Long time behaviour of kinetic equations

Chaoen Zhang

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Chaoen ZHANG

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Directeur de thèse : **Arnaud GUILLIN**
Co-encadrant de thèse : **Liming WU**

Jury de soutenance:

Mme Véronique BAGLAND,	Université Clermont Auvergne	Examinateur
M. François BOLLEY,	Sorbonne Université	Rapporteur
M. Andreas EBERLE,	Universität Bonn	Rapporteur
M. Arnaud GUILLIN,	Université Clermont Auvergne	Directeur de thèse
M. Clément MOUHOT,	University of Cambridge	Rapporteur
Mme Marjolaine PUEL,	Université Nice Sophia Antipolis	Examinateur
M. Liming WU,	Université Clermont Auvergne	Co-directeur de thèse

Université Clermont Auvergne
Laboratoire Mathématiques Blaise Pascal (LMBP)
UMR CNRS 6620, Clermont Ferrand, France

“How far can you go with the Cauchy-Schwarz inequality and integration by parts ? ”[1]

[1] Dominique Bakry, Ivan Gentil and Michel Ledoux to Leonard Gross in their monograph.

À LA MÉMOIRE DE MON PÈRE:

“Tu n’es plus là où tu étais,
mais tu es partout là où je suis.”[2]

[2] Une citation attribuée à Victor Hugo.

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Lastly I conclude by citing two stanzas from QU Yuan's *Li Sao*:
"朝发轫于苍梧兮，夕余至乎县圃。欲少留此灵琐兮，日忽忽其将暮。
吾令羲和弭节兮，望崦嵫而勿迫。路漫漫其修远兮，吾将上下而求索。"

Résumé

Cette thèse est consacrée au comportement à long terme de l'équation cinétique de Fokker-Planck et de l'équation de McKean-Vlasov. Le manuscrit est composé d'une introduction et de six chapitres.

L'équation cinétique de Fokker-Planck est un exemple de base de la théorie de l'hypocoercivité de Villani qui affirme la décroissance exponentielle dans le temps en l'absence de coercivité. Dans son mémoire AMS, Villani a prouvé l'hypocoercivité de l'équation cinétique de Fokker-Planck en $\mathrm{H}^1(\mu)$, $\mathrm{L}^2(\mu)$ ou entropie. Cependant, une condition sur la bornitude de l'Hessien de l'hamiltonien a été imposée dans le cas entropique. Nous montrons au chapitre 2 comment nous pouvons affaiblir cette hypothèse par des multiplicateurs bien choisis à l'aide d'une inégalitéde Sobolev logarithmique pondérée. Nous montrons que nos conditions sont satisfaites sous certaines conditions pratiques de fonction de Lyapunov.

Dans le chapitre 4, nous appliquons les idées de Villani et certaines conditions de Lyapunov pour prouver l'hypocoercivité en H^1 pondéré dans le cas d'une interaction de champ moyen avec un taux de convergence exponentielle indépendant du nombre de particules. Pour cet objectif nous devons établir l'inégalité de Poincaré uniforme (sur le nombre de particules) et rendre une estimation connue de Villani qui etait dimension-dépendante, dimension-indépendante.

Au chapitre 6, nous étudions la contraction hypocoercive de la distance L^2 -Wasserstein et nous retrouvons le taux optimal dans le cas du potentiel quadratique. La méthode est basée sur la dérivée en temps de la distance de Wasserstein. Au chapitre 7, le théorème d'hypoercivité de Villani dans l'espace H^1 pondéré est generalisé aux espaces H^k pondérés par une norm auxiliaire avec des termes mélangés bien choisis.

L'équation de McKean-Vlasov est une équation diffusive non linéaire non locale. Il est bien connu qu'il a une structure de gradient-flot. Cependant, les résultats connus dépendant fortement des hypothèses de convexité. De telles hypothèses sont notamment assouplies dans les chapitres 3 et 5 où nous prouvons la convergence exponentielle vers l'équilibre respectivement en énergie libre et la distance L^1-Wasserstain, sous la condition de Dobrushin-Zegarlinski de l'absence de phase de transition . Notre approche est basée sur la théorie de la limite de champ moyen. Autrement dit, nous étudions le système d'un grand nombre de particules avec une interaction du type champ-moyen, puis passons à la limite par la progation de chaos.

Mots clés: équation cinétique de Fokker-Planck, équation de McKean-Vlasov, convergence à l'équilibre, hypocoercivité, entropie, distance de Wasserstein, inégalité logarithmique de Sobolev, inégalité de Poincaré.

Abstract

This dissertation is devoted to the long time behaviour of the kinetic Fokker-Planck equation and of the McKean-Vlasov equation. The manuscript is composed of an introduction and six chapters.

The kinetic Fokker-Planck equation is a basic example for Villani's hypocoercivity theory which asserts the exponential decay in large time in the absence of coercivity. In his memoir, Villani proved the hypocoercivity for the kinetic Fokker-Planck equation in either weighted H^1, weighted L^2 or entropy.

However, a boundedness condition of the Hessian of the Hamiltonian was imposed in the entropic case. We show in Chapter 2 how we can get rid of this assumption by well-chosen multipliers with the help of a weighted logarithmic Sobolev inequality. Such a functional inequality can be obtained by some tractable Lyapunov condition.

In Chapter 4, we apply Villani's ideas and some Lyapunov conditions to prove hypocoercivity in weighted H^1 in the case of mean-field interaction with a rate of exponential convergence independent of the number N of particles. For proving this we should prove the Poincaré inequality with a constant independent of N, and rends a dimension dependent boundedness estimate of Villani dimension-free by means of the stronger uniform log-Sobolev inequality and Lyapunov function method. .

In Chapter 6, we study the hypocoercive contraction in L^2-Wasserstein distance and we recover the optimal rate in the quadratic potential case. The method is based on the temporal derivative of the Wasserstein distance. In Chapter 7, Villani's hypoercivity theorem in weighted H^1 space is extended to weighted H^k spaces by choosing carefully some appropriate mixed terms in the definition of norm of H^k.

The McKean-Vlasov equation is a nonlinear nonlocal diffusive equation. It is well-Known that it has a gradient flow structure. However, the known results strongly depend on convexity assumptions. Such assumptions are notably relaxed in Chapter 3 and Chapter 5 where we prove the exponential convergence to equilibrium respectively in free energy and the L^1-Wasserstein distance. Our approach is based on the mean field limit theory. That is, we study the associated system of a large number of particles with mean-field interaction and then pass to the limit by propagation of chaos.

Key words : kinetic Fokker-Planck equation, McKean-Vlasov equation, convergence to equilibrium, hypocoercivity, entropy, Wasserstein distance, logarithmic Sobolev inequality, Poincaré inequality.

Contents

Chapter 1

Introduction

This thesis is about the long time behaviour of kinetic equations. In the introduction we shall first introduce some important Lyapunov functionals with corresponding partial differential equations. Then we collect some basic results concerning functional inequalities with an emphasis on the two most important families of functional inequalities, namely the Poincaré inequality and the logarithmic Sobolev inequality. After this we present the Bakry-Émery theory, especially some proofs of the Poincaré inequality and the logarithmic Sobolev inequality under curvature-dimension conditions. We also present Villani's hypocoercivity theory in the particular case of kinetic Fokker-Planck equation. Lastly we briefly present our results.

1.1 Lyapunov functional

Let $f_t = f(t,\cdot)$ be a solution of an evolution equation

$$\partial_t f_t + \mathrm{L} f_t = 0$$

subject to certain initial condition, where L is some operator, linear or nonlinear, acting on some function space. Consider a functional E from some suitable function space to $[0,\infty]$. E is called a *Lyapunov functional* if

$$\frac{\mathrm{dE}(f_t)}{\mathrm{d}t} \le 0,$$

for a certain class of solutions, that is, the functional E is non-increasing along the flow generated by the evolution. Sometimes, if we are able to prove that there exists a constant $\mathrm{C} > 0$ such that

$$\frac{\mathrm{dE}(f_t)}{\mathrm{d}t} \le -\mathrm{CE}(f_t)$$

for all f_t in the specific class of solutions, then we have by Gronwall's lemma

$$\mathrm{E}(f_t) \le e^{-\mathrm{C}t}\mathrm{E}(f_0),$$

that is, we obtain an exponential decay of the solutions in the functional E. Sometimes, we may confront another situation: there exists a nonnegative increasing function Φ on $\mathbb{R}^+$ such that

$$\frac{\mathrm{dE}(f_t)}{\mathrm{d}t} \le -\mathrm{C}\Phi(\mathrm{E}(f_t))$$

or we may get even a system of differential inequalities about the functional E and other related functionals, then we might develop results of Gronwall type to obtain decay of solutions in the E functional.

Searching for such Lyapunov functionals for evolution equations is a fundamental problem towards the study of long time behaviour along the evolution. Such functionals can be used to describe the trend of a nonequilibrium state towards an equilibrium one, hence they are of great importance in partial differential equations arising from statistical physics (in particular, kinetic theory). Besides, it is usually not an easy task to find a good Lyapunov functional for nonlinear equations.

In this section, we shall present several important Lyapunov functionals which are concerned in this thesis,

- L^2-norms, H^1-norms;
- Boltzmann's entropy;
- free energy;
- Wasserstein distances.

We shall put an emphasis on entropy in the present section. And we refer to Villani's review [15] on kinetic theory for the literature on the whole subject.

To fix ideas, and to concentrate on the subject of long time behaviour, integrability and regularity issues are disregarded here. In other words, we shall always assume that the solutions are "smooth" enough in the sense that all manipulations needed (mainly integration by parts, differentiation and integration) in the discussion can be adequately justified.

(I). A classical example is the *heat equation* on Euclidean spaces and the square of L^2-norm or the Dirichlet energy, i.e. the case of $\mathrm{L} = -\Delta$ being the negative Laplace operator and E being the functional $\int f^2 \mathrm{d}x$ or $\int |\nabla f|^2 \mathrm{d}x$ where f is a solution of the heat equation. An integration by parts implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int f^2 \mathrm{d}x = \int f\Delta f \mathrm{d}x = -\int |\nabla f|^2 \mathrm{d}x \le 0,$$

so $\int f^2 \mathrm{d}x$ is nonincreasing along the heat equation.

Similarly, the functional $\int |\nabla f|^2 \mathrm{d}x$ is nonincreasing since we can compute that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla f|^2 \mathrm{d}x = \int \nabla f \cdot (\nabla \Delta f) \mathrm{d}x = -\int |\Delta f|^2 \mathrm{d}x \le 0$$

where an integration by parts can be performed by using derivatives inside Δ and it hence leads to another expression

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla f|^2 \mathrm{d}x = -\int |\nabla^2 f|^2 \mathrm{d}x \le 0$$

where $\nabla^2 f$ stands for the Hessian of V, and $|\nabla^2 f|^2 = \sum_{i,j} |\partial^2_{x_i x_j} f|^2$.

(II). Next we come to one of the most famous and inspiring example, the *Boltzmann equation* and *Boltzmann's* H *functional.* The concept of *entropy* was introduced by Clausius for the second law of thermodynamics which characterizes the irreversibility of spontaneous heat transmission. Later in 1872, in the study of ideal dilute gases, Boltzmann introduced a statistical definition of the entropy, namely the Boltzmann's H functional

$$\mathrm{H}(f) = \int f \log f \tag{1.1.1}$$

for a probability density function f. The purpose of this functional is to identify the physical phenomenon that any state of gas will approach the limit distribution, that is, the so-called Maxwellian distribution. Here is the Boltzmann equation which is used to model the evolution of ideal gases, let the unknown $f = f(t,x,v)$ be the distribution of the gas at time t with position x and velocity v,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \mathrm{Q}(f,f) \tag{1.1.2}$$

where $v \cdot \nabla_x$ stands for the free transport operator, while Q is the quadratic Boltzmann collision operator given by

$$\mathrm{Q}(f,f) = \int_{\mathbb{R}^d} \int_{\mathrm{S}^{d-1}} (f' f'_* - f f_*) \mathrm{B}(v - v_*, \omega) \mathrm{d}\omega \mathrm{d}v_*. \tag{1.1.3}$$

Here $\mathrm{Q}(f,f) = \mathrm{Q}(f,f)(t,x,v)$, $v_* \in \mathbb{R}^d$, $\omega \in \mathrm{S}^{d-1}$, $\mathrm{d}\omega$ stands for the volume element on the sphere, the Boltzmann's collision kernel $\mathrm{B}(v - v_*, \omega)$ is a nonnegative function which only depends on $|v - v_*|$ and $|\langle (v-v_*)/|v-v_*|, \omega\rangle|$, and we have used the standard abbreviations: $f = f(t,x,v)$, $f_* = f(t,x,v_*)$, $f' = f(t,x,v')$, $f'_* = f(t,x,v'_*)$ where v' and v'_* stand for the velocities before collision, v and v_* stand for the velocities after collision, that is,

$$\begin{cases} v' = v - \langle v - v_*, \omega\rangle\omega, \\ v'_* = v_* + \langle v - v_*, \omega\rangle\omega \end{cases} \tag{1.1.4}$$

(such that the collisions are elastic: $v + v_* = v' + v'_*$, $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$, which correspond to conservation of momentum and kinetic energy).

It is not difficult to verify for well-behaved solutions f that the mass, momentum and kinetic energy are conserved (see the proof below),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f \mathrm{d}v\mathrm{d}x = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\int f v \mathrm{d}v\mathrm{d}x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int f \frac{|v|^2}{2} \mathrm{d}v\mathrm{d}x = 0.$$

Now we state the celebrated Boltzmann's H theorem,

Theorem 1.1 (Boltzmann's *H* theorem)**.** *Let $(f_t)_{t\geq 0}$ be a smooth positive solution of the Boltzmann equation* (1.1.2)*, then Boltzmann's* H *functional* $\mathrm{H}(f)$ *is nonincreasing in time t. Indeed, it holds*

$$-\frac{d}{dt}\mathrm{H}(f) = \frac{1}{4}\int (f f_* - f' f'_*)(\log f f_* - \log f' f'_*)\mathrm{B}(v - v_*, \omega) d\omega dv_* dv dx \geq 0.$$

The Boltzmann's *H* theorem consisted of two parts: (i) the monotonicity of the entropy (as stated above), and (ii) the classification of the equilibrium states. Because of the great importance of the

Boltzmann's entropy, we think it is good to include the proof of the monotonicity, although it is well-known. About the second part, we only add a few words here. Define the *entropy production functional* $\mathrm{D}(f)$ as

$$\mathrm{D}(f) := -\int \mathrm{Q}(f,f)\log f \,\mathrm{d}v$$

or

$$\mathrm{D}(f) = \frac{1}{4}\int (ff_* - f'f'_*)(\log ff_* - \log f'f'_*)\mathrm{B}(v-v_*,\omega)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v$$

such that at least formally

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}(f) = \int \mathrm{D}(f)\,\mathrm{d}x.$$

The equilibrium states of gases are then related to the solutions of the equation $\mathrm{D}(f) = 0$. *Assume furthermore that the collision kernel* B *satisfies* $\mathrm{B} > 0$ *almost everywhere, Boltzmann proved that the local equilibrium state must be a Maxwellian distribution.* Indeed, it can be seen that $\mathrm{D}(f) = 0$ if and only if

$$f'f'_* = ff_*.$$

This equation implies that $\nabla \log f$ is proportional to v up to some additive constant vector. For more about the characterization of the equilibrium state, we refer to [15] for details and references.

Proof. (1). We claim that for a test function $\phi = \phi(v)$, the so-called *Boltzmann's weak formulation* holds

$$\int \mathrm{Q}(f,f)\phi\,\mathrm{d}v = \frac{1}{4}\int (f'f'_* - ff_*)(\phi + \phi_* - \phi' - \phi'_*)\mathrm{B}(v-v_*,\omega)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v \tag{1.1.5}$$

where the above-mentioned abbreviation $'$ and $_*$ was applied to ϕ. This weak formulation follows from the symmetries of the collision operator Q. Indeed, the symmetries allow us to perform changes of variables without much changes of the form of the integrand. Denote in this proof that

$$\Phi_* = \int (f'f'_* - ff_*)\phi_*\mathrm{B}(v-v_*,\omega)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v$$

and define Φ, Φ', Φ'_* in the same way (replacing ϕ_* in the integrand by respectively ϕ, ϕ', ϕ'_*). Now interchanging the variable v and v_* in Φ, we obtain

$$\begin{aligned}\Phi &= \int (f'_*f' - f_*f)\phi_*\mathrm{B}(v_*-v,\omega)\,\mathrm{d}\omega\,\mathrm{d}v\,\mathrm{d}v_*\\ &= \int (f'f'_* - ff_*)\phi_*\mathrm{B}(v-v_*,\omega)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v = \Phi_*\end{aligned}$$

since the collision kernel satisfies $\mathrm{B}(v_*-v,\omega) = \mathrm{B}(v-v_*,\omega)$. Now we perform a change of variables $(v,v_*) \mapsto (v',v'_*)$ to demonstrate that

$$\Phi = -\Phi', \quad \Phi_* = -\Phi'_*.$$

We compute the Jacobian first. Let the unit vector ω be fixed. By direct calculation, the Jacobian matrix is given by

$$\begin{pmatrix} \frac{\partial v'}{\partial v} & \frac{\partial v'}{\partial v_*} \\ \frac{\partial v'_*}{\partial v} & \frac{\partial v'_*}{\partial v_*} \end{pmatrix} = \begin{pmatrix} \mathrm{I} - \omega\otimes\omega & \omega\otimes\omega \\ \omega\otimes\omega & \mathrm{I}-\omega\otimes\omega \end{pmatrix}$$

where I is the identity matrix of size d. And so its determinant is

$$\begin{aligned}&\begin{vmatrix} \mathrm{I}-\omega\otimes\omega & \omega\otimes\omega \\ \omega\otimes\omega & \mathrm{I}-\omega\otimes\omega \end{vmatrix}\\ &= \begin{vmatrix} \mathrm{I} & \mathrm{I} \\ \omega\otimes\omega & \mathrm{I}-\omega\otimes\omega \end{vmatrix} = \begin{vmatrix} \mathrm{I} & 0 \\ \omega\otimes\omega & \mathrm{I}-2\omega\otimes\omega \end{vmatrix}\\ &= -1\end{aligned}$$

where the last equality follows from the fact that the matrix $\mathrm{I}-2\omega\otimes\omega$ is a reflection and thus has determinant -1.

Secondly we know that v, v_* are functions of v', v'_* and ω,

$$\begin{cases} v = v' - \langle v'-v'_*, \omega\rangle\omega, \\ v_* = v'_* + \langle v'-v'_*, \omega\rangle\omega \end{cases} \tag{1.1.6}$$

and

$$|v'-v'_*| = |v-v_*|, \quad \langle v'-v'_*,\omega\rangle = -\langle v-v_*,\omega\rangle$$

from which it follows that

$$\mathrm{B}(v-v_*,\omega) = \mathrm{B}(v'-v'_*,\omega).$$

Now by the change of variables $(v, v_*) \mapsto (v', v'_*)$, we have

$$\begin{aligned}\Phi &= \int \big(f(v')f(v'_*)-f(v)f(v_*)\big)\phi(v)\mathrm{B}(v-v_*,\omega)\mathrm{d}v_*\mathrm{d}v\mathrm{d}\omega\\ &= \int \big(f(v')f(v'_*)-f(v)f(v_*)\big)\phi(v)\mathrm{B}(v-v_*,\omega)\mathrm{d}v'_*\mathrm{d}v'\mathrm{d}\omega\\ &= \int \big(f(v')f(v'_*)-f(v)f(v_*)\big)\phi(v)\mathrm{B}(v'-v'_*,\omega)\mathrm{d}v'_*\mathrm{d}v'\mathrm{d}\omega.\end{aligned}$$

In sight of (1.1.6), we relabel the variables v'_*, v' by v_*, v and then arrive at

$$\Phi = \int \big(f(v)f(v_*)-f(v')f(v'_*)\big)\phi(v')\mathrm{B}(v-v_*,\omega)\mathrm{d}v_*\mathrm{d}v\mathrm{d}\omega = -\Phi'.$$

Similarly, it holds that $\Phi_* = -\Phi'_*$. Combined with the equalities $\Phi = \Phi_*$ and $\Phi = -\Phi'$, the equality (1.1.5) follows.

(2). We take $\phi = \log f$ in (1.1.5). This is justified by a standard approximation procedure provided certain integrability of f. Then we have

$$\int \mathrm{Q}(f,f)\log f\mathrm{d}v = \frac{1}{4}\int (f'f'_* - ff_*)(\log ff_* - \log f'f'_*)\mathrm{B}(v-v_*,\omega)\mathrm{d}\omega\mathrm{d}v_*\mathrm{d}v.$$

Thanks to the elementary inequality $(a-b)(\log a - \log b) \geq 0$ (for $a, b > 0$) with equality if and only if $a = b$, we see that

$$\int \mathrm{Q}(f,f)\log f\mathrm{d}v \leq 0.$$

Now we compute the time derivative of Boltzmann's H functional,

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\int f\log f\,\mathrm{d}x\mathrm{d}v &= -\int(\log f+1)\partial_t f\,\mathrm{d}x\mathrm{d}v\\
&= -\int(\log f+1)(\mathrm{Q}(f,f)-v\cdot\nabla_x f)\mathrm{d}x\mathrm{d}v.
\end{aligned}
$$

Note that by (1.1.5) with $\phi=1$, we have

$$\int \mathrm{Q}(f,f)\mathrm{d}x\mathrm{d}v=0.$$

The contribution of the transport operator is also zero since

$$
\begin{aligned}
\int(\log f+1)(v\cdot\nabla_x f)\mathrm{d}x\mathrm{d}v &= -\int f\,v\cdot\nabla_x(\log f+1)\mathrm{d}x\mathrm{d}v\\
&= -\int v\cdot\nabla_x f\,\mathrm{d}x\mathrm{d}v=0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\int f\log f\,\mathrm{d}x\mathrm{d}v &= -\int \mathrm{Q}(f,f)\log f\,\mathrm{d}x\mathrm{d}v\\
=\frac{1}{4}\int(ff_*-f'f'_*)(\log ff_*-\log f'f'_*)&\mathrm{B}(v-v_*,\omega)\mathrm{d}\omega\mathrm{d}v_*\mathrm{d}x\mathrm{d}v\geq 0 \qquad (1.1.7)
\end{aligned}
$$

$\square$

Remark 1.2. Another approach to prove nonnegativity of the entropy production $-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}(f)$ goes as follows: note that by change of variables

$$
\begin{aligned}
&\int ff_*(\log ff_*-\log f'f'_*)\mathrm{B}(v-v_*,\omega)\mathrm{d}\omega\mathrm{d}v_*\mathrm{d}x\mathrm{d}v\\
&=-\int f'f'_*(\log ff_*-\log f'f'_*)\mathrm{B}(v-v_*,\omega)\mathrm{d}\omega\mathrm{d}v_*\mathrm{d}x\mathrm{d}v,
\end{aligned}
$$

then

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}(f) &= \frac{1}{2}\int ff_*(\log ff_*-\log f'f'_*)\mathrm{B}(v-v_*,\omega)\mathrm{d}\omega\mathrm{d}v_*\mathrm{d}x\mathrm{d}v\\
&= \frac{1}{2}\int ff_*(\log\frac{ff_*}{f'f'_*}+\frac{f'f'_*}{ff_*}-1)\mathrm{B}(v-v_*,\omega)\mathrm{d}\omega\mathrm{d}v_*\mathrm{d}x\mathrm{d}v\geq 0
\end{aligned}
$$

by the elementary inequality $r-1-\log r\geq 0$ for all $r>0$.

(III). Entropy also serves as a Lyapunov functional for a number of equations. An example of interest is the *Landau equation*, a variant of the Boltzmann equation, which is of importance in plasma physics. In this equation, the Boltzmann collision operator $\mathrm{Q}(f,f)$ (see (1.1.3)) is replaced by the Landau collision operator,

$$
\begin{aligned}
\mathrm{Q_L}(f,f) &= \nabla_v\cdot\left(\int_{\mathbb{R}^d}a(v-v_*)\Big(f_*(\nabla f)-f(\nabla f)_*\Big)\mathrm{d}v_*\right) \qquad (1.1.8)\\
&= \sum_{i,j}\frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^d}\mathrm{d}v_*\,a_{ij}(v-v_*)\Big(f(v_*)\frac{\partial f}{\partial v_j}(v)-f(v)\frac{\partial f}{\partial v_j}(v_*)\Big)\right\}
\end{aligned}
$$

where the nonnegative symmetric matrix $a = (a_{ij})_{i,j}$ is given by the formula

$$a_{ij}(z) = \left(\delta_{ij} - \frac{z_i z_j}{|z|^2}\right)\Psi(|z|),$$

with the potential $\Psi \geq 0$ depending on the interactions between particles. For Coulomb interaction in dimension 3, $\Psi(z)$ is proportional to $1/|z|$. Like the Boltzmann equation, the Landau equation satisfies the conservation of mass, momentum and energy. Let f be a well-behaved solution of the Landau equation and let us consider the evolution of entropy $\mathrm{H}(f)$. The corresponding entropy production functional is given by

$$\mathrm{D_L}(f) = -\int \mathrm{Q_L}(f,f)\log f \,\mathrm{d}v$$

in such a way that

$$-\frac{\mathrm{dH}(f)}{\mathrm{d}t} = \int \mathrm{D_L}(f)\mathrm{d}x.$$

(Note that again the transport operator makes no contributions.) By an integration by parts and interchanging the variables v_* and v, $\mathrm{D_L}(f)$ takes the form

$$\mathrm{D_L}(f) = \frac{1}{2}\int \left|\sqrt{a(v-v_*)}\Big(\nabla(\log f) - \big[\nabla(\log f)\big]_*\Big)\right|^2 f f_* \mathrm{d}v\mathrm{d}v_*$$

and hence it is clear that $\mathrm{D_L}(f) \geq 0$. As a consequence,

$$-\frac{\mathrm{dH}(f)}{\mathrm{d}t} \leq 0.$$

i.e. the first part of the H theorem holds for the Landau equation as well.

(IV). Another Lyapunov functional we would like to introduce is the *free energy* for the *Fokker-Planck equation* which reads

$$\frac{\partial f}{\partial t} = \nabla\cdot(\nabla f + f v). \tag{1.1.9}$$

This equation occurs in various contexts, and it will be studied in the next sections in the more general form

$$\frac{\partial f}{\partial t} = \nabla\cdot(\nabla f + f\nabla \mathrm{V}). \tag{1.1.10}$$

Concerning the Fokker-Planck equation, there is only one conservation law, the conservation of mass. A natural Lyapunov functional is the sum of the Boltzmann's H functional and the kinetic energy, namely,

$$\mathrm{E}(f) = \int f\log f \,\mathrm{d}v + \int f\frac{|v|^2}{2}\mathrm{d}v.$$

We compute that

$$\begin{aligned}
-\frac{\mathrm{dE}(f)}{\mathrm{d}t} &= -\int(\log f + 1)\nabla\cdot(\nabla f + f v)\mathrm{d}v - \int\frac{|v|^2}{2}\nabla\cdot(\nabla f + f v)\mathrm{d}v\\
&= \int\frac{\nabla f}{f}\cdot(\nabla f + f v)\mathrm{d}v + \int v\cdot(\nabla f + f v)\mathrm{d}v\\
&= \int\left|\nabla(\log f) + v\right|^2 f\,\mathrm{d}v \geq 0.
\end{aligned}$$

We add a further remark on this free energy. Let h be the density function of $f\,\mathrm{d}v$ with respect to $\mathrm{d}\gamma(v) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|v|^2}{2}} \mathrm{d}v$, then the free energy described above becomes

$$\mathrm{E}(f) = \int h \log h \mathrm{d}\gamma(v) + \log(2\pi)^{-\frac{d}{2}} \int h \mathrm{d}\gamma(v).$$

Observe that the second term on the right hand side (RHS in short hereafter) is conserved, while the first one is the entropy of h with respect to γ (see Definition 3). So this free energy can be recovered by the notions of entropy. Moreover, the entropy production functional becomes

$$-\frac{\mathrm{d}\mathrm{E}(f)}{\mathrm{d}t} = \int \frac{|\nabla h|^2}{h} \mathrm{d}\gamma(v)$$

We shall see that the measure γ satisfies a logarithmic Sobolev inequality (see later in Theorem 1.19) so that we can apply Gronwall's lemma to obtain exponential decay in this Lyapunov functional.

(V). We present the *free energy* for the *McKean-Vlasov equation*. This equation reads

$$\frac{\partial f}{\partial t} = \Delta f + \nabla \cdot (f(\nabla \mathrm{V} + \nabla \mathrm{W} * f)) \tag{1.1.11}$$

where the unknown $f = f(t, \cdot)$ is a time-dependent probability density function on $\mathbb{R}^d$, V is a confining potential of class C^2, W is an interaction potential of class C^2, and $*$ stands for the convolution. It is also called *granular media equation* since it appears in the modelling of (space-homogeneous) granular flows. The potential W is assumed to be even in the sense $\mathrm{W}(-x) = \mathrm{W}(x)$ since the interactions are assumed to be symmetric between any two particles.

The free energy for McKean-Vlasov equation is given by

$$\mathrm{E}(f) = \int f \log f \mathrm{d}x + \int f(x)\mathrm{V}(x)\mathrm{d}x + \frac{1}{2}\int \mathrm{W}(x-y) f(x) f(y) \mathrm{d}x\mathrm{d}y \tag{1.1.12}$$

which is the sum of Boltzmann's H functional, a potential energy and an interaction energy. This functional can be viewed as a mean-field limit of entropy, c.f. Chapter 3. Let f be a smooth solution to the McKean-Vlasov equation. Denote $-\mathrm{L}f = \Delta f + \nabla \cdot (f(\nabla \mathrm{V} + \nabla \mathrm{W} * f))$. Then one can compute the time derivative of $\mathrm{E}(f)$,

$$\begin{aligned}
-\frac{\mathrm{d}\mathrm{E}(f)}{\mathrm{d}t} &= \int (\log f + 1)\mathrm{L}f \mathrm{d}x + \int \mathrm{V}(x)\mathrm{L}f \mathrm{d}x + \int \mathrm{W}(x-y) f(y) \mathrm{L}f(x) \mathrm{d}x\mathrm{d}by \\
&= \int \nabla(\log f + 1) \cdot (\nabla f + f(\nabla \mathrm{V} + \nabla \mathrm{W} * f))\mathrm{d}x \\
&\quad + \int \nabla \mathrm{V}(x) \cdot (\nabla f + f(\nabla \mathrm{V} + \nabla \mathrm{W} * f))\mathrm{d}x \\
&\quad + \int (\nabla \mathrm{W})(x-y) f(y) \cdot (\nabla f(x) + f(x)(\nabla \mathrm{V}(x) + (\nabla \mathrm{W} * f)(x)))\mathrm{d}x\mathrm{d}y \\
&= \int \left|\nabla(\log f) + \nabla \mathrm{V} + \nabla \mathrm{W} * f\right|^2 f \mathrm{d}x \\
&\geq 0.
\end{aligned}$$

(VI). The entropy also serves a Lyapunov functional for the Fokker-Planck equation for which we refer to the proof of Proposition 1.4. We end this section by the *Wasserstein distance*. The L^p-Wasserstein distance between two probability measures μ and ν on $\mathbb{R}^d$ is defined by

$$W_p(\mu,\nu) := \inf\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} d(x,y)^p \mathrm{d}\pi(x,y)\right)^{1/p} \tag{1.1.13}$$

where $d(x,y)$ stands for the distance between x and y, $p\in[1,\infty)$, and the infimum runs over all probability measure π on $\mathbb{R}^d\times\mathbb{R}^d$ which admits μ and ν as marginal measures, that is, for all nonnegative measurable function f,g on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(f(x)+g(y))\mathrm{d}\pi(x,y) = \int_{\mathbb{R}^d} f\mathrm{d}\mu + \int_{\mathbb{R}^d} g\mathrm{d}\nu.$$

In a more probabilistic formulation, it can be also defined by

$$W_p(\mu,\nu) = \inf[\mathbb{E}(d(X,Y)^p)]^{1/p} \tag{1.1.14}$$

where the infimum runs over all random variables (X,Y) such that law$(X)=\mu, Y=\nu$. Such a probability measure π or the couple of random variables (X,Y) above is called a *coupling* between the probability measure μ and ν. We shall work on *the space* $\mathscr{P}_p(\mathbb{R}^d)$ *of probability measures with finite* p*-th finite moment*. We refer to [16] for more information on Wasserstein metrics.

We present the following statement and a proof by stochastic calculus.

Proposition 1.3. *Suppose* $\nabla^2 V \geq c\,\mathrm{Id}$ *with* $c>0$*, then any two solutions* μ_t,ν_t *of Fokker-Planck equation* (1.1.10) *converge exponentially in* L^2*-Wasserstein distance,*

$$W_2(\mu_t,\nu_t)\leq e^{-ct}W_2(\mu_0,\nu_0)$$

where the initial data μ_0,ν_0 *are probability density functions with finite second moments.*

Proof. Let X_0,Y_0 be two random variables with respective distribution μ_0,ν_0. By Ito's formula, the solutions μ_t,ν_t are respectively the laws of X_t,Y_t which evolve according to the SDEs

$$\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t - \nabla V(X_t)\mathrm{d}t,$$
$$\mathrm{d}Y_t = \sqrt{2}\mathrm{d}B_t - \nabla V(Y_t)\mathrm{d}t$$

subject to the initial conditions X_0,Y_0 respectively. Here B_t is a standard Brownian motion on $\mathbb{R}^d$. In other words, (X_t,Y_t) is a coupling between μ_t and ν_t. Note that

$$\mathrm{d}(X_t-Y_t) = -(\nabla V(X_t)-\nabla V(Y_t))\mathrm{d}t.$$

Due to Ito's formula, it follows

$$\mathrm{d}|X_t-Y_t|^2 = -2\langle X_t-Y_t, \nabla V(X_t)-\nabla V(Y_t)\rangle\mathrm{d}t.$$

By the strict convexity of V, we obtain

$$\frac{\mathrm{d}|X_t-Y_t|^2}{\mathrm{d}t} = -2\langle X_t-Y_t,\nabla V(X_t)-\nabla V(Y_t)\rangle \leq -2c|X_t-Y_t|^2$$

which then implies an exponential decay

$$|X_t - Y_t|^2 \le e^{-2ct}|X_0 - Y_0|^2$$

and so we take expectations of both sides

$$\mathbb{E}(|X_t - Y_t|^2) \le e^{-2ct}\mathbb{E}(|X_0 - Y_0|^2).$$

By definition of Wasserstein distance, $W_2^2(\mu_t, \nu_t) \le \mathbb{E}(|X_t - Y_t|^2)$, therefore

$$W_2^2(\mu_t, \nu_t) \le e^{-2ct}\mathbb{E}(|X_0 - Y_0|^2).$$

It remains to take infimum over the random variables X_0, Y_0 to conclude that

$$W_2^2(\mu_t, \nu_t) \le e^{-2ct}W_2^2(\mu_0, \nu_0).$$

□

1.2 Functional inequalities

Functional inequalities are of great interest in the investigation of convergence to equilibrium. In this section, we shall begin by the Fokker-Planck equation and the McKean-Vlasov equation for which functional inequalities can be applied to obtain exponential decay in certain Lyapunov functionals. Then we collect the well-known results about two important families of functional inequalities, the Poincaré inequalities and the logarithmic Sobolev inequalities. The main reference of this and next section is the monograph [2] written by Bakry, Ledoux and Gentil.

Sometimes, the state space will not be specified but should be understood as some smooth metric spaces equipped with the Borel σ-algebra and differential structures; some results hold true for general complete metric spaces.

We shall denote by $\mathscr{P}(M)$ the space of probability measures on a metric space M. For a measure μ, we denote by $L^p(\mu)$ the L^p-space with respect to the reference measure μ, and by $H^k(\mu)$ the space of L^2-Sobolev space of order k. The following two *Orlicz spaces* are suitable for the study of entropy,

$$L\log L(\mu) := \left\{ f \in L^1(\mu) \,\middle|\, \int |f| \log(|f|) d\mu < \infty \right\}$$

and

$$L^2 \log L(\mu) := \left\{ f \in L^2(\mu) \,\middle|\, \int |f|^2 \log(|f|) d\mu < \infty \right\}.$$

Sometimes we also work in the space of bounded continuous functions $C_b(M)$ on a metric space M, and the space of Lipschitz functions. We denote by $||f||_{Lip}$ the Lipschitz seminorm of a Lipschitz function f.

Now we introduce the variance and Poincaré inequalities, one of the most well-known family of functional inequalities.

Definition 1. Given a probability measure μ, for any function $f \in L^2(\mu)$, the variance of f with respect to μ, denoted as $\mathrm{Var}_\mu(f)$, is defined by

$$\mathrm{Var}_\mu(f) := \int f^2 d\mu - \left(\int f d\mu\right)^2.$$

Definition 2. We say a probability measure μ satisfies a *Poincaré inequality* $P(C)$ with constant $C > 0$ if for any function $f \in H^1(\mu)$ it holds that

$$\mathrm{Var}_\mu(f) \le C \int |\nabla f|^2 d\mu. \tag{1.2.1}$$

The optimal constant C in (1.2.1) is called *the Poincaré constant* of μ, sometimes denoted by $C_P(\mu)$. A Poincaré inequality is also called a *spectral gap inequality*. In that occasion, a Poincaré inequality $P(C)$ is a spectral gap inequality with constant $\rho = \frac{1}{C}$.
Next we introduce the entropy and the logarithmic Sobolev inequalities.

Definition 3. For a probability measure μ and a positive function $f \in L\log L(\mu)$, the entropy of f with respect μ is defined as

$$\mathrm{Ent}_\mu(f) := \int f \log f d\mu - \int f d\mu \log \int f d\mu.$$

Definition 4. (i). We say a probability measure μ satisfies a *logarithmic Sobolev inequality* $LS(C)$ with constant $C > 0$ if for any function $f \in H^1(\mu)$ it holds that

$$\mathrm{Ent}_\mu(f^2) \le 2C \int |\nabla f|^2 d\mu. \tag{1.2.2}$$

(ii). We say a probability measure μ satisfies a *defective logarithmic Sobolev inequality* $LS(C,D)$ with constant $C, D > 0$ if for any function $f \in H^1(\mu)$ it holds that

$$\mathrm{Ent}_\mu(f^2) \le 2C \int |\nabla f|^2 d\mu + D \int f^2 d\mu. \tag{1.2.3}$$

Very often we shall use another formulation of the logarithmic Sobolev inequality. For $d\nu = f d\mu \in \mathscr{P}$, the *relative entropy* of ν with respect to μ is defined as

$$H(\nu|\mu) := \mathrm{Ent}_\mu(f) = \int f \log f d\mu,$$

the (relative) Fisher information is defined as

$$I(\nu|\mu) = I_\mu(f) := \int \frac{|\nabla f|^2}{f} d\mu,$$

and so the logarithmic Sobolev inequality $LS(C)$ can be rephrased as

$$H(\nu|\mu) \le \frac{C}{2} I(\nu|\mu). \tag{1.2.4}$$

Next we shall see that Poincaré inequalities and logarithmic Sobolev inequalities have direct applications to the long time behaviour of Fokker-Planck equation.

1.2.1 Convergence to equilibrium

A certain class of functional inequalities are of particular interest for their links with the convergence to equilibrium: the so-called *entropy-entropy production inequalities*. As in the previous section, consider a Lyapunov functional E for some PDE, one can define its production functional D given by

$$\mathrm{D}(f) = -\mathrm{d}\mathrm{E}(f)/\mathrm{d}t$$

for a well-behaved solution f. Assume that there exists a unique equilibrium state f_∞. Let $\mathrm{E}(f|f_\infty) := \mathrm{E}(f) - \mathrm{E}(\mathrm{F}_\infty)$ be the relative Lyapunov functional. The associated entropy-entropy production inequality takes the form

$$\mathrm{D}(f) \geq \Psi(\mathrm{E}(f|f_\infty)).$$

where Ψ is nonnegative on $[0,\infty)$ and it vanishes only at 0. When $\Psi(r) = \lambda r$ with $\lambda > 0$, then formally one derives exponential convergence to equilibrium in E with rate λ, due to Gronwall's lemma. When $\Psi(r) = \mathrm{K}r^{1+\alpha}$ with $\alpha > 0$, one can formally derive polynomial rate of convergence.

We now turn to two examples for which functional inequalities of entropy-entropy production type work well.

Example: Fokker-Planck equation. Let $\mathrm{V} = \mathrm{V}(x)$ be a smooth function such that $\int e^{-\mathrm{V}} \mathrm{d}x = 1$. Denote $\mathrm{d}\mu(x) := e^{-\mathrm{V}(x)}\mathrm{d}x$. Recall the Fokker-Planck equation reads

$$\partial_t h = \Delta h - \nabla \mathrm{V} \cdot \nabla h$$

subject to certain initial condition, where h is the density function w.r.t the invariant measure μ.

Proposition 1.4. *Consider the solutions to the Fokker-Planck equation. Then we have*

(1) A Poincaré inequality for μ *implies exponential convergence in variance;*

(2) A logarithmic Sobolev inequality for μ *implies exponential convergence in entropy.*

In particular, when $\nabla^2 \mathrm{V} \geq \rho\, \mathrm{Id} > 0$*, the solutions of Fokker-Planck equation converge to equilibrium exponential fast in variance and entropy with rate* 2ρ*.*

Proof. Observe that the mass is conserved for the Fokker-Planck equation. Let h be the solution to the Fokker-Planck equation with initial datum h_0 (in some suitable function space).

Assertion (1): Assume that the measure μ satisfies a Poincaré inequality with constant $\mathrm{C} > 0$. Let $h_t = h(t,\cdot)$ be the solution with initial condition $h_0 \in \mathrm{L}^2(\mu)$, then

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}_\mu(h_t) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int h^2 \mathrm{d}\mu = \int h \partial_t h \mathrm{d}\mu = \int h(\Delta h - \nabla \mathrm{V} \cdot \nabla h)\mathrm{d}\mu \\
&= -\int |\nabla h|^2 \mathrm{d}\mu \\
&\leq -\frac{1}{\mathrm{C}}\mathrm{Var}_\mu(h_t)
\end{aligned}$$

which, by Gronwall's lemma, implies that

$$\mathrm{Var}_\mu(h_t) \leq e^{-2t/\mathrm{C}}\mathrm{Var}_\mu(h_0). \tag{1.2.5}$$

Assertion (2): Assume that the measure μ satisfies a logarithmic Sobolev inequality with constant $C > 0$. Let $h_t = h(t, \cdot)$ be the solution with positive initial datum $h_0 \in L\log L(\mu)$, then

$$\begin{aligned}
\frac{d}{dt}\int h\log h d\mu &= \int (\log h + 1)\partial_t h d\mu = \int (\log h + 1)(\Delta h - \nabla V \cdot \nabla h) d\mu \\
&= -\int \langle \nabla(\log h + 1), \nabla h\rangle d\mu \\
&= -\int \frac{|\nabla h|^2}{h} d\mu \\
&\leq -\frac{2}{C}\mathrm{Ent}_\mu(h_t),
\end{aligned}$$

and so we have

$$\mathrm{Ent}_\mu(h_t) \leq e^{-2t/C}\mathrm{Ent}_\mu(h_0). \tag{1.2.6}$$

By Theorem 1.7 and Theorem 1.20(see later), a Poincaré inequality and a logarithmic Sobolev inequality (with constant $C = 1/\rho$) hold for the invariant measure whenever $\nabla^2 V \geq \rho \mathrm{Id} > 0$. So in that case, the assumptions in assertions (1) and (2) are verified and thus the conclusions follow. □

Actually it is also easy to see the converse implications hold true, by a Taylor expansion of both sides of (1.2.5) or (1.2.6) at time zero (i.e. taking first derivative of the functionals and evaluating at time zero, since both inequalities are equalities at the initial time).

Example: McKean-Vlasov equation. In this example we present a functional inequality which implies the exponential decay in free energy for the solutions of McKean-Vlasov equation. We shall consider time-dependent probability density solutions of the McKean-Vlasov equation

$$\frac{\partial f}{\partial t} = \Delta f + \nabla \cdot (f(\nabla V + \nabla W * f)).$$

Recall the associated free energy is given by

$$E(f) = \int f\log f dx + \int f(x)V(x)dx + \frac{1}{2}\int W(x-y)f(x)f(y)dxdy,$$

whereas its production functional is

$$-\frac{dE(f)}{dt} = \int \left|\nabla(\log f) + \nabla V + \nabla W * f\right|^2 f dx.$$

We quote the following result in [4, Theorem 2.1].

Theorem 1.5 (Carrillo, McCann, and Villani)**.** *Assume that there exists a constant $\rho > 0$ such that the potentials V and W satisfies*

$$\nabla^2 V \geq \rho \mathrm{Id} > 0, \quad \rho > ||(\nabla^2 W)^-||_{L^\infty}$$

where $(\nabla^2 W)^-$ is the negative part of the Hessian $\nabla^2 W$. Put

$$\lambda = \rho - ||(\nabla^2 W)^-||_{L^\infty}.$$

Then

(1) *There exists a unique minimizer f_∞ of the free energy, which turns out to be the unique stationary solution for the McKean-Vlasov equation.*

(2) *Whenever f is a smooth probability density satisfying $\mathrm{E}(f) < \infty$, it holds*

$$2\lambda(\mathrm{E}(f) - \mathrm{E}(f_\infty)) \leq \int \left|\nabla(\log f) + \nabla \mathrm{V} + \nabla \mathrm{W} * f\right|^2 f\,dx.$$

(3) *Let $f = (f_t)_{t\geq 0}$ be a solution of the McKean-Vlasov equation with finite initial free energy, then*

$$\mathrm{E}(f_t) - \mathrm{E}(f_\infty) \leq e^{-2\lambda t}(\mathrm{E}(f_0) - \mathrm{E}(f_\infty)).$$

The functional inequality in the second assertion is the entropy-entropy production inequality for the free energy associated to McKean-Vlasov equation. Later we shall see that it can be viewed as a mean-field limit of the logarithmic Sobolev inequality.

Sketch of proof. Let $\xi := \nabla(\log f) + \nabla \mathrm{V} + \nabla \mathrm{W} * f$. Using the gradient structure of the equation in Wasserstein space, it can be demonstrated that (see [4, Proposition 3.1])

$$\begin{aligned}\frac{\mathrm{d}^2\mathrm{E}(f)}{\mathrm{d}t^2} &= \int |\nabla\xi|^2 f\,\mathrm{d}x + 2\int \langle \nabla^2 \mathrm{V}\cdot\xi, \xi\rangle f\,\mathrm{d}x \\ &\quad + \int \langle \nabla^2 \mathrm{W}(x-y)\cdot[\xi(x)-\xi(y)], [\xi(x)-\xi(y)]\rangle f(x)f(y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

where $|\nabla\xi|$ stands for the Hilbert-Schmidt norm of the matrix $\nabla\xi$. By assumption we know that

$$\begin{aligned}&\int \langle \nabla^2 \mathrm{W}(x-y)\cdot[\xi(x)-\xi(y)], [\xi(x)-\xi(y)]\rangle f(x)f(y)\mathrm{d}x\mathrm{d}y \\ &\geq -||(\nabla^2\mathrm{W})^-||_{\mathrm{L}^\infty} \int |\xi(x)-\xi(y)|^2 f(x)f(y)\mathrm{d}x\mathrm{d}y \\ &= -2||(\nabla^2\mathrm{W})^-||_{\mathrm{L}^\infty}\left(\int |\xi|^2 f\,\mathrm{d}x - |\int \xi f\,\mathrm{d}x|^2\right) \\ &\geq -2||(\nabla^2\mathrm{W})^-||_{\mathrm{L}^\infty}\int |\xi|^2 f\,\mathrm{d}x.\end{aligned}$$

Combined with the condition on V, it follows that

$$\frac{\mathrm{d}^2\mathrm{E}(f)}{\mathrm{d}t^2} \geq 2\lambda \int |\xi|^2 f\,\mathrm{d}x = -2\lambda\frac{\mathrm{d}\mathrm{E}(f)}{\mathrm{d}t}.$$

By Gronwall's lemma, it yields

$$\frac{\mathrm{d}\mathrm{E}(f)}{\mathrm{d}t} \leq \left(\frac{\mathrm{d}\mathrm{E}(f)}{\mathrm{d}t}\right)\Big|_{t=0} e^{-2\lambda t}.$$

Integrating in time over $[0,\infty)$ gives the functional inequality in the second assertion. The third assertion follows from the second one, thanks to Gronwall's lemma. □

1.2.2 Poincaré inequality

This subsection is devoted to the Poincaré inequality. The next proposition concerns about the *standard Gaussian measure* γ on the Euclidean space $\mathbb{R}^d$ which is defined by

$$d\gamma(x) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2}} dx, \quad x \in \mathbb{R}^d.$$

Proposition 1.6 (Poincaré inequality for the standard Gaussian measure)**.** *For any function $f \in \mathrm{H}^1(\gamma)$,*

$$\int f^2 d\gamma - \left(\int f d\gamma\right)^2 \leq \int |\nabla f|^2 d\gamma. \tag{1.2.7}$$

And the constant 1 *above is the Poincaré constant for the standard Gaussian measure.*

This can be seen by spectral analysis, since it expresses that 1 is the smallest non-zero eigenvalue of $-\Delta + x \cdot \nabla_x$. The eigenfunctions of $-\Delta + x \cdot \nabla_x$ are given by multiple Hermite polynomials in the form

$$\mathrm{H}_{k_1}(x_1)\mathrm{H}_{k_2}(x_2)\cdots\mathrm{H}_{k_d}(x_d)$$

with the corresponding eigenvalue $k_1 + k_2 + \cdots + k_d$, where $\mathrm{H}_{k_i}(x_i)$ (for each i, $k_i \in \mathbb{N}$) is the k_i-th order Hermite orthonormal polynomial.
As a consequence, if μ is a centered Gaussian measure on $\mathbb{R}^d$ with covariance matrix Q, we perform a change of variables and obtain

$$\int f^2 d\mu - \left(\int f d\mu\right)^2 \leq \int \langle \mathrm{Q}\nabla f, \nabla f\rangle d\mu.$$

for all function $f \in \mathrm{H}^1(\mu)$. Such results can be reinforced by the Bakry-Émery Γ_2 criterion for Poincaré inequality and the matrix Brascamp-Lieb inequality.

Theorem 1.7 (Bakry-Émery Γ_2 criterion)**.** *Let* V *be a function of class* C^2 *on* $\mathbb{R}^d$ *such that* $d\mu(x) = e^{-\mathrm{V}} dx \in \mathscr{P}(\mathbb{R}^d)$. *Assume that* $\nabla^2\mathrm{V} \geq \rho\,\mathrm{Id}$ *for some* $\rho > 0$ *in the sense of symmetric matrices. Then the measure* μ *satisfies a Poincaré inequality with constant* $\frac{1}{\rho}$.

This criterion is a particular case of the Poincaré inequality under curvature-dimension conditions, which will be proved in the next section.

Theorem 1.8 (Brascamp-Lieb inequality)**.** *Let* $d\mu(x) = e^{-\mathrm{V}} dx \in \mathscr{P}(\mathbb{R}^d)$. *Assume that the smooth potential* V *is strictly convex, then*

$$\int f^2 d\mu - \left(\int f d\mu\right)^2 \leq \int \langle (\nabla^2\mathrm{V})^{-1}\nabla f, \nabla f\rangle d\mu.$$

for every function $f \in \mathrm{H}^1(\mu)$, *where* $(\nabla^2\mathrm{V})^{-1}$ *stands for the inverse of the Hessian* $\nabla^2\mathrm{V}$.

Another example is log-concave measures. Let $d\mu(x) = e^{-\mathrm{V}}dx \in \mathscr{P}(\mathbb{R}^d)$, then μ is called *log-concave* if V is a convex function. Such probability measures satisfy Poincaré inequalities. We quote the following theorem due to Bakry, Barthe, Cattiaux and Guillin (c.f. [1]),

Theorem 1.9. *Assume* $d\mu(x) = e^{-V}dx \in \mathscr{P}(\mathbb{R}^d)$ *where the function* V *is of class* C^2.

1. *If there exist* $\alpha > 0$ *and* $R \geq 0$ *such that for any* $|x| \geq R$,
$$\langle x, \nabla V(x)\rangle \geq \alpha |x|,$$
then μ *satisfies a Poincaré inequality.*

2. *If there exist* $a \in (0,1)$, $c > 0$ *and* $R \geq 0$ *such that for any* $|x| \geq R$,
$$a|\nabla V(x)|^2 - \Delta V(x) \geq c,$$
then μ *satisfies a Poincaré inequality.*

Corollary 1.10 (Kanan-Lovaász-Simonovits-Bobkov)**.** *If* $\mu \in \mathscr{P}(\mathbb{R}^d)$ *is log-concave, then* μ *satisfies a Poincaré inequality.*

The Poincaré inequalities satisfy tensorization properties, i.e. they are stable under product.

Proposition 1.11. *Let* M_1 *and* M_2 *be some Riemannian manifolds. If* $\mu_1 \in \mathscr{P}(M_1)$, $\mu_2 \in \mathscr{P}(M_2)$ *satisfy Poincaré inequalities with respective constants* c_1, c_2, *then the product measure* $\mu_1 \otimes \mu_2$ *satisfies a Poincaré inequality with constant* $c = \max\{c_1, c_2\}$. *Indeed, for all* $f \in H^1(\mu_1 \otimes \mu_2)$, *it holds*
$$\begin{aligned}\mathrm{Var}_{\mu_1\otimes\mu_2}(f) &\leq c_1 \int\int |\nabla_x f|^2 d\mu_1 d\mu_2 + c_2 \int\int |\nabla_y f|^2 d\mu_1 d\mu_2 \\ &\leq \max\{c_1, c_2\} \int\int (|\nabla_x f|^2 + |\nabla_y f|^2) d\mu_1 d\mu_2\end{aligned}$$
where ∇_x, ∇_y *stand for the gradient operator on* M_1, M_2 *respectively.*

Poincaré inequalities are also stable under bounded perturbations. Let us denote by $\mathrm{osc}(g)$ the oscillation of a bounded function g, i.e.
$$\mathrm{osc}(g) := \sup g - \inf g.$$

Theorem 1.12 (Bounded perturbations)**.** *Suppose the probability measure* μ *satisfies a Poincaré inequality* $P(C)$. *Let* $d\nu(x) = e^{g(x)}d\mu(x)$ *be a probability measure where the function g is a bounded function (that is,* $osc(g) < \infty$*). Then* ν *satisfies a Poincaré inequality with constant* $Ce^{osc(g)}$, *i.e.,*
$$\int f^2 d\nu - \left(\int f d\nu\right)^2 \leq Ce^{osc(g)} \int |\nabla f|^2 d\nu$$
for every function $f \in H^1(\nu)$.

This bounded perturbation theorem is a consequence of the following identity
$$\mathrm{Var}_\mu(f) = \frac{1}{2}\int\int |f(x) - f(y)|^2 \mathrm{d}\mu(x)\mathrm{d}\mu(y)$$
which is basic but useful in many occasions.

One of the important consequences of Poincaré inequality is the exponential integrability for Lipschitz functions. It can be shown, by approximation and Fatou's lemma, that

Lemma 1.13. *Lipschitz functions are integrable with respect to a probability measure which satisfies a Poincaré inequality.*

Moreover, we have the following

Theorem 1.14 (Exponential integrability)**.** *Suppose the probability measure* μ *satisfies a Poincaré inequality with constant* C*, then for any Lipschitz function* f*, it holds for* $0 \le s < \frac{2}{\sqrt{C}||f||_{Lip}}$*,*

$$\int e^{s(f-\int f d\mu)} d\mu \le \frac{2+\sqrt{C}s||f||_{Lip}}{2-\sqrt{C}s||f||_{Lip}}. \tag{1.2.8}$$

By Markov exponential inequality, it follows

Corollary 1.15 (Exponential measure concentration)**.** *Let* μ *be a probability measure satisfies a Poincaré inequality with constant* C*. Then for every Lipschitz function* f *and every* $r > 0$*, it holds*

$$\mu(f - \int f d\mu \ge r) \le 3\exp\left\{-\frac{r}{\sqrt{C}||f||_{Lip}}\right\};$$
$$\mu(|f - \int f d\mu| \ge r) \le 6\exp\left\{-\frac{r}{\sqrt{C}||f||_{Lip}}\right\}.$$

The following equivalent formulation of the Poincaré inequality is also very useful.

Theorem 1.16 (A dual description of Poincaré inequality)**.** *Consider a probability measure* $d\mu(x) = \frac{1}{Z}e^{-V(x)}dx$ *where* Z *is the normalizing constant and its associated generator* $L = \Delta - \nabla V \cdot \nabla$*. Then the following are equivalent:*

1. *The measure* μ *satisfies a Poincaré inequality* $P(C)$ *for some constant* $C > 0$*;*

2. *For all smooth compactly-supported function* f*,*

$$\int |\nabla f|^2 d\mu \le C \int (Lf)^2 d\mu. \tag{1.2.9}$$

The above equivalence can be demonstrated by spectral analysis as follows. By the spectral decomposition theorem for self-adjoint operators, there exists a spectral measure E_λ on the real line such that $-L = \int \lambda dE_\lambda$. By functional calculus, it then follows that

$$\int |\nabla f|^2 d\mu = \int \lambda d\langle E_\lambda f, f\rangle, \quad \int |Lf|^2 d\mu = \int \lambda^2 d\langle E_\lambda f, f\rangle.$$

So the second statement is equivalent to the statement that $-L$ admits a spectral gap $\frac{1}{C}$, hence it is equivalent to the first one. It can be proved by a semigroup interpolation as well, c.f. [2].

1.2.3 Logarithmic Sobolev Inequality

This subsection is devoted to the logarithmic Sobolev inequality. Let us discuss some properties of entropy first. The entropy is always nonnegative by Jensen's inequality, since the function $r\log r$ is strictly convex on $\mathbb{R}^+$. The next celebrated theorem expresses that L^1-norm of the difference of two probability density functions is controlled by the relative entropy. In particular, for an evolution equation, convergence to equilibrium in entropy implies convergence to equilibrium in L^1-norm.

Theorem 1.17 (The Pinsker-Csiszár-Kullback inequality)**.** *For two probability measures on the same state space,*

$$||\mu-\nu||_{\mathrm{TV}}^2 \leq \frac{1}{2}\mathrm{Ent}_\mu(\nu)$$

where $||\mu-\nu||_{\mathrm{TV}}$ *stands for the total variation defined by*

$$||\mu-\nu||_{\mathrm{TV}} = \frac{1}{2}\int |1-\frac{d\nu}{d\mu}|d\mu.$$

(Here as usual $\frac{d\nu}{d\mu}$ *stands for the Radon-Nikodym derivative.)*

Proposition 1.18 (The classical entropic inequality)**.** *Let* $\mu\in\mathscr{P}$*, it holds for any* $f\geq 0$ *and* g *suitably integrable that*

$$\int fg\,d\mu \leq \mathrm{Ent}_\mu(f)+\int f\,d\mu\log\Big(\int e^g\,d\mu\Big).$$

In other words, assume that $\int f\,d\mu = 1$*, then*

$$\mathrm{Ent}_\mu(f) = \sup\Big\{\int fg\,d\mu - \log\Big(\int e^g\,d\mu\Big)\Big\}$$

where the supremum runs over all function g *such that* $\int e^g\,d\mu<\infty$.

The logarithmic Sobolev inequality was introduced by L.Gross in 1975 for the standard Gaussian measure.

Theorem 1.19 (Gaussian logarithmic Sobolev inequality)**.** *The standard Gaussian measure* γ *on the Euclidean space satisfies a logarithmic Sobolev inequality with constant* 1*, i.e. for any function* $f\in\mathrm{H}^1(\gamma)$*, it holds*

$$\mathrm{Ent}_\gamma(f^2)\leq 2\int|\nabla f|^2\,dx.$$

Moreover, the constant is sharp.

Since this seminal work, various proofs and numerous applications have been extensively developed. For instance, it was used in Perelman's solution to the famous Poincaré conjecture in 2002 (It is not a coincidence that Bakry-Émery's theory plays an important role there). There are at least 15 proofs for this theorem. Gross's original proof is based on two-point space and central limit theorem. (Gross also prove the equivalence between the logarithmic Sobolev inequality and hypercontractivity there.) Other proofs involves very different theories such as hypercontractivity, semigroup theory, sharp Young inequality, optimal transportation and so on. We refer to the slides of M. Ledoux [13](Colloquium at the Technion, Haifa 2012) for a nice presentation on logarithmic Sobolev inequatlities.

In the next section, we shall present Bakry-Émery's proof using semigroup theory. A special case of their results asserts an important criterion for the logarithmic Sobolev inequality,

Theorem 1.20 (Bakry-Émery Γ_2 criterion)**.** *Let* $d\mu(x) = e^{-V}dx \in \mathscr{P}(\mathbb{R}^d)$ *where* V *is a function of class* C^2. *Assume that* $\nabla^2 V \geq \rho\,\mathrm{Id}$ *for some* $\rho > 0$ *in the sense of symmetric matrices. Then the measure* μ *satisfies a logarithmic Sobolev inequality with constant* $\frac{1}{\rho}$.

Then we collect some rather standard properties for the logarithmic Soblev inequalities, namely, bounded perturbation, tensorization, tightening properties, exponential integrability.

Proposition 1.21 (Bounded perturbation)**.** *Assume that* $\mu \in \mathscr{P}$ *satisfies a logarithmic Sobolev inequality* $LS(C,D)$. *Consider* $d\nu(x) = e^g d\mu(x) \in \mathscr{P}$ *with a bounded function* g. *Then for any compactly supported smooth function* f*,*

$$\mathrm{Ent}_\nu(f^2) \leq e^{osc(g)}\left\{C\int |\nabla f|^2 d\nu + D\int f^2 d\nu\right\},$$

i.e., ν *satisfies a logarithmic Sobolev inequality* $LS(e^{osc(g)}C, e^{osc(g)}D)$.

This follows from the following variational formulation,

$$\int \varphi(f) - \varphi(\int f d\mu) \leq \inf_{r\in I}\int [\varphi(f) - \varphi(r) - \varphi'(r)(f-r)]d\mu$$

for any C^2 real-valued convex function φ on some interval $I \subset \mathbb{R}$.

Proposition 1.22 (Tensorization)**.** *Let* M_1, M_2 *be two Riemannian manifolds. Assume that* $\mu_1 \in \mathscr{P}(M_1), \mu_2 \in \mathscr{P}(M_2)$ *satisfy logarithmic Sobolev inequalities* $LS(C_1,D_1)$ *and* $LS(C_2,D_2)$ *respectively. Then the product measure* $\mu_1 \otimes \mu_2$ *satisfies a logarithmic Sobolev inequality* $LS(\max(C_1,C_2), D_1 + D_2)$.

It follows from a Taylor expansion at a constant function that

Proposition 1.23. *A tight logarithmic Sobolev inequality* $LS(C)$ *implies a Poincaré inequality* $P(C)$.

Proposition 1.24 (Tightening with a Poincaré inequality)**.** *Assume a probability measure* μ *satisfies a defective logarithmic Sobolev inequality* $LS(C,D)$ *and a Poincaré inequality* $P(C')$*, then it also satisfies a tight logarithmic Sobolev inequality* $LS(C + C'(1 + \frac{D}{2}))$.

The proof of this proposition is based on

Lemma 1.25 (Rothaus' lemma)**.** *Let* $f \in L^2 \log L(\mu)$*, then for any* $a \in \mathbb{R}$*,*

$$\mathrm{Ent}_\mu((f+a)^2) \leq \mathrm{Ent}_\mu(f^2) + 2\int f^2 d\mu.$$

Theorem 1.26 (Exponential integrability)**.** *Assume that* μ *satisfies a logarithmic Sobolev inequality* LS(C)*, then for any Lipschitz function* f *and any* $s \in \mathbb{R}$,

$$\int e^{sf}\, d\mu \le e^{\frac{C\|f\|_{Lip}^2}{2}s^2} e^{s\int f d\mu}; \tag{1.2.10}$$

moreover, for any $\sigma^2 < \frac{1}{C\|f\|_{Lip}^2}$,

$$\int e^{\frac{\sigma^2}{2}f^2}\, d\mu \le \frac{1}{\sqrt{1 - C\|f\|_{Lip}^2 \sigma^2}} \exp\left\{ \frac{\sigma^2}{2(1 - C\|f\|_{Lip}^2 \sigma^2)} \left(\int f d\mu \right)^2 \right\}. \tag{1.2.11}$$

The method to prove this theorem is known as the Herbst's argument, attributed to a unpublished letter of I. Herbst to L. Gross in 1975. Although it seems less relevant to the topic of this thesis, we shall reproduce the proof here for two reasons: (1) Herbst's argument is a beautiful piece, neat and elegant; (2) in the core of the proof it is a differential inequality while differential inequalities are of great importance in the study of long time behaviour of evolution equations.

Proof: Herbst's argument. By approximation, it suffices to prove the result for any bounded Lipschitz function. We denote for a bounded Lipschitz function f and for $s \ge 0$,

$$Z(s) := \int e^{sf}\, d\mu.$$

To produce entropy-like terms, we take derivative with respect to s and get

$$Z'(s) = \int f e^{sf}\, d\mu = \frac{1}{s}\left(\mathrm{Ent}_\mu(e^{sf}) + Z\log Z \right).$$

So we may apply the logarithmic Sobolev inequality to e^{sf},

$$\mathrm{Ent}_\mu(e^{sf}) \le 2C \int |\nabla e^{\frac{s}{2}f}|^2 d\mu = \frac{Cs^2}{2} \int |\nabla f|^2 e^{sf}\, d\mu$$

Since f is Lipschitz, $|\nabla f| \le \|\nabla f\|_{L^\infty} = \|f\|_{Lip}$. Then it follows

$$sZ' - Z\log Z = \mathrm{Ent}_\mu(e^{sf}) \le \frac{Cs^2}{2} \int |\nabla f|^2 e^{sf}\, d\mu \le \frac{Cs^2\|f\|_{Lip}^2}{2} \int e^{sf}\, d\mu.$$

In other words, we thereby have a differential inequality for $Z = Z(s)$,

$$sZ' - Z\log Z \le \frac{Cs^2\|f\|_{Lip}^2}{2} Z.$$

Since

$$\frac{sZ' - Z\log Z}{Zs^2} = \frac{s(\log Z)' - \log Z}{s^2} = \frac{d}{ds}\left(\frac{\log Z}{s} \right),$$

the preceding inequality can be rewritten as

$$\frac{d}{ds}\left(\frac{\log Z}{s} \right) \le \frac{C\|f\|_{Lip}^2}{2}.$$

Notice by L'Hôpital's rule

$$\lim_{s\to 0}\frac{\log Z}{s} = \lim_{s\to 0}\frac{Z'(s)}{Z(s)} = \int f d\mu,$$

therefore, integrating the previous differential inequality in s, we get

$$\frac{\log Z}{s} - \int f d\mu \leq \frac{C||f||^2_{Lip}}{2} s$$

or equivalently (1.2.10)

$$\int e^{s(f-\int f d\mu)} d\mu \leq e^{\frac{C||f||^2_{Lip}}{2} s^2}.$$

Furthermore, as a function of s, the R.H.S. of the inequality (1.2.10) is integrable with respect the centered Gaussian measure with variance σ^2 whenever $C||f||^2_{Lip} < \frac{1}{\sigma^2}$. So the second assertion follows from an integration and Fubini's theorem. □

As for the Poincaré inequality, exponential integrability implies concentration inequalities.

Corollary 1.27 (Gaussian measure concentration)**.** *If* $\mu \in \mathscr{P}$ *satisfies a logarithmic Sobolev inequality* $LS(C)$ *with constant* $C > 0$*, then for any* $r > 0$ *and any Lipschitz function* f*,*

$$\mu(f - \int f d\mu \geq r) \leq \exp\left\{-\frac{r^2}{2C||f||^2_{Lip}}\right\},$$

$$\mu(|f - \int f d\mu| \geq r) \leq 2\exp\left\{-\frac{r^2}{2C||f||^2_{Lip}}\right\}.$$

1.3 Bakry-Émery theory

In this section we present some basic ideas and techniques of the Bakry-Émery theory. We shall prove the Poincaré and logarithmic Sobolev inequality mentioned in Theorem 1.7 and Theorem 1.20 first. Then we formally derive some basic rules in the Γ-calculus and present applications of Bakry-Émery's curvature-dimension condition $CD(\rho, n)$.

Bakry-Émery theory can be viewed as a systematic application of semigroup theory to functional inequalities. One of the techniques is the semigroup interpolation. Consider a semigroup $(P_t)_{t\geq 0}$ and some functional Φ, then $P_s\Phi(P_{t-s}f)$ (with s varying from 0 to t) is an interpolation between $\Phi(P_t f)$ and $\Phi(P_t f)$. Such interpolations have numerous applications. For instance, an interpolation by the heat semigroup can provide a proof of Hölder's inequality (see the preface of [2]).

In the discussion of Lyapunov functionals, we take the first derivative of the functional along an evolution equation. For instance, in the case of entropy, we control the entropy production in terms of entropy, so that we can apply Gronwall type lemma to derive convergence to equilibrium. One of the insights in Bakry-Émery's theory is taking one more derivative and then magnificent computations happen (the latter is in fact highly nontrivial). Bakry-Émery's curvature-dimension condition $CD(\rho, n)$ is a key in the whole theory. In particular, we shall present its applications to the Poincaré inequality and the logarithmic Sobolev inequality.

1.3.1 The fundamental example: the operator $\Delta - \nabla V \cdot \nabla$ with $\nabla^2 V \geq \rho \,\mathrm{Id} > 0$

This is in fact the example we shall use most frequently. The purpose here is to give a flavor of how Bakry-Émery theory works. We shall present a proof for Theorem 1.7 and Theorem 1.20. Indeed, we shall compute the second derivatives of the variance and entropy along the semigroup $P_t = e^{tL}$ generated by the operator $L = \Delta - \nabla V \cdot \nabla$. That way, a Poincaré inequality $P(\frac{1}{\rho})$ and a logarithmic Sobolev inequality $LS(\frac{1}{\rho})$ will be demonstrated for the invariant measure $d\mu = e^{-V(x)}dx \in \mathscr{P}$ where the potential V is uniformly strictly convex,

$$\nabla^2 V \geq \rho \,\mathrm{Id} > 0.$$

We shall see that this condition implies the Γ_2-criterion $CD(\rho, \infty)$ for $L = \Delta - \nabla V \cdot \nabla$. The systematic approach to perform the Γ-Calculus will be presented in the next subsection. The following explicit example of P_t might be helpful in understanding.

Example 1.1 (The Ornstein-Uhlenbeck semigroup)**.** When $V(x) = \frac{|x|^2}{2}$, for any bounded continuous functions h and $t \geq 0$,

$$P_t h(x) = \int h(e^{-t}x + \sqrt{1 - e^{-2t}}\, y) d\gamma(y)$$

where γ is the standard Gaussian measure on $\mathbb{R}^d$. This semigroup satisfies good properties. For instance, assume h is a smooth function with compact support, then

$$\nabla P_t h = e^{-t} P_t \nabla h.$$

Notice that $-L$ is a self-adjoint positive operator in $L^2(\mu)$. The invariant space of L consists of constant functions. By spectral analysis, we know that the semigroup P_t is $L^2(\mu)$-*ergodic*, in the sense that

Lemma 1.28. *$P_t h$ converges to $P_\infty h = \int h d\mu$ in $L^2(\mu)$ as $t \to \infty$, for any $h \in L^2(\mu)$.*

Now we are ready to prove Theorem 1.7 and Theorem 1.20.

Poincaré inequality: First proof of Theorem 1.7

Let us denote the scalar product in $L^2(\mu)$ by $\langle \cdot, \cdot \rangle$ and the norm by $|| \cdot ||$. As we have done in Section 1.2.1, we compute the time derivative of the variance along the semigroup

$$\frac{1}{2}\frac{d}{dt} \mathrm{Var}_\mu(P_t h) = \langle P_t h, L P_t h \rangle = -||\nabla P_t h||^2.$$

We take one more derivative and find

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} ||\nabla P_t h||^2 &= \langle \nabla P_t h, \nabla L P_t h \rangle \\
&= \langle \nabla P_t h, L \nabla P_t h - \nabla^2 V \cdot \nabla P_t h \rangle \\
&= -||\nabla^2 P_t h||^2 - \langle \nabla P_t h, \nabla^2 V \cdot \nabla P_t h \rangle \\
&\leq -\rho ||\nabla P_t h||^2
\end{aligned}$$

where $||\nabla^2 \mathrm{P}_t h||$ stands for the $\mathrm{L}^2(\mu)$-norm of the Hilbert-Schmidt norm of the matrix $\nabla^2 \mathrm{P}_t h$, and the following commutation relation was used,

$$\nabla \mathrm{L} - \mathrm{L}\nabla = -\nabla^2 \mathrm{V} \cdot \nabla.$$

By Gronwall's lemma, it follows that

$$||\nabla \mathrm{P}_t h||^2 \leq e^{-2\rho t} ||\nabla h||^2$$

and hence we obtain the Poincaré inequality

$$\mathrm{Var}_\mu(h) = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Var}_\mu(\mathrm{P}_t h) \mathrm{d}t \leq \int_0^\infty 2e^{-2\rho t} ||\nabla h||^2 \mathrm{d}t = \frac{1}{\rho} ||\nabla h||^2.$$

(Note the ergodicity is used in the first equality.)

Logarithmic Sobolev inequality: First proof of Theorem 1.20

As in Section 1.2.1, we compute the time derivative of the entropy along P_t,

$$-\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ent}_\mu(\mathrm{P}_t h) = -\int (\log(\mathrm{P}_t h) + 1) \mathrm{L}\mathrm{P}_t h \mathrm{d}\mu = \int \frac{|\nabla \mathrm{P}_t h|^2}{\mathrm{P}_t h} \mathrm{d}\mu$$

where h has finite entropy. Note that we can avoid the possible singularity in the denominator (in the integrand) being zero, via a approximation procedure: replacing h by $h + \epsilon$ and then let $\epsilon \searrow 0$. Next we compute the second derivative,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{|\nabla \mathrm{P}_t h|^2}{\mathrm{P}_t h} \mathrm{d}\mu = \frac{\mathrm{d}}{\mathrm{d}t} \int |\nabla \log(\mathrm{P}_t h)|^2 \mathrm{P}_t h \mathrm{d}\mu \\
&= 2\int \nabla \log(\mathrm{P}_t h) \cdot \nabla \left(\frac{\mathrm{L}\mathrm{P}_t h}{\mathrm{P}_t h} \right) \mathrm{P}_t h \mathrm{d}\mu + \int |\nabla \log(\mathrm{P}_t h)|^2 \mathrm{L}\mathrm{P}_t h \mathrm{d}\mu \\
&= 2\int \nabla \log(\mathrm{P}_t h) \cdot \nabla(\mathrm{L}\mathrm{P}_t h) \mathrm{d}\mu - \int |\nabla \log(\mathrm{P}_t h)|^2 \mathrm{L}\mathrm{P}_t h \mathrm{d}\mu \\
&= 2\int \nabla \log(\mathrm{P}_t h) \cdot \mathrm{L}\nabla(\mathrm{P}_t h) \mathrm{d}\mu - 2\int \nabla \log(\mathrm{P}_t h) \cdot \nabla^2 \mathrm{V} \nabla(\mathrm{P}_t h) \mathrm{d}\mu - \int |\nabla \log(\mathrm{P}_t h)|^2 \mathrm{L}\mathrm{P}_t h \mathrm{d}\mu
\end{aligned}$$

where again the commutation relation between ∇ and L was used. Observe that

$$\int \nabla \log(\mathrm{P}_t h) \cdot \nabla^2 \mathrm{V} \nabla(\mathrm{P}_t h) \mathrm{d}\mu = \left\langle \frac{\nabla(\mathrm{P}_t h)}{\mathrm{P}_t h}, \nabla^2 \mathrm{V} \cdot \nabla(\mathrm{P}_t h) \right\rangle \geq \rho \int \frac{|\nabla \mathrm{P}_t h|^2}{\mathrm{P}_t h} \mathrm{d}\mu$$

(where we have applied the convexity of V) and the sum of the other two terms is indeed nonpositive since

$$\begin{aligned}
&2\int \nabla \log(\mathrm{P}_t h) \cdot \mathrm{L}\nabla(\mathrm{P}_t h) \mathrm{d}\mu - \int |\nabla \log(\mathrm{P}_t h)|^2 \mathrm{L}\mathrm{P}_t h \mathrm{d}\mu \\
&= -2\int \left[\nabla^2 \log(\mathrm{P}_t h)\right] : \left[\nabla^2 (\mathrm{P}_t h)\right] \mathrm{d}\mu + 2\int \left[\nabla^2 \log(\mathrm{P}_t h)\right] \left[\nabla \log(\mathrm{P}_t h)\right] \cdot \nabla(\mathrm{P}_t h) \mathrm{d}\mu \\
&= -2\int |\nabla^2 \log(\mathrm{P}_t h)|^2 \mathrm{P}_t h \mathrm{d}\mu \leq 0.
\end{aligned}$$

We summarize the computation above as a formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\nabla \mathrm{P}_t h|^2}{\mathrm{P}_t h}\mathrm{d}\mu = -2\left\langle \frac{\nabla \mathrm{P}_t h}{\mathrm{P}_t h}, \nabla^2 \mathrm{V}\cdot\nabla(\mathrm{P}_t h)\right\rangle - 2\int |\nabla^2 \log(\mathrm{P}_t h)|^2 \mathrm{P}_t h \mathrm{d}\mu. \tag{1.3.1}$$

(Using the terminology in next subsection, $\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\nabla \mathrm{P}_t h|^2}{\mathrm{P}_t h}\mathrm{d}\mu = -2\int \mathrm{P}_t h \Gamma_2(\mathrm{P}_t h)\mathrm{d}\mu$.) It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\nabla \mathrm{P}_t h|^2}{\mathrm{P}_t h}\mathrm{d}\mu \le -2\rho \int \frac{|\nabla \mathrm{P}_t h|^2}{\mathrm{P}_t h}\mathrm{d}\mu.$$

Again, by Gronwall's lemma, we know

$$\int \frac{|\nabla \mathrm{P}_t h|^2}{\mathrm{P}_t h}\mathrm{d}\mu \le e^{-2\rho t}\int \frac{|\nabla h|^2}{h}\mathrm{d}\mu.$$

Integrating over time, it follows the logarithmic Sobolev inequality for μ,

$$\mathrm{Ent}_\mu(h) \le \frac{1}{2\rho}\int \frac{|\nabla h|^2}{h}\mathrm{d}\mu.$$

1.3.2 Γ-calculus

Definition 5. For a linear differential operator L, the *carré du champ operator* Γ is defined as

$$\Gamma(f,g) = \frac{1}{2}(\mathrm{L}(fg) - f\mathrm{L}g - g\mathrm{L}f), \tag{1.3.2}$$

while the *iterated carré du champ operator*, referred to as the Γ_2 *operator*, is defined by

$$\Gamma_2(f,g) = \frac{1}{2}(\mathrm{L}\Gamma(f,g) - \Gamma(f,\mathrm{L}g) - \Gamma(\mathrm{L}f,g)), \tag{1.3.3}$$

where the functions f,g are smooth functions. For simplicity, we shall write $\Gamma(f) = \Gamma(f,f)$ and $\Gamma_2(f) = \Gamma_2(f,f)$.

Now we present several examples.

Example 1.2 (Laplacian on Euclidean spaces)**.** Consider $\mathrm{L} = \Delta$ and smooth functions f,g. Then by definition we have

$$\Gamma(f) = \frac{1}{2}\big(\Delta(f^2) - 2f\Delta f\big) = |\nabla f|^2$$

and so

$$\begin{aligned}\Gamma_2(f) &= \frac{1}{2}\big(\Delta(|\nabla f|^2) - 2\langle\nabla f, \nabla\Delta f\rangle\big),\\ &= \langle\nabla^2 f, \nabla^2 f\rangle + \langle\nabla f, \Delta\nabla f\rangle - \langle\nabla f, \nabla\Delta f\rangle\\ &= |\nabla^2 f|^2\end{aligned}$$

where $|\nabla^2 f|$ stands for the Hilbert-Schmidt norm of the Hessian matrix $\nabla^2 f$,

Example 1.3 (Laplacian on Riemannian manifolds)**.** Now consider the Laplace-Beltrami operator Δ. While $\Gamma(f)$ takes the same form as in the case of Euclidean space,

$$\Gamma(f) = |\nabla f|^2,$$

the expression $\langle \nabla f, \Delta \nabla f\rangle - \langle \nabla f, \nabla \Delta f\rangle$ in the $\Gamma_2(f)$ no longer vanishes and in fact it turns out to be $\mathrm{Ric}(\nabla f, \nabla f)$. To see this, we shall use the following form of the *Bochner's formula*

$$\frac{1}{2}\Delta(|\nabla f|^2) = |\nabla^2 f|^2 - \langle \nabla f, \nabla \Delta f\rangle + \mathrm{Ric}(\nabla f, \nabla f)$$

where Ric stands for the *Ricci curvature tensor*. It then follows that

$$\Gamma_2(f) = |\nabla^2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f).$$

Example 1.4 (Fokker-Planck operator on Riemannian manifolds)**.** Consider $\mathrm{L} = \Delta - \nabla \mathrm{V} \cdot \nabla$. The derivatives of first order do not change the form of $\Gamma(f)$,

$$\Gamma(f) = \frac{1}{2}\big(\Delta(f^2) - \langle \nabla \mathrm{V}, \nabla(f^2)\rangle - 2f(\Delta - \nabla \mathrm{V}\cdot\nabla)f\big) = |\nabla f|^2.$$

But they result in a new term in $\Gamma_2(f)$,

$$\begin{aligned}
\Gamma_2(f) &= \frac{1}{2}\big(\Delta(|\nabla f|^2) - 2\langle \nabla f, \nabla \Delta f\rangle\big) - \langle \nabla \mathrm{V}, \nabla |\nabla f|^2\rangle + 2\langle \nabla f, \nabla(\nabla \mathrm{V}\cdot \nabla f)\rangle\big) \\
&= |\nabla^2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f) + \langle \nabla f, \nabla^2 \mathrm{V}\cdot\nabla f\rangle \\
&= |\nabla^2 f|^2 + (\mathrm{Ric} + \nabla^2 \mathrm{V})(\nabla f, \nabla f).
\end{aligned}$$

The expression $\mathrm{Ric} + \nabla^2\mathrm{V}$ is known as the *Bakry-Émery Ricci curvature tensor* on smooth metric measure spaces.

In general, one may consider the following differential operator L defined for smooth functions on $\mathbb{R}^d$

$$\mathrm{L}f = \sum_{i,j}(\sigma\sigma^*)_{ij}(x)\partial^2_{x_ix_j} f + \sum_i b_i(x)\partial_{x_i} f, \tag{1.3.4}$$

where $\sigma \in \mathrm{C}^2(\mathbb{R}^d;\mathbb{R}^{d\times m})$ and $b = (b_i) \in \mathrm{C}^1(\mathbb{R}^d;\mathbb{R}^d)$. The operator L is the infinitesimal generator of a diffusion process $(\mathrm{X}_t)_{t\geq 0}$ solving the SDE

$$d\mathrm{X}_t = \sqrt{2}\sigma(\mathrm{X}_t)d\mathrm{B}_t + b(\mathrm{X}_t)dt$$

where $(\mathrm{B}_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^m$. It corresponds to a Markov semigroup P_t given by

$$\mathrm{P}_t f(x) = \mathbb{E}(f(\mathrm{X}_t)|\mathrm{X}_0 = x), \qquad t \geq 0.$$

Note the density function ρ of the law of $(\mathrm{X}_t)_{t\geq 0}$ satisfies the PDE

$$\partial_t \rho = \mathrm{L}^*\rho$$

with L^* being the dual operator of L, i.e.

$$\mathrm{L}^* f = \sum_{i,j}\partial^2_{x_ix_j}\big((\sigma\sigma^*)_{ij}(x)f\big) - \sum_i \partial_{x_i}(b_i(x)f).$$

We shall assume that there exists a probability measure $d\mu(x) = e^{-\mathrm{V}(x)}dx$ such that

1. μ is *invariant* for the semigroup P_t, i.e.

$$\int Lf\,d\mu = 0, \quad \text{or equivalently for all } t \geq 0, \int P_t f\,d\mu = \int f\,d\mu;$$

2. L is μ-*symmetric* (or one may say μ is symmetric), i.e.

$$\int gLf\,d\mu = \int fLg\,d\mu$$

for f, g in the domain of L.

By these assumptions, it follows that

$$\int \Gamma(f,g)\,d\mu = -\frac{1}{2}\int (fLg + gLf)\,d\mu = -\int gLf\,d\mu,$$

and

$$\int \Gamma_2(f)\,d\mu = \int \frac{1}{2}(L\Gamma(f) - 2\Gamma(f, Lf))\,d\mu = -\int \Gamma(f, Lf)\,d\mu = \int (Lf)^2\,d\mu.$$

(Note that the latter identity is reminiscent of the classical identity $\int |\nabla^2 f|^2 dx = \int (\Delta f)^2 dx$ which we mentioned in section 1.1.)

In the next proposition we collect several useful formulas for Γ-calculus. They follow from the structure of the operator L, and sometimes they might be stated as assumptions. They will be referred to as *diffusion property* or *chain rule.* And such operators will be referred to as *diffusion operators.*

Proposition 1.29. *Let f, g, h be smooth functions on $\mathbb{R}^d$. Let ψ, ϕ be smooth functions as well.*

$$\Gamma(\psi(f_1, f_2, \cdots, f_k), g) = \sum_{j=1}^{k} \partial_j \psi(f_1, f_2, \cdots, f_k)\Gamma(f_j, g), \tag{1.3.5}$$

$$L\psi(f_1, f_2, \cdots, f_k) = \sum_{j=1}^{k} \partial_j \psi(f_1, f_2, \cdots, f_k) Lf_j + \sum_{i,j=1}^{k} \partial_{ij}^2 \psi(f_1, f_2, \cdots, f_k)\Gamma(f_i, f_j), \tag{1.3.6}$$

In particular,

$$\begin{aligned} \Gamma(fg, h) &= f\Gamma(g,h) + g\Gamma(f,h), \\ L(fg) &= fLg + gLf + 2\Gamma(f,g), \\ L(\phi(f)) &= \phi'(f)Lf + \phi''(f)\Gamma(f). \end{aligned}$$

Now we apply these identities to calculate the “chain rule" for Γ_2 operator.

Proposition 1.30. *Let ψ be a function of class C^3 in the range of a smooth function f, then*

$$\Gamma_2(\psi(f)) = (\psi'(f))^2\Gamma_2(f) + \psi'(f)\psi''(f)\Gamma(f, \Gamma(f)) + (\psi''(f))^2\Gamma(f)^2. \tag{1.3.7}$$

For instance,

$$\begin{aligned} \Gamma_2(\log f) &= \frac{\Gamma_2(f)}{f^2} - \frac{\Gamma(f, \Gamma(f))}{f^3} + \frac{\Gamma(f)^2}{f^4}, \\ \Gamma_2(e^{af}) &= a^2 e^{2af}\left[\Gamma_2(f) + a\Gamma(f, \Gamma(f)) + a^2\Gamma(f)^2\right]. \end{aligned}$$

Proof. This can be shown by the diffusion properties in the preceding proposition. Indeed,

$$\begin{aligned}\Gamma_2(\psi(f)) &= \frac{1}{2}\big[\mathrm{L}\Gamma(\psi(f)) - 2\Gamma(\psi(f), \mathrm{L}(\psi(f)))\big]\\ &= \frac{1}{2}\{\mathrm{L}\big[(\psi'(f))^2\Gamma(f)\big] - 2\psi'(f)\Gamma(f, \psi'(f)\mathrm{L}f + \psi''(f)\Gamma(f))\}.\end{aligned}$$

Then the equality (1.3.7) follows from

$$\begin{aligned}\mathrm{L}\big[(\psi'(f))^2\Gamma(f)\big] &= (\psi'(f))^2\mathrm{L}\Gamma(f) + 2\Gamma((\psi'(f))^2, \Gamma(f)) + \Gamma(f)\mathrm{L}\big[(\psi'(f))^2\big]\\ &= (\psi'(f))^2\mathrm{L}\Gamma(f) + 4\psi'(f)\psi''(f)\Gamma(f, \Gamma(f))\\ &\quad + \Gamma(f)\Big\{2\psi'(f)\psi''(f)\mathrm{L}f + 2\big[(\psi''(f))^2 + \psi'(f)\psi'''(f)\big]\Gamma(f)\Big\},\end{aligned}$$

and

$$\begin{aligned}-2\psi'(f)\Gamma(f, \psi'(f)\mathrm{L}f + \psi''(f)\Gamma(f)) &= -2\psi'(f)^2\Gamma(f, \mathrm{L}f) - 2\psi'(f)\psi''(f)\Gamma(f)\mathrm{L}f\\ &\quad - 2\psi'(f)\psi''(f)\Gamma(f, \Gamma(f)) - 2\psi'(f)\psi'''(f)\Gamma(f)^2.\end{aligned}$$

□

1.3.3 Curvature-dimension condition

Definition 6. We say the operator L (or the associated semigroup $\mathrm{P}_t = e^{t\mathrm{L}}$) satisfies the *curvature-dimension condition* $\mathrm{CD}(\rho, n)$ if for every smooth function f,

$$\Gamma_2(f) \geq \rho\Gamma(f) + \frac{1}{n}(\mathrm{L}f)^2. \tag{1.3.8}$$

We present some examples below while more can be find in [2, Chapter 2] where the whole chapter is devoted to various examples. It is good to test our intuition by the examples therein.

