Intégration convexe effective

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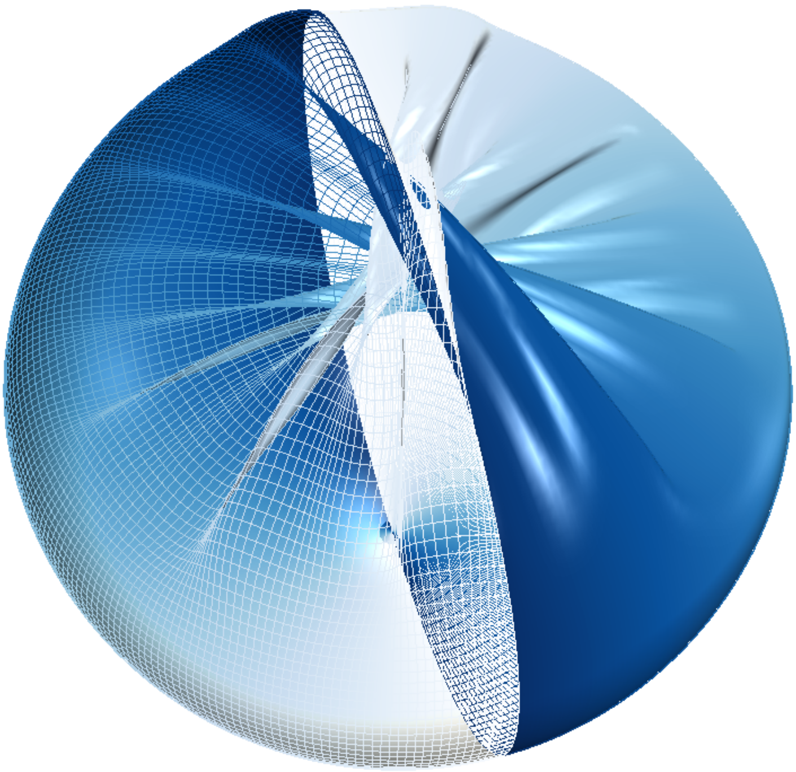

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Introduction en français

Le but de cette thèse est de proposer une version effective de la théorie de l'intégration convexe. Cette théorie, inventée par M. Gromov dans les années 70, permet de résoudre des relations différentielles, i.e. des équations / inéquations aux dérivées partielles. Elle est inspirée des travaux de J. Nash [Nas54] sur les plongements isométriques et des travaux de S. Smale [Sma58] sur les homotopies régulières. Elle est utilisée en géométrie différentielle [Spr98, EM02, Gei03] et a suscité depuis peu un regain d'intérêt en mécanique des fluides [DLS09, CDLDR17, BV19], en géométrie de contact [Mur18] et pour l'équation de Monge-Ampère [CL18]. Récemment, elle a également permis la construction explicite et la visualisation de plongements C^1-isométriques d'un tore plat et d'une sphère réduite effectuées par l'équipe Hévéa [BJLT13, BBD+18]. Ces travaux, qui se sont étalés sur dix ans, ont motivé la recherche d'une version plus effective de la méthode d'intégration convexe.

La théorie de l'intégration convexe repose sur une formule fondamentale construisant à partir d'une application une nouvelle application au moyen d'une intégrale. Cette formule est utilisée itérativement pour construire des plongements C^1-isométriques. Néanmoins son usage entraîne deux conséquences indésirables : d'une part, la présence d'une intégrale empêche d'avoir une expression locale de la fonction construite et, d'autre part, elle peut introduire des discontinuités qu'il faut ensuite corriger (voir la figure 1). Dans cette thèse, nous substituons à la formule fondamentale de l'intégration convexe une nouvelle formule appelée *procédé de corrugation* (définition 27). Cette nouvelle formule, tout en vérifiant les propriétés de la formule fondamentale (proposition 28), évite les effets indésirables : elle est locale et ne crée pas de discontinuités. Elle a également d'autres avantages, elle permet d'éviter le recours à des interpolations pour le recollement d'applications définies sur plusieurs cartes (voir le paragraphe *Relative property* de la sous-section 2.1.2) et s'exprime indépendamment d'un système de coordonnées locales (définition 29).

Etant données une relation différentielle d'ordre 1 et une sous-solution de cette relation (définition 38), la formule d'intégration convexe utilisée itérativement permet de construire des solutions de la relation différentielle considérée (voir la sous-section 1.2.3). Dans cette thèse, nous définissons un nouveau

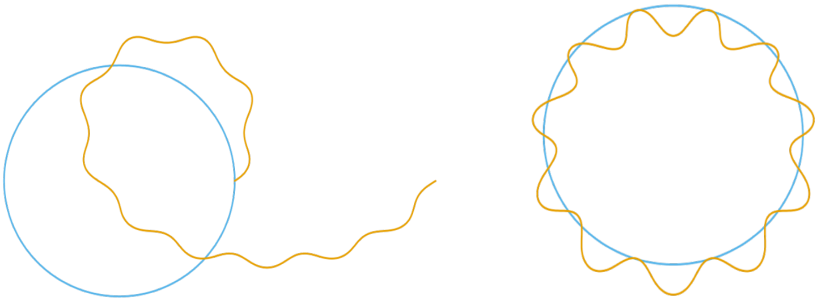


Figure 1 – **Illustration de la discontinuité induite par la formule d'intégration convexe :** en bleu le cercle $f_0 : t \mapsto \exp(i2\pi t)$, en orange, à gauche l'application construite par intégration convexe et à droite celle construite par le procédé de corrugation pour $N = 5$ et $\gamma(x,t) = (\cos(\alpha(x)\cos(2\pi t))f_0'(x) + \sin(\alpha(x)\cos(2\pi t))(-if_0'(x)))/J_0(\alpha(x)$, où $\alpha(x) = \cos(12\pi x)$.

type de relations différentielles que nous nommons *relations de Kuiper* (définition 45). Ce type de relations satisfait à une propriété de convexité particulière énoncée dans la section 2.2 et dont une reformulation informelle est la suivante : chaque point de l'enveloppe convexe de la relation est barycentre d'un lacet dans la relation dont la forme est définie par un patron donné (par exemple un arc de cercle ou un cercle). Nous montrons que ces relations se rencontrent en géométrie différentielle, par exemple chez les immersions, les immersions isométriques ou les applications totalement réelles. Pour ces relations, la formule du procédé de corrugation prend alors une expression analytique particulièrement simple : les intégrales disparaissent. Outre l'intérêt numérique évident, nous tirons avantage de ce fait pour construire une nouvelle immersion de $\mathbb{R}P^2$ par une application directe du procédé de corrugation (voir Figure 2).

Un autre avantage de la disparition des intégrales est qu'elle révèle un lien direct entre l'intégration convexe et la théorie des corrugations de Thurston. Cette théorie a été développée dans les années 1970 pour mieux comprendre les résultats de Smale et permet des constructions explicites à partir d'une formule de base. En particulier, elle a conduit à un nouveau retournement de la sphère décrit dans [Lev95] et visualisé en 1994 par le Geometric Center. En choisissant convenablement les paramètres dans le procédé de corrugation, on retrouve la formule générant les corrugations de Thurston (voir la sous-section 2.2.4). Le procédé de corrugation permet également de retrouver une autre formule : celle utilisée par S. Conti, C. De Lellis et L. Székelyhidi dans l'étude de la régularité $C^{1,\alpha}$ des immersions isométriques [CDLS12]. Avec un choix adéquat des paramètres, le procédé de corrugations coïncide avec l'ansatz de [CDLS12, Section 4.2] (voir la sous-section 2.2.5).

La construction explicite d'un tore plat et d'une sphère réduite a conduit à la mise en évidence d'une structure géométrique dite C^1-fractale : les différen-

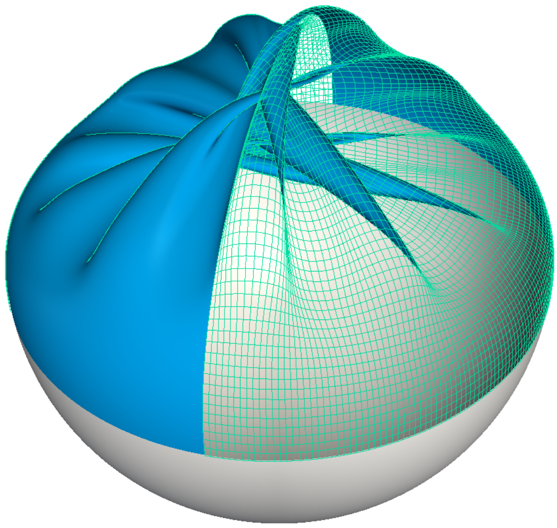


Figure 2 – **Une nouvelle immersion de $\mathbb{R}P^2$:** obtenue par le procédé de corrugation, voir la section 3.3.

tielles des plongements construits présentent des propriétés d'auto-similarité. L'examen de ces constructions montre que ces propriétés d'auto-similarité ne découlent pas directement de l'intégration convexe mais de choix spécifiques effectués et maintenus le long de toute la construction. Ces choix, au nombre de trois, sont les suivants : toutes les corrugations utilisées partagent la même forme, les directions de ces corrugations se répètent périodiquement, et les cartes sont choisies de façon à limiter les recollements entre les constructions locales.

La possibilité d'effectuer de tels choix dans le cas général, c'est-à-dire pour une variété riemannienne quelconque et un plongement court arbitraire, est loin d'être évidente. Il s'avère que là encore la notion de relation de Kuiper est centrale. Les propriétés "relatives" du procédé de corrugation (voir le paragraphe *Relative property* de la sous-section 2.1.2) permettent de traiter naturellement les problèmes de recollement, quant à la notion de relation de Kuiper, elle assure la possibilité de choisir des corrugations ayant toutes la même forme et ouvre ainsi la perspective de la construction de solutions C^1-fractales pour d'autres relations différentielles. Nous explorons plus spécifiquement le cas de la relation différentielle des applications totalement réelles. Cette relation est de Kuiper et nous montrons un théorème d'existence de plongements C^1-isométrique totalement réelles dans l'esprit de celui de Nash-Kuiper (théorème 61). Nous montrons que les plongements totalement réels construits par le procédé de corrugation satisfont à une propriété d'auto-similarité. Précisément, la composante de Maslov de son application de Gauss est similaire à une fonction de Weierstrass (proposition 65). Observons que la construction d'applications isométriques ayant de fortes propriétés d'auto-similarité pourrait offrir une voie vers une théorie rigide des C^1-isométries ([Gro17, p26]).

Résultats obtenus dans cette thèse

Avant de donner les résultats de cette thèse, nous donnons un aperçu de sa structure. Le premier chapitre présente les outils issus de la théorie de l'intégration convexe nécessaires au développement de cette thèse. Le deuxième chapitre présente le procédé de corrugation ainsi que les relations de Kuiper. Le troisième chapitre porte sur la relation des immersions et la relation des immersions isométriques et leur caractère de Kuiper. Nous construisons également une nouvelle immersion de $\mathbb{R}P^2$. Le dernier chapitre porte sur la relation des applications totalement réelles, le théorème de plongements C^1-isométriques totalement réels et l'étude de l'auto-similarité de la composante de Maslov.

Chapitre 2 - Procédé de corrugation et relations de Kuiper

La formule fondamentale de l'intégration convexe s'exprime comme suit : à partir d'une application $f_0 : [0,1]^m \to \mathbb{R}^n$, d'une famille de lacets $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ et d'un entier N, elle construit une nouvelle application

$$F_1(x_1, \dots, x_m) := f_0(0, x_2, \dots, x_m) + \int_{s=0}^{x_1} \gamma(s, x_2, \dots, x_m, Ns)ds.$$

On dit que F_1 a été obtenue par intégration convexe à partir de f_0 dans la direction x_1 (ou ∂_1). La théorie de l'intégration convexe est basée sur les propriétés suivantes vérifiées par F_1 :

Proposition 1. *Si la famille de lacets γ est choisie telle que pour tout $x \in [0,1]^m$ on a $\int_0^1 \gamma(x,t)dt = \partial_j f_0(x)$, alors l'application $F_1 = CI\gamma(f_0, \partial_j, N)$ vérifie*

(P_1) $\quad \|f_0 - F_1\|_{C^0} = O(1/N)$, *($C^0$-densité),*

(P_2) $\quad \|\partial_i f_0 - \partial_i F_1\|_{C^0} = O(1/N)$ *pour tout* $i \neq j$,

(P_3) $\quad \forall x \in [0,1]^m, \quad \partial_j F_1(x) = \gamma(x, Nx_j)$.

Procédé de corrugation Dans cette thèse, nous substituons à la formule ci-dessus la formule suivante :

$$f_1(x) := f_0(x) + \frac{1}{N} \int_{s=0}^{Nx_1} \gamma(x,s) - \overline{\gamma}(x)ds$$

où $\overline{\gamma}(x)$ désigne la moyenne du lacet $\gamma(x, \cdot)$. On dit que x est la *variable en espace* de γ et s la *variable en temps*. Nous appelons cette formule *procédé* *de corrugation* et nous notons $f_1 = CP_\gamma(f_0, \partial_1, N)$. Pour assurer de pouvoir développer la théorie de l'intégration convexe avec le procédé de corrugation, nous montrons que la fonction f_1 vérifie les propriétés suivantes :

Proposition 2. *L'application $f_1 = CP_\gamma(f_0, \partial_j, N)$ vérifie*

(P_1) $\|f_0 - f_1\|_\infty = O(1/N)$ *(C^0-densité),*

(P_2) $\|\partial_i f_0 - \partial_i f_1\|_\infty = O(1/N)$ *pour tout $i \neq j$,*

(P_3') *si la famille de lacets γ vérifie, pour tout $x \in [0,1]^m$, $\overline{\gamma}(x) = \partial_j f_0(x)$, alors $\partial_j f_1(x) = \gamma(x, Nx_j) + O(1/N)$ pour tout $x \in [0,1]^m$.*

Un avantage significatif du procédé de corrugation est que les variables en temps et en espace sont séparées : l'intégrale ne porte que sur le paramètre temporel. Ceci a une conséquence très importante : la valeur de f_1 en x dépend seulement des valeurs de f_0 et de γ en x, autrement dit la formule est locale. Un corollaire frappant de cette localité est que l'application construite par le procédé de corrugation préserve la 1-périodicité (voir le début de la sous-section 2.1.2), une propriété permettant de s'économiser certains recollements. Notons que, par une approche différente, Y. Eliashberg et N. Mishachev ont obtenu une formule partageant la même propriété de périodicité [EM02]. Leur formule dépend d'un choix spécifique de lacets (appelés *flowers*) et se prête moins facilement aux constructions décrites plus loin.

Expression indépendante d'un système de coordonnées locales Le procédé de corrugation a un autre avantage notable : il est possible d'en avoir une expression analytique sans avoir recours à un système de coordonnées locales[1]. Etant données M et W deux variétés, h une métrique de W, $f_0 : U \subset M \to (W,h)$ une application, $\pi : U \to \mathbb{R}$ une submersion, $u : M \to TM$ un champ de vecteur tangent tel que pour tout $x \in M$, $d\pi_x(u_x) = 1$ et $\gamma : U \times \mathbb{R}/\mathbb{Z} \to f_0^*TW$ une famille de lacets telle que $\gamma(x,.) : \mathbb{R}/\mathbb{Z} \to f_0^*TW_x$ pour tout $x \in U$ et que sa moyenne vérifie $\overline{\gamma}(x) = df_0(u_x)$, nous définissons le procédé de corrugation dans la "direction" π par la formule

$$f_1(x) := \exp_{f_0(x)}\left(\frac{1}{N}\int_{s=0}^{N\pi(x)} \gamma(x,s) - \overline{\gamma}(x)ds\right)$$

(définition 27). A notre connaissance, une telle expression indépendante d'un système de coordonnées n'existait pas dans les précédentes versions de la théorie de l'intégration convexe.

[1]Cet avantage a été mis en évidence lors d'un échange avec Patrick Massot.

Relations de Kuiper La théorie de l'intégration convexe propose une approche géométrique des inégalités aux dérivées partielles. Elle formalise une relation différentielle comme un sous-ensemble de l'espace des jets. Dans cette thèse nous nous intéressons aux relations différentielles d'ordre 1, i.e. un sous-ensemble $\mathscr{R}$ de l'espace des 1-jets $J^1(M,W)$, i.e. de l'espace des triplets (x,y,L) où $x \in M$, $y \in W$ et $L \in \mathscr{L}(T_xM, T_yW)$ est une application linéaire entre les espaces tangents. Une *solution formelle* $\mathfrak{S} : x \mapsto (x, f_0(x), L_x)$ est une section du fibré des 1-jets dont l'image est dans la relation différentielle $\mathscr{R}$. A partir de certaines solutions formelles appelées *sous-solutions* (définition 38), l'intégration convexe construit une solution de $\mathscr{R}$, i.e. une application f dont le 1-jet $x \mapsto (x, f(x), df_x)$ est à valeur dans $\mathscr{R}$. Dans cette construction, il est nécessaire de choisir la famille $x \mapsto \gamma(x, \cdot)$ telle que :

1) son image soit contenue dans un sous-ensemble de $\mathscr{R}$ dépendant de π et de u,

2) sa moyenne $\overline{\gamma}(x) = \int_0^1 \gamma(x,t)dt$ soit égale à $df_0(u_x)$,

3) son point de base soit $(x, f_0(x), L_x(u_x))$.

Traditionnellement l'existence d'une telle famille est déduite d'une construction barycentrique. Cependant le manque d'uniformité du procédé est un frein à l'effectivité des intégrations convexes ainsi obtenues. Nous proposons d'uniformiser la construction en définissant a priori une famille de lacets $\tilde{\gamma}(\sigma, w)$ indexée, non pas par $x \in M$, mais par un couple (σ, w) où $\sigma \in \mathscr{R}$ est le point de base du lacets et w est sa moyenne. On dira alors que cette famille de lacets est *entourante* (définition 40). Pour assurer que tous les lacets aient la même forme nous demandons en outre qu'ils soient tous modelés sur un même *patron* (définition 43). Un patron est une application $c : A \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^p$ où A est un espace de paramètres. Par exemple, on peut choisir une famille d'arcs de cercle indexée par l'angle, l'espace des paramètres sera alors l'ensemble des angles considérés. On dit qu'une famille de lacets $\tilde{\gamma}$ est *modelée sur* c s'il existe des vecteurs $e_1, \ldots, e_p$ de T_yW, où $\sigma = (x,y,L)$, permettant d'écrire $\tilde{\gamma}(\sigma, w)$ sous la forme :

$$\tilde{\gamma}(\sigma, w)(\cdot) = \sum_{i=1}^{p} c_i(a(\sigma, w), \cdot) e_i(\sigma, w)$$

où les c_i sont les fonctions coordonnées de c (voir la définition 44 pour une définition plus précise). Etant donnée une relation différentielle quelconque et un patron c, l'existence d'une telle famille $\tilde{\gamma}$ modelée sur c n'est pas garantie.

Définition 3. On dit que $\mathscr{R}$ est une *relation de Kuiper* suivant $(c, d\pi, u)$ s'il existe une famille de lacets entourante $\tilde{\gamma}$ modelée sur c.

Elimination de l'intégrale dans la formule du procédé de corrugation Pour les relations de Kuiper, la formule du procédé de corrugation ne fait plus apparaître que les primitives des fonctions coordonnées de c. Soit $C_i(a,\cdot)$ la fonction 1-périodique définie par $t \mapsto \int_0^t c_i(a,t) - \overline{c_i}(a)dt$. Dans le cas d'une relation de Kuiper $\mathscr{R}$ suivant $(c, d\pi, u)$, le procédé de corrugation s'écrit en fonction des C_i et des e_i. Précisément, on a la proposition suivante :

Proposition 4. *Soit $\mathfrak{S} = (x, f_0, L_0)$ une sous-solution. L'application $f_1 = CP_{\widehat{\gamma}}(\mathfrak{S}, \pi, N)$ peut s'écrire*

$$f_1(x) \quad = \quad \exp_{f_0(x)}\left(\frac{1}{N}\sum_{i=1}^{p} C_i(a(x), N\pi(x))e_i(x)\right)$$

où $a(x) := \boldsymbol{a}(\mathfrak{S}(x), df_0(u_x))$, $e(x) := \boldsymbol{e}(\mathfrak{S}(x), df_0(u_x))$ et $x \in U$.

Cette formule fournit une interprétation claire du procédé de corrugation : l'application f_1 est obtenue à partir de f_0 en ajoutant des oscillations dans les directions $e_1, \ldots, e_p$, directions qui ne dépendent que de la sous-solution $\mathfrak{S}$.

Cette formule permet aussi d'établir un lien direct avec d'autres formules présentes dans la littérature et qui génère des corrugations. En particulier, c'est le cas de la formule utilisée par Thurston dans sa théorie des corrugations [Lev95]. Soit la fonction $C(t) = (-h\sin(4\pi t), 2h\sin(2\pi t))$, un réel h et $e_1(x) = e(x)$, $e_2(x) = ie(x)$, où $e(x)$ est un vecteur complexe unitaire, le procédé de corrugation s'écrit

$$f_1(x) = f_0(x) + \frac{1}{N}\big(-h\sin(4\pi Nx)e(x) + 2h\sin(2\pi Nx)ie(x)\big).$$

Cette formule est, au coefficient $1/N$ prés, la formule de corrugations de Thurston (voir la sous-section 2.2.4). De même, si on considère la fonction

$$C(x,t) = \Big(\int_{s=0}^{t} \cos(\alpha(x)\sin(2\pi t)) - J_0(\alpha(x))ds, \int_{s=0}^{t} \sin(\alpha(x)\sin(2\pi t))ds, 0\Big),$$

alors l'expression sans intégrale du procédé de corrugation permet de retrouver l'ansatz de S. Conti, C. De Lellis et L. Székelyhidi (voir la sous-section 2.2.5).

Chapitre 3 - Immersions et immersions isométriques

Dans le chapitre 2, nous avons introduit la notion de relation de Kuiper. Il est maintenant naturel d'en chercher des exemples. Pour cela nous considérons les relations différentielles qui ont inspiré la théorie de l'intégration convexe : la relation des immersions et la relation des isométries. Nous montrons que la relation différentielle des immersions de codimension 1 est de Kuiper. Précisément nous montrons :

Théorème 5. *Soient M une variété orientable de dimension m et W une variété orientable riemannienne de dimension n telle que $n > m$. Si (m,n) vérifie l'une des conditions suivantes :*

$$\begin{array}{llll} (i) & n = m+1 & (ii) & m=1 \text{ et } n \text{ pair} \\ (iii) & m=2 \text{ et } n=7 & (iv) & m=3 \text{ et } n=8 \end{array}$$

alors la relation des immersions $\mathscr{I}(M,W)$ est une relation de Kuiper pour le patron c donné par

$$c(x,t) = \Big(\cos(\alpha(x)\cos(2\pi t)) - J_0(\alpha(x)), \sin(\alpha(x)\cos(2\pi t)), 1\Big).$$

Les conditions de codimension du théorème garantissent de l'existence de sections du fibré naturel $V_{m+1}(\mathbb{R}^n) \to V_m(\mathbb{R}^n)$, où $V_j(\mathbb{R}^k)$ est l'ensemble des familles de j vecteurs libres dans un espace de dimension k (variété de Stiefel). Pour les codimensions où ces sections n'existent pas, nous montrons que la relation des immersions satisfait à une forme plus faible de la condition de Kuiper que nous appelons *quasi-Kuiper* (voir la définition 49). Pour de telles relations, une formule du procédé de corrugation sans intégrale est toujours valable (proposition 50).

Théorème 6. *Soient M une variété orientable et W une variété orientable riemannienne telles que* $\dim(W) > \dim(M)$. *La relation $\mathscr{I}(M,W)$ est quasi-Kuiper.*

Ce théorème établit que, à condition d'avoir une sous-solution, on peut désingulariser des applications avec des expressions simples issues du procédé de corrugation. Cela nous permet de construire directement une nouvelle immersion de $\mathbb{R}P^2$ donnée par une inversion géométrique de l'application

$$f_1(x) = f_0(x) + \frac{1}{N}\Big(K_c(\alpha(x), Nx_2)rv_2(x) + K_s(\alpha(x), Nx_2)r(v_2(x) \wedge v_1(x))\Big)$$

avec

$$\begin{aligned} C(x,t) = \Big(\quad & K_c(\alpha(x),t) := \int_{s=0}^{t} \cos(a(x)\cos(2\pi s)) - J_0(\alpha(x))ds, \\ & K_s(\alpha(x),t) := \int_{s=0}^{t} \sin(a(x)\cos(2\pi s))ds, 0\Big) \end{aligned}$$

où f_0 est une paramétrisation du conoïde de Plücker et r et $\mathfrak{S} = (x, f_0, v_1, v_2)$ sont définis Section 3.3 (voir Figure 3 pour une représentation de f_1).

Nous considérons ensuite la relation des ϵ-isométries $\mathscr{I}\!s(\epsilon)$, où $\epsilon > 0$. En effet, ces relations sont un élément clef pour la résolution par intégration convexe de la relation des isométries (voir la section 1.2.4). Soient M et W deux variétés orientables riemanniennes. Une application $f : (M,g) \to (W,h)$ est ϵ-isométrique si pour tout $x \in M$ on a $\|(f^*h)_x - g_x\| < \epsilon$, où f^*h est le tiré en arrière de la métrique h par f. Nous montrons :

Figure 3 – **Le centre du conïde de Plücker corrugué (avant l'inversion géométrique)** qui permet d'obtenir une nouvelle immersion de $\mathbb{R}P^2$ par inversion géométrique (voir la section 3.3).

Théorème 7. *Soit* $\dim(W) > \dim(M)$. *La relation* $\mathscr{I}\!\mathscr{B}(\epsilon)$ *est quasi-Kuiper.*

Nous montrons également que la relation $\mathscr{I}\!\mathscr{B}(\epsilon)$ est de Kuiper pour les mêmes conditions de codimension que la relation des immersions $\mathscr{I}(M, W)$.

Chapitre 4 - Applications isométriques totalement réelles et auto-similarité

Les notions introduites jusqu'à présent dans cette thèse fournissent un cadre naturel pour étudier d'éventuelles propriétés d'auto-similarité. Ce type de propriétés a déjà été observé pour les plongements C^1-isométriques du tore plat et de la sphère réduite [BJLT13, BBD+18]. Dans ce chapitre, nous nous intéressons aux propriétés d'auto-similarité des plongements obtenus par application itérée du procédé de corrugation.

Applications isométriques totalement réelles Soient M une variété et (W, J) une variété presque complexe (i.e. J est un isomorphisme de fibré vectoriel tel que $J_x \circ J_x = -Id_{T_xW}$ pour tout $x \in W$) telles que $\dim(W) = 2\dim(M)$. Rappelons qu'une application $f : M \to (W, J)$ est dite totalement réelle si pour tout $x \in M$ on a $df(T_xM) \oplus Jdf(T_xM) = T_{f(x)}W$. Notons $\mathscr{I}_{TR}(M, W)$ la relation des applications totalement réelles. Nous montrons le théorème suivant :

Théorème 8. *La relation $\mathscr{I}_{TR}(M, W)$ est une relation de Kuiper.*

Nous établissons ensuite un théorème de type Nash-Kuiper pour les isométries totalement réelles :

Théorème 9. *Soient (M^m, g) une variété riemannienne compacte et $f_0 : (M^m, g) \to (W^{2m}, J, h)$ une immersion (resp. plongement) totalement réelle telle que $h - f_0^*$ est définie positif (i.e. strictement courte). Alors, pour tout $\epsilon > 0$, il existe une immersion (resp. plongement) isométrique totalement réelle de classe C^1 $f_\infty : (M^m, g) \to (W^{2m}, J, h)$ telle que $dist(f_\infty(x), f_0(x)) \leq \epsilon$ pour tout $x \in M^m$.*

Dans la construction de ces immersions isométriques, le procédé de corrugation est un élément clef qui permet de contrôler les dérivées partielles et plus précisément la J-densité (définie dans la sous-section 4.2.1) afin d'assurer qu'à la limite l'isométrie construite est bien totalement réelle. Notons que d'autres théorèmes de C^1-isométries dans l'esprit de celui de Nash ont été obtenus dans divers contextes : pour des métriques de Carnot-Carathéodory [D'A95], en géométrie de contact, en symplectique, en géométrie pseudo-riemannienne [D'A00, DL02, DD06] et pour des variétés sous-riemanniennes [LD13].

Auto-similarité Nous étudions ensuite les propriétés d'auto-similarité des immersions isométriques totalement réelles construites. Précisément, nous montrons que la composante de Maslov (voir la sous-section 4.2.2 pour une définition) de l'application de Gauss des immersions isométriques totalement réelles f_∞ construites en itérant le procédé de corrugation vérifient la proposition suivante :

Proposition 10. *Soient $\mathfrak{m}(f_0, f_\infty) = e^{i\mathscr{W}_\infty} : M \to \mathbb{S}^1$ l'application de Maslov de f_∞ et $\mathscr{W}_\infty = 2\sum_k \vartheta_k$ l'argument de Maslov. Alors si k est suffisamment grand*

$$\vartheta_k = \theta_k + \sum_{j \in I(k)} O\left(\frac{1}{N_{k,j}}\right) \quad \text{où } \theta_k := \sum_{j \in I(k)} \alpha_{k,j} \cos(2\pi N_{k,j} \pi_{k,j})$$

(si $x \in M$ n'est pas dans le domaine de $\pi_{k,j}$ il est entendu que le terme correspondant vaut zéro).

En particulier, la composante de Maslov de leur application de Gauss est similaire à une fonction de Weierstrass $x \mapsto \sum_k a^k \cos(b^k \pi x)$, $x \in \mathbb{R}$ et avec $0 < a < 1$, b un entier impair positif et $ab > 1 + 3\pi/2$. Rappelons que la dimension de Hausdorff du graphe de cette fonction est strictement plus grande que 1 et présente un caractère auto-similaire.

Perspectives

Les notions principales développées dans cette thèse sont le procédé de corrugation et les relations de Kuiper. Ces deux notions permettent de proposer une version plus effective de l'intégration convexe, ce qui ouvre la voie à la réalisation de nouveaux plongements isométriques. Une première perspective est de s'intéresser au cas non-compact et de construire un plongement isométrique explicite du plan hyperbolique. Cela soulève de nouveaux problèmes sur le choix des cartes et des constructions locales. Néanmoins les propriétés locales et relatives du procédé de corrugation rendent *a priori* l'exploration de ce problème plus simple.

Les notions développées dans cette thèse ont également permis d'établir une propriété d'auto-similarité pour des immersions isométriques totalement réelles. Cette propriété d'auto-similarité reste néanmoins plus faible que celles observées pour le tore plat et la sphère réduite. Une autre piste de recherche concerne l'étude de l'émergence de l'auto-similarité pour des relations différentielles fermées.

Les familles de lacets considérées dans cette thèse sont de classe C^0, C^1 ou C^∞. Il y a néanmoins un intérêt à considérer des familles dont la régularité est plus basse, i.e. des familles de lacets discontinues. Par exemple, le choix de lacets constants par morceaux dans le procédé de corrugation conduit à des solutions linéaires par morceaux et offre une voie pour résoudre, à l'aide de l'intégration convexe, des problèmes issus de la géométrie PL (piecewise linear). En particulier, nous pourrions nous intéresser à une intégration convexe PL avec comme application potentielle des plongements isométriques PL. Par le théorème de Burago-Zalgaller [BZ95], nous savons construire des plongements PL, mais ceux-ci ont un très grand nombre de sommets. Tanessi Quintanar Cortés [QC] a récemment construit un plongement PL du tore carré plat ayant seulement 40 sommets. Il est probable que l'intégration convexe offre un cadre universel pour les problèmes de construction explicite PL.

Introduction

The aim of this thesis is to propose an effective version of the Convex Integration Theory. This theory, developed by M. Gromov in the 70's, allows to solve differential relations, i.e. partial differential equalities / inequalities. This theory was originally inspired by the work of J. Nash [Nas54] on isometric embeddings and the work of S. Smale [Sma58] on regular homotopies. Over these last years, the Convex Integration Theory has been used in Differential Geometry [Spr98, EM02, Gei03] and more recently in Fluid Mechanics [DLS09, CDLDR17, BV19], in Contact Geometry [Mur18] and for the Monge-Ampère equation [CL18]. This theory has also allowed the explicit construction and the visualization of C^1-isometric embeddings of a flat torus and a reduced sphere realized by the french team Hevea [BJLT13, BBD+18]. These works, spread over ten years, have motivated the research of a more effective version of the Convex Integration method.

The Convex Integration Theory is based on a key formula that modifies a map using an integration. An iteration of this formula is used to build C^1-isometric embeddings. However this leads to two unintended consequences. Firstly the new map cannot be expressed locally because of the integration. Secondly discontinuities may appear and have to be handled (see Figure 4). In this thesis, we substitute the key formula of Convex Integration for a new

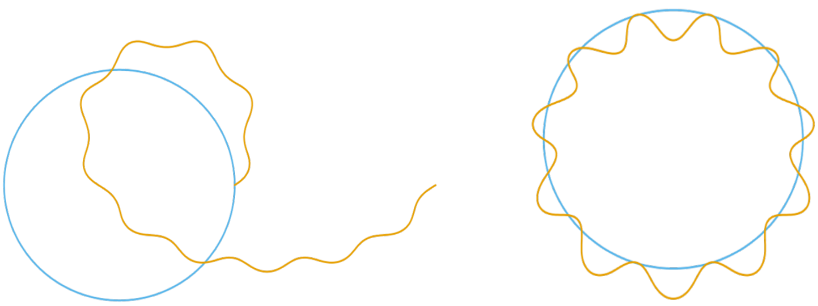


Figure 4 – **Illustration of discontinuity introduced by the convex integration formula:** in blue a graph of a circle given by $f_0 : t \mapsto \exp(i2\pi t)$, in orange, at left a map built by convex integration and at right by corrugation process for $N = 5$ and $\gamma(x,t) = (\cos(\alpha(x)\cos(2\pi t))f_0'(x) + \sin(\alpha(x)\cos(2\pi t))(-if_0'(x)))/J_0(\alpha(x)$, where $\alpha(x) = \cos(12\pi x)$.

formula called *Corrugation Process* (Definition 27). This new formula, which shares similar properties with the usual one (Proposition 28), avoids the two unintended effects: it is local and it does not create discontinuities. Other advantages also occur: it is not anymore necessary to interpolate along discontinuities (see the paragraph *Relative Property* of Subsection 2.1.2) and its expression can be given independently of a local system of coordinates (Definition 29).

Given a differential relation of order 1 and a subsolution of this relation (Definition 38), iterating the Convex Integration formula allows to build solutions of the given relation (Subsection 1.2.3). In this thesis, we define a new kind of differential relations that we call *Kuiper relations* (Definition 45). These relations satisfy a particular convex property given in Section 2.2 that can be informally recast as followings: each point of the convex hull of the relation is the barycenter of a loop belonging to the relation and whose shape is defined by a given generic pattern (for example an arc of circle or a circle). We show that these relations appear in differential geometry, for instance for immersions, isometric maps or totally real maps. For those relations, the expression of the Corrugation Process takes a really simple analytic expression: integrals vanish. Apart from the obvious numerical interest, we take advantage of this fact to build a new immersion of $\mathbb{R}P^2$ by a direct use of the Corrugation Process (see Figure 5).

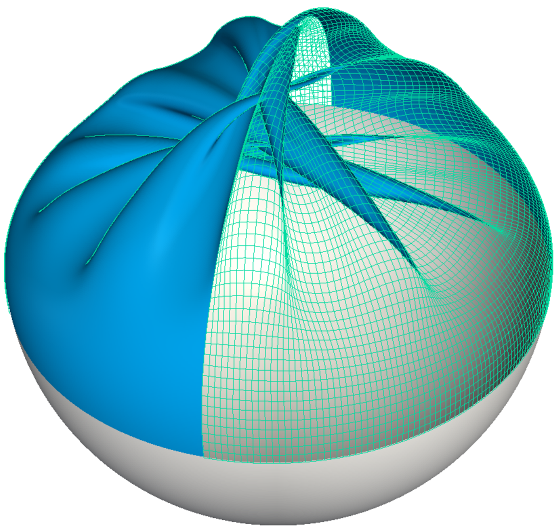


Figure 5 – **A new immersion of $\mathbb{R}P^2$:** obtained by the Corrugation Process, see Section 3.3.

The elimination of integrals also has another advantage: it reveals a direct connection between the Convex Integration and Thurston's theory of corrugations. This theory was developed in the 70's to gain a better understanding of Smale's results and relies on a simple formula generating corrugations. In particular this theory led to a new sphere eversion described in [Lev95] and visualized in 1994 by the Geometric Center. A suitable choice of parameters in the Corrugation Process allows to recover Thurston's corrugation formula (Subsection 2.2.4). The Corrugation Process also allows to recover another formula: the one used by S. Conti, C. De Lellis and L. Székelyhidi in their study of the $C^{1,\alpha}$-regularity of isometric immersions [CDLS12]. A suitable choice of parameters in the Corrugation Process leads to the ansatz of [CDLS12, Section 4.2] (see Subsection 2.2.5).

The explicit construction of both a flat torus and a reduced sphere has revealed a geometric structure called C^1-fractal: the differential of these embeddings exhibits a self-similarity property. The review of these constructions brings forward that self-similarity properties do not directly follow from the Convex Integration but from specific choices performed and maintained all along the construction. These choices, three in number, are as follows: every corrugation shares the same shape, the directions of the corrugations are repeated periodically and the charts are chosen to limit the number of gluings between the local constructions.

The possibility to make such choices in the general case, i.e. for any Riemannian manifold and any short embedding, is unclear. Here again it turns out that the notion of Kuiper relation plays a central role. The "relative property" of the Corrugation Process (see the paragraph *Relative Property* of Subsection 2.1.2) allows to handle easily the gluing problems while the Kuiper property ensures the possibility to choose corrugations sharing the same shape. This offers perspectives to build C^1-fractal solutions for other differential relations. We more specifically explore the relation of totally real maps. This relation is Kuiper and we state a theorem of existence for C^1-isometric totally real embeddings in the spirit of Nash-Kuiper (Theorem 61). We show that the totally real embeddings built by the Corrugation Process satisfy a self-similar property. Precisely, the Maslov component of its Gauss map is similar to a Weierstrass function (Proposition 65). Note that the construction of isometric maps with strong self-similar properties may open the way to a rigid theory of C^1-isometries ([Gro17, p26]).

Results obtained in this thesis

Before presenting the results of this thesis, we give an overview of its structure. The first chapter introduces the results stemming from the Convex Integration Theory which are required to the development of this thesis. The second chapter presents the Corrugation Process as well as the Kuiper relations. The third chapter focuses on the relation of immersions and the relation of isometric immersions. We also build a new immersion of $\mathbb{R}P^2$. The last chapter deals with the relation of totally real maps, the theorem of C^1-isometric totally real maps and the study of self-similarity of the Maslov component.

Chapter 2 - Corrugation Process and Kuiper relations

Given a map $f_0 : [0,1]^m \to \mathbb{R}^n$, a loop family $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ and an integer N, the fundamental formula of the Convex Integration Theory builds a map F_1 whose expression is the following

$$F_1(x_1, \dots, x_m) := f_0(0, x_2, \dots, x_m) + \int_{s=0}^{x_1} \gamma(s, x_2, \dots, x_m, Ns)ds.$$

We say that F_1 is obtained by a Convex Integration from f_0 in the direction x_1 (or ∂_1). The map F_1 satisfies the following properties on which the whole theory relies:

Proposition 1. *If the loop family γ is chosen such that for every $x \in [0,1]^m$ we have $\int_0^1 \gamma(x,t)dt = \partial_j f_0(x)$, then the map $F_1 = CI_\gamma(f_0, \partial_j, N)$ satisfies*

(P_1) $\|f_0 - F_1\|_{C^0} = O(1/N)$, *($C^0$-density),*

(P_2) $\|\partial_i f_0 - \partial_i F_1\|_{C^0} = O(1/N)$ *for every* $i \neq j$,

(P_3) $\forall x \in [0,1]^m, \quad \partial_j F_1(x) = \gamma(x, Nx_j)$.

