



# Cohomology of $\mathrm{PGL}_2$ over imaginary quadratic integers

Eduardo R. Mendoza

## ► To cite this version:

Eduardo R. Mendoza. Cohomology of  $\mathrm{PGL}_2$  over imaginary quadratic integers. K-Theory and Homology [math.KT]. Universität Bonn, 1980. English. NNT : . tel-02573006

**HAL Id: tel-02573006**

**<https://theses.hal.science/tel-02573006>**

Submitted on 14 May 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

COHOMOLOGY OF  $PGL_2$  OVER  
IMAGINARY QUADRATIC INTEGERS

Inaugural-Dissertation zur Erlangung des Doktorgrades  
der Hohen Mathematisch-Naturwissenschaftlichen Fakultät  
der Rheinischen Friedrich-Wilhelms-Universität zu Bonn

vorgelegt von

Eduardo R. Mendoza

Bonn 1979



Angefertigt mit Genehmigung  
der Mathematisch-Naturwissenschaftlichen Fakultät  
der Universität Bonn:

Referent: Prof. Dr. G. Harder  
Koreferent: Prof. Dr. F. Hirzebruch

2

-i-

Ang akdang ito'y alay  
sa gintong anibersaryo  
ng mga magulang ko

# TABLE OF CONTENTS

INTRODUCTION	
§ 1	REDUCTION THEORY ..... 1
§ 2	THE MINIMAL INCIDENCE SET OF AN IMAGINARY QUADRATIC FIELD ..... 16
§ 3	COMPARISON WITH THE CLASSICAL THEORY ..... 43
§ 4	THE QUOTIENT SPACE IN THE EUCLIDEAN CASES ..... 47
§ 5	COHOMOLOGY COMPUTATIONS ..... 67
	REFERENCES ..... 81

## INTRODUCTION

Let  $K$  be an imaginary quadratic field and  $\mathcal{O}$  its ring of integers.  $\Gamma = \text{PGL}(2, \mathcal{O})$  denotes the group of invertible  $2 \times 2$  matrices with coefficients in  $\mathcal{O}$  modulo its center. This paper deals with the problem of explicitly computing Eilenberg-MacLane cohomology groups  $H^n(\Gamma', M)$ , where  $\Gamma' \subset \Gamma$  is a subgroup of finite index and  $M$  a  $\Gamma'$ -module.

The basic approach to computing such cohomology groups is to find a contractible topological space  $X$  on which  $\Gamma'$  acts in a properly discontinuous manner. Well known results then relate group and quotient space cohomology: for example, if  $\Gamma'$  is torsionfree,  $H^n(\Gamma', M) = H^n(\Gamma' \backslash X, \tilde{M})$ , where  $\tilde{M}$  is the local system associated to  $M$ . One usually takes  $X$  to be the symmetric space associated to  $\text{PGL}(2, \mathbb{C})$ , which is the 3-dimensional upper half space  $H := \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta > 0\}$ . If  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,

$$g(z, \zeta) := \left( \frac{(az + b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}\zeta^2}{(cz + d)(\bar{c}\bar{z} + \bar{d}) + c\bar{c}\zeta^2}, \frac{\zeta}{(cz + d)(\bar{c}\bar{z} + \bar{d}) + c\bar{c}\zeta^2} \right)$$

However, for purposes of cohomology computations,  $H$  is "too big" for two reasons: first, the quotient  $\Gamma' \backslash H$  is noncompact and secondly,  $H$  is 3-dimensional whereas the cohomological dimension of  $\Gamma'$  is one less.

The main results of this paper are as follows: for each imaginary quadratic field  $K$ , there exists a close contractible 2-dimensional subspace  $I_K$  of  $H$  --called the minimal incidence set of  $K$ -- whose quotient  $\Gamma' \backslash I_K$  by any subgroup  $\Gamma' \subset \Gamma$  of finite index is compact.  $I_K$  is in fact a  $\Gamma$ -equivariant deformation retract of  $H$ . Moreover, it is, in a natural way,



a regular, locally finite CW complex and the group action is cellular. The CW structure has a subdivision (depending on  $\Gamma'$ ) such that  $\Gamma'$  acts without "inversion": the isotropy group of an open cell  $\sigma$  fixes each point in  $\sigma$ . These minimal incidence sets are 2-dimensional analogues of the infinite tree for  $\mathbb{Q}$  in the upper half-plane (cf. Serre (16), p.53).

The construction of  $I_K$  uses an explicit form of Harder's reduction theory for arithmetic groups ((7), (8)). In the case of  $\mathrm{PGL}_2$ , this is based on the following notion: the distance of  $(z, \zeta) \in H$  to  $\lambda \in K \cup \{\infty\}$  (a cusp) is

$$n_\lambda(z, \zeta) := \left( \frac{\alpha\bar{\alpha} - \bar{\alpha}\beta z - \alpha\bar{\beta}\bar{z} + \beta\bar{\beta}(z\bar{z} + \zeta\bar{\zeta})}{\zeta N(\alpha, \beta)} \right)^2$$

where  $\lambda = \frac{\alpha}{\beta}$  and  $N(\alpha, \beta)$  denotes the norm of the ideal in  $K$  generated by  $\alpha$  and  $\beta$ . The minimal sets relative to  $\lambda$   $H_\lambda := \{(z, \zeta) \in H \mid n_\lambda(z, \zeta) \leq n_\mu(z, \zeta) \text{ for all } \mu \in K \cup \{\infty\}\}$  are basic for the construction of a fundamental domain for  $\Gamma$  (in  $H$ ).  $I_K$  consists of the points where at least two minimal sets are incident, i.e.,  $I_K := \bigcup_{\lambda \neq \mu} H_\lambda \cap H_\mu$ . If the class number of  $K$  is 1, then this reduction theory coincides with classical reduction theory (for an account of this, cf. Swan (17)).

Thus, in this case,  $I_K = \Gamma \cdot B_K$ , where  $B_K$  is the "bottom" boundary of the classical fundamental domains of Bianchi and Humbert.

$\Gamma$  operates as a group of Möbius transformations on the set of cusps  $K \cup \{\infty\}$ . The stabilizer  $\Gamma_\lambda$  of a cusp  $\lambda$  no longer has finite index in  $\Gamma$ . However,  $I_{K, \lambda} = \bigcup_{\mu} H_\lambda \cap H_\mu$  is a contractible subcomplex of  $I_K$  on which  $\Gamma_\lambda$  acts with compact quotient. Thus we can also treat "cohomology at infinity" and investigate "restriction maps" (cf. Harder (9)).

Our construction also provides an affirmative answer for  $\Gamma' \subset \mathrm{PGL}(2, \mathcal{O})$  to the question of existence of a contractible CW complex with  $\Gamma'$ -action and dimension equal to the cohomological dimension of  $\Gamma'$ . (This is a general question for discrete groups, cf. e.g. Brown (2) pp. 3,7).

We discuss the explicit computation of cohomology when 2 and 3 (the only primes occurring in orders of stabilizers for the action on  $I$ ) are invertible in the coefficient module  $M$ . In this case,  $H^n(\Gamma, M) = H^n(\Gamma \backslash I, \underline{M}^{\Gamma'})$  where  $\underline{M}^{\Gamma'}$  is the sheaf associated to  $M$ . Here the basic fact to be used is that  $\underline{M}^{\Gamma'}$  is constant on open cells of the quotient cell structure (and all subdivisions thereof). The application we had in mind is the following: for any natural number  $p$ , let  $\mathcal{O}_p := \mathbb{Z} \left[ \frac{1}{2}, \dots, \frac{1}{k} \right]$  where  $k$  runs over all primes  $< 2p$ . The group  $\Gamma = \mathrm{PGL}(2, \mathbb{Z}[\frac{1}{p}])$  operates on the  $\mathcal{O}_p$ -algebra  $M_p = \{ \sum a_i x^i y^{2p-i} \mid a_i \in \mathcal{O}_p \}$  of homogeneous binary polynomials of degree  $2p$ . Explicit computation of  $H^n(\Gamma, M_p)$  and  $H^n(\Gamma_\infty, M_p)$  are of interest due to results of Harder relating these, for  $p \equiv 3 \pmod{4}$ , to values of  $L$ -functions expressed (in these cases) in terms of Hurwitz numbers. We do the computations for  $p = 3$ .

For an application of our results to the computation of integral cohomology of  $\mathrm{PSL}(2, \mathcal{O})$ , cf. (6).

The paper is divided into five sections. There is a detailed description of the contents at the beginning of each section.

Thanks are due to F. Grunewald, J. Rohlf, T. Schleich and J. Schwermer for stimulating discussions and useful comments on an earlier version of this paper. I would also like to ex-

press my deep gratitude to G. Harder for his advice and guidance during the preparation of this work.

## § 1. REDUCTION THEORY

1.0. Let  $K$  be an imaginary quadratic field and  $\mathcal{O}_K$  (or simply  $\mathcal{O}$ ) be its ring of integers. In this section, we shall discuss reduction theory for the action of  $\mathrm{PGL}(2, \mathcal{O}_K)$  on the upper half space  $H := \{(z, \zeta) \mid z \in \mathbb{C}, \zeta \in \mathbb{R}, \zeta > 0\}$ . Our version is based on the notion of "distance of a point from a cusp" (1.1.1.). If we interpret the point in  $H$  as a positive hermitian form and the cusp of  $K$  as a submodule of  $\mathcal{O}^2$  (1.1.5 ff), then the distance is essentially the square of the module's volume with respect to the form.

As far as we know, reduction theory was first formulated in this way by Siegel (14) for Hilbert modular groups of totally real fields. (Cf. pp. 270 - 273 for further historical remarks). Of course, in case the field is  $\mathbb{Q}$ , the fundamental domain obtained is the classical one. The general case of an arithmetic group was done by Harder (7), (8) in both number and function field cases. We redo the theory for our special case in order to get explicit values for the reduction constants (1.2.1.); we use this information for example for computations in §4. The connection of our results to the classical reduction theory of Bianchi (1) and Humbert (10), is discussed in detail in § 3.

In 1.1 we derive some basic properties of the distance function, in particular its invariance under  $\mathrm{PGL}(2, \mathcal{O}_K)$  and its finiteness property. In 1.2 the two main Reduction Theorems are established. Finally, we discuss briefly the construction of a (strict) fundamental domain in 1.3.

1.1. The group  $\mathrm{PGL}(2, K)$  operates on the set  $\bar{H} := \{(z, \zeta) \mid z \in \mathbb{C}, \zeta \in \mathbb{R}, \zeta > 0\} \cup \{(\infty, \infty)\}$  as follows:



for  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$g \cdot (z, \zeta) := \left( \frac{(az + b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}\zeta^2}{(cz + d)(\bar{c}\bar{z} + \bar{d}) + c\bar{c}\zeta^2}, \frac{\zeta |\det g|}{(cz + d)(\bar{c}\bar{z} + \bar{d}) + c\bar{c}\zeta^2} \right)$$

Note that if  $\zeta = \infty$  or  $\zeta = 0$  (i.e.  $(z, \zeta)$  lies on the boundary  $\bar{H} \setminus H = \mathbb{C} \cup \{\infty\}$ ), this is the usual Möbius transformation associated to  $g$ . In order to study the non-compact quotient space  $\text{PGL}(2, \mathbb{C}) \backslash H$ , we shall do reduction theory based on the notion of "distance from a cusp".

An element  $(z, \zeta) \in H$  is called a cusp if either  $(z, \zeta) = (\infty, \infty)$  or  $\zeta = 0$  and  $z \in K$ . We shall forget the second component and identify the set of cusps with  $K \cup \{\infty\}$ , the projective line over  $K$ . The operation defined above induces an action of  $\text{PGL}(2, K)$  on this set.

Now let  $(z, \zeta) \in H$  and write  $\lambda = \frac{\alpha}{\beta}$  as the quotient of two elements in  $K$ . (For  $\infty$ , we always choose  $\beta = 0$  and  $\alpha \neq 0$ ).  $(\alpha, \beta)$  denotes the fractional ideal in  $K$  generated by  $\alpha$  and  $\beta$ , and  $N(\alpha, \beta)$  is its norm.

**1.1.1. Definition** The distance of  $(z, \zeta)$  from the cusp  $\lambda$  is given by:

$$n_\lambda(z, \zeta) = \left( \frac{\alpha\bar{\alpha} - \bar{\alpha}\beta z - \alpha\bar{\beta}\bar{z} + \beta\bar{\beta}(z\bar{z} + \zeta^2)}{\zeta N(\alpha, \beta)} \right)^2$$

This is well defined since it doesn't depend on the particular description of  $\lambda$ . Note however that different expressions for  $\lambda$  lead to different ideals and hence to different norms. If we choose  $\alpha, \beta \in \mathcal{O}$ , then  $N(\alpha, \beta)$  is an integer. In fact, if  $K$  has class number equal to one, we can choose  $\alpha, \beta$  so that  $N(\alpha, \beta) = 1$ .

For each cusp  $\lambda$ , the function  $n_\lambda : H \rightarrow \mathbb{R}$  is obviously smooth. The level sets  $n_\lambda^{-1}(c)$  (with  $c > 0$ ) are easy to describe:

a)  $\lambda = \infty$ : choose  $\alpha = 1, \beta = 0$ , which implies that  $(\alpha, \beta) = \mathcal{O}$  and

$N(\alpha, \beta) = 1$ . Hence  $n_\infty(z, \zeta) = \left(\frac{1}{\zeta}\right)^2 = c$ , which means that  $n_\infty^{-1}(c)$  is a horizontal plane of "height"  $1/\sqrt{c}$ .

b)  $\lambda \neq \infty$ : let  $\lambda = \frac{\alpha}{\beta}$  and  $N_\lambda := N(\alpha, \beta) |\beta|^{-2}$ . By definition

$$n_\lambda(z, \zeta) = c \quad \text{iff} \quad \alpha\bar{\alpha} - \bar{\alpha}\beta z - \alpha\bar{\beta}\bar{z} + \beta\bar{\beta}(z\bar{z} + \zeta^2) = \sqrt{c} \zeta N(\alpha, \beta)$$

$$\quad \text{iff} \quad (\beta z - \alpha)(\bar{\beta}\bar{z} - \bar{\alpha}) + \beta\bar{\beta}\zeta^2 = \sqrt{c} \zeta N(\alpha, \beta)$$

$$\quad \text{iff} \quad \left(z - \frac{\alpha}{\beta}\right)\left(\bar{z} - \frac{\bar{\alpha}}{\bar{\beta}}\right) + \left(\zeta - \frac{\sqrt{c}}{2} N_\lambda\right)^2 = \left(\frac{\sqrt{c}}{2} N_\lambda\right)^2$$

In this case  $n_\lambda^{-1}(c)$  is a sphere with center at  $(\lambda, \frac{\sqrt{c}}{2} N_\lambda)$  and radius  $\frac{\sqrt{c}}{2} N_\lambda$ .

**1.1.2. Example** 0 and  $\infty$  are cusps for any imaginary quadratic field  $K$ .

Figure 1.1.2. gives a "side view" of the level sets  $n_\lambda^{-1}(1)$  for  $\lambda \in \{\infty, 0, \frac{1}{2}, \frac{1}{3}\}$ .

The unique point in  $n_\infty^{-1}(1) \cap n_0^{-1}(1)$  is  $((0, 0), 1)$ .

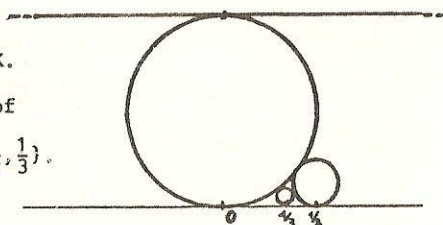


Figure 1.1.2.

For the further study of this distance function, it is useful to consider the following alternative description of  $H$  and the cusps of  $K$ . Let  $\underline{H}$  be the set of binary positive definite hermitian forms on  $\mathbb{C}^2$  with determinant equal to one, i.e., mappings of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto (x \ y) \begin{pmatrix} a & \bar{b} \\ b & c \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

with  $a, c > 0$  and  $ac - \bar{b}b = 1$ . We identify  $H$  and  $\underline{H}$  as follows:

$$(z, \zeta) \mapsto \frac{1}{\zeta} \begin{pmatrix} 1 & -\bar{z} \\ -z & z\bar{z} + \zeta^2 \end{pmatrix} \quad \text{and} \quad \left(-\frac{b}{a}, \frac{1}{a}\right) \longleftarrow \begin{pmatrix} a & \bar{b} \\ b & c \end{pmatrix}$$

The action of  $\text{PGL}(2, K)$  on  $H$  given above induces the following operation on  $\underline{H}$ :

$$g \cdot \begin{pmatrix} a & \bar{b} \\ b & c \end{pmatrix} = |\det g|^2 {}^t(g^{-1}) \begin{pmatrix} a & \bar{b} \\ b & c \end{pmatrix} (g^{-1})$$

We will be mainly concerned with the hermitian forms restricted to  $\mathcal{O}^2$ .

The first application of this "reinterpretation" of  $H$  as a set of hermitian forms is the following useful Transformation Rule:

1.1.3. Transformation Rule Let  $\lambda = \frac{\alpha}{\beta}$  be a cusp of  $K$  and  $(z, \zeta)$  be a point in  $H$ . For any  $g \in \text{PGL}(2, K)$ , we have:

$$n_{g\lambda}(g(z, \zeta)) = \left( |\det g|^2 \frac{N(\alpha, \beta)}{N(g(\alpha, \beta))} \right)^2 n_{\lambda}(z, \zeta)$$

(The important thing is that the factor doesn't depend on  $(z, \zeta)$ !)

Proof: Denote by  $h$  the matrix associated to  $(z, \zeta)$ , i.e.

$$\frac{1}{\zeta} \begin{pmatrix} 1 & -\bar{z} \\ -z & zz + \zeta^2 \end{pmatrix} \quad \text{Furthermore, let } x = (\alpha, \beta) \in K^2.$$

Then we can rewrite the distance of  $(z, \zeta)$  to  $\lambda$  as:

$$n_{\lambda}(z, \zeta) = \left( \frac{t_x h \bar{x}}{N(\alpha, \beta)} \right)^2$$

Doing the same for  $n_{g\lambda}(g(z, \zeta))$ , we see that it is equal to

$$\begin{aligned} \left( \frac{t(gx) |\det g|^2 t(g^{-1}) h (g^{-1}) \overline{(gx)}}{N(g(\alpha, \beta))} \right)^2 &= \left( \frac{|\det g|^2 t_x t(g^{-1}) h (g^{-1}) \overline{gx}}{N(g(\alpha, \beta))} \right)^2 \\ &= \left( \frac{|\det g|^2}{N(g(\alpha, \beta))} t_x h \bar{x} \right)^2 \\ &= \left( \frac{|\det g|^2 N(\alpha, \beta)}{N(g(\alpha, \beta))} \right)^2 n_{\lambda}(z, \zeta). \end{aligned}$$

Before we can state an important corollary, we have to introduce the following concept: if  $L$  is an  $\mathcal{O}$ -submodule of  $K^2$  and  $x \in \{ v l \mid v \in K, l \in L \}$ , we call the set  $L_x := \{ \mu \in K \mid v x \in L \}$  the coefficient of  $x$  in  $L$ . This is a non-zero fractional ideal in  $K$ , and, indeed, if  $L = \mathcal{O}^2$  and  $x \in K^2$  has components  $\alpha, \beta$ , then  $\mathcal{O}_x^2 = (\alpha, \beta)^{-1}$ , where again  $(\alpha, \beta)$  denotes the fractional ideal generated by  $\alpha$  and  $\beta$ . We refer to O'Meara (12) §81B, p. 210 for further properties.

1.1.4. Corollary (Invariance Property) If  $g \in \text{PGL}(2, \mathcal{O})$ , then

$$n_{g\lambda}(g(z, \zeta)) = n_{\lambda}(z, \zeta)$$

Proof: Choose a matrix in  $\text{GL}(2, \mathcal{O})$  representing the class  $g$ ; we

denote this by  $g$ , too. Since  $\det g$  is a unit,  $|\det g|^2 = 1$ . In order to see that  $N(g(\alpha, \beta)) = N(\alpha, \beta)$ , we show that, indeed, for  $x = (\alpha, \beta)$ ,  $\mathcal{O}_{gx}^2 = \mathcal{O}_x^2$ . If  $v \in \mathcal{O}_x^2$ , then  $vx \in \mathcal{O}^2$  and  $v(gx) = g(vx) \in \mathcal{O}^2$ , since  $\text{GL}(2, \mathcal{O})$  is the automorphism group of  $\mathcal{O}^2$ . Hence  $v \in \mathcal{O}_{gx}^2$ . Conversely, if  $v \in \mathcal{O}_{gx}^2$ , then  $vx = v(g^{-1}gx) = g^{-1}v(gx)$ . Now  $v(gx) \in \mathcal{O}^2$ , so that  $vx \in \mathcal{O}^2$ . This shows the equality.

Remark. The Transformation Rule (rather than just the Invariance Property) is useful for comparing geometric properties of certain sets associated to cusps not equivalent under  $\text{PGL}(2, \mathcal{O})$ , i.e., in the case of higher class number. (Cf. Proposition 2.2.1. p. 27).

We shall now derive another basic property of the distance function, its "finiteness" property. To be able to do this, we have to study the objects corresponding to cusps in the language of hermitian forms.

1.1.5. Definition A flag (in  $\mathcal{O}^2$ ) is an  $\mathcal{O}$ -submodule  $L \subset \mathcal{O}^2$  of rank 1 such that  $\mathcal{O}^2 / L$  is torsionfree.

This is the usual definition since  $L$  determines the object  $0 < L \subset \mathcal{O}^2$ . The next proposition gives a characterization of flags, which implies among other things their correspondence with cusps. Note that if an  $\mathcal{O}$ -submodule  $L$  has rank 1, then for any nonzero  $x \in L$ ,  $L = L_x x$ . Furthermore, if  $L \subset \mathcal{O}^2$ , then  $L \subset \mathcal{O}_x^2 x$ . Our claim is as follows:

1.1.6. Proposition Let  $L \subset \mathcal{O}^2$  be an  $\mathcal{O}$ -submodule of rank 1 and let  $x \in L$ ,  $x \neq 0$ . If  $L$  is a flag, then  $L = \mathcal{O}_x^2 x$ .

Proof: We want to compute the torsion submodule  $T(\mathcal{O}^2 / L)$ .  $\bar{y} \in T(\mathcal{O}^2 / L)$  implies that there exist  $\rho, \sigma \in \mathcal{O}$  such that



$\rho y = \sigma x$ , or equivalently,  $y = \frac{\sigma}{\rho} x$ . It follows that  $\frac{\sigma}{\rho} \in \mathcal{O}_x^2$ , since  $y \in \mathcal{O}^2$ . Hence,  $y \in \mathcal{O}_x^2$  and  $\bar{y} \in \mathcal{O}_x^2 / L$ . Conversely,  $\bar{y} \in \mathcal{O}_x^2 / L$  implies that  $y = \alpha x$ , with  $\alpha \in \mathcal{O}_x^2$ . Now there exists a  $\rho \in \mathcal{O}$  such that  $\rho \alpha \in \mathcal{O}$ , and hence  $\rho y = (\rho \alpha) x \in L_x \cdot x = L$ . In other words  $\bar{y} \in T(\mathcal{O}_x^2 / L)$ . Therefore,  $L$  is torsionfree iff  $L = \mathcal{O}_x^2 x$ .