Example 1.5. Consider the Laplcace-Beltrami operator on an n-dimensional Riemannian manifold with $\mathrm{Ric} \geq (n-1)\mathrm{K}$. Then

$$\begin{aligned}\Gamma_2(f) &= |\nabla^2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f)\\ &\geq \frac{1}{n}(\Delta f)^2 + (n-1)\mathrm{K}|\nabla f|^2\end{aligned}$$

i.e. the Laplace-Beltrami operator or the heat semigroup satisfies $\mathrm{CD}((n-1)\mathrm{K}, n)$. In particular, the heat semigroup on the n-dimensional sphere with constant curvature 1 satisfies $\mathrm{CD}(n-1, n)$; the heat semigroup on the n-dimensional Euclidean space satisfies $\mathrm{CD}(0, n)$; the heat semigroup on the n-dimensional hyperbolic space with constant curvature -1 satisfies $\mathrm{CD}(-(n-1), n)$.

Example 1.6. Consider $\mathrm{L} = \Delta - \nabla\mathrm{V}\cdot\nabla$ with $\nabla^2\mathrm{V} \geq \rho\,\mathrm{Id}$ on the Euclidean space, then

$$\begin{aligned}\Gamma_2(f) &= |\nabla^2 f|^2 + \langle\nabla^2\mathrm{V}\cdot\nabla f, \nabla f\rangle\\ &\geq \rho|\nabla f|^2\end{aligned}$$

i.e. L satisfies $\mathrm{CD}(\rho, \infty)$.

Example 1.7. Consider $\mathrm{L} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - a(x)\frac{\mathrm{d}}{\mathrm{d}x}$ on the real line, then the curvature dimension condition $\mathrm{CD}(\rho, n)$ is equivalent to

$$a' \geq \rho + \frac{a^2}{n-1}.$$

A special case of great interest is $\mathrm{CD}(\rho,\infty)$ which corresponds to $\Gamma_2(f) \geq \rho\Gamma(f)$. Now we present some equivalent descriptions of $\mathrm{CD}(\rho,\infty)$ in terms of the semigroup and local functional inequalities. We cite the following two results. Note that the quantities $\frac{1-e^{-2\rho t}}{\rho}$ and $\frac{e^{2\rho t}-1}{\rho}$ should be understood by convention as $2t$ when $\rho = 0$.

Theorem 1.31 (Local Poincaré inequalities)**.** *Let* L *be a diffusion operator with the carré du champ operator* Γ *and the iterated carré du champ operator* Γ_2. *Let* $(\mathrm{P}_t)_{t\geq 0}$ *denote the semigroup generated by* L. *Then the following assertions are equivalent.*

(i) The curvature condition $\mathrm{CD}(\rho,\infty)$ *holds for some* $\rho \in \mathbb{R}$.

(ii) For any function $f \in \mathrm{C}_c^\infty$*, and any* $t \geq 0$,

$$\Gamma(\mathrm{P}_t f) \leq e^{-2\rho t}\mathrm{P}_t(\Gamma(f)).$$

(iii) For any function $f \in \mathrm{C}_c^\infty$*, and any* $t \geq 0$,

$$\mathrm{P}_t(f^2) - (\mathrm{P}_t f)^2 \leq \frac{1-e^{-2\rho t}}{\rho}\mathrm{P}_t(\Gamma(f)).$$

(iv) For any function $f \in \mathrm{C}_c^\infty$*, and any* $t \geq 0$,

$$\mathrm{P}_t(f^2) - (\mathrm{P}_t f)^2 \geq \frac{e^{2\rho t}-1}{\rho}\Gamma(\mathrm{P}_t f).$$

Theorem 1.32 (Local logarithmic Sobolev inequalities)**.** *Under the same context as above. The following assertions are equivalent.*

(i) The curvature condition $\mathrm{CD}(\rho,\infty)$ *holds for some* $\rho \in \mathbb{R}$.

(ii) For any bounded function $f \in \mathrm{C}^\infty$*, and any* $t \geq 0$,

$$\sqrt{\Gamma(\mathrm{P}_t f)} \leq e^{-\rho t}\mathrm{P}_t\left(\sqrt{\Gamma(f)}\right).$$

(iii) For any positive bounded function $f \in \mathrm{C}^\infty$*, and any* $t \geq 0$,

$$\mathrm{P}_t(f\log f) - \mathrm{P}_t f \log \mathrm{P}_t f \leq \frac{1-e^{-2\rho t}}{2\rho}\mathrm{P}_t\left(\frac{\Gamma(f)}{f}\right).$$

(iv) For any positive bounded function $f \in \mathrm{C}^\infty$*, and any* $t \geq 0$,

$$\mathrm{P}_t(f\log f) - \mathrm{P}_t f \log \mathrm{P}_t f \geq \frac{e^{2\rho t}-1}{2\rho}\frac{\Gamma(\mathrm{P}_t f)}{\mathrm{P}_t f}.$$

Now we can present

Second proof of Theorem 1.7 and Theorem 1.20. Consider the operator $L = \Delta - \nabla V \cdot \nabla$ on $\mathbb{R}^d$ and the probability measure $d\mu(x) = \frac{1}{Z}e^{-V(x)}dx$. When $\nabla^2 V \geq \rho \mathrm{Id} > 0$, the operator L satisfies the $CD(\rho, \infty)$ criterion,

$$\Gamma_2(f) = |\nabla^2 f|^2 + \langle \nabla^2 V \cdot \nabla f, \nabla f \rangle \geq \rho |\nabla f|^2 = \rho \Gamma(f).$$

Therefore we can verify the local Poincaré inequalities and local logarithmic Sobolev inequalities asserted in the theorems above. It remains to let t tend to infinity and then apply the ergodicity of the associated semigroup. That way, we know the measure μ satisfies a Poincaré inequality $P(\frac{1}{\rho})$ and a logarithmic Sobolev inequality $LS(\frac{1}{\rho})$. □

Among all the assertions, we shall only include a unified proof of the implication of the curvature condition $CD(\rho, \infty)$ to the local functional inequalities. We adopt the terminology of the φ-entropy in [5] and the presentation in [3].

Definition 7. We say a function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ is *admissible* if φ is a strictly convex function of class C^4 such that $-1/\varphi''$ is convex. We define the notion of φ*-entropy* by

$$\mathrm{Ent}_\mu^\varphi(f) = \int \varphi(f) d\mu - \varphi(\int f d\mu)$$

for any function f with $\varphi(f)$ being integrable with respect to μ. Similarly, we define the φ-entropy with respect to an operator P_t by

$$\mathrm{Ent}_{P_t}^\varphi(f) = P_t(\varphi(f)) - \varphi(P_t f).$$

Definition 8. A probability measure μ is said to satisfy a φ-entropy inequality if there exists a constant $C > 0$ such that

$$\int \varphi(f) d\mu - \varphi(\int f d\mu) \leq \frac{C}{2} \int \varphi''(f) \Gamma(f) d\mu$$

for all smooth function f with compact support.

For instance, $\varphi(r) = r^2$ or $\varphi(r) = r \log r$ is admissible. More generally, for any $1 \leq p \leq 2$, the function

$$\varphi(r) = \begin{cases} \dfrac{r^p - r}{p(p-1)}, & \text{if } p \in (1,2]; \\ r \log r, & \text{if } p = 1 \end{cases}$$

is admissible. When $p = 2$, the φ-entropy corresponds to the variance and the φ-entropy inequality the Poincaré inequality. When $p = 1$, they correspond to the entropy and the logarithmic Sobolev inequality. The φ-entropy inequality for other $p \in (1,2)$ are sometimes referred to as *generalized Poincaré inequalities* or *Beckner's inequalities*. Among all φ-entropy inequalities, the Poincaré inequality is the weakest one in the sense that it can be deduced from all the other ones (by a Taylor expansion).

We reproduce the following theorem concerning local functional inequalities and curvature condition, c.f. Bolley and Gentil [3, Theorem 2].

Theorem 1.33. *Let φ be a admissible function. Then the following assertions are equivalent,*

(1) the semigroup $(\mathrm{P}_t)_{t\geq 0}$ satisfies the $\mathrm{CD}(\rho,\infty)$ criterion;

(2) the semigroup $(\mathrm{P}_t)_{t\geq 0}$ satisfies the local φ-entropy inequality

$$\mathrm{Ent}^{\varphi}_{\mathrm{P}_t}(f) \leq \frac{1-e^{-2\rho t}}{2\rho}\mathrm{P}_t(\varphi''(f)\Gamma(f))$$

for all positive time t and all function $f \in \mathrm{C}^{\infty}_c$.

If moreover the probability measure μ is ergodic for the semigroup $(\mathrm{P}_t)_{t\geq 0}$, and $\rho > 0$, then μ satisfies a φ-entropy inequality,

$$\mathrm{Ent}^{\varphi}_{\mu}(f) \leq \frac{1}{2\rho}\mu(\varphi''(f)\Gamma(f))$$

for all suitably-integrable function f.

Proof. We shall only prove the implication (1) $\Rightarrow$ (2). The converse implication is a consequence of a Taylor expansion. Now assume that the curvature dimension condition $\mathrm{CD}(\rho,\infty)$ holds. We consider the following quantity,

$$\Phi(s) = \mathrm{P}_s(\varphi(\mathrm{P}_{t-s}f)), \qquad 0 \leq s \leq t$$

which is a semigroup interpolation between $\Phi(0) = \varphi(\mathrm{P}_t f)$ and $\Phi(t) = \mathrm{P}_t(\varphi(f))$. The desired local φ-entropy inequality can be rephrased as

$$\Phi(t) - \Phi(0) \leq \frac{1-e^{-2\rho t}}{2\rho}\mathrm{P}_t(\varphi''(f)\Gamma(f))$$

and we shall find $\mathrm{P}_t(\varphi''(f)\Gamma(f)) = \Phi'(t)$. We denote $g := \mathrm{P}_{t-s}f$.
Step (i). We compute the derivatives of $\Phi(s)$:

(a) $\Phi'(s) = \mathrm{P}_s(\varphi''(\mathrm{P}_{t-s}f)\Gamma(\mathrm{P}_{t-s}f)) = \mathrm{P}_s\left(\frac{\Gamma(\varphi'(\mathrm{P}_{t-s}f))}{\varphi''(\mathrm{P}_{t-s}f)}\right)$;

(b) $\Phi''(s) = 2\mathrm{P}_s\left(\frac{\Gamma_2(\varphi'(\mathrm{P}_{t-s}f))}{\varphi''(\mathrm{P}_{t-s}f)}\right) + \mathrm{P}_s\left(\left(\frac{\Gamma(\varphi'(\mathrm{P}_{t-s}f))}{\varphi''(\mathrm{P}_{t-s}f)}\right)^2\left(\frac{-1}{\varphi''}\right)''(\mathrm{P}_{t-s}f)\right)$.

The first equality follows from the diffusion property,

$$\begin{aligned}\Phi'(s) &= \mathrm{P}_s(\mathrm{L}\varphi(g) - \varphi'(g)\mathrm{L}g)\\ &= \mathrm{P}_s(\varphi''(g)\Gamma(g)) = \mathrm{P}_s\left(\frac{\Gamma(\varphi'(g))}{\varphi''(g)}\right).\end{aligned}$$

Now we compute the second derivative,

$$\Phi''(s) = \mathrm{P}_s(\mathrm{L}[\varphi''(g)\Gamma(g)]) - \mathrm{P}_s(\varphi'''(g)\mathrm{L}g\Gamma(g) + 2\varphi''(g)\Gamma(g,\mathrm{L}g)).$$

By diffusion property,

$$\begin{aligned}
\mathrm{L}[\varphi''(g)\Gamma(g)] &= \varphi''(g)\mathrm{L}\Gamma(g) + \mathrm{L}\varphi''(g)\cdot\Gamma(g) + 2\Gamma(\varphi''(g),\Gamma(g)) \\
&= \varphi''(g)\mathrm{L}\Gamma(g) + \varphi'''(g)\mathrm{L}g\Gamma(g) + \varphi^{(4)}(g)\Gamma(g)^2 + 2\varphi'''(g)\Gamma(g,\Gamma(g)).
\end{aligned}$$

Substitute it into $\Phi''(s)$, and by the definition of Γ_2, we have

$$\begin{aligned}
\Phi''(s) &= \mathrm{P}_s(2\varphi''(g)\Gamma_2(g) + \varphi^{(4)}(g)\Gamma(g)^2 + 2\varphi'''(g)\Gamma(g,\Gamma(g))) \\
&= \mathrm{P}_s\left(\frac{2\Gamma_2(\varphi'(g))}{\varphi''(g)} + (\varphi^{(4)}(g) - \frac{2(\varphi'''(g))^2}{\varphi''(g)})\Gamma(g)^2\right)
\end{aligned}$$

where the chain rule for Γ_2 (see (1.3.7)) was applied. Therefore we obtain

$$\Phi''(s) = \mathrm{P}_s\left(\frac{2\Gamma_2(\varphi'(g))}{\varphi''(g)} + \Gamma(g)^2(\varphi''(g))^2\left(\frac{-1}{\varphi''}\right)''(g)\right).$$

Step (ii). Since $-1/\varphi''$ is convex, the second term on the RHS of the above equality is nonnegative. By $\mathrm{CD}(\rho,\infty)$ condition, it follows

$$\Phi''(s) \geq \mathrm{P}_s\left(\frac{2\Gamma_2(\varphi'(g))}{\varphi''(g)}\right) \geq \mathrm{P}_s\left(\frac{2\rho\Gamma(\varphi'(g))}{\varphi''(g)}\right) = 2\rho\Phi'(s).$$

By Gronwall's lemma we deduce that

$$\Phi'(s) \leq e^{-2\rho(t-s)}\Phi'(t).$$

Integrating in s over $(0,t)$ yields the desired inequality $\Phi(t) - \Phi(0) \leq \frac{1-e^{-2\rho t}}{2\rho}\Phi'(t)$. □

We end this section by quoting the Poincaré inequality and the logarithmic Sobolev inequality under the general curvature dimension condition $\mathrm{CD}(\rho,n)$.

Theorem 1.34 (Poincaré inequality under $\mathrm{CD}(\rho,n)$)**.** *Assume* L *satisfies the curvature dimension condition* $\mathrm{CD}(\rho,n)$*, then the associated invariant measure* μ *satisfies a Poincaré inequality with constant* $\frac{n-1}{n\rho}$.

Theorem 1.35 (Logarithmic Sobolev inequality under $\mathrm{CD}(\rho,n)$)**.** *Assume* L *satisfies the curvature dimension condition* $\mathrm{CD}(\rho,n)$*, then the associated invariant measure* μ *satisfies a logarithmic Sobolev inequality with constant* $\frac{n-1}{n\rho}$.

The constant in the above results is sharp. Due to Proposition 1.23, it suffices to consider the optimality of the Poincaré inequality. On a n-dimensional Riemannian manifolds with Ricci curvature $\mathrm{Ric} \geq (n-1)\mathrm{K}$, the Laplacian satisfies $\mathrm{CD}((n-1)\mathrm{K},n)$ and thus the Riemannian volume measure verifies a Poincaré inequality with constant $\frac{1}{n\mathrm{K}}$. Note that the Lichnerovich-Obata theorem expresses that the first eigenvalue of the Laplacian is greater than $\frac{1}{n\mathrm{K}}$ with equality if and only if the Riemannian manifold is the n-sphere with constant curvature K.

The proofs are based on respectively the Theorem 1.16 and the Theorem below. This theorem is proven by following the same line as in the first subsection in the proof of Theorem 1.20 (where we utilised the explicit form of Γ_2 instead).

Theorem 1.36. *Assume that there exists some constant* $C > 0$ *such that*

$$\int f\Gamma(\log f)d\mu \le C\int f\Gamma_2(\log f)d\mu$$

for all positive smooth function f *with* $f, \Gamma(f)$ *and* Lf *being bounded. Then the probability measure* μ *satisfies a logarithmic Sobolev inequality with constant* C.

It is an analogue of the dual description of Poincaré inequality (Theorem 1.16). But the inequality above is strictly stronger than the logarithmic Sobolev inequality. It would be interesting if one can find an equivalent integral formulation of the logarithmic Sobolev inequality in terms of curvature-dimension condition, as in the case of the Poincaré inequality.

1.4 Villani's hypocoercivity theory

In this section, we focus on Villani's hypocoercivity theory (see [17]) in the special case of kinetic Fokker-Planck equation. Let us introduce the notion of hypocoercivity in the Hilbert setting first. Let L be a linear operator generating s strong continuous semigroup e^{-tL} on some Hilbert space H with scalar product $\langle\cdot,\cdot\rangle_H$ (or sometimes abbreviated as $\langle\cdot,\cdot\rangle$) and norm $||\cdot||$. The operator L is said to be λ-*coercive* for some $\lambda > 0$ if

$$\langle h, Lh\rangle_H \ge \lambda||h||^2$$

for any function h in the domain of L and being orthogonal with respect to the kernel of L. By Gronwall's lemma, λ-coercivity implies exponential convergence of the semigroup e^{-tL} to the projection onto the invariant space of L.

However, in many cases, despite that coercivity does not hold, the exponential convergence is still valid. The notion of *hypocoercivity* is introduced for describing the exponential decay of the evolution in the absence of coercivity. The prefix *hypo* comes from *hypoellipticity* in order to highlight their links. Below is a definition of hypocoercivity in a Hilbert context.

Definition 9. Let H be a Hilbert space with norm $||\cdot||$, L an operator on H generating a strong continuous semigroup $(e^{-tL})_{t\ge0}$. The operator L is said to be λ-*hypocoercive* on H if there exists some constant $C > 0$ such that

$$||e^{-tL}h_0|| \le Ce^{-\lambda t}||h_0||$$

for any $h_0 \in \mathrm{Ker}^\perp(L)$ (the subspace perpendicular to the kernel of L).

The notion can be extended to general distance functions or other functionals.

Definition 10. Consider a distance function (or other functionals) d on some function space B. The semigroup $(e^{-tL})_{t\ge0}$ or the operator L is said to be λ-hypocoercive on B if there exists some constant $C > 0$ such that

$$d(e^{-tL}h_0, \mu_\infty^{h_0}) \le Ce^{-\lambda t}d(h_0, \mu_\infty^{h_0})$$

for any $h_0 \in B$, where $\mu_\infty^{h_0}$ stands for the invariant element corresponding to the initial datum h_0.

One of the main strategies to prove hypocoercivity is to find a distorted functional or distance or norm under which the operator is λ_0-coercive for some $\lambda_0 > 0$, and then to prove hypocoercivity under the original one by equivalence. We shall illustrate this idea in the setting of kinetic Fokker-Planck equation on $\mathbb{R}^{2d}$ which reads

$$\partial_t h + \mathrm{L}h = 0$$

where $\mathrm{L} = \mathrm{A}^*\mathrm{A} + \mathrm{B}$, V is a smooth potential on $\mathbb{R}^d$, and

$$\mathrm{A} = \nabla_v, \quad \mathrm{B} = v \cdot \nabla_x - \nabla \mathrm{V}(x) \cdot \nabla_v, \quad \mathrm{A}^* = -\mathrm{div}_v + v\cdot, \quad \mathrm{B}^* = -\mathrm{B}. \tag{1.4.1}$$

Here the dual operator is taken in $\mathrm{L}^2(\mu)$ with μ being the invariant measure,

$$\mathrm{d}\mu(x,v) = \mathrm{d}m(x)\mathrm{d}\gamma(v) = \frac{1}{\mathrm{Z}} e^{-\mathrm{V}(x) - \frac{|v|^2}{2}} \mathrm{d}x\mathrm{d}v$$

where γ is the standard Gaussian measure in the velocity space, Z is the normalizing constant. We shall use the notation from [17] and denote some commutators by

$$\mathrm{C} := [\mathrm{A},\mathrm{B}] = \nabla_x, \quad \mathrm{R} := [\mathrm{C},\mathrm{B}] = -\nabla^2 \mathrm{V}(x) \cdot \nabla_v. \tag{1.4.2}$$

Furthermore, by direct computation we know the commutation relation $[\mathrm{A},\mathrm{A}^*] = \mathrm{Id}$.

We shall present Villani's hypocoercivity theory in the case of the kinetic Fokker-Planck equation in $\mathrm{H}^1(\mu)$ and in entropy. We quote Villani's hypocoercivity theorem for the kinetic Fokker-Planck equation, see [17, Theorem 35, Theorem 39].

Theorem 1.37 (Villani)**.** *Let* V *be a potential on* $\mathbb{R}^d$ *of class* C^2*, satisfying*

$$|\nabla^2 \mathrm{V}| \le \mathrm{M}(1 + |\nabla \mathrm{V}|) \tag{1.4.3}$$

for some constant M*. Assume that the probability measure m satisfies a Poincaré inequality. Then there exist constant* $\mathrm{K} > 0$ *and* $\lambda > 0$*, explicitly computable, such that for all* $h_0 \in \mathrm{H}^1(\mu)$*,*

$$||e^{-t\mathrm{L}} h_0 - \int h_0 d\mu||_{\mathrm{H}^1(\mu)} \le \mathrm{K} e^{-\lambda t} ||h_0||_{\mathrm{H}^1(\mu)}.$$

Theorem 1.38 (Villani)**.** *Assume that*

(1) the potential $\mathrm{V} \in \mathrm{C}^2(\mathbb{R}^d)$ *with* $|\nabla^2 \mathrm{V}| \le \mathrm{M}$*;*

(2) the reference measure μ *satisfies a logarithmic Sobolev inequality;*

(3) the initial datum $h_0 d\mu(x,v)$ *is a probability measure with finite moment of order 2.*

Then the solution of the kinetic Fokker-Planck equation with initial datum h_0 *converges to 1 exponentially fast as* $t \to \infty$*, in the sense of entropy*

$$\int h_t \log h_t d\mu(x,v) = \mathrm{O}(e^{-\alpha t})$$

with explicit estimates.

Indeed, the proofs of the two proceeding theorems are very similar. So we shall present a unified proof (again only a sketch of proof) for them in the setting of φ-entropies. The proof below is essentially in the same spirit of Villani's original proofs, nevertheless, the presentation below is inspired by F. Bolley and I. Gentil [3]. Note that the convergence to equilibrium in φ-entropies has been considered by several other authors, see for instance P. Monmarché [12], J. Dolbeault and X. Li [8], J. Evans [9] and the references therein. In particular, J. Dolbeault and X. Li [8] studied hypocoercivity in a family of φ-entropies associated to Beckner's inequalities (with the optimal rate of convergence) for the kinetic Fokker-Planck equation when $V(x) = |x|^2/2$. Moreover, with P. Cattiaux, A. Guillin and P. Monmarché, we shall apply the entropic multipliers method to derive hypocoercive relaxation to equilibrium in φ-entropies, c.f. Chapter 2.

Recall that φ is a strictly convex function of class C^4 such that $-1/\varphi''$ is convex, and the φ-entropy is given by

$$\mathrm{Ent}_\mu^\varphi(h) = \int \varphi(h)\mathrm{d}\mu - \varphi(\int h\mathrm{d}\mu)$$

for any function h with $\varphi(h)$ being integrable with respect to μ (i.e. h is in some suitable Orlicz space). One can also introduce the associated φ-Fisher information

$$\mathrm{I}_\mu^\varphi(h) = \int \varphi''(h)|\nabla h|^2\mathrm{d}\mu.$$

Again we shall not discuss the regularity or integrability issues here, but focus on the convergence to equilibrium.

Theorem 1.39 (Hypocoercivity concerning φ-entropy)**.** *Consider solutions with finite φ-entropy and finite φ-Fisher information to the kinetic Fokker-Planck equation. Assume that (i) there exist constant M_1, M_2 such that for all suitable function h,*

$$\int \frac{1}{\varphi''(h)}|\nabla^2 V\cdot\nabla_v\varphi'(h)|^2 d\mu \le M_1\int \frac{1}{\varphi''(h)}|\nabla_v\varphi'(h)|^2 d\mu + M_2\int \frac{1}{\varphi''(h)}|\nabla^2_{xv}\varphi'(h)|^2_{HS}d\mu$$

where $|\cdot|_{HS}$ is the Hilbert-Schmidt norm of a square matrix. Assume that (ii) the invariant measure μ satisfies a φ-entropy inequality with constant $K > 0$, i.e.,

$$\int \varphi(h)d\mu - \varphi(\int hd\mu) \le K\int \varphi''(h)|\nabla h|^2 d\mu$$

for all suitable function h. Then there exist positive constants a, b, c and $\lambda > 0$, such that

$$\mathscr{E}(h_t) \le e^{-\lambda t}\mathscr{E}(h_0)$$

for all well-behaved solutions h, where $ac > b^2$ and

$$\mathscr{E}(h) = \mathrm{Ent}_\mu^\varphi(h) + a\int \varphi''(h)|\nabla_v h|^2 d\mu + 2b\int \varphi''(h)\langle\nabla_x h, \nabla_v h\rangle d\mu + c\int \varphi''(h)|\nabla_x h|^2 d\mu.$$

In particular, there exists $C > 0$ such that

$$\mathrm{Ent}_\mu^\varphi(h_t) + \mathrm{I}_\mu^\varphi(h_t) \le Ce^{-\lambda t}\left(\mathrm{Ent}_\mu^\varphi(h_t) + \mathrm{I}_\mu^\varphi(h_t)\right)$$

for all well-behaved solutions h.

Remark 1.40. When the φ-entropy is the variance (with respect to the invariant measure μ), the above result corresponds to the hypocoercivity in $H^1(\mu)$, i.e. Theorem 1.37. When the φ-entropy is the usual entropy (with repsect to μ), then it implies the decay estimate in Theorem 1.38 (Note that the initial conditions stated in Theorem 1.38 are more general).

Remark 1.41. The first assumption is satisfied with $M_1 = M$ and $M_2 = 0$ whenever $|\nabla^2 V| \le M$. Moreover, it becomes either

$$\int |\nabla^2 V \cdot \nabla_v h|^2 d\mu \le M_1 \int |\nabla_v h|^2 d\mu + M_2 \int |\nabla^2_{xv} h|^2_{HS} d\mu$$

in the variance case, or

$$\int |\nabla^2 V \cdot \nabla_v \log h|^2 h d\mu \le M_1 \int |\nabla_v \log h|^2 h d\mu + M_2 \int |\nabla^2_{xv} \log h|^2_{HS} h d\mu$$

in the entropic case (in that case, it seems awkward). But, to avoid further technique assumptions, we leave the assumption the form above.

Remark 1.42. Just as in the $H^1(\mu)$ and entropic case in [17], a regularization estimate of the φ-Fisher information can be proved. For instance, one may prove that for $0 < t < 1$,

$$\mathrm{Ent}^{\varphi}_{\mu}(h_t) + I^{\varphi}_{\mu}(h_t) \le \frac{C}{t^3} \mathrm{Ent}^{\varphi}_{\mu}(h_0).$$

In particular, the φ-Fisher information will be finite at any positive time even if we start with an initial datum with finite φ-entropy being required. Combined with the exponential convergence above, note also that the φ-entropy is nonincreasing along kinetic Fokker-Planck equation, one is led to hypocoercivity in the φ-entropy, namely, there exist contants $C, \lambda > 0$ such that

$$\mathrm{Ent}^{\varphi}_{\mu}(h_t) \le C e^{-\lambda t} \mathrm{Ent}^{\varphi}_{\mu}(h_0).$$

But we shall not discuss the regularization estimates here.

Sketch of Proof. First of all the mass $\int h d\mu$ is conserved. Then the entropy production functional is

$$\begin{aligned} -\frac{d}{dt} \int \varphi(h) d\mu &= \int \varphi'(h)(A^* A h + B h) d\mu \\ &= \int \langle A\varphi'(h), A h \rangle d\mu + \int B(\varphi(h)) d\mu \\ &= \int \varphi''(h) |\nabla_v h|^2 d\mu = \int \frac{|\nabla_v \varphi'(h)|^2}{\varphi''(h)} d\mu \end{aligned}$$

where $\int B(\varphi(h)) d\mu = 0$ since B is antisymmetric. We find that there is "missing of dissipation" in the ∇_x direction which results in the absence of entropy-entropy production inequalities. This is the reason for introducing a mixed term in the Fisher information: the mixed term leads to dissipation in the ∇_x direction. We *claim* that there exists some constants $a, b, c, \kappa > 0$ such that $ac > b^2$ and

$$-\frac{d}{dt} \mathscr{E}(h) \ge \kappa I^{\varphi}_{\mu}(h).$$

From this claim, the theorem follows by the second assumption and $ac > b^2$.

To this end, let us compute the time-derivatives of the other terms in $\mathscr{E}$. The computation can be done either as in the proofs in section 1.3.1 or section 1.3.3. We follow the former one. Since L is the sum of $\mathrm{A}^*\mathrm{A}$ and B, we shall deal with them separately. We adopt the following notation for the time-derivative of the functional $\mathscr{F}$ along the semigroup $e^{-t\mathrm{S}}$ generated by $-\mathrm{S}$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{S}} \mathscr{F}(h) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathscr{F}(e^{-t\mathrm{S}} h).$$

For instance, $\mathscr{F}$ might be Ent_μ; S will be $\mathrm{A}^*\mathrm{A}$, or B.

(1) Treatment of the operator B. We list the three equalities first.

$$\begin{aligned}
-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{B}} \int \varphi''(h)|\nabla_v h|^2 \mathrm{d}\mu &= 2\int \mathrm{d}\mu\Big\{\varphi''(h)\langle \nabla_v h, \nabla_x h\rangle\Big\};\\
-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{B}} \varphi''(h) \int |\nabla_x h|^2 \mathrm{d}\mu &= 2\int \mathrm{d}\mu\Big\{\varphi''(h)\langle \nabla_x h, -\nabla^2 \mathrm{V}\cdot\nabla_v h\rangle\Big\};\\
-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{B}} \int \varphi''(h)\langle \nabla_v h, \nabla_x h\rangle \mathrm{d}\mu &= \int \mathrm{d}\mu\Big\{\varphi''(h)\left(|\nabla_x h|^2 + \langle \nabla_v h, -\nabla^2\mathrm{V}\cdot\nabla_v h\rangle\right)\Big\}.
\end{aligned}$$

Note that the term $\varphi''(h)|\nabla_x h|^2$ arises in the last expression. That way, the anti-symmetric operator B plays an important role. Note also the usual φ-Fisher information I_μ^φ does not take advantage of the operator B.

Now we prove these three equalities. They can be treated in the same way. Consider two derivation operators $\mathrm{C}_1, \mathrm{C}_2$ (which will be either $\mathrm{A} = \nabla_v$ or $\mathrm{C} = \nabla_x$), then

$$\begin{aligned}
&-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{B}} \int \varphi''(h)\langle \mathrm{C}_1 h, \mathrm{C}_2 h\rangle \mathrm{d}\mu\\
&= \int \mathrm{d}\mu\Big\{\varphi''(h)(\langle \mathrm{C}_1\mathrm{B}h, \mathrm{C}_2 h\rangle + \langle \mathrm{C}_1 h, \mathrm{C}_2 \mathrm{B}h\rangle) + \varphi'''(h)\mathrm{B}h\langle \mathrm{C}_1 h, \mathrm{C}_2 h\rangle\Big\}\\
&= \int \mathrm{d}\mu\Big\{\varphi''(h)\Big(\langle [\mathrm{C}_1, \mathrm{B}]h, \mathrm{C}_2 h\rangle + \langle \mathrm{B}\mathrm{C}_1 h, \mathrm{C}_2 h\rangle + \langle \mathrm{C}_1 h, [\mathrm{C}_2, \mathrm{B}]h\rangle + \langle \mathrm{C}_1 h, \mathrm{B}\mathrm{C}_2 h\rangle\Big)\\
&\qquad + \mathrm{B}(\varphi''(h))\langle \mathrm{C}_1 h, \mathrm{C}_2 h\rangle\Big\}.
\end{aligned}$$

By the fact that B is anti-symmetric and it is a derivation operator, it holds

$$\int \mathrm{d}\mu\Big\{\varphi''(h)\Big(\langle \mathrm{B}\mathrm{C}_1 h, \mathrm{C}_2 h\rangle + \langle \mathrm{C}_1 h, \mathrm{B}\mathrm{C}_2 h\rangle\Big) + \mathrm{B}(\varphi''(h))\langle \mathrm{C}_1 h, \mathrm{C}_2 h\rangle\Big\} = 0.$$

Therefore

$$-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{B}} \int \varphi''(h)\langle \mathrm{C}_1 h, \mathrm{C}_2 h\rangle \mathrm{d}\mu = \int \mathrm{d}\mu\Big\{\varphi''(h)\Big(\langle [\mathrm{C}_1, \mathrm{B}]h, \mathrm{C}_2 h\rangle + \langle \mathrm{C}_1 h, [\mathrm{C}_2, \mathrm{B}]h\rangle\Big)\Big\}. \tag{1.4.4}$$

The three desired equalities are direct consequences of the above formula.

(2) Treatment of the operator $\mathrm{A}^*\mathrm{A}$**.** Denote $u := \varphi'(h)$. We shall prove the following equalities,

$$
\begin{aligned}
&-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{A}^*\mathrm{A}}\int \varphi''(h)|\nabla_v h|^2\mathrm{d}\mu \\
&= \int \mathrm{d}\mu\Big\{\left(\frac{-1}{\varphi''}\right)''(h)|\nabla_v h|^2|\nabla_v u|^2 + 2\varphi''(h)|\nabla_v h|^2 + \frac{2}{\varphi''(h)}|\nabla_v^2 u|^2\Big\}; \\
&-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{A}^*\mathrm{A}}\int \varphi''(h)|\nabla_x h|^2\mathrm{d}\mu \\
&= \int \mathrm{d}\mu\Big\{\left(\frac{-1}{\varphi''}\right)''(h)|\nabla_v h|^2|\nabla_x u|^2 + \frac{2}{\varphi''(h)}|\nabla_{xv}^2 u|^2\Big\}; \\
&-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{A}^*\mathrm{A}}\int \varphi''(h)\langle\nabla_v h, \nabla_x h\rangle\mathrm{d}\mu \\
&= \int \mathrm{d}\mu\Big\{\left(\frac{-1}{\varphi''}\right)''(h)|\nabla_v h|^2\langle\nabla_v u, \nabla_x u\rangle + \varphi''(h)\langle\nabla_v h, \nabla_x u\rangle + \frac{2}{\varphi''(h)}\langle\nabla_v^2 u, \nabla_{xv}^2 u\rangle\Big\}.
\end{aligned}
$$

Consider $\mathrm{C}_1, \mathrm{C}_2 \in \{\mathrm{A}, \mathrm{C}\}$, then $[\mathrm{C}_1, \mathrm{A}] = [\mathrm{C}_2, \mathrm{A}] = 0$. We compute that

$$
\begin{aligned}
&-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{A}^*\mathrm{A}}\int \varphi''(h)\langle \mathrm{C}_1 h, \mathrm{C}_2 h\rangle\mathrm{d}\mu = -\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathrm{A}^*\mathrm{A}}\int \frac{\langle \mathrm{C}_1 u, \mathrm{C}_2 u\rangle}{\varphi''(h)}\mathrm{d}\mu \\
&= \int \mathrm{d}\mu\Big\{\left(\frac{1}{\varphi''}\right)'(h)\mathrm{A}^*\mathrm{A}h\langle \mathrm{C}_1 h, \mathrm{C}_2 h\rangle \\
&\qquad + \frac{1}{\varphi''(h)}\Big(\langle \mathrm{C}_1(\varphi''(h)\mathrm{A}^*\mathrm{A}h), \mathrm{C}_2 u\rangle + \langle \mathrm{C}_1 u, \mathrm{C}_2(\varphi''(h)\mathrm{A}^*\mathrm{A}h)\rangle\Big)\Big\}.
\end{aligned}
$$

Note that $\mathrm{C}_1(\varphi''(h)\mathrm{A}^*\mathrm{A}h) = \varphi'''(h)\mathrm{C}_1 h\mathrm{A}^*\mathrm{A}h + \varphi''(h)\mathrm{C}_1\mathrm{A}^*\mathrm{A}h$, and

$$
\begin{aligned}
&\int \mathrm{d}\mu\Big\{\frac{1}{\varphi''(h)}\langle\varphi'''(h)\mathrm{C}_1 h\mathrm{A}^*\mathrm{A}h, \mathrm{C}_2 u\rangle\Big\} = \int \mathrm{d}\mu\Big\{\left(\frac{-1}{\varphi''}\right)'(h)\langle \mathrm{C}_1 u, \mathrm{C}_2 u\rangle\mathrm{A}^*\mathrm{A}h\Big\} \\
&= \int \mathrm{d}\mu\Big\{\left(\frac{-1}{\varphi''}\right)''(h)|\mathrm{A}h|^2\langle \mathrm{C}_1 u, \mathrm{C}_2 u\rangle \\
&\qquad + \left(\frac{-1}{\varphi''}\right)'(h)\Big(\langle \mathrm{C}_1\mathrm{A}u, \mathrm{C}_2 u\otimes \mathrm{A}h\rangle + \langle \mathrm{C}_2\mathrm{A}u, \mathrm{C}_1 u\otimes \mathrm{A}h\rangle\Big)\Big\},
\end{aligned}
$$

while

$$
\int \mathrm{d}\mu\Big\{\langle \mathrm{C}_1\mathrm{A}^*\mathrm{A}h, \mathrm{C}_2 u\rangle\Big\} = \int \mathrm{d}\mu\Big\{\langle[\mathrm{C}_1, \mathrm{A}^*]\mathrm{A}h, \mathrm{C}_2 u\rangle + \langle \mathrm{C}_1\mathrm{A}h, \mathrm{C}_2\mathrm{A}u\rangle\Big\}.
$$

Similarly we can do the computations for other terms by just exchanging the subscript in C_1 and

C_2. Then we have

$$
\begin{aligned}
&-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{A^*A}\int \varphi''(h)\langle C_1 h, C_2 h\rangle \mathrm{d}\mu\\
&=\int \mathrm{d}\mu\left\{\left(\frac{-1}{\varphi''}\right)'(h)\langle C_1 h, C_2 h\rangle A^*Ah+\langle C_1A^*Ah, C_2 u\rangle+\langle C_2A^*Ah, C_1 u\rangle\right\}\\
&=\int \mathrm{d}\mu\left\{\left(\frac{-1}{\varphi''}\right)''(h)|Ah|^2\langle C_1 u, C_2 u\rangle\right.\\
&\qquad+\left(\frac{-1}{\varphi''}\right)'(h)\Big(\langle C_1Au, C_2 u\otimes Ah\rangle+\langle C_2Au, C_1 u\otimes Ah\rangle\Big)\\
&\qquad\left.+\langle[C_1,A^*]Ah, C_2 u\rangle+\langle C_1Ah, C_2Au\rangle+\langle[C_2,A^*]Ah, C_1 u\rangle+\langle C_2Ah, C_1Au\rangle\right\}\\
&=\int \mathrm{d}\mu\left\{\left(\frac{-1}{\varphi''}\right)''(h)|Ah|^2\langle C_1 u, C_2 u\rangle+\langle[C_1,A^*]Ah, C_2 u\rangle+\langle[C_2,A^*]Ah, C_1 u\rangle\right.\\
&\qquad\left.+\frac{2}{\varphi''(h)}\langle C_1Au, C_2Au\rangle\right\}.
\end{aligned}
$$

Therefore we obtain the desired equalities.

(3) Conclusion. We arrive at the temporal derivative of $\mathscr{E}(h)$,

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(h)=\int \mathrm{d}\mu\Big\{&\varphi''(h)|\nabla_v h|^2+2a\varphi''(h)\langle\nabla_v h,\nabla_x h\rangle\\
&+2b\varphi''(h)\left(|\nabla_x h|^2+\langle\nabla_v h,-\nabla^2 V\cdot\nabla_v h\rangle\right)+2c\varphi''(h)\langle\nabla_x h,-\nabla^2V\cdot\nabla_v h\rangle\\
&+a\left(\left(\frac{-1}{\varphi''}\right)''(h)|\nabla_v h|^2|\nabla_v u|^2+2\varphi''(h)|\nabla_v h|^2+\frac{2}{\varphi''(h)}|\nabla_v^2 u|^2\right)\\
&+2b\left(\left(\frac{-1}{\varphi''}\right)''(h)|\nabla_v h|^2\langle\nabla_v u,\nabla_x u\rangle+\varphi''(h)\langle\nabla_v h,\nabla_x h\rangle+\frac{2}{\varphi''(h)}\langle\nabla_v^2 u,\nabla_{xv}^2 u\rangle\right)\\
&+c\left(\left(\frac{-1}{\varphi''}\right)''(h)|\nabla_v h|^2|\nabla_x u|^2+\frac{2}{\varphi''(h)}|\nabla_{xv}^2 u|^2\right)\Big\}. \qquad (1.4.5)
\end{aligned}
$$

Case (1): $\nabla^2 V$ *is bounded, say* $|\nabla^2 V|\leq M$. When $ac>b^2$ and $a>0, c>0$, it follows by the convexity of $-1/\varphi''$ that

$$\left(\frac{-1}{\varphi''}\right)''(h)\Big(a|\nabla_v h|^2|\nabla_v u|^2+2b|\nabla_v h|^2\langle\nabla_v u,\nabla_x u\rangle+c|\nabla_v h|^2|\nabla_x u|^2\Big)\geq 0;$$

and it holds

$$\frac{2a}{\varphi''(h)}|\nabla_v^2 u|^2+\frac{4b}{\varphi''(h)}\langle\nabla_v^2 u,\nabla_{xv}^2 u\rangle+\frac{2c}{\varphi''(h)}|\nabla_{xv}^2 u|^2\geq 0.$$

The remaining terms can be bounded from below by a quadratic form

$$Q(a,b,c):=\int \mathrm{d}\mu\Big\{\varphi''(h)\Big((1+2a-2bM)|\nabla_v h|^2+2b|\nabla_x h|^2-(2b+2cM)|\nabla_v h||\nabla_x h|\Big)\Big\}.$$

One can choose positive constants a,b,c such that $ac>b^2$ and

$$Q(a,b,c)\geq\kappa\int \mathrm{d}\mu\Big\{\varphi''(h)\Big(|\nabla_v h|^2+|\nabla_x h|^2\Big)\Big\}$$

for some constant $\kappa > 0$. Summering up, we then arrive at the claim

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(h) \geq \mathrm{Q}(a,b,c) \geq \kappa \mathrm{I}_\mu^\varphi(h).$$

Case (2): $\nabla^2 \mathrm{V}$ *satisfies the first assumption.* In that case, the terms involving $\nabla^2 \mathrm{V}$ can be controlled by the terms in φ-Fisher information and the terms involving $\nabla^2 u$. Compared with case (i), it is only a little more complicated in a technique level. Indeed, then it suffices to choose positive constants a, b, c such that the matrix

$$\begin{pmatrix} 1+2a-2b\sqrt{\mathrm{M}_1} & 0 & 0 & 0 \\ -2a-2b-2c\sqrt{\mathrm{M}_1} & 2b & 0 & 0 \\ 0 & 0 & 2a & 0 \\ -2b\sqrt{\mathrm{M}_2} & -2c\sqrt{\mathrm{M}_2} & -4b & 2c \end{pmatrix}$$

is strictly positive in the sense of quadratic forms. That way, one can prove the claim holds true as well. □

1.5 Contributions

1.5.1 Entropic multipliers method for Langevin diffusion and weighted log Sobolev inequalities (c.f. Chapter 2)

In this part, we are concerned with the convergence to equilibrium in entropy for the solutions to the kinetic Fokker-Planck equation which reads

$$\partial_t f_t = \mathrm{L} f_t$$

with L being given by

$$\mathrm{L} = -y \cdot \nabla_x + (\nabla \mathrm{U}(x) - y) \cdot \nabla_y + \Delta_y.$$

We denote by $\mathrm{H}(x,y) = \mathrm{U}(x) + \frac{1}{2}|y|^2$ the Hamiltonian. Assume that $\int e^{-\mathrm{U}} \mathrm{d}x < \infty$, we denote the unique invariant measure by μ, i.e. $\mathrm{d}\mu(x,y) = \frac{1}{\mathrm{Z}} e^{-\mathrm{H}(x,y)} \mathrm{d}x\mathrm{d}y$ where Z is the normalizing constant. Recall that in his hypocoercivity theorem for kinetic Fokker-Planck equation in the entropic case, Villani assumed the following two conditions on the potential U,

- $\nabla^2 \mathrm{U}$ is bounded;
- $e^{-\mathrm{U}(x)}\,\mathrm{d}x$ satisfies a log-Sobolev inequality.

Our purpose is to get rid of the boundedness condition. Here are our assumptions,

Assumption 1.1. Assume that there exists $\eta \geq 0$ such that $\mathrm{U}^{-2\eta}\nabla^2 \mathrm{U}$ is bounded.

Assumption 1.2. Assume that μ satisfies the following weighted log-Sobolev inequality: there exists $\rho > 0$ s.t. for all smooth enough g with $\int g^2 \,\mathrm{d}\mu = 1$:

$$\mathrm{Ent}_\mu(g^2) \leq \rho \int (\mathrm{H}^{-2\eta}|\nabla_x g|^2 + |\nabla_y g|^2)\mathrm{d}\mu. \tag{1.5.1}$$

With these assumptions, we can introduce a time-dependent gradient to study the long time behaviour. The method is called the multiplier method. It applies to entropy and the Ψ-entropy described in Section 3. And we get

Theorem 1.43. *Under Assumptions 1.1 and 1.2, let*

$$\begin{aligned}\lambda &= (\|H^{-2\eta}\nabla^2 U\|_\infty + 3)^2,\\ \kappa &= \frac{1}{16(d+1+5\eta^2+6\eta)^2}.\end{aligned}$$

Then for all initial probability density f,

$$\mathrm{Ent}_\mu(P_t f) \le \exp\left\{-\frac{\kappa}{1+8\lambda\rho}\int_0^t (1-e^{-s})^2 ds\right\} \mathrm{Ent}_\mu(f).$$

Theorem 1.44. *Let Ψ be an admissible function. Suppose that Assumption 1.1 is satisfied. If for any bounded density of probability f, the following inequality is satisfied*

$$\int \Psi(f)\, d\mu \le \rho \int \Psi''(f)\left(H^{-2\eta}|\nabla_x f|^2 + |\nabla_y f|^2\right) d\mu, \tag{1.5.2}$$

then

$$\int \Psi(P_t f) d\mu \le \exp\left\{-\frac{\kappa}{1+8\lambda\rho}\int_0^t (1-e^{-s})^2 ds\right\} \int \Psi(f) d\mu. \tag{1.5.3}$$

Next we furnish conditions under which the weighted logarithmic Sobolev inequality in the assumption can be verified. Our method is based on the Lyapunov function method. Let us introduce

$$L_\eta := H^{-2\eta}\Delta_x + \Delta_y - H^{-2\eta}\left(2\eta\frac{\nabla_x H}{H} + \nabla_x H\right)\cdot\nabla_x - \nabla_y H\cdot\nabla_y,$$

which is symmetric in $\mathbb{L}^2_\mu$ and satisfies

$$\int f L_\eta g\, d\mu = -\int (H^{-2\eta}\nabla_x f.\nabla_x g + \nabla_y f.\nabla_y g)\, d\mu. \tag{1.5.4}$$

Theorem 1.45. *Assume that U goes to infinity at infinity. Assume that $|\nabla H| \ge h > 0$ outside some large ball. Denote $A_r := \{(x,y) : H(x,y) \le r\}$, and*

$$\theta(r) = \sup_{z\in\partial A_r}\max_{i,j=1,\dots,2d} \left|\frac{\partial^2 H}{\partial z_i \partial z_j}\right|$$

Assume that $\theta(r) \le c e^{C_0 r}$ with some positive constants C_0 and c for r sufficiently large. Assume that there exists a Lyapunov function W with $W(x) \ge w > 0$ for all (x,y) and some $\lambda, b > 0$ satisfying

$$L_\eta W(x,y) \le -\lambda H(x,y) W(x,y) + b.$$

Then μ verifies a weighted logarithmic Sobolev inequality in Assumption 1.2.

And we have the following

Corollary 1.46. *Assume that the following conditions hold outside a compact domain:*

1. $\Delta_x U \le \kappa|\nabla_x U|^2$ *for some* $\kappa \in (0,1)$;
2. *a growth condition:* $|\nabla_x U|^2 \ge c U^{2\eta+1}$ *for some positive constant* c.

Then $d\mu = \frac{1}{Z} e^{-H(x,y)} dx dy$ satisfies a weighted logarithmic Sobolev inequality.

1.5.2 Uniform Poincaré and logarithmic Sobolev inequalities for mean field interactions and applications to McKean-Vlasov equation (c.f. Chapter 3)

In this part, we establish uniform functional inequalities for the following measure of mean field type

$$d\mu^{(N)}(x_1,\cdots,x_N)=\frac{1}{Z_N}\exp\{-H_N(x)\}\,dx_1\cdots dx_N$$

where

$$H_N(x_1,\cdots,x_N):=\sum_{i=1}^{N}V(x_i)+\frac{1}{N-1}\sum_{1\le i<j\le N}W(x_i,x_j)$$

is the Hamiltonian, Z_N is the normalization constant (assumed to be finite), the function $V:\mathbb{R}^d\to\mathbb{R}$ is an internal potential, and $W:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is an interaction potential so that $W(x,y)=W(y,x)$. The measure μ_N is the invariant measure for the mean-field particle system,

$$dX_i^N(t)=\sqrt{2}dB_i(t)-\nabla V(X_i^N(t))dt-\frac{1}{N-1}\sum_{j\neq i}\nabla_x W(X_i^N(t),X_j^N(t))dt,\ i=1,\cdots,N$$

where $B_i(1\le i\le N)$ are independent standard Brownian motions on $\mathbb{R}^d$. By propagation of chaos, as the number N of particles tends to infinity, for any $t>0$, the law of a single particle $X_1^N(t)$ converges to the one of the self-interacting diffusion X_t

$$dX_t=\sqrt{2}dB_t-\nabla V(X_t)dt-\int\nabla W(X_t,y)\nu_t(y)dydt$$

where ν_t is the law of the diffusion X_t which solves the McKean-Vlasov equation

$$\partial_t\nu_t=\Delta\nu_t+\nabla\cdot(\nu_t\nabla V)+\nabla\cdot(\nu_t\int\nabla W(x,y)\nu_t(y)dy).$$

Our method is based on a **Lipschtzian spectral gap condition for one particle**, that is,

Assumption 1.3. The following Lipschitzian constant is finite

$$c_{Lip,m}:=\frac{1}{4}\int_0^\infty\exp\left\{\frac{1}{4}\int_0^s b_0(u)du\right\}sds<+\infty$$

where $b_0(r)$ is the dissipativity rate of the drift of one particle in the system at distance $r>0$:

$$b_0(r)=\sup_{x,y,z\in\mathbb{R}^d:|x-y|=r}-\langle\frac{x-y}{|x-y|},(\nabla V(x)-\nabla V(y))+(\nabla_x W(x,z)-\nabla_x W(y,z))\rangle.$$

This assumption implies a spectral gap $1/c_{Lip,m}$ for the conditional measure $\mu_i:=\mu_i(dx_i|x^{\hat{i}})$ of x_i knowing $x^{\hat{i}}=(x_j)_{j\neq i}$.
Now we state our result for uniform Poncaré inequalities for the mean field measure $\mu^{(N)}$.

Theorem 1.47. *Suppose the Assumption 1.3. Assume that there is some constant $h > -1/c_{\mathrm{Lip},m}$ such that for any $(x_1, \cdots, x_N) \in (\mathbb{R}^d)^{\otimes N}$,*

$$\frac{1}{N-1}(\mathbb{1}_{i\neq j}\nabla^2_{x,y}W(x_i, x_j))_{1\leq i,j\leq N} \geq h I_{dN}$$

in the order of definite nonnegativity for symmetric matrices, where I_n is the identity matrix of size n. Then $\mu^{(N)}$ satisfies the following Poincaré inequality

$$\left(\frac{1}{c_{\mathrm{Lip},m}} + h\right)\mathrm{Var}_{\mu^{(N)}}(f) \leq \int |\nabla f|^2 d\mu^{(N)}, \ f \in C^1_b.$$

This result is a sharp estimate. It applies to the Curie-Weiss model and provides explicit estimates of the critical temperature of phase transition. Note that methods depending on convexity can not be applied to this model.

Corollary 1.48. *Assume that $W(x, y) = W_0(x - y)$ where $W_0 : \mathbb{R}^d \to \mathbb{R}$ is C^2, even. Assume*

1. *∇V is dissipative at infinity in the sense that*

$$\langle \nabla V(x) - \nabla V(y), x - y\rangle \geq c_V|x - y|^2 - c_1|x - y|\mathbb{1}_{[|x-y|\leq R]}$$

for some constants c_1, c_V and R;

2. *The Hessian matrix $\mathrm{Hess}W_0$ is bounded from below and from above :*

$$c_W I_d \leq \mathrm{Hess}W_0 \leq C_W I_d$$

and

$$c_W + c_V > 0.$$

Then for all $N \geq 2$, the invariant measure $\mu^{(N)}$ satisfies a Poincaré inequality with constant λ,

$$\lambda \geq \frac{1}{c_{\mathrm{Lip},m}} - \frac{N}{N-1}c^-_W - C_W$$

where c^-_W stands for the negative part of c_W.

Next we present our result concerning uniform logarithmic Sobolev inequalities for the mean field measure $\mu^{(N)}$.

Theorem 1.49. *Assume that*

1. *for some best constant $\rho_{\mathrm{LS,m}} > 0$, the conditional marginal distributions $\mu_i := \mu_i(dx_i|x^{\hat{i}})$ on $\mathbb{R}^d$ satisfy the log-Sobolev inequality :*

$$\rho_{\mathrm{LS,m}}\,\mathrm{Ent}_{\mu_i}(f^2) \leq 2\int |\nabla f|^2 d\mu_i, \ f \in C^1_b(\mathbb{R}^d)$$

for all i and $x^{\hat{i}}$;

2. *(a translation of Zegarlinski's condition)*

$$\gamma_0 = c_{Lip,m} \sup_{x,y\in\mathbb{R}^d, |z|=1} |\nabla^2_{x,y} W(x,y)z| < 1.$$

then $\mu^{(N)}$ *satisfies*

$$\rho_{\mathrm{LS,m}}(1-\gamma_0)^2 \operatorname{Ent}_{\mu^{(N)}}(f^2) \le 2\int_{(\mathbb{R}^d)^N} |\nabla f|^2 d\mu^{(N)}, \ f \in C^1_b((\mathbb{R}^d)^N).$$

We remark that the first assumption on $\rho_{\mathrm{LS,m}}$ in the above theorem can be verified independently. For instance, one of such an occasion is when $\nabla^2_x W$ is bounded from below and V is super-convex at infinity (i.e. the minimal eigenvalue of $\nabla^2 V(x)$ tends to $+\infty$ when $|x| \to \infty$). In that occasion, we can apply the classical Bakry-Émery Γ_2 criterion to obtain a uniform constant $\rho_{\mathrm{LS,m}}$ for the conditional marginal measures.

We then turn to the exponential convergence for the McKean-Vlasov equation in the free energy defined in (1.1.12). It will be denoted as E_f in this section. Under some technical conditions on the potentials V and W, it turns out that the free energy is the limit of the entropy with respect to the mean-field measure $\mu^{(N)}$. Then we can prove a functional inequality between the free energy and its "production functional" which is defined (up to some constant) as

$$I_W(\nu) := \frac{1}{4}\int \left|\nabla \log f + \nabla V(x) + (\nabla_x W \circledast \nu)(x)\right|^2 d\nu(x).$$

Define the relative free energy by

$$H_W(\nu) := E_f(\nu) - \inf_{\tilde{\nu}\in\mathscr{P}(\mathbb{R}^d)} E_f(\tilde{\nu}).$$

Theorem 1.50. *In the context in Theorem 1.49, assume furthermore that*

1. *For the confining potential* V, $\nabla^2 V$ *is bounded from below and there exists positive constants* c_1, c_2 *such that*
$$x\cdot\nabla V(x) \ge c_1|x|^2 - c_2, \quad \forall x\mathbb{R}^d;$$

2. *The interaction potential* W *satisfies*
$$\int\int \exp(-[V(x)+V(y)+\eta W(x,y)])dxdy < +\infty, \quad \forall \eta > 0,$$
and the Hessian $\nabla^2 W$ *is bounded.*

Then

1. *There exists a unique minimizer* ν_∞ *of* H_W *over* $\mathscr{P}(\mathbb{R}^d)$ *;*

2. *The following (nonlinear) log-Sobolev inequality*
$$\rho_{LS} H_W(\nu) \le 2 I_W(\nu), \ \nu \in \mathscr{P}(\mathbb{R}^d)$$
holds, where
$$\rho_{LS} \ge \rho_{LS,m}(1-\gamma_0)^2.$$

3. *The following Talagrand's transportation inequality holds*

$$\rho_{LS} W_2^2(\nu, \nu_\infty) \leq 2 H_W(\nu), \ \nu \in \mathscr{P}(\mathbb{R}^d)$$

where W_2 *is the* L^2*-Wasserstein distance.*

4. *For the solution* ν_t *of the McKean-Vlasov equation with the given initial distribution* ν_0 *of finite second moment,*

$$H_W(\nu_t) \leq e^{-t\cdot\rho_{LS}/2} H_W(\nu_0), \ t \geq 0$$

and in particular

$$W_2^2(\nu_t, \nu_\infty) \leq \frac{2}{\rho_{LS}} e^{-t\cdot\rho_{LS}/2} H_W(\nu_0), \ t \geq 0.$$

1.5.3 The kinetic Fokker-Planck equation with mean-field interaction (c.f. Chapter 4)

In this part we consider the kinetic Fokker-Planck equation on $(\mathbb{R}^{2d})^{\otimes N}$

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h - \nabla_x V(x) \cdot \nabla_v h = \Delta_v h - v \cdot \nabla_v h$$

with $x = (x_1, x_2, \cdots, x_N)$, $v = (v_1, v_2, \cdots, v_N)$, and the potential V given by

$$V(x_1, x_2, \cdots, x_N) = \sum_{1\leq i\leq N} U(x_i) + \frac{1}{2N} \sum_{1\leq i,j\leq N} W(x_i - x_j)$$

where $x_i \in \mathbb{R}^d$, the function $U : \mathbb{R}^d \to \mathbb{R}$ stands for the confining potential, and $W : \mathbb{R}^d \to \mathbb{R}$ stands for the interaction potential of mean-field type. The invariant measure is given by

$$\mu(dx, dv) = \frac{1}{Z} e^{-V(x)} \cdot (2\pi)^{-\frac{Nd}{2}} e^{-\frac{|v|^2}{2}} dx dv$$

and we shall denote $dm(x) := \frac{1}{Z} e^{-V(x)} dx$ for the part on the position variable. Uniform functional inequalities for such measures are studied in the previous subsection.

This kinetic Fokker-planck equation is associated to the distribution of the following system of N particles $(x_1, x_2, \cdots, x_N)$ moving in $\mathbb{R}^d$ according the stochastic differential equations

$$\begin{cases} dx_t^i = v_t^i dt \\ dv_t^i = \sqrt{2} dB_t^i - v_t^i dt - [\nabla U(x_t^i) + \frac{1}{N} \sum_{1\leq j\leq N} (\nabla W)(x_t^i - x_t^j)] dt \end{cases}$$

where $v = (v_1, v_2, \cdots, v_N)$ stands for the velocities, $(B_t^i)_{t\geq 0} (1 \leq i \leq N)$ are independent standard Brownian motions on $\mathbb{R}^d$. The mean-field limit of this system of particles is the self-interacting diffusion process $(\bar{x}_t, \bar{v}_t)_{t\geq 0}$ in $\mathbb{R}^d \times \mathbb{R}^d$ which evolves according to

$$\begin{cases} d\bar{x}_t = \bar{v}_t dt \\ d\bar{v}_t = \sqrt{2} d\bar{B}_t - \bar{v}_t dt - \left[\nabla U(\bar{x}_t) + \int \nabla W(\bar{x}_t - y) u_t(dy) \right] dt \end{cases}$$

where $u_t(\mathrm{d}y)$ is the law of $\bar{x}_t$, and $\bar{\mathrm{B}}$ is a standard Brownian motion on $\mathbb{R}^d$. By Ito's formula, this diffusion process corresponds to the following self-consistent Vlasov-Fokker-Planck equation on $\mathbb{R}^d \times \mathbb{R}^d$

$$\frac{\partial g}{\partial t} + \bar{v} \cdot \nabla_{\bar{x}} g - (\nabla \mathrm{U}(\bar{x}) + \nabla \mathrm{W} * \pi(g)) \cdot \nabla_{\bar{v}} g = \Delta_{\bar{v}} g + \nabla_{\bar{v}} \cdot (\bar{v} g)$$

where

$$\pi g(\bar{x}) = \int_{\mathbb{R}^d} g(t, \bar{x}, w) \mathrm{d}w$$

is the macroscopic density in the space of positions $\bar{x} \in \mathbb{R}^d$.

From the viewpoint of the mean-field limit, it is thus important to obtain results independent of the number of particles. We are concerned with convergence to equilibrium in $\mathrm{H}^1(\mu)$. When a Poincaré inequality for the measure m holds and some bounded condition for the Hessian $\nabla^2 \mathrm{V}$ is satisfied, Villani established exponential convergence in $\mathrm{H}^1(\mu)$. But his results, when applying to the equation with mean field interaction, depends on the number of particles.

To overcome such dependence, we introduce a Lyapunov condition on the confining potential, and assume the boundedness of the Hessian $\nabla^2 \mathrm{W}$ of the interaction potential.

Assumption 1.4. The functions U and W are twice continuously differentiable on $\mathbb{R}^d$, W is even (that is, $\mathrm{W}(x) = \mathrm{W}(-x)$ for all x), and

$$\mathrm{Z} = \mathrm{Z}_\mathrm{N} := \int_{\mathbb{R}^{\mathrm{N}d}} e^{-\mathrm{V}(x)} \mathrm{d}x < \infty, \quad \forall \mathrm{N} \geq 2.$$

Assumption 1.5. The following Lyapunov condition holds

$$|\nabla^2 \mathrm{U}|_{op} \leq \mathrm{K}_1 |\nabla \mathrm{U}| + \mathrm{K}_2$$

for some positive constants $\mathrm{K}_1, \mathrm{K}_2$.

Assumption 1.6. $\nabla^2 \mathrm{W}$ is bounded, i.e. there exists a positive constant K such that

$$-\mathrm{K}\mathrm{I}_d \leq \nabla^2 \mathrm{W} \leq \mathrm{K}\mathrm{I}_d$$

as quadratic forms on $\mathbb{R}^d$, where I_d is the identity matrix of size d.

Uniform functional inequalities for the measure m is important in our results. They are rather independent of the content and so they are stated as assumptions. Note that uniform Poincaré inequalities or uniform logarithmic Sobolev inequalities have been verified under various conditions of the previous work, or can be verified by Bakry-Émery Γ_2 criterion).

Assumption 1.7. The measure $\mathrm{d}m(x) = \frac{1}{\mathrm{Z}} e^{-\mathrm{V}(x)} \mathrm{d}x$ satisfies a uniform Poincaré inequality i.e. there exists a positive real number $\kappa > 0$ such that for any $\mathrm{N} \geq 2$, for all functions $g \in \mathrm{H}^1(m)$

$$\int \left(g - \int g \mathrm{d}m \right)^2 \mathrm{d}m \leq \kappa \int |\nabla g|^2 \mathrm{d}m.$$

Assumption 1.8. The mean field measure m satisfies a uniform log-Sobolev inequality with a constant $\mathrm{C}_{\mathrm{LS}} > 0$.

Below is our result:

Theorem 1.51. *Suppose Assumption 1.4, Assumption 1.5 and Assumption 1.6 holds. Suppose furthermore*

1. *either, the gradient of* W *is assumed bounded (i.e.* $|\nabla W| \le K'$*) and Assumption 1.7 holds;*
2. *or Assumption 1.8 holds.*

Then there exist explicitly computable constants C_0 *and* λ*, independent of the number* N *of the particles, such that*

$$||e^{-tL}h_0 - \int h_0 d\mu||_{H^1(\mu)} \le C_0 e^{-\lambda t}||h_0||_{H^1(\mu)} \tag{1.5.5}$$

for all $h_0 \in H^1(\mu)$*. Indeed,* C', λ *only depends on the constants* $C_{LS}, \kappa, K, K_1, K_2, K', d$ *stated in the assumptions.*

1.5.4 Long-time behavior of mean-field interacting particle systems related to McKean-Vlasov equation (c.f. Chapter 5)

In this part, we continue working under the context of subsection 1.5.2 which is devoted to the uniform functional inequalities for a mean-field interacting particle systems and McKean-Vlasov equation. Throughout this part, we assume $b_0(r)$ the dissipativity rate of the drift at distance $r > 0$ satisfies

Assumption 1.9. $b_0(r)$ is a continuous function on $(0, +\infty)$ such that

(1) $\limsup_{r\to+\infty} b_0(r)/r < 0$, i.e. the drift is dissipative at infinity;

(2) $\lim_{r\to 0+} b_0^+(r) = 0$.

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be the function determined by $h(0) = 0$ and

$$h'(r) = \frac{1}{4}\exp\left(-\frac{1}{4}\int_0^r b_0(s)ds\right)\int_r^{+\infty} s\cdot\exp\left(\frac{1}{4}\int_0^s b_0(u)du\right)ds.$$

Denote $\|h'\|_\infty := \sup_{r\ge 0} h'(r)$ and

$$||\nabla^2_{xy}W||_\infty := \sup_{x,y,z\in\mathbb{R}^d:|z|=1} |\nabla^2_{xy}W(x,y)z|.$$

The next assumption is a translation of Dobrushin-Zegarlinski's uniqueness condition which prevents phase transition.

Assumption 1.10. Assume $\gamma_0 := \|h'\|_\infty||\nabla^2_{xy}W||_\infty < 1$.

We consider the Wasserstein distance $W_{1,d_{l^1}}$ associated to the l^1 distance d_{l_1} on the configuration space $(\mathbb{R}^d)^{\otimes N}$,

$$d_{l^1}(x,y) = \sum_{i=1}^N |x^i - y^i|,\ \ x = (x^1, \cdots, x^N),\ y = (y^1, \cdots, y^N) \in (\mathbb{R}^d)^{\otimes N}.$$

Let $\{P_t^{(N)}\}_{t\geq 0}$ be the transition semigroup of the mean-field interacting particle system. Let $\mathbb{P}_x$ be the law of $X^{(N)} = (X_t^{(N)})_{t\geq 0}$ with $X_0^{(N)} = x \in (\mathbb{R}^d)^N$. On the space of continuous paths $C([0,T],(\mathbb{R}^d)^N)$ where $T \in (0,+\infty]$, we consider the L^1-metric

$$d_{L^1[0,T]}(\gamma_1,\gamma_2) := \int_0^T d_{l^1}(\gamma_1(t),\gamma_2(t))dt$$

where T might be infinity.

Theorem 1.52. *Under the assumption 1.9 and assumption 1.10. For any* $x_0 = (x_0^1,\cdots,x_0^N) \in (\mathbb{R}^d)^{\otimes N}$ *and* $y_0 = (y_0^1,\cdots,y_0^N) \in (\mathbb{R}^d)^{\otimes N}$, *we have*

$$\int_0^{+\infty} W_{d_{l^1}}(P_t^{(N)}(x_0,\cdot),P_t^{(N)}(y_0,\cdot))dt \leq \frac{1}{1-\gamma_0}\sum_{i=1}^N h(|x_0^i - y_0^i|).$$

Corollary 1.53. *Under the same assumptions as above. For any two solutions* μ_t,ν_t *of the McKean-Vlasov equation with the initial distributions* $\mu_0,\nu_0 \in \mathscr{P}_2(\mathbb{R}^d)$, *we have*

$$\int_0^\infty W_1(\mu_t,\nu_t)dt \leq \frac{\|h'\|_\infty}{1-\gamma_0}W_1(\mu_0,\nu_0).$$

Next we present an exponential convergence of the particle system in the $W_{d_{l^1}}$ distance and, as a corollary, an exponential convergence of the McKean-Vlasov equation.

Theorem 1.54. *Under the assumption 1.9 and assumption 1.10. Suppose furthermore that there exists a constant* $M \in \mathbb{R}$ *such that*

$$b_0(r) \leq rM, \forall r > 0.$$

Then for any $\varepsilon > 0$ *such that*

$$K_\varepsilon := \frac{1 - \|\nabla_{xy}^2 W\|_\infty \|h^{'}\|_\infty - \varepsilon(M + \|\nabla_{xy}^2 W\|_\infty)}{\|h'\|_\infty + \varepsilon} > 0,$$

we have for any $x_0, y_0 \in (\mathbb{R}^d)^N$

$$W_{d_{l^1}}(P_t^{(N)}(x_0,\cdot),P_t^{(N)}(y_0,\cdot)) \leq A_\varepsilon e^{-K_\varepsilon t} d_{l^1}(x_0,y_0),\ \forall t \geq 0,$$

where

$$A_\varepsilon := \sup_{r>0}\frac{r}{h(r)+\varepsilon r}\cdot\sup_{r>0}\frac{h(r)+\varepsilon r}{r}.$$

It follows that

Corollary 1.55. *Under the same assumptions as in the above theorem. Suppose* $\varepsilon > 0$ *such that* $K_\varepsilon > 0$. *For the solutions* μ_t,ν_t *of the McKean-Vlasov equation with the initial distributions* $\mu_0,\nu_0 \in \mathscr{P}_2(\mathbb{R}^d)$, *it holds*

$$W_1(\mu_t,\nu_t) \leq A_\varepsilon e^{-K_\varepsilon t}W_1(\mu_0,\nu_0),\ \forall t \geq 0.$$

Lastly in the part we are concerned with the uniform in time propagation of chaos.

Theorem 1.56. *Under the assumption 1.9 and assumption 1.10. Suppose that there exist some positive constants c_1, c_2, c_3 such that*

(1) $\langle x, \nabla V(x)\rangle \geq c_1|x|^2 - c_2, \forall x \in \mathbb{R}^d$;

(2) $\langle z, \nabla^2_{xx} W(x,y)z\rangle \geq -c_3|z|^2, \forall x, y, z \in \mathbb{R}^d$;

(3) $c_1 - c_3 - \|\nabla^2_{xy} W\|_\infty > 0$.

Then for any $\varepsilon > 0$ *such that* $K_\varepsilon > 0$*, and* $\tilde{\varepsilon} \in (0, c_1 - c_3 - \|\nabla^2_{xy} W\|_\infty)$*, the following estimates of propagation of chaos hold for the mean-field interacting particle system with any initial datum* $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$*,*

(a) **(path-type propagation of chaos)** *for any* $T > 0$, $1 \leq k \leq N$*, denote* $\mathbb{P}_\nu(\cdot) = \int_{(\mathbb{R}^d)^N} \mathbb{P}_x(\cdot) d\nu(x)$ *the law of* $(X_t^{(N)})_{t\geq 0}$ *with the initial distribution* ν*,* $\mathbb{P}_\nu^{[1,k],N}|_{[0,T]}$ *the joint law of paths of the k particles* $((X_t^{i,N})_{t\in[0,T]}, 1 \leq i \leq k)$ *in time interval* $[0,T]$*, and* $\mathbb{Q}_{\mu_0}$ *the law of the self-interacting diffusion* $(X_t)_{t\geq 0}$ *with the initial distribution* μ_0*. We have*

$$\frac{1}{k} W_{1,d_{L^1[0,T]}}(\mathbb{P}^{[1,k],N}_{\mu_0^{\otimes N}}|_{[0,T]}, \mathbb{Q}^{\otimes k}_{\mu_0}|_{[0,T]}) \leq \frac{T}{\sqrt{N-1}} \frac{\|\nabla^2_{xy} W\|_\infty \|h'\|_\infty}{1 - \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty} \cdot \max\{m_2(\mu_0), \hat{c}(\varepsilon)\}$$

where $m_2(\mu_0) = \left(\int |x|^2 d\mu_0(x)\right)^{1/2}$*, and*

$$\hat{c}(\varepsilon) = \left(\frac{d + c_2 + \frac{1}{4\tilde{\varepsilon}}|\nabla_x W(0,0)|}{c_1 - c_3 - \|\nabla^2_{xy} W\|_\infty - \tilde{\varepsilon}}\right)^{1/2}.$$

(b) **(Uniform in time propagation of chaos)** *for all time* $t > 0$ *and any* $1 \leq k \leq N$*:*

$$W_{1,d_{l^1}}(\mu_t^{[1,k],N}, \mu_t^{\otimes k}) \leq \frac{k}{\sqrt{N-1}} \frac{A_\varepsilon}{K_\varepsilon} \|\nabla^2_{xy} W\|_\infty \max\{m_2(\mu_0), \hat{c}(\varepsilon)\}$$

where $\mu_t = u_t dx$ *is the solution of the Mckean-Vlasov equation, and* $\mu_t^{[1,k],N}$ *is the joint law of the k particles* $(X_t^{i,N}, 1 \leq i \leq k)$ *in the mean-field system of interacting particles* $(X_t^{i,N})_{1\leq i\leq N}$ *with* $X_0^{i,N}, 1 \leq i \leq N$ *i.i.d. of law* μ_0 *(independent of the Brownian motions* $(B_t^{i,N})_{1\leq i\leq N, t\geq 0}$*).*

1.5.5 Convergence to equilibrium in Wasserstein distance for the kinetic Fokker-Planck equation (c.f. Chapter 6)

In this part, we consider convergence in L^2-Wasserstein distance for the kinetic Fokker-Planck equation,

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h - \nabla U(x) \cdot \nabla_v h = \Delta_v h - v \cdot \nabla_v h.$$

One of the issues we studied here is the optimal rate of convergence when the potential U is a quadratic potential, i.e. $U(x) = \frac{\omega_0^2}{2}|x|^2$. In that case, the optimal rate of convergence is known to be

$$\lambda_{optimal} = \begin{cases} 1/2, & \text{if } \omega_0^2 \geq 1/4; \\ 1/2 - \sqrt{1/4 - \omega_0^2}, & \text{if } 0 < \omega_0^2 \leq 1/4. \end{cases}$$

It is also true for the convergence in L^2-Wasserstein distance. Indeed, this can be shown by a synchronous coupling method using an explicit formula for solutions of certain linear ODE.
Then we study the following class of potentials,

Assumption 1.11. The confining potential U is a perturbation of quadratic potentials, namely, U takes the form

$$U(x) = \frac{\omega_0^2}{2}|x|^2 + \Psi(x)$$

where the constant $\omega_0 > 0$ and the function Ψ is of class C^2 such that the operator norm of its Hessian $\nabla^2\Psi$ is uniformly bounded, i.e.,

$$||\nabla^2\Psi||_{op} < \kappa$$

for some constant $\kappa \geq 0$.

In this setting, we are able to prove

Theorem 1.57. *Under the assumption 1.11, for any* $\omega_0 > 0$ *there exists a constant* $c > 0$ *such that, if* $\kappa \leq c$*, then there exist explicit computable constants* C *and* $\lambda > 0$ *such that*

$$W_2(g_t d\mu, h_t d\mu) \leq Ce^{-\lambda t} W_2(g_0 d\mu, h_0 d\mu). \tag{1.5.6}$$

for any two solutions $(g_t)_{t\geq 0}, (h_t)_{t\geq 0}$ *with respective initial data* g_0, h_0 *such that* $g_0 d\mu, h_0 d\mu \in \mathscr{P}_2(\mathbb{R}^{2n})$.
Moreover, concerning the quadratic confining potential $U(x) = \frac{|x|^2}{2}$, *we recover the optimal rate of convergence*

$$W_2(g_t d\mu, h_t d\mu) \leq \sqrt{3}e^{-\frac{1}{2}t} W_2(g_0 d\mu, h_0 d\mu). \tag{1.5.7}$$

1.5.6 Hypocoercivity in Sobolev spaces for the kinetic Fokker-Planck equation (c.f. Chapter 7)

In this part we consider the hypocoercivity for the kinetic Fokker-Planck equation on $\mathbb{R}^{2d}$,

$$\partial_t h_t + L h_t = 0$$

with L given by

$$L = -\Delta_v h + v \cdot \nabla_v h + v \cdot \nabla_x h - \nabla_x V(x) \cdot \nabla_v h.$$

The invariant measure is denoted by μ which is

$$d\mu(x, v) = \frac{1}{Z} e^{-V(x) - \frac{|v|^2}{2}} dx dv$$

where Z is the normalizing constant. We also denote $Z_1 = \int e^{-V(x)} dx$. We denote by $L^2(\mu)$ the L^2 function space with respect to the reference measure μ, and by $H^k(\mu)$ the L^2-Sobolev space of order k for which the scalar product is defined by

$$||h||_{H^k(\mu)} := \sum_{|\alpha|+|\beta|\leq k} \int |D_x^\alpha D_v^\beta h|^2 d\mu$$

where α,β are multi-indexes of respective order $|\alpha|$ and $|\beta|$, and $D_x^\alpha D_v^\beta h$ is given as usual by

$$D_x^\alpha D_v^\beta h = \frac{\partial^{|\alpha|+|\beta|} h}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}\partial v_1^{\beta_1}\cdots\partial v_d^{\beta_d}}.$$

Hypocoercivity for kinetic Fokker-Planck eqaution in $H^1(\mu)$ has been proved in $H^1(\mu)$ by Villani [17] under the Poincaré inequality and the following boundedness condition

$$|\nabla^2 V| \le C(1+|\nabla V|)$$

for some constant $C>0$. Here we extend his result to the convergence in higher order Sobolev spaces. In order to overcome the degeneracy in the x-variable and to obtain hypocoercive estimates, Villani introduced a mixed term $\langle \nabla_x h, \nabla_v h\rangle_{L^2(\mu)}$. In our work, we adopt the same strategy: the mixed term which proves helpful is $\langle \nabla_x^k h, \nabla_x^{k-1}\nabla_v h\rangle_{L^2(\mu)}$. We shall also assume some boundedness conditions of the derivatives of the potential V. We denote

$$|\nabla_x^l V\cdot\nabla_v g|^2 = \sum_{|\alpha|=l-1} |\sum_{j=1}^d D_x^\alpha \partial_{x_j} V(x)\partial_{x_j} g|^2.$$

Now we state our main results.

Theorem 1.58. *Assume that the measure $\frac{1}{Z_1}e^{-V(x)}dx$ satisfies a Poincaré inequality with constant κ. Assume furthermore that the confining potential $V\in C^\infty(\mathbb{R}^d)$ satisfies*

$$\int |\nabla_x^l V\cdot\nabla_v g|^2 d\mu \le M\Big(\int |\nabla_v g|^2 d\mu + \int |\nabla_{xv}^2 g|^2 d\mu\Big)$$

for $2\le l\le k+1$ and any function $g\in H^2(\mu)$. Then there exist explicitly computable constants C and $\lambda>0$ such that

$$||h_t - \int h_0 d\mu||_{H^k(\mu)} \le Ce^{-\lambda t}||h_0 - \int h_0 d\mu||_{H^k(\mu)}$$

where $h_t = h(t,x,v)$ is the solution to the kinetic Fokker-Planck equation with the initial condition $h_0\in H^k(\mu)$.

We remark that the set of conditions on $\nabla_x^l V$ $(2\le l\le k+1)$ can be viewed as a set of weighted Poincaré inequalities for which we may apply various criteria to obtain such inequalities. For instance, they can be deduced from the inequalities for all $g\in H^1(\mu)$

$$\int |\nabla_x^l V|^2 |g|^2 d\mu \le M\Big(\int |g|^2 d\mu + \int |\nabla_x g|^2 d\mu\Big)$$

where $|\nabla_x^l V|^2 = \sum_{\alpha:|\alpha|=l} |D_x^\alpha V|^2$ can be regarded as a weight function. Moreover, the assumption is satisfied when $\nabla^2 V\in W^{\infty,\infty}(\mathbb{R}^d)$ (i.e. all partial derivatives of the potential V of order not smaller than 2 is uniformly bounded in space.)

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Chapter 2

Entropic multipliers method for Langevin diffusion and weighted log Sobolev inequalities

This chapter is an article collaborated with Patrick Cattiaux, Arnaud Guillin, and Pierre Monmarché. In his work about hypocoercivity, Villani [20] considers in particular convergence to equilibrium for the kinetic Langevin process. While his convergence results in L^2 are given in a quite general setting, convergence in entropy requires some boundedness condition on the Hessian of the Hamiltonian. We will show here how to get rid of this assumption in the study of the hypocoercive entropic relaxation to equilibrium for the Langevin diffusion. Our method relies on a generalization to entropy of the multipliers method and an adequate functional inequality. As a byproduct, we also give tractable conditions for this functional inequality, which is a particular instance of a weighted logarithmic Sobolev inequality, to hold.

2.1 Settings and main results.

Let $U : \mathbb{R}^d \to \mathbb{R}$ be a smooth (C^∞) function such that $U \geq 1$, $U(x) \to +\infty$ as $|x| \to +\infty$ and $\int e^{-U(x)} dx$ is finite. U will represent the confinement potential for the Hamiltonian $H(x, y) = U(x) + \frac{1}{2}|y|^2$ defined on $\mathbb{R}^{2d}$. The associated Boltzmann-Gibbs (probability) measure is given by

$$d\mu = \frac{1}{Z}\, e^{-H(x,y)}\, dx\, dy$$

where Z is the normalizing constant $\int e^{-H(x,y)} dx\, dy$.
The Langevin dynamics associated to this measure is a flow of probability measures $d\mu_t = f_t\, d\mu$ for $t \geq 0$, where f_t solves (at least in a weak sense) the Langevin equation

$$\partial_t f_t = L f_t\,,$$

L being given by

$$L \quad = \quad -y.\nabla_x + \big(\nabla U(x) - y\big).\nabla_y + \Delta_y\,. \tag{2.1.1}$$

We are thus interested in solutions belonging to $\mathbb{L}^1(\mu)$. Of course, the Hörmander's sum of squares hypoelliptic theorem ensures that $(t,x,y) \mapsto f_t(x,y)$ is smooth on $\mathbb{R}_+^* \otimes \mathbb{R}^{2d}$, whatever the regularity of f_0. It is then easy to see that mass and positivity are preserved so that if $f_0 d\mu$ is a probability measure so is $f_t d\mu$ for any $t \geq 0$. We shall discuss below existence and uniqueness for (2.1.1).
The corresponding stochastic process is given by the S.D.E.

$$\begin{cases} dx_t = y_t dt \\ dy_t = -y_t dt - \nabla U(x_t) dt + \sqrt{2} dW_t \end{cases}$$

where (W_t) is an usual $d-$dimensional Wiener process. The infinitesimal generator of the process is thus $L^* = y.\nabla_x - \left(\nabla U(x) + y\right).\nabla_y + \Delta_y$. Since all coefficients are local Lipschitz, existence and strong uniqueness for the S.D.E. is ensured up to the explosion time τ. But thanks to our assumptions, $H(x,y) \to +\infty$ as $|x|+|y| \to +\infty$ and it is easily seen that $L^*H \leq M < +\infty$ for some constant M. It is then well known that τ is almost surely infinite, whatever the starting point (x_0, y_0), i.e. the diffusion process is conservative. According to what precedes, for any initial distribution, the distribution μ_t at time $t > 0$ of (x_t, y_t) admits a smooth density w.r.t. Lebesgue measure and since μ is equivalent to Lebesgue measure with a smooth density too, $d\mu_t = f_t d\mu$ where f_t is smooth (C^∞) and solves (2.1.1). It is easy to see that μ is the unique invariant but not reversible probability measure for the process (steady state).
We denote by $P_t = e^{tL}$ the semi-group on $\mathbb{L}^1(\mu)$ with generator $(L, D(L))$, i.e. $f_t = P_t f_0$. It is easy to see that for any solution g_t of (2.1.1) belonging to $\mathbb{L}^1(\mu)$, $Q_t f = \int f\, g_t d\mu$ is a Markov continuous semi-group on $L^\infty(\mu)$ whose generator coincides with L^* on the set C_0^∞ of smooth and compactly supported functions (just using integration by parts). The uniqueness of this semi-group implies that $g_t = f_t$ i.e. the uniqueness of the solutions of (2.1.1) in $\mathbb{L}^1(\mu)$.

We are interested in the long time behavior of the Langevin diffusion. The usual ergodic theorem tells us that $\frac{1}{t}\int_0^t \mu_s ds$ weakly converges to μ as t grows to infinity. One can thus ask for the convergence of f_t towards 1 as t goes to infinity.
This question has been investigated by many authors in recent years both in the P.D.E. community and the probability community. One of the main difference is of course the way to look at this convergence: total variation distance, $\mathbb{L}^2(\mu)$ norm, $\mathbb{H}^1(\mu)$ semi-norm, relative entropy, Wasserstein distance. Another associated problem is to get some bounds on the rate of convergence, once convergence holds true. Let's review some results in this direction.

More or less at the same time, both probabilists and PDE specialists have considered the problem of the speed of convergence to equilibrium. Talay [19] and Wu [22] have built Lyapunov functions and using Meyn-Tweedie's approach have established (non quantitative) exponential convergence to equilibrium (see also [3] for this approach for kinetic models) under quite general assumptions. Desvillettes and Villani [12] used an heavy Fourier machinery to established sub-exponential entropic convergence. Then Hérau and Nier [17] have carried out the spectral analysis of this equation and thus obtained a $\mathbb{L}^2$ exponential decay with quite sharp constants under general conditions. It has settled the bases for the theory of hypocercivity of Villani [20] for the $\mathbb{L}^2$ and the entropic convergence to equilibrium, when Hess(U) is bounded in the entropic case, see also [13] for a version without regularity issues. Let us also mention [1] where an unified approach dealing with various entropies (as we shall do) is performed, still for bounded Hessians for which explicit rates are given. Finally, and quite recently, coupling approaches, using synchronous coupling or coupling by reflection (see [7] or [14; 15]) have established exponential convergence to equilibrium in Wasserstein distance with sharp constants, once again when $\mathrm{Hess}(U)$ is bounded.

As we will adopt the terminology and adapt the methodology of hypocoercivity as in Villani [20], let us describe a little bit further the formalism of this setting. Recall that the variance of a squared integrable function g with respect to μ is defined by

$$\mathrm{Var}_\mu(g) := \int g^2 d\mu - \left(\int g d\mu\right)^2 = \int \left(g - \int g d\mu\right)^2 d\mu$$

while the entropy is defined for positive functions by

$$\mathrm{Ent}_\mu(f) := \int f \ln f d\mu - \int f d\mu \ln \int f d\mu\,.$$

The law μ is said to satisfy a Poincaré inequality if there exists a positive constant C_P such that for all smooth functions g

$$\mathrm{Var}_\mu(g) \le C_P \int |\nabla g|^2 d\mu\,.$$

Similarly, μ satisfies a logarithmic Sobolev (or log-Sobolev in short) inequality if there exists a constant C_{LS} such that for all smooth functions g,

$$\mathrm{Ent}_\mu(g^2) \le C_{LS} \int |\nabla g|^2 d\mu\,.$$

The natural $\mathbb{H}^1_\mu$ semi-norm is defined as $||g||_{\mathbb{H}^1_\mu} := ||\nabla g||_{\mathbb{L}^2_\mu}$. Exponential convergence of $P_t f_0$ to 1 in $\mathbb{H}^1_\mu$ and variance was proved by Villani [20] under two conditions:

(1-var) $|\nabla^2 U| \le c\,(1 + |\nabla U|)$;

(2-var) $e^{-U(x)} dx$ satisfies a Poincaré inequality.