Corrugation Process In this thesis, we substitute the previous formula for the following formula:

$$f_1(x) := f_0(x) + \frac{1}{N} \int_{s=0}^{Nx_1} \gamma(x,s) - \overline{\gamma}(x)ds$$

where $\overline{\gamma}(x)$ is the average of the loop $\gamma(x, \cdot)$. We say that x is the *space variable* of γ and s is the *time variable*. We call this formula *Corrugation Process* and we write $f_1 = CP_\gamma(f_0, \partial_1, N)$. The Convex Integration Theory can be used with the Corrugation Process because the map f_1 satisfies the following properties:

Proposition 2. *The map $f_1 = CP_\gamma(f_0, \partial_j, N)$ satisfies*

(P_1) $\|f_0 - f_1\|_\infty = O(1/N)$ *(C^0-density),*

(P_2) $\|\partial_i f_0 - \partial_i f_1\|_\infty = O(1/N)$ *for every* $i \neq j$,

(P_3') *if the loop family γ satisfies, for every $x \in [0,1]^m$, $\overline{\gamma}(x) = \partial_j f_0(x)$, then $\partial_j f_1(x) = \gamma(x, Nx_j) + O(1/N)$ for all $x \in [0,1]^m$.*

A significant advantage of the Corrugation Process is that the time and space variables are separated: the integral only involves the time variable. This has a main consequence: the value of f_1 at x only depends on the values of f_0 and of γ at x, in other words the formula is local. The locality of the map built by Corrugation Process leads to a useful corollary, it preserves the 1-periodicity (see the beginning of Subsection 2.1.2), a notable property allows to reduce the number of gluings. Note that by a different approach Y. Eliashberg and N. Mishachev obtained a formula sharing the same periodicity property [EM02]. Their formula depends on a specific choice of loops (called *flowers*) and is less suitable for to the constructions described later.

Expression independent of a system of local coordinates The Corrugation Process has another worthwhile advantage: it admits a coordinate-free analytic expression[2]. Let M and W be two manifolds, h be a metric on W, $f_0 : U \subset M \to (W,h)$ be a map, $\pi : U \to \mathbb{R}$ be a submersion, $u : M \to TM$ be a tangent vector field such that for every $x \in M$ $d\pi_x(u_x) = 1$ and $\gamma : U \times \mathbb{R}/\mathbb{Z} \to f_0^*TW$ be a loop family such that $\gamma(x,.) : \mathbb{R}/\mathbb{Z} \to f_0^*TW_x$ for every $x \in U$ and whose average satisfies $\overline{\gamma}(x) = df_0(u_x)$. We define the Corrugation Process in the "direction" π by the formula

$$f_1(x) := \exp_{f_0(x)}\left(\frac{1}{N}\int_{s=0}^{N\pi(x)} \gamma(x,s) - \overline{\gamma}(x)ds\right)$$

(see Definition 29). To our knowledge, such a coordinate-free expression does not exist in the previous versions of the Convex Integration Theory.

Kuiper Relations The Convex Integration Theory proposes a geometric approach to solve underdetermined partial differential inequalities. In this theory, a subset of the jet space encodes the differential relation. In this thesis, we consider differential relation of order 1, i.e. a subset $\mathscr{R}$ of the 1-jet space $J^1(M,W)$, i.e. of the space of 3-uples (x,y,L) where $x \in M$, $y \in W$ and $L \in \mathscr{L}(T_xM, T_yW)$ is a linear map between the tangent spaces. A *formal solution* $\mathfrak{S} : x \mapsto (x, f_0(x), L_x)$ is a section of the 1-jets bundle whose image belongs to the differential relation $\mathscr{R}$. From some formal solutions called *subsolutions* (Definition 38), the Convex Integration formula builds a solution of $\mathscr{R}$, i.e. a map f whose 1-jet $x \mapsto (x, f(x), df_x)$ belongs to $\mathscr{R}$. In this construction, we need to choose the family $x \mapsto \gamma(x,\cdot)$ such that:

1) its image belongs to a subset of $\mathscr{R}$ which depends on both π and u,

2) its average $\overline{\gamma}(x) = \int_0^1 \gamma(x,t)dt$ is equal to $df_0(u_x)$,

3) its base point is $(x, f_0(x), L_x(u_x))$.

[2]This advantage was brought into light during a conversation with Patrick Massot.

Usually the existence of such a family is deduced from a barycentric construction. However the non-uniformity of this process reduces the effectiveness of the convex integration. We propose a uniform loop family $\gamma(\sigma, w)$ indexed, not by $x \in M$, but by a couple (σ, w) where $\sigma \in \mathscr{R}$ is the base point of the loop and w is its average. In this case, we say that this loop is *surrounding* (Definition 40). To ensure that all the loops have a similar shape we impose them to be shaped on a common *pattern* (Definition 43). A pattern is a map $c : A \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^p$ where A is a space of parameters. For example, we can choose a family of arcs of circle where the space A is a set of angles. We say that a loop family γ is *c-shaped* if there exists vectors $e_1, \ldots, e_p$ of T_yW, with $\sigma = (x, y, L)$, such that

$$\gamma(\sigma, w)(\cdot) = \sum_{i=1}^{p} c_i(a(\sigma, w), \cdot) e_i(\sigma, w)$$

where the c_i are the coordinate maps of c (see Definition 44). Given any differential relation and a pattern c, the existence of such a family is not ensured.

Definition 3. We say that $\mathscr{R}$ is a *Kuiper relation* with respect to $(c, d\pi, u)$ if there exists a surrounding loop family γ which is c-shaped.

Corrugation Process with no integration For Kuiper relations, the Corrugation Process only involves primitive of the coordinate maps of c. Let $C_i(a, \cdot)$ be the 1-periodic map defined by $t \mapsto \int_0^t c_i(a, t) - \overline{c_i}(a) dt$. Given a Kuiper relation $\mathscr{R}$ with respect to $(c, d\pi, u)$, the Corrugation Process depends on C_1 and e_i. Precisely, we have the following proposition:

Proposition 4. *Let $\mathfrak{S} = (x, f_0, L_0)$ be a subsolution. The map $f_1 = CP_{\gamma}(\mathfrak{S}, \pi, N)$ has the following analytic expression*

$$f_1(x) \quad = \quad \exp_{f_0(x)} \left(\frac{1}{N} \sum_{i=1}^{p} C_i(a(x), N\pi(x)) e_i(x) \right)$$

where $a(x) := \boldsymbol{a}(\mathfrak{S}(x), df_0(u_x))$, $e(x) := \boldsymbol{e}(\mathfrak{S}(x), df_0(u_x))$ and $x \in U$.

This formula gives a clear interpretation of the Corrugation Process: the map f_1 is obtained from f_0 adding corrugations in the directions $e_1, \ldots, e_p$ which only depend on the subsolution $\mathfrak{S}$.

This formula also allows to give a direct connection with other formulas of the literature which generates corrugations. This is the case of Thurston's formula in his theory of corrugations [Lev95]. Let $C(t) = (-h \sin(4\pi t), 2h \sin(2\pi t))$, h be a real and $e_1(x) = e(x)$, $e_2(x) = ie(x)$, with $e(x)$ a unit complex vector. Then the Corrugation Process writes

$$f_1(x) = f_0(x) + \frac{1}{N}\big(-h \sin(4\pi N x) e(x) + 2h \sin(2\pi N x) ie(x)\big).$$

This formula is, up to a coefficient $1/N$, the Thurston's formula of corrugation (see Subsection 2.2.4). Similarly, if we consider the map

$$C(x,t) = \Big(\int_{s=0}^{t} \cos(\alpha(x)\sin(2\pi t)) - J_0(\alpha(x))ds, \int_{s=0}^{t} \sin(\alpha(x)\sin(2\pi t))ds, 0\Big),$$

then the integral free expression of the Corrugation Process allows to recover the ansatz of S. Conti, C. De Lellis and L. Székelyhidi (see Subsection 2.2.5).

Chapter 3 - Immersions and isometric immersions

In Chapter 2, we have introduced the notion of Kuiper relations. It is then natural to search examples of such relations. For that we consider the differential relations which inspired the Convex Integration Theory: the relation of immersions and the relation of isometries. We show that the relation of codimension 1 immersions is Kuiper. Precisely we show:

Theorem 5. *Let M be an orientable manifold of dimension m and W be a Riemannian orientable manifold of dimension n such that $n > m$. If (m,n) satisfies one of the following conditions:*

$$\begin{array}{llll} (i) & n = m+1 & (ii) & m = 1 \text{ and } n \text{ even} \\ (iii) & m = 2 \text{ and } n = 7 & (iv) & m = 3 \text{ and } n = 8 \end{array}$$

then the relation of immersions $\mathscr{I}(M,W)$ is a Kuiper relation for the pattern c given by

$$c(x,t) = \Big(\cos(\alpha(x)\cos(2\pi t)) - J_0(\alpha(x)), \sin(\alpha(x)\cos(2\pi t)), 1\Big).$$

In the theorem, the condition on the codimension ensures the existence of a section of the natural bundle $V_{m+1}(\mathbb{R}^n) \to V_m(\mathbb{R}^n)$, where $V_j(\mathbb{R}^k)$ is the set family of j-frames in a k-dimensional space (Stiefel manifold). For codimensions which do not have such section, we show that the relation of immersions satisfies a weaker property that we call *quasi-Kuiper* (see Definition 49). Such relations also admit a free integral Corrugation Process formula (Proposition 50).

Theorem 6. *Let M be an orientable manifold and W be a Riemannian orientable manifold such that $\dim(W) > \dim(M)$. The relation $\mathscr{I}(M,W)$ is quasi-Kuiper.*

If we have a subsolution, this theorem states that we can desingularize maps with simple expressions coming from the Corrugation Process. A new immersion of $\mathbb{R}P^2$ is then directly built from a sphere inversion of the desingularized map

$$f_1(x) = f_0(x) + \frac{1}{N}\Big(K_c(\alpha(x), Nx_2)rv_2(x) + K_s(\alpha(x), Nx_2)r(v_2(x)\wedge v_1(x))\Big)$$

with

$$C(x,t) = \Big(\quad K_c(\alpha(x),t) := \int_{s=0}^{t} \cos(a(x)\cos(2\pi s)) - J_0(\alpha(x))ds, \\ K_s(\alpha(x),t) := \int_{s=0}^{t} \sin(a(x)\cos(2\pi s))ds, 0\Big)$$

where f_0 is a parametrization of the Plücker's Conoid, r and $\mathfrak{S} = (x, f_0, v_1, v_2)$ are defined in Section 3.3 (see Figure 6 for an illustration of f_1). We then

Figure 6 – **The center of a corrugated Plucker's Conoid (before a sphere inversion)** which allows to obtained a new immersion of $\mathbb{R}P^2$ by a sphere inversion (see Section 3.3).

consider the relation of ϵ-isometries, where $\epsilon > 0$. Indeed, these relations are a key step to solve the relation of isometries by Convex Integration (see Section 1.2.4). Let M and W be two orientable Riemannian manifolds. A map

$f : (M, g) \to (W, h)$ is an ϵ-isometry if for each $x \in M$ we have $\|(f^*h)_x - g_x\| < \epsilon$, where f^*h is the pullback of the metric h by f. We show:

Theorem 7. *Let* $\dim(W) > \dim(M)$*. The relation* $\mathscr{I}\!s(\epsilon)$ *is quasi-Kuiper.*

We also show that the relation $\mathscr{I}\!s(\epsilon)$ is Kuiper for the same conditions on codimension as the ones for the relation of immersions $\mathscr{I}(M, W)$.

Chapter 4 - Totally real isometric maps and self-similarity

A natural framework to study potential self-similarity properties is given by notions introduced in this thesis. Self-similarity was already observed for the C^1-isometric embeddings both of the flat torus and the reduced sphere [BJLT13, BBD+18]. In this chapter, we are interested in self-similarity properties of embeddings obtained by an iteration of the Corrugation Process.

Totally real isometric maps Let M be a manifold and (W, J) be an almost complex manifold (i.e. J is a vector bundle isomorphism such that $J_x \circ J_x = -Id_{T_xW}$ for every $x \in W$) such that $\dim(W) = 2\dim(M)$. Recall that a map $f : M \to (W, J)$ is totally real if for every $x \in M$ we have $df(T_xM) \oplus Jdf(T_xM) = T_{f(x)}W$. Denote by $\mathscr{I}_{TR}(M, W)$ the relation of totally real maps. We show the following theorem:

Theorem 8. *The relation* $\mathscr{I}_{TR}(M, W)$ *is a Kuiper relation.*

We then state a theorem in the spirit of Nash-Kuiper for totally real isometries:

Theorem 9. *Let* (M^m, g) *be a compact Riemannian manifold and let* $f_0 : (M^m, g) \to (W^{2m}, J, h)$ *be a totally real immersion (resp. embedding) such that* $h - f_0^*g$ *is positive definite (i.e. strictly short). Then, for every* $\epsilon > 0$*, there exists a* C^1 *totally real isometric immersion (resp. embedding)* $f_\infty : (M^m, g) \to (W^{2m}, J, h)$ *such that* $dist(f_\infty(x), f_0(x)) \leq \epsilon$ *for every* $x \in M^m$*.*

In the construction of these isometric maps, the Corrugation Process is a key element to control partial derivatives, precisely the J-density (defined in 4.2.1) which ensures that the limit embedding obtained is totally real. Note that other C^1-isometric theorems in the spirit of Nash-Kuiper were also obtained in many contexts: for Carnot-Caratheodory metrics [D'A95], in contact, symplectic and pseudo-Riemannian geometries [D'A00, DL02, DD06] and for sub-Riemannian manifolds [D'A00, DL02, DD06].

Self-similarity We then study the self-similarity properties of totally real isometric maps obtained with our construction. Precisely we show that the Maslov component (see Subsection 4.2.2 for a definition) of the Gauss map of the totally real isometric maps f_∞ built by iteration of the Corrugation Process satisfies the following property:

Proposition 10. *Let $\mathfrak{m}(f_0, f_\infty) = e^{i\mathscr{W}_\infty} : M \to \mathbb{S}^1$ be the Maslov map of f_∞ and $\mathscr{W}_\infty = 2\sum_k \vartheta_k$ be the Maslov argument. Then if k is large enough*

$$\vartheta_k = \theta_k + \sum_{j\in I(k)} O\left(\frac{1}{N_{k,j}}\right) \textit{ where } \theta_k := \sum_{j\in I(k)} \alpha_{k,j}\cos(2\pi N_{k,j}\pi_{k,j})$$

(if $x \in M$ is not in the domain of $\pi_{k,j}$ it is understood that the corresponding term is zero).

Note that, the Maslov component of the Gauss map is similar to a Weierstrass function $x \mapsto \sum_k a^k \cos(b^k \pi x)$, $x \in \mathbb{R}$ where $0 < a < 1$, b is an odd integer and $ab > 1 + 3\pi/2$. Recall that the Hausdorff dimension of the graph of this function is strictly greater than 1 and have a self-similar behavior.

Future works

The key notions developed in this thesis are the Corrugation Process and the Kuiper relations. These two notions allow to propose an effective method of the Convex Integration, which opens the path to the realisation of new isometric embeddings. A first perspective is to consider the non-compact case and to build an explicit isometric embedding of the hyperbolic plane. This raises new questions on the choice of charts and of local constructions. Nevertheless the local and relative properties of the Corrugation Process make *a priori* the exploration of this problem simpler.

The notions developed in this thesis also allow to state a self-similarity property for totally real isometric immersions. However this self-similarity property is weaker than the one observed for the flat torus and the reduced sphere. An other line of research is the study of the emergence of self-similarity for closed differential relations.

The loop families considered in this thesis are of class C^0, C^1 or C^∞. Nevertheless considering loop families with low regularity, i.e. discontinue loop families, could have an interest. For example, the choice of piecewise constant loops in the Corrugation Process leads to piecewise linear (PL) solutions and opens a way to solve, with the Convex Integration, problems coming from the PL geometry. One perspective is therefore to develop a PL Convex Integration Theory, with a potential application for PL isometric embeddings. By the

Burago-Zalgaller Theorem [BZ95], we know the existence of PL isometric embeddings, but these embeddings have a very high number of vertices. Recently Tanessi Quintanar Cortés [QC] has built a PL isometric embedding of a flat square torus with only 40 vertices. It is likely that a PL Convex Integration Theory might provide a universal framework for the explicit construction of PL isometric embeddings.

Chapter 1

Differential relations, h-principles and Convex Integration

A central question for a differential problem is the existence of solutions. Mikhaïl Gromov remarked that for some problem in differential geometry, the existence of a solution is implied by the existence of an other object named formal solution. Let us explain that on differential problems of order 1, i.e. of the form $\Phi(x, f(x), \mathrm{d}f(x)) =, <,> 0$, where the map f is the unknown. Gromov noticed that for some differential problems, the existence of a *formal solution*, i.e. an uplet $(x, f(x), L(x))$ where L is any linear map of the same space of $\mathrm{d}f$, implies the existence of a solution. In the 1970s, he introduced this approach with the language of differential relations and h-principles [Gro73, Gro86].

In Section 1.1 we introduce the notions of 1-jet space and differential relation, and we give some h-principles. In Section 1.2, we present the Convex Integration Theory, which is a method that allows to prove these h-principles. At the end of this section, we recall how this theory can solve the relation of isometric immersions, following the proof of Nash's theorem of C^1-isometric embeddings [Nas54].

1.1 Differential relations and h-principles

1.1.1 The 1-jet space

Let M and W be two C^1-manifolds. The 1-jet space $J^1(M, W)$ is the set of triples defined by

$$J^1(M, W) := \{(x, y, L) \mid x \in M, y \in W, L \in \mathscr{L}(T_xM, T_yW)\}$$

where $\mathscr{L}(T_xM, T_yW)$ denotes the space of linear maps between the two vector spaces T_xM and T_yW. This set fibers over M by the natural map $(x, y, L) \mapsto x$.

The 1-*jet* of a C^1-map $f: M \to W$ is the section of $J^1(M,W)$ given by

$$\begin{array}{rccc} j^1 f: & M & \longrightarrow & J^1(M,W) \\ & x & \longmapsto & (x, f(x), df_x) \end{array}$$

where $df_x : T_xM \to T_{f(x)}W$ is the differential of f at the point x.

In the case where $M = [0,1]^m$ and $W = \mathbb{R}^n$, for m and n two non-negative integers, as the tangent spaces are trivial we have the identification

$$J^1([0,1]^m, \mathbb{R}^n) \simeq [0,1]^m \times \mathbb{R}^n \times (\mathbb{R}^n)^m .$$

Similarly the 1-jet of a C^1-map $f : [0,1]^m \to \mathbb{R}^n$ can be identified with the map

$$x \longmapsto (x, f(x), \partial_1 f(x), \ldots, \partial_m f(x)).$$

For a description of the topology and the fiber bundle structure of $J^1(M,W)$ see the book of M. Hirsch, *Differential Topology* [Hir76].

1.1.2 Differential relations

Definition 11. A *differential relation* of order 1 is a subset $\mathscr{R}$ of the jet space $J^1(M,W)$. We say that $\mathscr{R}$ is *open* or *closed* if it is an open or closed subset.

For the sake of simplicity, we will sometimes call *relation* a differential relation. Remark that any PDE can be seen as a differential relation. Indeed, let (S) be a system of PDE's of order 1 given by

$$(S) \quad \left\{ \begin{array}{l} \phi_1(x, f(x), \partial_1 f(x), \ldots, \partial_m f(x)) \ R_1 \ 0 \\ \ldots \\ \phi_\ell(x, f(x), \partial_1 f(x), \ldots, \partial_m f(x)) \ R_\ell \ 0 \end{array} \right.$$

where $R_1, \ldots, R_\ell$ are binary relations (for example $=$, $<$, $\ldots$), and $\phi_1, \ldots \phi_\ell$ are maps defined on $J^1([0,1]^m, \mathbb{R}^n)$. The differential relation associated to (S) is the subset of $J^1([0,1]^m, \mathbb{R}^n)$ defined by

$$\mathscr{R}_{(S)} := \left\{ (x, y, v_1, \ldots, v_m) \ \middle| \ \begin{array}{l} \phi_1(x, y, v_1, \ldots, v_m) \ R_1 \ 0 \\ \ldots \\ \phi_\ell(x, y, v_1, \ldots, v_m) \ R_\ell \ 0 \end{array} \right\}.$$

We give below several examples of differential relations that will be considered in this thesis.

Relation of Immersions. The relation of immersions of a manifold M into a manifold W is the set

$$\mathscr{I}(M,W) := \{(x, y, L) \mid \operatorname{rank} L \text{ maximal}\}.$$

It is an open relation because the condition on the rank is an open one. Remark that when $M = [0,1]^m$, the image of the 1-jet $j^1 f$ of a C^1-map f belongs to $\mathscr{I}([0,1]^m, W)$ if and only if for every $x \in [0,1]^m$ the partial derivatives $\partial_1 f(x)$, ..., $\partial_m f(x)$ are linearly independents. Then we have the identification

$$\mathscr{I}([0,1]^m, W) \simeq \{(x, y, v_1, \ldots, v_m) \mid v_1, \ldots, v_m \text{ are linearly independent}\}$$

where v_1, ..., v_m are vectors of $T_y W$.

Relation of Isometric Immersions. The relation of isometric immersions from (M, g) to (W, h), with g and h two metrics, is given by

$$\mathscr{I}\!s := \{(x, y, L) \mid L^* h = g\}$$

where $L^* h$ denotes the pullback by L of the metric h. As the condition on the linear map L is closed, the relation $\mathscr{I}\!s$ is closed.

As we will see later, the relation $\mathscr{I}\!s$ can be solved using inductively the Convex Integration formula on thickenings of $\mathscr{I}\!s$, i.e. relations of ϵ-isometric immersions

$$\mathscr{I}\!s(\epsilon) := \{(x, y, L) \mid |g - L^* h| < \epsilon\}$$

where $\epsilon > 0$.

Relation of Totally Real maps. Let M and W be two manifolds such that $\dim(W) = 2\dim(M)$ and $J : TW \to TW$ be an almost complex structure on W (i.e. a vector bundle isomorphism such that $J_x \circ J_x = -Id_{T_x W}$ for any $x \in W$). The differential relation associated to totally real maps is the following

$$\mathscr{I}_{TR} := \{(x, y, L) \mid L(T_x M) \oplus JL(T_x M) = T_y W\}.$$

Remark that if $(x, y, L) \in \mathscr{I}_{TR}$, then the rank of L is maximal. As the condition on the linear map is a combinaison of a maximal rank condition and a transversality condition, it is an open condition. So we have the inclusion $\mathscr{I}_{TR} \subset \mathscr{I}$. In particular if the 1-jet $j^1 f$ belongs to $\mathscr{I}_{TR}$, the map f is an immersion.

1.1.3 Formal and holonomic solutions

We denote by $\Gamma(J^1(M, W))$ the set of sections of the bundle $J^1(M, W) \to M$, i.e. the set of maps $\mathfrak{S}$

$$\begin{array}{rccc} \mathfrak{S}: & M & \longrightarrow & J^1(M, W) \\ & x & \longmapsto & (x, y(x), L_x) \end{array}$$

where $L_x : T_x M \to T_{y(x)} W$. For a section $\mathfrak{S}$, we denote by $bs\,\mathfrak{S} = y$ the *base map* of $\mathfrak{S}$. Note that, for a section $\mathfrak{S}$ of $J^1(M, W)$, the component L_x is not necessarily the differential of the base map y at the point x. In particular, a section whose image belongs to a differential relation is not necessarily the 1-jet of a map. The following definitions distinguish two cases.

Definition 12. Let $\mathscr{R}$ be a differential relation and $\mathfrak{S}$ be a section of $J^1(M,W)$. We say that $\mathfrak{S}$ is a *formal solution of* $\mathscr{R}$ if, for every x in M, $\mathfrak{S}(x) \in \mathscr{R}$.

We denote by $\Gamma(\mathscr{R})$ the set of formal solutions of $\mathscr{R}$.

Definition 13. Let $\mathscr{R}$ be a differential relation and $\mathfrak{S}$ be a formal solution of $\mathscr{R}$. If there exists a C^1-map $f : M \to W$ such that $j^1 f = \mathfrak{S}$, we say that $\mathfrak{S}$ is a *holonomic solution of* $\mathscr{R}$.

We denote by $Sol(\mathscr{R})$ the set of holonomic solutions of $\mathscr{R}$. We clearly have $Sol(\mathscr{R}) \subset \Gamma(\mathscr{R})$. Note that if the 1-jet $j^1 f$ is a holonomic solution of $\mathscr{R}$, then the map f is a solution of the differential problem associated to $\mathscr{R}$.

1.1.4 h-principles

We say that a differential relation satisfies to a property of h-principle if we can deform any formal solution to a holonomic solution among formal solutions. We briefly present different flavors of h-principle connected to the Convex Integration Theory (for more details see [Gro86, Spr98, EM02]).

The homotopy principle (h-principle). We say that a differential relation $\mathscr{R}$ satisfies the *homotopy principle* if any formal solution of $\mathscr{R}$ is homotopic in $\Gamma(\mathscr{R})$ to a holonomic solution of $\mathscr{R}$. This definition amounts to saying that the inclusion $Sol(\mathscr{R}) \subset \Gamma(\mathscr{R})$ induces a surjective map $\pi_0(Sol(\mathscr{R})) \twoheadrightarrow \pi_0(\Gamma(\mathscr{R}))$, where $\pi_0(\cdot)$ denotes the set of path-connected components.

The one-parametric h-principle. We say that a relation $\mathscr{R}$ satisfies the *one-parametric h-principle* if for any homotopy $h : [0,1] \to \Gamma(\mathscr{R})$ between two 1-jets $h(0) = j^1 f_0$ and $h(1) = j^1 f_1$ of C^1-maps f_0 and f_1, there exists a homotopy $H : [0,1]^2 \to \Gamma(\mathscr{R})$ such that

(i) $H(0,t) = h(t)$ for every $t \in [0,1]$

(ii) $H(s,0) = j^1 f_0$ and $H(s,1) = j^1 f_1$ for each $s \in [0,1]$

(iii) $H(1,t) \in Sol(\mathscr{R})$ for each $t \in [0,1]$.

It means that any path connecting two holonomic solutions in $\Gamma(\mathscr{R})$ can be deformed into a path in $Sol(\mathscr{R})$ without modify the end points.

The (multi)parametric h-principle. We say that a relation $\mathscr{R}$ satisfies the *(multi)parametric h-principle* if the inclusion $\imath : Sol(\mathscr{R}) \to \Gamma(\mathscr{R})$ is a weak homotopy equivalence, i.e. if the inclusion $\imath$ induces a one-to-one correspondence between the path-connected components of $Sol(\mathscr{R})$ and of $\Gamma(\mathscr{R})$ and if for each holonomic solution $\mathfrak{S}$ and for each natural integer $k \geq 1$, the map $\pi_k(\imath) : \pi_k(Sol(\mathscr{R}), \mathfrak{S}) \to \pi_k(\Gamma(\mathscr{R}), \imath(\mathfrak{S}))$ is a group isomorphism.

The C^0-dense h-principle. We say that a relation $\mathscr{R}$ satisfies the C^0-*dense h-principle* if for any formal solution $\mathfrak{S}$ of base map f_0 and for any neighborhood U of f_0 there is a homotopy $H : [0,1] \to \Gamma(\mathscr{R})$ such that

(i) $H(0) = \mathfrak{S}$

(ii) for all $t \in [0,1]$ the base map $f_t = bs\, H(t)$ is in U,

(iii) the section $H(1)$ is holonomic.

1.2 Convex Integration Theory

The Convex Integration Theory is a method that allows to prove some h-principles. This method works for differential relations on general compact manifolds, i.e. when $\mathscr{R} \subset J^1(M, W)$, but it fundamentally relies on a formula expressed on charts. Therefore the Convex Integration formula usually deals with charts $[0,1]^m$ and a cubical covering of the compact manifold M is considered to build a global solution.

In this section, we introduce the main formula and the convex hull condition (on the differential relation) that allows to apply this formula. Then, for an open relation, we present a way to deform homotopically a formal solution to a holonomic solution on a chart using the Convex Integration formula. At the end, we discuss the case of the isometric relation and the Nash-Kuiper theorem on C^1-isometric immersions in terms of differential relations.

1.2.1 Convex Integration Formula

We first recall the Convex Integration formula [Spr98, EM02, BJLT13]. Let $f_0 : [0,1]^m \to \mathbb{R}^n$ be a map of class C^k with $k \geq 1$, ∂_j be a direction, $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ be a loop family of class C^{k-1} and $N \in]0, +\infty[$ be a number. We define a map $F_1 : [0,1]^m \to \mathbb{R}^n$ by setting :

$$\begin{aligned} F_1(x) \quad := \quad & f_0(x_1, \ldots x_{j-1}, 0, x_{j+1}, \ldots, x_m) \\ & + \int_{s=0}^{x_j} \gamma(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_m, Ns)\, ds \end{aligned} \tag{1.1}$$

which is of class C^{k-1}.

Definition 14. We say that F_1 is obtained from f_0 by *Convex Integration* in the direction ∂_j and write $F_1 = CI_\gamma(f_0, \partial_j, N)$.

The Convex Integration formula relies on the choice of a loop family γ. The following property will help us to control the components of the 1-jet of the map F_1:

Definition 15. A loop family $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ satisfies the *Average Constraint with respect to f_0 in the direction ∂_j* if

$$(AC) \qquad \forall x \in [0,1]^m, \quad \partial_j f_0(x) = \overline{\gamma}(x)$$

where $\overline{\gamma}(x) := \int_0^1 \gamma(x,t)dt$ denotes the average of γ on its time parameter.

Note that by the Average Constraint if $x \mapsto \partial_j f_0(x)$ is C^k then $x \mapsto \overline{\gamma}(x)$ is C^k and $x \mapsto \gamma(x,t)$ is C^k. In general, if f_0 is C^k we assume that the loop family is C^{k-1} so that F_1 is C^{k-1}. The following proposition shows that the map F_1 is close to the map f_0 if N is large enough and has γ as partial derivative in the direction ∂_j (see [BJLT13] Lemma 2 and 3):

Proposition 16. *Let $f_0 : [0,1]^m \to \mathbb{R}^n$ be a map of class C^k with $k \geq 2$ and $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ be a loop family of class C^{k-1} which satisfies the average constraint (AC). Then the map $F_1 = CI_\gamma(f_0, \partial_j, N)$ is of class C^{k-1} and satisfies*

(P_1) $\quad \|f_0 - F_1\|_{C^0} = O(1/N)$, *($C^0$-density),*

(P_2) $\quad \|\partial_i f_0 - \partial_i F_1\|_{C^0} = O(1/N)$ *for every $i \neq j$,*

(P_3) $\quad \forall x \in [0,1]^m, \quad \partial_j F_1(x) = \gamma(x, Nx_j)$.

Property (P_3) directly follows from the Convex Integration formula. The constants involved in the notation $O(1/N)$ can be taken to be

$$C_{P_1} = 2\|\gamma\|_\infty + \|\partial_j\gamma\|_\infty \quad \text{and} \quad C_{P_2} = 2\|\partial_i\gamma\|_\infty + \|\partial_i\partial_j\gamma\|_\infty$$

see [BJLT12]. The Convex Integration formula allows to modify one partial derivative with a controlled error on the other components of the 1-jet, the base map and the other partial derivatives. This implies that if the relation is open, we can build a solution by iterating this formula for each partial derivatives.

1.2.2 Convex hull condition

The previous subsection gives a condition on the loop family γ to have a control on the map build by Convex Integration. Here we give a condition on an open differential relation $\mathscr{R}$ of $J^1([0,1]^m, \mathbb{R}^n)$ to have the existence of such a loop family.

Let $\sigma = (x, y, v_1, \dots, v_m)$ be a point of $\mathscr{R}$. For $j \in \{1, \dots, m\}$, we define the slice

$$\mathscr{R}(\sigma, \partial_j) := Conn_{v_j}\{\mathbf{v} \in \mathbb{R}^n | (x, y, v_1, \dots, v_{j-1}, \mathbf{v}, v_{j+1}, \dots, v_m) \in \mathscr{R}\}$$

where $Conn_a A$ denotes the path-connected component of A containing a. Note that the slice $\mathscr{R}(\sigma, \partial_j)$ can not be empty as $\sigma \in \mathscr{R}$. We then denote by $\operatorname{IntConv} \mathscr{R}(\sigma, \partial_j)$ the interior of the convex hull of $\mathscr{R}(\sigma, \partial_j)$.

Definition 17. Let $x \mapsto \mathfrak{S}(x) = (x, f_0(x), v_1(x), \ldots, v_m(x))$ be a formal solution of $\mathscr{R}$ whose base map $f_0 = \operatorname{bs} \mathfrak{S}$ is of class C^1. We say that $\mathfrak{S}$ is a *subsolution of $\mathscr{R}$ with respect to ∂_j* if for all $x \in [0,1]^m$ the map f_0 satisfies

$$\partial_j f_0(x) \in \operatorname{IntConv} \mathscr{R}(\mathfrak{S}(x), \partial_j).$$

Definition 18. The relation $\mathscr{R}$ is an *ample relation* if for every $\sigma \in \mathscr{R}$ and every $j \in \{1, \ldots, m\}$ we have

$$\operatorname{IntConv} \mathscr{R}(\sigma, \partial_j) = \mathbb{R}^n.$$

Remark that for an ample relation every formal solution is a subsolution. The relation $\mathscr{I}$ of immersions of codimension greater than 1 and the relation $\mathscr{I}_{TR}$ of totally real maps are ample because, in the two cases, the slice $\mathscr{R}(\sigma, \partial_j)$ is the complement in $\mathbb{R}^n$ of a vector space of codimension greater than 2 (see the geometric description of $\mathscr{I}$ in Section 3.1 and of $\mathscr{I}_{TR}$ in Section 4.1).

The following lemma gives the existence of a loop family γ which allows to use the Convex Integration formula (for a proof see [Gro86, p169] or [Spr98, p29]).

Lemma 19 (Integral Representation)**.** *Let $\mathscr{R}$ be a differential relation and $\mathfrak{S}$ be a subsolution of $\mathscr{R}$ in the direction ∂_j. Then there exists a C^{k-1}-loop family $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ such that, for each x in $[0,1]^m$,*

(i) the image of $\gamma(x, \cdot)$ lies inside $\mathscr{R}(\mathfrak{S}(x), \partial_j)$;

(ii) we have $\gamma(x, 0) = \gamma(x, 1) = v_j(x)$;

(iii) the average of the loop $t \mapsto \gamma(x, t)$ satisfies $\overline{\gamma}(x) = \partial_j f_0(x)$;

(iv) each loop $t \mapsto \gamma(x, t)$ is contractible in $\mathscr{R}(\mathfrak{S}(x), \partial_j)$ to $v_j(x)$.

We will see later that the properties of the loop family γ given in Points (ii) and (iv) allow to show that a section built by Convex Integration is homotopic to $\mathfrak{S}$ in $\Gamma(\mathscr{R})$.

1.2.3 The homotopy principle by Convex Integration

In the previous subsection, we have seen a condition on the open differential relation $\mathscr{R}$ allowing to use the Convex Integration formula. Recall that this formula allows to modify one partial derivative with a controlled error on the other partial derivatives and the base map. In this subsection, we consider any

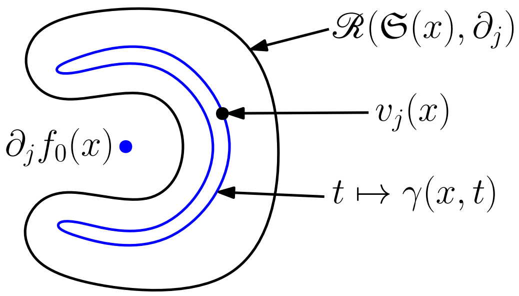


Figure 1.1 – **Illustration of Lemma 19:** example of a loop $\gamma(x, \cdot)$ that belongs to $\mathscr{R}(\mathfrak{S}(x), \partial_j)$ and whose average is exactly $\partial_j f_0(x)$.

formal solution of an open differential relation $\mathscr{R}$ and we modify iteratively its partial derivatives to build a holonomic solution. Moreover we can guarantee that the section built at each step is homotopic to the previous one.

Let $\mathfrak{S}_0 : x \mapsto (x, f_0(x), v_1(x), \ldots, v_m(x))$ be a formal solution of $\mathscr{R}$. With the Convex Integration formula, we build a sequence of formal solutions $(\mathfrak{S}_j)_{1 \leq j \leq m}$ as in the following lemma:

Lemma 20. *Let $\mathscr{R}$ be an open relation,*

$$\mathfrak{S}_j : x \mapsto (x, F_j(x), \partial_1 F_j(x), \ldots, \partial_j F_j(x), v_{j+1}, \ldots, v_m(x))$$

be a subsolution of $\mathscr{R}$ in the direction ∂_{j+1}. Then the section

$$\mathfrak{S}_{j+1} : x \mapsto (x, F_{j+1}(x), \partial_1 F_{j+1}(x), \ldots, \partial_{j+1} F_{j+1}(x), v_{j+2}, \ldots, v_m(x))$$

with $F_{j+1} = CI_{\gamma_{j+1}}(F_j, \partial_{j+1}, N_{j+1})$ and γ_{j+1} given in Lemma 19, is a formal solution of $\mathscr{R}$ for N_{j+1} large enough. Moreover, the two sections $\mathfrak{S}_j$ and $\mathfrak{S}_{j+1}$ are homotopics.

Remark in general that the regularity of the sequence $(F_j)_{1 \leq j \leq m}$ decreases along the iteration. Indeed, for $k \geq 2$, if the base map F_j of $\mathfrak{S}_j$ is of class C^k, if $\mathfrak{S}_j$ is of class C^{k-1} and if the loop family γ_{j+1} is such that $x \mapsto \gamma_{j+1}(x, \cdot)$ is C^{k-1} and $t \mapsto \gamma_{j+1}(\cdot, t)$ is C^{k-2}, the new map $F_{j+1} = CI_{\gamma_{j+1}}(F_j, \partial_{j+1}, N_{j+1})$ is of class C^{k-1}.

For the sake of completeness, we give here the idea of the proof of Lemma 20:

Proof of Lemma 20.– By property (P_3) of Proposition 16, we have for every $x \in [0,1]^m$

$$\partial_{j+1} F_{j+1}(x) = \gamma(x, N_{j+1} x_{j+1}) \in \mathscr{R}(\mathfrak{S}(x), \partial_{j+1})$$

so the intermediary section

$$\mathfrak{S}_j^{1/2} : x \mapsto (x, F_j(x), \partial_1 F_j(x), \ldots, \partial_j F_j(x), \gamma(x, N_{j+1} x_{j+1}), v_{j+2}, \ldots, v_m(x))$$

is a formal solution of $\mathscr{R}$. As the map $x \mapsto \gamma(x,\cdot)$ is continuous and as each loop $\gamma(x,\cdot)$ is contractible to $v_j(x)$ in $\mathscr{R}(\mathfrak{S}(x),\partial_j)$, the two formal solutions $\mathfrak{S}_j$ and $\mathfrak{S}_j^{1/2}$ are homotopic.
By properties (P_1) and (P_2) of Proposition 16, as $\mathscr{R}$ is open and $\mathfrak{S}_j^{1/2}$ is a formal solution of $\mathscr{R}$, for N_{j+1} large enough, the section $\mathfrak{S}_{j+1}$ is a formal solution too. Moreover we can homotope linearly $\mathfrak{S}_{j+1}^{1/2}$ to $\mathfrak{S}_{j+1}$. Then the two formal solutions $\mathfrak{S}_j$ and $\mathfrak{S}_{j+1}$ are homotopic. □

Remark that if a relation $\mathscr{R}$ is ample, each formal solution is a subsolution, so we can iterate the construction from $\mathfrak{S}_0$ to $\mathfrak{S}_m$. A direct consequence of Lemma 20 is the following proposition:

Proposition 21. *An open and ample relation in $J^1([0,1]^m,\mathbb{R}^n)$ satisfies the homotopy principle.*

In fact, this result is more general and can be stated for manifolds. The following theorem can be found in [Gro86, Chap. 2.4] or [Spr98, Chap. 4].