We have, in fact, shown more: we now know that, for any  $x \in K^2$  (not just those in  $\mathcal{O}^2$ ), the  $\mathcal{O}$ -module  $L = \mathcal{O}_x^2 x$  is indeed a flag. This observation allows us to define a  $\text{PGL}(2, K)$ -action on flags, namely:

$$g \cdot (\mathcal{O}_x^2 x) := \mathcal{O}_{gx}^2 gx.$$

This doesn't depend on the choice of the element  $x$ :

if  $\mathcal{O}_y^2 y = \mathcal{O}_x^2 x$ , then  $x = \alpha y$ , with  $\alpha \in K$ . Hence

$$g \cdot (\mathcal{O}_y^2 y) = \mathcal{O}_{g(\alpha^{-1}x)}^2 g(\alpha^{-1}x) = \alpha^{-1} \mathcal{O}_{\alpha^{-1}gx}^2 gx = \mathcal{O}_{gx}^2 gx.$$

We now formulate the correspondence between flags in  $\mathcal{O}^2$  and the cusps of  $K$ .

1.1.7. Proposition There is a bijection

$$\{\text{flags in } \mathcal{O}^2\} \xrightarrow{\sim} \{\text{cusps of } K\}$$

compatible with the  $\text{PGL}(2, K)$ -actions.

Proof: The map is given by: for  $x = (\alpha, \beta) \in K^2$

$$L = \mathcal{O}_x^2 x \longmapsto \frac{\alpha}{\beta}.$$

The verifications are obvious.

1.1.8. Example If  $K$  has class number equal to one, then each flag  $L$  in  $\mathcal{O}^2$  can be written as  $L = \mathcal{O}x$  with  $x \in \mathcal{O}^2$ .

We shall now consider the interpretation of the distance function in terms of hermitian forms and flags. We begin by recalling

the concept of "volume" of a module.

Let  $L$  be an  $\mathcal{O}$ -submodule of rank 1 in  $K$ .  $L$  has rank 2 as a  $\mathbb{Z}$ -module, and we denote by  $x, y$  base elements over  $\mathbb{Z}$ . Let  $V$  be the real subspace generated by  $x, y$  in  $\mathbb{R}^4$  and  $e_1, e_2$  be an orthonormal basis of  $V$  with respect to the real scalar product  $\underline{h} := \text{Re } h$  induced by  $h$ . Denote by  $A$  the transition matrix from  $\{e_1, e_2\}$  to  $\{x, y\}$ .

1.1.9. Definition The volume of  $L$  with respect to  $h$  is given

$$\text{by: } \text{vol}_h L := |\det A|$$

A result from linear algebra tells us:

1.1.10. Proposition Let  $L, h, x, y$  be as in the preceding

discussion. Then  $\text{vol}_h L = (\underline{h}(x, x) \underline{h}(y, y) - \underline{h}(x, y)^2)^{1/2}$

Now if  $L, L'$  are  $\mathbb{Z}$ -modules of rank two in  $\mathbb{R}^4$ , then for any scalar product  $b$  on  $\mathbb{R}^4$ , we get  $\text{vol}_b L' = \text{vol}_b L \cdot \det T$ , where  $T$  is a transition matrix from a basis of  $L$  to one of  $L'$ .

1.1.11. Proposition Let  $(z, \zeta) \in H$  and  $\lambda$  be a cusp of  $K$ . Denote by  $h$  and  $L$  the corresponding hermitian form and flag, respectively.

Let  $D$  be the discriminant of  $K$ . Then:

$$n_\lambda(z, \zeta) = \frac{4}{|D|} (\text{vol}_h L)^2$$

Proof: We do the proof in several steps. First of all, viewing  $L = \mathcal{O}_x^2 x$  and  $\mathcal{O}x$  as abelian groups and recalling the definition of volume, we get

$$\text{vol}_h(\mathcal{O}_x^2 x) = \frac{\text{vol}_h(\mathcal{O}x)}{[\mathcal{O}_x^2 x : \mathcal{O}x]}$$

where the square bracket denotes the index for abelian groups.

We compute the volume of  $\mathcal{O}x$  as follows: a  $\mathbb{Z}$ -basis for  $\mathcal{O}x$  is



given by  $\{x, \omega x\}$ , where  $\omega = \frac{1}{2}(\alpha + \sqrt{D})$  and  $\alpha = 1$  for  $D \equiv 1 \pmod{4}$  and  $\alpha = 0$  otherwise. Proposition 1.1.10 tells us that

$$\begin{aligned} \text{vol}_h(\mathcal{O}x) &= (\underline{h}(x, x) \underline{h}(\omega x, \omega x) - \underline{h}(x, \omega x)^2)^{1/2} \\ &= (\underline{h}(x, x)^2 |\omega|^2 - \underline{h}(x, x)^2 (\text{Re } \bar{\omega})^2)^{1/2} \\ &= (\underline{h}(x, x)^2 |\text{Im } \omega|^2)^{1/2} \\ &= \frac{1}{2} \underline{h}(x, x) \sqrt{|D|} \end{aligned}$$

Hence

$$\frac{2}{\sqrt{|D|}} \text{vol}_h L = \frac{\underline{h}(x, x)}{[L : \mathcal{O}x]}$$

We have to reinterpret the index  $[L : \mathcal{O}x]$  in a second step.

We claim that  $[L : \mathcal{O}x] = [\mathcal{O}_x^2 : \mathcal{O}]$ . The index on the right hand side is the determinant of a transition matrix from a basis of  $\mathcal{O}_x^2$  to one of  $\mathcal{O}$ ; analogously, that on the left hand side is from a basis of  $\mathcal{O}_x^2 x$  to  $\mathcal{O}x$ . Now if  $\alpha_1, \alpha_2$  is a  $\mathbb{Z}$ -basis of  $\mathcal{O}_x^2$ , then  $\alpha_1 x, \alpha_2 x$  is a  $\mathbb{Z}$ -basis of  $\mathcal{O}_x^2 x$ ; similarly for  $\mathcal{O}$  and  $\mathcal{O}x$ . The claim is now obvious.

Finally, since  $\mathcal{O}_x^2 = (\alpha, \beta)^{-1}$  where  $\alpha, \beta$  are the components of  $x$ , then  $[\mathcal{O}_x^2 : \mathcal{O}] = N(\alpha, \beta)$ . This completes the proof.

We can now show the "finiteness" property of the distance function.

**1.1.12. Theorem (Finiteness Property)** Let  $(z, \zeta) \in H$ . For any real number  $c > 0$ , there is just a finite number of cusps  $\lambda$  such that  $n_\lambda(z, \zeta) \leq c$ .

**Proof:** By the preceding proposition, this is equivalent to showing that there are only finitely many flags  $L$  with  $\text{vol}_h L$  less than or equal to  $c' = \frac{\sqrt{c|D|}}{2}$ . Now, it is well known that there are just finitely many rank one  $\mathcal{O}$ -submodules in  $\mathcal{O}^2$  satisfying this condition. Note that if  $M$  is such a non-zero

$\mathcal{O}$ -module and  $x \in M$  is non-zero, then  $L_M := \mathcal{O}_x^2 x$  is a flag with  $\text{vol}_h(L_M) \leq \text{vol}_h M$ .

**1.2.** To simplify the statement of the two main theorems of reduction theory, let us introduce the following notions.

**1.2.1. Definition** A positive real number  $c$  is called an upper reduction constant (for  $K$ ) if for each  $(z, \zeta) \in H$ , there is at least one cusp  $\lambda$  of  $K$  such that  $n_\lambda(z, \zeta) \leq c$ . A positive real  $d$  is a lower reduction constant (for  $K$ ) if for each  $(z, \zeta) \in H$ , there is at most one cusp  $\mu$  of  $K$  such that  $n_\mu(z, \zeta) < d$ , i.e.  $n_\mu(z, \zeta) < d$  and  $n_{\mu'}(z, \zeta) < d \implies \mu = \mu'$ . Note that if  $c$  is an upper reduction constant, each  $c' > c$  has also this property, so that we are interested in getting as low a value as possible. The optimal upper reduction constant is  $\inf\{c \mid c \text{ upper reduction constant}\}$ . Similarly we are interested in getting the highest possible value of  $d$ .

The main problem of course is to show that such constants exist at all. These statements are the two main theorems of reduction theory.

**1.2.2. First Reduction theorem**  $\frac{|D|}{2}$  is an upper reduction constant for any imaginary quadratic field  $K$ .

**1.2.3. Second Reduction theorem** 1 is the optimal lower reduction constant for any imaginary quadratic field  $K$ .

**Proof of 1.2.2.:** Let  $h$  be the hermitian form associated to  $(z, \zeta)$  and  $x$  be a minimal vector of  $h$  in  $\mathcal{O}^2$ . By a result of Korkine-Zolotarev (13) we have  $h(x, x) \leq \sqrt{\frac{|D|}{2}}$ . Choose  $\lambda$

to be the cusp associated to  $\sigma_x^2 x$ .

$$n_\lambda(z, \zeta) = \left( \frac{h(x, x)}{N(\alpha, \beta)} \right)^2 \leq (h(x, x))^2 \leq \frac{|D|}{2}.$$

1.2.4. Remark As one would expect,  $\frac{|D|}{2}$  is not the optimal upper reduction constant in general. It is, however, optimal for all imaginary quadratic fields with class number equal to one, except  $Q(\sqrt{-7})$ . This follows from the computations of Bianchi ( ) and was proved independently by Speiser, Perron, Oberseider and Oppenheim (cf. (13), (11) and the references given there.) Actually, the last group of authors (1932 - 1934) computed minima of positive hermitian forms and were not aware of the connection with reduction theory. It was Mahler (11) who pointed this out more than five years later.

In order to prove the Second Reduction Theorem, we derive the following easy but useful result:

1.2.5. Proposition (Incidence Criterion) Let  $\lambda, \mu$  be different cusps of  $K$  and let  $c_1, c_2$  be positive real numbers. Then:

$$n_\lambda^{-1}(c_1) \cap n_\mu^{-1}(c_2) \neq \emptyset \text{ iff } \sqrt{c_1 c_2} N_\lambda N_\mu \geq N(\lambda - \mu)$$

Proof: We set  $N(\infty) := 1$ . We have to consider two cases:

Case 1.  $\lambda, \mu$  are both different from  $\infty$ . The level sets are spheres; now two spheres have a non-empty intersection if and only if the sum of their radii is greater than or equal to the (euclidean) distance between their centers. If  $S_1$  and  $S_2$  have the centers  $(z_1, r_1)$  and  $(z_2, r_2)$  and radii  $r_1, r_2$  respectively,

$$S_1 \cap S_2 \neq \emptyset \text{ iff } r_1 + r_2 \geq (|z_1 - z_2|^2 + |r_1 - r_2|^2)^{1/2}$$

$$\text{iff } (r_1 + r_2)^2 - (r_1 - r_2)^2 \geq |z_1 - z_2|^2$$

$$\text{iff } 4r_1 r_2 \geq |z_1 - z_2|^2$$

Substituting  $S_1 = n_\lambda^{-1}(c_1)$  and  $S_2 = n_\mu^{-1}(c_2)$ , we obtain the condition

$$\frac{\sqrt{c_1}}{2} N_\lambda \frac{\sqrt{c_2}}{2} N_\mu \geq |\lambda - \mu|^2 = N(\lambda - \mu)$$

Case 2. We can assume that  $\lambda = \infty$ . The horizontal plane  $\zeta = \frac{1}{\sqrt{c_1}}$  intersects the sphere  $n_\mu^{-1}(c_2)$  if and only if the "height" of the plane is less than or equal to the diameter of the sphere.

This means that  $\frac{1}{\sqrt{c_1}} \leq \sqrt{c_2} N_\mu$

$$\text{or } 1 \leq \sqrt{c_1 c_2} N_\mu$$

Now  $N_\infty = 1$  and by convention  $N(\infty - \mu) = N(\infty) = 1$ .

1.2.6. Remark If we choose  $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathcal{O}$  such that  $\lambda = \frac{\alpha}{\beta}, \mu = \frac{\gamma}{\delta}$  we get the equivalent condition

$$c_1 c_2 N(\alpha, \beta) N(\gamma, \delta) \geq N(\alpha\delta - \gamma\beta).$$

Here we don't need the convention that  $N(\infty) = 1$ .

Proof of 1.2.3. Let  $c_1, c_2$  be positive real numbers both less than 1. Let  $\lambda, \mu$  be different cusps such that there is a point  $(z, \zeta) \in n_\lambda^{-1}(c_1) \cap n_\mu^{-1}(c_2)$ . If we choose  $\alpha, \beta, \gamma, \delta$  as in the preceding remark, we get the condition

$$\sqrt{c_1 c_2} N(\alpha, \beta) N(\gamma, \delta) \geq N(\alpha\delta - \gamma\beta).$$

Since  $\alpha\delta - \gamma\beta$  is in the product ideal  $(\alpha, \beta) \cdot (\gamma, \delta)$ , we have

$$N(\alpha\delta - \gamma\beta) \geq N(\alpha, \beta) \cdot N(\gamma, \delta).$$

But  $\sqrt{c_1 c_2} < 1$ , and this gives the contradiction.

To show the optimality, we note that for any imaginary quadratic field  $K$ , the point  $(z, \zeta) = ((0, 0), 1)$  has distance 1 to both cusps  $O_\infty$  (cf Example 1.1.2.)

1.2.7. Remark One can also prove the Second Reduction Theorem by showing that, for any two flags  $L, L' \subset \mathcal{O}$

$$\text{vol}_h(L + L') \leq \text{vol}_h L \cdot \text{vol}_h L'$$



(this generalizes the inequality  $|x \times y| \leq |x||y|$ !) Since  $\text{vol}_H(L + L') = [\mathcal{O}^2 : L + L'] \cdot \text{vol}_H(\mathcal{O}^2)$ , one easily obtains the result.

1.3. From now on,  $\Gamma := \text{PGL}(2, \mathcal{O})$ . We shall now discuss the construction of a (strict) fundamental domain for  $\Gamma$  in  $H$ . We shall not provide details, as the arguments in our case are basically the same as those in Siegel's lectures (14) treating the case of Hilbert modular groups. The main purpose of our discussion is simply to collect more or less well known facts needed later.

Let  $\lambda$  be a cusp of  $K$ . We have the following important concept:

1.3.1. Definition  $H_\lambda := \{(z, \zeta) \in H \mid n_\lambda(z, \zeta) \leq n_\mu(z, \zeta)\}$  for all cusps  $\mu$  of  $K$  is called the minimal set of  $\lambda$  in  $H$ .

First of all,  $H_\lambda$  is never empty, since, by the Second Reduction Theorem, it contains the open ball (or open half-space)  $\{(z, \zeta) \in H \mid n_\lambda(z, \zeta) < 1\}$ . Moreover, each  $H_\lambda$  is a closed subset of the upper half space: if a sequence  $((z_n, \zeta_n))$  in  $H_\lambda$  converges to  $(z, \zeta)$  in  $H$ , then by definition, for all cusps  $\mu$  and for all  $n$ ,  $n_\lambda(z_n, \zeta_n) \leq n_\mu(z_n, \zeta_n)$ . By the continuity of the cusp distances,  $n_\lambda(z, \zeta) \leq n_\mu(z, \zeta)$  for all cusps  $\mu$ . We denote the boundary of  $H_\lambda$  by  $I_\lambda$ .

It is easy to see that  $H_\lambda$  is a (closed) covering of  $H$ : for any upper reduction constant  $c$ , there is at least one and at most finitely many cusps  $\mu$  with  $n_\mu(z, \zeta) \leq c$ . Choose a cusp  $\lambda$  to which  $(z, \zeta)$  has minimal distance: by definition  $(z, \zeta) \in H_\lambda$ .

Further properties of the minimal sets will be studied in § 2.

1.3.2. Example It is very useful to draw the picture for the classical case  $K = \mathbb{Q}$  in order to get a (geometric) feeling for these minimal sets. The  $H_\lambda$ 's we are studying are just somewhat more complicated (at least more difficult to draw) 3-dimensional analogues.

For a point  $z = x + iy$  in the upper half plane, the distance to a cusp  $\lambda = \frac{\alpha}{\beta}$  in  $\mathbb{Q} \cup \{\infty\}$  is given by:

$$n_\lambda(z) := \frac{(-\beta x + \alpha)^2 + \beta^2 y^2}{y}$$

The level sets are the well known Farey circles for  $\lambda \neq \infty$  and horizontal lines for  $\lambda = \infty$ .

We have the following picture of the minimal sets:

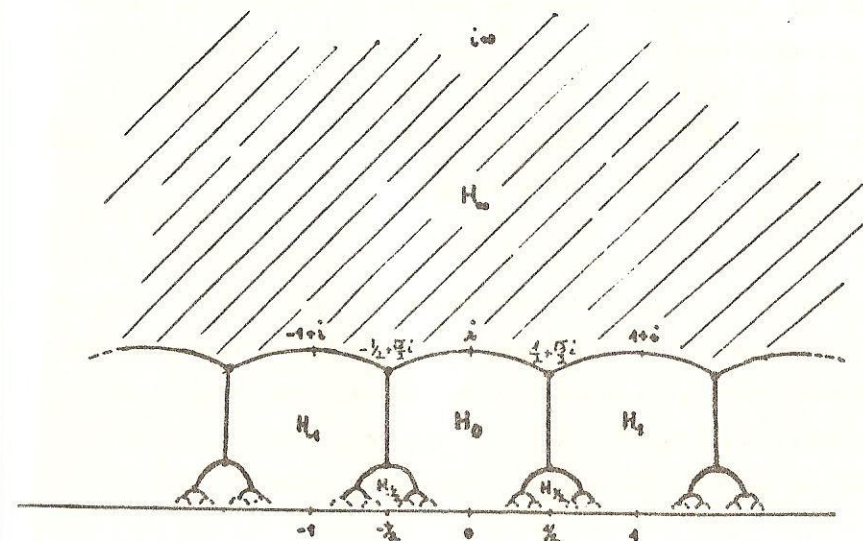


Figure 1.3.2.

The (infinite) tree in the drawing is the union of the boundary sets  $I_\lambda$ . One finds the same object in (16), p. 53, where it is studied from a slightly different point of view.

Note that the minimal sets transform nicely under  $\Gamma$ : for any cusp  $\lambda$  of  $K$  and  $g \in \Gamma$ ,

$$g H_\lambda = H_{g\lambda}.$$

This follows immediately from the invariance property of cusp distances. In particular the image of a minimal set is again a minimal set. This is no longer true in general if  $g \notin \Gamma$ . For example, consider the matrix  $\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Q})$ ; this induces the translation  $z \mapsto z + 1/2$  and the image of, say  $H_\infty$  is not a minimal set.

We can now start with the construction of the fundamental domain. Denote by  $h_K$  (or simply  $h$ ) the class number of  $K$ . It is well known that there are exactly  $h_K$   $\Gamma$ -orbits of cusps, the correspondence being as follows: write  $\lambda = \frac{\alpha}{\beta}$  with  $\alpha, \beta \in \mathcal{O}$  and form the ideal class  $[(\alpha, \beta)]$ . The map  $\Gamma\lambda \mapsto [(\alpha, \beta)]$  is well defined and is indeed a bijection. (Compare (14), Proposition 20, p. 242.) Hence we get

$$H = \Gamma \cdot (H_{\lambda_1} \cup \dots \cup H_{\lambda_h})$$

where the  $\lambda_i$ 's are representatives of the  $\Gamma$ -orbits. However, this finite union is not yet a fundamental domain since any minimal set  $H_\lambda$  is stabilized by the isotropy group  $\Gamma_\lambda$  of  $\lambda$  in  $\Gamma$  (invariance property again!). The following well known result can be used to construct a fundamental domain  $T_{\lambda_1}$  for  $\Gamma_{\lambda_1}$  in  $H$ :

**1.3.3. Proposition** Let  $\lambda$  be a cusp. Choose  $\alpha, \beta \in \mathcal{O}$  with  $\lambda = \frac{\alpha}{\beta}$  and fix a matrix  $A$  which maps  $\infty$  to  $\lambda$ . Then:  $\Gamma_\lambda$  consists of elements of the form

$$A \begin{pmatrix} \varepsilon & \rho\varepsilon' \\ 0 & \varepsilon' \end{pmatrix} A^{-1}$$

where  $\rho \in (\alpha, \beta)^{-2}$  and  $\varepsilon, \varepsilon'$  are units of  $\mathcal{O}$ .

(Compare (14), Proposition, p. 246) For example, for  $\lambda = 0$ , we can set  $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Hence, any  $g \in \Gamma_0$  is of the form

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon & \rho\varepsilon' \\ 0 & \varepsilon' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 \\ \rho\varepsilon' & \varepsilon' \end{pmatrix}.$$

We now let  $F_i := H_{\lambda_i} \cap T_{\lambda_i}$ . The set  $F := F_1 \cup \dots \cup F_h$  is a fundamental domain for the action of  $\Gamma$  in  $H$ . Again, we refer to Siegel's notes for details (14), pp. 261 - 269.

To finish this section, we state another important result, a compactness criterion:

**1.3.4. Proposition** Let  $r_1, \dots, r_h$  be positive real numbers. The set

$F(r_1, \dots, r_h) := \{(z, \zeta) \in F \mid n_{\lambda_i}(z, \zeta) \geq r_i \text{ for all } 1 \leq i \leq h\}$  is compact.

One easily proves by an explicit construction of  $T_{\lambda_i}$  that the set  $B_i := \{(z, \zeta) \in T_{\lambda_i} \mid c \geq n_{\lambda_i}(z, \zeta) \geq r_i\}$  is compact (here  $c$  is an upper reduction constant). Hence  $B = B_1 \cup \dots \cup B_h$  is also compact, and this is also the case for the closed subset  $F(r_1, \dots, r_h)$ . (Compare (14), p. 253 and Proposition 22, p. 270).



## § 2. THE MINIMAL INCIDENCE SET OF AN IMAGINARY QUADRATIC FIELD

Remember:  $\Gamma := \text{PGL}(2, \mathcal{O})$ .

2.0. In this section, we shall use reduction theory to find a "good" space for the group  $\Gamma$  - "good" in the sense that it provides an effective method for computing cohomology. This is the minimal incidence set  $I$  (2.1.6.) - the union of the boundaries  $I_\lambda$  of the minimal sets  $H_\lambda$ .  $I$  is a contractible subspace of  $H$  on which  $\Gamma$  still acts in a properly discontinuous manner and the quotient  $\Gamma \backslash I$  is compact. Moreover,  $I$  has a 2-dimensional cell structure compatible with the action of  $\Gamma$ . The cell structure also has the following nice "cohomological" property: the (finite) isotropy groups remain the same for points in an open cell.

In case the class number of  $K$  is 1,  $I = \Gamma \cdot I_\infty$  and the set  $I_\infty$  has been studied in the classical theory (see §3 for a detailed discussion and references.)

In 2.1. we introduce a number of concepts involving "minimality" - minimal cusp distance, minimal cusp set of a point (both 2.1.1.), minimal incidence set of  $K$  (2.1.6.) - and study their basic properties and mutual interrelationships. We show that the quotient  $\Gamma \backslash I$  is compact (2.1.9.). The cardinality of the minimal cusp set (2.1.11.) - viewed as a function  $d: H \rightarrow \mathbb{N}$  - presents a number of interesting problems; we study its values at fixed points of the  $\Gamma$ -action. In 2.2. we describe the cell structure on  $I$  and prove the properties mentioned above. Finally, in 2.3., we show that the minimal incidence set is a contractible space, being a (strong) deformation retract of  $H$ . In fact, we prove that each  $I_\lambda$  is a  $\Gamma_\lambda$ -equivariant deformation retract of  $H_\lambda$ .