Remark that (2-var) is equivalent to the fact that μ satisfies a Poincaré inequality, thanks to the tensorization property of the latter, since the gaussian measure satisfies a Poincaré inequality.

For convergence in entropy, the assumptions made by Villani are much stronger:

(1-ent) $\nabla^2 U$ is bounded;

(2-ent) $e^{-U(x)} dx$ satisfies a log-Sobolev inequality.

Again, (2-ent) is equivalent to the fact that μ satisfies a log-Sobolev inequality, thanks to a similar argument of tensorization.
When both these assumptions are satisfied, Villani showed that, for any initial probability density f_0 with finite moments of order 2, the entropy of $P_t f_0$ converges to 0 exponentially fast (see Villani [20] Theorem 39).

Our main goal in this paper is to get rid of the boundedness assumption (1-ent) for $\nabla^2 U$, replacing it by

Assumption 2.1. In addition to our assumptions on U, we assume that there exists $\eta \geq 0$ such that $U^{-2\eta}\nabla^2 U$ is bounded.

A typical situation where Assumption 2.1 is satisfied is when both U and $\nabla^2 U$ have polynomial growth at infinity, i.e. $U(x) \geq c_1\,(1+|x|)^l$ and $|\nabla^2 U| \leq c_2\,(1+|x|)^j$ so that we may choose $\eta \geq \frac{j}{2l}$. In particular if $j = l-2 \geq 0$ as it is the case for true polynomials of degree at least 2, we may choose $\eta = \frac{1}{2} - \frac{1}{l}$.

The counterpart is that we have to reinforce (2-ent) replacing it by the stronger

Assumption 2.2. μ satisfies the following weighted log-Sobolev inequality: there exists $\rho > 0$ s.t. for all smooth enough g with $\int g^2\, d\mu = 1$:

$$\mathrm{Ent}_\mu(g^2) \leq \rho \int (H^{-2\eta}|\nabla_x g|^2 + |\nabla_y g|^2) d\mu. \tag{2.1.2}$$

Once both Assumptions 2.1 and 2.2 are satisfied, we can prove exponential decay in entropy for the Langevin diffusion. Our approach is based on the multiplier method. More precisely we will prove the following:

Theorem 2.1. *Under Assumptions 2.1 and 2.2, let*

$$\begin{aligned} \lambda &= \left(\|H^{-2\eta}\nabla^2 U\|_\infty + 3\right)^2, \\ \kappa &= \frac{1}{16\left(d+1+5\eta^2+6\eta\right)^2}. \end{aligned}$$

Then for all initial probability density f,

$$\mathrm{Ent}_\mu(P_t f) \leq \exp\left(-\frac{\kappa}{1+8\lambda\rho}\int_0^t (1-e^{-s})^2 ds\right)\mathrm{Ent}_\mu(f).$$

Section 2 is devoted to the proof of this theorem which contains Villani's result in the case $\eta = 0$. Actually as in [1] (also see [8] in the non degenerate case) we shall prove a more general statement including both the variance and the entropic case. To this end introduce an admissible function Ψ, that is

$$\Psi \geq 0\,,\ \Psi \in C^4 \text{ and } \frac{1}{\Psi''} \text{ is positive and concave,} \tag{2.1.3}$$

as in [1; 18]. Theorem 2.1 corresponds to

$$\Psi : \mathbb{R}^+ \to \mathbb{R}, u \mapsto u \ln u + 1 - u,$$

while the $\mathbb{L}^2_\mu$ case corresponds to $\Psi(u) = (u-1)^2$. We also denote $\psi = \Psi''$. The general statement is the following

Theorem 2.2. *Let Ψ be an admissible function. Suppose that Assumption 2.1 is satisfied. If for any bounded density of probability f, the following inequality is satisfied*

$$\int \Psi(f)\, d\mu \leq \rho \int \psi(f)\left(H^{-2\eta}|\nabla_x f|^2 + |\nabla_y f|^2\right) d\mu, \tag{2.1.4}$$

then

$$\int \Psi(P_t f) d\mu \leq \exp\left(-\frac{\kappa}{1+8\lambda\rho}\int_0^t (1-e^{-s})^2 ds\right)\int \Psi(f) d\mu. \tag{2.1.5}$$

Remark 2.3. For (2.1.4) to be satisfied it is immediate that $\Psi(1)$ has to be equal to 0.
A natural family of admissible functions, namely $\Psi_p(u) = u^p - 1 - p(u-1)$ defined for $1 < p \leq 2$, is introduced in [1]. Remark that, up to the constants, a Taylor expansion shows that (2.1.4) is satisfied for Ψ_2 as soon as it is satisfied for Ψ_p.
Also notice that, if $1/\psi$ is concave at infinity (i.e. outside some compact interval) one can modify it and introduce some $\tilde{\psi}$ which satisfies all the required properties and coincides with ψ outside some larger compact interval. The corresponding $\tilde{\Psi}$ will behave like Ψ at infinity, which is the interesting property for controlling the convergence. ♢

The key idea for proving Theorem 2.2 is to use a twisted gradient depending on time, see lemma 2.6. An important aspect of our result is that the bounded Hessian condition in Villani's approach is relaxed as Assumption 2.1. In fact it was a major issue raised by Villani [20] concerning the entropic convergence. Indeed, his L^2 multiplier method, at the basis of the entropic hypocercivity, does not rely on a Poincaré inequality but on a Brascamp-Lieb inequality. It was thus thought that for the multiplier method to hold for entropy, an entropic Brascamp-Lieb inequality was needed. However Bobkov-Ledoux [6] proved that this inequality is false in general, and true in very particular setting. Our strategy is then to show that it is not an entropic Brascamp-Lieb inequality that we need but a particular weighted logarithmic Sobolev inequality. Note also that a first attempt to skip the boundedness assumption for the Hessian is contained in [3] Theorem 6.10, but the statement therein is much weaker than the one of the present theorem and most importantly not at all quantitative. One can also look at [1] for a quantitative result in the bounded Hessian case.

Next we shall show that, similarly to the non weighted case studied in [9] (see also [2; 10]), the weighted log Sobolev inequality in Assumption 2.2 is equivalent to some Lyapunov type condition. To this end we introduce the natural second order operator

$$\mathrm{L}_\eta := \mathrm{H}^{-2\eta}\Delta_x + \Delta_y - \mathrm{H}^{-2\eta}\left(2\eta\frac{\nabla_x \mathrm{H}}{\mathrm{H}} + \nabla_x \mathrm{H}\right).\nabla_x - \nabla_y \mathrm{H}.\nabla_y\,,$$

which is symmetric in $\mathbb{L}^2_\mu$ and satisfies

$$\int f\,\mathrm{L}_\eta g\,d\mu = -\int (\mathrm{H}^{-2\eta}\nabla_x f.\nabla_x g + \nabla_y f.\nabla_y g)\,d\mu\,. \tag{2.1.6}$$

Theorem 2.4. *Recall that* U *goes to infinity at infinity. Assume that* $|\nabla \mathrm{H}| \geq h > 0$ *outside some large ball. Denote* $\mathrm{A}_r := \{(x,y) : \mathrm{H}(x,y) \leq r\}$, *and*

$$\theta(r) = \sup_{z\in\partial \mathrm{A}_r}\ \max_{i,j=1,\dots,2d} \left|\frac{\partial^2 \mathrm{H}}{\partial z_i \partial z_j}\right|$$

Assume that $\theta(r) \leq ce^{\mathrm{C}_0 r}$ *with some positive constants* C_0 *and* c *for* r *sufficiently large. Assume that there exists a Lyapunov function* W *with* $\mathrm{W}(x) \geq w > 0$ *for all* (x,y) *and some* $\lambda, b > 0$ *satisfying*

$$\mathrm{L}_\eta \mathrm{W}(x,y) \leq -\lambda \mathrm{H}(x,y)\,\mathrm{W}(x,y) + b\,.$$

Then μ *verifies a weighted logarithmic Sobolev inequality* (2.1.2).

Remark that the condition $\theta(r) \leq ce^{C_0 r}$ is trivially verified when both U and Hess(U) have a polynomial growth. Also, a Lyapunov function exists if U satisfies the conditions in the following corollary:

Corollary 2.5. *Assume that the following conditions hold outside a compact domain:*

1. $\Delta_x U \leq \kappa |\nabla_x U|^2$ *for some* $\kappa \in (0,1)$*;*
2. *a growth condition:* $|\nabla_x U|^2 \geq c U^{2\eta+1}$ *for some positive constant* c*.*

Then $d\mu = \frac{1}{Z} e^{-H(x,y)} dx dy$ *satisfies a weighted logarithmic Sobolev inequality.*
Moreover, if we assume that $U^{-2\eta}\nabla^2 U$ *is bounded, then we may apply Theorem 2.1.*

The next section will present the proof of Theorem 2.2, where the entropic multipliers method is presented. In Section 3, the treatment via Lyapunov condition of weigthed log-Sobolev inequality, i.e. Theorem 2 and Corollary 3, is done.
The final section discusses some additional points on weighted inequalities. Indeed, the proof of weighted Poincaré inequality used by Villani relies solely on some Poincaré inequality for each measure and adapt the usual argument of tensorization, using heavily the orthogonality inherited from the $\mathbb{L}^2_\mu$ structure. However, in the entropic case, starting with a log-Sobolev for each marginal, we are only able to recover a weaker (but interesting) inequality for the product measure.

2.2 Proof of Theorem 2.2.

This section is devoted to the proof of Theorem 2.2.
We only consider the case where f_0 is bounded away from zero. Indeed, if it is not the case, writing $g_0 = (1-\delta) f_0 + \delta$ for some $\delta > 0$, then we may prove the theorem for $g_t = (1-\delta) f_t + \delta$ and let δ go to zero to recover the result for f_t.

The key point of the proof is to introduce a time and space-dependent twisted gradient. Consider $r \in \mathbb{N}$ and for $0 \leq i \leq r$, $x \mapsto b_i(x) \in \mathbb{R}^d$ a smooth vector field, $C_i = b_i.\nabla$, $Cf = (C_0 f, \ldots, C_r f)$, $t, x \mapsto M_t(x)$ a smooth function from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathcal{M}^{sym+}_{r\times r}(\mathbb{R})$ the set of positive semi-definite symmetric real matrices of size r, and

$$F(t) = \int \psi(P_t f) \left(CP_t f\right)^T M_t C P_t f d\mu,$$

where A^T stands for the transpose of the matrix A and vectors are seen as 1-column matrices. The coefficients of M_t are the so-called multipliers in the eponymous method introduced in [20, Section I.8].
The following results holds for any diffusion operator:

Lemma 2.6. *Let* $L = L_s + L_a$*, where* $L_s = \frac{1}{2}(L + L^*)$ *and* $L_a = \frac{1}{2}(L - L^*)$ *stand for the symmetric and antisymmetric part of* L *in* $\mathbb{L}^2_\mu$*. Then*

$$F'(t) \leq \int \psi(P_t f) \left(CP_t f\right)^T \Big(2M_t [C, L] + ((2L_s - L)M_t + \partial_t M_t) C\Big) P_t f d\mu,$$

where $[C_i, L] = C_i L - L C_i$ *is the (generalized) Lie bracket of* C_i *and* L *and* $[C, L] = ([C_0, L], \ldots, [C_r, L])$.

Proof. In the following we write f for $P_t f$ and $M_t(x) = (m_{i,j}(t,x))_{0\leq i,j\leq r}$. First it holds

$$\partial_t\left(\int \psi(f) m_{i,j} C_i f C_j f \mathrm{d}\mu\right) \quad = \quad \int \psi(f)\partial_t(m_{i,j}) C_i f C_j f + m_{i,j}\partial_t\left(\psi(f) C_i f C_j f\right) \mathrm{d}\mu.$$

This derivation is justified by the fact that f_0 is uniformly strictly positive and so is f_t, by hypoellipticity and the control of the growth of the derivative of f_t, using Villani [20, Sect. A.21] or [16]. Denote as usual the Carré-du-Champ operator $2\Gamma(g,h) = L(gh) - gLh - hLg$. Next, μ being invariant for L, and using the diffusion property, i.e. that the chain rule property $L\Psi(f_1,...,f_d) = \sum_1^d \partial_i \Psi(f) L f_i + \sum_{i,j} \partial_{i,j}\Psi(f)\Gamma(f_i,f_j)$ holds for all nice Ψ and f,

$$\begin{aligned}
0 &= \int L\left(m_{i,j}\psi(f) C_i f C_j f\right) \mathrm{d}\mu \\
&= \int L\left(m_{i,j}\right)\psi(f) C_i f C_j f \mathrm{d}\mu + \int m_{i,j} L\left(\psi(f) C_i f C_j f\right) \mathrm{d}\mu \\
&\quad +2\int \Gamma\left(m_{i,j}, \psi(f) C_i f C_j f\right) \mathrm{d}\mu \\
&= \int (L - 2L_s)\left(m_{i,j}\right)\psi(f) C_i f C_j f \mathrm{d}\mu + \int m_{i,j} L\left(\psi(f) C_i f C_j f\right) \mathrm{d}\mu\,.
\end{aligned}$$

The case where M is constant (and symmetric semi-definite positive) is already treated in [18, Lemma 8] where it is shown that

$$\sum_{i,j} m_{i,j}\left(L\left(\psi(f) C_i f C_j f\right) - \partial_t\left(\psi(f) C_i f C_j f\right)\right) \quad \geq \quad 2\psi(f)\sum_{i,j} m_{i,j}\left(C_i f\right)\left[L, C_j\right] f\,.$$

The proof follows by taking the integral of both sides. $\square$

Proof of Theorem 2.2. Now consider the case of the Langevin diffusion, namely L is given by (2.1.1). Note that

$$\left[L, \nabla_y\right] = \nabla_x + \nabla_y \qquad\qquad [L, \nabla_x] = -\nabla^2 U(x).\nabla_y.$$

The operator L is decomposed as $L = L_s + L_a$ where

$$L_s = -y.\nabla_y + \Delta_y \qquad\qquad L_a = -y.\nabla_x + \nabla U(x).\nabla_y.$$

Recalling $H(x,y) = U(x) + \frac{1}{2}|y|^2$, then $L_a H = 0$ and more generally $L_a(g\circ H) = 0$ for any smooth $g:\mathbb{R}\to\mathbb{R}$. In particular for $\eta > 0$,

$$\begin{aligned}
\left|(2L_s - L)\left(H^{-\eta}\right)\right| &= \left|L_s\left(H^{-\eta}\right)\right| \\
&= \left|\eta(|y|^2 - d)H^{-\eta-1} + \eta(\eta+1)|y|^2 H^{-\eta-2}\right| \\
&\leq (d + \eta^2 + 2\eta)H^{-\eta}.
\end{aligned}$$

Let a,b,c depend on t and $H(x,y)$, and let $M = \begin{pmatrix} a & b \\ b & c\end{pmatrix}$ and $C = \nabla$, so that Lemma 2.6 reads

$$F'(t) \quad \leq \quad -2\int \psi(P_t f)\left(\nabla P_t f\right)^T N \nabla P_t f \mathrm{d}\mu$$

with

$$N = \begin{pmatrix} b - \frac{1}{2}(L_s + \partial_t) a & -a\nabla^2 U + b - \frac{1}{2}(L_s+\partial_t) b \\ \\ c - \frac{1}{2}(L_s+\partial_t) b & -b\nabla^2 U + c - \frac{1}{2}(L_s+\partial_t) c \end{pmatrix}.$$

In the top left corner b is good news since it gives some coercivity in the x variable. Nevertheless as soon as $b \neq 0$, $b\nabla^2 U$ in the bottom right corner is an annoying term that can only be controlled by the entropy production if it is bounded (which is where, in the previous studies, the assumption that $\nabla^2 U$ is bounded barged in).

Writing $\alpha(t) = (1 - e^{-t})$, set

$$c = 2\varepsilon\alpha H^{-\eta} \qquad b = \varepsilon^2\alpha^2 H^{-2\eta} \qquad a = \varepsilon^3 \alpha^3 H^{-3\eta}$$

for some $\varepsilon \in (0,1)$. In other words,

$$(\nabla f)^T M \nabla f \quad = \quad \varepsilon\alpha H^{-\eta} |\nabla_y f|^2 + \varepsilon \alpha H^{-\eta} |\nabla_y f + \varepsilon\alpha H^{-\eta}\nabla_x f|^2,$$

so that, in particular, M is positive definite. In that case we bound

$$\begin{aligned} b - \frac{1}{2}(L_s + \partial_t) a &\geq \varepsilon^2\alpha^2 H^{-2\eta} - \frac{1}{2}(d + 9\eta^2 + 6\eta)\varepsilon^3\alpha^3 H^{-3\eta} - \frac{3}{2}\varepsilon^3\alpha^2 e^{-t} H^{-3\eta} \\ &\geq \varepsilon^2\alpha^2 H^{-2\eta}\left(1 - (d + 1 + 5\eta^2 + 6\eta)\varepsilon\right), \end{aligned}$$

$$\begin{aligned} -b\nabla^2 U + c - \frac{1}{2}(L_s+\partial_t) c &\geq -\varepsilon^2\alpha^2 \|H^{-2\eta}\nabla^2 U\|_\infty + \left(2\alpha - \frac{1}{2}(d + \eta^2 + 2\eta)\alpha - e^{-t}\right)\varepsilon H^{-\eta} \\ &\geq -\varepsilon^2 \|H^{-2\eta}\nabla^2 U\|_\infty - \varepsilon\left(d + \eta^2 + \eta\right), \end{aligned}$$

$$\begin{aligned} |b + c - a\nabla^2 U - (L_s + \partial_t) b| &\leq |\varepsilon^2\alpha^2 H^{-2\eta} + 2\varepsilon\alpha H^{-\eta} - 2e^{-t}\varepsilon^2\alpha H^{-2\eta}| \\ &\quad + |\varepsilon^3\alpha^3 H^{-3\eta}\nabla^2 U| + (d + 4\eta^2 + 4\eta)\varepsilon^2\alpha^2 H^{-2\eta} \\ &\leq \varepsilon\alpha H^{-\eta}\left(\varepsilon^2 \|H^{-2\eta}\nabla^2 U\|_\infty + 2 + \varepsilon(d + 4\eta^2 + 4\eta)\right), \end{aligned}$$

which implies for $\varepsilon = (d + 1 + 5\eta^2 + 6\eta)^{-1}/4$ that

$$(\nabla f)^T N \nabla f \quad \geq \quad \frac{1}{4}\varepsilon^2\alpha^2 H^{-2\eta}|\nabla_x f|^2 - A|\nabla_y f|^2$$

with

$$\begin{aligned} A &= \left(\varepsilon^2 \|H^{-2\eta}\nabla^2 U\|_\infty + 2 + \varepsilon(d + 4\eta^2 + 4\eta)\right)^2 + \varepsilon^2\|H^{-2\eta}\nabla^2 U\|_\infty + \varepsilon\left(d + \eta^2 + \eta\right) \\ &\leq \frac{1}{2}\left(\|H^{-2\eta}\nabla^2 U\|_\infty + 3\right)^2 := \frac{1}{2}\lambda. \end{aligned}$$

Writing

$$G(t) \quad = \quad \frac{1}{2\lambda}F(t) + \int \Psi(P_t f) d\mu,$$

we have obtained

$$\begin{aligned} G'(t) &\leq -\int \psi(P_t f)\left(\frac{\alpha^2\varepsilon^2}{4\lambda}H^{-2\eta}|\nabla_x P_t f|^2 + \left(1-\frac{A}{\lambda}\right)|\nabla_y P_t f|^2\right)d\mu \\ &\leq -\frac{\alpha^2\varepsilon^2}{4\lambda}\int \psi(P_t f)\left(H^{-2\eta}|\nabla_x P_t f|^2 + |\nabla_y P_t f|^2\right)d\mu. \end{aligned}$$

On the one hand,

$$F(t) \leq 3\varepsilon\alpha\int \psi(P_t f)\left(H^{-2\eta}|\nabla_x P_t f|^2 + |\nabla_y P_t f|^2\right)d\mu,$$

and on the other hand, using the inequality (2.1.4),

$$\int \Psi(P_t f)d\mu \leq \rho\int \psi(P_t f)\left(H^{-2\eta}|\nabla_x P_t f|^2 + |\nabla_y P_t f|^2\right)d\mu,$$

which implies

$$G'(t) \leq -\frac{\alpha^2\varepsilon^2}{1+4\lambda\rho}G(t),$$

where we have used that $6\varepsilon\alpha \leq 1$ for simplicity. Hence,

$$\int \Psi(P_t f)d\mu \leq G(t) \leq G(0)\exp\left(-\frac{\varepsilon^2}{1+4\lambda\rho}\int_0^t \alpha^2(s)ds\right),$$

and $G(0) = \int \Psi(f)d\mu$. The proof is complete. □

2.3 Weighted Functional Inequalities with $\eta \geq 0$.

We turn to the study of the functional inequality (2.1.4). For simplicity we shall only consider the cases $\Psi(u) = (u-1)^2$ (Variance) and $\Psi(u) = u\ln u - u + 1$ (Entropy).
Recall the definition of L_η,

$$L_\eta := H^{-2\eta}\Delta_x + \Delta_y - H^{-2\eta}\left(2\eta\frac{\nabla_x H}{H} + \nabla_x H\right).\nabla_x - \nabla_y H.\nabla_y,$$

which satisfies

$$-\int f L_\eta f\, d\mu = \int (H^{-2\eta}|\nabla_x f|^2 + |\nabla_y f|^2)\, d\mu := \mathcal{E}_\eta(f). \tag{2.3.1}$$

Let us state our first main results

Theorem 2.7. *The weighted Poincaré inequality*

$$Var_\mu(g) \leq \rho\int \left(H^{-2\eta}|\nabla_x g|^2 + |\nabla_y g|^2\right) d\mu$$

is satisfied if and only if there exists a Lyapunov function, i.e. a smooth function W *such that* $W(x,y) \geq w > 0$ *for all* (x,y)*, a constant* $\lambda > 0$ *and a bounded open set* A *such that*

$$L_\eta W \leq -\lambda W + \mathbf{1}_{\bar{A}}.$$

We provide then the equivalent result for the logarithmic Sobolev inequality.

Theorem 2.8. *Assume that* H *goes to infinity at infinity and that there exists* $a > 0$ *such that* $e^{aH} \in \mathbb{L}^1(\mu)$.

1. *If* μ *satisfies the weighted log-Sobolev inequality* (2.1.2)*, then, there exists a Lyapunov function, i.e. a smooth function* W *such that* $W(x,y) \geq w > 0$ *for all* (x,y)*, two positive constants* λ *and* b *such that*
$$L_\eta W \leq -\lambda H W + b. \tag{2.3.2}$$

2. *Conversely, assume that there exists a Lyapunov function satisfying* (2.3.2) *and that* $|\nabla H|(x,y) \geq c > 0$ *for* $|(x,y)|$ *large enough. Define*
$$\theta(r) = \sup_{z \in \partial A_r} \max_{i,j=1,\ldots,2d} \left|\frac{\partial^2 H}{\partial z_i \partial z_j}\right|$$
and assume that $\theta(r) \leq c e^{C_0 r}$ *with some positive constants* C_0 *and* c *for* r *sufficiently large. Then* μ *satisfies the weighted log-Sobolev inequality* (2.1.2)*.*

These theorems are the analogues, in the weighted situation we are looking at, of (part of) Theorem 1.1 and Theorem 1.2 in [9]. Their proofs are very similar concerning the part 1) of the previous theorem and we shall only give some details in the entropic case. Let us begin by a simple and crucial Lemma, at the basis of the use of Lyapunov type condition. Note that it can also be proved via large deviations argument.

Lemma 2.9. *For every continuous function* $W \geq 1$ *in the domain of* L_η *such that* $-L_\eta W/W$ *is* μ*-a.e. lower bounded, for all* g *in the domain of* L_η*,*
$$\int -\frac{L_\eta W}{W} g^2\, d\mu \leq \int \left(H^{-2\eta}|\nabla_x g|^2 + |\nabla_y g|^2\right) d\mu. \tag{2.3.3}$$

Proof. This follows from integration by parts and Cauchy-Schwarz inequality. Indeed,
$$\begin{aligned}
\int -\frac{L_\eta W}{W} g^2\, d\mu &= \int H^{-2\eta}\langle \nabla_x W, \nabla_x \frac{g^2}{W}\rangle + \langle \nabla_y W, \nabla_y \frac{g^2}{W}\rangle d\mu \\
&= \int H^{-2\eta}\left(-\frac{g^2}{W^2}|\nabla_x W|^2 + 2\frac{g}{W}\langle \nabla_x W, \nabla_x g\rangle\right) \\
&\qquad + \left(-\frac{g^2}{W^2}|\nabla_y W|^2 + 2\frac{g}{W}\langle \nabla_y W, \nabla_y g\rangle\right) d\mu \\
&\leq \int \left(H^{-2\eta}|\nabla_x g|^2 + |\nabla_y g|^2\right) d\mu.
\end{aligned}$$
□

Let us now prove Theorem 2.8.

Proof. For a given function ϕ, introduce the operator G_η via $G_\eta h = -L_\eta h + \phi h$. For any h in the domain of L_η, $\int h G_\eta h d\mu = \mathcal{E}_\eta(h) + \int h^2 \phi \, d\mu$. Choosing $\phi = -c + \mathbf{1}_A$, for some set A to be defined, in the variance case and $\phi = \rho(b - H)$ in the entropic case, one deduces that G_η is continuous for the norms whose square are respectively $\mathcal{E}_\eta(h) + \int_A h^2 \, d\mu$ and $\mathcal{E}_\eta(h) + \int h^2 \, d\mu$. If a weighted Poincaré inequality (resp. weighted log-Sobolev inequality) is satisfied, following the proof of Theorem 2.1 (resp. Proposition 3.1) in [9], we get that the form $\int h G_\eta h \, d\mu$ is also coercive so that the Lax-Milgram theorem gives a solution to $G_\eta h = 1$, which furnishes the desired Lyapunov function (see [9] for the details).

For the converse, we revisit the proof of [9] Proposition 3.5 in order to adapt it to our case. As usual, we will rather prove the (weighted) log-Sobolev inequality in its equivalent (weighted) Super Poincaré inequality form, i.e. there exist $c, \beta > 0$ such that for all smooth f and $s > 0$,

$$\int f^2 d\mu \le s \int (H^{-2\eta} |\nabla_x f|^2 + |\nabla_y f|^2) d\mu + c\, e^{\beta/s} \left(\int |f| d\mu \right)^2 .$$

Indeed, the latter implies a defective (weighted) log-Sobolev inequality and a weighted Poincaré inequality (choosing s such that $ce^{\beta/s} = 1$) and we obtain a tight (weighted) log-Sobolev inequality by using Rothaus lemma (see [4] p.239), which states that

$$\mathrm{Ent}_\mu(f^2) \le \mathrm{Ent}_\mu(\tilde{f}^2) + 2\mathrm{Var}_\mu(f) , \tag{2.3.4}$$

where $\tilde{f} = f - \int f \, d\mu$. For all this we refer to [10; 11; 21].

Recall $A_r = \{H \le r\}$. For r_0 large enough and some $\lambda' < \lambda$ we have

$$L_\eta W \le -\lambda' H W + b \mathbf{1}_{A_{r_0}} ,$$

so that we may assume that

$$\frac{L_\eta W}{W}(x, y) \le -\lambda H(x, y) + \frac{b}{w} \mathbf{1}_{A_{r_0}} .$$

For $r > r_0$,

$$\begin{aligned}
\int f^2 \, d\mu &\le \int_{A_r} f^2 \, d\mu + \int_{A_r^c} \frac{\lambda H}{\lambda r} f^2 \, d\mu \\
&\le \int_{A_r} f^2 \, d\mu + \int \frac{\lambda H}{\lambda r} f^2 \, d\mu \\
&\le \int_{A_r} f^2 \, d\mu + \frac{1}{\lambda r} \int \left(\frac{-L_\eta W}{W} + \frac{b \mathbf{1}_{A_{r_0}}}{w} \right) f^2 \, d\mu \\
&\le \left(1 + \frac{b}{\lambda r w} \right) \int_{A_r} f^2 \, d\mu + \frac{1}{\lambda r} \int \left(H^{-2\eta} |\nabla_x f|^2 + |\nabla_y f|^2 \right) d\mu .
\end{aligned}$$

It remains to control the integral in A_r. It is in fact a simple consequence of Nash inequalities for the Lebesgue measure rewritten in its Super Poincaré form (c.f. [10, Prop 3.8]): there exists c_d such that for all r large enough, all smooth f and $s > 0$

$$\begin{aligned}
\int_{A_r} f^2 dx dy &\le s \int_{A_r} |\nabla f|^2 dx dy + c_d \theta^d(r)(1 + s^{-2d}) \left(\int |f| dx dy \right)^2 \\
&\le s \int_{A_r} |\nabla f|^2 dx dy + c_d c e^{2dC_0 r}(1 + s^{-2d}) \left(\int |f| dx dy \right)^2 .
\end{aligned}$$

Recall that $\mathrm{H} \geq 1$. We thus have

$$\begin{aligned}\int_{\mathrm{A}_r} f^2 \mathrm{d}\mu &\leq \frac{1}{e\mathrm{Z}} \int_{\mathrm{A}_r} f^2 \mathrm{d}x\mathrm{d}y \\ &\leq \frac{r^{2\eta} e^r}{e} s \int \left(\mathrm{H}^{-2\eta} |\nabla_x f|^2 + |\nabla_y f|^2\right) \mathrm{d}\mu + \mathrm{Z} c_d c e^{2d\mathrm{C}_0 r} (1 + s^{-2d}) e^{2r} \left(\int_{\mathrm{A}_r} |f| d\mu\right)^2 .\end{aligned}$$

Letting $u = se^{r-1} r^{2\eta}$ and $\mathrm{C}' = \mathrm{Z}cc_d$, and considering integral on the whole space in the right hand side, we have thus obtained (for r large enough)

$$\int_{\mathrm{A}_r} f^2 \mathrm{d}\mu \leq u \int \left(\mathrm{H}^{-2\eta} |\nabla_x f|^2 + |\nabla_y f|^2\right) \mathrm{d}\mu + \mathrm{C}' r^{4d\eta} (1 + u^{-2d}) e^{2(1+d\mathrm{C}_0+d)r} \left(\int |f| d\mu\right)^2 .$$

Denoting $c = 1 + \frac{b}{\lambda r_0 w}$, and $\beta_d = 2 + d + 2d\mathrm{C}_0$, we thus have, for all $u > 0$ and r large enough ,

$$\int f^2 \mathrm{d}\mu \leq \left(uc + \frac{1}{\lambda r}\right) \int \left(\mathrm{H}^{-2\eta} |\nabla_x f|^2 + |\nabla_y f|^2\right) \mathrm{d}\mu + \mathrm{C}' (1 + u^{-2d}) r^{2d\eta} c e^{\beta_d r} \left(\int |f| d\mu\right)^2 . \tag{2.3.5}$$

Choosing $r\lambda = (uc)^{-1}$ and $s = 2uc$, we have thus proved the existence of some β'_d such that

$$\int f^2 \mathrm{d}\mu \leq s \int \left(\mathrm{H}^{-2\eta} |\nabla_x f|^2 + |\nabla_y f|^2\right) \mathrm{d}\mu + \mathrm{C}'' e^{\beta'_d / s} \left(\int |f| d\mu\right)^2 ,$$

and the proof is complete. $\square$

Remark 2.10. For a general weighted logarithmic Sobolev inequality with the weighted energy

$$\int \left(w_1 |\nabla_x f|^2 + w_2 |\nabla_y f|^2\right) d\mu,$$

we can introduce the symmetric generator

$$\mathrm{L}_{w_1,w_2} := w_1 \Delta_x + w_2 \Delta_y - w_1 \left(-\frac{\nabla_x w_1}{w_1} + \nabla_x \mathrm{H}\right).\nabla_x - w_2 \left(-\frac{\nabla_y w_2}{w_2} + \nabla_y \mathrm{H}\right).\nabla_y.$$

If a Lyapunov function (as in Theorem 2.4 but for L_{w_1,w_2}) exists, then following the same line, we can obtain (with the required additional assumptions on the weights) a weighted logarithmic Sobolev inequality. $\diamondsuit$

We now proceed to the

Proof of Corollary 2.5. Consider a smooth function $\mathrm{W}(x, y) = e^{\alpha \mathrm{U}(x) + \frac{\beta}{2}|y|^2}$ with two constants $\alpha, \beta \in (0, 1)$ to be determined. Then for $|(x, y)| \geq \mathrm{R}$,

$$\begin{aligned}\frac{\mathrm{L}_\eta \mathrm{W}}{\mathrm{W}} &= \alpha \mathrm{H}^{-2\eta} \left[\Delta_x \mathrm{U} + \left(\alpha - \frac{2\eta}{\mathrm{H}} - 1\right) |\nabla_x \mathrm{U}|^2\right] + \beta(d - (1 - \beta)|y|^2) \\ &\leq \beta d - \alpha(1 - \alpha - \kappa) |\nabla_x \mathrm{U}|^2 \mathrm{H}^{-2\eta} - \beta(1 - \beta)|y|^2 ,\end{aligned}$$

where we used the first condition in the assumption of the corollary.

To bound the last term by some $C - \lambda H$, we consider $\alpha \in (0, 1-\kappa), \beta \in (0,1)$, and divide it into two cases. If $\frac{|y|^2}{2} \geq \frac{H}{2}$, then

$$-\alpha(1-\alpha-\kappa)|\nabla_x U|^2 H^{-2\eta} - \beta(1-\beta)|y|^2 \leq -\beta(1-\beta)H$$

Otherwise,we have $U \geq \frac{H}{2}$. Combined with the second condition, it follows

$$-\frac{|\nabla_x U|^2}{H^{2\eta}} \leq -\frac{cU^{2\eta+1}}{2^{2\eta}U^{2\eta}} \leq -\frac{c}{2^{2\eta+1}}H,$$

which completes the proof of the Lyapunov condition. Since the second condition implies that U goes to infinity at infinity and $|\nabla_x U| \geq u \geq 0$, we get a weighted logarithmic Sobolev inequality for μ by the previous theorem. □

The next example, which is the simple polynomial case, will show the adequacy of our conditions on weighted log-Sobolev inequality with the Assumption 2.1.

Example 2.1. Let us consider the example where $U(x) = |x|^l$ with $l > 2$ for $|x|$ large enough, that is, $H(x,y) = |x|^l + \frac{|y|^2}{2}$. Then $\Delta_x U = (dl + l^2 - 2l))|x|^{l-2}$ and $|\nabla_x U|^2 = l^2|x|^{2l-2}$. The first condition is satisfied since $l > 2$, while the second condition requires

$$\eta \leq \frac{1}{2} - \frac{1}{l}.$$

Note that $||U^{-2\eta}\nabla^2 U||_\infty \sim |x|^{l-2-2l\eta}$ so that, to ensure that $U^{-2\eta}\nabla^2 U$ is bounded, we have to choose $\eta = \frac{1}{2} - \frac{1}{l}$. With the case $l = 2$ we recover Villani's result.

Let us give another example which will show that our limit growth for the potential U is below the exponential growth

Example 2.2. Choose now $U(x) = e^{a|x|^b}$ for $a, b > 0$ for $|x|$ large enough. Then $\Delta_x U \sim a^2 b^2 |x|^{2(b-1)} e^{a|x|^b}$ and $|\nabla_x U|^2 \sim a^2 b^2 e^{2a|x|^b}$. The first condition is thus satisfied , while the second one imposes once again that $2\eta + 1 \leq 2$. Now, Assumption 1 imposes that $2\eta > 1$ if $b \geq 1$, leading to an impossible adequacy of the two sets of conditions, and to $2\eta \geq 1$ if $b < 1$, in which case the choice of $\eta = 1/2$ is admissible.

Let us end this section by a remark

Remark 2.11. For the multipliers method in the variance case, Villani does not use $H^{-2\eta}$ in the energy to get his inequality but, as will be seen in the next section, proves a rather stronger inequality with weight $U(x)^{-2\eta}(1+|y|^2)^{-2\eta}$ in the derivative in x. The fact that he deals with the variance helps him enough to prove such a weighted Poincaré inequality. We may also consider a weighted logarithmic Sobolev inequality with such a weight. However, via the Lyapunov condition approach, the condition on η is then too strong to match with Assumption 1. It is thus crucial to have a weighted inequality with weight $H^{-2\eta}$ for Theorem 1. ♢

The next section presents an alternative approach, trying to provide an answer to the problem alluded in the previous remark. Is it possible to provide a 'tensorization-like" approach to provide a weighted logarithmic Sobolev inequality as in Villani's paper, thus giving an alternative to Lyapunov conditions?

2.4 Some further remarks on weighted inequalities.

In this final section we shall try to understand whether it is possible to impose conditions on U solely in order to get weighted inequalities. We shall use several times the following elementary inequalities, true for all $\eta \geq 0$, all x and y (recall that $U \geq 1$)

$$U^{-\eta}(x)\left(1+\frac{1}{2}|y|^2\right)^{-\eta} \leq H^{-\eta}(x,y) \leq \min\left(U^{-\eta}(x), \left(1+\frac{1}{2}|y|^2\right)^{-\eta}\right). \tag{2.4.1}$$

We shall use in the sequel the notations $U^{-2\eta}(x) = \phi_1(x)$ and $\left(1+\frac{1}{2}|y|^2\right)^{-2\eta} = \phi_2(y)$.

2.4.1 The case of weighted Poincaré inequalities.

Assume that μ satisfies a weighted Poincaré inequality. If we choose an f that only depends on x and use that $H^{-2\eta}(x,y) \leq U^{-2\eta}(x)$ for all y, we immediately see that the first marginal of μ, i.e. $d\mu_1(x) := \frac{1}{Z_1}e^{-U(x)}dx$ also satisfies the weighted Poincaré inequality

$$\mathrm{Var}_{\mu_1}(f) \leq C\int U^{-2\eta}|\nabla f|^2 d\mu_1\,. \tag{2.4.2}$$

Conversely we have,

Theorem 2.12. *Write $\mu(dx,dy) = \mu_1(dx)\otimes\mu_2(dy)$. If $\mu_1(dx) = \frac{1}{Z_1}e^{-U(x)}dx$ satisfies the weighted Poincaré inequality (2.4.2) with constant C_1, then μ satisfies the following weighted Poincaré inequality*

$$Var_\mu(h) \leq C'\int (H^{-2\eta}|\nabla_x h|^2 + |\nabla_y h|^2)d\mu$$

with

$$C' \leq \max\left(\left(2+\frac{4}{M_2}\right), \frac{4C_1}{M_2}\right) \quad \textit{where}\ M_2 = \int \left(1+\frac{1}{2}|y|^2\right)^{-2\eta}\mu_2(dy)\,.$$

Proof. A proof is given in Villani [20] Theorem A.3. It uses extensively the spectral theory of the sum of operators. We shall give a more pedestrian (similar) proof.

The first point is that, since we assumed that $U \geq 1$,

$$H^{-2\eta}(x,y) \geq \phi_1(x)\phi_2(y) := U^{-2\eta}(x)\left(1+\frac{1}{2}|y|^2\right)^{-2\eta}. \tag{2.4.3}$$

Thus, if we decompose $\mu(dx,dy) = \mu_1(dx)\otimes\mu_2(dy)$ we have

$$\begin{aligned}
\int H^{-2\eta}|\nabla_x h|^2\,\mu(dx,dy) &\geq \int \phi_1(x)\phi_2(y)|\nabla_x h|^2\,\mu_1(dx)\otimes\mu_2(dy)\\
&\geq \frac{1}{C_1}\int \phi_2(y)\left(h(x,y)-\int h(u,y)\mu_1(du)\right)^2\mu(dx,dy)\,.
\end{aligned}$$

Now, write

$$\begin{aligned} h(x,y) - \int h(u,y)\mu_1(du) &= \left(h(x,y) - \int h(u,y)\mu_1(du) - \int h(x,v)\mu_2(dv) + \int\int h d\mu_1 d\mu_2\right) \\ &\quad + \left(\int h(x,v)\mu_2(dv) - \int\int h d\mu_1 d\mu_2\right) \\ &= g_1(x,y) + g_2(x) \end{aligned}$$

and use

$$(a+b)^2 \geq \frac{1}{2}\, b^2 - a^2 \,.$$

This yields, since $\phi_2(y) \leq 1$,

$$\int H^{-2\eta}\, |\nabla_x h|^2\, \mu(dx,dy) \geq \frac{1}{2C_1}\left(\int \phi_2\, d\mu_2\right)\left(\int g_2^2(x)\mu_1(dx)\right) - \frac{1}{C_1}\int\int g_1^2(x,y)\,\mu_1(dx)\mu_2(dy)\,.$$

Notice that for all y,

$$\int g_1^2(x,y)\,\mu_1(dx) = \mathrm{Var}_{\mu_1}\left(h(\cdot,y) - \int h(\cdot,v)\,\mu_2(dv)\right),$$

so that

$$\int g_1^2(x,y)\,\mu_1(dx) \leq \int\left(h(x,y) - \int h(x,v)\,\mu_2(dv)\right)^2 \mu_1(dx)\,.$$

We can thus integrate this inequality w.r.t. μ_2, use Fubini's theorem, then for each fixed x use the usual Poincaré inequality for the standard gaussian measure μ_2 and finally integrate with respect to μ_1. This yields

$$\begin{aligned} \int\int g_1^2(x,y)\,\mu_1(dx)\mu_2(dy) &\leq \int\int\left(h(x,y) - \int h(x,v)\,\mu_2(dv)\right)^2 \mu(dx,dy) \\ &\leq \int\int |\nabla_y h|^2(x,y)\,\mu(dx,dy)\,. \end{aligned}$$

Gathering all this we have obtained

$$\int g_2^2(x)\mu_1(dx) \leq \frac{2\,C_1}{M_2}\int H^{-2\eta}|\nabla_x h|^2\, d\mu + \frac{2}{M_2}\int |\nabla_y h|^2\, d\mu\,. \tag{2.4.4}$$

Finally,

$$\begin{aligned} \mathrm{Var}_\mu(h) &= \int\left(h(x,y) - \int h(x,v)\mu_2(dv) + \int h(x,v)\mu_2(dv) - \int\int h d\mu\right)^2 \mu(dx,dy) \\ &\leq 2\int\int\left(h(x,y) - \int h(x,v)\,\mu_2(dv)\right)^2 \mu(dx,dy) + 2\int g_2^2(x)\mu_1(dx) \\ &\leq 2\int\int |\nabla_y h|^2(x,y)\,\mu(dx,dy) + 2\int g_2^2(x)\mu_1(dx)\,, \end{aligned}$$

and the result follows from (2.4.4). □

As a conclusion the weighted Poincaré inequality on $\mathbb{R}^{2d}$ reduces to a weighted Poincaré inequality on $\mathbb{R}^d$ (up to some constant). One should think that the previous result is a kind of weighted tensorization property. This is not the case due to the fact that the weight in front of ∇_x depends on both variables x and y.
There are many ways to obtain such an inequality. Of course since it is stronger than the usual Poincaré inequality, our result is weaker than the one of Villani (but with a simpler proof and explicit bounds for the constants), and we will only describe a typical situation where this equality can be obtained.
As we have seen in the previous section, this weighted Poincaré inequality is equivalent to the existence of some Lyapunov function for $L_{1,\eta}$ which is built similarly to L_η replacing H by U. We can also obtain a slightly different condition. Introduce the probability measure $\mu_1^\phi(dy) = \frac{\phi_1(y)}{M_1}\mu_1(dy)$ and the μ_1^ϕ symmetric operator

$$G_1^\phi = \Delta_x - \left(1 + \frac{2\eta}{U}\right)\nabla U.\nabla\,.$$

Assume that we can find a Lyapunov function $W \geq 1$ such that

$$\frac{G_1^\phi W(x)}{W(x)} \leq -a\,U^{2\eta}(x)$$

for $|x|$ larger than some $R > 0$. If h is compactly supported in $|x| > R$, we may write

$$\int h^2\, d\mu_1 \leq -\frac{M_1}{a}\int \frac{G_1^\phi W}{W}\, h^2\, d\mu_1^\phi \leq \frac{M_1}{a}\int |\nabla h|^2\, d\mu_1^\phi = \frac{M_1}{a}\int |\nabla h|^2 U^{-2\eta}\, d\mu_1$$

according to the computations in [2] p.64. Following the method introduced in [2] we then obtain that μ_1 satisfies the desired weighted Poincaré inequality. According to [9] Theorem 4.4, the existence of such a Lyapunov function is linked to the fact that μ_1 satisfies some F-Sobolev inequality, with $F = \ln_+^{2\eta}$. This is for instance the case when $U(x) = 1 + |x|^\alpha$ and $\eta = 1 - \alpha^{-1}$.

2.4.2 The case of weighted log-Sobolev inequalities.

We look now at the similar weighted logarithmic Sobolev inequality, namely,

$$\mathrm{Ent}_\mu(f^2) \leq \rho \int (H^{-2\eta}|\nabla_x f|^2 + |\nabla_y f|^2)\,d\mu.$$

As in the L^2 setting, it implies a weighted log Sobolev inequality for μ_1 on $\mathbb{R}^d$ i.e.

$$\mathrm{Ent}_{\mu_1}(f^2) \leq C\int U^{-2\eta}|\nabla_x f|^2\,d\mu_1\,. \tag{2.4.5}$$

Since the standard gaussian measure μ_2 satisfies a log-Sobolev inequality too (with optimal constant 2), one should expect to obtain the analogue of theorem 2.12. This is not so easy (actually we did not succeed in proving such a result and believe that it is wrong) and certainly explains the limitation of Villani's approach, since this property reduces to the well known tensorization property of the logarithmic Sobolev inequality only in the case $\eta = 0$. The best we are able to do is to prove that, in this situation

Theorem 2.13. *Write* $\mu(dx,dy) = \mu_1(dx) \otimes \mu_2(dy)$. *If* $\mu_1(dx) = \frac{1}{Z_1} e^{-U(x)} dx$ *satisfies the weighted log-Sobolev inequality* (2.4.5)*, then* μ *satisfies* (2.1.4) *with an admissible function* $u \mapsto \Psi(u)$ *behaving like* $u \ln^{\frac{1+4\eta}{1+2\eta}}(u)$ *at infinity.*

Combined with theorem 2.2 which deals with a decay for more general functionals than the variance or entropy, we are thus able to prove under such conditions an exponential decay for Ψ behaving like $u \ln^{\frac{1+4\eta}{1+2\eta}}(u)$ at infinity.
Notice that for a bounded Hessian we recover the weighted log-Sobolev inequality, i.e. we recover Villani's result.

Proof. The first step of the proof is the following

Lemma 2.14. *Define the probability measure* $\mu_2^\phi(dy) = \frac{\phi_2(y)}{M_2} \mu_2(dy)$. *Then* μ_2^ϕ *satisfies a log-Sobolev inequality.*

An immediate consequence is the following inequality for $\mu^\phi(dx,dy) = \mu_1(dx) \otimes \mu_2^\phi(dy)$,

$$\mathrm{Ent}_{\mu^\phi}(h^2) \le C \int (\phi_1 |\nabla_x h|^2 + |\nabla_y h|^2) d\mu^\phi, \tag{2.4.6}$$

which follows from the tensorization property of the log-Sobolev inequality.

Proof of Lemma 2.14. Write

$$\mu_2^\phi(dy) = Z^\phi e^{-\left(\frac{|y|^2}{2} + 2\eta \ln(1+|y|^2/2)\right)} dy = Z^\phi e^{-V_2(y)} dy.$$

A simple calculation shows that

$$\mathrm{Hess} V_2(y) = \left(1 + \frac{2\eta}{1+|y|^2/2}\right) Id - \frac{2\eta}{(1+|y|^2/2)^2} M(y)$$

where $M_{i,j}(y) = y_i y_j$. Hence,

$$\mathrm{Hess} V_2(y) \ge \left(1 + \frac{2\eta}{1+|y|^2/2} - \frac{2\eta d |y|^2}{(1+|y|^2/2)^2}\right) Id$$

in the sense of quadratic forms. Hence for $|y|$ large enough (of order $c\sqrt{d}$), the potential $V_2(y)$ is uniformly convex, uniformly in y. This proves (combining Bakry-Emery criterion and Holley-Stroock perturbation argument) the Lemma. $\square$

As we recalled, the weighted log-Sobolev inequality is equivalent to a (weighted) super Poincaré inequality, for all smooth h and all $s>0$,

$$\int h^2 d\mu^{\phi} \le s\int (\phi_1\,|\nabla_x h|^2 + |\nabla_y h|^2) d\mu^{\phi} + c\,e^{\beta/s}\left(\int |h|\, d\mu^{\phi}\right)^2 . \tag{2.4.7}$$

Since $\phi_2 \le 1$, it follows

$$\int h^2 d\mu^{\phi} \le \frac{s}{M_2}\int (H^{-2\eta}|\nabla_x h|^2 + |\nabla_y h|^2) d\mu + \frac{c}{M_2}\, e^{\beta/s}\left(\int |h|\, d\mu\right)^2 . \tag{2.4.8}$$

For $R>1$, introduce the 1-Lipschitz function

$$\varphi(r) = (r-R)\,\mathbf{1}_{R<r<R+1} + \mathbf{1}_{R+1\le r} .$$

One can write

$$\begin{aligned}
\int h^2 d\mu &\le \int_{|y|\le R+1} h^2\, d\mu + \int h^2\,\varphi^2(|y|)\, d\mu \\
&\le \frac{M_2}{\phi_2(R+1)}\int_{|y|\le R+1} h^2 d\mu^{\phi} + \int h^2\,\varphi^2(|y|)\, d\mu \\
&\le \frac{M_2}{\phi_2(R+1)}\int h^2 d\mu^{\phi} + \int h^2\,\varphi^2(|y|)\, d\mu .
\end{aligned}$$

The first term in the sum will be controlled thanks to (2.4.8). In order to control the second term, we introduce,once again, some Lyapunov function.
Denote by G the Ornstein-Uhlenbeck operator $G = \Delta_y - y.\nabla_y$ and consider $W(y) = e^{|y|^2/4}$. A simple calculation shows that

$$\frac{GW}{W} \le \frac{1}{4}\,(2d - |y|^2)$$

for $|y| > \sqrt{2d}$. Hence if $R > \sqrt{2d}$, we get for $|y| > R$,

$$1 \le 4\left(\frac{-GW}{W}\right)\frac{1}{|y|^2 - 2d} \le 4\left(\frac{-GW}{W}\right)\frac{1}{R^2 - 2d}$$

and finally

$$\int h^2\,\varphi^2(|y|)\, d\mu \le \frac{4}{R^2 - 2d}\int \left(\frac{-GW}{W}\right) h^2\,\varphi^2(|y|)\, d\mu . \tag{2.4.9}$$

Integrating by parts, and after some easy manipulations (see [2] for the details), we will thus obtain for well chosen constants C, C' all $s>0$ and large enough R (only depending on d),

$$\int h^2 d\mu \le C\,(sR^{4\eta} + R^{-2})\int (\phi_1\,|\nabla_x h|^2 + |\nabla_y h|^2) d\mu^{\phi} + C'\,R^{4\eta}\,e^{\beta/s}\left(\int |h|\, d\mu\right)^2 . \tag{2.4.10}$$

Choosing $u = s^{\frac{1}{1+2\eta}} = R^{-2}$, we obtain a super Poincaré inequality

$$\int h^2 d\mu \le C\,u\int (\phi_1\,|\nabla_x h|^2 + |\nabla_y h|^2) d\mu^{\phi} + C'\,e^{\beta'/u^{1+2\eta}}\left(\int |h|\, d\mu\right)^2 . \tag{2.4.11}$$

which furnishes a $F=\ln_+^{\frac{1}{1+2\eta}}$-Sobolev inequality, i.e. if $\int h^2\,d\mu=1$,

$$\int h^2 \ln_+^{\frac{1}{1+2\eta}} h^2\,d\mu \le C\int (\phi_1|\nabla_x h|^2+|\nabla_y h|^2)d\mu^{\phi}.$$

Notice that, since $\phi_2\le 1$, the previous inequality is stronger than

$$\int h^2 \ln_+^{\frac{1}{1+2\eta}} h^2\,d\mu \le C\int (H^{-2\eta}|\nabla_x h|^2+|\nabla_y h|^2)d\mu. \tag{2.4.12}$$

It remains to link (2.4.12) to (2.1.4). Actually, as explained in [5] section 7, one can replace $\ln_+$ by smooth functions F with a similar behaviour at infinity (and satisfying $F(1)=0$).
Let

$$\psi(u)=\frac{\ln^{\frac{2\eta}{1+2\eta}}(1+u)}{1+u}$$

defined for $u\ge e$. It is easily seen that $1/\psi$ is concave at infinity. Hence as explained in Remark 2.3, we may modify ψ and consider some admissible Ψ such that $\Psi(u)$ behaves like $(1+u)\ln^{\frac{1+4\eta}{1+2\eta}}(1+u)$ at infinity. If we define $h^2=f\ln^{\frac{2\eta}{1+2\eta}}(1+f)$, we get from (2.4.12) that μ satisfies (2.1.4) with this Ψ and some ρ, completing the proof. □

For a discussion about the connections between F-Sobolev inequalities and the Ψ entropic inequalities one can look at [8] subsection 3.2.

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Chapter 3

Uniform Poincaré and logarithmic Sobolev inequalities for mean field particles systems

This chapter is an article collaborated with Arnaud Guillin, Wei Liu and Liming Wu. In this paper we establish some explicit and sharp estimates of the spectral gap and the log-Sobolev constant for mean field particles system, uniform in the number of particles, when the confinement potential have many local minimums. Our uniform log-Sobolev inequality, based on Zegarlinski's theorem for Gibbs measures, allows us to obtain the exponential convergence in entropy of the McKean-Vlasov equation with an explicit rate constant, generalizing the result of [10] by means of the displacement convexity approach, or [20; 21] by Bakry-Emery technique or the recent [9] by dissipation of the Wasserstein distance.

3.1 Introduction

Functional inequalities such as Poincaré or logarithmic Sobolev inequalities have nowadays an important impact on various fields of mathematics (probability, PDE, statistics,...) due to their various properties such as convergence to equilibrium (in L^2 or in entropy) or concentration of measure (exponential or gaussian). We refer to the beautiful book [3] for an introduction (and more) to the subject as well as bibliographical references. Let us introduce these two inequalities. Let μ be a probability measure on $\mathbb{R}^d$, we say that the probability measure μ satisfies a Poincaré (or equivalently spectral gap) inequality with (optimal) constant λ_μ if for all smooth functions f we have

$$\text{(PI)} \qquad \lambda_\mu \operatorname{Var}_\mu(f) \le \int |\nabla f|^2 d\mu, \tag{3.1.1}$$

where $\operatorname{Var}_\mu(f) := \int f^2 d\mu - \left(\int f d\mu\right)^2$ denotes the variance of f with respect to (w.r.t in short) μ and a logarithmic Sobolev inequality with (optimal) constant ρ_μ if for all smooth functions f we have

$$\text{(LSI)} \qquad \rho_\mu \operatorname{Ent}_\mu(f^2) = \int f^2 \log f^2 d\mu - \int f^2 d\mu \log \int f^2 d\mu \le \int |\nabla f|^2 d\mu, \tag{3.1.2}$$

where $\mathrm{Ent}_\mu(f^2) := \int f^2 \log(f^2/\int f^2 d\mu) d\mu$ denotes the entropy of f^2 w.r.t. μ. A famous condition to verify those inequalities is the Bakry-Emery Γ_2 condition, i.e. Hess $V \geq \kappa Id > 0$ whenever $d\mu = e^{-V}dx$ in which case $\rho_\mu, \lambda_\mu \geq \kappa$.
One crucial property of these two inequalities is the tensorization, i.e. if μ satisfies a Poincaré or a logarithmic Sobolev inequality then $\mu^{\otimes N}$ satisfies the same inequality with the same constant (and thus independent of N) leading for example to (non asymptotic) gaussian deviation inequalities refining central limit inequalities or convergence to equilibrium independent of the number of particles. However interesting physical systems are far from being independent, so that there exists a huge literature devoted to the obtention of functional inequalities such as Poincaré or logarithmic Sobolev inequalities, in particular to assess convergence to equilibrium, in various dependent settings such as (discrete or continuous) spin systems [4; 7; 8; 19; 23–25; 30–33] (see also [18] for a survey) or mean field models [11; 15; 16; 20; 21] with a particular emphasis on the dependence on the number of spins or particles.
We will focus our attention on mean field particles system. To this end, consider the $N(\geq 2)$ interacting particles system of mean-field type :

$$dX_i^N(t) = \sqrt{2}dB_i(t) - \nabla V(X_i^N(t))dt - \frac{1}{N-1}\sum_{j\neq i}\nabla_x W(X_i^N(t), X_j^N(t))dt,\ i = 1, \cdots, N \tag{3.1.3}$$

where the confinement potential V is a function on $\mathbb{R}^d$ of class C^2 and the interaction potential W is a function on $\mathbb{R}^d \times \mathbb{R}^d$ of class C^2, and $B_i (1 \leq i \leq N)$ are independent standard Brownian motions on $\mathbb{R}^d$. Its infinitesimal generator $\mathscr{L}^{(N)}$ is given by

$$\begin{aligned}
\mathscr{L}^{(N)} f(x_1, \cdots, x_N) &= \sum_{i=1}^n \mathscr{L}_i^{(N)} f(x_1, \cdots, x_N) \\
\mathscr{L}_i^{(N)} f(x_1, \cdots, x_N) &:= \Delta_i f(x_1, \cdots, x_N) - \nabla_i V(x_i) \cdot \nabla_i f(x_1, \cdots, x_N) \\
&\quad - \frac{1}{N-1}\sum_{j\neq i}(\nabla_x W)(x_i, x_j) \cdot \nabla_i f(x_1, \cdots, x_N)
\end{aligned} \tag{3.1.4}$$

for any smooth function f on $(\mathbb{R}^d)^N$, where ∇_i denotes the gradient with respect to x_i, Δ_i the Laplacian w.r.t. x_i, and $x \cdot y = \langle x, y\rangle$ denotes the Euclidean inner product.
The unique invariant probability measure of (3.1.3) is

$$d\mu^{(N)}(x_1, \cdots, x_N) = \frac{1}{Z_N}\exp\{-H_N(x)\}\, dx_1 \cdots dx_N \tag{3.1.5}$$

where

$$H_N(x_1, \cdots, x_N) := \sum_{i=1}^N V(x_i) + \frac{1}{N-1}\sum_{1\leq i<j\leq N} W(x_i, x_j)$$

is the Hamiltonian, Z_N is the normalization constant called *partition function* in statistical mechanics, which is assumed to be finite throughout the paper. Without interaction (i.e. $W = 0$ or constant), $\mu^{(N)} = \alpha^{\otimes N}$ (i.e. the particles are independent), where

$$d\alpha(x) = \frac{1}{C}e^{-V(x)}dx,\ C = \int e^{-V(x)}dx.$$

Our first major goal is to get uniform (in the number of particles N) Poincaré or logarithmic Sobolev inequalities for the measure $\mu^{(N)}$ under tractable conditions. Malrieu [20] used Bakry-Emery's Γ_2 technique to establish a logarithmic Sobolev inequality for the mean field case thus requiring uniform convexity assumption for V and W. Recent techniques such as Lyapunov conditions (see [1; 2] for example) are usually unefficient to get adimensional results. For each of these inequalities we require a uniform bound for the spectral gap or the logarithmic Sobolev constants of the one particle conditional distribution. To bypass the perturbation techniques, our main assumptions for Poincaré inequality will be of two sorts: for the confinement potential we will need some linear growth at infinity as well as a lipschitzian spectral gap property (see Section 2 for details) which will be sufficient to get a Poincaré inequality for the one particle conditional distribution, and for the interaction potential a lower bound on the "extra diagonal" hessian of W, leading to new and sharp results. A particular emphasis will be made on Curie-Weiss model and on interaction potential of the form $W(x,y) = W_0(x-y)$. The proof will consider refinements of the ideas of Ledoux [19]. For the logarithmic Sobolev inequality we will consider a translation of Zegarlinski's condition (see [32]) for mean field model which relies on the smallness of the product of the Lipschitzian spectral gap and of the infinite norm of the Hessian of the interaction potential. One of our interest to consider logarithmic Sobolev inequality for mean field particles system is to get an (generalized) entropic decay for the limit nonlinear SDE of McKean-Vlasov type. Indeed, consider the nonlinear McKean-Vlasov equation with an internal potential $V : \mathbb{R}^d \to \mathbb{R}$ and an interaction potential $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ (between two particles) so that $W(x,y) = W(y,x)$:

$$\partial_t \nu_t = \Delta \nu_t + \nabla \cdot (\nu_t \nabla V) + \nabla \cdot (\nu_t \nabla (W \circledast \nu_t)) \tag{3.1.6}$$

where $(\nu_t)_{t\geq 0}$ is a flow of probability measures on $\mathbb{R}^d$ with ν_0 given, ∇ is the gradient, $\nabla\cdot$ is the divergence, and

$$(W \circledast \nu)(x) = \int_{\mathbb{R}^d} W(x,y)\,d\nu(y). \tag{3.1.7}$$

It corresponds to the self-interacting diffusion

$$dX_t = \sqrt{2}dB_t - \nabla V(X_t)dt - \nabla W \circledast \nu_t(X_t)dt$$

where ν_t is the law of X_t. It can be seen through the propagation of chaos phenomenon (see [26] for example) that the law of $X_1^N(t)$ converges to the one of X_t as the number of particles N tends to infinity(for each $t > 0$) . Via the logarithmic Sobolev inequality for the mean field particles system and a quite technical passage to the limit, we will be able to prove entropic convergence to equilibrium for the non linear McKean-Vlasov SDE generalizing results of [9; 10].

Let us finish this introduction by the plan of the paper. In the next section, we will present our set of assumptions and the main results of the paper concerning uniform Poincaré or logarithmic Sobolev inequality of mean field particles system as well as exponential convergence to equilibrium for McKean-Vlasov SDE. Section 3 presents the Lipschitzian spectral gap for conditional distribution needed in the proof of the uniform Poincaré inequality detailed in Section 4. The translation of Zegarlinski's condition and thus the proof of uniform logarithmic Sobolev inequality are the core of Section 5. The exponential convergence to equilibrium of McKean-Vlasov SDE is finally detailed in the last Section 6.

3.2 Main results

3.2.1 Framework and main assumptions.

Throughout the paper we work in the following framework.

(H1) The confinement potential $V : \mathbb{R}^d \to \mathbb{R}$ is C^2-smooth, its Hessian $\nabla^2 V = (\partial_{x_k}\partial_{x_l} V)_{1\le k,l\le d}$ of V is bounded from below and there are two positive constants c_1, c_2 such that

$$x\cdot\nabla V(x) \ge c_1|x|^2 - c_2,\ x\in\mathbb{R}^d. \tag{3.2.1}$$

(H2) The pairwise interaction potential $W : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ is C^2-smooth such that its Hessian $\nabla^2 W$ is bounded and

$$\iint \exp\left(-[V(x)+V(y)+\lambda W(x,y)]\right) dx dy < +\infty,\ \forall\lambda>0.$$

(H3) **(Lipschitzian spectral gap condition for one particle)** The following Lipschitzian constant (for the marginal conditional distribution of one particle) is finite

$$c_{Lip,m} := \frac{1}{4}\int_0^\infty \exp\left\{\frac{1}{4}\int_0^s b_0(u)\mathrm{d}u\right\} s ds < +\infty \tag{3.2.2}$$

where $b_0(r)$ is the dissipativity rate of the drift of one particle in the system (3.1.3) at distance $r>0$:

$$b_0(r) = \sup_{x,y,z\in\mathbb{R}^d:|x-y|=r} -\langle\frac{x-y}{|x-y|}, (\nabla V(x)-\nabla V(y)) + (\nabla_x W(x,z)-\nabla_x W(y,z))\rangle. \tag{3.2.3}$$

This last condition is of course reminiscent of the work of Eberle [15; 16] without the interaction potential for convergence to equilibrium in L^1-Wasserstein distance. However in their work the interaction potential is seen only as a perturbation.

3.2.2 Uniform Poincaré inequality for mean-field $\mu^{(N)}$

In the sequel we shall use the notation $\nabla^2_{x_i,x_j}H$ for a C^2-function H on $(\mathbb{R}^d)^N$, defined by

$$\nabla^2_{x_i,x_j}H := (\partial^2_{x_{ik}x_{jl}}H)_{1\le k,l\le d}.$$

where $x_i = (x_{i1}, x_{i2},\cdots, x_{id})\in\mathbb{R}^d$. Let

$$\lambda_{1,m} = \inf_{N\ge2}\inf_{1\le i\le N}\lambda_1(\mu_i) \tag{3.2.4}$$

where $\lambda_1(\mu_i)$ is the spectral gap of the conditional distribution $\mu_i = \mu_i(dx_i|x^{\hat{i}})$ of x_i knowing $x^{\hat{i}} = (x_j)_{j\neq i}$, i.e. the best constant such that the following Poincaré inequality

$$\lambda_1(\mu_i)\mathrm{Var}_{\mu_i}(f) \le \int_{\mathbb{R}^d}|\nabla_i f|^2 d\mu_i,\ \forall f\in C_b^1(\mathbb{R}^d)$$

holds.

Theorem 3.1. *In the framework described above, we have always*

$$\lambda_{1,m} \ge \frac{1}{c_{Lip,m}}. \tag{3.2.5}$$

Assume that there is some constant $h > -\lambda_{1,m}$ such that for any $(x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$,

$$\frac{1}{N-1}(\mathbb{1}_{i\neq j}\nabla^2_{x,y}W(x_i,x_j))_{1\le i,j\le N} \ge h I_{dN} \tag{3.2.6}$$

in the order of definite nonnegativity for symmetric matrices, where I_n is the identity matrix of size n. Then $\mu^{(N)}$ satisfies the following Poincaré inequality

$$\big(\lambda_{1,m} + h\big)\mathrm{Var}_{\mu^{(N)}}(f) \le \int_{(\mathbb{R}^d)^N} |\nabla f|^2 d\mu^{(N)}, \ f \in C^1_b(\mathbb{R}^{dN}) \tag{3.2.7}$$

or equivalently the spectral gap $\lambda_1(\mu^{(N)})$ of $\mathscr{L}^{(N)}$ on $L^2(\mu^{(N)})$, defined as the infimum of those spectral points $\lambda > 0$ of $\mathscr{L}^{(N)}$ on $L^2(\mu^{(N)})$, verifies

$$\lambda_1(\mu^{(N)}) \ge \lambda_{1,m} + h \ge \frac{1}{c_{Lip,m}} + h. \tag{3.2.8}$$

Its proof will be given in §3.

The uniform Poincaré inequality in Theorem 3.1 gives us the following explicit correlation inequality. For any C^1-function f on $\mathbb{R}^d$, denote $\|f\|^2_{\mathrm{Lip}}$ by its Lipschitzian norm w.r.t. the Euclidean metric on $\mathbb{R}^d$.

Corollary 3.2. *Under the conditions of Theorem 3.1, for any two bounded Lipschitzian functions f, g on $\mathbb{R}^d$ and $i \neq j$*

$$\mathrm{Cov}_{\mu^{(N)}}(f(x_i), g(x_j)) \le \frac{c_{\mathrm{Lip,m}}}{(1 + c_{\mathrm{Lip,m}}h)(N-1)}\left(\|f\|^2_{\mathrm{Lip}} + \|g\|^2_{\mathrm{Lip}}\right) \tag{3.2.9}$$

where $\mathrm{Cov}_{\mu^{(N)}}(\cdot,\cdot)$ denotes the covariance of two functions under the probability measure $\mu^{(N)}$. Roughly speaking, two particles x_i and x_j become asymptotically independent at the rate $1/N$.

Proof. The l.h.s of (3.2.9) does not depend on (i, j). Applying the Poincaré inequality to $F := \frac{1}{\sqrt{N}}\sum_{i=1}^N f(x_i)$, we have

$$\begin{aligned}\mathrm{Var}_{\mu^{(N)}}(F) &= \mathrm{Var}_{\mu^{(N)}}(f(x_1)) + (N-1)\mathrm{Cov}_{\mu^{(N)}}(f(x_1), f(x_2)) \\ &\le \frac{1}{\lambda_1(\mu^{(N)})}\int |\nabla F|^2 d\mu^{(N)} \le \frac{1}{\lambda_1(\mu^{(N)})}\|f\|^2_{\mathrm{Lip}}\end{aligned}$$

then by (3.2.8)

$$\mathrm{Cov}_{\mu^{(N)}}(f(x_1), f(x_2)) \le \frac{c_{\mathrm{Lip,m}}}{(1 + c_{\mathrm{Lip,m}}h)(N-1)}\|f\|^2_{\mathrm{Lip}}. \tag{3.2.10}$$

Note that $\mathrm{Var}_{\mu^{(N)}}(F) \geq 0$, then by Poincaré inequality again, it holds

$$\begin{aligned} -(N-1)\mathrm{Cov}_{\mu^{(N)}}(f(x_1), f(x_2)) &\leq \mathrm{Var}_{\mu^{(N)}}(f(x_1)) \leq \frac{1}{\lambda_1(\mu^{(N)})}\int |\nabla f(x_1)|^2 d\mu^{(N)} \\ &\leq \frac{c_{\mathrm{Lip,m}}}{1+c_{\mathrm{Lip,m}}h}\|f\|_{\mathrm{Lip}}^2. \end{aligned}$$

Combined with the previous inequality (3.2.10), we obtain

$$|\mathrm{Cov}_{\mu^{(N)}}(f(x_1), f(x_2))| \leq \frac{c_{\mathrm{Lip,m}}}{(1+c_{\mathrm{Lip,m}}h)(N-1)}\|f\|_{\mathrm{Lip}}^2. \tag{3.2.11}$$

Now

$$\begin{aligned} \mathrm{Cov}_{\mu^{(N)}}(f(x_1), g(x_2)) &= \frac{1}{4}\Big[\mathrm{Cov}_{\mu^{(N)}}((f+g)(x_1),(f+g)(x_2)) - \mathrm{Cov}_{\mu^{(N)}}((f-g)(x_1),(f-g)(x_2))\Big] \\ &\leq \frac{c_{\mathrm{Lip,m}}}{4(1+c_{\mathrm{Lip,m}}h)(N-1)}\left(\|f+g\|_{\mathrm{Lip}}^2 + \|f-g\|_{\mathrm{Lip}}^2\right) \\ &\leq \frac{c_{\mathrm{Lip,m}}}{(1+c_{\mathrm{Lip,m}}h)(N-1)}\left(\|f\|_{\mathrm{Lip}}^2 + \|g\|_{\mathrm{Lip}}^2\right) \end{aligned}$$

the desired (3.2.9). □

Remark 3.3. The Poincaré inequality (3.2.7) is sharp. In fact, let $d = 1$, $V(x) = x^2/2$, $W(x, y) = \beta xy$. In that case $b_0(r) = -r$ (such W does not change b_0), $1/c_{Lip,m} = 1 = \lambda_{1,m}$. Note that $\lambda_0 := \min\left\{1+\beta,\ 1-\dfrac{\beta}{N-1}\right\}$ is the smallest eigenvalue of the symmetric matrix

$$\frac{1}{N-1}(\beta \mathbb{1}_{i\neq j}) + I_N = \frac{1}{N-1}(\beta \mathbb{1}_{i\neq j}) + \lambda_{1,m} I_N.$$

Our condition (3.2.6) for the Poincaré inequality becomes

$$\lambda_0 > 0.$$

This is necessary even for well defining $\mu^{(N)}$. And our estimate (3.2.8) says that $\lambda_1(\mu^{(N)}) \geq \lambda_0$. As the matrix $\frac{1}{N-1}(\beta\mathbb{1}_{i\neq j}) + I_N$ is exactly the inverse of the covariance matrix of the centered gaussian distribution $\mu^{(N)}$, its spectral gap is exactly λ_0, showing so the sharpness of this theorem.

Remark 3.4. Here we give an explicit estimate of $c_{Lip,m}$ under the following assumptions. Assume there are some constants $c_V, c_1, c_W, c_2 \in \mathbb{R}$ and $R \geq 0$ such that

$$\langle \nabla V(x) - \nabla V(y), x-y\rangle \geq c_V|x-y|^2 - c_1|x-y|\mathbb{1}_{[|x-y|\leq R]} \tag{3.2.12}$$

$$\langle \nabla_x W(x,z) - \nabla_x W(y,z), x-y\rangle \geq c_W|x-y|^2 - c_2|x-y|\mathbb{1}_{[|x-y|\leq R]}; \tag{3.2.13}$$

for all $x, y \in \mathbb{R}^d$, and $c_V + c_W > 0$, then we have for any $r > 0$,

$$\begin{aligned} b_0(r) &= \sup_{|x-y|=r,z} \langle \frac{x-y}{|x-y|}, -[(\nabla V(x) - \nabla V(y)) + (\nabla_x W(x,z) - \nabla_x W(y,z))]\rangle \\ &\leq -(c_V + c_W)r + (c_1+c_2)\mathbb{1}_{[r\leq R]} \end{aligned}$$

which implies that

$$\begin{aligned} c_{Lip,m} &\le \frac{1}{4}\int_0^\infty \exp\left\{\frac{1}{4}\int_0^s [-(c_V+c_W)u+(c_1+c_2)\mathbb{1}_{[0,R]}(u)]du\right\} s ds \\ &\le \frac{1}{4}\int_0^\infty \exp\left\{-\frac{1}{8}(c_V+c_W)s^2+\frac{1}{4}(c_1+c_2)R]\right\} s ds \\ &= \frac{1}{c_V+c_W}\exp\left(\frac{1}{4}(c_1+c_2)R\right). \end{aligned}$$

Example 3.1. (Curie-Weiss model) Let . This model is ferromagnetic or anti-ferromagnetic according to $K>0$ or $K<0$.

For this example, we find by elementary analysis

$$b_0(r) = -2V'(r/2) = -2\beta(r^3/8 - r/2),\ r>0.$$

then

$$\begin{aligned} c_{Lip,m} &= \frac{1}{4}\int_0^\infty \exp\left\{\frac{\beta}{4}\int_0^s (r-\frac{r^3}{4})dr\right\} s ds \\ &= \frac{1}{4}\int_0^\infty \exp\left\{\frac{\beta}{4}(\frac{s^2}{2}-\frac{s^4}{16})\right\} s ds \\ &= e^{\beta/4}\int_0^\infty e^{-\beta(1/2-u)^2} du \le \frac{\sqrt{\pi}}{\sqrt{\beta}}e^{\beta/4} \end{aligned}$$

Let $\lambda(\beta)=\frac{1}{c_{Lip,m}}$. By Theorem 3.1, if there exists $h>-\lambda(\beta)$ such that

$$-\frac{\beta K}{N-1}(\mathbb{1}_{i\neq j}) \ge h I_N$$

then $\lambda_1(\mu^{(N)}) \ge h+\lambda(\beta)$. Note that $(\mathbb{1}_{i\neq j})$ has two eigenvalues, $N-1$ and -1. Hence

$$-\frac{\beta K}{N-1}(\mathbb{1}_{i\neq j}) \ge \begin{cases} \frac{\beta K}{N-1} I_N, & \text{if } K<0, \\ -\beta K I_N, & \text{if } K>0 \end{cases}$$

so taking

$$h = \begin{cases} \frac{\beta K}{N-1}, & \text{if } K<0, \\ -\beta K, & \text{if } K>0 \end{cases}$$

we get by Theorem 3.1,

$$\lambda_1(\mu^{(N)}) \ge \begin{cases} \frac{\sqrt{\beta}}{\sqrt{\pi}}e^{-\beta/4}+\frac{\beta K}{N-1}, & \text{if } K<0, \\ \frac{\sqrt{\beta}}{\sqrt{\pi}}e^{-\beta/4}-\beta K, & \text{if } K>0. \end{cases} \tag{3.2.14}$$

(It holds automatically if the right hand side above is ≤ 0.)

In particular in the anti-ferromagnetic case (i.e. $K<0$), for any $\varepsilon>0$ small enough, $\lambda_1(\mu^{(N)}) \ge \pi^{-1/2}\beta^{1/2}e^{-\beta/4}-\varepsilon>0$ when the number N of particles is big enough : the mean field should have no phase transition.

Corollary 3.5. *Assume that* $W(x,y) = W_0(x-y)$ *where* $W_0 : \mathbb{R}^d \to \mathbb{R}$ *is* C^2*, even. If*

(1) ∇V *is dissipative at infinity in the sense of (3.2.12), and*

(2) The Hessian matrix $\mathrm{Hess} W_0$ *of* W_0 *is bounded from below and from above :*

$$c_W I_d \le \mathrm{Hess} W_0 \le C_W I_d \tag{3.2.15}$$

and

$$c_W + c_V > 0.$$

Then for all $N \ge 2$,

$$\lambda_1(\mu^{(N)}) \ge \lambda_{1,m} - \frac{N}{N-1} c_W^- - C_W \tag{3.2.16}$$

where c_W^- *stands for the negative part of* c_W.

Remark 3.6. Let us see what the Bakry-Emery Γ_2-criterion yields. If $\nabla^2 W_0 \ge c_W I_d$ and $\nabla^2 V \ge c_V I_d$, by following the proof of the corollary above, we have $\nabla^2 H \ge (c_V - \frac{N}{N-1} c_W^-) I_{dN}$. Thus by the Bakry-Emery Γ_2-criterion,

$$\lambda_1(\mu^{(N)}) \ge \rho_{LS}(\mu^{(N)}) \ge c_V - \frac{N}{N-1} c_W^-$$

where $\rho_{LS}(\mu^{(N)})$ is the log-Sobolev constant, given in the next subsection.

Remark 3.7. We notice that if V is super-convex at infinity (i.e. the minimal eigenvalue of $\nabla^2 V(x)$ tends to $+\infty$ when $|x| \to \infty$), then c_V can be taken arbitrarily large, so the condition $c_W + c_V > 0$ is always satisfied. In particular, if $W_0(x) = \frac{c_W}{2}|x|^2$ with $c_W < 0$ (then concave and $C_W = c_W$), the uniform Poincaré inequality will hold for all big N by (3.2.16) since, in this case,

$$\lambda_{1,m} - \frac{N}{N-1} c_W^- - C_W = \lambda_{1,m} + \frac{1}{N-1} c_W.$$

This phenomenon, apparently strange, can be intuitively explained as follows. The confinement potential, being super-convex, pushes strongly all particles towards some bounded domain ; and the interaction potential W_0, being concave, pushes every particle far away from others. This creates an equilibrium : the meaning of our spectral gap estimate (3.2.16) for the concave potential W_0.

We now present an example for which some much better estimates (than those in Corollary 3.5) can be obtained.

Example 3.2. Let $W(x,y) = W_0(x-y)$ where

$$W_0(x) = \int_{\mathbb{R}^d} e^{-\sqrt{-1}\langle x,y\rangle} d\nu(y) + \frac{c}{2}|x|^2$$

where ν is some bounded symmetric (i.e. $\nu(-A) = \nu(A)$ for any Borel subset A of $\mathbb{R}^d$) positive measure on $\mathbb{R}^d$ with finite second moment. Let $\Gamma_\nu = (\int y_k y_l d\nu(y))_{1\le k,l\le d}$ be the covariance matrix of ν, and $\lambda_{\max}(\Gamma_\nu)$ (resp. $\lambda_{\min}(\Gamma_\nu)$) its maximal (resp. minimal) eigenvalue.
In §4, we will show the following result :

$$\lambda_1(\mu^{(N)}) \ge \lambda_{1,m} + \frac{1}{N-1}\left(\min\{c, -c(N-1)\} - \lambda_{\max}(\Gamma_\nu)\right). \tag{3.2.17}$$

In particular, if $c \le 0$, this implies that the spectral gap of $\mu^{(N)}$ is always uniformly lower bounded.

3.2.3 Uniform log-Sobolev inequality for the mean field $\mu^{(N)}$

Recall that for some nonnegative function $f \in L\log L(\mu)$, its entropy w.r.t. the probability measure μ is defined by

$$\mathrm{Ent}_\mu(f) := \int f \log f d\mu - \mu(f)\log\mu(f),\ \mu(f) := \int f d\mu.$$

Theorem 3.8. *Assume that*

(1) for some best constant $\rho_{LS,m} > 0$*, the conditional marginal distributions* $\mu_i := \mu_i(dx_i|x^{\hat{i}})$ *on* $\mathbb{R}^d$ *satisfy the log-Sobolev inequality :*

$$\rho_{LS,m}\mathrm{Ent}_{\mu_i}(f^2) \le 2\int |\nabla f|^2 d\mu_i,\ f \in C_b^1(\mathbb{R}^d) \tag{3.2.18}$$

for all i and $x^{\hat{i}}$ *;*

(2) (a translation of Zegarlinski's condition)

$$\gamma_0 = c_{Lip,m} \sup_{x,y\in\mathbb{R}^d,|z|=1} |\nabla^2_{x,y}W(x,y)z| < 1. \tag{3.2.19}$$

then $\mu^{(N)}$ *satisfies*

$$\rho_{LS,m}(1-\gamma_0)^2\mathrm{Ent}_{\mu^{(N)}}(f^2) \le 2\int_{(\mathbb{R}^d)^N} |\nabla f|^2 d\mu^{(N)},\ f \in C_b^1((\mathbb{R}^d)^N)$$

i.e. the log-Sobolev constant of $\mu^{(N)}$ *verifies*

$$\rho_{LS}(\mu^{(N)}) \ge \rho_{LS,m}(1-\gamma_0)^2. \tag{3.2.20}$$

Remark 3.9. In this remark we present one approach to establish the first assumption. Suppose that V is super-convex in the sense that for any $K > 0$ there exists $R > 0$ such that

$$\nabla^2 V(x) \ge K I_d,\ \text{for } |x| \ge R,$$

and suppose that

$$\nabla_x^2 W(x,y) \ge -K_0 I_d,\ \text{for all } x, y,$$

then V can be decomposed as the sum of a uniform convex function V_c and a bounded function V_b such that

$$\nabla^2 V_c \ge (K_1 + K_0) I_d$$

for some constant $K_1 > 0$. Therefore, thanks to Bakry-Emery criterion, the probability measure

$$\frac{1}{\tilde{Z}}\exp\left(-V_c(x_i) - \frac{1}{N-1}\sum_{j:j\neq i} W(x_i,x_j)\right)dx_i$$

satisfies a log Sobolev inequality with constant K_1. By the bounded perturbation theorem, the conditional measures $\mu_i = \mu_i(\cdot|x^{\hat{i}}), i = 1,\cdots,N$, satisfy a log Sobolev inequality with a uniform constant

$$\rho_{LS,m} \ge K_1\exp(-(\sup V_b - \inf V_b))$$

which does not depend on i, x, N.

Example 3.3. Let us go back to the Curie-Weiss example in dimension 1: $d = 1$, $V(x) = \beta(x^4/4 - x^2/2)$, $W(x, y) = -\beta K x y$ where $\beta > 0$. As given before we have

$$c_{Lip,m} \le \sqrt{\frac{\pi}{\beta}} e^{\beta/4}.$$

So that

$$\gamma_0 \le c_{Lip,m} \|\nabla^2_{x,y} W\|_\infty \le \sqrt{\pi\beta} e^{\beta/4} |K|$$

which will be smaller than 1 if β or K is sufficiently small.

3.2.4 Exponential convergence of McKean-Vlasov equation in entropy and in the Wasserstein metric W_2

We present now an application of the uniform log-Sobolev inequality in Theorem 3.8 to the non-linear McKean-Vlasov equation.

Recall at first the relative entropy of a probability measure ν w.r.t. the given probability measure μ on $\mathbb{R}^d$:

$$H(\nu|\mu) := \begin{cases} \int f \log f d\mu = \mathrm{Ent}_\mu(f), & \text{if } \nu \ll \mu, f := \frac{d\nu}{d\mu} \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.2.21}$$

The L^p-Wasserstein distance $W_p(\nu, \mu)$ is defined by

$$W_p(\mu, \nu) = \inf_{(X,Y)} \left(\mathbb{E}|X - Y|^p\right)^{1/p}$$

where the infimum is taken over all couples (X, Y) of random variables defined on some probability space, such that the laws of X, Y are respectively μ, ν (a such couple as well as their joint law is called a *coupling of* (μ, ν). Recall that the space $\mathbb{M}_1^p(\mathbb{R}^d)$ of probability measures with finite p-moment, equipped with L^p-Wasserstein distance W_p, is complete and separable (Villani [27]).

The Fisher-Donsker-Varadhan's information of ν w.r.t. μ is defined by

$$I(\nu|\mu) := \begin{cases} \int |\nabla \sqrt{f}|^2 d\mu, & \text{if } \nu \ll \mu, \sqrt{f} := \sqrt{\frac{d\nu}{d\mu}} \in H^1_\mu \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.2.22}$$

where H^1_μ is the domain of the Dirichlet form $\mathcal{E}_\mu[g] = \int |\nabla g|^2 d\mu$ (well defined if μ has C^1-density w.r.t. dx). Recall that the log-Sobolev inequality for $\mu^{(N)}$ can be rewritten in

$$\rho_{LS} H(\nu|\mu^{(N)}) \le 2I(\nu|\mu^{(N)}), \ \nu \in \mathbb{M}_1((\mathbb{R}^d)^N). \tag{3.2.23}$$

What replaces the role of the relative entropy in linear interacting particle system for the nonlinear McKean-Vlasov equation is the free energy of a probability measure ν on $\mathbb{R}^d$:

$$E_f(\nu) := \begin{cases} H(\nu|\alpha) + \frac{1}{2} \iint W(x, y) d\nu(x) d\nu(y), & \text{if } H(\nu|\alpha) < +\infty \\ +\infty & \text{otherwise} \end{cases} \tag{3.2.24}$$

or more precisely the corresponding mean-field entropy

$$H_W(\nu) := E_f(\nu) - \inf_{\tilde{\nu}\in\mathbb{M}_1(\mathbb{R}^d)} E_f(\tilde{\nu}). \tag{3.2.25}$$

And the substituter of the Fisher-Donsker-Varadhan information is: if $\nu = f(x)dx, \int |x|^2 d\nu(x) < +\infty$ and $\nabla f \in L^1_{loc}(\mathbb{R}^d)$ in the distribution sense,

$$I_W(\nu) := \frac{1}{4}\int \left| \frac{\nabla f(x)}{f(x)} + \nabla V(x) + (\nabla_x W \circledast \nu)(x) \right|^2 d\nu(x), \tag{3.2.26}$$

and $+\infty$ otherwise. Those two objects appeared both in Carrillo-McCann-Villani [10]. The following result generalizes the main result of [10] from the convex framework to the more general non-convex case.

Theorem 3.10. *Assume the uniform marginal log-Sobolev inequality, i.e.* (3.2.18) *with* $\rho_{LS,m} > 0$, *and the uniqueness condition of Zegarlinski (3.2.19). Then*

(1) There exists a unique minimizer ν_∞ *of* H_W *over* $\mathbb{M}_1(\mathbb{R}^d)$ *;*

(2) The following (nonlinear) log-Sobolev inequality

$$\rho_{LS} H_W(\nu) \le 2 I_W(\nu), \ \nu \in \mathbb{M}_1(\mathbb{R}^d) \tag{3.2.27}$$

holds, where

$$\rho_{LS} := \limsup_{N\to\infty} \rho_{LS}(\mu^{(N)}) \ge \rho_{LS,m}(1-\gamma_0)^2.$$

(3) The following Talagrand's transportation inequality holds

$$\rho_{LS} W_2^2(\nu, \nu_\infty) \le 2 H_W(\nu), \ \nu \in \mathbb{M}_1(\mathbb{R}^d). \tag{3.2.28}$$

(4) For the solution ν_t *of the McKean-Vlasov equation with the given initial distribution* ν_0 *of finite second moment,*

$$H_W(\nu_t) \le e^{-t\cdot\rho_{LS}/2} H_W(\nu_0), \ t \ge 0 \tag{3.2.29}$$

and in particular

$$W_2^2(\nu_t, \nu_\infty) \le \frac{2}{\rho_{LS}} e^{-t\cdot\rho_{LS}/2} H_W(\nu_0), \ t \ge 0. \tag{3.2.30}$$

Remark 3.11. In the work by Carrillo-McCann-Villani [10], presuming the presence of confining potential, such results were obtained in the case where $W(x,y) = W_0(x-y)$ and

(a) either $\nabla^2 V > ||(\nabla^2 W)^-||_{L^\infty}$ (in particular, V is uniformly strictly convex);

(b) or W is strictly convex at infinity, and both V and W are strictly convex (possibly degenerate at the origin).