Theorem 22 (Gromov)**.** *An open and ample relation of $J^1(M,W)$, with M compact, satisfies the homotopy principle.*

Note that in the books of Gromov and Spring the theorem is given for parametric and C^0-dense h-principles, and is proved by the Convex Integration Theory. Since the relation of immersions of codimension greater than one and the relation of totally real maps are ample, this theorem implies the following result:

Corollary 23. *The relation of immersions of codimension greater than one and the relation of totally real maps satisfy the homotopy principle.*

We present here the idea of the proof of Theorem 22 given in [Spr98, Chap. 4]. The proof relies on suitable finite coverings of the compact C^1-manifold M and on the following lemma (see the property [Spr98, Chap. 3, Complement 3.5 (iii)] called *relative theorem*):

Lemma 24 (Spring)**.** *Let $\mathscr{R}$ be a differential relation in $J^1([0,1]^m,\mathbb{R}^n)$. Let $\mathfrak{S}$ be a subsolution of $\mathscr{R}$. Let $K \subset [0,1]^m$ be a closed subset such that $\mathfrak{S} = j^1 f_0$ on an open neighborhood $\mathfrak{Op}K$ of K. Then there exists a holonomic solution $j^1 f$ and a smaller neighborhood $\mathfrak{Op}_1 K$ of K (i.e. $K \subset \mathfrak{Op}_1 K \subset \mathfrak{Op}K$) such that $j^1 f = \mathfrak{S} = j^1 f_0$ on $\mathfrak{Op}_1 K$.*
Moreover there is a homotopy between $\mathfrak{S}$ and $j^1 f$ whose image belongs to $\Gamma(\mathscr{R})$ and which is constant on $\mathfrak{Op}_1 K$.

The proof of Theorem 22 builds inductively a holonomic solution. At the nth stage, Lemma 24 allows to modify the section of a holonomic one on the nth chart and which coincides suitably on the overlap which the sections constructed during previous steps.

1.2.4 Isometric case

The Nash-Kuiper C^1-Embedding Theorem was a prominent inspiration for the conception of Convex Integration Theory (see the interview of Gromov for the Abel Prize celebration [RS09]). In the language of differential relation, the proof of this theorem solves the relation of isometric immersions $\mathscr{I}\!s$. However this relation is neither open nor ample, so it cannot be proved directly by the homotopy principle for this relation by Convex Integration.

We decide to present here ideas of Nash's approach to solve the relation of isometric immersions $\mathscr{I}\!s$, because we will adapt it later to the relation of isometric totally real maps. For a proof, see [Nas54]or [EM02]. To deal with the closed relation of isometries, we consider open relations of the form

$$\mathscr{I}\!s(g,\epsilon) := \{(x,y,v_1,\ldots,v_m) \,|\, \forall i,j \in \{1,\ldots,m\},\ |g_{i,j} - \langle v_i, v_j\rangle| < \epsilon\}$$

where g is a metric on $[0,1]^m$ and $\epsilon > 0$. Then the idea is to consider a sequence $(\mathscr{I}\!s(g,\epsilon_k))_k$ where $\epsilon_k \to 0$.

In the proof of the C^1-isometric embeddings Theorem [Nas54], Nash starts with an immersion $f_0 : ([0,1]^m, g) \to (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, with $n \geq m+2$, which is strictly short. A map f_0 is said to be *strictly short* if the difference between the metric g and the induced metric $f_0^*\langle\cdot,\cdot\rangle$ is positive definite. He then builds a sequence of metrics $g_k = g - \delta_k\Delta$ converging to g, where $\delta_k \to 0$ and $\Delta = g - f_0^*\langle\cdot,\cdot\rangle$ is the isometric default. Finally he builds a sequence of maps f_k which are ϵ_k-isometric for the metric g_k and where ϵ_k is chosen such that f_k is strictly short for g_{k+1} and converging to zero.

In the language of differential relations, an associated sequence $(\mathscr{I}\!s(g_k,\epsilon_k))_k$ of open relations is considered, with $g_k \to g$ and $\epsilon_k \to 0$. If we choose the sequence $(\epsilon_k)_k$ with ϵ_k sufficiently small then every map in $\mathscr{I}\!s(g_k,\epsilon_k)$ is in the convex hull of $\mathscr{I}\!s(g_{k+1})$, so at each step k, the map f_k is a subsolution for the following relation. A condition on the sequence $(\delta_k)_{k\geq 1}$ gives the C^1-convergence (see [EM02, p192]).

Let us describe the construction of the map f_{k+1} from the map f_k. Since f_k is strictly short for the metric g_{k+1}, the isometric default $\Delta_k = g_{k+1} - f_k^*\langle\cdot,\cdot\rangle$ is positive definite and can be decomposed as a finite sum of squares of linear forms

$$\Delta_k = \sum_{i=1}^{N(k)} \rho_{k,i}\ell_{k,i} \otimes \ell_{k,i}$$

with $\rho_{k,i} : [0,1]^m \to \mathbb{R}_+$ (see [EM02] Lemma 21.3.1). From the map f_k, we build a finite sequence of maps $(f_{k,i})_{0\leq i\leq N(k)}$ such that $f_{k,0} = f_k$ and $f_{k,i+1}$ is built by Convex Integration from $f_{k,i}$ to be an $\epsilon_{k,i}$-isometry for the metric $g_{k,i} = f_{k,i-1}^*\langle\cdot,\cdot\rangle + \rho_{k,i}\ell_{k,i} \otimes \ell_{k,i}$. Then, for a sequence of $(\epsilon_{k,i})_{1\leq i\leq N(k)}$ small

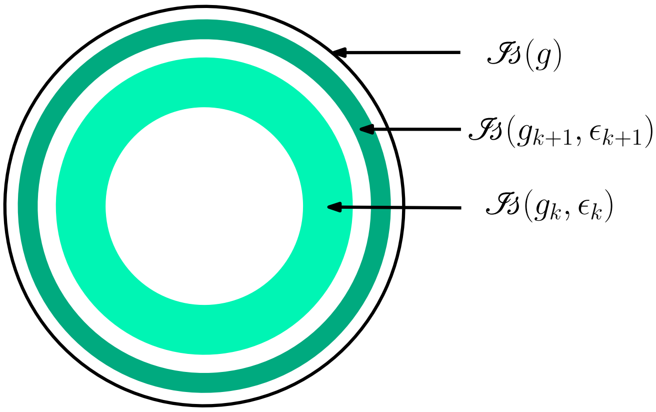


Figure 1.2 – **Approach the relation of isometries.** The relation of isometries for g and two open relations of the sequence of ϵ_k-isometries for g_k. For $m = 1$ and $n = 2$, the relation $\mathscr{I}_\mathfrak{Z}$ is a circle, for a general geometric description, see Section 3.4.

enough, we obtain a map $f_{k,N(k)}$ which is an ϵ_{k+1}-isometry for the metric g_{k+1}, and we set $f_{k+1} := f_{k,N(k)}$. So at each step i, the considered relation is the open relation of $\epsilon_{k,i}$-isometries for the metric $f^*_{k,i-1}\langle \cdot, \cdot \rangle$ to which we have added the primitive metric.

We now state the famous Nash-Kuiper theorem.

Theorem 25 (Nash-Kuiper [Nas54, Kui55]). *Let M and W be two compact C^1-manifolds such that $\dim W > \dim M$ and let g and h be two metrics. Let $f_0 : (M, g) \to (W, h)$ be a strictly short C^1-embedding, i.e. $g - f_0^* h$ is positive definite. Let $\epsilon > 0$. Then there exist an isometric C^1-embedding $f : (M, g) \to (W, h)$ such that $\|f - f_0\|_\infty < \epsilon$.*

In fact, J. Nash stated the following theorem for codimension 2 manifolds using a spiraling process instead of the Convex Integration formula in 1954. N. Kuiper generalized it to codimension 1 manifolds using oscillations in 1955. Note that Nash's result is given for embedding instead of immersions, but we do not discuss this part of the proof here. In the book *Partial Differential Relations* of Gromov, the Theorem of Nash-Kuiper is given with the language of h-principles as follow:

Theorem 26 ([Gro86, p203]). *The h-principle for isometric C^1-immersions $M \to W$ is C^0-dense in the space of strictly short maps $M \to W$, provided $\dim W > \dim M$.*

Chapter 2

Corrugation Process and Kuiper relations

A fundamental tool on which is based the Convex Integration Theory is the integral formula (1.1). In this chapter we propose an alternative formula that we called Corrugation Process (2.1). Similarly to the Convex Integration formula, the Corrugation Process depends on a loop family γ, a map f_0, a direction ∂_j and an integer N. The map built by this new formula is denoted $f_1 = CP\gamma(f_0, \partial_j, N)$. We will show in this chapter that this formula bears similar properties with the ones of the Convex Integration formula. Therefore the theory of Convex Integration can be indifferently developed with any one of the two formulas. A first significant advantage of the Corrugation Process is the following local property: at a point x, the map f_1 depends on x whereas the map $IC\gamma(f_0, \partial_j, N)$ obtained by the Convex Integration formula depends on the whole segment $[(x_1, \ldots, 0, \ldots, x_m), x]$. This locality property allows to define the Corrugation Process directly on a manifold, independently of a system of coordinates (see Definition 29).

Similarly to the Convex Integration formula, the Corrugation Process relies on a loop family γ, which depends on the subsolution under consideration (see Lemma 19). However the construction of a holonomic solution often requires to repeat the Corrugation Process in several directions ∂_j, and consequently needs to re-build at each step the loop family γ on a different subsolution at each time. We address in this chapter the question of finding a universal loop family. The interest of having such a generic loop family is to automatically provide a loop family for each subsolution. We also introduce the notion of Kuiper relation, for which there exist a universal loop family and a notion of a generic pattern (see Subsection 2.2.2). For such a relation, the loop family γ is automatically given and has the same "shape" for each subsolution. These two properties will be crucial in Chapter 4 to solve the relation of isometric totally real maps and to analyze a self-similarity behavior of isometric embeddings obtained with the Corrugation Process.

When a differential relation is a Kuiper one, the Corrugation Process has a simple expression. In the particular case of isometric embeddings, this formula is similar to the integral free formulas used by Nash and Kuiper to iteratively modify a short embedding into a C^1-isometric embedding in [Nas54, Kui55]. It is also similar to the *ansatz* used by Conti, De Lellis and Székelyhidi to construct isometric embeddings and to study their $C^{1,\alpha}$-regularity [CDLS12].

In the first section, we define the Corrugation Process and derive several properties. In the second section, we introduce the notion of universal loop family and Kuiper relation. In the third section, we extend the notion of Kuiper relation. This notion will be useful in Chapter 3 for the relation of immersions of codimension strictly greater than 1.

2.1 Corrugation Process formula

2.1.1 Coordinate expression

Let $f_0 : [0,1]^m \to \mathbb{R}^n$ be a map, ∂_j be a direction, $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ be a loop family and $N \in]0,+\infty[$.

Definition 27. We define the map $f_1 : [0,1]^m \to \mathbb{R}^n$ by

$$f_1(x) := f_0(x) + \frac{1}{N}\Gamma(x, Nx_j) \quad \text{with} \quad \Gamma(x,t) = \int_{s=0}^{t} \gamma(x,s) - \overline{\gamma}(x)ds \tag{2.1}$$

and $\overline{\gamma}(x) = \int_0^1 \gamma(x,s)ds$ denotes the average of the loop $t \mapsto \gamma(x,t)$. We say that f_1 is obtained from f_0 by a *Corrugation Process* in the direction ∂_j and we denote $f_1 = CP_\gamma(f_0, \partial_j, N)$.

The real number N is called the *number of corrugations*. If $x \mapsto \gamma(x,\cdot)$ is a C^k-map and $t \mapsto \gamma(\cdot,t)$ is a C^{k-1}-map, then $(x,t) \mapsto \Gamma(x,t)$ is a C^k-map. Remark that this formula involves the same ingredients as the usual Convex Integration formula but the variables x and s are separated and the integral relates only on the variable s. It ensues a local expression: the value of f_1 at x only depends on the values of f_0 and of a loop family γ at x.

Proposition 28. *Let $f_0 : [0,1]^m \to \mathbb{R}^n$ be a map of class C^k with $k \geq 2$ and $\gamma : [0,1]^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ be a loop family such that $x \mapsto \gamma(x,\cdot)$ is C^{k-1} and $t \mapsto \gamma(\cdot,t)$ is C^{k-2}. Then the map $f_1 = CP_\gamma(f_0, \partial_j, N)$ is of class C^{k-1} and we have*

(P_1) $\quad \|f_0 - f_1\|_\infty = O(1/N)$ *(C^0-density),*

(P_2) $\quad \|\partial_i f_0 - \partial_i f_1\|_\infty = O(1/N)$ *for every $i \neq j$,*

(P_3') *if γ satisfies the Average Constraint (AC) with respect to f_0 in the direction ∂_j then $\partial_j f_1(x) = \gamma(x, Nx_j) + O(1/N)$ for all $x \in [0,1]^m$.*

Propositions 16 and 28 show that both the Convex Integration formula and the Corrugation Process share the same properties (P_1), (P_2) and (P_3) up to an $O(1/N)$. In practice, the error in (P_3') is not a restriction since the two processes already introduce an error term in the other derivatives (property (P_2)). The theory of the Convex Integration is based on these properties (see Chapter 1 or [Gro86, Spr98]). It ensues that any one of the two formulas can be used in the Convex Integration Theory.

Proof.– Property (P_1) is obvious since Γ is a periodic map, so bounded. Regarding the derivatives in the direction $\partial_i \neq \partial_j$, since Γ is C^1, we have

$$\begin{aligned} \partial_i f_1(x) &= \partial_i f_0(x) + \frac{1}{N}\partial_i \left(\Gamma(x, Nx_j)\right) \\ &= \partial_i f_0(x) + O\left(\frac{1}{N}\right) \end{aligned}$$

and this shows property (P_2). For the property (P_3'), since γ and $\overline{\gamma}$ are C^1, we have:

$$\begin{aligned} \partial_j f_1(x) &= \partial_j f_0(x) + \frac{1}{N}\partial_j \left(\int_{t=0}^{Nx_j} \gamma(x,t) - \overline{\gamma}(x) dt\right) \\ &= \partial_j f_0(x) + \gamma(x, Nx_j) - \overline{\gamma}(x) + \frac{1}{N}\int_{t=0}^{Nx_j} \partial_j \Big(\gamma(x,t) - \overline{\gamma}(x)\Big) dt \\ &= \partial_j f_0(x) + \gamma(x, Nx_j) - \overline{\gamma}(x) + O\left(\frac{1}{N}\right). \end{aligned}$$

Since $\overline{\gamma}(x) - \partial_j f_0(x) = 0$ by the Average Constraint, we obtain (P_3'). □

From the proof above, it is readily seen that the constants involved in the notation $O(1/N)$ can be taken to be $K_{P_1} = 2\|\gamma\|_\infty$, $K_{P_2} = 2\|\partial_i\gamma\|_\infty$ and $K_{P_3'} = 2\|\partial_j\gamma\|_\infty$.

Note that, by a different approach, Y. Eliashberg and N. Mishachev obtain a formula with the same above local property [EM02]. Nevertheless, their construction depends on a specific choice of paths (called *flowers*) to build the loop family γ and is less amenable to the uniform construction described in the next subsections.

2.1.2 Coordinate-free expression and extra properties

Let M be a manifold and W be a complete manifold endowed with a Riemannian metric h. Let $\exp : TW \to W$ be the exponential map induced by the metric h. The formula (2.1) of the Corrugation Process can be rewritten without using coordinates as follows:

Definition 29. Let $f_0 : U \to (W, h)$ be a map from an open set $U \subset M$, $\pi : U \to \mathbb{R}$ be a submersion and $\gamma : U \times \mathbb{R}/\mathbb{Z} \to f_0^*TW$ be a C^{k-1} loop family such that $\gamma(x, .) : \mathbb{R}/\mathbb{Z} \to f_0^*TW_x$ for every $x \in U$ the map defined by Corrugation Process is defined by

$$f_1 = CP_\gamma(f_0, \pi, N) : x \mapsto \exp_{f_0(x)} \frac{1}{N}\Gamma(x, N\pi(x)) \tag{2.2}$$

where $\Gamma(x, s) = \int_0^s (\gamma(x, t) - \overline{\gamma}(x))dt$.

This expression reduces to the one in Definition 27 if $M = [0, 1]^m$, $W = \mathbb{R}^n$ and $\pi(x) = x_j$. In that case $d\pi = dx_j = \langle \partial_j, \cdot\rangle$ and $\exp_{f_0(x)} y = f_0(x) + y$ for every $x \in [0, 1]^m$ and $y \in \mathbb{R}^n$. Properties (P_1), (P_2) and (P_3') may also be reformulated:

Proposition 30. *Let g be a metric of M and h be a metric on W. We denote by hs the Sasaki metric on TW. We have*

(P_1) $d_h(f_1(x), f_0(x)) = O(1/N)$ *for every* $x \in U$,

(P_2) $d_{hs}((df_1)_x(u_0), (df_0)_x(u_0)) = O(1/N)$ *for every* $u_0 \in \ker d\pi_x$ *such that* $\|u_0\|_g \leq 1$.

(P_3') $d_{hs}\Big((df_1)_x(u), \gamma(x, N\pi(x))\Big) = O(1/N)$ *for* $u \in T_xM$ *such that* $d\pi_x(u) = 1$ *and* γ *satisfies the Average Constraint* $\overline{\gamma}(x) = (df_0)_x(u)$.

Relative property.– The Corrugation Process has a nice property: the map f_1 might be equal to f_0 at points where the differential relation is satisfied. Indeed at a point x if the 1-jet of f_0 is in the differential relation and if we consider the constant loop $t \mapsto \gamma(x, t) = \overline{\gamma}(x)$, then the map $f_1 = CP_\gamma(f_0, \pi, N)$ coincides with f_0 at x. In particular, if the 1-jet of f_0 satisfies the differential relation on a set K, and coincides on K with a formal solution $\mathfrak{S}$, this property allows to not modify the map f_0 on K. That is illustrated in Figure 2.1 with a desingularization of a cone in a neighborhood of its vertex. The terminology *relative* is used in the book of Spring (see *relative theorem* [Spr98, Chap. 3, Complement 3.5 (iii)] recalled in Chapter 1 Lemma 24).

Periodicity property.– Let $f_1 = CP_\gamma(f_0, \pi, N)$. If N is a non-zero integer and $x \in U$ is such that $\pi(x) \in \mathbb{Z}$ then $f_1(x) = f_0(x)$. Indeed, the map $t \mapsto \Gamma(x, t)$ is 1-periodic and $\Gamma(x, N) = 0$.

To prove Proposition 30, we need the following lemma:

Lemma 31. *For any $u_0 \in T_xM$ such that $\|u_0\|_g \leq 1$, we have*

$$d_{hs}\Big((df_1)_x(u_0), (df_0)_x(u_0) + (\gamma(x, N\pi(x)) - \overline{\gamma}(x))d\pi_x(u_0)\Big) = O(1/N).$$

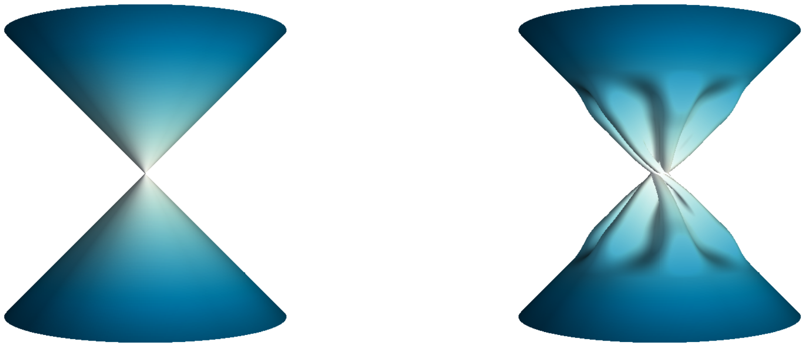


Figure 2.1 – **A cone and a desingularization of its.** Left: a cone with one singular point, Right: a local desingularization obtained by a Corrugation Process. This example is detailed in Subsection 3.2.2.

Proof of Proposition 30.– We first prove Point (P_1). By definition of f_1, we have $f_1(x) = \exp_{f_0(x)} v$ where $v = (1/N)\Gamma(x, N\pi(x))$. By definition of the exponential map, the point $\exp_{f_0(x)}(v)$ on the manifold M is given by the end point $\tau(1)$ of the constant speed geodesic $\tau : [0,1] \mapsto M$ such that $\tau(0) = f_0(x)$ and $\tau'(0) = v$. For N large, τ is a minimizing geodesic. Therefore the length of τ coincides with the distance $d_h(f_0(x), \exp_{f_0(x)}(v))$, i.e.

$$d_h(f_0(x), f_1(x)) = \ \mathit{length}(\tau) = \int_0^1 \|\tau'(t)\|_h dt.$$

Since τ has a constant speed equals to $\|\tau'(0)\|_h = \|v\|_h$, we have

$$d_h(f_0(x), f_1(x)) = \frac{1}{N}\|\Gamma(x, N\pi(x))\|_h.$$

As the map $s \mapsto \Gamma(x,s)$ is bounded, we obtain the desired property.
The point (P_2) of the proposition is proved by evaluating the bound of Lemma 31 for $u_0 \in \ker d\pi_x$, and the point (P_3') for $u \in T_xM$ such that $d\pi_x(u) = 1$ and assuming the loop family γ satisfies the Average Constraint. $\square$

Proof of Lemma 31.– We first want to have an expression of $(df_1)_x$. We consider a chart $U \subset M$ and we set

$$\begin{array}{rccc} \exp : & f_0(U)\times\mathbb{R}^n & \to & W \\ & (y,v) & \mapsto & \exp(y,v) := \exp_y(v) \end{array} \qquad \begin{array}{rccc} \psi : & U & \to & U\times\mathbb{R} \\ & x & \mapsto & (x, N\pi(x)). \end{array}$$

For any $x \in U$, we have $f_1(x) = \exp(\Phi(x))$ with $\Phi(x) := (f_0(x), \frac{1}{N}\Gamma\circ\psi(x))$. Let us calculate df_1. For any $x \in U$ we have

$$\begin{aligned} (df_1)_x &= (d\exp)_{\Phi(x)} \circ d\Phi_x \\ &= (\partial_1 \exp)_{\Phi(x)} \circ (d\Phi^1)_x + (\partial_2 \exp)_{\Phi(x)} \circ (d\Phi^2)_x \\ &= (\partial_1 \exp)_{\Phi(x)} \circ (df_0)_x + \frac{1}{N}(\partial_2 \exp)_{\Phi(x)} \circ d\left(\Gamma\circ\psi(x)\right)_x \qquad (2.3) \end{aligned}$$

where Φ^i denotes the i-th component of Φ. First note that for any $y \in f_0(U)$ we have $\exp(y,0) = y$, thus $(\partial_1 \exp)_{(y,0)}(Y,V) = Y$. Since $t \mapsto \Gamma(x,t)$ is bounded for any x, the second component Φ^2 of $\Phi(x)$ satisfies $\|\Phi^2(x)\|_h = O(1/N)$. By the C^1 continuity of exp, for any $Y \in \mathbb{R}^n$ such that $\|Y\|_h = 1$ we have

$$d_{hs}\Big((\partial_1 \exp)_{(y,\Phi^2(x))}Y, Y\Big) = d_{hs}\Big((\partial_1 \exp)_{(y,\Phi^2(x))}Y, (\partial_1 \exp)_{(y,0)}Y\Big) = O\left(\frac{1}{N}\right) \quad (2.4)$$

Now we want to give an expression of $(\partial_2 \exp)_{(y,v)}$. As before we first express $(\partial_2 \exp)_{(y,0)}$. Let $V \in \mathbb{R}^n$, then

$$(\partial_2 \exp)_{(y,0)}V = \left(\frac{d}{dt}\exp(y,tV)\right)\Big|_{t=0}.$$

If we denote by τ_{tV} the geodesic curve of constant speed tV such that $\tau_{tV}(0) = y$ and $\tau_{tV}(1) = \exp(y,tV)$, then

$$(\partial_2 \exp)_{(y,0)}V = \left(\frac{d}{dt}\tau_{tV}(1)\right)\Big|_{t=0} = \left(\frac{d}{dt}\tau_V(t)\right)\Big|_{t=0} = V.$$

By the C^1-continuity of exp, for any $V \in \mathbb{R}^n$ such that $\|V\|_{hs} = 1$ we have

$$d_{hs}\Big((\partial_2 \exp)_{(y,\Phi^2(x))}V, V\Big) = O(1/N). \quad (2.5)$$

Let us now calculate the differential of $\Gamma \circ \psi$. We can write

$$\begin{aligned} d(\Gamma \circ \psi(x))_x &= \partial_1\Gamma(x, N\pi(x)) + N\partial_2\Gamma(x, N\pi(x)) \circ d\pi_x \\ &= \partial_1\Gamma(x, N\pi(x)) + N(\gamma(x, N\pi(x)) - \overline{\gamma}(x)) \circ d\pi_x. \end{aligned}$$

Finally, by combining (2.3), (2.4) and (2.5), we have for any $u_0 \in \mathbb{R}^m$ such that $\|u_0\|_{gs} \leq 1$:

$$d_{hs}\Big((df_1)_x(u_0), (df_0)_x(u_0) + (\gamma(x, N\pi(x)) - \overline{\gamma}(x)) \circ d\pi_x(u_0)\Big) = O(1/N).$$

□

2.1.3 Quotient condition

In the previous subsection, we have seen that the Corrugation Process formula satisfies a periodicity property. In this subsection we consider manifolds given by a quotient of the form $M = \mathbb{R}^m/G$ with G a group.

Proposition 32. *Let G be a group acting on $\mathbb{R}^m$ with $[0,1]^m$ as fundamental domain. Let $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ be a G-invariant map. If for some $N \in \mathbb{R}^*$ we have the quotient condition*

$$\forall g \in G, \quad \forall x \in [0,1]^m, \quad \Gamma(x, Nx_j) = \Gamma(g \cdot x, N(g \cdot x)_j),$$

where $(g \cdot x)_j$ is the j-th component of $g \cdot x$, then $f_1 = CP_\gamma(f_0, \partial_j, N)$ is a G-invariant map.

In general, it is not enough to require $\gamma : \mathbb{R}^m \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ to be G-invariant (i.e. $\gamma(x,t) = \gamma(g \cdot x, t)$) to obtain the G-invariance for f_1. The invariance condition on γ only implies $\Gamma(x,t) = \Gamma(g \cdot x, t)$.

Proof of Proposition 32.– By the Corrugation Process we have

$$f_1(x) = f_0(x) + \frac{1}{N}\Gamma(x, Nx_j).$$

Let $g \in G$, then

$$f_1(g \cdot x) = f_0(g \cdot x) + \frac{1}{N}\Gamma(g \cdot x, N(g \cdot x)_j).$$

As f_0 is G-invariant and with the hypothesis on Γ, we obtain

$$f_1(g \cdot x) = f_0(x) + \frac{1}{N}\Gamma(x, Nx_j) = f_1(x).$$

So the map f_1 is G-invariant. $\square$

The quotient condition involves both the function Γ and the number of corrugations N. We give below two examples of G-invariant maps.

The torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$.– In that case $G = \mathbb{Z}^m$ is acting on the obvious way on $\mathbb{R}^m$. Let $f_0 : \mathbb{T}^m \to \mathbb{R}^n$. We shall show that if γ is G-invariant and N is a non zero integer, then $f_1 = CP_\gamma(f_0, \partial_j, N)$ defines a map from $\mathbb{T}^m$ to $\mathbb{R}^n$. This is a consequence of the following lemma:

Lemma 33. *If γ is $\mathbb{Z}^m$-invariant and $N \in \mathbb{Z}^*$ then the quotient condition of Proposition 32 is fulfilled.*

Proof.– Indeed, if γ is $\mathbb{Z}^m$-invariant then the map $\Gamma(\cdot, t) : \mathbb{R}^m \to \mathbb{R}^n$ is also $\mathbb{Z}^m$-invariant, i.e. $\Gamma(x,t) = \Gamma(g \cdot x, t)$. In particular $\Gamma(g \cdot x, N(g \cdot x)_j) = \Gamma(x, N(g \cdot x)_j)$. Let $g = (g_1, \ldots, g_m) \in G$. We have $(g \cdot x)_j = x_j + g_j$ and $N(x_j + g_j) = Nx_j + Ng_j$. The quotient condition to satisfy is

$$\Gamma(x, Nx_j + Ng_j) = \Gamma(x, Nx_j).$$

Since $t \mapsto \Gamma(\cdot, t)$ is 1-periodic, it is enough to choose $N \in \mathbb{Z}^*$ to fulfill the quotient condition. $\square$

The Mobius strip $\mathbb{M}^2 = \mathbb{R}^2/G$.– We assume $m = 2$ and $n = 3$. We consider the action of $G = \mathbb{Z}$ on $\mathbb{R}^2$ given by $k \cdot (x_1, x_2) = ((-1)^k x_1, x_2 + k)$. A fundamental domain for this action is $\mathbb{R} \times [0,1]$ and the quotient $\mathbb{M}^2 = \mathbb{R}^2/G$ is a Mobius strip. In particular, if f_0 is a G-invariant map for this action then its partial derivatives satisfy

$$\partial_1 f_0(k \cdot x) = (-1)^k \partial_1 f_0(x), \quad \partial_2 f_0(k \cdot x) = \partial_2 f_0(x).$$

Therefore, the action of G on the 1-jet space $J^1(\mathbb{R}^2, \mathbb{R}^3)$ is non trivial:

$$k \cdot (x, y, v_1, v_2) = (k \cdot x, y, (-1)^k v_1, v_2).$$

Thus, the presence of any term involving v_1 in the analytic expression of γ could compromise its G-invariance. This is the reason why we require a weaker hypothesis on γ, namely its $2G$-invariance. Indeed, $2G$ acts trivially on $J^1(\mathbb{R}^2, \mathbb{R}^3)$.

Lemma 34. *Let $\partial_j = \partial_2$ and γ be $2G$-invariant. If there exists $p \in \mathbb{Z}^*$ such that*

$$\forall x \in [0,1]^2, \quad \Gamma(x, \frac{px_2}{2}) = \Gamma(1 \cdot x, \frac{px_2}{2} + \frac{p}{2})$$

then the quotient condition of Proposition 32 is fulfilled for $N = p/2$.

Proof.– Let $N = p/2$, $p \in \mathbb{Z}^*$ and $k \in G$. We first assume that k is even. As $x \mapsto \gamma(x, \cdot)$ is $2G$-invariant, then $\Gamma(k \cdot x, p(x_2 + k)/2) = \Gamma(x, p(x_2 + k)/2)$. Moreover $t \mapsto \Gamma(\cdot, t)$ is 1-periodic, so $\Gamma(x, p(x_2 + k)/2) = \Gamma(x, px_2/2)$ and the quotient condition of Proposition 32 is fulfilled. In the case where k is odd, the condition given in the lemma allows to fulfill the quotient condition. $\square$

2.1.4 Comparaison with the Convex Integration formula

The Convex Integration formula and the Corrugation Process can both be used in the Convex Integration Theory. In this section, we compare the two formulas.

Proposition 35. *Let $f_0 : [0,1]^m \to \mathbb{R}^n$ be a C^k-map, $k \geq 3$, $F_1 = CI_\gamma(f_0, \partial_j, N)$ and $f_1 = CP_\gamma(f_0, \partial_j, N)$. For all $x \in [0,1]^m$, we have*

$$F_1(x) = f_1(x) - \frac{1}{N}\Big[\overline{\Gamma}(s)\Big]_{s=x\widehat{\jmath}}^{s=x} + O\left(\frac{1}{N^2}\right)$$

with the notation $\Big[\overline{\Gamma}(s)\Big]_{s=x\widehat{\jmath}}^{s=x} = \overline{\Gamma}(x_1, \ldots, x_j, \ldots, x_m) - \overline{\Gamma}(x_1, \ldots, 0, \ldots, x_m)$ *and* $\overline{\Gamma}(x) = \int_0^1 \Gamma(x,t)dt$. *In particular, for all $x \in [0,1]^m$ we have*

$$F_1(x) = f_1(x) + O(1/N).$$

The proof is postponed to the end of this subsection.

Illustration with curves.– In this paragraph, we present pictures illustrating the proximity between F_1 and f_1. We consider the case of ϵ-isometric relation for plane curves:

$$\mathscr{I}(\epsilon) = \{(x, y, v) \in J^1([0,1], \mathbb{C}) \mid r(x) - \epsilon < \|v\| < r(x) + \epsilon\}$$

where $\epsilon > 0$ and $r : [0,1] \mapsto \mathbb{R}_+$ is a given map. A formal solution $x \mapsto (x, f_0(x), v(x))$ is a subsolution if for any $x \in [0,1]$ we have $\|f_0'(x)\| < r(x) + \epsilon$. To solve this relation we introduce the loop family

$$\gamma(\cdot, t) \quad = \quad r\left(\cos(\alpha\cos(2\pi t))\frac{f_0'}{\|f_0'\|} + \sin(\alpha\cos(2\pi t))\frac{if_0'}{\|if_0'\|}\right)$$

where $\alpha(x) = J_0^{-1}(\|f_0'(x)\|/r(x))$ and J_0 denotes the Bessel function of order zero.

The image of γ is an arc of circle of amplitude 2α. The value of α is chosen such that $\overline{\gamma} = f_0'$ (see also Subsection 2.2.5, Equation (2.8)). We start with a subsolution of the form $x \mapsto (x, f_0(x), r(x)f_0'(x)/\|f_0'(x)\|)$ where f_0 is a parametrization of the circle.

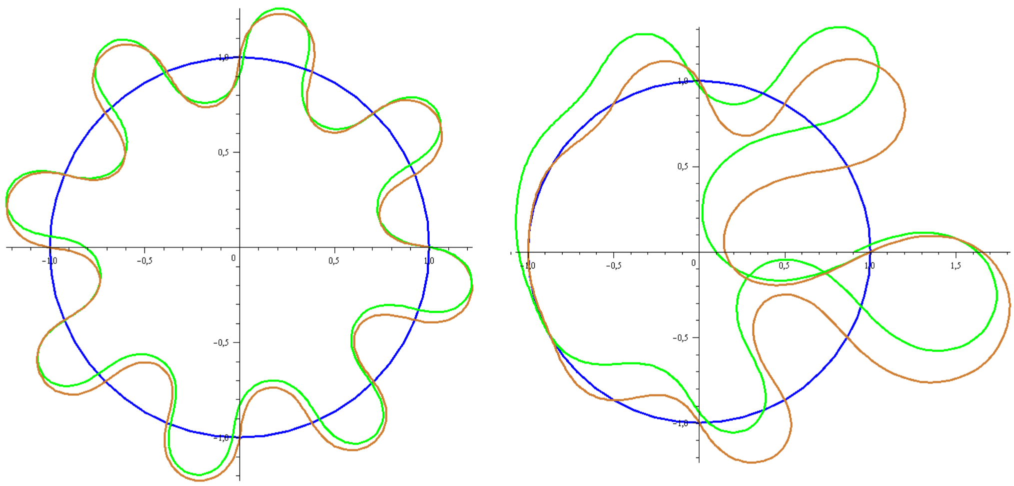


Figure 2.2 – **Comparaison of the two formulas for the ϵ isometric relation.** On the two figures, $f_0(x) = e^{i2\pi x}$ is plotted in blue, the map built by Convex Integration in green and the one built by the Corrugation Process in brown. Left: $N = 8$, $\alpha(x) = 1.4$, Right: $N = 6$, $\alpha(x) = 1 + \cos(2\pi x)$.

Note the proximity of the two curves even for small values of N. Observe that the curves obtained by the Convex Integration formula are not closed (see Figure 2.4).

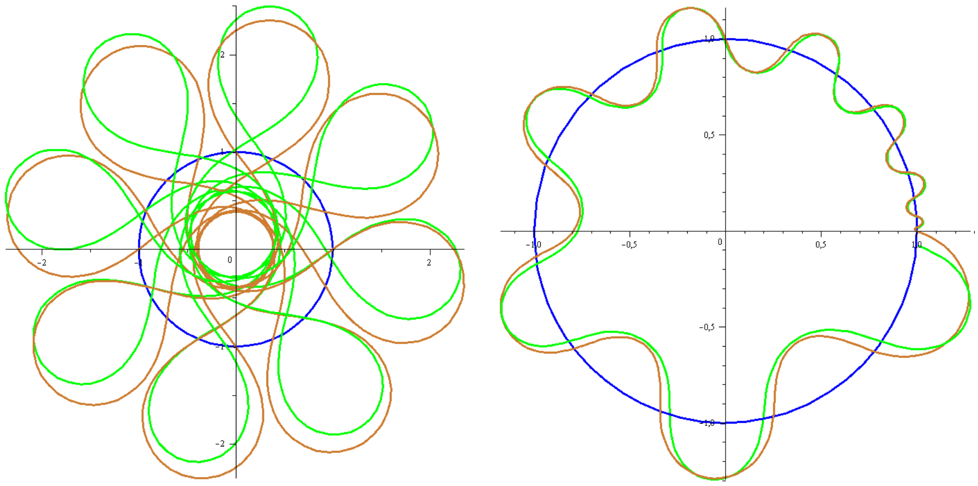


Figure 2.3 – **Comparaison of the two formulas for the ϵ isometric relation.** On the two figures, f_0 is plotted in blue, the map built by Convex Integration in green and the one built by the Corrugation Process in brown. Left: $f_0(x) = e^{i2\pi x}$, $N = 8$, $\alpha = 2.2$, Right: $f_0(x) = e^{i2\pi x^2}$, $N = 9$, $\alpha = 1.2$.

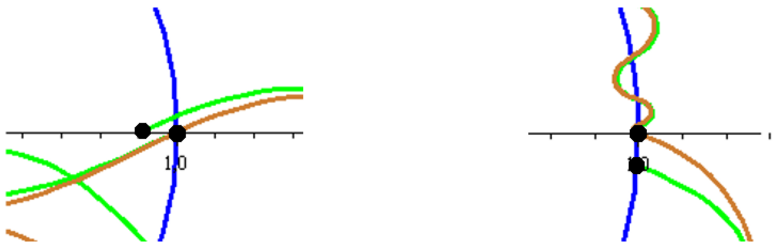


Figure 2.4 – **Periodicity.** End points of the green curve of the two previous figures ($N = 6$ and $N = 9$). The curves obtained by the Corrugation Process are closed accordingly to the Periodicity property (see Subsection 2.1.2).