2.1. We proved in §1 that for a given element  $(z, \zeta) \in H$  and a fixed constant  $c$ , there is just a finite number of cusps to which  $(z, \zeta)$  has distance less than or equal to  $c$ . This fact allows us to define the following function:

2.1.1. Definition The minimal cusp distance of  $(z, \zeta)$  is given by:

$$n(z, \zeta) := \min\{n_\lambda(z, \zeta) \mid \lambda \in K \cup \{\infty\}\}$$

We denote by  $M(z, \zeta)$  the (finite) set of cusps  $\lambda$  such that  $n_\lambda(z, \zeta) = n(z, \zeta)$ . An element of  $M(z, \zeta)$  is called a minimal cusp of  $(z, \zeta)$ .

The concept of "minimal cusp of a point" is the dual notion to that of "minimal set of a cusp", i.e., we have:

$$H_\lambda = \{(z, \zeta) \mid \lambda \in M(z, \zeta)\} \quad \text{and} \quad M(z, \zeta) = \{\lambda \in K \cup \{\infty\} \mid (z, \zeta) \in H_\lambda\}$$

We shall now describe some basic properties of the function  $n$ . It is obviously invariant under  $\Gamma$ . For any cusp  $\lambda$ , it is smooth on the set  $H_\lambda = I_\lambda$  (it's simply the restriction of  $n_\lambda$  to this open set.) We will now show that it is in fact continuous on the whole upper half-space. We need the following easy (but important) Lemma:

2.1.2. Lemma For each  $(z, \zeta) \in H$ , there is an open neighborhood  $U$  such that

$$U \cap H_\lambda \neq \emptyset \quad \text{iff} \quad \lambda \in M(z, \zeta)$$

In particular,  $\{H_\lambda\}$  is a locally finite closed covering of  $H$ .

Proof: The sufficiency of the condition is obvious. To prove the necessity, note that  $\lambda \notin M(z, \zeta)$  means that  $n_\lambda(z, \zeta) > n(z, \zeta)$ . Hence, we only need to show that  $\inf\{n_\lambda(z, \zeta) \mid \lambda \notin M(z, \zeta)\}$  is (strictly) greater than  $n(z, \zeta)$ . Now, there are just finitely many cusps  $\lambda_1, \dots, \lambda_n$  such that  $n_{\lambda_1}(z, \zeta) \leq n(z, \zeta) + 1$ . We



have two cases to consider:

Case 1: Not all  $\lambda_i$ 's are contained in  $M(z, \zeta)$ . In this case

$$\inf n_\lambda(z, \zeta) | \lambda \notin M(z, \zeta) = \min\{n_{\lambda_i}(z, \zeta) \mid \lambda_i \notin M(z, \zeta) \ 1 \leq i \leq n\} > n(z, \zeta).$$

Case 2: All  $\lambda_i$ 's are minimal for  $(z, \zeta)$ . Then we have

$$\inf\{n_\lambda(z, \zeta) \mid \lambda \notin M(z, \zeta)\} > n(z, \zeta) + 1 > n(z, \zeta).$$

2.1.3. Proposition The function  $n: H \rightarrow \mathbb{R}$  is continuous.

Proof: By the preceding Lemma,  $\{H_\lambda\}$  is a locally finite closed covering of  $H$ . Since  $n$  restricted to  $H_\lambda$  is simply  $n_\lambda$  restricted to  $H_\lambda$ , by elementary topology, the claim follows.

We consider now the image of  $n$ . Minimal cusp distance can be arbitrarily small—we shall now see that it cannot be arbitrarily large. First we need the following Lemma:

2.1.4. Lemma Let  $H' = \{(z, \zeta) \mid n(z, \zeta) \geq 1\}$ . Denote by  $F$  the fundamental domain described in §1.3. Then  $F' := F \cap H'$  is compact and  $H' = \Gamma \cdot F'$ .

Proof: The compactness follows immediately from Proposition 1.3.4. since  $n_{\lambda_i}(z, \zeta) \geq n(z, \zeta) \geq 1$  for each  $(z, \zeta)$ , whereby  $\lambda_1, \dots, \lambda_n$  are representatives of  $\Gamma$ -orbits of cusps. The second assertion is a consequence of the  $\Gamma$ -invariance of  $n$ .

We now claim:

2.1.5. Proposition  $n$  attains an absolute maximum  $m$  on  $H$ , which is equal to  $\inf\{c \mid c \text{ upper reduction constant}\}$ .

Proof: Since  $n$  is continuous, it has a maximum on the compact set  $F'$ . This value is indeed the maximum on  $H'$  due to the  $\Gamma$ -in-

variance of  $n$ . Now, by definition of  $H'$ , any value of  $n$  on  $H'$  is greater than or equal to any value on  $H \setminus H'$ ; it follows that the maximum of  $n$  on  $H'$  is the absolute maximum  $m$  on  $H$ .

We want to show that  $m =$  the optimal upper reduction constant  $\inf\{c \mid c \text{ upper reduction constant}\}$ . For any such  $c$ ,  $m \leq c$ ; this is because there is at least one point  $(z, \zeta)$  with  $n(z, \zeta) = m$ . For such a point, there is at least one cusp  $\lambda$  such that  $n_\lambda(z, \zeta) \leq c$ , whence the claim follows. Therefore,  $m \leq \{\inf c \mid c \text{ upper reduction constant}\}$ . Suppose that strict inequality holds; this implies, in particular, that  $m$  is not an upper reduction constant. Thus, there is a  $(z_0, \zeta_0)$  which has cusp distance  $n_\lambda(z_0, \zeta_0) > m$  for all cusps  $\lambda$ . On the other hand, there is at least one cusp  $\lambda_0$  such that  $n(z_0, \zeta_0) = n_{\lambda_0}(z_0, \zeta_0)$ . This leads to the contradiction  $n(z_0, \zeta_0) > m$ .

By virtue of the First Reduction Theorem, we have the general estimate  $m \leq \frac{|D|}{2}$ .

We now want to specify where this maximum value  $m$  is attained:

2.1.6. Definition The minimal incidence set  $I_K$  (or simply  $I$ ) of an imaginary quadratic field  $K$  is defined as:

$$I_K := \bigcup_{\substack{(\lambda, \mu) \\ \lambda, \mu \text{ distinct cusps of } K}} H_\lambda \cap H_\mu$$

$I_K$  consists of all points in  $H$  with at least two minimal cusps. Alternatively, we could write  $I_K = \bigcup \{I_\lambda \mid \lambda \text{ cusp of } K\}$ . Note that  $I$  is a closed set, since the sets  $H_\lambda \cap H_\mu$  form a locally finite closed covering of  $I$ .

Notation We will be dealing in the following sections with a number of sets which, like  $I_K$ , are indexed by pairs of distinct cusps of  $K$ . To save space, we shall just write  $(\lambda, \mu)$  to denote such pairs, and no longer explicitly state that the cusps  $\lambda, \mu$  have to be distinct.

The minimal incidence set will play a central role in our further investigations. At this juncture, we just want to show the following:

2.1.7. Proposition  $n$  attains its maximum only at points in  $I$ . Moreover, the image of  $I$  under  $n$  is the closed interval  $[1, m]$ .

Proof: Let  $(z_0, \zeta_0) \in H$  with  $n(z_0, \zeta_0) = m$ . Suppose that it is in  $H \setminus I$ . Since  $I$  is closed, there is an open ball  $D((z_0, \zeta_0), r_0)$  contained in  $H \setminus I$ . Let  $\lambda$  be the unique cusp such that  $n(z_0, \zeta_0) = n_\lambda(z_0, \zeta_0)$ . Now there is a point  $(z_1, \zeta_1)$  with  $n(z_1, \zeta_1) = n_\lambda(z_1, \zeta_1) > n_\lambda(z_0, \zeta_0) = n(z_0, \zeta_0)$ , a contradiction.

According to the Second Reduction Theorem, there are points in  $I$  with minimal cusp distance equal to 1 (Cf. Example 1.1.2.) The claim follows from the intermediate value theorem of calculus

2.1.8. Example Let  $K = \mathbb{Q}(i)$  and  $\mathcal{O} = \mathbb{Z}[i]$ , the ring of Gaussian integers. As we have remarked in 1.2.4., the optimal upper reduction constant in this case is equal to  $\frac{|D|}{2}$ , i.e. 2. By 2.1.5. this is also the maximal minimal cusp distance.  $n$  attains it at  $(\frac{1+i}{2}, \frac{\sqrt{2}}{2})$  and, by  $\Gamma$ -invariance, at all images of it under  $\Gamma$ . Our computations in §4 show that these are the only points in  $H$  where the maximal value is attained.

It is obvious that  $I$  is stable under the action of  $\Gamma$ . The

first cohomologically relevant property of the quotient space  $\Gamma \backslash I$  is the following:

2.1.9. Theorem  $\Gamma \backslash I$  is compact.

Proof: The quotient space  $\Gamma \backslash I$  is compact if and only if there is a non-empty compact subset  $C$  of  $I$  such that  $I = \Gamma \cdot C$ . Take  $C = I \cap F$ , where  $F$  is the fundamental domain for  $\Gamma$  constructed in § 1.3. This set is non-empty, closed (being the intersection of two closed sets!) and, by virtue of the Second Reduction Theorem, contained in the compact set  $F'$  of Lemma 2.1.4. Hence  $C$  is compact. It is clear that  $I = \Gamma \cdot C$ .

2.1.10. Remark With this result, we have already overcome one difficulty in the cohomology computation, the non-compactness of the original quotient space  $\Gamma \backslash H$ . Of course, there are other  $\Gamma$ -invariant subsets of  $H$  with compact quotient (for example,  $H'$  in Lemma 2.1.4.). But as we shall see in §4 and §5, working with  $I$  has a number of advantages.  $I$ , for example, has dimension equal to 2, the lowest possible.

By means of the minimal cusp distance, we are able to attach to each point  $(z, \zeta)$  in  $H$  a finite set of cusps, namely  $M(z, \zeta)$ . We will now investigate questions concerning its cardinality.

2.1.11. Definition The cusp degree of  $(z, \zeta)$  is defined as:

$$d(z, \zeta) := |M(z, \zeta)|$$

We denote by  $d: H \longrightarrow \mathbb{N}$  the cusp degree function.

The following proposition describes the properties of



$M(z, \zeta)$  and  $d$  with respect to  $\Gamma$ -equivalence:

2.1.12. Proposition i) If  $(z', \zeta') = \gamma \cdot (z, \zeta)$ , then there is a natural bijection  $\tilde{\gamma}: M(z, \zeta) \longrightarrow M(z', \zeta')$ .

ii) The cusp degree is  $\Gamma$ -invariant, i.e.,  $d(\gamma(z, \zeta)) = d(z, \zeta)$ .

Proof: ii) follows immediately from i). The natural map  $\tilde{\gamma}$  is defined as follows:  $\lambda \longmapsto \gamma\lambda$ . Now  $n_{\gamma\lambda}(z', \zeta') = n_{\gamma\lambda}(\gamma(z, \zeta)) = n_{\lambda}(z, \zeta) = n(z, \zeta) = n(\gamma(z, \zeta)) = n(z', \zeta')$ . Hence  $\gamma\lambda \in M(z', \zeta')$ . The map is obviously bijective.

The fibres  $d^{-1}(i)$ , where  $i \in \mathbb{N}$ , are also  $\Gamma$ -invariant sets; they form another interesting decomposition of the upper half-space. If we write  $H_1 := d^{-1}(1)$ , then we have

$$H = \bigcup_{i \geq 1} H_i$$

and

$$I = \bigcup_{i \geq 2} H_i$$

The first question that comes to mind is the following: is this a finite decomposition? In other words, is  $d$  a bounded function? The positive answer is implied by the following result:

2.1.13. Proposition  $d$  is an upper semi-continuous function.

Proof: Since  $\mathbb{N}$  is a discrete set,  $d$  is upper semi-continuous at  $(z, \zeta)$  if and only if there is a neighborhood of  $(z, \zeta)$  such that  $d(z', \zeta') < d(z, \zeta)$  for all  $(z', \zeta')$  in the neighborhood. Take  $U$  as in Lemma 2.1.2. to be the neighborhood.

Since  $\bigwedge_{\Gamma} I$  is compact, there is a compact subset  $C \subset I$

with  $I = \Gamma \cdot C$ . By elementary calculus, it follows that  $d$  has a maximum on this set; due to  $\Gamma$ -invariance, this is also a maximum for  $I$ , and hence for  $H$ , too. We summarize this as follows:

2.1.14. Proposition There is a natural number  $d_{\max}$  (called the maximal degree of  $K$ ) such that for all  $i > d_{\max}$ ,  $H_i = \emptyset$ .

The degree function assumes only a finite number of values. One would, of course, like to more about these values, i.e., which invariants of the quadratic field determine them. To be more specific, here are some open questions:

1. What is the maximal degree  $d_{\max}$  for a given imaginary quadratic field?
2. Which values between 2 and  $d_{\max}$  occur?
3. At which points does the maximal value of  $d$  occur? At points with maximal minimal cusp distance?

A sort of starting point for studying such questions is to look at the values of  $d$  at fixed points of  $\Gamma$ . First of all, we characterize the fixed points which do not lie in  $I$  and their stabilizers.

2.1.15. Proposition Let  $K$  be an imaginary quadratic field. Let  $(z, \zeta)$  be a fixed point of  $\Gamma$  with  $M(z, \zeta) = \{ \frac{\alpha}{\beta} \}$ . Let  $A$  be a fixed matrix mapping  $\infty$  to  $\frac{\alpha}{\beta}$ . If  $\gamma \in \Gamma(z, \zeta)$ , then  $\gamma$  is induced by an element of the form

$$A \begin{pmatrix} \epsilon & (1-\epsilon)z' \\ 0 & 1 \end{pmatrix} A^{-1}$$

where  $\epsilon$  is a unit in  $O$  and  $A^{-1}(z, \zeta) = (z', \zeta')$ . The order of  $\gamma$  is equal to the order of  $\epsilon$ . If  $n$  is the maximal order of a stabilizer in  $\Gamma(z, \zeta)$ , then  $\Gamma(z, \zeta) = \frac{\mathbb{Z}}{n\mathbb{Z}}$ .

Proof: If  $\gamma$  stabilizes  $(z, \zeta)$ , then  $\gamma \frac{\alpha}{\beta} = \frac{\alpha}{\beta}$ . Hence,  $\gamma$  is an element of finite order in  $\Gamma_{\frac{\alpha}{\beta}}$ . By Proposition 1.3.3., any element of  $\Gamma_{\frac{\alpha}{\beta}}$  has the form

$$A \begin{pmatrix} \epsilon & r \\ 0 & 1 \end{pmatrix} A^{-1}$$

with  $\epsilon$  a unit in  $\mathcal{O}$ ,  $A$  as above and  $r \in (\alpha, \beta)^{-2}$ . Since

$$\left( A \begin{pmatrix} \epsilon & r \\ 0 & 1 \end{pmatrix} A^{-1} \right)^m = \begin{pmatrix} \epsilon & r \\ 0 & 1 \end{pmatrix}^m = \begin{pmatrix} \epsilon^m & r \left( \sum_{i=0}^{m-1} \epsilon^i \right) \\ 0 & 1 \end{pmatrix}$$

such a transformation  $\neq$  identity has infinite order if  $\epsilon = 1$  and order equal to that of  $\epsilon$  if  $\epsilon \neq 1$ . Now  $(z', \zeta') := A^{-1}(z, \zeta)$  is a fixed point of  $\begin{pmatrix} \epsilon & r \\ 0 & 1 \end{pmatrix}$  so that  $\epsilon z' + r = z'$  and  $r = (\epsilon - 1)z'$ .

The last statement is now evident.

2.1.16. Remarks a) A fixed point  $(z, \zeta)$  not in  $I$  thus lies on a whole fixed geodesic with an "endpoint" in  $\frac{\alpha}{\beta}$ . Such a geodesic intersects  $I_{\frac{\alpha}{\beta}}$  at a single point (cf. the application of Lemma 2.3.3., p. 40).

b) For all fields  $K \neq \mathbb{Q}(i)$ ,  $\mathbb{Q}(\sqrt{-3})$ , the proposition tells us that the stabilizer of a fixed point not in  $I$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . This also follows from that fact such a fixed point has trivial stabilizer in  $\text{PSL}(2, \mathcal{O})$ . If  $\gamma_1, \gamma_2$  are stabilizers different from the identity, then  $\gamma_1 \gamma_2, \gamma_1^2, \gamma_2^2$  are also stabilizers, but which belong to  $\text{PSL}(2, \mathcal{O})$ . Hence,  $\gamma_1 \gamma_2 = \gamma_1^2 = \gamma_2^2$ , and this implies that  $\gamma_1 = \gamma_2$ .

2.1.17. Example Let  $K = \mathbb{Q}(i)$  and  $\mathcal{O} = \mathbb{Z}[i]$ . The points  $(0, \epsilon)$  with  $\epsilon > 1$  lie in  $H_{\infty} \setminus I_{\infty}$  and those with  $\epsilon < 1$  lie in  $H_0 \setminus I_0$ . Such points are stabilized by the map induced by  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

The stabilizer  $\Gamma(z, \zeta)$  of a point acts on the minimal cusp set as follows:  $(\gamma, \lambda) \mapsto \gamma\lambda$ . Similarly, we have an action of  $\Gamma(z, \zeta)$ , the stabilizer in  $\text{PSL}(2, \mathcal{O})$ . Using this action, we can derive the relationship between the stabilizer orders and the cusp degree of a point:

2.1.18. Proposition Let  $(z, \zeta)$  be any point in  $H$  and  $\tilde{\Gamma}(z, \zeta)$ ,  $\Gamma(z, \zeta)$  as above. Then:

- $|\tilde{\Gamma}(z, \zeta)|$  divides  $\frac{|\mathcal{O}^*|}{2} d(z, \zeta)$ . Moreover,  $|\tilde{\Gamma}(z, \zeta)| = \frac{|\mathcal{O}^*|}{2} d(z, \zeta)$  implies that the above action is transitive.
- $|\Gamma(z, \zeta)|$  divides  $|\mathcal{O}^*| d(z, \zeta)$  and similarly, equality implies transitivity.

Proof: We will just prove a) in detail since analogous arguments hold for b). If  $|\tilde{\Gamma}(z, \zeta)| = 1$ , then the claim is trivial. If  $\frac{\alpha}{\beta} = \lambda \in M(z, \zeta)$ , then we have a bijection:  $\frac{\tilde{\Gamma}(z, \zeta)}{\tilde{\Gamma}(z, \zeta) \cap \Gamma_{\frac{\alpha}{\beta}}} \cong \Gamma(z, \zeta) \lambda$ . We claim that  $|\tilde{\Gamma}(z, \zeta) \cap \Gamma_{\frac{\alpha}{\beta}}| = 1$  or  $\frac{|\mathcal{O}^*|}{2}$ . Now an element of  $\Gamma_{\frac{\alpha}{\beta}}$  with determinant = 1 can be written as  $A \begin{pmatrix} \epsilon & r \\ 0 & \epsilon'$   $A^{-1}$  with  $A, \epsilon, r$  as above. If  $K \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ , the only such element of finite order is the identity. For  $K = \mathbb{Q}(i)$ , the transformation of order 2 induced by  $A \begin{pmatrix} 1 & -2iz' \\ 0 & -1 \end{pmatrix} A^{-1}$  is in  $\tilde{\Gamma}(z, \zeta) \cap \Gamma_{\frac{\alpha}{\beta}}$  iff  $-2iz' \in (\alpha, \beta)^{-2}$ . For  $K = \mathbb{Q}(\sqrt{-3})$  and  $\omega := \frac{1 + \sqrt{3}i}{2}$ , a similar statement holds for the map of order 3 induced by  $A \begin{pmatrix} \omega & -3iz' \\ 0 & -\omega^2 \end{pmatrix} A^{-1}$ . In both cases,  $|\Gamma(z, \zeta) \cap \Gamma_{\frac{\alpha}{\beta}}| = 1$  or  $\frac{|\mathcal{O}^*|}{2}$ . Since

$$d(z, \zeta) = |M(z, \zeta)| = \sum_{\text{orbits}} |\tilde{\Gamma}(z, \zeta) \lambda| = \sum_{\lambda} \frac{|\tilde{\Gamma}(z, \zeta)|}{|\tilde{\Gamma}(z, \zeta) \cap \Gamma_{\lambda}|} = |\tilde{\Gamma}(z, \zeta)| \sum \frac{1}{|\tilde{\Gamma}(z, \zeta) \cap \Gamma_{\lambda}|}$$

Multiplying by  $\frac{|\mathcal{O}^*|}{2}$ , we get  $|\Gamma(z, \zeta)| \cdot n = \frac{|\mathcal{O}^*|}{2} d(z, \zeta)$  with  $n \in \mathbb{N}$ .

Note that  $n \geq$  number of orbits of  $\tilde{\Gamma}(z, \zeta)$ , whence  $n = 1$  implies transitivity. If  $K \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ , then  $n =$  number of orbits.



2.1.19. Remarks It is easy to see that there are points  $(z, \zeta)$  in  $I$  with trivial stabilizer, so that in general the actions of  $\Gamma(z, \zeta)$  and  $\Gamma(z, \zeta)$  are non-transitive. However, there are also fixed points in  $I$  with non-transitive action. For example, for  $K = \mathbb{Q}(\sqrt{-7})$ ,  $\tilde{\Gamma}(z, \zeta) = \Gamma(z, \zeta) \cong \mathbb{Z}/2\mathbb{Z}$  for any  $(z, \zeta)$  with  $z\bar{z} + \zeta^2 = 1$ ,  $0 < \operatorname{Re} z < \frac{1}{2}$  and  $\operatorname{Im} z = \frac{-\operatorname{Re} z}{7} + \frac{2}{\sqrt{7}}$ . (Cf. §4.2 for the stabilizer computations). However,  $d(z, \zeta) = 4$ , so that there are 2 orbits.

2.2. We will now prove that  $I$  has a cell structure with the property: the stabilizers in  $\Gamma$  remain the same for points in an open cell. This will be done in two steps: first of all, we show that for each pair of distinct cusps  $(\lambda, \mu)$ , the set  $H_\lambda \cap H_\mu$  has a natural cell structure. This induces a cell structure on  $I$  with the following property: the minimal cusp sets remain the same for points in an open cell. Then we show how to refine this cellular subdivision in order to get a new one with the nice "cohomological" property stated above.