In particular, V was required to be convex in both situations. If we consider the case in dimension one, $V(x) = \beta(x^4/4 - x^2/2)$ and $W_0(x) = -\beta K x^2/2$ with $K \geq 0$. Then by analogous calculations than for the Curie-Weiss model, we have $c_{Lip,m} \leq \sqrt{\pi/\beta} e^{\beta(1+K)^2/4}$ so that $\gamma_0 \leq \sqrt{\pi\beta} K e^{\beta(1+K)^2/4}$ and thus the conditions (3.2.18), (3.2.19) are verified for β or K small enough for example, cases not covered in [10]. Our conditions are quite comparable with the results obtained in [16] but they only consider convergence in L^1-Wasserstein distance. Remark also that the conditions are comparable to the assumptions made in [14] to get an uniform in time propagation of chaos (but in L^1-Wasserstein distance) which explains why we may pass to the limit in the number of particles.

3.3 Lipschitzian spectral gap for conditional distribution

Notice that the conditional distribution $\mu_i(dx_i) := \mu_i(dx_i | x_j, j \neq i)$ of x_i knowing $x^{\hat{i}} := (x_j)_{j\neq i}$ of our mean field measure $\mu^{(N)}$ defined in (3.1.5) is given by

$$d\mu_i(x_i) = \frac{1}{Z_i} \exp\left\{ -V(x_i) - \frac{1}{N-1} \sum_{j:\, j\neq i} W(x_i, x_j) \right\} dx_i$$

where $Z_i = Z_i(x^{\hat{i}})$ is the normalization factor. Let

$$H_i(x_i) := V(x_i) + \frac{1}{N-1} \sum_{j:\, j\neq i} W(x_i, x_j)$$

be the potentiel associated with μ_i. The generator $\mathscr{L}_i^{(N)} = \Delta_i - \nabla_i H_i \cdot \nabla_i$ given in (3.1.4), with $(x_j)_{j\neq i}$ fixed, is symmetric w.r.t. μ_i. By the definition (3.2.3) of $b_0(r)$, for all $x, y \in (\mathbb{R}^d)^N$,

$$\begin{aligned}
&\langle \frac{x_i - y_i}{|x_i - y_i|}, -[\nabla_i H_i(x) - \nabla_i H_i(x^{\hat{i}, y_i})] \rangle \\
&= \frac{1}{N-1} \sum_{j\neq i} \langle \frac{x_i - y_i}{|x_i - y_i|}, -[(\nabla V(x_i) + \nabla_x W(x_i, x_j)) - (\nabla V(y_i) + \nabla_x W(y_i, x_j)] \rangle \\
&\leq b_0(|x_i - y_i|)
\end{aligned}$$

where $x^{\hat{i}, y_i} \in (\mathbb{R}^d)^N$ is given by $(x^{\hat{i}, y_i})_j = x_j, j \neq i$, $(x^{\hat{i}, y_i})_i = y_i$. So we have the following result (due to the third named author [28]), which is the starting point of our investigation.

Lemma 3.12. *Assume (3.2.2). Then the Poisson operator $(-\mathscr{L}_i)^{-1}$ on the Banach space $C_{\mathrm{Lip},0}(\mathbb{R}^d)$ of Lipschitzian continuous functions f on $\mathbb{R}^d$ with $\mu_i(f) = 0$, equipped with the norm $\|f\|_{\mathrm{Lip}}$, is bounded and its norm*

$$||(-\mathscr{L}_i)^{-1}||_{Lip} \leq c_{Lip,m} \tag{3.3.1}$$

where $c_{Lip,m}$ is given in (3.2.2). In particular the spectral gap $\lambda_1(\mu_i)$ of $\mathscr{L}_i$ on $L^2(\mu_i)$ satisfies

$$\lambda_1(\mu_i) \geq \frac{1}{c_{Lip,m}}. \tag{3.3.2}$$

For the completeness and the convenience of the reader, a sketch of proof is given in the appendix.

3.4 Uniform Poincaré inequality : proof of Theorem 3.1

Let $V \in C^2(\mathbb{R}^d)$ be the confinement potential, U a C^2-potential of interaction on $(\mathbb{R}^d)^N$ and let $H(x_1, \cdots, x_N) = \sum_{i=1}^N V(x_i) + U(x_1, \cdots, x_N)$ be the Hamiltonian. Now consider the probability measure

$$d\mu := \frac{1}{Z} e^{-H} dx_1 \cdots dx_N$$

where $Z = \int e^{-H} dx$ is the normalization constant (called often *partition function*), assumed to be finite. We denote by $\mu_i = \mu(dx_i | x^{\hat{i}})$ the conditional distribution of x_i given $x^{\hat{i}}$ under μ, where

$$x^{\hat{i}} := (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N).$$

It is given by

$$\mu_i(\mathrm{d}x_i) = \frac{1}{Z_i} e^{-U(x) - V(x_i)} \mathrm{d}x_i, \; Z_i = Z_i(x^{\hat{i}}) := \int e^{-U(x) - V(x_i)} \mathrm{d}x_i < +\infty (\text{assumed}).$$

We shall describe below conditions on the Hamiltonian H such that μ satisfies a Poincaré inequality, namely for some positive constant ρ,

$$\rho \int f^2 \mathrm{d}\mu \leq \int |\nabla f|^2 \mathrm{d}\mu$$

for every smooth function $f \in C_b^1((\mathbb{R}^d)^N)$. The largest ρ is called the spectral gap of μ, denoted as $\lambda_1(\mu)$.

Proposition 3.13. *Assume that* $Z = \int_{(\mathbb{R}^d)^N} e^{-H} dx < +\infty, Z_i(x^{\hat{i}}) < +\infty$ *for all* $i, x^{\hat{i}}$. *If*

(1) the marginal conditional distributions μ_i *satisfy the uniform Poincaré inequality, i.e.*

$$\lambda_{1,m} := \inf_{1 \leq i \leq N, x^{\hat{i}} \in (\mathbb{R}^d)^{N-1}} \lambda_1(\mu_i) > 0, \tag{3.4.1}$$

(2) for some constant $h \in \mathbb{R}$,

$$(\mathbb{1}_{i \neq j} \nabla^2_{x_i, x_j} U) \geq h I_{dN}, \tag{3.4.2}$$

in the sense of nonnegative definiteness of symmetric matrices;

then

$$\lambda_1(\mu) \geq h + \lambda_{1,m}.$$

This result is essentially due to Ledoux [19]. Indeed, in the case of $d = 1$, if $\mathrm{Hess}(U) \geq \underline{\lambda} I_N$ and $\partial_{ii} U(x) \leq \bar{\lambda}$ for all i and $x^{\hat{i}}$, then for every $v = (v_1, v_2, \cdots, v_N) \in \mathbb{R}^N$,

$$\sum_{i \neq j} v_i \partial^2_{ij} U v_j = \langle \mathrm{Hess}(U) v, v \rangle - \sum_i v_i^2 \partial^2_{ii} U \geq (\underline{\lambda} - \bar{\lambda}) |v|^2$$

i.e. the assumption (3.4.2) holds with $h = \underline{\lambda} - \overline{\lambda}$. This proposition gives $\lambda_1(\mu) \geq \lambda_{1,m} + \underline{\lambda} - \overline{\lambda}$, which is the original result of Ledoux [19].

For the convenience of the reader, we reproduce the beautiful proof of Ledoux [19, Prop. 3.1].

Proof. Of course we may and will assume that $\lambda_{1,m}+h>0$. Let $\mathscr{L}=\Delta-\nabla H\cdot\nabla$ be the symmetric generator associated with the probability measure μ. By the dual description of Poincaré inequality [3, Prop. 4.8.3], the conclusion above is equivalent to

$$\int(\mathscr{L}f)^2 d\mu \geq (\lambda_{1,m}+h)\int|\nabla f|^2 d\mu.$$

Thanks to the Bakry-Emery's formulae $\int\Gamma_2(f)d\mu=\int(\mathscr{L}f)^2 d\mu$ and

$$\Gamma_2(f)=\|\nabla^2 f\|_{HS}^2+\langle\nabla^2 H\nabla f,\nabla f\rangle$$

where $||A||_{HS}:=(\sum_{i,j}|a_{ij}|^2)^{1/2}$ is the Hilbert-Schmidt norm of a matrix $A=(a_{ij})$, we have

$$\begin{aligned}
\int(\mathscr{L}f)^2 d\mu &= \int\left(||\nabla^2 f||_{HS}^2+\langle\nabla^2 H\nabla f,\nabla f\rangle\right)d\mu\\
&= \int\left(||\nabla^2 f||_{HS}^2+\sum_{i=1}^n\langle \mathrm{Hess}\,(V)(x_i)\nabla_{x_i}f,\nabla_{x_i}f\rangle+\langle\mathrm{Hess}\,(U)\nabla f,\nabla f\rangle\right)d\mu\\
&\geq \sum_{1\leq i\leq N}\int\int_{\mathbb{R}^d}\left(||\nabla_{x_i}^2 f||_{HS}^2+\langle(\mathrm{Hess}\,(V)(x_i)+\nabla_{x_i,x_i}^2 U)\nabla_{x_i}f,\nabla_{x_i}f\rangle\right)d\mu_i\,d\mu\\
&\quad+\int\sum_{i\neq j}\langle\nabla_{x_i,x_j}^2 U\nabla_{x_i}f,\nabla_{x_j}f\rangle d\mu
\end{aligned}$$

Applying the above characterization of the Poincaré inequality but to the conditional measures μ_i, we have

$$\int\left[||\nabla_{x_i}^2 f||_{HS}^2+\langle(\mathrm{Hess}\,(V)(x_i)+\nabla_{x_i,x_i}^2 U)\nabla_{x_i}f,\nabla_{x_i}f\rangle\right]d\mu_i\geq\lambda_{1,m}\int|\nabla_{x_i}f|^2 d\mu_i$$

for any i and any given $x^{\hat{i}}$. Moreover by the assumption (3.4.2),

$$\int\sum_{i\neq j}\langle\nabla_{x_ix_j}^2 U\nabla_{x_i}f,\nabla_{x_j}f\rangle d\mu\geq h\int|\nabla f|^2 d\mu.$$

This, combined with the previous inequality, yields the desired inequality

$$\int(\mathscr{L}f)^2 d\mu\geq(\lambda_{1,m}+h)\int|\nabla f|^2 d\mu.$$

□

We come back to the mean-field setting.

Proof of Theorem 3.1. We shall apply Proposition 3.13 to $\mu=\mu^{(N)}$. With the notations above, the interaction potential U is then given by

$$U(x)=\frac{1}{N-1}\sum_{1\leq i<j\leq N}W(x_i,x_j)=\frac{1}{2}\sum_{i=1}^N U_i(x) \tag{3.4.3}$$

where $U_i(x) = \frac{1}{N-1}\sum_{j:j\neq i} W(x_i, x_j)$. For $i \neq j$,

$$\nabla^2_{x_i,x_j} U = \frac{1}{N-1}(\nabla^2_{x,y} W)(x_i, x_j)$$

therefore the assumption (3.2.6) implies the condition (3.4.2) with constant h in Proposition 3.13. On the other hand, since $\mu_i(dx_i|x^{\hat{i}}) = e^{-[V(x_i)+U_i(x)]}dx_i/Z_i(x^{\hat{i}})$ and

$$-\langle \frac{x_i - y_i}{|x_i - y_i|}, \nabla_{x_i}[V(x_i) + U_i(x)] - \nabla_{x_i}[V(y_i) + U_i(x^{\hat{i},y_i})]\rangle \le b_0(|x_i - y_i|)$$

as noted in §3, thanks to the assumption (3.2.2), Lemma 3.12 yields $\lambda_1(\mu_i) \ge 1/c_{\mathrm{Lip,m}}$. Hence we can apply Proposition 3.13 to the invariant measure $\mu^{(N)}$, and obtain (3.2.8). □

Proof of Corollary 3.5. In this particular context $W(x, y) = W_0(x - y)$, for $U(x)$ given by (3.4.3),

$$\nabla^2_{x_i,x_i} U(x) = \frac{1}{N-1}\sum_{j\neq i}(\nabla^2 W_0)(x_i - x_j);\ \nabla^2_{x_i,x_j} U = -\frac{1}{N-1}(\nabla^2 W_0)(x_i - x_j) \text{ for } i \neq j$$

i.e. $\nabla^2 U = -\frac{1}{N-1}(A_{ij})$ where $A_{ij} = (\nabla^2 W_0)(x_i - x_j)$ for $i \neq j$ and $A_{ii} = -\sum_{j:j\neq i} A_{ij}$. As A_{ij} is symmetric and $A_{ij} = A_{ji}$, we have for any $u = (u_1, \cdots, u_N)$ in $(\mathbb{R}^d)^N$,

$$\begin{aligned}
-\sum_{i,j}\langle u_i, A_{ij}u_j\rangle &= \sum_{i\neq j}\langle -u_i, A_{ij}(u_j - u_i)\rangle = \sum_{i\neq j}\langle u_j, A_{ij}(u_j - u_i)\rangle \\
&= \frac{1}{2}\sum_{i\neq j}\langle (u_j - u_i), A_{ij}(u_j - u_i)\rangle \\
&\ge \frac{c_W}{2}\sum_{i\neq j}|u_j - u_i|^2 = c_W\sum_{i,j}\langle u_j, u_j - u_i\rangle \text{ (by the previous equality with } A_{ij} = I) \\
&= c_W N\left(|u|^2 - N|\bar{u}|^2\right) = c_W N|u - \bar{u}|^2 \\
&\ge \begin{cases} c_W N|u|^2, & \text{if } c_W \le 0. \\ 0 & \text{if } c_W > 0 \end{cases}
\end{aligned}$$

Therefore $\nabla^2 U \ge -c_W^- \frac{N}{N-1} I_{dN}$. Obviously $\nabla^2_{x_i,x_i} U \le C_W I_d$. Then

$$(1_{i\neq j}\nabla^2_{x_i,x_j} U) = \nabla^2 U - (1_{i=j}\nabla^2_{x_i,x_i} U) \ge -\left(c_W^- \frac{N}{N-1} + C_W\right) I_{dN}$$

It remains to apply Proposition 3.13 to get the desired spectral gap estimate (3.2.16). □

Proof of (3.2.17) in Example 2. Notice that

$$\begin{aligned}
(\mathbb{1}_{i\neq j}\nabla^2_{x_i,x_j} W(x_i, x_j)) &= \left(\mathbb{1}_{i\neq j}[-cI_d + \int e^{-\sqrt{-1}(x_i - x_j)\cdot y} yy^\top d\nu(y)]\right) \\
&= -c(\mathbb{1}_{i\neq j} I_d) + \left(\int e^{-\sqrt{-1}(x_i - x_j)\cdot y} yy^\top d\nu(y)\right) - (\mathbb{1}_{i=j}\int yy^\top d\nu(y)) \\
&\ge cP_{\mathbf{H}} - c(N-1)P_{\mathbf{H}^\perp} - \lambda_{\max}(\Gamma_\nu) I_{dN}
\end{aligned}$$

where the second expression in the second line is a positive-definite matrix since it holds for all $u = (u_1, u_2, \cdots, u_N) \in (\mathbb{R}^d)^N$ that

$$\left\langle u\Big(\int e^{-\sqrt{-1}(x_i-x_j)\cdot y} y^T y d\nu(y)\Big)_{i,j}, u\right\rangle = \sum_{i,j} \int \langle u_i, y\rangle\langle y, u_j\rangle e^{-\sqrt{-1}(x_i-x_j)\cdot y} d\nu(y)$$
$$= \int |\sum \langle u_i, y\rangle e^{-x_i\cdot y}|^2 d\nu(y) \geq 0;$$

and $P_{\mathbf{H}}, P_{\mathbf{H}^\perp}$ are respectively the orthogonal projection from $(\mathbb{R}^d)^N$ to $\mathbf{H}$ and to its orthogonal complement $\mathbf{H}^\perp$,

$$\mathbf{H} = \{x = (x_1, \cdots, x_N); \bar{x} := \frac{1}{N}\sum_{i=1}^{N} x_i = 0\},$$
$$\mathbf{H}^\perp = \{x = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N; x_1 = x_2 = \cdots = x_N\}.$$

Thus we obtain from Theorem 3.1

$$\lambda_1(\mu^{(N)}) \geq \lambda_{1,m} + \frac{1}{N-1}(\min\{c, -c(N-1)\} - \lambda_{\max}(\Gamma_\nu))$$

which is the desired inequality (3.2.17). □

3.5 Uniform log-Sobolev inequality

Inspired by Dobrushin's uniqueness condition for the Gibbs measures, Zegarlinski [32, Theorem 0.1] proved a criterion about the logarithmic Sobolev inequality for the Gibbs measure $\mu = e^{-H}dx/Z$ on $(\mathbb{R}^d)^N$ in terms of the conditional marginal distributions $\mu_i = \mu(dx_i|x^{\hat{i}})$.
Let us introduce at first Zegarlinski's dependence coefficient $c^{\mathbf{Z}}_{ij}$ of μ_j upon x_i : this is the best nonnegative constant such that

$$|\nabla_i(\mu_j(f^2))^{1/2}| \leq (\mu_j(|\nabla_i f|^2))^{1/2} + c^{\mathbf{Z}}_{ij}(\mu_j(|\nabla_j f|^2))^{1/2} \tag{3.5.1}$$

for all smooth strictly positive functions $f(x_1, \cdots, x_N)$. Obviously $c^{\mathbf{Z}}_{ii} = 0$. The matrix $c^{\mathbf{Z}} := (c^{\mathbf{Z}}_{ij})_{1\leq i,j\leq N}$ will be called Zegarlinski's matrix of interdependence in the sequel.

Theorem 3.14. **(Zegarlinski [32, Theorem 0.1])** *If*

(1) μ_i *satisfies a uniform log-Sobolev inequality (LSI in short), i.e.*

$$\rho_{\mathrm{LS,m}} := \inf_{i,x^i} \rho_{\mathrm{LS}}(\mu_i) > 0.$$

(2) The following ***Zegarlinski's condition*** *is verified*

$$\gamma := \sup_i \max\Big(\sum_j c^{\mathbf{Z}}_{ji}, \sum_j c^{\mathbf{Z}}_{ij}\Big) < 1. \tag{3.5.2}$$

Then the Gibbs measure μ *satisfies the logarithmic Sobolev inequality*

$$\rho_{\mathrm{LS,m}}(1-\gamma)^2\,\mathrm{Ent}_\mu(f^2) \le 2\mu(|\nabla f|^2) \tag{3.5.3}$$

for all smooth bounded functions f *on* $(\mathbb{R}^d)^{\mathrm{N}}$, *i.e.*

$$\rho_{\mathrm{LS}}(\mu) \ge \rho_{\mathrm{LS,m}}(1-\gamma)^2.$$

Our objective is to estimate $c^{\mathbf{Z}}_{ij}$. We begin with a simple observation :

Lemma 3.15. *If for any function* $g = g(x_j) \in \mathrm{C}^1_b(\mathbb{R}^d)$ *on the single particle* x_j,

$$|\nabla_i \mu_j(g)| \le c_{ij}\mu_j(|\nabla g|), \tag{3.5.4}$$

then $c^{\mathbf{Z}}_{ij} \le c_{ij}$.

Proof. For any $0 < g \in \mathrm{C}^1_b((\mathbb{R}^d)^{\mathrm{N}})$, by the condition (3.5.4), we have for all $i \neq j$,

$$\begin{aligned}
|\nabla_i \sqrt{\mu_j(g)}| &= \frac{1}{2\sqrt{\mu_j(g)}}\cdot\left|\mu_j(\nabla_i g) + (\nabla_{x_i}\int g(x_j, y^{\hat{j}})\mathrm{d}\mu_j(x_j|x^{\hat{j}}))|_{y^{\hat{j}}=x^{\hat{j}}}\right| \\
&\le \frac{1}{2\sqrt{\mu_j(g)}}\left[\mu_j(|\nabla_i g|) + c_{ij}\mu_j(|\nabla_j g|)\right].
\end{aligned}$$

When $g = f^2$ with $f > 0$, we have by the Cauchy-Schwarz inequality for all i, j,

$$\mu_j(|\nabla_i g|) = 2\mu_j(f|\nabla_i f|) \le 2\sqrt{\mu_j(f^2)\mu_j(|\nabla_i f|^2)}.$$

Substituting it into the previous inequality we get

$$|\nabla_i\sqrt{\mu_j(f^2)}| \le \sqrt{\mu_j(|\nabla_i f|^2)} + c_{ij}\sqrt{\mu_j(|\nabla_j f|^2)}$$

so it follows $c^{\mathbf{Z}}_{ij} \le c_{ij}$. □

Lemma 3.16. *For the mean field Gibbs measure* $\mu = \mu^{(\mathrm{N})}$, *the interdependence coefficient* $c^{\mathbf{Z}}_{ji}$ *satisfies*

$$c^{\mathbf{Z}}_{ji} \le \frac{1}{\mathrm{N}-1}c_{\mathrm{Lip,m}}\|\nabla^2_{x,y}\mathrm{W}\|_\infty,\ i \neq j$$

where $c_{\mathrm{Lip,m}}$ *is given by (3.2.2),*

$$\|\nabla^2_{x,y}\mathrm{W}\|_\infty := \sup_{x,y\in\mathbb{R}^d}\ \sup_{z\in\mathbb{R}^d,|z|=1}|\nabla^2_{x,y}\mathrm{W}(x,y)z|.$$

Proof. For any $z \in \mathbb{R}^d$ with $|z| = 1$ and $g = g(x_i) \in \mathrm{C}^2_0(\mathbb{R}^d)$,

$$\begin{aligned}
\nabla_{x_j}\mu_i(g) &= \nabla_{x_j}\Big(\int g(x_i)e^{-\mathrm{H}(x_1,x_2,\cdots,x_\mathrm{N})}dx_i / \int e^{-\mathrm{H}(x_1,x_2,\cdots,x_\mathrm{N})}dx_i\Big) \\
&= \frac{\int g(x_i)(-\nabla_{x_j}\mathrm{H})e^{-\mathrm{H}}dx_i}{\int e^{-\mathrm{H}}dx_i} + \frac{\int g(x_i)e^{-\mathrm{H}}dx_i \int \nabla_{x_j}\mathrm{H}e^{-\mathrm{H}}dx_i}{(\int e^{-\mathrm{H}}dx_i)^2} \\
&= -\int g(x_i)\nabla_{x_j}\mathrm{H}d\mu_i + \int g(x_i)d\mu_i\int\nabla_{x_j}\mathrm{H}d\mu_i \\
&= \mathrm{Cov}_{\mu_i}(g, -\nabla_{x_j}\mathrm{H}) = \mathrm{Cov}_{\mu_i}(g, -\frac{1}{\mathrm{N}-1}(\nabla_y\mathrm{W})(x_i,x_j))
\end{aligned}$$

and so

$$
\begin{aligned}
z\cdot\nabla_{x_j}\mu_i(g) &= \mathrm{Cov}_{\mu_i}(g, -\frac{1}{N-1}(\nabla_y W)(x_i,x_j)\cdot z)\\
&= -\frac{1}{N-1}\langle(-\mathscr{L}_i)g, (-\mathscr{L}_i)^{-1}((\nabla_y W)(\cdot,x_j)\cdot z-\mu_i((\nabla_y W)(\cdot,x_j)\cdot z)\rangle_{\mu_i}\\
&= -\frac{1}{N-1}\int\nabla_i g\cdot\nabla_i(-\mathscr{L}_i)^{-1}[(\nabla_y W)(\cdot,x_j)\cdot z-\mu_i((\nabla_y W)(\cdot,x_j)\cdot z)]\,d\mu_i.
\end{aligned}
$$

By Lemma 3.12,

$$
\begin{aligned}
&\|\nabla_i(-\mathscr{L}_i)^{-1}((\nabla_y W)(\cdot,x_j)\cdot z-\mu_i((\nabla_y W)(\cdot,x_j)\cdot z))\|_{L^\infty(\mu_i)}\\
&\le c_{\mathrm{Lip,m}}\sup_{x_i,x_j}|\nabla_{x_i}((\nabla_y W)(x_i,x_j)\cdot z)|\\
&= c_{\mathrm{Lip,m}}\sup_{x,y\in\mathbb{R}^d}|\nabla^2_{x,y}W(x,y)z|\\
&\le c_{\mathrm{Lip,m}}\|\nabla^2_{x,y}W\|_\infty.
\end{aligned}
$$

Plugging it into the previous inequality, we obtain

$$
|\nabla_{x_j}\mu_i(g)| = \sup_{|z|=1}|z\cdot\nabla_{x_j}\mu_i(g)| \le \frac{1}{N-1}c_{\mathrm{Lip,m}}\|\nabla^2_{x,y}W\|_\infty|\mu_i(\nabla_{x_i}g)|
$$

which, by Lemma 3.15, completes the proof. □

Proof of Theorem 3.8. By Lemma 3.16,

$$
\gamma = \sup_{1\le i\le N}\max\{\sum_{1\le j\le N}c^{\mathbf{Z}}_{ji}, \sum_{1\le j\le N}c^{\mathbf{Z}}_{ij}\} \le c_{\mathrm{Lip,m}}\|\nabla^2_{x,y}W\|_\infty = \gamma_0 < 1.
$$

Then Theorem 3.8 follows directly from Theorem 3.14. □

3.6 Exponential convergence of McKean-Vlasov equation

Assume that $\mu^{(N)}$ satisfies a uniform log-Sobolev inequality with constant

$$
\rho_{LS} = \limsup_{N\to\infty}\rho_{LS}(\mu^{(N)}) > 0.
$$

That is the case if $c_{\mathrm{Lip,m}}\|\nabla^2_{x,y}W\|_\infty < 1$ by Theorem 3.8,

$$
\rho_{LS} \ge \rho_{LS,m}(1-c_{\mathrm{Lip,m}}\|\nabla^2_{x,y}W\|_\infty)^2.
$$

3.6.1 Free energy, entropy related to the McKean-Vlasov equation

The entropy $H_W(\nu)$ can be identified as the mean relative entropy per particle of $\nu^{\otimes N}$ w.r.t. the mean field Gibbs measure $\mu^{(N)}$:

Lemma 3.17. *For any probability measure ν on $\mathbb{R}^d$ such that $H(\nu|\alpha) < +\infty$,*

$$
\frac{1}{N}H(\nu^{\otimes N}|\mu^{(N)}) \to H_W(\nu). \tag{3.6.1}
$$

Proof. Recall that $\alpha = \frac{1}{C} e^{-V} dx$. By (H1), it is known that ([12])

$$\int e^{\lambda_0 |x|^2} d\alpha(x) < +\infty \text{ for some } \lambda_0 > 0. \tag{3.6.2}$$

Let

$$\tilde{Z}_N := \int \exp\left(-\frac{1}{2(N-1)} \sum_{i \neq j} W(x_i, x_j)\right) d\alpha^{\otimes N}$$

so that

$$d\mu^{(N)} = \frac{1}{\tilde{Z}_N} \exp\left(-\frac{1}{2(N-1)} \sum_{i \neq j} W(x_i, x_j)\right) d\alpha^{\otimes N}.$$

Let $\nu \in \mathbb{M}_1(\mathbb{R}^d)$ such that $H(\nu|\alpha) < +\infty$. Since $H(\nu^{\otimes 2}|\alpha^{\otimes 2}) = 2H(\nu|\alpha) < +\infty$, by Donsker-Varadhan's variational formula of entropy, (3.6.2) and the fact that $|W(x, y)| \le C(1 + |x|^2 + |y|^2)$ (for $\nabla^2 W$ is bounded), we have $W \in L^1(\nu^{\otimes 2})$. Therefore

$$\begin{aligned}
\frac{1}{N} H(\nu^{\otimes N}|\mu^{(N)}) &= \frac{1}{N} \int \log \frac{d\nu^{\otimes N}}{d\mu^{(N)}} d\nu^{\otimes N} \\
&= \frac{1}{N} \int \sum_{i=1}^{N} \log \frac{d\nu}{d\alpha}(x_i) d\nu^{\otimes N} + \int \frac{1}{2N(N-1)} \sum_{i \neq j} W(x_i, x_j) d\nu^{\otimes N} + \frac{1}{N} \log \tilde{Z}_N \\
&= H(\nu|\alpha) + \frac{1}{2} \iint W(x, y) d\nu(x) d\nu(y) + \frac{1}{N} \log \tilde{Z}_N
\end{aligned}$$

By [29, (3.30)],

$$\lim_{N \to \infty} \frac{1}{N} \log \tilde{Z}_N = -\inf_{\nu} E_f(\nu).$$

Combining those two equalities we obtain (3.6.1). □

The following super-additivity of the relative entropy w.r.t. a product probability measure should be known.

Lemma 3.18. *Let $\prod_{i=1}^N \alpha_i, Q$ be respectively a product probability measure and a probability measure on $E_1 \times \cdots \times E_N$ where E_i's are Polish spaces, and Q^i the marginal distribution of x_i under Q. Then*

$$H(Q|\prod_{i=1}^N \alpha_i) \ge \sum_{i=1}^N H(Q^i|\alpha_i).$$

Proof. Let $Q_i(\cdot|x_{[1,i-1]})$ be the conditional distribution of x_i knowing $x_{[1,i-1]} = (x_1, \cdots, x_{i-1})$ (knowing nothing if $i = 1$). We have

$$\begin{aligned}
H(Q|\prod_{i=1}^N \alpha_i) &= \mathbb{E}^Q \log \frac{dQ}{d\prod_{i=1}^N \alpha_i} = \mathbb{E}^Q \sum_{i=1}^N \log \frac{Q_i(dx_i|x_{[1,i-1]})}{\alpha_i(dx_i)} \\
&= \mathbb{E}^Q \sum_{i=1}^n H(Q_i(\cdot|x_{[1,i-1]})|\alpha_i).
\end{aligned}$$

Since $\mathbb{E}^Q Q_i(\cdot|x_{[1,i-1]}) = Q^i(\cdot)$, we obtain by the convexity of the relative entropy

$$\mathbb{E}^Q H(Q_i(\cdot|x_{[1,i-1]})|\alpha_i) \geq H(Q^i|\alpha_i)$$

where the desired super-additivity follows. □

Lemma 3.19. *Let* μ *be a probability measure on some Polish space* S *and* $U : S \to (-\infty, +\infty]$ *a measurable potential satisfying*

$$\int e^{-pU} d\mu < +\infty$$

for some $p > 1$. *Consider the Boltzmann probability measure* $\mu_U = e^{-U} d\mu/C$. *If* $H(\nu|\mu_U) < +\infty$, *then* $H(\nu|\mu) < +\infty$ *and* $U \in L^1(\nu)$, *and*

$$H(\nu|\mu_U) = H(\nu|\mu) + \int U d\nu - \log \int e^{-U} d\mu.$$

Proof. For any measurable function f on S, let

$$\Lambda_\mu(f) := \log \int e^f d\mu \in (-\infty, +\infty]$$

be the log-Laplace transform w.r.t. μ, which is convex in f (by Hölder's inequality). Then

$$\Lambda_{\mu_U}(f) = \log \int e^f d\mu_U = \Lambda_\mu(-U + f) - \Lambda_\mu(-U) \leq \frac{1}{p}\Lambda_\mu(-pU) + \frac{1}{q}\Lambda_\mu(qf) - \Lambda_\mu(-U)$$

where $q = p/(p-1)$. By Donsker-Varadhan's variational formula,

$$\begin{aligned}
H(\nu|\mu_U) &= \sup_{f \in b\mathscr{B}} \left(\nu(f) - \Lambda_{\mu_U}(f)\right) \\
&\geq \sup_{f \in b\mathscr{B}} \left(\nu(f) - \frac{1}{q}\Lambda_\mu(qf)\right) + \Lambda_\mu(-U) - \frac{1}{p}\Lambda_\mu(-pU) \\
&= \frac{1}{q} H(\nu|\mu) + \Lambda_\mu(-U) - \frac{1}{p}\Lambda_\mu(-pU).
\end{aligned}$$

Hence if $H(\nu|\mu_U) < +\infty$, $H(\nu|\mu) < +\infty$ or equivalently $\log \frac{d\nu}{d\mu} \in L^1(\nu)$, and $\log \frac{d\nu}{d\mu_U} = \log \frac{d\nu}{d\mu} + U + \Lambda_\mu(-U) \in L^1(\nu)$. This completes the proof of the Lemma. □

Lemma 3.20. (propagation of chaos) *Let* $(\nu_t)_{t\geq 0}$ *be the solution of the McKean-Vlasov equation with the given initial distribution* ν_0 *such that* $\int |x|^2 d\nu_0(x) < +\infty$. *Let* μ_t^N *be the law of* $X^N(t) = (X_1^N(t), \cdots, X_N^N(t))$ *solving* (3.1.3) *such that* $\mu_0^N = \nu_0^{\otimes N}$, *and* $\mu_t^{N,I}$ *the law of the particles* $(X_i^N(t))_{i\in I}$ *for any index set* $I \subset \mathbb{N}^*$. *Then for each* $t \in \mathbb{R}$ *and each finite subset* I *of* $\mathbb{N}^*$, $\mu_t^{N,I} \to \nu_t^{\otimes I}$ *in the* L^2-*Wasserstein metric* W_2 *as* $N \to \infty$.

This is well known, see [26] or [11].

Lemma 3.21. (uniqueness of the minimizer of H_W**)** *If* $c_{Lip,m}\|\nabla_{xy}^2 W\|_\infty < 1$, *then the minimizer* ν_∞ *of the free energy* $E_f(\nu)$ *is unique.*

Proof. By [29], under (H2), if $H(\nu|\alpha) < +\infty$, $\iint W^-(x,y)d\nu(x)d\nu(y) < +\infty$; and $E_f : \mathbb{M}_1(\mathbb{R}^d) \to \mathbb{R}$ is inf-compact (i.e. the set $\{\nu \in \mathbb{M}_1(\mathbb{R}^d) : E_f(\nu) \le r\}$ is compact for any real number r). Then a minimizer ν_∞ of E_f exists.
If a probability measure ν is a minimizer of E_f, $H(\nu|\alpha) < +\infty$, and then $\int |x|^2 d\nu < +\infty$ by (H1). Regarding the Gateaux-derivative, we see that ν must be a fixed point of the mapping Φ defined by

$$\Phi(\nu) := \frac{1}{Z'}\exp(-V - W \circledast \nu)dx.$$

where Z' is the normalizing constant. Here $W \circledast \nu$ is well defined because $|W(x,y)| \le C(1+|x|^2+|y|^2)$ by the boundedness of the second derivatives of W.
We claim that $\Phi : \mathbb{M}_1^2(\mathbb{R}^d) \to \mathbb{M}_1^2(\mathbb{R}^d)$. Indeed, since the hamiltonian $H_\nu = V + W \circledast \nu$ (for any $\nu \in \mathbb{M}_1^2(\mathbb{R}^d)$) satisfies again the dissipative rate condition

$$-\langle \frac{x-y}{|x-y|}, \nabla H_\nu(x) - \nabla H_\nu(y)\rangle \le b_0(|x-y|),\ x,y \in \mathbb{R}^d$$

(as in §3), the associated generator $\mathscr{L}_\nu = \Delta - \nabla H_\nu \cdot \nabla$ satisfies the Lipschitzian spectral gap estimate (3.3.1) by Lemma 3.12. That implies a spectral gap inequality for $\nu' = \Phi(\nu)$, in particular $\int e^{\delta|x|}d\nu' < +\infty$ for some $\delta > 0$ ([5]). Then if $\nu \in \mathbb{M}_1^2(\mathbb{R}^d)$, $\Phi(\nu) \in \mathbb{M}_1^2(\mathbb{R}^d)$.
Now for the uniqueness of the minimizer of E_f, it suffices to show that Φ is contractive on $(\mathbb{M}_1^2(\mathbb{R}^d), W_1)$. Let $\mu_k = \Phi(\nu_k), k = 0,1$, and

$$\nu_t := (1-t)\nu_0 + t\nu_1,\ \mu_t = \Phi(\nu_t).$$

For any 1-Lipschitzian function f, we have

$$\begin{aligned}\frac{d}{dt}\mu_t(f) &= \mathrm{Cov}_{\mu_t}(f, -\partial_t(W \circledast \nu_t)) \\ &= \mathrm{Cov}_{\mu_t}(f, -W \circledast (\nu_1 - \nu_0))\end{aligned}$$

and

$$|\nabla_x[W \circledast (\nu_1 - \nu_0)]| = |(\nabla_x W) \circledast (\nu_1 - \nu_0)| \le \|\nabla^2_{yx}W\|_\infty W_1(\nu_0,\nu_1).$$

Therefore using the Lipschitzian spectral gap estimate (3.3.1) in Lemma 3.12 for the generator $\mathscr{L}_{\nu_t}$,

$$\begin{aligned}\mathrm{Cov}_{\mu_t}(f, -W \circledast (\nu_1 - \nu_0)) &= \langle (-\mathscr{L}_{\nu_t})^{-1}f, \mathscr{L}_{\nu_t} W \circledast (\nu_1 - \nu_0)\rangle_{\mu_t} \\ &= \int \langle \nabla(-\mathscr{L}_{\nu_t})^{-1}f, \nabla W \circledast (\nu_1 - \nu_0)\rangle d\mu_t \\ &\le c_{Lip,m}\|\nabla^2_{xy}W\|_\infty W_1(\nu_0,\nu_1)\end{aligned}$$

Thus we have

$$\mu_1(f) - \mu_0(f) = \int_0^1 \frac{d}{dt}\mu_t(f)dt \le c_{Lip,m}\|\nabla^2_{xy}W\|_\infty W_1(\nu_0,\nu_1).$$

This means that $W_1(\Phi(\nu_0),\Phi(\nu_1)) \le c_{Lip,m}\|\nabla^2_{xy}W\|_\infty W_1(\nu_0,\nu_1)$ by Kantorovitch-Rubinstein's duality relation. The proof is so completed. □

Remark 3.22. Though $(M_1^2(\mathbb{R}^d), W_1)$ is not complete, the Banach's fixed point theorem works for the essential: let ν_∞ be the unique minimizer of E_f, then for any $\nu \in M_1^2(\mathbb{R}^d)$,

$$W_1(\Phi^n(\nu),\nu_\infty) \le [c_{Lip,m}\|\nabla_{xy}W\|_\infty]^n \cdot W_1(\nu,\nu_\infty), n \ge 0.$$

As for the mean-field relative entropy, the Fisher-Donsker-Varadhan's information $I_W(\nu)$ can be also interpreted as the mean Fisher-Donsker-Varadhan's information per particle.

Lemma 3.23. (convergence of the Fisher information) *If* $I(\nu|\alpha) < +\infty$,

$$\frac{1}{N} I(\nu^{\otimes N}|\mu^{(N)}) \to I_W(\nu). \tag{3.6.3}$$

Proof. For every probability measure ν on $\mathbb{R}^d$ such that $I(\nu|\alpha) < +\infty$, by the Lyapunov function condition (H1) on V ([17]),

$$c_1 \int |x|^2 d\nu \le c_2 + I(\nu|\alpha) < +\infty.$$

As W has bounded second order derivatives, $\nabla_x W$ is of linear growth. Then $\nabla_x W \in L^2(\nu^{\otimes 2})$ and we have

$$\begin{aligned}
\frac{1}{N} I(\nu^{\otimes N}|\mu^{(N)}) &= \frac{1}{4N} \int |\nabla \log \frac{d\nu^{\otimes N}}{d\mu^{(N)}}|^2 d\nu^{\otimes N} \\
&= \frac{1}{4N} \int \sum_{i=1}^{N} |\nabla_{x_i} \log \frac{d\nu^{\otimes N}}{d\alpha^{\otimes N}} + \frac{1}{N-1} \sum_{j\neq i} \nabla_x W(x_i, x_j)|^2 d\nu^{\otimes N} \\
&= \frac{1}{4} \int |\nabla \log \frac{d\nu}{d\alpha}(x_1) + \frac{1}{N-1} \sum_{j=2}^{N} \nabla_x W(x_1, x_j)|^2 d\nu^{\otimes N} \\
&\to \frac{1}{4} \int |\nabla \log \frac{d\nu}{d\alpha}(x_1) + \int \nabla_x W(x_1, y) d\nu(y)|^2 d\nu(x_1) = I_W(\nu).
\end{aligned}$$

where the passing to the limit follows from the symmetry of product measure $\nu^{\otimes N}$. $\square$

3.6.2 Proof of Theorem 3.10

(1). At first the minimizer ν_∞ of H_W is unique by Lemma 3.21.

(2). We may assume that $I(\nu|\alpha) < +\infty$, otherwise (3.2.27) is trivial for $I_W(\nu) = +\infty$. Since the Hessian $\nabla^2 V$ is lower bounded, and V satisfies the Lyapunov function condition (3.2.1), by Cattiaux-Guillin-Wu [12], α satisfies a log-Sobolev inequality. Then $H(\nu|\alpha) < +\infty$. By the log-Sobolev inequality of $\mu^{(N)}$ in Theorem 3.8,

$$\rho_{LS}(\mu^{(N)}) H(\nu^{\otimes N}|\mu^{(N)}) \le 2I(\nu^{\otimes N}|\mu^{(N)})$$

and $\rho_{LS}(\mu^{(N)}) \ge \rho_{LS,m}/(1-\gamma_0)^2 > 0$. Dividing the two sides by N and letting N go to infinity, we get by Lemma 3.17 and Lemma 3.23,

$$\rho_{LS} H_W(\nu) \le 2I_W(\nu).$$

(3). By Otto-Villani [22] or Bobkov-Gentil-Ledoux [6], the log-Sobolev inequality implies the Talagrand's T_2 transportation inequality, i.e.

$$\rho_{LS}(\mu^{(N)}) W_2^2(Q, \mu^{(N)}) \le 2H(Q|\mu^{(N)}),\ Q \in \mathbb{M}_1((\mathbb{R}^d)^N).$$

Applying it to $Q = \nu^{\otimes N}$ with $H(\nu|\alpha) < +\infty$, we obtain

$$\rho_{LS}(\mu^{(N)})\frac{1}{N}W_2^2(\nu^{\otimes N}, \mu^{(N)}) \leq \frac{1}{N}H(\nu^{\otimes N}|\mu^{(N)}).$$

Notice that

$$W_2^2(\nu^{\otimes N}, \mu^{(N)}) \geq \sum_{i=1}^{N} W_2^2(\nu, \mu^{(N,i)}) = NW_2^2(\nu, \mu^{(N,1)})$$

where $\mu^{(N,i)}$ is the marginal distribution of x_i under $\mu^{(N)}$, which are all the same by the symmetry of $\mu^{(N)}$. Moreover by the uniqueness of ν_∞ and the large deviation principle of $\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$ under $\mu^{(N)}$ ([29]), for any $f \in C_b(\mathbb{R}^d)$,

$$\mu^{(N,1)}(f) = \int \frac{1}{N}\sum_{i=1}^{N} f(x_i)d\mu^{(N)} \to \nu_\infty(f),$$

i.e. $\mu^{(N,1)}$ converges weakly to ν_∞. We obtain by Lemma 3.17 and the lower semi-continuity of W_2,

$$\rho_{LS}W_2^2(\nu, \nu_\infty) \leq \rho_{LS}\liminf_{N\to\infty} W_2^2(\nu, \mu^{(N,1)}) \leq 2H_W(\nu)$$

the desired Talagrand's type T_2-inequality for McKean-Vlasov equation.

(4). The exponential convergence in entropy (3.2.29) should be equivalent to the mean-field log-Sobolev inequality (3.2.27) in part (2), basing on

$$-\frac{d}{dt}H_W(\nu_t) = 4I_W(\nu_t) \tag{3.6.4}$$

noted by Carrillo-McCann-Villani [10] in their convex framework. The proof of (3.6.4) demands the regularity of ν_t which requires the PDE theory of the McKean-Vlasov equation. That is why we prefer to give a rigorous probabilistic proof based directly on the log-Sobolev inequality of $\mu^{(N)}$ in Theorem 3.8.

For the exponential convergence (3.2.29), we may and will assume that $H_W(\nu_0) < +\infty$ and we fix the time $t > 0$. By Lemma 3.17,

$$\lim_{N\to\infty}\frac{1}{N}H(\nu_0^{\otimes N}|\mu^{(N)}) = H_W(\nu_0).$$

Moreover by the equivalence between the log-Sobolev inequality for $\mu^{(N)}$ and the exponential convergence in entropy of the law μ_t^N of $X_t^N = (X_t^{N,i})_{1\leq i\leq N}$ to $\mu^{(N)}$,

$$\begin{aligned}\frac{1}{N}H(\mu_t^N|\mu^{(N)}) &\leq e^{-\rho_{LS}(\mu^{(N)})t/2}\frac{1}{N}H(\mu_0^N|\mu^{(N)}) \\ &= e^{-\rho_{LS}(\mu^{(N)})t/2}\frac{1}{N}H(\nu_0^{\otimes N}|\mu^{(N)}) < +\infty.\end{aligned} \tag{3.6.5}$$

Therefore $H(\mu_t^N|\alpha^{\otimes N}) < +\infty$ by Lemma 3.19, since the condition

$$\int e^{-p\frac{1}{N-1}\sum_{i<j}W(x_i,x_j)}d\alpha^{\otimes N} < \infty$$

holds by Hölder's inequality and Assumption H2. Note that μ_t^N has finite second moment (easy from the SDE theory), and W has at most quadratic growth,

$$W(x_i, x_j) \in L^1(\mu_t^N).$$

From Lemma 3.18, we have

$$\frac{1}{N} H(\mu_t^N | \alpha^{\otimes N}) \geq H(\mu_t^{N,1} | \alpha).$$

And by the propagation of chaos (Lemma 3.20) and the lower semi-continuity of the relative entropy $\nu \to H(\nu|\alpha)$, $\liminf_{N\to\infty} H(\mu_t^{N,1}|\alpha) \geq H(\nu_t|\alpha)$.
So we get by Lemma 3.19

$$\begin{aligned}
\liminf_{N\to\infty} \frac{1}{N} H(\mu_t^N|\mu^{(N)}) &= \liminf_{N\to\infty} \left(\frac{1}{N} H(\mu_t^N|\alpha^{\otimes N}) + \int \frac{1}{N(N-1)} \sum_{1\leq i<j\leq N} W(x_i, x_j) d\mu_t^N + \frac{1}{N} \log \tilde{Z}_N \right) \\
&\geq H(\nu_t|\alpha) + \liminf_{N\to\infty} \frac{1}{2} \int W(x_1, x_2) d\mu_t^N - \inf_{\nu \in \mathbb{M}_1(\mathbb{R}^d)} E_f(\nu) \\
&= H(\nu_t|\alpha) + \frac{1}{2} \iint W(x_1, x_2) d\nu_t(x_1) d\nu_t(x_2) - \inf_{\nu\in\mathbb{M}_1(\mathbb{R}^d)} E_f(\nu) \\
&= H_W(\nu_t)
\end{aligned}$$

by the W_2-propagation of chaos in Lemma 3.20. Plugging it into (3.6.5), we obtain the exponential convergence in entropy (3.2.29). That implies the W_2-exponential convergence (3.2.30) by Talagrand's type T_2-inequality (3.2.28). □

3.7 Appendix: Proof of Lemma 3.12

For the convenience of the reader, we reproduce the proof of Lemma 3.12 in Wu[28] but only sketch the main ideas in this appendix.

Proof. To avoid heavy notations (also because the lemma holds in a general setting), in this proof, we replace $\mathscr{L}_i$ by $\mathscr{L}$, μ_i by μ, and $b(x_i) = -\nabla_i H_i$. Recall the fact (see Wu[28], Remark 3.3)

$$\frac{1}{\lambda_1} = \|(-\mathscr{L})^{-1}\|_{L_0^2(\mu)} \leq \|(-\mathscr{L})^{-1}\|_{Lip}$$

(In fact, the last inequality holds for any Banach norm, not only for the Liptchitz norm.), it is sufficient to prove an estimate of the Lipschtz norm of $(-\mathscr{L})^{-1}$.
Chen and Wang [13] studied the reflection couplings (X_t, Y_t) with initial datum $(X_0, Y_0) = (x, y)$

$$\begin{cases} dX_t &= \sqrt{2} dB_t + b(X_t) dt; \\ dY_t &= \sqrt{2} R(X_t, Y_t) dB_t + b(Y_t) dt \end{cases} \tag{3.7.1}$$

where R is a reflection given by

$$R(x, y) = I - 2\frac{xx^T}{|x|^2}.$$

Then we have (by Itô's formula)

$$\mathrm{d}|X_t - Y_t| \le 2\sqrt{2}d\beta_t + b_0(|X_t - Y_t|)\mathrm{d}t$$

where β is some standard Brownian motion, and $b_0(r)$ satisfies

$$\langle \frac{x-y}{|x-y|}, -(b(x)-b(y))\rangle \le b_0(|x-y|)$$

for any x, y.
Assume that for some bounded 1-Lipschitz function g so that $\mu(g) = 0$, and

$$-\mathscr{L}G = g,$$

then

$$G(x) = \int_0^\infty \mathbb{E}g(X_t)\mathrm{d}t.$$

It follows that

$$\begin{aligned} |\nabla G|(x) = \sup_{y\to x}\frac{|G(x)-G(y)|}{|x-y|} &= \sup_{y\to x}\frac{\int_0^\infty |\mathbb{E}(g(X_t)-g(Y_t))|\mathrm{d}t}{|x-y|} \\ &\le \sup_{y\to x}\frac{\int_0^\infty \mathbb{E}|X_t-Y_t|\mathrm{d}t}{|x-y|} \\ &= \sup_{y\to x}\frac{1}{|x-y|}\mathbb{E}\int_0^\infty |X_t-Y_t|\mathrm{d}t. \end{aligned}$$

Consider a one-dimensional diffusion ρ_t killed at 0, valued in $\mathbb{R}^+$, which is generated by L_0

$$L_0\phi(r) = 4\phi''(r) + b_0(r)\phi'(r). \tag{3.7.2}$$

Let h be the solution of

$$-L_0 h(r) = r$$

with Dirichlet boundary condition at 0 and Neumann boundary condition at infinity, namely

$$h(0) = 0, \lim_{r\to\infty} h'(r) = 0$$

which is given by

$$h(r) = \mathbb{E}^{\rho(0)=r}\int_0^\infty \rho_t\mathrm{d}t.$$

Thanks to the method of variation of constant, as in [28], it can be solved explicitly that

$$h'(r) = \frac{1}{4}\exp\left(-\frac{1}{4}\int_0^r b_0(s)\mathrm{d}s\right)\int_r^\infty \exp\left(\frac{1}{4}\int_0^s b_0(u)\mathrm{d}u\right)s\mathrm{d}s,$$

hence

$$h'(0) = \frac{1}{4}\int_0^\infty \exp\left(\frac{1}{4}\int_0^s b_0(u)\mathrm{d}u\right)s\mathrm{d}s.$$

We observe that $|X_t - Y_t| \le \rho_t$ if $\rho_0 = |x-y|$. It follows that

$$\begin{aligned}
|\nabla G|(x) &\le \sup_{y\to x} \frac{1}{|x-y|}\mathbb{E}^{\rho(0)=|x-y|}\int_0^\infty \rho(t)\mathrm{d}t \\
&\le \sup_{y\to x} \frac{h(|x-y|)}{|x-y|} \\
&\le h'(0).
\end{aligned}$$

As a consequence, we obtain that

$$||G||_{\mathrm{Lip}} = ||\nabla G||_\infty \le h'(0)$$

and therefore an estimate for the Lipschitz norm of $(-\mathscr{L})^{-1}$,

$$||(-\mathscr{L})^{-1}||_{\mathrm{Lip}} \le h'(0)$$

which completes the proof. □

3.8 Références

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Chapter 4

The kinetic Fokker-Planck equation with mean-field interaction

This chapter is an article collaborated with Arnaud Guillin, Wei Liu and Liming Wu. We study the long time behaviour of the kinetic Fokker-Planck equation with mean field interaction, whose limit is often called Vlasov-Fokker-Planck equation. We prove a uniform (in the number of particles) exponential convergence to equilibrium for the solutions in the weighted Sobolev space $H^1(\mu)$ with a rate of convergence which is explicitly computable and independent of the number of particles. The originality of the proof relies on functional inequalities and hypocoercivity with Lyapunov type conditions, usually not suitable to provide adimensional results.

4.1 Introduction

In this paper we are interested in the system of N particles moving in $\mathbb{R}^d$ with mean field interaction

$$\begin{cases} dx_t^i = v_t^i dt \\ dv_t^i = \sqrt{2}dB_t^i - v_t^i dt - \nabla U(x_t^i) - \frac{1}{N}\sum_{1\le j\le N} \nabla W(x_t^i - x_t^j)dt \end{cases} \tag{4.1.1}$$

where x_t^i, v_t^i are respectively the position and the velocity of the i-th particle, and $(B_t^i)_{t\ge0}(1\le i\le N)$ are independent standard Brownian motions on $\mathbb{R}^d$, $U:\mathbb{R}^d\to\mathbb{R}$ is the confinement potential, and $W:\mathbb{R}^d\to\mathbb{R}$ is the interaction potential. Equivalently, denote $(x_t, v_t) = ((x_t^1, x_t^2, \cdots, x_t^N), (v_t^1, v_t^2, \cdots, v_t^N))$, the particle system can be rewritten in a more compact form

$$\begin{cases} dx_t = v_t dt \\ dv_t = \sqrt{2}dB_t - v_t dt - \nabla V(x_t)dt \end{cases} \tag{4.1.2}$$

where $B_t = (B_t^1, B_t^2, \cdots, B_t^N)$ and the function V is the whole potential with mean field interaction given by

$$V(x_1, x_2, \cdots, x_N) = \sum_{1\le i\le N} U(x_i) + \frac{1}{2N}\sum_{1\le i,j\le N} W(x_i - x_j). \tag{4.1.3}$$

This damping stochastic Newton equation, though non-elliptic, is hypoelliptic. It has a unique invariant probability measure $\mu(dx, dv)$ on $\mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$ given by

$$\mu(dx, dv) = \frac{1}{Z} e^{-V(x)} \cdot (2\pi)^{-\frac{Nd}{2}} e^{-\frac{|v|^2}{2}} dx dv$$

where $x = (x_1, x_2, \cdots, x_N)$, $v = (v_1, v_2, \cdots, v_N)$ with $x_i, v_i \in \mathbb{R}^d$ for $1 \leq i \leq N$, and Z is the normalization constant (called often the partition function). Denote

$$dm(x) = \frac{1}{Z} e^{-V(x)} dx, \quad d\gamma(v) = (2\pi)^{-\frac{Nd}{2}} e^{-\frac{|v|^2}{2}} dv$$

and so $\mu(dx, dv) = dm(x) d\gamma(v)$.

The density function $h_t(x, v) = d\mu_t(x, v)/d\mu(x, v)$ of the law μ_t of the diffusion process (x_t, v_t) with respect to the equilibrium measure μ satisfies the kinetic Fokker-Planck equation on $\mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h - \nabla_x V(x) \cdot \nabla_v h = \Delta_v h - v \cdot \nabla_v h \tag{4.1.4}$$

subject to the initial condition $h_0(x, v) = d\mu_0(x, v)/d\mu(x, v)$. Here $a \cdot b$ denotes the Euclidean inner product of two vectors a and b, ∇_x stands for the gradient with respect to the position variable $x \in \mathbb{R}^{Nd}$, whereas ∇_v and Δ_v stand for the gradient and the Laplacian with respect to the velocity variable $v \in \mathbb{R}^{Nd}$, respectively. And we shall adopt the notation ∇^2 for the Hessian operator, and $\nabla^2_{xv} = (\partial^2/\partial x_k \partial v_l)_{1 \leq k, l \leq Nd}$ for the mixed Hessian operator.

We denote by $L^2(\mu)$ the weighted L^2 space with respect to the reference measure μ for which $||\cdot||$ is the $L^2(\mu)$-norm and $\langle \cdot, \cdot \rangle$ is the associated inner product. Denote by $H^1(\mu)$ the weighted L^2-Sobolev space of order 1 with respect to μ, and the norm $||\cdot||_{H^1(\mu)}$ is given by

$$||h||^2_{H^1(\mu)} := \int h^2 d\mu + \int \left(|\nabla_x h|^2(x, v) + |\nabla_v h|^2(x, v) \right) d\mu(x, v). \tag{4.1.5}$$

When the probability measure m satisfies a Poincaré inequality, and when $\nabla^2 V$ satisfies some "boundedness" condition (see the condition (4.2.8) below), C. Villani [32] established the exponential convergence of h_t in $H^1(\mu)$. This is the starting of the term "hypocoercivity" method, which was before initiated by [14; 22; 24]. An other approach was initiated by Dolbeault-Mouhot-Schmeiser [15; 16] with the advantage of not needing a priori regularity results. Their H^1-convergence holds under the same assumptions. Noete that is has triggered quite a lot of results for kinetic equations [9–11; 17; 21; 28]. However Both Villani's and DMS's approach on the exponential convergence rate depends highly on the number N of particles. To complete this review on the speed to equilibrium for the Langevin equation, let us mention that a probabilistic approach based on coupling [20] or Lyapunov conditions [31; 33] was also developed but, as is often usual for Meyn-Tweedie's approach relying on Lyapunov conditions, the rate also depends (even more dramatically) on the dimension. Note however that, under very strong convexity assumptions, Bolley&-al [7] obtained a uniform decay in Wasserstein distance for the mean field Langevin equation by a coupling approach. Very recently, an interesting work by Monmarché [29] established an entropic decay, using Villani's hypocoercivity, but still under strong convexity assumptions, and Baudoin&-al [6] mixed Bakry's Γ_2 approach with hypocoercivity to obtain H^1 exponential decay even in a non regular case, i.e. Lennard-Jones potential, but with a rate still depending on the dimension. Note also that for a non mean-field case but oscillators Menegaki [27] obtianed a dimension dependent convergence to equilibrium. The objective of this work is to establish, and it seems to be the first result under non convexity assumptions on the potential, some exponential convergence in $H^1(\mu)$, uniform in the number N of particles. The originality of our approach is that we will combine Villani's hypocoercivity with recent uniform functional inequality and Lyapunov conditions (usually not suitable to provide adimensional results).

As an other motivation to get uniform in the number of particles result, the linear diffusion process $(x_t, v_t)_{t\geq 0}$ in $\mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$ is the mean field approximation of the self-interacting diffusion process $(\bar{x}_t, \bar{v}_t)_{t\geq 0}$ in $\mathbb{R}^d \times \mathbb{R}^d$ which evolves according to

$$\begin{cases} \mathrm{d}\bar{x}_t = \bar{v}_t \mathrm{d}t \\ \mathrm{d}\bar{v}_t = \sqrt{2}\mathrm{d}\bar{B}_t - \bar{v}_t \mathrm{d}t - \left[\nabla U(\bar{x}_t) + \int \nabla W(\bar{x}_t - y) u_t(\mathrm{d}y)\right] \mathrm{d}t \end{cases} \tag{4.1.6}$$

where $u_t(\mathrm{d}y)$ is the law of $\bar{x}_t$, and $\bar{B}$ is a standard Brownian motion on $\mathbb{R}^d$. Its equivalent analytic version is: the density function $g_t = g(t, \bar{x}, \bar{v})$ of the law of $(\bar{x}_t, \bar{v}_t)_{t\geq 0}$ with respect to the Lebesgue measure $\mathrm{d}\bar{x}\mathrm{d}\bar{v}$ satisfies the following self-consistent Vlasov-Fokker-Planck equation on $\mathbb{R}^d \times \mathbb{R}^d$

$$\frac{\partial g}{\partial t} + \bar{v} \cdot \nabla_{\bar{x}} g - (\nabla U(\bar{x}) + \nabla W * \pi g) \cdot \nabla_{\bar{v}} g = \Delta_{\bar{v}} g + \nabla_{\bar{v}} \cdot (\bar{v} g) \tag{4.1.7}$$

subject to the initial condition that $g_0(\bar{x}, \bar{v})$ is given by the law of (x_0^1, v_0^1), where

$$\pi g(\bar{x}) = \int_{\mathbb{R}^d} g(t, \bar{x}, w) \mathrm{d}w$$

is the macroscopic density in the space of positions $\bar{x} \in \mathbb{R}^d$. This kinetic equation describes the evolution of clouds of charged particles, and it is significant in plasma physics (see Villani [32] and references therein). Only very few results on the long time behavior of this nonlinear equation is known, see however [32] in the compact valued case, Bolley *et al.*[7] in the strictly convex case (see also [29]), Hérau [22] or Hérau *et al.* [25] in the case of mollified or small Coulomb interactions, and Addala *et al.* [1] for the linearized equation. Our results are a first step towards such a long time behavior but the H^1 convergence does not behave well with respect to the dimension. We thus plan for a future work to consider entropic convergence and propagation of chaos for the mean field Langevin equation.

Let us finish this introduction with the plan of our paper. The next Section presents the main assumptions and the main results, i.e. a uniform exponential convergence to equilibrium in H^1 under non convex assumptions. It also presents a crucial tool: Villani's hypocoercivity theorem. Its details will be given in Section 3. Section 4 contains useful lemmas in the case where the interaction potential has a bounded hessian. The next sections present the proofs of our main results: Theorem 3 in Section 5 and Theorem 4 in Section 6. The final Section presents a discussion on an improvement on the rate of convergence.

4.2 Main results

4.2.1 Framework

As in the introduction, $\mathrm{d}m(x) = \frac{1}{Z}e^{-V(x)}\mathrm{d}x$ is the probability measure on the position space $\mathbb{R}^{Nd}$ and will be referred as the mean field measure later. Let $\mathrm{d}\gamma(v)$ be the standard gaussian measure on the velocity space $\mathbb{R}^{Nd}$, so $\mathrm{d}\mu(x,v) = \mathrm{d}m(x)\mathrm{d}\gamma(v)$.
Now we introduce our assumptions.

Assumption 4.1. (A1) *The functions* U *and* W *are twice continuously differentiable on* $\mathbb{R}^d$, W *is even (that is,* $W(x) = W(-x)$ *for all* x*), and*

$$Z = \int_{\mathbb{R}^{Nd}} e^{-V(x)}dx < \infty, \quad \forall N \geq 2.$$

i.e. m *is always assumed to be a probability measure.*

Assumption 4.2. (A2)$\nabla^2 W$ *is bounded, i.e. there exists a positive constant* K *such that*

$$-KI_d \leq \nabla^2 W \leq KI_d$$

as quadratic forms on $\mathbb{R}^d$, *where* I_d *is the identity matrix of size* d.

This assumption, which of course relaxes convexity, has been also considered in the propagation of chaos problem as well as the convergence of the (non kinetic) McKean-Vlasov equation in [18; 19].

Assumption 4.3. UPI *The measure* $dm(x) = \frac{1}{Z}e^{-V(x)}dx$ *satisfies a uniform Poincaré inequality i.e. there exists a positive real number* $\kappa > 0$ *such that for any* $N \geq 2$, *and any compact-supported smooth function* h *on* $\mathbb{R}^{Nd}$, *it holds*

$$\kappa \int \left(h - \int h dm\right)^2 dm \leq \int |\nabla_x h|^2 dm. \tag{4.2.1}$$

The most easy-to-check criterion might be the Bakry-Emery curvature-dimension condition $CD(\kappa,\infty)$ (see for instance [4]). It says that both Poincaré inequality and logarithmic Sobolev inequality (see (4.2.12) below) hold true for $\mathrm{d}m(x) = \frac{1}{Z}e^{-V(x)}\mathrm{d}x$ as soon as

$$\nabla^2 V(x) \geq \kappa I_{Nd}$$

in the sense of quadratic forms on $\mathbb{R}^{Nd}$. It can be verified if there exist constants κ_1, κ_2 such that

$$\nabla^2 U \geq \kappa_1 I_d > 0, \nabla^2 W \geq \kappa_2 I_d \tag{4.2.2}$$

as quadratic forms on $\mathbb{R}^d$, with $\kappa = \kappa_1 - \kappa_2^- > 0$ where κ_2^- is the negative part of κ_2. Indeed, by Lemma 4.6 below, the above inequalities imply that the contribution of the interaction potential W in $\nabla^2 V$ is bounded from below by $-\kappa_2^- I_d$, and the contribution of the confinement potential U is bounded from below by $\kappa_1 I_d$. Hence we have that $\nabla^2 V \geq (\kappa_1 - \kappa_2^-)I_{Nd}$ as quadratic forms. It should be noted that κ is then independent of the number N of particles, i.e. we obtain a family of uniform functional inequalities for the mean field measure. Note that this strong convexity assumptions are the one employed in [7] for convergence in Wasserstein distance and by [29] for entropic convergence.

Other assumptions, more specified to the mean field measure m for the uniform Poincaré inequalities and logarithmic Sobolev inequalities, can be found in another work [26] of the authors. Indeed they proved these two functional inequalities with uniform (with respect to the number N of particles) constants under various conditions on the confinement and interaction potentials, even when U has two or more wells, and no convexity conditions on W. The methods used there depend on some dissipativity rate of the drift at distance $r > 0$, defined by

$$b_0(r) = \sup_{x,y,z\in\mathbb{R}^d:|x-y|=r} -\langle\frac{x-y}{|x-y|}, \nabla U(x) - \nabla U(y) + \nabla W(x-z) - \nabla W(y-z)\rangle. \tag{4.2.3}$$

Theorem 4.1. *Assume that the following Lipschitzian constant $c_{Lip,m}$ is finite*

$$c_{Lip,m} := \frac{1}{4}\int_0^\infty \exp\left\{\frac{1}{4}\int_0^s b_0(u)du\right\} sds < \infty. \tag{4.2.4}$$

Assume that there exists some constant $h > -1/c_{Lip,m}$ such that for any $(x_1, x_2, \cdots, x_N) \in \mathbb{R}^{Nd}$,

$$\frac{1}{N}(-1_{i\neq j}\nabla^2 W(x_i - x_j))_{1\le i,j\le N} \ge hI_{Nd} \tag{4.2.5}$$

as quadratic forms. Then the mean field measure m satisfies the following Poincaré inequality

$$(h + 1/c_{Lip,m})\int\left(h - \int hdm\right)^2 dm \le \int |\nabla_x h|^2 dm.$$

for any function $h \in H^1(m)$.

Recall that some nonnegative function $f \in L\log L(\mu)$, its entropy w.r.t. the probability measure μ is defined by

$$\mathrm{Ent}_\mu(f) := \int f\log f d\mu - \mu(f)\log\mu(f),\ \mu(f) := \int f d\mu.$$

Theorem 4.2. *Assume that*

(1) There exists a constant $\rho_{LS,m} > 0$ such that for all i and $x^{\hat{i}}$, m_i, the conditional marginal distributions $m_i := m_i(dx_i|x^{\hat{i}})$ of $x_i \in \mathbb{R}^d$ knowing $x^{\hat{i}} = (x_j)_{j\neq i}$, satisfies the log-Sobolev inequality :

$$\rho_{LS,m}\mathrm{Ent}_{m_i}(f^2) \le 2\int |\nabla f|^2 dm_i,\ f \in C_b^1(\mathbb{R}^d). \tag{4.2.6}$$

(2) (a translation of Zegarlinski's condition)

$$\gamma_0 = c_{Lip,m}K < 1.$$

then m satisfies

$$\rho_{LS,m}(1-\gamma_0)^2\mathrm{Ent}_m(f^2) \le 2\int_{(\mathbb{R}^d)^N} |\nabla f|^2 dm,\ f \in C_b^1((\mathbb{R}^d)^N)$$

i.e. the log-Sobolev constant of m verifies

$$\rho_{LS}(m) \ge \rho_{LS,m}(1-\gamma_0)^2.$$

We remark that the assumptions can be verified in various settings for which we refer to [26]. For instance, the uniform logarithmic Sobolev inequalities for the conditional marginal measure can be verified by the Bakry-Émery Γ_2-criterion and the bounded perturbation theorem.
We will provide later explicit conditions on V and W to get such a result.

4.2.2 Villani's hypocoercivity theorem

We shall present Villani's hypocoercivity theorem for kinetic Fokker-Planck equation concerning the convergence to equilibrium (c.f. [32] Theorem 35, Theorem 18). In the sequel we shall adopt the semigroup formulation. Set

$$-\mathrm{L} := \Delta_v - v\cdot\nabla_v - v\cdot\nabla_x + \nabla \mathrm{V}(x)\cdot\nabla_v, \tag{4.2.7}$$

then the kinetic Fokker-Planck equation can be rewritten as

$$\partial_t h + \mathrm{L}h = 0.$$

The associated semigroup will be denoted as $e^{-t\mathrm{L}}$ and a solution could be represented by

$$h(t,x,v) = e^{-t\mathrm{L}}h(0,\cdot,\cdot).$$

We shall use the notation $|\mathrm{S}|^2_{\mathrm{HS}} := \sum_{i,j} |\mathrm{S}^2_{ij}h|^2$ for the square of the Hilbert-Schimidt norm of the square matrix $\mathrm{S} = (\mathrm{S}_{ij})$. For instance, $|\nabla^2_{xv}h|^2_{\mathrm{HS}} := \sum_{i,j} |\partial^2_{x_i v_j} h|^2$. And for a square matrix S, $|\mathrm{S}|_{op}$ stands for its operator norm.

Villani's Hypocoercivity theorem in $\mathrm{H}^1(\mu)$ (see [32, Theorem 35]) states,

Theorem 4.3. *Let* V *be a* C^2 *function on* $\mathbb{R}^{\mathrm{N}d}$*, satisfying the condition 4.3. Suppose that there exists a positive real number* M *such that*

$$\int |\nabla^2_x \mathrm{V}(x)\cdot\nabla_v h|^2 d\mu \le \mathrm{M}\left(\int |\nabla_v h|^2 d\mu + \int |\nabla^2_{xv} h|^2_{\mathrm{HS}} d\mu\right). \tag{4.2.8}$$

for any $h \in \mathrm{H}^2(\mu)$*. Then there are constants* $\mathrm{C}_0 > 0$ *and* $\lambda > 0$*, explicitly computable, such that for all* $h_0 \in \mathrm{H}^1(\mu)$

$$\|e^{-t\mathrm{L}}h_0 - \int h_0 d\mu\|_{\mathrm{H}^1(\mu)} \le \mathrm{C}_0 e^{-\lambda t}\|h_0\|_{\mathrm{H}^1(\mu)}. \tag{4.2.9}$$

The idea in Villani's proof of Theorem 4.3 is as follows: if one could find a Hilbert space such that the operator L is coercive with respect to its norm, then one has exponential convergence for the semigroup $e^{-t\mathrm{L}}$ under such a norm; If, in addition, this norm is equivalent to some usual norm (such as $\mathrm{H}^1(\mu)$-norm), then one obtains exponential convergence under the usual norm as well.
We shall refer to the condition (4.2.8) as the boundedness condition (4.2.8) on $\nabla^2\mathrm{V}$. In his statement of [32, Theorem 35], this boundedness condition is verified by $|\nabla^2_x\mathrm{V}| \le \mathrm{C}(1+|\nabla\mathrm{V}|)$ with a constant M depending unfortunately on the dimension.
In the setting with mean field interaction, the constants C_0 and λ given in [32] depend on the number N of particles, through the dependence of M (in (4.2.8)) on N. In fact, by a careful analysis of the study in [32], we are led to the following observation: in [32, Theorem 35, Lemma A.24], as $\mathrm{N}\to\infty$, λ decays faster than N^{-2}, while C_0 grows faster than $\mathrm{N}^{3/2}$. We will give conditions under which we may bypass this dependence in the number of particles.

4.2.3 Main results

We have two different assumptions on the interaction potential ensuring an H^1 convergence to equilibrium. The first one is quite strong, namely that W is a Lipschitzian function but we only assume a uniform Poincaré inequality (UPI).

case UPI and $|\nabla \mathrm{W}|$ bounded

Theorem 4.4. *Assume 4.1,4.2 and the condition 4.3. Suppose furthermore that* $|\nabla \mathrm{W}| \leq \mathrm{K}'$ *and the following Lyapunov condition holds*

$$|\nabla^2 \mathrm{U}|_{op} \leq \mathrm{K}_1 |\nabla \mathrm{U}| + \mathrm{K}_2 \tag{4.2.10}$$

for some positive constants $\mathrm{K}', \mathrm{K}_1, \mathrm{K}_2$. *Then there exist explicitly computable constants* C_0 *and* λ, *independent of the number* N *of the particles, such that*

$$||e^{-t\mathrm{L}} h_0 - \int h_0 d\mu||_{\mathrm{H}^1(\mu)} \leq \mathrm{C}_0 e^{-\lambda t} ||h_0||_{\mathrm{H}^1(\mu)} \tag{4.2.11}$$

for all $h_0 \in \mathrm{H}^1(\mu)$.

case Uniform Logarithmic Sobolev Inequality and (A2)

In the next theorem, we shall release the boundedness assumption on $\nabla \mathrm{W}$, but reinforce the condition 4.3 as

Assumption 4.4. ULSI *The mean field measure m satisfies a uniform log-Sobolev inequality with a constant* $\mathrm{C}_{\mathrm{LS}} > 0$, *i.e. for all* $\mathrm{N} \geq 2$ *and for all smooth compactly-supported function g on* $\mathbb{R}^{\mathrm{N}d}$, *it holds*

$$\mathrm{Ent}_m(g^2) := \int g^2 \log g^2 dm - \int g^2 dm \log\left(\int g^2 dm\right) \leq 2\mathrm{C}_{\mathrm{LS}} \int |\nabla g|^2 dm. \tag{4.2.12}$$

In [26] practical conditions are given to ensure such a condition, see example below.

Theorem 4.5. *Assume 4.1,4.2 and the condition 4.4. Suppose furthermore that the Lyapunov condition (4.2.10) holds for some positive constants* K_1 *and* K_2. *Then there exist explicitly computable constants* C_0 *and* λ, *independent of the number* N *of the particles, such that*

$$||e^{-t\mathrm{L}} h_0 - \int h_0 d\mu||_{\mathrm{H}^1(\mu)} \leq \mathrm{C}_0 e^{-\lambda t} ||h_0||_{\mathrm{H}^1(\mu)} \tag{4.2.13}$$

for all $h_0 \in \mathrm{H}^1(\mu)$.

We relax in this theorem the strong assumption concerning the boundedness of $|\nabla \mathrm{W}|$ but we reinforce the functional inequality needed to ensure the adimensional result.

4.2.4 Examples

UPI and Theorem 3

Let assume the following convexity at infinity assumptions on U: there exists constants c_U, c_1 and $R \geq 0$ such that

$$\langle \nabla U(x) - \nabla U(y), x-y \rangle \geq c_U |x-y|^2 - c|x-y| 1_{|x-y| \leq R}. \tag{4.2.14}$$

By following [26, Cor. 5, Rem. 4], then assuming **(A2)**, if we suppose moreover

$$(c_U - K)e^{-cR/4} - 2K > 0$$

then **UPI** holds. The Lyapunov condition (4.2.10), expressing that U cannot grow too fast (more than exponentially) and the boundedness condition of $|\nabla W|$ are easy to verify.

ULSI and Theorem 4

For simplicity, we will suppose that U is super convex at infinity, i.e. for any $\tilde{K} > 0$ there exists $R > 0$ such that

$$\nabla^2 U \geq \tilde{K} I, \qquad \forall |x| \geq R.$$

Note that it implies (4.2.14). Suppose also

$$\frac{e^{cR/4}}{(c_U - K)} K < 1$$

where $c_U (> K)$ and c are described in (4.2.14), then a ULSI holds and once again the Lyapunov condition can be easily verified on examples.

4.3 Villani's hypocoercivity theorem

This section is devoted to Villani's hypocoercivity theorem. The following outline of the proof of [32, Theorem 35] further details the use of the condition 4.3 and the boundedness condition (4.2.8),

(1) Introduce an inner product $((\cdot,\cdot))$ in the form of

$$((h,h)) = ||h||^2 + a||\nabla_v h||^2 + 2b\langle \nabla_v h, \nabla_x h\rangle + c||\nabla_x h||^2 \tag{4.3.1}$$

where the coefficients a, b, c will be specified later such that

$$c_1 ||h||_{H^1(\mu)} \leq ((h,h))^{1/2} \leq c_2 ||h||_{H^1(\mu)}, \quad \forall h \in H^1(\mu) \tag{4.3.2}$$

for some constants $c_1 > 0, c_2 > 0$.

(2) Prove a coercivity estimate for L under the new inner product. Thanks to the boundedness condition (4.2.8), one can choose appropriately the constants a, b and c such that

$$((h, Lh)) \geq \lambda_0(||\nabla_x h||^2 + ||\nabla_v h||^2), \quad \text{if} \int h d\mu = 0 \tag{4.3.3}$$

for some constant $\lambda_0 > 0$ depending only on the constant M. By the tensorization property of Poincaré inequality, the condition 4.3 implies that

$$((h,h)) \leq (2a+1)||\nabla_v h||^2 + (2c+\kappa^{-1})||\nabla_x h||^2$$

for all function $h \in \mathrm{H}^1(\mu)$ with $\int h d\mu = 0$, and hence

$$((h,\mathrm{L}h)) \geq \lambda((h,h)), \qquad \text{if} \int h d\mu = 0 \tag{4.3.4}$$

where λ can be given by

$$\lambda = \lambda_0 \min\left\{\frac{1}{2a+1}, \frac{\kappa}{2c\kappa+1}\right\}. \tag{4.3.5}$$

(3) Apply Gronwall's lemma and deduce exponential decay in the new inner product,

$$((e^{-t\mathrm{L}}h, e^{-t\mathrm{L}}h)) \leq e^{-2\lambda t}((h,h)), \qquad \text{if} \int h d\mu = 0$$

which, due to the equivalence of the two inner products, implies exponential decay in $\mathrm{H}^1(\mu)$-norm

$$||e^{-t\mathrm{L}}h - \int h d\mu||_{\mathrm{H}^1(\mu)} \leq \frac{c_2}{c_1} e^{-\lambda t}||h - \int h d\mu||_{\mathrm{H}^1(\mu)}$$

and so the theorem follows by taking $\mathrm{C}_0 = c_2/c_1$.

In the coercivity estimate (4.3.4), a vital technical point is the introduction of the mixed term $\langle\nabla_x h, \nabla_v h\rangle$. And one has to bound the terms involving $\nabla_x^2 \mathrm{V}$ since it appears naturally in the computations. To see this, recall the following expression taken from [32],

$$\begin{aligned} ((h,\mathrm{L}h)) \quad = \quad & ||\nabla_v h||^2 + a(||\nabla_v^2 h||^2 + ||\nabla_v h||^2 + \langle\nabla_v h, \nabla_x h\rangle) \\ & + b(2\langle\nabla_v^2 h, \nabla_{xv}^2 h\rangle + \langle\nabla_v h, \nabla_x h\rangle + ||\nabla_x h||^2 - \langle\nabla_v h, \nabla_x^2 \mathrm{V}\cdot\nabla_v h\rangle) \\ & + c(||\nabla_{xv}^2 h||^2 - \langle\nabla_x h, \nabla_x^2\mathrm{V}\cdot\nabla_v h\rangle). \end{aligned} \tag{4.3.6}$$

It is then clear that, without the mixed term $\langle\nabla_x h, \nabla_v h\rangle$ (i.e. let $b = 0$), there would be no dissipation in the ∇_x direction, and so it would be impossible to get a coercivity estimate. That way, the inner products $((\cdot,\cdot))$ and $\langle\cdot,\cdot\rangle_{\mathrm{H}^1(\mu)}$, though being equivalent, are quite different in coercivity. And we see that the mixed term really helps to get coercivity.

As the computation (4.3.6) shows, in order to obtain a coercivity estimate in the form of (4.3.3) or (4.3.4), we need to bound the terms involving $\nabla_x^2\mathrm{V}(x)\cdot\nabla_v h$ which occur in $((h,\mathrm{L}h))$, namely, $-\langle\nabla_v h, \nabla_x^2\mathrm{V}(x)\cdot\nabla_v h\rangle$ and $-\langle\nabla_x h, \nabla_x^2\mathrm{V}(x)\cdot\nabla_v h\rangle$, in terms of the L^2-norm of $\nabla_v h$, $\nabla_v^2 h$, $\nabla_x h$, and $\nabla_{xv}^2 h$. And it then becomes natural to consider boundedness conditions in the form of (4.2.8).

Moreover, assuming the condition (4.2.8) holds with a constant M, by Cauchy-Schwarz inequality, we have

$$((h,\mathrm{L}h)) \geq \langle\mathrm{Z},\mathrm{TZ}\rangle$$

with the vector $Z = (||\nabla_v h||, ||\nabla_v^2 h||, ||\nabla_x h||, ||\nabla_{xv}^2 h||) \in \mathbb{R}^4$ and the symmetric 4×4 matrix T given by

$$T = \begin{pmatrix} 1 + a - b\sqrt{M} & 0 & -(a + b + c\sqrt{M})/2 & -b\sqrt{M}/2 \\ 0 & a & 0 & -b \\ -(a + b + c\sqrt{M})/2 & 0 & b & -c\sqrt{M}/2 \\ -b\sqrt{M}/2 & -b & -c\sqrt{M}/2 & c \end{pmatrix}. \tag{4.3.7}$$

To ensure the coercivity estimate (4.3.3), it suffices to choose a, b, c such that

$$T \geq \mathrm{Diag}(\lambda_0, 0, \lambda_0, 0) \tag{4.3.8}$$

as bilinear forms. In doing so, the constants a, b, c and λ_0 depend only on M (and so does C_0). For instance, assuming that $M \geq 1$, we could take $a = \frac{1}{25M}$, $b = \frac{1}{200M^2}$, $c = \frac{1}{800M^3}$ and $\lambda_0 = \frac{1}{440M^2}$. Then, following the outline above, we obtain a rate of convergence λ given by (4.3.5) which depends only on M and the spectral gap constant κ.

This shows that we can get rid of the dependence of the number N of particles, if we can verify the boundedness condition (4.2.8) with a constant M independent of N.

4.4 Bounded interaction assumption

We compute at first the Hessian of the interaction potential:

$$\nabla^2_{x_i x_j} \left(\frac{1}{2N} \sum_{1 \leq k, l \leq N} W(x_k - x_l) \right) = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{k: k \neq i} \nabla^2 W(x_i - x_k), & \text{if } i = j; \\ -\dfrac{1}{N} \nabla^2 W(x_i - x_j), & \text{if } i \neq j. \end{cases}$$

Denote it by H_{ij} for $1 \leq i, j \leq N$. It is clear that $H_{ii} = -\sum_{j: j \neq i} H_{ij}$. Put

$$H_W := (H_{ij})_{1 \leq i, j \leq N},$$

$$H_U := \mathrm{Diag}(\nabla^2 U(x_1), \nabla^2 U(x_2), \cdots, \nabla^2 U(x_N)).$$

Then we get

$$\nabla^2 V(x) = (\nabla^2_{x_i x_j} V(x))_{1 \leq i, j \leq N} = H_U + H_W. \tag{4.4.1}$$

We begin by giving an upper bound for the operator norm of the matrix $H_W(x)$. For a real number r, as usual, we denote its positive part by r^+ and its negative part by r^-.

Lemma 4.6. *If $|\nabla^2 W(y)|_{op} \leq K$ for all $y \in \mathbb{R}^d$, then*

$$|H_W(x)|_{op} \leq K$$

for all $x \in \mathbb{R}^{Nd}$. More precisely, it holds

(1) If $\nabla^2 W \leq \lambda_M I_d$, then $H_W(x) \leq \lambda_M^+ I_{Nd}$;

(2) If $\nabla^2 W \geq \lambda_m I_d$, then $H_W(x) \geq -\lambda_m^- I_{Nd}$.

where the inequalities are understood in the sense of quadratic forms.

Remark 4.7. The coefficient in the above lemma is in fact optimal. Consider $d = 1$ and $W(y) = \frac{1}{2}y^2$. In this case, set $p = (1, 1, \cdots, 1)^T \in \mathbb{R}^N$, and the matrix $NH_W = NI_{Nd} - pp^T = N\Pi_{p^\perp}$ where $\Pi_{p^\perp}$ denotes the projection onto the subspace which is perpendicular to p. Hence H_W has two eigenvalues, 1 and 0. It follows that the operator norm of H_W is 1.

Proof. Here we use the notation $\langle \cdot, \cdot \rangle$ for the scalar product in the Euclidean spaces. Fix $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^{Nd}$. Let $z = (z_1, z_2, \cdots, z_N)$ where $z_i \in \mathbb{R}^d$ for $1 \le i \le N$. Since $H_{ii} = -\sum_{j: j \neq i} H_{ij}$ and $H_{ij} = H_{ji}$, we have

$$\begin{aligned} \langle z, H_W z \rangle &= \sum_{j \neq i} \langle z_i, H_{ij}(z_j - z_i) \rangle = \sum_{i \neq j} \langle z_j, H_{ji}(z_i - z_j) \rangle \\ &= -\frac{1}{2} \sum_{i \neq j} \langle z_i - z_j, H_{ji}(z_i - z_j) \rangle \\ &= \frac{1}{2N} \sum_{i \neq j} \langle z_i - z_j, \nabla^2 W(x_i - x_j) \cdot (z_i - z_j) \rangle. \end{aligned}$$

(1) Assume $\nabla^2 W \le \lambda_M I_d$, then

$$\langle z_i - z_j, \nabla^2 W(x_i - x_j) \cdot (z_i - z_j) \rangle \le \lambda_M |z_i - z_j|^2$$

and therefore

$$\begin{aligned} \langle z, H_W z \rangle &\le \frac{\lambda_M}{2N} \sum_{i \neq j} |z_i - z_j|^2 = \frac{\lambda_M}{N} \left(N|z|^2 - |\sum_i z_i|^2 \right) \\ &\le \lambda_M^+ |z|^2. \end{aligned}$$

(2) Assume $\nabla^2 W \ge \lambda_m I_d$,then

$$\langle z_i - z_j, \nabla^2 W(x_i - x_j) \cdot (z_i - z_j) \rangle \ge \lambda_m |z_i - z_j|^2$$

and therefore

$$\begin{aligned} \langle z, H_W z \rangle &\ge \frac{\lambda_m}{2N} \sum_{i \neq j} |z_i - z_j|^2 = \frac{\lambda_m}{N} \left(N|z|^2 - |\sum_i z_i|^2 \right) \\ &\ge -\lambda_m^- |z|^2. \end{aligned}$$

(3) $|\nabla^2 W|_{op} \le K$ means that $-KI_d \le \nabla^2 W \le KI_d$. By parts (1) and (2), this implies that $-KI_{Nd} \le H_W \le KI_{Nd}$ as quadratic forms and hence $|H_W|_{op} \le K$.

