, we obtain the same symmetries as before. So the formula describing the cardinality of (H,n) is exactly the same as (A4), except that $|(G,n)|$ is replaced with $|(H,n)|$ and each $G^{\alpha\beta}$ (α, β in $\{b,c,d\}$ and $\alpha \neq \beta$) is replaced with $H^{\alpha\beta}$. By applying the method based on LattE and Lrs (and Normaliz for the triple intersections), we calculate the volumes of all the polytopes associated with the cardinalities involved in the second member of this formula. We then get:

<table>
<tr><th></th><th>volume of the associated polytope</th></tr>
<tr><td>H^{bc}</td><td>1122570228285484416840414329038859444597 | 2203251766912513538732851200000000000000000</td></tr>
<tr><td>H^{cb}</td><td>4815102613831086008681845331774767357 | 275406470864064192341606400000000000000000</td></tr>
<tr><td>H^{cd}</td><td>4859814977353382934768278656966765458531281829079037659495505447570096549131 | 2636152533698047281461986677988445611214432454458846070397992960000000000000000</td></tr>
<tr><td>$H^{bc} \cap H^{bd}$</td><td>6833482604299574922319228350257359109218343485126709565221 | 2287938039920377319816594876291319070720000000000000000000</td></tr>
<tr><td>$H^{cb} \cap H^{db}$</td><td>5049881685008345434475345496190114508046016393 | 104085197425154650591762896125952000000000000000</td></tr>
</table>

$H^{bc} \cap H^{db}$	675827524940449071804022414564519275245666212472817163 │ 83197746906195538981151254095604796620800000000000000000
$H^{bc} \cap H^{cd}$	54527549003500598877896693999359561653981068895014160097048506522520901040183861295097633826231569261 │ 46107870190868020671174591850139263031166900356375104588254062448121512540343640260608000000000000000000
$H^{bc} \cap H^{dc}$	8308091188337554623944084932295328721407928052921637390327492939829616693208962854674889 │ 6253787586569434428002409935673694836855808351567447350530718940594490572800000000000000000
$H^{cb} \cap H^{cd}$	536467444833359046580624594631977457827077563033371231007512557540128954421668859753948816150778793 │ 11294304867447584918473102060096821240242724955020356797044400952410717357520977920000000000000000000000
$H^{cb} \cap H^{dc}$	8829802817815659461561114981596342439514389717111976228094675237975903263366333445667 │ 24510239414342286607887164160978619779956136984391328044408069530058752000000000000000000
$H^{bc} \cap H^{bd} \cap H^{cd}$	6526360468678669861037589052034470126857779241331314839826230041617566152568440550895761885023785390756518058117863413 │ 7216599736503166408006324991787941691613794336591761487594800388300067534635371463393316550373015552000000000000000000000000
$H^{bc} \cap H^{db} \cap H^{dc}$	987760979695491729713131336886222849186173537468343019201090980213224186369431218746360706439232516885140014895617 │ 32997712558313518097879858215765622732573362307232562814791039727023628416256842539521337678888960000000000000000000000000
$H^{cb} \cap H^{db} \cap H^{dc}$	5404398077528054472521504460835385332865736755704757514533585222718698855027095804933932537271634529787736605385657 │ 381830673889056423704038359353859348762634620983691083999724888269844843102400606528746907427143680000000000000000000000000
$H^{cb} \cap H^{bd} \cap H^{dc}$	141767031443658485490372638618892830741501772123401878931601845278076851988166862281062607360239016458736050692837 │ 76366134777811284740807671870771869752526924196738216799944977653968968620480121305749381485428736000000000000000000000000

After calculation, we obtain

$$\Pr(H,\infty)=\frac{86196191235167272312652407600525591350591906553065213523273899326331417319}{99689834966898486128082471771173815713892564017830952960000000000000000000000}$$

Finally, going to the limit in (A1), we have:

$$\Pr(E_3,\infty) = 3(|(G,\infty)| + |(H,\infty)|) = \frac{1087728064806496337719968633307455328929405251956556660146836615246691931}{28884683852842846824715253851562078123198903658479616000000000000000000000}$$

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Chapter 3 : On the Condorcet Efficiency of Evaluative Voting (and other Voting Rules) with Trichotomous Preferences

Abstract. We investigate the propensity of Evaluative Voting $(2, 1, 0)$ to fulfill Condorcet majority conditions in a framework where preferences are supposed to be trichotomous and only three candidates are in contention. In this framework, we also compare Evaluative Voting to other voting rules, including Borda Rule, Plurality Rule and Approval Voting.

Keywords: Voting by evaluation - Three-valued scale - Approval voting - Scoring rules - Probability.

1. Introduction and motivation

The *three-valued scale evaluative voting* is a new voting rule recently considered by voting theorists[18]. It proceeds as follows: each voter evaluates each candidate and gives her a score belonging to $\{0, 1, 2\}$. In other words, each voter is given the possibility to form three groups of candidates: those she appreciates and all these candidates receive 2 points, those she does not appreciate who receives 0 point and an intermediate group of candidates who obtain 1 point[19]. The voters' preferences are said to be *trichotomous*. Of course, one of these groups can be empty (we ignore in what fallows the case where two groups are empty, i.e. the case where all the candidates belong to the same group). The winner is the candidate obtaining the highest number of points. Notice that what we call here Evaluative Voting $(2,1,0)$ can be considered as a particular case of *Range Voting* (where the set of ratings that a voter can give is not limited to $\{0, 1, 2\}$) or an extension of *Approval Voting* (where the set of marks is $\{0,1\}$, which implies that preferences are dichotomous). According to Hillinger (2005), the motivation for advocating for a three-valued scale is twofold: (i) the two-valued range of *Approval Voting* is not discriminating enough (in addition to feeling positive or negative about candidates, one may

[18] This voting rule has been introduced by Felsenthal (1989) and Hillinger (2004, 2005).

[19] Condorcet (1793) was the first to propose a voting rule in which voters were required to partition the candidates into three groups according to their preferences.

also feel neutral); (ii) the electorate is generally poorly informed on candidates and issues, and a finer division of the voting scale appears to be both unnecessary and possibly confusing to the voters. Three-values scale evaluative voting will be simply denoted by EV in the remainder of this paper[20].

A number of studies have been recently conducted to analyze EV, both from an empirical point of view (Baujard et al. 2013, 2014; Baujard and Igerscheim 2009; Igersheim et al. 2015; Lebon et al. 2015) and a theoretical perspective (Smaoui and Lepelley 2013; Alcantud and Laruelle 2014; see also Felsenthal 2012, Pivato 2014 and Macé 2015 who adopt the more general perspective of Range Voting). Most of these studies have demonstrated that EV has many good properties. However, Felsenthal (2012) and Smaoui and Lepelley (2013) emphasize some possible difficulties with EV: this voting rule does not always choose the Condorcet Winner (CW) when such a candidate exists (a Condorcet Winner beats each of the other candidates in pairwise majority comparisons) and is susceptible to select the Condorcet Loser (a candidate who loses each of her majority comparisons, denoted by CL). In other words, EV violates both the CW condition (a CW should be selected when such a candidate exists) and the CL condition (a CL should not be elected when such a candidate exists). According to Smaoui and Lepelley (2013), these violations constitute the main flaw of EV and Felsenthal (2012) considers the possible CL election as "intolerable". It is thus of interest to try to evaluate the likelihood of these violations. Also, we would like to know how EV performs when compared to usual voting rules, such as Plurality Rule or Approval Voting. The current paper, where we limit our investigation to three candidate elections, is a first analytical step in this direction[21].

We consider an appropriate framework where preferences are trichotomous and in which the EV winner can be easily identified and compared to the CW (or the CL), when such a candidate exists. With three candidates, the number of possible trichotomous preferences is $3^3 = 27$ (number of ways to put three objects in three cells); as we ignore the case where all the candidates are put in a same group, $27 - 3 = 24$ possible preferences are left and we can enumerate and number these preferences as indicated in Figure 1.

[20] The score vector $(2, 1, 0)$ can be replaced with any positive linear transformation without changing the election winner. Some authors consider the score vector $(1, 0, -1)$; Alcantud and Laruelle (2014) refer to this rule as *Dis&approval Voting*.

[21] Smaoui and Lepelley (2013) have run simulations to obtain some information on this issue. We will go back to their findings in the discussion of our results.

A	A	B	B	C	C	AB	AB	AC	AC	BC	BC
B	C	A	C	A	B	C	$-$	B	$-$	A	$-$
C	B	C	A	B	A	$-$	C	$-$	B	$-$	A
R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}

A	A	B	B	C	C	$-$	$-$	$-$	$-$	$-$	$-$
BC	$-$	AC	$-$	AB	$-$	A	BC	B	AC	C	AB
$-$	BC	$-$	AC	$-$	AB	BC	A	AC	B	AB	C
R_{13}	R_{14}	R_{15}	R_{16}	R_{17}	R_{18}	R_{19}	R_{20}	R_{21}	R_{22}	R_{23}	R_{24}

Figure 1. The possible trichotomous preferences on three candidates

We suppose that voter's preferences are anonymous and we denote by n_i $(1 \leq i \leq 24)$ the number of voters with preference R_i on the three candidates. Thus, n_7 denotes the number of voters who put A and B in the first group (with EV, each of them gives two points to A and to B), C in the intermediate group (with EV, C receives 1 point from each of these n_7 voters) and no candidate in the third group. A *(trichotomous) voting situation* reports the value of each n_i and can be represented by a 24-tuple $x = (n_1, n_2, \cdots, n_{24})$ such that $n_i \geq 0$ $(1 \leq i \leq 24)$ and $\sum_{i=1}^{24} n_i = n$, where n is the total number of voters. We denote by $V(n)$ the set of all voting situations with n voters.

To illustrate the violation of Condorcet conditions by the EV rule, consider the following simple example with 5 voters and two voting situations : x (3 voters with preference R_1 and 2 voters with preference R_{15}) and y (2 voters with preference R_{15} and 3 voters with preference R_{22}) :

3	2		2	3
A	B		B	$-$
B	$-$		$-$	AC
C	AC		AC	B
x			y	

In x, candidate A is the CW[22]. When EV is applied, the scores of A, B and C are respectively 6, 7 and 0 points. It is therefore B and not A who wins the election. In y, candidate B is the CL. It

[22] Observe that A is here an "absolute" Condorcet Winner, *i.e.* a candidate ranked in first position by more than one half of the voters. We shall come back to that notion in Section 4.2.

is, however, B who wins the election, the scores of A, B and C with EV being respectively 3, 4 and 3.

To calculate the likelihood of Condorcet condition violations, we will assume that all the possible (trichotomous) voting situations in $V(n)$ are equally likely to occur: it is an IAC-like assumption, where IAC stands for *Impartial Anonymous Culture*, a model very often used in this kind of investigation. We assume very large electorate (i.e. $n \to \infty$) and we define the CW Efficiency of a voting rule as the probability of electing the CW, given that such a candidate exists. For a voting rule F, $CWE(F)$ will denote this CW Efficiency. Similarly, $CLE(F)$ denotes the CL Efficiency of F, i.e. the probability of electing a candidate different from the CL, given that a CL exists.

Our study is organized as follows. We start by extending the four rules we would like to compare with EV to our trichotomous preference framework (Section 2). Then we present in Section 3 and Section 4 the probabilistic results we have obtained for the CW Efficiency and the CL Efficiency of the voting rules under study, by considering the case of large electorates. Our results are summarized and discussed in Section 5.

2. Extending Scoring Rules and Approval Voting to the trichotomous framework

In addition to EV, four voting rules will be studied (and compared to EV) in this paper: Plurality Rule (PR), Negative Plurality Rule (NPR), Borda Rule (BR) and Approval Voting (AV). The first three rules belong to the class of (simple) scoring rules that are usually defined in a context where each voter expresses his (her) preferences by a strict order on the set of candidates. A scoring rule gives points to candidates according to their rank in voters' preference orders. Each candidate obtains a score equal to the total of points she received and the candidate with the highest score is selected. For elections with three candidates, a scoring rule can be represented by a scoring vector $(\lambda_1, \lambda_2, \lambda_3)$, with $\lambda_3 \leq \lambda_2 \leq \lambda_1$ and $\lambda_3 < \lambda_1$, indicating that every candidate receives λ_i points each time she is ranked in position i in an individual preference (strict) ranking. PR, NPR and BR use respectively the scoring vectors $(1, 0, 0)$, $(1, 1, 0)$ and $(2, 1, 0)$. Approval Voting (AV) is defined in a different context where all individual preferences are dichotomous. Each individual preference consists of two groups, that of the approved candidates and that of the disapproved candidates. Each candidate receives 1 point each time she is approved and 0 point whenever she is disapproved; the winner is the candidate with the highest total number of points.

In our framework, the score of each candidate under EV for a voting situation x in $V(n)$ is easily calculated as follows:

$S_{EV}(A,x) = 2(n_1 + n_2 + n_7 + n_8 + n_9 + n_{10} + n_{13} + n_{14})+ (n_3 + n_5 + n_{11} + n_{15} + n_{17} + n_{19} + n_{22} + n_{24})$

$S_{EV}(B,x) = 2(n_3 + n_4 + n_7 + n_8 + n_{11} + n_{12} + n_{15} + n_{16})+ (n_1 + n_6 + n_9 + n_{13} + n_{17} + n_{20} + n_{21} + n_{24})$

$S_{EV}(C,x) = 2(n_5 + n_6 + n_9 + n_{10} + n_{11} + n_{12} + n_{17} + n_{18})+ (n_2 + n_4 + n_7 + n_{13} + n_{15} + n_{20} + n_{22} + n_{23})$

As our aim in this note is not only to compute the Condorcet Efficiency of EV but also to compare its performance to the one of other voting rules ($PR,\ NPR,\ BR$ and AV), and since these rules are generally introduced in a context where voters' preferences are (strict) linear orders (with or without ties), we have to consider how these rules can be implemented in the trichotomous framework.

In order to adapt $PR,\ NPR$ and BR to our framework, we make use of a general method that could be applied to any positional voting rule using a scoring vector $(\lambda_1,\ \lambda_2,\ \lambda_3)$. This method was first proposed by Black (1976) and has been recently used by Diss et al. (2010) and Gehrlein and Lepelley (2015) to extend positional voting rules to dichotomous preferences. We start by reducing each trichotomous individual preference to a weak order on the three candidates, ignoring the empty group when the trichotomous preference does not correspond to a strict order. This way of doing is justified by the observation that the scoring (positional) rules are based on the notion of an ordinal ranking and not on an evaluation (or rating) principle: the only relevant information for applying these voting rules is the ranking of each candidate in the voters' preferences; the strength of preferences is only a consequence of this ordinal ranking. After this reduction, we compute the number of points that each candidate receives from each voter in the following way. If the order resulting from the reduction process of the trichotomous preference is a strict order, then the candidates receive, as explained above, α_i points for a ith position. If this order is a weak order like $X \succ Y \sim Z$ (X is preferred to Y and Z, Y and Z are ex aequo), as for example in R_{13}, then X receives λ_1 points and Y and Z receive an average $(\lambda_2 + \lambda_3)/2$ points each. If this order is a weak order like $X{\sim}Y \succ Z$ (X and Y are ex aequo and both are preferred to Z), as for example in R_7 , then X and Y receive an average $(\lambda_1 + \lambda_2)/2$ points each and Z receives λ_3 points. Therefore, our extended scoring rules treat all voters equally, since they consistently allocate the same $\lambda_1 + \lambda_2 + \lambda_3$ points to the candidates for each voter, and this extension allows to take into consideration the various types of preferences we consider here.

With these assumptions, the scores of A, B and C under PR are:

$$S_{PR}(A,x) = (n_1 + n_2 + n_{13} + n_{14} + n_{19}) + \frac{1}{2}(n_7 + n_8 + n_9 + n_{10} + n_{22} + n_{24})$$
$$S_{PR}(B,x) = (n_3 + n_4 + n_{15} + n_{16} + n_{21}) + \frac{1}{2}(n_7 + n_8 + n_{11} + n_{12} + n_{20} + n_{24})$$
$$S_{PR}(C,x) = (n_5 + n_6 + n_{17} + n_{18} + n_{23}) + \frac{1}{2}(n_9 + n_{10} + n_{11} + n_{12} + n_{20} + n_{22})$$

Under the Negative Plurality Rule (NPR), we easily obtain:

$$S_{NPR}(A,x) = n - [(n_4 + n_6 + n_{11} + n_{12} + n_{20}) + \frac{1}{2}(n_{15} + n_{16} + n_{17} + n_{18} + n_{21} + n_{23})]$$
$$S_{NPR}(B,x) = n - [(n_2 + n_5 + n_9 + n_{10} + n_{22}) + \frac{1}{2}(n_{13} + n_{14} + n_{17} + n_{18} + n_{19} + n_{23})]$$
$$S_{NPR}(C,x) = n - [(n_1 + n_3 + n_7 + n_8 + n_{24}) + \frac{1}{2}(n_{13} + n_{14} + n_{15} + n_{16} + n_{19} + n_{21})]$$

The scores under the Borda Rule (BR) are given as:

$$S_{BR}(A,x) = 2(n_1 + n_2 + n_{13} + n_{14} + n_{19}) + \frac{3}{2}(n_7 + n_8 + n_9 + n_{10} + n_{22} + n_{24})$$
$$+(n_3 + n_5) + \frac{1}{2}(n_{15} + n_{16} + n_{17} + n_{18} + n_{21} + n_{23})$$
$$S_{BR}(B,x) = 2(n_3 + n_4 + n_{15} + n_{16} + n_{21}) + \frac{3}{2}(n_7 + n_8 + n_{11} + n_{12} + n_{20} + n_{24})$$
$$+(n_1 + n_6) + \frac{1}{2}(n_{13} + n_{14} + n_{17} + n_{18} + n_{19} + n_{23})$$
$$S_{BR}(C,x) = 2(n_5 + n_6 + n_{17} + n_{18} + n_{23}) + \frac{3}{2}(n_9 + n_{10} + n_{11} + n_{12} + n_{20} + n_{22})$$
$$+(n_2 + n_4) + \frac{1}{2}(n_{13} + n_{14} + n_{15} + n_{16} + n_{19} + n_{21})$$

AV is easy to implement when the trichotomous preference is R_7 to R_{24}: in these cases, we ignore the empty group and the preferences are actually dichotomous. When the trichotomous preference corresponds to a strict order (R_1 to R_6), we assume that the voter will approve the candidate ranked in first position (with certainty) and that she will approve the candidate ranked in second position with probability $1/2$ (We shall come back to that assumption in the last Section). Thus, we obtain for Approval Voting (AV) the following scores:

$$S_{AV}(A,x) = (n_1 + n_2 + n_7 + n_8 + n_9 + n_{10} + n_{13} + n_{14} + n_{19} + n_{22} + n_{24}) + \frac{1}{2}(n_3 + n_5)$$
$$S_{AV}(B,x) = (n_3 + n_4 + n_7 + n_8 + n_{11} + n_{12} + n_{15} + n_{16} + n_{20} + n_{21} + n_{24}) + \frac{1}{2}(n_1 + n_6)$$
$$S_{AV}(C,x) = (n_5 + n_6 + n_9 + n_{10} + n_{11} + n_{12} + n_{17} + n_{18} + n_{20} + n_{22} + n_{23}) + \frac{1}{2}(n_2 + n_4)$$

3. Results on the Condorcet winner efficiency with trichotomous preferences

Let $CWE(F,n)$ be the CW Efficiency of the voting rule F, when all considered voting situations are in $V(n)$. According to the definition given in Section 2, we have: $CWE(F) = \lim_{n\to\infty} CWE(F,n)$. To compute this limiting probability for the different rules under consideration in this paper, we will use a method based on Ehrhart theory and on algorithms for counting integer points in rational polytopes. This method, which is partly original, will be explained and illustrated in the following subsection; it will then be applied throughout the rest of the paper. We first recall in the following paragraph some definitions related to the notions of polytope and quasi-polynomial and we give a brief overview of Ehrhart theory. These mathematical tools are now well known and widely used in probability calculations in voting theory[23].

A rational polytope P in $\mathbb{R}^d$ is a bounded subset of $\mathbb{R}^d$ defined by a system of integer linear (in)equalities. Note that P could be not full-dimensional (this is the case when the linear system describing P contains some equalities) and could be semi-open (this is the case when some of the inequalities describing P are strict). A parametric polytope (with a single parameter n) of dimension d is a sequence of (possibly empty) d-dimensional rational polytopes P_n ($n \in \mathbb{N}$) of the form $P_n = \{x \in \mathbb{R}^d: Mx \leq bn + c\}$, where M is an $m \times d$ integer matrix and b and c are two integer vectors with m components. An important instance of parametric polytopes is obtained when the constant term c is equal to the zero vector. In this case, P_n is denoted nP and corresponds to the dilatation, by the positive integer factor n, of the rational polytope P defined by $P = \{x \in \mathbb{R}^d: Mx \leq b\}$. Calculating the number of integer solutions of a parametric linear system ($Mx \leq bn + c$) amounts to calculating the number of integer coordinate points belonging to the parametric polytope P_n defined by this system. By Ehrhart theory, we know that this number is a quasi-polynomial in n, of degree d, i.e. a polynomial expression $f(n)$ of the form $f(n) = \sum_{k=0}^{d} c_k(n)n^k$, where the coefficients $c_k(n)$ are rational periodic numbers in n. A rational periodic number of period q on the integer variable n is a function $u: \mathbb{Z} \to \mathbb{Q}$ such that $u(n) = u(n')$ whenever $n \equiv n'$ (mod q). Each coefficient $c_k(n)$ can have its own period, but we can always write $f(n)$ in a form where the coefficients have a common period called the period of the quasi-polynomial $f(n)$ and defined as the least common multiple (*lcm*) of the periods of all coefficients.

[23] For a general background on Ehrhart theory and on the general problem of counting integer points in rational polytopes, see for example Beck and Robins (2015). We refer also to Lepelley et al. (2008) and Wilson and Pritchard (2007) for more details on the use of these tools in probability calculations under IAC hypothesis in voting theory.

3.1. Existence of a Condorcet Winner

For a voting situation x in $V(n)$ and two different candidates w, w', we denote by $P_x(w, w')$ the number of voters that prefer w to w' (i.e. the number of voters who put w in a group higher than the group in which they put w'). The numbers involved in the binary comparisons between A and B and between A and C are:

$$\begin{gathered}
P_x(A,B) = n_1 + n_2 + n_5 + n_9 + n_{10} + n_{13} + n_{14} + n_{19} + n_{22} \\
P_x(B,A) = n_3 + n_4 + n_6 + n_{11} + n_{12} + n_{15} + n_{16} + n_{20} + n_{21} \\
P_x(A,C) = n_1 + n_2 + n_5 + n_9 + n_{10} + n_{13} + n_{14} + n_{19} + n_{22} \\
P_x(C,A) = n_3 + n_4 + n_6 + n_{11} + n_{12} + n_{15} + n_{16} + n_{20} + n_{21}
\end{gathered}$$

With this notation, A is the CW if and only if $P_x(A,B) > P_x(B,A)$ and $P_x(A,C) > P_x(C,A)$. We denote by $CW(A,n)$ the event "A is the CW", when the number of voters is equal to n. Let $\Pr(CW(A,n))$ and $\Pr(CW(A))$ be the probability and the limiting probability (when $\to \infty$) of this event. We have:

$$\Pr\big(CW(A,n)\big) = \frac{|CW(A,n)|}{|V(n)|}$$

In this identity, $|V(n)|$ and $|CW(A,n)|$ denote the cardinalities of sets $V(n)$ and $CW(A,n)$ respectively. The number $|V(n)|$ is well known and is given by $|V(n)| = \binom{n+23}{23}$. Indeed, with n voters and 24 possible (trichotomous) individual preferences, the number of voting situations is equal to the number of ways n objects can be chosen from a set of 24 objects, where repetition is allowed. Note that $|V(n)|$ is a polynomial of degree 23 and that the coefficient of the leading term of this polynomial is equal to $1/23!$. The computation of $|CW(A,n)|$ as a function of n is more involved. By definition, $CW(A,n)$ is the set of all integer solutions, $x = (n_i)_{i=1}^{24}$, of the following system:

$$S_n: \begin{cases} n_i \geq 0, \;\; i = 1, \dots, 24 \\ n_1 + n_2 + \cdots + n_{24} = n \\ P_x(A,B) - P_x(B,A) > 0 \\ P_x(A,C) - P_x(C,A) > 0 \end{cases}$$

All (in)equalities in this system are linear and have integer coefficients on the variables n_i and on the integer parameter n. This defines, for each value of n, a parametric (semi-open) rational polytope (of dimension 23), P_n, which is the dilatation, by the factor n, of the (semi-open) rational polytope P_1. To obtain $|CW(A,n)|$, we have to compute the quasi-polynomial describing the number of integer points belonging to P_n. In voting theory, to perform this type of calculation, we usually resort to a computer program based on (parameterized) Barvinok algorithm (see [barvinok]). This program performs very well when the calculations concern voting events with three candidates and individual preferences are expressed by strict orders. In this case there are only 6 variables and the quasi-polynomials are generally of degree 5. With 24 variables and a degree 23, as in the case we are studying, the use of Barvinok algorithm does not make it possible to obtain the desired results. Other software such as LattE with its new version Latte integrale (see [latte]) and Normaliz ([normaliz]) allow, in some cases, to calculate quasi-polynomials corresponding to polytopes of dimension 23. However, as our goal is to calculate the limit value of $\Pr\big(CW(A,n)\big)$, we do not need to know the exact expression of $|CW(A,n)|$. Indeed, $\Pr\big(CW(A,n)\big)$ is the quotient of a quasi-polynomial ($|CW(A,n)|$) by a polynomial ($|V(n)|$). Since these two polynomial expressions are of the same degree (23) and since we already know the expression of $|V(n)|$, we have only to calculate the coefficient of the leading term of $|CW(A,n)|$. We know, again by Ehrhart theory, that this coefficient is independent of n and is equal to the volume[24] of P_1. So we have:

$$\Pr\big(CW(A)\big) = \lim_{n\to+\infty} \frac{|CW(A,n)|}{|V(n)|} = \frac{\mathrm{Vol}(P_1)}{1/23!} = 23!\,\mathrm{Vol}(P_1) \qquad (1)$$

Now, it only remains to calculate $\mathrm{Vol}(P_1)$. In general, algorithms that compute the volume of polytopes are not very efficient when, as in all the cases studied in this paper, the number of variables is equal to 24. However, recent improvements in algorithms such as LattE and Normaliz have made it possible to obtain some results describing the probability of voting events with four candidates, requiring the calculation of the volumes of certain polytopes of dimension 24 (see Schürmann, 2013 ; Bruns and Söger, 2015 ; Brandt et al., 2016). To compute $\mathrm{Vol}(P_1)$ and all the other volumes involved in the calculations developed in the remainder of this paper, we will not use any algorithm of direct volume computation. Instead, we will apply a new method based on the Ehrhart theory and on the combined use of two software, LattE integrale and lrs ([lrs]).