We begin by studying the set of points in  $H$  which are equidistant to two distinct cusps  $\lambda, \mu$ . We define:

$$S(\lambda, \mu) := \bigcup_{0 < c < \infty} n_\lambda^{-1}(c) \cap n_\mu^{-1}(c)$$

This set is easy to picture geometrically. If  $N_\lambda = N_\mu$ , and both cusps are different from  $\infty$ , then  $n_\lambda^{-1}(c) \cap n_\mu^{-1}(c)$  is just a vertical circle for each positive number  $c$ . These circles are concentric and lie in the same vertical plane. Hence,  $S(\lambda, \mu)$  is just a vertical plane in this case. If  $N_\lambda \neq N_\mu$ , then we have the following result:

2.2.1. Proposition  $S(\lambda, \mu)$  is a hemisphere with center in the plane  $\zeta = 0$ .

Proof: We first consider the case  $\mu = \infty$ . We choose  $\alpha, \beta \in \mathcal{O}$  such that  $\lambda = \frac{\alpha}{\beta}$ . Let  $k$  be a positive number which doesn't depend on  $(z, \zeta)$ . The set  $n_\lambda^{-1}(kc) \cap n_\infty^{-1}(c)$ , where  $c$  is any positive constant, is described by the equation:

$$k \frac{1}{\zeta^2} = \left( \frac{\alpha\alpha - \alpha\beta z - \alpha\beta\bar{z} + \beta\beta(z\bar{z} + \zeta^2)}{N(\alpha, \beta)\zeta} \right)^2$$

Simplifying both sides, we get

$$\sqrt{k} N(\alpha, \beta) = (\beta z - \alpha)(\overline{\beta z - \alpha}) + \beta\bar{\beta}\zeta^2$$

$$\text{or } \sqrt{k} N_\lambda = (z - \lambda)(\overline{z - \lambda}) + \zeta^2$$

This last equation describes a hemisphere with center  $(\lambda, 0)$  and radius  $\sqrt{k} N_\lambda^{\frac{1}{2}}$ . Taking  $k = 1$  settles the claim for  $S(\lambda, \infty)$ . For arbitrary  $\mu$ , we take a  $g \in \operatorname{SL}(2, K)$  which maps  $\mu$  to  $\infty$ . By the Transformation Rule, we obtain for  $(z, \zeta) \in S(\lambda, \mu)$ :

$$n_{g\mu}(g(z, \zeta)) = n_\infty(g(z, \zeta)) = \left( \frac{N(\gamma, \delta)}{N(g(\gamma, \delta))} \right)^2 n_\mu(z, \zeta)$$

and

$$n_{g\lambda}(g(z, \zeta)) = \left( \frac{N(\alpha, \beta)}{N(g(\alpha, \beta))} \right)^2 n_\lambda(z, \zeta)$$

Let  $k := \left( \frac{N(\alpha, \beta)}{N(\gamma, \delta)} \cdot \frac{N(g(\gamma, \delta))}{N(g(\alpha, \beta))} \right)^2$ . Hence we get:

$(z, \zeta) \in S(\lambda, \mu)$  iff  $g(z, \zeta) \in \bigcup_c n_\lambda^{-1}(kc) \cap n_\infty^{-1}(c) =: S$ . Now the hemisphere  $S$  is mapped onto  $S(\lambda, \mu)$  by  $g^{-1}$ , it follows that  $S(\lambda, \mu)$  too is a hemisphere.

We denote the set of equidistant points with distance less than or equal to  $r$  with  $S(\lambda, \mu)_r$ . This is a closed set



(continuity of cusp distances again!) and we want to show that it is indeed compact. Again, we choose a  $g \in SL(2, K)$  which maps  $\mu$  to  $\infty$ . By the Transformation Rule, we obtain for  $(z, \zeta) \in S(\lambda, \mu)_x$ :

$$n_\infty(g(z, \zeta)) = \left( \frac{N(\gamma, \delta)}{N(g(\gamma, \delta))} \right)^2 n_\mu(z, \zeta) \leq \left( \frac{N(\gamma, \delta)}{N(g(\gamma, \delta))} \right)^2 x$$

Writing  $(z', \zeta')$  for  $g(z, \zeta)$ , we obtain the inequality

$$\frac{1}{(\zeta')^2} \leq \left( \frac{N(\gamma, \delta)}{N(g(\gamma, \delta))} \right)^2 x \quad \text{or} \quad \frac{N(g(\gamma, \delta))}{N(\gamma, \delta)\sqrt{x}} \leq \zeta'$$

Now the set  $\{(z', \zeta') \in S \mid \zeta' \geq \frac{N(g(\gamma, \delta))}{N(\gamma, \delta)\sqrt{x}}\}$  is compact and  $g, g^{-1}$  being homeomorphisms,  $gS(\lambda, \mu)_x$  and  $S(\lambda, \mu)_x$  are compact, too.

We will now discuss the natural cell structure on the intersection of two minimal sets. (Remember that these intersections form a locally-finite closed covering of the minimal incidence set!) Let  $\lambda, \mu$  be distinct cusps. We shall describe the inequalities that determine  $H_\lambda \cap H_\mu$  as a subset of  $S(\lambda, \mu)$ .

First of all,  $H_\lambda \cap H_\mu$  is in fact a closed subset of  $S(\lambda, \mu)_m$ , where  $m$  is the maximal value of the minimal distance function  $n$ , since the image of  $n$  is precisely  $[1, m]$ . It follows that  $H_\lambda \cap H_\mu$  is compact and  $n$  has maximal value on the intersection, which we will call  $m(\lambda, \mu)$ . Again,  $H_\lambda \cap H_\mu \subset S(\lambda, \mu)_{m(\lambda, \mu)}$ .

Now, by definition,  $(z, \zeta) \in H_\lambda \cap H_\mu$  if and only if for all cusps  $v \neq \lambda, \mu$ ,  $n_\lambda(z, \zeta) = n_\mu(z, \zeta) \leq n_v(z, \zeta)$ . Geometrically, these inequalities mean that the point  $(z, \zeta)$  lies on  $S(\lambda, \mu)$  or "outside" of it and on  $S(\mu, \nu)$  or "outside" of it for all cusps  $v \neq \lambda, \mu$ . For cusps  $\nu$  such that  $S(\lambda, \nu) \cap S(\lambda, \mu)_{m(\lambda, \mu)} = \emptyset$  and

$S(\mu, \nu) \cap S(\lambda, \mu)_{m(\lambda, \mu)} = \emptyset$ , these conditions are automatically fulfilled. We just have to consider those cusps  $\nu$  such that

$$(X) \quad S(\lambda, \nu) \cap S(\lambda, \mu)_{m(\lambda, \mu)} \neq \emptyset \text{ or } S(\mu, \nu) \cap S(\lambda, \mu)_{m(\lambda, \mu)} \neq \emptyset$$

The following Proposition implies that there are just finitely many cusps satisfying the above condition (X):

**2.2.2. Proposition** Let  $\alpha$  be a fixed cusp of  $K$ . Define for any  $(z, \zeta) \in H$  the following set:

$$M_\alpha(z, \zeta) := \{ \tau \mid n_\tau(z, \zeta) \leq n_\alpha(z, \zeta) \}$$

(This is clearly a finite set!) Then there is an open neighborhood  $V$  of  $(z, \zeta)$  such that

$$V \cap S(\alpha, \nu) \neq \emptyset \text{ iff } \nu \in M_\alpha(z, \zeta)$$

**Proof:** The proof is analogous to that of Lemma 2.1.2. We have to show that the infimum of the distances of  $(z, \zeta)$  to cusps which are not in  $M_\alpha(z, \zeta)$  is still strictly greater than the distance to  $\alpha$ . By the Finiteness Property, there are just finitely many cusps  $\sigma_1, \dots, \sigma_r$  such that  $n_{\sigma_1}(z, \zeta) \leq n_\alpha(z, \zeta) + 1$ . We have two cases to consider:

**Case 1:** Not all  $\sigma_i$ 's are in  $M_\alpha(z, \zeta)$ . Then we have

$$\inf_{\sigma \notin M_\alpha(z, \zeta)} n_\sigma(z, \zeta) = \min_{\sigma_i \notin M_\alpha(z, \zeta)} n_{\sigma_i}(z, \zeta) > n_\alpha(z, \zeta)$$

**Case 2:** All  $\sigma_i$ 's are in  $M_\alpha(z, \zeta)$ . Then

$$\inf_{\sigma \notin M_\alpha(z, \zeta)} n_\sigma(z, \zeta) \geq n_\alpha(z, \zeta) + 1 > n_\alpha(z, \zeta)$$

This proves the Proposition.

Cover the set  $S(\lambda, \mu)_{m(\lambda, \mu)}$  with neighborhoods as in the preceding Proposition with  $\alpha = \lambda$ . A finite number of them  $V(z_1, \zeta_1), \dots, V(z_s, \zeta_s)$  will cover the whole set. This means that

$$S(\lambda, \nu) \cap S(\lambda, \mu)_{m(\lambda, \mu)} \neq \emptyset \text{ implies } \nu \in \bigcup_{i=1}^s M_\lambda(z_i, \zeta_i)$$

and the latter union is a finite set. Similarly, there is only a finite number of cusps  $\nu$  with  $S(\mu, \nu) \cap S(\lambda, \mu)_{m(\lambda, \mu)} \neq \emptyset$ .

We can hence picture  $H_\lambda \cap H_\mu$  as follows: we intersect the set  $S(\lambda, \mu)$  with every  $S(\lambda, \nu)$  and  $S(\mu, \nu)$ , where  $\nu$  satisfies the inequalities (X). In each case, the intersection is a geodesic (a vertical half-line or semi-circle in this case). What obviously results is either empty or a closed  $n$ -cell ( $n = 0, 1, 2$ ). Finally, we take those points  $(z, \zeta)$  with cusp distances  $n_\lambda(z, \zeta) = n_\mu(z, \zeta) \leq m(\lambda, \mu)$ . This amounts to intersecting the closed  $n$ -cell with a cupola. It is also clear that the  $n$ -cells are regular.

One can easily see that the circle  $n_\lambda^{-1}(m(\lambda, \mu)) \cap n_\mu^{-1}(m(\lambda, \mu))$  intersects  $H_\lambda \cap H_\mu$  only at a finite number (and at at least one) of vertices. This implies in particular that  $n$  has its maximal value  $m$  at a vertex, so that the set of points where the maximal value occurs is discrete.

**2.2.3. Example** We have the following pictures of  $H_0 \cap H_\infty$  for  $K = Q(\sqrt{-1})$  and  $K = Q(\sqrt{-7})$ . In both cases, the 2-cell lies on the hemisphere  $\{(z, \zeta) \mid z\bar{z} + \zeta^2 = 1\}$ . The square and the hexagon are the (vertical) projections to the plane  $\zeta = 0$ . The computational justification for the drawings can be found in 4.1. and 4.2.

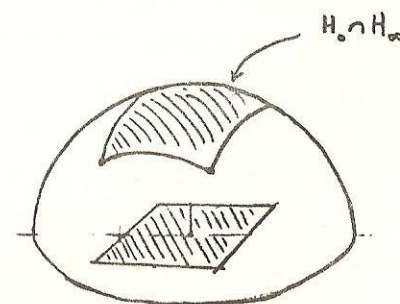


Figure 2.2.3a  $K = Q(\sqrt{-1})$

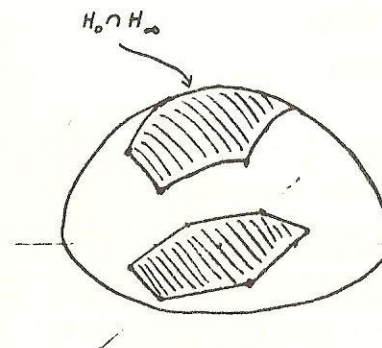


Figure 2.2.3b  $K = Q(\sqrt{-7})$



It is clear from the preceding description of  $H_\lambda \cap H_\mu$  as a subset of  $S(\lambda, \mu)$  that it is a closed 2-cell if and only if there are points in it, whose minimal cusp sets consist of just  $\lambda$  and  $\mu$ . Indeed, the open 2-cell consists precisely of these points. This implies that the intersection of two distinct closed cells is either empty, a 0-cell or a 1-cell. Since the intersections of minimal sets form a locally-finite covering of the Hausdorff space  $I$ , we see that the minimal incidence set is a 2-dimensional regular CW complex. Note that, for each fixed cusp  $\lambda$ , we also have

$$I_\lambda = \bigcup_{\mu \neq \lambda} H_\lambda \cap H_\mu$$

Hence,  $I_\lambda$  has a natural subcomplex structure.

We will now derive a very important property of the mapping  $(z, \zeta) \mapsto M(z, \zeta)$ , namely, that it is constant on open cells. This is clearly the case for open 2-cells, as we have seen in the preceding paragraph. We now consider the case of the open 1-cells:

**2.2.4. Proposition** Let  $e$  be an open 1-cell in  $I$ . Suppose that  $(z, \zeta)$  lies in  $e$ . If  $(z, \zeta) \in H_\lambda$ , then  $e \subset H_\lambda$ .

**Proof:** We shall also call a 1-cell (or open 1-cell) an edge (or open edge). Since  $e$  is an open edge in  $I$ , there are at least two cusps  $\mu, \nu$ , both different from  $\lambda$ , such that  $e$  is contained in  $H_\mu \cap H_\nu$ . The claim is equivalent to the statement that if  $(z, \zeta) \in H_\mu \cap H_\lambda$ , then  $e \subset H_\mu \cap H_\lambda$ . Now, if  $e$  were not contained in  $H_\mu \cap H_\lambda$ , then there are points  $(z', \zeta')$  in  $e$  such that

$$n_\mu(z', \zeta') = n_\nu(z', \zeta') > n_\lambda(z', \zeta')$$

(those "inside" the hemisphere  $S(\mu, \lambda)$ !) This is a contradiction to the fact  $e \subset H_\mu \cap H_\nu$ .

It follows immediately that  $(z, \zeta) \mapsto M(z, \zeta)$  is constant on open edges. In particular, the cusp degree  $d(z, \zeta)$  remains the same for points in an open edge, too.

We summarize the above discussion in the following Theorem:

**2.2.5. Theorem** The minimal incidence set  $I$  is a 2-dimensional regular CW complex. For each cusp  $\lambda$ ,  $I_\lambda$  is a subcomplex. If  $(z, \zeta)$  and  $(z', \zeta')$  lie in the same open  $n$ -cell, then  $M(z, \zeta) = M(z', \zeta')$ .

Let us now consider the operation of  $\Gamma$  in terms of this cell structure. Our claim is that the  $\Gamma$ -operation is cellular:

**2.2.6. Proposition** An element  $\gamma$  of  $\Gamma$  maps an open  $n$ -cell ( $n = 0, 1, 2$ ) to an open  $n$ -cell.

**Proof:** We start with  $n = 2$ . If  $H_\lambda \cap H_\mu$  is a 2-cell, then there is a point  $(z, \zeta)$  with minimal cusp set  $\{\lambda, \mu\}$ . The image of this point under  $\gamma \in \Gamma$  has the minimal cusp set  $\{\gamma\lambda, \gamma\mu\}$ . Hence,  $\gamma \cdot (H_\lambda \cap H_\mu) = H_{\gamma\lambda} \cap H_{\gamma\mu}$  has an element with exactly two minimal cusps, which implies that it is a 2-cell. The argument applies also to the open cells.

Let  $e$  be an open edge. The set  $\gamma e$  lies in the 1-skeleton of  $I$  (due to the argument just sketched), and we claim that no vertex (or corner) lies on  $\gamma e$ . Assume the contrary, and let  $(z_0, \zeta_0)$  be a vertex on  $\gamma e$ . The part of  $\gamma e$  near  $(z_0, \zeta_0)$  and on one side of  $(z_0, \zeta_0)$  must lie on some open edge  $e'$  with

one end at  $(z_0, \zeta_0)$ . Since  $(z_0, \zeta_0)$  is a vertex, there is also some other open edge  $e''$  with one end at  $(z_0, \zeta_0)$  such that the union  $e' \cup e''$  doesn't lie on a geodesic. Now consider a neighborhood  $U$  of  $(z_0, \zeta_0)$  as in Lemma 2.1.2. For any element  $(z, \zeta) \in U \cap \gamma e$ , we have  $M(z, \zeta) \subset M(z_0, \zeta_0)$ . By the preceding Theorem, the cusp degree is constant on  $e$ , and hence, by  $\Gamma$ -invariance, on  $\gamma e$

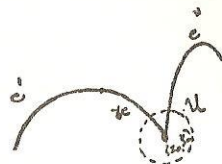


Figure 2.2.6.

too. This shows that  $d(z, \zeta) = d(z_0, \zeta_0)$ , implying in turn that  $M(z, \zeta) = M(z_0, \zeta_0)$ . Again, by the preceding Theorem, this equality is true for all points  $(z', \zeta')$  in  $e'$ . Analogously, we conclude that  $M(z'', \zeta'') \subset M(z_0, \zeta_0)$  for all  $(z'', \zeta'') \in e''$ . But this means that the edges  $e'$  and  $e''$  are contained in  $S(\lambda, \lambda') \cap S(\lambda', \lambda'')$ , where  $\lambda, \lambda', \lambda''$  are three distinct cusps in  $M(z'', \zeta'')$ ; since the latter set is a geodesic, this is a contradiction. Therefore,  $\gamma e$  is a connected subset of the 1-skeleton of  $I$  containing no vertex. Thus  $\gamma e$  lies in an open edge  $\bar{e}$ .  $\gamma^{-1}\bar{e}$  is also a connected subset of the 1-skeleton of  $I$  containing no vertex (using the same argument as in the case of  $\gamma e$ ), and it lies in an open edge  $\bar{\bar{e}}$ . Now  $e \subset \bar{\bar{e}}$  implies that  $e = \bar{\bar{e}}$ , and hence,  $\gamma e = e'$ .

Finally, the image of a vertex cannot lie in an open 2-cell or in an open edge, because of the above arguments. It has no other choice but to be a vertex, too.

Although the minimal cusp set  $M(z, \zeta)$  is the same for all points in an open cell, this is in general no longer true

for the stabilizer  $\Gamma(z, \zeta)$ . We cite an example which is computed in detail in §4.

**2.2.7. Example** Let  $K = \mathbb{Q}(\sqrt{-1})$ . The open 2-cell in  $H_0 \cap H_\infty$  consists of all points on the hemisphere  $\{(z, \zeta) \mid z\bar{z} + \zeta^2 = 1\}$  with  $|\operatorname{Re} z| < \frac{1}{2}$  and  $|\operatorname{Im} z| < \frac{1}{2}$ . (Cf. the picture on p. 31.) However, we have the following stabilizers:

$$\Gamma(0, 1) \cong \text{dihedral group with 8 elements}$$

$$\Gamma\left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right) \cong \text{cyclic group with 2 elements}$$

$$\Gamma\left(\frac{1+i}{4}, \frac{\sqrt{14}}{4}\right) \cong \text{trivial group}$$

to name a few points in the open cell. Similar remarks apply to  $\tilde{\Gamma} = \operatorname{PSL}(2, \mathbb{C})$  and other subgroups  $\Gamma'$  of  $\Gamma$ . For example,  $\tilde{\Gamma}(0, 1) = \text{dihedral group with 4 elements}$ ,  $\tilde{\Gamma}\left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right)$  is cyclic of order 2. This difference in terms of stabilizers is due to the fact that there are  $\gamma \in \Gamma'$  such that  $\gamma(H_0 \cap H_\infty) = H_0 \cap H_\infty$  and each of these elements fixes a point or geodesic segment in the open 2-cell.

Let  $\Gamma'$  be a subgroup of  $\Gamma$ . It is crucial for the cohomology computations in §5 to have a cell structure for which the stabilizers in  $\Gamma'$  remain the same for all points in an open cell. We shall now show how to modify the above cell structure (which we call the natural cell structure) to yield one (depending on  $\Gamma'$ ) with the desired property.

Denote by  $(H_\lambda \cap H_\mu)^0$  the open cell in the natural  $n$ -cell  $H_\lambda \cap H_\mu$  ( $n = 1, 2$ ). If  $\gamma \in \Gamma'$  stabilizes a point in  $(H_\lambda \cap H_\mu)^0$ ,



then obviously

$$\gamma(H_\lambda \cap H_\mu) \cap (H_\lambda \cap H_\mu) \neq \emptyset \quad (X)$$

Since  $\Gamma'$  is discrete and  $H_\lambda \cap H_\mu$  is compact, there is just a finite number of elements of  $\Gamma'$  satisfying condition (X)--- hence, also just finitely many  $\gamma$ 's fixing a point in  $(H_\lambda \cap H_\mu)^0$ . Now, the set of points fixed by a Möbius transformation is a geodesic, and this either pierces  $S(\lambda, \mu)$  or lies entirely in it (Cf. ( ), Lemma 4.4. p. 25). If  $H_\lambda \cap H_\mu$  is an edge, this means that either  $\gamma$  fixes the whole edge or a single point on it. In the latter case, we add the fixed point as a new vertex. We obtain a subdivision of the edge into a finite number of new edges. If  $H_\lambda \cap H_\mu$  is a 2-cell, we have to consider two cases:

Case 1. There are no isolated fixed points in  $(H_\lambda \cap H_\mu)^0$ .

In this case, by adding the intersections of the closed geodesic segments with themselves and the natural 1-cells in  $H_\lambda \cap H_\mu$  as additional vertices and all resulting open geodesic segments as open 1-cells, we obtain the necessary "modifications" on the set  $(H_\lambda \cap H_\mu)^0$ .

Case 2. If there is an isolated fixed point  $(z, \zeta)$ , then a stabilizer  $\gamma$  also fixes points not in  $I$ . Any geodesic in  $H_\lambda \cap H_\mu$  on which  $(z, \zeta)$  lies is invariant under  $\gamma$  (since  $\gamma$  is an isometry). If  $(z, \zeta)$  is fixed by some other elements  $\gamma'$  in  $\Gamma$ , we add the cells induced by  $\gamma'$  and the translates of these under  $\Gamma'$ . If  $(z, \zeta)$  isn't fixed by any other element in  $\Gamma$ , then we add any geodesic thru  $(z, \zeta)$  whose endpoints in  $H_\lambda \cap H_\mu$  are vertices and then take the  $\Gamma'$ -translates.

2.2.8. Remark We don't know whether there really are isolated fixed points for  $\Gamma$ .

We shall call the cell structure resulting from all these modifications the  $\Gamma'$ -induced cell structure on  $I$ .  $\Gamma'$ -induced cells are still regular, and it is clear that points in the same open cell have the same stabilizer in  $\Gamma'$ . In particular, open  $\Gamma'$ -induced cells have trivial stabilizer in  $\Gamma'$ .

2.2.9. Example For  $H_0 \cap H_\infty$  in the case of  $K = Q(1)$ , we have the following natural and  $\Gamma'$ -induced cell structures (viewed from "above"):

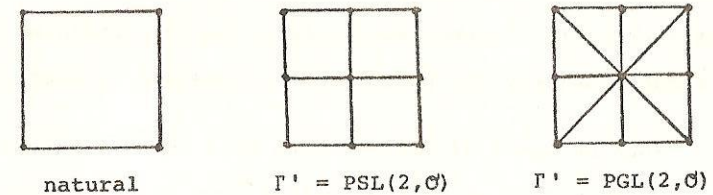


Figure 2.2.9a

For  $\Gamma' = \Gamma_\infty$ , we have an isolated fixed point  $(0, 1)$ . We illustrate two ways of completing the cell structure:

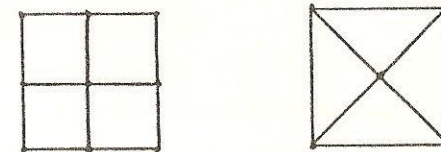


Figure 2.2.9b

$I$  together with  $\Gamma'$ -induced cell structure is still a 2-dimensional regular CW complex. Furthermore,  $\Gamma'$  still acts in a cellular fashion:

2.2.10. Proposition An element  $\gamma \in \Gamma'$  maps an open  $\Gamma'$ -induced  $n$ -cell to an open  $\Gamma'$ -induced  $n$ -cell ( $n = 0, 1, 2$ ).



Proof: The proof is direct but rather tedious since it involves a case by case consideration. We indicate it briefly for the vertices. An induced vertex must belong to one of the following categories: a) a natural vertex, b) the intersection of two fixed geodesics, c) the intersection of a fixed geodesic and a natural edge and d) an isolated fixed point. By 2.2.6. the image of a vertex in case a) also belongs to case a). By  $\Gamma'$ -invariance and the fact that  $\Gamma'(\gamma(z, \zeta))$  equals  $\gamma\Gamma'(z, \zeta)\gamma^{-1}$ , the same is true for the remaining cases. We argue similarly for open induced edges and induced 2-cells.