□

Lemma 4.6 allows us to reduce the boundedness condition (4.2.8) to a simpler one,

Lemma 4.8. *Suppose that* $|\nabla^2 W|_{op} \leq K$. *Suppose that there exist positive constants* C_1, C_2 *such that for each* i *and for all* $g \in H^1(m)$,

$$\int |\nabla^2 U(x_i)|^2_{op} g^2 dm \leq C_1 \int |\nabla_x g|^2 dm + C_2 \int g^2 dm. \tag{4.4.2}$$

Then the boundedness condition (4.2.8) *is satisfied with a constant* M *given by*

$$M = \max\{2C_1, 2C_2 + 2K^2\}. \tag{4.4.3}$$

Proof. Under the assumptions and using $\nabla^2 V = H_U + H_W$, by Lemma 4.6, we have

$$\begin{aligned}
\int |\nabla_x^2 V \cdot \nabla_v h|^2 d\mu &\leq 2\int \left(|H_U \cdot \nabla_v h|^2 + |H_W \cdot \nabla_v h|^2\right) d\mu \\
&\leq 2\int \sum_{1\leq i\leq N} |\nabla^2 U(x_i)|^2_{op} |\nabla_{v_i} h|^2 d\mu + 2K^2 \int |\nabla_v h|^2 d\mu.
\end{aligned}$$

We estimate these terms separately. Apply the inequality (4.4.2) with $g = \partial_{v_{il}} h$ (here v_{il} is the l-th variable of $v_i \in \mathbb{R}^d$) for $1 \leq i \leq N$ and $1 \leq l \leq d$, we get

$$\int |\nabla^2 U(x_i)|^2_{op} |\partial_{v_{il}} h|^2 d\mu \leq \int \left[C_1 \int |\nabla_x \partial_{v_{il}} h|^2 dm(x) + C_2 \int |\partial_{v_{il}} h|^2 dm(x)\right] d\gamma(v)$$

Summing over i and l, we have

$$\int \sum_{1\leq i\leq N} |\nabla^2 U(x_i)|^2_{op} |\nabla_{v_i} h|^2 d\mu \leq C_1 \int |\nabla^2_{xv} h|^2_{HS} d\mu + C_2 \int |\nabla_v h|^2 d\mu.$$

and so

$$\int |\nabla_x^2 V \cdot \nabla_v h|^2 d\mu \leq 2C_1 \int |\nabla^2_{xv} h|^2_{HS} d\mu + (2C_2 + 2K^2) \int |\nabla_v h|^2 d\mu.$$

i.e. the boundedness condition (4.2.8) is satisfied with the constant M given in (4.4.3). □

4.5 Proof of Theorem 4.4

Let $\mathcal{H}$ be the elliptic generator associated to the mean field measure m, that is,

$$\begin{aligned}
\mathcal{H} &= \Delta_x - \nabla V(x) \cdot \nabla_x \\
&= \Delta_x - \sum_{1\leq i\leq N} \left(\nabla U(x_i) + \frac{1}{N} \sum_{1\leq j\leq N} \nabla W(x_i - x_j)\right) \cdot \nabla_{x_i} \\
&= \sum_{1\leq i\leq N} \mathcal{H}_i
\end{aligned}$$

where

$$\mathcal{H}_i = \Delta_{x_i} - \nabla U(x_i) \cdot \nabla_{x_i} - \frac{1}{N} \sum_{1\leq j\leq N} \nabla W(x_i - x_j) \cdot \nabla_{x_i}.$$

The following known lemma is a key to the Lyapunov type conditions, it was initially proved in [5] to get a Poincaré inequality. We include its simple proof for completeness.

Lemma 4.9. *Let $\mathcal{H}$ and m be defined as above, then for all twice-differentiable function* $S > 0$ *and for all* $g \in H^1(m)$,

$$\int -\frac{\mathcal{H}S}{S} g^2 dm \leq \int |\nabla g|^2 dm. \tag{4.5.1}$$

Proof. Indeed, an integration by parts gives

$$\begin{aligned} \int -\frac{\mathcal{H}S}{S} g^2 \mathrm{d}m &\leq \int \langle \nabla S, \nabla \frac{g^2}{S} \rangle \mathrm{d}m(x) \\ &\leq \int \langle \nabla S, \frac{2g\nabla g}{S} - \frac{g^2 \nabla S}{S^2} \rangle \mathrm{d}m(x) \\ &\leq \int |\nabla g|^2 \mathrm{d}m \end{aligned}$$

where the last inequality follows from

$$\langle 2g\nabla g, \frac{\nabla S}{S} \rangle \leq \frac{g^2 |\nabla S|^2}{S^2} + |\nabla g|^2.$$

□

This second lemma is the heart of the proof. It uses Lyapunov conditions, yet well know for being highly dimensional, but at the marginal level, thus providing results independent of the number of particles.

Lemma 4.10. *Suppose that the Lyapunov condition* (4.2.10) *holds, i.e. there exists positive constants* K_1, K_2 *such that*

$$|\nabla^2 U|_{op} \leq K_1 |\nabla U| + K_2.$$

Then for all $g \in H^1(m)$,

$$\int |\nabla^2 U(x_i)|^2_{op} g^2 dm \leq C_1 \int |\nabla_x g|^2 dm + C_2 \int g^2 dm$$

with C_1, C_2 *given by*

$$C_1 = 50K_1^2, \quad C_2 = 4K_2^2 + \frac{25K_1^4 d^2}{4} + \frac{25K'^2 K_1^2}{2}. \tag{4.5.2}$$

Proof. Step 1: We show that the Lyapunov condition $|\nabla^2 U|_{op} \leq K_1 |\nabla U| + K_2$ implies

$$|\nabla^2 U|^2_{op} \leq \eta_1((1-\alpha)|\nabla U|^2 - \Delta U) + \eta_2. \tag{4.5.3}$$

where

$$\eta_1 = 5K_1^2, \eta_2 = 4K_2^2 + \frac{25K_1^4 d^2}{4}, \text{ and } \alpha = \frac{1}{5}.$$

Indeed, note that

$$C\Delta U \leq Cd|\nabla^2 U|_{op} \leq \epsilon |\nabla^2 U|^2_{op} + \frac{C^2 d^2}{4\epsilon}$$

for $\epsilon > 0$ and $C > 0$. And the condition $|\nabla^2 U|_{op} \leq K_1|\nabla U| + K_2$ implies

$$|\nabla^2 U|_{op}^2 \leq 2K_1^2|\nabla U|^2 + 2K_2^2$$

Then we have

$$\begin{aligned}|\nabla^2 U|_{op}^2 + C\Delta U &\leq (1+\epsilon)|\nabla^2 U|_{op}^2 + \frac{C^2 d^2}{4\epsilon}\\ &= 2(1+\epsilon)K_1^2|\nabla U|^2 + 2(1+\epsilon)K_2^2 + \frac{C^2 d^2}{4\epsilon}\end{aligned}$$

or

$$|\nabla^2 U|_{op}^2 \leq C\left[\frac{2(1+\epsilon)K_1^2}{C}|\nabla U|^2 - \Delta U\right] + 2(1+\epsilon)K_2^2 + \frac{C^2 d^2}{4\epsilon} \tag{4.5.4}$$

The desired inequality (4.5.3) follows by taking $\epsilon = 1, C = 5K_1^2$.
Step 2. We take $S(x) = e^{\alpha U(x_i)/2}$ and compute

$$\frac{\mathscr{H}S}{S} = \frac{\mathscr{H}_i S}{S} = \frac{\alpha}{2}\left(\Delta U(x_i) + (\frac{\alpha}{2} - 1)|\nabla U|^2(x_i) - \frac{1}{N}\sum_j \nabla W(x_i - x_j)\cdot\nabla U(x_i)\right)$$

Since $|\nabla W| \leq K'$, we have

$$\begin{aligned}-\frac{1}{N}\sum_j \nabla W(x_i - x_j)\cdot\nabla U(x_i) &\leq K'|\nabla U|(x_i)\\ &\leq \frac{K'^2}{2\alpha} + \frac{\alpha}{2}|\nabla U|^2(x_i)\end{aligned}$$

and so

$$\frac{2\mathscr{H}S}{\alpha S} \leq \Delta U(x_i) + (\alpha - 1)|\nabla U|^2(x_i) + \frac{K'^2}{2\alpha}$$

or

$$(1-\alpha)|\nabla U|^2(x_i) - \Delta U(x_i) \leq -\frac{2\mathscr{H}S}{\alpha S} + \frac{K'^2}{2\alpha}$$

Therefore, by the inequality obtained in Step 1,

$$|\nabla^2 U(x_i)|_{op}^2 \leq \eta_1(-\frac{2\mathscr{H}S}{\alpha S} + \frac{K'^2}{2\alpha}) + \eta_2$$

Integrating with respect to $g^2 \mathrm{d}m$, we obtain

$$\begin{aligned}\int |\nabla^2 U(x_i)|_{op}^2 g^2 \mathrm{d}m &\leq \frac{2\eta_1}{\alpha}\int -\frac{\mathscr{H}S}{S}g^2\mathrm{d}m + (\eta_2 + \frac{K'^2\eta_1}{2\alpha})\int g^2\mathrm{d}m\\ &\leq \frac{2\eta_1}{\alpha}\int |\nabla g|^2\mathrm{d}m + (\eta_2 + \frac{K'^2\eta_1}{2\alpha})\int g^2\mathrm{d}m\end{aligned}$$

where the last inequality follows from Lemma 4.9. □

Proof of Theorem 4.4. By the Lyapunov condition (4.2.10) in the assumptions, we can apply Lemma 4.10 and obtain that for any $g \in H^1(m)$, it holds

$$\int |\nabla^2 U(x_i)|_{op}^2 g^2 dm \leq C_1 \int |\nabla_x g|^2 dm + C_2 \int g^2 dm$$

with C_1, C_2 given by (4.5.2) for instance which are independent of the number N of particles. Next, using Lemma 4.8, the boundedness condition (4.2.8) holds with M given by

$$M = \max\{2C_1, 2C_2 + 2K^2\}.$$

We apply Villani's Hypocoercivity theorem 4.3 and then obtain the result. □

4.6 Proof of Theorem 4.5

The next results extend the ones in the previous section to unbounded ∇W. Instead, we shall require that the mean field measure m satisfies the Uniform Logarithmic Sobolev Inequality. We prove the following estimate first, relying only on the variational formulation of entropy.

Lemma 4.11. *Assume that the measure m satisfies a log-Sobolev inequality with a constant* C_{LS}*. For* $0 < \tau < \frac{1}{4C_{LS}}$ *given and for each i fixed, it holds for all suitably integrable function g that*

$$\int \frac{1}{N-1} \sum_{j:j\neq i} |x_i - x_j|^2 g^2 dm \leq \frac{2C_{LS}}{\tau} \int |\nabla g|^2 dm + \frac{d \ln(1-4\tau C_{LS})^{-1}}{2\tau} \int g^2 dm. \tag{4.6.1}$$

In particular, taking $\tau = \frac{1}{8C_{LS}}$*, it holds*

$$\int \frac{1}{N-1} \sum_{j:j\neq i} |x_i - x_j|^2 g^2 dm \leq 16C_{LS}^2 \int |\nabla g|^2 dm + 4 \ln 2 \cdot dC_{LS} \int g^2 dm. \tag{4.6.2}$$

Proof. Put

$$F(x) = \frac{1}{N-1} \sum_{j:j\neq i} |x_i - x_j|^2$$

Since the measure m satisfies a log-Sobolev inequality, we can apply the classical entropy inequality

$$\int f g^2 dm \leq \mathrm{Ent}_m(g^2) + \int g^2 dm \log \int e^f dm$$

with $f = \tau F$. Then, for any $\tau > 0$ such that $c_2 = \log \int e^{\tau F} dm$ is finite, we obtain

$$\begin{aligned} \int F g^2 dm &\leq \frac{1}{\tau} \mathrm{Ent}_m(g^2) + \frac{1}{\tau} \int g^2 dm \log \int e^{\tau F} dm \\ &\leq \frac{2C_{LS}}{\tau} \int |\nabla_x g|^2 dm + \frac{c_2}{\tau} \int g^2 dm \end{aligned}$$

where the last inequality follows from the log Sobolev inequality for m.

Now it remains to give an upper bound of $\int e^{\tau F} dm$. Thanks to the symmetry of $m(dx_1, dx_2, \cdots, dx_N)$, we find

$$\begin{aligned}\int e^{\tau F} dm &\leq \int \frac{1}{N-1} \sum_{j:j\neq i} e^{\tau |x_i - x_j|^2} dm(x) \\ &= \int e^{\tau |x_1 - x_2|^2} dm(x)\end{aligned}$$

Let $d\gamma_1(y) = (2\pi)^{-d/2} e^{-|y|^2/2} dy$ be the standard gaussian measure on $\mathbb{R}^d$. Due to the identity $e^{\tau|x|^2} = \int e^{\sqrt{2\tau}x\cdot y} d\gamma_1(y)$, we have

$$\begin{aligned}\int e^{\tau |x_1 - x_2|^2} dm(x) &= \int\int e^{\sqrt{2\tau}(x_1 - x_2)\cdot y} d\gamma_1(y) dm(x) \\ &= \int d\gamma_1(y) \int e^{\sqrt{2\tau}(x_1 - x_2)\cdot y} dm(x)\end{aligned}$$

For any given $y \in \mathbb{R}^d$, the function $\sqrt{2\tau}(x_1 - x_2)\cdot y$ has mean zero w.r.t the measure m. Indeed this is a consequence of symmetry,

$$\int (x_1 - x_2)\cdot y dm(x) = \int x_1 \cdot y dm(x) - \int x_2 \cdot y dm(x) = 0.$$

And note that $\sqrt{2\tau}(x_1 - x_2)\cdot y$ is a Lipschitz function of x with Lipschitz constant $2\sqrt{\tau}|y|$. Therefore, according to the exponential integrability under a logarithmic Sobolev inequality (see [4, Chapter 5] for instance), the function $\sqrt{2\tau}(x_1 - x_2)\cdot y$ satisfies

$$\int e^{\sqrt{2\tau}(x_1 - x_2)\cdot y} dm(x) \leq e^{2\tau|y|^2 C_{LS}}$$

for any $y \in \mathbb{R}^d$. Hence, if $0 < \tau < 1/(4C_{LS})$, we obtain

$$\begin{aligned}\int e^{\tau F} dm &\leq \int e^{2\tau C_{LS}|y|^2} d\gamma_1(y) \\ &= (1 - 4\tau C_{LS})^{-d/2}\end{aligned}$$

and then the desired estimate follows. □

Lemma 4.12. *Suppose that the mean field measure m satisfies a log-Sobolev inequality with a constant* C_{LS}. *Suppose the Lyapunov condition* (4.2.10) *and*

$$|\nabla^2 W|_{op} \leq K.$$

Then, for all $g \in H^1(m)$,

$$\int |\nabla^2 U(x_i)|_{op}^2 g^2 dm \leq C_1 \int |\nabla g|^2 dm + C_2 \int g^2 dm$$

with the constants C_1, C_2 *given by*

$$C_1 = 50K_1^2(1 + 4K^2 C_{LS}^2), \quad C_2 = 4K_2^2 + \frac{25K_1^4 d^2}{4} + 50 \ln 2 \cdot dK^2 K_1^2 C_{LS}. \tag{4.6.3}$$

Proof. As in the proof of Lemma 4.10, the Lyapunov condition (4.2.10) implies

$$|\nabla^2 U|_{op}^2 \le \eta_1((1-\alpha)|\nabla U|^2 - \Delta U) + \eta_2 \tag{4.6.4}$$

with $\eta_1 = 5K_1^2, \eta_2 = 4K_2^2 + \frac{25K_1^4 d^2}{4}$ and $\alpha = \frac{1}{5}$.
Consider $S(x) = e^{\alpha U(x_i)/2}$ and compute

$$\frac{\mathcal{H}S}{S} = \frac{\alpha}{2}\left(\Delta U(x_i) + (\frac{\alpha}{2}-1)|\nabla U|^2(x_i) - \frac{1}{N}\sum_{j:j\neq i}\nabla W(x_i-x_j)\cdot\nabla U(x_i)\right)$$

By Cauchy-Schwarz inequality, it holds

$$\begin{aligned}-\frac{1}{N}\sum_{1\le j\le N}\nabla W(x_i-x_j)\cdot\nabla U(x_i) &\le \frac{1}{2\alpha}|\frac{1}{N}\sum_{1\le j\le N}\nabla W(x_i-x_j)|^2 + \frac{\alpha}{2}|\nabla U|^2(x_i)\\ &\le \frac{1}{2\alpha N}\sum_{1\le j\le N}|\nabla W(x_i-x_j)|^2 + \frac{\alpha}{2}|\nabla U|^2(x_i)\end{aligned}$$

and so

$$\frac{2\mathcal{H}S}{\alpha S} \le \Delta U(x_i) + (\alpha-1)|\nabla U|^2(x_i) + \frac{1}{2\alpha N}\sum_{1\le j\le N}|\nabla W(x_i-x_j)|^2$$

Using the assumption on $\nabla^2 U$, we have

$$|\nabla^2 U(x_i)|_{op}^2 \le \eta_1\left(-\frac{2\mathcal{H}S}{\alpha S} + \frac{1}{2\alpha N}\sum_{1\le j\le N}|\nabla W(x_i-x_j)|^2\right) + \eta_2$$

Integrating with respect to $g^2 dm$, we obtain by lemma 4.9

$$\begin{aligned}\int |\nabla^2 U(x_i)|_{op}^2 g^2 dm &\le \frac{2\eta_1}{\alpha}\int -\frac{\mathcal{H}S}{S}g^2 dm + \eta_2\int g^2 dm + \frac{\eta_1}{2\alpha}\Theta\\ &\le \frac{2\eta_1}{\alpha}\int|\nabla g|^2 dm + \eta_2\int g^2 dm + \frac{\eta_1}{2\alpha}\Theta\end{aligned}$$

with

$$\Theta := \frac{1}{N}\int\sum_{1\le j\le N}|\nabla W(x_i-x_j)|^2 g^2 dm.$$

To prove the lemma, it remains to show that

$$\Theta \le 16K^2C_{LS}^2\int|\nabla g|^2 dm + 4\ln 2\cdot dK^2C_{LS}\int g^2 dm \tag{4.6.5}$$

Since W is even, we see that $\nabla W(0) = 0$. Then it follows from the assumption $|\nabla^2 W|_{op} \le K$ that

$$|\nabla W(z)| \le |\nabla W(0)| + K|z| \le K|z|$$

therefore

$$\Theta \le \int\frac{1}{N-1}\sum_{j:j\neq i}|\nabla W(x_i-x_j)|^2 g^2 dm \le K^2\int\frac{1}{N-1}\sum_{j:j\neq i}|x_i-x_j|^2 g^2 dm$$

So we can apply the lemma 4.11 to get the inequality (4.6.5) and the proof is then complete. □

Now we turn to the

Proof of Theorem 4.5. By the Lyapunov condition (4.2.10) in the assumptions, we can apply Lemma 4.12 and obtain that for any $g \in H^1(m)$, it holds

$$\int |\nabla^2 U(x_i)|_{op}^2 g^2 \mathrm{d}m \le C_1 \int |\nabla_x g|^2 \mathrm{d}m + C_2 \int g^2 \mathrm{d}m$$

with C_1, C_2 given by (4.6.3). Note that these constants are independent of the number N of particles.

Next, owing to Lemma 4.8, we know the boundedness condition (4.2.8) holds with

$$M = \max\{2C_1, 2C_2 + 2K^2\}$$

We apply Villani's Hypocoercivity theorem 4.3 and then obtain the convergence with rates independent of the number N of particles. □

4.7 An improvement on the rate of convergence

The boundedness conditions proved in the previous sections share the following form

$$\int |\nabla_x^2 V \cdot \nabla_v h|^2 \mathrm{d}\mu \le M_1 \int |\nabla_{xv}^2 h|_{HS}^2 \mathrm{d}\mu + M_2 \int |\nabla_v h|^2 \mathrm{d}\mu.$$

where the coefficients M_1 and M_2 might be

$$M_1 = 2C_1, \quad M_2 = 2C_2 + 2K^2$$

with constants C_1 and C_2 being given in (4.5.2) or (4.6.3). Note that C_1 and C_2 depend on K_1 and K_2 in the Lyapunov condition (4.2.10)

$$|\nabla^2 U|_{op} \le K_1 |\nabla U| + K_2.$$

It is clear that K_1 is related to the asymptotic behaviour of $\nabla^2 U$ and ∇U at infinity, while K_2 is more relevant to the local properties. For instance, when U behaves as a polynomial at infinity, K_1 can be taken to be arbitrarily close to zero (with the price of K_2 being large); consequently, M_1 might be very small while M_2 might be large. This suggests that in general we can obtain a boundedness condition with very different M_1 and M_2.

In this section, we shall take advantage of this fact and get a slight improvement on the rate of convergence λ. As mentioned before, the rate of convergence in [32, Theorem 35] is of order M^{-2}, as $M \to \infty$ with $M = \max\{1, M_1, M_2\}$. However, by distinguishing the two constants M_1 and M_2, the rate can be improved to be of order $M_2^{-1/2}$ for small M_1 and big M_2.

Proposition 4.13. *If the following boundedness condition holds,*

$$\int |\nabla_x^2 V \cdot \nabla_v h|^2 d\mu \le M_1 \int |\nabla_{xv}^2 h|_{HS}^2 d\mu + M_2 \int |\nabla_v h|^2 d\mu,$$

then the rate of convergence λ can be taken to be of order $\frac{1}{\sqrt{M_2}}$ for small M_1 and big M_2.

Remark 4.14. We consider mainly the behaviour of λ when M_2 is large while M_1 is small. For specific M_1 and M_2, an refinement of the method is always needed to get a better rate of convergence.

Proof. We set in this proof that $M = \max\{1, M_2\}$. By Cauchy-Schwarz inequality and the boundedness condition above,

$$\begin{aligned}
-\langle \nabla_v h, \nabla_x^2 V \cdot \nabla_v h\rangle &\geq -\|\nabla_v h\|\|\nabla_x^2 V \cdot \nabla_v h\| \\
&\geq -\|\nabla_v h\|\sqrt{M_1\|\nabla_{xv}^2 h|^2 + M_2\|\nabla_v h\|^2} \\
&\geq -\|\nabla_v h\|(\sqrt{M_1}\|\nabla_{xv}^2 h| + \sqrt{M_2}\|\nabla_v h\|).
\end{aligned}$$

Similarly,

$$-\langle \nabla_x h, \nabla_x^2 V \cdot \nabla_v h\rangle \geq -\|\nabla_x h\|(\sqrt{M_1}\|\nabla_{xv}^2 h| + \sqrt{M_2}\|\nabla_v h\|).$$

These inequalities lead to

$$((h, Lh)) \geq \langle Z, T'Z\rangle$$

with a matrix T' given by

$$T' = \begin{pmatrix} 1 + a - b\sqrt{M_2} & 0 & -(a+b+c\sqrt{M_2})/2 & -b\sqrt{M_2}/2 \\ 0 & a & 0 & -b \\ -(a+b+c\sqrt{M_2})/2 & 0 & b & -c\sqrt{M_1}/2 \\ -b\sqrt{M_1}/2 & -b & -c\sqrt{M_1}/2 & c \end{pmatrix}.$$

Denote

$$S = (S_{ij})_{1\leq i,j\leq 4} := T' - \mathrm{Diag}(\lambda_0, 0, \lambda_0, 0),$$

$$Z = (Z_1, Z_2, Z_3, Z_4).$$

The object is then choose a, b, c such that S is positive definite. If now it is assumed that

$$b\sqrt{M_2} \leq \frac{1}{4}, \quad \lambda_0 = \frac{b}{4} \leq \frac{1}{4}, \tag{4.7.1}$$

then

$$S_{11} = 1 + a - b\sqrt{M_2} - \lambda_0 \geq a + \frac{1}{2}, \quad S_{33} = b - \lambda_0 = \frac{3}{4}b. \tag{4.7.2}$$

And if we impose furthermore the conditions below

$$\frac{1}{2}\cdot\frac{b}{2} \geq \left(\frac{a+b+c\sqrt{M_2}}{2}\right)^2, \quad a\cdot\frac{c}{8} \geq \left(\frac{b\sqrt{M_1}}{2}\right)^2, \quad a\cdot\frac{c}{2} \geq b^2, \quad \frac{b}{4}\cdot\frac{3c}{8} \geq \left(\frac{c\sqrt{M_1}}{2}\right)^2, \tag{4.7.3}$$

then we have

$$\frac{1}{2}Z_1^2 + \frac{b}{2}Z_3^2 \geq |2S_{13}Z_1Z_3|, \quad aZ_1^2 + \frac{c}{8}Z_4^2 \geq |2S_{14}Z_1Z_4|,$$

$$aZ_2^2 + \frac{c}{2}Z_4^2 \geq |2S_{24}Z_2Z_4|, \quad \frac{b}{4}Z_3^2 + \frac{3c}{8}Z_4^2 \geq |2S_{34}Z_3Z_4|,$$

and it follows that

$$\begin{aligned}\langle Z, SZ\rangle &= S_{11}Z_1^2+S_{22}Z_2^2+S_{33}Z_3^2+S_{44}Z_4^2+2S_{13}Z_1Z_3+2S_{14}Z_1Z_4+2S_{24}Z_2Z_4+2S_{34}Z_3Z_4\\ &\geq S_{11}Z_1^2+S_{22}Z_2^2+S_{33}Z_3^2+S_{44}Z_4^2-(\frac{1}{2}Z_1^2+\frac{b}{2}Z_3^2)-(aZ_1^2+\frac{c}{8}Z_4^2)\\ &\quad -(aZ_2^2+\frac{c}{2}Z_4^2)-(\frac{b}{4}Z_3^2+\frac{3c}{8}Z_4^2)\\ &= (S_{11}-\frac{1}{2}-a)Z_1^2+(S_{33}-\frac{3b}{4})Z_3^2\\ &\geq 0\end{aligned}$$

where the last inequality follows from (4.7.2).

Case 1: To fix ideas, we consider the case $M_1 \leq 1$ first. In this case, we may take M_1 as 1, then the conditions (4.7.3) become

$$b \geq \left(a+b+c\sqrt{M_2}\right)^2, \quad ac \geq 2b^2, \quad \frac{3}{8}b \geq c. \tag{4.7.4}$$

For the moment let α, β, γ be the constants such that

$$a = \alpha/\sqrt[4]{M}, \quad b = \beta/\sqrt[4]{M}^2, c = \gamma/\sqrt[4]{M}^3.$$

then, since $M = \max\{1, M_2\} \geq 1$, it suffices that

$$\beta \geq (\alpha+\beta+\gamma)^2, \quad \alpha\gamma \geq 2\beta^2, \quad \frac{3}{8}\beta \geq \gamma \tag{4.7.5}$$

where $\beta \leq 1/4$ (so that $b\sqrt{M_2} \leq 1/4$). To conclude we may take all these inequalities to be equalities, and in this case

$$\beta = \frac{576}{25921}, \quad \alpha = \frac{16}{3}\beta = \frac{3072}{25921}, \quad \gamma = \frac{3}{8}\beta = \frac{216}{25921}.$$

and then

$$\lambda_0 = \frac{b}{4} = \frac{144}{25921\sqrt{M}}.$$

Recall the equality (4.3.5) says that

$$\lambda = \lambda_0 \min\{\frac{1}{2a+1}, \frac{\kappa}{2c\kappa+1}\} = \frac{144}{25921\sqrt{M}}\min\{\frac{1}{\frac{6144}{25921\sqrt[4]{M}}+1}, \frac{\kappa}{\frac{512\kappa}{25921\sqrt[4]{M}^3}+1}\}.$$

In particular, note that $M = \max\{1, M_2\} = M_2$ for $M_2 \geq 1$, hence the rate of convergence λ is of order $1/\sqrt{M_2}$ for large M_2.

Case 2: Now we consider the case $M_1 > 1$. The conditions (4.7.3) become

$$b \geq \left(a+b+c\sqrt{M_2}\right)^2, \quad ac \geq 2M_1 b^2, \quad \frac{3}{8}b \geq M_1 c. \tag{4.7.6}$$

The solution to the corresponding system of equalities is given by

$$b = \frac{1}{\left(\frac{16}{3}M_1^2 + 1 + \frac{3\sqrt{M_2}}{8M_1}\right)^2}, \quad a = \frac{16}{3}M_1^2 b, \quad c = \frac{3}{8M_1}b.$$

which gives a rate of convergence of order M_2^{-1} for large M_2. Or we can proceed as in Case 1, and we may take

$$b = \frac{1}{\left(\frac{16}{3}M_1^2 + 1 + \frac{3}{8M_1}\right)^2 \sqrt{M}}, \quad a = \frac{16}{3\sqrt[4]{M}}M_1^2, \quad c = \frac{3}{8M_1\sqrt[4]{M}^3}.$$

which gives a rate of convergence of order $1/\sqrt{M_2}$ for large M_2. □

Remark 4.15. To ensure the positiveness of the matrix T' in the proof, the constant λ_0 must satisfy

$$1 + a - b\sqrt{M_2} - \lambda_0 \geq 0, \text{ and } b - \lambda_0 \geq 0.$$

Assume $a \leq 1$, then the first inequality implies that $b \leq 2/\sqrt{M_2}$ while the second one implies $\lambda_0 \leq b$. As a consequence, $\lambda \leq \lambda_0$ is at most of order $1/\sqrt{M_2}$. The rate of convergence stated in the proposition is sharp in this sense.

Consider the matrix T given in (4.3.7) in section 3, similarly the rate of convergence λ is at most of order $1/\sqrt{M}$. Furthermore, a fine argument shows that the positiveness of T requires that λ_0 is at most of order M^{-2} as M tends to infinity. So the distinction between M_1 and M_2 allows us to get a better growth control for λ (for large M_2).

4.8 Références

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Chapter 5

Long-time behavior of mean-field interacting particle systems related to McKean-Vlasov equation

This chapter is an article collaborated with Wei Liu and Liming Wu. In this paper, we investigate gradient estimate of the Poisson equation, the exponential convergence in the Wasserstein metric $W_{1,d_{l^1}}$ and uniform in time propagation of chaos for the mean-field weakly interacting particle system related to McKean-Vlasov equation. By means of the known approximate componentwise reflection coupling and with the help of some new cost function, we obtain explicit estimates for those three problems, avoiding the technical conditions in the known results. Our results apply when the confinement potential V has many wells, the interaction potential W has bounded second mixed derivative $\nabla^2_{xy}W$ which should be not too big so that there is no phase transition. As application, we obtain the concentration inequality of the mean-field interacting particle system with explicit and sharp constants, uniform in time. Several examples are provided to illustrate the results.

5.1 Introduction

In this paper, we consider the following nonlinear McKean-Vlasov equation with initial condition u_0

$$\partial_t u_t = \nabla \cdot [\nabla u_t + u_t \nabla V + u_t(\nabla_x W \circledast u_t)], \tag{5.1.1}$$

where the unknown u_t is a time dependent probability density on $\mathbb{R}^d$ $(d \geq 1)$, $V : \mathbb{R}^d \to \mathbb{R}$ is a confinement potential and $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is an interaction potential. Here ∇ and $\nabla \cdot$ (applied to a vector field) denote the gradient operator and the divergence operator respectively, while $\nabla_x W$ stands for the gradient of W with respect to (w.r.t. in short) the first variable, and

$$\nabla_x W \circledast u_t(x) := \int_{\mathbb{R}^d} \nabla_x W(x,y) u_t(y) dy.$$

When $W(x,y) = W_0(x-y)$ for some even potential W_0, $\nabla_x W \circledast u = \nabla W_0 * u$ (the usual convolution).

The probabilistic equivalent version of (5.1.1) is the following self-interacting stochastic differential equation (SDE in short):

$$\begin{cases} dX_t = \sqrt{2}dB_t - \nabla V(X_t)dt - \nabla_x W \circledast \mu_t(X_t)dt, \\ X_0 \sim u_0(x)dx, \end{cases} \tag{5.1.2}$$

where μ_t is the law of X_t. The density u_t of the law μ_t of X_t at time t is the solution of the McKean-Vlasov equation (5.1.1) and *vice versa*. The existence and uniqueness of the solution of the SDE (5.1.2) and the McKean-Vlasov equation (5.1.1) have been extensively studied. The reader is referred to [16; 24; 25; 27] and recent works [7; 18; 26] as well as the references therein. For the convergence to equilibrium of solution μ_t as $t \to +\infty$, it is worth mentioning that Carrillo, McCann and Villani [8] obtained the explicit exponential convergence in entropy under various kinds of convexity conditions on the potentials V and W, via their enlightening idea of interpreting the McKean-Vlasov equation as the gradient descent flow of the free energy on the space of probability measures equipped with the L^2-Wasserstein metric. Eberle et al. [15] got the quantitative bounds on the exponential convergence in some appropriate transport cost to equilibrium for McKean-Vlasov equations by using Lyapunov condition and reflection coupling. Eberle [14] showed the exponential contractivity for diffusion semigroups w.r.t. Kantorovich distance by using componentwise reflection coupling methods and choosing appropriate distance functions. One can also refer to [20] for the exponential convergence of diffusion semigroups w.r.t. the L^p-Wasserstein distance for all $p \geq 1$.

The McKean-Vlasov equation (5.1.1) or (5.1.2) is the idealization of the following interacting particle system of mean-field type when the number N of particles goes to infinity:

$$\begin{cases} dX_t^{i,N} = \sqrt{2}dB_t^i - \nabla V(X_t^{i,N})dt - \frac{1}{N-1}\sum_{j:j\neq i, 1\leq j\leq N} \nabla_x W(X_t^{i,N}, X_t^{j,N})dt, \\ X_0^{i,N} = X_0^i,\ i = 1, \cdots, N, \end{cases} \tag{5.1.3}$$

where the initial values $X_0^1, \cdots, X_0^N$ are i.i.d. random variables with common law $\mu_0(dx) = u_0(x)dx$, and $B_t^1 \cdots, B_t^N$ are N independent Brownian motions taking values in $\mathbb{R}^d$, independent of $X_0^i, 1 \leq i \leq N$. In fact it is the goal of the so-called propagation of chaos: when the number N of particles goes to infinity, the empirical measures $\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ of the particle system (5.1.3) (or the law of a single particle) converge weakly to the solution μ_t of the self-interacting diffusion (5.1.2).

The propagation of chaos for the mean-field interacting particle systems has been widely studied during the last forty years. The early studies were concentrated on the propagation of chaos in bounded time intervals, see [21; 24; 27] and the references therein. The study on the propagation of chaos in the whole time interval $\mathbb{R}^+$ is much more difficult and recent. When the confinement potential V is strictly convex and the interaction potential $W(x, y) = W_0(x-y)$ with W_0 strictly convex, Malrieu [22] showed the uniform in time propagation of chaos by applying the logarithmic Sobolev inequality. In the case that there is no confinement (i.e. $V \equiv 0$) and the interaction potential W_0 is strictly convex, Benachour et al. [1; 2] proved propagation of chaos (but not uniform in time) and polynomial convergence to equilibrium; Malrieu [23] obtained the uniform in time propagation of chaos and exponential convergence to equilibrium for the particle system viewed from the center, by using functional inequalities. When W_0 is degenerately convex and $V = 0$, Cattiaux et al. [9] showed the uniform in time propagation of chaos and exponential convergence to equilibrium by using synchronous coupling.

Without the convexity of V and W_0, recently Durmus, Eberle, Guillin and Zimmer [13] use the componentwise reflection coupling introduced in [14] to prove the exponential convergence in some Wasserstein metric and uniform in time propagation of chaos for weakly interacting mean-field particle system. For more results about propagation of chaos, we refer the reader to [12; 19].
The main purpose of this paper is to investigate the exponential convergence in L^1-Wasserstein metric in the purpose of refining the previous results in [13; 14], the concentration inequalities and the propagation of chaos of the mean-field weakly interacting particle system. Although we use the same approximate componentwise reflection coupling ([14]), our next approach will be quite different from [13; 14]:

(1) our starting point is some explicit gradient estimate of the Poisson equation, which implies moreover the concentration inequalities of the empirical mean of the interacting particle system, useful for numerical computation of solution μ_t of the McKean-Vlasov equation;

(2) we will choose a different metric from that in [13; 14], which allows us to obtain some explicit and almost sharp estimate of the exponential rate in the convergence of the interacting particles system to its equilibrium in the W_1−metric, uniform in the number N of the particles.

(3) As a by-product, we obtain some explicit estimate on the propagation of chaos, uniform in time.

The paper is organized as follows. In the next section, we will present our framework and main results. The proofs are provided in Section 3 and section 4. The applications to concentration inequalities are given in the last section.

5.2 Main resutls

5.2.1 Framework: notations and conditions

Conditions on the dissipativity rate of a single particle

First we introduce the dissipative rate $b_0(r)$ of the drift of one single particle in (5.1.3) at distance $r > 0$,

$$\langle x-y, -[\nabla V(x) - \nabla V(y)] - [\nabla_x W(x,z) - \nabla_x W(y,z)]\rangle \le b_0(r)|x-y| \tag{5.2.1}$$

holds for any $x, y, z \in \mathbb{R}^d$ with $|x-y| = r$. Throughout this paper we assume that $b_0(r)$ is a continuous function on $(0, +\infty)$ satisfying

$$\limsup_{r\to+\infty} \frac{b_0(r)}{r} < 0, \tag{5.2.2}$$

i.e. the drift of one particle is dissipative at infinity.
We also assume that

$$\lim_{r\to 0+} b_0^+(r) = 0. \tag{5.2.3}$$

Next we introduce an important reference function h which enables us to obtain some new results, avoiding the technical parameters in [13; 14]. For any function $f \in C^2(0,+\infty)$ and $r > 0$, let $\mathscr{L}_{ref}$ be the generator defined by

$$\mathscr{L}_{ref} f(r) := 4f''(r) + b_0(r) f'(r). \tag{5.2.4}$$

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be the function determined by: $h(0) = 0$ and

$$h'(r) = \frac{1}{4}\exp\left(-\frac{1}{4}\int_0^r b_0(s)ds\right)\int_r^{+\infty} s\cdot\exp\left(\frac{1}{4}\int_0^s b_0(u)du\right)ds. \tag{5.2.5}$$

It is a well defined C^2 function by the dissipative condition (5.2.2). It is a solution of the one-dimensional Poisson equation

$$\mathscr{L}_{ref} h(r) = 4h''(r) + b_0(r)h'(r) = -r, \; r > 0 \tag{5.2.6}$$

with $h(0) = 0$. This function was used by the second named author [28] for functional and isoperimetric inequalities on Riemmanian manifolds.

Kantorovich-Wasserstein W_1-metric

For the configuration space $(\mathbb{R}^d)^N$, instead of the usual Euclidean metric, we will use the l^1-metric

$$d_{l^1}(x,y) = \sum_{i=1}^N |x^i - y^i|, \;\; x = (x^1,\cdots,x^N), \; y = (y^1,\cdots,y^N) \in (\mathbb{R}^d)^N.$$

We consider the Kantorovich-Wasserstein distance w.r.t. d_{l^1} metric on $(\mathbb{R}^d)^N$, i.e., for any two probability measures μ and ν on $(\mathbb{R}^d)^N$,

$$W_{1,d_{l^1}}(\mu,\nu) = \inf_{P\in\Pi(\mu,\nu)} \iint_{(\mathbb{R}^d)^N\times(\mathbb{R}^d)^N} d_{l^1}(x,y)P(dx,dy)$$

where $\Pi(\mu,\nu)$ is the set of all couplings of μ,ν, i.e. the set of all probability measures on $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ whose marginal distributions of x and y are respectively μ and ν.

Notice that for a C^1-function g on $(\mathbb{R}^d)^N$, its Lipschitzian norm $\|g\|_{Lip(d_{l^1})}$ w.r.t. d_{l^1} coincides with $\max_{1\le i\le N}\|\nabla_i g\|_\infty$ where ∇_i is the gradient w.r.t. x_i. By Kantorovich-Rubinstein duality relation,

$$W_{d_{l^1}}(\mu,\nu) = \sup_{g\in C_b^1((\mathbb{R}^d)^N):\max_{1\le i\le N}\|\nabla_i g\|_\infty\le 1}\left(\int g d\mu - \int g d\nu\right)$$

When $N = 1$, we write simply W_1 for $W_{1,d_{l^1}}$.

We notice that for two probability measures μ,ν on $(\mathbb{R}^d)^N$,

$$\sum_{i=1}^N W_1(\mu^i,\nu^i) \le W_{d_{l^1}}(\mu,\nu) \tag{5.2.7}$$

where μ^i (resp. ν^i) is the marginal distribution of x_i of μ (resp. ν). In fact if $X = (X^1,\cdots,X^N), Y = (Y^1,\cdots,Y^N)$ are two random vectors such that the law of (X,Y) is an optimal coupling of (μ,ν) in $W_{1,d_{l^1}}$, then for each i, the law of (X^i,Y^i) is a coupling of (μ^i,ν^i), so

$$W_{d_{l^1}}(\mu,\nu) = \mathbb{E}d_{l^1}(X,Y) = \sum_{i=1}^N \mathbb{E}|X^i - Y^i| \ge \sum_{i=1}^N W_1(\mu^i,\nu^i).$$

5.2.2 An explicit gradient estimate of the Poisson equation and its applications in concentration inequalities

Let $\{P_t^{(N)}\}_{t\geq 0}$ be the transition semigroup of the mean-field interacting particle system (5.1.3), whose generator is given by

$$\mathscr{L}^{(N)} f(x^1,\cdots,x^N) = \sum_{i=1}^{N}\left(\Delta_i f - \nabla V(x^i)\cdot\nabla_i f - \frac{1}{N-1}\sum_{j\neq i}\nabla_x W(x^i,x^j)\cdot\nabla_i f\right).$$

Its unique invariant probability measure is given by

$$\mu^{(N)}(dx^1,\cdots,dx^N) = \frac{1}{C_N}\exp\left(-\sum_{i=1}^{N}V(x^i) - \frac{1}{N-1}\sum_{1\leq i<j\leq N}W(x^i,x^j)\right)dx^1\cdots dx^N,$$

where C_N is the normalization constant.
We introduce the following key assumption on the interaction potential:

$$(\mathbf{H}):\ \|\nabla^2_{xy}W\|_\infty\|h'\|_\infty < 1$$

where h is given by (5.2.5), $\|h'\|_\infty := \sup_{r\geq 0} h'(r)$, and $\nabla^2_{xy}W$ stands for the second order gradient of W w.r.t. the first variable and the second variable,

$$\|\nabla^2_{xy}W\|_\infty := \sup_{x,y\in\mathbb{R}^d}\ \sup_{z\in\mathbb{R}^d,|z|=1} |\nabla^2_{xy}W(x,y)z|.$$

This assumption is a translation of Dobrushin-Zegarlinski's uniqueness condition in the framework of mean field, and it implies that the mean field has no phase transition (see [17]).
Notice that under the assumption (**H**) and (5.2.2), both the equations (5.1.2) and (5.1.3) have unique strong solution. On the space of continuous paths $C([0,T],(\mathbb{R}^d)^N)$ where $T\in(0,+\infty]$, we consider the L^1-metric

$$d_{L^1[0,T]}(\gamma_1,\gamma_2) := \int_0^T d_{l^1}(\gamma_1(t),\gamma_2(t))dt \tag{5.2.8}$$

(may be infinite). Given the starting point $x\in(\mathbb{R}^d)^N$, let $\mathbb{P}_x$ be the law of $X^{(N)} = (X_t^{(N)})_{t\geq 0}$ with $X_0^{(N)} = x$.

Theorem 5.1. *Assume (5.2.2), (5.2.3) and* (**H**). *For any $x_0 = (x_0^1,\cdots,x_0^N)\in(\mathbb{R}^d)^N$ and $y_0 = (y_0^1,\cdots,y_0^N)\in(\mathbb{R}^d)^N$, we have*

$$\begin{aligned}\int_0^{+\infty} W_{d_{l^1}}(P_t^{(N)}(x_0,\cdot),P_t^{(N)}(y_0,\cdot))dt &\leq W_{1,d_{L^1[0,\infty]}}(\mathbb{P}_{x_0},\mathbb{P}_{y_0})\\ &\leq \frac{1}{1-\|\nabla^2_{xy}W\|_\infty\|h'\|_\infty}\sum_{i=1}^{N}h(|x_0^i-y_0^i|).\end{aligned} \tag{5.2.9}$$

In particular for any $g\in C_b^1((\mathbb{R}^d)^N)$ with $\mu^{(N)}(g)=0$, the solution G of the Poisson equation $-\mathscr{L}^{(N)}G = g$ with $\mu(G)=0$ satisfies

$$\|\nabla_i G\|_\infty \leq c_{Lip}\cdot\max_{1\leq j\leq N}\|\nabla_j g\|_\infty, 1\leq i\leq N, \tag{5.2.10}$$

where

$$c_{Lip} := \frac{h'(0)}{1 - \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty} \tag{5.2.11}$$

and

$$h'(0) = \frac{1}{4}\int_0^{+\infty} s \cdot \exp\left(\frac{1}{4}\int_0^s b_0(u)\,du\right) ds.$$

By the theorem above we can immediately obtain the following result about the nonlinear McKean-Vlasov equation (5.1.1).

Corollary 5.2. *Under the same assumptions as in Theorem 5.1, for any two solutions μ_t, ν_t of the self-interacting diffusion (5.1.2) with the initial distributions μ_0, ν_0 with finite second moment respectively, we have*

$$\int_0^\infty W_1(\mu_t, \nu_t)\,dt \le \frac{\|h'\|_\infty}{1 - \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty} W_1(\mu_0, \nu_0). \tag{5.2.12}$$

Proof. By (5.2.9) in Theorem 5.1 and the fact that

$$h(r) \le h(0) + \|h'\|_\infty \cdot r = \|h'\|_\infty \cdot r, \ \forall r \ge 0$$

we have

$$\int_0^\infty W_{1,d_{l^1}}(\mu_0^{\otimes N} P_t^{(N)}, \nu_0^{\otimes N} P_t^{(N)})\,dt \le \frac{\|h'\|_\infty}{1 - \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty} W_{1,d_{l^1}}(\mu_0^{\otimes N}, \nu_0^{\otimes N}). \tag{5.2.13}$$

Notice that $\mu_t^{(N)} := \mu_0^{\otimes N} P_t^{(N)}$ and $\nu_t^{(N)} := \nu_0^{\otimes N} P_t^{(N)}$ are symmetric probability measures on $(\mathbb{R}^d)^N$ and their marginal distributions $\mu_t^{(i,N)}$, $\nu_t^{(i,N)}$ of x_i converge weakly to μ_t, ν_t (respectively) by the finite time propagation of chaos. By using (5.2.7) we have

$$N W_1(\mu_t^{(1,N)}, \nu_t^{(1,N)}) = \sum_{i=1}^N W_1(\mu_t^{(i,N)}, \nu_t^{(i,N)}) \le W_{1,d_{l^1}}(\mu_t^{(N)}, \nu_t^{(N)})$$

and then

$$W_1(\mu_t, \nu_t) \le \liminf_{N\to+\infty} W_1(\mu_t^{(1,N)}, \nu_t^{(1,N)}) \le \liminf_{N\to+\infty} \frac{1}{N} W_{1,d_{l^1}}(\mu_t^{(N)}, \nu_t^{(N)}). \tag{5.2.14}$$

Combining (5.2.13) and (5.2.14) together, we obtain

$$\begin{aligned}
\int_0^\infty W_1(\mu_t, \nu_t)\,dt &\le \liminf_{N\to+\infty} \frac{1}{N} \int_0^\infty W_{1,d_{l^1}}(\mu_t^{(N)}, \nu_t^{(N)})\,dt \\
&\le \frac{\|h'\|_\infty}{1 - \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty} \liminf_{N\to+\infty} \frac{1}{N} W_{1,d_{l^1}}(\mu_0^{\otimes N}, \nu_0^{\otimes N}) \\
&= \frac{\|h'\|_\infty}{1 - \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty} W_1(\mu_0, \nu_0)
\end{aligned}$$

which completes the proof. □

As an application of Theorem 5.1 to the concentration inequality, we have the following result about the Gaussian concentration of the U-statistics, which is a straightforward application of a general result in Proposition 5.18. The proofs are given in the last section.
For any $1 \le m \le \mathrm{N}$, let $f_m : (\mathbb{R}^d)^m \to \mathbb{R}$ be a measurable function. The U-statistic of order m with kernel f_m is defined by

$$\mathrm{U_N}(f_m)(x^1,\cdots,x^{\mathrm{N}}) = \frac{1}{|\mathrm{I}_{\mathrm{N}}^m|}\sum_{(i_1,\cdots,i_m)\in \mathrm{I}_{\mathrm{N}}^m} f_m(x^{i_1},\cdots,x^{i_m}),\ \forall (x^1,\cdots,x^{\mathrm{N}})\in(\mathbb{R}^d)^{\mathrm{N}}, \tag{5.2.15}$$

where

$$\mathrm{I}_{\mathrm{N}}^m := \{(i_1,\cdots,i_m)\in\mathbb{N}^k | i_1,\cdots,i_m \text{ are different }, 1\le i_1,\cdots,i_m\le \mathrm{N}\} \tag{5.2.16}$$

and $|\mathrm{I}_{\mathrm{N}}^m|$ denotes the number of elements in $\mathrm{I}_{\mathrm{N}}^m$ (equal to $\mathrm{N}!/(\mathrm{N}-m)!$).
Next we introduce the following *Gaussian integrability* assumption of the initial distribution μ_0:

$$\int_{\mathbb{R}^d} e^{\lambda_0|x|^2}\mu_0(dx) < +\infty, \text{ for some } \lambda_0 > 0 \tag{5.2.17}$$

which is equivalent to say that there is some Gaussian concentration constant $c_{\mathrm{G}}(\mu_0) > 0$ such that

$$\int_{\mathbb{R}} e^{f(x)-\mu_0(f)}d\mu_0(x) \le \exp\left(\frac{c_{\mathrm{G}}(\mu_0)}{2}\|f\|_{\mathrm{L}ip}^2\right) \tag{5.2.18}$$

for all Lipschitzian functions f on $\mathbb{R}^d$ (w.r.t. the usual Euclidean distance).

Remark 5.3. The equivalence between the *Gaussian integrability* (5.2.17) and the *Gaussian concentration inequality* (5.2.18) was established by H. Djellout, A. Guillin and the second named author [10], and (5.2.18) is the famous characterization of Bobkov-Götze [3] of the transport-entropy inequality. By the tensorization of the transport-entropy inequality for product measure, (5.2.18) implies that for any $\mathrm{N}\ge 1$,

$$\int_{(\mathbb{R}^d)^{\mathrm{N}}} e^{g(x)-\mu_0^{\otimes \mathrm{N}}(g)}d\mu_0^{\otimes\mathrm{N}}(x) \le \exp\left(\frac{\mathrm{N}}{2}c_{\mathrm{G}}(\mu_0)\|g\|_{\mathrm{L}ip(d_{l^1})}^2\right) \tag{5.2.19}$$

for all Lipschitzian functions g on $(\mathbb{R}^d)^{\mathrm{N}}$.

Corollary 5.4. *Assume the conditions in Theorem 5.1 and the Gaussian integrability (5.2.17) of the initial distribution μ_0. Let $f_m \in \mathrm{C}^2((\mathbb{R}^d)^m,\mathbb{R})$ be a 1-Lipschitzian function w.r.t the d_{l^1}-metric on $(\mathbb{R}^d)^m$, i.e. $\max_i\|\nabla_i f\|_\infty \le 1$. Then for any $\lambda,\mathrm{T}>0$, we have*

$$\begin{aligned}&\mathbb{E}\exp\left(\frac{\lambda}{\mathrm{T}}\left[\int_0^{\mathrm{T}}\mathrm{U_N}(f_m)(\mathrm{X}_t^{1,\mathrm{N}},\cdots,\mathrm{X}_t^{\mathrm{N},\mathrm{N}})dt - \int_0^{\mathrm{T}}\mathbb{E}f_m(\mathrm{X}_t^{1,\mathrm{N}},\cdots,\mathrm{X}_t^{m,\mathrm{N}})dt\right]\right)\\ &\le \exp\left(\frac{m^2\lambda^2c_{\mathrm{L}ip}^2}{2\mathrm{NT}}\left(1+\frac{c_{\mathrm{G}}(\mu_0)}{\mathrm{T}}\right)\right),\end{aligned} \tag{5.2.20}$$

where $c_{\mathrm{L}ip}$ is the same as given in (5.2.11). In particular we have for any $\delta>0$

$$\begin{aligned}&\mathbb{P}\left\{\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\mathrm{U_N}(f_m)(\mathrm{X}_t^{1,\mathrm{N}},\cdots,\mathrm{X}_t^{\mathrm{N},\mathrm{N}})dt - \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\mathbb{E}f_m(\mathrm{X}_t^{1,\mathrm{N}},\cdots,\mathrm{X}_t^{m,\mathrm{N}})dt > \delta\right\}\\ &\le \exp\left(-\frac{(1-\|\nabla_{xy}^2\mathrm{W}\|_\infty\|h'\|_\infty)^2}{2m^2(h'(0))^2(1+c_{\mathrm{G}}(\mu_0)/\mathrm{T})}\mathrm{NT}\delta^2\right).\end{aligned} \tag{5.2.21}$$

The concentration inequality (5.2.21) is sharp when V is quadratic and $W = 0$, see Example 5.1 for explicit expression of all involved constants in the Gaussian case.

5.2.3 Exponential convergence of the particle system in the $W_{1,d_{l^1}}$-metric

Theorem 5.5. *Assume (5.2.2) and* (**H**). *Suppose that there exists a constant* $M \in \mathbb{R}$ *such that*

$$b_0(r) \le rM, \forall r > 0 \tag{5.2.22}$$

(this condition is stronger than (5.2.3)), then for any $\varepsilon > 0$ *such that*

$$K_\varepsilon := \frac{1 - \|\nabla^2_{xy}W\|_\infty \|h'\|_\infty - \varepsilon(M + \|\nabla^2_{xy}W\|_\infty)}{\|h'\|_\infty + \varepsilon} > 0, \tag{5.2.23}$$

we have for any $x_0, y_0 \in (\mathbb{R}^d)^N$

$$W_{d_{l^1}}(P_t^{(N)}(x_0,\cdot), P_t^{(N)}(y_0,\cdot)) \le A_\varepsilon e^{-K_\varepsilon t} d_{l^1}(x_0,y_0),\ \forall t \ge 0, \tag{5.2.24}$$

where

$$A_\varepsilon = \sup_{r>0} \frac{r}{h(r)+\varepsilon r} \cdot \sup_{r>0} \frac{h(r)+\varepsilon r}{r}. \tag{5.2.25}$$

Remark 5.6. An easy estimate of A_ε is $A_\varepsilon \le \frac{\sup_{r\ge 0} h'(r)+\varepsilon}{\inf_{r\ge 0} h'(r)+\varepsilon}$. Note that the exponential rate K_ε increases (then better and better) as ε decreases to 0, but A_ε may explode once if $\inf_{r\ge 0} h'(r) = 0$.

Remark 5.7. Notice that (5.2.22) is equivalent to say that

$$\nabla^2 V(x) + \nabla^2_{xx} W(x,y) \ge -MI,\ x,y \in \mathbb{R}^d.$$

When $\kappa := -M - \|\nabla^2_{xy}W\|_\infty > 0$, we see that the Hessian of the Hamiltonian

$$H(x^1,\cdots,x^N) = \sum_{i=1}^N V(x^i) + \frac{1}{N-1} \sum_{1\le i<j\le N} W(x^i,x^j)$$

is bounded from below by κI (this estimate of the lower bound of the Bakry-Emery curvature is sharp if $\nabla^2_{xy}W$ is constant and definitely nonnegative). Notice that when $M < 0$, we can take $b_0(r) = Mr$, so $h'(r) = -1/M$. Then $\kappa > 0$ if and only if **(H)** is satisfied. The advantage of our condition **(H)** (w.r.t. the positive curvature condition) is: it does not depend on the curvature but on the dissipativity and holds even if V has many wells once if the interaction is weak enough.
In the positive curvature $\kappa > 0$ case we have by Bakry-Emery's curvature characterization

$$W_1(P_t^{(N)}(x,\cdot), P_t^{(N)}(y,\cdot)) \le e^{-\kappa t}|x-y|$$

in the Euclidean metric on $(\mathbb{R}^d)^N$. On the other hand as above $b_0(r) = Mr, h(r) = -r/M$, we see that $K_\varepsilon \to -M - \|\nabla^2_{xy}W\|_\infty = \kappa$ as $\varepsilon \to +\infty$, and $A_\varepsilon = 1$, so (5.2.24) yields

$$W_{1,d_{l^1}}(P_t^{(N)}(x,\cdot), P_t^{(N)}(y,\cdot)) \le e^{-\kappa t} d_{l^1}(x,y). \tag{5.2.26}$$

Theorem 5.5 above will give us an explicit exponential convergence in W_1 of the nonlinear McKean-Vlasov equation (5.1.1). For the exponential convergence in entropy of the nonlinear McKean-Vlasov equation (5.1.1) under the condition **(H)**, see Guillin *et al.* [17].

Corollary 5.8. *Under the same assumptions as in Theorem 5.5, for any $\varepsilon > 0$ so that $K_\varepsilon > 0$ (i.e. (5.2.23)), we have for the solutions μ_t, ν_t of the self-interacting diffusion (5.1.2) with the initial distributions μ_0, ν_0 with finite second moment respectively,*

$$W_1(\mu_t, \nu_t) \le A_\varepsilon e^{-K_\varepsilon t} W_1(\mu_0, \nu_0), \ \forall t \ge 0, \tag{5.2.27}$$

where K_ε and A_ε are given by (5.2.23) *and (5.2.25) respectively.*

Proof. The proof of this corollary is similar to that of Corollary 5.2, and we utilize the same notations as in the Corollary 5.2. First by Theorem 5.5, we have for any $t \ge 0$

$$W_{1,d_{l^1}}(\mu_0^{\otimes N} P_t^{(N)}, \nu_0^{\otimes N} P_t^{(N)}) \le A_\varepsilon e^{-K_\varepsilon t} W_{1,d_{l^1}}(\mu_0^{\otimes N}, \nu_0^{\otimes N}).$$

Combining the inequality above with (5.2.14), we obtain

$$\begin{aligned} W_1(\mu_t, \nu_t) &\le A_\varepsilon e^{-K_\varepsilon t} \liminf_{N \to +\infty} \frac{1}{N} W_{1,d_{l^1}}(\mu_0^{\otimes N}, \nu_0^{\otimes N}) \\ &= A_\varepsilon e^{-K_\varepsilon t} W_1(\mu_0, \nu_0) \end{aligned}$$

the desired result. □

5.2.4 Propagation of chaos in large time

We have the following uniform in time propagation of chaos.

Theorem 5.9. *Assume (5.2.2), (5.2.22) and* (**H**). *Suppose that there exist some positive constants c_1, c_2, c_3 such that*

$$\langle x, \nabla V(x) \rangle \ge c_1 |x|^2 - c_2, \forall x \in \mathbb{R}^d \tag{5.2.28}$$

and

$$\langle z, \nabla^2_{xx} W(x,y) z \rangle \ge -c_3 |z|^2, \forall x, y, z \in \mathbb{R}^d. \tag{5.2.29}$$

Assume

$$c_1 - c_3 - \|\nabla^2_{xy} W\|_\infty > 0. \tag{5.2.30}$$

Then for any $\varepsilon > 0$ such that $K_\varepsilon > 0$, and $\tilde{\varepsilon} \in (0, c_1 - c_3 - \|\nabla^2_{xy} W\|_\infty)$, the following estimates of propagation of chaos hold for the mean-field interacting particle system (5.1.3) with any initial probability measure μ_0 having finite second moment:

(a) **(path-type propagation of chaos)** *for any $T > 0$, $1 \le k \le N$, denote $\mathbb{P}_\nu(\cdot) = \int_{(\mathbb{R}^d)^N} \mathbb{P}_x(\cdot) d\nu(x)$ the law of $(X_t^{(N)})_{t \ge 0}$ with the initial distribution ν, $\mathbb{P}_\nu^{[1,k],N}|_{[0,T]}$ the joint law of paths of the k particles $((X_t^{i,N})_{t \in [0,T]}, 1 \le i \le k)$ in time interval $[0,T]$, and $\mathbb{Q}_{\mu_0}$ the law of the self-interacting diffusion $(X_t)_{t \ge 0}$ with the initial distribution μ_0. We have*

$$\frac{1}{k} W_{1,d_{L^1[0,T]}}(\mathbb{P}^{[1,k],N}_{\mu_0^{\otimes N}}|_{[0,T]}, \mathbb{Q}_{\mu_0}^{\otimes k}|_{[0,T]}) \le \frac{T}{\sqrt{N-1}} \frac{\|\nabla^2_{xy} W\|_\infty \|h'\|_\infty}{1 - \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty} \cdot \max\{m_2(\mu_0), \hat{c}(\varepsilon)\} \tag{5.2.31}$$

where

$$\begin{aligned} m_2(\mu_0) &= \left(\int |x|^2 d\mu_0(x)\right)^{\frac{1}{2}}, \\ \hat{c}(\varepsilon) &= \left(\frac{d + c_2 + \frac{1}{4\tilde{\varepsilon}}|\nabla_x W(0,0)|}{c_1 - c_3 - \|\nabla^2_{xy} W\|_\infty - \tilde{\varepsilon}}\right)^{\frac{1}{2}}. \end{aligned} \tag{5.2.32}$$

(b) **(Uniform in time propagation of chaos)** *for all time $t > 0$ and any $1 \le k \le N$:*

$$W_{1,d_{l^1}}(\mu_t^{[1,k],N}, \mu_t^{\otimes k}) \le \frac{k}{\sqrt{N-1}} \frac{A_\varepsilon}{K_\varepsilon} \|\nabla^2_{xy} W\|_\infty \max\{m_2(\mu_0), \hat{c}(\varepsilon)\} \tag{5.2.33}$$

where $\mu_t = u_t dx$ is the solution of the Mckean-Vlasov equation (5.1.1), and $\mu_t^{[1,k],N}$ is the joint law of the k particles $(X_t^{i,N}, 1 \le i \le k)$ in the mean-field system (5.1.3) of interacting particles $(X_t^{i,N})_{1\le i\le N}$ with $X_0^{i,N}, 1 \le i \le N$ i.i.d. of law μ_0 (independent of $(B_t^{i,N})_{1\le i\le N, t\ge 0}$), and the constants K_ε, A_ε, $m_2(\mu_0)$ and $\hat{c}(\varepsilon)$ are given in (5.2.23), (5.2.25) and (5.2.32) respectively.

Remark 5.10. The time-uniform propagation of chaos is much more difficult than the bounded time propagation of chaos, accomplished in the 80-90's of the last century. The physical reason is that the time-uniform propagation of chaos fails in the regime of phase transition. That is why we impose the condition **(H)**, which excludes the phase transition.

The reader is referred to [5; 9; 12; 13; 19] and the references therein for recent studies and progresses on this subject. The main new point here is that our estimate (5.2.33) is explicit and relatively neat.

Remark 5.11. All the results presented in this paper can be extended to more general case:

$$dX_t = \sqrt{2}dB_t + b(X_t, \mu_t)dt$$

where μ_t is the law of X_t, if b satisfies some dissipative condition in x (uniformly in μ) and a Lipchitzian condition in μ with sufficiently small Lipschitzian constant. For the sake of clarity, we deal only with the case of $b(X_t, \mu_t) = -\nabla V(X_t) - \nabla_x W \circledast \mu_t(X_t)$ in this paper.

5.2.5 Examples

We first present the Gaussian model for which the constants in Theorem 5.1 and Theorem 5.5 become exact, showing their sharpness.

Example 5.1. **(Gaussian model)** Let $d = 1$, and

$$V(x) = \beta\frac{x^2}{2}, \ W(x, y) = -\beta Kxy$$

where $\beta > 0$ is the inverse temperature, $K \ge 0$.

For this model, by some simple calculations we have

$$b_0(r) = -\beta r, \ \forall r > 0.$$

and

$$h'(r) \equiv \beta^{-1},\ \forall r \geq 0.$$

It is obvious that conditions (5.2.2) and (5.2.3) hold, and the assumption (**H**) holds once if

$$K < 1. \tag{5.2.34}$$

But this condition is equivalent to say that the matrix $A = (a_{ij})_{1\leq i,j\leq N}$ is positively definite, where

$$a_{ii} = \beta, a_{ij} = \frac{-\beta K}{N-1}, i \neq j.$$

A must be the inverse of the covariance matrix of the Gaussian measure $\mu^{(N)}$. In other words **(H)** is equivalent to well defining the equilibrium probability measure $\mu^{(N)}$.
Note that $\|\nabla^2_{xy} W\|_\infty = \beta K$, so we have $c_{Lip} = \frac{1}{\beta(1-K)}$ under (5.2.34). Moreover (5.2.22) is satisfied with $M = -\beta$.

• *Sharpness of Theorem 5.1*. The gradient estimate (5.2.10) in Theorem 5.1 tells us: if $-\mathscr{L}^{(N)} G = g$, then

$$\|\nabla_i G\|_\infty \leq \frac{1}{\beta(1-K)} \max_i \|\nabla_i g\|_\infty.$$

Let us show that it becomes equality for $g(x^1, \cdots, x^N) = \sum_{i=1}^N x^i$. In fact

$$\mathscr{L}^{(N)} g(x^1, \cdots, x^N) = -\sum_i \beta x^i + \sum_i \frac{1}{N-1} \sum_{j\neq i} \beta K x^j = -\beta(1-K) g.$$

In other words $G = \frac{1}{\beta(1-K)} g$ for which the gradient estimate above becomes equality. As the gradient estimate (5.2.10) comes from (5.2.9), the process level $W_{1,d_{L^1}}$ estimate (5.2.9) is sharp too.

• *Sharpness of Theorem 5.5*. As $\varepsilon \to \infty$ in (5.2.23), we have by Theorem 5.5

$$W_{1,d_{l^1}}(P_t^{(N)}(x_0,\cdot), P_t^{(N)}(y_0,\cdot)) \leq e^{-\beta(1-K)t} d_{l^1}(x_0, y_0).$$

This is equivalent to say that

$$\max \|\nabla_i P_t^{(N)} g\|_\infty \leq e^{-\beta(1-K)t} \max \|\nabla_i g\|_\infty.$$

But it becomes equality for $g = \sum_{i=1}^N x^i$: in fact as $\mathscr{L}^{(N)} g = -\beta(1-K) g$,

$$P_t^{(N)} g = e^{-\beta(1-K)t} g.$$

Hence the exponential convergence result (5.2.24) in Theorem 5.5 is sharp.
Of course for this Gaussian model all results in Theorems 5.1 and 5.5 can be derived easily by using the synchronous coupling, or from the commutativity relation

$$\nabla P_t^{(N)} g = e^{-At} P_t^{(N)} \nabla g$$

which is one of the origins of the Bakry-Emery curvature.

Next we give another two typical models to illustrate our results.

Example 5.2. (Curie-Weiss mean-field lattice model) Let $d = 1$, and

$$V(x) = \beta(x^4/4 - x^2/2), \ W(x, y) = -\beta K x y$$

where $\beta = \frac{1}{\kappa T} > 0$ (κ is the Boltzmann constant) is the inverse temperature, $K \in \mathbb{R}^*$. This model is ferromagnetic or anti-ferromagnetic according to $K > 0$ or $K < 0$.
By an elementary calculation, we get

$$b_0(r) = \beta r(1 - r^2/4), \ \forall r > 0.$$

It is obvious that conditions (5.2.2) and (5.2.3) are satisfied and (5.2.22) holds with $M = \beta$.
For the assumption (**H**), first notice that $\|\nabla^2_{xy} W\|_\infty = |K|\beta$. Next we estimate $\|h'\|_\infty$. By (5.2.5) and some calculations, we have for any $r \geq 0$

$$\begin{aligned} h'(r) &= \frac{1}{4}\exp(\beta(r^4 - 8r^2)/64)\int_r^{+\infty} s\cdot\exp(\beta(8s^2 - s^4)/64)\,ds \\ &= \frac{1}{4}e^{\beta/4}\exp(\beta(r^4 - 8r^2)/64)\int_{r^2/2}^{+\infty}\exp(-\beta(u-2)^2/16)\,du. \end{aligned}$$

When $\frac{r^2}{2} > 2$, i.e. $r > 2$, we have

$$h'(r) \leq \frac{1}{4}e^{\beta/4}\exp(\beta(r^4 - 8r^2)/64)\sqrt{2\pi\frac{8}{\beta}}\exp(-\beta(\frac{r^2}{2} - 2)^2/16) = \frac{\sqrt{\pi}}{\sqrt{\beta}}. \tag{5.2.35}$$

When $0 \leq r \leq 2$, by (5.2.6) we have

$$4h''(r) = -r - \beta r(1 - r^2/4)h'(r) \leq 0,$$

hence

$$h'(r) \leq h'(0) = \frac{1}{4}e^{\beta/4}\int_0^{+\infty}\exp(-\beta(u-2)^2/16)\,du < e^{\beta/4}\frac{\sqrt{\pi}}{\sqrt{\beta}}. \tag{5.2.36}$$

Combining (5.2.35) and (5.2.36), we obtain $\|h'\|_\infty < e^{\beta/4}\frac{\sqrt{\pi}}{\sqrt{\beta}}$. Thus assumption (**H**) holds once if

$$|K|\sqrt{\pi\beta}e^{\beta/4} \leq 1 \tag{5.2.37}$$

and then the conclusions of Theorem 5.1 and Theorem 5.5 hold under (5.2.37).
For the result of propagation of chaos, we can take $c_1 = |K|\beta + \varepsilon'$, $c_2 = \frac{\beta}{4}(1 + |K| + \frac{\varepsilon'}{\beta})^2$ for any $\varepsilon' > 0$, and $c_3 = 0$. Then condition (5.2.30) is satisfied and then the conclusion of Theorem 5.9 holds under (5.2.37).

Example 5.3. (Double-Well confinement potential and quadratic interaction) Let $d = 1$, and

$$V(x) = \beta(x^4/4 - x^2/2), \ W(x, y) = \beta K(x - y)^2$$

where $\beta > 0$ is the inverse temperature, $K \in \mathbb{R}$. This model has the double-well confinement potential and quadratic interaction potential.
For this model, we have

$$b_0(r) = \beta r(1 - 2K - r^2/4), \ \forall r > 0.$$

So conditions (5.2.2) and (5.2.3) are satisfied. By the similar calculations as in Example 5.2, we get

$$\|h'\|_\infty < \begin{cases} e^{(1-2K)^2\beta/4}\frac{\sqrt{\pi}}{\sqrt{\beta}}, & \text{if } K \le \frac{1}{2}, \\ \frac{\sqrt{\pi}}{\sqrt{\beta}}, & \text{if } K > \frac{1}{2}. \end{cases}$$

Since $\|\nabla^2_{xy}W\|_\infty = 2|K|\beta$, assumption (**H**) holds once if

$$\begin{cases} 2|K|\sqrt{\pi\beta}e^{(1-2K)^2\beta/4} \le 1, & \text{if } \ K \le \frac{1}{2} \\ 2|K|\sqrt{\pi\beta} \le 1, & \text{if } \ K > \frac{1}{2}, \end{cases} \tag{5.2.38}$$

and then the conclusion of Theorem 5.1 holds under (5.2.38).
Furthermore, note that (5.2.22) holds with $M = \beta(1 - 2K)$, and

$$M + \|\nabla^2_{xy}W\|_\infty = \begin{cases} \beta, & \text{if } K \ge 0, \\ \beta(1-4K), & \text{if } K < 0 \end{cases}$$

which is strictly positive. Then the conclusion of Theorem 5.5 holds.
For the result of propagation of chaos in Theorem 5.9, we can take $c_3 = 0$ when $K \ge 0$, and $c_3 = -2K\beta$ when $K < 0$. To ensure that conditions (5.2.28) and (5.2.30) are satisfied, one can take $c_1 = 2|K|\beta + \varepsilon'$, $c_2 = \frac{\beta}{4}(1 + 2|K| + \frac{\varepsilon'}{\beta})^2$ in the case of $K > 0$ and $c_1 = -4K\beta + \varepsilon'$, $c_2 = \frac{\beta}{4}(1 - 4K + \frac{\varepsilon'}{\beta})^2$ in the case of $K < 0$, for any $\varepsilon' > 0$.

5.3 Proofs of Theorems 5.1 and 5.5

5.3.1 Coupling

We first introduce the approximate componentwise reflection coupling by following A. Eberle [14]. Given $\delta > 0$, let $\lambda_\delta, \pi_\delta : \mathbb{R}^+ \to [0, 1]$ be two Lipschitz continuous functions such that

$$\lambda_\delta(r)^2 + \pi_\delta(r)^2 = 1, \ \forall r \in \mathbb{R}^+ \tag{5.3.1}$$

and

$$\lambda_\delta(r) = \begin{cases} 1, \text{ if } r \ge \delta, \\ 0, \text{ if } r \le \delta/2. \end{cases} \tag{5.3.2}$$

Then a coupling of two solutions of the mean-field interacting particle system (5.1.3) with initial values $x_0, y_0 \in (\mathbb{R}^d)^N$ is given by a strong solution of the system

$$
\begin{aligned}
dX_t^{i,N} &= \sqrt{2}[\lambda_\delta(|Z_t^i|)dB_t^{1,i} + \pi_\delta(|Z_t^i|)dB_t^{2,i}] - \nabla V(X_t^{i,N})dt \\
&\qquad - \frac{1}{N-1}\sum_{j:j\neq i,1\le j\le N} \nabla_x W(X_t^{i,N}, X_t^{j,N})dt, \\
dY_t^{i,N} &= \sqrt{2}[\lambda_\delta(|Z_t^i|)R_t^i dB_t^{1,i} + \pi_\delta(|Z_t^i|)dB_t^{2,i}] - \nabla V(Y_t^{i,N})dt \\
&\qquad - \frac{1}{N-1}\sum_{j:j\neq i,1\le j\le N} \nabla_x W(Y_t^{i,N}, Y_t^{j,N})dt,
\end{aligned}
\tag{5.3.3}
$$

$1 \le i \le N$. Here $Z_t^i := X_t^{i,N} - Y_t^{i,N}$ and $R_t^i := I_d - 2e_t^i(e_t^i)^T$, where I_d is the d-dimensional unit matrix and $e_t^i(e_t^i)^T$ is the orthogonal projection onto the unit vector $e_t^i := Z_t^i/|Z_t^i|$ if $|Z_t^i| \neq 0$. $B_t^{1,i}$ and $B_t^{2,i}, 1 \le i \le N$, are independent standard Brownian motions taking values in $\mathbb{R}^d$. We will denote $X_t^{(N)} = (X_t^{1,N}, \cdots, X_t^{N,N})$, $Y_t^{(N)} = (Y_t^{1,N}, \cdots, Y_t^{N,N})$ and $Z_t^{(N)} := X_t^{(N)} - Y_t^{(N)}$.
To see that $(X_t^{(N)}, Y_t^{(N)})$ is a coupling process, it is enough to notice that

$$
\begin{aligned}
\hat{B}_t^i &:= \int_0^t \lambda_\delta(|Z_s^i|)dB_s^{1,i} + \int_0^t \pi_\delta(|Z_s^i|)dB_s^{2,i} \\
\check{B}_t^i &:= \int_0^t \lambda_\delta(|Z_s^i|)R_t^i dB_s^{1,i} + \int_0^t \pi_\delta(|Z_s^i|)dB_s^{2,i}, 1 \le i \le N,
\end{aligned}
\tag{5.3.4}
$$

are standard Brownian motions on $(\mathbb{R}^d)^N$.

Remark 5.12. (1) The coupling (5.3.3) behaves as a reflection coupling when the distance between the two particles $X_t^{i,N}$ and $Y_t^{i,N}$ are larger than δ. When the particles are very close (with distance less than $\frac{1}{2}\delta$), they are driven by the same Brownian motion, i.e., it is a synchronous coupling. And when the distance is between $\frac{1}{2}\delta$ and δ, it is a mixture of reflection coupling and synchronous coupling. The aim is to make λ_δ and π_δ globally Lipschitz continuous, so that the coupling SDE has a unique strong solution, given the independent Brownian motions $B_t^{1,i}, B_t^{2,i}$, $1 \le i \le N$.

(2) If one adopts the componentwise reflection coupling (i.e. the limit coupling when $\delta \to 0$), since $X^{i,N}$, $Y^{i,N}$ will separate after the time that they meet (i.e. $X_t^{i,N} = Y_t^{i,N}$), the local times will appear when Itô's formula is applied for $|X_t^{i,N} - Y_t^{i,N}|$. This makes the control of $\sum_{i=1}^N |X_t^{i,N} - Y_t^{i,N}|$ difficult to deal with. That is the reason why A. Eberle [14] introduced the synchronous coupling when $|X_t^{i,N} - Y_t^{i,N}|$ is small.

5.3.2 Proofs of Theorem 5.1

Proof of Theorem 5.1. **1). Proof of (5.2.9).** The first inequality in (5.2.9) is trivial, and next we prove the second inequality. By doing subtraction of the equations in (5.3.3), we have

$$
\begin{aligned}
dZ_t^i &= 2\sqrt{2}\lambda_\delta(|Z_t^i|)e_t^i d\tilde{B}_t^i - [\nabla V(X_t^{i,N}) - \nabla V(Y_t^{i,N})]dt \\
&\qquad - \frac{1}{N-1}\sum_{j:j\neq i,1\le j\le N} [\nabla_x W(X_t^{i,N}, X_t^{j,N}) - \nabla_x W(Y_t^{i,N}, Y_t^{j,N})]dt, \\
Z_0^i &= x_0^i - y_0^i,
\end{aligned}
\tag{5.3.5}
$$

where the processes $\tilde{B}^i_t = \int_0^t (e^i_s)^T dB^{1,i}_s, 1 \le i \le N$, are one-dimensional standard Brownian motions such that $\langle \tilde{B}^i, \tilde{B}^j \rangle_t = 0$ for $i \neq j$.
Let $r^i_t = |Z^i_t|, 1 \le i \le N$. By applying Itô's formula, we have

$$\begin{aligned} dr^i_t =& 1_{\{r^i_t \neq 0\}} 2\sqrt{2}\lambda_\delta(r^i_t) d\tilde{B}^i_t - 1_{\{r^i_t \neq 0\}} \langle e^i_t, \nabla V(X^{i,N}_t) - \nabla V(Y^{i,N}_t) \rangle dt \\ & - 1_{\{r^i_t \neq 0\}} \langle e^i_t, \frac{1}{N-1} \sum_{j: j\neq i, 1\le j \le N} [\nabla_x W(X^{i,N}_t, X^{j,N}_t) - \nabla_x W(Y^{i,N}_t, Y^{j,N}_t)] \rangle dt \\ & + 1_{\{r^i_t \neq 0\}} \sum_{k,l=1}^d [1_{\{k=l\}} (r^i_t)^{-1} - (X^{i,N,k}_t - Y^{i,N,k}_t)(X^{i,N,l}_t - Y^{i,N,l}_t)(r^i_t)^{-3}] \lambda_\delta(r^i_t)^2 (I_d - R^i_t)^2_{kl} dt, \end{aligned} \tag{5.3.6}$$

where $X^{i,N,k}_t$ and $Y^{i,N,k}_t$ denote the k-th coordinate of $X^{i,N}_t$ and $Y^{i,N}_t$ respectively, $1 \le k \le d$. Notice that the last term in the right hand side of the above equation equals to 0 by an easy calculation. Hence we get

$$\begin{aligned} dr^i_t =& 1_{\{r^i_t \neq 0\}} 2\sqrt{2}\lambda_\delta(r^i_t) d\tilde{B}^i_t - 1_{\{r^i_t \neq 0\}} \langle e^i_t, \frac{1}{N-1} \sum_{j: j\neq i, 1\le j \le N} [\nabla_x W(X^{i,N}_t, X^{j,N}_t) - \nabla_x W(X^{i,N}_t, Y^{j,N}_t)] \rangle dt \\ & - 1_{\{r^i_t \neq 0\}} \langle e^i_t, \nabla V(X^{i,N}_t) - \nabla V(Y^{i,N}_t) + \frac{1}{N-1} \sum_{j: j\neq i, 1\le j \le N} [\nabla_x W(X^{i,N}_t, Y^{j,N}_t) - \nabla_x W(Y^{i,N}_t, Y^{j,N}_t)] \rangle dt \\ \le & 1_{\{r^i_t \neq 0\}} 2\sqrt{2}\lambda_\delta(r^i_t) d\tilde{B}^i_t + \frac{1}{N-1} \|\nabla^2_{xy} W\|_\infty \sum_{j: j\neq i, 1\le j \le N} r^j_t dt + 1_{\{r^i_t \neq 0\}} b_0(r^i_t) dt, \end{aligned} \tag{5.3.7}$$

where we use the definition (5.2.1) of b_0 in the last inequality. Here $d\xi_t \le d\eta_t$ means that $\eta_t - \xi_t$ is a non-decreasing process.
Let L_{λ_δ} be the generator defined by for any function $f \in C^2(0, +\infty)$ and $r > 0$,

$$L_{\lambda_\delta} f(r) := 4\lambda^2_\delta(r) f''(r) + b_0(r) f'(r). \tag{5.3.8}$$

Note that L_{λ_δ} equals $\mathscr{L}_{ref}$ when $\lambda_\delta \equiv 1$.
Applying Itô's formula to the function $h(r^i_t)$ and using (5.3.7) and the fact that $h'(r) > 0$, we get for any $t > 0$ and $i = 1, \cdots, N$,

$$\begin{aligned} dh(r^i_t) \le & 2\sqrt{2}\lambda_\delta(r^i_t) h'(r^i_t) d\tilde{B}^i_t + h'(r^i_t) b_0(r^i_t) dt + 4h'' \lambda_\delta(r^i_t)^2 dt \\ & + \frac{1}{N-1} \|\nabla^2_{xy} W\|_\infty h'(r^i_t) \sum_{j: j\neq i, 1\le j \le N} r^j_t dt \\ =& 2\sqrt{2}\lambda_\delta(r^i_t) h'(r^i_t) d\tilde{B}^i_t + L_{\lambda_\delta} h(r^i_t) dt + \frac{1}{N-1} \|\nabla^2_{xy} W\|_\infty h'(r^i_t) \sum_{j: j\neq i, 1\le j \le N} r^j_t dt. \end{aligned} \tag{5.3.9}$$

Notice that by the definition of L_{λ_δ} and the Poisson equation (5.2.6),

$$L_{\lambda_\delta} h(r) = \mathscr{L}_{ref} h(r) + 4(\lambda^2_\delta - 1) h''(r) = -r + (1 - \lambda^2_\delta)(r + b_0(r) h'(r)). \tag{5.3.10}$$

Then

$$
\begin{aligned}
-\sum_{i=1}^{N}&\left(L_{\lambda_\delta}h(r_t^i)+\frac{1}{N-1}\|\nabla^2_{xy}W\|_\infty h'(r_t^i)\sum_{j:j\neq i,1\le j\le N}r_t^j\right)\\
&\ge(1-\|\nabla^2_{xy}W\|_\infty\|h'\|_\infty)\sum_{i=1}^{N}r_t^i-\sum_{i=1}^{N}(1-\lambda_\delta(r_t^i)^2)(r_t^i+b_0(r_t^i)h'(r_t^i))
\end{aligned}
$$

which is bounded from below by $-N(\delta+\sup_{r\in(0,\delta)}b_0^+(r)\|h'\|_\infty)$ according to the conditions **(H)** and (5.2.3). By integrating from 0 to T and taking expectation in the previous inequality (5.3.9) for $dh(r_t^i)$ and using Fatou's lemma, we have for any $T>0$,

$$
\begin{aligned}
&\mathbb{E}\int_0^T\left\{(1-\|\nabla^2_{xy}W\|_\infty\|h'\|_\infty)\sum_{i=1}^{N}r_t^i-\sum_{i=1}^{N}(1-\lambda_\delta(r_t^i)^2)(r_t^i+b_0(r_t^i)h'(r_t^i))\right\}dt\\
&\le\sum_{i=1}^{N}h(|x_0^i-y_0^i|).
\end{aligned}
\tag{5.3.11}
$$

Letting $\mathbb{P}_x|_{[0,T]}$ be the law of $(X_t^{(N)})_{t\in[0,T]}$, we obtain by assumption (**H**) and (5.3.11)

$$
\begin{aligned}
&W_{1,d_{L^1[0,T]}}(\mathbb{P}_{x_0}|_{[0,T]},\mathbb{P}_{y_0}|_{[0,T]})\le\mathbb{E}\int_0^T d_{l^1}(X_t^{(N)},Y_t^{(N)})dt=\mathbb{E}\int_0^T\sum_{i=1}^{N}r_t^i dt\\
&\le\frac{1}{1-\|\nabla^2_{xy}W\|_\infty\|h'\|_\infty}\left\{\sum_{i=1}^{N}h(|x_0^i-y_0^i|)+\sum_{i=1}^{N}\mathbb{E}\int_0^T(1-\lambda_\delta(r_t^i)^2)(r_t^i+b_0^+(r_t^i)h'(r_t^i))dt\right\}.
\end{aligned}
\tag{5.3.12}
$$

By the definition of λ_δ and the assumption $\lim_{r\to0}b_0^+(r)=0$, the second term in the right hand side of the inequality above converges to 0, a.s., as $\delta\downarrow0$. Hence

$$
W_{1,d_{L^1[0,T]}}(\mathbb{P}_{x_0}|_{[0,T]},\mathbb{P}_{y_0}|_{[0,T]})\le\frac{1}{1-\|\nabla^2_{xy}W\|_\infty\|h'\|_\infty}\sum_{i=1}^{N}h(|x_0^i-y_0^i|).\tag{5.3.13}
$$

Let Q_n be an optimization coupling of $(\mathbb{P}_{x_0}|_{[0,n]},\mathbb{P}_{y_0}|_{[0,n]})$ for $W_{1,d_{L^1[0,n]}}(\mathbb{P}_{x_0}|_{[0,n]},\mathbb{P}_{y_0}|_{[0,n]})$. Then $\{Q_n|_{[0,T]};n\ge T\}$ is tight for any finite time T (because their marginal distributions are respectively $\mathbb{P}_{x_0}|_{[0,T]}$ and $\mathbb{P}_{y_0}|_{[0,T]}$), hence one can find a probability measure Q on $C(\mathbb{R}^+,(\mathbb{R}^d)^N)^2$ such that $Q_n|_{[0,T]}\to Q|_{[0,T]}$ weakly for all $T>0$. Thus

$$
\begin{aligned}
W_{1,d_{L^1[0,\infty]}}(\mathbb{P}_{x_0},\mathbb{P}_{y_0})&\le\mathbb{E}^Q\int_0^\infty d_{l^1}(\gamma_1(t),\gamma_2(t))dt\\
&=\lim_{T\to+\infty}\mathbb{E}^Q\int_0^T d_{l^1}(\gamma_1(t),\gamma_2(t))dt\\
&\le\lim_{T\to\infty}\lim_{T\le n\to+\infty}\mathbb{E}^{Q_n}\int_0^T d_{l^1}(\gamma_1(t),\gamma_2(t))dt\\
&\le\lim_{n\to\infty}W_{1,d_{L^1[0,n]}}(\mathbb{P}_{x_0}|_{[0,n]},\mathbb{P}_{y_0}|_{[0,n]}).
\end{aligned}
$$

The converse inequality is evident. Therefore we have

$$
W_{1,d_{L^1}}(\mathbb{P}_{x_0},\mathbb{P}_{y_0})=\lim_{n\to\infty}W_{1,d_{L^1[0,n]}}(\mathbb{P}_{x_0}|_{[0,n]},\mathbb{P}_{y_0}|_{[0,n]}).
$$

From this and (5.3.13) we obtain (5.2.9).

2). Proof of (5.2.10). Note that for any Lipschitzian function g w.r.t the d_{l^1}-metric on $(\mathbb{R}^d)^N$, g is $\mu^{(N)}$-integrable because $\int \sum_{i=1}^N |x^i| d\mu^{(N)}(x) < +\infty$. So we can assume $\mu^{(N)}(g) = 0$ without loss of generality. Moreover we have

$$\begin{aligned}
\int_0^{+\infty} |P_t^{(N)} g(x)| dt &= \int_0^{+\infty} |P_t^{(N)} g(x) - \int P_t^{(N)} g(y) d\mu^{(N)}(y)| dt \\
&\leq \|g\|_{Lip(d_{l^1})} \int_{(\mathbb{R}^d)^N} \int_0^{+\infty} W_{d_{l^1}}(P_t^{(N)}(x,\cdot), P_t^{(N)}(y,\cdot)) dt d\mu^{(N)}(y) \\
&\leq \frac{\|g\|_{Lip(d_{l^1})}}{1 - \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty} \int_{(\mathbb{R}^d)^N} \sum_{i=1}^N h(|x^i - y^i|) d\mu^{(N)}(y) \\
&< +\infty,
\end{aligned}$$

then the unique solution of the Poisson equation $-\mathscr{L}^{(N)} G = g$ with $\mu^{(N)}(G) = 0$ is given by $G(x) = \int_0^{+\infty} P_t^{(N)} g(x) dt, \forall x \in (\mathbb{R}^d)^N$.