The command (count --ehrhart-polynomial) in the first program makes it possible to calculate in a reasonable time (from a few seconds to a few hours) the quasi-polynomial describing the number of integer points belonging to the dilatation nP of a polytope P by an integer factor n.

[24] Here, by the volume of P_1 we mean its relative volume, i.e. the volume of P_1 relative to its affine span.

With LattE integrale, this computation is possible only when P is an integral polytope (i.e. when all its vertices have integer coordinates). In this case, the quasi-polynomial associated with nP has period equal to 1 and hence is simply a polynomial. The second program, lrs, allows to obtain (usually within seconds) the coordinates of all vertices of a rational polytope. To calculate $\mathrm{Vol}(P_1)$, the idea is then the following. We start by dilating P_1 by a positive integer factor k such that the obtained polytope kP_1 is integral; for this, k must be a multiple of the period of P_1. Now, we know by Ehrhart that the period of P_1 is a divisor of the *lcm* of the denominators of the vertices of P_1. It suffices then to take k equal to this number that we can easily obtain by applying lrs. After this step, we apply LattE integrale to the integral polytope kP_1 and we obtain the polynomial associated with the dilated polytope nkP_1. It is obvious that if δ is the coefficient of the leading term of this polynomial, then $\delta = \mathrm{Vol}(kP_1) = k^{24}\,\mathrm{Vol}(P_1)$. Finally, we have:

$$\mathrm{Vol}(P_1) = \frac{\delta}{k^{24}}$$

Applying this method, we obtained:

$$\mathrm{Vol}(P_1) = \frac{6914641}{55063252024302917174427648000 0}$$

Multiplying this number by 23!, as indicated in (1), we obtain the probability of having candidate A as CW:

$$\Pr\big(CW(A)\big) = \frac{2233429043}{6879707136} = 0.3246401$$

Since each of the three candidates A, B and C can be a CW, multiplying $\Pr\big(CW(A)\big)$ by 3 gives the probability $\Pr(CW)$ that a CW exists:

$$\Pr(CW) = \frac{2233429043}{2293235712} = 0.9739204$$

<u>Remark</u>: Candidate A is the CL if and only if $P_x(A,B) < P_x(B,A)$ and $P_x(A,C) < P_x(C,A)$. Given the symmetry of the notions of Condorcet Winner and Condorcet Loser, it can be noticed that

$$|CW(A,n)| = |CL(A,n)|$$

and this implies that:

$$\Pr(CL) = \Pr(CW).$$

3.2. Condorcet Winner Efficiency of voting rules

Let $CWE(F, n)$ be the CW Efficiency of the voting rule F, when all considered voting situations are in $V(n)$ and recall that $CWE(F) = \lim_{n\to\infty} CWE(F, n)$. By definition, $CWE(F, n)$ is the conditional probability to have the CW elected under F, given that a CW exists. We denote by $CW(F, A, n)$ the event "A is the CW and A is elected under F" and, as in the previous subsection, we denote by $CW(A, n)$ the event "A is a CW". As the three candidates A, B and C are symmetric in our probabilistic model, it is possible to assume without loss of generality that A is the CW, and we have:

$$CWE(F, n) = \frac{\Pr(CW(F, A, n))}{\Pr(CW(A, n))}$$

Since $\Pr(CW(F, A, n)) = |CW(F, A, n)|/|V(n)|$ and $\Pr(CW(A, n)) = |CW(A, n)|/|V(n)|$, we obtain:

$$CWE(F, n) = \frac{|CW(F, A, n)|}{|CW(A, n)|} \qquad (2)$$

Voting situations belonging to $CW(F, A, n)$ are the integer solutions of the following linear system:

$$S_n^F \quad \begin{cases} n_i \geq 0, \ \ i = 1, \dots, 24 \\ n_1 + n_2 + \cdots n_{24} = n \\ P_x(A, B) - P_x(B, A) > 0 \\ P_x(A, C) - P_x(C, A) > 0 \\ S_x(F, A) - S_x(F, B) > 0 \\ S_x(F, A) - S_x(F, C) > 0 \end{cases}$$

Let P_1^F be the (semi-open) polytope defined by S_1^F (the system obtained when $n = 1$) and let P_1 be the polytope defined in the previous subsection. Taking the limit in equality (2) and reasoning as in the previous subsection, we obtain:

$$CWE(F) = \frac{\mathrm{Vol}(P_1^F)}{\mathrm{Vol}(P_1)} \qquad (3)$$

We have already calculated $\mathrm{Vol}(P_1)$. To calculate $\mathrm{Vol}(P_1^F)$ for the five rules under consideration, we replace successively, in system S_n^F, the voting rule F by EV, PR, NPR, BR and AV (by referring to the scores defined in section 2) and then we apply, as in the previous subsection, the calculation method based on the LattE and lrs algorithms. We then apply formula 3. The results we have obtained are summarized in the following table which gives, for each F, the exact values of $CWE(F)$ and $\Pr(CW(F,A))$, the limiting probability of the event "A is the CW and A is elected under F.

	$\Pr(CW(F,A)) = 23!\,\mathrm{Vol}(P_1^F)$	$CWE(F)$
EV	$\frac{14202256426752049}{50565847449600000}$	$\frac{747487180355371}{863984392950000} \approx \mathbf{0.8651628}$
PR	$\frac{6188893189678377041}{22463437455746924544}$	$\frac{6188893189678377041}{7292533334267676672} \approx \mathbf{0.8486616}$
NPR	$\frac{2175614180234674750 1}{85691213438976000000}$	$\frac{4351228360469349500 2}{55637613939420140625} \approx \mathbf{0.7820660}$
BR	$\frac{6099120993963576543216 71}{20116230858591436800000 00}$	$\frac{4940288005110497000005 5351}{5289734055704062782125 0000} \approx \mathbf{0.9339388}$
AV	$\frac{72152487916719941809 7}{24281719945297920000 00}$	$\frac{72152487916719941809 7}{78828208043972100000 0} \approx \mathbf{0.9153131}$

Table 1. Condorcet Winner Efficiencies of five voting rules

4. Condorcet Loser Election and other results with trichotomous preferences

4.1. Condorcet Loser Efficiency

The computations are very similar to those we have conducted in Section 3: denoting the CL Efficiency of rule F by $CLE(F,n)$, the event "A is the CL and A is elected under F" by $CL(F,A,n)$ and the event "A is a CL" by $CL(A,n)$, we have just to replace (2) with

$$CLE(F,n) = 1 - \frac{|CL(F,A,n)|}{|CL(A,n)|}$$

and

$$P_x(A,B) - P_x(B,A) > 0$$
$$P_x(A,C) - P_x(C,A) > 0$$

with,

$$P_x(A,B) - P_x(B,A) < 0$$
$$P_x(A,C) - P_x(C,A) < 0$$

in the definition of S_n^F.

As BR never selects the CL (Fishburn and Gehrlein, 1976), we only consider $EV,\ PR,\ NPR$ and AV in this subsection. The results are displayed in Table 2.

	$Pr(CL(F,A))$	$CLE(F)$
EV	$\frac{161266847754571}{505658474496000000}$	$\frac{162544366182954 29}{16415703466050000} \approx \mathbf{0.99017606}$
PR	$\frac{634316125185941653924 61}{877478025615114240000000 0}$	$\frac{27852142211797170346075 39}{284864583369831120000000 0} \approx \mathbf{0.97773271}$
NPR	$\frac{385366400414044979 09}{548423766009446400000 0}$	$\frac{162544366182954 29}{178040364606144450000 0} \approx \mathbf{0.9783551}$
AV	$\frac{371518918953857}{1011316948992000000}$	$\frac{327942550402046143}{328314069321000000} \approx \mathbf{0.99886840}$

Table 2. Condorcet Loser Efficiencies of four voting rules

4.2. Other Results

A strengthening of the Condorcet Winner condition sometimes used in the literature is based on the notion of an Absolute CW (ACW): a candidate is an ACW when more than one half of the voters rank this candidate (and only this candidate) in first position. In our framework, A is a ACW iff:

$$n_1 + n_2 + n_{13} + n_{14} + n_{19} > n/2.$$

One can define in the same way the notion of Absolute CL (ACL): a candidate is an ACL when more than one half of the voters rank this candidate (and only this candidate) in last position. Thus, A is an ACL when:

$$n_4 + n_5 + n_{11} + n_{12} + n_{20} > n/2.$$

Of course, when such candidates exist, the ACW should be elected and the ACL should not.

Felsenthal (2012) and Smaoui and Lepelley (2013) observe that EV violates these two conditions[25] and we are interested in this subsection in the computation of the likelihood of such violations. It is of interest to notice that AV also violates both the ACW and the ACL conditions; and among the scoring rules, PR is the only rule verifying the ACW condition (Lepelley, 1992), whereas BR and NPR verifies the ACL condition.

Given the symmetry of our model, the probability of having an ACW is equal to the probability of having an ACL and it turns out that this probability is very low. Using the same technique as above, we obtain:

$$Pr(ACW) = Pr(ACL) = 32709 / 8388608 = 0.0038992.$$

Denoting by $ACWE(F)$ the probability of having the ACW elected under F, given that such a candidate exists, we obtain the following results:

$$ACWE(EV) = 20508589358593657 / 20510395637760000 = 0.9999119.$$

$$ACWE(NPR) = 13945016265602573 / 14598301564455552 = 0.9552492.$$

$$ACWE(BR) = 2045074032821477360350 1 / 2045081233754181000000 0$$
$$= 0.9999965.$$

$$ACWE(AV) = 12863582803932107 / 12864120144003072 = 0.9999582.$$

Regarding the election of the ACL, the probabilities we obtain are close to 0 and, consequently, the ACL Efficiencies are close to 1:

$$Pr(ACL, EV) = 7414393657 / 20510395637760000 = 0.0000004$$
and
$$ACLE(EV) = 20510388223366343 / 20510395637760000 = 0.99999964.$$

$$Pr(ACL, PR) = 42835450539276625 / 119589286416019881894 = 0.0003582$$
and
$$ACLE(PR) = 119546450965480605359 / 119589286416019881984 = 0.99964181.$$

$$Pr(ACL, AV) = 292081 / 8576080096002048 = 3.405763\ldots \times 10^{-11}$$
and
$$ACLE(AV) = 8576080095709967 / 8576080096002048 = 0.999999999.$$

[25] As the CL election, the ACL election and the ACW non election are qualified as "intolerable" by Felsenthal (2012).

5. Conclusions and final remark

All the results we have obtained in this study are summarized in Table 3. The following conclusions emerge from the examination of Table 3:

	EV	PR	NPR	BR	AV
CW Eff.	0.8651628	0.8486616	0.7820660	0.9339388	0.9153131
CL Eff.	0.9901761	0.9777327	0.9783551	1	0.9988684
ACW Eff.	0.9999119	1	0. 9552492	0.9999965	0. 9999582
ACL Eff.	0.9999996	0.9996418	1	1	0.9999999

Table 3. Condorcet Efficiencies of five voting rules with trichotomous preferences.

- The hierarchy of PR, NPR and BR regarding CW Efficiency is consistent with what we could expect from previous studies (Gehrlein and Lepelley, 2011); in other words, moving from linear orders to trichotomous preferences does not modify the ranking of the scoring rules: BR is better than PR, itself better than NPR. Moreover, our study confirms the **superiority of BR** over all the other (one-stage) voting rules in terms of Condorcet Efficiency in three-candidate elections.

- The **Condorcet Winner Efficiency of EV holds a middle position** between the CW Efficiencies of PR and NPR on the one hand, and the CW Efficiencies of AV and BR on the other hand.[26]

- Compared to PR and NPR, **EV reduces the risk of electing the CL** and the performance of AV on this issue is even better. This observation, added to the preceding one, gives a strong argument for using EV or AV instead of PR in political elections.

- The probability of not electing the Absolute Condorcet Winner when such a candidate exists appears to be **very low**, except for NPR; and it turns out that, if the election of an Absolute Condorcet Loser can occur under EV, PR and AV, such an event is **highly unlikely** in our framework. Consequently, we should not worry about these possibilities as such.

[26] The CW Efficiency ranking of these five voting rules obtained from the simulations conducted by Smaoui and Lepelley (2013) is similar with however one exception: they obtain that EV stands before AV. Their simulations are based on a framework completely different from the one used in the current study.

- Generally speaking, the comparison of **EV and AV in terms of Condorcet Efficiency is to the advantage of AV**. Notice that our framework and our probabilistic assumption play an important role in this conclusion: among the 24 possible trichotomous preferences, 18 are actually dichotomous preferences; as we consider every possible preference as equally likely, this peculiarity is in favor of AV since we know that when all the preferences are dichotomous, AV always selects the CW (Brams and Fishburn, 1983). Let $\alpha = (n_1 + n_2 + \cdots + n_6)/n$ be the proportion of voters having strict preference orders (or linear orders) in the electorate (our study assumes that α is on average equal to 6/24=1/4). It can be expected that the CW Efficiency of EV increases when α increases. To check this conjecture, we have computed the CW Efficiency of EV for various values of α, assuming that for each specific value under consideration, the corresponding voting situations are equally likely to occur. For comparison, we have also computed the CW Efficiency of AV under the same assumptions. The results are shown in Table 4 (exact fractions are here omitted).

α	0	1/4	1/3	1/2	2/3	3/4	1
$CWE_\alpha(EV)$	0.854666	0.863551	0.873434	0.894253	0.905886	0.908669	0.911111
$CWE_\alpha(AV)$	1	0.910172	0.893512	0.890087	0.901906	0.906364	0.911111

Table 4. CW Efficiency of $EV$and AVas function of the proportion of voters with linear orders

We observe the expected increase for the Condorcet Efficiency of EV with an upper limit at $\frac{41}{45} = 0.911111$, which corresponds to the CW Efficiency of BR with strict preferences. For AV, the Condorcet Efficiency decreases when α moves from 0 to $1/2$ and then increases when α moves from $1/2$ to 1. We note that for $1/2 \leq \alpha < 1$, EV performs slightly better than AV.

The good performance of AV could also be due to our assumption that voters with R_1 to R_6 (strict) preferences approve their candidate ranked in second position with probability $1/2$. If, following Gehrlein and Lepelley (2015), we reject this assumption[27], the AVscores become:

$$S'_{AV}(A,x) = (n_1 + n_2 + n_7 + n_8 + n_9 + n_{10} + n_{13} + n_{14} + n_{19} + n_{22} + n_{24})$$
$$S'_{AV}(B,x) = (n_3 + n_4 + n_7 + n_8 + n_{11} + n_{12} + n_{15} + n_{16} + n_{20} + n_{21} + n_{24})$$
$$S'_{AV}(C,x) = (n_5 + n_6 + n_9 + n_{10} + n_{11} + n_{12} + n_{17} + n_{18} + n_{20} + n_{22} + n_{23})$$

[27] This assumption is tantamount to consider that AV works as BR in presence of linear orders, whereas Gehrlein and Lepelley (2015) consider that, in this context, AV works as PR.

and we obtain:

$$CWE'(AV) = \frac{21895717301068657}{25126076733750000} = 0.87143399.$$

The CW Efficiency of AV is reduced but remains slightly higher than the CW Efficiency of EV (0.8651628), leading to the conclusion that, in our framework, the superiority of AV on EV seems to be rather robust.

Finally, it is of interest to notice that one can easily compute from our results the CW Efficiency of PER and $NPER$, where PER stands for Plurality Elimination Rule and $NPER$ for Negative Plurality Elimination Rule. PER and $NPER$ are two-stage voting rules where the candidate with the lowest score under PR and $NPER$ (respectively) is eliminated at the first stage. While never electing the CL, PER and $NPER$ are susceptible to elect a candidate different from the CW and $NPER$ can even elect a candidate different from the ACW. To obtain $CWE(PER)$, it is easily observed that (i) starting from a voting situation in which a CL exists and is elected under NPR, if we inverse the preference orders of every voter, then we obtain a voting situation in which a CW exists and is ranked first by the smallest number of voters and hence is eliminated under PER; (ii) similarly, if we inverse the voters' preferences in a voting situation where the CW gets the minimum number of first ranks (and hence is eliminated under PER), then we obtain a voting situation in which the CL is elected under NPR. Using these symmetry arguments, it can be concluded that:

$$CWE(PER) = CLE(NPR) = 0.9783551 .$$

And from similar arguments, we obtain:

$$CWE(NPER) = CLE(PR) = 0.9777327$$

and

$$ACWE(NPER) = ACLE(PR) = 0.9996418.$$

We conclude that, in our trichotomous framework as well as in the usual framework where only linear orders are considered (see Gehrlein and Lepelley, 2011), two-stage voting rules perform better than one-stage rules in electing the CW in three-alternative elections.

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Chapitre 4 : Manipulabilité coalitionnelle du vote par note à trois niveaux

Résumé. *Nous prolongeons notre analyse probabiliste de la règle de vote par note à trois niveaux (EV) par le calcul de sa vulnérabilité à la manipulation stratégique par une coalition de votants. Pour pouvoir comparer les performances de EV à celles des règles de la pluralité, de l'antipluralité et de Borda, nous calculons aussi les fréquences théoriques des situations de vote instables sous chacune des extensions de ces trois règles au cadre des préférences trichotomiques.*

1. Introduction

Nous savons, depuis le théorème de Gibbard (1973) et Satterthwaite (1975), que toutes les règles de vote non dictatoriales sont manipulables. Cela signifie que, pour toutes ces règles, il existe des situations pour lesquelles certains votants ont la possibilité d'exprimer une préférence non sincère de manière à obtenir un résultat collectif qu'ils préfèrent à celui qu'ils obtiendraient en votant sincèrement. L'étude de la fréquence théorique de ce type de situations a fait l'objet d'une abondante littérature en théorie du vote. A titre d'exemples, on peut citer les travaux de Peleg (1979), Chamberlin (1985), Nitzan (1985), Lepelley et Mbih (1987, 1994), Lepelley et Valognes (2003), Saari (1990), Kelly (1993), Kim et Roush (1996), Favardin et Lepelley (2006), Pritchard et Wilson (2007)[28]. Ces études différent par les hypothèses sur la distribution des préférences des votants (IC ou IAC), mais aussi par la manière de définir et de mesurer la manipulation stratégique. Globalement, on peut distinguer deux définitions : la manipulation individuelle (seules les situations où la règle de vote est manipulable par un seul votant sont prises en compte) et la manipulation coalitionnelle (qui retient les situations où la règle de vote est manipulable par un groupe constitué d'un ou de plusieurs votants). La mesure de la fréquence des situations de vote propices à la manipulation (situations instables) dépend en général du seuil k que la taille de la coalition manipulatrice (minimale) ne doit pas dépasser :

[28] Ces études considèrent des règles de vote qui choisissent un seul gagnant (les cas d'égalité sont tranchés en se référant à une règle de tie-break). Il existe une autre littérature qui s'intéresse aux règles à choix multiple (possibilité d'avoir plusieurs candidats gagnants) ; voir par exemple Aleskerov et al. (2011).

on a ainsi $k = 1$ pour la manipulation individuelle, et $k = \infty$ pour la manipulation par une coalition de taille quelconque[29].

Tous les travaux que nous venons de citer se sont intéressés à des règles de vote par classement (i.e., où les préférences individuelles sont représentées par des ordres linéaires sur l'ensemble des candidats). A notre connaissance, il n'existe à ce jour aucune étude permettant de quantifier la manipulabilité (théorique) des règles de vote par évaluation. Dans ce qui suit, nous nous proposons d'étendre la notion de manipulation stratégique (par une coalition de votants) au cadre des préférences trichotomiques, dans le but d'évaluer et de comparer la manipulabilité des quatre règles de vote suivantes : le vote par note à trois niveaux (EV), et les règles de la pluralité (PR), de l'antipluralité (NPR) et de Borda (BR). Notre motivation principale étant d'associer un ordre de grandeur à l'une des critiques (intuitives) les plus fréquentes de la règle de vote par note EV, selon laquelle cette règle est « très manipulable ».

La suite de ce papier est organisée comme suit. Dans la section 2, nous introduisons quelques définitions et notations, nous adaptons la notion de manipulation stratégique au contexte des préférences trichotomiques. Dans la section 3, nous caractérisons les situations de vote instables sous chacune des quatre règles étudiées et nous évaluons les fréquences théoriques de ces situations. Enfin, dans la section 4, nous donnons une courte description d'une (nouvelle) méthode de calcul de probabilités sous IAC, que nous avons utilisée pour obtenir nos résultats et dont la portée nous semble potentiellement importante.

2. Préférences trichotomiques et manipulation stratégique

Nous reprenons le cadre des préférences trichotomiques introduit dans Smaoui et Lepelley (2013) et El Ouafdi et al. (2017). On considère donc des élections avec un ensemble N de n votants et un ensemble de trois candidats $X = \{A, B, C\}$, et on suppose que chaque votant exprime son avis sur les trois candidats en choisissant l'une des 24 préférences trichotomiques suivantes :

[29] Pour une description plus complète des différentes manières d'évaluer la manipulation stratégique, voir Pritchard et Wilson (2007).

A	A	B	B	C	C	AB	AB	AC	AC	BC	BC
B	C	A	C	A	B	C	$-$	B	$-$	A	$-$
C	B	C	A	B	A	$-$	C	$-$	B	$-$	A
R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}

A	A	B	B	C	C	$-$	$-$	$-$	$-$	$-$	$-$
BC	$-$	AC	$-$	AB	$-$	A	BC	B	AC	C	AB
$-$	BC	$-$	AC	$-$	AB	BC	A	AC	B	AB	C
R_{13}	R_{14}	R_{15}	R_{16}	R_{17}	R_{18}	R_{19}	R_{20}	R_{21}	R_{22}	R_{23}	R_{24}

Dans la suite de ce document, pour des raisons pratiques, nous utiliserons aussi une notation sur une seule ligne pour désigner ces préférences trichotomiques : par exemple, ABC pour R_1, $(AB)C-$ pour R_8, $(AC)-B$ pour R_{10}, $-A(BC)$ pour R_{19}, etc. Rappelons aussi que, dans toutes les préférences trichotomiques, il y a toujours trois classes (groupes, catégories), que tout candidat d'une classe supérieure est préféré à tout candidat d'une classe inférieure, et que l'une des classes peut être vide (nous ignorons le cas où deux classes sont vides, c'est-à dire le cas où les trois candidats se trouvent dans la même classe). Ainsi, dans R_1, les candidats A, B et C se trouvent respectivement dans la première, la deuxième et la troisième classe ; alors que dans R_{10}, A et B sont dans la première classe, aucun candidat n'est dans la deuxième classe, et B est seul à la dernière classe.