When we speak of the quotient as a CW complex, we mean the cell structure on  $\Gamma' \backslash I$  inherited from the  $\Gamma'$ -induced cell structure on  $I$ . We shall also call this quotient structure  $\Gamma'$ -induced. We have the following corollary:

2.2.11. Corollary The quotient space  $\Gamma' \backslash I$  is a normal CW complex.

Proof: Since  $\Gamma'$  operates discontinuously on  $I$ , the quotient space is a hausdorff space. By the preceding proposition,  $\Gamma'$ -equivalence is a cellular equivalence relation on  $I$ , so that quotient space has a normal cell structure.

If the quotient is compact, then of course it is a finite complex.

In § 4 the quotient spaces  $\Gamma \backslash I$  and  $\tilde{\Gamma} \backslash I$  are computed for the euclidean cases. These computations also yield the cell stabilizers.

2.3. The cell structure on  $I$  implies that it is a locally contractible space. In this paragraph we shall show that it is in fact contractible - a property crucial for the cohomology considerations of §5. Contractibility of  $I$  is an immediate corollary to the following Theorem:

2.3.1. Theorem The minimal incidence set  $I$  is a deformation retract of  $H$ .

We shall give an explicit construction of the deformation retraction  $r: H \longrightarrow I$ . The proof can also be stated in terms of Morse theory, but we decided to give an elementary presentation since the underlying geometry is easy to visualize.

The first step is to reduce the problem to showing that for each cusp  $\lambda$ ,  $I_\lambda$  is a deformation retract of the minimal set  $H_\lambda$ .

2.3.2. Lemma For each cusp  $\lambda$ , let  $r_\lambda: H_\lambda \longrightarrow I_\lambda$  be a deformation retraction and  $f_\lambda: H_\lambda \times [0, 1] \longrightarrow H_\lambda$  the associated deformation homotopy. Then there is a deformation retraction  $r: H \longrightarrow I$ .

Proof: Recall that the minimal sets  $H_\lambda$  (their boundaries  $I_\lambda$  respectively) cover  $H$  ( $I$  respectively). We define  $r$  as follows:

$$r(z, \zeta) := \begin{cases} (z, \zeta) & \text{if } (z, \zeta) \in I \\ r_\lambda(z, \zeta) & \text{if } (z, \zeta) \notin I \end{cases}$$

(if  $(z, \zeta) \notin I$ , then there is a unique cusp  $\lambda$  such that  $(z, \zeta)$  lies in  $H_\lambda$ .) Now  $r$  is continuous on each set of the locally-finite covering  $\{H_\lambda\}$  of  $H$ , so that it is continuous on  $H$ .  $r$  is clearly a retraction, i.e.,  $r \circ i = \text{id}_I$  (where  $i: I \hookrightarrow H$  is the

inclusion). Similarly, the deformation homotopy is given by

$$f(z, \zeta) := \begin{cases} (z, \zeta) & \text{if } (z, \zeta) \in I \\ f_\lambda(z, \zeta) & \text{if } (z, \zeta) \notin I \text{ and } \lambda \text{ as above} \end{cases}$$

Again  $f$  is continuous by the same reasoning as in the case of  $r$ .

Proof of 2.3.1. We now proceed with the construction of the deformation retraction  $r_\lambda: H_\lambda \rightarrow I_\lambda$  (for an arbitrary cusp  $\lambda$ ). If  $(z, \zeta)$  lies on the boundary, then, of course,  $r_\lambda(z, \zeta) := (z, \zeta)$ . If  $(z, \zeta)$  doesn't lie on the boundary, consider the unique geodesic from  $\lambda$  which contains the point  $(z, \zeta)$ . (For  $\lambda \neq \infty$ , this is a vertical semicircle, perpendicular to the level set of  $n_\lambda$  containing  $(z, \zeta)$ ; we

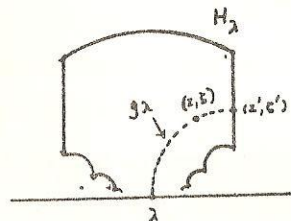


Figure 2.3.1.

consider it without endpoints in the plane  $\zeta = 0$  and denote it by  $g_\lambda$ . For  $\lambda = \infty$ , this is a vertical half-line).  $n_\lambda$  is clearly injective when restricted to  $g_\lambda$  and moreover,  $n_\lambda(g_\lambda) = (0, \infty)$ . (One way to see this is to transport the cusp  $\lambda$  to  $\infty$  via a  $\gamma \in \text{SL}(2, K)$  and apply the Transformation Rule!)

Denote by  $g_\lambda(c)$  the compact set  $\{(z, \zeta) \in g_\lambda \mid 1 \leq n_\lambda(z, \zeta) \leq c\}$  and by  $H_\lambda^0$  the set  $H_\lambda \setminus I_\lambda$ . Since  $H \setminus H_\lambda^0$  is closed, the set  $g_\lambda(m) \cap (H \setminus H_\lambda^0)$ , where  $m$  is the maximal value of  $n$ , is non-empty and compact too. The function  $n_\lambda$  has a minimum on this set, say at the point  $(z', \zeta')$ . Since  $n_\lambda$  is injective on  $g_\lambda$ ,  $(z', \zeta')$  is unique. Furthermore,  $(z', \zeta')$  lies in  $I$ : any neighborhood contains points of  $H_\lambda^0$ . We set:  $r_\lambda(z, \zeta) := (z', \zeta')$ .

$(z', \zeta')$  is, as a matter of fact, the only element in  $g_\lambda \cap I_\lambda$ : this results from the following Lemma:

2.3.3. Lemma Let  $S$  be a hemisphere in  $H$  with center in the plane  $\zeta = 0$  or a vertical half-plane. Let  $e$  be an open geodesic segment of  $H$  which meets  $S$  but does not pierce  $S$ . Then  $e \subset S$ .

For a proof cf. ( ), p. 25. In order to apply the Lemma, choose  $S$  to be  $S(\lambda, \mu)$  for any  $\mu \in M(z', \zeta')$  different from  $\lambda$ . Since  $g_\lambda \not\subset S(\lambda, \mu)$  (for example,  $(z, \zeta) \notin S(\lambda, \mu)$ ) but meets it, then  $g_\lambda$  must pierce it. Hence  $(z', \zeta')$  is unique.

$r_\lambda$  is easily seen to be continuous since the geodesic depends in a continuous manner on  $(z, \zeta)$  in the sense that the radius and the coordinates of the center of  $g_\lambda$  are continuous functions of  $(z, \zeta)$ .  $r_\lambda$  is obviously a retraction. The idea for constructing the deformation homotopy is clear: we push  $(z, \zeta)$  along  $g_\lambda$  to  $r_\lambda(z, \zeta)$ . This works since we have shown that the points of  $g_\lambda$  between  $(z, \zeta)$  and  $r_\lambda(z, \zeta)$  all lie in  $H_\lambda$ . If  $\sigma: [0, 1] \rightarrow [n_\lambda(z, \zeta), n_\lambda(r_\lambda(z, \zeta))]$  is a homeomorphism, we let

$$f_\lambda((z, \zeta), t) := g_\lambda \cap n_\lambda^{-1}(\sigma(t))$$

2.3.4. Corollary The minimal incidence set  $I$  is a contractible space.

Proof: Since  $H$  is a convex subspace of  $\mathbb{R}^3$ , we can contract it to any of its points. We choose a  $(z, \zeta)$  in  $I$ . Let  $r: H \rightarrow I$  be the retraction constructed above, and  $f$  be the associated homotopy. Furthermore, denote by  $s: H \rightarrow \{(z, \zeta)\}$  and  $g: H \times [0, 1] \rightarrow H$  the contraction and its homotopy, respectively. We define:

$$\bar{s} := s|_I: I \rightarrow \{(z, \zeta)\}$$



and

$$h := r \circ g|_{I \times [0,1]}$$

$\bar{s}$  is a contraction of  $I$  to  $\{(z, \zeta)\}$  and  $h$  is its associated homotopy.

2.3.5. Remarks It is easy to verify that the deformation retraction  $r : H \longrightarrow I$  is  $\Gamma$ -equivariant.

This implies, for example, that

$$H^1(\Gamma' \backslash H, Q) \cong H^1(\Gamma' \backslash I, Q)$$

for any subgroup  $\Gamma'$  of  $\Gamma$ .

b) Similar considerations show that  $H_\lambda$  and  $I_\lambda$  are contractible. A ball "sitting" at  $\lambda$  with radius  $< 1$  and punctured at  $\lambda$  is clearly homotopically equivalent to  $H_\lambda$ . The same is true for a punctured sphere and  $I_\lambda$ . (Cf Figure 2.3.5.)

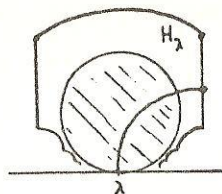


Figure 2.3.5.

### §3. COMPARISON WITH THE CLASSICAL THEORY

3.0. We would like to discuss some connections of the theory developed in the preceding sections to the theory of Bianchi and Humbert. Bianchi (1) determined the fundamental domain for small values of the discriminant and Humbert (10) extended the construction to the general case. Swan (17) has given a detailed account of the Bianchi-Humbert theory, filling in some gaps and beyond that, discussing methods for the effective determination of the fundamental domain. (Compare also (10)). We shall follow Swan's notation in the following account of the classical theory.

3.1. The classical construction is as follows: for each pair  $\alpha, \beta, \beta \neq 0$ , both elements of  $\mathcal{O}$  such that  $(\alpha, \beta) = \mathcal{O}$ , define the following set:

$$S_{\beta, \alpha} := \{(z, \zeta) \in H \mid |\beta z - \alpha|^2 + |\beta|^2 \zeta^2 = 1\}$$

-this is the hemisphere with center  $(\frac{\alpha}{\beta}, 0)$  and radius  $\frac{1}{|\beta|}$ . Now let  $B$  be the set of points in the upper half-space, which lie above or on  $S_{\beta, \alpha}$  for all such pairs, i.e.,

$$B := \{(z, \zeta) \in H \mid |\beta z - \alpha|^2 + |\beta|^2 \zeta^2 \geq 1 \text{ for all } \alpha, \beta \text{ as above}\}$$

A fundamental domain for  $\text{PGL}(2, \mathcal{O})$  (actually the classical theory is concerned with  $\text{PSL}(2, \mathcal{O})$ ) is then obtained by taking that part of  $B$  which lies above a fundamental domain for the translations and furthermore taking into account identifications due elements of the form  $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$ , where  $\varepsilon \in \mathcal{O}^\times$ .

$S_{\beta, \alpha}$  is clearly the set  $S(\infty, \frac{\alpha}{\beta})$  we introduced in 2.2., and the condition on the pair  $\alpha, \beta$  is equivalent to the statement that the cusp  $\frac{\alpha}{\beta}$  is  $\Gamma$ -equivalent to  $\infty$ . Hence  $B$  can be rewritten as

$$B = \{ (z, \zeta) \in H \mid n_{\infty}(z, \zeta) \leq n_{\lambda}(z, \zeta) \text{ for all cusps } \lambda \text{ which are } \Gamma\text{-equivalent to } \infty \}$$

Using this description of  $B$  and the properties of the cusp distances (§ 1), we can easily derive many properties of  $S_{\beta, \alpha}$  and  $B$ . An example:

3.1.1. Proposition  $H = \text{PSL}(2, \mathcal{O}) \cdot B$

Proof: Let  $(z, \zeta) \in H \setminus B$ . By definition, there is a cusp  $\frac{\alpha}{\beta}$ ,  $(\alpha, \beta) = \mathcal{O}$  with  $n_{\frac{\alpha}{\beta}}(z, \zeta) < n_{\infty}(z, \zeta)$ . There is just a finite number of such cusps: we choose  $\mu = \frac{\gamma}{\delta}$  with  $n_{\mu}(z, \zeta)$  minimal. Now there is a  $\phi \in \text{PSL}(2, \mathcal{O})$  such that  $\phi\mu = \infty$  so that  $n_{\mu}(z, \zeta) = n_{\infty}(\phi(z, \zeta))$ . Hence  $n_{\infty}(\phi(z, \zeta)) \leq n_{\nu}(\phi(z, \zeta))$  for all  $\nu = \frac{\sigma}{\tau}$  and  $(\sigma, \tau) = \mathcal{O}$ , i.e.,  $\phi(z, \zeta)$  is in  $B$ .

In general, we have the inclusion:  $H_{\infty} \subset B$ . In case the class number of  $K$  is 1, then we have equality  $H_{\infty} = B$ , since each cusp is  $\Gamma$ -equivalent to  $\infty$ . We also have  $I_{\infty} = \partial B$ , where  $\partial B = \{ (z, \zeta) \in B \mid \text{there exist } \alpha, \beta \text{ } (\alpha, \beta) = \mathcal{O} \text{ such that } |\beta z - \alpha|^2 + |\beta|^2 \zeta^2 = 1 \}$ . Clearly,  $I = \Gamma \cdot I_{\infty}$ . It is also clear that in this case, the fundamental domain  $F$  constructed in §1.3 is the same as the one defined previously. If the class number of  $K$  is greater than 1, then there are always points in  $B \setminus H_{\infty}$ , and the corresponding sets differ.

Swan ([17], p.23) also uses a "degree function"  $\bar{d}$  which plays a role analogous to that of  $d$  in the investigation of a

cell structure on  $\partial B$ . It is defined as follows: let  $\Phi =$

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathcal{O}) \mid c = 0 \right\} \text{ and define for } (z, \zeta) \in H$$

$$\Gamma_B(z, \zeta) := \{ \sigma \in \Gamma \mid \sigma(z, \zeta) \in B \}$$

It is easy to see that  $\Gamma_B(z, \zeta)$  is invariant under left multiplication by elements of  $\Phi$ . There are just finitely many orbits, and  $\bar{d}(z, \zeta) := \text{number of orbits} - 1$  (cf. (17) Lemma 4.1., p. 23). In case the class number of  $K$  is 1, then we have

$$d(z, \zeta) = \bar{d}(z, \zeta) + 1$$

(Lemma 4.2a, p. 24). Again, in the general case, there seems to be no direct comparison.

The analogue of the minimal incidence set  $I$  in the classical theory is the set  $\bar{I} := \Gamma \cdot \partial B$ . As we have already said, in the class number one case,  $I = \bar{I}$ . Note that for class number greater than one, neither is the quotient space  $\mathbb{H}^n / \Gamma$  compact, nor is  $\bar{I}$  a deformation retract of the upper half space  $H$ .

3.2. From the point of view of hermitian forms, the minimal incidence set  $I$  consists of all forms relative to which at least two different flags have minimal volume. We have an analogous interpretation for the set  $\bar{I}$ . Recall that the proper minimum  $\bar{m}(h)$  of a positive definite hermitian form is defined as

$$\bar{m}(h) := \min_{\substack{(x, y) \in \mathcal{O}^2 \\ (x, y) \neq \mathcal{O}}} h(x, y)$$

We have following proposition:

3.2.1. Proposition  $\bar{I}$  consists of all forms which have two  $K$ -linearly independent proper minimal vectors.

Proof:  $(x, y)$  and  $(x', y')$  are  $K$ -linearly dependent if and only



if  $xy' = x'y$ . This is equivalent to saying  $\partial_{(x,y)}^2(x,y) = \partial_{(x',y')}^2(x',y')$ . Since  $\partial_v^2 = \partial$  for a proper minimal vector  $v$  in  $\partial^2$ , we easily see the equality.

3.2.2. Remark If  $m(h) := \min_{(x,y) \neq 0} h(x,y)$  is the true minimum of the hermitian form  $h$ , one has in general the inequality  $m(h) \leq \bar{m}(h)$ . For class number = 1,  $m(h) = \bar{m}(h)$ . In general, it might be of interest to study

$\bar{I} :=$  the set of all forms which have two  $K$ -linearly independent true minimal vectors

and its relationship to  $I$ .

#### § 4 THE QUOTIENT SPACE IN THE EUCLIDEAN CASES

4.0. In this section we will describe the quotient space  $\Gamma \backslash I$  for five imaginary quadratic fields whose ring of integers is euclidean, namely  $\mathbb{Q}(\sqrt{d})$  for  $d = -1, -2, -3, -7$  and  $-11$ . At the same time, we shall compute the isotropy groups of points. Our goals are twofold: we would like to illustrate the theory developed so far by discussing the simplest cases and to do some preparatory work for the cohomology computations in the next section.

4.1. We begin with a notion of fundamental domain which takes the cell structure on  $I$  into account:

4.1.1. Definition A subset  $C \subset I$  is called a fundamental cellular domain (or simply cellular domain) for  $\Gamma$  if  $C$  is a finite subcomplex of  $I$  and has the following properties:

- i)  $I = \Gamma \cdot C$
- ii) Points in open induced 2-cells are not  $\Gamma$ -equivalent.

With the help of a cellular domain, we obtain a space homeomorphic to  $\Gamma \backslash I$  which is better to "visualize":

4.1.2. Proposition Let  $C$  be a cellular domain for  $\Gamma$ . Denote by " $\sim$ " the cellular equivalence relation on  $C$  induced by identifications of 0-cells or 1-cells. Then  $\Delta^C$  and  $\Gamma \backslash I$  are isomorphic CW complexes.

Proof: The inclusion  $i: C \hookrightarrow I$  is obviously continuous and compatible with " $\sim$ " and  $\Gamma$ -equivalence: it induces a conti-

nuous regular cellular map  $\bar{I}: \Delta^C \rightarrow \Delta^I$ . By definition of a cellular domain, this is a bijection and hence an isomorphism of CW complexes.

For the remainder of this section, we will be concerned with determining cellular domains in the five simplest cases, i.e., those with the lowest discriminant values. Before we plunge into these computations, we should observe that cellular domains exist in all cases. If  $C'$  is a compact subset of  $I$  such that  $I = \Gamma \cdot C'$ , then the carrier of  $C'$  is a finite subcomplex from which we can delete a finite number of open cells if necessary to get rid of identifications between points in open 2-cells. Furthermore, there are only finitely many cells meeting such a compact  $C'$ . This means that there are only finitely many cusps which are minimal for points in  $C'$ .

Let  $K$  be an imaginary quadratic field with class number equal to 1. In this case,  $I = \Gamma \cdot I_\infty$  and translations by elements of  $O$  leave  $I_\infty$  invariant. If  $T$  is a fundamental domain for these in  $I_\infty$ , then  $I = \Gamma(I_\infty \cap T)$ , with  $I_\infty \cap T$  compact. In fact, any closed subset  $D \subset T$  with the property  $I = \Gamma(I_\infty \cap D)$  is a good first approximation to a cellular domain. Such sets are well known in the cases under consideration (cf (1), (17)); we shall nevertheless discuss the computation for  $Q(\sqrt{-1})$  in detail, since our method is slightly different and we also want additional information on minimal cusp sets (in order to compute isotropy groups.) This information is not explicit (especially for vertices) in the classical literature.

Our problem is to determine the finite set of cusps which occur as minimal cusps of points in  $I_\infty \cap T$ . The first

step is to find a finite set of cusps which include these minimal ones. The starting point is the following easy remark: if  $\lambda = \frac{\alpha}{\beta}$  is such a minimal cusp, then in particular,  $I_\infty$  meets  $S_{\beta, \alpha}$ , and this means that there are level sets  $n_\infty^{-1}(c)$  and  $n_\lambda^{-1}(c)$  which meet at  $c \leq m \leq \frac{|D|}{2}$  (by Proposition 2.1.5. and First Reduction Theorem). On the other hand, from the incidence criterion, we get

$$\frac{1}{\sqrt{c}} \leq \frac{\sqrt{c}}{|\beta|^2} \text{ or } |\beta|^2 \leq c.$$

Combining both statements, we get the following denominator estimate for a minimal cusp:  $N(\beta) \leq \frac{|D|}{2}$ .

4.1.3. Example For  $K = Q(\sqrt{-1})$ , we have  $N(\beta) \leq 2$ . If  $N(\beta) = 1$ , then  $\beta$  is a unit. In this case,  $S_{\beta, \alpha} = S_{1, \beta^{-1}\alpha}$ . If  $N(\beta) = 2$ , then  $\beta \in \{1+i, 1-i, -1+i, -1-i\}$ .

We get an estimate on numerators by the following geometric consideration: if 1 and  $b$  form a basis for  $O$  as a  $\mathbb{Z}$ -module and  $b = b_1 + ib_2$ , then  $T$  can be chosen as  $\{z \mid z = z_1 + iz_2, |z_1| < \frac{1}{2} \text{ and } |z_2| < \frac{b_2}{2}\} \times \mathbb{R}_+^\times$ . Let  $d = \sqrt{1+b_2^2}$  be the diameter of the base rectangle of  $T$ . For a given  $\beta$ ,  $(I_\infty \cap T) \cap S_{\beta, \alpha} = \emptyset$  unless the distance from  $\frac{\alpha}{\beta}$  to the base rectangle is less than or equal to  $\frac{1}{|\beta|}$ . Hence, if  $I_\infty \cap T$  meets  $S_{\beta, \alpha}$ , then  $|\frac{\alpha}{\beta}| \leq d + \frac{1}{|\beta|}$  or equivalently,  $|\alpha| \leq |\beta|d + 1$ . Thus  $N(\alpha) \leq [(|\beta|d + 1)^2]$ . We can of course improve the estimate a bit by choosing a suitable  $D \subset T$  to get a smaller diameter.

The finite set we get by forming quotients  $\frac{\alpha}{\beta}$  for all values  $\alpha, \beta$  satisfying the estimates given and adding  $\infty$  certainly includes all those which occur as minimal cusps for points in  $I_\infty \cap T$ .

4.1.4. Example (cont'd) For  $Q(\sqrt{-1})$ , the base rectangle for  $T$



is the square  $\{z \mid |z_1| \leq \frac{1}{2}, |z_2| \leq \frac{1}{2}\}$  and  $d = \sqrt{2}$ . Now there are obvious further identifications: rotations by  $\frac{\pi}{2}$  and  $\pi$  around the  $\zeta$ -axis. If we set  $D := \{z \mid 0 \leq z_1 \leq \frac{1}{2}\} \times \mathbb{R}_+^*$ , then we get the estimate

$$(I_\infty \cap D) \cap S_{\beta, \alpha} \neq \emptyset \text{ implies } N(\alpha) \leq \left[ \left( |\beta| \frac{2}{2} + 1 \right)^2 \right].$$

Hence if  $|\beta| = 1$ ,  $N(\alpha) \leq 2$  and if  $|\beta| = \sqrt{2}$ ,  $N(\alpha) \leq 4$ . We obtain the following set of cusps:  $\{\infty, 0, \pm 1, \pm i, 1 \pm i, -1 \pm i, \frac{1 \pm i}{2}, \frac{-1 \pm i}{2}\}$ .

The set of cusps obtained is in general too large, since we used conditions involving the norm of elements instead of such involving the elements themselves. Hence a second step consists in eliminating those cusps from the set, whose hemisphere projections do not meet the interior of the projection of  $D \cap T$  at all. We shall call the remaining cusps admissible.

**4.1.5. Example (cont'd)** For  $K = \mathbb{Q}(\sqrt{-1})$ , after the second step, we are left with the cusps  $0, 1, i, 1+i, \frac{1+i}{2}$  as candidates for minimal ones ( $\infty$  is by definition minimal). It turns out that in this case all admissible cusps occur as minimal cusps for points in  $I_\infty \cap T$ .