For each $1 \leq i \leq N$, letting $\tilde{x}^i \neq x^i$ and $\tilde{x} \in (\mathbb{R}^d)^N$ so that $(\tilde{x})^j = x^j$ for $j \neq i$ and $(\tilde{x})^i = \tilde{x}^i$, we have

$$\begin{aligned}
|\nabla_i G(x)| &\leq \limsup_{\tilde{x}^i \to x^i} \frac{|G(x) - G(\tilde{x})|}{|x^i - \tilde{x}^i|} \\
&\leq \limsup_{\tilde{x}^i \to x^i} \frac{1}{|x^i - \tilde{x}^i|} \int_0^{+\infty} |P_t^{(N)} g(x) - P_t^{(N)} g(\tilde{x})| dt \\
&\leq \limsup_{\tilde{x}^i \to x^i} \frac{1}{|x^i - \tilde{x}^i|} \|g\|_{Lip(d_{l^1})} \int_0^{+\infty} W_{d_{l^1}}(P_t^{(N)}(x,\cdot), P_t^{(N)}(\tilde{x},\cdot)) dt \qquad (5.3.14) \\
&\leq \frac{1}{1 - \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty} \|g\|_{Lip(d_{l^1})} \lim_{\tilde{x}^i \to x^i} \frac{h(|x^i - \tilde{x}^i|)}{|x^i - y^i|} \\
&= \frac{h'(0)}{1 - \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty} \|g\|_{Lip(d_{l^1})},
\end{aligned}$$

where the fourth inequality follows from (5.3.13). □

5.3.3 Proof of Theorem 5.5

Proof. Here we also adopt the coupling (5.3.3) . Let h be defined as in (5.2.5). Define for any $\varepsilon > 0$,

$$h_\varepsilon(r) := h(r) + \varepsilon r, \forall r \geq 0, \qquad (5.3.15)$$

and

$$H_t^\varepsilon := e^{K_\varepsilon t} \sum_{i=1}^N h_\varepsilon(r_t^i),$$

where $r_t^i = |X_t^{i,N} - Y_t^{i,N}|, 1 \le i \le N$, as in the proof of Theorem 5.1. By using Ito's formula and (5.3.7), we get for any $t \ge 0$,

$$\begin{aligned} dH_t^\varepsilon \le\ & 2\sqrt{2}e^{K_\varepsilon t}\sum_{i=1}^N \lambda_\delta(r_t^i)d\tilde{B}_t^i + K_\varepsilon H_t^\varepsilon dt + e^{K_\varepsilon t}\sum_{i=1}^N (L_{\lambda_\delta} h(r_t^i) + \varepsilon b_0(r_t^i))dt \\ & + e^{K_\varepsilon t}\sum_{i=1}^N (h'(r_t^i)+\varepsilon)\sum_{j:j\neq i}\frac{1}{N-1}\|\nabla_{xy}^2 W\|_\infty r_t^j dt \end{aligned} \tag{5.3.16}$$

Let

$$D_t^\varepsilon := K_\varepsilon H_t^\varepsilon + e^{K_\varepsilon t}\sum_{i=1}^N (L_{\lambda_\delta} h(r_t^i) + \varepsilon b_0(r_t^i)) + \frac{1}{N-1}e^{K_\varepsilon t}\|\nabla_{xy}^2 W\|_\infty \sum_{i\neq j, 1\le i,j\le N}(h'(r_t^i)+\varepsilon) r_t^j \tag{5.3.17}$$

be the drift term at the right hand side above. Calculating as in the proof of Theorem 5.1, we have

$$\begin{aligned} D_t^\varepsilon \le\ & e^{K_\varepsilon t}\sum_{i=1}^N [1-\lambda_\delta(r_t^i)^2][r_t^i + b_0(r_t^i)h'(r_t^i)] \\ & + e^{K_\varepsilon t}\sum_{i=1}^N \{K_\varepsilon h_\varepsilon(r_t^i) - [1-(\|h'\|_\infty+\varepsilon)\|\nabla_{xy}^2 W\|_\infty] r_t^i + \varepsilon b_0(r_t^i)\} \\ \le\ & e^{K_\varepsilon t}\sum_{i=1}^N [1-\lambda_\delta(r_t^i)^2][r_t^i + b_0(r_t^i)h'(r_t^i)] \\ & + e^{K_\varepsilon t}\sum_{i=1}^N \{K_\varepsilon(\|h'\|_\infty+\varepsilon) + \varepsilon M - [1-(\|h'\|_\infty+\varepsilon)\|\nabla_{xy}^2 W\|_\infty]\} r_t^i, \end{aligned} \tag{5.3.18}$$

where we use the assumption $b_0(r) \le Mr, \forall r > 0$.

By taking

$$K_\varepsilon = \frac{1-\|\nabla_{xy}^2 W\|_\infty \|h'\|_\infty - \varepsilon(M+\|\nabla_{xy}^2 W\|_\infty)}{\|h'\|_\infty + \varepsilon}, \tag{5.3.19}$$

the second term in the right hand of the inequality above vanishes. Then by taking expectation in (5.3.16) and the localization stopping time technique, we have for any $t \ge 0$,

$$\mathbb{E}e^{K_\varepsilon t}\sum_{i=1}^N h_\varepsilon(r_t^i) \le \sum_{i=1}^N h_\varepsilon(|x_0^i - y_0^i|) + \mathbb{E}\int_0^t e^{K_\varepsilon s}[1-\lambda_\delta(r_t^i)^2][r_t^i + b_0^+(r_t^i)h'(r_t^i)]ds. \tag{5.3.20}$$

Note that the second term in the right hand side of the above inequality converges to 0 as $\delta \downarrow 0$, under the assumption (5.2.3). Therefore we get

$$\begin{aligned} W_{1,d_{l^1}}(P_t^{(N)}(x_0,\cdot), P_t^{(N)}(y_0,\cdot)) &\le \lim_{\delta\to 0}\mathbb{E}\sum_{i=1}^N r_t^i \\ &\le \sup_{r>0}\frac{r}{h(r)+\varepsilon r}\lim_{\delta\to 0}\mathbb{E}\sum_{i=1}^N h_\varepsilon(r_t^i) \\ &\le \sup_{r>0}\frac{r}{h(r)+\varepsilon r} e^{-K_\varepsilon t}\sum_{i=1}^N h_\varepsilon(|x_0^i - y_0^i|) \\ &\le \sup_{r>0}\frac{r}{h(r)+\varepsilon r}\cdot \sup_{r>0}\frac{h(r)+\varepsilon r}{r} e^{-K_\varepsilon t}\sum_{i=1}^N |x_0^i - y_0^i| \end{aligned} \tag{5.3.21}$$

where the third inequality above follows by (5.3.20). □

5.4 Propagation of chaos

We begin with a uniform in time control of the second moment, which is more or less known, see e.g. Cattiaux *et al.* [9].

Lemma 5.13. *Suppose that there exist some positive constants c_1, c_2, c_3 such that*

$$\langle x, \nabla V(x)\rangle \geq c_1|x|^2 - c_2, \forall x \in \mathbb{R}^d \tag{5.4.1}$$

and

$$\langle z, \nabla^2_{xx} W(x,y)z\rangle \geq -c_3|z|^2, \forall x, y, z \in \mathbb{R}^d. \tag{5.4.2}$$

Assume

$$c_1 - c_3 - \|\nabla^2_{xy}W\|_\infty > 0. \tag{5.4.3}$$

Let X_t be a solution of (5.1.2) with $\mathbb{E}|X_0|^2 < \infty$, then for any $\varepsilon \in (0, c_1 - c_3 - \|\nabla^2_{xy}W\|_\infty)$,

$$\sup_{t\geq 0} \mathbb{E}(|X_t|^2)^{\frac{1}{2}} \leq \max\{m_2(\mu_0), \hat{c}(\varepsilon)\}, \tag{5.4.4}$$

where $m_2(\mu_0)$ and $\hat{c}(\varepsilon)$ are given in (5.2.32).

Proof. By Itô's formula, we have

$$d|X_t|^2 = -2\langle X_t, \nabla V(X_t)\rangle dt - 2\langle X_t, \nabla_x W \circledast \mu_t(X_t)\rangle dt + 2d\cdot dt + 2\sqrt{2}\langle X_t, dB_t\rangle$$

Notice that for any $x \in \mathbb{R}^d$, we have

$$\begin{aligned}\langle x, \nabla_x W \circledast \mu_t(x) - \nabla_x W \circledast \mu_t(0)\rangle &= \langle x, \int_0^1 \frac{d}{ds}\nabla_x W \circledast \mu_t(sx)ds\rangle \\ &= \langle x, \int_0^1 \frac{d}{ds}\int_{\mathbb{R}^d} \nabla_x W(sx,y)\mu_t(dy)ds\rangle \\ &= \int_0^1\int_{\mathbb{R}^d} \langle x, \nabla^2_{xx}W(sx,y)x\rangle \mu_t(dy)ds \\ &\geq -c_3|x|^2,\end{aligned} \tag{5.4.5}$$

where the last inequality follows from (5.4.2).
On the other hand,

$$\begin{aligned}|\nabla_x W \circledast \mu_t(0)\rangle| &\leq |\nabla_x W(0,0)| + \int_{\mathbb{R}^d} |\nabla_x W(0,y) - \nabla_x W(0,0)|\mu_t(dy) \\ &\leq |\nabla_x W(0,0)| + \|\nabla^2_{xy}W\|_\infty \mathbb{E}|X_t|.\end{aligned} \tag{5.4.6}$$

Therefore we have

$$\begin{aligned}\frac{d}{dt}|X_t|^2 &\le \Big(-2c_1|X_t|^2+2[c_3|X_t|^2+\|\nabla^2_{xy}W\|_\infty\cdot|X_t|^2+|\nabla_x W(0,0)||X_t|]+2(d+c_2)\Big)dt\\ &\qquad +2\sqrt{2}\langle X_t,dB_t\rangle\\ &\le -2(c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon)|X_t|^2dt+2(d+c_2+\frac{1}{4\varepsilon}|\nabla_x W(0,0)|)dt\\ &\qquad +2\sqrt{2}\langle X_t,dB_t\rangle\end{aligned}$$

where $0<\varepsilon<c_1-c_3-\|\nabla^2_{xy}W\|_\infty$. By the previous stochastic differential inequality,

$$|X_t|^2+\int_0^t 2(c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon)|X_t|^2-2(d+c_2+\frac{1}{4\varepsilon}|\nabla_x W(0,0)|)dt$$

is a local supermartingale, then a supermartingale by Fatou's lemma. Then $\mathbb{E}\int_0^T|X_t|^2dt<+\infty$. In other words $\int_0^t 2\sqrt{2}\langle X_s,dB_s\rangle$ is a L^2-martingale. By taking expectation in (5.4.1) we obtain by (5.4.5) and (5.4.6),

$$\begin{aligned}\frac{d}{dt}\mathbb{E}|X_t|^2 &\le -2c_1\mathbb{E}|X_t|^2+2[c_3\mathbb{E}|X_t|^2+\|\nabla^2_{xy}W\|_\infty(\mathbb{E}|X_t|)^2+|\nabla_x W(0,0)|\mathbb{E}|X_t|]+2(d+c_2)\\ &\le -2(c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon)\mathbb{E}|X_t|^2+2(d+c_2+\frac{1}{4\varepsilon}|\nabla_x W(0,0)|)\end{aligned}\tag{5.4.7}$$

where $0<\varepsilon<c_1-c_3-\|\nabla^2_{xy}W\|_\infty$. By Gronwall's lemma we get for any $t\ge 0$

$$\begin{aligned}\mathbb{E}|X_t|^2 &\le e^{-2(c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon)t}\left(\mathbb{E}|X_0|^2-\frac{d+c_2+\frac{1}{4\varepsilon}|\nabla_x W(0,0)|}{c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon}\right)\\ &\quad+\frac{d+c_2+\frac{1}{4\varepsilon}|\nabla_x W(0,0)|}{c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon}\\ &\le \max\left\{\mathbb{E}|X_0|^2,\frac{d+c_2+\frac{1}{4\varepsilon}|\nabla_x W(0,0)|}{c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon}\right\}\end{aligned}$$

the desired result. □

Following the proof above we have the much stronger uniform Gaussian integrability for X_t, which should be of independent interest.

Lemma 5.14. *Assume (5.4.1), (5.4.2) and (5.4.3). Let X_t be a solution of (5.1.2) with*

$$\mathbb{E}\exp\left(\lambda_0|X_0|^2\right)<\infty,\quad \textit{for some } \lambda_0>0.$$

If

$$0<\lambda\le\min\{\lambda_0;\frac{1}{2}(c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon)\}$$

for some $\varepsilon>0$, then

$$\sup_{t\ge0}\mathbb{E}\exp(\lambda|X_t|^2)<+\infty.$$

Proof. By Itô's formula, we have by the estimates leading to (5.4.7) in the proof of Lemma 5.13,

$$\begin{aligned} d\exp(\lambda|X_t|^2) &= \lambda\exp(\lambda|X_t|^2)\left([2d-2\langle X_t,\nabla V(X_t)+\nabla_x W\circledast\mu_t(X_t)\rangle]dt+2\sqrt{2}\langle X_t,dB_t\rangle\right)\\ &\quad+4\lambda^2|X_t|^2\exp(\lambda|X_t|^2)dt\\ &\le \lambda\exp(\lambda|X_t|^2)\left[-2(c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon-2\lambda)|X_t|^2+2(d+c_2+\frac{1}{4\varepsilon}|\nabla_x W(0,0)|)\right]dt\\ &\quad+\lambda\exp(\lambda|X_t|^2)2\sqrt{2}\langle X_t,dB_t\rangle \end{aligned}$$

where $\varepsilon>0,\lambda>0$ verify $c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon-2\lambda>0$. Taking $L>0$ sufficiently large so that

$$c_5:=2(c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon-2\lambda)L^2-2(d+c_2+\frac{1}{4\varepsilon}|\nabla_x W(0,0)|>0,$$

and noting that

$$-ax^2+b\le-(aL^2-b)+aL^2 1_{|x|\le L},\ a>0,x\in\mathbb{R},$$

we obtain by following the same argument as in Lemma 5.13

$$\frac{d}{dt}\mathbb{E}\exp(\lambda|X_t|^2)\le-\lambda c_5\mathbb{E}\exp(\lambda|X_t|^2)dt+2(c_1-c_3-\|\nabla^2_{xy}W\|_\infty-\varepsilon-2\lambda)L^2\lambda e^{\lambda L^2}.$$

Therefore by Gromwall's inequality

$$\sup_{t\ge0}\mathbb{E}\exp(\lambda|X_t|^2)<+\infty.$$

□

Next we present the proof of Theorem 5.9, which is quite close to those of Theorems 5.1 and 5.5 .

Proof of Theorem 5.9. Let λ_δ and π_δ be defined as in Section 3.1. Consider the following coupling between the independent copies $\bar X^i_t, 1\le i\le N$ of the nonlinear diffusion processes (5.1.2) and the mean-field interacting particle system (5.1.3):

$$\begin{aligned} d\bar X^i_t &= \sqrt{2}[\lambda_\delta(|Z^i_t|)dB^{1,i}_t+\pi_\delta(|Z^i_t|)dB^{2,i}_t]-\nabla V(\bar X^i_t)dt-\nabla_x W\circledast\mu_t(\bar X^i_t)dt,\\ dX^{i,N}_t &= \sqrt{2}[\lambda_\delta(|Z^i_t|)R^i_t dB^{1,i}_t+\pi_\delta(|Z^i_t|)dB^{2,i}_t]-\nabla V(X^{i,N}_t)dt\\ &\quad-\frac{1}{N-1}\sum_{j:j\ne i,1\le j\le N}\nabla_x W(X^{i,N}_t,X^{j,N}_t)dt. \end{aligned} \tag{5.4.8}$$

Here $Z^i_t:=\bar X^i_t-X^{i,N}_t$ and $R^i_t:=I_d-2e^i_te^{i,T}_t$, where I_d is the d-dimensional unit matrix and $e^i_te^{i,T}_t$ is the orthogonal projection onto the unit vector $e^i_t:=Z^i_t/|Z^i_t|$ if $|Z^i_t|\ne0$. $B^{1,i}_t$ and $B^{2,i}_t, 1\le i\le N$, are independent standard Brownian motions in $\mathbb{R}^d$. We assume that $\bar X^i_t$ and $X^{i,N}_t$, $1\le i\le N$ have the same starting points $X^i_0, 1\le i\le N$, i.i.d. of law μ_0. The independence of $\bar X^i_t, 1\le i\le N$ comes from the fact that the Brownian motions $\{\int_0^t\lambda_\delta(|Z^i_s|)dB^{1,i}_s+\int_0^t\pi_\delta(|Z^i_s|)dB^{2,i}_s,\ 1\le i\le N\}$ are independent because their inter-brackets are zero.

By doing subtraction of the equations in (5.4.8), we have

$$dZ_t^i = 2\sqrt{2}\lambda_\delta(|Z_t^i|)e_t^i d\tilde{B}_t^i - [\nabla V(\bar{X}_t^i) - \nabla V(X_t^{i,N})]dt - \nabla_x W \circledast \mu_t(\bar{X}_t^i)dt + \frac{1}{N-1}\sum_{j:j\neq i,1\le j\le N} \nabla_x W(X_t^{i,N}, X_t^{j,N})dt,$$

where the processes $\tilde{B}_t^i = \int_0^t (e_s^i)^T dB_s^{1,i}$, $1 \le i \le N$, are one-dimensional standard Brownian motions such that $\langle \tilde{B}^i, \tilde{B}^j\rangle_t = 0$ for $i \neq j$.

Let $r_t^i = |Z_t^i|, 1 \le i \le N$. By applying Itô's formula, we have

$$\begin{aligned}
dr_t^i &= 1_{\{r_t^i\neq 0\}} 2\sqrt{2}\lambda_\delta(r_t^i)d\tilde{B}_t^i - 1_{\{r_t^i\neq 0\}}\langle e_t^i, \nabla V(\bar{X}_t^i) - \nabla V(X_t^{i,N})\rangle dt \\
&\quad - 1_{\{r_t^i\neq 0\}}\langle e_t^i, \nabla_x W \circledast \mu_t(\bar{X}_t^i) - \frac{1}{N-1}\sum_{j:j\neq i,1\le j\le N} \nabla_x W(X_t^{i,N}, X_t^{j,N})\rangle dt \\
&= 1_{\{r_t^i\neq 0\}} 2\sqrt{2}\lambda_\delta(r_t^i)d\tilde{B}_t^i \\
&\quad - 1_{\{r_t^i\neq 0\}}\langle e_t^i, \nabla V(\bar{X}_t^i) - \nabla V(X_t^{i,N})\rangle dt \\
&\quad - 1_{\{r_t^i\neq 0\}}\langle e_t^i, \nabla_x W \circledast \mu_t(\bar{X}_t^i) - \frac{1}{N-1}\sum_{j:j\neq i,1\le j\le N} \nabla_x W(\bar{X}_t^i, \bar{X}_t^j)\rangle dt \\
&\quad - 1_{\{r_t^i\neq 0\}}\langle e_t^i, \frac{1}{N-1}\sum_{j:j\neq i,1\le j\le N} [\nabla_x W(\bar{X}_t^i, \bar{X}_t^j) - \nabla_x W(\bar{X}_t^i, X_t^{j,N})]\rangle dt \\
&\quad - 1_{\{r_t^i\neq 0\}}\langle e_t^i, \frac{1}{N-1}\sum_{j:j\neq i,1\le j\le N} [\nabla_x W(\bar{X}_t^i, X_t^{j,N}) - \nabla_x W(X_t^{i,N}, X_t^{j,N})]\rangle dt.
\end{aligned}$$

Remark that the sum of the first and the fourth drift terms above is $\le b_0(r_t^i)dt$, the third drift term above is $\le \frac{1}{N-1}\|\nabla^2_{xy}W\|_\infty \sum_{j:j\neq i,1\le j\le N} r_t^j dt$, and the second drift term is bounded by $I_t^i dt$, where

$$I_t^i := |\nabla_x W \circledast \mu_t(\bar{X}_t^i) - \frac{1}{N-1}\sum_{j:j\neq i,1\le j\le N} \nabla_x W(\bar{X}_t^i, \bar{X}_t^j)|. \tag{5.4.9}$$

Therefore we obtain

$$dr_t^i \le 2\sqrt{2}\lambda_\delta(r_t^i)d\tilde{B}_t^i + b_0(r_t^i)dt + \frac{1}{N-1}\|\nabla^2_{xy}W\|_\infty \sum_{j:j\neq i,1\le j\le N} r_t^j dt + I_t^i dt. \tag{5.4.10}$$

Recall that for any $\varepsilon \ge 0$, $h_\varepsilon(r) = h(r) + \varepsilon r, \forall r \ge 0$. By using (5.4.10) and Itô's formula again, we get

$$\begin{aligned}
dh_\varepsilon(r_t^i) &\le 2\sqrt{2}\lambda_\delta(r_t^i)h'_\varepsilon(r_t^i)d\tilde{B}_t^i + 4\lambda_\delta^2(r_t^i)h''_\varepsilon(r_t^i)dt + b_0(r_t^i)h'_\varepsilon(r_t^i)dt \\
&\quad + \frac{1}{N-1}\|\nabla^2_{xy}W\|_\infty h'_\varepsilon(r_t^i)\sum_{j:j\neq i,1\le j\le N} r_t^j dt + h'_\varepsilon(r_t^i)I_t^i dt \\
&= 2\sqrt{2}\lambda_\delta(r_t^i)h'_\varepsilon(r_t^i)d\tilde{B}_t^i + [4\lambda_\delta^2(r_t^i)h''(r_t^i) + b_0(r_t^i)h'(r_t^i)]dt + \varepsilon b_0(r_t^i)dt \\
&\quad + \frac{1}{N-1}\|\nabla^2_{xy}W\|_\infty (h'(r_t^i)+\varepsilon)\sum_{j:j\neq i,1\le j\le N} r_t^j dt + (h'(r_t^i)+\varepsilon)I_t^i dt \\
&\le 2\sqrt{2}\lambda_\delta(r_t^i)h'_\varepsilon(r_t^i)d\tilde{B}_t^i + [1-\lambda_\delta^2(r_t^i)][r_t^i + b_0(r_t^i)h'(r_t^i)]dt - (1-\varepsilon M)r_t^i dt \\
&\quad + \frac{1}{N-1}\|\nabla^2_{xy}W\|_\infty (\|h'\|_\infty+\varepsilon)\sum_{j:j\neq i,1\le j\le N} r_t^j dt + (\|h'\|_\infty+\varepsilon)I_t^i dt,
\end{aligned} \tag{5.4.11}$$

where the last inequality follows from (5.3.8), (5.3.10) and (5.2.22).
Taking expectation in the inequality above and using the fact that $r_t^i, 1 \le i \le \mathrm{N}$ have the same law, and setting

$$c_\varepsilon := 1 - \|\nabla^2_{xy}\mathrm{W}\|_\infty \|h'\|_\infty - \varepsilon(\mathrm{M} + \|\nabla^2_{xy}\mathrm{W}\|_\infty),$$

we have

$$d\mathbb{E}h_\varepsilon(r_t^1) \le \mathbb{E}[1-\lambda_\delta(r_t^1)^2][r_t^1 + b_0^+(r_t^1)h'(r_t^1)]dt + (\|h'\|_\infty + \varepsilon)\mathbb{E}\mathrm{I}_t^1 dt - c_\varepsilon \mathbb{E}r_t^1 dt \tag{5.4.12}$$

Proof of part (a). Choose $\varepsilon = 0$, $c_0 = 1 - \|\nabla^2_{xy}\mathrm{W}\|_\infty \|h'\|_\infty$. For any $1 \le k \le \mathrm{N}$, by (5.4.12) we have

$$\begin{aligned}
\frac{1}{k}\mathrm{W}_{1,d_{\mathrm{L}^1[0,\mathrm{T}]}}(\mathbb{P}^{[1,k],\mathrm{N}}_{\mu_0^{\otimes \mathrm{N}}}|_{[0,\mathrm{T}]}, \mathbb{Q}^{\otimes k}_{\mu_0}|_{[0,\mathrm{T}]}) &\le \frac{1}{k}\mathbb{E}\int_0^{\mathrm{T}} \sum_{i=1}^k r_t^i dt = \int_0^{\mathrm{T}} \mathbb{E}r_t^1 dt \\
&\le \frac{1}{c_0}\|h'\|_\infty \int_0^{\mathrm{T}} \mathbb{E}\mathrm{I}_t^1 dt dt + \frac{1}{c_0}\mathbb{E}\int_0^{\mathrm{T}} [1-\lambda_\delta(r_t^1)^2][r_t^1 + b_0^+(r_t^1)h'(r_t^1)]dt.
\end{aligned}$$

Letting $\delta \to 0+$, the last term tends to zero. Hence

$$\frac{1}{k}\mathrm{W}_{1,d_{\mathrm{L}^1[0,\mathrm{T}]}}(\mathbb{P}^{[1,k],\mathrm{N}}_{\mu_0^{\otimes \mathrm{N}}}|_{[0,\mathrm{T}]}, \mathbb{Q}^{\otimes k}_{\mu_0}|_{[0,\mathrm{T}]}) \le \frac{\|h'\|_\infty}{c_0}\int_0^{\mathrm{T}} \mathbb{E}\mathrm{I}_t^1 dt. \tag{5.4.13}$$

Next we estimate $\mathbb{E}\mathrm{I}_t^1$, which is the only new point w.r.t. the proofs in Theorems 5.1 and 5.5. Note that $\bar{\mathrm{X}}_t^j, 2 \le j \le \mathrm{N}$ are independent copies of $\bar{\mathrm{X}}_t^1$, and

$$\mathbb{E}[\nabla_x\mathrm{W}(\bar{\mathrm{X}}_t^1, \bar{\mathrm{X}}_t^j)|\bar{\mathrm{X}}_t^1] = \nabla_x\mathrm{W} \circledast \mu_t(\bar{\mathrm{X}}_t^1).$$

Thus by using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\mathbb{E}\mathrm{I}_t^1 &\le \left(\mathbb{E}\left\{\mathbb{E}\left[|\nabla_x\mathrm{W} \circledast \mu_t(\bar{\mathrm{X}}_t^1) - \frac{1}{\mathrm{N}-1}\sum_{2\le j\le \mathrm{N}} \nabla_x\mathrm{W}(\bar{\mathrm{X}}_t^1, \bar{\mathrm{X}}_t^j)|^2 |\bar{\mathrm{X}}_t^1\right]\right\}\right)^{\frac{1}{2}} \\
&= \left(\mathbb{E}\frac{1}{\mathrm{N}-1}\int |\nabla_x\mathrm{W}(\bar{\mathrm{X}}_t^1, y) - \nabla_x\mathrm{W} * \mu_t(\bar{\mathrm{X}}_t^1)|^2 d\mu_t(y)\right)^{\frac{1}{2}} \\
&\le \frac{1}{\sqrt{\mathrm{N}-1}}\|\nabla^2_{xy}\mathrm{W}\|_\infty \left(\int_{x\in\mathbb{R}^d} |x - \mu_t(\bar{\mathrm{X}}_t^1)|^2 \mu_t(dx)\right)^{\frac{1}{2}} \\
&\le \frac{1}{\sqrt{\mathrm{N}-1}}\|\nabla^2_{xy}\mathrm{W}\|_\infty \sup_{t\ge 0}(\mathbb{E}|\mathrm{X}_t|^2)^{\frac{1}{2}}.
\end{aligned} \tag{5.4.14}$$

Plugging (5.4.14) into (5.4.13), we get

$$\frac{1}{k}\mathrm{W}_{1,d_{\mathrm{L}^1[0,\mathrm{T}]}}(\mathbb{P}^{[1,k],\mathrm{N}}_{\mu_0^{\otimes \mathrm{N}}}|_{[0,\mathrm{T}]}, \mathbb{Q}^{\otimes k}_{\mu_0}|_{[0,\mathrm{T}]}) \le \frac{\mathrm{T}}{\sqrt{\mathrm{N}-1}} \frac{\|\nabla^2_{xy}\mathrm{W}\|_\infty \|h'\|_\infty}{1 - \|\nabla^2_{xy}\mathrm{W}\|_\infty \|h'\|_\infty} \sup_{t\ge 0}(\mathbb{E}|\mathrm{X}_t|^2)^{\frac{1}{2}}.$$

Then by using Lemma 5.13, we obtain the desired result (5.2.31).

Proof of part (b). For any $\varepsilon > 0$, by (5.4.12) we have

$$d\mathbb{E}h_\varepsilon(r_t^1) \le \mathbb{E}[1-\lambda_\delta(r_t^1)^2][r_t^1 + b_0^+(r_t^1)h'(r_t^1)]dt + (\|h'\|_\infty + \varepsilon)\mathbb{E}I_t^1 dt - c_\varepsilon \cdot \inf_{r>0} \frac{r}{h(r)+\varepsilon r}\mathbb{E}h_\varepsilon(r_t^1)dt \tag{5.4.15}$$

Plugging (5.4.14) into (5.4.15), we obtain by Gronwall's inequality that for any $\varepsilon > 0$ so that $\beta = c_\varepsilon \cdot \inf_{r>0} \frac{r}{h(r)+\varepsilon r} > 0$ (i.e. $K_\varepsilon > 0$),

$$\begin{aligned}\inf_{r>0}\frac{h_\varepsilon(r)}{r}\cdot\mathbb{E}|\bar{X}_t^1 - X_t^{1,N}| &\le \mathbb{E}h_\varepsilon(|\bar{X}_t^1 - X_t^{1,N}|)\\ &\le \int_0^t e^{-\beta(t-s)}\frac{1}{\sqrt{N-1}}(\|h'\|_\infty+\varepsilon)\|\nabla^2_{xy}W\|_\infty \sup_{t\ge 0}(\mathbb{E}|X_t|^2)^{\frac12}ds\\ &\quad + \int_0^t e^{-\beta(t-s)}\mathbb{E}[1-\lambda_\delta(r_s^1)^2][r_s^1 + b_0^+(r_s^1)h'(r_s^1)]ds.\end{aligned} \tag{5.4.16}$$

By letting $\delta \to 0+$, the last term tends to zero. As the joint law of $(\bar{X}_t^i, 1\le i\le k)$ is $\mu_t^{\otimes k}$, we get for any $1 \le k \le N$,

$$\begin{aligned}W_{1,d_{l^1}}(\mu_t^{\otimes k}, \mu_t^{[1,k],N}) &\le \limsup_{\delta\to 0}\mathbb{E}\sum_{i=1}^k |\bar{X}_t^i - X_t^{i,N}| = k\cdot\limsup_{\delta\to 0}\mathbb{E}|\bar{X}_t^1 - X_t^{1,N}|\\ &\le k\cdot\sup_{r>0}\frac{r}{h(r)+\varepsilon r}\frac{1}{\beta\sqrt{N-1}}(\|h'\|_\infty+\varepsilon)\|\nabla^2_{xy}W\|_\infty\sup_{t\ge0}(\mathbb{E}|X_t|^2)^{\frac12}\\ &= \frac{k}{\sqrt{N-1}}\frac{A_\varepsilon}{K_\varepsilon}\|\nabla^2_{xy}W\|_\infty\sup_{t\ge0}(\mathbb{E}|X_t|^2)^{\frac12},\end{aligned} \tag{5.4.17}$$

which completes the proof by using Lemma 5.13. □

The proof above yields also a control of $W_1\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}, \mu_t\right)$.

Proposition 5.15. *Under the conditions of Theorem 5.9, we have*

$$\mathbb{E}W_1\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}, \mu_t\right) \le \frac{1}{\sqrt{N-1}}\frac{A_\varepsilon}{K_\varepsilon}\|\nabla^2_{xy}W\|_\infty\sup_{t\ge0}(\mathbb{E}|X_t|^2)^{\frac12} + \mathbb{E}W_1\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}, \mu_t\right) \tag{5.4.18}$$

where $(X_t^i)_{t\ge0}, i\ge 1$ are independent copies of the solution $(X_t)_{t\ge0}$ of the McKean-Vlasov equation (5.1.2).

Proof. At first by the triangle inequality,

$$W_1\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}, \mu_t\right) \le W_1\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}, \frac{1}{N}\sum_{i=1}^N \delta_{\bar{X}_t^i}\right) + W_1\left(\frac{1}{N}\sum_{i=1}^N \delta_{\bar{X}_t^i}, \mu_t\right).$$

Notice that

$$W_1\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}, \frac{1}{N}\sum_{i=1}^N \delta_{\bar{X}_t^i}\right) \le \frac{1}{N}\sum_{i=1}^N |X_t^{i,N} - \bar{X}_t^i| = \frac{1}{N}\sum_{i=1}^N r_t^i.$$

And by (5.4.17),

$$\limsup_{\delta\to 0}\mathbb{E}\frac{1}{N}\sum_{i=1}^{n} r_t^i = \limsup_{\delta\to 0}\mathbb{E}r_t^1 \le \frac{1}{\sqrt{N-1}}\frac{A_\varepsilon}{K_\varepsilon}\|\nabla^2_{xy}W\|_\infty \sup_{t\ge 0}(\mathbb{E}|X_t|^2)^{\frac{1}{2}}.$$

Therefore we obtain (5.4.18) by noting that $\bar{X}_t^i, 1\le i\le N$ and $X_t^i, 1\le i\le N$ are of the same law μ_t. □

Remark 5.16. In one-dimensional case, i.e. $d=1$, it is well known that

$$W_1\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i},\mu_t\right)=\int_{-\infty}^{\infty}|\frac{1}{N}\sum_{i=1}^{N}1_{(-\infty,x]}(X_t^i)-\mu_t(-\infty,x]|dx.$$

Then letting $F_t(x)=\mu_t(-\infty,x]$ (the distribution function), we have by Cauchy-Schwarz,

$$\begin{aligned}\mathbb{E}W_1\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i},\mu_t\right)&\le\int_{\mathbb{R}}\sqrt{\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N}1_{(-\infty,x]}(X_t^i)\right)}dx\\&=\frac{1}{\sqrt{N}}\int_{\mathbb{R}}\sqrt{F_t(x)(1-F_t(x))}dx.\end{aligned}$$

where the last factor is uniformly bounded in time $t>0$ by K once if $\sup_{t\ge 0}\mathbb{E}|X_t|^{2+\varepsilon}<+\infty$ for some $\varepsilon>0$. The latter uniform $2+\varepsilon$-moment condition is verified once if μ_0 has the $2+\varepsilon$-moment by the argument in Lemmas 5.13 and 5.14. In other words if μ_0 has the $2+\varepsilon$-moment, there is some constant $K>0$ such that

$$\mathbb{E}W_1\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{\bar{X}_t^i},\mu_t\right)\le\frac{K}{\sqrt{N}},\ \forall t>0 \tag{5.4.19}$$

and then the same type bound holds for $\mathbb{E}W_1\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^{i,N}},\mu_t\right)$.

5.5 Quantitative Concentration inequalities

This section is devoted to the concentration inequalities of the mean-field interaction particle system (5.1.3), as applications of our main theorems. This kind of concentration estimate are useful to numerical simulations and mean-field limit. Under the conditions that V is uniformly convex and W is convex, Malrieu [22] established logarithmic Sobolev inequality and then used its connection with optimal transport and concentration of measure to get the following non-asymptotic bounds on the deviation of the empirical mean of an observable f from its true mean,

$$\sup_{\|f\|_{Lip}\le 1}\mathbb{P}\left\{|\frac{1}{N}\sum_{i=1}^{N}f(X_t^{i,N})-\mu_t(f)|>\frac{A}{\sqrt{N}}+\delta\right\}\le 2e^{-\lambda N\delta^2}, \tag{5.5.1}$$

where A and λ are positive constants depending on the particle system.

As pointed out in [6], this approach can lead to nice bounds but it is limited to a finite number of observables. Bolley-Guillin-Villani [6, Theorem 2.9] obtained for any $t>0$ fixed and $\delta>0$

$$\mathbb{P}\left\{\sup_{\|f\|_{Lip}\le 1}|\frac{1}{N}\sum_{i=1}^{N}f(X_t^{i,N})-\mu_t(f)|>\delta\right\}\le C(1+t\delta^{-2})e^{-K(t)N\delta^2}, \tag{5.5.2}$$

for all N big enough (quantifiable), where $K(t)$ depending on t is some explicitly computable constant. Furthermore, Bolley [4] got quantitative concentration inequalities on the sample path space with uniform norm, on a given time interval $[0,T]$, which implies (5.5.2) by projection at time $t \in [0,T]$.

5.5.1 Uniform in time concentration inequality

Our previous general results allow us to generalize (5.5.1) and to reinforce (5.5.2) (under stronger conditions of course).

Proposition 5.17. *Assume* **(H)**, *(5.2.2) and (5.2.22). Then for any Lipschitzian function* F *on* $(\mathbb{R}^d)^N$, *we have for any lower bounded convex function* φ *on* $\mathbb{R}$,

$$\mathbb{E}_x\varphi\left(F(X_T^{(N)}) - \mathbb{E}_xF(X_T^{(N)})\right) \le \mathbb{E}\varphi\left(\alpha A_\varepsilon\sqrt{\frac{N}{2K_\varepsilon}}\xi\right),\ \forall x \in (\mathbb{R}^d)^N,\ \forall T>0 \tag{5.5.3}$$

where ξ *is some standard real Gaussian random variable of law* $\mathcal{N}(0,1)$, $\alpha := \|F\|_{Lip(d_{l^1})} = \max_{1\le i\le N}\|\nabla_iF\|_\infty$, A_ε *and* K_ε *are given in Theorem 5.5.*
In particular for any initial distribution μ_0 *satisfying the Gaussian integrability assumption on* $\mathbb{R}^d$, *we have for any* $\delta, T>0$

$$\mathbb{P}_{\mu_0^{\otimes N}}\left\{F(X_T^{(N)}) - \mathbb{E}_{\mu_0^{\otimes N}}F(X_T^{(N)}) > \delta\right\} \le \exp\left(-\frac{K_\varepsilon\delta^2}{N\alpha^2A_\varepsilon^2\left[1+2c_G(\mu_0)K_\varepsilon e^{-2K_\varepsilon T}\right]}\right). \tag{5.5.4}$$

Proof. Without loss of generality we may assume that $\alpha = \max_{1\le i\le N}\|\nabla_iF\|_\infty = 1$.
By approximation we may assume that F is C^2-smooth with bounded derivatives of the first and the second order. For any initial position $x \in (\mathbb{R}^d)^N$, let $M_t = \mathbb{E}_x(F(X_T^{(N)})|\mathscr{F}_t)$, $0 \le t \le T$. Then by applying Itô's formula to $u(t,x) = P_{T-t}F(x)$, we have

$$F(X_T^{(N)}) - \mathbb{E}_xF(X_T^{(N)}) = M_T - M_0 = \sum_{i=1}^N \int_0^T \nabla_iP_{T-t}F(X_t^{(N)})dB_t^i, \tag{5.5.5}$$

Note that by Theorem 5.5, for any $\varepsilon>0$ such that $K_\varepsilon>0$, we have

$$W_{d_{l^1}}(P_t^{(N)}(x,\cdot), P_t^{(N)}(y,\cdot)) \le A_\varepsilon e^{-K_\varepsilon t}d_{l^1}(x,y), \forall x,y \in (\mathbb{R}^d)^N, \tag{5.5.6}$$

which implies that

$$|\nabla_iP_{T-t}F| \le A_\varepsilon e^{-K_\varepsilon(T-t)},\ 1\le i\le N, \tag{5.5.7}$$

where A_ε and K_ε are the same as given in Theorem 5.5.
Since $M_t = \xi_{\tau_t}$ where (ξ_t) is a real valued Brownian motion w.r.t. some new filtration $(\tilde{\mathscr{F}}_t)$ and

$$\tau_t = \langle M\rangle_t = \int_0^t \sum_{i=1}^N |\nabla_iP_{t-s}F(X_s^{(N)})|^2ds \le \frac{A_\varepsilon^2}{2K_\varepsilon}N =: CN$$

is a stopping time w.r.t. $(\tilde{\mathscr{F}}_t)$, we obtain

$$\begin{aligned}\mathbb{E}_x\varphi\left(F(X_T^{(N)})-\mathbb{E}_xF(X_T^{(N)})\right)&=\mathbb{E}\varphi\left(M_T-M_0\right)=\mathbb{E}\varphi(\xi_{\tau_T})\\&=\mathbb{E}\varphi\left(\mathbb{E}(\xi_{CN}|\tilde{\mathscr{F}}_{\tau_T})\right)\\&\le\mathbb{E}\varphi\left(\xi_{CN}\right)\quad\text{(by Jensen's inequality)}\\&=\mathbb{E}\varphi\left(A_\varepsilon\sqrt{\frac{N}{2K_\varepsilon}}\xi_1\right)\end{aligned}\tag{5.5.8}$$

the desired result (5.5.3).

Letting $g(x):=\mathbb{E}_xF(X_T^{(N)})$, $\forall x\in(\mathbb{R}^d)^N$. By (5.5.7) we have

$$\|g\|_{Lip(d_{l^1})}=\max_{1\le i\le N}\|\nabla_i g\|_\infty\le A_\varepsilon e^{-K_\varepsilon T}.\tag{5.5.9}$$

Applying (5.5.3) to $\varphi(z)=e^{\lambda z}$ $(\lambda\in\mathbb{R})$, we get

$$\begin{aligned}&\mathbb{E}_{\mu_0^{\otimes N}}\exp\left(\lambda[F(X_T^{(N)})-\mathbb{E}_{\mu_0^{\otimes N}}F(X_T^{(N)})]\right)\\&=\int_{(\mathbb{R}^d)^N}\mathbb{E}_x\exp\left(\lambda[F(X_T^{(N)})-\mathbb{E}_xF(X_T^{(N)})]\right)\cdot\exp\left(\lambda[g(x)-\mu_0^{\otimes N}(g)]\right)d\mu_0^{\otimes N}(x)\\&\le\int_{(\mathbb{R}^d)^N}\mathbb{E}\exp\left(\lambda A_\varepsilon\sqrt{\frac{N}{2K_\varepsilon}}\xi_1\right)\cdot\exp\left(\lambda[g(x)-\mu_0^{\otimes N}(g)]\right)d\mu_0^{\otimes N}(x)\\&\le\exp\left(\frac{NA_\varepsilon^2\lambda^2}{4K_\varepsilon}\right)\exp\left(\frac{\lambda^2}{2}Nc_G(\mu_0)\|g\|^2_{Lip(d_{l^1})}\right)\\&\le\exp\left(\frac{N\lambda^2A_\varepsilon^2}{2}\left[\frac{1}{2K_\varepsilon}+c_G(\mu_0)e^{-2K_\varepsilon T}\right]\right)\end{aligned}$$

where the third and the last inequality follows from the Gaussian concentration condition on the initial distribution μ_0 (see (5.2.19) in Remark 5.3) and (5.5.9) respectively.

Finally the concentration inequality (5.5.4) is derived from the above inequality by the standard procedure of Chebyshev's inequality and optimization over λ. □

Example 5.4. Given a Lipschitzian observable $f:\mathbb{R}^d\to\mathbb{R}$ with $\|f\|_{Lip}=1$ and $N\ge 2$, let $F(x)=\frac{1}{N}\sum_{i=1}^N f(x^i)$. For any $T>0$,

$$F(X_T^{(N)})=\frac{1}{N}\sum_{i=1}^N f(X_T^{i,N})$$

is the empirical mean of f at time T. Since

$$\|F\|_{Lip(d_{l^1})}=\frac{1}{N}\|f\|_{Lip}=\frac{1}{N}$$

we obtain by (5.5.4) for any $\delta>0$,

$$\mathbb{P}_{\mu_0^{\otimes N}}\left\{\frac{1}{N}\sum_{i=1}^N f(X_T^{i,N})-\mathbb{E}_{\mu_0^{\otimes N}}f(X_T^{1,N})>\delta\right\}\le\exp\left(-\frac{NK_\varepsilon\delta^2}{A_\varepsilon^2\left[1+2c_G(\mu_0)K_\varepsilon e^{-2K_\varepsilon T}\right]}\right).\tag{5.5.10}$$

As the absolute value of the bias $|\mathbb{E}_{\mu_0^{\otimes N}}f(X_T^{1,N})-\mu_T(f)|\le W_1(\mu_T^{1,N},\mu_T)\le A/\sqrt{N}$ by (5.2.33), our result above generalizes Malrieu's result (5.5.1) to the case that V may have many wells.

5.5.2 Concentration for time average

The counterpart of Proposition 5.17 for the empirical time average is presented in the following result.

Proposition 5.18. *Assume* **(H)**, *(5.2.2) and (5.2.3). Given any* $T \in (0, +\infty]$, *let* F *be any* $d_{L^1[0,T]}$*-Lipschitzian continuous function on* $C([0,T],(\mathbb{R}^d)^N)$, *given by*

$$F(X^{(N)}_{[0,T]}) := G\left(\int_0^T g_1(X^{(N)}_t)dt, \cdots, \int_0^T g_n(X^{(N)}_t)dt\right),$$

where $G \in C^2(\mathbb{R}^n)$, $g_i \in C^2((\mathbb{R}^d)^N, \mathbb{R})$, $1 \le i \le n$. *Then for any convex function* φ *on* $\mathbb{R}$ *and any starting point* $X^{(N)}_0 = x \in (\mathbb{R}^d)^N$, *we have*

$$\mathbb{E}_x\varphi\left(F(X^{(N)}_{[0,T]}) - \mathbb{E}_xF(X^{(N)}_{[0,T]})\right) \le \mathbb{E}\varphi\left(\sqrt{NT}\|F\|_{Lip(d_{L^1[0,T]})}c_{Lip}\xi\right), \tag{5.5.11}$$

where ξ *is some standard real Gaussian random variable of law* $\mathcal{N}(0,1)$, *and*

$$c_{Lip} = \frac{h'(0)}{1 - \|\nabla^2_{xy}W\|_\infty \|h'\|_\infty}.$$

Proof. Let $\{\mathscr{F}_t\}_{t\ge0}$ be the filtration generated by the process $(X^{(N)}_t)_{t\ge0}$ and

$$M_t = \mathbb{E}(F(X^{(N)}_{[0,T]})|\mathscr{F}_t),\ 0 \le t \le T.$$

Then by the martingale representation theorem, we have

$$F(X^{(N)}_{[0,T]}) - \mathbb{E}F(X^{(N)}_{[0,T]}) = M_T - M_0 = \sum_{i=1}^N \int_0^T \beta^i_t dB^i_t, \tag{5.5.12}$$

where $\beta^i_t, 1 \le i \le N$ are adapted processes w.r.t. $\mathscr{F}_t$, and $B^i_t, 1 \le i \le N$ are N independent standard Brownian motions on $\mathbb{R}^d$.
Let $A^k_t = \int_0^t g_k(X^{(N)}_s)ds$, $1 \le k \le n$, and $A_t = (A^1_t, \cdots, A^n_t)$. Note that

$$M_t = \phi(A_t, X^{(N)}_t)$$

where

$$\phi(a,x) := \mathbb{E}\left(G\left(a_1 + \int_t^T g_1(X^{(N)}_s)ds, \cdots, a_n + \int_t^T g_N(X^{(N)}_s)ds\right)|X^{(N)}_t = x\right),$$

for $a = (a_1, \cdots, a_n) \in \mathbb{R}^n, x \in (\mathbb{R}^d)^N$. Since φ is C^2 (for V,W are C^2), we can apply Itô's formula to obtain that

$$\beta^i_t = \partial_{x_i}\varphi(A_t, X^{(N)}_t).$$

For any $x = (x^1, \cdots, x^i, \cdots, x^N) \in (\mathbb{R}^d)^N$, denote $y = (x^1, \cdots, y^i, \cdots, x^N)$ which only differs from x at the i-th coordinate. Let $(X^{(N)}_t)_{t\ge0}, (Y^{(N)}_t)_{t\ge0}$ be an optimal coupling of $\mathbb{P}_x, \mathbb{P}_y$ for $W_{1,d_{L^1[0,T]}}(\mathbb{P}_x, \mathbb{P}_y)$

(that exists because $d_{L^1[0,T]}$ is lower semi-continuous from $(C(\mathbb{R}^+,(\mathbb{R}^d)^N))^2$ to $[0,+\infty]$). Then for any $0\le t\le T$ and $i=1,\cdots,N$, we have

$$\begin{aligned}
&|\partial_{x_i}\phi(a,x)|\le \limsup_{y^i\to x^i}\frac{|\phi(a,x)-\phi(a,y)|}{|x^i-y^i|}| \\
&=\limsup_{y^i\to x^i}\frac{1}{|x^i-y^i|}|\mathbb{E}[G(a+\int_0^{T-t}g(X_s^{(N)}ds)]-\mathbb{E}[G(a+\int_0^{T-t}g(Y_s^{(N)}ds)]| \\
&\le \limsup_{y^i\to x^i}\frac{\|F\|_{Lip(d_{L^1[0,T]})}}{|x^i-y^i|}\mathbb{E}\int_0^\infty d_{l^1}(X_s^{(N)},Y_s^{(N)})ds \\
&=\|F\|_{Lip(d_{L^1[0,T]})}\limsup_{y^i\to x^i}\frac{W_{1,d_{L^1}}(\mathbb{P}_x,\mathbb{P}_y)}{|x^i-y^i|} \\
&\le \|F\|_{Lip(d_{L^1[0,T]})}\cdot c_{Lip}
\end{aligned}\tag{5.5.13}$$

where the last inequality follows by Theorem 5.1.

We now repeat the argument in the proof of Proposition 5.17. Since $\sum_{i=1}^N\int_0^T\beta_t^i dB_t^i=\xi_{\tau_T}$ where (ξ_t) is a real valued Brownian motion w.r.t. some new filtration $(\tilde{\mathscr{F}}_t)$ and $\tau_T=\int_0^T\sum_{i=1}^N|\beta_t^i|^2dt\le \|F\|^2_{Lip(d_{L^1[0,T]})}c^2_{Lip}NT=:CNT$ is a stopping time w.r.t. $(\tilde{\mathscr{F}}_t)$, we obtain

$$\begin{aligned}
\mathbb{E}_x\varphi\left(F(X_{[0,T]}^{(N)})-\mathbb{E}F(X_{[0,T]}^{(N)})\right)&=\mathbb{E}\varphi\left(\sum_{i=1}^N\int_0^T\beta_t^i dB_t^i\right)=\mathbb{E}\varphi(\xi_{\tau_T}) \\
&=\mathbb{E}\varphi\left(\mathbb{E}(\xi_{CNT}|\tilde{\mathscr{F}}_{\tau_T})\right) \\
&\le \mathbb{E}\varphi(\xi_{CNT})\quad\text{(by Jensen's inequality)} \\
&=\mathbb{E}\varphi\left(\sqrt{NT}\|F\|_{Lip(d_{L^1[0,T]})}c_{Lip}\xi_1\right)
\end{aligned}\tag{5.5.14}$$

the desired result. □

Next we give the proof of Corollary 5.4.

Proof of Corollary 5.4. For any given $\lambda,T>0$, let

$$F(X_{[0,T]}^{(N)})=\frac{1}{T}\int_0^T U_N(f_m)(X_t^{(N)})dt.$$

Since f_m is 1-Lipschitzian w.r.t the d_{l^1}-metric on $(\mathbb{R}^d)^m$, by an easy calculation we have

$$\|F\|_{Lip_{(d_{L^1[0,T]})}}\le\frac{m}{NT}.$$

Let $g(x)=\mathbb{E}_xF$, $\forall x\in(\mathbb{R}^d)^N$. For any fixed initial value $x\in(\mathbb{R}^d)^N$, by applying Proposition 5.18 with $\varphi(z)=e^{\lambda z}$, we get

$$\begin{aligned}
&\mathbb{E}_x\exp\left(\lambda\left[\frac{1}{T}\int_0^T U_N(f_m)(X_t^{(N)})dt-g(x)\right]\right) \\
\le&\mathbb{E}\exp\left(\frac{m\lambda}{\sqrt{NT}}c_{Lip}\xi_1\right)=\exp\left(\frac{m^2\lambda^2c^2_{Lip}}{2NT}\right).
\end{aligned}\tag{5.5.15}$$

By the proof of Proposition 5.18,

$$\|g\|_{Lip(d_{l^1})} \le c_{Lip}\|F\|_{d_{L^1[0,T]}} \le \frac{mc_{Lip}}{NT}.$$

By the condition (5.2.18) and its consequence (5.2.19), the product measure $\mu_0^{\otimes N}$ satisfies

$$\begin{aligned}\int_{(\mathbb{R}^d)^N} e^{\lambda(g-\mu_0^{\otimes N}(g))}d\mu_0^{\otimes N} &\le \exp\left(\frac{1}{2}Nc_G(\mu_0)\lambda^2\|g\|^2_{Lip(d_{l^1})}\right)\\ &\le \exp\left(\frac{1}{2NT^2}c_G(\mu_0)\lambda^2 m^2 c^2_{Lip}\right).\end{aligned} \tag{5.5.16}$$

Hence for the i.i.d. initial values $X_0^{1,N}, \cdots, X_0^{N,N}$ with the common law μ_0, noting that

$$\mathbb{E}\frac{1}{T}\int_0^T U_N(f_m)(X_t^{(N)})dt = \mu_0^{\otimes N}(g)$$

we have

$$\begin{aligned}&\mathbb{E}\exp\left(\lambda\left[\frac{1}{T}\int_0^T U_N(f_m)(X_t^{(N)})dt - \mathbb{E}\frac{1}{T}\int_0^T U_N(f_m)(X_t^{(N)})dt\right]\right)\\ &= \int_{(\mathbb{R}^d)^N}\mathbb{E}_x\left[\exp\left(\lambda\left[\frac{1}{T}\int_0^T U_N(f_m)(X_t^{(N)})dt - g(x)\right]\right)\right]e^{\lambda(g(x)-\mu_0^{\otimes N}(g))}d\mu_0^{\otimes N}(x)\\ &\le \exp\left(\frac{m^2\lambda^2c^2_{Lip}}{2NT}\right)\int_{(\mathbb{R}^d)^N} e^{\lambda(g(x)-\mu_0^{\otimes N}(g))}d\mu_0^{\otimes N}(x)\\ &\le \exp\left(\frac{m^2\lambda^2c^2_{Lip}}{2NT}\left(1+\frac{c_G(\mu_0)}{T}\right)\right)\end{aligned} \tag{5.5.17}$$

where the second inequality follows from (5.5.15), and the last inequality is a consequence of (5.5.16). This gives us (5.2.20). Finally (5.2.21) follows from (5.2.20), by the standard procedure of Chebyshev inequality and optimization over $\lambda > 0$. □

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Chapter 6

Hypocoercivity in the Wasserstein distance for the kinetic Fokker-Planck equation

This chapter is the result of a collaboration with Arnaud Guillin. We study the long time behaviour of the kinetic Fokker-Planck equation in Wasserstein distance. An exponential convergence to equilibrium is proved for those potentials which are perturbations of the quadratic potentials. All the estimates can be made explicit. Our approach is based on investigations of temporal derivative of the Wasserstein distance along the evolution and thus provides an alternative method to prove the related results in [5]. Moreover, one novelty is that we are able to recover the optimal rate of convergence $\frac{1}{2}$ in the case of the confining potential is given by $\frac{1}{2}|x|^2$.

6.1 Introduction

Consider a smooth function $\mathrm{U} : \mathbb{R}^n \to \mathbb{R}$ such that $\int_{\mathbb{R}^n} e^{-\mathrm{U}(x)}\mathrm{d}x < \infty$ which will represent the confining potential. For $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$, set

$$\mathrm{M}(x, v) = \frac{1}{\mathrm{Z}} e^{-\mathrm{U}(x) - \frac{|v|^2}{2}}, \quad \mathrm{d}\mu(x, v) = \mathrm{M}(x, v)\mathrm{d}x\mathrm{d}v,$$

where Z is the normalizing constant $\int e^{-\mathrm{U}(x) - \frac{|v|^2}{2}} \mathrm{d}x\mathrm{d}v$. The probability measure μ is the unique invariant measure of the kinetic Fokker-Planck equation in the form

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla \mathrm{U}(x) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v f) \tag{6.1.1}$$

where $f_t = f(t, x, v)$ can be interpreted as the density of particles at position x and velocity v (when $f \geq 0$). This equation preserves mass and positivity. We shall be concerned with the solutions subject to the initial condition that f_0 is a probability density, and hence the solution $(f_t)_{t\geq 0}$ will be a flow of probability densities.

Thanks to Ito's formula, the equation (6.1.1) has the following stochastic interpretation: the probability measure $f(t,x,v)\mathrm{d}x\mathrm{d}v$ is the law of the stochastic process $(\mathrm{X}_t, \mathrm{V}_t)_{t\geq 0}$ on $\mathbb{R}^{2n}$ driven by the following stochastic differential equation

$$\begin{cases} \mathrm{dX}_t = \mathrm{V}_t \mathrm{d}t \\ \mathrm{dV}_t = -\mathrm{V}_t \mathrm{d}t - \nabla \mathrm{U}(\mathrm{X}_t)\mathrm{d}t + \sqrt{2}\mathrm{dB}_t \end{cases} \tag{6.1.2}$$

subject to the initial condition that $(\mathrm{X}_0, \mathrm{V}_0)$ is a random variable with its law $f_0 \mathrm{d}x\mathrm{d}v$, where $(\mathrm{B}_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^n$.

With the change of unknown $h := f\mathrm{M}^{-1}$, the equation (6.1.1) becomes

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h - \nabla \mathrm{U}(x) \cdot \nabla_v h = \Delta_v h - v \cdot \nabla_v h. \tag{6.1.3}$$

The Wasserstein distance is an important tool to quantify the discrepancy between probability measures. Let d be a distance function on $\mathbb{R}^m$, and let $p \in [1, \infty)$. Let μ and ν be two probability measures on $\mathbb{R}^m$. The L^p-Wasserstein distance between μ and ν is defined by

$$\mathrm{W}_p(\mu,\nu) \quad := \quad \inf \left(\int_{\mathbb{R}^m \times \mathbb{R}^m} d(x,y)^p \mathrm{d}\pi(x,y) \right)^{1/p} \tag{6.1.4}$$

where the infimum runs over all probability measures π on $\mathbb{R}^m \times \mathbb{R}^m$ which admit μ and ν as marginal measures, that is, for all nonnegative measurable functions f, g on $\mathbb{R}^m$,

$$\int_{\mathbb{R}^m \times \mathbb{R}^m} (f(x) + g(y))\mathrm{d}\pi(x,y) = \int_{\mathbb{R}^m} f \mathrm{d}\mu + \int_{\mathbb{R}^m} g \mathrm{d}\nu.$$

In a more probabilistic formulation, it can be also defined by

$$\mathrm{W}_p(\mu,\nu) \quad = \quad \inf[\mathbb{E}(d(\mathrm{X},\mathrm{Y})^p)]^{1/p} \tag{6.1.5}$$

where the infimum runs over all random variables (X,Y) such that $\mathrm{law}(\mathrm{X}) = \mu, \mathrm{law}(\mathrm{Y}) = \nu$. Such a probability measure π or the couple of random variables (X,Y) above is called a *coupling* between the probability measures μ and ν. By definition, any construction of a coupling between μ and ν yields an upper bound for the Wasserstein distances between μ and ν. This feature makes the Wasserstein distances well-adapted to derive quantitative rates of convergence to equilibrium for a large class of evolution equations or Markov processes.

In this paper, we are mainly concerned with the L^2-Wasserstein distance associated to the Euclidean distance (i.e. $p = 2$, and $d(x,y) = |x-y|$ for any $x,y \in \mathbb{R}^m$). We shall work on the space $\mathscr{P}_2(\mathbb{R}^m)$ of probability measures μ on $\mathbb{R}^m$ with finite second moment $\int |x|^2 \mathrm{d}\mu(x) < \infty$, between which the L^2-Wasserstein distance is always finite. Its subspace $\mathscr{P}_2^{ac}(\mathbb{R}^m)$ is consisted of those probability measures in $\mathscr{P}_2(\mathbb{R}^m)$ which are absolutely continuous with respect to the Lebesgue measure. Very often we shall not distinguish a probability measure and its density function. For instance, $\mathrm{W}_2(f,g)$ may stand for $\mathrm{W}_2(f\mathrm{d}x, g\mathrm{d}x)$ where $\mathrm{d}x$ is the Lebesgue measure in $\mathbb{R}^m$. We shall collect some results concerning Wasserstein distances later, but we refer to the monographs [12] or [1] for further notions and properties.

Assumption 6.1. The confining potential U is a perturbation of quadratic potentials, namely, U takes the form

$$U(x) = \frac{\omega_0^2}{2}|x|^2 + \Psi(x)$$

where the constant $\omega_0 > 0$ and the function Ψ is of class C^2 such that the operator norm of its Hessian $\nabla^2\Psi$ is uniformly bounded, i.e.,

$$||\nabla^2\Psi||_{op} < \kappa$$

for some constant $\kappa \geq 0$.

One typical example would be $\omega_0 = 1$ and in that case κ could be $1/\sqrt{3}$. Now we state our main result,

Theorem 6.1. *Under the assumption 6.1, for any $\omega_0 > 0$ there exists a constant $c > 0$ such that, if $\kappa \leq c$, then there exist explicit computable constants C and $\lambda > 0$ such that*

$$W_2(g_t d\mu, h_t d\mu) \leq Ce^{-\lambda t} W_2(g_0 d\mu, h_0 d\mu). \tag{6.1.6}$$

for any two solutions $(g_t)_{t\geq 0}, (h_t)_{t\geq 0}$ to the equation (6.1.3) with respective initial data g_0, h_0 such that $g_0 d\mu, h_0 d\mu \in \mathscr{P}_2(\mathbb{R}^{2n})$.

Moreover, concerning the kinetic Fokker-Planck equation with quadratic confining potential (i.e. $U(x) = \frac{|x|^2}{2}$), we recover the optimal rate of convergence

$$W_2(g_t d\mu, h_t d\mu) \leq \sqrt{3}e^{-\frac{1}{2}t} W_2(g_0 d\mu, h_0 d\mu). \tag{6.1.7}$$

This article is organized as follows. The section 2 is devoted to a discussion about the optimality of the convergence rate when the confining potential $U(x) = \frac{1}{2}|x|^2$ or $U(x) = \frac{\omega_0^2}{2}|x|^2$. In section 3, we recall Kantorovich duality theorem, Brenier's theorem and a result concerning the time-derivative of Wasserstein distance along evolution equations. A lemma of Gronwall type , involving a weaker derivative notion, will also be proved there. In Section 4, we prove the Lemma 6.7. In Section 5, we shall prove our main result Theorem 6.1 by using the approach developed in section 3. We shall also discuss the possibility of an improvement of the Lemma 6.7 in the appendix.

6.2 Optimal rate for quadratic potential

To fix ideas, we consider in this section the case when the confining potential $U(x)$ is quadratic, i.e. $U(x) = \frac{1}{2}|x|^2$ or $\nabla U(x) = x$. In this simple case, the fundamental solution can be explicitly computed (see for instance [10, p.238-239]), and the true rate of convergence is $\frac{1}{2}$. We shall use a synchronous coupling to recover this sharp rate of convergence in Wasserstein distance. This approach is based on explicit solutions to certain ODE.

Proposition 6.2. *Assuming that the confining potential $U(x) = \frac{1}{2}|x|^2$, then two solutions f_t, g_t with finite second moments to the kinetic Fokker-Planck equation with initial data f_0, g_0 respectively satisfy*

$$W_2(f_t, g_t) \leq sqrt3e^{-\frac{1}{2}t} W_2(f_0, g_0). \tag{6.2.1}$$

Moreover, the rate $\frac{1}{2}$ is sharp for convergence in L^2-Wasserstein distance.

Remark 6.3. In general, for potential in the form of $U(x) = \frac{\omega_0^2}{2}|x|^2$ with $\omega_0 > 0$, it can be proved by following the same line below that for any solutions f_t, g_t with finite second moments,

$$W_2(f_t, g_t) \le Ce^{-\lambda t} W_2(f_0, g_0)$$

where λ is the optimal rate of convergence in L^2-Wasserstein distance given by

$$\lambda = \begin{cases} 1/2, & \text{if } \omega_0^2 \ge 1/4; \\ 1/2 - \sqrt{1/4 - \omega_0^2}, & \text{if } 0 < \omega_0^2 \le 1/4. \end{cases}$$

It is the rate of convergence of the semigroup $e^{S_{\omega_0} t}$ with the following matrix

$$S_{\omega_0} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -1 \end{pmatrix}$$

of which the eigenvalues are either $-1/2 \pm i\sqrt{\omega_0^2 - 1/4}$ (when $\omega_0^2 \ge 1/4$) or $-1/2 \pm \sqrt{1/4 - \omega_0^2}$ (when $0 < \omega_0^2 \le 1/4$).

Proof. Let $(B_t)_{t\ge0}$ be a standard Brownian motion on $\mathbb{R}^n$. Let $(X_t, V_t)_{t\ge0}$ be the stochastic process on $\mathbb{R}^{2n}$ driven by the following stochastic differential equation

$$\begin{cases} dX_t = V_t dt \\ dV_t = -V_t dt - X_t dt + \sqrt{2} dB_t \end{cases} \tag{6.2.2}$$

subject to the initial condition that (X_0, V_0) is a random variable with its law $f_0 dx dv$. Owing to Ito's formula, the density function f_t of $(X_t, V_t)_{t\ge0}$ is the solution of the kinetic Fokker-Planck equation (6.1.1) with initial condition f_0. Now consider another stochastic process $(Y_t, W_t)_{t\ge0}$ on $\mathbb{R}^{2n}$ driven by the same SDE (6.2.2) but with initial distribution $g_0 dx dv$. Let $g_t dx dv$ be the law of $(Y_t, W_t)_{t\ge0}$. Then $(X_t, V_t)_{t\ge0}$ and $(Y_t, W_t)_{t\ge0}$ form a *coupling* between the probability measures $f_t dx dv$ and $g_t dx dv$. Since the two stochastic processes share the same Brownian motion, it is called a *synchronous coupling*.

(1). We prove the rate of convergence $1/2$. If we set

$$x_t := X_t - Y_t, \quad v_t := V_t - W_t,$$

then we arrive at a linear ordinary equation,

$$d\begin{pmatrix} x_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_t \\ v_t \end{pmatrix} dt := S\begin{pmatrix} x_t \\ v_t \end{pmatrix} dt. \tag{6.2.3}$$

where we denote the 2-by-2 matrix by S. Indeed, the matrix S has two distinct complex eigenvalues, namely, $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, where i is the imaginary unit. Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$, then its complex conjugate $\bar\omega = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$, and the vectors $\xi = (\bar\omega, 1)^T$ and $\bar\xi = (\omega, 1)^T$ are associated eigenvectors. The solution to the linear equation $dz(t) = Sz(t)dt$ is given by $z(t) = e^{St} z(0)$ in the form

$$z(t) = ce^{\omega t}\xi + \bar{c} e^{\bar\omega t}\bar\xi \tag{6.2.4}$$

where $c = \frac{1}{2}z_2 + \frac{i}{\sqrt{3}}(\frac{z_2}{2} + z_1)$ for the initial condition $z(0) = (z_1, z_2)$.
It is then clear that the rate of decay for $z(t)$ is $\frac{1}{2}$. Indeed, we have

$$\begin{aligned}|z(t)|^2 &= 4\left|\mathrm{Re}(ce^{\omega t}\xi)\right|^2 = 4e^{-t}\left|\mathrm{Re}(ce^{\frac{\sqrt{3}}{2}ti}\xi)\right|^2 \\ &\le 4e^{-t}\left|ce^{\frac{\sqrt{3}}{2}ti}\xi\right|^2 = 4e^{-t}|c|^2|\xi|^2\end{aligned}$$

where $\mathrm{Re}(c)$ stands for the real part of a complex number c. Note that

$$|\xi|^2 = |\bar{\omega}|^2 + 1 = 2,$$

$$|c|^2 = \frac{1}{3}\left(z_2^2 + z_1 z_2 + z_1^2\right) \le \frac{2}{3}(z_1^2 + z_2^2),$$

it then follows that

$$|z(t)|^2 \le \frac{16}{3}e^{-t}|z(0)|^2. \tag{6.2.5}$$

Consider $z(t) = (x(t), v(t))$ and apply the above inequality. After taking expectations and by the very definition of Wasserstein distance, we then know the rate $1/2$ of convergence in Wasserstein distance.

(2). We prove that $1/2$ **is sharp.** We shall use the following explicit formula for the W_2 distance between two Gaussian distributions (see for instance [6]),

$$W_2^2(\mathcal{N}(m_1,\Sigma_1), \mathcal{N}(m_2,\Sigma_2)) = ||m_1 - m_2||_2^2 + \mathrm{Tr}\,(\Sigma_1 + \Sigma_2 - 2(\Sigma_1^{1/2}\Sigma_2\Sigma_1^{1/2})^{1/2})$$

where $\mathcal{N}(m,\Sigma)$ stands for the Gaussian distribution with mean vector m and covariance matrix Σ.
Let us take the initial datum being

$$(X_0, V_0) = (\delta_{x_0}, \delta_{v_0}), \quad (Y_0, W_0) = (\delta_{y_0}, \delta_{w_0}),$$

Then from the SDE we see that $m_1(t) := \mathbb{E}(X_t, V_t)$ and $m_2(t) := \mathbb{E}(Y_t, W_t)$ are solutions to

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = Sz(t)$$

and so $m_1(t) - m_2(t) = e^{St}(m_1(0) - m_2(0))$. And (X_t, V_t) and (Y_t, W_t) have the same covariance matrix. It then follows that

$$W_2^2(law(X_t, V_t), law(Y_t, W_t)) = |m_1(t) - m_2(t)|^2 = |e^{St}(m_1(0) - m_2(0))|^2.$$

According the spectral analysis of S, the rate of convergence of $W_2(law(X_t, V_t), law(Y_t, W_t))$ is $1/2$. Although $law(X_t, V_t)$ and $law(Y_t, W_t)$ start from measure-valued initial data, they admit smooth density functions at $t > 0$. So the optimal rate of convergence is not greater than $1/2$. $\square$

6.3 Temporal derivative of Wasserstein distance

This section is devoted to the temporal derivative of Wasserstein distance along evolution equations, namely, the theorem 6.4. Before the exact statement, we need some preliminary. Let $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^m)$. According to Kantorovich duality theorem (see for instance [12, Thm 5.10] or [1, Thm 6.1.1]), the minimization problem in the definition (6.1.4) of Wasserstein distance has a dual formulation, namely,

$$\mathrm{W}_2^2(\mu,\nu) = \sup_{\phi(y)+\psi(x)\le|x-y|^2}\left\{\int \phi(y)\mathrm{d}\nu(y) + \int \psi(x)\mathrm{d}\mu(x)\right\} \tag{6.3.1}$$

$$= \sup_{\psi\in \mathrm{L}^1(\mu)}\left\{\int \psi^c(y)\mathrm{d}\nu(y) + \int \psi(x)\mathrm{d}\mu(x)\right\} \tag{6.3.2}$$

where the first supremum runs over all pairs $(\phi,\psi) \in \mathrm{L}^1(\nu)\times \mathrm{L}^1(\mu)$ such that the inequality $\phi(y) + \psi(x) \le |x-y|^2$ holds $\mu\otimes\nu$-almost-everywhere, the second runs over $\psi \in \mathrm{L}^1(\mu)$ and

$$\psi^c(y) := \inf_{x\in\mathbb{R}^m}\{|x-y|^2 - \psi(x)\}, \quad \nu\text{-a.e..}$$

A maximizer $\psi \in \mathrm{L}^1(\mu)$ of the supremum above, usually called a *(maximal) Kantorovich potential*, satisfies

$$\psi^c(y) + \psi(x) = |x-y|^2, \quad \pi\text{-a.e..} \tag{6.3.3}$$

for some optimal coupling π between μ and ν (in the definition (6.1.4) of W_2). This relation leads to Brenier theorem which gives a explicit form of a (the) optimal coupling and the Wasserstein distance.

Brenier's theorem asserts that *if* $\mu \in \mathscr{P}_2^{ac}(\mathbb{R}^m)$, $\nu \in \mathscr{P}_2(\mathbb{R}^m)$ *, then there exists a convex function* φ *on* $\mathbb{R}^m$ *such that*

$$\mathrm{W}_2^2(\mu,\nu) = \int_{\mathbb{R}^m} |\nabla\varphi(x) - x|^2 d\mu(x),$$

and the map $\nabla\varphi : \mathbb{R}^m \to \mathbb{R}^m$ *transports* μ *onto* ν*, i.e.* $(\nabla\varphi)_\#\mu = \nu$*, or equivalently,*

$$\int_{\mathbb{R}^m} g d\nu = \int_{\mathbb{R}^m} g(\nabla\varphi) d\mu \tag{6.3.4}$$

for any bounded continuous functions g*. Furthermore, the map* $\nabla\varphi$ *is uniquely determined* μ*-almost-everywhere.* In other terms, the measure $(\mathrm{Id},\nabla\varphi)_\#\mu$ on $\mathbb{R}^m\times\mathbb{R}^m$ is a (the) minimizer of the infimum in the definition (6.1.4). The map $\nabla\varphi$ is often referred to as the *Brenier map* or the *optimal transport map* from μ to ν. Moreover, if we introduce the Legendre-Fenchel conjugate of φ defined by

$$\varphi^*(y) := \sup_{x\in\mathbb{R}^m}\{x\cdot y - \varphi(x)\}$$

and if $\nu \in \mathscr{P}_2^{ac}(\mathbb{R}^m)$, then the map $\nabla\varphi^*$ transports ν onto μ, i.e. $(\nabla\varphi^*)_\#\nu = \mu$, and it holds almost everywhere

$$\nabla\varphi^*(\nabla\varphi(x)) = x, \quad \nabla\varphi(\nabla\varphi^*(y)) = y \tag{6.3.5}$$

and

$$\nabla^2\varphi^*(\nabla\varphi(x))\nabla^2\varphi(x) = \mathrm{Id} \tag{6.3.6}$$

(Note that here the notion of Hessian is in the sense of Alexandrov with the help of the convexity). Here is the relationship between Kantorovich potentials and the Brenier map,

$$\nabla\varphi(x) = x - \frac{\nabla\psi(x)}{2}, \quad \mu\text{-a.e.}$$

which is based on the equality (6.3.3). That is, the pair

$$(\psi^c(y), \psi(x)) := (|y|^2 - 2\varphi^*(y), |x|^2 - 2\varphi(x))$$

is a maximizer of the supremum in (6.3.1). We shall use this fact to derive Theorem 6.4 below.

We shall use the Dini derivatives with respect to the time variable. Let u be a real-valued function on $[0,\infty)$. *The upper left derivative* of u at time $t>0$ is defined by

$$\frac{\mathrm{d}^-}{\mathrm{d}t}u(t) := \limsup_{h\searrow 0+} \frac{u(t)-u(t-h)}{h}. \tag{6.3.7}$$

It is easy to see that if u has a C^1 support function from below at $t = t_0 > 0$, say w, i.e.

$$w(t) \le u(t) \text{ in a neighborhood of } t_0,$$

and $w(t_0) = u(t_0)$, then

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\Big|_{t=t_0} u(t) = \limsup_{h\searrow 0+} \frac{u(t_0)-u(t_0-h)}{h} \le \limsup_{h\searrow 0+} \frac{w(t_0)-w(t_0-h)}{h} = w'(t_0).$$

It turns out by Lemma 6.6 that the upper left derivative is sufficient to prove decay results.

Theorem 6.4. *Consider two continuous curves μ_t, ν_t in $\mathscr{P}_2^{ac}(\mathbb{R}^m)$. Assume that $\partial_t\mu_t, \partial_t\nu_t$ exist (which usually follows from the evolution equations they satisfy). Let $\nabla\varphi_t$ be the Brenier map from μ_t to ν_t, then it holds for any $t>0$*

$$\frac{1}{2}\frac{d^-}{dt}\mathrm{W}_2^2(\mu_t,\nu_t) \le \int \left(\frac{|z|^2}{2} - \varphi_t(z)\right) d\partial_t\mu_t + \int \left(\frac{|z|^2}{2} - \varphi_t^*(z)\right) d\partial_t\nu_t. \tag{6.3.8}$$

Remark 6.5. In general, under some conditions, one can prove $\mathrm{W}_2^2(\mu_t,\nu_t)$ is differentiable at almost any t, then the upper left derivative becomes the true derivative at almost any $t>0$, and the inequality (6.3.8) is indeed a equality, i.e., for almost every $t>0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{W}_2^2(\mu_t,\nu_t) = \int \left(\frac{|z|^2}{2} - \varphi_t(z)\right) \mathrm{d}\partial_t\mu_t + \int \left(\frac{|z|^2}{2} - \varphi_t^*(z)\right) \mathrm{d}\partial_t\nu_t.$$

One approach to prove this result goes as follows: Assuming μ_t, ν_t are two absolutely continuous curves in $\mathscr{P}_2(\mathbb{R}^m)$, then, due to [1, Thm 8.3.1], there exists a Borel vector field $v : (t,x) \mapsto v_t(x)$ such that $v_t \in \mathrm{L}^2(\mu_t)$ and the continuity equation

$$\partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0$$

holds in the sense of distributions. (Intuitively, v_t is the tangent vector to the curve μ_t, in terms of [1, Thm 8.4.5]). With the help of v_t, one can then apply the arguments in [1, Thm 8.4.7] or apply [12, Thm 23.9] to obtain that such a equality holds for almost any $t>0$.

However, since the weaker inequality (6.3.8) is sufficient to help us to obtain our result, we shall not prove the general formula for the temporal derivative of Wasserstein distance.

A heuristic proof of such a result can be found in [4, Section 2].

Proof. For fixed time t_0, let φ be a convex function such that $\nabla\varphi$ is the Brenier map transporting μ_{t_0} onto ν_{t_0}, then $\nabla\varphi^*$ is the Brenier map transporting ν_{t_0} onto μ_{t_0}. And the pair $(\psi^c,\psi) := (|y|^2 - 2\varphi^*(y), |x|^2 - 2\varphi(x))$ is a maximizer of the supremum in (6.3.1), i.e.

$$\mathrm{W}_2^2(\mu_{t_0},\nu_{t_0}) = \int \psi^c \mathrm{d}\nu_{t_0} + \int \psi \mathrm{d}\mu_{t_0},$$

meanwhile, by the Kantorovich dual formulation, it holds

$$\mathrm{W}_2^2(\mu_t,\nu_t) \geq \int \psi^c \mathrm{d}\nu_t + \int \psi \mathrm{d}\mu_t.$$

(In other terms, the function $\int \psi^c \mathrm{d}\nu_t + \int \psi \mathrm{d}\mu_t$ is a support function from below for the function $\mathrm{W}_2^2(\mu_t,\nu_t)$ at time $t = t_0$.) This implies that, as a function of time $t \geq 0$, $\mathrm{W}_2^2(\mu_t,\nu_t)$ is subdifferentiable and the *upper left derivative* can be bounded from above

$$\begin{aligned}
\frac{\mathrm{d}^-}{\mathrm{d}t}\Big|_{t=t_0} \mathrm{W}_2^2(\mu_t,\nu_t) &\leq \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0}\left(\int \psi(x)\mathrm{d}\mu_t(x) + \int \psi^c(y)\mathrm{d}\nu_t(y)\right) \\
&= \left(\int (|x|^2 - 2\varphi(x))\mathrm{d}\partial_t\mu_t + \int (|y|^2 - 2\varphi^*(y))\mathrm{d}\partial_t\nu_t\right)\Big|_{t=t_0}
\end{aligned}$$

which completes the proof. □

Now we prove a lemma of Gronwall type under certain conditions on *the upper left derivatives.* This lemma will help us to prove exponential convergence to equilibrium.

Lemma 6.6. *Let u be a continuous function on $[0,+\infty)$. Let C be some real constant.*

(1) Assume that $\frac{d^-}{dt}u(t) \leq 0$ for any $t > 0$, then: $u(t) \leq u(0)$;

(2) Assume that $\frac{d^-}{dt}u(t) \leq -\mathrm{C}u$ for any $t > 0$, then: $u(t) \leq e^{-\mathrm{C}t}u(0)$.

Proof of Lemma 6.6. (1). Given $t_0 > 0$. By definition of $\frac{\mathrm{d}^-}{\mathrm{d}t}u(t) \leq 0$, for any $\epsilon > 0$ and for any $t > 0$, there exists $\delta(t) > 0$, such that

$$u(t) - u(s) \leq \epsilon(t-s), \quad \forall s \in [t - \delta(t), t].$$

Let us define

$$\mathrm{I}_\epsilon(t_0) := \{t \,|\, 0 \leq t \leq t_0, u(t_0) - u(s) \leq \epsilon(t_0 - s), \forall s \in [t, t_0]\}$$

which is not empty, since $[t_0 - \delta(t_0), t_0] \subset \mathrm{I}_\epsilon(t_0)$. Note that $\mathrm{I}_\epsilon(t_0)$ is bounded, we can set t_1 as the infimum in $\mathrm{I}_\epsilon(t_0)$, then $(t_1, t_0] \subset \mathrm{I}_\epsilon(t_0)$ and moreover $t_1 \in \mathrm{I}_\epsilon(t_0)$ since by the continuity of u,

$$u(t_0) - u(t_1) \leq \epsilon(t_0 - t_1).$$

Assume that $t_1 > 0$, then there exists $\delta(t_1) > 0$, such that

$$u(t_1) - u(s) \leq \epsilon(t_1 - s), \quad \forall s \in [t_1 - \delta(t_1), t_1].$$

It then follows that

$$u(t_0) - u(s) = [u(t_0) - u(t_1)] + [u(t_1) - u(s)] \le \epsilon(t_0 - s), \quad \forall s \in [t_1 - \delta(t_1), t_1].$$

Together with the fact $[t_1, t_0] \subset \mathrm{I}_\epsilon(t_0)$, we obtain that $t_1 - \delta(t_1) \in \mathrm{I}_\epsilon(t_0)$, which contradicts with the assumption that t_1 is the infimum in $\mathrm{I}_\epsilon(t_0)$. Therefore 0 is the infimum of $\mathrm{I}_\epsilon(t_0)$, in particular,

$$u(t_0) - u(0) \le \epsilon t_0.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $u(t_0) \le u(0)$.
(2). By (1), it suffices to prove that

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\left(e^{\mathrm{C}t}u(t)\right) \le 0.$$

In fact, we have

$$\begin{aligned}
\frac{\mathrm{d}^-}{\mathrm{d}t}\left(e^{\mathrm{C}t}u(t)\right) &:= \limsup_{h\searrow 0+} \frac{e^{\mathrm{C}t}u(t) - e^{\mathrm{C}(t-h)}u(t-h)}{h} \\
&= \limsup_{h\searrow 0+} \left(\frac{[e^{\mathrm{C}t} - e^{\mathrm{C}(t-h)}]u(t)}{h} + \frac{e^{\mathrm{C}(t-h)}[u(t) - u(t-h)]}{h}\right) \\
&= \mathrm{C}e^{\mathrm{C}t}u(t) + \limsup_{h\searrow 0+} \frac{e^{\mathrm{C}(t-h)}[u(t) - u(t-h)]}{h} \\
&= \mathrm{C}e^{\mathrm{C}t}u(t) + e^{\mathrm{C}t}\limsup_{h\searrow 0+} \frac{[u(t) - u(t-h)]}{h} \\
&\le \mathrm{C}e^{\mathrm{C}t}u(t) + e^{\mathrm{C}t} \times (-\mathrm{C}u(t)) = 0.
\end{aligned}$$

□

6.4 A lemma concerning positive definite matrices

Before going into the proof of our main result, we state the following lemma which will be useful. We denote by I_m the identity matrix of size m for a positive integer m. The matrix $\mathrm{M}^\top$ stands for the transpose of the matrix M, and Tr (M) stands for the trace of M.

Lemma 6.7. *Let* $\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{P}', \mathrm{Q}', \mathrm{R}'$ *be matrices of size n such that*

$$\begin{pmatrix} \mathrm{P} & \mathrm{Q} \\ \mathrm{Q}^\top & \mathrm{R} \end{pmatrix}\begin{pmatrix} \mathrm{P}' & \mathrm{Q}' \\ \mathrm{Q}'^\top & \mathrm{R}' \end{pmatrix} = \mathrm{I}_{2n}$$

and that the matrix $\begin{pmatrix} \mathrm{P} & \mathrm{Q} \\ \mathrm{Q}^\top & \mathrm{R} \end{pmatrix}$ *is symmetric positive definite. Then it holds*

$$\mathrm{Tr}\,(\mathrm{P} + \mathrm{P}' - 2\mathrm{I}_n)\,\mathrm{Tr}\,(\mathrm{R} + \mathrm{R}' - 2\mathrm{I}_n) \ge (\mathrm{Tr}\,(\mathrm{Q} + \mathrm{Q}'))^2 \tag{6.4.1}$$

Consequently, for any real number α, β*, it holds*

$$\alpha^2\,\mathrm{Tr}\,(\mathrm{P} + \mathrm{P}' - 2\mathrm{I}_n) + 2\alpha\beta\,\mathrm{Tr}\,(\mathrm{Q} + \mathrm{Q}') + \beta^2\,\mathrm{Tr}\,(\mathrm{R} + \mathrm{R}' - 2\mathrm{I}_n) \ge 0. \tag{6.4.2}$$

Remark 6.8. An application of Lemma 6.7, which will be used later, is for the Hessian matrices of size $2n$

$$\begin{pmatrix} \mathrm{P} & \mathrm{Q} \\ \mathrm{Q}^\top & \mathrm{R} \end{pmatrix} = \nabla^2\varphi, \quad \begin{pmatrix} \mathrm{P}' & \mathrm{Q}' \\ \mathrm{Q}'^\top & \mathrm{R}' \end{pmatrix} = \nabla^2\varphi^*(\nabla\varphi),$$

where $\nabla\varphi$ and $\nabla\varphi^*$ are some Brenier maps as described in Section 2. In that setting, $\nabla^2\varphi$ and $\nabla^2\varphi^*(\nabla\varphi)$ are inverse matrix of each other almost everywhere (see the equality (6.3.6)). Let us work on the variable (x,v) in the phase space, then: P,P' will be $\nabla^2_x\varphi$ and $\nabla^2_x\varphi^*(\nabla\varphi)$, respectively; R,R' will be $\nabla^2_v\varphi$ and $\nabla^2_v\varphi^*(\nabla\varphi)$, respectively; while Q,Q' will be $\nabla^2_{xv}\varphi$ and $\nabla^2_{xv}\varphi^*(\nabla\varphi)$, respectively.

It follows from Lemma 6.7 that

$$\begin{aligned} &\alpha^2[\Delta_x\varphi + (\Delta_x\varphi^*)(\nabla\varphi) - 2n] + \beta^2[\Delta_v\varphi + (\Delta_v\varphi^*)(\nabla\varphi) - 2n] \\ &+2\alpha\beta[\mathrm{Tr}(\nabla^2_{xv}\varphi) + (\mathrm{Tr}(\nabla^2_{xv}\varphi^*))(\nabla\varphi)] \geq 0 \end{aligned} \tag{6.4.3}$$

for any real numbers α,β.

Proof of Lemma 6.7. Step (1). We prove first the following observation,

Claim: *Let* $\mathrm{M}_1,\mathrm{M}_2,\mathrm{M}_3$ *be matrices of size d such that the matrix* $\begin{pmatrix} \mathrm{M}_1 & \mathrm{M}_2 \\ \mathrm{M}_2^\top & \mathrm{M}_3 \end{pmatrix}$ *is symmetric positive semi-definite, then it holds*

$$\mathrm{Tr}(\mathrm{M}_1)\,\mathrm{Tr}(\mathrm{M}_3) \geq |\mathrm{Tr}(\mathrm{M}_2)|^2. \tag{6.4.4}$$

Indeed, let us set

$$\mathrm{M}_1 = (a_{ij})_{1\leq i,j\leq n}, \quad \mathrm{M}_2 = (b_{ij})_{1\leq i,j\leq n}, \quad \mathrm{M}_3 = (c_{ij})_{1\leq i,j\leq n}.$$

then, by the assumption, we know that the 2×2 submatrix $\begin{pmatrix} a_{ii} & b_{ii} \\ b_{ii} & c_{ii} \end{pmatrix}$ is also positive semi-definite for each given i. It follows that

$$a_{ii} \geq 0, \quad c_{ii} \geq 0, \text{ and } |b_{ii}| \leq \sqrt{a_{ii}c_{ii}}.$$

So we have

$$\begin{aligned} |\mathrm{Tr}(\mathrm{M}_2)|^2 &= |\sum_{1\leq i\leq n} b_{ii}|^2 \leq |\sum_{1\leq i\leq n} \sqrt{a_{ii}c_{ii}}|^2 \\ &\leq \sum_{1\leq i\leq n} a_{ii} \sum_{1\leq i\leq n} c_{ii} = \mathrm{Tr}(\mathrm{M}_1)\,\mathrm{Tr}(\mathrm{M}_3) \end{aligned}$$

which is the desired inequality (6.4.4) in the Claim.