Pour une préférence trichotomique R_i $(1 \leq i \leq 24)$ on notera $P(R_i)$ et $I(R_i)$ la relation de préférence stricte et la relation d'indifférence qui découlent naturellement de R_i. Rappelons qu'un profil de taille n est une liste ordonnée de n préférences individuelles et qu'une situation de vote de taille n est un profil anonyme, c'est-à-dire une répartition de n préférences individuelles en un 24-uplet $x = (n_1, n_2, \cdots, n_{24})$ où n_i est le nombre de votants ayant la préférence trichotomique R_i. L'ensemble des situations de vote de taille n et l'ensemble de toutes les situations de vote (de taille quelconque) sont notés $V(n)$ et V respectivement. Nous nous nous intéressons ici à des procédures électorales anonymes, nous définissons donc une règle de vote comme une application F qui à chaque situation x dans V associe un unique candidat gagnant $F(x)$ dans X. La règle de vote par note à trois niveaux, EV, peut être définie de la manière suivante. Pour chaque préférence individuelle trichotomique, les candidats de la première classe (ceux qui sont préférés à tous les autres candidats) se voient attribuer 2 points chacun, les candidats de la deuxième classe 1 point chacun, et les candidats de la troisième classe 0 point chacun. Le gagnant est le candidat qui obtient le total de points (score) le plus élevé.

Pour adapter PR, NPR et BR (et plus généralement n'importe quelle règle positionnelle simple) au cadre des préférences trichotomiques, nous procédons de la même manière que dans El Ouafdi et al. (2016). Soit F une règle positionnelle simple utilisant le vecteur points $v = (\lambda_1, \lambda_2, \lambda_3)$, avec $\lambda_1 \geq \lambda_2 \geq \lambda_3$ et $\lambda_1 > \lambda_3$; on a $v = (1,0,0)$ pour PR, $v = (1,1,0)$ pour NPR et $v = (2,1,0)$ pour BR. On commence par transformer chaque préférence trichotomique, R_i, en un classement des trois candidats, en ignorant les classes vides et en utilisant les composantes $P(R_i)$ et $I(R_i)$ (par exemple, R_8 devient $A \sim B \succ C$: A et B sont équivalents et sont préférés à C). Les scores des candidats sous F se calculent alors de la manière suivante. Si le classement obtenu, après la transformation de R_i, est un ordre strict, alors chaque candidat reçoit λ_k points lorsqu'il occupe la position k dans cet ordre. Si ce classement est un ordre faible du type $X \succ Y \sim Z$, alors X reçoit λ_1 points, et Y et Z reçoivent chacun $(\lambda_2 + \lambda_3)/2$ points. Enfin, si ce classement est un ordre faible du type $X \sim Y \succ Z$, alors X et Y reçoivent chacun $(\lambda_1 + \lambda_2)/2$ points et Z reçoit λ_3 points. La candidat gagnant par la règle positionnelle étendue F est alors celui qui obtient le plus grand total de points.

Nous allons maintenant nous intéresser à la vulnérabilité à la manipulation coalitionnelle des règles de vote décrites ci-dessus. Dans la définition suivante nous n'imposons aucune limite à la taille de la coalition manipulatrice, nous supposons que l'information est parfaite (chaque votant connait parfaitement les préférences des autres votants), et nous ignorons les situations d'ex aequo (qui conduisent à l'application d'une règle de tie-break).

Définition. Soient F une règle de vote et $x = (n_1, \cdots, n_{24})$ une situation de vote. On dit que F est manipulable en x (ou que x est instable sous F) s'il existe une situation de vote de même taille, $y = (m_1, \cdots, m_{24})$ telle que pour tout $i = 1, \ldots, 24$, on a : $(n_i > 0$ et $n_i < m_i) \Longrightarrow F(y)P(R_i)F(x)$.

Il est important de souligner que, dans cette définition formelle, les situations instables sont décrites comme des situations *pouvant donner lieu à* un comportement stratégique de la part de certains votants. Plus précisément, une situation de vote x est instable si elle offre à un groupe de votants (coalition) la possibilité de changer de préférences (au moment du vote) pour transformer x en une situation de vote y dont le gagnant, $F(y)$, est préféré par tous les membres de la coalition au gagnant dans x, $F(x)$. Les deux exemples suivants peuvent aider à rendre encore plus claire cette notion d'instabilité et donnent une idée sur les possibilités d'action stratégique que ces situations peuvent offrir aux éventuels groupes de manipulateurs sous chacune des règles de vote étudiées.

Exemple 1. Considérons la situation de vote x définie par :

A	C	AB	BC	$-$
C	B	$-$	$-$	BC
B	A	C	A	A
4	4	12	1	1

On peut voir facilement que cette situation est instable lorsque la règle de vote par note à trois niveaux, EV, est utilisée. En effet, les scores des trois candidats sont donnés par : $S_{EV}(A,x) = 32$, $S_{EV}(B,x) = 31$ et $S_{EV}(C,x) = 15$. Le gagnant est donc le candidat A ($EV(x) = A$). Supposons que deux des quatre votants ayant la préférence R_6 (CBA) décident de ne pas voter sincèrement, et inversent leur préférence entre C et B, passant ainsi de R_6 à R_4 (BCA). On obtient alors la situation de vote y définie par :

A	B	C	AB	BC	$-$
C	C	B	$-$	$-$	BC
B	A	A	C	A	A
4	2	2	12	1	1

Les scores deviennent alors : $S_{EV}(A,y) = 32$, $S_{EV}(B,y) = 33$ et $S_{EV}(C,y) = 13$, et B devient gagnant ($EV(y) = B$). Ainsi, deux votants (parmi les quatre du type R_6) peuvent coordonner leur action et agir en coalition pour voter stratégiquement et changer le résultat en faveur d'un candidat qu'ils préfèrent au vainqueur du vote sincère (pour ces deux votants on a $EV(y)P(R_6)EV(x)$).

Notons que la situation de vote x peut donner lieu à d'autres possibilités de manipulation coalitionnelle, toujours en faveur de B, contre A. En effet, si le votant du type R_{20} ($-(BC)A$) et l'un des quatre votants du type R_6 (CBA) changent leurs préférences en R_4 (BCA), alors on obtient une nouvelle situation de vote z où les scores sont donnés par : $S_{EV}(A,z) = 32$,

$S_{EV}(B, z) = 33$ et $S_{EV}(C, z) = 13$. Le candidat B, qui est préféré à A par le groupe des deux votants manipulateurs, devient gagnant ($EV(z) = B$, $BP(R_{20})A$ et $BP(R_6)A$). En revanche, aucune coalition (de votants qui préfèrent C à A) ne peut, en exprimant des votes non sincères, changer le résultat collectif en faveur de C : la différence de score entre A et C étant de 17 points, en manipulant tous contre A, les votants qui préfèrent C à A ne peuvent réduire cette différence que de 2 points. Notons aussi que si l'ordre lexicographique est retenu pour trancher les cas d'ex aequo, alors x n'offre aucune possibilité de manipulation individuelle contre A en faveur de B : la différence de score entre A et B est de 1 point, et chacun des votants qui ont intérêt à voter stratégiquement en faveur de B (ceux du type R_6 ou de type R_{20}) ne peut réduire cette différence que d'un seul point (l'ordre lexicographique désignera alors A comme vainqueur).
Supposons maintenant que la règle de la pluralité (étendue aux préférences trichotomiques) est appliquée. Les scores sont alors : $S_{PR}(A, x) = 10$, $S_{PR}(B, x) = 7$ et $S_{PR}(C, x) = 5$. Il suffit que les 4 votants du type CBA optent pour le vote stratégique BCA pour changer le résultat en faveur de B. La situation x est donc instable sous PR (d'autres coalitions peuvent aussi manipuler PR en x : par exemple, le groupe constitué du votant du type $(BC) - A$, du votant du type $-(BC)A$, et de deux des quatre votants du type CBA).

Pour finir, on peut vérifier facilement que la règle de l'antipluralité n'est pas manipulable en x et que la règle de Borda est manipulable en x. Sous NPR, nous avons les scores suivants : $S_{NPR}(A, x) = 16$, $S_{NPR}(B, x) = 18$ et $S_{NPR}(C, x) = 10$. On a donc $NPR(x) = B$. Aucune coalition de votants (parmi les 6 préférant A à B) ne peut, en votant stratégiquement, faire gagner plus de points à B ou faire perdre des points à A. De même, les votants qui préfèrent C à B peuvent au plus réduire la différence de score entre B et C de 4 points (en faisant passer B à la dernière place dans les 4 préférences CBA). Avec la règle de Borda, les scores sont : $S_{BR}(A, x) = 26$, $S_{BR}(B, x) = 25$ et $S_{BR}(C, x) = 15$; donc $BR(x) = A$. Pour manipuler BR en x, en faveur de B contre A, il suffit par exemple que deux des 4 votants du type CBA changent leur préférence en BCA.

Exemple 2. Considérons la situation de vote suivante :

		x			
A	B	C	AB	BC	$-$
C	A	B	$-$	A	BC
B	C	A	C	$-$	A
8	8	2	4	2	1

Intéressons-nous d'abord à la manipulabilité de NPR en x. Nous avons les scores suivants : $S_{NPR}(A,x) = 20$, $S_{NPR}(B,x) = 17$ et $S_{NPR}(C,x) = 13$; on a donc $NPR(x) = A$. Parmi tous les votants qui préfèrent B à A, les seuls qui ont la possibilité de manipuler (contre A, en faveur de B) sont les 8 du type BAC (R_3) (dans les préférences CBA, $(BC)A-$ et $-(BC)A$, il n'est pas possible d'augmenter le score de B ou de diminuer le score de A). Ces votants ne peuvent pas augmenter le score de B, mais chacun d'eux est capable de faire baisser le score de A, d'un point ou d'un demi-point (en passant à BCA, $B(AC)-$, ou $B-(AC)$). Tout point (ou demi-point) perdu par A sera gagné par C, il faut donc que l'action de ces votants stratégiques soit suffisante pour que le score de A devienne inférieur au score de B, sans que C passe devant B. On peut alors voir facilement que les seules coalitions possibles qui peuvent changer le résultat du vote en faveur de B sont celles qui font perdre exactement 3,5 points à A (par exemple, 4 votants du type BAC dont trois qui votent BCA et un qui vote $B(AC)-$). En conclusion, NPR est manipulable en x en faveur de A, à condition que seule une partie des manipulateurs potentiels votent stratégiquement, et qu'ils se concertent entre eux avant d'agir, pour éviter de faire gagner C (par exemple si 4 votants du type BAC votent tous BCA, le résultat final sera en faveur de C).

Examinons maintenant les possibilités de manipulation de BR en x. Nous avons les scores suivants : $S_{BR}(A,x) = 30$, $S_{BR}(B,x) = 28{,}5$ et $S_{BR}(C,x) = 16{,}5$; on a donc $BR(x) = A$. Il y a ici plusieurs possibilités de manipulation stratégique en faveur de B. Par exemple, on peut vérifier facilement que chacune des coalitions suivantes peut faire gagner B : le groupe formé des deux votants du type CBA, celui formé des deux votants du type $(BC)A-$ et d'un votant du type CBA, ou encore celui constitué d'un votant de chacun des types $(BC)A-$, $(BC)A-$ et CBA. Dans toutes ces configurations, la stratégie des manipulateurs est plutôt simple, car elle n'exige pas que ces derniers veillent à ce que le score de C ne dépasse pas celui de B : il s'agit simplement de faire gagner des points à B en le mettant seul à la première place (ce qui au passage fait perdre des points à C). Une autre possibilité de manipulation stratégique s'offre aux votants du type BAC. Il suffit que deux d'entre eux votent BCA pour enlever deux points à A et faire gagner B, augmentant au passage le score de C de deux points. En fait, dans ce cas, même si les 8 votants du type BAC optent pour le vote stratégique BCA, il n'y a aucun risque que C passe devant B (au maximum C gagne 8 points et la différence de score entre B et C est de 12 points).

3. Vulnérabilité de EV, PR NPR et BR à la manipulation coalitionnelle

Les deux exemples précédents nous ont permis d'avoir une première idée sur la nature des situations de vote instables sous chacune des quatre règles de vote étudiées. Cependant, cette idée reste très partielle, et il nous faut la description complète et rigoureuse de l'ensemble de ces situations, dans chaque cas, pour pouvoir ensuite évaluer leur fréquence de manière exacte.

3.1. Caractérisation des situations instables

Dans cette sous-section, nous considérons des situations de vote de taille n, nous supposons, sans perte de généralité, que le candidat A est le vainqueur, et nous déterminons les conditions qui caractérisent les situations instables. Rappelons que notre analyse ne porte que sur le cas d'un très grand nombre de votants, et que par conséquent nous ignorons les situations qui donnent lieu à des égalités entre les scores de deux ou trois candidats (le nombre de ces situations tend vers 0 quand n tend vers l'infini).

Pour une situation de vote $x = (n_1, \cdots, n_{24})$, une règle de vote $F \in \{EV, PR, NPR, BR\}$ et deux candidats X et Y dans $\{A, B, C\}$, la différence de score, $\Delta_F(XY, x)$, est définie par :

$$\Delta_F(XY, x) = S_F(X, x) - S_F(Y, x)$$

Notre premier résultat de caractérisation (Proposition 1) concerne la règle de vote par évaluation et la règle de la pluralité. Nous nous intéressons uniquement aux cas où la manipulation a pour objectif de changer le résultat du vote en faveur de B (au détriment de A) ; la symétrie entre les trois candidats nous permettra ensuite de connaitre toutes les situations de vote instables (sous EV, puis sous PR).

Proposition 1.

1) Soit x une situation de vote dans $V(n)$ telle que $EV(x) = A$. Alors EV est manipulable en x, en faveur de B, si et seulement si :

$$\Delta_{EV}(AB, x) < N_1 + N_2 \text{ et } \Delta_{EV}(CB, x) < N_2 + N_3$$

avec $N_1 = n_3 + n_{11} + n_{15}$, $N_2 = n_6 + n_{20} + n_{21}$,

et $N_3 = n_4 + 2n_6 + 2n_{11} + 2n_{12} + n_{15} + n_{20}$

2) Soit x une situation de vote dans $V(n)$ telle que $PR(x) = A$. Alors PR est manipulable en x, en faveur de B, si et seulement si :

$$\Delta_{PR}(AB, x) < N_4$$

avec $N_4 = n_6 + \frac{1}{2}(n_{11} + n_{12} + n_{20})$

Preuve

1) Soit x une situation de vote telle que $EV(x) = A$. Les stratégies possibles en faveur de B doivent, bien sûr, viser à faire passer B devant A, mais il peut aussi être nécessaire de faire baisser le score de C (car si $S_{EV}(C, x) > S_{EV}(B, x)$, il n'est pas sûr que l'augmentation du score de B suffirait à elle seule pour que ce score dépasse celui de C) . A l'exception de ceux du type R_{16} ($B - (AC)$), tous les votants qui préfèrent B à A ont la possibilité de manipuler en faveur de B. Pour chacun de ces votants stratégiques, la manipulation maximale consiste à remplacer sa préférence sincère par la préférence R_{16}. On vérifie alors facilement qu'au maximum, le score de B peut augmenter de N_2 points et que les scores de A et de C peuvent baisser de N_1 et N_3 points respectivement.

Supposons que EV soit manipulable en x en faveur de B. Par définition, il existe une situation de vote, , qui résulte de l'action d'une partie des votants stratégiques et qui vérifie

$$S_{EV}(A, y) < S_{EV}(B, y) \quad (1)$$

$$S_{EV}(C, y) < S_{EV}(B, y) \quad (2)$$

Or, les scores dans y sont donc tels que

$$S_{EV}(A, y) \geq S_{EV}(A, x) - N_1 \quad (3)$$

$$S_{EV}(B, y) \leq S_{EV}(B, x) + N_2 \quad (4)$$

$$S_{EV}(C, y) \geq S_{EV}(C, x) - N_3 \quad (5)$$

Les inégalités (1), (3) et (4) donnent $\Delta_{EV}(AB, x) < N_1 + N_2$, et les inégalités (2), (4) et (5) donnent $\Delta_{EV}(CB, x) < N_2 + N_3$.

Réciproquement, supposons que $\Delta_{EV}(AB, x) < N_1 + N_2$ (1) et $\Delta_{EV}(CB, x) < N_2 + N_3$ (2). L'action de tous les votants stratégiques conduit à une situations y où les scores sont tels que

$$S_{EV}(A, y) = S_{EV}(A, x) - N_1 \quad (3)$$

$$S_{EV}(B,y) = S_{EV}(B,x) + N_2 \quad (4)$$
$$S_{EV}(C,y) = S_{EV}(C,x) - N_3 \quad (5)$$

Par (1), (3) et (4), on obtient $S_{EV}(A,y) < S_{EV}(B,y)$, et par (2), (4) et (5), on obtient $S_{EV}(C,x) < S_{EV}(A,x)$. Donc $EV(y) = B$, ce qui montre que EV est manipulable en x en faveur de B.

2) Soit x une situation de vote telle que $PR(x) = A$. Les votants qui préfèrent B à A et qui ont la possibilité de manipuler en faveur de B sont ceux du type R_6, qui peuvent chacun augmenter le score de B d'un point (au maximum), et ceux du type R_{11}, R_{12} et R_{20}, qui peuvent chacun augmenter le score de B d'un demi-point. L'action de ces votants stratégiques laisse le score de A inchangé et enlève à C les points qu'elle fait gagner à B. Au maximum, le nombre de ces points est $N_4 = n_6 + \frac{1}{2}(n_{11} + n_{12} + n_{20})$.

Supposons que PR soit manipulable en x en faveur de B. Il existe une situation de vote, , qui résulte de l'action d'une partie des votants stratégiques et qui vérifie $PR(y) = B$. En particulier, on a

$$S_{PR}(A,y) < S_{PR}(B,y) \quad (1)$$

Or, les scores de A et B dans y sont donc tels que

$$S_{PR}(A,y) = S_{PR}(A,x) \qquad (2)$$
$$S_{PR}(B,y) \leq S_{PR}(B,x) + N_4 \quad (3)$$

Il vient alors, de (1), (2) et (3), que $\Delta_{PR}(AB,x) < N_4$.

Réciproquement, supposons que $\Delta_{PR}(AB,x) < N_4$ (1). Si tous les votants stratégiques agissent pour augmenter au maximum le score de B, alors on obtient une situation y où les scores sont tels que :

$$S_{PR}(A,y) = S_{PR}(A,x) \qquad (2)$$
$$S_{PR}(B,y) = S_{PR}(B,x) + N_4 \quad (3)$$
$$S_{PR}(C,y) \leq S_{PR}(C,x) \qquad (4)$$

Par (1), (2) et (3), on obtient $S_{PR}(A,y) < S_{PR}(B,y)$ (5). D'autre part, on sait que $S_{PR}(C,x) \leq$ $< S_{PR}(A,x)$, car $PR(x) = A$. Par (4) et (5), on obtient alors $S_{PR}(C,y) < S_{PR}(B,y)$ (6). Les inégalités (5) et (6) montrent que $PR(y) = B$, et donc que PR est manipulable en x.

Notre deuxième résultat de caractérisation (Proposition 2) concerne la règle de l'antipluralité et la règle de Borda. Comme pour le premier résultat, il permet, en utilisant la symétrie entre les trois candidats, d'identifier toutes les situations de vote où chacune de ces deux règles de vote est manipulable par une coalition de votants.

Proposition 2.

1) Soit x une situation de vote dans $V(n)$ telle que $NPR(x) = A$. Alors NPR est manipulable en x, en faveur de B, si et seulement si :

$\Delta_{NPR}(AB, x) < \Delta_{NPR}(BC, x)$ et $\Delta_{NPR}(AB, x) < N_1$, avec $N_1 = n_3 + \frac{1}{2}(n_{15} + n_{16} + n_{21})$.

2) Soit x une situation de vote dans $V(n)$ telle que $BR(x) = A$. Alors BR est manipulable en x, en faveur de B, si et seulement si :

$\Delta_{BR}(AB, x) < N_2$ ou

($\Delta_{BR}(AB, x) \geq N_2$ et $\Delta_{BR}(AB, x) < N_2 + N_3$ et $\Delta_{BR}(AB, x) \leq \Delta_{BR}(BC, x) + 3N_2$),

avec $N_2 = n_6 + \frac{1}{2}(n_{11} + n_{12} + n_{20})$ et $N_3 = n_3 + \frac{1}{2}(n_{15} + n_{16} + n_{21})$

Preuve

1) Soit x une situation de vote telle que $NPR(x) = A$. La seule stratégie possible en faveur de B est de faire baisser le score de A. En effet, on vérifie facilement qu'il n'est pas possible pour les votants qui préfèrent B à A d'augmenter le score de B (car B n'est jamais classé dernier), et qu'il est alors inutile d'enlever des points à C (car ces points seront gagnés par A). Les votants stratégiques sont donc ceux du type R_3 qui peuvent chacun faire baisser le score de A d'un point au maximum (en passant à R_4), et ceux du type R_{15}, R_{16} et R_{21}, qui peuvent chacun faire baisser le score de A d'un demi-point (en passant à R_4). Ainsi, suite à l'action d'un groupe manipulateur, B conserve son score, et A perd des points que C récupère ; le nombre maximum de ces points est $N_1 = n_3 + \frac{1}{2}(n_{15} + n_{16} + n_{21})$.

Supposons que NPR soit manipulable en x en faveur de B. Par définition, il existe une situation de vote, , qui résulte de l'action d'une partie des votants stratégiques et qui vérifie

$$S_{NPR}(A, y) < S_{NPR}(B, y) \quad (1)$$

$$S_{NPR}(C, y) < S_{NPR}(B, y) \quad (2)$$

Soit δ le nombre de points perdus par A et gagnés par C. On a $0 < \delta \leq N_1$, et

$$S_{NPR}(A, y) = S_{NPR}(A, x) - \delta \quad (3)$$

$$S_{NPR}(B, y) = S_{NPR}(B, x) \quad (4)$$

$$S_{NPR}(C, y) = S_{NPR}(C, x) + \delta \quad (5)$$

En utilisant (1), (3) et (4) on obtient $\Delta_{NPR}(AB, x) < \delta$ (6). Or $\delta \leq N_1$, donc (6) nous donne $\Delta_{NPR}(AB, x) < N_1$. D'autre part, (2), (4) et (5) donnent $\Delta_{NPR}(BC, x) > \delta$. En tenant compte de (6), on obtient alors $\Delta_{NPR}(BC, x) > \Delta_{NPR}(AB, x)$.

Réciproquement, supposons que $\Delta_{NPR}(AB, x) < N_1$ (1) et $\Delta_{NPR}(AB, x) < \Delta_{NPR}(BC, x)$ (2). L'action de tous les votants stratégiques conduit à une situation y où les scores sont

$$S_{NPR}(A,y) = S_{NPR}(A,x) - N_1 \quad (3)$$
$$S_{NPR}(B,y) = S_{NPR}(B,x) \qquad (4)$$
$$S_{NPR}(C,y) = S_{NPR}(C,x) + N_1 \quad (5)$$
Par (1), (3) et (4), on obtient $S_{NPR}(A,y) < S_{NPR}(B,y)$ (6). Par (4) et (5), on a $S_{NPR}(B,y) - S_{NPR}(C,y) = \Delta_{NPR}(BC,x) - N_1$. En utilisant (1), on alors $S_{NPR}(B,y) - S_{NPR}(C,y) > \Delta_{NPR}(BC,x) - \Delta_{NPR}(AB,x)$, et en utilisant (2), on obtient $S_{NPR}(B,y) - S_{NPR}(C,y) > 0$. (6) et (7) montrent que $NPR(x) = B$, est donc que NPR est manipulable en x en faveur de B.