Illustration with a surface.– We present a picture showing the proximity of f_1 and F_1 after three steps of corrugation (for the construction of a flat torus). The differential relation under consideration here is the relation of isometric immersion (see Subsection 2.2.5 below). In the following table, we give the computation time at each step on a processor core, with a mesh of 10^4 x 10^4:

N	**Corrugation Process**	**Convex Integration formula**
12	18s	162s
80	125s	410s
500	129s	1658s
Total:	**272s**	**2230s**

Note that the first step is really fast because, as the analytic expression of f_0 is known, the differential of f_0 is not computed with the values of f_0 on the mesh. The time difference between the two formula comes from the evaluation of the integral in the Convex Integration formula and also because we need to evaluate the map on points which are not on the initial mesh. The algorithm which uses the Corrugation Process is based on the algorithm of the Convex Integration formula. Note that, since the formula is local, the algorithm of the Corrugation Process could be parallelized by trivially splitting the mesh.

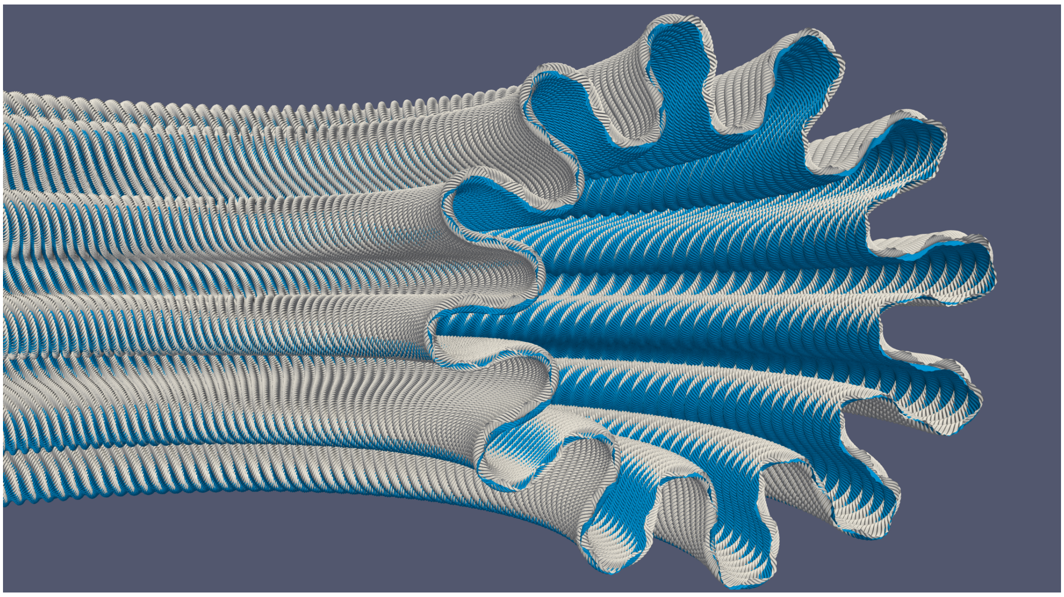


Figure 2.5 – **Comparaison of the two formulas for the flat torus.** The three first corrugations of the construction of a flat torus following [BJLT13] with the Convex Integration formula (in white) and with the Corrugation Process (in blue) (courtesy of Roland Denis).

Before proving Proposition 35, we need the two following lemmas:

Lemma 36. *Let $g : [a, b] \times \mathbb{R} \to \mathbb{R}^n$ be a continuous map which is of class C^1 with respect to the first variable. Then*

$$\int_{s=a}^{b} g(s, Ns)ds = \frac{G(b, Nb) - G(a, Na)}{N} - \frac{1}{N}\int_{s=a}^{b} (\partial_1 G)(s, Ns)ds.$$

where $G(x, t) = \int_{s=0}^{t} g(x, s)ds$ and ∂_1 denotes the partial derivative with respect to the first variable.

Note that if G and $\partial_1 G$ are bounded over $[a, b] \times \mathbb{R}$ then the lemma implies that

$$\int_{s=a}^{b} g(s, Ns)ds = O\left(\frac{1}{N}\right).$$

Proof.– Let Φ denote the map $s \mapsto \Phi(s) = (s, Ns)$. We have

$$\begin{aligned}(G \circ \Phi)'(s) &= (\partial_1 G) \circ \Phi(s) + N(\partial_2 G) \circ \Phi(s) \\ &= (\partial_1 G) \circ \Phi(s) + Ng \circ \Phi(s).\end{aligned}$$

Thus

$$g \circ \Phi(s) = \frac{1}{N}(G \circ \Phi)'(s) - \frac{1}{N}(\partial_1 G) \circ \Phi(s)$$

and

$$\int_{s=a}^{b} g \circ \Phi(s) ds = \frac{G \circ \Phi(b) - G \circ \Phi(a)}{N} - \frac{1}{N} \int_{s=a}^{b} (\partial_1 G) \circ \Phi(s) ds.$$

$\square$

Lemma 37. *Let $g : [a,b] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ be a continuous map which is of class C^2 with respect to its first variable and 1-periodic with respect to the second one. We further assume that, for all $x \in [a,b]$, the average $\overline{g}(x) = \int_{t=0}^{1} g(x,t)dt$ vanishes. Then*

$$\int_{s=a}^{b} g(s, Ns) ds = \frac{G(b, Nb) - G(a, Na)}{N} - \frac{\overline{G}(b) - \overline{G}(a)}{N} + O\left(\frac{1}{N^2}\right)$$

where $G(x,t) = \int_{s=0}^{t} g(x,s)ds$ and $\overline{G}(x) = \int_{t=0}^{1} G(x,t)dt$. The constant involved in the $O\left(1/N^2\right)$ can be taken to be $C = 4\|\partial_1 g\|_\infty + 2|b-a|\|\partial_1^2 g\|_\infty$.

Proof.– The proof reduces to apply Lemma 36 twice. First we apply the lemma to the map g to obtain

$$\int_{s=a}^{b} g(s, Ns) ds = \frac{G(b, Nb) - G(a, Na)}{N} - \frac{1}{N} \int_{s=a}^{b} (\partial_1 G)(s, Ns) ds. \tag{2.6}$$

As we have assumed $\overline{g}(x) = 0$ for all $x \in [a,b]$, the map $t \mapsto G(x,t)$ is 1-periodic too but without a vanishing average. Let h be the map defined by

$$h : (x,t) \mapsto (\partial_1 G)(x,t) - (\partial_1 \overline{G})(x)$$

and $H(x,t) = \int_{s=0}^{t} h(x,s)ds$. We apply Lemma 36 once more to the map h :

$$\int_{s=a}^{b} h(s, Ns) ds = \frac{H(b, Nb) - H(a, Na)}{N} - \frac{1}{N} \int_{s=a}^{b} (\partial_1 H)(s, Ns) ds.$$

As $\partial_1 \overline{G} = \overline{\partial_1 G}$ we have $\overline{h} = 0$, that implies the map $t \mapsto H(x,t)$ is 1-periodic. In particular, the maps $t \mapsto H(x,t)$ and $t \mapsto \partial_1 H(x,t)$ are bounded, so we can write

$$\int_{s=a}^{b} h(s, Ns) ds = O\left(\frac{1}{N}\right).$$

Since

$$\int_{s=a}^{b} (\partial_1 G)(s, Ns) ds = \int_{s=a}^{b} (\partial_1 \overline{G})(s) ds + \int_{s=a}^{b} h(s, Ns) ds$$

we deduce that

$$\int_{s=a}^{b} (\partial_1 G)(s, Ns) ds \quad = \quad \overline{G}(b) - \overline{G}(a) + O\left(\frac{1}{N}\right).$$

We report this last relation to Equation (2.6) to obtain the desired result. □

Proof of Proposition 35.– To simplify the writing of formulas, we do the proof in the one-dimensional setting ($m = 1$). The general case easily follows by considering for any multi-variable map $F : [0,1]^m \to \mathbb{R}^n$ the following one-dimensional restriction

$$x_j \longmapsto F(x_1, \dots, x_j, \dots, x_m).$$

Let $F_1 = CI(f_0, \gamma, j, N)$ with $f_0 : [0,1] \to \mathbb{R}^n$. The proof relies on the evaluation of the difference $F_1 - f_0$. Indeed, for all $x \in [0,1]$, we shall prove that

$$F_1(x) - f_0(x) = \frac{\Gamma(x, Nx)}{N} - \frac{\overline{\Gamma}(x) - \overline{\Gamma}(0)}{N} + O\left(\frac{1}{N^2}\right)$$

Since $f_1(x) = f_0(x) + \frac{\Gamma(x,Nx)}{N}$, this formula will prove the theorem. We have

$$F_1(x) - f_0(x) = \int_{s=0}^{x} F_1'(s) - f_0'(s) ds.$$

The Average Constraint (AC) implies that

$$F_1(x) - f_0(x) = \int_{s=0}^{x} F_1'(s) - \overline{\gamma}(s) ds = \int_{s=0}^{x} \gamma(s, Ns) - \overline{\gamma}(s) ds.$$

Let $g(s,t) := \gamma(s,t) - \overline{\gamma}(s)$. Since g is 1-periodic with respect to the second variable and the average $\overline{g}(s)$ vanishes for every $s \in [0,1]$, we deduce from Lemma 37 :

$$\begin{aligned} F_1(x) - f_0(x) &= \frac{\Gamma(x, Nx) - \Gamma(0,0)}{N} - \frac{\overline{\Gamma}(x) - \overline{\Gamma}(0)}{N} + O\left(\frac{1}{N^2}\right) \\ &= \frac{\Gamma(x, Nx)}{N} - \frac{\overline{\Gamma}(x) - \overline{\Gamma}(0)}{N} + O\left(\frac{1}{N^2}\right) \end{aligned}$$

with $\Gamma(s,t) = \int_{w=0}^{t} g(s,w) dw$. □

2.2 Kuiper relations

2.2.1 Surrounding loop families

Let $J^1(M, W)$ endowed with a distance function *dist* and $\mathscr{R}$ be a differential relation. Let $\sigma = (x, y, L) \in \mathscr{R}$ and $(\lambda, u) \in T_x^*M \times T_xM$ such that $\lambda(u) = 1$. In the beginning of Subsection 1.2.2, we have defined the slice $\mathscr{R}(\sigma, \partial_j)$ in the coordinate case, we give here a more general definition. We set

$$\mathscr{R}(\sigma, \lambda, u) := Conn_{L(u)}\{v \in T_yW \mid (x, y, L + (v - L(u)) \otimes \lambda) \in \mathscr{R}\}$$

where $Conn_a A$ denotes the path connected component of A that contains a. Note that the linear map $L + (v - L(u)) \otimes \lambda$ coincides with L over $\ker \lambda$ and maps u to v. We then denote by $\mathrm{IntConv}\, \mathscr{R}(\sigma, \lambda, u)$ the interior of the convex hull of $\mathscr{R}(\sigma, \lambda, u_x)$.

Definition 38. Let $U \subset M$, $\pi : U \to \mathbb{R}$ be a submersion and $u : U \to TM$ be a vector field such that $d\pi_x(u_x) = 1$. Let $x \mapsto \mathfrak{S}(x) = (x, f_0(x), L(x))$ be a formal solution of $\mathscr{R}$ over U. If for all x in U the base map $f_0 = \mathrm{bs}\, \mathfrak{S}$ satisfies

$$df_0(u_x) \in \mathrm{IntConv}\, \mathscr{R}(\mathfrak{S}(x), d\pi_x, u_x)$$

then the formal solution $\mathfrak{S}$ is called a *subsolution of $\mathscr{R}$ with respect to* $(d\pi, u)$.

In a same way, if $L(x)(u_x) = df_0(u_x)$ we say that $\mathfrak{S}$ is *holonomic with respect to* $(d\pi, u)$. The following lemma is a straightforward consequence of Proposition 30.

Lemma 39. *Let $\mathscr{R}$ be an open differential relation and let $\mathfrak{S}$ be a subsolution of $\mathscr{R}$ with respect to $(d\pi, u)$ and with base map $f_0 = \mathrm{bs}\ \mathfrak{S}$. If γ satisfies the Average Constraint with respect to f_0 in the direction u, i.e. $df_0(u_x) = \overline{\gamma}(x)$, and if*

$$\forall (x,t) \in U \times \mathbb{R}/\mathbb{Z}, \qquad \gamma(x,t) \in \mathscr{R}(\mathfrak{S}(x), d\pi_x, u_x)$$

then, for N large enough, $f_1 := CP_\gamma(f_0, \pi, N)$ satisfies

$$\forall x \in U, \qquad df_1(u_x) \in \mathscr{R}(\mathfrak{S}(x), d\pi_x, u_x).$$

Given a subsolution of a differential relation $\mathscr{R}$ with respect to $(d\pi, u)$, the Corrugation Process builds a solution of $\mathscr{R}$ in the direction u provided that the loop family γ satisfies the conditions of Lemma 39. In general finding such a loop family is an issue which depends on the topology and the geometry of $\mathscr{R}$, on the direction u and on the point x. In this section, we distinguish a class of differential relations (see Definition 40) for which the existence of such a family is uniformly given by a map $\mathfrak{\gamma}$ depending on a base point σ in $\mathscr{R}$ and a barycentric point w in $\mathrm{IntConv}\, \mathscr{R}(\sigma, d\pi_x, u_x)$.

We consider the bundle p_y^*TW over $\mathscr{R}$ induced by the projection $p_y : \mathscr{R} \to W$, $\sigma = (x, y, L) \mapsto y$, and we define

$$\mathrm{IntConv}(\mathscr{R}, d\pi, u) := \{(\sigma, w) \in p_y^*TW \mid w \in \mathrm{IntConv}\, \mathscr{R}(\sigma, d\pi_x, u_x)\}.$$

Definition 40. Let $k \geq 1$ and $\mathscr{R}$ be a differential relation of $J^1(U, W)$. We say that a loop family

$$\begin{array}{rccc} \mathfrak{\gamma}: & \mathrm{IntConv}(\mathscr{R}, d\pi, u) & \xrightarrow{C^k} & C^{k-1}(\mathbb{R}/\mathbb{Z}, TW) \\ & (\sigma, w) & \longmapsto & \mathfrak{\gamma}(\sigma, w)(\cdot) \end{array}$$

is *surrounding with respect to* $(d\pi, u)$ if for every (σ, w) we have

(1) $t \mapsto \mathscr{\gamma}(\sigma, w)(t)$ is a loop in $\mathscr{R}(\sigma, d\pi_x, u_x)$,

(2) the average of $t \mapsto \mathscr{\gamma}(\sigma, w)(t)$ is w,

(3) there exists a continuous homotopy $H : \mathrm{IntConv}(\mathscr{R}, d\pi, u) \times [0,1] \to TW$ such that $H(\sigma, w, 0) = \mathscr{\gamma}(\sigma, w)(0)$, $H(\sigma, w, 1) = L(u_x)$ and $H(\sigma, w, t) \in \mathscr{R}(\sigma, d\pi_x, u_x)$ for all $t \in [0,1]$.

If there exists such a $\mathscr{\gamma}$, we say that $\mathscr{R}$ is a *surrounding relation with respect to* $(d\pi, u)$. The third point is a homotopic properties needed to state a potential h-principle for $\mathscr{R}$. In the traditional approach of the Convex Integration Theory, it is required to have $L(u_x)$ as the base point of the loop under consideration. Here we relax this condition to a more flexible one which is however enough to ensure the h-principle for $\mathscr{R}$.

Let $\mathfrak{S}$ be a C^{k-1} subsolution of $\mathscr{R}$ with respect to $(d\pi, u)$ with base map $f_0 = \mathrm{bs}\, \mathfrak{S}$ of class C^k $(k \geq 2)$. Let $\mathscr{\gamma} : \mathrm{IntConv}(\mathscr{R}, d\pi, u) \to C^{k-1}(\mathbb{R}/\mathbb{Z}, TW)$ be a surrounding loop family with respect to $(d\pi, u)$. We define a loop family γ by

$$\gamma(x,t) := \mathscr{\gamma}(\mathfrak{S}(x), df_0(u_x))(t) \in \mathscr{R}(\sigma, d\pi_x, u_x)$$

for every $(x,t) \in U \times \mathbb{R}/\mathbb{Z}$. Note that $x \mapsto \gamma(x, \cdot)$ is C^{k-1} and $t \mapsto \gamma(\cdot, t)$ is C^{k-2}. It ensues that the map $CP_\gamma(f_0, \pi, N)$ is a C^{k-1} solution of $\mathscr{R}$ in the direction u. We denote by $CP_{\mathscr{\gamma}}(\mathfrak{S}, \pi, N)$ the map built by the Corrugation Process $CP_\gamma(f_0, \pi, N)$ given by the previous choice of γ. Observe that, by Definition 40, if $\mathfrak{S}$ and $\mathfrak{S}'$ coincide on U then the maps $CP_{\mathscr{\gamma}}(\mathfrak{S}, \pi, N)$ and $CP_{\mathscr{\gamma}}(\mathfrak{S}', \pi, N)$ coincides on U too.

Definition 41. Let $\delta>0$ and $\mathscr{R}$ be a surrounding differential relation with respect to $(d\pi, u)$. We say that a loop family $\mathscr{\gamma}_\delta : \mathrm{IntConv}(\mathscr{R}, d\pi, u) \to C^k(\mathbb{R}/\mathbb{Z}, TW)$ is *δ-relative with respect to* $(d\pi, u)$ if for every couples $(\sigma = (x, y, L), w) \in \mathrm{IntConv}(\mathscr{R}, d\pi, u)$ such that

$$L(u) = w \;\text{ and }\; dist(w, \mathscr{R}(\sigma, d\pi_x, u_x)^C) \geq \delta$$

where $\mathscr{R}(\sigma, d\pi_x, u_x)^C$ denotes the complement of $\mathscr{R}(\sigma, d\pi_x, u_x)$ in T_yW, we have $\mathscr{\gamma}_\delta(\sigma, w) \equiv w$. If for every $\delta \in \,]0, \delta_0]$ (for some $\delta_0 > 0$) there exists a δ-relative loop family $\mathscr{\gamma}_\delta$ we say that $\mathscr{\gamma} :]0, \delta_0] \times \mathrm{IntConv}(\mathscr{R}, d\pi, u) \to C^k(\mathbb{R}/\mathbb{Z}, TW)$ is *relative.*

By convention the distance function to an empty set is infinite, so if the set $\mathscr{R}(\sigma, d\pi_x, u_x)^C = \emptyset$, then the above condition on $dist(w, \mathscr{R}(\sigma, d\pi_x, u_x)^C)$ is fulfilled.

Note that if δ is greater than the diameter of $\mathscr{R}$ then the δ-relativity condition is empty. Therefore, if this diameter is not infinite, the definition is meaningful only when δ is sufficiently small.

If $dist(w, \mathscr{R}(\sigma, d\pi_x, u_x)^C) > 0$, then the point w lies inside the differential relation and is the average of the constant loop $\mathscr{Y}(\sigma, w) \equiv w$. The constant loops condition in the previous definition allows to build, from a subsolution, a holonomic section such that the two base maps coincide outside a δ-neighborhood of $\mathscr{R}(\sigma, d\pi_x, u_x)^C$. The following lemma is a direct consequence of the relative property of the Corrugation Process (cf. Subsection 2.1.2).

Lemma 42. *Let $\mathscr{R}$ be a surrounding open differential relation with respect to $(d\pi, u)$ and $x \mapsto \mathfrak{S}(x)$ be a subsolution of $\mathscr{R}$ for $(d\pi, u)$. Let $\mathscr{Y}$ be a δ-relative loop family with respect to $(d\pi, u)$ and $f_0 = \mathrm{bs}\, \mathfrak{S}$, $f_1 = CP_{\mathscr{Y}}(\mathfrak{S}, \pi, N)$. Then*

$$f_1(x) = f_0(x)$$

for every $x \in U$ such that $dist(df_0(u_x), \mathscr{R}(\sigma, d\pi_x, u_x)^C) \geq \delta$.

2.2.2 Shaped loop families and Kuiper relations

A key ingredient of the Convex Integration Theory is the Integral Representation (see Lemma 24). This lemma states that, in the conditions of the theory, there exists a loop family γ which satisfies the Average Constraint (see Definition 15). In a surrounding relation, the loop family $\mathscr{Y}$ is defined on the far larger space $\mathrm{IntConv}(\mathscr{R}, d\pi, u)$ and uniformly generates a γ for every subsolution $\mathfrak{S}$ with respect to $(d\pi, u)$. This procedure allows to simplify the expression of the Corrugation Process if the family of loops $\mathscr{Y}$ is c-shaped, i.e. built from a single closed curve $c : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^p$ seen as a pattern.

Definition 43. Let $p, r > 0$ be two natural numbers and $A \subset \mathbb{R}^r$ be a parameter space. A *loop pattern* is a map $c : A \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^p$ such that $a \mapsto c(a, \cdot)$ is C^{k-1} and $t \mapsto c(\cdot, t)$ is C^{k-2} for some $k \geq 2$.

Notation.– If c is a loop pattern, we set

$$C(a, t) := \int_{u=0}^{t} \Big(c(a, u) - \overline{c}(a)\Big) du.$$

As the loop $t \mapsto c(a, t)$ is 1-periodic and $\overline{c}$ is its average, the map $t \mapsto C(a, t)$ is 1-periodic. We denote by $(C_1(a, \cdot), \dots, C_p(a, \cdot))$ the components of $C(a, \cdot)$.

We denote by $E \to W$ the fiber bundle over W with fiber $\mathscr{L}(\mathbb{R}^p, T_yW) = (\mathbb{R}^p)^* \otimes T_yW$ over y and we consider its pull back by the projection $q : \mathrm{IntConv}(\mathscr{R}, d\pi, u) \to W$, $(\sigma, w) \mapsto y$. A section e of q^*E defines a family of linear maps $e(\sigma, w) : \mathbb{R}^p \to T_yW$.

Definition 44. Let $\mathscr{R}$ be a surrounding differential relation with respect to $(d\pi, u)$ and let $c : A \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^p$ be a loop pattern for some $k \geq 2$. A surrounding loop family $\mathscr{\gamma}: \mathrm{IntConv}(\mathscr{R}, d\pi, u) \to C^{k-2}(\mathbb{R}/\mathbb{Z}, TW)$ is said to be *c-shaped* if there exist a section $\mathscr{e}$ of $q^*E \to \mathrm{IntConv}(\mathscr{R}, d\pi, u)$ and a map $\mathbf{a} : \mathrm{IntConv}(\mathscr{R}, d\pi, u) \to A$ such that

$$\mathscr{\gamma}(\sigma, w)(t) = \mathscr{e}(\sigma, w) \circ c(\mathbf{a}(\sigma, w), t)$$

for all $((\sigma, w), t) \in \mathrm{IntConv}(\mathscr{R}, d\pi, u) \times \mathbb{R}/\mathbb{Z}$.

Notation.– If $(c_1, \ldots, c_p)$ denote the components of c in the standard basis of $\mathbb{R}^p$ and if $\mathbf{e}_1, \ldots, \mathbf{e}_p$ denote the image of this basis by $\mathscr{e}$, we write

$$\mathscr{\gamma}(\sigma, w)(t) = c(\mathbf{a}(\sigma, w), t) \cdot \mathbf{e}(\sigma, w) = \sum_{i=1}^{p} c_i(\mathbf{a}(\sigma, w), t)\, \mathbf{e}_i(\sigma, w).$$

Definition 45. Let c be a loop pattern. We say that $\mathscr{R}$ is a *Kuiper relation with respect to* $(c, d\pi, u)$ if there exists a surrounding loop family $\mathscr{\gamma}$ which is c-shaped. Moreover, if $\mathscr{\gamma}$ is relative we say that $\mathscr{R}$ is a *relative Kuiper relation.*

2.2.3 Corrugation Process with no integration

Proposition 46. *Let c be a loop pattern, $\mathscr{R}$ be an open Kuiper relation with respect to $(c, d\pi, u)$, $\mathfrak{S} = (x, f_0, L_0)$ be a subsolution and $\mathscr{\gamma}$ be a c-shaped surrounding loop family. Then $f_1 = CP_{\mathscr{\gamma}}(\mathfrak{S}, \pi, N)$ has the following analytic expression*

$$f_1(x) \quad = \quad \exp_{f_0(x)}\left(\frac{1}{N} C(a(x), N\pi(x)) \cdot e(x)\right) \tag{2.7}$$

where $a(x) := \mathbf{a}(\mathfrak{S}(x), df_0(u_x))$, $e(x) := \mathscr{e}(\mathfrak{S}(x), df_0(u_x))$ and $x \in U$. Moreover, if N is large enough, the section

$$x \mapsto \mathfrak{S}_1 := (x, f_1, L_1 = L_0 + (df_1(u_x) - L_0(u_x)) \otimes d\pi)$$

is a section in $\mathscr{R}$ holonomic with respect to $(d\pi, u)$. If $\mathscr{R}$ is δ-relative then $\mathfrak{S}_1(x) = \mathfrak{S}_0(x)$ for every $x \in U$ such that $dist(df_0(u_x), \mathscr{R}(\sigma, d\pi_x, u_x)^C) \geq \delta$.

Note that in a coordinate system the map $f_1 = CP_{\mathscr{\gamma}}(\mathfrak{S}, \partial_j, N)$ is given by

$$f_1(x) = f_0(x) + \frac{1}{N}\Big(C(a(x), Nx_j) \cdot e(x)\Big).$$

Proof of Proposition 46.– By the non-coordinate expression of the Corrugation Process of Subsection 2.1.2, we have

$$\forall x \in U, \;\; f_1(x) = \exp_{f_0(x)} \frac{1}{N} \int_0^{N\pi(x)} (\gamma(x, t) - \overline{\gamma}(x))dt.$$

The loop family γ is given by $\gamma(x,t) := \tilde{\gamma}(\mathfrak{S}(x), df_0(u_x))(t)$ and since the map $\tilde{\gamma}$ is c-shaped and surrounding, we have

$$\gamma(x,t) := c(a(x),t) \cdot e(x)$$

with $\overline{\gamma}(x) = df_0(u_x)$. Then

$$\int_{s=0}^{t} \gamma(x,s) - \overline{\gamma}(x) ds = \left(\int_{s=0}^{t} c(a(x),s) - \overline{c}(a(x)) ds \right) \cdot e(x) = C(a(x),t) \cdot e(x)$$

It ensues that, for all $x \in U$, we have

$$f_1(x) = \exp_{f_0(x)} \left(\frac{1}{N} C(a(x), N\pi(x)) \cdot e(x) \right).$$

According to property (P_3') of Proposition 30, for every $x \in U$, we have

$$d_{hs}\Big((df_1)_x(u), \gamma(x, N\pi(x)) \Big) = O(1/N)$$

where hs is the Sasaki metric associated to the metric h of W. Since $\mathscr{R}$ is open and $\gamma(\cdot,\cdot) \in \mathscr{R}(\mathfrak{S}(x), d\pi_x, u_x)$, if N is large enough, we have

$$x \mapsto (x, f_0(x), L_0 + (df_1(u_x) - L_0(u_x)) \otimes d\pi) \in \mathscr{R}$$

for all $x \in U$. According to property (P_1) of the same proposition, we also have

$$x \mapsto \mathfrak{S}_1(x) = (x, f_1(x), L_0 + (df_1(u_x) - L_0(u_x)) \otimes d\pi) \in \mathscr{R}.$$

If $\mathscr{R}$ is relative then the equality $f_1(x) = f_0(x)$ holds for all x such that $dist(df_0(u_x), \mathscr{R}(\sigma, d\pi_x, u_x)^C) \geq \delta$ (cf Lemma 42). □

2.2.4 Connection with Thurston's corrugations

In this subsection, we present the link between the corrugation used by W. Thurston in the construction of the sphere eversion *Outside In* and the Corrugation Process. The idea of Thurston is presented in [Lev95] and recalled here. From a map $f_0 : [0,1] \to \mathbb{C}$, he builds the map $x \mapsto f_1(x) = f_0(x) + \text{eight}(x)$ where

$$\text{eight}(x) = -h \sin(4\pi n x) e(x) + 2h \sin(2\pi n x) i e(x)$$

with h a scalar to choose large enough to have an immersion, n an integer and the number of oscillations, and $x \mapsto e(x)$ a map whose values are in the unit circle (see [Lev95, p 35-36]). The name of this last curve comes from the shape of its graph (see Figure 2.6). Note that the map f_1 can be expressed as

$$f_1(x) = f_0(x) + C(x, nx) \cdot e(x)$$

and, up to a factor $1/n$, has the same expression as the one of Proposition 46 for a particular c-shape. The oscillations which appeared in the construction of f_1 are named *corrugations*, that's why we have chosen the name of Corrugation Process.

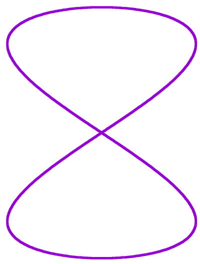


Figure 2.6 – **An eight curve** for $h = 1$, $n = 1$ and $e(x) = 1$.

2.2.5 Connection with the ansatz formula for isometric embeddings

Here we present the connection between the ansatz formula used by S. Conti, C. De Lellis and L. Székelyhidi [CDLS12] to build isometric embeddings and the Corrugation Process. To this end, we first give a short geometric description of the relation $\mathscr{I}_S$ which will be developed later.

Let $f : [0,1]^m \to (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be an immersion. The proof of the Nash-Kuiper Theorem relies on solving intermediary relations of ϵ-isometries of the form

$$\mathscr{I}_S(g,\epsilon) = \{(x, y, v_1, \dots, v_m) \mid \|g - f^*\langle\cdot,\cdot\rangle\| < \epsilon\}$$

where $\epsilon > 0$ and g is a metric on $[0,1]^m$ of the form $g = f^*\langle\cdot,\cdot\rangle + \rho\, d_1 \otimes d_1$, where $\rho : [0,1]^m \to \mathbb{R}^*_+$ (see Subsection 1.2.4). This relation is a thickening of the relation of isometries $\mathscr{I}_S(g,0)$. For short we write $\mathscr{I}_S$ instead of $\mathscr{I}_S(g,0)$.

Geometric description of $\mathscr{I}_S$.– As the description of $\mathscr{I}_S$ is easier, we just give it instead of the description of $\mathscr{I}_S(g,\epsilon)$ (see also [Gro86, p202], [Spr98, p194] or for a coordinate-free geometric description, see Section 3.4). Let $\sigma \in \mathscr{I}_S$, the space $\mathscr{I}_S(\sigma,\partial_1)$ is the intersection of the $(n-1)$-dimensional sphere of radius $\sqrt{g_{11}}$ with the affine $(n-m+1)$-plane

$$\Pi := \{v \in \mathbb{R}^n \mid \langle v, v_i\rangle = g_x(\partial_1, \partial_i)\ \forall i \in \{2, \dots, m\}\}$$

where as usual $\sigma = (x, y, v_1, \dots, v_m)$. It ensues that $\mathscr{I}_S(\sigma,\partial_1)$ is a $(n-m)$-dimensional sphere of $\mathbb{R}^n$. Let $P := Span(v_2, \dots, v_m)$. For every $v \in \mathbb{R}^n$ we write $v = [v]^P + [v]^{P^\perp}$ the decomposition of v in $P \oplus P^\perp$.

The subsolution.– We consider the section $\mathfrak{S}$ given by

$$\mathfrak{S}(x) = (x, f(x), [\partial_1 f(x)]^P + r(x)[\partial_1 f(x)]^{P^\perp}, \partial_2 f(x), \dots, \partial_m f(x))$$

where f is assumed to be an immersion, as in [Nas54, CDLS12], and $r(x)$ is such that the norm of $[\partial_1 f(x)]^P + r(x)[\partial_1 f(x)]^{P^\perp}$ equals $\sqrt{g_x(\partial_1,\partial_1)}$. If the map f is not an immersion, this construction is impossible because in

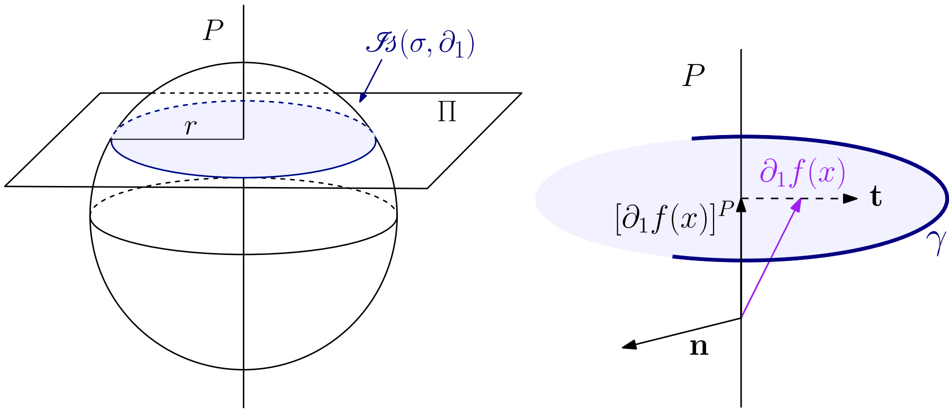


Figure 2.7 – **A geometric description of $\mathscr{I}\!s(\sigma, \partial_1)$. Left:** The slice of $\mathscr{I}\!s$ and its Convex hull, **Right:** the loop γ.

this case $[\partial_1 f(x)]^{P^\perp} = 0$. We set $\mathbf{t}(x) = [\partial_1 f(x)]^{P^\perp} / \|[\partial_1 f(x)]^{P^\perp}\|$ and $r(x) = \sqrt{g_x(\partial_1, \partial_1) - \|[\partial_1 f(x)]^P\|^2}$. By the analytical expression of g, r is well-defined and the section $\mathfrak{S}$ is a subsolution of $\mathscr{I}\!s$.

The loop family.– We consider the loop family chosen in [BJLT13] and used in Chapter 4. Every loop of this loop family has the shape of an arc of circle. Let

$$\gamma(\cdot, t) \quad := \quad r\Big(\cos(\alpha\cos(2\pi t))\mathbf{t} + \sin(\alpha\cos(2\pi t))\mathbf{n}\Big) + [\partial_1 f]^P \qquad (2.8)$$

where $\mathbf{n}$ is any unit normal to $Span(\partial_1 f, \ldots, \partial_m f)$. Observe that the existence of such a normal ensues from the contractibility of $[0,1]^m$. Remark that, for any (x,t), we have $\gamma(x,t) \in \mathscr{I}\!s(\mathfrak{S}(x), \partial_1)$. Now we have to choose α to ensure the average condition $\overline{\gamma}(x) = \partial_1 f(x)$. As

$$\int_0^1 \cos(\alpha\cos(2\pi t))dt = J_0(\alpha), \quad \int_0^1 \sin(\alpha\cos(2\pi t))dt = 0$$

where J_0 denotes the Bessel function of order 0, we choose $\alpha := J_0^{-1}(\|[\partial_1 f]^{P^\perp}\|/r)$. Note that $\|[\partial_1 f]^{P^\perp}\|/r$ lies between 0 and 1 because

$$\|[\partial_1 f(x)]^{P^\perp}\|^2 = \|\partial_1 f(x)\|^2 - \|[\partial_1 f(x)]^P\|^2 \leq g_x(\partial_1, \partial_1) - \|[\partial_1 f(x)]^P\|^2 = r(x)^2$$

so α is well-defined.

Analytic expression of the Corrugation Process.– As $r = \|[\partial_1 f]^{P^\perp}\|/J_0(\alpha)$ and by our choice of γ, it ensues the following proposition:

Proposition 47. *If N large enough, the Corrugation Process builds an ϵ-isometry $f_1 = CP_\gamma(f, \partial_1, N)$ given by*

$$f_1(x) = f(x) + \frac{\|[\partial_1 f(x)]^{P^\perp}\|}{N J_0(\alpha(x))}\Big(K_c(\alpha(x), Nx_1)\boldsymbol{t}(x) + K_s(\alpha(x), Nx_1)\boldsymbol{n}(x)\Big)$$

with

$$\begin{aligned} K_c(\alpha,t) &:= \int_{u=0}^{t} \Big(\cos(\alpha\cos 2\pi u) - J_0(\alpha)\Big)du \\ K_s(\alpha,t) &:= \int_{u=0}^{t} \sin(\alpha\cos 2\pi u)du. \end{aligned}$$

Ansatz formula.– In [CDLS12, Section 4.2] (and in [Was16, p27]) the map considered is given by

$$\widetilde{f}_1(x) = f(x) + \frac{\|[\partial_1 f(x)]^{P^\perp}\|}{N}\Big(\Gamma_1(\alpha(x),Nx_1)\mathbf{t}(x) + \Gamma_2(\alpha(x),Nx_1)\mathbf{n}(x)\Big)$$

with

$$\Gamma(\alpha,t) := \int_0^t \frac{1}{J_0(\alpha)}\begin{pmatrix}\cos(\alpha\sin 2\pi u)\\ \sin(\alpha\sin 2\pi u)\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}du.$$

It is readily checked that this formula is a Corrugation Process with a pattern given by $u\mapsto(\cos(\alpha\sin 2\pi u),\sin(\alpha\sin 2\pi u))$, which is an other parametrization of the arc of circle of the pattern of Proposition 47.

2.3 Quasi-Kuiper relations

Let $m=\dim M$ and $n=\dim W$. To define a c-shaped loop family, we usually use a normal vector field of the vector space $L(T_xM)$ given by a section $\mathfrak{S}=(x,y,L)$ (see Sections 3.1 and 3.4 for example). For example, in the case of the relation of immersions of codimension 1 and with ∂_1, ..., ∂_m a local base of $U\subset M$, a normal vector field is given by $L(\partial_1)\wedge\ldots\wedge L(\partial_m)$. In any codimension, we do not always have a formula which allows to define a normal vector fields. The question is equivalent to finding a section of the natural fibration $V_{m+1}(\mathbb{R}^n)\to V_m(\mathbb{R}^n)$. It is known (see [Bor53, Ada62, Whi63]) that such a section exists if and only if (m,n) satisfies one of the following conditions

(i)	$n=m+1$	(ii)	$m=1$ and n even
(iii)	$m=2$ and $n=7$	(iv)	$m=3$ and $n=8$.

In the other cases, the lack of a section of $V_{m+1}(\mathbb{R}^n)\to V_m(\mathbb{R}^n)$ can be sorted out by adding to the formal solution $\mathfrak{S}: M\to\mathscr{R}$ a transverse direction, i.e. a continuous map $\nu: M\to TW$ such that

$$\forall x\in M,\quad \nu(x)\notin L(T_xM)$$

or locally not in $Span(L_x(\partial_1),\ldots,L_x(\partial_m))$. Nevertheless, the map $(\mathfrak{S},\nu): M\to\mathscr{R}\times TW$ is no longer a section of the 1-jet bundle $J^1(M,W)\to M$. One way to address this problem is to consider a thickening $M\times]-\epsilon,\epsilon[$ of M and then to work with microextensions of $(\mathfrak{S},\nu)$ over this thickening. Any such microextension then defines a section of the 1-jet bundle over $M\times]-\epsilon,\epsilon[$. This motivates the definition of quasi-Kuiper relation.

We set the projection

$$\begin{array}{rccc} \widetilde{p}: & J^1(M\times]-\epsilon,\epsilon[,W) & \to & J^1(M,W) \\ & (\widetilde{x},y,L) & \mapsto & (x,y,L|_{T_xM}) \end{array}$$

where $\widetilde{x}=(x,t)$.