Step (2). Denote $\mathrm{S} = \begin{pmatrix} \mathrm{P} & \mathrm{Q} \\ \mathrm{Q}^\top & \mathrm{R} \end{pmatrix}$, and so $\mathrm{S}^{-1} = \begin{pmatrix} \mathrm{P}' & \mathrm{Q}' \\ \mathrm{Q}'^\top & \mathrm{R}' \end{pmatrix}$. We set

$$\mathrm{M} := \mathrm{S} + \mathrm{S}^{-1} - 2\mathrm{I}_{2n} = \begin{pmatrix} \mathrm{P}+\mathrm{P}'-2\mathrm{I}_n & \mathrm{Q}+\mathrm{Q}' \\ \mathrm{Q}^\top + \mathrm{Q}'^\top & \mathrm{R}+\mathrm{R}'-2\mathrm{I}_n \end{pmatrix}.$$

Since S is a strictly positive symmetric operator and the function $\theta(t) = t + t^{-1} - 2$ is a nonnegative continuous function on $(0, +\infty)$, with the help of spectral decomposition, the operator $\theta(L) = S + S^{-1} - 2I_{2n}$ is also a positive operator. In other words, M is a positive semi-definite matrix (i.e. $M \geq 0$).
Applying the Claim in Step (1), we obtain the inequality (6.4.1).
Note that the fact $M \geq 0$ implies

$$\mathrm{Tr}\,(P + P' - 2I_n) \geq 0, \quad \mathrm{Tr}\,(R + R' - 2I_n) \geq 0.$$

So the inequality (6.4.2) is a direct consequence of the inequality (6.4.1). □

6.5 Proof of the main Theorem

Let the derivation operators A, B on $L^2(\mu)$ be as follows

$$A := \nabla_v, \quad B := v \cdot \nabla_x - \nabla U(x) \cdot \nabla_v. \tag{6.5.1}$$

Then B^*, the dual operator of B, satisfies $B^* = -B$, i.e. B is anti-symmetric. While A^* is given by

$$A^* = -\nabla_v \cdot + v \cdot .$$

The kinetic Fokker-Planck equation (6.1.3) then takes the form

$$\partial_t h + Lh = 0 \tag{6.5.2}$$

with the operator L defined as

$$L := A^* A + B = -\Delta_v + v \cdot \nabla_v + v \cdot \nabla_x - \nabla U(x) \cdot \nabla_v .$$

6.5.1 Change of coordinates

Let α, α', β be some constants (to be specified later) satisfying

$$\beta > 0, \quad \beta > \alpha\alpha'.$$

We shall perform a change of coordinates,

$$\begin{pmatrix} \tilde{x} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ \alpha' & \beta \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}, \tag{6.5.3}$$

and so

$$\begin{pmatrix} x \\ v \end{pmatrix} = \frac{1}{\beta - \alpha\alpha'} \begin{pmatrix} \beta & -\alpha \\ -\alpha' & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{v} \end{pmatrix}.$$

We shall consider the Wasserstein distance $\tilde{W}_2$ associated to the Euclidean distance in this new coordinates $(\tilde{x}, \tilde{v})$, i.e. the distance induced by a twisted inner product

$$|\tilde{x}_t|^2 + |\tilde{v}_t|^2 \quad = \quad (1 + \alpha'^2)|x|^2 + 2(\alpha + \alpha'\beta)\, x \cdot v + (\alpha^2 + \beta^2)|v|^2 \tag{6.5.4}$$

which is equivalent to the standard Euclidean inner product $|x|^2+|v|^2$ since $\beta>\alpha\alpha'$. It follows that the Wasserstein distances $\tilde{W}_2$ and W_2 are equivalent, namely, there exists positive constants $c_1, c_2>0$ such that

$$c_1 W_2(\nu_1,\nu_2)\le \tilde{W}_2(\nu_1,\nu_2)\le c_2 W_2(\nu_1,\nu_2) \tag{6.5.5}$$

for any $\nu_1,\nu_2\in\mathscr{P}_2(\mathbb{R}^{2n})$. The constants c_1 and c_2 can be explicitly computed.

A typical case, which will be used in the particular setting when $U(x)=\frac{1}{2}|x|^2$, is the case of $\alpha=\alpha'=2-\sqrt{3}$ and $\beta=1$. (*Keeping this case in mind would make it easier to read the proof.*) In this case, the new inner product

$$|\tilde{x}_t|^2+|\tilde{v}_t|^2=4(2-\sqrt{3})\left(|x|^2+x\cdot v+|v|^2\right)$$

satisfies the following inequality

$$2(2-\sqrt{3})\left(|x|^2+|v|^2\right)\le|\tilde{x}_t|^2+|\tilde{v}_t|^2\le 6(2-\sqrt{3})\left(|x|^2+|v|^2\right),$$

and so

$$(\sqrt{3}-1)W_2(\nu_1,\nu_2)\le\tilde{W}_2(\nu_1,\nu_2)\le(3-\sqrt{3})W_2(\nu_1,\nu_2).$$

Now we reformulate the kinetic Fokker-Planck equation in the new coordinates $(\tilde{x},\tilde{v})$. For any function $F(x,v)$, we denote by $\tilde{F}(\tilde{x},\tilde{v})$ the push-forward function of $F(x,v)$ by the change of coordinates, that is,

$$\tilde{F}(\tilde{x},\tilde{v}):=F\left(\frac{\beta\tilde{x}-\alpha\tilde{v}}{\beta-\alpha\alpha'},\frac{-\alpha'\tilde{x}+\tilde{v}}{\beta-\alpha\alpha'}\right),$$

In this way, $F(x,v)=\tilde{F}(x+\alpha v,\alpha' x+\beta v)=\tilde{F}(\tilde{x},\tilde{v})$. For instance, the invariant measure $\tilde{\mu}$ is then given by

$$d\tilde{\mu}=\frac{1}{Z(\beta-\alpha\alpha')}e^{-\tilde{U}(\tilde{x},\tilde{v})-\frac{|-\alpha'\tilde{x}+\tilde{v}|^2}{2(\beta-\alpha\alpha')}}\,d\tilde{x}d\tilde{v}$$

We compute that

$$\begin{aligned}
A &= \alpha\nabla_{\tilde{x}}+\beta\nabla_{\tilde{v}}, &(6.5.6)\\
A^* &= -\left(\alpha\nabla_{\tilde{x}}\cdot+\beta\nabla_{\tilde{v}}\cdot\right)+\frac{\tilde{v}-\alpha'\tilde{x}}{\beta-\alpha\alpha'}\cdot\quad, &(6.5.7)\\
\nabla_x U(x) &= \left(1-\frac{\alpha\alpha'}{\beta}\right)\nabla_{\tilde{x}}\tilde{U}=-\frac{\beta-\alpha\alpha'}{\alpha}\nabla_{\tilde{v}}\tilde{U}, &(6.5.8)\\
B &= \frac{\tilde{v}-\alpha'\tilde{x}}{\beta-\alpha\alpha'}\cdot(\nabla_{\tilde{x}}+\alpha'\nabla_{\tilde{v}})-\left(1-\frac{\alpha\alpha'}{\beta}\right)\nabla_{\tilde{x}}\tilde{U}\cdot(\alpha\nabla_{\tilde{x}}+\beta\nabla_{\tilde{v}}) &(6.5.9)
\end{aligned}$$

Note that B is still an anti-symmetric operator in new coordinates (such properties do not change since we perform the change of coordinates on the invariant measure at the same time). The operator L becomes

$$\begin{aligned}
L &= -\alpha^2\Delta_{\tilde{x}}-\beta^2\Delta_{\tilde{v}}-2\alpha\beta\,\mathrm{Tr}\,(\nabla_{\tilde{x}}\nabla_{\tilde{v}})+\frac{\tilde{v}-\alpha'\tilde{x}}{\beta-\alpha\alpha'}\cdot(\alpha\nabla_{\tilde{x}}+\beta\nabla_{\tilde{v}})\\
&\quad+\frac{\tilde{v}-\alpha'\tilde{x}}{\beta-\alpha\alpha'}\cdot(\nabla_{\tilde{x}}+\alpha'\nabla_{\tilde{v}})-\left(1-\frac{\alpha\alpha'}{\beta}\right)\nabla_{\tilde{x}}\tilde{U}\cdot(\alpha\nabla_{\tilde{x}}+\beta\nabla_{\tilde{v}}) \qquad (6.5.10)
\end{aligned}$$

in which we obtain diffusion terms (or the Laplacians) in either the direction $\tilde{x}$ or the direction $\tilde{v}$, unlike the one in the coordinates (x, v) in which only the diffusion in the direction v is present. One may expect this form would help us to give better estimates.
As a summary, the function $\tilde{h}_t = \tilde{h}_t(\tilde{x}, \tilde{v})$ satisfies the equation

$$\partial_t \tilde{h} + \mathrm{L}\tilde{h} = 0 \tag{6.5.11}$$

with the initial condition $\tilde{h}_0$ and with L given in (6.5.10). So the question reduces to the convergence of $\tilde{h}_t$ to the invariant measure $\tilde{\mu}$ in terms of the Wasserstein distance $\tilde{\mathrm{W}}_2$.

6.5.2 The proof

Now we turn to

Proof of Theorem 6.1. Since $\mathrm{U}(x) = \frac{\omega_0^2}{2}|x|^2 + \Psi(x)$, we have

$$\nabla \mathrm{U}(x) = \omega_0^2 x + \nabla\Psi(x) := \omega_0^2 x + \mathrm{G}(x). \tag{6.5.12}$$

And so the operator B becomes

$$\begin{aligned} \mathrm{B} &= \left[\frac{\tilde{v} - \alpha'\tilde{x}}{\beta - \alpha\alpha'} \cdot (\nabla_{\tilde{x}} + \alpha'\nabla_{\tilde{v}}) - \omega_0^2 \frac{\beta\tilde{x} - \alpha\tilde{v}}{\beta - \alpha\alpha'} \cdot (\alpha\nabla_{\tilde{x}} + \beta\nabla_{\tilde{v}})\right] + \left[\mathrm{G}(\frac{\beta\tilde{x} - \alpha\tilde{v}}{\beta - \alpha\alpha'}) \cdot (\alpha\nabla_{\tilde{x}} + \beta\nabla_{\tilde{v}})\right] \\ &:= \mathrm{B}_0 + \mathrm{B}_{\mathrm{G}}. \end{aligned} \tag{6.5.13}$$

Now consider two solutions $(g_t)_{t\geq 0}, (h_t)_{t\geq 0}$ to the equation (6.1.3) with respective initial data g_0, h_0 such that $g_0 \mathrm{d}\mu, h_0 \mathrm{d}\mu \in \mathscr{P}_2(\mathbb{R}^{2n})$. Then, after the change of coordinates (6.5.3) we obtain two solutions $\tilde{g}_t$, $\tilde{h}_t$ to the equation (6.5.11) with respective initial data $\tilde{g}_0$ and $\tilde{h}_0$, and it holds that

$$\mathrm{W}_2(\tilde{g}_t \mathrm{d}\tilde{\mu}, \tilde{h}_t \mathrm{d}\tilde{\mu}) = \tilde{\mathrm{W}}_2(g_t \mathrm{d}\mu, h_t \mathrm{d}\mu).$$

Thanks to the equivalence (6.5.5) of W_2 and $\tilde{\mathrm{W}}_2$, *it suffices to prove the following contraction result in the new Wasserstein distance* $\tilde{\mathrm{W}}_2$,

$$\mathrm{W}_2(\tilde{g}_t d\tilde{\mu}, \tilde{h}_t d\tilde{\mu}) \leq e^{-\lambda t} \mathrm{W}_2(\tilde{g}_0 d\tilde{\mu}, \tilde{h}_0 d\tilde{\mu}). \tag{6.5.14}$$

in which the rate λ *can be taken as* $\frac{1}{2}$ *when the confining potential* $\mathrm{U}(x) = \frac{1}{2}|x|^2$.
Let $\tilde{\varphi}_t$ be a convex function such that $\nabla\tilde{\varphi}_t$ is the Brenier map from $\tilde{g}_t \mathrm{d}\tilde{\mu}$ to $\tilde{h}_t \mathrm{d}\tilde{\mu}$, i.e.

$$(\nabla\tilde{\varphi}_t)_\#(\tilde{g}_t \mathrm{d}\tilde{\mu}) = \tilde{h}_t \mathrm{d}\tilde{\mu},$$

and

$$\mathrm{W}_2^2(\tilde{g}_t \mathrm{d}\tilde{\mu}, \tilde{h}_t \mathrm{d}\tilde{\mu}) = \int |\nabla\tilde{\varphi}_t - z|^2 \tilde{g}_t(z) \mathrm{d}\tilde{\mu}(z).$$

It follows from Theorem 6.4 that

$$\frac{1}{2}\frac{\mathrm{d}^-}{\mathrm{d}t} \mathrm{W}_2^2(\tilde{g}_t \mathrm{d}\tilde{\mu}, \tilde{h}_t \mathrm{d}\tilde{\mu}) \leq -\int (\frac{|z|^2}{2} - \tilde{\varphi}_t(z)) \mathrm{L}\tilde{g} \mathrm{d}\tilde{\mu} - \int (\frac{|z|^2}{2} - \tilde{\varphi}_t^*(z)) \mathrm{L}\tilde{h} \mathrm{d}\tilde{\mu}. \tag{6.5.15}$$

We shall perform integrations by parts. This is a little bit tricky because of the production of the Hessian of convex functions. The easy part is the integration by parts for the operator B, since B involves only first derivatives,

$$-\int (\frac{|z|^2}{2} - \tilde{\varphi}_t(z))\mathrm{B}\tilde{g}\mathrm{d}\tilde{\mu} = \int [\mathrm{B}(\frac{|z|^2}{2} - \tilde{\varphi}_t(z))]\tilde{g}\mathrm{d}\tilde{\mu},$$

$$-\int (\frac{|z|^2}{2} - \tilde{\varphi}_t^*(z))\mathrm{L}\tilde{h}\mathrm{d}\tilde{\mu} = \int [\mathrm{B}(\frac{|z|^2}{2} - \tilde{\varphi}_t^*(z))]\tilde{h}\mathrm{d}\tilde{\mu}.$$

While the integration by parts for the operator $\mathrm{A}^*\mathrm{A}$, will produce second derivatives of φ and φ^* defined in the sense of Aleksandrov. To handle this, we have to use the fact that *the Laplacian of a convex function in the sense of Aleksandrov is not greater than the one in the sense of distribution, i.e. for a nonnegative function f and a convex function ϕ, it holds*

$$-\int \phi\partial_{z_i}^2 f = \int \partial_{z_i} f \partial_{z_i}\phi \leq -\int f\partial_{z_i}^2\phi \tag{6.5.16}$$

where $\partial_{z_i}^2\phi$ is defined in the sense of Aleksandrov. (See for instance [7, Lemma 1, Section 2 and Appendix].) We can rewrite $\int \tilde{\varphi}_t\mathrm{A}^*\mathrm{A}\tilde{g}\mathrm{d}\tilde{\mu}$ in the coordinates (x, v), namely $\int \tilde{\varphi}_t\mathrm{A}^*\mathrm{A}g\mathrm{d}\mu$. Note that, in the coordinates (x, v), $\tilde{\varphi}$ is still a convex function and $\mathrm{A}^*\mathrm{A} = -\Delta_v + v\cdot\nabla_v$, then, due to (6.5.16), we know

$$\int \tilde{\varphi}_t\mathrm{A}^*\mathrm{A}g\mathrm{d}\mu \leq \int [\mathrm{A}^*\mathrm{A}(\tilde{\varphi}_t)]g\mathrm{d}\mu,$$

or equivalently,

$$\int \tilde{\varphi}_t\mathrm{A}^*\mathrm{A}\tilde{g}\mathrm{d}\tilde{\mu} \leq \int [\mathrm{A}^*\mathrm{A}(\tilde{\varphi}_t)]\tilde{g}\mathrm{d}\tilde{\mu}.$$

Similarly, we have

$$\int \tilde{\varphi}_t^*\mathrm{A}^*\mathrm{A}\tilde{h}\mathrm{d}\tilde{\mu} \leq \int [\mathrm{A}^*\mathrm{A}(\tilde{\varphi}_t^*)]\tilde{h}\mathrm{d}\tilde{\mu}.$$

Therefore we deduce from the inequality (6.5.15) that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}^-}{\mathrm{d}t}\mathrm{W}_2^2(\tilde{g}_t\mathrm{d}\tilde{\mu}, \tilde{h}_t\mathrm{d}\tilde{\mu}) &\leq -\int [\mathrm{A}^*\mathrm{A}(\frac{|z|^2}{2} - \tilde{\varphi}_t(z))]\tilde{g}\mathrm{d}\tilde{\mu} + \int [\mathrm{B}(\frac{|z|^2}{2} - \tilde{\varphi}_t(z))]\tilde{g}\mathrm{d}\tilde{\mu} \\ &\quad -\int [\mathrm{A}^*\mathrm{A}(\frac{|z|^2}{2} - \tilde{\varphi}_t^*(z))]\tilde{h}\mathrm{d}\tilde{\mu} + \int [\mathrm{B}(\frac{|z|^2}{2} - \tilde{\varphi}_t^*(z))]\tilde{h}\mathrm{d}\tilde{\mu}. \\ &:= (\mathrm{I})_\mathrm{A} + (\mathrm{I})_{\mathrm{B}_0} + (\mathrm{I})_{\mathrm{B}_\mathrm{G}} + (\mathrm{II})_\mathrm{A} + (\mathrm{II})_{\mathrm{B}_0} + (\mathrm{II})_{\mathrm{B}_\mathrm{G}}\end{aligned} \tag{6.5.17}$$

where the terms $(\mathrm{I})_\mathrm{A}, (\mathrm{I})_{\mathrm{B}_0}, (\mathrm{I})_{\mathrm{B}_\mathrm{G}}, (\mathrm{II})_\mathrm{A}, (\mathrm{II})_{\mathrm{B}_0}, (\mathrm{II})_{\mathrm{B}_\mathrm{G}}$ are defined in a manner as follows

$$(\mathrm{I})_\mathrm{A} := -\int [\mathrm{A}^*\mathrm{A}(\frac{|z|^2}{2} - \tilde{\varphi}_t(z))]\tilde{g}\mathrm{d}\tilde{\mu},$$

$$(\mathrm{I})_{\mathrm{B}_0} := \int [\mathrm{B}_0(\frac{|z|^2}{2} - \tilde{\varphi}_t(z))]\tilde{g}\mathrm{d}\tilde{\mu}, \quad (\mathrm{I})_{\mathrm{B}_\mathrm{G}} := \int [\mathrm{B}_\mathrm{G}(\frac{|z|^2}{2} - \tilde{\varphi}_t(z))]\tilde{g}\mathrm{d}\tilde{\mu}.$$

We then compute these integrals. Indeed, we have

$$
\begin{aligned}
(\mathrm{I})_{\mathrm{A}} + (\mathrm{II})_{\mathrm{A}} &= -\int \tilde{g}\,\mathrm{d}\tilde{\mu}\left[Diff(\tilde{\varphi},\alpha,\beta) + \frac{\beta}{\beta-\alpha\alpha'}|\nabla_{\tilde{v}}\tilde{\varphi} - \tilde{v}|^2 \right.\\
&\quad \left. -\frac{\alpha\alpha'}{\beta-\alpha\alpha'}|\nabla_{\tilde{x}}\tilde{\varphi} - \tilde{x}|^2 + \frac{\alpha-\alpha'\beta}{\beta-\alpha\alpha'}(\nabla_{\tilde{x}}\tilde{\varphi}-\tilde{x})\cdot(\nabla_{\tilde{v}}\tilde{\varphi}-\tilde{v})\right], \qquad (6.5.18)\\
(\mathrm{I})_{\mathrm{B}_0} + (\mathrm{II})_{\mathrm{B}_0} &= \frac{1}{\beta-\alpha\alpha'}\int \tilde{g}\,\mathrm{d}\tilde{\mu}\left[(\alpha' + \omega_0^2\alpha\beta)(|\nabla_{\tilde{v}}\tilde{\varphi}-\tilde{v}|^2 - |\nabla_{\tilde{x}}\tilde{\varphi}-\tilde{x}|^2)\right.\\
&\quad \left. + (1+\omega_0^2\alpha^2 - \alpha'^2 - \omega_0^2\beta^2)(\nabla_{\tilde{x}}\tilde{\varphi}-\tilde{x})\cdot(\nabla_{\tilde{v}}\tilde{\varphi}-\tilde{v})\right], \qquad (6.5.19)\\
|(\mathrm{I})_{\mathrm{B}_{\mathrm{G}}} + (\mathrm{II})_{\mathrm{B}_{\mathrm{G}}}| &\leq ||\nabla^2\Psi||\frac{(\alpha+\beta)^2}{2(\beta-\alpha\alpha')}\mathrm{W}_2^2(\tilde{g}_t\,\mathrm{d}\tilde{\mu}, \tilde{h}_t\,\mathrm{d}\tilde{\mu}) \qquad (6.5.20)
\end{aligned}
$$

where the term $Diff(\tilde{\varphi},\alpha,\beta)$ is given by

$$
\begin{aligned}
Diff(\tilde{\varphi},\alpha,\beta) &= \alpha^2[\Delta_{\tilde{x}}\tilde{\varphi} + (\Delta_{\tilde{x}}\tilde{\varphi}^*)(\nabla\tilde{\varphi}) - 2n]\\
&\quad +\beta^2[\Delta_{\tilde{v}}\tilde{\varphi} + (\Delta_{\tilde{v}}\tilde{\varphi}^*)(\nabla\tilde{\varphi}) - 2n]\\
&\quad +2\alpha\beta[\mathrm{Tr}\,(\nabla^2_{\tilde{x}\tilde{v}}\tilde{\varphi}) + (\mathrm{Tr}\,(\nabla^2_{\tilde{x}\tilde{v}}\tilde{\varphi}^*))(\nabla\tilde{\varphi})]. \qquad (6.5.21)
\end{aligned}
$$

We put off the proof of these identities to next subsection, see Lemma 6.9. Note that, almost everywhere, the matrices $\nabla^2\tilde{\varphi}$ and $(\nabla^2\tilde{\varphi}^*)(\nabla\varphi)$ are symmetric positive-definite matrices, and they are inverse matrix of each other. This fact implies that the term $Diff(\tilde{\varphi},\alpha,\beta)$ is nonnegative, by Lemma 6.7 (as explained in the remark there). As a consequence, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}^-}{\mathrm{d}t}\mathrm{W}_2^2(\tilde{g}_t\,\mathrm{d}\tilde{\mu}, \tilde{h}_t\,\mathrm{d}\tilde{\mu}) \leq (\mathrm{I})_{\mathrm{A}} + (\mathrm{I})_{\mathrm{B}_0} + (\mathrm{I})_{\mathrm{B}_{\mathrm{G}}} + (\mathrm{II})_{\mathrm{A}} + (\mathrm{II})_{\mathrm{B}_0} + (\mathrm{II})_{\mathrm{B}_{\mathrm{G}}}\\
\leq\ & -\int \tilde{g}\,\mathrm{d}\tilde{\mu}\,\mathrm{Q}(\alpha,\alpha',\beta,\nabla\tilde{\varphi}-z) + ||\nabla^2\Psi||\frac{(\alpha+\beta)^2}{2(\beta-\alpha\alpha')}\mathrm{W}_2^2(\tilde{g}_t\,\mathrm{d}\tilde{\mu}, \tilde{h}_t\,\mathrm{d}\tilde{\mu}) \qquad (6.5.22)
\end{aligned}
$$

where $\mathrm{Q}(\alpha,\alpha',\beta,\nabla\tilde{\varphi}-z)$ is a quadratic form of $\nabla\tilde{\varphi}-z$, defined by

$$
\begin{aligned}
\mathrm{Q}(\alpha,\alpha',\beta,(x,v)) &:= \frac{1}{\beta-\alpha\alpha'}\Big\{(\beta-\alpha'-\omega_0^2\alpha\beta)|v|^2 + (\alpha'+\omega_0^2\alpha\beta-\alpha\alpha')|x|^2\\
&\quad +(\alpha-\alpha'\beta-1-\omega_0^2\alpha^2+\alpha'^2+\omega_0^2\beta^2)x\cdot v\Big\}
\end{aligned}
$$

(1). A special case: $\omega_0 = 1$**.** We consider first the case when the confining potential is given by $\mathrm{U}(x) = \frac{1}{2}|x|^2$, i.e. $\Psi = 0$. Then we take $\alpha = \alpha' = 2-\sqrt{3}$ and $\beta = 1$, we have

$$
\mathrm{Q}(\alpha,\alpha',\beta,\nabla\tilde{\varphi}-z) = \frac{1}{2}(|\nabla_{\tilde{v}}\tilde{\varphi}-\tilde{v}|^2 + |\nabla_{\tilde{x}}\tilde{\varphi}-\tilde{x}|^2).
$$

This implies

$$
\frac{1}{2}\frac{\mathrm{d}^-}{\mathrm{d}t}\mathrm{W}_2^2(\tilde{g}_t\,\mathrm{d}\tilde{\mu}, \tilde{h}_t\,\mathrm{d}\tilde{\mu}) \leq -\frac{1}{2}\int |\nabla\tilde{\varphi}_t - z|^2\tilde{g}_t\,\mathrm{d}\tilde{\mu} = -\frac{1}{2}\mathrm{W}_2^2(\tilde{g}_t\,\mathrm{d}\tilde{\mu}, \tilde{h}_t\,\mathrm{d}\tilde{\mu})
$$

By Lemma 6.6, we deduce that

$$
\mathrm{W}_2^2(\tilde{g}_t\,\mathrm{d}\tilde{\mu}, \tilde{h}_t\,\mathrm{d}\tilde{\mu}) \leq e^{-t}\mathrm{W}_2^2(\tilde{g}_0\,\mathrm{d}\tilde{\mu}, \tilde{h}_0\,\mathrm{d}\tilde{\mu})
$$

thus we have recovered the optimal rate of convergence in this case.
When there is a perturbation Ψ in the potential U, according to (6.5.22),

$$\frac{1}{2}\frac{d^-}{dt}W_2^2(\tilde{g}_t d\tilde{\mu}, \tilde{h}_t d\tilde{\mu}) \le -\left(\frac{1}{2} - \frac{\sqrt{3}}{2}||\nabla^2\Psi||\right) W_2^2(\tilde{g}_t d\tilde{\mu}, \tilde{h}_t d\tilde{\mu}).$$

This implies that, under the Assumption 6.1 with $\kappa = 1/\sqrt{3}$, there exists $\lambda > 0, C > 0$ such that the exponential convergence (6.1.6) holds true.
(2). General case. For any $\omega_0 > 0$, we can take

$$\alpha = \min\{\omega_0, \frac{1}{2\omega_0^2}\}, \quad \alpha' = 0, \quad \beta = \frac{1}{\omega_0},$$

then one can find that

$$Q(\alpha, \alpha', \beta, \nabla\tilde{\varphi} - z) \ge \frac{\lambda_0}{2}(|\nabla\tilde{\varphi} - z|^2)$$

with

$$\lambda_0 := \min\{\omega_0^2, \frac{1}{2}, \frac{1}{2\omega_0}\}.$$

By Lemma 6.6, whenever the Assumption 6.1 holds with $\kappa \le \frac{\lambda_0\beta}{(\alpha+\beta)^2}$, there exists a constant $\lambda > 0$ such that the exponential convergence (6.5.14) holds true. Then the convergence (6.1.6) follows. We end with a remark that although these constants may be far from optimal rates of convergence, they are always explicitly computable.

□

6.5.3 The computations of the integrals $(\mathrm{I})_A, (\mathrm{I})_{B_0}, (\mathrm{I})_{B_G}, (\mathrm{II})_A, (\mathrm{II})_{B_0}, (\mathrm{II})_{B_G}$

Lemma 6.9. *With the notations as in the previous subsection, it holds*

$$\begin{aligned}
(\mathrm{I})_A \quad &:= \quad -\int [A^*A(\frac{|z|^2}{2} - \tilde{\varphi}(z))]\tilde{g}\,d\tilde{\mu} \\
&= \quad -\int \tilde{g}\,d\tilde{\mu}\Big\{\alpha^2\Delta_{\tilde{x}}\tilde{\varphi} + \beta^2\Delta_{\tilde{v}}\tilde{\varphi} + 2\alpha\beta\,\mathrm{Tr}\,(\nabla^2_{\tilde{x}\tilde{v}}\tilde{\varphi}) - (\alpha^2+\beta^2)n \\
&\quad + \frac{1}{\beta - \alpha\alpha'}(\beta|\tilde{v}|^2 - \alpha\alpha'|\tilde{x}|^2 + (\alpha - \alpha'\beta)\,\tilde{v}\cdot\tilde{x}) \\
&\quad - \frac{1}{\beta - \alpha\alpha'}(\beta\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{\varphi} - \alpha\alpha'\tilde{x}\cdot\nabla_{\tilde{x}}\tilde{\varphi} + \alpha\tilde{v}\cdot\nabla_{\tilde{x}}\tilde{\varphi} - \alpha'\beta\tilde{x}\cdot\nabla_{\tilde{v}}\tilde{\varphi})\Big\}, \qquad (6.5.23) \\
(\mathrm{II})_A \quad &:= \quad -\int [A^*A(\frac{|z|^2}{2} - \tilde{\varphi}^*(z))]\tilde{h}\,d\tilde{\mu} \\
&= \quad -\int \tilde{g}\,d\tilde{\mu}\Big\{\alpha^2(\Delta_{\tilde{x}}\tilde{\varphi}^*)(\nabla\tilde{\varphi}) + \beta^2(\Delta_{\tilde{v}}\tilde{\varphi}^*)(\nabla\tilde{\varphi}) + 2\alpha\beta\,\mathrm{Tr}\,(\nabla^2_{\tilde{x}\tilde{v}}\tilde{\varphi}^*)(\nabla\tilde{\varphi}) - (\alpha^2+\beta^2)n \\
&\quad + \frac{1}{\beta - \alpha\alpha'}(\beta|\nabla_{\tilde{v}}\tilde{\varphi}|^2 - \alpha\alpha'|\nabla_{\tilde{x}}\tilde{\varphi}|^2 + (\alpha - \alpha'\beta)\nabla_{\tilde{v}}\tilde{\varphi}\cdot\nabla_{\tilde{x}}\tilde{\varphi}) \\
&\quad - \frac{1}{\beta - \alpha\alpha'}(\beta\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{\varphi} - \alpha\alpha'\tilde{x}\cdot\nabla_{\tilde{x}}\tilde{\varphi} + \alpha\tilde{x}\cdot\nabla_{\tilde{v}}\tilde{\varphi} - \alpha'\beta\tilde{v}\cdot\nabla_{\tilde{x}}\tilde{\varphi})\Big\}, \qquad (6.5.24)
\end{aligned}$$

$$\begin{aligned}
(\mathrm{I})_{\mathrm{B}_0} &:= \int [\mathrm{B}_0(\frac{|z|^2}{2} - \tilde{\varphi}(z))]\tilde{g}\,d\tilde{\mu} \\
&= \frac{1}{\beta-\alpha\alpha'}\int \tilde{g}\,d\tilde{\mu}\Big\{(\alpha'+\alpha\omega_0^2\beta)(|\tilde{v}|^2-|\tilde{x}|^2)+(1+\omega_0^2\alpha^2-\alpha'^2-\omega_0^2\beta^2)\tilde{x}\cdot\tilde{v} \\
&\quad -(\alpha'+\alpha\omega_0^2\beta)(\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{\varphi}-\tilde{x}\cdot\nabla_{\tilde{x}}\tilde{\varphi}) \\
&\quad +(\alpha'^2+\omega_0^2\beta^2)\tilde{x}\cdot\nabla_{\tilde{v}}\tilde{\varphi}-(1+\omega_0^2\alpha^2)\tilde{v}\cdot\nabla_{\tilde{x}}\tilde{\varphi}\Big\}, \qquad (6.5.25)
\end{aligned}$$

$$\begin{aligned}
(\mathrm{II})_{\mathrm{B}_0} &:= \int [\mathrm{B}_0(\frac{|z|^2}{2} - \tilde{\varphi}^*(z))]\tilde{h}\,d\tilde{\mu} \\
&= \frac{1}{\beta-\alpha\alpha'}\int \tilde{g}\,d\tilde{\mu}\Big\{(\alpha'+\alpha\omega_0^2\beta)(|\nabla_{\tilde{v}}\tilde{\varphi}|^2-|\nabla_{\tilde{x}}\tilde{\varphi}|^2) \\
&\quad +(1+\omega_0^2\alpha^2-\alpha'^2-\omega_0^2\beta^2)\nabla_{\tilde{v}}\tilde{\varphi}\cdot\nabla_{\tilde{x}}\tilde{\varphi}-(\alpha'+\alpha\omega_0^2\beta)(\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{\varphi}-\tilde{x}\cdot\nabla_{\tilde{x}}\tilde{\varphi}) \\
&\quad +(\alpha'^2+\omega_0^2\beta^2)\tilde{v}\cdot\nabla_{\tilde{x}}\tilde{\varphi}-(1+\omega_0^2\alpha^2)\tilde{x}\cdot\nabla_{\tilde{v}}\tilde{\varphi}\Big\}. \qquad (6.5.26)
\end{aligned}$$

Consequently,

$$\begin{aligned}
(I)_{\mathrm{A}} + (II)_{\mathrm{A}} &= -\int \tilde{g}\,d\tilde{\mu}\Big[\mathit{Diff}(\tilde{\varphi},\alpha,\beta)+\frac{\beta}{\beta-\alpha\alpha'}|\nabla_{\tilde{v}}\tilde{\varphi}-\tilde{v}|^2 \\
&\quad -\frac{\alpha\alpha'}{\beta-\alpha\alpha'}|\nabla_{\tilde{x}}\tilde{\varphi}-\tilde{x}|^2+\frac{\alpha-\alpha'\beta}{\beta-\alpha\alpha'}(\nabla_{\tilde{x}}\tilde{\varphi}-\tilde{x})\cdot(\nabla_{\tilde{v}}\tilde{\varphi}-\tilde{v})\Big], \qquad (6.5.27)
\end{aligned}$$

where $\mathit{Diff}(\tilde{\varphi},\alpha,\beta)$ *is defined as in* (6.5.21).

$$\begin{aligned}
(I)_{\mathrm{B}_0} + (II)_{\mathrm{B}_0} &= \frac{1}{\beta-\alpha\alpha'}\int \tilde{g}\,d\tilde{\mu}\,[(\alpha'+\alpha\omega_0^2\beta)(|\nabla_{\tilde{v}}\tilde{\varphi}-\tilde{v}|^2-|\nabla_{\tilde{x}}\tilde{\varphi}-\tilde{x}|^2) \\
&\quad +(1+\omega_0^2\alpha^2-\alpha'^2-\omega_0^2\beta^2)(\nabla_{\tilde{x}}\tilde{\varphi}-\tilde{x})\cdot(\nabla_{\tilde{v}}\tilde{\varphi}-\tilde{v})], \qquad (6.5.28)
\end{aligned}$$

And it holds for $(I)_{\mathrm{B_G}} + (II)_{\mathrm{B_G}}$ *that*

$$|(I)_{\mathrm{B_G}} + (II)_{\mathrm{B_G}}| \le ||\nabla^2\Psi||\frac{(\alpha+\beta)^2}{2(\beta-\alpha\alpha')}\int|\nabla\tilde{\varphi}-z|^2\tilde{g}\,d\tilde{\mu}. \qquad (6.5.29)$$

Proof of Lemma 6.9. **To alleviate the heavy notations, we shall use symbols** $x, v, g, h, \varphi, \varphi^*$ μ **to replace** $\tilde{x}, \tilde{v}, \tilde{g}, \tilde{h}, \tilde{\varphi}, \tilde{\varphi}^*, \tilde{\mu}$ **throughout the proof of Lemma 6.9.**

(1)The terms $(\mathrm{I})_{\mathrm{A}}, (\mathrm{II})_{\mathrm{A}}$**.** We recall first $\mathrm{A}^*\mathrm{A}$ is given by

$$\mathrm{A}^*\mathrm{A} = -\alpha^2\Delta_x-\beta^2\Delta_v-2\alpha\beta\,\mathrm{Tr}\,(\nabla_{xv}^2)+\frac{v-\alpha' x}{\beta-\alpha\alpha'}\cdot(\alpha\nabla_x+\beta\nabla_v),$$

so we obtain

$$\begin{aligned}
-\mathrm{A}^*\mathrm{A}\frac{|z|^2}{2} &= (\alpha^2+\beta^2)n-\frac{v-\alpha' x}{\beta-\alpha\alpha'}\cdot(\alpha x+\beta v) \\
&= (\alpha^2+\beta^2)n-\frac{\beta|v|^2-\alpha\alpha'|x|^2+(\alpha-\alpha'\beta)v\cdot x}{\beta-\alpha\alpha'},
\end{aligned}$$

$$
\begin{aligned}
A^*A\varphi &= -\alpha^2\Delta_x\varphi-\beta^2\Delta_v\varphi-2\alpha\beta\,\mathrm{Tr}\,(\nabla^2_{xv}\varphi)+\frac{v-\alpha' x}{\beta-\alpha\alpha'}\cdot(\alpha\nabla_x\varphi+\beta\nabla_v\varphi)\\
&= -\alpha^2\Delta_x\varphi-\beta^2\Delta_v\varphi-2\alpha\beta\,\mathrm{Tr}\,(\nabla^2_{xv}\varphi)\\
&\quad+\frac{1}{\beta-\alpha\alpha'}\left(-\alpha\alpha' x\cdot\nabla_x\varphi+\beta v\cdot\nabla_v\varphi-\alpha'\beta x\cdot\nabla_v\varphi+\alpha v\cdot\nabla_x\varphi\right).
\end{aligned}
$$

Therefore it holds

$$
\begin{aligned}
(\mathrm{I})_A &= -\int g\,d\mu\Big\{\alpha^2\Delta_x\varphi+\beta^2\Delta_v\varphi+2\alpha\beta\,\mathrm{Tr}\,(\nabla^2_{xv}\varphi)-(\alpha^2+\beta^2)n\\
&\quad+\frac{1}{\beta-\alpha\alpha'}(\beta|v|^2-\alpha\alpha'|x|^2+(\alpha-\alpha'\beta)\,v\cdot x)\\
&\quad-\frac{1}{\beta-\alpha\alpha'}(-\alpha\alpha' x\cdot\nabla_x\varphi+\beta v\cdot\nabla_v\varphi-\alpha'\beta x\cdot\nabla_v\varphi+\alpha v\cdot\nabla_x\varphi)\Big\}.
\end{aligned}
$$

In the same way (replacing g by h, φ by φ^*), we obtain

$$
\begin{aligned}
(\mathrm{II})_A &= -\int h\,d\mu\Big\{\alpha^2\Delta_x\varphi^*+\beta^2\Delta_v\varphi^*+2\alpha\beta\,\mathrm{Tr}\,(\nabla^2_{xv}\varphi^*)-(\alpha^2+\beta^2)n\\
&\quad+\frac{1}{\beta-\alpha\alpha'}(\beta|v|^2-\alpha\alpha'|x|^2+(\alpha-\alpha'\beta)\,v\cdot x)\\
&\quad-\frac{1}{\beta-\alpha\alpha'}(-\alpha\alpha' x\cdot\nabla_x\varphi^*+\beta v\cdot\nabla_v\varphi^*-\alpha'\beta x\cdot\nabla_v\varphi^*+\alpha v\cdot\nabla_x\varphi^*)\Big\}.
\end{aligned}
$$

Thanks to the change of variables $(\nabla\varphi)_\#(g\,d\mu)=h\,d\mu$(see the equality (6.3.4)), using the fact that $\nabla\varphi^*(\nabla\varphi(z))=z$ (see (6.3.5)), $(\mathrm{II})_A$ can be written as

$$
\begin{aligned}
(\mathrm{II})_A &= -\int g\,d\mu\Big\{\alpha^2\Delta_x\varphi^*(\nabla\varphi)+\beta^2\Delta_v\varphi^*(\nabla\varphi)+2\alpha\beta\,\mathrm{Tr}\,(\nabla^2_{xv}\varphi^*)(\nabla\varphi)-(\alpha^2+\beta^2)n\\
&\quad+\frac{1}{\beta-\alpha\alpha'}(\beta|\nabla_v\varphi|^2-\alpha\alpha'|\nabla_x\varphi|^2+(\alpha-\alpha'\beta)\nabla_v\varphi\cdot\nabla_x\varphi)\\
&\quad-\frac{1}{\beta-\alpha\alpha'}(-\alpha\alpha'\nabla_x\varphi\cdot x+\beta\nabla_v\varphi\cdot v-\alpha'\beta\nabla_x\varphi\cdot v+\alpha\nabla_v\varphi\cdot x)\Big\}.
\end{aligned}
$$

Adding it with $(\mathrm{I})_A$ and rearranging the terms, the identity (6.5.27) follows.

(2)The terms $(\mathrm{I})_{B_0}$, $(\mathrm{II})_{B_0}$. The operator B_0 was introduced in (6.5.13) (together with B_G),

$$
\begin{aligned}
B_0 &= \frac{v-\alpha' x}{\beta-\alpha\alpha'}\cdot(\nabla_x+\alpha'\nabla_v)-\omega_0^2\frac{\beta x-\alpha v}{\beta-\alpha\alpha'}\cdot(\alpha\nabla_x+\beta\nabla_v)\\
&= \frac{1}{\beta-\alpha\alpha'}\Big\{(\alpha'+\alpha\omega_0^2\beta)(v\cdot\nabla_v-x\cdot\nabla_x)-(\alpha'^2+\omega_0^2\beta^2)x\cdot\nabla_v+(1+\omega_0^2\alpha^2)v\cdot\nabla_x\Big\}.
\end{aligned}
$$

We compute that

$$
\begin{aligned}
(\beta-\alpha\alpha')B_0\frac{|z|^2}{2} &= (\alpha'+\alpha\omega_0^2\beta)(|v|^2-|x|^2)-(\alpha'^2+\omega_0^2\beta^2)x\cdot v+(1+\omega_0^2\alpha^2)v\cdot x\\
(\beta-\alpha\alpha')B_0\varphi &= (\alpha'+\alpha\omega_0^2\beta)(v\cdot\nabla_v\varphi-x\cdot\nabla_x\varphi)-(\alpha'^2+\omega_0^2\beta^2)x\cdot\nabla_v\varphi\\
&\quad+(1+\omega_0^2\alpha^2)v\cdot\nabla_x\varphi
\end{aligned}
$$

and so

$$
\begin{aligned}
(\beta-\alpha\alpha')(\mathrm{I})_{B_0} \;=\; & \int g d\mu \Big\{(\alpha'+\alpha\omega_0^2\beta)(|v|^2-|x|^2)+(1+\omega_0^2\alpha^2-\alpha'^2-\omega_0^2\beta^2)x\cdot v \\
& -(\alpha'+\alpha\omega_0^2\beta)(v\cdot\nabla_v\varphi - x\cdot\nabla_x\varphi)+(\alpha'^2+\omega_0^2\beta^2)x\cdot\nabla_v\varphi \\
& -(1+\omega_0^2\alpha^2)v\cdot\nabla_x\varphi\Big\}.
\end{aligned}
$$

In the same manner, we compute $(\beta-\alpha\alpha')(\mathrm{II})_{B_0}$ and then apply $(\nabla\varphi)_{\#}(g d\mu) = h d\mu$ (the change of variables),

$$
\begin{aligned}
(\beta-\alpha\alpha')(\mathrm{II})_{B_0} \;=\; & \int h d\mu \Big\{(\alpha'+\alpha\omega_0^2\beta)(|v|^2-|x|^2)+(1+\omega_0^2\alpha^2-\alpha'^2-\omega_0^2\beta^2)x\cdot v \\
& -(\alpha'+\alpha\omega_0^2\beta)(v\cdot\nabla_v\varphi^* - x\cdot\nabla_x\varphi^*)+(\alpha'^2+\omega_0^2\beta^2)x\cdot\nabla_v\varphi^* \\
& -(1+\omega_0^2\alpha^2)v\cdot\nabla_x\varphi^*\Big\} \\
=\; & \int g d\mu \Big\{(\alpha'+\alpha\omega_0^2\beta)(|\nabla_v\varphi|^2-|\nabla_x\varphi|^2) \\
& +(1+\omega_0^2\alpha^2-\alpha'^2-\omega_0^2\beta^2)\nabla_x\varphi\cdot\nabla_v\varphi-(\alpha'+\alpha\omega_0^2\beta)(\nabla_v\varphi\cdot v-\nabla_x\varphi\cdot x) \\
& +(\alpha'^2+\omega_0^2\beta^2)\nabla_x\varphi\cdot v-(1+\omega_0^2\alpha^2)\nabla_v\varphi\cdot x\Big\}.
\end{aligned}
$$

Then identities concerning B_0 in the lemma are direct consequences of the above expressions of $(\beta-\alpha\alpha')(\mathrm{I})_{B_0}$ and $(\beta-\alpha\alpha')(\mathrm{II})_{B_0}$.

(3)The term (I)$_{B_G}$ + (II)$_{B_G}$. Recall that

$$
B_G = G\Big(\frac{\beta x-\alpha v}{\beta-\alpha\alpha'}\Big)\cdot(\alpha\nabla_x+\beta\nabla_v),
$$

hence

$$
\begin{aligned}
|(\mathrm{I})_{B_G}+(\mathrm{II})_{B_G}| \;=\; & \Big|\int g d\mu\Big\{B_G\Big(\frac{|z|^2}{2}-\varphi\Big)\Big\}+\int h d\mu\Big\{B_G\Big(\frac{|z|^2}{2}-\varphi^*\Big)\Big\}\Big| \\
=\; & \Big|\int g d\mu G\Big(\frac{\beta x-\alpha v}{\beta-\alpha\alpha'}\Big)\cdot(\alpha x+\beta v-\alpha\nabla_x\varphi-\beta\nabla_v\varphi) \\
& +\int h d\mu G\Big(\frac{\beta x-\alpha v}{\beta-\alpha\alpha'}\Big)\cdot(\alpha x+\beta v-\alpha\nabla_x\varphi^*-\beta\nabla_v\varphi^*)\Big| \\
=\; & \Big|\int g d\mu G\Big(\frac{\beta x-\alpha v}{\beta-\alpha\alpha'}\Big)\cdot(\alpha x+\beta v-\alpha\nabla_x\varphi-\beta\nabla_v\varphi) \\
& +\int g d\mu G\Big(\frac{\beta\nabla_x\varphi-\alpha\nabla_v\varphi}{\beta-\alpha\alpha'}\Big)\cdot(\alpha\nabla_x\varphi+\beta\nabla_v\varphi-\alpha x-\beta v)\Big| \\
:=\; & \Big|\int g d\mu F(\alpha,\alpha',\beta,\varphi)\Big|
\end{aligned}
$$

where the third equality follows from the change of variables $(\nabla\varphi)_{\#}(g d\mu) = h d\mu$. Now we give an estimate of the integrand $F(\alpha,\alpha',\beta,\varphi)$,

$$
\begin{aligned}
|F(\alpha,\alpha',\beta,\varphi)| \;=\; & \Big|\Big[G\Big(\frac{\beta x-\alpha v}{\beta-\alpha\alpha'}\Big)-G\Big(\frac{\beta\nabla_x\varphi-\alpha\nabla_v\varphi}{\beta-\alpha\alpha'}\Big)\Big]\cdot(\alpha x+\beta v-\alpha\nabla_x\varphi-\beta\nabla_v\varphi)\Big| \\
\le\; & \Big|G\Big(\frac{\beta x-\alpha v}{\beta-\alpha\alpha'}\Big)-G\Big(\frac{\beta\nabla_x\varphi-\alpha\nabla_v\varphi}{\beta-\alpha\alpha'}\Big)\Big|\times|\alpha p+\beta q|
\end{aligned}
$$

where we denote

$$p := \nabla_x \varphi - x, \quad q := \nabla_v \varphi - v.$$

Recall that $G = \nabla\Psi$, then it follows

$$\begin{aligned} |F(\alpha,\alpha',\beta,\varphi)| &\le \|\nabla^2\Psi\|_{op}\left|\frac{\beta p - \alpha q}{\beta - \alpha\alpha'}\right| \times |\alpha p + \beta q| \\ &\le \|\nabla^2\Psi\|_{op}\frac{\beta|p| + \alpha|q|}{\beta - \alpha\alpha'} \times (\alpha|p| + \beta|q|) \\ &\le \|\nabla^2\Psi\|_{op}\frac{(\beta+\alpha)^2}{2(\beta - \alpha\alpha')}(|p|^2 + |q|^2). \end{aligned}$$

from which the inequality (6.5.29) follows. □

6.6 Appendix: A remark on Lemma 6.7

In this section we discuss the possibility of improvements of Lemma 6.7. We shall use the notations of Lemma 6.7 with $P, Q, R, P', Q', R', M_1, M_2, M_3$ being matrices of size n. Recall that S is the symmetric positive definite matrix of size $2n$ such that

$$S = \begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} P' & Q' \\ Q'^T & R' \end{pmatrix},$$

and M is also a symmetric positive semi-definite matrix of size $2n$,

$$M := \begin{pmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{pmatrix} := S + S^{-1} - 2I_{2n} = \begin{pmatrix} P + P' - 2I_n & Q + Q' \\ Q^T + Q'^T & R + R' - 2I_n \end{pmatrix}. \tag{6.6.1}$$

Lemma 6.7 says that for any real number α it holds

$$\alpha^2 \operatorname{Tr}(P + P' - 2I_n) + 2\alpha \operatorname{Tr}(Q + Q') + \operatorname{Tr}(R + R' - 2I_n) \ge 0 \tag{6.6.2}$$

In terms of M, it is equivalent to

$$\alpha^2 \operatorname{Tr}(M_1) + 2\alpha \operatorname{Tr}(M_2) + \operatorname{Tr}(M_3) \ge 0$$

The following question arises naturally: *For some positive real number* α, *does there exist a constant* $c_\alpha > 0$ *such that, for any symmetric positive definite matrix* S,

$$\alpha^2 \operatorname{Tr}(M_1) + 2\alpha \operatorname{Tr}(M_2) + \operatorname{Tr}(M_3) \ge c_\alpha \lambda_1(M)?$$

where M *is defined in* (6.6.1) *and* $\lambda_1(M)$ *is the largest eigenvalue of* M.

Remark 6.10. If such a constant $c_\alpha > 0$ existed, then we could apply the argument in the proof of [4, Proposition 3.4] where a key ingredient is an inequality in the form of

$$|(\nabla^2\varphi)^{\frac{1}{2}} - ((\nabla^2\varphi^*)(\nabla\varphi))^{\frac{1}{2}}|^2 \le \Delta\varphi + \Delta\varphi^*(\nabla\varphi) - 2n$$

where φ is a convex function on $\mathbb{R}^n$. (The expression on the left hand side is the operator norm of the matrix $\nabla^2\varphi + \nabla^2\varphi^*(\nabla\varphi) - 2I_n$, i.e. its largest eigenvalue; While the expression on the right is the trace of $\nabla^2\varphi + (\nabla^2\varphi^*)(\nabla\varphi) - 2I_n$.) Doing so, the diffusion term *Diff*$(\tilde\varphi, \alpha, \beta)$ would produce dissipation in the Wasserstein distance in a local manner.

Lemma 6.11. *Given $\alpha \in \mathbb{R}$, assume for any symmetric positive definite matrix S of size $2n$ it holds true that*

$$\alpha^2 \operatorname{Tr}(\mathrm{M}_1) + 2\alpha \operatorname{Tr}(\mathrm{M}_2) + \operatorname{Tr}(\mathrm{M}_3) \geq c_\alpha \lambda_1(\mathrm{M}) \tag{6.6.3}$$

where $\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3$ are matrices of size n, and the matrices $\mathrm{M}, \mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3$ are given by

$$\mathrm{M} := \begin{pmatrix} \mathrm{M}_1 & \mathrm{M}_2 \\ \mathrm{M}_2^\top & \mathrm{M}_3 \end{pmatrix} := \mathrm{S} + \mathrm{S}^{-1} - 2\mathrm{I}_{2n},$$

then

$$c_\alpha \leq 0.$$

Proof. We shall prove it in two steps,

(1) The map $\mathrm{S} \mapsto \mathrm{S} + \mathrm{S}^{-1} - 2\mathrm{I}_{2n}$, from symmetric positive definite matrices to symmetric positive semi-definite matrices, is surjective;

(2) We can find certain symmetric positive semi-definite matrices (for the matrix M in (6.6.3)) to show that $c_\alpha \leq 0$.

Step 1: The map $\mathrm{S} \mapsto \mathrm{S} + \mathrm{S}^{-1} - 2\mathrm{I}_{2n}$ is surjective. In fact, since for any real symmetric positive semi-definite matrix M , there exists a real orthogonal matrix P such that

$$\mathrm{P}\mathrm{M}\mathrm{P}^\top = \mathrm{Diag}(\lambda_1, \lambda_2, \cdots, \lambda_{2n})$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n} \geq 0$ are the eigenvalues of M. Note that for each eigenvalue λ_i, there exists a positive number η_i such that

$$\eta_i + \eta_i^{-1} - 2 = \lambda_i.$$

Take $\mathrm{S} = \mathrm{P}^\top \mathrm{Diag}(\eta_1, \eta_2, \cdots, \eta_{2n})\mathrm{P}$, which is a symmetric positive definite matrix, then we have

$$\begin{aligned} \mathrm{S} + \mathrm{S}^{-1} - 2\mathrm{I}_{2n} &= \mathrm{P}^\top \mathrm{Diag}(\eta_1, \eta_2, \cdots, \eta_{2n})\mathrm{P} + \mathrm{P}^\top \mathrm{Diag}(\eta_1^{-1}, \eta_2^{-1}, \cdots, \eta_{2n}^{-1})\mathrm{P} - 2\mathrm{I}_{2n} \\ &= \mathrm{P}^\top \mathrm{Diag}(\eta_1 + \eta_1^{-1} - 2, \eta_2 + \eta_2^{-1} - 2, \cdots, \eta_{2n} + \eta_{2n}^{-1} - 2)\mathrm{P} \\ &= \mathrm{P}^\top \mathrm{Diag}(\lambda_1, \lambda_2, \cdots, \lambda_{2n})\mathrm{P} = \mathrm{M} \end{aligned}$$

from which we conclude that the map $\mathrm{S} + \mathrm{S}^{-1} - 2\mathrm{I}_{2n}$ is surjective.

Step 2: Now we are ready to prove that $c_\alpha \leq 0$. Due to Step 1, it suffices to prove that a constant c_α cannot be greater than 0 if it holds for any positive semi-definite matrix M of size $2n$ that

$$\alpha^2 \operatorname{Tr}(\mathrm{M}_1) + 2\alpha \operatorname{Tr}(\mathrm{M}_2) + \operatorname{Tr}(\mathrm{M}_3) \geq c_\alpha \lambda_1(\mathrm{M}) \tag{6.6.4}$$

where

$$\mathrm{M} = \begin{pmatrix} \mathrm{M}_1 & \mathrm{M}_2 \\ \mathrm{M}_2^\top & \mathrm{M}_3 \end{pmatrix}.$$

Consider

$$\mathrm{M} = \begin{bmatrix} a & & & & b & & & \\ & 0 & & & & 0 & & \\ & & \ddots & & & & \ddots & \\ & & & 0 & & & & 0 \\ b & & & & c & & & \\ & 0 & & & & 0 & & \\ & & \ddots & & & & \ddots & \\ & & & 0 & & & & 0 \end{bmatrix}$$

where a, b, c will be determined later. Then we compute that

$$\alpha^2 \operatorname{Tr} \mathrm{M}_1 + 2\alpha \operatorname{Tr} \mathrm{M}_2 + \operatorname{Tr} \mathrm{M}_3 = \alpha^2 a + 2\alpha b + c,$$

and

$$\lambda_1(\mathrm{M}) = \frac{a+c}{2} + \sqrt{b^2 - ac + \frac{(a+c)^2}{4}}.$$

We consider three cases according to the sign of α,

(1) **Case 1:** $\alpha = 0$. Let $a > 0, b = c = 0$. The inequality (6.6.4) turns to be

$$0 \geq c_\alpha a$$

which yields $c_\alpha \leq 0$.

(2) **Case 2:** $\alpha > 0$. Let a, c be strictly positive numbers and $b = -\sqrt{ac} + \epsilon$ with $0 \leq \epsilon \ll \sqrt{ac}$. Then

$$\alpha^2 \operatorname{Tr} \mathrm{M}_1 + 2\alpha \operatorname{Tr} \mathrm{M}_2 + \operatorname{Tr} \mathrm{M}_3 = (\alpha\sqrt{a} - \sqrt{c})^2 + 2\alpha\epsilon.$$

And we can compute the largest eigenvalue of M,

$$\lambda_1(\mathrm{M}) = \frac{a+c}{2} + \sqrt{\epsilon^2 - 2\sqrt{ac}\epsilon + \frac{(a+c)^2}{4}}.$$

Therefore, if the inequality (6.6.3) holds, letting $\epsilon \to 0+$, we could get

$$(\alpha\sqrt{a} - \sqrt{c})^2 \geq c_\alpha(a+c).$$

Since a, c are arbitrary strictly positive numbers, we can take a, c such that $c = \alpha^2 a$, so we have $\alpha\sqrt{a} - \sqrt{c} = 0$ and then

$$c_\alpha \leq 0.$$

(3) **Case 3:** $\alpha < 0$. Replace b in Case 2 by $\sqrt{ac} - \epsilon$. Following the same line as in Case 2, we can show that

$$(\alpha\sqrt{a} + \sqrt{c})^2 \geq c_\alpha(a+c).$$

Choose $a, c > 0$ such that $\alpha\sqrt{a} + \sqrt{c} = 0$ and it follows that $c_\alpha \leq 0$.

Note in all three cases the matrix M above is positive semi-definite, the proof is completed. □

Remark 6.12. (a) Combined with Lemma 6.7, the optimal constant c_α satisfying (6.6.3) is 0.

(b) We actually proved in Step 2 the following fact: 0 *is the optimal constant for* c_α *such that for any symmetric positive semi-definite matrix* M*,*

$$\alpha^2 \operatorname{Tr}(M_1) + 2\alpha \operatorname{Tr}(M_2) + \operatorname{Tr}(M_3) \geq c_\alpha \lambda_1(M) \tag{6.6.5}$$

where M_1, M_2, M_3 *are matrices of size* n*, and* M *is given by*

$$M := \begin{pmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{pmatrix}.$$

6.7 Références

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Chapter 7

Hypocoercivity in higher order L^2-Sobolev Spaces for the kinetic Fokker-Planck equation

In this chapter, we study the long time behaviour of the kinetic Fokker-Planck equation. The purpose of this paper is to extend the hypocoercivity results in L^2-Sobolev space of order 1 in Villani's memoir [11] to higher order L^2-Sobolev spaces. We construct some twisted L^2-Sobolev norms and prove coercivity estimates under them. Our results are based on assumptions on some relative boundedness of higher order derivatives of the confining potential. Moreover, in the future, we shall prove global hypoellipticity in short time in higher order L^2-Sobolev spaces in a similar manner.

7.1 Introduction

We are concerned in this paper with the kinetic Fokker-Planck equation which takes the form

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (vf), \quad t \geq 0 \tag{7.1.1}$$

subject to the initial condition $f(0,x,v) = f_0(x,v)$, where the unknown function $f(t,x,v)$ stands for the density function at time t with position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$, and the function $V = V(x) : \mathbb{R}^d \to \mathbb{R}$ is a confining potential. We shall always assume that $\int e^{-V(x)} dx < \infty$ and thus the Fokker-Planck equation admits an unique invariant measure

$$d\mu(x,v) = \frac{1}{Z_0} e^{-V(x) - \frac{|v|^2}{2}} dx dv$$

where $Z_0 = \int\int e^{-V(x)-\frac{|v|^2}{2}} dx dv$ is the normalizing constant. We also denote $Z_1 = \int e^{-V(x)} dx$.
This evolution preserves mass and positivity. Assuming that the initial datum f_0 is a probability density function, the solution f_t will be a probability density function (for each $t > 0$) as well. It is

indeed the law of a stochastic process $(X_t, Y_t)_{t\geq 0}$ on $\mathbb{R}^d \times \mathbb{R}^d$ evolving according to the S.D.E.

$$\begin{cases} dX_t = Y_t dt \\ dY_t = -Y_t dt - \nabla V(X_t) dt + \sqrt{2} dB_t \end{cases} \tag{7.1.2}$$

where the initial distribution is assumed to be $f_0(x,v)\mathrm{d}x\mathrm{d}v$, and $(B_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^d$.

7.1.1 Basic notations

Let $\mathscr{H}$ be a Hilbert space equipped with the Hilbert norm $||\cdot||$ and the scalar product $\langle\cdot,\cdot\rangle$. Let $A : \mathscr{H} \to \mathscr{H}^m$ be a linear densely-defined operator with domain $\mathscr{D}(A)$. So the operator A could be written as a m-tuple vector of linear operators on $\mathscr{H}$, say

$$A = (A_i^*)_{1\leq i\leq m}^{\top} = (A_1, A_2, \cdots, A_m)^{\top}, \quad \text{with } A_i : \mathscr{H} \to \mathscr{H}.$$

And its adjoint operator $A^* : \mathscr{H}^m \to \mathscr{H}$ is then given by

$$A^* = (A_i^*)_{1\leq i\leq m} = (A_1^*, A_2^*, \cdots, A_m^*),$$

or more explicitly, for a vector $g = (g_1, g_2, \cdots, g_m)^{\top} \in \mathscr{H}^m$,

$$A^* g = (A_1^*, A_2^*, \cdots, A_m^*)(g_1, g_2, \cdots, g_m)^{\top} = \sum_{1\leq i\leq m} A_i^* g_i.$$

Therefore the linear operator A^*A has the form

$$A^*A = (A_1^*, A_2^*, \cdots, A_m^*)(A_1, A_2, \cdots, A_m)^{\top} = \sum_{1\leq i\leq m} A_i^* A_i.$$

Given two operators $B_1, B_2 : \mathscr{H} \to \mathscr{H}$, their commutator is defined by $[B_1, B_2] := B_1B_2 - B_2B_1$. We stress that the commutator $[A, B]$ of $A : \mathscr{H} \to \mathscr{H}^m$ and $B : \mathscr{H} \to \mathscr{H}$, should be understood as the m-tuple vector $([A_i, B])_{1\leq i\leq m}$; similarly, the commutator of two operator-valued vectors should be understood as an operator-valued matrix in the same way.

We shall mainly work on the space $L^2(\mu)$ and the Sobolev spaces $H^k(\mu)$ with respect to the reference measure μ. In the sequel, we set $\mathscr{H} = L^2(\mu)$, and so the notations $||\cdot||$ and $\langle\cdot,\cdot\rangle$ are rather reserved for the Hilbert space $L^2(\mu)$, with only exception in local computations or linear algebra where $\langle\cdot,\cdot\rangle$ might be used for the scalar product in Euclidean spaces. Other norms or scalar products are indicated by a self-explanatory subscript, for instance, $\langle\cdot,\cdot\rangle_{H^k}$ stands for the scalar product in $H^k(\mu)$. The homogeneous $H^k(\mu)$ seminorm is denoted by $\dot{H}^k(\mu)$, while the associated scalar product by $\langle\cdot,\cdot\rangle_{\dot{H}^k(\mu)}$.

Let h be the density function with respect to the invariant measure μ, i.e.,

$$h(t,x,v) = Zf(t,x,v)e^{V(x)+\frac{|v|^2}{2}},$$

then the evolution equation (7.1.1) becomes

$$\partial_t h_t + L h_t = 0 \tag{7.1.3}$$

with L given by

$$\mathrm{L} = -\Delta_v h + v \cdot \nabla_v h + v \cdot \nabla_x h - \nabla_x \mathrm{V}(x) \cdot \nabla_v h.$$

Set two operators $\mathrm{A} : \mathscr{H} \to \mathscr{H}^d$ and $\mathrm{B} : \mathscr{H} \to \mathscr{H}$ as

$$\mathrm{A} := (\mathrm{A}_i)_{1 \le i \le d} = \nabla_v, \qquad \mathrm{B} := v \cdot \nabla_x - \nabla \mathrm{V}(x) \cdot \nabla_v,$$

then

$$\mathrm{A}^* = -\mathrm{Div}_v + v \cdot, \qquad \mathrm{B}^* = -\mathrm{B},$$

and

$$\mathrm{A}^*\mathrm{A} = -\Delta_v + v \cdot \nabla_v.$$

The operator L can be then written in Hörmander form

$$\mathrm{L} = \mathrm{A}^*\mathrm{A} + \mathrm{B}.$$

We denote $\nabla^2_{xv} := (\partial_{x_i}\partial_{v_j})_{(i,j):1\le i,j\le d}$ and similarly we can define ∇^2_{vx}. Note that these two matrices are generically not identical. They will be referred to as mixed Hessian.

We denote

$$\nabla_x^{l+1}\mathrm{V} \cdot \nabla_v h := \textstyle\sum_{j=1}^d \left(\nabla_x^l \partial_{x_j}\mathrm{V}\right)\partial_{v_j} h,$$

$$|\nabla_x^l \mathrm{V} \cdot \nabla_v g|^2 = \sum_{|\alpha|=l-1} |\sum_{j=1}^d \mathrm{D}_x^\alpha \partial_{x_j} \mathrm{V}(x) \partial_{x_j} g|^2.$$

$$||\nabla_x^i \nabla_v^j h||^2 := \sum_{|\alpha|=i, |\beta|=j} \int |\mathrm{D}_x^\alpha \mathrm{D}_v^\beta h|^2 d\mu$$

where α, β are multi-indexes of respective order $|\alpha|$ and $|\beta|$, and $\mathrm{D}_x^\alpha \mathrm{D}_v^\beta h$ is given as usual by

$$\mathrm{D}_x^\alpha \mathrm{D}_v^\beta h = \frac{\partial^{|\alpha|+|\beta|} h}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d} \partial v_1^{\beta_1} \cdots \partial v_d^{\beta_d}}.$$

Recall that a probability measure ν is said to satisfy a Poincaré inequality with constant $\kappa > 0$ if it holds

$$||g - \int g d\nu||^2_{\mathrm{L}^2(\nu)} \le \kappa ||\nabla g||^2_{\mathrm{L}^2(\nu)}$$

for all functions $g \in \mathrm{H}^1(\nu)$. An importance consequence of Poincaré inequalities is the equivalence of the $\mathrm{H}^1(\mu)$-norm and the $\mathrm{H}^1(\mu)$-seminorm.

7.1.2 Main results

We shall prove hypocoercivity and global hypoellipticity in $\mathrm{H}^k(\mu)$ under conditions on the derivatives of the potential V. Although the hypocoercivity result in $\mathrm{H}^2(\mu)$ is contained in Theorem 7.3 about hypocoercivity in $\mathrm{H}^k(\mu)$, we shall state it independently. Moreover, an explicit choice of the convergence rate in $\mathrm{H}^2(\mu)$ is given.

Theorem 7.1. *Assume that the measure* $\frac{1}{Z_1}e^{-V(x)}dx$ *satisfies a Poincaré inequality with constant* κ. *Assume furthermore that the confining potential* $V \in C^{\infty}(\mathbb{R}^d)$ *satisfies*

$$\int |\nabla^2 V \cdot \nabla_v g|^2 d\mu \leq M\Big(\int |\nabla_v g|^2 d\mu + \int |\nabla^2_{xv} g|^2 d\mu\Big), \tag{7.1.4}$$

and

$$\int |\nabla^3 V \cdot \nabla_v g|^2 d\mu \leq M(\int |\nabla_v g|^2 d\mu + \int |\nabla^2_{xv} g|^2 d\mu) \tag{7.1.5}$$

for all $g \in H^2(\mu)$. *Then there exist explicitly computable constants* C *and* $\lambda > 0$ *such that*

$$||h_t - \int h_0 d\mu||_{H^2(\mu)} \leq Ce^{-\lambda t}||h_0 - \int h_0 d\mu||_{H^2(\mu)} \tag{7.1.6}$$

where $h_t = h(t,x,v)$ *is the solution to the kinetic Fokker-Planck equation with the initial condition* $h_0 \in H^2(\mu)$. *For instance, the convergence rate may be chosen as*

$$\lambda = \left(\frac{7}{8}\right)^{12} \cdot \frac{1}{16(1+16\kappa \max(M^2,1))}.$$

Remark 7.2. Roughly speaking, the inequalities (7.1.4) and (7.1.5) require relative boundedness of $\nabla^2 V$ and $\nabla^3 V$ as operators. Of course, these conditions are satisfied when $\nabla^2 V$ and $\nabla^3 V$ are uniformly pointwise bounded. But they hold true for a much general class of potentials V. For instance, Villani shows that (7.1.4) holds when there exists some positive constant such that

$$|\nabla^2 V| \leq C(1 + |\nabla V|).$$

These inequalities can be also regarded as some weighted Poincaré inequalities, for which various conditions on the potential V may apply.

It is possible to extend Theorem 7.1 in higher order Sobolev spaces presuming conditions on higher order derivatives of the potential V.

Theorem 7.3. *Assume that the measure* $\frac{1}{Z_1}e^{-V(x)}dx$ *satisfies a Poincaré inequality with constant* κ. *Assume furthermore that the confining potential* $V \in C^{\infty}(\mathbb{R}^d)$ *satisfies*

$$\int |\nabla^l_x V \cdot \nabla_v g|^2 d\mu \leq M\Big(\int |\nabla_v g|^2 d\mu + \int |\nabla^2_{xv} g|^2 d\mu\Big) \tag{7.1.7}$$

for $2 \leq l \leq k+1$ *and any function* $g \in H^2(\mu)$. *Then there exist explicitly computable constants* C *and* $\lambda > 0$ *such that*

$$||h_t - \int h_0 d\mu||_{H^k(\mu)} \leq Ce^{-\lambda t}||h_0 - \int h_0 d\mu||_{H^k(\mu)} \tag{7.1.8}$$

where $h_t = h(t,x,v)$ *is the solution to the kinetic Fokker-Planck equation with the initial condition* $h_0 \in H^k(\mu)$.

Remark 7.4. The conditions on $\nabla^l V(x)$ $(2 \leq l \leq k+1)$ might seem doomed at first glance. But they are just relative boundedness as operators. Moreover, some criteria for them will be provided in this paper. On the other hand, uniform pointwise boundedness of $\nabla^l V(x)$ $(2 \leq l \leq k+1)$ was imposed in the regularity results presented in [11, Theorem A.15] as well, where the global hypoellipticity in high order Sobolev spaces are concerned.

7.1.3 Plan of the paper

We prove theorem 7.1 in Section 2 and 3. In section 2, we calculate the temporal derivatives of the $\mathrm{H}^2(\mu)$ scalar product along the kinetic Fokker-Planck equation. Then we are led to introduce a mixed term to overcome the degeneracy of the dissipation. In Section 3, we construct a twisted $\mathrm{H}^2(\mu)$ -norm and choose carefully the coefficients in it such that a coercivity estimate holds for the operator L. That way, we obtain convergence to equilibrium under the twisted $\mathrm{H}^2(\mu)$-norm and thus the usual $\mathrm{H}^2(\mu)$-norm. The Section 4 is devoted to the proof of Theorem 7.3. Its structure is the same as the one of the proof of theorem 7.1. In spite of that, it is based on an induction argument.

A future perspective is that we shall develop global hypoellipticity estimates by Hérau's method [8]. As a complement, we shall then furnish conditions under which the inequalities (7.1.4), (7.1.5), (7.1.7) can be verified.

7.2 Temporal derivative of $\mathrm{H}^2(\mu)$ norm

7.2.1 A twisted $\mathrm{H}^1(\mu)$ norm

In his hypocoercivity theory, Villani [11, Theorem 18, Theorem 35] introduced a twisted $\mathrm{H}^1(\mu)$ norm which takes the form

$$((h,h))_{\mathrm{H}^1} = ||h||^2 + a||\nabla_v h||^2 + 2b\langle\nabla_v h, \nabla_x h\rangle + c||\nabla_x h||^2 \tag{7.2.1}$$

in order to get coercivity estimates. Following his proof of [11, Theorem 18], one obtains the following result for the kinetic Fokker-Planck equation.

Lemma 7.5. *It holds for the twisted norm (7.2.1) that*

$$\begin{aligned}((h,\mathrm{L}h))_{\mathrm{H}^1} = {} & ||\nabla_v h||^2 + a(||\nabla_v^2 h||^2 + ||\nabla_v h||^2 + \langle\nabla_v h, \nabla_x h\rangle) \\ & + b(2\langle\nabla_v^2 h, \nabla_{xv}^2 h\rangle + \langle\nabla_v h, \nabla_x h\rangle + ||\nabla_x h||^2 - \langle\nabla_v h, \nabla^2\mathrm{V}\cdot\nabla_v h\rangle) \\ & + c(||\nabla_{xv}^2 h||^2 - \langle\nabla_x h, \nabla^2\mathrm{V}\cdot\nabla_v h\rangle).\end{aligned} \tag{7.2.2}$$

In the calculations here and below, the following commutation relations play an essential role. By direct computation,

(1) $[\mathrm{A},\mathrm{A}^*] = \mathrm{I}$, i.e. $[\mathrm{A}_i,\mathrm{A}_j^*] = \delta_{ij}$;

(2) $\mathrm{C} := [\mathrm{A},\mathrm{B}] = \nabla_x$;

(3) $\mathrm{R} := [\mathrm{C},\mathrm{B}] = [\nabla_x, v\cdot\nabla_x - \nabla\mathrm{V}(x)\cdot\nabla_v] = -\nabla^2\mathrm{V}(x)\cdot\nabla_v$.

Note also that A commutes with both itself and C.

7.2.2 The second order terms

First of all, we consider the temporal derivatives of second order terms in the $\mathrm{H}^2(\mu)$ scalar product. Set

$$\mathrm{T}_1 := \left\langle \nabla_v^2 \mathrm{L}h, \nabla_v^2 h \right\rangle, \quad \mathrm{T}_2 := \left\langle \nabla_{xv}^2 \mathrm{L}h, \nabla_{xv}^2 h \right\rangle, \quad \mathrm{T}_3 := \left\langle \nabla_x^2 \mathrm{L}h, \nabla_x^2 h \right\rangle.$$

Lemma 7.6. *Let h be a rapidly decreasing function. It holds that*

$$\mathrm{T}_1 = ||\nabla_v^3 h||^2 + 2||\nabla_v^2 h||^2 + 2\langle \nabla_{xv}^2 h, \nabla_v^2 h\rangle; \tag{7.2.3}$$

$$\mathrm{T}_2 = ||\nabla_x \nabla_v^2 h||^2 + ||\nabla_{xv}^2 h||^2 + \langle \nabla_x^2 h, \nabla_{xv}^2 h\rangle - \langle \nabla^2 \mathrm{V} \cdot \nabla_v^2 h, \nabla_{xv}^2 h\rangle; \tag{7.2.4}$$

$$\mathrm{T}_3 = ||\nabla_x^2 \nabla_v h||^2 - \left\langle \nabla^2 \mathrm{V} \cdot \nabla_{vx}^2 h, \nabla_x^2 h \right\rangle - \left\langle \nabla_x (\nabla^2 \mathrm{V} \cdot \nabla_v h), \nabla_x^2 h \right\rangle. \tag{7.2.5}$$

Proof. Using repeatedly the aforementioned commutation relations between $\mathrm{A}, \mathrm{A}^*, \mathrm{B}$ and C, we compute

$$\begin{aligned}
\mathrm{T}_{1\mathrm{A}} := \left\langle \nabla_v^2 \mathrm{A}^* \mathrm{A} h, \nabla_v^2 h \right\rangle &= \sum_{i,j,k} \left\langle \mathrm{A}_i \mathrm{A}_j \mathrm{A}_k^* \mathrm{A}_k h, \mathrm{A}_i \mathrm{A}_j h \right\rangle \\
&= \sum_{i,j,k} \langle \mathrm{A}_i (\mathrm{A}_k^* \mathrm{A}_j + [\mathrm{A}_j, \mathrm{A}_k^*]) \mathrm{A}_k h, \mathrm{A}_i \mathrm{A}_j h\rangle \\
&= \sum_{i,j,k} \langle \left(\mathrm{A}_k^* \mathrm{A}_i + [\mathrm{A}_i, \mathrm{A}_k^*]\right) \mathrm{A}_j \mathrm{A}_k h + \delta_{jk} \mathrm{A}_i \mathrm{A}_k h, \mathrm{A}_i \mathrm{A}_j h\rangle \\
&= \sum_{i,j,k} \langle \mathrm{A}_i \mathrm{A}_j \mathrm{A}_k h, \mathrm{A}_k \mathrm{A}_i \mathrm{A}_j h\rangle + \sum_{i,j} \langle \mathrm{A}_j \mathrm{A}_i h, \mathrm{A}_i \mathrm{A}_j h\rangle + \sum_{i,j} \langle \mathrm{A}_i \mathrm{A}_j h, \mathrm{A}_i \mathrm{A}_j h\rangle \\
&= \sum_{i,j,k} \left\|\mathrm{A}_i \mathrm{A}_j \mathrm{A}_k h\right\|^2 + 2\sum_{i,j} \left\|\mathrm{A}_i \mathrm{A}_j h\right\|^2 = ||\mathrm{A}^3 h||^2 + 2||\mathrm{A}^2 h||^2 \\
&= \left\|\nabla_v^3 h\right\|^2 + 2\left\|\nabla_v^2 h\right\|^2.
\end{aligned}$$

Recall that B is anti-symmetric, i.e. $\langle \mathrm{B}g, g\rangle = 0$ for any suitably integrable function g. In particular, it holds that

$$\langle \mathrm{B}\mathrm{A}_i \mathrm{A}_j h, \mathrm{A}_i \mathrm{A}_j h\rangle = 0 \tag{7.2.6}$$

Therefore

$$\begin{aligned}
\mathrm{T}_{1\mathrm{B}} := \langle \nabla_v^2 \mathrm{B} h, \nabla_v^2 h\rangle &= \sum_{i,j} \langle \mathrm{A}_i \mathrm{A}_j \mathrm{B} h, \mathrm{A}_i \mathrm{A}_j h\rangle \\
&= \sum_{i,j} \langle \mathrm{A}_i \left(\mathrm{B}\mathrm{A}_j + [\mathrm{A}_j, \mathrm{B}]\right) h, \mathrm{A}_i \mathrm{A}_j h\rangle \\
&= \sum_{i,j} \left\langle (\mathrm{B}\mathrm{A}_i \mathrm{A}_j + [\mathrm{A}_i, \mathrm{B}]\mathrm{A}_j + \mathrm{A}_i \mathrm{C}_j) h, \mathrm{A}_i \mathrm{A}_j h \right\rangle \\
&= \sum_{i,j} \left\langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{A}_i \mathrm{A}_j h \right\rangle + \sum_{i,j} \left\langle \mathrm{A}_i \mathrm{C}_j h, \mathrm{A}_i \mathrm{A}_j h \right\rangle \qquad \text{by (7.2.6)} \\
&= 2\sum_{i,j} \langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{A}_i \mathrm{A}_j h\rangle = 2\left\langle \mathrm{C}\mathrm{A} h, \mathrm{A}^2 h \right\rangle \\
&= 2\langle \nabla_{xv}^2 h, \nabla_v^2 h\rangle.
\end{aligned}$$

Combined with the calculation for T_{1A}, it follows that

$$T_1 = T_{1A} + T_{1B} = ||\nabla_v^3 h||^2 + 2||\nabla_v^2 h||^2 + 2\langle \nabla_{xv}^2 h, \nabla_v^2 h\rangle \tag{7.2.7}$$

which is the first equality (7.2.3) in this lemma.
The other two equalities (7.2.4) and (7.2.5) can be obtained in the very same way. Note that A^* also commutates with C, we get

$$\begin{aligned}
T_{2A} &= \left\langle \nabla_{xv}^2 A^* A h, \nabla_{xv}^2 h\right\rangle = \sum_{i,j,k} \langle C_i A_j A_k^* A_k h, C_i A_j h\rangle \\
&= \sum_{i,j,k} \langle C_i A_k^* A_j A_k h + \delta_{jk} C_i A_k h, C_i A_j h\rangle \\
&= \sum_{i,j,k} \langle A_k^* C_i A_j A_k h, C_i A_j h\rangle + \sum_{i,j} \langle C_i A_j h, C_i A_j h\rangle \\
&= \sum_{i,j,k} \langle C_i A_j A_k h, A_k C_i A_j h\rangle + \sum_{i,j} \langle C_i A_j h, C_i A_j h\rangle = ||CA^2 h||^2 + ||CAh||^2 \\
&= ||\nabla_x \nabla_v^2 h||^2 + ||\nabla_{xv}^2 h||^2.
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
T_{2B} &= \left\langle \nabla_{xv}^2 B h, \nabla_{xv}^2 h\right\rangle = \sum_{i,j} \langle C_i A_j B h, C_i A_j h\rangle \\
&= \sum_{i,j} \left\langle C_i \left(BA_j + C_j\right) h, C_i A_j h\right\rangle \\
&= \sum_{i,j} \left\langle BC_i A_j h + R_i A_j h + C_i C_j h, C_i A_j h\right\rangle \\
&= \sum_{i,j} \left\langle R_i A_j h, C_i A_j h\right\rangle + \sum_{i,j} \left\langle C_i C_j h, C_i A_j h\right\rangle = \langle RAh, CAh\rangle + \left\langle C^2 h, CAh\right\rangle \\
&= -\left\langle \nabla^2 V \cdot \nabla_v^2 h, \nabla_{xv}^2 h\right\rangle + \left\langle \nabla_x^2 h, \nabla_{xv}^2 h\right\rangle
\end{aligned}$$

where again the anti-symmetry of B was used when passing from the third line to the fourth. Therefore we obtain the second equality (7.2.4).
Similarly, the third equality (7.2.5) follows from

$$\begin{aligned}
T_{3A} &= \left\langle \nabla_x^2 A^* A h, \nabla_x^2 h\right\rangle = \sum_{i,j,k} \left\langle C_i C_j A_k^* A_k h, C_i C_j h\right\rangle \\
&= \sum_{i,j,k} \left\langle A_k^* C_i C_j A_k h, C_i C_j h\right\rangle \\
&= \sum_{i,j,k} \left\langle C_i C_j A_k h, A_k C_i C_j h\right\rangle = \sum_{i,j,k} \left\langle C_i C_j A_k h, C_i C_j A_k h\right\rangle \\
&= ||C^2 A h||^2 = ||\nabla_x^2 \nabla_v h||^2,
\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{T}_{3\mathrm{B}} = \left\langle \nabla_x^2 \mathrm{B}h, \nabla_x^2 h \right\rangle &= \sum_{i,j} \left\langle \mathrm{C}_i \mathrm{C}_j \mathrm{B}h, \mathrm{C}_i \mathrm{C}_j h \right\rangle \\
&= \sum_{i,j} \left\langle \mathrm{C}_i (\mathrm{B}\mathrm{C}_j + \mathrm{R}_j) h, \mathrm{C}_i \mathrm{C}_j h \right\rangle \\
&= \sum_{i,j} \left\langle \mathrm{B}\mathrm{C}_i \mathrm{C}_j h + \mathrm{R}_i \mathrm{C}_j h + \mathrm{C}_i \mathrm{R}_j h, \mathrm{C}_i \mathrm{C}_j h \right\rangle \\
&= \left\langle \mathrm{RC}h, \mathrm{C}^2 h \right\rangle + \left\langle \mathrm{CR}h, \mathrm{C}^2 h \right\rangle \\
&= -\left\langle \nabla^2 \mathrm{V} \cdot \nabla_{vx}^2 h, \nabla_x^2 h \right\rangle - \left\langle \nabla_x (\nabla^2 \mathrm{V} \cdot \nabla_v h), \nabla_x^2 h \right\rangle .
\end{aligned}
$$

□

Putting all the three terms $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3$ together, we have got two terms $||\nabla_v^2 h||^2$ and $||\nabla_{xv}^2 h||^2$ which occur in the usual $\mathrm{H}^2(\mu)$-seminorm, however, the other term $||\nabla_x^2 h||^2$ is still missing. To be able to bound the temporal derivative of the $\mathrm{H}^2(\mu)$-scalar-product along the evolution equation (7.1.3), we are led to introduce a mixed term, just as in Villani [11].