2) Soit x une situation de vote telle que $BR(x) = A$. Les stratégies possibles en faveur de B sont celles qui permettent d'augmenter le score de B ou de baisser le score de A. Notons que les points perdus par C ne sont utiles (pour faire gagner B) que lorsqu'ils sont récupérés par B (dans le cas contraire, ces points seront gagnés par A).
Les votants qui préfèrent B à A et qui ont la possibilité d'augmenter le score de B sont ceux du type R_6, R_{11}, R_{12} et R_{20}. L'action de ces votants stratégiques laisse le score de A inchangé et enlève à C les points qu'elle fait gagner à B. Au maximum, le nombre de ces points est $N_2 = n_6 + \frac{1}{2}(n_{11} + n_{12} + n_{20})$. Les votants qui préfèrent B à A et qui ont la possibilité de baisser le score de A sont ceux du type R_3, R_{15}, R_{16} et R_{21}. L'action de ces votants stratégiques laisse le score de B inchangé et fait gagner à C les points qu'elle enlève à A. Au maximum, le nombre de ces points est $N_3 = n_3 + \frac{1}{2}(n_{15} + n_{16} + n_{21})$.
Supposons que BR soit manipulable en x en faveur de B. Par définition, il existe une situation de vote, , qui résulte de l'action d'une partie des votants stratégiques et dans laquelle B est le gagnant :
$$S_{BR}(A,y) < S_{BR}(B,y) \quad (1)$$
$$S_{BR}(C,y) < S_{BR}(B,y) \quad (2)$$
On distingue deux cas : soit l'action des votants du type R_6, R_{11}, R_{12} et R_{20} suffit à elle seule à faire gagner B, soit cette action n'est pas suffisante. Dans le premier cas, le score de A ne change pas, et donc la différence de scores entre A et B doit être inférieure au nombre maximum de points que B peut gagner. Il est donc nécessaire d'avoir $\Delta_{BR}(AB,x) < N_2$. Dans le second cas, on a $\Delta_{BR}(AB,x) \geq N_2$; la coalition manipulatrice doit donc contenir une partie des votants du type R_3, R_{15}, R_{16} et R_{21}, en plus de tous ceux du type R_6, R_{11}, R_{12} et R_{20}.
Soit δ le nombre de points perdus par A et gagnés par C. On a $0 < \delta \leq N_3$, et
$$S_{BR}(A,y) = S_{BR}(A,x) - \delta \quad (3)$$
$$S_{BR}(B,y) = S_{BR}(B,x) + N_2 \qquad (4)$$

$$S_{BR}(C,y) = S_{BR}(C,x) - N_2 + \delta \qquad (5)$$

En utilisant (1), (3) et (4) on obtient $\Delta_{BR}(AB,x) < N_2 + \delta$ (6). Or $\delta \leq N_3$, donc (6) nous donne $\Delta_{BR}(AB,x) < N_2 + N_3$. D'autre part, (2), (4) et (5) donnent $\Delta_{BR}(BC,x) + 2N_2 > \delta$. En tenant compte de (6), on obtient alors $\Delta_{BR}(BC,x) + 2N_2 > \Delta_{BR}(AB,x) - N_2$, d'où $\Delta_{BR}(AB,x) < \Delta_{BR}(AB,x) + 3N_2$.

Réciproquement, supposons d'abord que $\Delta_{BR}(AB,x) < N_2$ (1). L'action de tous les votants du type R_6, R_{11}, R_{12} et R_{20} conduit à une situation de vote y où les scores sont tels que :

$$S_{BR}(A,y) = S_{BR}(A,x) \qquad (3)$$

$$S_{BR}(B,y) = S_{BR}(B,x) + N_2 \qquad (4)$$

$$S_{BR}(C,y) = S_{BR}(C,x) - N_2 \qquad (5)$$

Par (1), (3) et (4), on obtient $\Delta_{BR}(AB,y) < 0$ (6). Par (3), (5) et par le fait que $S_{BR}(C,x) < S_{BR}(A,x)$ (car $BR(x) = A$), on obtient $S_{BR}(C,y) < S_{BR}(A,y)$; et en utilisant (6) on obtient $S_{BR}(C,y) < S_{BR}(B,y)$ (7). Enfin, d'après (6) et (7), on a $BR(y) = B$, ce qui montre que BR est manipulable en x.

Supposons maintenant que $\Delta_{BR}(AB,x) \geq N_2$ (1), $\Delta_{BR}(AB,x) < N_2 + N_3$ (2) et $\Delta_{BR}(AB,x) \leq \Delta_{BR}(BC,x) + 3N_2$ (3). L'action de tous les votants stratégiques conduit à une situation de vote y où les scores sont tels que :

$$S_{BR}(A,y) = S_{BR}(A,x) - N_2 \qquad (4)$$

$$S_{BR}(B,y) = S_{BR}(B,x) + N_2 \qquad (5)$$

$$S_{BR}(C,y) = S_{BR}(C,x) - N_2 + N_3 \qquad (6)$$

Par (2), (4) et (5), on obtient $\Delta_{BR}(AB,y) < 0$ (7). Par (5) et (6), on obtient $\Delta_{BR}(BC,y) = \Delta_{BR}(BC,x) + 2N_2 + N_3$. En utilisant (3), on a alors $\Delta_{BR}(BC,y) > \Delta_{BR}(AB,x) - N_2 - N_3$; et, par (1) on a $\Delta_{BR}(BC,y) > 0$ (8). Ainsi, par (7) et (8), on a $BR(y) = B$, ce qui monte que BR est manipulable en x.

3.2. Résultats probabilistes : fréquences théoriques des situations instables

Pour une règle vote F et un entier naturel n $(n \geq 2)$, nous noterons $\Pr(Manip, F, n)$ la probabilité que F soit manipulable par une coalition de votants en une situation de vote de taille n. Cette probabilité définit la vulnérabilité de F à la manipulation coalitionnelle (en présence de n votants). La limite de $\Pr(Manip, F\, n)$ quand n tend vers l'infini sera notée $\Pr(Manip, F, \infty)$. Nous supposons que toutes les situations de vote sont équiprobables (Impartial Anonymous Culture, IAC).

Comme nous l'avons indiqué dans El Ouafdi et al. (2017), même lorsqu'il est possible d'obtenir les expressions exactes, en fonction du paramètre n, des probabilités des évènements de vote pour trois candidats dans le cadre des préférences trichotomiques, ces représentations analytiques sont généralement peu exploitables (en raison de la très grande période des quasi-polynômes obtenus). Par conséquent, nous nous intéressons, dans ce qui suit, uniquement à la valeur limite de la vulnérabilité des quatre règles étudiées. Tous les résultats présentés dans cette sous-section ont été obtenus en appliquant une nouvelle méthode de calcul (qui sera décrite dans la section 4) et vérifiés par l'application du programme [Normaliz].

- Vote par évaluation :

Rappelons que, pour une situation de vote $x = (n_1, \cdots, n_{24})$, les scores des trois candidats sont donnés par :

$$S_{EV}(A,x) = 2(n_1 + n_2 + n_7 + n_8 + n_9 + n_{10} + n_{13} + n_{14}) + (n_3 + n_5 + n_{11} + n_{15} + n_{17} + n_{19} + n_{22} + n_{24})$$

$$S_{EV}(B,x) = 2(n_3 + n_4 + n_7 + n_8 + n_{11} + n_{12} + n_{15} + n_{16}) + (n_1 + n_6 + n_9 + n_{13} + n_{17} + n_{20} + n_{21} + n_{24})$$

$$S_{EV}(C,x) = 2(n_5 + n_6 + n_9 + n_{10} + n_{11} + n_{12} + n_{17} + n_{18}) + (n_2 + n_4 + n_7 + n_{13} + n_{15} + n_{20} + n_{22} + n_{23})$$

D'après la proposition 1, l'ensemble des situations de vote, $x \in V(n)$, où EV est manipulable en faveur de B (et contre A) est caractérisé par le système suivant :

$$(S^{EV}(n)) \quad \begin{cases} n_i \geq 0, \;\; i = 1, \dots, 24 \\ n_1 + n_2 + \cdots + n_{24} = n \\ \Delta_{EV}(AB,x) > 0 \\ \Delta_{EV}(AC,x) > 0 \\ \Delta_{EV}(AB,x) < (n_3 + n_{11} + n_{15}) + (n_6 + n_{20} + n_{21}) \\ \Delta_{EV}(CB,x) < (n_6 + n_{20} + n_{21}) + (n_4 + 2n_6 + +2n_{11} + 2n_{12} + n_{15} + n_{20}) \end{cases}$$

En utilisant à nouveau la proposition 1, et la symétrie entre les candidats B et C, on obtient la caractérisation des situations de vote x où EV est manipulable en faveur de C (contre A) :

$$(T^{EV}(n)) \quad \begin{cases} n_i \geq 0, \;\; i = 1, \dots, 24 \\ n_1 + n_2 + \cdots + n_{24} = n \\ \Delta_{EV}(AB,x) > 0 \\ \Delta_{EV}(AC,x) > 0 \\ \Delta_{EV}(AC,x) < (n_5 + n_{11} + n_{17}) + (n_4 + n_{20} + n_{23}) \\ \Delta_{EV}(BC,x) < (n_4 + n_{20} + n_{23}) + (2n_4 + n_6 + 2n_{11} + 2n_{12} + n_{17} + n_{20}) \end{cases}$$

Soit V_1^{EV} le volume associé à $(S^{EV}(n))$, V_2^{EV} le volume associé à $(T^{EV}(n))$, et V_3^{EV} le volume associé à l'intersection caractérisée par $(S^{EV}(n)$ et $)$ $T^{EV}(n)$. Ici, on prend en compte le volume

V_3^{EV}, car (*a priori*) on peut trouver des situations de vote où EV est à la fois manipulable en faveur de B (contre A) et manipulable en faveur de C (contre A).

Soit V_s le volume décrivant toutes les situations de vote possibles. On a alors, par symétrie entre les trois candidats, puis par symétrie entre B et C :

$$\Pr(Manip, EV, \infty) = \frac{3(V_1^{EV} + V_2^{EV} - V_3^{EV})}{V_s} \qquad (E_1)$$

$$= \frac{3(2V_1^{EV} - V_3^{EV})}{V_s}$$

On sait que $V_s = 1/23!$, et on obtient en appliquant notre nouvelle méthode :

$$V_1^{EV} = \frac{16574448349185206114759149651 3}{24142160016462623142975523180781764608000000000000000}$$

$$V_3^{EV} = \frac{27515306132866533654184695769946215 21}{6598073698310849474396692018645671259228078080000000000000000}$$

En remplaçant dans (E_1), on obtient la valeur limite de la vulnérabilité de EV à la manipulation coalitionnelle :

$$\Pr(Manip, EV3, \infty) = \frac{281611258533551712264451104736644 7}{3797986884438825459140760000000000} \approx 74.14\%$$

- Pluralité :

Pour une situation de vote $x = (n_1, \cdots, n_{24})$, les scores des trois candidats sont donnés par :

$$S_{PR}(A, x) = (n_1 + n_2 + n_{13} + n_{14} + n_{19}) + \frac{1}{2}(n_7 + n_8 + n_9 + n_{10} + n_{22} + n_{24})$$
$$S_{PR}(B, x) = (n_3 + n_4 + n_{15} + n_{16} + n_{21}) + \frac{1}{2}(n_7 + n_8 + n_{11} + n_{12} + n_{20} + n_{24})$$
$$S_{PR}(C, x) = (n_5 + n_6 + n_{17} + n_{18} + n_{23}) + \frac{1}{2}(n_9 + n_{10} + n_{11} + n_{12} + n_{20} + n_{22})$$

D'après la proposition 1, l'ensemble des situations de vote, $x \in V(n)$, où PR est manipulable en faveur de B (et contre A) est caractérisé par le système suivant :

$$(S^{PR}(n)) \quad \begin{cases} n_i \geq 0, \ \ i = 1, \ldots, 24 \\ n_1 + n_2 + \cdots + n_{24} = n \\ \Delta_{PR}(AB, x) > 0 \\ \Delta_{PR}(AC, x) > 0 \\ \Delta_{PR}(AB, x) < n_6 + \frac{1}{2}(n_{11} + n_{12} + n_{20}) \end{cases}$$

En utilisant à nouveau la proposition 1, et la symétrie entre les candidats B et C, on obtient la caractérisation des situations de vote x où PR est manipulable en faveur de C (contre A) :

$$(T^{PR}(n)) \quad \begin{cases} n_i \geq 0, \;\; i = 1, \ldots, 24 \\ n_1 + n_2 + \cdots + n_{24} = n \\ \Delta_{PR}(AB, x) > 0 \\ \Delta_{PR}(AC, x) > 0 \\ \Delta_{PR}(AC, x) < n_4 + \frac{1}{2}(n_{11} + n_{12} + n_{20}) \end{cases}$$

Soient V_1^{PR} le volume associé à $(S^{PR}(n))$, V_2^{PR} le volume associé à $(T^{PR}(n))$, et V_3^{PR} le volume associé à l'intersection caractérisée par $(S^{PR}(n)$ et $(T^{PR}(n))$. Comme pour EV, le volume V_3^{PR} est (*a priori*) non nul. Par symétrie entre les trois candidats, puis par symétrie entre B et C, nous avons :

$$\Pr(Manip, PR, \infty) = \frac{3(2V_1^{PR} - V_3^{PR})}{V_s} \qquad (E_2)$$

$$= 23! \times 3(2V_1^{PR} - V_3^{PR})$$

Les calculs nous donnent :

$$V_1^{PR} = \frac{33963995124768598967076553943}{673983968000622762554239248808297428418560000000000}$$

$$V_3^{PR} = \frac{655855723741860559762193240147 1061}{2149587617941986223321426804167963610762444800000000000000}$$

En remplaçant dans (E_2), on obtient la valeur limite de la vulnérabilité de PR à la manipulation coalitionnelle :

$$\Pr(Manip, PR, \infty) = \frac{4539130454565253746545079460613}{8328295976442566604480000000000} \approx 54.5\%$$

- Antipluralité :

Pour une situation de vote $x = (n_1, \cdots, n_{24})$, les scores des trois candidats sont donnés par :

$$S_{NPR}(A, x) = n - [(n_4 + n_6 + n_{11} + n_{12} + n_{20}) + \frac{1}{2}(n_{15} + n_{16} + n_{17} + n_{18} + n_{21} + n_{23})]$$
$$S_{NPR}(B, x) = n - [(n_2 + n_5 + n_9 + n_{10} + n_{22}) + \frac{1}{2}(n_{13} + n_{14} + n_{17} + n_{18} + n_{19} + n_{23})]$$
$$S_{NPR}(C, x) = n - [(n_1 + n_3 + n_7 + n_8 + n_{24}) + \frac{1}{2}(n_{13} + n_{14} + n_{15} + n_{16} + n_{19} + n_{21})]$$

D'après la proposition 2, l'ensemble des situations de vote, $x \in V(n)$, où NPR est manipulable en faveur de B (et contre A) est caractérisé par le système suivant :

$$(S^{NPR}(n)) \quad \begin{cases} n_i \geq 0, \;\; i = 1, \dots, 24 \\ n_1 + n_2 + \cdots + n_{24} = n \\ \Delta_{NPR}(AB, x) > 0 \\ \Delta_{NPR}(AC, x) > 0 \\ \Delta_{NPR}(AB, x) < \Delta_{NPR}(BC, x) \\ \Delta_{NPR}(AB, x) < n_3 + \dfrac{1}{2}(n_{15} + n_{16} + n_{21}) \end{cases}$$

De même, les situations de vote, $x \in V(n)$, où NPR est manipulable en faveur de C (et contre A) est caractérisé par le système suivant :

$$(T^{NPR}(n)) \quad \begin{cases} n_i \geq 0, \;\; i = 1, \dots, 24 \\ n_1 + n_2 + \cdots + n_{24} = n \\ \Delta_{NPR}(AB, x) > 0 \\ \Delta_{NPR}(AC, x) > 0 \\ \Delta_{NPR}(AC, x) < \Delta_{NPR}(CB, x) \\ \Delta_{NPR}(AC, x) < n_5 + \dfrac{1}{2}(n_{17} + n_{18} + n_{23}) \end{cases}$$

Soient V_1^{NPR} le volume associé à $S^{NPR}(n)$ et V_2^{NPR} le volume associé à $T^{NPR}(n)$. L'intersection caractérisée par ($S^{NPR}(n)$ et $T^{NPR}(n)$) est vide. En effet, comme $\Delta_{NPR}(BC, x) = -\Delta_{NPR}(CB, x)$, l'une des différences, $\Delta_{NPR}(BC, x)$ ou $\Delta_{NPR}(CB, x)$, est négative ; et comme A est le gagnant dans x, on ne peut avoir à la fois $\Delta_{NPR}(AB, x) < \Delta_{NPR}(BC, x)$ et $\Delta_{NPR}(AC, x) < \Delta_{NPR}(CB, x)$.

D'autre part, la symétrie entre B et C nous donne a $V_1^{NPR} = V_2^{NPR}$. En utilisant la symétrie entre les trois candidats, on a alors :

$$\Pr(Manip, NPR, \infty) = \frac{3(2V_1^{NPR})}{V_S} = 23! \times 6V_1^{NPR} \qquad (E_3)$$

Après calcul, on obtient :

$$V_1^{NPR} = \frac{15415206887}{48675820697748914408064752025600000}$$

En remplaçant dans (E_3), on obtient la valeur limite de la vulnérabilité de NPR à la manipulation coalitionnelle :

$$\Pr(Manip, NPR, \infty) = \frac{15415206887}{31381059609} \approx 49.12\%$$

- Borda :

Pour une situation de vote $x = (n_1, \cdots, n_{24})$, les scores des trois candidats sont donnés par :

$$\begin{aligned}
S_{BR}(A,x) &= 2(n_1+n_2+n_{13}+n_{14}+n_{19})+\tfrac{3}{2}(n_7+n_8+n_9+n_{10}+n_{22}+n_{24}) \\
&\quad +(n_3+n_5)+\tfrac{1}{2}(n_{15}+n_{16}+n_{17}+n_{18}+n_{21}+n_{23}) \\
S_{BR}(B,x) &= 2(n_3+n_4+n_{15}+n_{16}+n_{21})+\tfrac{3}{2}(n_7+n_8+n_{11}+n_{12}+n_{20}+n_{24}) \\
&\quad +(n_1+n_6)+\tfrac{1}{2}(n_{13}+n_{14}+n_{17}+n_{18}+n_{19}+n_{23}) \\
S_{BR}(C,x) &= 2(n_5+n_6+n_{17}+n_{18}+n_{23})+\tfrac{3}{2}(n_9+n_{10}+n_{11}+n_{12}+n_{20}+n_{22}) \\
&\quad +(n_2+n_4)+\tfrac{1}{2}(n_{13}+n_{14}+n_{15}+n_{16}+n_{19}+n_{21})
\end{aligned}$$

D'après la proposition 2, les situations de vote, $x \in V(n)$, où BR est manipulable en faveur de B (et contre A) sont caractérisées par :

$$(S_1^{BR}(n)) \begin{cases} n_i \geq 0, \ \ i = 1, \dots, 24 \\ n_1 + n_2 + \cdots + n_{24} = n \\ \Delta_{BR}(AB,x) > 0 \\ \Delta_{BR}(AC,x) > 0 \\ \Delta_{BR}(AB,x) < n_6 + \frac{1}{2}(n_{11}+n_{12}+n_{20}) \end{cases}$$

Ou

$$(S_2^{BR}(n)) \begin{cases} n_i \geq 0, \ \ i = 1, \dots, 24 \\ n_1 + n_2 + \cdots + n_{24} = n \\ \Delta_{BR}(AB,x) > 0 \\ \Delta_{BR}(AC,x) > 0 \\ \Delta_{BR}(AB,x) \geq n_6 + \dfrac{1}{2}(n_{11}+n_{12}+n_{20}) \\ \Delta_{BR}(AB,x) < n_3 + \dfrac{1}{2}(n_{15}+n_{16}+n_{21}) + n_6 + \dfrac{1}{2}(n_{11}+n_{12}+n_{20}) \\ \Delta_{BR}(AB,x) \leq \Delta_{BR}(BC,x) + 3(n_6 + \dfrac{1}{2}(n_{11}+n_{12}+n_{20})) \end{cases}$$

De même, les situations de vote, $x \in V(n)$, où BR est manipulable en faveur de C (et contre A) sont caractérisées par :

$$(T_1^{BR}(n)) \begin{cases} n_i \geq 0, \ \ i = 1, \dots, 24 \\ n_1 + n_2 + \cdots + n_{24} = n \\ \Delta_{BR}(AB,x) > 0 \\ \Delta_{BR}(AC,x) > 0 \\ \Delta_{BR}(AC,x) < n_4 + \frac{1}{2}(n_{11}+n_{12}+n_{20}) \end{cases}$$

Ou

$$
(T_2^{BR}(n)) \begin{cases} n_i \geq 0, \ \ i = 1, \dots, 24 \\ n_1 + n_2 + \cdots + n_{24} = n \\ \Delta_{BR}(AB, x) > 0 \\ \Delta_{BR}(AC, x) > 0 \\ \Delta_{BR}(AC, x) \geq n_4 + \dfrac{1}{2}(n_{11} + n_{12} + n_{20}) \\ \Delta_{BR}(AC, x) < n_4 + \dfrac{1}{2}(n_{11} + n_{12} + n_{20}) + n_5 + \dfrac{1}{2}(n_{17} + n_{18} + n_{23}) \\ \Delta_{BR}(AC, x) \leq \Delta_{BR}(CB, x) + 3(n_4 + \dfrac{1}{2}(n_{11} + n_{12} + n_{20})) \end{cases}
$$

Soient V_1^{BR}, V_2^{BR}, V_3^{BR}et V_4^{BR} les volumes associés à $(S_1^{BR}(n))$, $(S_2^{BR}(n))$, $(T_1^{BR}(n))$ et $(T_2^{BR}(n))$, respectivement. On vérifie facilement que les intersections caractérisées par ($S_1^{BR}(n)$ et $S_2^{BR}(n)$) et par ($T_1^{BR}(n)$ et $T_2^{BR}(n)$) sont vides. Notons V_5^{BR}, V_6^{BR}, V_7^{BR} et V_8^{BR} les volumes associés aux intersections caractérisées respectivement par ($S_1^{BR}(n)$ et $T_1^{BR}(n)$), ($S_1^{BR}(n)$ et $T_2^{BR}(n)$), ($S_2^{BR}(n)$ et $T_1^{BR}(n)$) et ($S_2^{BR}(n)$ et $T_2^{BR}(n)$). Par symétrie entre B et C, on a : $V_1^{BR} = V_3^{BR}$, $V_2^{BR} = V_4^{BR}$ et $V_6^{BR} = V_7^{BR}$. En utilisant la symétrie entre les trois candidats, on obtient :

$$
\Pr(Manip, BR, \infty) = 23! \times 3(2V_1^{BR} + 2V_2^{BR} - V_5^{BR} - 2V_6^{BR} - V_8^{BR}) \qquad (E_4)
$$

Nous avons calculé tous les volumes qui apparaissent dans cette formule. Les valeurs exactes de ces volumes et la valeur exacte de $\Pr(Manip, BR, \infty)$ comportent un très grand nombre de chiffres et sont données dans l'annexe de ce document. On obtient la valeur approchée suivante :

$$
\Pr(Manip, BR, \infty) \approx 65.36\%
$$

Les chiffres obtenus confirment le caractère "très" manipulable de la règle EV puisque les 3/4 des situations de vote sont instables pour cette règle, contre (environ) 2/3 pour BR et 1/2 pour PR et NPR. Si l'on compare nos résultats à ceux qu'ont obtenus Favardin et Lepelley (2006) dans le cadre de préférences linéaires, on note aussi que le passage aux préférences trichotomiques modifie très nettement la hiérarchie des règles positionnelles, au détriment de BR et (surtout) de PR, et au profit de NPR, comme l'indique la table ci-dessous.

	PR	NPR	BR	EV
Préférences trichotomiques	54.5%	49.1%	65.4%	74.1%
Préférences linéaires	29.2%	51.9%	50.2%	-

Table 1. Vulnérabilité à la manipulation coalitionnelle

Il convient enfin de souligner que nous avons supposé ici un comportement "naïf" de la part des votants, qui ne réagissent pas lorsqu'ils sont en présence d'une menace de manipulation. On sait (Favardin et Lepelley, 2006) que la prise en compte des ces "réactions" est susceptible de changer la hiérarchie des règles ; par exemple, elle réduit la manipulabilité de BR à un niveau inférieur à celui de PR. Nous conjecturons que cette prise en compte pourrait, de la même manière, réduire significativement la vulnérabilité de EV à la manipulation coalitionnelle.

4. Considérations méthodologiques

Dans cette section, nous donnons une brève description de la méthode que nous avons utilisée pour obtenir les résultats de la section précédente. Comme nous l'avons déjà indiqué, ces résultats limites ont été vérifiés et confirmés en recalculant, avec [Normaliz], tous les volumes associés aux événements de vote étudiés dans ce papier. Dans toute la suite, nous noterons M2 la méthode que nous présentons ici, et M1 la méthode proposée dans El Ouafdi et al. (2019). En suggérant la méthode M2, après avoir proposé M1, nous souhaitons contribuer à élargir l'ensemble des techniques de calcul de probabilités sous la condition IAC et proposer des pistes de recherche qui pourraient conduire à des méthodes applicables à des problèmes de vote avec cinq candidats (dans le cas des préférences linéaires). La technique de calcul que nous présentons ici (M2) nous parait à la fois simple et prometteuse. Tout comme M1, elle permet de calculer des volumes de polytopes rationnels de dimension 23, et repose sur un usage combiné des programmes [LattE] et [Lrs]. Cependant, à la différence de M1, le volume d'un polytope rationnel Q, associé un système linéaire paramétrique $S(n)$, s'obtient (avec M2) simplement en calculant le nombre de solutions entières de $S(n)$ pour une seule valeur (bien choisie) du paramètre n. Pour rendre plus précise cette idée, nous avons besoin de la notation suivante. Pour un nombre réel x, nous désignerons par $\lfloor x \rceil$ l'entier le plus proche de x : soit n l'unique entier tel que $n \leq x < n + 1$, alors $\lfloor x \rceil = n$ si $x - n < 1/2$ et $\lfloor x \rceil = n + 1$ si $x - n > 1/2$ (nous n'aurons pas besoin de définir $\lfloor x \rceil$ pour $x - n = 1/2$). Nous pouvons maintenant formuler l'idée de base de la méthode M2.