Definition 48. From any $\mathscr{R}\subset J^1(M,W)$ we define its normal extension to be the differential relation

$$\mathscr{R}^N := \{(\widetilde{x},y,L)\in\widetilde{p}^{-1}(\mathscr{R}) \mid L(\partial_t)\notin L(T_xM)\}.$$

In this definition, $\widetilde{p}^{-1}(\mathscr{R})$ denotes the preimage of $\mathscr{R}$ by $\widetilde{p}$, that is the subset of $J^1(M\times]-\epsilon,\epsilon[,W)$ defined by

$$\widetilde{p}^{-1}(\mathscr{R}) = \{(\widetilde{x},y,L) \mid (x,y,L|_{T_xM})\in\mathscr{R}\}.$$

A formal solution of $\mathscr{R}^N$ is thus a section $\widetilde{x}\mapsto\mathfrak{S}^\nu(\widetilde{x}) = (\widetilde{x},y(\widetilde{x}),L_{\widetilde{x}})$ where $L(\partial_t)\notin L(T_xM)$ and $(\widetilde{x},y(\widetilde{x}),L_{\widetilde{x}}|_{T_xM})\in\widetilde{p}^{-1}(\mathscr{R})$. The wanted transverse vector field is given by $\nu=L(\partial_t)$.

Example.– If $\mathscr{R}$ is the immersion relation in codimension $s\geq 1$ then $\mathscr{R}^N$ is the immersion relation in codimension $s-1$.

Let $\mathscr{R}^1\subset J^1(M\times]-\epsilon,\epsilon[,W)$ be a relation. For a submersion $\pi: U\subset M\to\mathbb{R}$ and a vector field $u: U\to TM$ such that $d\pi_x(u_x)=1$, we define $\widetilde{\pi}$ and $\widetilde{u}$ on $U\times]\epsilon,\epsilon[$ by $\widetilde{\pi}(x,t)=\pi(x)$ and $\widetilde{u}(x,t)=u(x)$. We make this choice of $\widetilde{\pi}$ and $\widetilde{u}$ to ensure that the image of $\widetilde{u}$ belongs to TM. For every subrelation $\mathscr{R}^0\subset\mathscr{R}^1$ we put

$$\begin{aligned} \mathrm{IntConv}_{\mathscr{R}^0}(\mathscr{R}^1,d\widetilde{\pi},\widetilde{u}) &:= \{(\sigma,w)\in p_y^*TW \mid w\in\mathrm{IntConv}\,\mathscr{R}^1(\sigma,d\widetilde{\pi}_x,\widetilde{u}_x),\ \sigma\in\mathscr{R}^0\} \\ &:= \bigcup_{\sigma\in\mathscr{R}^0}\mathrm{IntConv}\,\mathscr{R}^1(\sigma,d\widetilde{\pi},\widetilde{u}) \end{aligned}$$

Note that $\mathrm{IntConv}_{\mathscr{R}^0}(\mathscr{R}^1,d\widetilde{\pi},\widetilde{u})$ is a subset of the convex hull $\mathrm{IntConv}(\mathscr{R}^1,d\pi,u)$ defined in Section 2.2.1.

Definition 49. Let $\mathscr{R}\subset J^1(M,W)$, $\mathscr{R}^1=\widetilde{p}^{-1}(\mathscr{R})$ and $\mathscr{R}^0=\mathscr{R}^N$. We say that $\mathscr{R}$ is *quasi-surrounding with respect to* $(d\pi,u)$ if there exists a surrounding family $\widetilde{\gamma}:\mathrm{IntConv}_{\mathscr{R}^0}(\mathscr{R}^1,d\widetilde{\pi},\widetilde{u})\to C^k(\mathbb{R}/\mathbb{Z},\mathscr{R}^1)$. If moreover, the surrounding family is c-shaped for some loop pattern c, we say that $\mathscr{R}$ is a *quasi-Kuiper relation with respect to* $(c,d\pi,u)$.

Obviously, when $\mathscr{R}^1$ is surrounding, the restriction of the loop family $\widetilde{\gamma}$: $\mathrm{IntConv}(\mathscr{R}^1, d\widetilde{\pi}, \widetilde{u}) \to C^k(\mathbb{R}/\mathbb{Z}, \mathscr{R}^1)$ over $\mathrm{IntConv}_{\mathscr{R}^0}(\mathscr{R}^1, d\widetilde{\pi}, \widetilde{u})$ is surrounding too. The introduction of the subspace $\mathrm{IntConv}_{\mathscr{R}^0}(\mathscr{R}^1, d\widetilde{\pi}, \widetilde{u})$ is relevant when $\mathscr{R}^1$ is not surrounding: in that case, if there exists a subrelation $\mathscr{R}^0 \subset \mathscr{R}^1$ and a family of surrounding loops

$$\widetilde{\gamma} : \mathrm{IntConv}_{\mathscr{R}^0}(\mathscr{R}^1, d\widetilde{\pi}, \widetilde{u}) \longrightarrow C^k(\mathbb{R}/\mathbb{Z}, \mathscr{R}^1)$$

then we can use the machinery of Section 2.2 leading to a formula comparable to the one of Proposition 46:

Proposition 50. *Let c be a loop pattern, $\mathscr{R}$ be an open quasi-Kuiper relation of $J^1(M,W)$ with respect to $(c, d\pi, u)$, $\mathfrak{S}^\nu$ be a C^{k-1} subsolution of $\mathscr{R}^N$ with respect to $(d\widetilde{\pi}, \widetilde{u})$ with base map $f_0 : M\times]-\epsilon, \epsilon[\to W$ of class C^k. Then the analytic expression of the C^{k-1}-map $f_1 = CP_{\widetilde{\gamma}}(\mathfrak{S}^\nu, \widetilde{\pi}, N)$ writes*

$$f_1(x) \quad = \quad \exp_{f_0(x,0)}\left(\frac{1}{N} C(a(x), N\pi(x)) \cdot e(x)\right)$$

where $a(x) := \boldsymbol{a}(\mathfrak{S}^\nu(x), df_0(u_x))$, $e(x) := \boldsymbol{e}(\mathfrak{S}^\nu(x), df_0(u_x))$ and $x \in U \subset M$. Moreover, if N is large enough the section

$$x \mapsto \mathfrak{S}_1(x) := (x, f_1(x), L|_{T_xM} + (df_1(u_x) - L|_{T_xM}(u_x)) \otimes d\pi)$$

is a C^{k-2} holonomic solution of $\mathscr{R}$ with respect to $(d\pi, u)$.

Proof of Proposition 50.– The proof is an obvious adaptation of the one of Proposition 46. The only change in this case is that $\widetilde{\gamma}$ depends on $L(\partial_t)$ too. □

If $\mathscr{R}$ is the differential relation $\mathscr{I}(m,n)$ of m-dimensional immersions inside a n-dimensional manifold and if (m,n) does not satisfy one of the conditions $(i) - (iv)$, then the relation $\mathscr{R}^1 = \widetilde{p}^{-1}(\mathscr{R})$ is not surrounding (see Proposition 52) but a family of surrounding loops $\widetilde{\gamma}$ does exist over $\mathrm{IntConv}_{\mathscr{R}^0}(\mathscr{R}^1, d\widetilde{\pi}, \widetilde{u})$. Any value $\widetilde{\gamma}(\sigma, w)$ of this family is a loop starting at some point $\sigma \in \mathscr{R}^0 = \mathscr{I}(m+1, n)$, going through $\mathscr{R}^1$ to surround w and ending at σ. We will see in Chapter 3 that this implies that the relation of immersions is a Kuiper relation in codimension 1, and is a quasi-Kuiper relation for any codimension greater than 1.

Chapter 3

Immersions and isometric immersions

In this chapter, we are interested in the relation of immersions $\mathscr{I}(M, W)$ between two manifolds M and W, and the relation of isometric immersions $\mathscr{I}s(M, W)$. In the first section, we prove that the relation $\mathscr{I}(M, W)$ is a Kuiper relation in codimension 1, and is a quasi-Kuiper relation in codimension greater than one. In the second section, for pedagogical reason we desingularize the cone by the Corrugation Process. In the third section, we propose a new desingularization of the projective plane built by the Corrugation Process. We first give a short historic on this surface and on the different parametrizations and representations obtained until now, then we give a parametrization obtained by the Corrugation Process (solving the relation of immersion by the Corrugation Process for a surface closed to the crosscap). In the last section, we prove that $\mathscr{I}s(M, W)$ is a Kuiper relation in codimension 1, and is a quasi-Kuiper relation in codimension greater than one.

3.1 The relation of immersions

3.1.1 Codimension 1

Theorem 51. *Let M be an orientable manifold and W be a Riemannian orientable manifold such that $\dim W = \dim M + 1$. The relation $\mathscr{I}(M, W)$ of immersions is a relative Kuiper relation with respect to the loop pattern $c : [0, \alpha_0] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ defined by*

$$c(\alpha, t) = \Big(\cos(\alpha \cos 2\pi t) - J_0(\alpha), \sin(\alpha \cos 2\pi t), 1\Big)$$

where $J_0(\alpha) = \int_0^1 \cos(\alpha \cos 2\pi t)dt$ denotes the Bessel function of order zero and $\alpha_0 \simeq 2.4$ its first positive root.

The remainder of the section is devoted to the proof of this theorem. We put $\mathscr{I} = \mathscr{I}(M, W)$, $m = \dim M$ and $n = \dim W = m + 1$. The space $\mathscr{I}$ for codimension 1 immersions is given by

$$\mathscr{I} := \{(x, y, L) \mid \text{rank } L = m\} \subset J^1(M, W).$$

Let $\sigma = (x, y, L) \in \mathscr{I}$. For $\lambda \in T_x^*M$ and $u \in T_xM$ such that $\lambda(u) = 1$, we define the slice

$$\mathscr{I}(\sigma, \lambda, u) := Conn_{L(u)}\{v \in T_yW \mid (x, y, L_v) \in \mathscr{I}\}$$

where $L_v := L+(v-L(u))\otimes\lambda$. It is readily seen that $L_v(T_xM) = L(\ker\lambda)+\mathbb{R}v$ and thus $\mathscr{I}(\sigma, \lambda, u)$ is the complementary of $P := L(\ker\lambda)$ in T_yW. Note that as $n > m$, the convex hull $\text{IntConv}\,\mathscr{I}(\sigma, \lambda, u)$ is the whole space T_yW, so each formal solution σ is a subsolution. Let $\pi : U \subset M \to \mathbb{R}$ be a submersion and $u : U \to TM$ be such that $d\pi_x(u_x) = 1$.

Supporting plane of the loop family.– Let $(\sigma, w) \in \text{IntConv}\,\mathscr{I}(\sigma, d\pi_x, u_x)$. Since $\sigma = (x, y, L)$ is in the relation $\mathscr{I}$, the rank of L is maximal, then

$$P := L(\ker d\pi_x)$$

is a $(m-1)$-dimensional vector subspace of T_yW, thus P is of codimension 2 and $L(T_xM)$ is of codimension 1. Observe that $L(T_xM)^\perp$ inherits an orientation from the one of $L(T_xM)$ and T_yW. Let ν be the unique unit vector of $L(T_xM)^\perp$ inducing its orientation. Let Π be the subspace spanned by $L(u)$ and ν, we have $P \oplus \Pi = T_yW$. We will build a loop family $\mathscr{\gamma}$ such that $\mathscr{\gamma}(\sigma, w)$ belongs to the affine space $w + \Pi$.

Expression of the loop family.– Let $(\sigma, w) \mapsto \mathscr{e}(\sigma, w)$ be the section of $q^*E \to \text{IntConv}(\mathscr{I}, d\pi, u)$ (see Definition 44) defined by

$$\begin{array}{rccl} \mathscr{e}(\sigma, w) : & \mathbb{R}^3 & \longrightarrow & T_yW \\ & e_1 & \longmapsto & \mathbf{r}L(u)/\|L(u)\| \\ & e_2 & \longmapsto & \mathbf{r}\nu \\ & e_3 & \longmapsto & w \end{array}$$

and where $\mathbf{r} = \mathbf{r}(\sigma, w)$ is any C^{k-1} map strictly greater than the distance between w and its P-component $\{[w]^P\} = P \cap (w + \Pi)$, in particular $\mathbf{r}$ never vanishes. This condition on $\mathbf{r}$ ensures that the circle C_0 of center w and of radius $\mathbf{r}$ parametrized by

$$t \mapsto \cos(2\pi t)\mathbf{e}_1 + \sin(2\pi t)\mathbf{e}_2 + \mathbf{e}_3$$

with $\mathbf{e}_i := \mathscr{e}(\sigma, w)(e_i)$ does not intersect the vector space P (see Figure 3.1). Let α_0 the first zero of the Bessel function J_0, we consider the loop pattern given by

$$\begin{array}{rccc} c : & [0, \alpha_0] \times \mathbb{R}/\mathbb{Z} & \longrightarrow & \mathbb{R}^3 \\ & (\alpha, t) & \longmapsto & \Big(\cos(\alpha\cos 2\pi t) - J_0(\alpha), \sin(\alpha\cos 2\pi t), 1\Big) \end{array}$$

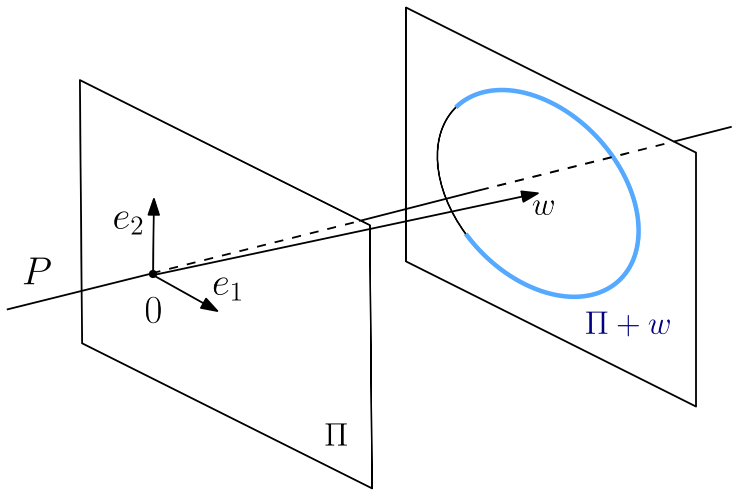


Figure 3.1 – **Proof of Theorem 51:** in blue the arc of circle of angle α_0 on the circle C_0 of center w and of radius r inside the 2-plane $\Pi + w$.

and the loop family

$$\gamma(\sigma, w)(t) := (\cos(\alpha \cos 2\pi t) - J_0(\alpha))\,\mathbf{e}_1 + \sin(\alpha \cos 2\pi t)\,\mathbf{e}_2 + \mathbf{e}_3. \qquad (3.1)$$

Note that if $\alpha = \alpha_0$, the image of γ is an arc of circle of angle α_0 whose image lies inside the circle C_0 of center $\mathbf{e}_3 = w$. Moreover if $\alpha = 0$, we have $c(0, t) = (0, 0, 1)$, then $\gamma(\sigma, w) \equiv w$.

Relative property.– Let $\delta > 0$. The image of γ is an arc of circle of angle 2α and of center $\Omega = \mathbf{e}_3 - J_0(\alpha)\mathbf{e}_1$. Our goal is to define a function α : $\mathrm{IntConv}(\mathscr{I}, d\pi_x, u_x) \to [0, \alpha_0]$ such that γ is δ-relative. To do so we introduce three subspaces of $\mathrm{IntConv}(\mathscr{I}, d\pi_x, u_x)$:

$$\begin{array}{lcl} Z_\Delta & = & \{(\sigma, w) \mid \delta \le d(w) \text{ and } w = L(u)\} \\ Z_1(\epsilon) & = & \{(\sigma, w) \mid \frac{\delta}{2} < d(w) \text{ and } dist(L(u), w) < \epsilon\} \\ Z_0 & = & Z_1(\epsilon)^C. \end{array}$$

where $d(w) := dist(w, \mathscr{I}(\sigma, d\pi_x, u_x)^C) = dist(w, P)$ and $\epsilon > 0$ will be chosen latter. We consider any smooth map $\alpha : \mathrm{IntConv}(\mathscr{I}, d\pi_x, u_x) \to [0, \alpha_0]$ such that

$$\begin{cases} \alpha(\sigma, w) = 0 & \text{if } (\sigma, w) \in Z_\Delta \\ \alpha(\sigma, w) = \alpha_0 & \text{if } (\sigma, w) \in Z_0. \end{cases}$$

By construction of α, it is immediate that the loop family γ is δ-relative. Indeed, $t \mapsto \gamma(\sigma, w)(t)$ is constant equal to w if $dist(w, P) \ge \delta$ and $w = L(u)$. Remark also that $\gamma(\sigma, w)$ is an arc of circle of angle $2\alpha_0$ if $(\sigma, w) \in Z_0$.

Proof of the surrounding property.– To prove that the loop family built with the previous pattern c and the previous map e is surrounding, we have

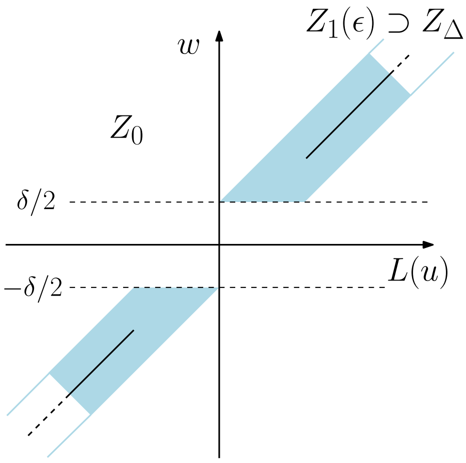


Figure 3.2 – **Values of α in** $\mathrm{IntConv}(\mathscr{I}, d\pi_x, u_x)$: $\alpha = \alpha_0$ over Z_0 and $\alpha = 0$ over Z_Δ. In between, α is a smooth interpolation.

to prove Points (1), (2) and (3) of Definition 40.

Proof of Point (1).– We first check Point (1) of Definition 40 i.e. that $t \mapsto \gamma(\sigma, w)(t)$ is a loop in $\mathscr{I}(\sigma, d\pi_x, u_x)$. Observe that if $(\sigma, w) \in Z_\Delta$ then $\gamma(\sigma, w)$ is a constant map whose value w is in $\mathscr{I}(\sigma, d\pi_x, u_x)$ by definition of Z_Δ. Observe also that over Z_0, the image of $\gamma(\sigma, w)$ lies on the circle C_0 of radius $\mathbf{r}$ and center w. The choice of r implies that this circle is included in $\mathscr{I}(\sigma, d\pi_x, u_x)$. It remains to show that if ϵ is small enough and if $(\sigma, w) \in Z_1(\epsilon)$, the image of $\gamma(\sigma, w)$ lies inside $\mathscr{I}(\sigma, d\pi_x, u_x)$. We first consider the punctured diagonal $\Delta^* = \{(L(u), L(u)), L(u) \neq 0\}$ (note that for every $(\sigma, w) \in Z_\Delta$ we have $(L(u_x), w) \in \Delta^*$). Over Δ^*, we have

$$\gamma(\sigma, w)(t) = \left(1 + \Big(\cos(\alpha \cos 2\pi t) - J_0(\alpha)\Big)\frac{\mathbf{r}}{\|L(u)\|}\right) L(u) + \sin(\alpha \cos 2\pi t)\mathbf{r}\nu$$

Observe that $\gamma(\sigma, w)(t) \notin P \iff \gamma(\sigma, w)(t) \neq 0$ and it is readily checked that $\gamma(\sigma, w)(t)$ never vanishes. By continuity, this is still true on a neighborhood of Z_Δ and thus on $Z_1(\epsilon)$ for $\epsilon > 0$ small enough (see Figure 3.2). This finishes the proof of Point (1).

Proof of Point (2).– By the definition of the Bessel function, $\overline{\gamma}(\sigma, w) = \mathbf{e}_3 = w$ and therefore γ satisfies the Average Constraint, i.e. Point (2).

Proof of Point (3).– To prove this point, we describe a canonical homotopy between the base point

$$\gamma(\sigma, w)(0) = (\cos(\alpha) - J_0(\alpha))\mathbf{e}_1 + \sin(\alpha)\mathbf{e}_2 + w$$

and $L(u)$. To do so, we distinguish two cases whether (σ, w) lies inside Z_0 or $Z_1(\epsilon)$.

Case $(\sigma, w) \in Z_0$. In that case $\alpha(L(u), w) = \alpha_0$. We first consider a homotopy $h_{1/4}$ between $\bar{\gamma}(\sigma, w)(0)$ and $\bar{\gamma}(\sigma, w)(1/4)$ given on $\bar{\gamma}$ by a homotopy between 0 and $1/4$. Remark that $\bar{\gamma}(\sigma, w)(1/4) = \mathbf{e}_1 + w$ is the point equidistant to the end points of the loop. Then we can define the obvious affine homotopy from $\bar{\gamma}(\sigma, w)(1/4)$ to $L(u)$ by setting

$$h(t) := t\,(\mathbf{e}_1 + w) + (1-t)L(u).$$

We decompose $w = [w]^P + w_1\mathbf{e}_1 + e^\perp$ with $e^\perp \in (P \oplus Span(L(u)))^\perp$. Because $\mathbf{r} > dist([w]^P, w)$, we have $\|w_1\mathbf{e}_1\| < \mathbf{r}$ and thus $|w_1| < 1$. We now can write

$$h(t) = (t + tw_1 + (1-t)\mathbf{r}^{-1}\|L(u)\|)\,\mathbf{e}_1 + te^\perp.$$

Since $t + tw_1 > 0$, the $\mathbf{e}_1$-component of $h(t)$ never vanishes so $h(t)$ is a canonical homotopy of $\mathscr{I}(\sigma, d\pi_x, u_x)$ joining the base point $\bar{\gamma}(\sigma, w)(1/4)$ to $L(u)$. It is enough to consider the homotopy $h \bullet h_{1/4}$ to conclude.

Case $(\sigma, w) \in Z_1(\epsilon)$. The previously defined homotopy $h_{1/4}$ allows to join $\bar{\gamma}(\sigma, w, \alpha)(0)$ to $\bar{\gamma}(\sigma, w, \alpha)(1/4)$ while staying in $\mathscr{I}(\sigma, d\pi_x, u_x)$. But over $Z_1(\epsilon)$, the function $J_0(\alpha)$ varies from 1 to 0 and thus $\bar{\gamma}(1/4) = (1 - J_0(\alpha))\mathbf{e}_1 + w$ is not $\mathbf{e}_1 + w$ in general. We thus introduce an extra homotopy h_{α_0} defined by

$$h_{\alpha_0}(t) := \bar{\gamma}(\sigma, w, (1-t)\alpha + t\alpha_0)(1/4)$$

to join $\bar{\gamma}(\sigma, w, \alpha)(1/4)$ to $\bar{\gamma}(\sigma, w, \alpha_0)(1/4)$. Note that our choice of ϵ in the definition of $Z_1(\epsilon)$ ensures that the image of h_{α_0} lies inside $\mathscr{I}(\sigma, d\pi_x, u_x)$. The homotopy $h \bullet h_{\alpha_0} \bullet h_{1/4}$ joins $\bar{\gamma}(\sigma, w)(0)$ to $L(u)$. This concludes Point (3) of Definition 40.

It ensues that $\bar{\gamma}$ is a relative surrounding loop family which is c-shaped. This proves Theorem 51.

3.1.2 Codimension greater than 1

We have shown that the relation of codimension one immersions is a Kuiper relation but -except for some particular dimensions- this result no longer holds in greater codimensions. The reason is the impossibility of finding a normal section to create a surrounding loop family. Indeed, the image of any loop of the surrounding families defined in Subsection 3.1.1 is lying inside an affine two dimensional plane spanned by $L(u)$ and a normal vector ν. Therefore, these surrounding families could be generalized to higher codimension provided that the natural fibration $V_{m+1}(\mathbb{R}^n) \to V_m(\mathbb{R}^n)$, where $m = \dim M$ and $n = \dim W$, admits a section. It is known (see [Whi63, Ada62, Bor53]) that such a section exists if and only if (m, n) satisfies one of the following conditions

(i)	$n = m + 1$	(ii)	$m = 1$ and n even
(iii)	$m = 2$ and $n = 7$	(iv)	$m = 3$ and $n = 8$.

Conversely, the following proposition proves that in other cases, the relation of immersions is not a Kuiper one.

Proposition 52. *The relation $\mathscr{I}(M,W)$, where M and W are orientable and $m = \dim M < \dim W = n$, is a Kuiper relation if and only if*

(i)	$n = m+1$	(ii)	$m = 1$ *and* n *even*
(iii)	$m = 2$ *and* $n = 7$	(iv)	$m = 3$ *and* $n = 8$.

The proof of this proposition relies on this lemma:

Lemma 53. *If the relation $\mathscr{I}(M,W)$ $(n > m)$ is a Kuiper relation, then we can build a continuous map*

$$\begin{array}{rccc} N: & V_m(\mathbb{R}^n) & \longrightarrow & \mathbb{E}^n \setminus \{0\} \\ & (v_1,\ldots,v_m) & \longmapsto & N(v_1,\ldots,v_m) \end{array}$$

such that $\langle N(v_1,\ldots,v_m), v_i\rangle = 0$ for every $i \in \{1,\ldots,m\}$.

Proof of Lemma 53.– If the relation $\mathscr{I}(M,W)$ is a Kuiper relation, then there exists a pattern c and a map $\mathbf{e}$ such that the loop family $\gamma : \mathrm{IntConv}(\mathscr{I}, d\pi, u) \to C^k(\mathbb{R}/\mathbb{Z}, TW)$ is given by

$$\gamma(\sigma, w)(t) = \sum_{i=1}^{p} c_i(\mathbf{a}(\sigma,w),t)\mathbf{e}_i(\sigma,w).$$

Recall that $\sigma = (x,y,L)$ and $w \in \mathrm{IntConv}\,\mathscr{I}(\sigma, d\pi_x, u_x)$ is the complementary of the space $P := L(\ker d\pi_x)$ (of $\dim m-1$) in T_yW (of $\dim n$). Since $n > m$, the convex hull $\mathrm{IntConv}\,\mathscr{I}(\sigma, d\pi_x, u_x)$ is the whole space T_yW. So there is no condition on the vector w. For any $x \in M$ and $y \in W$, we set

$$\mathscr{L}_{x,y} = \{(x,y,L,w=0) = (\sigma, w=0) |\, \mathrm{rank}\, L = m\} \simeq V_m(\mathbb{R}^n)$$

which is a subset of $\mathrm{IntConv}\,(\mathscr{I}, d\pi, u)$. This choice of set is made to avoid the case where the loop family is reduced to a point. Indeed, otherwise $\overline{\gamma}(\sigma,w) = w = L(u) = 0$ that is impossible because the rank of L is maximal. We define the map $N : \mathscr{L}_{x,y} \to T_yW$ by setting

$$N(\sigma, w=0) = \sum_{i=1}^{p} \int_{t=0}^{1} |c_i(\mathbf{a}(\sigma,0),t)| dt \;\; \mathrm{proj}^{\perp}\left(\mathbf{e}_i(\sigma,0)\right)$$

where $\mathrm{proj}^{\perp}$ is the orthogonal projection on $L(T_xM)^{\perp}$. We shall show that the map N never vanishes over $\mathscr{L}_{x,y}$. Indeed as we consider the relation of immersions, for every $t \in \mathbb{R}/\mathbb{Z}$ we have $\gamma(\sigma,0)(t) \notin P = L(\ker d\pi_x)$, and by the Average Constraint $\overline{\gamma}(\sigma,w) = w$. As we have considered $w = 0$ we have

$$\int_0^1 \mathrm{proj}_R \gamma(\sigma,0)(t) dt = 0$$

where proj_R is the orthogonal projection on $R := Span(L(u_x))$, then there exists a $t_0 \in \mathbb{R}/\mathbb{Z}$ such that $\mathrm{proj}_R \mathscr{Y}(\sigma,0)(t_0) = 0$. As for every t we have $\mathscr{Y}(\sigma,0)(t) \notin P$, at t_0 we have

$$\mathscr{Y}(\sigma,0)(t_0) \notin P \oplus R = L(T_x M)$$

so $\mathrm{proj}^{\perp} \mathscr{Y}(\sigma,0)(t_0) \neq 0$. Because

$$\mathrm{proj}^{\perp} \mathscr{Y}(\sigma,0)(t_0) = \sum_{i=1}^{p} c_i(\mathbf{a}(\sigma,0),t_0)\ \mathrm{proj}^{\perp}(\mathbf{e}_i(\sigma,0)) \neq 0$$

we deduce that $N(\sigma,0) \neq 0$. It ensues that

$$\begin{array}{ccc} \mathscr{L}_{x,y} \simeq V_m(\mathbb{R}^n) & \longrightarrow & V_{m+1}(\mathbb{R}^n) \\ (L(\partial_1),\ldots,L(\partial_m)) & \longmapsto & (L(\partial_1),\ldots,L(\partial_m),N(\sigma,0)) \end{array}$$

where $(\partial_1,\ldots,\partial_m)$ is a base of $T_x M$, provides a (continuous) section of the bundle $V_{m+1}(\mathbb{R}^n) \to V_m(\mathbb{R}^n)$. □

Proof of Proposition 52.– The relative loop family built in Subsection 3.1.1 shows that the relation of immersion is a Kuiper relation for $n = m+1$. For greater codimensions, this construction is still valid provided that we can find a continuous map

$$\begin{array}{cccc} N: & V_m(\mathbb{R}^n) & \longrightarrow & \mathbb{E}^n \setminus \{0\} \\ & (v_1,\ldots,v_m) & \longmapsto & N(v_1,\ldots,v_m) \end{array}$$

such that $\langle N(v_1,\ldots,v_m), v_i\rangle = 0$ for all $i \in \{1,\ldots,m\}$ to define the component $\mathbf{e}_3(\sigma,w)$ of the map $\mathscr{e}$. This amounts to finding a section of the natural fibration

$$\mathbb{R}^n \setminus \mathbb{R}^m \hookrightarrow V_{m+1}(\mathbb{R}^n) \longrightarrow V_m(\mathbb{R}^n).$$

It is known that such a section exists if and only if (m,n) satisfies one of the four conditions $(i)-(iv)$ (see [Whi63, Ada62, Bor53]). Thus, in these four cases the relation of immersions is a Kuiper relation. The converse follows from Lemma 53. □

In the dimension cases which do not satisfy the conditions $(i)-(iv)$, the lack of a section of $V_{m+1}(\mathbb{R}^n) \to V_m(\mathbb{R}^n)$ can be sorted out by adding to the formal solution $\mathfrak{S}: M \to \mathscr{I}$ a transverse direction, i.e. a continuous map $\nu: M \to TW$ such that

$$\forall x \in M, \quad \nu(x) \in T_{y(x)}W \setminus L(T_x M).$$

Theorem 54. *For* $\dim(M) < \dim(W)$ *and M and W orientable, the relation of immersions $\mathscr{I}(M,W)$ is a relative quasi-Kuiper relation with respect to the loop pattern $c : [0,\alpha_0] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ defined by*

$$c(\alpha,t) = \Big(\cos(\alpha\cos 2\pi t) - J_0(\alpha), \sin(\alpha\cos 2\pi t), 1\Big).$$

Proof of Theorem 54.– In the proof of Theorem 51 in Subsection 3.1.1, it is enough to define the component $\mathbf{e}_3$ of the map $\mathscr{e}$ with the transverse vector field ν given by the normal extension $\mathscr{I}^N$ of the relation of immersions. Following Section 2.3, for a formal solution $\mathfrak{S}^\nu$ of $\mathscr{I}^N$, the map ν is given by $L(\partial_t)$. □

3.1.3 Expression of the Corrugation Process formula

As we have build an explicit loop family in Subsection 3.1.1, it could be interesting to apply the Corrugation Process with this loop family to desingularize surfaces, that it the purpose of Sections 3.2 and 3.3. In all cases, according to Proposition 46, the expression of the Corrugation Process applied on any subsolution $\mathfrak{S}$ is given by

$$CP_\gamma(f_0, \pi, N)(x) \quad = \quad \exp_{f_0(x)}\left(\frac{1}{N}C(\alpha(x), N\pi(x)) \cdot e(x)\right)$$

where $f_0 = bs\,\mathfrak{S} : M \to W$, $e(x) := \mathbf{e}(\mathfrak{S}(x), df_0(u_x))$ and

$$\begin{aligned} C(\alpha, t) &= \int_{u=0}^{t} \Big(c(\alpha, u) - (0,0,1)\Big)du \\ &= \left(\int_{u=0}^{t} \Big(\cos(\alpha\cos 2\pi u) - J_0(\alpha)\Big)du, \int_{u=0}^{t} \sin(\alpha \cos 2\pi u)du, 0\right) \\ &= (K_c(\alpha, t), K_s(\alpha, t), 0). \end{aligned}$$

Observe that K_c is $\frac{1}{2}$-periodic in t and that $K_s(\alpha, t + \frac{p}{2}) = (-1)^p K_s(\alpha, t)$ for $p \in \mathbb{N}$.

3.2 A toy example: the cone

In this section and the following one, we write a section $\mathfrak{S} = (x, y, L)$ under the form $x \mapsto (x, y(x), v_1(x), v_2(x))$ where $v_1 = L(\partial_1)$, $v_2 = L(\partial_2)$ and (∂_1, ∂_2) is the canonical base of $\mathbb{R}^2$.

3.2.1 A global desingularization of a truncated cone

We first test the Corrugation Process formula with a naive loop family. As the formula is defined for compact sets, we consider here a truncated cone.

Initial map.– Let $f_0 : \mathbb{R} \times [-1, 1] \to \mathbb{R}^3$ be the parametrization of a truncated cone given by

$$f_0(x_1, x_2) = (x_2 \cos(2\pi x_1), x_2 \sin(2\pi x_1), x_2).$$

Under the obvious action of $\mathbb{Z}$ on $\mathbb{R}$, the map f_0 induces a map from $\mathbb{T}^1 \times [-1, 1]$ to $\mathbb{R}^3$ (which is still denoted by f_0).

Direction.– The singular point of the is the vertex of the cone given by the image of the segment $[0, 1] \times \{0\}$ by f_0 because the partial derivative

$$\partial_1 f_0(x_1, x_2) = (-2\pi x_2 \sin(2\pi x_1),\ 2\pi x_2 \cos(2\pi x_1),\ 0)$$

vanishes on this segment. Since the second partial derivative

$$\partial_2 f_0(x_1, x_2) = (\cos(2\pi x_1),\ \sin(2\pi x_1),\ 1)$$

never vanishes, we shall use the Corrugation Process in the direction $u = \partial_1$.

Subsolution.– Let $\mathfrak{S}$ be the subsolution given by

$$x = (x_1, x_2) \longmapsto \mathfrak{S}(x) = (x, f_0(x), v_1(x), \partial_2 f_0(x))$$

with $v_1(x) = (-\sin(2\pi x_1), \cos(2\pi x_1), 0)$.

Loop family.– Following the construction of a loop family in Subsection 3.1.1, we set the vector fields

$$\begin{array}{rcl} e_1(x) & := & r(x)\dfrac{v_1(x)}{\|v_1(x)\|} \\ e_2(x) & := & r(x)\dfrac{v_1(x) \wedge v_2(x)}{\|v_1(x) \wedge v_2(x)\|} = r(x)\dfrac{1}{\sqrt{2}}(\cos(2\pi x_1), \sin(2\pi x_1), -1) \end{array}$$

and we have to choose a function $\mathbf{r}$ such that $\mathbf{r}(x)$ is greater than the distance between the point $\partial_1 f_0(x)$ and the vector space $Span(\partial_2 f_0(x))$. As the vectors $\partial_1 f_0(x)$ and $\partial_2 f_0(x)$ are orthogonal, the previous distance is equal to $\|\partial_1 f_0(x)\| = 2\pi |x_2|$, and we choose

$$\mathbf{r}(\mathfrak{S}(x), \partial_1 f_0(x)) = 2\pi x_2^2 + 2\pi.$$

We have seen in Subsection 3.1.1 that the circle C_0 of center w lies inside $\mathscr{I}(\sigma, d\pi_x, u_x)$. We then set the naive loop family

$$\gamma(\sigma, w)(t) := \cos(2\pi t)\mathbf{e}_1 + \sin(2\pi t)\mathbf{e}_2 + \mathbf{e}_3$$

given by a parametrization of C_0. We can prove that is a surrounding family which is non-relative, i.e. we cannot reduce it to a point without intersecting $\mathscr{I}(\sigma, d\pi_x, u_x)^C$. In this case, the map C writes:

$$C(t) = \int_{u=0}^{t} \Big(c(u) - (0, 0, 1)\Big) du = \left(\frac{\sin(2\pi t)}{2\pi}, \frac{1 - \cos(2\pi t)}{2\pi}, 0\right).$$

Quotient condition.– As the map f_0 is $\mathbb{Z}$-invariant on x_1, and as the loop family $\gamma(\mathfrak{S}(x), \partial_1 f_0(x))$ is $\mathbb{Z}$-invariant in x_1, if N is a non zero integer, the

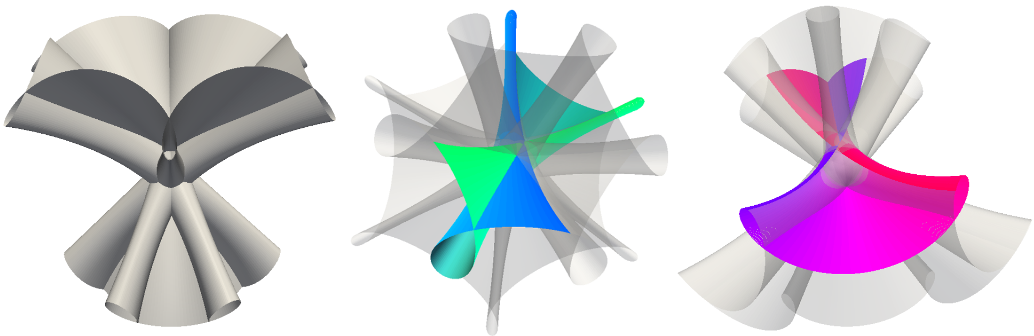


Figure 3.3 – **Desingularisation of a truncated cone** obtained by a Corrugation Process, with $N = 6$.

map built by a Corrugation Process will be $\mathbb{Z}$-invariant in x_1 too (see Subsection 2.1.3). So to desingularize f_0, we choose such a N.

Desingularized map.– By the Corrugation Process, the map f_1 is given by

$$\begin{aligned} f_1(x) &= x_2 \begin{pmatrix} \cos(2\pi x_1) \\ \sin(2\pi x_1) \\ 1 \end{pmatrix} + \frac{x_2^2+1}{N}\sin(2\pi N x_1)\begin{pmatrix} -\sin(2\pi x_1) \\ \cos(2\pi x_1) \\ 0 \end{pmatrix} \\ &+ \frac{x_2^2+1}{\sqrt{2}N}(1-\cos(2\pi N x_1))\begin{pmatrix} \cos(2\pi x_1) \\ \sin(2\pi x_1) \\ -1 \end{pmatrix} \end{aligned}$$

for every $x \in [0,1] \times [-1,1]$. According to Proposition 46 the map f_1 is an immersion if N is large enough (a direct computation shows that it is indeed the case provided that $N > 2$). Note that here only one Corrugation Process is needed since the initial subsolution $\mathfrak{S}$ is already holonomic in the direction ∂_2.

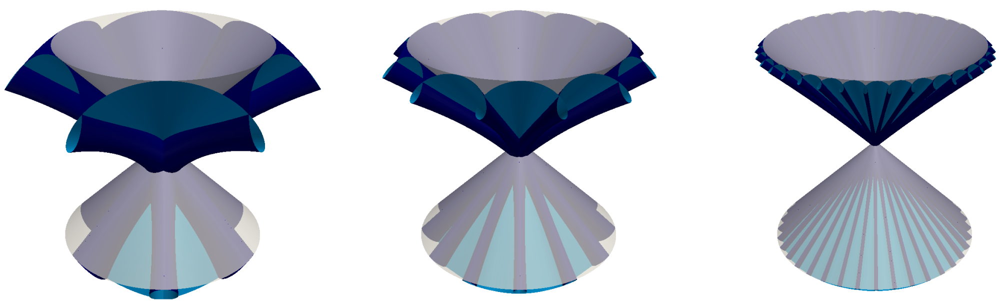


Figure 3.4 – **The C^0-density phenomenon:** Corrugation Process with $N = 6,\ 12,\ 36$ (see point (P_1) of Proposition 28).