Finally we study the intersections  $S_{\beta, \alpha} \cap S_{\beta', \alpha'}$  for all admissible cusps  $\frac{\alpha}{\beta}, \frac{\alpha'}{\beta'}$  in order to determine the exact fundamental cellular domain, i.e., all points which lie on at least one  $S_{\beta, \alpha}$  and not inside  $S_{\beta', \alpha'}$  for all other admissible cusps  $\frac{\alpha'}{\beta'}$ . This last step is analogous to the classical computations.

**4.1.6. Remark** The fastest way to carry out the last two steps is to make a (relatively) accurate drawing and do the comparison optically, and then check numerically those (few) cases where the inaccuracies of the drawing might play a role.

For the remainder of §4.1.,  $K := \mathbb{Q}(\sqrt{-1})$  and  $\Gamma := \text{PGL}(2, \mathbb{Z}[\frac{1}{2}])$ .

We have the following minimal cusp sets for  $(z, \zeta)$  in  $I_\infty \cap D$ :

- a) if  $\text{Re } z < \frac{1}{2}$  and  $\text{Im } z < \frac{1}{2}$ , then  $M(z, \zeta) = \{0, \infty\}$ .
- b) if  $\text{Re } z = \frac{1}{2}$  and  $\text{Im } z < \frac{1}{2}$ , then  $M(z, \zeta) = \{0, 1, \infty\}$ .
- c) if  $\text{Re } z < \frac{1}{2}$  and  $\text{Im } z = \frac{1}{2}$ , then  $M(z, \zeta) = \{0, 1, \infty\}$ .
- d) if  $\text{Re } z = \frac{1}{2}$  and  $\text{Im } z = \frac{1}{2}$ , then  $M(z, \zeta) = \{0, 1, 1+i, \frac{1+i}{2}, \infty\}$ .

With this information, we can now determine a cellular domain for  $\Gamma$  and the stabilizers of its cells:

**4.1.7. Proposition** Let  $C := \{(z, \zeta) \in S_{1,0} \mid \frac{1}{2} > \text{Re } z \geq \text{Im } z \geq 0\}$ .

Then  $C$  is a fundamental cellular domain for  $\Gamma$ ; in fact,  $C$

and  $\bigwedge_{\Gamma}^I$  are isomorphic CW complexes.

**Proof:** Consider the set  $C' = \{(z, \zeta) \in S_{1,0} \mid \frac{1}{2} > \text{Re } z \geq \text{Im } z \geq 0\}$ ,

i.e., the set of all points with minimal cusp set equal to

$\{0, \infty\}$ . If  $\gamma(z, \zeta) = (z', \zeta')$  with both  $(z, \zeta)$  and  $(z', \zeta')$  in  $C'$ ,

then either  $\gamma^\infty = \infty$  or  $\gamma^\infty = 0$ . If  $\gamma^\infty = \infty$ , then  $\gamma$  is of the

form  $\begin{pmatrix} \epsilon_1 & t \\ 0 & \epsilon_2 \end{pmatrix}$  with  $\epsilon_1 \in \mathbb{O}^*$ ,  $t \in \mathbb{O}$ . Now  $\gamma 0 = 0$  implies that  $t = 0$ . Such a matrix induces the transformation  $(z, \zeta)$

$\mapsto (\epsilon_1 \epsilon_2^{-1} z, \zeta)$ , so that the possible maps are  $(z, \zeta) \mapsto (i^r z, \zeta)$ ,

with  $0 \leq r \leq 3$ ; all these stabilize the point  $(0, 1)$  but don't

cause other identifications. If  $\gamma^\infty = 0$ , then we can write it

in the form  $\begin{pmatrix} 0 & \epsilon_2 \\ \epsilon_1 & t \end{pmatrix}$  with  $\epsilon_1$  and  $t$  as above. Again  $\gamma 0 = \infty$  implies  $t=0$ . The possible maps are  $(z, \zeta) \mapsto (\frac{i^r \bar{z}}{z\bar{z} + \zeta^2}, \frac{\zeta}{z\bar{z} + \zeta^2})$

$0 \leq r \leq 3$ ; the case  $r=0$  stabilizes the edge with  $\text{Im } z = 0$ ;

the cases  $r=2,3$  the point  $(0, 1)$  while the case  $r=1$  the edge

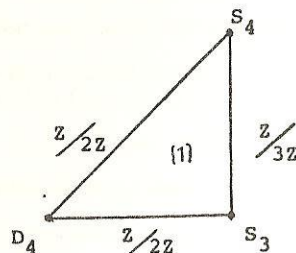
with  $\text{Re } z = \text{Im } z$ . Otherwise, there are no further identifications.

This shows that  $C$  is in fact a fundamental cellular domain for  $\Gamma$ .

To prove the isomorphism claim, note that there are no

identifications between points in  $C'$  and  $C - C'$  since the former have cusp degree 2 and the latter degree  $\geq 3$ . Furthermore, two distinct points  $(z, \zeta)$ ,  $(z', \zeta')$  in  $C - C'$  are not  $\Gamma$ -equivalent since  $\zeta \neq \zeta'$  and hence  $n_\infty(z, \zeta) \neq n_\infty(z', \zeta')$ .

**4.1.8. Proposition** The isotropy groups in  $\Gamma$  for points in  $C$  are as follows:



**Proof:** We have already shown the following (cf. the proof of part a) of the preceding proposition):

$(z, \zeta)$ mit	$\Gamma(z, \zeta)$	Generator(s)
1. $0 < \text{Im } z < \text{Re } z < \frac{1}{2}$	$\{1\}$	
2. $0 < \text{Im } z = \text{Re } z < \frac{1}{2}$	$\frac{z}{2z}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3. $0 = \text{Im } z < \text{Re } z < \frac{1}{2}$	$\frac{z}{2z}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
4. $z = 0, \zeta = 1$	$D_4$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

To complete the proof, we have to consider two further cases:

5.  $0 \leq \text{Im } z < \text{Re } z = \frac{1}{2}$  and  
 6.  $z = \frac{1+i}{2}, \zeta = \frac{\sqrt{2}}{2}$

Case 5 encompasses all those points with minimal cusp set equal to  $\{0, 1, \infty\}$ . Any stabilizer must map the minimal cusp set onto itself and we have to consider six possibilities:

Since the computations are straightforward, we will just list the results:

( $\epsilon$  always denotes a unit)

	$\gamma$ is of the form	$(z, \zeta)$ is mapped to	fixed points
1. $\gamma_\infty = \infty$ $\gamma_0 = 0$ $\gamma_1 = 1$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$(z, \zeta)$	the whole edge
2. $\gamma_\infty = \infty$ $\gamma_0 = 1$ $\gamma_1 = 0$	$\begin{pmatrix} \epsilon & -\epsilon \\ 0 & -\epsilon \end{pmatrix}$	$(-z+1, \zeta)$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
3. $\gamma_\infty = 0$ $\gamma_0 = \infty$ $\gamma_1 = 1$	$\begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}$	$(\frac{\bar{z}}{z\bar{z}+\zeta^2}, \frac{\zeta}{z\bar{z}+\zeta^2})$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
4. $\gamma_\infty = 0$ $\gamma_0 = 1$ $\gamma_1 = \infty$	$\begin{pmatrix} 0 & -\epsilon \\ \epsilon & -\epsilon \end{pmatrix}$	$(\frac{-\bar{z}+1}{z\bar{z}+\zeta^2+1}, \frac{\zeta}{z\bar{z}+\zeta^2+1})$	the whole edge
5. $\gamma_\infty = 1$ $\gamma_0 = 0$ $\gamma_1 = \infty$	$\begin{pmatrix} \epsilon & 0 \\ \epsilon & -\epsilon \end{pmatrix}$	$(\frac{-z+1}{z\bar{z}+\zeta^2+1}, \frac{\zeta}{z\bar{z}+\zeta^2+1})$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
6. $\gamma_\infty = 1$ $\gamma_0 = \infty$ $\gamma_1 = 0$	$\begin{pmatrix} \epsilon & -\epsilon \\ \epsilon & 0 \end{pmatrix}$	$(\frac{-\bar{z}+z\bar{z}+\zeta^2}{z\bar{z}+\zeta^2}, \frac{\zeta}{z\bar{z}+\zeta^2})$	the whole edge

As for Case 6, we can compute the stabilizers of  $(\frac{1+i}{2}, \frac{\sqrt{2}}{2})$  explicitly in a similar manner. However, there are  $6 \cdot 5 \cdot 4 = 120$  cases to consider (a fractional transformation is determined by its values at three points!). Fortunately, we just have to count elements in order to show that  $\Gamma(\frac{1+i}{2}, \frac{\sqrt{2}}{2}) \simeq S_4$ . The preceding computations imply that the order of the stabilizer



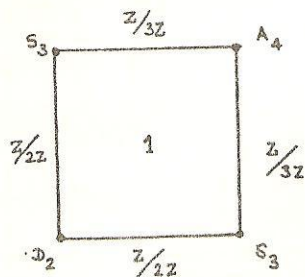
is at least 24. To get an upper bound, we use the following easy result: for a fixed point  $(z, \zeta)$  and  $\lambda \in M(z, \zeta)$ , the inequality  $|\Gamma(z, \zeta)| \leq |\text{Stab}_{\Gamma(z, \zeta)} \lambda| |M(z, \zeta)|$ . If  $\lambda = \infty$ , then the induced transformation must be of the type  $(z, \zeta) \mapsto (\epsilon_2'(\epsilon z + t), \zeta)$ . If  $\gamma$  stabilizes the point in question, then  $\epsilon_2'(\epsilon z + t) = z$  or  $t = \epsilon z - \epsilon z$ . Hence there are just at most 7 possibilities, which implies that

$$|\Gamma(z, \zeta)| \leq 7 \cdot 6 = 42$$

By a general theorem on finite subgroups of  $\text{PGL}_2$ , this implies that  $\Gamma(z, \zeta) \simeq S_4$ .

With some additional computation, we get the following result:

**4.1.9. Proposition** Let  $\tilde{\Gamma}$  be the group  $\text{PSL}(2, \mathbb{Z}[i])$ . The space  $\{(z, \zeta) \in S_{1,0} \mid 0 \leq \text{Re } z \leq \frac{1}{2} \text{ and } 0 \leq \text{Im } z \leq \frac{1}{2}\}$  is isomorphic to  $\tilde{\Gamma} \backslash \mathbb{H}$  (as CW complexes). The stabilizers in are as follows:



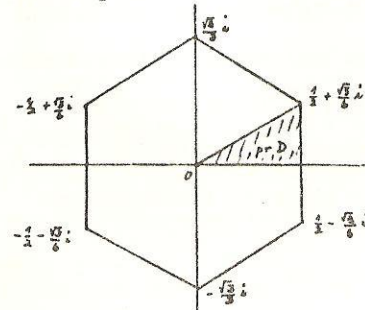
Finally, we have the following corollary:

**4.1.10. Corollary** Let  $X = \tilde{\Gamma} \backslash \mathbb{H}$  and  $Y = \tilde{\Gamma} \backslash \mathbb{H}$ . Then

$$H^1(X, \mathbb{Q}) = H^1(Y, \mathbb{Q}) = 0 \text{ for } i=1,2.$$

**4.2.** We will now discuss the quotient space for the remaining cases. Since the computations are tedious and analogous to those for  $K = \mathbb{Q}(i)$ , we will just state the main results and comment on them briefly.

We start with the other "exceptional" imaginary quadratic field  $\mathbb{Q}(\sqrt{-3})$ . Let  $\omega := \frac{1 + \sqrt{-3}}{2}$  (a primitive 6th root of unity). Since the lattice  $\mathcal{O} = \mathbb{Z}[\omega]$  has hexagonal symmetry, we can take the set of points lying above the hexagon in Figure 4.2.1. as a fundamental domain for the translations by elements of  $\mathcal{O}$ . (It turns out that  $H_0 \cap H_\infty$  consists of all points in this set which lie on the unit hemisphere, i.e., satisfy  $z\bar{z} + \zeta^2 = 1$ ). Since rotation by  $\frac{\pi}{3}$  around the  $\zeta$ -axis



and reflection on the  $x$ -axis (for  $z = x + iy$ ) are possible, we can take  $D$  to be the set above the triangle with vertices  $0, \frac{1}{2}$  and  $\frac{1}{2} + \frac{3i}{6}$ . To be precise,  $D = \{(z, \zeta) \mid \frac{1}{2} \geq \text{Re } z \geq \sqrt{3} \text{Im } z \geq 0\}$ . The set of admissible cusps

with respect to  $C = I_\infty \cap D$  is  $\{0, 1, \omega, \infty\}$ . We have the following sets of minimal cusps for points in  $C$ :

- a) if  $\text{Re } z = \frac{1}{2}$  and  $\text{Im } z < \frac{3}{6}$ , then  $M(z, \zeta) = \{0, 1, \infty\}$ .
- b) if  $\text{Re } z = \frac{1}{2}$  and  $\text{Im } z = \frac{3}{6}$ , then  $M(z, \zeta) = \{0, 1, \omega, \infty\}$ .
- c) for all other  $(z, \zeta)$  in  $C$ , we have  $M(z, \zeta) = \{0, \infty\}$ .

With this information, we get the following result concerning the fundamental cellular domain:

**4.2.2. Proposition** Let  $C = \{(z, \zeta) \in S_{1,0} \mid \frac{1}{2} \geq \text{Re } z \geq \sqrt{3} \text{Im } z \geq 0\}$ .

Then  $C$  is a fundamental cellular domain for  $PGL(2, \mathbb{Z}[\omega])$ . The quotient complex  $X$  is isomorphic to  $C$ . In particular,  $H^1(X, \mathbb{Q}) = H^2(X, \mathbb{Q}) = 0$ .

The corresponding result for  $PSL(2, \mathbb{Z}[\omega])$  is as follows:

**4.2.3. Proposition** Let  $\tilde{C} = \{(z, \zeta) \in S_{1,0} \mid \frac{1}{2} \geq \operatorname{Re} z \geq \sqrt{3} \mid \operatorname{Im} z > 0\}$ . Then  $\tilde{C}$  is a cellular domain for  $PSL(2, \mathbb{Z}[\omega])$ . The quotient complex  $Y$  is isomorphic to  $\tilde{C}$ , and hence  $H^1(Y, \mathbb{Q}) = H^2(Y, \mathbb{Q}) = 0$ .

The domain  $\tilde{C}$  consists simply of those points on the unit hemisphere lying above the triangle of Figure 4.2.1. and its conjugate. The minimal cusp sets for this bigger domain are easily calculated from those for points in  $C$ . Note that, since we take coefficients in a field,  $H^1(X, \mathbb{Q})$  and  $H^1(Y, \mathbb{Q})$  express the respective group cohomologies (Cf. §5.1).

The results concerning the stabilizers are tabulated in Figure 4.2.14.

The cases  $K = \mathbb{Q}(\sqrt{-2})$  and  $K = \mathbb{Q}(\sqrt{-7})$  are more interesting because some new phenomena regarding the induced cell structure and the isotropy groups occur. Among other things we see that

- a) the actions of the stabilizers on minimal cusp sets are in general not transitive and
- b) for the quotient  $Y$  with respect to the special linear group,  $H^1(Y, \mathbb{Q}) \neq H^2(Y, \mathbb{Q})$  in general.

We consider first the case  $K = \mathbb{Q}(\sqrt{-2})$ . A fundamental domain for the translations is given by the set  $\{(z, \zeta) \mid |\operatorname{Re} z| \leq \frac{1}{2}$

and  $|\operatorname{Im} z| < \frac{\sqrt{2}}{2}\}$ . The matrices  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  allow reflections on the  $x$ - and  $y$ -axis respectively, so we can take  $D$  to be the set  $\{(z, \zeta) \mid 0 \leq \operatorname{Re} z \leq \frac{1}{2} \text{ and } 0 \leq \operatorname{Im} z \leq \frac{\sqrt{2}}{2}\}$ . We have the following minimal cusp sets for points in  $I_\infty \cap D$ :

- a) if  $0 \leq \operatorname{Re} z < \frac{1}{2}$  and  $0 \leq \operatorname{Im} z < \frac{\sqrt{2}}{2}$ , then  $M(z, \zeta) = \{0, \infty\}$ .
- b) if  $\operatorname{Re} z = \frac{1}{2}$  and  $0 \leq \operatorname{Im} z < \frac{\sqrt{2}}{2}$ , then  $M(z, \zeta) = \{0, 1, \infty\}$ .
- c) if  $0 \leq \operatorname{Re} z < \frac{1}{2}$  and  $\operatorname{Im} z = \frac{\sqrt{2}}{2}$ , then  $M(z, \zeta) = \{0, i\sqrt{2}, i\frac{\sqrt{2}}{2}, \infty\}$ .
- d) if  $\operatorname{Re} z = \frac{1}{2}$  and  $\operatorname{Im} z = \frac{\sqrt{2}}{2}$ , then  $M(z, \zeta) = \{0, 1, \infty, i\sqrt{2}, 1 + i\sqrt{2}, i\frac{\sqrt{2}}{2}, \frac{1 + i\sqrt{2}}{2}, \frac{2 + i\sqrt{2}}{2}, \frac{1 + i\sqrt{2}}{3}, \frac{2 + i\sqrt{2}}{3}, \frac{1 + 2i\sqrt{2}}{3}, \frac{2 + 2i\sqrt{2}}{3}\}$  and  $d(z, \zeta) = 12$ .

Again, these computations yield results on the structure of the quotient space:

**4.2.4. Proposition** Let  $C = \{(z, \zeta) \in S_{1,0} \mid 0 \leq \operatorname{Re} z \leq \frac{1}{2} \text{ and } 0 \leq \operatorname{Im} z \leq \frac{\sqrt{2}}{2}\}$ . Then  $C$  is a cellular domain for  $PGL(2, \mathbb{Z}[\sqrt{-2}])$ . The quotient complex  $X$  is isomorphic to  $C$ , and hence  $H^1(X, \mathbb{Q}) = H^2(X, \mathbb{Q}) = 0$ .

In the case of  $PSL(2, \mathbb{Z}[\sqrt{-2}])$ , we obtain the following more interesting results:

**4.2.5. Proposition** Let  $\tilde{C} = \{(z, \zeta) \in S_{1,0} \mid 0 \leq \operatorname{Re} z \leq \frac{1}{2} \text{ and } |\operatorname{Im} z| \leq \frac{\sqrt{2}}{2}\}$ . Then  $\tilde{C}$  is a cellular domain for  $PSL(2, \mathbb{Z}[\sqrt{-2}])$ . The quotient complex  $Y$  is isomorphic to  $\sim \tilde{C}$ , where  $\sim$  is given by the identification of the lower horizontal edge with the upper one by  $\begin{pmatrix} 1 & i\sqrt{2} \\ 0 & 1 \end{pmatrix}$ . In particular,  $H^1(Y, \mathbb{Q}) \cong \mathbb{Q}$  and  $H^2(Y, \mathbb{Q}) = 0$ .

**4.2.6. Remark** The results concerning the stabilizers can





Figure 4.2.7.

be found in Figure 4.2.15. The computations show that the operation of the stabilizer of a point in  $PSL(2,0)$  on the minimal set of the point need not be transitive. For any point on the upper horizontal edge, we have  $\Gamma(z, \zeta) \cong \mathbb{Z}/2\mathbb{Z}$  but  $d(z, \zeta) = 4$ .

We now turn to  $K = Q(\sqrt{-7})$ . Denote by  $\omega = \frac{1 + i\sqrt{7}}{2}$ . Just as in the case of  $Q(\sqrt{-3})$ , we can take the set of points lying above the hexagon in Figure 4.2.8. as the fundamental domain for the translations. The reflections on the x- and y-axes allow us to take D as the set  $\{(z, \zeta) \mid 0 \leq \operatorname{Re} z \leq \frac{1}{2}, 0 \leq \operatorname{Im} z \leq \frac{-\operatorname{Re} z}{7} + \frac{2}{\sqrt{7}}\}$ . Our table of minimal cusp sets for points  $(z, \zeta)$  in  $D \cap I_\infty$  is as follows:

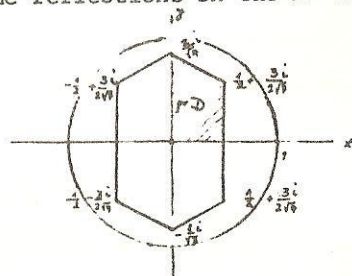


Figure 4.2.8.

- a) if  $\operatorname{Re} z = 0$  and  $\operatorname{Im} z = \frac{2}{\sqrt{7}}$ , then  $M(z, \zeta) = \{0, \infty, \omega, \frac{\omega}{2}, \omega^{-1}, \frac{1}{\omega}\}$
- b) if  $\operatorname{Re} z = \frac{1}{2}$  and  $\operatorname{Im} z = \frac{3}{2\sqrt{7}}$ , then  $M(z, \zeta) = \{0, \infty, \omega, \frac{\omega}{2}, 1, \frac{\omega+1}{2}\}$
- c) if  $0 < \operatorname{Re} z < \frac{1}{2}$  and  $\operatorname{Im} z = \frac{-\operatorname{Re} z}{7} + \frac{2}{\sqrt{7}}$ , then  $M(z, \zeta) = \{0, \infty, \omega, \frac{\omega}{2}\}$
- d) if  $\operatorname{Re} z = \frac{1}{2}$  and  $0 \leq \operatorname{Im} z < \frac{3}{2\sqrt{7}}$ , then  $M(z, \zeta) = \{0, 1, \infty\}$ .
- e) for all other points in C,  $M(z, \zeta) = \{0, \infty\}$ .

**4.2.9. Proposition** Let  $C = \{(z, \zeta) \in S_{1,0} \mid 0 \leq \operatorname{Re} z \leq \frac{1}{2} \text{ and } 0 \leq \operatorname{Im} z \leq \frac{-\operatorname{Re} z}{7} + \frac{2}{\sqrt{7}}\}$ . Then C is a cellular domain for  $PGL(2, \mathbb{Z}[\omega])$ . The quotient complex X is isomorphic to  $\sim C$ , where  $\sim$  is given by the identification along the

edge c). In particular,  $H^1(X, Q) = H^2(X, Q) = 0$ .

**4.2.10. Remarks** This case gives an example of an identification within a natural 1-cell. The induced cell structure is indicated in Figure 4.2.10b for  $H_0 \cap H_\infty$ . This is also the first example for non-transitivity of the action of stabilizers in  $PGL(2, Q)$  on the corresponding minimal cusp sets.



Figure 4.2.10a

The points on the edge c) between  $(\frac{\omega}{2}, \frac{\sqrt{2}}{2})$  and  $(\frac{21}{17}, \frac{\sqrt{3}}{7})$  have isotropy groups isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  but cusp degree equal to 4. The complete results for the stabilizers are tabulated in Figure 4.2.16.

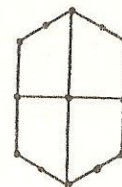


Figure 4.2.10b

The corresponding result for  $PSL(2, \mathbb{Z}[\omega])$  is:

**4.2.11. Proposition** Let  $\tilde{C} = \{(z, \zeta) \in S_{1,0} \mid 0 \leq \operatorname{Re} z \leq \frac{1}{2} \text{ and } 0 \leq |\operatorname{Im} z| \leq \frac{-\operatorname{Re} z}{7} + \frac{2}{\sqrt{7}}\}$ . Then  $\tilde{C}$  is a cellular domain for  $PSL(2, \mathbb{Z}[\omega])$ . The quotient complex Y is isomorphic to  $\sim \tilde{C}$  with the identifications " $\sim$ " as in Figure 4.2.11. In particular,  $H^1(Y, Q) \cong Q$  and  $H^2(Y, Q) = 0$  (Y is homeomorphic to a Möbius strip).