Proposition 3. Soit $F(t)$ un polynôme de degré d à coefficients dans $\mathbb{Z}$:

$$F(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0$$

Soit M un majorant de toutes les valeurs absolues des coefficients c_r, pour $0 \leq r \leq d - 1$. On a alors : $\lfloor F(t)/t^d \rceil = c_d, \forall\ t \geq 2M+1$.

Preuve

Pour $t > 0$, on a :

$$\frac{F(t)}{t^d} = c_d + \frac{c_{d-1}t^{d-1} + \cdots + c_1 t + c_0}{t^d} \qquad (1)$$

Posons $R(t) = \frac{c_{d-1}t^{d-1}+\cdots+c_1t+c_0}{t^d}$. Comme $|c_r| \leq M$, pour $0 \leq r \leq d-1$, on a, pour $t > 1$:

$$|R(t)| \leq \frac{|c_{d-1}|t^{d-1} + \cdots + |c_1|t + |c_0|}{t^d} \leq \frac{M\sum_{i=0}^{d-1} t^i}{t} = \frac{M(t^d - 1)}{t^d(t-1)} < \frac{M}{t-1}$$

La fonction φ définie par $\varphi(t) = M/(t-1)$ est strictement décroissante sur $]1, +\infty[$ et vérifie $\varphi(2M+1) = 1/2$. On peut donc écrire : $\forall\, t \geq 2M+1, 0 \leq |R(t)| < 1/2$. En prenant l'entier le proche dans les deux membres de l'égalité (1), on obtient : $\left\lfloor \frac{F(t)}{t^d} \right\rceil = c_d, \forall\, t \geq 2M+1$. ■

Considérons maintenant un polytope rationnel, Q, associé à un système paramétrique, $S(n)$, décrivant un évènement de vote $E(n,m)$, impliquant n votants et m candidats (avec m fixé). Comme nous l'avons déjà vu, $S(n)$ est constitué de l'égalité $\sum_{i=1}^{m!} n_i = 1$, des contraintes de non négativité ($n_i \geq 0, \forall i = 1, \ldots, m!$), et d'un certain nombre d'inégalités linéaires à coefficients rationnels en n_1, …, $n_{m!}$ et n. Le polytope Q est décrit par le système $S(1)$, sa dimension est $d = m! - 1$ et il est inclus dans le simplexe standard, $\Delta = \left\{(x_0, \ldots, x_d) \colon \sum_{i=0}^{d} x_i = 1, x_i \geq 0, \forall i = 0, \ldots, d\right\}$ dont le volume est égal à $1/d!$. On a donc :

$$\mathrm{Vol}(Q) \leq 1/d! \qquad (1)$$

Pour obtenir la valeur exacte de $\mathrm{Vol}(Q)$, nous procédons de la manière suivante :

1. Comme dans la méthode M1, nous considérons le polytope P défini par $P = \delta Q$ (dilatation de Q par le facteur δ), où δ est un multiple de la période du quasi-polynôme décrivant le nombre de solutions entières de $S(n)$. Le nombre δ peut-être obtenu en appliquant l'algorithme [Lrs]. Le polytope P est alors entier (i.e., tous ses sommets sont à coordonnées entières). On sait, d'après le théorème d'Ehrhart (1962), que le nombre de points entiers du polytope dilaté nP est décrit par un simple polynôme (polynôme d'Ehrhart) à coefficients rationnels, que nous désignerons par $G(n)$.
2. En multipliant $G(n)$ par $d!$, on obtient un polynôme, $F(n)$, à coefficients dans $\mathbb{Z}$, $F(n) = d!\,G(n)$. On a alors $F(n) = c_d n^d + c_{d-1}n^{d-1} + \cdots + c_1 n + c_0$, avec :

   $$c_d = d!\,\delta^d\,\mathrm{Vol}(Q) \qquad (2)$$

   On peut maintenant appliquer la proposition 3 au polynôme $F(n)$. Pour cela, nous utilisons une majoration issue des résultat de Beck et al. (2005) sur les coefficients des polynômes d'Ehrhart : pour $0 \leq r \leq d-1$, $|c_r| \leq c_d(d+1)!$. En utilisant (1) et (2), on obtient : $|c_r| \leq M = \delta^d(d+1)!, \quad \forall\, 0 \leq r \leq d-1$. On a alors, d'après la proposition 3 :

$$c_d = \left\lfloor \frac{F(n)}{n^d} \right\rceil, \forall\, n > k = 2\delta^d(d+1)! + 1 \qquad (3)$$

3. Pour trouver la valeur de l'entier positif c_d, on applique le programme [LattE] (commande « count ») pour calculer $G(n)$ pour un entier n dépassant le seuil k et on retient l'entier le plus proche du nombre $F(n)/n^d$ ($F(n) = d!\,G(n)$). Il suffit ensuite d'utiliser l'égalité (2) pour déduire la valeur exacte du volume du polytope Q : $\mathrm{Vol}(Q) = c_d/(d!\,\delta^d)$ (4)

Pour obtenir les résultats de la sous-section 3.2, en utilisant la méthode M2, nous avons, à chaque fois, appliqué la formule (3) avec $n = 3\delta^d(d+1)!$ (ainsi, n est un multiple de δ, et donc un multiple de la période de Q , ce qui permet de faciliter le calcul de $G(n)$ par le programme [LattE]. Par souci de simplicité, nous ne donnons ici que les détails du calcul de la vulnérabilité limite de NPR à la manipulation coalitionnelle (c'est le cas le plus simple), les autres résultats s'obtiennent de la même manière.

Soit Q le polytope rationnel associé au système $S^{NPR}(n)$, i.e., Q est décrit par le système $S^{NPR}(1)$. Dans la sous-section 3.2, nous avons noté V_1^{NPR} le volume de Q, et nous avons montré que : $\Pr(Manip, NPR, \infty) = 23! \times 6V_1^{NPR}$ (E_3). Pour calculer V_1^{NPR}, nous avons procédé comme suit :

- On applique le programme [Lrs] pour trouver un multiple δ de la période de Q (Ici Q est de dimension $d = 23$). On trouve $\delta = 6$.
- On applique |LattE] pour calculer $G(n)$ pour $n = 3\delta^d(d+1)! = 3 \times 6^{23} \times (24!)$. On trouve une valeur de $G(n)$ qu'il n'est pas possible d'exhiber ici (elle comporte 947 chiffres, voir Annexe). Avec cette valeur, on a :

$$c_d = \left\lfloor \frac{F(n)}{n^d} \right\rceil = \left\lfloor \frac{d!\,G(3\delta^d(d+1)!)}{(3\delta^d(d+1)!)^d} \right\rceil = \left\lfloor \frac{23!\,G(3 \times 6^{23} \times (24!))}{(3 \times 6^{23} \times 24!)^{23}} \right\rceil$$

  On trouve : $c_d = 64656063906971648$
- En appliquant la formule (4), on a alors :

$$V_1^{NPR} = \frac{c_d}{d!\,\delta^{23}} = \frac{15415206887}{486758206977489144080647520256000 0}$$

- Enfin, en utilisant la formule (E_3), on trouve :

$$\Pr(Manip, NPR, \infty) = \frac{15415206887}{31381059609} \approx 49.12\%$$

Nous clôturons cette section par quelques remarques sur la méthode M2 et ses possibles généralisations :

- Soit Q un polytope rationnel et soit $H(n)$ le quasi-polynôme décrivant le nombre de points entiers dans la dilatation nQ. On montre facilement que les étapes de calcul du volume de Q par la méthode M2 peuvent être résumées par la formule générale suivante :

$$\mathrm{Vol}(Q) = \frac{\left\lceil \frac{d!\,H(2\,t\,\delta^{d+1}\,d!\,(d+1)!+\delta)}{(2\,t\,\delta^{d}\,d!\,(d+1)!+1)^{d}} \right\rfloor}{d!\,\delta^{d}}$$

  où δ est un multiple quelconque de la période de H et t un majorant quelconque du volume de Q.

- La méthode M2 peut être appliquée (de manière récursive) pour obtenir tous les coefficients de $H(n)$ lorsque n est un multiple de la période de H.
- La proposition 3, sur laquelle est basée la méthode M2, peut se généraliser aux cas des polynômes à plusieurs variables. Cette généralisation, sur laquelle nous continuons de travailler, peut avoir des applications intéressantes dans le calcul de la probabilité d'occurrence des évènements de vote dépendant d'un paramètre k, en plus du paramètre n qui représente le nombre de votants (dans beaucoup de problèmes considérés par Gehrlein et Lepelley (2017), k est une mesure de la cohérence mutuelle des préférences individuelles).

Conclusion

Dans ce papier, nous avons pu obtenir le premier résultat probabiliste sur la vulnérabilité (théorique) du vote par note à trois niveaux à la manipulation coalitionnelle. Bien que la valeur que nous avons trouvée confirme clairement le caractère « très » manipulable de ce mode de scrutin, il est important de préciser que ce résulat a été obtenu en supposant que les votants (non stratégiques) ne réagissent pas à la menace de manipulation et qu'il serait utile de le compléter par une analyse plus « réaliste » qui prend en compte ces réactions. Une autre piste de recherche qui pourrait conduire à une meilleure évaluation de la manipulabilité de EV, serait de calculer la fréquence des situation instables, sous cette règle, en fonction du degré d'homogénéité des préférences individuelles.

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[Normaliz] Normaliz by Bruns W, Ichim B, and Söger C, ver. 3.6.2 (2018), http://www.math matik.uni-osnabrueck.de/normaliz/

[lrs] lrs by Avis D, ver. 6.2 (2016), http://cgm.cs.mcgill.ca/~avis/C/lrs.html.

Annexe

V_1^{BR} =12660788040788908665157908971366342943135363036624540848008081116655967222137/381367191440050344960811019479091799093521094736028034751075

V_2^{BR} =86418621505721509890668331861221837345654001241260411329753/37978907183544221330316943827900660875791986574887582264714510336000000000000000000

V_5^{BR} =1266078804078890866515790897136634294313536303662454084800808111665596 7222137/98591110167697328971764129453425846988978217790852197902106573076962 79862235383528947712000000000000

V_6^{BR} =1238808125883667824886104916856789900835811755259754967984158937312143 278202960277134443434785268696840435464451461/392314186005816497106685386304

726157417949881993916768078995544013935065156009915675370414778246539228677
87857920000000000000000000000000000

V_8^{BR} =1386419294095133167352635217087839108418931200604804816089674670481152
52552839065900942732829070415790568970017947631518864360903/3261995940618986
505805126586924673259509462602916031429786950146813780725171556839988059367
734266809576937557987907224909905920000000000000000000000000

$\Pr(Manip, BR, \infty) =$
134440078869971253875350187983191390247575002676451784653207456958300024521
59856780019406505745005269940510690508465763511 | 205691530169077334731752169
468128854811656763077900969666506528978974197743703432750407876324408927020
751257600000000000000

$G(n) =$
223185923745860386699068040609425011527941786423677302338776481272728664514
631347271841274769353663115765214764626330249323540599631233873452568828361
503759053300248128791928057956438291479367405055103424537200260065899338720
738190367398696793831009542501569773069134737498626470406255179724292477267
914904208651705984107560956523248417484689438368843707183347464463854729072
428304931422909677525027853080026877492406600969873986825331409149088996305
070477414342515794567306796007519810551059931164258446256673041681859939342
275782464952484839099914606363588157934732645837447727749828678181815201056
053502587679786706686100511834898991547094066263368573906516277376880138794
595461508399818335322872021049618739107522941450667101634100821656755255091
403865526923463045936892176172860660860726734695180372418617463859034517064
352301541349110653616954947577408101441112009059211360809483582156596458264
00202911729560911966609398921453665513373696001

Conclusion générale

Lorsque nous avons commencé cette thèse, il était devenu aisé (techniquement) de calculer, sous l'hypothèse IAC, la fréquence théorique des événements de vote impliquant trois candidats et des votants exprimant des préférences linéaires (dans ce cas, il n'y a que 6 ordres de préférence possibles). En revanche, il était encore difficile d'obtenir des résultats probabilistes exacts pour des élections avec quatre candidats. La motivation principale de notre travail de recherche était alors de contribuer aux efforts qui visait à dépasser cette difficulté pour étendre l'analyse probabiliste des règles de vote aux cas où quatre candidats sont soumis à la décision collective. Nous souhaitions aussi appliquer cette approche probabiliste, dans le cas de trois candidats et avec des préférences individuelles trichotomiques, à un (nouveau) mode de scrutin, le Vote par note (ou par évaluation), qui suscitait un intérêt croissant parmi les théoriciens du vote, mais pour lequel il n'y avait aucune étude permettant d'associer un ordre de grandeur à ses défauts théoriques majeurs. Même s'ils sont de nature différente, ces deux objectifs nous ont conduit à proposer des solutions à un même et unique problème technique : calculer la probabilité limite d'un événement de vote lorsqu'il y a 24 préférences individuelles possibles. Les réponses que nous avons essayé d'apporter à cette question méthodologique constituent, avec les nombreux résultats décrivant la fréquence théorique de différents évènements électoraux, les deux principales contributions de notre travail de thèse à l'analyse probabiliste en théorie du vote.

L'ensemble des résultats numériques que nous avons obtenus est présenté dans la Table 1 (voir le récapitulatif des résultats de la thèse, à la fin de cette conclusion). L'examen de cette Table montre que l'essentiel de nos contributions concerne, d'une part, l'aptitude des règles de vote à satisfaire aux critères majoritaires de Condorcet et, d'autre part, la vulnérabilité de certaines de ces règles à la manipulation par des coalitions d'électeurs. Ces deux thèmes sont les plus courants dans l'étude probabiliste des règles de vote et nos résultats complètent les analyses existantes dans deux directions.

Nous avons en premier lieu étendu les résultats relatifs à l'efficacité de Condorcet et à la manipulabilité des règles de vote <u>aux élections à quatre candidats</u>. Nous montrons notamment que le passage de trois à quatre candidats s'accompagne d'une quasi-disparition du risque d'élire un Perdant de Condorcet Absolu, ou du risque de ne pas élire un Vainqueur de Condorcet Absolu. Cet avantage est cependant largement compensé par une augmentation très significative de la manipulabilité, au moins pour les deux règles de vote que nous avons étudiées (règle de la Pluralité et Vote majoritaire à deux tours).

Nous avons en second lieu entrepris des calculs de probabilité pour une règle de vote, le Vote par évaluation (2,1,0), qui n'avait pas été étudiée d'un point de vue probabiliste jusqu'ici. Nos résultats montrent que l'aptitude de cette règle à satisfaire aux conditions de Condorcet est plutôt bonne ; en revanche, et comme l'on pouvait s'y attendre, sa vulnérabilité aux manipulations stratégiques est élevée.

Sur le plan technique, à quelques rares exceptions, nos résultats ont été obtenus grâce à l'introduction de deux nouvelles méthodes de calcul (de volume de polytopes). Nous avons développé ces deux techniques de calcul dès le début de ce travail de recherche, à un moment où l'application directe des algorithmes les plus utilisés en théorie du vote, tels que *Barvinok* et *LattE*, ne permettaient pas d'obtenir des résultats probabilistes pour des événements de vote avec quatre candidats, et avant que le calcul des volumes de polytopes de dimension 24 ne devienne possible avec les versions améliorées des algorithmes *Normaliz* et *Convex*. En suggérant ces deux procédures de calcul, qui reposent sur un usage combiné des programmes *LattE* et *Lrs*, nous avons essayé de contribuer à enrichir l'ensemble des outils de calcul de probabilités sous la condition IAC et à proposer des pistes de recherche qui pourraient conduire à des méthodes applicables à des problèmes de vote avec cinq candidats (dans le cas des préférences linéaires).

Bien entendu, nos apports à l'approche probabiliste en théorie du vote, à travers cette thèse, ne peuvent être que limités et incomplets. Le premier prolongement de notre recherche serait de compléter la table 1 (du récapitulatif) en calculant les valeurs manquantes, notamment celles correspondant à la vulnérabilité coalitionnelle de la règle de l'Antipluralité itérative et de la règle de Borda itérative dans le cas d'élections à quatre candidats. Un autre prolongement possible serait de reprendre les calculs effectués (notamment pour l'efficacité de Condorcet) en tenant compte du degré de cohérence mutuelle des préférences individuelles et de comparer les résultats obtenus avec ceux déjà connus dans le cas de trois candidats. Enfin, il y a le problème, toujours ouvert, du passage du cas de quatre candidats à celui de cinq candidats (qui correspond à une transition de 24 à 120 variables), et aussi celui du passage, dans le contexte des préférences trichotomique, de trois à quatre candidats (on passe alors de 24 à 102 variables).

Récapitulatif des résultats de la thèse

- **Règles de vote étudiées** : Pluralité (PR), Antipluralité (NPR), Borda (BR), Pluralité itérative (PER), Antipluralité itérative (NPER), Borda itérative (BER), Vote par note à trois niveaux (EV), Vote par approbation (AV).
- **Probabilités calculées** : CW Eff (élection, lorsqu'il existe, du vainqueur de Condorcet), ACW Eff (élection, lorsqu'il existe, du vainqueur de Condorcet absolu), CL Eff (non-élection, lorsqu'il existe, du perdant de Condorcet), ACL Eff (non-élection, lorsqu'il existe, du perdant de Condorcet absolu), VM (vulnérabilité à la manipulation coalitionnelle).
- Dans la Table 1, ci-dessous, les probabilités sont exprimées en pourcentages, les valeurs soulignées concernent le cas de quatre candidats, en supposant que les préférences des votants sont linéaires. Les valeurs en gras sont obtenues dans le cas de trois candidats, pour des préférences individuelles trichotomiques.

	PR	NPR	BR	PER	NPER	BER	EV	AV
CW Eff	74.26 **84.86**	55.16 **78.20**	87.06 **93.39**	91.16	84.50	99.61	**86.51**	**91.53**
ACW Eff	100 **100**	62.30 **95.52**	99.83 **99.99**	100	90.84	99.99	**99.99**	**99.99**
CL Eff	97.73 **97.77**	97.62 **97.83**	100 **100**	100	100	100	**99.01**	**99.88**
ACL Eff	99.65 **99.96**	100 **100**	100 **100**	100	100	100	**99.99**	**99.99**
VM	87.28 **54.50**	**49.12**	**65.36**	38.63	-	-	**74.14**	-

Table 1. Performances majoritaires et vulnérabilité à la manipulation coalitionnelle

- **Deux autres résultats** : dans des élections avec quatre candidats,

- La probabilité que toutes les règles positionnelles simples (généralisation de PR, NPR et BR) choisissent le même vainqueur est de 26,22%,
- La probabilité que toutes les règles positionnelles simples élisent le vainqueur de Condorcet, lorsqu'un tel candidat existe, est de 24,64%.

Annexe A : Code - Trichotomous preferences

Condorcet winner

```
## Code : Condorcet winner (trichotomous preferences)
3 25

## Condorcet winner

## (A>B) : n1+n2+n5+n9+n10+n13+n14+n19+n22 > n3+n4+n6+n11+n12+n15+n16+n20+n21
## (A>C) : n1+n2+n3+n7+n8+n13+n14+n19+n24  > n4+n5+n6+n11+n12+n17+n18+n20+n23

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1  -1  -1   1  -1   0   0   1   1  -1  -1   1   1  -1  -1   0   0   1  -1  -1   1   0   0

0   1   1   1  -1  -1  -1   1   1   0   0  -1  -1   1   1   0   0  -1  -1   1  -1   0   0  -1   1

1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet winner – Borda

```
## Code : Condorcet winner - Borda score  (trichotomous preferences)

5 25
## Condorcet winner

## (A>B) : n1+n2+n5+n9+n10+n13+n14+n19+n22 > n3+n4+n6+n11+n12+n15+n16+n20+n21
## (A>C) : n1+n2+n3+n7+n8+n13+n14+n19+n24  > n4+n5+n6+n11+n12+n17+n18+n20+n23

## Borda score

## score(A) = 2(n1+n2+n13+n14+n19) + 3(n7+n8+n9+n10+n22+n24)/2 + (n3+n5) + (n15+n16+n17+n18+n21+n23)/2
## score(B) = 2(n3+n4+n15+n16+n21) + 3(n7+n8+n11+n12+n20+n24)/2 + (n1+n6) + (n13+n14+n17+n18+n19+n23)/2
## score(C) = 2(n5+n6+n17+n18+n23) + 3(n9+n10+n11+n12+n20+n22)/2 + (n2+n4) + n13+n14+n15+n16+n19+n21)/2

## score(A)>score(B) : 4n2 + 3(n9+n10+n13+n14+n19+n22) + 2(n1+n5) > 4n4 + 3(n11+n12+n15+n16+n20+n21) +2(n3+n6)
## score(A)>score(C) : 4n1 + 3(n7+n8+n13+n14+n19+n24) + 2(n2+n3)  > 4n6 + 3(n11+n12+n17+n18+n20+n23) + 2(n4+n5)


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1  -1  -1   1  -1   0   0   1   1  -1  -1   1   1  -1  -1   0   0   1  -1  -1   1   0   0

0   1   1   1  -1  -1  -1   1   1   0   0  -1  -1   1   1   0   0  -1  -1   1  -1   0   0  -1   1


0   2   4  -2  -4   2  -2   0   0   3   3  -3  -3   3   3  -3  -3   0   0   3  -3  -3   3   0   0

0   4   2   2  -2  -2  -4   3   3   0   0  -3  -3   3   3   0   0  -3  -3   3  -3   0   0  -3   3


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet winner – Plurality

```
## Code : Condorcet winner - Plurality  (trichotomous preferences) - V2

5 25
## Condorcet winner

## (A>B) : n1+n2+n5+n9+n10+n13+n14+n19+n22 > n3+n4+n6+n11+n12+n15+n16+n20+n21
## (A>C) : n1+n2+n3+n7+n8+n13+n14+n19+n24  > n4+n5+n6+n11+n12+n17+n18+n20+n23

## Plurality

## (A>B) : n1+n2+n13+n14+n19+(n9+n10+n22)/2 > n3+n4+n15+n16+n21+(n11+n12+n20)/2
## (A>C) : n1+n2+n13+n14+n19+(n7+n8+n24)/2  > n5+n6+n17+n18+n23+(n11+n12+n20)/2
## Donc :
## (A>B) : 2n1+2n2+2n13+2n14+2n19+n9+n10+n22 > 2n3+2n4+2n15+2n16+2n21+n11+n12+n20
## (A>C) : 2n1+2n2+2n13+2n14+2n19+n7+n8+n24  > 2n5+2n6+2n17+2n18+2n23+n11+n12+n20


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1  -1  -1   1  -1   0   0   1   1  -1  -1   1   1  -1  -1   0   0   1  -1  -1   1   0   0

0   1   1   1  -1  -1  -1   1   1   0   0  -1  -1   1   1   0   0  -1  -1   1  -1   0   0  -1   1


0   2   2  -2  -2   0   0   0   0   1   1  -1  -1   2   2  -2  -2   0   0   2  -1  -2   1   0   0

0   2   2   0   0  -2  -2   1   1   0   0  -1  -1   2   2   0   0  -2  -2   2  -1   0   0  -2   1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet winner – Negative plurality

```
## Code : Condorcet winner - Negative plurality  (trichotomous preferences)

5 25
## Condorcet winner

## (A>B) : n1+n2+n5+n9+n10+n13+n14+n19+n22 > n3+n4+n6+n11+n12+n15+n16+n20+n21
## (A>C) : n1+n2+n3+n7+n8+n13+n14+n19+n24  > n4+n5+n6+n11+n12+n17+n18+n20+n23

## Negative plurality

## A bat B : n4+n6+n11+n12+n20+(n15+n16+n21)/2 < n2+n5+n9+n10+n22+(n13+n14+n19)/2
## A bat C : n4+n6+n11+n12+n20+(n17+n18+n23)/2 < n1+n3+n7+n8+n24+(n13+n14+n19)/2
## donc
## (A<B) : 2n4+2n6+2n11+2n12+2n20+n15+n16+n21  < 2n2+2n5+2n9+2n10+2n22+n13+n14+n19
## (A<C) : 2n4+2n6+2n11+2n12+2n20+n17+n18+n23  < 2n1+2n3+2n7+2n8+2n24+n13+n14+n19