3.2.2 A local desingularization of the cone

In this subsection, we test the relative loop family built in Subsection 3.1.1 to desingularize locally the cone.

Initial map and direction.– We consider the same initial map f_0 and the same direction $u = \partial_1$ as in the previous subsection but we enlarge the domain of f_0 to be $D := [0,1] \times [-3,3]$. Moreover we set

$$K := [0,1] \times \Big([-3,-2] \cup [2,3]\Big) \text{ and } \mathfrak{Op}(K) := [0,1] \times \Big([-3,-1[\, \cup\,]1,3]\Big).$$

We will build a section $\mathfrak{S} : D \mapsto \mathscr{I}$ coinciding with $j^1 f_0$ on $\mathfrak{Op}(K)$.

Subsolution.– We first consider the family of oriented planes

$$\Pi_1(x_1, \pm 1) = Span(\partial_1 f_0, \partial_1 f_0 \wedge \partial_2 f_0)(x_1, \pm 1).$$

This family has the property that $\Pi_1(x_1, 1) = \Pi_1(x_1, -1)$ for every $x_1 \in [0,1]$. We then denote by $R_{x_1,\theta}$ the rotation in the plane $\Pi_1(x_1, -1)$ of angle θ. Let $\theta : [-3,3] \to [0,\pi]$ be any smooth map such that $\theta \equiv 0$ on $[-3,-1]$ and $\theta \equiv \pi$ on $[1,3]$. We set $v_1(x) := R_{x_1,\theta(x_2)}(\partial_1 f_0(x_1,-1))$ and $v_2(x) := \partial_2 f_0(x)$. Observe that

$$v_1(x_1, 1) = R_{x_1,\theta(1)}(\partial_1 f_0(x_1,-1)) = -v_1(x_1,-1)$$

since $\theta(1) = \pi$. Thus v_1 is continuous on $[0,1] \times [-3,3]$. The map $x \mapsto \mathfrak{S}(x) :=$

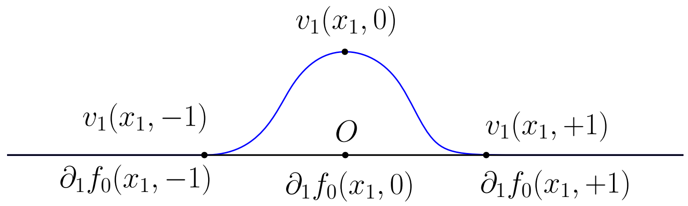


Figure 3.5 – **Construction of v_1:** for a given x_1, the image of the map $x_2 \mapsto \partial_1 f_0(x_1, x_2)$ is a right line crossing zero and the image of $x_2 \mapsto v_1(x_1, x_2)$ (in blue) coincides with the first map for $x_2 \leq -1$ or $x_2 \geq 1$.

$(x, f_0(x), v_1(x), \partial_2 f_0(x))$ is a continuous section of $\mathscr{I}$ which coincides with $j^1 f_0$ on $\mathfrak{Op}(K)$.

Relative loop family.– From the relative loop family $\tilde{\gamma}$ of Subsection 3.1.1, we derive the following expression

$$\begin{aligned} \gamma(x,t) \;&:=\; \Big(\cos(\alpha(x)\cos 2\pi t) - J_0(\alpha(x))\Big) e_1(x) \\ &\quad + \sin(\alpha(x)\cos 2\pi t) e_2(x) + \partial_1 f_0(x) \end{aligned}$$

with

$$r := \sup_{x \in D \setminus \mathfrak{Op}(K)} \|\partial_1 f_0(x)\| = 2\pi, \; e_1 := r\frac{v_1}{\|v_1\|}, \; e_2 := r\frac{v_1 \wedge v_2}{\|v_1 \wedge v_2\|}$$

and α is any smooth interpolating function such that $\alpha \equiv 0$ on K and $\alpha \equiv \alpha_0$ on $D \setminus \mathfrak{Op}(K) = [0,1] \times [-1,1]$.

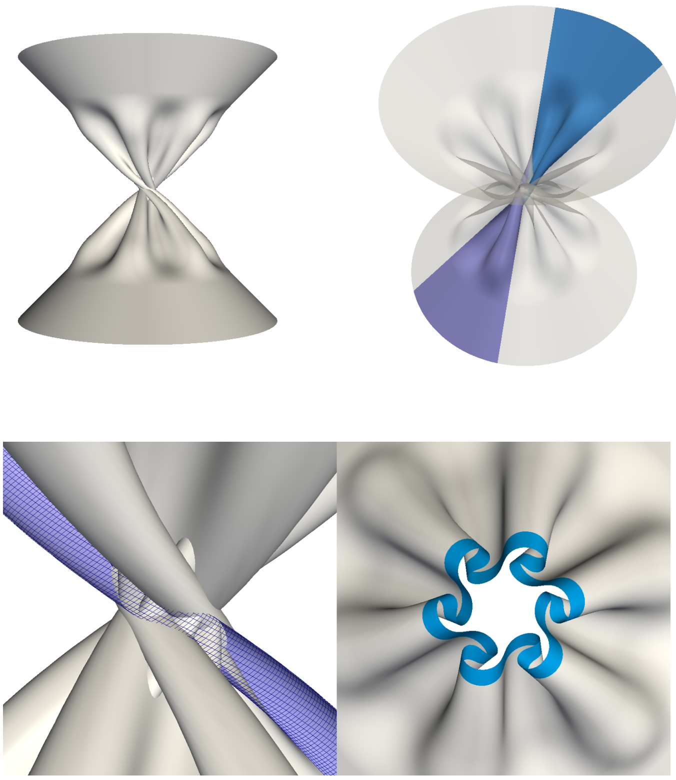


Figure 3.6 – **Local desingularization of a cone** obtained by a Corrugation Process with $\theta(x_2) = 0.5\pi(\sin(0.5\pi x_2) + 1$, and $\alpha(x_2) = \frac{\alpha_0}{2}(\cos(\pi x_2 + \pi) + 1)$ on $[-1,1]$. In the pictures, $N = 6$.

Desingularized map.– The map $f_1 = CP_{\widetilde{\gamma}}(\mathfrak{S}, \partial_1, N)$ is given by

$$f_1(x) = f_0(x) + \frac{1}{N}K_c(\alpha(x), Nx_1)e_1(x) + \frac{1}{N}K_s(\alpha(x), Nx_1)e_2(x)$$

where K_c and K_s are the functions defined in Subsection 3.1.3. By construction, f_1 and f_0 coincides on K. According to Proposition 46, and since $\mathfrak{S}$ is holonomic in the direction ∂_2, the map f_1 is an immersion if N is large enough.

3.3 New Immersions of $\mathbb{R}P^2$

Here we propose an explicit construction of an immersion of $\mathbb{R}P^2$. We choose a singular representation of it, and we desingularize it by the Corrugation Process. One of the representation with singular points of the real projective plane is the Plücker's conoid because we can view this surface as a sphere inversion of the cross-cap (the definition of an inversion is recalled in Subsection 3.3.3). Then in this section we propose an immersion of $\mathbb{R}P^2$ using the Corrugation Process to desingularize the Plücker's conoid.

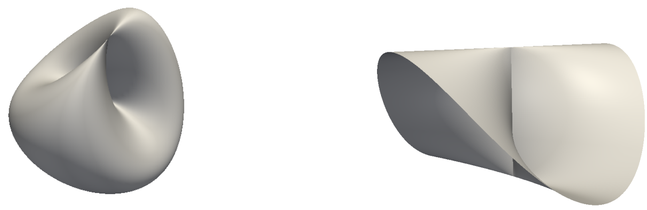


Figure 3.7 – **Cross-cap and (truncated) Plücker's conoid**

3.3.1 A short history of immersions of the projective plane

The projective plane $\mathbb{R}P^2$ is the quotient of the sphere by the antipodal relation $\mathbb{S}^2/(x \sim -x)$. It is a non-orientable compact surface without boundary and consequently cannot be embedded in $\mathbb{R}^3$. In 1844, J. Steiner built a map from $\mathbb{R}P^2$ to $\mathbb{R}^3$ which fails to be an immersion at only 6 points: the Roman surface.

In 1901, W. Boy, with the help of a sequence of drawings, described an immersion of the projective plane now called the Boy surface [Boy01]. However the question of finding an explicit analytic expression for the immersion remained open, as mentioned by Cohen-Vossen and D. Hilbert in [HCV32][1]. Many years after, in 1978 B. Morin [Mor78] gives an immersion of the Boy's surface as a central model of a sphere eversion, but his parametrization is C^1 and not C^2 at one point. Shortly after, J-P. Petit, who was working together with Morin in the construction of the sphere eversion of Morin-Petit, realized a model of $\mathbb{R}P^2$ as a family of ellipses made of iron wires. By an empirical evaluation of the parameters, he gives in 1981 with J. Souriau an other parametrization of $\mathbb{R}P^2$ [PS81]. Unfortunately the article does not contained any formal proof.

In 1986 F. Apéry builds a smooth immersion of $\mathbb{R}P^2$ [Ape86]. His parametrization is given by

$$f(x,y,z) = \begin{bmatrix} ((2x^2-y^2-z^2)(x^2+y^2+z^2)+2yz(y^2-z^2)+xy(y^2-x^2))/2 \\ ((y^2-z^2)(x^2+y^2+z^2)+zx(z^2-x^2)+xy(y^2-x^2))\sqrt{3}/2 \\ (x+y+z)((x+y+z)^3/4+(y-x)(z-y)(x-z)) \end{bmatrix}$$

where $(x,y,z) \in \mathbb{S}^2$. Note that F. Apéry proposes a detailed history of the immersions of $\mathbb{R}P^2$ unitl 1986 in [Ape87]. In 1987, based on a work of R. Bryant [Bry84], R. Kusner gives a parametrization of $\mathbb{R}P^2$ which achieves the

[1] At the end of §48 in its english translation *Geometry and the imagination.*

minimum of the Willmore functional [Kus87] among all immersions of $\mathbb{R}P^2$ into $\mathbb{R}^3$, the so called Bryant-Kusner immersion. Recently, S. Goodman and G. Howard had explored another immersion of the projective plane that they called the Girl's surface. This immersion has a unique triple point but its image is not diffeomorphic to the Boy's surface (see [GK09, GH12]).

In the following, we describe a new explicit immersion of $\mathbb{R}P^2$ built by the Corrugation Process. Note that the projective plane can be viewed as a truncated Plücker's conoid closed by a disk. Then we first desingularize locally the Plücker's conoid and we close it with a disk.

3.3.2 A local desingularization of the Plücker's conoid

Initial map.– We consider the following parametrization $f_0 : D = [-3,3] \times [0,1] \to \mathbb{R}^3$ of the Plücker's conoid

$$f_0(x_1,x_2) = \left(x_1\cos(\pi x_2),\ x_1\sin(\pi x_2),\ \frac{1}{2}\cos(2\pi x_2)\right).$$

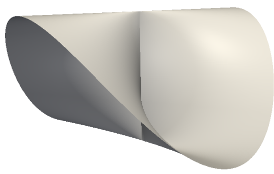


Figure 3.8 – **Plücker's conoid**

Direction.– The map f_0 has two singular points $x=(0,0)$ and $x=(0,\frac{1}{2})$ on which $\partial_2 f_0$ is zero. The removal of these singularities will be performed by a relative Corrugation Process in the direction $u=\partial_2$. Let

$$K := \Big([-3,-2] \cup\ [2,3]\Big) \times [0,1] \ \text{ and } \ \mathfrak{Op}(K) := \Big([-3,-1[\cup\,]1,3]\Big) \times [0,1].$$

Precisely, we shall built a section $\mathfrak{S} : D \to \mathscr{I}$ coinciding with $j^1 f_0$ on $\mathfrak{Op}(K)$.

Subsolution.– For each $x_2 \in [0,1]$, we consider the rotation $R_{x_2,\theta}$ of angle θ in the oriented plane

$$\Pi_2(-1,x_2) := Span(\partial_2 f_0(-1,x_2),(\partial_2 f_0 \wedge \partial_1 f_0)(-1,x_2)).$$

Let θ be a smooth map such that $x_1 \mapsto \theta(x_1,x_2)$ is an interpolation between 0 and $\theta_{max}(x_2)$ where

$$\theta_{max}(x_2) := \mathrm{angle}(\partial_2 f_0(-1,x_2),\partial_2 f_0(1,x_2)).$$

We define v_1 and v_2 to be

$$\begin{aligned} v_1(x_1,x_2) &:= \partial_1 f_0(x_1,x_2),\\ v_2(x_1,x_2) &:= R_{x_2,\theta(x_1,x_2)}(\partial_2 f_0(-1,x_2)) \end{aligned}$$

on $D\backslash\mathfrak{Op}(K)$ and to be $v_i = \partial_i f_0$ elsewhere. Since $\partial_2 f_0(1,x_2)$ is in the plane $\Pi_2(-1,x_2)$, the map v_2 is continuous on D. It implies that $x \mapsto \mathfrak{S}(x) = (x, f_0(x), \partial_1 f_0(x), v_2(x))$ is a continuous section of $\mathscr{I}$ which coincide with $j^1 f_0$ over $\mathfrak{Op}(K)$.

Relative loop family.– We set

$$r := \sup_{x\in D\backslash\mathfrak{Op}(K)} \|\partial_2 f_0(x)\| + \frac{1}{2} = \sqrt{2}\pi + \frac{1}{2},\ e_1 := r\frac{v_2}{\|v_2\|},\quad e_2 := r\frac{v_2\wedge v_1}{\|v_2\wedge v_1\|}.$$

Let α be any interpolating smooth function such that $\alpha \equiv 0$ on K and $\alpha \equiv \alpha_0$ on $D \setminus \mathfrak{Op}(K)$. From the relative loop family $\tilde{\gamma}$ of Subsection 3.1.1, we derive the following expression for γ:

$$\begin{aligned} \gamma(x,t) := &\Big(\cos(\alpha(x)\cos 2\pi t) - J_0(\alpha(x))\Big)e_1(x)\\ &+\sin(\alpha(x)\cos 2\pi t)e_2(x) + \partial_2 f_0(x). \end{aligned}$$

Quotient condition.– We consider the action of $G = \mathbb{Z}$ on $[-3,3]\times\mathbb{R}$ given by $k\cdot(x_1,x_2) = ((-1)^k x_1, x_2+k)$. A fundamental domain for this action is $[-3,3]\times[0,1]$ and its quotient $\mathbb{M}^2 = \mathbb{R}^2/G$ is a Mobius strip. Observe that the parametrization f_0 is G-invariant and thus descends to the quotient. It is readily seen that $f_1 = CP_\gamma(f_0,\partial_2,N)$ descends to the quotient if

$$\Gamma(1\cdot x, Nx_2+N) = \Gamma(x, Nx_2). \tag{3.2}$$

By definition of v_1 and v_2, we have

$$e_1(k\cdot x) = e_1(x),\ \ e_2(k\cdot x) = (-1)^k e_2(x).$$

Thus, if we choose α to be G-invariant, we obtain

$$\begin{aligned} \Gamma\left(1\cdot x, N(x_2+1)\right) &= K_c\left(\alpha(x), N(x_2+1)\right)e_1(1\cdot x) + K_s\left(\alpha(x), N(x_2+1)\right)e_2(1\cdot x)\\ &= K_c\left(\alpha(x), N(x_2+1)\right)e_1(x) - K_s\left(\alpha(x), N(x_2+1)\right)e_2(x) \end{aligned}$$

Since $K_c(\alpha, t+\frac{p}{2}) = K_c(\alpha,t)$ and $K_s(\alpha, t+\frac{p}{2}) = (-1)^p K_s(\alpha,t)$, it is enough to choose $N \in \mathbb{N}+1/2$ to fulfill Condition 3.2 (see Subsection 2.1.3).

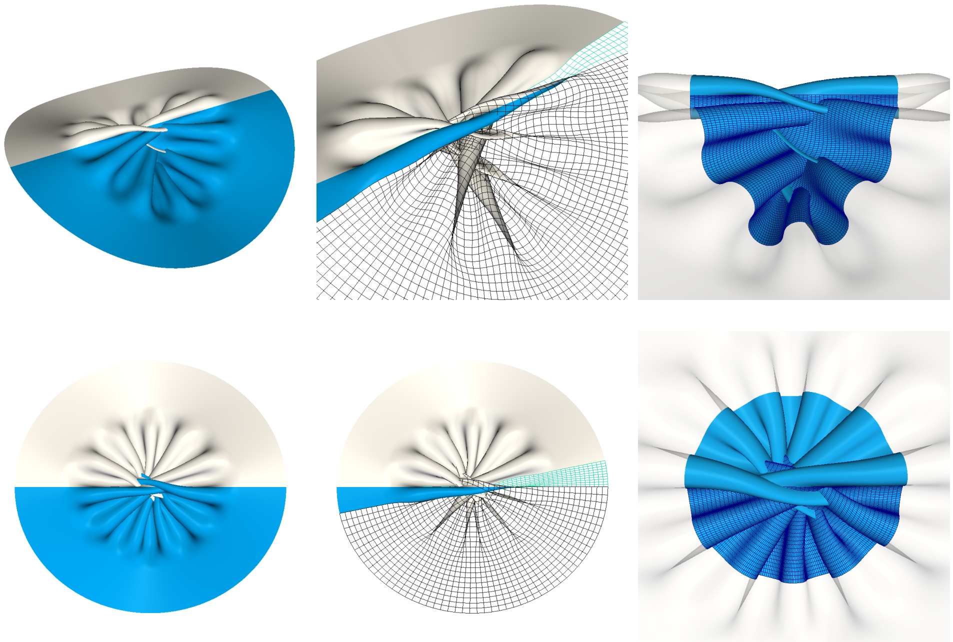
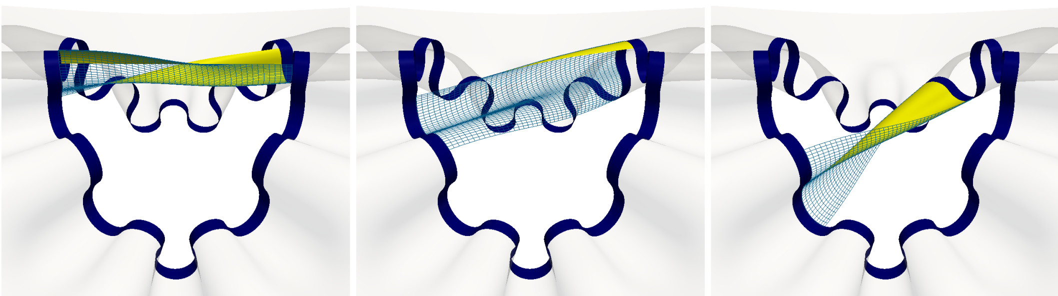


Figure 3.9 – **Desingularization of the Plücker's Conoid** obtained as an image of $\mathbb{M}^2$ by f_1 with $N = 5.5$ and $\theta(x_1, x_2) = 0.5\ (\sin(0.5\pi x_1) + 1)\theta_{max}(x_2)$, $\alpha(x) = \frac{\alpha_0}{2}(\cos(\pi x_1 + \pi) + 1)$. Note that θ and α are not C^∞ on the boundary of the set where this map are constant. So here f_1 is only C^1.

Desingularized map.– This map is given by

$$f_1(x) = f_0(x) + \frac{1}{N} K_c(\alpha(x), Nx_2)e_1(x) + \frac{1}{N} K_s(\alpha(x), Nx_2)e_2(x) \tag{3.3}$$

for all $x \in D$. The maps K_c and K_s are the functions defined in Subsection 3.1.3. According to Proposition 46 the map f_1 is an immersion if N is large enough.



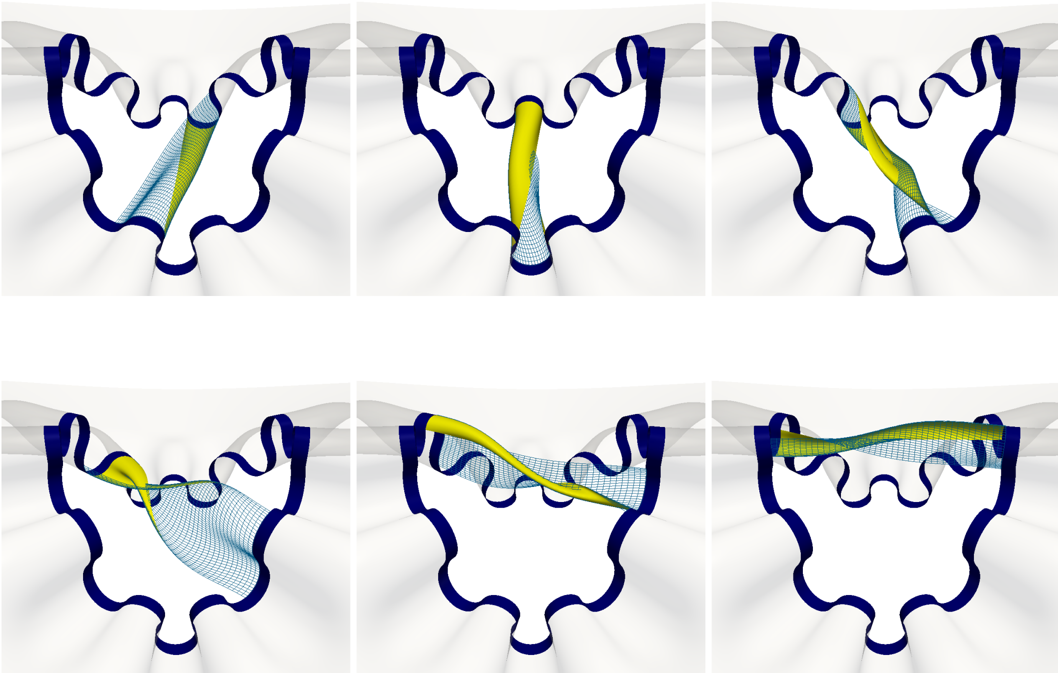


Figure 3.10 – **The spinning of corrugation:** note the similarity with the "tobacco pouch" surfaces of [Fra87, p 115].

3.3.3 Immersions of $\mathbb{R}P^2$ by the Corrugation Process

We have seen that only one of the two partial derivatives of the Plücker's conoid vanishes, so only one Corrugation Process was needed to desingularize it. Nevertheless the Plücker's conoid is not bounded. To build a bounded immersion of $\mathbb{R}P^2$ we have two choices: use an inversion or close the desingularized truncated Plücker's conoid with a disk. Note that we do not have desingularize directly the cross-cap (which can be obtained by an inversion of the Plücker's conoid) because its two partial derivatives vanish, so two Corrugation Process will be needed.

Immersions of $\mathbb{R}P^2$ via an inversion.– First recall that the inversion of center O_I and radius k is given on $\mathbb{E}^3 \cup \{\infty\}$ by

$$\begin{array}{rccc} I_{O_I,k}: & \mathbb{E}^3 \cup \{\infty\} & \longrightarrow & \mathbb{E}^3 \cup \{\infty\} \\ & O_I & \longmapsto & \infty \\ & \infty & \longmapsto & O_I \\ & M & \longmapsto & M' = O_I + \dfrac{k}{O_I M^2}\overrightarrow{O_I M}. \end{array}$$

Let us consider the entire Plücker conoid, that is the image of f_0 on the set $\mathbb{R} \times [0,1]$ (where f_0 is well-defined). We then consider $\widetilde{f_1} := f_1$ on D and $\widetilde{f_1} = f_0$ on $(\mathbb{R} \times [0,1]) \backslash D$. To obtain an immersion of the real projective space it is enough to apply an inversion to $\widetilde{f_1}$ and to take its closure, see Figures 3.11, 3.12 and 3.13.

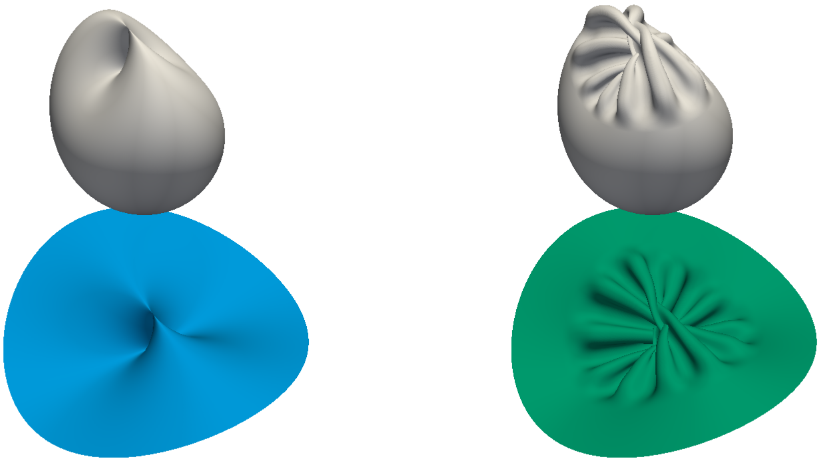


Figure 3.11 – **Immersions of $\mathbb{R}P^2$ via an inversion.** Left: the center of a Plücker's Conoid and its inversion for $O_I = (0,0,4)$ and $k = -12$. Right: the center of a desinfularized Plücker's conoid and its inversion with the same parameters.

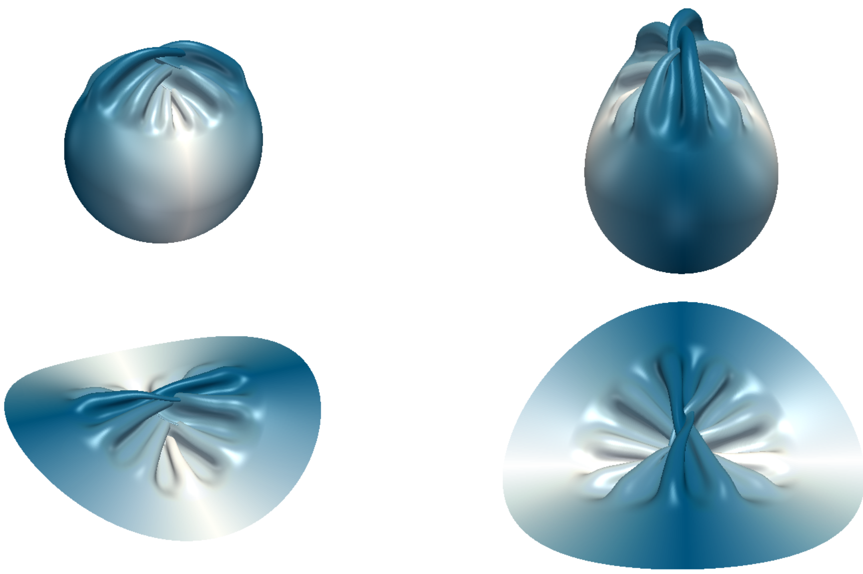


Figure 3.12 – **Immersions of $\mathbb{R}P^2$ via an inversion.** A desingularization of the Plücker's conoid and its inversion.

Immersions of $\mathbb{R}P^2$ via an extension of f_1.– Here, for purely aesthetic reasons, we avoid the use of an inversion and we choose to extend the corrugated

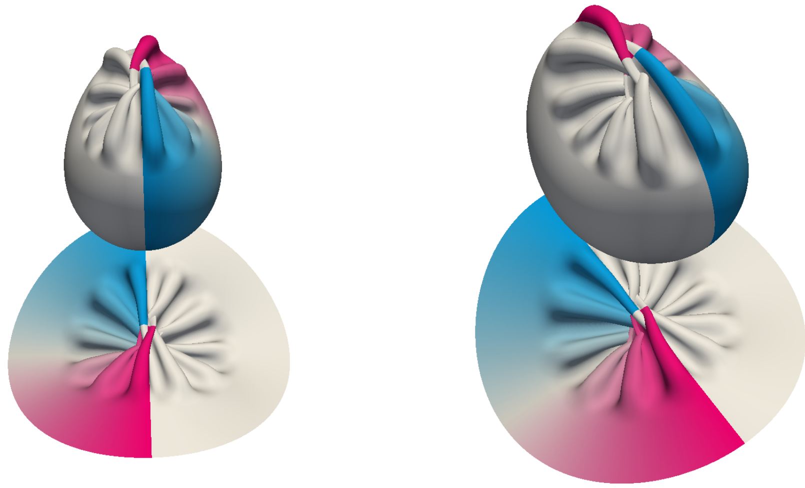


Figure 3.13 – **Immersions of $\mathbb{R}P^2$ via an inversion.** A desingularization of the Plücker's conoid and its inversion.

Plücker conoid in such a way that it closes up on itself like an hemisphere. Specifically, we parametrize the sphere of radius 2.5 by a map S defined on $[-5, 5] \times [0, 1]$ and given by

$$S : (x_1, x_2) \mapsto 2.5 \left(\cos \pi x_2 \sin \frac{\pi x_1}{5}, \sin \pi x_2 \sin \frac{\pi x_1}{5}, \cos \frac{\pi x_1}{5} \right).$$

We then define a map F_1 on the same domain by setting:

$$\left\{ \begin{array}{lcl} F_1(x_1, x_2)_{XY} & = & S(x_1, x_2)_{XY} + f_1(x_1, x_2)_{XY} - f_0(x_1, x_2)_{XY} \\ F_1(x_1, x_2)_Z & = & S(x_1, x_2)_Z + \beta(x_1)(f_1(x_1, x_2)_Z - 1) \end{array} \right. \tag{3.4}$$

if $(x_1, x_2) \in [-2.5, 2.5] \times [0, 1]$ and $F_1 = S$ otherwise. In this formula, f_{XY} means the X, Y components of f, f_Z means the Z components and the map β is a smooth interpolation between $\beta(0) = 1$ and $\beta(\pm 2.5) = 0$, see Figures 3.14 and 3.15.

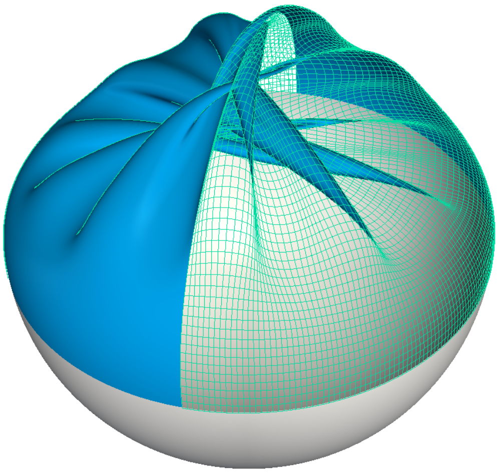


Figure 3.14 – **An immersion of $\mathbb{R}P^2$ obtained by a Corrugation Process.** Image of the map F_1 with $\beta(x_1) = \left(\dfrac{1+\cos(2\pi x_1/5)}{2}\right)^{0.75}$ and the same θ, α, N as in Figure 3.9.

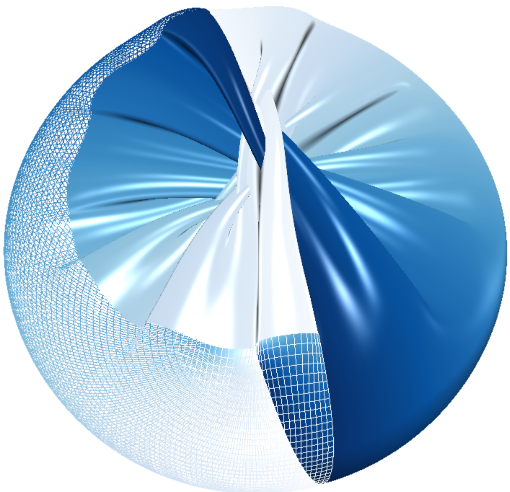


Figure 3.15 – **An immersion of $\mathbb{R}P^2$ obtained by a Corrugation Process.** Image of the map F_1 as in Figure 3.14.

3.4 The relation of ϵ-isometric immersions

A famous relation treated by the Convex Integration Theory is the relation of isometries. The resolution of this relation by the Convex Integration is based on the fact that the relation of ϵ-isometries can be solved by Convex Integration (see Subsections 1.2.4 and 2.2.5). A natural question is to know if the relation of ϵ-isometries is Kuiper.

Let $\epsilon > 0$. Recall that the relation of ϵ-isometric immersions from (M, g) to (W, h), with g and h two metrics, is defined by

$$\mathscr{I}(\epsilon) := \{(x, y, L) \in J^1(M, W) \mid |L^*h - g| < \epsilon\}.$$

We show in this section the following theorem:

Theorem 55. *Let M and W be orientable Riemannian manifolds, and let $\epsilon > 0$. The relation $\mathscr{I}(\epsilon)$ is a relative Kuiper relation in codimension 1 and a relative quasi-Kuiper one in codimension greater than 1.*

The key point of the proof of this theorem is to build a loop family $\tilde{\gamma}$ c-shaped for all couples (σ, w) such that σ belongs to $\mathscr{I}(\epsilon)$ and w belongs to the convex hull of the slice $\mathscr{I}(\epsilon)(\sigma, \lambda, u)$, for some λ, u. To understand the slice $\mathscr{I}(\epsilon)(\sigma, \lambda, u)$ and its convex hull, we first present its geometric description. Then we give a proof of Theorem 55.

3.4.1 Preliminary results on $\mathscr{I}$

Geometric description of $\mathscr{I}$

Such a description can be found in [Gro86, p202], [Spr98, p194], and a short one in Subsection 2.2.5. For the sake of completeness we recall this description here in the coordinate-free case and we give some extra details needed for our construction of a surrounding loop family.

Let $\sigma = (x, y, L) \in \mathscr{I}$. Let $\lambda \in T_x^*M$ and $u \in T_xM$ such that $\lambda(u) = 1$. For every $v \in T_yW$, we set $L_v := L + (v - L(u)) \otimes \lambda$. We have

$$\begin{aligned} \mathscr{I}(\sigma, \lambda, u) &:= Conn_{L(u)}\{v \in T_yW \mid (x, y, L_v) \in \mathscr{I}\} \\ &= Conn_{L(u)}\{v \in T_yW \mid g_x = L_v^*h_y\}. \end{aligned}$$

Note that, by the definition of L_v, we have $L_v(u) = v$ and for every $u_0 \in \ker\lambda$ we have $L_v(u_0) = L(u_0)$, in particular $L_v(\ker\lambda) = L(\ker\lambda)$. Let $w_1 = \alpha_1 u + a_1$ and $w_2 = \alpha_2 u + a_2$ with $\alpha_1, \alpha_2 \in \mathbb{R}$ and $a_1, a_2 \in \ker\lambda$. As $g = L^*h$, we have

$$\begin{aligned} (g - L_v^*h)(w_1, w_2) &= \alpha_1\alpha_2(g(u, u) - h(v, v)) \\ &\quad + \alpha_1 h(L(u) - v, L(a_2)) + \alpha_2 h(L(u) - v, L(a_1)). \end{aligned}$$

From this expression it is readily seen that $g = L_v^*h$ if and only if $g(u, u) = h(v, v)$ and $v \in L(u) + L(\ker\lambda)^\perp$. So v lies inside the $(n-1)$-dimensional sphere S_u of radius $\|u\|_g$ and inside the affine $(n - m + 1)$-plane

$$P_u := L(u) + L(\ker\lambda)^\perp.$$

Thus $\mathscr{I}(\sigma, \lambda, u) = S_u \cap P_u$ is a $(n - m)$-dimensional sphere of T_yW and its convex hull is a ball of the same dimension. Since we have assumed $n > m$, the

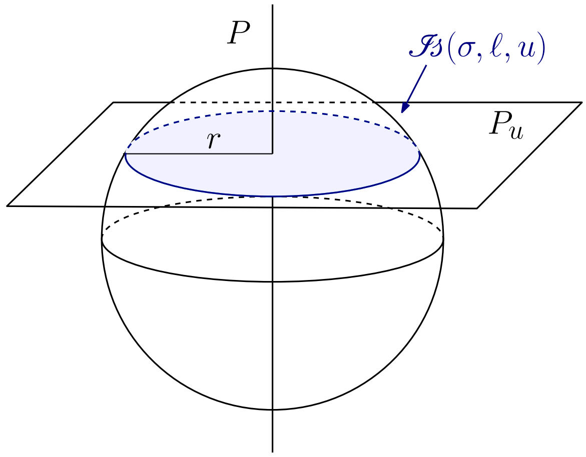


Figure 3.16 – **The slice $\mathscr{I}\!s(\sigma,\lambda,u)$ and its convex hull:** the $(n-m)$-dimensional sphere in dark blue is $\mathscr{I}\!s(\sigma,\lambda,u)$ and the convex hull IntConv $\mathscr{I}\!s(\sigma,\lambda,u)$ is the $(n-m+1)$-ball in light blue.

space $\mathscr{I}\!s(\sigma,\lambda,u)$ is arc-connected. Recall we denote by P the $(m-1)$-plane $L(\ker\lambda)$.

In the following proposition, we give an expression of the radius r of $\mathscr{I}\!s(\sigma,\lambda,u)$ which does not depend of σ. To do that we denote by proj_0 the orthogonal projection on $\ker\lambda$ in T_xM, and by proj_P the orthogonal projection on $P = L(\ker\lambda)$ in T_yW.

Proposition 56. *The radius r of $\mathscr{I}\!s(\sigma,\lambda,u)$ is given by $r^2 = \|u\|_g^2 - \|proj_0 u\|_g^2$.*

To prove Proposition 56, we need this lemma:

Lemma 57. *We have $L(proj_0 u) = proj_P L(u)$.*

Proof of Lemma 57.– We first decompose T_xM as

$$T_xM = \ker\lambda \oplus^{\perp} V_x$$

where V_x is the orthogonal complement of $\ker\lambda$ for the metric g. As L is isometric, we have

$$T_yW = L(\ker\lambda) \oplus^{\perp} L(V_x) \oplus^{\perp} R_y$$

where R_y is the orthogonal complement for the metric h of $L(\ker\lambda) \oplus^{\perp} L(V_x)$. Then we have

$$L(u) = L\Big(\mathrm{proj}_0 u + (u - \mathrm{proj}_0 u)\Big) = L(\mathrm{proj}_0 u) + L(u - \mathrm{proj}_0 u)$$

so $\mathrm{proj}_P L(u) = L(\mathrm{proj}_0 u)$. $\square$

Proof of Proposition 56.– Let $v \in \mathscr{I}\!s(\sigma,\lambda,u)$. The radius r of $\mathscr{I}\!s(\sigma,\lambda,u)$ is given by

$$r = \sqrt{\|v\|_h^2 - \|\mathrm{proj}_P v\|_h^2}.$$

Since $v \in \mathscr{I}\!s(\sigma, \lambda, u)$, we have $\|v\|_h = \|u\|_g$. Moreover, by Lemma 57 and as L is isometric, we have

$$\|\mathrm{proj}_P v\|_h = \|\mathrm{proj}_P L(u)\|_h = \|L(\mathrm{proj}_0 u)\|_h = \|\mathrm{proj}_0 u\|_g$$

so we have the result. $\square$

Since $\mathscr{I}\!s(\sigma, \lambda, u)$ is a $(n-m)$-dimensional sphere, IntConv $\mathscr{I}\!s(\sigma, \lambda, u)$ is a $(n-m+1)$-ball of P_u.