The results for  $K = Q(\sqrt{-11})$  are quite similar to those for the preceding case. The pictures one obtains are indeed (viewed topologically) identical--the principal differences lie in the cusp degrees associated to cells and thus in the cell stabilizers. To be precise, we have the following proposition:

**4.2.12. Proposition** Let  $K = Q(\sqrt{-11})$  and  $\omega := \frac{1+\sqrt{11}i}{2}$ . Denote by  $C$  the intersection of the unit hemisphere  $S_{1,0}$  with the set  $\{(z, \zeta) \mid 0 \leq \operatorname{Re} z \leq \frac{1}{2} \text{ and } 0 \leq \operatorname{Im} z \leq \frac{-\operatorname{Re} z + 3}{\sqrt{11}}\}$ .

1) The minimal cusp sets for points in  $C$  are as follows:

for $(z, \zeta)$ with	$M(z, \zeta)$ is equal to
a) $\operatorname{Re} z = 0$ and $\operatorname{Im} z = \frac{3}{\sqrt{11}}$	$\infty, 0, \omega, \omega^{-1}, \frac{1}{1-\omega}, \frac{2}{1-\omega}, \frac{-1}{\omega}, \frac{-2}{\omega}, \frac{\omega}{2}, \frac{\omega-1}{2}, \frac{\omega-2}{\omega+1}, \frac{\omega+1}{2-\omega}$
b) $0 < \operatorname{Re} z < \frac{1}{2}$ and $\operatorname{Im} z = \frac{-\operatorname{Re} z + 3}{\sqrt{11}}$	$\infty, 0, \omega, \frac{\omega}{2}, \frac{1}{1-\omega}, \frac{2}{1-\omega}$
c) $\operatorname{Re} z = \frac{1}{2}$ and $\operatorname{Im} z = \frac{5}{2\sqrt{11}}$	$\infty, 0, 1, \omega, \frac{1}{1-\omega}, \frac{2}{1-\omega}, \frac{\omega-1}{\omega}, \frac{\omega-2}{\omega}, \frac{\omega}{2}, \frac{\omega+1}{2}, \frac{\omega-1}{\omega+1}, \frac{2}{2-\omega}$
d) $\operatorname{Re} z = \frac{1}{2}$ and $0 \leq \operatorname{Im} z < \frac{5}{2\sqrt{11}}$	$\infty, 0, 1$
e) otherwise	$\infty, 0$

2)  $C$  is a fundamental cellular domain for  $\operatorname{PGL}(2, Z[\omega])$ . The quotient complex is isomorphic to  $\sim C$  where  $\sim$  is given by identification along the edge b) thru  $\begin{pmatrix} 1 & -\omega \\ 0 & -1 \end{pmatrix}$ . In particular,

$$H^1(X, Q) = H^2(X, Q) = 0.$$

3) Let  $\tilde{C}$  be the intersection of the unit hemisphere with the set  $\{(z, \zeta) \mid 0 \leq \operatorname{Re} z \leq \frac{1}{2} \text{ and } |\operatorname{Im} z| \leq \frac{-\operatorname{Re} z + 3}{\sqrt{11}}\}$ . Then  $\tilde{C}$  is a cellular fundamental domain for  $\operatorname{PSL}(2, Z[\omega])$ . The quotient complex  $Y$  is isomorphic to  $\sim \tilde{C}$  with the identifications " $\sim$ "

as in Figure 4.2.12. In particular,  $H^1(Y, Q) \cong Q$  and

$H^2(Y, Q) = 0$  ( $Y$  is homeomorphic to a Möbius strip).

$$\begin{pmatrix} \omega & -1 \\ 1 & 0 \end{pmatrix}$$

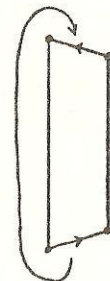


Figure 4.2.12.

**4.2.13.** The complete results for the stabilizers are tabulated in Figure 4.2.17.

In all the five cases considered here, the action of  $\Gamma(z, \zeta)$  on the minimal cusp sets of vertices  $(z, \zeta)$  is transitive. We do not know whether this is a general phenomenon; in fact, it isn't clear whether the vertices of the natural cell structure are always fixed points.

We now summarize the stabilizer computations for the cases considered so far. The column on the right of each diagram consists of matrices inducing generators of edge stabilizers. A convenient way of checking the computations is to calculate the virtual Euler characteristic of the corresponding group (cf. (45), p. 93) which has to be zero.

The case  $d = -1$  has already been treated in detail in Propositions 4.1.8. and 4.1.9. We just complete the information concerning the generators: for  $\operatorname{PGL}(2, Z[i])$ , the vertical edge stabilizer is generated by  $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ . For  $\operatorname{PSL}(2, Z[i])$ ,



numbering the vertices as follows  $\begin{smallmatrix} 4 & 2 \\ 1 & 3 \end{smallmatrix}$ , we get as generator of the stabilizer of the edge (ij):

$$(12): \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (23): \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad (34): \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad (14): \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\underline{d = -3} \quad \omega := \frac{1 + \sqrt{3}i}{2}$$

$\text{PGL}(2, \mathbb{Z}[\omega])$

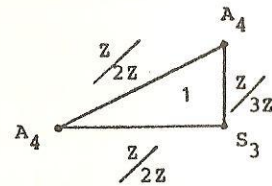


Figure 4.2.14a

$\text{PSL}(2, \mathbb{Z}[\omega])$

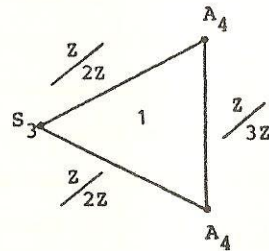


Figure 4.2.14b

$$(12): \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(13): \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$$

$$(23): \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(12): \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$$

$$(13): \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}$$

$$(23): \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\underline{d = -2} \quad \omega := \sqrt{2}i$$

$\text{PGL}(2, \mathbb{Z}[\omega])$

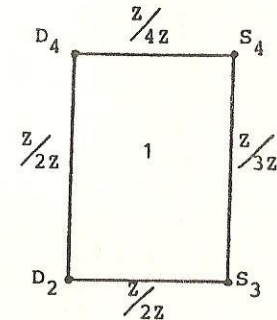


Figure 4.2.13a

$$(12): \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(23): \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(34): \begin{pmatrix} \omega & 1 \\ 1 & 0 \end{pmatrix}$$

$$(14): \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\text{PSL}(2, \mathbb{Z}[\omega])$

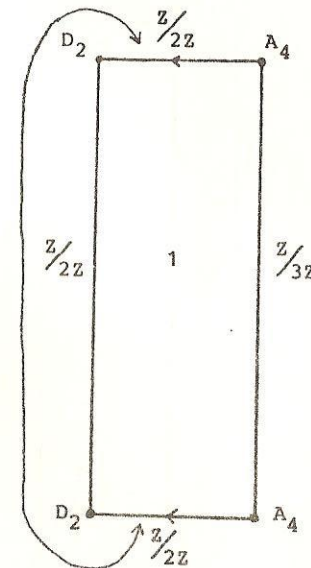


Figure 4.2.13b

$$(12): \begin{pmatrix} 1 & \omega \\ \omega & -1 \end{pmatrix}$$

$$(23): \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(34): \begin{pmatrix} 1 & -\omega \\ -\omega & -1 \end{pmatrix}$$

$$(14): \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\underline{d} = -7 \quad \omega := \frac{1 + \sqrt{7}i}{2}$$

$\text{PGL}(2, \mathbb{Z}[\omega])$

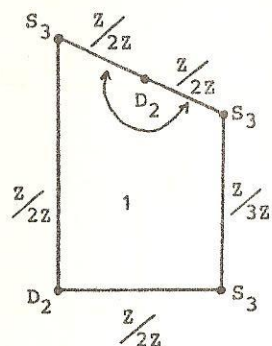


Figure 4.2.14a

$\text{PSL}(2, \mathbb{Z}[\omega])$

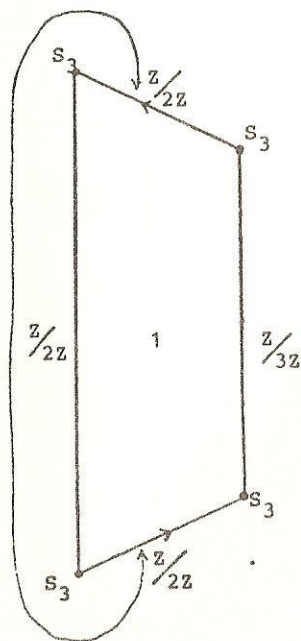


Figure 4.2.14b

$$(12): \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(23): \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(34): \begin{pmatrix} 1 & -\omega \\ \bar{\omega} & -1 \end{pmatrix}$$

$$(45): \begin{pmatrix} 1 & -\omega \\ \bar{\omega} & -1 \end{pmatrix}$$

$$(15): \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(12): \begin{pmatrix} 1 & -\bar{\omega} \\ \omega & 1 \end{pmatrix}$$

$$(23): \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(34): \begin{pmatrix} 1 & -\omega \\ \bar{\omega} & -1 \end{pmatrix}$$

$$(14): \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\underline{d} = -11$$

$$\omega := \frac{1 + \sqrt{11}i}{2}$$

$\text{PGL}(2, \mathbb{Z}[\omega])$

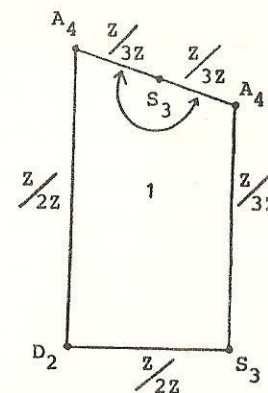


Figure 4.2.17a

$$(12): \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(23): \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(34): \begin{pmatrix} 1 & -\omega \\ \bar{\omega} & -2 \end{pmatrix}$$

$$(15): \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\text{PSL}(2, \mathbb{Z}[\omega])$

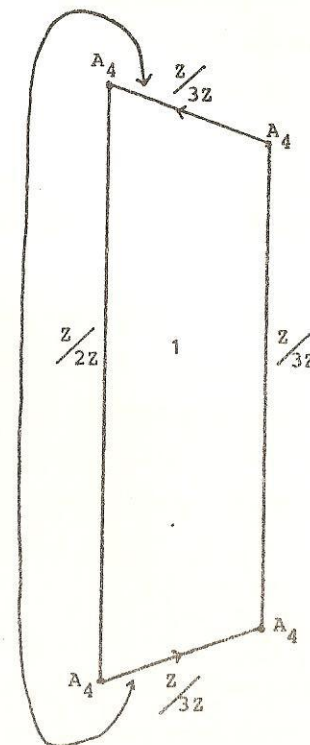


Figure 4.2.17b

$$(12): \begin{pmatrix} 1 & -\bar{\omega} \\ \omega & -2 \end{pmatrix}$$

$$(23): \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(34): \begin{pmatrix} 1 & -\omega \\ \bar{\omega} & -2 \end{pmatrix}$$

$$(14): \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



4.2.18. Remark After I had made these computations, I received a manuscript of Flügge (3) where the cellular domains and stabilizers in the euclidean cases were determined by similar methods. However, he considered only  $PSL(2, \mathcal{O})$  and also did not determine the quotient spaces. Since he used classical reduction theory, his method works only for class number equal to 1.

## § 5. COHOMOLOGY COMPUTATIONS

5.0. In this section we shall use the action of  $\Gamma$  on the minimal incidence set  $I$  to yield a procedure for explicitly computing certain cohomology groups of  $\Gamma$ . To be precise, our method works for all  $\Gamma$ -modules over rings where 2 and 3 are invertible. This restriction is due to the Theorem (5.1.2.) we use to pass from the topological to the algebraic situation, and is not relevant to the applications we had in mind.

The topological problem consists of computing cohomology groups of  $\bigwedge^I$  with coefficients in a sheaf  $\underline{M}^\Gamma : \underline{M}^\Gamma$ , although in general not locally constant, has the nice property that it is constant on open cells of the finite quotient complex. This cell structure has a subdivision with the property that the cohomology of the covering by open stars of vertices is the same as that of  $\bigwedge^I$ . These results constitute 5.1.

In the last two subsections 5.2. and 5.3. we apply the method to the action of  $PGL(2, \mathbb{Z}[i])$  on the binary homogeneous polynomials of degree  $2p$  over  $\mathcal{O}_p$  and compute the case  $p = 3$ . We also investigate the "cohomology at infinity"  $H^n(\Gamma_\infty, M_p) \approx H^n(\bigwedge_{\infty}^I, M_p^{\Gamma_\infty})$  and the "restriction mapping"  $r_n: H^n(\Gamma, M_p) \longrightarrow H^n(\Gamma_\infty, M_p)$ .

Remember:  $\Gamma := PGL(2, \mathcal{O})$ .

5.1. We have to broaden our framework slightly in order to accommodate the application considered in 5.3.

Let  $\Gamma' \subset \Gamma$  be a subgroup and  $I' \subset I$  a 2-dimensional regular contractible subcomplex (relative to the natural cell structure) such that  $\Gamma' \cdot I' = I'$  and  $\Gamma' \backslash I'$  is compact. It is clear that the considerations of §2 and §4 carry over to this more general situation. When we speak of the quotient as a CW complex, we mean the cell structure on  $\Gamma' \backslash I'$  inherited from the  $\Gamma'$ -induced structure on  $I'$ .

5.1.1. Example Let  $K = Q(i)$  and  $\Gamma := PGL(2, \mathbb{Z}[i])$ . The example considered in detail in 5.3. is  $\Gamma' = \Gamma_\infty$  and  $I' = I_\infty$ .  $I_\infty$  is easily seen to be contractible: indeed the vertical projection onto the plane  $\zeta = 0$  is a homeomorphism. Due to  $\Gamma$ -invariance of distance functions and cusp degrees,  $\Gamma_\infty \cdot I_\infty = I_\infty$ . We shall determine the structure of the finite complex  $\Gamma_\infty \backslash I_\infty$  in 5.3.

Let  $R$  be a commutative ring and  $M$  an  $R$ -module on which  $\Gamma'$  operates. We associate to  $M$  the following sheaf  $\underline{M}^{\Gamma'}$  on  $\Gamma' \backslash I'$ : if  $p: I' \rightarrow \Gamma' \backslash I'$  is the projection map and  $U$  is an open set in  $\Gamma' \backslash I'$ , then

$$\underline{M}^{\Gamma'}(U) := \{s: p^{-1}(U) \rightarrow M \mid s \text{ is locally constant and } \Gamma\text{-invariant, i.e., } \gamma s(z, \zeta) = s(\gamma(z, \zeta))\}$$

Since  $\Gamma'$  operates properly (discontinuously) on the contractible space  $I'$ , a well known theorem ( (5), p. 204 Corollaire) yields the following result:

5.1.2. Theorem Let  $M$  be a  $\Gamma'$ -module with the following property: the map  $M \rightarrow M$  given by  $m \mapsto |\Gamma'(z, \zeta)|m$  is an automorphism for each  $(z, \zeta)$  in  $I'$ . Then:

$$H^n(\Gamma', M) = H^n(\Gamma' \backslash I', \underline{M}^{\Gamma'})$$

As Bianchi (1) p. 297-298 has observed, the only

primes which occur in stabilizer orders are 2 and 3; hence the method we will develop cannot handle the 2- and 3-torsion in the cohomology. However, this isn't important for the application we will discuss.

Our task is now a topological one, i.e., computing the sheaf cohomology. However,  $\underline{M}^{\Gamma'}$  is in general not locally constant, and it is at this point that the cellular structure becomes very handy. The reason is the following proposition:

5.1.3. Proposition Let  $\sigma$  and  $\sigma'$  be open cells in  $\Gamma' \backslash I'$  and  $I'$  respectively, such that  $p$  maps  $\sigma'$  homeomorphically to  $\sigma$ . Let  $\Gamma_\sigma$  be the (common) stabilizer of points in  $\sigma'$ . Then  $\underline{M}^{\Gamma'}|_\sigma$  is isomorphic to the constant sheaf on  $\sigma$  with fiber  $\underline{M}^{\Gamma_\sigma}$ .

Proof: Every section  $s$  over a set  $U$  open in  $\sigma$  can be represented as  $s': p^{-1}(U) \rightarrow M$ , with  $s'$  locally constant and  $\Gamma'$ -invariant. The isomorphism is given by  $s \mapsto s'|_{\sigma' \cap p^{-1}(U)} \circ (p|_{\sigma'})^{-1}|_U =: \phi(s)$ . It is clear that  $\phi$  is well-defined and injective; the surjectivity follows from the fact that, due to the discontinuous action, each open cell  $\sigma'$  in  $I'$  has an open neighborhood  $U'$  such that  $\gamma U' \cap U' \neq \emptyset$  if and only if  $\gamma \in \Gamma_\sigma$ .

5.1.4. Corollary The stalk  $\underline{M}_u^{\Gamma'}$  is isomorphic to  $M^{\Gamma(z, \zeta)}$ , where  $p(z, \zeta) = u$ .

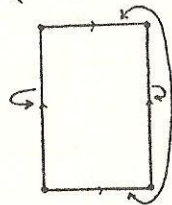
Recall that for a normal CW complex  $X$ , the (open) star  $st(A)$  of a subset  $A \subset X$  is defined as the union of all open cells  $\sigma$  such that  $\bar{\sigma} \cap A \neq \emptyset$ .  $st(A)$  is clearly an open set in  $X$ , and the sets  $\{st(v) \mid v \text{ 0-cells}\}$  form an open covering



of  $X$ , which we call the star covering of (the complex)  $X$ . If  $F$  is a sheaf on  $X$ , then an open covering  $(U_i)$  is called  $F$ -acyclic if  $H^n(U_{i_0} \cap \dots \cap U_{i_p}, F) = 0$  for all  $n \geq 1$  and tuples  $(i_0, \dots, i_p)$  with  $p \geq 0$ .

**5.1.5. Remarks** a) For any point  $u \in \Gamma' \backslash I'$ , the module of  $\underline{M}^{\Gamma'}$ -sections over  $\text{st}(u)$  is clearly isomorphic to  $M^{\Gamma'} \sigma'$ , where the open cell  $p(\sigma')$  contains  $u$ .

b) The star covering of  $\Gamma' \backslash I'$  (with the  $\Gamma'$ -induced cell structure) is not necessarily  $\underline{M}^{\Gamma'}$ -acyclic. Let  $\mathcal{O} = \mathbb{Z}[\sqrt{-2}]$  and consider  $\Gamma' = \Gamma_\infty \cap \text{PSL}(2, \mathcal{O})$ , the group of translations by elements of  $\mathcal{O}$ .  $\Gamma'$  acts on  $I_\infty$  and the quotient is compact. Since  $\Gamma'$  has no elements of finite order,  $\underline{M}^{\Gamma'} = \underline{M}$  (constant sheaf!). If  $v$  is the unique vertex in the  $\Gamma'$ -induced structure, then  $\text{st}(v) = \Gamma' \backslash I'$  which is a torus, so that  $H^1(\text{st}(v), \underline{M}) \neq 0$ .



cellular domain for  $\Gamma'$

Figure 5.1.5.

We shall now give a sufficient condition for the acyclicity of the star covering. This amounts to a sort of "simplicial" condition on the cell structure:

**5.1.6. Proposition** If  $\Gamma' \backslash I'$  is regular, then the star covering  $\{\text{st}(v)\}$  is acyclic. In this case,  $H^n(\{\text{st}(v)\}, \underline{M}^{\Gamma'}) = H^n(\Gamma' \backslash I', \underline{M}^{\Gamma'})$ .

**Proof:** Let  $U := \text{st}(v)$ . Denote by  $A$  the 1-skeleton in  $U$ : this

set is closed in  $U$ . We have an exact sequence of sheaves:

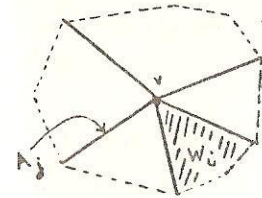


Figure 5.1.6.

$$0 \rightarrow \underline{M}_{U-A}^{\Gamma'} \rightarrow \underline{M}^{\Gamma'} \rightarrow \underline{M}_A^{\Gamma'} \rightarrow 0$$

( (4), p. 140). This induces a long exact sequence for cohomology groups:

$$\dots \rightarrow H^1(U, \underline{M}_{U-A}^{\Gamma'}) \rightarrow H^1(U, \underline{M}^{\Gamma'}) \rightarrow H^1(U, \underline{M}_A^{\Gamma'}) \rightarrow H^2(U, \underline{M}_{U-A}^{\Gamma'}) \rightarrow \dots$$

Our first claim is:  $H^n(U, \underline{M}_{U-A}^{\Gamma'}) = 0$  for all  $n \geq 1$ . It is clear that  $U - A$  = the disjoint union of the open 2-cells  $W_i$  in  $U$ . Thus:  $\underline{M}_{U-A}^{\Gamma'} = \bigoplus \underline{M}_{W_i}^{\Gamma'} = \bigoplus \underline{M}_{W_i}$  and this implies that  $H^n(U, \underline{M}_{U-A}^{\Gamma'}) = \bigoplus H^n(U, \underline{M}_{W_i})$ . Since the quotient  $\Gamma' \backslash I'$  is regular,  $U$  is contractible and each direct summand is zero.

To show that  $H^n(U, \underline{M}_A^{\Gamma'}) = 0$  for  $n \geq 1$ , we do the analogous construction for  $A$  and the closed subset  $\{v\}$ . Note that if  $G := (\underline{M}_A^{\Gamma'})|_A$ , then  $H^n(A, G) = H^n(U, \underline{M}_A^{\Gamma'})$  (cf (4), p. 188). The exact sequence  $0 \rightarrow G_{A-\{v\}} \rightarrow G \rightarrow G_{\{v\}} \rightarrow 0$  induces the long exact sequence

$$\dots \rightarrow H^1(A, G_{A-\{v\}}) \rightarrow H^1(A, G) \rightarrow H^1(A, G_{\{v\}}) \rightarrow H^2(A, G_{A-\{v\}}) \rightarrow \dots$$

Obviously  $H^n(A, G_{\{v\}}) = 0$  for  $n \geq 1$ . Again  $A - \{v\}$  is the disjoint union of the open 1-cells  $A_j$  in  $U$ , so that  $H^n(A, G_{A-\{v\}}) = \bigoplus H^n(A, G_{A_j})$ . Since regularity implies that  $A$  is contractible, each summand is zero.

The arguments for the groups  $H^n(U_{i_0} \cap \dots \cap U_{i_p}, \underline{M}^{\Gamma'})$  with  $n, p \geq 1$  are analogous.

If  $\Gamma' \backslash I'$  is not regular, we can find a regular subdivision of its  $\Gamma'$ -induced cell structure since we know the

(finitely many) identifications to be done on a cellular domain in order to get the quotient space. We illustrate this with a simple example:

5.1.7. Example Let  $\mathcal{O} = \mathbb{Z}[\sqrt{-2}]$  and  $\Gamma' = \text{PSL}(2, \mathcal{O})$ . Set  $I' = I$ . The cellular domain determined in 4.2. is the leftmost figure.

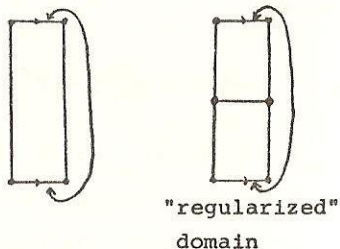


Figure 5.1.7.



alternative covering  
Figure 5.1.8.