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1  -1  -1   1  -1   0   0   1   1  -1  -1   1   1  -1  -1   0   0   1  -1  -1   1   0   0

0   1   1   1  -1  -1  -1   1   1   0   0  -1  -1   1   1   0   0  -1  -1   1  -1   0   0  -1   1


0   0   2   0  -2   2  -2   0   0   2   2  -2  -2   1   1  -1  -1   0   0   1  -2  -1   2   0   0

0   2   0   2  -2   0  -2   2   2   0   0  -2  -2   1   1   0   0  -1  -1   1  -2   0   0  -1   2


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet winner – EV

```
## Code : Condorcet winner - Evaluative Voting  (trichotomous preferences)

5 25
## Condorcet winner

## (A>B) : n1+n2+n5+n9+n10+n13+n14+n19+n22 > n3+n4+n6+n11+n12+n15+n16+n20+n21
## (A>C) : n1+n2+n3+n7+n8+n13+n14+n19+n24  > n4+n5+n6+n11+n12+n17+n18+n20+n23

## Borda score

## score(A):=2*(n1+n2+n7+n8+n9+n10+n13+n14)+(n3+n5+n11+n15+n17+n19+n22+n24)
## score(B):=2*(n3+n4+n7+n8+n11+n12+n15+n16)+(n1+n6+n9+n13+n17+n20+n21+n24)
## score(C):=2*(n5+n6+n9+n10+n11+n12+n17+n18)+(n2+n4+n7+n13+n15+n20+n22+n23)

## score(A)>score(B) : n1+2*n2-n3-2*n4+n5-n6+n9+2*n10-n11-2*n12+n13+2*n14-n15-2*n16+n19-n20-n21+n22 > 0
## score(A)>score(C) : 2n1+n2+n3-n4-n5-2n6+n7+2n8-n11-2n12+n13+2n14-n17-2n18+n19-n20-n23+n24   > 0


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1  -1  -1   1  -1   0   0   1   1  -1  -1   1   1  -1  -1   0   0   1  -1  -1   1   0   0   #

0   1   1   1  -1  -1  -1   1   1   0   0  -1  -1   1   1   0   0  -1  -1   1  -1   0   0  -1   1   #


0   1   2  -1  -2   1  -1   0   0   1   2  -1  -2   1   2  -1  -2   0   0   1  -1  -1   1   0   0   #

0   2   1   1  -1  -1  -2   1   2   0   0  -1  -2   1   2   0   0  -1  -2   1  -1   0   0  -1   1   #


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet Loser

```
## Code : Condorcet loser - (trichotomous preferences)
3 25

## Condorcet loser

## (A<B) : n1+n2+n5+n9+n10+n13+n14+n19+n22 < n3+n4+n6+n11+n12+n15+n16+n20+n21
## (A<C) : n1+n2+n3+n7+n8+n13+n14+n19+n24  < n4+n5+n6+n11+n12+n17+n18+n20+n23

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0  -1  -1   1   1  -1   1   0   0  -1  -1   1   1  -1  -1   1   1   0   0  -1   1   1  -1   0   0

0  -1  -1  -1   1   1   1  -1  -1   0   0   1   1  -1  -1   0   0   1   1  -1   1   0   0   1  -1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet Loser – Plurality

```
## Code : Condorcet loser - Plurality   (trichotomous preferences)

5 25
## Condorcet loser

## (A<B) : n1+n2+n5+n9+n10+n13+n14+n19+n22 < n3+n4+n6+n11+n12+n15+n16+n20+n21
## (A<C) : n1+n2+n3+n7+n8+n13+n14+n19+n24  < n4+n5+n6+n11+n12+n17+n18+n20+n23

## Plurality

## (A>B) : n1+n2+n13+n14+n19+(n9+n10+n22)/2 > n3+n4+n15+n16+n21+(n11+n12+n20)/2
## (A>C) : n1+n2+n13+n14+n19+(n7+n8+n24)/2  > n5+n6+n17+n18+n23+(n11+n12+n20)/2
## Donc :
## (A>B) : 2n1+2n2+2n13+2n14+2n19+n9+n10+n22 > 2n3+2n4+2n15+2n16+2n21+n11+n12+n20
## (A>C) : 2n1+2n2+2n13+2n14+2n19+n7+n8+n24  > 2n5+2n6+2n17+2n18+2n23+n11+n12+n20

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0  -1  -1   1   1  -1   1   0   0  -1  -1   1   1  -1  -1   1   1   0   0  -1   1   1  -1   0   0

0  -1  -1  -1   1   1   1  -1  -1   0   0   1   1  -1  -1   0   0   1   1  -1   1   0   0   1  -1


0   2   2  -2  -2   0   0   0   0   1   1  -1  -1   2   2  -2  -2   0   0   2  -1  -2   1   0   0

0   2   2   0   0  -2  -2   1   1   0   0  -1  -1   2   2   0   0  -2  -2   2  -1   0   0  -2   1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet Loser – Negative plurality

```
## Code : Condorcet loser - Negative plurality   (trichotomous preferences)

5 25
## Condorcet loser

## (A<B) : n1+n2+n5+n9+n10+n13+n14+n19+n22 < n3+n4+n6+n11+n12+n15+n16+n20+n21
## (A<C) : n1+n2+n3+n7+n8+n13+n14+n19+n24  < n4+n5+n6+n11+n12+n17+n18+n20+n23

## Negative plurality

## A bat B : n4+n6+n11+n12+n20+(n15+n16+n21)/2 < n2+n5+n9+n10+n22+(n13+n14+n19)/2
## A bat C : n4+n6+n11+n12+n20+(n17+n18+n23)/2 < n1+n3+n7+n8+n24+(n13+n14+n19)/2
## donc
## (A<B) : 2n4+2n6+2n11+2n12+2n20+n15+n16+n21  < 2n2+2n5+2n9+2n10+2n22+n13+n14+n19
## (A<C) : 2n4+2n6+2n11+2n12+2n20+n17+n18+n23  < 2n1+2n3+2n7+2n8+2n24+n13+n14+n19

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0  -1  -1   1   1  -1   1   0   0  -1  -1   1   1  -1  -1   1   1   0   0  -1   1   1  -1   0   0

0  -1  -1  -1   1   1   1  -1  -1   0   0   1   1  -1  -1   0   0   1   1  -1   1   0   0   1  -1


0   0   2   0  -2   2  -2   0   0   2   2  -2  -2   1   1  -1  -1   0   0   1  -2  -1   2   0   0

0   2   0   2  -2   0  -2   2   2   0   0  -2  -2   1   1   0   0  -1  -1   1  -2   0   0  -1   2


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet Loser – Borda

```
## Code : Condorcet loser - Borda score  (trichotomous preferences)

5 25
## Condorcet loser

## (A<B) : n1+n2+n5+n9+n10+n13+n14+n19+n22 < n3+n4+n6+n11+n12+n15+n16+n20+n21
## (A<C) : n1+n2+n3+n7+n8+n13+n14+n19+n24  < n4+n5+n6+n11+n12+n17+n18+n20+n23

## Borda score

## score(A):=2*(n1+n2+n7+n8+n9+n10+n13+n14)+(n3+n5+n11+n15+n17+n19+n22+n24)
## score(B):=2*(n3+n4+n7+n8+n11+n12+n15+n16)+(n1+n6+n9+n13+n17+n20+n21+n24)
## score(C):=2*(n5+n6+n9+n10+n11+n12+n17+n18)+(n2+n4+n7+n13+n15+n20+n22+n23)

## score(A)>score(B) : n1+2*n2-n3-2*n4+n5-n6+n9+2*n10-n11-2*n12+n13+2*n14-n15-2*n16+n19-n20-n21+n22 > 0
## score(A)>score(C) : n1-n2+2*n3+n4-2*n5-n6+n7+2*n8-n9-2*n10+n15+2*n16-n17-2*n18+n21-n22-n23+n24   > 0


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0  -1  -1   1   1  -1   1   0   0  -1  -1   1   1  -1  -1   1   1   0   0  -1   1   1  -1   0   0

0  -1  -1  -1   1   1   1  -1  -1   0   0   1   1  -1  -1   0   0   1   1  -1   1   0   0   1  -1


0   1   2  -1  -2   1  -1   0   0   1   2  -1  -2   1   2  -1  -2   0   0   1  -1  -1   1   0   0

0   2   1   1  -1  -1  -2   1   2   0   0  -1  -2   1   2   0   0  -1  -2   1  -1   0   0  -1   1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet winner absolu

```
## Code : Condorcet winner (trichotomous preferences) - V2
2 25

## Condorcet winner

## Condorcet winner Absolu

## (A> n/2) : n1+n2+n13+n14+n19 > n4+n5+n6+n7+n8+n9+n10+n11+n12+n15+n16+n17+n18+n20+n21+n22+n23+n24


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1   1   1  -1  -1  -1  -1   1  -1  -1  -1  -1  -1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet winner absolu – Plurality

```
## Code : Condorcet winner absolu - Plurality  (trichotomous preferences) - V2

4 25
## Condorcet winner Absolu

## (A> n/2) : n1+n2+n13+n14+n19 > n4+n5+n6+n7+n8+n9+n10+n11+n12+n15+n16+n17+n18+n20+n21+n22+n23+n24

## Plurality

## (A>B) : n1+n2+n13+n14+n19+(n9+n10+n22)/2 > n3+n4+n15+n16+n21+(n11+n12+n20)/2
## (A>C) : n1+n2+n13+n14+n19+(n7+n8+n24)/2  > n5+n6+n17+n18+n23+(n11+n12+n20)/2
## Donc :
## (A>B) : 2n1+2n2+2n13+2n14+2n19+n9+n10+n22 > 2n3+2n4+2n15+2n16+2n21+n11+n12+n20
## (A>C) : 2n1+2n2+2n13+2n14+2n19+n7+n8+n24  > 2n5+2n6+2n17+2n18+2n23+n11+n12+n20


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1   1   1  -1  -1  -1  -1   1  -1  -1  -1  -1  -1


0   2   2  -2  -2   0   0   0   0   1   1  -1  -1   2   2  -2  -2   0   0   2  -1  -2   1   0   0

0   2   2   0   0  -2  -2   1   1   0   0  -1  -1   2   2   0   0  -2  -2   2  -1   0   0  -2   1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet winner absolu – Negative plurality

```
## Code : Condorcet winner absolu - Negative plurality  (trichotomous preferences)

4 25
## Condorcet winner Absolu

## (A> n/2) : n1+n2+n13+n14+n19 > n4+n5+n6+n7+n8+n9+n10+n11+n12+n15+n16+n17+n18+n20+n21+n22+n23+n24


## Negative plurality

## A bat B : n4+n6+n11+n12+n20+(n15+n16+n21)/2 < n2+n5+n9+n10+n22+(n13+n14+n19)/2
## A bat C : n4+n6+n11+n12+n20+(n17+n18+n23)/2 < n1+n3+n7+n8+n24+(n13+n14+n19)/2
## donc
## (A<B) : 2n4+2n6+2n11+2n12+2n20+n15+n16+n21  < 2n2+2n5+2n9+2n10+2n22+n13+n14+n19
## (A<C) : 2n4+2n6+2n11+2n12+2n20+n17+n18+n23  < 2n1+2n3+2n7+2n8+2n24+n13+n14+n19


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1   1   1  -1  -1  -1  -1   1  -1  -1  -1  -1  -1


0   0   2   0  -2   2  -2   0   0   2   2  -2  -2   1   1  -1  -1   0   0   1  -2  -1   2   0   0

0   2   0   2  -2   0  -2   2   2   0   0  -2  -2   1   1   0   0  -1  -1   1  -2   0   0  -1   2


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet winner absolu – Borda

```
## Code : Condorcet winner Absolu - Borda score  (trichotomous preferences)

4 25
## Condorcet winner Absolu

## (A> n/2) : n1+n2+n13+n14+n19 > n4+n5+n6+n7+n8+n9+n10+n11+n12+n15+n16+n17+n18+n20+n21+n22+n23+n24


## Borda score

## score(A) = 2(n1+n2+n13+n14+n19) + 3(n7+n8+n9+n10+n22+n24)/2 + (n3+n5) + (n15+n16+n17+n18+n21+n23)/2
## score(B) = 2(n3+n4+n15+n16+n21) + 3(n7+n8+n11+n12+n20+n24)/2 + (n1+n6) + (n13+n14+n17+n18+n19+n23)/2
## score(C) = 2(n5+n6+n17+n18+n23) + 3(n9+n10+n11+n12+n20+n22)/2 + (n2+n4) + n13+n14+n15+n16+n19+n21)/2

## score(A)>score(B) : 4n2 + 3(n9+n10+n13+n14+n19+n22) + 2(n1+n5) > 4n4 + 3(n11+n12+n15+n16+n20+n21) +2(n3+n6)
## score(A)>score(C) : 4n1 + 3(n7+n8+n13+n14+n19+n24) + 2(n2+n3)  > 4n6 + 3(n11+n12+n17+n18+n20+n23) + 2(n4+n5)


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1   1   1  -1  -1  -1  -1   1  -1  -1  -1  -1  -1


0   2   4  -2  -4   2  -2   0   0   3   3  -3  -3   3   3  -3  -3   0   0   3  -3  -3   3   0   0

0   4   2   2  -2  -2  -4   3   3   0   0  -3  -3   3   3   0   0  -3  -3   3  -3   0   0  -3   3


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet winner absolu – EV

```
## Code : Condorcet winner - Evaluative Voting  (trichotomous preferences)

4 25
## Condorcet winner Absolu

## (A> n/2) : n1+n2+n13+n14+n19 > n4+n5+n6+n7+n8+n9+n10+n11+n12+n15+n16+n17+n18+n20+n21+n22+n23+n24

## Borda score

## score(A):=2*(n1+n2+n7+n8+n9+n10+n13+n14)+(n3+n5+n11+n15+n17+n19+n22+n24)
## score(B):=2*(n3+n4+n7+n8+n11+n12+n15+n16)+(n1+n6+n9+n13+n17+n20+n21+n24)
## score(C):=2*(n5+n6+n9+n10+n11+n12+n17+n18)+(n2+n4+n7+n13+n15+n20+n22+n23)

## score(A)>score(B) : n1+2*n2-n3-2*n4+n5-n6+n9+2*n10-n11-2*n12+n13+2*n14-n15-2*n16+n19-n20-n21+n22 > 0
## score(A)>score(C) : 2n1+n2+n3-n4-n5-2n6+n7+2n8-n11-2n12+n13+2n14-n17-2n18+n19-n20-n23+n24   > 0


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1   1   1  -1  -1  -1  -1   1  -1  -1  -1  -1  -1   #


0   1   2  -1  -2   1  -1   0   0   1   2  -1  -2   1   2  -1  -2   0   0   1  -1  -1   1   0   0   #

0   2   1   1  -1  -1  -2   1   2   0   0  -1  -2   1   2   0   0  -1  -2   1  -1   0   0  -1   1   #


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Annexe B : Code - Probabilities of electoral outcomes : From three candidate to four candidate elections

Condorcet winner

```
## Code : Condorcet winner
4 25

## Condorcet winner
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   1   1   1   1  -1  -1  -1  -1  -1  -1   1   1  -1  -1   1  -1   1   1  -1  -1   1  -1

0   1   1   1   1   1   1   1   1  -1  -1   1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1

0   1   1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet winner – Borda

```
## Code : Condorcet winner - Borda score
7 25

## Condorcet winner
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   1   1   1   1  -1  -1  -1  -1  -1  -1   1   1  -1  -1   1  -1   1   1  -1  -1   1  -1

0   1   1   1   1   1   1   1   1  -1  -1   1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1

0   1   1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1


## Borda score
## score(A):=3*(n1+n2+n3+n4+n5+n6)+2*(n7+n8+n13+n14+n19+n20)+(n9+n11+n15+n17+n21+n23)
## score(B):=3*(n7+n8+n9+n10+n11+n12)+2*(n1+n2+n15+n16+n21+n22)+(n3+n5+n13+n18+n19+n24)
## score(C):=3.(n13+n14+n15+n16+n17+n18)+2*(n3+n4+n9+n10+n23+n24)+(n1+n6+n7+n12+n20+n22)
## score(D):=3*(n19+n20+n21+n22+n23+n24)+2*(n5+n6+n11+n12+n17+n18)+(n2+n4+n8+n10+n14+n16)

## score(A)>score(B) : n1+n2+2*n3+3*n4+2*n5+3*n6-n7-n8-2*n9-3*n10-2*n11-3*n12+n13+2*n14-n15-2*n16
+n17-n18+n19+2*n20-n21-2*n22+n23-n24 > 0
## score(A)>score(C) : 2*n1+3*n2+n3+n4+3*n5+2*n6+n7+2*n8-n9-2*n10+n11-n12-n13-n14-2*n15-3*n16-2
*n17-3*n18+2*n19+n20+n21-n22-n23-2*n24 > 0
## score(A)>score(D) : 3*n1+2*n2+3*n3+2*n4+n5+n6+2*n7+n8+n9-n10-n11-2*n12+2*n13+n14+n15-n16-n17
-2*n18-n19-n20-2*n21-3*n22-2*n23-3*n24 > 0

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   2   3   2   3  -1  -1  -2  -3  -2  -3   1   2  -1  -2   1  -1   1   2  -1  -2   1  -1

0   2   3   1   1   3   2   1   2  -1  -2   1  -1  -1  -1  -2  -3  -2  -3   2   1   1  -1  -1  -2

0   3   2   3   2   1   1   2   1   1  -1  -1  -2   2   1   1  -1  -1  -2  -1  -1  -2  -3  -2  -3


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet winner – Negative Plurality

```
## Code : Condorcet winner - Negative plurality
7 25

## Condorcet winner
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   1   1   1   1  -1  -1  -1  -1  -1  -1   1   1  -1  -1   1  -1   1   1  -1  -1   1  -1

0   1   1   1   1   1   1   1   1  -1  -1   1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1

0   1   1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1

## Negative plurality

## (A<B) : n10+n12+n16+n18+n22+n24 < n4+n6+n14+n17+n20+n23
## (A<C) : n10+n12+n16+n18+n22+n24 < n2+n5+n8+n11+n19+n21
## (A<D) : n10+n12+n16+n18+n22+n24 < n1+n3+n7+n9+n13+n15

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   0   0   0   1   0   1   0   0   0  -1   0  -1   0   1   0  -1   1  -1   0   1   0  -1   1  -1

0   0   1   0   0   1   0   0   1   0  -1   1  -1   0   0   0  -1   0  -1   1   0   1  -1   0  -1

0   1   0   1   0   0   0   1   0   1  -1   0  -1   1   0   1  -1   0  -1   0   0   0  -1   0  -1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet loser

```
## Code : Condorcet loser
4 25

## Condorcet winner
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0  -1  -1  -1  -1  -1  -1   1   1   1   1   1   1  -1  -1   1   1  -1   1  -1  -1   1   1  -1   1

0  -1  -1  -1  -1  -1  -1  -1  -1   1   1  -1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1

0  -1  -1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1   1   1   1   1   1   1   1   1   1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet loser – Negative plurality

```
## Code : Condorcet loser - Negative plurality
7 25

## Condorcet loser
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0  -1  -1  -1  -1  -1  -1   1   1   1   1   1   1  -1  -1   1   1  -1   1  -1  -1   1   1  -1   1

0  -1  -1  -1  -1  -1  -1  -1  -1   1   1  -1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1

0  -1  -1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1   1   1   1   1   1   1   1   1   1

## Negative plurality

## (A<B) : n10+n12+n16+n18+n22+n24 < n4+n6+n14+n17+n20+n23
## (A<C) : n10+n12+n16+n18+n22+n24 < n2+n5+n8+n11+n19+n21
## (A<D) : n10+n12+n16+n18+n22+n24 < n1+n3+n7+n9+n13+n15

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   0   0   0   1   0   1   0   0   0  -1   0  -1   0   1   0  -1   1  -1   0   1   0  -1   1  -1

0   0   1   0   0   1   0   0   1   0  -1   1  -1   0   0   0  -1   0  -1   1   0   1  -1   0  -1

0   1   0   1   0   0   0   1   0   1  -1   0  -1   1   0   1  -1   0  -1   0   0   0  -1   0  -1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet loser – Plurality

```
## Code : Condorcet loser - plurality
7 25

## Condorcet loser
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0  -1  -1  -1  -1  -1  -1   1   1   1   1   1   1  -1  -1   1   1  -1   1  -1  -1   1   1  -1   1

0  -1  -1  -1  -1  -1  -1  -1  -1   1   1  -1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1

0  -1  -1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1   1   1   1   1   1   1   1   1   1

## Plurality

## (A>B) :  n1+n2+n3+n4+n5+n6 > n7+n8+n9+n10+n11+n12
## (A>C) :  n1+n2+n3+n4+n5+n6 > n13+n14+n15+n16+n17+n18
## (A>D) :  n1+n2+n3+n4+n5+n6 > n19+n20+n21+n22+n23+n24


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   1   1   1   1  -1  -1  -1  -1  -1  -1   0   0   0   0   0   0   0   0   0   0   0   0

0   1   1   1   1   1   1   0   0   0   0   0   0  -1  -1  -1  -1  -1  -1   0   0   0   0   0   0

0   1   1   1   1   1   1   0   0   0   0   0   0   0   0   0   0   0   0  -1  -1  -1  -1  -1  -1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Condorcet loser – Borda score

```
## Code : Condorcet loser - Borda score
7 25

## Condorcet loser
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0  -1  -1  -1  -1  -1  -1   1   1   1   1   1   1  -1  -1   1   1  -1   1  -1  -1   1   1  -1   1

0  -1  -1  -1  -1  -1  -1  -1  -1   1   1  -1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1

0  -1  -1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1   1   1   1   1   1   1   1   1   1


## Borda score
## score(A):=3*(n1+n2+n3+n4+n5+n6)+2*(n7+n8+n13+n14+n19+n20)+(n9+n11+n15+n17+n21+n23)
## score(B):=3*(n7+n8+n9+n10+n11+n12)+2*(n1+n2+n15+n16+n21+n22)+(n3+n5+n13+n18+n19+n24)
## score(C):=3.(n13+n14+n15+n16+n17+n18)+2*(n3+n4+n9+n10+n23+n24)+(n1+n6+n7+n12+n20+n22)
## score(D):=3*(n19+n20+n21+n22+n23+n24)+2*(n5+n6+n11+n12+n17+n18)+(n2+n4+n8+n10+n14+n16)

## score(A)>score(B) : n1+n2+2*n3+3*n4+2*n5+3*n6-n7-n8-2*n9-3*n10-2*n11-3*n12+n13+2*n14-n15-2
*n16+n17-n18+n19+2*n20-n21-2*n22+n23-n24 > 0
## score(A)>score(C) : 2*n1+3*n2+n3+n4+3*n5+2*n6+n7+2*n8-n9-2*n10+n11-n12-n13-n14-2*n15-3*n16
-2*n17-3*n18+2*n19+n20+n21-n22-n23-2*n24 > 0
## score(A)>score(D) : 3*n1+2*n2+3*n3+2*n4+n5+n6+2*n7+n8+n9-n10-n11-2*n12+2*n13+n14+n15-n16
-n17-2*n18-n19-n20-2*n21-3*n22-2*n23-3*n24 > 0

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   2   3   2   3  -1  -1  -2  -3  -2  -3   1   2  -1  -2   1  -1   1   2  -1  -2   1  -1

0   2   3   1   1   3   2   1   2  -1  -2   1  -1  -1  -1  -2  -3  -2  -3   2   1   1  -1  -1  -2

0   3   2   3   2   1   1   2   1   1  -1  -1  -2   2   1   1  -1  -1  -2  -1  -1  -2  -3  -2  -3


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

PER

```
## Code : I - PER - Plurality Elimination Rule (A CW & A PRW)
7 25

## Condorcet winner
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   1   1   1   1  -1  -1  -1  -1  -1  -1   1   1  -1  -1   1  -1   1   1  -1  -1   1  -1

0   1   1   1   1   1   1   1   1  -1  -1   1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1

0   1   1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1

## Plurality

## (A>B) :  n1+n2+n3+n4+n5+n6 > n7+n8+n9+n10+n11+n12
## (A>C) :  n1+n2+n3+n4+n5+n6 > n13+n14+n15+n16+n17+n18
## (A>D) :  n1+n2+n3+n4+n5+n6 > n19+n20+n21+n22+n23+n24


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   1   1   1   1  -1  -1  -1  -1  -1  -1   0   0   0   0   0   0   0   0   0   0   0   0

0   1   1   1   1   1   1   0   0   0   0   0   0  -1  -1  -1  -1  -1  -1   0   0   0   0   0   0

0   1   1   1   1   1   1   0   0   0   0   0   0   0   0   0   0   0   0  -1  -1  -1  -1  -1  -1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
## Code : II - PER - Plurality Elimination Rule (A CW & B>A, A>C, A>D PRW)
7 25