Characterization of subsolutions of $\mathscr{I}\!s$

We characterize subsolutions of $\mathscr{I}\!s$ with respect to $(d\pi, u)$, for a submersion $\pi : U \subset M \to \mathbb{R}$ and a tangent vector field $u : U \to TM$ such that $d\pi(u) = 1$, in the following proposition:

Proposition 58. *Let $f_0 : M \to W$ be a C^1-map and $P := df_0(\ker d\pi)$ such that $\dim\ P(x) = m - 1$ for all $x \in U$. If f_0 satisfies $g|_{\ker d\pi} = f_0^* h|_{\ker d\pi}$, then a section*

$$x \mapsto \mathfrak{S}(x) = (x, f_0(x), L_x := (df_0)_x + (v_x - (df_0)_x(u_x)) \otimes d\pi_x)$$

is a formal solution of $\mathscr{I}\!s$ with respect to $(d\pi, u)$ if and only if, for every x, the vector v_x can be written in the form $v_x = proj_{P(x)} L_x(u_x) + \tau_x$ where $\tau_x \in P(x)^\perp$ and $\|\tau_x\|_h = r(x) = \sqrt{\|u_x\|_g^2 - \|proj_0 u_x\|_g^2}$.

Proof.– Recall that $v_x \in \mathscr{I}\!s(\mathfrak{S}(x), d\pi_x, u_x)$ if and only if $v_x \in S_{u(x)} \cap P_{u(x)}$ i.e.

$$\|v_x\|_h^2 = \|u_x\|_g^2 \quad \text{and} \quad \mathrm{proj}_{P(x)} v_x = \mathrm{proj}_{P(x)} L(u_x).$$

If we decompose v_x in $P(x) \oplus P(x)^\perp$, we so have $v_x = \mathrm{proj}_{P(x)} L(u_x) + \widetilde{\tau}_x$, where $\widetilde{\tau}_x$ is a vector of $P(x)^\perp$. As $\|v_x\|_h = \|u_x\|_g$ by the Pythagorean theorem we have

$$\|\widetilde{\tau}_x\|_h^2 = \|u_x\|_g^2 - \|\mathrm{proj}_{P(x)} L(u_x)\|_h^2.$$

Then, by Lemma 57 and Proposition 56, we conclude that $\|\widetilde{\tau}_x\|_h^2 = r(x)^2$. So $\widetilde{\tau}_x$ is the wanted vector τ_x. $\square$

Note that in Subsection 2.2.5, we have recalled the construction of the ansatz formula for isometric embeddings of Conti, De Lellis and Székelyhidi [CDLS12]. Their construction builds loop families for specific subsolutions. We recall in the following paragraph the context of their construction.

Previous constructions of solutions of $\mathscr{I}_3$

In 1954 J. Nash gave an explicit way to build in codimension greater than 2 an isometric immersion from a strictly short immersion [Nas54]. This result was extended to codimension 1 by N. Kuiper in 1955 [Kui55]. They both start with an initial immersion and correct iteratively the isometric default (see Subsection 1.2.4). Note that since the initial map is an immersion, it does not have singular point.

In the framework of the h-principle, we cannot disregard singular points. Indeed the convex hull of the slice $\mathscr{I}_3(\sigma, \lambda, u)$ contains a point of $P = L(\ker \lambda)$, where $\sigma = (x, y, L)$ (see the geometric description above). This implies the set of subsolutions of $\mathscr{I}_3$ with respect to (λ, u) contains subsolutions which have a singular base map. Therefore to prove that the relation of isometries is Kuiper, we cannot use the construction of Nash or of Kuiper. Note also that, in the constructions of Conti and al. [CDLS12] and of the Hevea team [BJLT13] inspired by the work of Nash and Kuiper, the direction of oscillation is given by the vector $e_1 = [df_x(u_x)]^{P^\perp}$. Obviously this direction cannot be defined if f is singular at x.

In the following section, we prove that the relation of isometries is Kuiper relative. To do that we build a relative loop family for all subsolutions. In codimension 1, the construction is not hard but time-consuming. In codimension greater than 2, it is much easier. In that case, following Nash, we consider n_1 and n_2 two normal unit vector fields linearly independent and we define the following family of loops

$$\gamma(x, t) := df_x(u_x) + r(x)\cos(2\pi t)n_1(x) + r(x)\sin(2\pi t)n_2(x)$$

Using such a loop family in the Corrugation Process in the direction ∂_1, we obtain

$$f_1(x) = f(x) + \frac{r(x)}{2\pi N}\sin(2\pi N x_1)n_1(x) - \frac{r(x)}{2\pi N}\cos(2\pi N x_1)n_2(x) + \frac{r(x)}{2\pi N}n_2(x).$$

Up to the last term, this expression matches with the formula of Nash.

3.4.2 Proof of Theorem 55

For the sake of simplicity, we illustrate in Figure 3.17 a slice of $\mathscr{I}_3(\epsilon)$ built as a thickening of the slice of $\mathscr{I}_3$ described in Subsection 3.4.1.

Step 1: proof of the Kuiper property (in codimension 1)

We begin with a preparatory lemma, then describe $\mathrm{IntConv}\, \mathscr{I}_3(\epsilon)(\sigma, \lambda, u) \cap P_u(w)$ and define a c-shaped loop family for the relation $\mathscr{I}_3(\epsilon)$. We finally

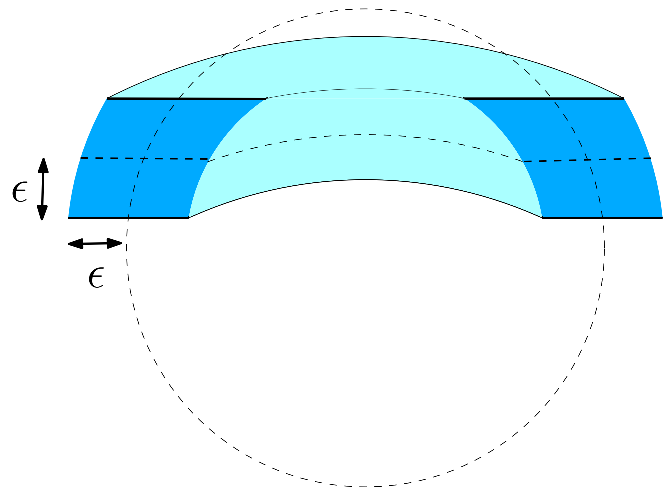


Figure 3.17 – **Illustration of a slice of $\mathscr{I}\!s(\epsilon)$:** in blue, a cut of $\mathscr{I}\!s(\epsilon)(\sigma, \lambda, u)$, in dashed, the two sets S_u and P_u described in Figure 3.16 to define the slice $\mathscr{I}\!s(\sigma, \lambda, u)$.

construct $\tilde{\gamma}$ and prove that it is surrounding.

Let $\sigma = (x, y, L) \in \mathscr{I}\!s(\epsilon)$. Let $\lambda \in T_x^*M$, $u \in T_xM$ such that $\lambda(u) = 1$, and let $w \in \mathrm{IntConv}\, \mathscr{I}\!s(\epsilon)(\sigma, \lambda, u)$. Note that as $\mathscr{I}\!s(\epsilon)$ is a thickening of $\mathscr{I}\!s$ and by definition of σ and w, the distance (for the metric h) between w and P_u is less than 2ϵ, but w does not belong necessarily to P_u. We denote by $P_u(w)$ the affine $(n - m + 1)$-plane that contains w and which is a translation of P_u:

$$P_u(w) := \{v \in T_yW \mid \mathrm{proj}_P w = \mathrm{proj}_P v\}$$

where P denotes $L(\ker \lambda)$. Thanks to the following lemma, we can assume that w belongs to P_u:

Lemma 59. *Let* $(\sigma, w) \in \mathrm{IntConv}(\mathscr{I}\!s(\epsilon), \lambda, u)$ *with* $\sigma = (x, y, L)$. *There exists a homotopy* $\sigma_t = (x, y, L_t)$ *such that* $\sigma_0 = \sigma$, $\sigma_t \in \mathscr{I}\!s(\epsilon)(\sigma, \lambda, u)$ *for all* $t \in [0, 1]$, *and* $proj_P L_1(u) = proj_P w$.

Proof.– We set $v_0 = L_0(u) = L(u)$. We can assume that $\|v_0\|_h \geq \|w\|_h$. Indeed, if $\|v_0\|_h < \|w\|_h$ we perform a first homotopy. Let $\widetilde{L_t} = L + (\widetilde{v_t} - v_0) \otimes \lambda$ where

$$\widetilde{v_t} := \mathrm{proj}_P v_0 + \left((1-t) + t \frac{\sqrt{\|w\|_h^2 - \|\mathrm{proj}_P v_0\|_h^2}}{\|v_0 - \mathrm{proj}_P v_0\|_h} \right) (v_0 - \mathrm{proj}_P v_0).$$

This homotopy joins v_0 to $\widetilde{L_1}(u) = \widetilde{v_1}$ where $\|\widetilde{v_1}\|_h = \|w\|_h$. Let $V_0 = v_0$ if $\|v_0\|_h \geq \|w\|_h$, and $V_0 = \widetilde{v_1}$ if $\|v_0\|_h < \|w\|_h$. In both cases, we consider the homotopy $L_t = L + (v_t - V_0) \otimes \lambda$ with:

$$v_t := t\,\mathrm{proj}_P w + (1-t)\,\mathrm{proj}_P V_0 + \varphi(t)(V_0 - \mathrm{proj}_P V_0)$$

and

$$\varphi(t) = \sqrt{\frac{\|V_0\|_h^2 - \|t\,\mathrm{proj}_P w + (1-t)\,\mathrm{proj}_P V_0\|_h^2}{\|V_0 - \mathrm{proj}_P V_0\|_h^2}}.$$

Since $\|V_0\|_h \geq \|w\|_h$ the numerator is positive and φ is well defined. By definition of φ, for every t, we have $\|v_t\|_h = \|V_0\|_h$. This property ensures that $\sigma_t = (x, y, L_t) \in \mathscr{I}\!s(\epsilon)(\sigma, \lambda, u)$ for all $t \in [0,1]$. By the expression of v_t, we obviously have $\mathrm{proj}_P v_1 = \mathrm{proj}_P w$. □

This lemma and Point (3) of Definition 40 imply that it is enough to construct the loop family γ for every couple (σ, w) such that $\mathrm{proj}_P L(u) = \mathrm{proj}_P w$. We assume in the sequel that this last condition is fulfilled together with the fact that the codimention is one.

Description of $\mathrm{IntConv}\,\mathscr{I}\!s(\epsilon)(\sigma, \lambda, u) \cap P_u(w)$**.–** By assumption $n = m+1$ therefore the space $P_u(w)$ is a 2-plane. We denote by $D(\rho)$ the open disk of $P_u(w)$ with radius ρ and center $\mathrm{proj}_P(L(u))$ and by $A(\rho_{min}, \rho_{max})$ the open annulus $D(\rho_{max}) \setminus \overline{D(\rho_{min})}$. The intersection of the thickened relation $\mathscr{I}\!s(\epsilon)(\sigma, \lambda, u)$ with $P_u(w)$ is either an annulus or a disk depending on the value of ϵ. Precisely, let

$$\begin{aligned} r^2_{min}(\epsilon) &:= \min\left((\|u\|_g - \epsilon)^2 - \|\mathrm{proj}_P w\|_h^2, 0\right) \\ r^2_{max}(\epsilon) &:= (\|u\|_g + \epsilon)^2 - \|\mathrm{proj}_P w\|_h^2. \end{aligned}$$

because the sphere S_u of Subsection 3.4.1 is of radius $\|u\|_g$. A computation shows that $\mathscr{I}\!s(\epsilon)(\sigma, \lambda, u) \cap P_u(w)$ is the annulus $A(r_{min}(\epsilon), r_{max}(\epsilon))$ if $r_{min}(\epsilon) > 0$ and the disk $D(r_{max}(\epsilon))$ if $r_{min}(\epsilon) = 0$. In any case,

$$\mathrm{IntConv}\,\mathscr{I}\!s(\epsilon)(\sigma, \lambda, u) \cap P_u(w) = D(r_{max}(\epsilon)).$$

In particular, we have $w \in D(r_{max}(\epsilon))$ and $L(u) \in A(r_{min}(\epsilon), r_{max}(\epsilon))$. We want to build a c-shape loop family inside $A(r_{min}(\epsilon), r_{max}(\epsilon))$, for that we define a disk which will support γ and such that a neighborhood of this disk will be in $A(r_{min}(\epsilon), r_{max}(\epsilon))$ too. Let $D(\widetilde{r})$ a disk where

$$\widetilde{r} = \max\left(\sqrt{\|L(u)\|_h^2 - \|\mathrm{proj}_P L(u)\|_h^2}, \sqrt{\|w\|_h^2 - \|\mathrm{proj}_P w\|_h^2} + \frac{1}{3} d_1(w)\right)$$

where

$$d_1(w) := dist(w, \partial(\mathrm{IntConv}\,\mathscr{I}\!s(\epsilon)(\sigma, \lambda, u)))$$

is the distance between w and the boundary of the convex hull of $\mathscr{I}\!s(\epsilon)(\sigma, \lambda, u)$. Obviously $w \in D(\widetilde{r})$ and $\partial D(\widetilde{r}) \subset A(r_{min}(\epsilon), r_{max}(\epsilon))$.

Parametrization of $D(\widetilde{r})$.– Let ν be th unique unit normal vector of $L(T_xM)$ induced by the orientation of M and W. We see $P_u(w)$ as the complex plane $\mathbb{C}$ by identifying the base $(\nu, (L(u) - \mathrm{proj}_P L(u))/\|L(u) - \mathrm{proj}_P L(u)\|_h)$ with $(1, i)$ and we define a parametrization of $\overline{D(\widetilde{r})}$ by

$$\begin{array}{llll} b: & [0,\pi] \times [0,1] & \longrightarrow & \overline{D(\widetilde{r})} \\ & (\theta, \beta) & \longmapsto & \mathrm{proj}_P L(u) + \beta \widetilde{r} e^{i\theta} + (1-\beta)\widetilde{r} e^{-i\theta}. \end{array}$$

This parametrization is 1-to-1 except over points of the form $(0, \beta)$ and (π, β). It maps the boundary of the square $[0,\pi] \times [0,1]$ onto the circle $\partial D(\widetilde{r})$.

The shape.– We first define the parameter space A to be

$$A := \{(\eta, \theta, \beta) \in]0, \tfrac{1}{2}[\times[0,\pi] \times [0,1] \mid \eta \leq \beta \leq 1 - \eta\}.$$

and then the shape $c : A \times \mathbb{R}/\mathbb{Z} \to \mathbb{C} \times \mathbb{R}$ by

$$c(\eta, \theta, \beta, t) := (\exp(i g_{\theta,\beta}(t)) + \eta \cos\theta, 1).$$

The image of $c(\eta, \theta, \beta, \cdot)$ is a whole circle of center $(\eta\cos\theta, 1)$ and radius 1. Let $\beta' = \beta - \frac{\eta}{2}$, the angular function $g_{\theta,\beta}$ is the piecewise linear map given by

(i) $g_{\theta,\beta}(0) = 0$ and $g_{\theta,\beta}(\frac{1}{2}) = 2\pi$

(ii) $g_{\theta,\beta}(t) = \theta$ on $\left[\dfrac{\eta\theta}{4\pi}, \dfrac{\beta'}{2} + \dfrac{\eta\theta}{4\pi}\right]$

(iii) $g_{\theta,\beta}(t) = 2\pi - \theta$ on $\left[\dfrac{\beta'}{2} + \dfrac{\eta(2\pi - \theta)}{4\pi}, \dfrac{1}{2} - \dfrac{\eta\theta}{4\pi}\right]$

on $[0, \frac{1}{2}]$ and such that $g_{\theta,\beta}(t) = g_{\theta,\beta}(1-t)$ for all $t \in [0, \frac{1}{2}]$ (see its graph on Figure 3.18). A computation shows that

$$\overline{c(\eta, \theta, \beta)} \quad = \quad (\beta e^{i\theta} + (1-\beta)e^{-i\theta}, 1)$$

(for the details of calculations, see the Appendix).

The loop family.– Since b induces a bijection between $]0,\pi[\times]0,1[$ and $D(\widetilde{r})$, there exists a unique couple $(\theta, \beta) \in]0,\pi[\times]0,1[$ such that $b(\theta, \beta) = w$. We define two functions c_1 and c_2 by the equality

$$c(\eta, \theta, \beta, \cdot) = (c_1(\cdot) + i c_2(\cdot), 1)$$

(η will be chosen later). We put

$$\mathbf{e}_1 := \widetilde{r}\frac{\nu}{\|\nu\|_h}, \quad \mathbf{e}_2 := \widetilde{r}\frac{L(u) - \mathrm{proj}_P L(u)}{\|L(u) - \mathrm{proj}_P L(u)\|_h} \quad \text{and} \quad \mathbf{e}_3 := \mathrm{proj}_P L(u)$$

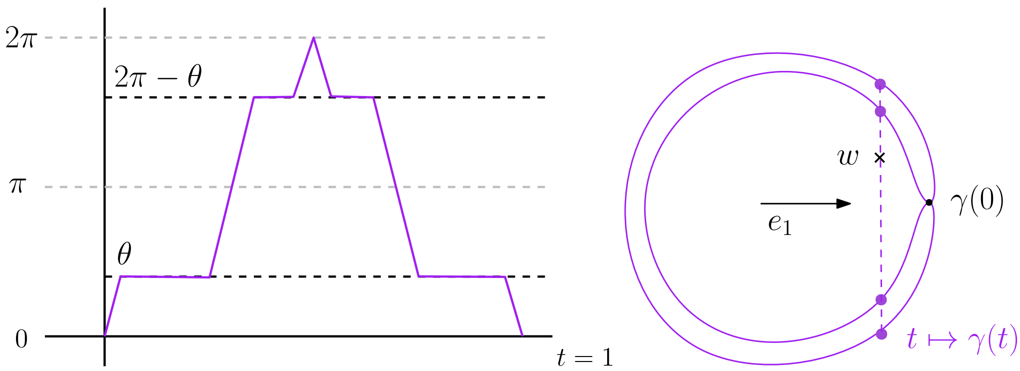


Figure 3.18 – **Proof of Theorem 55**: Left: the graph of the function $g_{\theta,\beta}$, Right: the image of the loop γ in the affine plane $P_u(w)$, the two circles visualise the round-trip of the loop.

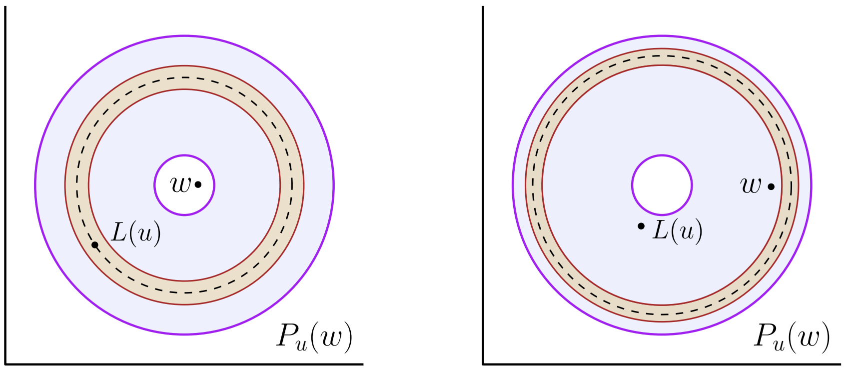


Figure 3.19 – **Proof of Theorem 55**: In purple, the trace of the relation on $P_u(w)$, in black (dashed line) the boundary of disk $D(\tilde{r})$ and in brown, the annulus $A(\tilde{r}(1-\eta), \tilde{r}(1+\eta))$ depending on whether $\sqrt{\|L(u)\|_h^2 - \|\mathrm{proj}_P L(u)\|_h^2} > \sqrt{\|w\|_h^2 - \|\mathrm{proj}_P w\|_h^2} + \frac{1}{3}d_1(w)$ (left) or not (right), see the definition of $\tilde{r}$.

and we define the loop family γ by

$$\gamma(\sigma, w)(t) := c_1(t)\mathbf{e}_1 + c_2(t)\mathbf{e}_2 + \mathbf{e}_3.$$

The image of the loop $\gamma(\sigma, w)$ is the translated circle $\partial D(\tilde{r}) + \tilde{r}\eta\cos\theta\mathbf{e}_1$ which lies inside the annulus $A(\tilde{r}(1-\eta), \tilde{r}(1+\eta))$ of $P_u(w)$. Consequently, to ensure that the image of $\gamma(\sigma, w)$ is in the relation, it is enough to choose η such that $A(\tilde{r}(1-\eta), \tilde{r}(1+\eta)) \subset A(r_{min}(\epsilon), r_{max}(\epsilon)) = \mathscr{I}_3(\epsilon)(\sigma, \lambda, u) \cap P_u(w)$. It is readily checked that the choice

$$\eta := \frac{1}{3}\min(d_1(w), d_2(L(u)))$$

where $d_2(L(u)) = dist(L(u), \partial A(r_{min}(\epsilon), r_{max}(\epsilon)))$ is convenient. It is also straightforward to see that this loop family satisfies the Average Constraint: $\overline{\gamma}(\sigma, w) = w$. The base point of the loop is $\gamma(\sigma, w)(0) = (1 + \eta\cos\theta)\mathbf{e}_1 + \mathbf{e}_3$. The homotopy $H(s) := (\cos s\ \mathbf{e}_1 + \sin s\ \mathbf{e}_2) + \eta\cos\theta\mathbf{e}_1 + \mathbf{e}_3$ with $s \in\ [0, \frac{\pi}{2}]$

connects $\tilde{\gamma}(\sigma, w)(0)$ with $\eta \cos\theta \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ A linear homotopy joins this last point to $L(u) = (\|L(u) - \mathrm{proj}_P L(u)\|_h)\mathbf{e}_2/\widetilde{r} + \mathbf{e}_3$. Consequently, the loop family $\tilde{\gamma}$ is c-shaped and surrounding (see Definition 40). This proves that $\mathscr{I}(\epsilon)$ is a Kuiper relation. The relative property will be proved in Step 3.

Step 2: proof of quasi-Kuiper property (in codimension greater than 1)

In greater codimension, the space $P_u(w) = \mathrm{proj}_P w + P^{\perp}$ is no longer a 2-plane. Nevertheless, since Lemma 59 still holds, we can continue to assume that $\mathrm{proj}_P w = \mathrm{proj}_P L(u)$. Let ν be the normal direction given by the formal solution of the thickened relation (see Section 2.3). We decompose w as $w = \mathrm{proj}_P L(u) + a_1\nu + a_2(L(u) - \mathrm{proj}_P L(u)) + a_3\, n$ where $n \in P^{\perp}$. It is then enough to repeat the former construction in the 2-plane $\mathrm{proj}_P L(u) + a_3 n + Span(L(u) - \mathrm{proj}_P L(u), \nu) \subset P_u(w)$ to obtain a c-shaped surrounding family.

Step 3: proof of relative property

To deal with this case, we consider the shape $c : A \times [0,1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{C} \times \mathbb{R}$ given by

$$c(\eta, \theta, \beta, \rho, t) := (\exp(i\alpha_{\theta,\beta,\rho}(t)) + \tau(\eta, \theta, \beta, \rho), 1)$$

where $\alpha_{\theta,\beta,\rho}$ is a piecewise linear angular function and $\tau(\eta, \theta, \beta, \rho)$ a "small" translation term given by the difference between the barycenter of the two points $e^{\pm i\theta}$ with the weights β and $1 - \beta$ and the average of $\exp(i\alpha_{\theta,\beta,\rho})$:

$$\tau(\eta, \theta, \beta, \rho) := (\beta e^{i\theta} + (1 - \beta)e^{-i\theta}) - \overline{\exp(i\alpha_{\theta,\beta,\rho})}.$$

We define the angular function by distinguishing between the case $\rho \leq \frac{1}{2}$ and the case $\rho \geq \frac{1}{2}$:

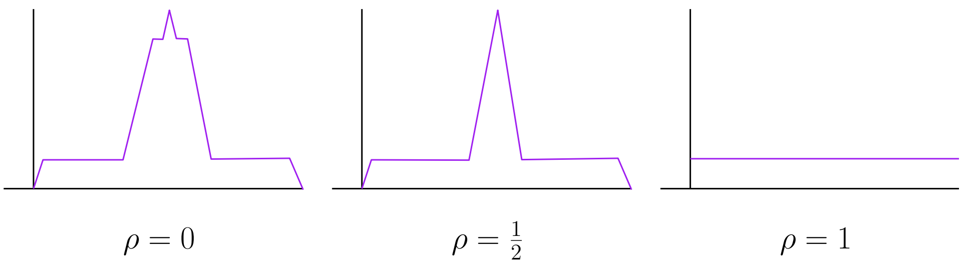


Figure 3.20 – **Relative case**: the graph of α for $\rho = 0,\ 1/2,\ 1$.

First case: If $\rho \in [0, \frac{1}{2}]$.– We then put $\alpha_{\theta,\beta,\rho} := g_{\theta,\beta(\rho)}$ where $\beta(\rho) = (1-2\rho)\beta + \rho(2 - \eta)$. Obviously $\alpha_{\theta,\beta,0} = \alpha_{\theta,\beta}$. The β function is chosen such that the two plateaus of altitude $2\pi - \theta$ of the graph of the angular function reduces to a point while ρ is increasing from 0 to $\frac{1}{2}$. For every $0 \leq \rho \leq \frac{1}{2}$, we have

$$\begin{array}{rcl}\overline{\exp(i\alpha_{\theta,\beta,\rho}(t))} & = & \beta(\rho)e^{i\theta} + (1-\beta(\rho))e^{-i\theta} - \eta\cos\theta \\ & = & (\beta e^{i\theta} + (1-\beta)e^{-i\theta}) - \eta\cos\theta + 2i(\beta(\rho)-\beta)\sin\theta.\end{array}$$

In particular $\tau(\eta,\theta,\beta,\rho) = \eta\cos\theta - 2i(\beta(\rho)-\beta)$.

Second case: If $\rho \in [\frac{1}{2}, 1]$.– We define the angular function $\alpha_{\theta,\beta,\rho}$ to be the piecewise linear map such that $\alpha_{\theta,\beta,\rho}(t) = \alpha_{\theta,\beta,\rho}(1-t)$ for all $t \in [0, \frac{1}{2}]$ and satisfying

(i) $\alpha_{\theta,\beta,\rho}(0) = \min(\theta, 4\pi\rho - 2\pi)$ and $\alpha_{\theta,\beta,\rho}(\frac{1}{2}) = \max(\theta, -4\pi\rho + 4\pi)$

(ii) $\alpha_{\theta,\beta,\rho}(t) = \theta$ on $\left[\frac{\eta}{4\pi}\big(\theta - \alpha_{\theta,\beta,\rho}(0)\big), \frac{1}{2} - \frac{\eta}{4\pi}\big(\theta - \alpha_{\theta,\beta,\rho}(1/2)\big)\right]$.

The graph of $\alpha_{\theta,\beta,\rho}$ shows two plateaus of altitude θ and four segments with non-zero slope. As ρ increases from $\frac{1}{2}$ to 1, these segments become smaller and smaller to disappear when $\rho = 1$. The angular function $\alpha_{\theta,\beta,1}$ is then constant equal to θ. A computation shows that

$$\begin{array}{rcl}\overline{\exp(i\alpha_{\theta,\beta,\rho}(t))} & = & e^{i\theta} - \dfrac{\eta}{2\pi}\big(\alpha_{\theta,\beta,\rho}(1/2) - \alpha_{\theta,\beta,\rho}(0)\big)e^{i\theta} \\ & & -i\dfrac{\eta}{2\pi}(e^{i\alpha_{\theta,\beta,\rho}(1/2)} - e^{i\alpha_{\theta,\beta,\rho}(0)})\end{array}.$$

To sum up: as ρ goes from 0 to 1, the image of the shape c retracts on a single point while its average remains constant equal to $(\beta e^{i\theta} + (1-\beta)e^{-i\theta}, 1)$. This image lies inside a translated circle by a vector $\tau(\eta,\theta,\beta,\rho)$ of the unit circle. When constructing $\tilde{\gamma}$, this translation may locate the image of the loop family outside $A(r_{min}(\epsilon), r_{max}(\epsilon))$. Nevertheless, when w tends towards $L(u)$ the coefficient β tends towards 1 and the error term vanishes. To obtain a δ-relative family, one has to choose ρ according to the distance between w and $L(u)$ and the distance $dist(L(u), \mathscr{I}\!\mathscr{J}(\epsilon)(\sigma,\lambda,u)^C)$ between $L(u)$ and the complement of the slice of the relation. Details are left to the reader.

Chapter 4

Totally real isometric maps and self-similarity

In Chapter 3 we have considered the differential relations of immersions and of ϵ-isometries. We have proven that these two relations are relative Kuiper in codimension 1 and relative quasi-Kuiper in greater codimension. In this chapter, we consider immersions and isometric maps with the additional constraint of being totally real. We prove that similar results than the ones for immersions and ϵ-isometries still hold. We first state that the relation of totally real maps is a relative Kuiper relation in codimension 1 and a relative quasi-Kuiper one in greater codimension (see Section 4.1). Then we show that a theorem similar to the C^1-Isometric Embedding Theorem of Nash-Kuiper holds despite the additional constraint of being totally real (see Theorem 61).

The proof of Theorem 61 uses the notions developed in this thesis. The Corrugation Process formula allows to control the partial derivatives and thus the J-density. This allows to ensure that isometric embeddings built by iteration of the Corrugation Process are totally real. Moreover, we take advantages in this chapter of the explicit expression of the pattern of the loop family to give a description of the Maslov component of the Gauss map of isometric totally real maps built by our iterated process.

4.1 Totally real maps

We consider (M, g) a Riemannian manifold and (W, J, h) a (complete) almost Hermitian manifold with $\dim\ W = 2\dim\ M$, i.e. (W, J) is an almost complex manifold (see Subsection 1.1.2) such that the metric h is compatible with the complex structure J (this implies that we have $h(v_1, v_2) = h(Jv_1, Jv_3)$ for any v_1, $v_2 \in T_yW$ for $y \in W$). Recall that the relation of totally real maps is given by

$$\mathscr{I}_{TR} := \{(x, y, L) \mid L(T_xM) \oplus JL(T_xM) = T_yW\}.$$

This condition means that each subspace $L(T_xM)$ of T_yW contains no complex line. Observe that the totally real condition is a relaxation of the Lagrangian condition $\{L(T_xM) \oplus^{\perp} JL(T_xM) = T_yW\}$. Then we have the following theorem:

Theorem 60. *The relation $\mathscr{I}_{TR}(M,W)$ is a relative Kuiper relation with respect to the loop pattern $c : [0,\alpha_0] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ defined by*

$$c(\alpha,t) = \Big(\cos(\alpha\cos 2\pi t) - J_0(\alpha), \sin(\alpha\cos 2\pi t), 1\Big).$$

Note that, if f is a totally real map, i.e. $j^1 f \in \mathscr{I}_{TR}$, then f is an immersion. In the proof of the theorem, we use the same loop family as the one used to prove that the relation $\mathscr{I}$ is Kuiper in codimension 1 (see Theorem 51).

Proof of Theorem 60.– Let $m = \dim(M)$, $\sigma = (x,y,L) \in \mathscr{I}_{TR}$, $\lambda \in T_x^*M$ and $u \in T_xM$ such that $\lambda(u) = 1$.
Step 1: description of the slice $\mathscr{I}_{TR}(\sigma,\lambda,u)$: We have

$$\mathscr{I}_{TR}(\sigma,\lambda,u) = Conn_{L(u)}\{v \in T_yW \mid (x,y,L_v) \in \mathscr{I}_{TR}\}$$

where $L_v := L - (v - L(u)) \otimes \lambda$. Let $P := L(\ker\lambda) + JL(\ker\lambda)$. Observe that $L_v(\ker\lambda) = L(\ker\lambda) \subset P$, $L_v(u) = v$ and $JP = P$. If $v \in P$ then $L_v(T_xM) = L_v(\ker\lambda) + \mathbb{R}v \subset P$, so

$$L_v(T_xM) + JL_v(T_xM) \subset P \subsetneq T_yW$$

and L_v is not a totally real linear map. If $v \notin P$, we have $Jv \notin P$, then L_v is totally real. We conclude that $v \in \mathscr{I}_{TR}(\sigma,\lambda,u)$ if and only if $v \notin P$. We have $\dim\ P = 2m-2$ because $L : T_xM \to T_yW$ is a totally real linear map. It ensues that the space $\mathscr{I}_{TR}(\sigma,\lambda,u)$ is the complementary of the codimension 2 linear subspace P in T_yW.

Step 2: the loop family: The slice $\mathscr{I}_{TR}(\sigma,\lambda,u)$ is thus completely analoguous to the slices of the relation of codimension 1 immersions. As a consequence, the construction of $\tilde{\gamma}$ done for codimension 1 immersions fully applies for totally real immersions: we replace the unit normal vector ν by $\mathbf{r}JL(u)/\|JL(u)\|_h$. This shows that $\mathscr{I}_{TR}$ is a Kuiper relation with respect to c and that $CP_{\tilde{\gamma}}(f,\pi,N)$ is totally real if N is large enough. $\square$

4.2 Totally real isometric embeddings

According to Theorems 51 and 60, the relation of immersions in codimension 1 and the relation of totally real maps are relative Kuiper relations for the same loop pattern, i.e. even if we add the constraint of being totally real we can use the same pattern in the two cases. A loop pattern is also one of the ingredients of the proof of the C^1-Isometric Embedding Theorem of Nash-Kuiper. Adding the constraint of being totally real to the base map, we state this theorem:

Theorem 61. *Let (M^m, g) be a compact Riemannian manifold and let $f_0 : (M^m, g) \to (W^{2m}, J, h)$ be a strictly short totally real immersion (resp. embedding). Then, for every $\epsilon > 0$, there exists a C^1 totally real isometric immersion (resp. embedding) $f_\infty : (M^m, g) \to (W^{2m}, J, h)$ such that $dist(f_\infty(x), f_0(x)) \leq \epsilon$ for every $x \in M^m$.*

Since the relation of totally real maps is open, it is unclear that the map f_∞, built as a limit in the proof, is totally real. Note that, the Nash-Kuiper Theorem implies the existence of isometric embeddings of a two dimensional flat torus in $\mathbb{R}^3$. In 2012, such an embedding was built by V. Borrelli, S. Jabrane, F. Lazarus and B. Thibert [BJLT13]. This embedding revealed a C^1-fractal behavior, i.e. the differential of this embedding exhibits a self-similarity property. Following this idea, it is interesting to study the normal of isometric maps built by iterating the Corrugation Process (as in the proof of Theorem 61). We take advantage of the explicit expression of the pattern c to provide in the following proposition a description of the Maslov component of the Gauss map of f_∞ (see Section 4.2.2 for the definition of the Maslov map).

Proposition 62. *Let $\mathfrak{m}(f_0, f_\infty) = e^{i\mathscr{W}_\infty} : M \to \mathbb{S}^1$ be the Maslov map of f_∞ and $\mathscr{W}_\infty = 2\sum_\ell \vartheta_\ell$ be the Maslov argument. Then if ℓ is large enough*

$$\vartheta_\ell = \alpha_\ell \cos(2\pi N_\ell \pi_\ell) + O\left(\frac{1}{N_\ell}\right).$$

4.2.1 Proof of Theorem 61

The proof is divided into two parts. The first one relies on the arguments of Nash-Kuiper [Nas54, Kui55, EM02] to construct a sequence of maps $(f_k)_k$ converging toward an C^1 isometric map $f_\infty : (M, g) \to (W, J, h)$. Note that we replace the process of the fundamental step by a Corrugation Process to construct iteratively the sequence $(f_k)_k$. We use the pattern c of Theorem 60 to solve both the totally real constraint and the isometric constraint with $\mu = f^*h + \rho \, \mathrm{d}\pi \otimes d\pi$ (see Section 2.2.5). In a second part, we show that the limit map f_∞ is totally real. For that, we use the fact that $\mathscr{I}_{TR}$ is Kuiper with respect to c to control the geometry of each f_k.

Construction of an isometric embedding f_∞. We recall here briefly the Nash construction. Let $\Delta := g - f_0^*h$ be the isometric default, $(\delta_k)_{k\in\mathbb{N}^*}$ be an increasing sequence of positive numbers converging to 1 and

$$g_k = f_0^*h + \delta_k \Delta$$

be an increasing sequence of metrics converging toward g. The Nash-Kuiper proof consists in building an infinite sequence $(f_k)_{k\in\mathbb{N}}$ of maps such that each f_k is approximatively isometric for g_k and short for g_{k+1}, i.e.

$$f_k^*h \approx g_k \quad \text{and} \quad f_k^*h \leq g_{k+1}.$$

This sequence is obtained by a succession of local deformations performed to reduce the isometric default in one direction (approximatively). This generates a finite number of recursively defined intermediary maps

$$f_k = f_{k,0}, f_{k,1}, \ldots, f_{k,I(k)} = f_{k+1}$$

where $I(k) < +\infty$ depends on $\dim M = m$ and on the number of charts of a finite atlas $\{(U_a, \varphi_a) \mid a \in A\}$ of M. More precisely, let $(\psi_a)_{a\in A}$ be a partition of unity associated to $(U_a)_{a\in A}$. A first collection of the intermediary maps achieves the approximation of the increase $\psi_1(g_{k+1} - f_k^*h)$ over U_1, a second portion achieves the increase $\psi_2(g_{k+1} - f_k^*h)$ over U_2, etc. To do so, on each U_a the desired increase is decomposed as a finite combination of squares of constant linear forms:

$$\psi_a(g_{k+1} - f_k^*h) = \sum_{j=1}^{j_{max}(a)} \rho_{a,j}\ell_{a,j} \otimes \ell_{a,j}$$

with positive coefficients $\rho_{a,j} : U_a \to \mathbb{R}_{\geq 0}$. The corrugated map $f_{k,1}$ is built to satisfy $f_{k,1}^*h \approx f_k^*h + \rho_{1,1}\ell_1 \otimes \ell_1$, the map $f_{k,2}$ to satisfy $f_{k,2}^*h \approx f_{k,1}^*h + \rho_{1,2}\ell_2 \otimes \ell_2$, etc. In particular, for $I = \sum_{a\in A} j_{max}(a)$ the map $f_{k,I}$ satisfies

$$f_{k,I}^*h \approx f_k^*h + (g_{k+1} - f_k^*h) = g_{k+1}.$$

In fine, the fundamental step in this approach is the following: given a map $f : U \to W$, a positive coefficient $\rho : U \to \mathbb{R}_{\geq 0}$, a chart $(U, \varphi = (\varphi^1, \ldots, \varphi^m))$ and a constant linear form $\ell = \sum_{i=1}^m c_i d\varphi^i$, this step builds an ϵ-isometric map $f_\epsilon : (U, \mu) \to (W, h)$ where $\mu := f^*h + \rho\ell \otimes \ell$. Eventually, the C^1 convergence of the sequence $(f_k)_{k\in\mathbb{N}}$ towards a C^1 isometric map $f_\infty : (M, g) \to (W, h)$ is guaranteed by choosing a sequence of oscillation numbers $(N_{k,i})$ that increases rapidly and an increasing sequence of real numbers $(\delta_k)_{k\in\mathbb{N}^*}$ such that $\sum \sqrt{\delta_k - \delta_{k-1}} < +\infty$.