5.1.8. Remark If  $\frac{I'}{\Gamma'}$  is not regular, one can also obtain an  $\underline{M}^{\Gamma'}$ -acyclic covering indexed by the set of all cells. For each 0- or 1-cell  $\sigma$ , we take the intersection of  $\text{st}(\sigma)$  with a small open neighborhood. For each 2-cell, we take the (open) 2-cell itself. (cf. Figure 5.1.8.) This covering usually has more sets than the star covering of a "regularization".

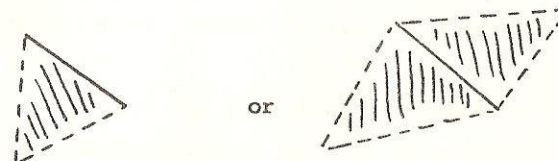
We now assume that  $\frac{I'}{\Gamma'}$  has a regular cell structure (this no longer need be the  $\Gamma'$ -induced structure). The complex of alternating Cech-cochains has a particularly simple form under a certain "simplicial" condition:

5.1.9. Proposition Suppose that each 2-cell in  $\frac{I'}{\Gamma'}$  is a triangle, i.e., there are exactly 3 0-cells on its boundary. Then the alternating Cech-complex of the star covering has the form

$$0 \longrightarrow \bigoplus_{\sigma \text{ 0-cell}} M^{\Gamma'}_{\sigma} \longrightarrow \bigoplus_{\tau \text{ 1-cell}} M^{\Gamma'}_{\tau} \longrightarrow \bigoplus_{\text{no of 2-cells}} M \longrightarrow 0$$

where  $\sigma'$  is a cell of  $I'$  with  $p(\sigma') = \sigma$ .

Proof: The intersection  $\text{st}(v) \cap \text{st}(u)$  is a disjoint union of open sets of the form



Each set corresponds to a 1-cell with endpoints  $u$  and  $v$ . Moreover, the intersection of 3 open stars is a disjoint union of open 2-cells. Because of "triangularity", each open 2-cell appears exactly once.

Again it is clear that the cell structure of  $\frac{I'}{\Gamma'}$  can be refined to yield a "triangular" one as above.

We close this paragraph by defining the restriction maps  $r_n: H^n(\Gamma, M) \longrightarrow H^n(\Gamma_\lambda, M)$  in our context. We have a commutative diagram:

$$\begin{array}{ccc} I_\lambda & \hookrightarrow & I \\ P_\lambda \downarrow & & \downarrow P \\ \frac{I_\lambda}{\Gamma_\lambda} & \xrightarrow{\phi_\lambda} & \frac{I}{\Gamma} \end{array}$$

where  $\phi_\lambda(\Gamma_\lambda(z, \zeta)) = \Gamma(z, \zeta)$ . Since  $\phi_\lambda^{-1}U = p_\lambda(p^{-1}U \cap I_\lambda)$ ,  $\phi_\lambda$  is a continuous map. If  $s \in \underline{M}^\Gamma(U)$ , i.e.,  $s: p^{-1}U \longrightarrow M$  is locally constant and  $\Gamma$ -invariant, then the restriction  $s|_{p^{-1}U \cap I_\lambda} \in \underline{M}^{\Gamma_\lambda}(\phi_\lambda^{-1}U)$ . This map is clearly a  $\phi_\lambda$ -cohomomorphism from  $\underline{M}^\Gamma$  to  $\underline{M}^{\Gamma_\lambda}$ , and hence gives rise to homomorphisms



$r_n: H^n(\Gamma \backslash I, \underline{M}^\Gamma) \longrightarrow H^n(\Gamma_\lambda \backslash I_\lambda, \underline{M}^{\Gamma_\lambda})$ . We shall examine these maps in a special case in the next paragraph.

5.2. From now on,  $\mathcal{O} = \mathbb{Z}[i]$  and  $\Gamma := \text{PGL}(2, \mathbb{Z}[i])$ . Let  $p$  be a natural number. For each integer  $j$  with  $-p \leq j \leq p$ , define  $e_j := X^{p+j} Y^{p-j}$  where  $X, Y$  are variables. Denote by  $M_p^{(o)}$  the  $\mathcal{O}$ -algebra

$$M_p^{(o)} := \left\{ \sum_{j=-p}^p a_j e_j \mid a_j \in \mathcal{O} \right\}$$

The group  $\Gamma$  operates on  $M_p^{(o)}$  as follows: if  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , then

$$\gamma \cdot e_j := \frac{(aX + cY)^{p+j} (bX + dY)^{p-j}}{(ad - bc)^p}$$

This operation is irreducible and thus  $H^0(\Gamma, M_p^{(o)}) = 0$ .

We denote by  $\mathcal{O}_p$  the ring  $\mathbb{Z}[i, \frac{1}{2}, \dots, \frac{1}{k}]$  where  $k$  runs thru all prime numbers  $< 2p$ .  $M_p := M_p^{(o)} \otimes \mathcal{O}_p$ . Since  $\Gamma \backslash I$  is isomorphic to a 2-simplex (Prop. 4.1.8.), we can use Prop. 5.1.9. to compute the cohomology groups  $H^n(\Gamma, M_p)$  for  $p \geq 2$ . Before we start with some of these computations, let us fix some notation. We denote the cells in  $\Gamma \backslash I$  as in Figure 5.2.1.

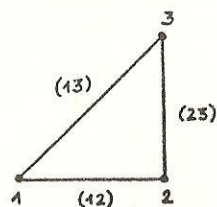


Figure 5.2.1.

The cell stabilizers are:

$$\Gamma_1 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong D_4$$

$$\Gamma_2 = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong S_3$$

$$\Gamma_3 = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong S_4$$

$$\Gamma_{12} = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad \Gamma_{23} = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad \Gamma_{13} = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

We shall write  $M_p^i$  and  $M_p^{ij}$  for  $M_p^1$  and  $M_p^{ij}$  respectively. If the context is clear, we shall also drop the index  $p$ .

In this case, the Čech complex of 5.1.9. is as follows:

$$0 \longrightarrow M_p^1 \oplus M_p^2 \oplus M_p^3 \xrightarrow{d_0} M_p^{12} \oplus M_p^{23} \oplus M_p^{13} \xrightarrow{d_1} M_p \longrightarrow 0$$

$$\text{where } d_0(m_1, m_2, m_3) = (m_2 - m_1, m_3 - m_2, m_3 - m_1) \\ d_1(m_{12}, m_{23}, m_{13}) = m_{12} + m_{23} - m_{13}.$$

We shall now compute  $H^1(\Gamma, M_p)$  for  $p = 3$ . This is the lowest value of the interesting case  $p \equiv 3 \pmod{4}$  and is still accessible to computation by hand.

Let  $\tilde{M} := M^{12} \oplus M^{23} \oplus M^{13}$ . We have an exact sequence:

$$0 \longrightarrow \text{Ker } d_1 / \text{Im } d_0 \longrightarrow \tilde{M} / \text{Im } d_0 \longrightarrow \tilde{M} / \text{Ker } d_1 \longrightarrow 0$$

Since  $\tilde{M} / \text{Ker } d_1 \cong \text{Im } d_1$  is free, we have a decomposition

$$\tilde{M} / \text{Im } d_0 = H^1(\Gamma, M) \oplus F$$

where  $F$  is a free submodule of  $\tilde{M} / \text{Im } d_0$  isomorphic to  $\tilde{M} / \text{Ker } d_1$ . First we determine bases for  $M^{12}$ ,  $M^{13}$  and  $M^{23}$  in order to obtain one for  $\tilde{M}$ .

a)  $M^{12}$ : we have in general  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e_j = (-1)^p e_{-j}$ . Hence, a basis is given by

$$\{e_j + (-1)^p e_{-j} \neq 0 \mid 0 \leq j \leq p\}.$$

In particular, for  $p = 3$ :

$A_1 = e_3 - e_{-3}$ ,  $A_2 = e_2 - e_{-2}$  and  $A_3 = e_1 - e_{-1}$  form a basis for  $M^{12}$ .

b)  $M^{13}$ : again, in general,  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e_j = (-1)^{p+j} e_{-j}$ . For odd  $p$ , the set

$$\{e_j + (-1)^{j+1} e_{-j} \neq 0 \mid 0 \leq j \leq p\}$$

is a basis. For  $p = 3$ , we have:

$$C_1 = e_3 - ie_{-3}, \quad C_2 = e_2 + e_{-2} \quad \text{and} \quad C_3 = e_1 + ie_{-1}.$$

c)  $M^{23}$ : in this case,  $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} e_j = (-1)^{p-j} (e_p + \binom{p+j}{1} e_{p-1} + \dots + e_{-j})$ . The linear equation for getting a basis isn't as simple as in the previous two cases. For  $p = 3$ , we obtain the following basis elements:

$$B_1 = e_3 + 6e_2 + 15e_1 + 10e_0 + e_{-3}$$

$$B_2 = -e_2 - 5e_1 - 5e_0 + e_{-2}$$

$$B_3 = e_1 + 2e_0 + e_{-1}$$

The basis for  $M$  is  $\{(A_1, 0, 0), (0, B_j, 0), (0, 0, C_k) \mid 1 \leq j, k \leq 3\}$ .

The second step is to obtain a basis for  $\text{Im } d_0$ . Since  $d_0$  is injective, it suffices to get a basis for  $M^1 \oplus M^2 \oplus M^3$ . Again we consider  $M^1$ ,  $M^2$  and  $M^3$  separately.

a)  $M^1$ : since  $\Gamma_1$  is generated by  $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , we have  $M^1 = M^{12} \cap M^{\gamma}$ . Since  $\gamma e_j = i^j e_j$ ,

$v = \sum_{j=-3}^3 a_j e_j \in M^{\gamma}$  iff for all  $-3 \leq j \leq 3$ ,  $i^j a_j = a_j$ , or equivalently,  $a_j \neq 0$  iff  $j \equiv 0 \pmod{4}$ . This is true only for  $j = 0$ . Since  $\{e_j - e_{-j} \mid 0 < j \leq 3\}$  is a basis for  $M^{12}$ ,  $M^1 = 0$ .

b)  $M^3$ : since  $\Gamma_3$  contains  $\Gamma_{23}$  and  $\Gamma_{13}$ , we have  $M^3 \subset M^{23} \cap M^{13}$ .

We claim that this intersection is already 0. Suppose that  $z = b_1 B_1 + b_2 B_2 + b_3 B_3 = c_1 C_1 + c_2 C_2 + c_3 C_3$ , where  $B_j, C_k$  form the bases for  $M^{23}$  and  $M^{13}$  respectively. By comparison of coefficients, we get  $c_1 = b_1 = -ic_1$ ,  $c_2 = b_2 = -c_2$  and

$c_3 = b_3 = ic_3$ . Hence all coefficients are zero. Thus,  $M^3 = 0$ .

c)  $M^2$ :  $M^2$  is generated by  $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and thus  $M^2 = M^{12} \cap M^{23}$ .

For an element in the intersection, we get the equations:

$a_1 = b_1 = -a_1$  (this implies  $a_1 = b_1 = 0$ ),  $a_2 = -b_2$ ,  $a_3 = -b_3$  and  $b_2 = \frac{2}{5} b_3$ . Let  $D := \frac{2}{5} A_2 + A_3$ . This is a base vector for  $M^2$ .

Our basis for  $\text{Im } d_0$  is  $d_0(D) = (\frac{2}{5} A_2 + A_3, \frac{2}{5} B_2 + B_3, 0)$ .

By elementary divisors theory, one sees that  $M / \text{Im } d_0$  is free. Since the dimension of  $F$  is 7, we can conclude that  $H^1(\Gamma, M)$  is free of rank 1 over  $\mathcal{O}_3 = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]$ .

5.3. We shall now consider the "cohomology at infinity", i.e.,  $H^n(\Gamma_\infty, M_p)$ . First of all, we determine a cellular domain for  $\Gamma_\infty$  in  $I_\infty$  and compute stabilizers:

5.3.1. Proposition a) The set  $C_\infty = \{(z, \zeta) \in S_{1,0} \mid 0 \leq \text{Re } z \leq \frac{1}{2} \text{ and } 0 \leq \text{Im } z \leq \frac{1}{2}\}$  is a cellular fundamental domain for  $\Gamma_\infty$  in  $I_\infty$ .

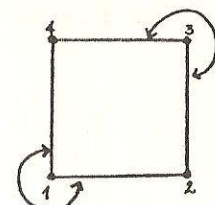


Figure 5.3.1.

The edge identifications are  $((ij) \text{ denotes the closed cell})$ :

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \text{ maps } (23) \text{ to } (34) \quad \text{and}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ maps } (12) \text{ to } (14)$$

b) Let  $\Gamma_\infty^1$  and  $\Gamma_\infty^{ij}$  denote the stabilizers of the open cells 1 and  $(ij)$  respectively. Then:

$$\Gamma_\infty^1 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle = \mathbb{Z} / 4\mathbb{Z} \quad \Gamma_\infty^{23} = \left\langle \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle = \mathbb{Z} / 2\mathbb{Z}$$

$$\Gamma_\infty^{34} = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle = \mathbb{Z} / 4\mathbb{Z} \quad \Gamma_\infty^{12} = \Gamma_\infty^{13} = \Gamma_\infty^{23} = \{1\}$$

Proof: The proof is immediate from the computations in 4.1.7. and 4.1.8.

Since  $\Gamma_\infty / I_\infty$  (with the induced cell structure) is not regular, we add a 1-cell connecting the vertices 1 and 3. Thus Prop. 5.1.9. is also applicable and we have the following complex for computing cohomology:

$$0 \longrightarrow \bar{M}_p^1 \oplus \bar{M}_p^2 \oplus \bar{M}_p^3 \xrightarrow{\bar{d}_0} M_p \oplus M_p \oplus M_p \xrightarrow{\bar{d}_1} M_p \oplus M_p \longrightarrow 0$$



where  $\bar{M}_p^i = M_p^i$  and

$$\bar{d}_0(m_1, m_2, m_3) = (m_2 - m_1, m_3 - m_2, m_3 - m_1)$$

$$\bar{d}_1(m_{12}, m_{23}, m_{13}) = (m_{12} + m_{23} - m_{13}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} m_{12} + \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} m_{23} - m_{13})$$

The first question we would like to answer is: What is the kernel of  $\bar{d}_0$ ?

5.3.2. Proposition If  $p \not\equiv 0 \pmod{4}$ , then  $\bar{d}_0$  is injective.

Proof: Let  $v_p^1 := M_p^1 \otimes Q(i)$  and extend  $\bar{d}_0$  by the same rule to  $\delta_0: v_p^1 \oplus v_p^2 \oplus v_p^3 \rightarrow v_p \oplus v_p \oplus v_p$ . Clearly  $\text{Ker } \bar{d}_0 = \text{Ker } \delta_0 \cap \bar{M}_p^1 \oplus \bar{M}_p^2 \oplus \bar{M}_p^3$ . Now

$$\delta_0(v_1, v_2, v_3) = 0 \text{ iff } v_1 = v_2 = v_3 \text{ or } \text{Ker } \delta_0 = v_p^1 \cap v_p^2 \cap v_p^3$$

We claim that  $v_p^1 \cap v_p^3 = 0$ . Recall that  $\{e_j \mid j \equiv 0 \pmod{4}\}$  is a basis for  $v_p^1$ . Since

$$\begin{pmatrix} 1 & \frac{1+i}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1+i}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix}, \quad v_p^3 = \begin{pmatrix} 1 & \frac{1+i}{2} \\ 0 & 1 \end{pmatrix} v_p^1.$$

Suppose that  $v = \sum a_k e_k$  is in  $v_p^1$ , and take  $a_j \neq 0$ ,  $j$  minimal.

Then  $\begin{pmatrix} 1 & \frac{1+i}{2} \\ 0 & 1 \end{pmatrix} e_j = e_j + \frac{1+i}{2}(p-j)e_{j+1} + \dots$

Since  $p \not\equiv 0 \pmod{4}$ ,  $p \neq j$ . However  $j+1 \not\equiv 0 \pmod{4}$ , and this implies that  $\begin{pmatrix} 1 & \frac{1+i}{2} \\ 0 & 1 \end{pmatrix} v \notin v_p^1$ . Hence  $\text{Ker } \delta_0 = 0$  and  $\bar{d}_0$  is injective.

For  $p \not\equiv 0 \pmod{4}$ ,  $H^0(\Gamma_\infty, M) = 0$ . We will now compute  $H^1(\Gamma_\infty, M)$  for  $p = 3$  with the same method used for  $H^1(\Gamma, M)$ . Let  $\bar{M} := M \oplus M \oplus M$ . We have the exact sequence:

$$0 \longrightarrow \text{Ker } \bar{d}_1 / \text{Im } \bar{d}_0 \longrightarrow \bar{M} / \text{Im } \bar{d}_0 \longrightarrow \bar{M} / \text{Ker } \bar{d}_1 \longrightarrow 0$$

We just have to determine a basis for  $\text{Im } \bar{d}_0$  and the dimension of  $\text{Im } \bar{d}_1$ . We already know a basis for  $\bar{M}^1$ , namely  $A = e_0$ .

a)  $\bar{M}^2: \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} e_j = (-1)^j (e_j + \begin{pmatrix} p-j \\ 1 \end{pmatrix} e_{j+1} + \dots + e_p)$

Solving the linear equation given by this relation, we obtain a basis for  $\bar{M}^2$ :

$$B_1 = \frac{1}{2}e_3 - \frac{5}{2}e_1 + \frac{5}{2}e_{-1} + e_{-2}$$

$$B_2 = -\frac{1}{4}e_3 + \frac{3}{2}e_1 + e_0$$

$$B_3 = \frac{1}{2}e_3 + e_2$$

b)  $\bar{M}^3: \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} e_j = i^j (e_p + \begin{pmatrix} p-j \\ 1 \end{pmatrix} e_{p-1} + \dots + e_j)$  A basis vector is

$$C = \frac{1-i}{4}e_3 + \frac{3i}{2}e_2 + \frac{3(1+i)}{2}e_1 + e_0$$

The basis for  $\text{Im } \bar{d}_0$  is  $\{(-A, 0, A), (B_j, -B_j, 0) \mid 1 \leq j \leq 3\}$ .

Again by computing elementary divisors, we obtain the result that  $\bar{M} / \text{Im } \bar{d}_0$  is free.  $\dim \text{Im } \bar{d}_1 = 14$  implies that  $H^1(\Gamma_\infty, M)$  is free of rank 2 over  $\mathcal{O}_3$ .

Our final computation deals with the restriction maps in this case. In terms of Cech complexes, the "restrictions" turn into "inclusions":

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_p^1 \oplus M_p^2 \oplus M_p^3 & \xrightarrow{\bar{d}_0} & M_p^{12} \oplus M_p^{23} \oplus M_p^{13} & \xrightarrow{\bar{d}_1} & M_p \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow f \\ 0 & \longrightarrow & \bar{M}_p^1 \oplus \bar{M}_p^2 \oplus \bar{M}_p^3 & \xrightarrow{\bar{d}_0} & M_p \oplus M_p \oplus M_p & \xrightarrow{\bar{d}_1} & M_p \oplus M_p \longrightarrow 0 \end{array}$$

where  $f$  denotes the mapping  $f(m) = (m, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} m)$ , which is also injective. Both squares commute, as an easy computation shows.

Let  $p = 3$ . The image of the restriction map

$r_1: H^1(\Gamma, M) \longrightarrow H^1(\Gamma_\infty, M)$  is free of rank 1: the last state-

ment follows from the existence of an element in  $\text{Ker } d_1$  which doesn't lie in the image of  $\bar{d}_0$ . We would like to know whether  $\text{Coker } r_1$  has torsion.

Consider the following diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & H^1(r, M) & \longrightarrow & M / \text{Im } d_0 & \longrightarrow & M / \text{Ker } d_1 \longrightarrow 0 \\
 & & \downarrow r_1 & & \downarrow s_1 & & \downarrow t_1 \\
 0 & \longrightarrow & H^1(r_0, M) & \longrightarrow & \tilde{M} / \text{Im } \bar{d}_0 & \longrightarrow & \tilde{M} / \text{Ker } \bar{d}_1 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Coker } r_1 & \longrightarrow & \text{Coker } s_1 & \longrightarrow & \text{Coker } t_1 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

All squares commute, the first two rows and the last vertical column are exact. It follows that the last row is exact too. Now

$$s_1 \left( \tilde{M} / \text{Im } d_0 \right) = \tilde{M} / \tilde{M} \cap \text{Im } \bar{d}_0 = \tilde{M} + \text{Im } \bar{d}_0 / \text{Im } \bar{d}_0$$

so that  $\text{Coker } s_1 = \tilde{M} / \tilde{M} + \text{Im } \bar{d}_0$ .

We already have bases for  $\tilde{M}$  and  $\text{Im } \bar{d}_0$ , and the computation of elementary divisors show that  $\text{Coker } s_1$  is torsionfree, and hence  $\text{Coker } r_1$  too (as  $\mathcal{O}_3$ -modules).

# REFERENCES

1. BIANCHI, L. Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginari  
Math. Annalen 40 (1892) S. 332 - 412
2. BROWN, K. Groups of virtually finite dimension to appear in: Proceedings of the 1977 Durham Conference on Homological and Combinatorial Techniques in Group Theory
3. FLÜGGE, Eine Beschreibung der  $SL_2$  über den euklidischen imaginär-quadratischen Zahlringen als amalgamiertes Produkt bzw. als HNN-Erweiterung eines amalgamierten Produkts  
Diplomarbeit, Frankfurt (1979)
4. GODEMENT, R. Topologie algébrique et théorie des faisceaux  
Hermann (1958)
5. GROTHENDIECK, A. Sur quelques points d'algèbre homologique  
Tohoku Math. Journal (1957) p. 119 -221
6. GRUNEWALD, F. On the integral cohomology of  $PSL_2$  over imaginary quadratic integers  
MENDOZA, E. R. (in preparation)
7. HARDER, G. Minkowski'sche Reduktionstheorie über Funktionenkörpern  
Inventiones mathematicae 7 (1969) p.33-54
8. HARDER, G. A Gauß-Bonnet formula for discrete arithmetically defined groups  
Ann. Scient. Ec. Norm. Sup. t. 4 (1971) p. 409 -455



9. HARDER, G.      Period Integrals of Cohomology Classes  
which are represented by Eisenstein series  
to appear in: Proceedings of the 1979  
Tata Institute Conference on Automorphic  
Forms
  
10. HUMBERT, G.    Sur la réduction des forms d'Hermite dans  
un corps quadratiques imaginaire  
Comptes Rendus t. 161 (1915)    190 -196
  
11. MAHLER, K.      On the minimum of positive definite her-  
mitian forms  
Proceedings of the London Math Society (1939)
  
12. O'MEARA, O.T.   Introduction to Quadratic Forms  
Grundlehren der Math Wiss. Bd. 117  
Springer (1963)
  
13. OPPENHEIM, A.   The minimum of positive definite hermitian  
binary quadratic forms  
Math. Zeitschrift 38 (1934)    S. 538 - 545
  
14. SIEGEL, C. L.   Lectures on Advanced Analytic Number Theory  
Tata Institute Bombay (1961)
  
15. SERRE, J. P.    Cohomologie des groupes discrets  
Prospects in Mathematics, Annals of Math.  
Studies 70 (1971)    p. 77 - 169
  
16. SERRE, J. P.    Arbres, amalgames et  $SL_2$   
Astérisque 46 (1977)
  
17. SWAN, R.        Generators and relations for certain  
special linear groups  
Advances in Math. 6 (1971)    p. 1-78

18. WOODRUFF, W.M.   Singular points of the fundamental  
domains for the groups of Bianchi  
Ph.D. Thesis (1967) University of  
Arizona