## Condorcet winner
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   1   1   1   1  -1  -1  -1  -1  -1  -1   1   1  -1  -1   1  -1   1   1  -1  -1   1  -1

0   1   1   1   1   1   1   1   1  -1  -1   1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1

0   1   1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1

## Plurality

## (A>B) :  n7+n8+n9+n10+n11+n12 > n1+n2+n3+n4+n5+n6
## (A>C) :  n1+n2+n3+n4+n5+n6 > n13+n14+n15+n16+n17+n18
## (A>D) :  n1+n2+n3+n4+n5+n6 > n19+n20+n21+n22+n23+n24


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0  -1  -1  -1  -1  -1  -1   1   1   1   1   1   1   0   0   0   0   0   0   0   0   0   0   0   0

0   1   1   1   1   1   1   0   0   0   0   0   0  -1  -1  -1  -1  -1  -1   0   0   0   0   0   0

0   1   1   1   1   1   1   0   0   0   0   0   0   0   0   0   0   0   0  -1  -1  -1  -1  -1  -1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

NPER

```
## Code : I - NPER - Negative Plurality Elimination Rule (A CW & A NPRW)
7 25

## Condorcet winner
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   1   1   1   1  -1  -1  -1  -1  -1  -1   1   1  -1  -1   1  -1   1   1  -1  -1   1  -1

0   1   1   1   1   1   1   1   1  -1  -1   1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1

0   1   1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1

## Negative plurality

## (A<B) : n10+n12+n16+n18+n22+n24 < n4+n6+n14+n17+n20+n23
## (A<C) : n10+n12+n16+n18+n22+n24 < n2+n5+n8+n11+n19+n21
## (A<D) : n10+n12+n16+n18+n22+n24 < n1+n3+n7+n9+n13+n15

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   0   0   0   1   0   1   0   0   0  -1   0  -1   0   1   0  -1   1  -1   0   1   0  -1   1  -1

0   0   1   0   0   1   0   0   1   0  -1   1  -1   0   0   0  -1   0  -1   1   0   1  -1   0  -1

0   1   0   1   0   0   0   1   0   1  -1   0  -1   1   0   1  -1   0  -1   0   0   0  -1   0  -1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

```
## Code : II - NPER - Negative Plurality Elimination Rule (A CW & B>A, A>C, A>D NPRW)
7 25

## Condorcet winner
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   1   1   1   1  -1  -1  -1  -1  -1  -1   1   1  -1  -1   1  -1   1   1  -1  -1   1  -1

0   1   1   1   1   1   1   1   1  -1  -1   1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1

0   1   1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1

## Negative plurality

## (B<A) : n4+n6+n14+n17+n20+n23 < n10+n12+n16+n18+n22+n24
## (A<C) : n10+n12+n16+n18+n22+n24 < n2+n5+n8+n11+n19+n21
## (A<D) : n10+n12+n16+n18+n22+n24 < n1+n3+n7+n9+n13+n15

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   0   0   0  -1   0  -1   0   0   0   1   0   1   0  -1   0   1  -1   1   0  -1   0   1  -1   1

0   0   1   0   0   1   0   0   1   0  -1   1  -1   0   0   0  -1   0  -1   1   0   1  -1   0  -1

0   1   0   1   0   0   0   1   0   1  -1   0  -1   1   0   1  -1   0  -1   0   0   0  -1   0  -1


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

BER

```
## Code : I - Condorcet winner - Borda score (A CW & A Borda Winner)
7 25

## Condorcet winner
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   1   1   1   1  -1  -1  -1  -1  -1  -1   1   1  -1  -1   1  -1   1   1  -1  -1   1  -1

0   1   1   1   1   1   1   1   1  -1  -1   1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1

0   1   1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1


## Borda score
## score(A):=3*(n1+n2+n3+n4+n5+n6)+2*(n7+n8+n13+n14+n19+n20)+(n9+n11+n15+n17+n21+n23)
## score(B):=3*(n7+n8+n9+n10+n11+n12)+2*(n1+n2+n15+n16+n21+n22)+(n3+n5+n13+n18+n19+n24)
## score(C):=3*(n13+n14+n15+n16+n17+n18)+2*(n3+n4+n9+n10+n23+n24)+(n1+n6+n7+n12+n20+n22)
## score(D):=3*(n19+n20+n21+n22+n23+n24)+2*(n5+n6+n11+n12+n17+n18)+(n2+n4+n8+n10+n14+n16)

## score(A)>score(B) : n1+n2+2*n3+3*n4+2*n5+3*n6-n7-n8-2*n9-3*n10-2*n11-3*n12+n13+2*n14-n15-2*n16
+n17-n18+n19+2*n20-n21-2*n22+n23-n24 > 0
## score(A)>score(C) : 2*n1+3*n2+n3+n4+3*n5+2*n6+n7+2*n8-n9-2*n10+n11-n12-n13-n14-2*n15-3*n16-
2*n17-3*n18+2*n19+n20+n21-n22-n23-2*n24 > 0
## score(A)>score(D) : 3*n1+2*n2+3*n3+2*n4+n5+n6+2*n7+n8+n9-n10-n11-2*n12+2*n13+n14+n15-n16-n17
-2*n18-n19-n20-2*n21-3*n22-2*n23-3*n24 > 0

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   2   3   2   3  -1  -1  -2  -3  -2  -3   1   2  -1  -2   1  -1   1   2  -1  -2   1  -1

0   2   3   1   1   3   2   1   2  -1  -2   1  -1  -1  -1  -2  -3  -2  -3   2   1   1  -1  -1  -2

0   3   2   3   2   1   1   2   1   1  -1  -1  -2   2   1   1  -1  -1  -2  -1  -1  -2  -3  -2  -3


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1

## Code : I - Condorcet winner - Borda score (A CW & A Borda Winner)
7 25

## Condorcet winner
## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0   1   1   1   1   1   1  -1  -1  -1  -1  -1  -1   1   1  -1  -1   1  -1   1   1  -1  -1   1  -1

0   1   1   1   1   1   1   1   1  -1  -1   1  -1  -1  -1  -1  -1  -1  -1   1   1   1  -1  -1  -1

0   1   1   1   1   1   1   1   1   1  -1  -1  -1   1   1   1  -1  -1  -1  -1  -1  -1  -1  -1  -1


## Borda score
## score(A):=3*(n1+n2+n3+n4+n5+n6)+2*(n7+n8+n13+n14+n19+n20)+(n9+n11+n15+n17+n21+n23)
## score(B):=3*(n7+n8+n9+n10+n11+n12)+2*(n1+n2+n15+n16+n21+n22)+(n3+n5+n13+n18+n19+n24)
## score(C):=3.(n13+n14+n15+n16+n17+n18)+2*(n3+n4+n9+n10+n23+n24)+(n1+n6+n7+n12+n20+n22)
## score(D):=3*(n19+n20+n21+n22+n23+n24)+2*(n5+n6+n11+n12+n17+n18)+(n2+n4+n8+n10+n14+n16)

## score(A)<score(B) : n1+n2+2*n3+3*n4+2*n5+3*n6-n7-n8-2*n9-3*n10-2*n11-3*n12+n13+2*n14-n15-2*n16
+n17-n18+n19+2*n20-n21-2*n22+n23-n24 < 0
## score(A)>score(C) : 2*n1+3*n2+n3+n4+3*n5+2*n6+n7+2*n8-n9-2*n10+n11-n12-n13-n14-2*n15-3*n16
-2*n17-3*n18+2*n19+n20+n21-n22-n23-2*n24 > 0
## score(A)>score(D) : 3*n1+2*n2+3*n3+2*n4+n5+n6+2*n7+n8+n9-n10-n11-2*n12+2*n13+n14+n15-n16-n17
-2*n18-n19-n20-2*n21-3*n22-2*n23-3*n24 > 0

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

0  -1  -1  -2  -3  -2  -3   1   1   2   3   2   3  -1  -2   1   2  -1   1  -1  -2   1   2  -1   1

0   2   3   1   1   3   2   1   2  -1  -2   1  -1  -1  -1  -2  -3  -2  -3   2   1   1  -1  -1  -2

0   3   2   3   2   1   1   2   1   1  -1  -1  -2   2   1   1  -1  -1  -2  -1  -1  -2  -3  -2  -3


1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1  -1
```

Annexe C : Code - Manipulabilité coalitionnelle du vote par note à trois niveaux

Manipulabilité – Borda

```
## Borda S1BR : volume V1

## Score

SBR(A,x) = 2(n1 + n2 + n13 + n14 + n19) + 3/2 (n7 + n8 + n9 + n10 + n22 + n24) +(n3 + n5) + 1/2 (n15 + n16 + n17 + n18 + n21 + n23)
SBR(B,x) = 2(n3 + n4 + n15 + n16 + n21) + 3/2 (n7 + n8 + n11 + n12 + n20 + n24) +(n1 + n6) + 1/2 (n13 + n14 + n17 + n18 + n19 + n23)
SBR(C,x) = 2(n5 + n6 + n17 + n18 + n23) + 3/2 (n9 + n10 + n11 + n12 + n20 + n22) +(n2 + n4) + 1/2 (n13 + n14 + n15 + n16 + n19 + n21)

s1 =  n6+(1/2)(n11+n12+n20)

## score en maple code :
          n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

SBR_A := [2,  2,  1,  0,  1,  0,3/2,3/2,3/2,3/2,  0,  0,  2,  2,1/2,1/2,1/2,1/2,  2,  0,1/2,3/2,1/2,3/2];
SBR_B := [1,  0,  2,  2,  0,  1,3/2,3/2,  0,  0,3/2,3/2,1/2,1/2,  2,  2,1/2,1/2,1/2,3/2,  2,  0,1/2,3/2];
SBR_C := [0,  1,  0,  1,  2,  2,  0,  0,3/2,3/2,3/2,3/2,1/2,1/2,1/2,1/2,  2,  2,1/2,3/2,1/2,3/2,  2,  0];

s1       := [0,  0,  0,  0,  0,  1,  0,  0,  0,  0,1/2,1/2,  0,  0,  0,  0,  0,  0,  0,1/2,  0,  0,  0,  0];

donc

DBR_AB := SBR_A-SBR_B := [1,  2, -1, -2,  1, -1,  0,  0,3/2,3/2,-3/2,-3/2,3/2,3/2,-3/2,-3/2,0,0,3/2,-3/2,-3/2,3/2,0,  0];

## Systeme :
On a :

DNPR_AB > 0
DNPR_AC > 0
s1-DBR_AB > 0

il est equivalent à :

2*DNPR_AB > 0
2*DNPR_AC > 0
2*s1-2*DBR_AB > 0

donc
                      n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

2*DNPR_AB          := [0,  2,  0, -2,  2, -2,  0,  0,  2,  2, -2, -2,  1,  1, -1, -1,  0,  0,  1, -2, -1,  2,  0,  0]
2*DNPR_AC          := [2,  0,  2, -2,  0, -2,  2,  2,  0,  0, -2, -2,  1,  1,  0,  0, -1, -1,  1, -2,  0,  0, -1,  2]
2*s1-2*DBR_AB      :=[-2, -4,  2,  4, -2,  4,  0,  0, -3, -3,  4,  4, -3, -3,  3,  3,  0,  0, -3,  4,  3, -3,  0,  0]


##                    n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24


                       0   2   0  -2   2  -2   0   0   2   2  -2  -2   1   1  -1  -1   0   0   1  -2  -1   2   0   0
                       2   0   2  -2   0  -2   2   2   0   0  -2  -2   1   1   0   0  -1  -1   1  -2   0   0  -1   2
                      -2  -4   2   4  -2   4   0   0  -3  -3   4   4  -3  -3   3   3   0   0  -3   4   3  -3   0   0

############### Code sous Normaliz :

amb_space 24
inequalities 3

     0   2   0  -2   2  -2   0   0   2   2  -2  -2   1   1  -1  -1   0   0   1  -2  -1   2   0   0
     2   0   2  -2   0  -2   2   2   0   0  -2  -2   1   1   0   0  -1  -1   1  -2   0   0  -1   2
    -2  -4   2   4  -2   4   0   0  -3  -3   4   4  -3  -3   3   3   0   0  -3   4   3  -3   0   0

nonnegative
total_degree
Multiplicity

Mult 156514175989879855880607379079108344670579353/19475394160008297348420342891492160800000000000
Mult (float) 0.0803650877122
```

```
## Borda S2BR : volume V2

## Score

SBR(A,x) = 2(n1 + n2 + n13 + n14 + n19) + 3/2 (n7 + n8 + n9 + n10 + n22 + n24) +(n3 + n5) + 1/2 (n15 + n16 + n17 + n18 + n21 + n23)
SBR(B,x) = 2(n3 + n4 + n15 + n16 + n21) + 3/2 (n7 + n8 + n11 + n12 + n20 + n24) +(n1 + n6) + 1/2 (n13 + n14 + n17 + n18 + n19 + n23)
SBR(C,x) = 2(n5 + n6 + n17 + n18 + n23) + 3/2 (n9 + n10 + n11 + n12 + n20 + n22) +(n2 + n4) + 1/2 (n13 + n14 + n15 + n16 + n19 + n21)

ss1 = n6 + 1/2 (n11 + n12 + n20)
ss2 = n3 + 1/2 (n15 + n16 + n21) + n6 + 1/2 (n11 + n12 + n20)
ss3 = 3 (n6 + 1/2 (n11 + n12 + n20))

## score en maple code :
          n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

SBR_A := [2,  2,  1,  0,  1,  0,3/2,3/2,3/2,3/2,  0,  0,  2,  2,1/2,1/2,1/2,1/2,  2,  0,1/2,3/2,1/2,3/2];
SBR_B := [1,  0,  2,  2,  0,  1,3/2,3/2,  0,  0,3/2,3/2,1/2,1/2,  2,  2,1/2,1/2,1/2,3/2,  2,  0,1/2,3/2];
SBR_C := [0,  1,  0,  1,  2,  2,  0,  0,3/2,3/2,3/2,3/2,1/2,1/2,1/2,1/2,  2,  2,1/2,3/2,1/2,3/2,  2,  0];

ss1   := [0,  0,  0,  0,  0,  1,  0,  0,  0,  0,1/2,1/2,  0,  0,  0,  0,  0,  0,  0,1/2,  0,  0,  0,  0];
ss2   := [0,  0,  1,  0,  0,  1,  0,  0,  0,  0,1/2,1/2,  0,  0,1/2,1/2,  0,  0,  0,1/2,1/2,  0,  0,  0];
ss3   := [0,  0,  0,  0,  0,  3,  0,  0,  0,  0,3/2,3/2,  0,  0,  0,  0,  0,  0,  0,3/2,  0,  0,  0,  0];

donc

DBR_AB := SBR_A-SBR_B := [1,  2, -1, -2,  1, -1,  0,  0,3/2,3/2,-3/2,-3/2,3/2,3/2,-3/2,-3/2,0,0,3/2,-3/2,-3/2,3/2,0,  0];
DBR_BC := SBR_B-SBR_C := [1, -1,  2,  1, -2, -1,3/2,3/2,-3/2,-3/2,0,  0,  0,  0,3/2,3/2,-3/2,-3/2,0,  0,3/2,-3/2,-3/2,3/2];

## Systeme :
On a :

DNPR_AB > 0
DNPR_AC > 0
DBR_AB-ss1 > 0
ss2-DBR_AB > 0
ss3+DBR_BC-DBR_AB > 0

il est equivalent à :

2*DNPR_AB > 0
2*DNPR_AC > 0
2*DBR_AB-2*ss1 > 0
2*ss2-2*DBR_AB > 0
2*(ss3+DBR_BC-DBR_AB) > 0

donc
                         n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

2*DNPR_AB            := [0,  2,  0, -2,  2, -2,  0,  0,  2,  2, -2, -2,  1,  1, -1, -1,  0,  0,  1, -2, -1,  2,  0,  0]
2*DNPR_AC            := [2,  0,  2, -2,  0, -2,  2,  2,  0,  0, -2, -2,  1,  1,  0,  0, -1, -1,  1, -2,  0,  0, -1,  2]
2*DBR_AB-2*ss1       := [2,  4, -2, -4,  2, -4,  0,  0,  3,  3, -4, -4,  3,  3, -3, -3,  0,  0,  3, -4, -3,  3,  0,  0]
2*ss2-2*DBR_AB       :=[-2, -4,  4,  4, -2,  4,  0,  0, -3, -3,  4,  4, -3, -3,  4,  4,  0,  0, -3,  4,  4, -3,  0,  0]
2*(ss3+DBR_BC-DBR_AB):= [0, -6,  6,  6, -6,  6,  3,  3, -6, -6,  6,  6, -3, -3,  6,  6, -3, -3, -3,  6,  6, -6, -3,  3]

##                      n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

                         0   2   0  -2   2  -2   0   0   2   2  -2  -2   1   1  -1  -1   0   0   1  -2  -1   2   0   0
                         2   0   2  -2   0  -2   2   2   0   0  -2  -2   1   1   0   0  -1  -1   1  -2   0   0  -1   2
                         2   4  -2  -4   2  -4   0   0   3   3  -4  -4   3   3  -3  -3   0   0   3  -4  -3   3   0   0
                        -2  -4   4   4  -2   4   0   0  -3  -3   4   4  -3  -3   4   4   0   0  -3   4   4  -3   0   0
                         0  -6   6   6  -6   6   3   3  -6  -6   6   6  -3  -3   6   6  -3  -3  -3   6   6  -6  -3   3
## Borda T1BR : volume V3

## Score

SBR(A,x) = 2(n1 + n2 + n13 + n14 + n19) + 3/2 (n7 + n8 + n9 + n10 + n22 + n24) +(n3 + n5) + 1/2 (n15 + n16 + n17 + n18 + n21 + n23)
SBR(B,x) = 2(n3 + n4 + n15 + n16 + n21) + 3/2 (n7 + n8 + n11 + n12 + n20 + n24) +(n1 + n6) + 1/2 (n13 + n14 + n17 + n18 + n19 + n23)
SBR(C,x) = 2(n5 + n6 + n17 + n18 + n23) + 3/2 (n9 + n10 + n11 + n12 + n20 + n22) +(n2 + n4) + 1/2 (n13 + n14 + n15 + n16 + n19 + n21)

t1 = n4+ 1/2 (n11+n12+n20)

## score en maple code :
          n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

SBR_A := [2,  2,  1,  0,  1,  0,3/2,3/2,3/2,3/2,  0,  0,  2,  2,1/2,1/2,1/2,1/2,  2,  0,1/2,3/2,1/2,3/2];
SBR_B := [1,  0,  2,  2,  0,  1,3/2,3/2,  0,  0,3/2,3/2,1/2,1/2,  2,  2,1/2,1/2,1/2,3/2,  2,  0,1/2,3/2];
SBR_C := [0,  1,  0,  1,  2,  2,  0,  0,3/2,3/2,3/2,3/2,1/2,1/2,1/2,1/2,  2,  2,1/2,3/2,1/2,3/2,  2,  0];

t1    := [0,  0,  0,  1,  0,  0,  0,  0,  0,  0,1/2,1/2,  0,  0,  0,  0,  0,  0,  0,1/2,  0,  0,  0,  0];

donc

DBR_AC := SBR_A-SBR_C := [2,  1,  1, -1, -1, -2,3/2,3/2,  0,  0,-3/2,-3/2,3/2,3/2,0,  0,-3/2,-3/2,3/2,-3/2,0, 0,-3/2,3/2]

## Systeme :
On a :

DNPR_AB > 0
DNPR_AC > 0
t1-DBR_AC > 0

il est equivalent à :

2*DNPR_AB > 0
2*DNPR_AC > 0
2*t1-2*DBR_AC > 0
```

```
donc
                     n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

2*DNPR_AB        := [0,  2,  0, -2,  2, -2,  0,  0,  2,  2, -2, -2,  1,  1, -1, -1,  0,  0,  1, -2, -1,  2,  0,  0]
2*DNPR_AC        := [2,  0,  2, -2,  0, -2,  2,  2,  0,  0, -2, -2,  1,  1,  0,  0, -1, -1,  1, -2,  0,  0, -1,  2]
2*t1-2*DBR_AC    :=[-4, -2, -2,  4,  2,  4, -3, -3,  0,  0,  4,  4, -3, -3,  0,  0,  3,  3, -3,  4,  0,  0,  3, -3]


##                   n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

                      0   2   0  -2   2  -2   0   0   2   2  -2  -2   1   1  -1  -1   0   0   1  -2  -1   2   0   0
                      2   0   2  -2   0  -2   2   2   0   0  -2  -2   1   1   0   0  -1  -1   1  -2   0   0  -1   2
                     -4  -2  -2   4   2   4  -3  -3   0   0   4   4  -3  -3   0   0   3   3  -3   4   0   0   3  -3
```

```
## Borda T2BR : volume V4

## Score

SBR(A,x) = 2(n1 + n2 + n13 + n14 + n19) + 3/2 (n7 + n8 + n9 + n10 + n22 + n24) +(n3 + n5) + 1/2 (n15 + n16 + n17 + n18 + n21 + n23)
SBR(B,x) = 2(n3 + n4 + n15 + n16 + n21) + 3/2 (n7 + n8 + n11 + n12 + n20 + n24) +(n1 + n6) + 1/2 (n13 + n14 + n17 + n18 + n19 + n23)
SBR(C,x) = 2(n5 + n6 + n17 + n18 + n23) + 3/2 (n9 + n10 + n11 + n12 + n20 + n22) +(n2 + n4) + 1/2 (n13 + n14 + n15 + n16 + n19 + n21)

tt1 = n4 + 1/2 (n11 + n12 + n20)
tt2 = n4 + 1/2 (n11 + n12 + n20) + n5 + 1/2 (n17 + n18 + n23)
tt3 = 3(n4 + 1/2 (n11 + n12 + n20))

## score en maple code :
          n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

SBR_A := [2,  2,  1,  0,  1,  0,3/2,3/2,3/2,3/2,  0,  0,  2,  2,1/2,1/2,1/2,1/2,  2,  0,1/2,3/2,1/2,3/2];
SBR_B := [1,  0,  2,  2,  0,  1,3/2,3/2,  0,  0,3/2,3/2,1/2,1/2,  2,  2,1/2,1/2,1/2,3/2,  2,  0,1/2,3/2];
SBR_C := [0,  1,  0,  1,  2,  2,  0,  0,3/2,3/2,3/2,3/2,1/2,1/2,1/2,1/2,  2,  2,1/2,3/2,1/2,3/2,  2,  0];

tt1   := [0,  0,  0,  1,  0,  0,  0,  0,  0,  0,1/2,1/2,  0,  0,  0,  0,  0,  0,  0,1/2,  0,  0,  0,  0];
tt2   := [0,  0,  0,  1,  1,  0,  0,  0,  0,  0,1/2,1/2,  0,  0,  0,  0,1/2,1/2,  0,1/2,  0,  0,1/2,  0];
tt3   := [0,  0,  0,  3,  0,  0,  0,  0,  0,  0,3/2,3/2,  0,  0,  0,  0,  0,  0,  0,3/2,  0,  0,  0,  0];

donc

DBR_AC := SBR_A-SBR_C := [2,  1,  1, -1, -1, -2,3/2,3/2,  0,  0,-3/2,-3/2,3/2,3/2,0,  0,-3/2,-3/2,3/2,-3/2,0, 0,-3/2,3/2];
DBR_CB := SBR_C-SBR_B :=[-1,  1, -2, -1,  2,  1,-3/2,-3/2,3/2,3/2,0,  0,  0,  0,-3/2,-3/2,3/2,3/2,0,  0,-3/2,3/2,3/2,-3/2];

## Systeme :
On a :

DNPR_AB > 0
DNPR_AC > 0
DBR_AC-tt1 > 0
tt2-DBR_AC > 0
DBR_CB+tt3-DBR_AC > 0
```

```
donc
                          n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

2*DNPR_AB             := [0,  2,  0, -2,  2, -2,  0,  0,  2,  2, -2, -2,  1,  1, -1, -1,  0,  0,  1, -2, -1,  2,  0,  0]
2*DNPR_AC             := [2,  0,  2, -2,  0, -2,  2,  2,  0,  0, -2, -2,  1,  1,  0,  0, -1, -1,  1, -2,  0,  0, -1,  2]
2*DBR_AC-2*tt1        := [4,  2,  2, -4, -2, -4,  3,  3,  0,  0, -4, -4,  3,  3,  0,  0, -3, -3,  3, -4,  0,  0, -3,  3]
2*tt2-2*DBR_AC        :=[-4, -2, -2,  4,  4,  4, -3, -3,  0,  0,  4,  4, -3, -3,  0,  0,  4,  4, -3,  4,  0,  0,  4, -3]
2*(DBR_CB+tt3-DBR_AC) :=[-6,  0, -6,  6,  6,  6, -6, -6,  3,  3,  6,  6, -3, -3, -3, -3,  6,  6, -3,  6, -3,  3,  6, -6]

##                     n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

                        0   2   0  -2   2  -2   0   0   2   2  -2  -2   1   1  -1  -1   0   0   1  -2  -1   2   0   0
                        2   0   2  -2   0  -2   2   2   0   0  -2  -2   1   1   0   0  -1  -1   1  -2   0   0  -1   2
                        4   2   2  -4  -2  -4   3   3   0   0  -4  -4   3   3   0   0  -3  -3   3  -4   0   0  -3   3
                       -4  -2  -2   4   4   4  -3  -3   0   0   4   4  -3  -3   0   0   4   4  -3   4   0   0   4  -3
                       -6   0  -6   6   6   6  -6  -6   3   3   6   6  -3  -3  -3  -3   6   6  -3   6  -3   3   6  -6
```