The fundamental step can also be achieved by a Corrugation Process as explained in Subsection 2.2.5. We now assume that f is a totally real map. We put $\pi = \sum_{i=1}^m c_i\varphi^i$, so that $\ell = d\pi$ and u is any vector field such that $\ell(u) = 1$. Let

$$P_x := df(\ker d\pi_x) + Jdf(\ker d\pi_x) \subset T_{f(x)}W \quad \text{and} \quad \mathbf{t}(x) := \frac{[df(u_x)]^{P_x^\perp}}{\|[df(u_x)]^{P_x^\perp}\|_h}$$

where $[v]^{P_x^\perp}$ denotes the $P_x^\perp$ component of any vector v. Observe that $\mathbf{t}(x)$ is a unit vector normal to P_x. As $JP_x = P_x$, $\mathbf{n}(x) := J\mathbf{t}(x)$ is normal to $P_x + \mathbb{R}\mathbf{t}(\mathrm{x})$ and thus to $df(T_xM)$. By Subsection 2.2.5, the corrugation

$$CP_\gamma(f, \pi, N) = \exp_f\left(\frac{r}{N}K_c(\alpha, N\pi)\mathbf{t} + \frac{r}{N}K_s(\alpha, N\pi)\mathbf{n}\right)$$

produces the requested ϵ-isometric totally real map $f_\epsilon : (U, \mu) \to (W, h, J)$ if N is large enough.

The isometric map f_∞ is totally real. It remains to show that the limit map f_∞ of the sequence of totally real maps $(f_{k,i})$ generated by the above Corrugation Process is totally real as well. To do so, we consider the notion of J-density: given a map $f : M \to (W, J, h)$ its J-density is the map $\kappa(f) : M \to \mathbb{R}_{\geq 0}$ defined by

$$\kappa(f)(x) = \sqrt{|vol_W(df(e_1), \ldots, df(e_m), Jdf(e_1), \ldots, Jdf(e_m))|}$$

where vol_W is the volume form of W and $(e_1, \ldots, e_m)$ is any orthonormal basis of T_xM for the pull back metric $(f^*h)_x$ (see [Bor98]). Obviously, $\kappa(f)(x)$ does not depend on the chosen orthonormal basis and $df(T_xM)$ is totally real in $T_{f(x)}W$ if and only if $\kappa(f)(x) > 0$. It is Lagrangian if $\kappa(f)(x) = 1$. Our goal is to show that $\kappa(f_\infty) > 0$.

Lemma 63. *If the vector field u is chosen to be f^*h-orthogonal to $\ker d\pi$ on every point $x \in U$ then the J-density of the map $f_\epsilon = CP_\gamma(f, \pi, N)$ satisfies*

$$\kappa(f_\epsilon) \geq \frac{1}{\sqrt{\mu(u^*, u^*)}}\kappa(f) + O(1/N)$$

where $u^* := u/\|u\|_{f^*h}$ is the normalized vector u for the metric f^*h.

Proof of Lemma 63.– Let $(e_1, \ldots, e_m)$ be a local f^*h-orthonormal basis of TM over $U \subset M$ such that $Span(e_1, \ldots, e_{m-1}) := \ker \ell$. We choose the vector field u to be f^*h-orthogonal to $\ker \ell$ and such that $\ell(u) = 1$. Observe that $e_m = \pm u/\|df(u)\|_h = \pm u^*$. Let $\widetilde{e_m} = \frac{e_m}{\sqrt{\mu(e_m, e_m)}} = \pm\frac{u}{\sqrt{\mu(u,u)}}$. Since

$$f_\epsilon^*h = \mu + O(1/N) \text{ with } \mu = f^*h + \rho\ell \otimes \ell + O(1/N)$$

by Subsection 2.2.5, we deduce that $(e_1, \ldots, e_{m-1}, \widetilde{e_m})$ is μ-orthogonal and thus approximatively f_ϵ^*h-orthonormal. In particular

$$\begin{aligned} \kappa(f_\epsilon)^2 &= |vol_W(df_\epsilon(e_1), \ldots, df_\epsilon(\widetilde{e}_m), Jdf_\epsilon(e_1), \ldots, Jdf_\epsilon(\widetilde{e}_m))| + O(1/N) \\ &= \frac{1}{\mu(e_m, e_m)}|vol_W(df_\epsilon(e_1), \ldots, df_\epsilon(e_m), Jdf_\epsilon(e_1), \ldots, Jdf_\epsilon(e_m))| + O(1/N) \end{aligned}$$

The volume form $vol_W : \Lambda^{2m}(TW) \to \mathbb{R}$ is a C^∞ function and from

$$d_{TW}(df_\epsilon(u), \gamma(\cdot, N\pi)) = O(1/N) \quad \text{and} \quad d_{TW}(df_\epsilon(e_j), df(e_j)) = O(1/N)$$

for $j \in \{1, \ldots, m-1\}$, we deduce

$$\begin{aligned} \kappa(f_\epsilon)^2 &= \frac{1}{\mu(u, u)}|vol_W(df(e_1), \ldots, df(e_{m-1}), \gamma, Jdf(e_1), \ldots, Jdf(e_{m-1}), J\gamma)| \\ &\quad + O(1/N). \end{aligned}$$

Here γ stands for $\gamma(\cdot, N\pi)$. Recalling that

$$\gamma(\cdot, Nt) = r\cos(\alpha\cos(2\pi Nt))\mathbf{t} + r\sin(\alpha\cos(2\pi Nt))J\mathbf{t} + [df(u)]^P$$

and replacing in the above expression we obtain

$$\kappa(f_\epsilon)^2 = \frac{r^2}{\mu(u,u)}|vol_W(df(e_1),\ldots,df(e_{m-1}),\mathbf{t},Jdf(e_1),\ldots,Jdf(e_{m-1}),J\mathbf{t})| + O(1/N)$$

As

$$\mathbf{t} = \frac{[df(u)]^{P^\perp}}{\|[df(u)]^{P^\perp}\|_h} = \pm\frac{[df(e_m)]^{P^\perp}}{\|[df(e_m)]^{P^\perp}\|_h}$$

we have

$$\kappa(f_\epsilon)^2 = \frac{\kappa(f)^2}{\mu(u,u)}\frac{r^2}{\|[df(e_m)]^{P^\perp}\|_h^2} + O(1/N) = \frac{\kappa(f)^2}{\mu(e_m,e_m)}\frac{r^2}{\|[df(u)]^\perp\|_h^2} + O(1/N).$$

We then observe that

$$r^2 = \mu(u,u) - \|[df(u)]^P\|_h^2 = \|df(u)\|_h^2 + \rho - \|[df(u)]^P\|_h^2 = \rho + \|[df(u)^{P^\perp}]\|_h^2$$

to obtain

$$\kappa(f_\epsilon) \geq \frac{1}{\sqrt{\mu(e_m,e_m)}}\kappa(f) + O(1/N).$$

□

In the sequel, it is invariably decided to choose the vector field u to be f^*h-orthogonal to $\ker d\pi = \ker \ell$. To apply Lemma 63 into the body of the work, we need to rephrase it. This is the purpose of the next lemma.

Lemma 64. *We have*

$$\kappa(f_{k,i+1}) \geq \frac{1}{1 + (\delta_{k+1} - \delta_k)\|\Delta\|_{f_0^*h}}\kappa(f_{k,i}) + O(1/N_{k,i})$$

where $\|\Delta\|_{f_0^*h} = \sup_{\{v_x \in TM \mid \|v\|_{f_0^*h}=1\}} \Delta(v_x, v_x).$

Proof of Lemma 64.– Let $\mu_{k,i} := f_{k,i}^*h + \rho_{k,i}\ell_{k,i}\otimes\ell_{k,i}$. We have

$$\begin{aligned} 0 \leq \mu_{k,i}(u_{k,i}^*, u_{k,i}^*) - 1 &= \mu_{k,i}(u_{k,i}^*, u_{k,i}^*) - f_{k,i}^*h(u_{k,i}^*, u_{k,i}^*) \\ &\leq g_{k+1}(u_{k,i}^*, u_{k,i}^*) - g_k(u_{k,i}^*, u_{k,i}^*) \\ &\leq (\delta_{k+1} - \delta_k)\Delta(u_{k,i}^*, u_{k,i}^*) \end{aligned}$$

We observe that $1 = \|u_{k,i}^*\|_{f_{k,i}^*h} \geq \|u_{k,i}^*\|_{f_0^*h}$ because $f_{k,i}^*h \geq f_0^*h$, thus

$$\Delta(u_{k,i}^*, u_{k,i}^*) \leq \|\Delta\|_{f_0^*h}$$

and

$$\mu_{k,i}(u_{k,i}^*, u_{k,i}^*) \leq 1 + (\delta_{k+1} - \delta_k)\|\Delta\|_{f_0^*h}.$$

Lemma 64 is now a straightforward consequence of Lemma 63. □

So, if the corrugation numbers $(N_{k,i})$ and the sequence $(\delta_k)_k$ are conveniently chosen, we can insure that $\kappa(f_\infty) \geq C\kappa(f_0)$ for some $0 < C < 1$ which shows that f_∞ is totally real.

4.2.2 Gauss and Maslov maps

Totally real Grassmannian and Maslov map.– We denote by $TR(m)$ the Grassmannian of totally real m planes of $\mathbb{C}^m$. Given a totally real m-plane Π_0 of $\mathbb{C}^m$, this Grassmannian is identified with the homogeneous space $GL(m,\mathbb{C})/GL(m,\mathbb{R})$ via the map $i_{\Pi_0} : \Pi \mapsto [L]$ where L is any $\mathbb{C}$-linear map such that $L(\Pi_0) = \Pi$. This homogeneous space admits a fibration $\phi : GL(m,\mathbb{C})/GL(m,\mathbb{R}) \to \mathbb{S}^1$ given by

$$[L] \longmapsto \frac{\det^2 L}{|\det^2 L|}.$$

Let $p : TR(W) \to W$ be the totally real Grassman bundle of (W,J) (where $\dim_{\mathbb{R}} W = 2m$). Any choice of a totally real m-plane $\Pi_0(y) \subset T_yW$ induces an identification between the fiber $p^{-1}(y)$ and $GL(m,\mathbb{C})/GL(m,\mathbb{R})$. Thus any local section $\Pi_0 : V \subset W \to TR(W)$ allows to define a map $\phi \circ i_{\Pi_0} : p^{-1}(V) \to \mathbb{S}^1$. Given such a section Π_0, the Gauss map

$$G_f : M \longrightarrow TR(W), \quad x \longmapsto (f(x), df(T_xM))$$

of any totally real embedding $f : M \to (W,J)$ such that $f(M) \subset V$ induces a map $\mathfrak{m}(\Pi_0, f) := \phi \circ i_{\Pi_0} \circ G_f : M \to \mathbb{S}^1$ that we call the *Maslov map*. Observe that a local section Π_0 can be constructed from a totally real embedding $f_0 : M \to (W,J)$: V is a tubular neighborhood of $f_0(M)$ and Π_0 is any extension of $f_0(x) \mapsto df_0(T_xM)$. In that case, if $f(M) \subset V$, we denote by $\mathfrak{m}(f_0, f)$ the corresponding Maslov map.

Maslov map of f_∞.– In the above proof, every map $f_{k,j}$ as well as f_∞ have images lying inside an ϵ-tubular neighborhood $V(\epsilon)$ of $f_0(M)$. If ϵ is small enough, this neighborhood retracts by deformation on $f_0(M)$ and it can be used to construct a local section Π_0 extending the one induced by f_0. In that case, for every (k,j) we write

$$x \mapsto \mathfrak{m}(f_0, f_{k,j})(x) = e^{2i\vartheta_{k,j}(x)}\mathfrak{m}(f_0, f_{k,j-1})(x)$$

where $\vartheta_{k,j}$ is some angle function. If k is large enough we choose this angle function such that $\vartheta_{k,j}(x) \in \,]-\frac{\pi}{2}, \frac{\pi}{2}[$ for all $x \in M$ (this is always possible by the convergence of $(f_{k,j})$). We define inductively a sequence of maps $\mathscr{W}_k : M \to \mathbb{R}$ by $\mathscr{W}_0 = 0$, $\mathscr{W}_{k+1} := \mathscr{W}_k + 2\vartheta_k$ with $\vartheta_k := \sum_{j\in I(k)} \vartheta_{k,j}$. Since the f_k's are C^1 converging toward f_∞, the maps $\mathscr{W}_k$ also C^0 converge toward a map $\mathscr{W}_\infty$ such that

$$\mathfrak{m}(f_0, f_\infty) = e^{i\mathscr{W}_\infty}.$$

The following proposition matches with Proposition 62 for $\ell = (k,j)$.

Proposition 65. *Let* $\mathfrak{m}(f_0, f_\infty) = e^{i\mathscr{W}_\infty} : M \to \mathbb{S}^1$ *be the Maslov map of* f_∞ *and* $\mathscr{W}_\infty = 2\sum_k \vartheta_k$ *be the Maslov argument defined above. Then if* k *is large enough*

$$\vartheta_k = \theta_k + \sum_{j\in I(k)} O\left(\frac{1}{N_{k,j}}\right) \text{ where } \theta_k := \sum_{j\in I(k)} \alpha_{k,j}\cos(2\pi N_{k,j}\pi_{k,j})$$

(if $x \in M$ *is not in the domain of* $\pi_{k,j}$ *it is understood that the corresponding term is zero)*

Observe that $\mathscr{W}_\infty$ is closed to a Weierstrass function $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$, where $0 < a < 1$, b is a positive odd integer, and $ab > 1 + 3\pi/2$. The graph of this function is known to have a self-similar behavior (see Figure 4.1). The proof of Proposition 65 is a straightforward consequence of the following

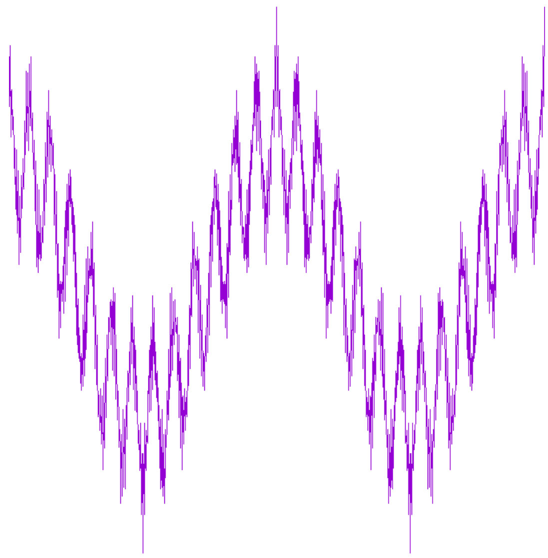


Figure 4.1 – **The graph of a Weierstrass function:** for $a = 0.5$ and $b = 13$.

lemma.

Lemma 66. *Let* $f : U \to W$ *be a totally real map. Let* $\pi : U \to \mathbb{R}$ *be a submersion,* $u : U \to TU$ *a vector field chosen to be* f^*h*-orthogonal to* $\ker d\pi$ *on every point* $x \in U$ *and such that* $d\pi(u) = 1$*. Then, for all* $x \in U$*:*

$$\mathfrak{m}(\Pi_0, f_\epsilon)(x) = e^{2i(\alpha(x)\cos(2\pi N\pi(x)) + O(1/N))}\mathfrak{m}(\Pi_0, f)(x)$$

where $f_\epsilon = CP_\gamma(f, \pi, N)$*.*

Proof of the lemma.– Let $(e_1, \ldots, e_m)$ be a local basis of TM over $U \subset M$ such that $Span(e_1, \ldots, e_{m-1}) := \ker d\pi$ and $e_m = u$. Let $(\epsilon_1, \ldots, \epsilon_m)$ be a local basis of the m-plane field Π_0. We consider the square of the determinant map $z(f) : U \to \mathbb{C}$:

$$z(f) := \det\nolimits_{\mathbb{C}}^2(df(e_1), \dots, df(e_m))$$

where $df(e_1), \dots, df(e_m)$ are seen as complex vectors with complex coordinates relative to the basis $(\epsilon_1, \dots, \epsilon_m)$. Since $df(T_xM)$ is totally real for every x, the complex number $z(f)(x)$ never vanishes. It is readily seen that its argument is $\mathfrak{m}(\Pi_0, f)(x)$. Similarly, the argument of

$$z(f_\epsilon) = \det\nolimits_{\mathbb{C}}^2(df_\epsilon(e_1), \dots, df_\epsilon(e_m))$$

is $\mathfrak{m}(\Pi_0, f_\epsilon)$. From the smoothness of $\det_{\mathbb{C}}^2$ and from

$$d_{TW}(df_\epsilon(u), \gamma(\cdot, N\pi)) = O(1/N) \quad \text{and} \quad d_{TW}(df_\epsilon(e_j), df(e_j)) = O(1/N)$$

for $j \in \{1, \dots, m-1\}$, we deduce

$$z(f_\epsilon) = \det\nolimits_{\mathbb{C}}^2(df(e_1), \dots, df(e_{m-1}), \gamma(\cdot, N\pi)) + O(1/N).$$

As

$$\gamma(\cdot, t) = r\, e^{i\alpha\cos(2\pi N t)}\mathbf{t} + [df(u)]^P \quad \text{and} \quad \mathbf{t} = \frac{[df(u)]^{P^\perp}}{\|[df(u)]^{P^\perp}\|}$$

we have

$$z(f_\epsilon) = \frac{r^2}{\|[df(u)]^{P^\perp}\|^2} e^{2i\alpha\cos(2\pi N\pi(\cdot))} z(f) + O(1/N).$$

□

4.3 Self-similarities and isometric maps

We put in perspective the result obtained for totally real maps with the ones obtained for the isometric embedding of the square flat torus and the reduced sphere in [BJLT13, BBD+18]. In these papers, the differential of the isometric embedding is expressed by an infinite product of *corrugation matrices* and a self similarity property ensues from a *Corrugation Theorem* (see [BJLT13, Theorem 21]). We put into light in this subsection that we have similar infinite product of rotations and self similarities behavior.

Corrugation matrices.– We assume $n = m + 1$. Let $U \simeq [0,1]^m$ be a chart of a m-dimensional oriented manifold and let $f_{k,j} : U \to \mathbb{E}^{m+1}$ be a sequence obtained iteratively by the Nash process (see Subsection 4.2.1). We recall the definition of the corrugation matrices. We denote by $\ell_{k,j}$ the successive linear forms. We are going to build two basis of $\mathbb{E}^{m+1}$ to express the corrugation matrices. We denote by $u_{k,j+1}$ any vector field such that $\ell_{k,j+1}(u_{k,j+1}) = 1$ and by $\mathbf{t}_{k,j}$ the normalisation of $df_{k,j}(u_{k,j+1})$. Let $(V^1_{k,j+1}, \cdots, V^{m-1}_{k,j+1})$ be any local basis of Ker $\ell_{k,j+1}$. We put

$$\mathbf{n}_{k,j} := \frac{\mathbf{t}_{k,j} \wedge df_{k,j}(V^1_{k,j+1}) \wedge \dots \wedge df_{k,j}(V^{m-1}_{k,j+1})}{\|\mathbf{t}_{k,j} \wedge df_{k,j}(V^1_{k,j+1}) \wedge \dots \wedge df_{k,j}(V^{m-1}_{k,j+1})\|}.$$

Note that $\mathbf{n}_{k,j}$ is normal to $df_{k,j}(TU)$. We denote by $v^1_{k,j}, \cdots, v^{m-1}_{k,j}$ the Gram-Schmidt orthonormalisation of $df_{k,j}(V^1_{k,j}), \cdots, df_{k,j}(V^{m-1}_{k,j})$ and we put

$$v^\perp_{k,j} := v^1_{k,j} \wedge \cdots v^{m-1}_{k,j} \wedge \mathbf{n}_{k,j}.$$

Let $x \in U$. Observe that $\mathscr{B}_{k,j}(x) := (v^\perp_{k,j}, v^1_{k,j}, \cdots, v^{m-1}_{k,j}, \mathbf{n}_{k,j})(x)$ is an orthonormal basis of $\mathbb{E}^{m+1}$. We also introduce $m-1$ vectors $v^{1+}_{k,j}, \cdots, v^{(m-1)+}_{k,j}$ as the Gram-Schmidt orthonormalisation of $df_{k,j}(V^1_{k,j+1}), \cdots, df_{k,j}(V^{m-1}_{k,j+1})$ and we define a second orthonormal basis $\mathscr{B}^+_{k,j}(x) := (\mathbf{t}_{k,j}, v^{1+}_{k,j}, \cdots, v^{(m-1)+}_{k,j}, \mathbf{n}_{k,j})(x)$. We denote by $\mathscr{R}_{k,j}(x)$ the rotation matrix that maps $\mathscr{B}_{k,j}(x)$ to $\mathscr{B}^+_{k,j}(x)$ and by $\mathscr{L}_{k,j+1}(x, N_{k,j+1})$ the rotation matrix that maps $\mathscr{B}^+_{k,j}(x)$ to $\mathscr{B}_{k,j+1}(x)$. The *corrugation matrix* is defined by the product

$$\mathscr{M}_{k,j+1}(x, N_{k,j+1}) := \mathscr{L}_{k,j+1}(x, N_{k,j+1})\mathscr{R}_{k,j}(x)$$

and maps $\mathscr{B}_{k,j}(x)$ to $\mathscr{B}_{k,j+1}(x)$.

Corrugation theorem.– Corrugation matrices encode both the differential and the Gauss map of the limit embedding, via the following infinite product:

$$\begin{pmatrix} v^\perp_\infty \\ v^1_\infty \\ \vdots \\ v^{m-1}_\infty \\ \mathbf{n}_\infty \end{pmatrix}(x) = \left(\prod_{k,j} \mathscr{M}_{k,j+1}(x, N_{k,j+1})\right) \begin{pmatrix} v^\perp_0 \\ v^1_0 \\ \vdots \\ v^{m-1}_0 \\ \mathbf{n}_0 \end{pmatrix}(x)$$

In their construction of an isometric embedding of the flat torus, the authors succeed in reducing the number of the direction of corrugation to three. As a consequence, they could prove that the matrices $\mathscr{R}_{k,j}$ converge toward three constant matrices (depending on the value of j) when k goes to infinity. Regarding the matrices $\mathscr{L}_{k,j}$ they show that they are equal to a simple rotation of angle $\alpha_{k,j+1}(x)\cos(2\pi N_{k,j+1}\pi_{k,j+1}(x))$ modulo $O(1/N_{k,j+1})$. Therefore, up to the three constant matrices, the infinite product of the $\mathscr{M}_{k,j}$ is similar to a product of rotations whose angles oscillate with increasing frequencies. Here, we observe that if the property on $\mathscr{R}_{k,j}$ is specific to their construction, the asymptotic behavior of $\mathscr{L}_{k,j}$ still holds in a general context.

Proposition 67. *Let $f_{k,j} : U \simeq [0,1]^m \to \mathbb{E}^{m+1}$ be a sequence obtained iteratively by the Nash process. If $f_{k,j+1}$ is obtained from $f_{k,j}$ by the corrugation process of Proposition 47, then for every $x \in U$, we have*

$$\mathscr{L}_{k,j+1}(x, N_{k,j+1}) = \begin{pmatrix} \cos\theta_{k,j+1} & 0 & \sin\theta_{k,j+1} \\ 0 & Id & 0 \\ -\sin\theta_{k,j+1} & 0 & \cos\theta_{k,j+1} \end{pmatrix} + O\left(\frac{1}{N_{k,j+1}}\right)$$

where $\theta_{k,j+1} := \alpha_{k,j+1}(x)\cos(2\pi N_{k,j+1}\pi_{k,j+1}(x))$.

Proof.– The proof is a straightforward adaptation of Lemma 20 from [BJLT13].

□

The totally real case.– In the totally real case, the above approach can be adapted as follows. Let $(V^1, \dots, V^m)$ be a local basis of U and $f_{k,j}$ be a totally real map. We put $v^i_{k,j} = df_{k,j}(V^i)$ and observe that the family $\mathscr{B}_{k,j} = (v^1_{k,j}, \dots, v^m_{k,j}, Jv^1_{k,j}, \dots, Jv^m_{k,j})$ is a basis of $\mathbb{E}^{2m}$. The differential of the limit embedding f_∞ is determined by

$$\begin{pmatrix} v^1_\infty \\ \vdots \\ v^m_\infty \\ Jv^1_\infty \\ \vdots \\ Jv^m_\infty \end{pmatrix} = \left(\prod_{k,j} \mathscr{M}_{k,j+1}\right) \begin{pmatrix} v^1_0 \\ \vdots \\ v^m_0 \\ Jv^1_0 \\ \vdots \\ Jv^m_0 \end{pmatrix}$$

where $\mathscr{M}_{k,j+1}$ is the matrix that maps the basis $\mathscr{B}_{k,j}$ to the basis $\mathscr{B}_{k,j+1}$. Each $\mathscr{M}_{k,j}$ is a $2m \times 2m$ real matrix which commutes with J. We denote by $\mathscr{M}^{\mathbb{C}}_{k,j} \in GL(m, \mathbb{C})$ its complexification. The Maslov map $\mathfrak{m}(f_0, f_\infty)$ is the square of the determinant of the infinite product of the $\mathscr{M}^{\mathbb{C}}_{k,j}$'s divided by its module (so that its image lies in $\mathrm{U}(1)$). Observe that this Maslov map is not affected by the rotations in the tangent spaces encoded by the matrices $\mathscr{R}_{k,j}$. As stated in Proposition 65, this fact allows to obtain an analogy with a Weierstrass function even if the directions of the corrugations are not under control.

Appendix A

Details of calculations of the proof of Theorem 55

We give here the details of calculations of the average of the shape $c : A \times \mathbb{R}/\mathbb{Z} \to \mathbb{C} \times \mathbb{R}$ defined by

$$c(\eta, \theta, \beta, t) := (\exp(i\alpha_{\theta,\beta}(t)) + \eta \cos\theta, 1).$$

Let $\beta' = \beta - \frac{\eta}{2}$. The expression of the angular function $\alpha_{\theta,\beta}$ is given by

$$\alpha_{\theta,\beta}(t) = \begin{cases} \dfrac{4\pi}{\eta} t & \text{on } \left[0, \dfrac{\eta\theta}{4\pi}\right] \\ \theta & \text{on } \left[\dfrac{\eta\theta}{4\pi}, \dfrac{\beta'}{2} + \dfrac{\eta\theta}{4\pi}\right] \\ \dfrac{4\pi}{\eta} t - \dfrac{4\pi}{\eta}\dfrac{\beta'}{2} & \text{on } \left[\dfrac{\beta'}{2} + \dfrac{\eta\theta}{4\pi}, \dfrac{\beta'}{2} + \dfrac{\eta(2\pi - \theta)}{4\pi}\right] \\ 2\pi - \theta & \text{on } \left[\dfrac{\beta'}{2} + \dfrac{\eta(2\pi - \theta)}{4\pi}, \dfrac{1}{2} - \dfrac{\eta\theta}{4\pi}\right] \\ \dfrac{4\pi}{\eta} t + 2\pi - \dfrac{2\pi}{\eta} & \text{on } \left[\dfrac{1}{2} - \dfrac{\eta\theta}{4\pi}, \dfrac{1}{2}\right] \end{cases}$$

on $[0, \frac{1}{2}]$ and satisfies $\alpha_{\theta,\beta}(t) = \alpha_{\theta,\beta}(1-t)$ for all $t \in [0, \frac{1}{2}]$. Intervals are chosen to the function will not be constant for a time of $\eta/2$. For short we denote $\alpha(\cdot) = \alpha_{\theta,\beta}(\cdot)$.

We first show that know the value of the integral of $\exp(i\alpha(\cdot))$ on $[0, \frac{1}{2}]$ is sufficient. Note that

$$\int_0^{1/2} \exp(i\alpha(s))ds \quad = \quad \int_{1/2}^{1} \exp(i\alpha(s))ds.$$

Indeed, by substitution and the properties of $\alpha(\cdot)$, we have

$$\begin{aligned}\int_0^{1/2}\exp(i\alpha(s))ds &= \int_0^{1/2}\exp(i\alpha(1-s))ds \\ &= \int_1^{1/2}\exp(i\alpha(t))(-dt) \\ &= \int_{1/2}^{1}\exp(i\alpha(t))dt\end{aligned}$$

So we only calculate the average of $s\mapsto\exp(i\alpha(s))$ on $[0,1/2]$. We split the integral on the five intervals which define α on $[0,\frac{1}{2}]$, and we calculate the three where α is not constant.

On $\left[0,\frac{\eta\theta}{4\pi}\right]$ we have

$$I_1 := \int_0^{\eta\theta/4\pi}\exp\left(i\frac{4\pi}{\eta}t\right)dt.$$

On $\left[\frac{\beta'}{2}+\frac{\eta\theta}{4\pi},\frac{\beta'}{2}+\frac{\eta(2\pi-\theta)}{4\pi}\right]$ we have

$$\begin{aligned}I_3 &:= \int_{\beta'/2+\eta\theta/4\pi}^{\beta'/2+\eta(2\pi-\theta)/4\pi}\exp\left(i\left(\frac{4\pi}{\eta}t-\frac{4\pi}{\eta}\frac{\beta'}{2}\right)\right)dt \\ &= \int_{\eta\theta/4\pi}^{\eta(2\pi-\theta)/4\pi}\exp\left(i\left(\frac{4\pi}{\eta}\left(s+\frac{\beta'}{2}\right)-\frac{4\pi}{\eta}\frac{\beta'}{2}\right)\right)ds \\ &= \int_{\eta\theta/4\pi}^{\eta(2\pi-\theta)/4\pi}\exp\left(i\frac{4\pi}{\eta}s\right)ds.\end{aligned}$$

On $\left[\frac{1}{2}-\frac{\eta\theta}{4\pi},\frac{1}{2}\right]$ we have

$$\begin{aligned}I_5 &:= \int_{1/2-\eta\theta/4\pi}^{1/2}\exp\left(i\left(\frac{4\pi}{\eta}t+2\pi-\frac{2\pi}{\eta}\right)\right)dt \\ &= \int_{1/2-\eta\theta/4\pi}^{1/2}\exp\left(i\left(\frac{4\pi}{\eta}\left(t+\frac{\eta}{2}-\frac{1}{2}\right)\right)\right)dt \\ &= \int_{\eta(2\pi-\theta)/4\pi}^{\eta/2}\exp\left(i\frac{4\pi}{\eta}s\right)ds.\end{aligned}$$

The sum of this three integrals is

$$I_1+I_3+I_5 = \int_0^{\eta/2}\exp\left(i\frac{4\pi}{\eta}t\right)dt = \int_0^1\exp(i2\pi s)ds = 0.$$

So the average of c is

$$\begin{aligned}\overline{c}(\eta,\theta,\beta) &= (\beta' e^{i\theta}+(1-\eta-\beta')e^{i(2\pi-\theta)}+\eta\cos\theta,1) \\ &= (\beta e^{i\theta}+(1-\beta)e^{-i\theta},1).\end{aligned}$$

Bibliography

[Ada62] J. F. Adams. Vector fields on spheres. *Ann. of Math. (2)*, 75:603–632, 1962.

[Ape86] F. Apery. La surface de Boy. (Boy's surface). *Adv. Math.*, 61:185–266, 1986.

[Ape87] F. Apery. Models of the real projective plane. Computer graphics of Steiner and Boy surfaces. Computer Graphics and Mathematical Models. Braunschweig/Wiesbaden: Friedr. Vieweg & Sohn. XI, 1987.

[BBD+18] E. Bartzos, V. Borrelli, R. Denis, F. Lazarus, D. Rohmer, and B. Thibert. An explicit isometric reduction of the unit sphere into an arbitrarily small ball. *Found. of Comput. Math.*, 109(4):1015–1042, 2018.

[BJLT12] V. Borrelli, S. Jabrane, F. Lazarus, and B. Thibert. Flat tori in three-dimensional space and convex integration. *Proc. Natl. Acad. Sci. USA*, 109(19):7218–7223, 2012.

[BJLT13] V. Borrelli, S. Jabrane, F. Lazarus, and B. Thibert. *Isometric embeddings of the square flat torus in ambient space*, volume 24 of *Ensaios Matemáticos*. Sociedade Brasileira de Matemática, Rio de Janeiro, 2013.

[Bor53] A. Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953.

[Bor98] V. Borrelli. Maslov form and J-volume of totally real immersions. *J. Geom. Phys.*, 25(3-4):271–290, 1998.

[Boy01] W. Boy. Über die Abbildung der projektiven Ebene auf eine im Endlichen geschlossene singularitätenfreie Fläche. *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.*, 1901:20–33, 1901.

[Bry84] R. Bryant. A duality theorem for Willmore surfaces. *J. Differ. Geom.*, 20:23–53, 1984.

[BV19] T. Buckmaster and V. Vicol. Nonuniqueness of weak solutions to the Navier-Stokes equation. *Ann. Math. (2)*, 189(1):101–144, 2019.

[BZ95] D. Burago and V. Zalgaller. Isometric piecewise-linear immersions of two-dimensional manifolds with polyhedral metric into $\mathbb{R}^3$. *St. Petersbg. Math. J.*, 7(3):76–95, 1995.

[CDLDR17] M. Colombo, C. De Lellis, and L. De Rosa. Ill-posedeness of leray solutions for the ipodissipative navier–stokes equations. ArXiv, 2017.

[CDLS12] S. Conti, C. De Lellis, and L. Székelyhidi, Jr. h-principle and rigidity for $C^{1,\alpha}$ isometric embeddings. In *Nonlinear partial differential equations*, volume 7 of *Abel Symp.*, pages 83–116. Springer, Heidelberg, 2012.

[CL18] L. Codenotti and M. Lewicka. Visualization of the convex integration solutions to the Monge-Ampère equation. *ArXiv e-prints*, July 2018.

[D'A95] G. D'Ambra. Nash C^1-embedding theorem for Carnot-Carathéodory metrics. *Differ. Geom. Appl.*, 5(2):105–119, 1995.

[D'A00] G. D'Ambra. An application to the h-principle to C^1-isometric immersions in contact manifolds. *Commun. Anal. Geom.*, 8(2):347–373, 2000.

[DD06] G. D'Ambra and M. Datta. Isotropic C^1-immersions in a pseudo-Riemannian manifold. *Differ. Geom. Appl.*, 24(2):142–149, 2006.

[DL02] G. D'Ambra and A. Loi. A symplectic version of Nash C^1-isometric embedding theorem. *Differ. Geom. Appl.*, 16(2):167–179, 2002.

[DLS09] C. De Lellis and L. Székelyhidi, Jr. The euler equation as a differential inclusion. *Annals of Math.*, 170(3), 2009.

[EM02] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. A.M.S., Providence, 2002.

[Fra87] G. Francis. *A topological picturebook.* Springer-Verlag, New York, 1987.

[Gei03] H. Geiges. h-principles and flexibility in geometry. *Mem. Amer. Math. Soc.*, 164(779), 2003.

[GH12] S. Goodman and G. Howard. Generic maps of the projective plane with a single triple point. *Math. Proc. Camb. Philos. Soc.*, 152(3):455–472, 2012.

[GK09] S. Goodman and M. Kossowski. Immersions of the projective plane with one triple point. *Differ. Geom. Appl.*, 27(4):527–542, 2009.

[Gro73] M. Gromov. Convex integration of differential relations. I. *Math USSR Izvestia*, volume 7:329–343, 1973.

[Gro86] M. Gromov. *Partial differential relations.* Springer-Verlag, Berlin, 1986.

[Gro17] M. Gromov. Geometric, algebraic, and analytic descendants of Nash isometric embedding theorems. *Bull. Am. Math. Soc., New Ser.*, 54(2):173–245, 2017.

[HCV32] D. Hilbert and S. Cohn-Vossen. *Anschauliche Geometrie.*, volume 37. Springer, Berlin, 1932.

[Hir76] M. Hirsch. *Differential topology.* Springer-Verlag, New York-Heidelberg, 1976. Graduate Texts in Mathematics, No. 33.

[Kui55] N. Kuiper. On C^1-isometric imbeddings. *Indag. Math.*, volume 17:545–556, 1955.

[Kus87] R. Kusner. Conformal geometry and complete minimal surfaces. *Bull. Am. Math. Soc., New Ser.*, 17:291–295, 1987.

[LD13] E. Le Donne. Lipschitz and path isometric embeddings of metric spaces. *Geom. Dedicata*, 166:47–66, 2013.

[Lev95] S. Levy. *Making waves. A guide to the ideas behind Outside In. With an article by Bill Thurston and an afterword by Albert Marden.* Wellesley, MA: A K Peters, 1995.

[Mor78] Bernard Morin. Equations du retournement de la sphère. *C. R. Acad. Sci., Paris, Sér. A*, 287:879–882, 1978.

[Mur18] E. Murphy. Loose Legendrian Embeddings in High Dimensional Contact Manifold. arXiv, 2018.

[Nas54] J. Nash. C^1 isometric imbeddings. *Ann. of Math.*, volume 60:383–396, 1954.

[PS81] J.-P. Petit and J. Souriau. Une représentation analytique de la surface de Boy. *C. R. Acad. Sci. Paris Sér. I Math.*, 293(4):269–272, 1981.

[QC] P. T. Quintanar Cortés. An explicit pl-embedding of the flat square torus into $\mathbb{E}^3$. preprint.

[RS09] M. Raussen and C. Skau. Interview with Mikhail Gromov. *Newsletter of the European Mathematical Society*, 2009.

[Sma58] S. Smale. A classification of immersions of the two-sphere. *Trans. Amer. Math. Soc.*, 90:281–290, 1958.

[Spr98] D. Spring. *Convex integration theory*, volume 92 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1998. Solutions to the h-principle in geometry and topology.

[Was16] M. Wasem. *Convex Integration, Isometric Extensions and Approximations of Curves*. PhD thesis, ETH-Zurich, 2016.

[Whi63] G. W. Whitehead. Note on cross-sections in Stiefel manifolds. *Comment. Math. Helv.*, 37:239–240, 1962/1963.

List of Figures

Intégration convexe effective

Résumé : Le but de cette thèse est de proposer une version effective de la théorie de l'intégration convexe. Cette théorie, inventée par M. Gromov dans les années 70, permet de résoudre des relations différentielles, i.e. des équations / inéquations aux dérivées partielles. Dans cette thèse, nous introduisons une formule appelée *procédé de corrugation*. Cette formule peut se substituer à la formule principale de la théorie de l'intégration convexe. L'expression de cette nouvelle formule est particulièrement intéressante pour des relations que nous caractérisons dans cette thèse : les *relations de Kuiper*. Nous montrons que ce type de relation se rencontre en géométrie différentielle, par exemple pour les immersions, les immersions isométriques et les applications totalement réelles. En particulier, les résultats obtenus dans cette thèse nous permettent de construire directement une nouvelle immersion de $\mathbb{R}P^2$. Le procédé de corrugation et les relations de Kuiper fournissent également un cadre propice à l'étude des propriétés d'auto-similarités observées dans les constructions de plongements C^1-isométriques d'un tore plat et d'une sphère réduite effectuées par l'équipe Hévéa. Précisément, nous montrons une propriété d'auto-similarité pour des plongements C^1-isométriques totalement réels.

Mots clés : intégration convexe, immersions, isométries, applications totalement réelles, C^1-fractal.

Effective Convex Integration

Abstract : The aim of this thesis is to propose an effective version of Convex Integration Theory. This theory, developed by M. Gromov in the 70's, allows to solve differential relations, i.e. partial differential equalities / inequalities. In this thesis, we introduce a formula called *Corrugation Process*. The key formula of the Convex Integration Theory can be substituted by this new formula. The expression of the Corrugation Process is interesting for the relations characterized in this thesis : the *Kuiper relations*. We show that this kind of relation appears in differential geometry, for example for immersions, for isometric immersions and for totally real maps. In particular, the results obtained in this thesis allow to build directly a new immersion of $\mathbb{R}P^2$. The Corrugation Process and the notion of Kuiper relation offer a natural framework to study potential self-similarity properties. Such properties were already observed for the C^1-isometric embedding of a flat torus and of a reduced sphere built by the Hevea team. Precisely, in this thesis, we show a self-similarity property for some C^1-isometric totally real embeddings.

Keywords : Convex Integration, immersions, isometric maps, totally real maps, C^1-fractal.

Image en couverture : une nouvelle immersion de $\mathbb{R}P^2$ construite grâce au procédé de corrugations.

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INFOMATHS

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