Manipulabilité – Pluralité

```
## Pluralité SPR : volume V1

## Score

SPR(A,x) = (n1 + n2 + n13 + n14 + n19) + (1/2)*(n7 + n8 + n9 + n10 + n22 + n24)
SPR(B,x) = (n3 + n4 + n15 + n16 + n21) + (1/2)*(n7 + n8 + n11 + n12 + n20 + n24)
SPR(C,x) = (n5 + n6 + n17 + n18 + n23) + (1/2)*(n9 + n10 + n11 + n12 + n20 + n22)

s1 =  n6+(1/2)*(n11+n12+n20)


## score en maple code :
          n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

SPR_A := [1,  1,  0,  0,  0,  0,1/2,1/2,1/2,1/2,  0,  0,  1,  1,  0,  0,  0,  0,  1,  0,  0,1/2,  0,1/2];
SPR_B := [0,  0,  1,  1,  0,  0,1/2,1/2,  0,  0,1/2,1/2,  0,  0,  1,  1,  0,  0,  0,1/2,  1,  0,  0,1/2];
SPR_C := [0,  0,  0,  0,  1,  1,  0,  0,1/2,1/2,1/2,1/2,  0,  0,  0,  0,  1,  1,  0,1/2,  0,1/2,  1,  0];

s1 :=    [0,  0,  0,  0,  0,  1,  0,  0,  0,  0,1/2,1/2,  0,  0,  0,  0,  0,  0,  0,1/2,  0,  0,  0,  0];

DPR_AB := SPR_A-SPR_B := [1,  1, -1, -1,  0,  0,  0,  0,1/2,1/2,-1/2,-1/2,1,  1, -1, -1,  0,  0,  1,-1/2,-1,1/2,  0,  0];

## Systeme :
On a :

DEV_AB > 0
DEV_AC > 0
s1-DPR_AB > 0

il est equivalent à :

DEV_AB > 0
DEV_AC > 0
2*s1-2*DPR_AB > 0
donc
                    n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

DEV_AB :=          [1,  2, -1, -2,  1, -1,  0,  0,  1,  2, -1, -2,  1,  2, -1, -2,  0,  0,  1, -1, -1,  1,  0,  0]
DEV_AC :=          [2,  1,  1, -1, -1, -2,  1,  2,  0,  0, -1, -2,  1,  2,  0,  0, -1, -2,  1, -1,  0,  0, -1,  1]
2*s1-2*DPR_AB :=[-2, -2,  2,  2,  0,  2,  0,  0, -1, -1,  2,  2, -2, -2,  2,  2,  0,  0, -2,  2,  2, -1,  0,  0]


## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

    1   2  -1  -2   1  -1   0   0   1   2  -1  -2   1   2  -1  -2   0   0   1  -1  -1   1   0   0
    2   1   1  -1  -1  -2   1   2   0   0  -1  -2   1   2   0   0  -1  -2   1  -1   0   0  -1   1
   -2  -2   2   2   0   2   0   0  -1  -1   2   2  -2  -2   2   2   0   0  -2   2   2  -1   0   0
## Pluralité TPR : volume V2

## Score

SPR(A,x) = (n1 + n2 + n13 + n14 + n19) + (1/2)*(n7 + n8 + n9 + n10 + n22 + n24)
SPR(B,x) = (n3 + n4 + n15 + n16 + n21) + (1/2)*(n7 + n8 + n11 + n12 + n20 + n24)
SPR(C,x) = (n5 + n6 + n17 + n18 + n23) + (1/2)*(n9 + n10 + n11 + n12 + n20 + n22)

t1 :=  n4+(1/2)*(n11+n12+n20)

## score en maple code :
          n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

SPR_A := [1,  1,  0,  0,  0,  0,1/2,1/2,1/2,1/2,  0,  0,  1,  1,  0,  0,  0,  0,  1,  0,  0,1/2,  0,1/2];
SPR_B := [0,  0,  1,  1,  0,  0,1/2,1/2,  0,  0,1/2,1/2,  0,  0,  1,  1,  0,  0,  0,1/2,  1,  0,  0,1/2];
SPR_C := [0,  0,  0,  0,  1,  1,  0,  0,1/2,1/2,1/2,1/2,  0,  0,  0,  0,  1,  1,  0,1/2,  0,1/2,  1,  0];

t1    := [0,  0,  0,  1,  0,  0,  0,  0,  0,  0,1/2,1/2,  0,  0,  0,  0,  0,  0,  0,1/2,  0,  0,  0,  0];

DPR_AC := SPR_A-SPR_C := [1,  1,  0,  0, -1, -1,1/2,1/2,  0,  0,-1/2,-1/2,1,  1,  0,  0, -1, -1,  1,-1/2, 0,  0, -1,1/2]

## Systeme :
On a :

DEV_AB > 0
DEV_AC > 0
t1-DPR_AC > 0

il est equivalent à :

DEV_AB > 0
DEV_AC > 0
2*t1-2*DPR_AC > 0
donc
                    n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

DEV_AB :=          [1,  2, -1, -2,  1, -1,  0,  0,  1,  2, -1, -2,  1,  2, -1, -2,  0,  0,  1, -1, -1,  1,  0,  0]
DEV_AC :=          [2,  1,  1, -1, -1, -2,  1,  2,  0,  0, -1, -2,  1,  2,  0,  0, -1, -2,  1, -1,  0,  0, -1,  1]
2*t1-2*DPR_AC :=[-2, -2,  0,  2,  2,  2, -1, -1,  0,  0,  2,  2, -2, -2,  0,  0,  2,  2, -2,  2,  0,  0,  2, -1]

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

    1   2  -1  -2   1  -1   0   0   1   2  -1  -2   1   2  -1  -2   0   0   1  -1  -1   1   0   0
    2   1   1  -1  -1  -2   1   2   0   0  -1  -2   1   2   0   0  -1  -2   1  -1   0   0  -1   1
   -2  -2   0   2   2   2  -1  -1   0   0   2   2  -2  -2   0   0   2   2  -2   2   0   0   2  -1
```

Manipulabilité – Vote par note

```
## Vote par évaluation SEV : volume V1

## Score

SEV(A,x) = 2(n1 + n2 + n7 + n8 + n9 + n10 + n13 + n14)+ (n3 + n5 + n11 + n15 + n17 + n19 + n22 + n24)
SEV(B,x) = 2(n3 + n4 + n7 + n8 + n11 + n12 + n15 + n16)+ (n1 + n6 + n9 + n13 + n17 + n20 + n21 + n24)
SEV(C,x) = 2(n5 + n6 + n9 + n10 + n11 + n12 + n17 + n18)+ (n2 + n4 + n7 + n13 + n15 + n20 + n22 + n23)

s1 = (n3 + n11 + n15) + (n6 + n20 + n21)
s2 = (n6 + n20 + n21) + (n4 + 2n6 + +2n11 + 2n12 + n15 + n20)


## score en maple code :
           n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

SEV_A := [2,  2,  1,  0,  1,  0,  2,  2,  2,  2,  1,  0,  2,  2,  1,  0,  1,  0,  1,  0,  0,  1,  0,  1];
SEV_B := [1,  0,  2,  2,  0,  1,  2,  2,  1,  0,  2,  2,  1,  0,  2,  2,  1,  0,  0,  1,  1,  0,  0,  1];
SEV_C := [0,  1,  0,  1,  2,  2,  1,  0,  2,  2,  2,  2,  1,  0,  1,  0,  2,  2,  0,  1,  0,  1,  1,  0];

s1 :=    [0,  0,  1,  0,  0,  1,  0,  0,  0,  0,  1,  0,  0,  0,  1,  0,  0,  0,  0,  1,  1,  0,  0,  0];
s2 :=    [0,  0,  0,  1,  0,  3,  0,  0,  0,  0,  2,  2,  0,  0,  1,  0,  0,  0,  0,  2,  1,  0,  0,  0];

DEV_AB := SEV_A-SEV_B := [1,  2, -1, -2,  1, -1,  0,  0,  1,  2, -1, -2,  1,  2, -1, -2,  0,  0,  1, -1, -1,  1,  0,  0]
DEV_AC := SEV_A-SEV_C := [2,  1,  1, -1, -1, -2,  1,  2,  0,  0, -1, -2,  1,  2,  0,  0, -1, -2,  1, -1,  0,  0, -1,  1]
DEV_CB := SEV_C-SEV_B :=[-1,  1, -2, -1,  2,  1, -1, -2,  1,  2,  0,  0,  0,  0, -1, -2,  1,  2,  0,  0, -1,  1,  1, -1]

## Systeme :
                n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

DEV_AB      := [1,  2, -1, -2,  1, -1,  0,  0,  1,  2, -1, -2,  1,  2, -1, -2,  0,  0,  1, -1, -1,  1,  0,  0]
DEV_AC      := [2,  1,  1, -1, -1, -2,  1,  2,  0,  0, -1, -2,  1,  2,  0,  0, -1, -2,  1, -1,  0,  0, -1,  1]
s1-DEV_AB   :=[-1, -2,  2,  2, -1,  2,  0,  0, -1, -2,  2,  2, -1, -2,  2,  2,  0,  0, -1,  2,  2, -1,  0,  0]
s2-DEV_CB   := [1, -1,  2,  2, -2,  2,  1,  2, -1, -2,  2,  2,  0,  0,  2,  2, -1, -2,  0,  2,  2, -1, -1,  1]

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

    1   2  -1  -2   1  -1   0   0   1   2  -1  -2   1   2  -1  -2   0   0   1  -1  -1   1   0   0
    2   1   1  -1  -1  -2   1   2   0   0  -1  -2   1   2   0   0  -1  -2   1  -1   0   0  -1   1
   -1  -2   2   2  -1   2   0   0  -1  -2   2   2  -1  -2   2   2   0   0  -1   2   2  -1   0   0
    1  -1   2   2  -2   2   1   2  -1  -2   2   2   0   0   2   2  -1  -2   0   2   2  -1  -1   1
```

```
## Vote par évaluation TEV : volume V2

## Score

SEV(A,x) = 2(n1 + n2 + n7 + n8 + n9 + n10 + n13 + n14)+ (n3 + n5 + n11 + n15 + n17 + n19 + n22 + n24)
SEV(B,x) = 2(n3 + n4 + n7 + n8 + n11 + n12 + n15 + n16)+ (n1 + n6 + n9 + n13 + n17 + n20 + n21 + n24)
SEV(C,x) = 2(n5 + n6 + n9 + n10 + n11 + n12 + n17 + n18)+ (n2 + n4 + n7 + n13 + n15 + n20 + n22 + n23)


t1 = (n5 + n11 + n17) + (n4 + n20 + n23)
t2 = (n4 + n20 + n23) + (2n4 + n6 + 2n11 + 2n12 + n17 + n20)


## score en maple code :
           n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

SEV_A := [2,  2,  1,  0,  1,  0,  2,  2,  2,  2,  1,  0,  2,  2,  1,  0,  1,  0,  1,  0,  0,  1,  0,  1];
SEV_B := [1,  0,  2,  2,  0,  1,  2,  2,  1,  0,  2,  2,  1,  0,  2,  2,  1,  0,  0,  1,  1,  0,  0,  1];
SEV_C := [0,  1,  0,  1,  2,  2,  1,  0,  2,  2,  2,  2,  1,  0,  1,  0,  2,  2,  0,  1,  0,  1,  1,  0];

s1 :=    [0,  0,  0,  1,  1,  0,  0,  0,  0,  0,  1,  0,  0,  0,  0,  0,  1,  0,  0,  1,  0,  0,  1,  0];
s2 :=    [0,  0,  0,  3,  0,  1,  0,  0,  0,  0,  2,  2,  0,  0,  0,  0,  1,  0,  0,  2,  0,  0,  1,  0];

DEV_AB := SEV_A-SEV_B := [1,  2, -1, -2,  1, -1,  0,  0,  1,  2, -1, -2,  1,  2, -1, -2,  0,  0,  1, -1, -1,  1,  0,  0]
DEV_AC := SEV_A-SEV_C := [2,  1,  1, -1, -1, -2,  1,  2,  0,  0, -1, -2,  1,  2,  0,  0, -1, -2,  1, -1,  0,  0, -1,  1]
DEV_BC := SEV_B-SEV_C := [1, -1,  2,  1, -2, -1,  1,  2, -1, -2,  0,  0,  0,  0,  1,  2, -1, -2,  0,  0,  1, -1, -1,  1]

## Systeme :
                n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

DEV_AB      := [1,  2, -1, -2,  1, -1,  0,  0,  1,  2, -1, -2,  1,  2, -1, -2,  0,  0,  1, -1, -1,  1,  0,  0]
DEV_AC      := [2,  1,  1, -1, -1, -2,  1,  2,  0,  0, -1, -2,  1,  2,  0,  0, -1, -2,  1, -1,  0,  0, -1,  1]
s1-DEV_AC   :=[-2, -1, -1,  2,  2,  2, -1, -2,  0,  0,  2,  2, -1, -2,  0,  0,  2,  2, -1,  2,  0,  0,  2, -1]
s2-DEV_BC   :=[-1,  1, -2,  2,  2,  2, -1, -2,  1,  2,  2,  2,  0,  0, -1, -2,  2,  2,  0,  2, -1,  1,  2, -1]

## n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

    1   2  -1  -2   1  -1   0   0   1   2  -1  -2   1   2  -1  -2   0   0   1  -1  -1   1   0   0
    2   1   1  -1  -1  -2   1   2   0   0  -1  -2   1   2   0   0  -1  -2   1  -1   0   0  -1   1
   -2  -1  -1   2   2   2  -1  -2   0   0   2   2  -1  -2   0   0   2   2  -1   2   0   0   2  -1
   -1   1  -2   2   2   2  -1  -2   1   2   2   2   0   0  -1  -2   2   2   0   2  -1   1   2  -1
```

Manipulabilité – Négative pluralité

```
## Antipluralité SNPR : volume V1

## Score

SNPR(A,x) = n - [(n4 + n6 + n11 + n12 + n20) + (1/2)*(n15 + n16 + n17 + n18 + n21 + n23)]
SNPR(B,x) = n - [(n2 + n5 + n9 + n10 + n22) + (1/2)*(n13 + n14 + n17 + n18 + n19 + n23)]
SNPR(C,x) = n - [(n1 + n3 + n7 + n8 + n24) + (1/2)*(n13 + n14 + n15 + n16 + n19 + n21)]

s1 = n3+(1/2)(n15+n16+n21)

## score en maple code :
               n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

SNPR_A := n-[0,  0,  0,  1,  0,  1,  0,  0,  0,  0,  1,  1,  0,  0,1/2,1/2,1/2,1/2,  0,  1,1/2,  0,1/2,  0];
SNPR_B := n-[0,  1,  0,  0,  1,  0,  0,  0,  1,  1,  0,  0,1/2,1/2,  0,  0,1/2,1/2,1/2,  0,  0,  1,1/2,  0];
SNPR_C := n-[1,  0,  1,  0,  0,  0,  1,  1,  0,  0,  0,  0,1/2,1/2,1/2,1/2,  0,  0,1/2,  0,1/2,  0,  0,  1];

s1     :=   [0,  0,  1,  0,  0,  0,  0,  0,  0,  0,  0,  0,  0,  0,1/2,1/2,  0,  0,  0,  0,1/2,  0,  0,  0];
n      :=   [1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1];

donc :

SNPR_A :=  [1,  1,  1,  0,  1,  0,  1,  1,  1,  1,  0,  0,  1,  1,1/2,1/2,1/2,1/2,  1,  0,1/2,  1,1/2, 1];
SNPR_B :=  [1,  0,  1,  1,  0,  1,  1,  1,  0,  0,  1,  1,1/2,1/2,  1,  1,1/2,1/2,1/2,  1,  1,  0,1/2, 1];
SNPR_C :=  [0,  1,  0,  1,  1,  1,  0,  0,  1,  1,  1,  1,1/2,1/2,1/2,1/2,  1,  1,1/2,  1,1/2,  1,  1, 0];


DNPR_AB := SNPR_A-SNPR_B := [0,  1,  0, -1,  1, -1,  0,  0,  1,  1, -1, -1,1/2,1/2,-1/2,-1/2,0,  0,1/2, -1,-1/2, 1,  0,  0];
DNPR_AC := SNPR_A-SNPR_C := [1,  0,  1, -1,  0, -1,  1,  1,  0,  0, -1, -1,1/2,1/2,  0,  0,-1/2,-1/2,1/2,-1, 0,  0,-1/2, 1];
DNPR_BC := SNPR_B-SNPR_C := [1, -1,  1,  0, -1,  0,  1,  1, -1, -1,  0,  0,  0,  0,1/2,1/2,-1/2,-1/2,0,  0,1/2, -1,-1/2, 1];

## Systeme :
On a :

DNPR_AB > 0
DNPR_AC > 0
DNPR_BC-DNPR_AB > 0
s1-DNPR_AB > 0
il est equivalent à :

2*DNPR_AB > 0
2*DNPR_AC > 0
2*DNPR_BC-2*DNPR_AB > 0
2*s1-2*DNPR_AB > 0

donc
                           n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

2*DNPR_AB              := [0,  2,  0, -2,  2, -2,  0,  0,  2,  2, -2, -2,  1,  1, -1, -1,  0,  0,  1, -2, -1,  2,  0,  0]
2*DNPR_AC              := [2,  0,  2, -2,  0, -2,  2,  2,  0,  0, -2, -2,  1,  1,  0,  0, -1, -1,  1, -2,  0,  0, -1,  2]
2*DNPR_BC-2*DNPR_AB := [2, -4,  2,  2, -4,  2,  2,  2, -4, -4,  2,  2, -1, -1,  2,  2, -1, -1, -1,  2,  2, -4, -1,  2]
2*s1-2*DNPR_AB         := [0, -2,  2,  2, -2,  2,  0,  0, -2, -2,  2,  2, -1, -1,  2,  2,  0,  0, -1,  2,  2, -2,  0,  0]


##                         n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

                            0   2   0  -2   2  -2   0   0   2   2  -2  -2   1   1  -1  -1   0   0   1  -2  -1   2   0   0
                            2   0   2  -2   0  -2   2   2   0   0  -2  -2   1   1   0   0  -1  -1   1  -2   0   0  -1   2
                            2  -4   2   2  -4   2   2   2  -4  -4   2   2  -1  -1   2   2  -1  -1  -1   2   2  -4  -1   2
                            0  -2   2   2  -2   2   0   0  -2  -2   2   2  -1  -1   2   2   0   0  -1   2   2  -2   0   0

## Antipluralité TNPR : volume V2

## Score

SNPR(A,x) = n - [(n4 + n6 + n11 + n12 + n20) + (1/2)*(n15 + n16 + n17 + n18 + n21 + n23)]
SNPR(B,x) = n - [(n2 + n5 + n9 + n10 + n22) + (1/2)*(n13 + n14 + n17 + n18 + n19 + n23)]
SNPR(C,x) = n - [(n1 + n3 + n7 + n8 + n24) + (1/2)*(n13 + n14 + n15 + n16 + n19 + n21)]

t1 =  n5+(1/2)(n17+n18+n23)

## score en maple code :
               n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

SNPR_A := n-[0,  0,  0,  1,  0,  1,  0,  0,  0,  0,  1,  1,  0,  0,1/2,1/2,1/2,1/2,  0,  1,1/2,  0,1/2,  0];
SNPR_B := n-[0,  1,  0,  0,  1,  0,  0,  0,  1,  1,  0,  0,1/2,1/2,  0,  0,1/2,1/2,1/2,  0,  0,  1,1/2,  0];
SNPR_C := n-[1,  0,  1,  0,  0,  0,  1,  1,  0,  0,  0,  0,1/2,1/2,1/2,1/2,  0,  0,1/2,  0,1/2,  0,  0,  1];

t1     :=   [0,  0,  0,  0,  1,  0,  0,  0,  0,  0,  0,  0,  0,  0,  0,  0,1/2,1/2,  0,  0,  0,  0,1/2,  0];
n      :=   [1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1,  1];
```

```
donc :

SNPR_A :=  [1,  1,  1,  0,  1,  0,  1,  1,  1,  1,  0,  0,  1,  1,1/2,1/2,1/2,1/2,  1,  0,1/2,  1,1/2, 1];
SNPR_B :=  [1,  0,  1,  1,  0,  1,  1,  1,  0,  0,  1,  1,1/2,1/2,  1,  1,1/2,1/2,1/2,  1,  1,  0,1/2, 1];
SNPR_C :=  [0,  1,  0,  1,  1,  1,  0,  0,  1,  1,  1,  1,1/2,1/2,1/2,1/2,  1,  1,1/2,  1,1/2,  1,  1, 0];


DNPR_AB := SNPR_A-SNPR_B := [0,  1,  0, -1,  1, -1,  0,  0,  1,  1, -1, -1,1/2,1/2,-1/2,-1/2,0,  0,1/2, -1,-1/2, 1,  0,  0];
DNPR_AC := SNPR_A-SNPR_C := [1,  0,  1, -1,  0, -1,  1,  1,  0,  0, -1, -1,1/2,1/2,  0,  0,-1/2,-1/2,1/2,-1, 0,  0,-1/2, 1];
DNPR_CB := SNPR_C-SNPR_B :=[-1,  1, -1,  0,  1,  0, -1, -1,  1,  1,  0,  0,  0,  0,-1/2,-1/2,1/2,1/2 0,  0,-1/2, 1,1/2, -1]

## Systeme :
On a :

DNPR_AB > 0
DNPR_AC > 0
DNPR_CB-DNPR_AC > 0
t1-DNPR_AC > 0

il est equivalent à :

2*DNPR_AB > 0
2*DNPR_AC > 0
2*DNPR_CB-2*DNPR_AC > 0
2*t1-2*DNPR_AC > 0

donc
                        n1  n2  n3  n4  n5  n6  n7  n8  n9 n10 n11 n12 n13 n14 n15 n16 n17 n18 n19 n20 n21 n22 n23 n24

2*DNPR_AB           := [0,  2,  0, -2,  2, -2,  0,  0,  2,  2, -2, -2,  1,  1, -1, -1,  0,  0,  1, -2, -1,  2,  0,  0]
2*DNPR_AC           := [2,  0,  2, -2,  0, -2,  2,  2,  0,  0, -2, -2,  1,  1,  0,  0, -1, -1,  1, -2,  0,  0, -1,  2]
2*DNPR_CB-2*DNPR_AC :=[-4,  2, -4,  2,  2,  2, -4, -4,  2,  2,  2,  2, -1, -1, -1, -1,  2,  2, -1,  2, -1,  2,  2, -4]
2*t1-2*DNPR_AC      :=[-2,  0, -2,  2,  2,  2, -2, -2,  0,  0,  2,  2, -1, -1,  0,  0,  2,  2, -1,  2,  0,  0,  2, -2]
```

Résumé

La motivation principale de cette thèse est de contribuer à étendre l'analyse probabiliste des règles de vote d'une part aux cas où quatre candidats sont soumis à la décision collective et d'autre part à l'étude de l'efficacité majoritaire du vote par évaluation à 3 niveaux et de sa manipulabilité par une coalition de votants. Ces deux problématiques nous ont conduit à un même et unique problème technique : calculer la probabilité limite d'un événement de vote lorsqu'il y a 24 préférences individuelles possibles. Les réponses que nous avons essayé d'apporter à cette question méthodologique constituent, avec les nombreux résultats décrivant la fréquence théorique de différents événements électoraux, les deux principales contributions de notre travail de thèse à l'analyse probabiliste en théorie du vote.

Abstract

Our main purpose in this thesis is to contribute to extending the probabilistic analysis of voting rules to the consideration of four candidate elections, on the one hand, and to the study of the majority efficiency of Evaluative Voting with 3 levels and its manipulability by coalitions of voters, on the other hand. These two issues lead to a unique technical problem, consisting in computing the limiting probability of an electoral event when 24 possible individual preferences have to be considered. The answers we have tried to bring to this methodological question constitute, together with the numerous results describing the theoretical likelihood of various electoral outcomes, the main contributions of our study to probabilistic analysis in voting theory.

POLE RECHERCHE
Ecoles Doctorales

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