7.2.3 A mixed term

We shall add the following mixed term in the usual $\mathrm{H}^2(\mu)$-norm,

$$\langle \nabla_{xv}^2 h, \nabla_x^2 h \rangle,$$

and we shall compute

$$\mathrm{T}_{\mathrm{mix}} := \langle \nabla_{xv}^2 \mathrm{L}h, \nabla_x^2 h \rangle + \langle \nabla_{xv}^2 h, \nabla_x^2 \mathrm{L}h \rangle$$

Lemma 7.7. *For any rapidly decreasing function $h \in \mathscr{S}(\mathbb{R}^{2d})$, it holds*

$$
\begin{aligned}
\mathrm{T}_{mix} \quad = \quad & 2\langle \nabla_x \nabla_v^2 h, \nabla_x^2 \nabla_v h \rangle + \langle \nabla_{xv}^2 h, \nabla_x^2 h \rangle + ||\nabla_x^2 h||^2 - \langle \nabla^2 \mathrm{V} \cdot \nabla_v^2 h, \nabla_x^2 h \rangle \\
& - \langle \nabla^2 \mathrm{V} \cdot \nabla_{vx}^2 h, \nabla_{xv}^2 h \rangle - \langle \nabla_x (\nabla^2 \mathrm{V} \cdot \nabla_v h), \nabla_{xv}^2 h \rangle .
\end{aligned}
\tag{7.2.8}
$$

Proof. As in the proof of Lemma 7.6, $\mathrm{T}_{\mathrm{mix}}$ can be written as the sum of $\mathrm{T}_{\mathrm{A}}, \mathrm{T}_{\mathrm{B}}$ defined by

$$\mathrm{T}_{\mathrm{A}} := \langle \nabla_{xv}^2 \mathrm{A}^* \mathrm{A}h, \nabla_x^2 h \rangle + \langle \nabla_{xv}^2 h, \nabla_x^2 \mathrm{A}^* \mathrm{A}h \rangle$$

and

$$\mathrm{T}_{\mathrm{B}} := \langle \nabla_{xv}^2 \mathrm{B}h, \nabla_x^2 h \rangle + \langle \nabla_{xv}^2 h, \nabla_x^2 \mathrm{B}h \rangle .$$

As before, we compute these two terms separately. For T_{A}, we obtain

$$
\begin{aligned}
\mathrm{T}_{\mathrm{A}} &= \sum_{i,j,k} \Big(\langle \mathrm{C}_i \mathrm{A}_j \mathrm{A}_k^* \mathrm{A}_k h, \mathrm{C}_i \mathrm{C}_j h \rangle + \langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{C}_i \mathrm{C}_j \mathrm{A}_k^* \mathrm{A}_k h \rangle \Big) \\
&= \sum_{i,j,k} \Big(\langle \mathrm{C}_i (\mathrm{A}_k^* \mathrm{A}_j + [\mathrm{A}_j, \mathrm{A}_k^*]) \mathrm{A}_k h, \mathrm{C}_i \mathrm{C}_j h \rangle + \langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{A}_k^* \mathrm{C}_i \mathrm{C}_j \mathrm{A}_k h \rangle \Big) \\
&= \sum_{i,j,k} \Big(\langle \mathrm{A}_k^* \mathrm{C}_i \mathrm{A}_j \mathrm{A}_k h + \delta_{jk} \mathrm{C}_i \mathrm{A}_k h, \mathrm{C}_i \mathrm{C}_j h \rangle + \langle \mathrm{A}_k \mathrm{C}_i \mathrm{A}_j h, \mathrm{C}_i \mathrm{C}_j \mathrm{A}_k h \rangle \Big) \\
&= \sum_{i,j,k} \langle \mathrm{C}_i \mathrm{A}_j \mathrm{A}_k h, \mathrm{A}_k \mathrm{C}_i \mathrm{C}_j h \rangle + \sum_{i,j} \langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{C}_i \mathrm{C}_j h \rangle + \sum_{i,j,k} \langle \mathrm{C}_i \mathrm{A}_j \mathrm{A}_k h, \mathrm{C}_i \mathrm{C}_j \mathrm{A}_k h \rangle \\
&= 2 \langle \mathrm{CA}^2 h, \mathrm{C}^2 \mathrm{A}h \rangle + \langle \mathrm{CA}h, \mathrm{C}^2 h \rangle ,
\end{aligned}
$$

or equivalently,

$$\mathrm{T_A} = 2\langle \nabla_x \nabla_v^2 h, \nabla_x^2 \nabla_v h\rangle + \langle \nabla_{xv}^2 h, \nabla_x^2 h\rangle.$$

For the term $\mathrm{T_B}$, we compute that

$$\begin{aligned}
\langle \nabla_{xv}^2 \mathrm{B}h, \nabla_x^2 h\rangle &= \sum_{i,j} \langle \mathrm{C}_i \mathrm{A}_j \mathrm{B}h, \mathrm{C}_i \mathrm{C}_j h\rangle = \sum_{i,j} \langle \mathrm{C}_i (\mathrm{B}\mathrm{A}_j + [\mathrm{A}_j, \mathrm{B}])h, \mathrm{C}_i \mathrm{C}_j h\rangle \\
&= \sum_{i,j} \langle (\mathrm{B}\mathrm{C}_i \mathrm{A}_j + [\mathrm{C}_i, \mathrm{B}]\mathrm{A}_j + \mathrm{C}_i \mathrm{C}_j)h, \mathrm{C}_i \mathrm{C}_j h\rangle \\
&= \sum_{i,j} \Big(\langle \mathrm{B}\mathrm{C}_i \mathrm{A}_j h, \mathrm{C}_i \mathrm{C}_j h\rangle + \langle \mathrm{R}_i \mathrm{A}_j h, \mathrm{C}_i \mathrm{C}_j h\rangle + \langle \mathrm{C}_i \mathrm{C}_j h, \mathrm{C}_i \mathrm{C}_j h\rangle \Big),
\end{aligned}$$

and

$$\begin{aligned}
\langle \nabla_{xv}^2 h, \nabla_x^2 \mathrm{B}h\rangle &= \sum_{i,j} \langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{C}_i \mathrm{C}_j \mathrm{B}h\rangle = \sum_{i,j} \langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{C}_i (\mathrm{B}\mathrm{C}_j + [\mathrm{C}_j, \mathrm{B}])h\rangle \\
&= \sum_{i,j} \langle \mathrm{C}_i \mathrm{A}_j h, (\mathrm{B}\mathrm{C}_i \mathrm{C}_j + [\mathrm{C}_i, \mathrm{B}]\mathrm{C}_j + \mathrm{C}_i \mathrm{R}_j)h\rangle \\
&= \sum_{i,j} \Big(\langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{B}\mathrm{C}_i \mathrm{C}_j h\rangle + \langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{R}_i \mathrm{C}_j h\rangle + \langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{C}_i \mathrm{R}_j h\rangle \Big).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\mathrm{T_B} &= \sum_{i,j} \Big(\langle \mathrm{B}\mathrm{C}_i \mathrm{A}_j h, \mathrm{C}_i \mathrm{C}_j h\rangle + \langle \mathrm{R}_i \mathrm{A}_j h, \mathrm{C}_i \mathrm{C}_j h\rangle + \langle \mathrm{C}_i \mathrm{C}_j h, \mathrm{C}_i \mathrm{C}_j h\rangle \\
&\quad + \langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{B}\mathrm{C}_i \mathrm{C}_j h\rangle + \langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{R}_i \mathrm{C}_j h\rangle + \langle \mathrm{C}_i \mathrm{A}_j h, \mathrm{C}_i \mathrm{C}_j h\rangle \Big) \\
&= \langle \mathrm{RA}h, \mathrm{C}^2 h\rangle + ||\mathrm{C}^2 h||^2 + \langle \mathrm{CA}h, \mathrm{RC}h\rangle + \langle \mathrm{CA}h, \mathrm{CR}h\rangle \\
&= -\langle \nabla^2 \mathrm{V} \cdot \nabla_v^2 h, \nabla_x^2 h\rangle + ||\nabla_x^2 h||^2 - \langle \nabla_{xv}^2 h, \nabla^2 \mathrm{V} \cdot \nabla_{vx}^2 h\rangle - \langle \nabla_{xv}^2 h, \nabla_x(\nabla^2 \mathrm{V} \cdot \nabla_v h)\rangle.
\end{aligned}$$

Then the equality (7.2.8) follows. □

7.3 Coercivity estimate under a twisted norm

In this section, we shall search for a twisted $\mathrm{H}^2(\mu)$-norm such that the operator L is coercive in this norm. Consider a quadratic form given by

$$\begin{aligned}
((h,h))_{\mathrm{H}^2} &:= ||h||^2 + a||\nabla_v h||^2 + 2b\langle \nabla_v h, \nabla_x h\rangle + c||\nabla_x h||^2 \\
&\quad + e_1 ||\nabla_v^2 h||^2 + e_2 ||\nabla_{xv}^2 h||^2 + e_3 ||\nabla_x^2 h||^2 + 2e_4 \langle \nabla_{xv}^2 h, \nabla_x^2 h\rangle \qquad (7.3.1) \\
&=: ((h,h))_{\mathrm{H}^1} + ((h,h))_{\dot{\mathrm{H}}^2} \qquad (7.3.2)
\end{aligned}$$

where $a, b, c, e_1, e_2, e_3, e_4$ are positive constants to be determined later. $((\cdot,\cdot))_{\mathrm{H}^2}^{1/2}$ (with both entries being the same) will be a norm equivalent to the usual one in $\mathrm{H}^2(\mu)$ whenever

$$ac > b^2, \quad e_2 e_3 > e_4^2. \qquad (7.3.3)$$

These conditions will be satisfied by a delicate choice of the constants. In this case of (7.3.3) being satisfied, it is not difficult to check that $((\cdot,\cdot))_{\dot{\mathrm{H}}^2}^{1/2}$ will be a seminorm.
Collecting the results in Lemma 7.6 and Lemma 7.7, we know that

Lemma 7.8. *For the seminorm $((\cdot,\cdot))^{1/2}_{\dot{\mathrm{H}}^2}$ it holds*

$$\begin{aligned}((h,\mathrm{L}h))_{\dot{\mathrm{H}}^2} =& e_1\big(\|\nabla_v^3 h\|^2 + 2\|\nabla_v^2 h\|^2 + 2\langle\nabla_{xv}^2 h, \nabla_v^2 h\rangle\big)\\ &+ e_2\big(\|\nabla_x\nabla_v^2 h\|^2 + \|\nabla_{xv}^2 h\|^2 + \langle\nabla_x^2 h, \nabla_{xv}^2 h\rangle - \langle\nabla^2\mathrm{V}\cdot\nabla_v^2 h, \nabla_{xv}^2 h\rangle\big)\\ &+ e_3\big(\|\nabla_x^2\nabla_v h\|^2 - \big\langle\nabla^2\mathrm{V}\cdot\nabla_{vx}^2 h, \nabla_x^2 h\big\rangle - \big\langle\nabla_x(\nabla^2\mathrm{V}\cdot\nabla_v h), \nabla_x^2 h\big\rangle\big)\\ &+ e_4\big(2\langle\nabla_x\nabla_v^2 h, \nabla_x^2\nabla_v h\rangle + \langle\nabla_{xv}^2 h, \nabla_x^2 h\rangle + \|\nabla_x^2 h\|^2 - \langle\nabla^2\mathrm{V}\cdot\nabla_v^2 h, \nabla_x^2 h\rangle\\ &- \langle\nabla^2\mathrm{V}\cdot\nabla_{vx}^2 h, \nabla_{xv}^2 h\rangle - \langle\nabla_x(\nabla^2\mathrm{V}\cdot\nabla_v h), \nabla_{xv}^2 h\rangle\big).\end{aligned} \tag{7.3.4}$$

Next we shall establish a coercivity estimate for the operator L.

Proposition 7.9. *Under the assumptions in Theorem 7.1, then there exist positive constants a, b, c, e_1, e_2, e_3, e_4, satisfying (7.3.3), and some explicitly computable constant $\lambda > 0$, such that for any rapidly decaying function h , it holds*

$$((h,\mathrm{L}h))_{\mathrm{H}^2} \geq \lambda((h - \int h\,d\mu, h - \int h\,d\mu))_{\mathrm{H}^2}. \tag{7.3.5}$$

For instance, assuming that $\mathrm{M} \geq 1$ *(otherwise one may replace* M *by* $\max\{\mathrm{M}, 1\}$*), we may take*

$$a = \left(\frac{7}{8}\right)^7 \cdot \frac{1}{8\mathrm{M}}, \quad b = \left(\frac{7}{8}\right)^{12} \frac{1}{16\mathrm{M}^2}, \quad c = \left(\frac{7}{8}\right)^{15} \frac{1}{16\mathrm{M}^3},$$
$$e_2 = \frac{c}{30720\mathrm{M}}, \quad e_1 = \frac{e_2}{2}, \quad e_4 = \frac{e_2}{64\mathrm{M}}, \quad e_3 = \frac{e_2}{1024\mathrm{M}^2},$$

and then λ can be taken as

$$\lambda = \left(\frac{7}{8}\right)^{12} \cdot \frac{1}{16(16\kappa\mathrm{M}^2 + 1)}.$$

Proof of Proposition 7.9. The proof will be divided into four steps. We shall apply Cauchy-Schwarz inequality and the assumptions in Theorem 7.1 to get some lower bound of the temporal derivative of the twisted $\mathrm{H}^2(\mu)$ scalar product. Then we choose the constants such that the coercivity estimate (7.3.5) holds.

For convenience, we denote in this proof

$$\mathrm{Z} := (\|\nabla_v h\|, \|\nabla_v^2 h\|, \|\nabla_x h\|, \|\nabla_{xv}^2 h\|)^{\mathsf{T}} \in \mathbb{R}^4,$$

$$\mathrm{W} := (\|\nabla_x^2 h\|, \|\nabla_v^3 h\|, \|\nabla_x\nabla_v^2 h\|, \|\nabla_x^2\nabla_v h\|)^{\mathsf{T}} \in \mathbb{R}^4.$$

Step 1. We begin with the terms related to the twisted $\mathrm{H}^1(\mu)$-norm. Note first that it holds

$$\|\nabla^2\mathrm{V}\cdot\nabla_v h\| \leq \sqrt{\mathrm{M}}(\|\nabla_{xv}^2 h\| + \|\nabla_v h\|).$$

thanks to inequality (7.1.4). And so by Cauchy-Schwarz inequality and the Lemma 7.5, we see that

$$\begin{aligned}
((h,\mathrm{L}h))_{\mathrm{H}^1} \geq{} & \|\nabla_v h\|^2 + a\big(\|\nabla_v^2 h\|^2 + \|\nabla_v h\|^2 - \|\nabla_v h\|\|\nabla_x h\|\big) \\
& + b\big(-2\|\nabla_v^2 h\|\|\nabla_{xv}^2 h\| - \|\nabla_v h\|\|\nabla_x h\| + \|\nabla_x h\|^2 - \|\nabla_v h\|\|\nabla^2 \mathrm{V}\cdot\nabla_v h\|\big) \\
& + c\big(\|\nabla_{xv}^2 h\|^2 - \|\nabla_x h\|\|\nabla^2\mathrm{V}\cdot\nabla_v h\|\big) \\
\geq{} & \|\nabla_v h\|^2 + a\big(\|\nabla_v^2 h\|^2 + \|\nabla_v h\|^2 - \|\nabla_v h\|\|\nabla_x h\|\big) \\
& + b\big(-2\|\nabla_v^2 h\|\|\nabla_{xv}^2 h\| - \|\nabla_v h\|\|\nabla_x h\| + \|\nabla_x h\|^2\big) \\
& - b\sqrt{\mathrm{M}}\|\nabla_v h\|(\|\nabla_{xv}^2 h\| + \|\nabla_v h\|) \\
& + c\big(\|\nabla_{xv}^2 h\|^2 - \sqrt{\mathrm{M}}\|\nabla_x h\|(\|\nabla_{xv}^2 h\| + \|\nabla_v h\|)\big).
\end{aligned} \tag{7.3.6}$$

In other terms,

$$((h,\mathrm{L}h))_{\mathrm{H}^1} \geq \langle \mathrm{Z}, \mathrm{S}_1 \mathrm{Z}\rangle$$

where S_1 is a 4-by-4 matrix given by

$$\mathrm{S}_1 = \begin{pmatrix} 1+a-b\sqrt{\mathrm{M}} & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ -(a+b+c\sqrt{\mathrm{M}}) & 0 & b & 0 \\ -b\sqrt{\mathrm{M}} & -2b & -c\sqrt{\mathrm{M}} & c \end{pmatrix}. \tag{7.3.7}$$

Step 2. Then we deal with the terms associated to $((h,h))_{\dot{\mathrm{H}}^2}$. Let us apply the inequality (7.1.4) with $\partial_{x_i} h$, then

$$\int |\nabla^2\mathrm{V}\cdot\nabla_v \partial_{x_i} h|^2 d\mu \leq \mathrm{M}\left(\int |\nabla_{xv}^2 \partial_{x_i} h|^2 d\mu + \int |\nabla_v \partial_{x_i} h|^2 d\mu\right).$$

Summing over i, we arrive at

$$\int |\nabla^2\mathrm{V}\cdot\nabla_{vx}^2 h|^2 d\mu \leq \mathrm{M}\left(\int |\nabla_x^2\nabla_v h|^2 d\mu + \int |\nabla_{xv}^2 h|^2 d\mu\right)$$

and so

$$\|\nabla^2\mathrm{V}\cdot\nabla_{vx}^2 h\| \leq \sqrt{\mathrm{M}}\big(\|\nabla_x^2\nabla_v h\| + \|\nabla_{xv}^2 h\|\big).$$

In the same manner, it holds

$$\int |\nabla^2\mathrm{V}\cdot\nabla_v^2 h|^2 d\mu \leq \mathrm{M}\left(\int |\nabla_x\nabla_v^2 h|^2 d\mu + \int |\nabla_v^2 h|^2 d\mu\right)$$

and so

$$\|\nabla^2\mathrm{V}\cdot\nabla_v^2 h\| \leq \sqrt{\mathrm{M}}\big(\|\nabla_x\nabla_v^2 h\| + \|\nabla_v^2 h\|\big).$$

Therefore the expressions $\mathrm{T}_1, \mathrm{T}_2$ can be treated by Cauchy-Schwarz inequality,

$$\mathrm{T}_1 \geq \|\nabla_v^3 h\|^2 + 2\|\nabla_v^2 h\|^2 - 2\|\nabla_{xv}^2 h\|\|\nabla_v^2 h\|,$$

$$\begin{aligned}
\mathrm{T}_2 &\geq \|\nabla_x\nabla_v^2 h\|^2 + \|\nabla_{xv}^2 h\|^2 - \|\nabla_x^2 h\|\|\nabla_{xv}^2 h\| - \|\nabla^2\mathrm{V}\cdot\nabla_v^2 h\|\|\nabla_{xv}^2 h\| \\
&\geq \|\nabla_x\nabla_v^2 h\|^2 + \|\nabla_{xv}^2 h\|^2 - \|\nabla_x^2 h\|\|\nabla_{xv}^2 h\| - \sqrt{\mathrm{M}}\|\nabla_{xv}^2 h\|(\|\nabla_x\nabla_v^2 h\| + \|\nabla_v^2 h\|).
\end{aligned}$$

For T_3, we compute

$$-\left\langle \nabla_x(\nabla^2\mathrm{V}\cdot\nabla_v h), \nabla_x^2 h\right\rangle = -\left\langle \nabla^3\mathrm{V}\cdot\nabla_v h + \nabla_{xv}^2 h\cdot\nabla^2\mathrm{V}, \nabla_x^2 h\right\rangle$$

where $\nabla^3\mathrm{V}\cdot\nabla_v h$ stands for a square matrix with (i,j)-element $\sum_k \partial^3_{x_i x_j x_k}\mathrm{V}\partial_{v_k} h$. So we know

$$\begin{aligned}
\mathrm{T}_3 &\geq \|\nabla_x^2\nabla_v h\|^2 - \|\nabla^2\mathrm{V}\cdot\nabla_{vx}^2 h\|\|\nabla_x^2 h\| - \|\nabla_x^2 h\|\big(\|\nabla^3\mathrm{V}\cdot\nabla_v h\| + \|\nabla_{xv}^2 h\cdot\nabla^2\mathrm{V}\|\big) \\
&= \|\nabla_x^2\nabla_v h\|^2 - \|\nabla_x^2 h\|\big(2\|\nabla^2\mathrm{V}\cdot\nabla_{vx}^2 h\| + \|\nabla^3\mathrm{V}\cdot\nabla_v h\|\big) \\
&\geq \|\nabla_x^2\nabla_v h\|^2 - \sqrt{\mathrm{M}}\|\nabla_x^2 h\|\big(2\|\nabla_x^2\nabla_v h\| + 3\|\nabla_{vx}^2 h\| + \|\nabla_v h\|\big)
\end{aligned}$$

where the last line follows from the presumed inequalities (7.1.4) and (7.1.5).
Now it remains to control the expression $\mathrm{T}_{\mathrm{mix}}$. As above, we have

$$\begin{aligned}
-\left\langle \nabla_x(\nabla^2\mathrm{V}\cdot\nabla_v h), \nabla_{xv}^2 h\right\rangle &= -\left\langle \nabla^3\mathrm{V}\cdot\nabla_v h + \nabla_{xv}^2 h\cdot\nabla^2\mathrm{V}, \nabla_{xv}^2 h\right\rangle \\
&\geq -\big(\|\nabla^3\mathrm{V}\cdot\nabla_v h\| + \|\nabla_{xv}^2 h\cdot\nabla^2\mathrm{V}\|\big)\|\nabla_{xv}^2 h\|,
\end{aligned}$$

and hence

$$\begin{aligned}
\mathrm{T}_{\mathrm{mix}} \geq& -2\|\nabla_x\nabla_v^2 h\|\|\nabla_x^2\nabla_v h\| - \|\nabla_{xv}^2 h\|\|\nabla_x^2 h\| + \|\nabla_x^2 h\|^2 - \|\nabla^2\mathrm{V}\cdot\nabla_v^2 h\|\|\nabla_x^2 h\| \\
&- \|\nabla^2\mathrm{V}\cdot\nabla_{vx}^2 h\|\|\nabla_{xv}^2 h\| - \big(\|\nabla^3\mathrm{V}\cdot\nabla_v h\| + \|\nabla_{xv}^2 h\cdot\nabla^2\mathrm{V}\|\big)\|\nabla_{xv}^2 h\| \\
\geq& -2\|\nabla_x\nabla_v^2 h\|\|\nabla_x^2\nabla_v h\| - \|\nabla_{xv}^2 h\|\|\nabla_x^2 h\| + \|\nabla_x^2 h\|^2 \\
&- \sqrt{\mathrm{M}}\big(\|\nabla_v^2 h\| + \|\nabla_x\nabla_v^2 h\|\big)\|\nabla_x^2 h\| \\
&- \sqrt{\mathrm{M}}\|\nabla_{xv}^2 h\|\big(\|\nabla_v h\| + 3\|\nabla_{xv}^2 h\| + 2\|\nabla_x^2\nabla_v h\|\big).
\end{aligned}$$

Now we conclude that

$$\begin{aligned}
((h,\mathrm{L}h))_{\dot{\mathrm{H}}^2} \geq\ & e_1\big(\|\nabla_v^3 h\|^2 + 2\|\nabla_v^2 h\|^2 - 2\|\nabla_{xv}^2 h\|\|\nabla_v^2 h\|\big) \\
&+ e_2\big\{\|\nabla_x\nabla_v^2 h\|^2 + \|\nabla_{xv}^2 h\|^2 - \|\nabla_x^2 h\|\|\nabla_{xv}^2 h\| \\
&\quad - \sqrt{\mathrm{M}}\|\nabla_{xv}^2 h\|(\|\nabla_x\nabla_v^2 h\| + \|\nabla_v^2 h\|)\big\} \\
&+ e_3\big\{\|\nabla_x^2\nabla_v h\|^2 - \sqrt{\mathrm{M}}\|\nabla_x^2 h\|\big(2\|\nabla_x^2\nabla_v h\| + 3\|\nabla_{vx}^2 h\| + \|\nabla_v h\|\big)\big\} \\
&+ e_4\big\{-2\|\nabla_x\nabla_v^2 h\|\|\nabla_x^2\nabla_v h\| - \|\nabla_{xv}^2 h\|\|\nabla_x^2 h\| + \|\nabla_x^2 h\|^2 \\
&\quad - \sqrt{\mathrm{M}}\big(\|\nabla_v^2 h\| + \|\nabla_x\nabla_v^2 h\|\big)\|\nabla_x^2 h\| \\
&\quad - \sqrt{\mathrm{M}}\|\nabla_{xv}^2 h\|\big(\|\nabla_v h\| + 3\|\nabla_{xv}^2 h\| + 2\|\nabla_x^2\nabla_v h\|\big)\big\}. \qquad (7.3.8)
\end{aligned}$$

Equivalently,

$$((h,\mathrm{L}h))_{\dot{\mathrm{H}}^2} \geq \left\langle \begin{pmatrix}\mathrm{Z}\\ \mathrm{W}\end{pmatrix}, \mathrm{S}_2'\begin{pmatrix}\mathrm{Z}\\ \mathrm{W}\end{pmatrix}\right\rangle \qquad (7.3.9)$$

where S_2' is the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2e_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -e_4\sqrt{M} & -2e_1-e_2\sqrt{M} & 0 & e_2-3e_4\sqrt{M} & 0 & 0 & 0 & 0 \\ -e_3\sqrt{M} & -e_4\sqrt{M} & 0 & -e_2-3e_3\sqrt{M}-e_4 & e_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_1 & 0 & 0 \\ 0 & 0 & 0 & -e_2\sqrt{M} & -e_4\sqrt{M} & 0 & e_2 & 0 \\ 0 & 0 & 0 & -2e_4\sqrt{M} & -2e_3\sqrt{M} & 0 & -2e_4 & e_3 \end{pmatrix}.$$

Step 3. Without loss of generality, we may assume that $M \ge 1$ in this and next step. Put S_2 as the matrix given by

$$\begin{pmatrix} 1+a-b\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a+2e_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a-b-c\beta & 0 & b & 0 & 0 & 0 & 0 & 0 \\ -b\beta-e_4\beta & -2b-2e_1-e_2\beta & -c\beta & c+e_2-3e_4\beta & 0 & 0 & 0 & 0 \\ -e_3\beta & -e_4\beta & 0 & -e_2-3e_3\beta-e_4 & e_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_1 & 0 & 0 \\ 0 & 0 & 0 & -e_2\beta & -e_4\beta & 0 & e_2 & 0 \\ 0 & 0 & 0 & -2e_4\beta & -2e_3\beta & 0 & -2e_4 & e_3 \end{pmatrix}$$

with $\beta = \sqrt{M}$ (to save space in the matrix above). Combining the results in Step 1 and Step 2 together, we find

$$((h, Lh))_{H^2} \ge \left\langle \begin{pmatrix} Z \\ W \end{pmatrix}, S_2 \begin{pmatrix} Z \\ W \end{pmatrix} \right\rangle. \tag{7.3.10}$$

We shall prove in this step that there exists constants a, b, c, e_1, e_2, e_3 and e_4, which may be taken as

$$a = \left(\frac{7}{8}\right)^7 \cdot \frac{1}{8M}, \quad b = \left(\frac{7}{8}\right)^{12} \frac{1}{16M^2}, \quad c = \left(\frac{7}{8}\right)^{15} \frac{1}{16M^3},$$
$$e_2 = \frac{c}{30720M}, \quad e_1 = \frac{e_2}{2}, \quad e_4 = \frac{e_2}{64M}, \quad e_3 = \frac{e_2}{1024M^2},$$

satisfying (7.3.3), such that

$$S_2 \ge \mathrm{Diag}(\frac{1}{16}, \frac{a}{16}, \frac{b}{16}, \frac{c}{16}, \frac{e_4}{4}, \frac{e_1}{2}, \frac{e_2}{8}, \frac{e_3}{8}) \tag{7.3.11}$$

in the sense of quadratic forms. In particular, it implies

$$S_2 \ge \mathrm{Diag}(\frac{1}{16}, \frac{a}{16}, \frac{b}{16}, \frac{c}{16}, \frac{e_4}{4}, 0, 0, 0).$$

To see (7.3.11), the following fact might be useful: a real-valued matrix $S = (s_{ij})_{1 \le i,j \le N}$, with positive diagonal elements, is positive in the sense of quadratic forms whenever there exists constants $\{k_{ij} \ge 0 \mid 1 \le i, j \le N, i \ne j\}$ such that $\sum_{j: j \ne i} k_{ij} \le 1$ and

$$|s_{ij}|^2 \le 4 k_{ij} s_{ii} k_{ji} s_{jj}, \text{ for all } i \ne j.$$

We illustrate its utilisation by an example which we shall use soon after. Let $e_1 \geq 0$, $e_2 \geq 0$, $e_4 = \frac{e_2}{64\mathrm{M}}$ and $e_3 = \frac{e_2}{1024\mathrm{M}^2}$ as above, then the matrix

$$\begin{pmatrix} \frac{1}{2}e_4 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ -e_4\sqrt{\mathrm{M}} & 0 & \frac{3}{4}e_2 & 0 \\ -2e_3\sqrt{\mathrm{M}} & 0 & -2e_4 & \frac{3}{4}e_3 \end{pmatrix}$$

is positive in the sense of quadratic forms. In this case we may take $k_{13} = k_{14} = \frac{1}{2}$, $k_{31} = \frac{1}{3}$, $k_{34} = \frac{2}{3}$, $k_{41} = \frac{1}{3}$, $\mathrm{K}_{43} = \frac{2}{3}$ (with all other k_{ij}'s being zero) and it is easy to verify the set of inequalities

$$\begin{aligned} e_4^2\mathrm{M} &\leq 4 \times k_{13} \cdot \frac{1}{2}e_4 \times k_{31} \cdot \frac{3}{4}e_2 = \frac{1}{4}e_4 e_2, \\ 4e_3^2\mathrm{M} &\leq 4 \times k_{14} \cdot \frac{1}{2}e_4 \times k_{41} \cdot \frac{3}{4}e_3 = \frac{1}{4}e_4 e_3, \\ 4e_4^2 &\leq 4 \times k_{34} \cdot \frac{3}{4}e_2 \times k_{43} \cdot \frac{3}{4}e_3 = e_2 e_3 \end{aligned}$$

which is equivalent to the set of inequalities $4\mathrm{M}e_4 \leq e_2$, $16\mathrm{M}e_3 \leq e_4$, $4e_4^2 \leq e_2 e_3$.

Now let us set $\mathrm{S}_z, \mathrm{S}_w, \mathrm{S}_m$ as the 4-by-4 matrix such that

$$\mathrm{S}_2 = \begin{pmatrix} \mathrm{S}_z & 0 \\ \mathrm{S}_m & \mathrm{S}_w \end{pmatrix},$$

and let $z = (z_1, z_2, z_3, z_4)$, $w = (w_1, w_2, w_3, w_4)$ be vectors in $\mathbb{R}^4$.
First, the preceding example shows that

$$\mathrm{S}_w \geq \mathrm{Diag}(\frac{1}{2}e_4, e_1, \frac{1}{4}e_2, \frac{1}{4}e_3) := \mathrm{S}'_w \tag{7.3.12}$$

in the sense of quadratic forms.
Next, we prove that

$$\mathrm{S}_1 \geq \mathrm{Diag}(\frac{1}{8}, \frac{1}{8}a, \frac{1}{8}b, \frac{1}{8}c) := \mathrm{S}'_1 \tag{7.3.13}$$

as quadratic forms under the condition that

$$a = \left(\frac{7}{8}\right)^7 \cdot \frac{1}{8\mathrm{M}}, \quad b = \left(\frac{7}{8}\right)^{12} \frac{1}{16\mathrm{M}^2}, \quad c = \left(\frac{7}{8}\right)^{15} \frac{1}{16\mathrm{M}^3}.$$

Actually, for (7.3.13), by the fact mentioned before, it suffices to verify that

$$\begin{aligned} (a+b+c\sqrt{\mathrm{M}})^2 &\leq 4 \times \frac{1}{2}(\frac{7}{8} + a - b\sqrt{\mathrm{M}}) \times \frac{1}{2} \cdot \frac{7b}{8}, \\ b^2\mathrm{M} &\leq 4 \times \frac{1}{2}(\frac{7}{8} + a - b\sqrt{\mathrm{M}}) \times \tau \cdot \frac{7c}{8}, \\ 4b^2 &\leq 4 \times \frac{7}{8}a \times \frac{1}{2} \cdot \frac{7c}{8}, \\ c^2\mathrm{M} &\leq 4 \times \frac{1}{2} \cdot \frac{7b}{8} \times (\frac{1}{2} - \tau) \cdot \frac{7c}{8} \end{aligned}$$

for some $0 < \tau = a\mathrm{M}/4 \le 1/2$. Note that in the situation we have $a \ge b + c\sqrt{\mathrm{M}}$, $a \ge b\sqrt{\mathrm{M}}$ and $1-2\tau = 1 - \frac{a\mathrm{M}}{2} \le \frac{7}{8}$ (since $a \le \frac{1}{4\mathrm{M}}$). Therefore the preceding inequalities can by deduced respectively from

$$a^2 \le \frac{1}{4}\cdot\left(\frac{7}{8}\right)^2 b, \quad b^2\mathrm{M} \le \left(\frac{7}{8}\right)^2 \cdot \frac{ac}{2}\mathrm{M}, \quad b^2 \le \left(\frac{7}{8}\right)^2 \cdot \frac{ac}{2}, \quad c\mathrm{M} \le \left(\frac{7}{8}\right)^3 b$$

which hold true for a, b, c given as above.

Then we consider all the remaining terms

$$\begin{aligned}\mathrm{Q}(z,w) = &-e_4\sqrt{\mathrm{M}}z_1z_4 - e_3\sqrt{\mathrm{M}}z_1w_1 + 2e_1z_2^2 - (2e_1 + e_2\sqrt{\mathrm{M}})z_2z_4 - e_4\sqrt{\mathrm{M}}z_2w_1 \\ &+ (e_2 - 3e_4\sqrt{\mathrm{M}})z_4^2 - (e_2 + 3e_3\sqrt{\mathrm{M}} + e_4)z_4w_1 - e_2\sqrt{\mathrm{M}}z_4w_3 - 2e_4\sqrt{\mathrm{M}}z_4w_4.\end{aligned}$$

and we claim that

$$\mathrm{Q}(z,w) \ge -\frac{1}{2}\big(\langle \mathrm{S}_1' z, z\rangle + \langle \mathrm{S}_w' w, w\rangle\big) \tag{7.3.14}$$

where we recall that $\mathrm{S}_1' = \frac{1}{8}\mathrm{Diag}(1, a, b, c)$, $\mathrm{S}_w' = \mathrm{Diag}(\frac{1}{2}e_4, e_1, \frac{1}{4}e_2, \frac{1}{4}e_3)$. Again, we apply the aforementioned fact to the lower triangle 8-by-8 matrix corresponding to the quadratic form

$$\mathrm{Q}(z,w) + \frac{1}{2}\big(\langle \mathrm{S}_1' z, z\rangle + \langle \mathrm{S}_w' w, w\rangle\big) - 2e_1z_2^2 - (e_2 - 3e_4\sqrt{\mathrm{M}})z_4^2.$$

It suffices to verify the set of inequalities below,

$$\begin{aligned} e_4^2\mathrm{M} &\le 4\times\frac{1}{2}\cdot\frac{1}{16}\times\frac{1}{5}\cdot\frac{c}{16} = \frac{c}{640}, \\ e_3^2\mathrm{M} &\le 4\times\frac{1}{2}\cdot\frac{1}{16}\times\frac{1}{3}\cdot\frac{e_4}{4} = \frac{e_4}{96}, \\ (2e_1 + e_2\sqrt{\mathrm{M}})^2 &\le 4\times\frac{1}{2}\cdot\frac{a}{16}\times\frac{1}{5}\cdot\frac{c}{16} = \frac{ac}{640}, \\ e_4^2\mathrm{M} &\le 4\times\frac{1}{2}\cdot\frac{a}{16}\times\frac{1}{3}\cdot\frac{e_4}{4} = \frac{ae_4}{96}, \\ (e_2 + 3e_3\sqrt{\mathrm{M}} + e_4)^2 &\le 4\times\frac{1}{5}\cdot\frac{c}{16}\times\frac{1}{3}\cdot\frac{e_4}{4} = \frac{ce_4}{240}, \\ e_2^2\mathrm{M} &\le 4\times\frac{1}{5}\cdot\frac{c}{16}\times\frac{e_2}{8} = \frac{ce_2}{160}, \\ 4e_4^2\mathrm{M} &\le 4\times\frac{1}{5}\cdot\frac{c}{16}\times\frac{e_3}{8} = \frac{ce_3}{160}. \end{aligned}$$

Note that all these inequalities express that e_1, e_2, e_3, e_4 can be bounded from above in terms of $1, a, b$ and c, thanks to the relations between e_1, e_2, e_3, e_4. Therefore, roughly speaking, when e_2 is chosen small enough compared to a, b, c, these inequalities will be satisfied. This is the case for instance when $e_2 = \frac{c}{30720\mathrm{M}}$.

At last, summarizing(7.3.12),(7.3.13),(7.3.14), we obtain for any $z, w \in \mathbb{R}^4$

$$\begin{aligned}\left\langle \begin{pmatrix} z \\ w\end{pmatrix}, \mathrm{S}_2\begin{pmatrix} z \\ w\end{pmatrix}\right\rangle &= \langle \mathrm{S}_z z, z\rangle + \langle \mathrm{S}_m z, w\rangle + \langle \mathrm{S}_w w, w\rangle \\ &= \langle \mathrm{S}_1 z, z\rangle + \mathrm{Q}(z,w) + \langle \mathrm{S}_w w, w\rangle \\ &\ge \langle \mathrm{S}_1' z, z\rangle - \frac{1}{2}\big(\langle \mathrm{S}_1' z, z\rangle + \langle \mathrm{S}_w' w, w\rangle\big) + \langle \mathrm{S}_w' w, w\rangle \\ &\ge \frac{1}{2}\big(\langle \mathrm{S}_1' z, z\rangle + \langle \mathrm{S}_w' w, w\rangle\big).\end{aligned}$$

Thus we obtain (7.3.11) since z, w are arbitrary vectors in $\mathbb{R}^4$. As a consequence, we arrive at

$$\begin{aligned}((h, \mathrm{L}h))_{\mathrm{H}^2} \geq &\frac{1}{16}||\nabla_v h||^2 + \frac{a}{16}||\nabla_v^2 h||^2 + \frac{b}{16}||\nabla_x h||^2 + \frac{c}{16}||\nabla_{xv}^2 h||^2 + \frac{e_4}{4}||\nabla_x^2 h||^2 \\ &+ \frac{e_1}{2}||\nabla_v^3 h||^2 + \frac{e_2}{8}||\nabla_x \nabla_v^2 h||^2 + \frac{e_3}{8}||\nabla_x^2 \nabla_v h||^2. \end{aligned} \tag{7.3.15}$$

Step 4. As we can see from the preceding inequality, it remains to demonstrate that

$$\frac{1}{16}||\nabla_v h||^2 + \frac{a}{16}||\nabla_v^2 h||^2 + \frac{b}{16}||\nabla_x h||^2 + \frac{c}{16}||\nabla_{xv}^2 h||^2 + \frac{e_4}{4}||\nabla_x^2 h||^2 \geq \lambda((h - \int h \mathrm{d}\mu, h - \int h \mathrm{d}\mu))_{\mathrm{H}^2} \tag{7.3.16}$$

for some explicitly computable constant λ. By tensorization property of Poincaré inequality, it holds

$$||h - \int h \mathrm{d}\mu||^2 \leq ||\nabla_v h||^2 + \kappa ||\nabla_x h||^2.$$

Note by Cauchy-Schwarz inequality it holds as well

$$|2b\langle \nabla_v h, \nabla_x h\rangle| \leq 2 \cdot \frac{7}{8} \cdot \sqrt{\frac{ac}{2}} ||\nabla_v h|| ||\nabla_x h|| \leq \frac{2}{3}\left(a||\nabla_v h||^2 + c||\nabla_x h||^2\right)$$

$$|2e_4 \langle \nabla_{xv}^2 h, \nabla_x^2 h\rangle| \leq \sqrt{e_2 e_3} ||\nabla_{xv}^2 h|| ||\nabla_x^2 h|| \leq \frac{1}{2}\left(e_2 ||\nabla_{xv}^2 h||^2 + e_3 ||\nabla_x^2 h||^2\right).$$

The three previous inequalities imply that

$$\begin{aligned} &((h - \int h \mathrm{d}\mu, h - \int h \mathrm{d}\mu))_{\mathrm{H}^2} \\ &\leq (\frac{5}{3}a + 1)||\nabla_v h||^2 + (\frac{5}{3}c + \kappa)||\nabla_x h||^2 + e_1 ||\nabla_v^2 h||^2 + \frac{3}{2} e_2 ||\nabla_{xv}^2 h||^2 + \frac{3}{2} e_3 ||\nabla_x^2 h||^2. \end{aligned}$$

So (7.3.16) follows if we set λ as

$$\lambda = \left(\frac{7}{8}\right)^{12} \cdot \frac{1}{16(16\kappa \mathrm{M}^2 + 1)}$$

since

$$\frac{1}{16} \geq \lambda(\frac{5}{3}a + 1), \quad \frac{b}{16} \geq \lambda(\frac{5}{3}c + \kappa), \quad \frac{a}{16} \geq \lambda e_1, \quad \frac{c}{16} \geq \lambda \frac{3}{2} e_2, \quad \frac{e_4}{4} \geq \lambda \frac{3}{2} e_3.$$

(Actually, the three last inequalities always hold true when $\lambda \leq 1$, while the first one holds when $\lambda \leq \frac{1}{32}$.)

The proof of the coercivity estimate (7.3.5) is then completed. □

Proof of Theorem 7.1. Note that as a solution to the kinetic Fokker-Planck equation, h_t is smooth at positive time $t > 0$. By Proposition 7.9 and standard approximation procedure, we know that at positive time

$$((h, \mathrm{L}h))_{\mathrm{H}^2(\mu)} \geq \lambda((h - \int h \mathrm{d}\mu, h - \int h \mathrm{d}\mu))_{\mathrm{H}^2(\mu)}.$$

Consequently Gronwall's lemma implies that

$$((h - \int h \mathrm{d}\mu, h - \int h \mathrm{d}\mu))_{\mathrm{H}^2(\mu)} \leq e^{-2\lambda t}((h_0 - \int h_0 \mathrm{d}\mu, h_0 - \int h_0 \mathrm{d}\mu))_{\mathrm{H}^2(\mu)}$$

As we see in the proof of the previous proposition, the norm induced by $((\cdot, \cdot))_{\mathrm{H}^2(\mu)}$ is equivalent to the usual $\mathrm{H}^2(\mu)$ norm with explicit constants. The theorem then follows. □

7.4 Convergence to equilibrium in $\mathrm{H}^k(\mu)$ spaces

In the present section we study convergence in higher order Sobolev spaces. We shall proceed as in the previous section: We calculate temporal derivatives, choose a twisted Sobolev norm under which we are able to prove a coercivity estimate and thus a exponential convergence, finally we translate the convergence into a convergence under the usual Sobolev norm.
Let us introduce $||\cdot||_{\dot{\mathrm{H}}^l}$, the $\mathrm{H}^l(\mu)$-seminorm, which will be defined by

$$||h||^2_{\dot{\mathrm{H}}^l} := \sum_{i=0}^{l} ||\nabla_x^i \nabla_v^{l-i} h||^2_{\mathrm{L}^2(\mu)} = \sum_{i=0}^{l} \sum_{|\alpha|=i,|\beta|=l-i} \int |\mathrm{D}_x^\alpha \mathrm{D}_v^\beta h|^2 \mathrm{d}\mu$$

where α, β are multi-indexes.
The following coercivity estimate is the essential result of this section.

Proposition 7.10. *Under the assumptions in Theorem 7.3. There exists a twisted* $\mathrm{H}^k(\mu)$*-norm, denoted by* $((\cdot,\cdot))^{\frac{1}{2}}_{\mathrm{H}^k}$*, which is equivalent to the usual* $\mathrm{H}^k(\mu)$*-norm and satisfies an estimate*

$$((h,\mathrm{L}h))_{\mathrm{H}^k} \geq \lambda_{k,0}\big(\sum_{1\leq l\leq k} ||h||^2_{\dot{\mathrm{H}}^l} + \sum_{0\leq l\leq k} ||\nabla_x^l \nabla_v^{k+1-l} h||^2\big) \tag{7.4.1}$$

for some constant $\lambda_{k,0} > 0$ *and for any rapidly decreasing function* h *(i.e.* $h \in \mathscr{S}(\mathbb{R}^{2d})$*). As a consequence, it holds the following coercivity estimate*

$$((h,\mathrm{L}h))_{\mathrm{H}^k} \geq \lambda_k((h - \int h d\mu, h - \int h d\mu))_{\mathrm{H}^k} \tag{7.4.2}$$

for some constant $\lambda_k > 0$ *and for all function* $h \in \mathscr{S}(\mathbb{R}^{2d})$.

Remark 7.11. As we have proved (see (7.3.13),(7.3.15)), this proposition holds true in the particular cases of $k=1$ and $k=2$. Note also that the only difference between $\sum_{l=0}^{k} ||\nabla_x^l \nabla_v^{k+1-l} h||^2$ and $||h||^2_{\dot{\mathrm{H}}^{k+1}}$ is that the former expression does not contain the term $||\nabla_x^{k+1} h||^2$ while the latter one does.

Define a twisted $\mathrm{H}^k(\mu)$-seminorm $((h,h))^{\frac{1}{2}}_{\dot{\mathrm{H}}^k}$ by

$$((h,h))_{\dot{\mathrm{H}}^k} := \sum_{0\leq i\leq k} \omega_{k,i} ||\nabla_v^{k-i} \nabla_x^i h||^2 + 2\omega_k \langle \nabla_x^{k-1} \nabla_v h, \nabla_x^k h\rangle \tag{7.4.3}$$

with all the constants $\omega_{k,i}$ $(0 \leq i \leq k)$, ω_k being strictly positive and satisfying

$$\omega_k^2 < \omega_{k,k-1}\omega_{k,k}. \tag{7.4.4}$$

It is then clear that such a seminorm is equivalent to the usual $\mathrm{H}^k(\mu)$-seminorm in the sense that there exist constants $c_1, c_2 > 0$ such that

$$c_1 ||h||^2_{\dot{\mathrm{H}}^k} \leq ((h,h))_{\dot{\mathrm{H}}^k} \leq c_2 ||h||^2_{\dot{\mathrm{H}}^k}.$$

As an example, we have taken in the previous section

$$((h,h))_{\dot{\mathrm{H}}^2} := e_1\|\nabla_v^2 h\|^2 + e_2\|\nabla_{xv}^2 h\|^2 + e_3\|\nabla_x^2 h\|^2 + 2e_4\langle\nabla_{xv}^2 h, \nabla_x^2 h\rangle,$$

or in other terms, we set $\omega_2 = e_4, \omega_{2,0} = e_1, \omega_{2,1} = e_2, \omega_{2,2} = e_3$. And recall as well

$$((h,h))_{\dot{\mathrm{H}}^1} := a\|\nabla_v h\|^2 + c\|\nabla_x h\|^2 + 2b\langle\nabla_v h, \nabla_x h\rangle,$$

so we may set $\omega_1 = b$, $\omega_{1,0} = a$, $\omega_{1,1} = c$. We also set $\omega_{0,0} = 1$ as a convention.
We then define that

$$((h,h))_{\mathrm{H}^k} := \|h\|^2 + \sum_{l=1}^{k} ((h,h))_{\dot{\mathrm{H}}^l} \tag{7.4.5}$$

$$= \sum_{0\le i\le j\le k} \omega_{j,i}\|\nabla_v^{j-i}\nabla_x^i h\|^2 + \sum_{1\le j\le k} 2\omega_j\langle\nabla_x^{j-1}\nabla_v h, \nabla_x^j h\rangle \tag{7.4.6}$$

with suitable coefficients ω_j, $\omega_{j,i}$ ($1 \le j \le k$, $0 \le i \le j$) to be determined later (by an induction argument on k).

7.4.1 Preliminary computations

As above, we may start with the computations of $((h, \mathrm{L}h))_{\dot{\mathrm{H}}^k}$. Let m_1, m_2 be nonnegative integers such that $m_1 + m_2 = k$. Set

$$\mathrm{T}^{\mathrm{A}}_{m_1,m_2} := \langle\nabla_x^{m_1}\nabla_v^{m_2}\mathrm{A}^*\mathrm{A}h, \nabla_x^{m_1}\nabla_v^{m_2}h\rangle, \tag{7.4.7}$$

$$\mathrm{T}^{\mathrm{B}}_{m_1,m_2} := \langle\nabla_x^{m_1}\nabla_v^{m_2}\mathrm{B}h, \nabla_x^{m_1}\nabla_v^{m_2}h\rangle, \tag{7.4.8}$$

$$\mathrm{T}^{\mathrm{A}}_{mix} := \langle\nabla_x^{k-1}\nabla_v\mathrm{A}^*\mathrm{A}h, \nabla_x^k h\rangle + \langle\nabla_x^{k-1}\nabla_v h, \nabla_x^k\mathrm{A}^*\mathrm{A}h\rangle, \tag{7.4.9}$$

$$\mathrm{T}^{\mathrm{B}}_{mix} := \langle\nabla_x^{k-1}\nabla_v\mathrm{B}h, \nabla_x^k h\rangle + \langle\nabla_x^{k-1}\nabla_v h, \nabla_x^k\mathrm{B}h\rangle, \tag{7.4.10}$$

then we have the following result.

Lemma 7.12. *Let $h \in \mathscr{S}(\mathbb{R}^{2d})$ be a rapidly decreasing function. Then*

$$\mathrm{T}^{\mathrm{A}}_{m_1,m_2} = \|\nabla_x^{m_1}\nabla_v^{m_2+1}h\|^2 + m_2\|\nabla_x^{m_1}\nabla_v^{m_2}h\|^2, \tag{7.4.11}$$

$$\begin{aligned}\mathrm{T}^{\mathrm{B}}_{m_1,m_2} = &\sum_{l=1}^{m_2}\langle\nabla_x^{m_1}\nabla_v^{m_2-l}\nabla_x\nabla_v^{l-1}h, \nabla_x^{m_1}\nabla_v^{m_2}h\rangle\\ &+ \sum_{l=1}^{m_1}\langle\nabla_x^{m_1-l}(-\nabla^2\mathrm{V}\cdot\nabla_v)\nabla_x^{l-1}\nabla_v^{m_2}h, \nabla_x^{m_1}\nabla_v^{m_2}h\rangle,\end{aligned} \tag{7.4.12}$$

$$\mathrm{T}^{\mathrm{A}}_{mix} = 2\langle\nabla_x^{k-1}\nabla_v^2 h, \nabla_x^k\nabla_v h\rangle + \langle\nabla_x^{k-1}\nabla_v h, \nabla_x^k h\rangle, \tag{7.4.13}$$

$$\begin{aligned}\mathrm{T}^{\mathrm{B}}_{mix} = \|\nabla_x^k h\|^2 &+ \sum_{l=1}^{k-1}\langle\nabla_x^{k-l-1}(-\nabla^2\mathrm{V}\cdot\nabla_v)\nabla_x^{l-1}\nabla_v h, \nabla_x^k h\rangle\\ &+ \sum_{l=1}^{k-1}\langle\nabla_x^{k-1}\nabla_v h, \nabla_x^{k-l}(-\nabla^2\mathrm{V}\cdot\nabla_v)\nabla_x^{l-1}h\rangle\end{aligned} \tag{7.4.14}$$

where $-\nabla^2\mathrm{V}\cdot\nabla_v$ is understood as a d-tuple operator-valued vector in the pairing.

We may find that $\mathrm{T}^{\mathrm{A}}_{m_1,m_2}$ does not contain the term $||\nabla_x^{m_1+m_2}h||^2$ for all m_1, m_2 with a given sum $m_1+m_2=k$. This is the reason to introduce a mixed term $\langle\nabla_x^{k-1}\nabla_v h, \nabla_x^k h\rangle$ in $((h,h))_{\dot{\mathrm{H}}^k}$. Note also that in the case of $\mathrm{V}(x)$ being a quadratic potential, say $\frac{1}{2}|x|^2$ for instance, one can not expect the expressions in $\mathrm{T}^{\mathrm{B}}_{m_1,m_2}$ to be positive for a general function h: The pairing is given by

$$-\langle\nabla_x^{m_1-l}\nabla_v\nabla_x^{l-1}\nabla_v^{m_2}h, \nabla_x^{m_1}\nabla_v^{m_2}h\rangle = -\sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}\langle\nabla_x^{\alpha_1}\nabla_v^{\alpha_2}\nabla_x^{\alpha_3}\nabla_v^{\alpha_4}h, \nabla_x^{\alpha_1}\nabla_x^{\alpha_2}\nabla_x^{\alpha_3}\nabla_v^{\alpha_4}h\rangle$$

where α_1, α_2 α_3, α_4 run over multi-indexes of respective order $m_1-l, 1, l-1, m_2$; and we adopt the notation $\nabla_x^{\alpha} := \nabla_{x_{i_1}}\nabla_{x_{i_2}}\cdots\nabla_{x_{i_{|\alpha|}}}$ for $\alpha=(i_1,i_2,\cdots,i_{|\alpha|})$ and so on.
Now we turn to the proof of Lemma 7.12.

Proof. (0). We collect some commutation relations first. Let l_1, l_2 be positive integers, then

$$[\mathrm{A}^{l_2},\mathrm{B}] = \sum_{l=1}^{l_2}\mathrm{A}^{l_2-l}\mathrm{C}\mathrm{A}^{l-1}; \tag{7.4.15}$$

$$[\mathrm{C}^{l_1},\mathrm{B}] = \sum_{l=1}^{l_1}\mathrm{C}^{l_1-l}\mathrm{R}\mathrm{C}^{l-1}; \tag{7.4.16}$$

$$[\mathrm{C}^{l_1}\mathrm{A}^{l_2},\mathrm{B}] = \sum_{l=1}^{l_2}\mathrm{C}^{l_1}\mathrm{A}^{l_2-l}\mathrm{C}\mathrm{A}^{l-1} + \sum_{l=1}^{l_1}\mathrm{C}^{l_1-l}\mathrm{R}\mathrm{C}^{l-1}\mathrm{A}^{l_2}. \tag{7.4.17}$$

They may be deduced by induction from commutation relations between B and A or C . We only provide a proof for (7.4.16) here, since (7.4.15) may be proved in the very same manner, and (7.4.17) follows from the two preceding equalities. For (7.4.16), the case of $l_1=1$ is exactly the commutation relation between B and C. Now assume that (7.4.16) holds for l_1-1 ($l_1\geq 2$), i.e.

$$[\mathrm{C}^{l_1-1},\mathrm{B}] = \sum_{l=1}^{l_1-1}\mathrm{C}^{l_1-1-l}\mathrm{R}\mathrm{C}^{l-1}$$

then we obtain

$$\begin{aligned}[\mathrm{C}^{l_1},\mathrm{B}] &= \mathrm{C}(\mathrm{C}^{l_1-1}\mathrm{B}) - \mathrm{B}\mathrm{C}^{l_1} = \mathrm{C}[\mathrm{C}^{l_1-1},\mathrm{B}] + [\mathrm{C},\mathrm{B}]\mathrm{C}^{l_1-1} \\ &= \mathrm{C}\sum_{l=1}^{l_1-1}\mathrm{C}^{l_1-1-l}\mathrm{R}\mathrm{C}^{l-1} + \mathrm{R}\mathrm{C}^{l_1-1} \text{ (by assumption for } l_1-1\text{)} \\ &= \sum_{l=1}^{l_1}\mathrm{C}^{l_1-l}\mathrm{R}\mathrm{C}^{l-1}\end{aligned}$$

as desired. Therefore (7.4.16) holds for any positive integer l_1 by induction.
Another commutation relation might also be useful, namely,

$$\sum_{j=1}^{d}\mathrm{A}_{i_1}\mathrm{A}_{i_2}\cdots\mathrm{A}_{i_m}\mathrm{A}_j^*\mathrm{A}_j = \sum_{j=1}^{d}\mathrm{A}_j^*\mathrm{A}_{i_1}\mathrm{A}_{i_2}\cdots\mathrm{A}_{i_m}\mathrm{A}_j + m\mathrm{A}_{i_1}\mathrm{A}_{i_2}\cdots\mathrm{A}_{i_m}. \tag{7.4.18}$$

Omitting the subscripts $i_1, i_2, \cdots, i_m$, it may be written as

$$\sum_{j=1}^{d}\mathrm{A}^m\mathrm{A}_j^*\mathrm{A}_j = \sum_{j=1}^{d}\mathrm{A}_j^*\mathrm{A}^m\mathrm{A}_j + m\mathrm{A}^m.$$

Again it is a simple application of the commutation relation mentioned earlier and it may be proved by induction on m.

(1). Now we compute $\mathrm{T}^{\mathrm{A}}_{m_1,m_2}$ and $\mathrm{T}^{\mathrm{B}}_{m_1,m_2}$. Let us prove (7.4.11) first,

$$
\begin{aligned}
\mathrm{T}^{\mathrm{A}}_{m_1,m_2} &= \langle \mathrm{C}^{m_1}\mathrm{A}^{m_2}\mathrm{A}^*\mathrm{A}h, \mathrm{C}^{m_1}\mathrm{A}^{m_2}h\rangle = \sum_{j=1}^{d} \langle \mathrm{C}^{m_1}\mathrm{A}^{m_2}\mathrm{A}_j^*\mathrm{A}_j h, \mathrm{C}^{m_1}\mathrm{A}^{m_2}h\rangle \\
&= \sum_{j=1}^{d} \langle \mathrm{C}^{m_1}\mathrm{A}_j^*\mathrm{A}^{m_2}\mathrm{A}_j h, \mathrm{C}^{m_1}\mathrm{A}^{m_2}h\rangle + m_2\langle \mathrm{C}^{m_1}\mathrm{A}^{m_2}h, \mathrm{C}^{m_1}\mathrm{A}^{m_2}h\rangle \qquad \text{by (7.4.18)} \\
&= \sum_{j=1}^{d} \langle \mathrm{C}^{m_1}\mathrm{A}^{m_2}\mathrm{A}_j h, \mathrm{A}_j\mathrm{C}^{m_1}\mathrm{A}^{m_2}h\rangle + m_2||\mathrm{C}^{m_1}\mathrm{A}^{m_2}h||^2 \\
&= ||\mathrm{C}^{m_1}\mathrm{A}^{m_2+1}\mathrm{A}_j h||^2 + m_2||\mathrm{C}^{m_1}\mathrm{A}^{m_2}h||^2
\end{aligned}
$$

where the two last equality follow from the fact both A^* and A commutate with C.
Then we prove (7.4.12). As a consequence of (7.4.17), we find

$$
\begin{aligned}
\mathrm{T}^{\mathrm{B}}_{m_1,m_2} &= \langle \mathrm{C}^{m_1}\mathrm{A}^{m_2}\mathrm{B}h, \mathrm{C}^{m_1}\mathrm{A}^{m_2}h\rangle \\
&= \sum_{l=1}^{m_1} \langle \mathrm{C}^{m_1}\mathrm{A}^{m_2-l}\mathrm{C}\mathrm{A}^{l-1}h, \mathrm{C}^{m_1}\mathrm{A}^{m_2}h\rangle \\
&\quad + \sum_{l=1}^{m_2} \langle \mathrm{C}^{m_1-l}\mathrm{R}\mathrm{C}^{l-1}\mathrm{A}^{m_2}h, \mathrm{C}^{m_1}\mathrm{A}^{m_2}h\rangle + \langle \mathrm{B}\mathrm{C}^{m_1}\mathrm{A}^{m_2}h, \mathrm{C}^{m_1}\mathrm{A}^{m_2}h\rangle
\end{aligned}
$$

Then (7.4.12) follows since B is anti-symmetric,

$$\langle \mathrm{B}\mathrm{C}^{m_1}\mathrm{A}^{m_2}h, \mathrm{C}^{m_1}\mathrm{A}^{m_2}h\rangle = 0.$$

(2). Similarly we proceed with the computations of $\mathrm{T}^{\mathrm{A}}_{mix}$ and $\mathrm{T}^{\mathrm{B}}_{mix}$. Thanks to (7.4.18) and $[\mathrm{A},\mathrm{C}] = [\mathrm{A}^*,\mathrm{C}] = 0$, we know

$$
\begin{aligned}
\mathrm{T}^{\mathrm{A}}_{mix} &= \sum_{j=1}^{d} \left(\langle \mathrm{C}^{k-1}\mathrm{A}\mathrm{A}_j^*\mathrm{A}_j h, \mathrm{C}^k h\rangle + \langle \mathrm{C}^{k-1}\mathrm{A}h, \mathrm{C}^k\mathrm{A}_j^*\mathrm{A}_j h\rangle\right) \\
&= \langle \mathrm{C}^{k-1}\mathrm{A}h, \mathrm{C}^k h\rangle + \sum_{j=1}^{d} \langle \mathrm{A}_j^*\mathrm{C}^{k-1}\mathrm{A}\mathrm{A}_j h, \mathrm{C}^k h\rangle + \sum_{j=1}^{d} \langle \mathrm{C}^{k-1}\mathrm{A}h, \mathrm{A}_j^*\mathrm{C}^k\mathrm{A}_j h\rangle \\
&= \langle \mathrm{C}^{k-1}\mathrm{A}h, \mathrm{C}^k h\rangle + \sum_{j=1}^{d} \langle \mathrm{C}^{k-1}\mathrm{A}\mathrm{A}_j h, \mathrm{C}^k\mathrm{A}_j h\rangle + \sum_{j=1}^{d} \langle \mathrm{C}^{k-1}\mathrm{A}\mathrm{A}_j h, \mathrm{C}^k\mathrm{A}_j h\rangle \\
&= \langle \mathrm{C}^{k-1}\mathrm{A}h, \mathrm{C}^k h\rangle + 2\langle \mathrm{C}^{k-1}\mathrm{A}^2 h, \mathrm{C}^k\mathrm{A}h\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{T}^{\mathrm{B}}_{mix} &= \langle \mathrm{C}^{k-1}\mathrm{A}\mathrm{B}h, \mathrm{C}^k h\rangle + \langle \mathrm{C}^{k-1}\mathrm{A}h, \mathrm{C}^k\mathrm{B}h\rangle \\
&= \langle \mathrm{C}^k h, \mathrm{C}^k h\rangle + \sum_{l=1}^{k-1} \langle \mathrm{C}^{k-1-l}\mathrm{R}\mathrm{C}^{l-1}\mathrm{A}h, \mathrm{C}^k h\rangle + \langle \mathrm{B}\mathrm{C}^{k-1}\mathrm{A}h, \mathrm{C}^k h\rangle \\
&\quad + \sum_{l=1}^{k} \langle \mathrm{C}^{k-1}\mathrm{A}h, \mathrm{C}^{k-l}\mathrm{R}\mathrm{C}^{l-1}h\rangle + \langle \mathrm{C}^{k-1}\mathrm{A}h, \mathrm{B}\mathrm{C}^k h\rangle \qquad \text{by (7.4.17)} \\
&= ||\mathrm{C}^k h||^2 + \sum_{l=1}^{k-1} \langle \mathrm{C}^{k-1-l}\mathrm{R}\mathrm{C}^{l-1}\mathrm{A}h, \mathrm{C}^k h\rangle + \sum_{l=1}^{k} \langle \mathrm{C}^{k-1}\mathrm{A}h, \mathrm{C}^{k-l}\mathrm{R}\mathrm{C}^{l-1}h\rangle
\end{aligned}
$$

where the last equality holds since B is anti-symmetric. □

7.4.2 Proof of Proposition 7.10 and Theorem 7.3

Proof of Proposition 7.10. We prove it by induction on k. The proposition for $k = 1$ and $k = 2$ has been demonstrated in the previous section. Let us assume that the coercivity estimate holds true for $k-1$, i.e., there exists a seminorm $((\cdot,\cdot))^{\frac{1}{2}}_{\mathrm{H}^{k-1}}$ and $\lambda_{k-1,0} > 0$ such that

$$((h,\mathrm{L}h))_{\mathrm{H}^{k-1}} \geq \lambda_{k-1,0}\Big(\sum_{1\leq l\leq k-1} \|h\|^2_{\dot{\mathrm{H}}^l} + \sum_{0\leq l\leq k-1} \|\nabla^l_x \nabla^{k-l}_v h\|^2\Big). \tag{7.4.19}$$

We shall prove the existence of $\omega_k, \omega_{k,0}, \omega_{k,1}, \cdots, \omega_{k,k}$ (recall (7.4.3)) such that the norm defined by

$$((h,\mathrm{L}h))_{\mathrm{H}^k} = ((h,\mathrm{L}h))_{\mathrm{H}^{k-1}} + ((h,\mathrm{L}h))_{\dot{\mathrm{H}}^k}$$

satisfies a coercivity estimate

$$((h,\mathrm{L}h))_{\mathrm{H}^k} \geq \lambda_{k,0}\Big(\sum_{1\leq l\leq k} \|h\|^2_{\dot{\mathrm{H}}^l} + \sum_{0\leq l\leq k} \|\nabla^l_x \nabla^{k+1-l}_v h\|^2\Big) \tag{7.4.20}$$

for some $\lambda_{k,0} > 0$.

Now we rephrase (7.4.19) and (7.4.20). In this proof, we put

$$\mathrm{Z} := \Big(\sum_{1\leq l\leq k-1} \|h\|^2_{\dot{\mathrm{H}}^l} + \sum_{0\leq l\leq k-1} \|\nabla^l_x \nabla^{k-l}_v h\|^2\Big)^{\frac{1}{2}}$$

and $\mathrm{W} := (\mathrm{W}_x, \mathrm{W}_0, \mathrm{W}_1, \cdots, \mathrm{W}_k)^{\mathsf{T}} \in \mathbb{R}^{k+2}$ where

$$\mathrm{W}_x = \|\nabla^k_x h\|, \quad \mathrm{W}_l = \|\nabla^l_x \nabla^{k+1-l}_v h\|, \quad 0 \leq l \leq k.$$

So we have $\mathrm{Z}^2 + \mathrm{W}_x^2 = \sum_{l=1}^k \|h\|^2_{\dot{\mathrm{H}}^l}$. Moreover, in terms of Z and W, the induction hypothesis (7.4.19) is equivalent to

$$((h,\mathrm{L}h))_{\mathrm{H}^{k-1}} \geq \lambda_{k-1,0}\mathrm{Z}^2,$$

and the desired estimate (7.4.20) is equivalent to

$$((h,\mathrm{L}h))_{\mathrm{H}^k} \geq \lambda_{k,0}(\mathrm{Z}^2 + |\mathrm{W}|^2).$$

The idea here is to distinguish, on the one hand, the derivatives of order not greater than k excluding ∇^k_x, and on the other hand, ∇^k_x and the derivatives of order $k+1$ excluding ∇^{k+1}_x. The former collection of derivatives already existed in the coercivity estimate of $((h,\mathrm{L}h))_{\mathrm{H}^{k-1}}$, while the latter one are the newly appeared coming from $((h,\mathrm{L}h))_{\dot{\mathrm{H}}^k}$. Such a division will prove helpful in the induction procedure .

Then we shall bound $((h,\mathrm{L}h))_{\dot{\mathrm{H}}^k}$ from below in terms of Z and W, more precisely, we shall prove

$$((h,\mathrm{L}h))_{\dot{\mathrm{H}}^k} \geq -\mathrm{K}_1\omega_k\mathrm{Z}^2 - \mathrm{K}_2\omega_k\mathrm{Z}|\mathrm{W}| + \eta\omega_k|\mathrm{W}|^2 \tag{7.4.21}$$

for some constants $\mathrm{K}_1, \mathrm{K}_2$, and $\eta = \frac{1}{64(k-1)^2\mathrm{M}}$. Later in the proof, we shall see that ω_k can be chosen as small as one desires (without any modification of $\mathrm{K}_1, \mathrm{K}_2$ and η), and so that we are able to obtain a coercivity estimate in form of (7.4.20).

Recall that (7.4.3)

$$
\begin{aligned}
((h,\mathrm{L}h))_{\dot{\mathrm{H}}^k} &:= \sum_{0\le i\le k} \omega_{k,i}\langle \nabla_v^{k-i}\nabla_x^i \mathrm{L}h, \nabla_v^{k-i}\nabla_x^i h\rangle \\
&\quad + \omega_k\big(\langle \nabla_x^{k-1}\nabla_v \mathrm{L}h, \nabla_x^k h\rangle + \langle \nabla_x^{k-1}\nabla_v h, \nabla_x^k \mathrm{L}h\rangle\big) \\
&= \sum_{0\le i\le k} \omega_{k,i}(\mathrm{T}^{\mathrm{A}}_{i,k-i} + \mathrm{T}^{\mathrm{B}}_{i,k-i}) + \omega_k(\mathrm{T}^{\mathrm{A}}_{mix} + \mathrm{T}^{\mathrm{B}}_{mix}).
\end{aligned}
$$

Step 0. We set some relation between the coefficients in $((h,\mathrm{L}h))_{\dot{\mathrm{H}}^k}$. Indeed, we shall adopt the relations of e_2, e_3, e_4 (in the proof of Proposition 7.9) for the ones of $\omega_{k,k-1}, \omega_{k,k}, \omega_k$ (replacing M there by $(k-1)^2\mathrm{M}$)

$$
\omega_{k,i} = \begin{cases} 64(k-1)^2\mathrm{M}\omega_k, \text{ for } 0\le i\le k-1; \\ \dfrac{1}{16(k-1)^2\mathrm{M}}\omega_k, \text{ for } i=k, \end{cases}
$$

where ω_k will be determined later. Then by (7.3.12), we know that in the sense of quadratic forms

$$
\begin{pmatrix} \omega_k & 0 & 0 & 0 \\ 0 & \omega_{k,0} & 0 & 0 \\ -(k-1)\omega_k\sqrt{\mathrm{M}} & 0 & \omega_{k,k-1} & 0 \\ -k\omega_{k,k}\sqrt{\mathrm{M}} & 0 & -2\omega_k & \omega_{k,k} \end{pmatrix} \ge \begin{pmatrix} \omega_k & 0 & 0 & 0 \\ 0 & \omega_{k,0} & 0 & 0 \\ -(k-1)\omega_k\sqrt{\mathrm{M}} & 0 & \omega_{k,k-1} & 0 \\ -2(k-1)\omega_{k,k}\sqrt{\mathrm{M}} & 0 & -2\omega_k & \omega_{k,k} \end{pmatrix}
$$
$$
\ge \mathrm{Diag}(\frac{\omega_k}{2}, \omega_{k,0}, \frac{\omega_{k,k-1}}{4}, \frac{\omega_{k,k}}{4}). \tag{7.4.22}
$$

It follows that

$$
\begin{aligned}
\mathrm{Q}(\mathrm{W}) &:= \omega_k \mathrm{W}_x^2 + \sum_{i=0}^k \omega_{k,i}\mathrm{W}_i^2 - (k-1)\omega_k\sqrt{\mathrm{M}}\mathrm{W}_x\mathrm{W}_{k-1} - k\omega_{k,k}\sqrt{\mathrm{M}}\mathrm{W}_x\mathrm{W}_k - 2\omega_k\mathrm{W}_{k-1}\mathrm{W}_k \\
&\ge \frac{1}{4}\big(\omega_k\mathrm{W}_x^2 + \sum_{i=0}^k \omega_{k,i}\mathrm{W}_i^2\big).
\end{aligned} \tag{7.4.23}
$$

In particular, it holds

$$
\mathrm{Q}(\mathrm{W}) \ge \frac{1}{4}\cdot\frac{\omega_k}{16(k-1)^2\mathrm{M}}|\mathrm{W}|^2 = \eta\omega_k|\mathrm{W}|^2. \tag{7.4.24}
$$

where $\eta = \frac{1}{64(k-1)^2\mathrm{M}}$ as in (7.4.21).

Step 1. We deal with the terms coming from the usual $\mathrm{H}^k(\mu)$ seminorm, namely the terms $\mathrm{T}^{\mathrm{A}}_{m_1,m_2}$ and $\mathrm{T}^{\mathrm{B}}_{m_1,m_2}$ with $m_1+m_2=k$. By (7.4.11) in Lemma 7.12, we know for $0\le i\le k$

$$
\mathrm{T}^{\mathrm{A}}_{i,k-i} = ||\nabla_x^i\nabla_v^{k-i+1}h||^2 + (k-i)||\nabla_x^i\nabla_v^{k-i}h||^2 \ge \mathrm{W}_i^2 \tag{7.4.25}
$$

where $(k-i)||\nabla_x^i\nabla_v^{k-i}h||^2 (0\le i\le k)$ are discarded here since they correspond to $k||\nabla_v^k h||^2$, $(k-1)||\nabla_x^1\nabla_v^{k-1}h||^2, \cdots, ||\nabla_x^{k-1}\nabla_v^1 h||^2$ and 0, which are positive and do not relate to W.
Thanks to the (generalized) Leibniz rule, it holds that

$$
\nabla_x^{m_1-l}(-\nabla^2\mathrm{V}\cdot\nabla_v)_i\nabla_x^{l-1}\nabla_v^{m_2}h = \sum_{l_1=0}^{m_1-l}\binom{m_1-l}{l_1}\sum_{j=1}^d \big(\nabla_x^{l_1}\partial^2_{x_ix_j}\mathrm{V}\big)(\partial_{v_j}\nabla_x^{m_1-l_1-1}\nabla_v^{m_2}h) \tag{7.4.26}
$$

where $\binom{m_1-l}{l_1} = \frac{n!}{k!(n-k)!}$ is a binomial coefficient. Substitute it into (7.4.12), we therefore have

$$
\begin{aligned}
\mathrm{T}^{\mathrm{B}}_{i,k-i} &= \sum_{l=1}^{k-i} \langle \nabla_x^i \nabla_v^{k-i-l} \nabla_x \nabla_v^{l-1} h, \nabla_x^i \nabla_v^{k-i} h \rangle \\
&\quad + \sum_{l=1}^{i} \sum_{l_1=0}^{i-l} \binom{i-l}{l_1} \langle \sum_{j=1}^{d} \left(\nabla_x^{l_1} \nabla_x \partial_{x_j} \mathrm{V}\right)(\partial_{v_j} \nabla_x^{i-l_1-1} \nabla_v^{k-i} h), \nabla_x^i \nabla_v^{k-i} h \rangle \\
&\geq -\sum_{l=1}^{k-i} ||\nabla_x^i \nabla_v^{k-i-l} \nabla_x \nabla_v^{l-1} h|| ||\nabla_x^i \nabla_v^{k-i} h|| \\
&\quad - \sum_{l=1}^{i} \sum_{l_1=0}^{i-l} \binom{i-l}{l_1} || \sum_{j=1}^{d} \left(\nabla_x^{l_1} \nabla_x \partial_{x_j} \mathrm{V}\right)\left(\partial_{v_j} \nabla_x^{i-l_1-1} \nabla_v^{k-i} h\right)|| ||\nabla_x^i \nabla_v^{k-i} h|| \\
&\geq -(k-i) ||\nabla_x^{i+1} \nabla_v^{k-i-1} h|| ||\nabla_x^i \nabla_v^{k-i} h|| \\
&\quad - \sum_{l=1}^{i} \sum_{l_1=0}^{i-l} \binom{i-l}{l_1} \sqrt{\mathrm{M}} \left(||\nabla_x^{i-l_1-1} \nabla_v^{k-i+1} h|| + ||\nabla_x^{i-l_1} \nabla_v^{k-i+1} h|| \right) ||\nabla_x^i \nabla_v^{k-i} h|| \qquad (7.4.27)
\end{aligned}
$$

where in the last inequality we have applied

$$
\begin{aligned}
|| \sum_{j=1}^{d} \left(\nabla_x^{l_1} \nabla_x \partial_{x_j} \mathrm{V}\right)\left(\partial_{v_j} \nabla_x^{i-l_1-1} \nabla_v^{k-i} h\right)|| &\leq \sqrt{\mathrm{M}} \left(||\nabla_v \nabla_x^{i-l_1-1} \nabla_v^{k-i} h|| + ||\nabla_v \nabla_x^{i-l_1-1+1} \nabla_v^{k-i} h|| \right) \\
&= \sqrt{\mathrm{M}} \left(||\nabla_x^{i-l_1-1} \nabla_v^{k-i+1} h|| + ||\nabla_x^{i-l_1} \nabla_v^{k-i+1} h|| \right)
\end{aligned}
$$

which holds true thanks to our assumption (7.1.7). (Note that if $i = 0$, the summation in the last line of (7.4.27) is over a empty set hence equals to zero.)
Let us highlight certain features of the lower bound (7.4.27). The expression

$$-(k-i) ||\nabla_x^{i+1} \nabla_v^{k-i-1} h|| ||\nabla_x^i \nabla_v^{k-i} h||$$

depends only on the terms in Z, except the case when $i = k-1$. In that exceptional case, $(k-i) ||\nabla_x^{i+1} \nabla_v^{k-i-1} h|| ||\nabla_x^i \nabla_v^{k-i} h|| = ||\nabla_x^k h|| ||\nabla_x^{k-1} \nabla_v h||$ which also depends on $\mathrm{W}_x = ||\nabla_x^k h||$. So we see

$$-(k-i) ||\nabla_x^{i+1} \nabla_v^{k-i-1} h|| ||\nabla_x^i \nabla_v^{k-i} h|| \geq -2k\mathrm{Z}^2 - \mathrm{Z}\mathrm{W}_x. \qquad (7.4.28)$$

While inside the expression

$$\left(||\nabla_x^{i-l_1-1} \nabla_v^{k-i+1} h|| + ||\nabla_x^{i-l_1} \nabla_v^{k-i+1} h|| \right) ||\nabla_x^i \nabla_v^{k-i} h||,$$

there are more interesting terms involving W, the term $\mathrm{W}_k \mathrm{W}_x$. Note that $||\nabla_x^{i-l_1-1} \nabla_v^{k-i+1} h||$ contains derivatives in v-direction and its highest order of derivatives of h is k, so it depends only on Z (W is not involved at all). While the highest order of derivatives of h in $||\nabla_x^{i-l_1} \nabla_v^{k-i+1} h||$ is not greater than $k+1$ with equality if and only if $l_1 = 0$; moreover, in that case, $||\nabla_x^{i-l_1} \nabla_v^{k-i+1} h||$ becomes W_i. If $l_1 \geq 1$, then $i - l_1 + k - i + 1 \leq k$ and $k - i + 1 \geq 1$, so it follows that $||\nabla_x^{i-l_1} \nabla_v^{k-i+1} h||$ occurs in Z and thus can be bounded by Z. The other factor $||\nabla_x^i \nabla_v^{k-i} h||$ either becomes W_x (if

$i=k$), or occurs in Z and thus can be bounded by Z (if $i \le k-1$). That way, in the above lower bound, the only terms independent of Z are in forms of

$$\|\nabla_x^{i-l_1}\nabla_v^{k-i+1}h\|\|\nabla_x^i\nabla_v^{k-i}h\|, \text{ with } i=k, l_1=0$$

(i.e. $\mathrm{W}_k\mathrm{W}_x$) and its pre-coefficient is

$$-\sum_{l=1}^{k}\sum_{l_1=0}\binom{k-l}{l_1}\sqrt{\mathrm{M}} = -k\sqrt{\mathrm{M}}.$$

Therefore for $1 \le i \le k-1$, it holds

$$\begin{aligned}
&-\sum_{l=1}^{i}\sum_{l_1=0}^{i-l}\binom{i-l}{l_1}\sqrt{\mathrm{M}}\Big(\|\nabla_x^{i-l_1-1}\nabla_v^{k-i+1}h\|+\|\nabla_x^{i-l_1}\nabla_v^{k-i+1}h\|\Big)\|\nabla_x^i\nabla_v^{k-i}h\| \\
\ge& -\sum_{l=1}^{i}\sum_{l_1=0}^{i-l}\binom{i-l}{l_1}\sqrt{\mathrm{M}}\Big(2\mathrm{Z}+\mathrm{W}_i\Big)\mathrm{Z} \\
\ge& -2^i\sqrt{\mathrm{M}}(2\mathrm{Z}^2+\mathrm{ZW}_i).
\end{aligned} \tag{7.4.29}$$

Whereas for $i=k$, it holds

$$\begin{aligned}
&-\sum_{l=1}^{i}\sum_{l_1=0}^{i-l}\binom{i-l}{l_1}\sqrt{\mathrm{M}}\Big(\|\nabla_x^{i-l_1-1}\nabla_v^{k-i+1}h\|+\|\nabla_x^{i-l_1}\nabla_v^{k-i+1}h\|\Big)\|\nabla_x^i\nabla_v^{k-i}h\| \\
\ge& -\Big\{\sum_{l=1}^{k}\sum_{l_1=1}^{k-l}\binom{k-l}{l_1}\sqrt{\mathrm{M}}\cdot 2\mathrm{ZW}_x+\sum_{l=1}^{k}\sum_{l_1=0}\binom{k-l}{l_1}\sqrt{\mathrm{M}}(\mathrm{Z}+\mathrm{W}_k)\mathrm{W}_x\Big\} \\
\ge& -2^{k+1}\sqrt{\mathrm{M}}\mathrm{ZW}_x-k\sqrt{\mathrm{M}}\mathrm{W}_k\mathrm{W}_x.
\end{aligned} \tag{7.4.30}$$

Combining the lower bounds in (7.4.25),(7.4.27), (7.4.28), (7.4.29) and (7.4.30) in this step, we then discover that

$$\begin{aligned}
\sum_{i=0}^{k}\omega_{k,i}(\mathrm{T}^{\mathrm{A}}_{i,k-i}+\mathrm{T}^{\mathrm{B}}_{i,k-i}) \ge& \sum_{i=0}^{k}\omega_{k,i}\mathrm{W}_i^2-k\sum_{i=0}^{k-1}\mathrm{Z}^2\omega_{k,i}-\sum_{i=0}^{k-1}\omega_{k,i}2^i\sqrt{\mathrm{M}}(2\mathrm{Z}^2+\mathrm{ZW}_i) \\
&-2^{k+1}\sqrt{\mathrm{M}}\omega_{k,k}\mathrm{ZW}_x-k\sqrt{\mathrm{M}}\omega_{k,k}\mathrm{W}_k\mathrm{W}_x.
\end{aligned} \tag{7.4.31}$$

So there exist constants K_1' and K_2' such that

$$\sum_{i=0}^{k}\omega_{k,i}(\mathrm{T}^{\mathrm{A}}_{i,k-i}+\mathrm{T}^{\mathrm{B}}_{i,k-i}) \ge \sum_{i=0}^{k}\omega_{k,i}\mathrm{W}_i^2-\mathrm{K}_1'\omega_k\mathrm{Z}^2-\mathrm{K}_2'\omega_k\mathrm{Z}|\mathrm{W}|-k\sqrt{\mathrm{M}}\omega_{k,k}\mathrm{W}_k\mathrm{W}_x \tag{7.4.32}$$

We stress that the constants K_1' and K_2' depend only on k and M, since, by Step 0, the ratio of $\omega_{k,i}$ and ω_k is a constant depending only on k and M, and it is independent of ω_k.

Step 2. Next we give lower bounds for the mixed terms $\mathrm{T}^{\mathrm{A}}_{mix}$ and $\mathrm{T}^{\mathrm{B}}_{mix}$, and thus for $\mathrm{T}^{\mathrm{A}}_{mix}+\mathrm{T}^{\mathrm{B}}_{mix} = \langle\nabla_x^{k-1}\nabla_v\mathrm{L}h, \nabla_x^k h\rangle+\langle\nabla_x^{k-1}\nabla_v h, \nabla_x^k\mathrm{L}h\rangle$. By (7.4.13), we find

$$\begin{aligned}
\mathrm{T}^{\mathrm{A}}_{mix} &\ge -2\|\nabla_x^{k-1}\nabla_v^2 h\|\|\nabla_x^k\nabla_v h\|-\|\nabla_x^{k-1}\nabla_v h\|\|\nabla_x^k h\| \\
&\ge -2\mathrm{W}_{k-1}\mathrm{W}_k-\mathrm{ZW}_x.
\end{aligned} \tag{7.4.33}$$

By (7.4.14) and (7.4.26), we have

$$\mathrm{T}^{\mathrm{B}}_{mix} = ||\nabla_x^k h||^2 + (\mathrm{I}) + (\mathrm{II}) \tag{7.4.34}$$

where I), (II) are given and bounded from below as follows,

$$\begin{aligned}
(\mathrm{I}) &= -\sum_{l=1}^{k-1}\sum_{l_1=0}^{k-l-1}\binom{k-1-l}{l_1}\langle\sum_{j=1}^{d}\big(\nabla_x^{l_1}\nabla_x\partial_{x_j}\mathrm{V}\big)\big(\partial_{v_j}\nabla_x^{k-l_1-2}\nabla_v h\big),\nabla_x^k h\rangle\\
&\geq -\sum_{l=1}^{k-1}\sum_{l_1=0}^{k-l-1}\binom{k-1-l}{l_1}||\sum_{j=1}^{d}\big(\nabla_x^{l_1}\nabla_x\partial_{x_j}\mathrm{V}\big)\big(\partial_{v_j}\nabla_x^{k-l_1-2}\nabla_v h\big)||\cdot||\nabla_x^k h||\\
&\geq -\sum_{l=1}^{k-1}\sum_{l_1=0}^{k-l-1}\binom{k-1-l}{l_1}\sqrt{\mathrm{M}}\big(||\nabla_x^{k-l_1-2}\nabla_v^2 h||+||\nabla_x^{k-l_1-1}\nabla_v^2 h||\big)||\nabla_x^k h||,
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{II}) &= -\sum_{l=1}^{k}\sum_{l_1=0}^{k-l}\binom{k-l}{l_1}\langle\nabla_x^{k-1}\nabla_v h,\sum_{j=1}^{d}\big(\nabla_x^{l_1}\nabla_x\partial_{x_j}\mathrm{V}\big)\big(\partial_{v_j}\nabla_x^{k-l_1-1}h\big)\rangle\\
&\geq -\sum_{l=1}^{k}\sum_{l_1=0}^{k-l}\binom{k-l}{l_1}||\nabla_x^{k-1}\nabla_v h||\cdot||\sum_{j=1}^{d}\big(\nabla_x^{l_1}\nabla_x\partial_{x_j}\mathrm{V}\big)\big(\partial_{v_j}\nabla_x^{k-l_1-1}h\big)||\\
&\geq -\sum_{l=1}^{k}\sum_{l_1=0}^{k-l}\binom{k-l}{l_1}||\nabla_x^{k-1}\nabla_v h||\cdot\sqrt{\mathrm{M}}\big(||\nabla_v\nabla_x^{k-l_1-1}h||+||\nabla_v\nabla_x^{k-l_1}h||\big).
\end{aligned}$$

Now we give lower bounds of (I) and (II) in terms of Z and W. Note that

$$||\nabla_x^{k-l_1-2}\nabla_v^2 h||+||\nabla_x^{k-l_1-1}\nabla_v^2 h|| \leq \begin{cases} 2\mathrm{Z}, & \text{if } l_1\geq 1;\\ \mathrm{Z}+\mathrm{W}_{k-1}, & \text{if } l_1=0.\end{cases}$$

Hence we have

$$\begin{aligned}
(\mathrm{I}) &\geq -\sum_{l=1}^{k-1}\sum_{l_1=0}^{k-l-1}\binom{k-1-l}{l_1}\sqrt{\mathrm{M}}\cdot 2\mathrm{Z}\mathrm{W}_x-\sum_{l=1}^{k-1}\sum_{l_1=0}\binom{k-1-l}{l_1}\sqrt{\mathrm{M}}\mathrm{W}_{k-1}\mathrm{W}_x\\
&\geq -2^k\sqrt{\mathrm{M}}\mathrm{Z}\mathrm{W}_x-(k-1)\sqrt{\mathrm{M}}\mathrm{W}_{k-1}\mathrm{W}_x.
\end{aligned} \tag{7.4.35}$$

Similarly, for (II), since

$$||\nabla_v\nabla_x^{k-l_1-1}h||+||\nabla_v\nabla_x^{k-l_1}h|| \leq \begin{cases} 2\mathrm{Z}, & \text{if } l_1\geq 1;\\ \mathrm{Z}+\mathrm{W}_k, & \text{if } l_1=0,\end{cases}$$

we obtain

$$\begin{aligned}
(\mathrm{II}) &\geq -\sum_{l=1}^{k}\sum_{l_1=0}^{k-l}\binom{k-l}{l_1}\mathrm{Z}\cdot\sqrt{\mathrm{M}}\big(2\mathrm{Z}+\mathrm{W}_k\big).\\
&\geq -2^k\sqrt{\mathrm{M}}\mathrm{Z}(2\mathrm{Z}+\mathrm{W}_k).
\end{aligned} \tag{7.4.36}$$

We then deduce from (7.4.34),(7.4.35) and (7.4.36) that

$$T^B_{mix} \geq W_x^2 - 2^k\sqrt{M}ZW_x - (k-1)\sqrt{M}W_{k-1}W_x - 2^{k-1}\sqrt{M}Z(2Z+W_k). \tag{7.4.37}$$

Combined with (7.4.33), we see that

$$\begin{aligned} T^A_{mix} + T^B_{mix} \geq & -2W_{k-1}W_k - ZW_x + W_x^2 - 2^k\sqrt{M}ZW_x \\ & - (k-1)\sqrt{M}W_{k-1}W_x - 2^k\sqrt{M}Z(2Z+W_k). \end{aligned} \tag{7.4.38}$$

In particular, there exist constants K_1'' and K_2'', depending only on k and M (independent of ω_k), such that

$$T^A_{mix} + T^B_{mix} \geq W_x^2 - 2W_{k-1}W_k - (k-1)\sqrt{M}W_{k-1}W_x - K_1''Z^2 - K_2''Z|W|. \tag{7.4.39}$$

Step 3. We may now conclude from the lower bounds (7.4.32) and (7.4.39) in Step 1 and Step 2 that the desired estimate (7.4.21) holds true for some constants K_1 and K_2 given by

$$K_1 = K_1' + K_1'', \quad K_2 = K_2' + K_2''$$

which depend only on k, M and can be explicitly computed. Indeed,

$$\begin{aligned} ((h, Lh))_{\dot{H}^k} &= \sum_{0\leq i\leq k} \omega_{k,i}(T^A_{i,k-i} + T^B_{i,k-i}) + \omega_k(T^A_{mix} + T^B_{mix}) \\ &\geq \sum_{i=0}^{k} \omega_{k,i}W_i^2 - K_1'\omega_k Z^2 - K_2'\omega_k Z|W| - k\sqrt{M}\omega_{k,k}W_kW_x \\ &\quad + W_x^2 - 2W_{k-1}W_k - (k-1)\sqrt{M}W_{k-1}W_x - K_1''Z^2 - K_2''Z|W| \\ &= Q(W) - \omega_k K_1 Z^2 - \omega_k K_2 Z|W| \\ &\geq \omega_k\{\eta|W|^2 - K_1Z^2 - K_2Z|W|\} \end{aligned} \tag{7.4.40}$$

where Q(W) was introduced in Step 0 and it satisfies (7.4.23). By induction hypothesis,

$$((h, Lh))_{H^{k-1}} \geq \lambda_{k-1,0}Z^2,$$

and so we obtain

$$\begin{aligned} ((h, Lh))_{H^k} &= ((h, Lh))_{H^{k-1}} + ((h, Lh))_{\dot{H}^k} \\ &\geq \lambda_{k-1,0}Z^2 + \omega_k\{\eta|W|^2 - K_1Z^2 - K_2Z|W|\}. \end{aligned}$$

Then it holds that

$$((h, Lh))_{H^k} \geq \lambda_{k,0}(Z^2 + W^2)$$

where

$$\omega_k = \min\left\{\frac{\lambda_{k-1,0}}{2K_1}, \frac{3\eta\lambda_{k-1,0}}{4K_2^2}\right\}, \quad \lambda_{k,0} = \min\left\{\frac{\lambda_{k-1,0}}{4}, \frac{\omega_k\eta}{4}\right\}.$$

By now, the proof of (7.4.1) is thus finished.

Step 4. To prove its consequence (7.4.2), it suffices to observe that the constructed twisted $\mathrm{H}^l(\mu)$-seminorm associated to $((\cdot,\cdot))_{\dot{\mathrm{H}}^l}$ is bounded by (in fact, equivalent to) the usual $\mathrm{H}^l(\mu)$-seminorm up to a constant, for each $1 \le l \le k$. Indeed, in the setting of Step 0, we may find $\omega_{l,l}\omega_{l,l-1} = 4\omega_l^2$ for each $2 \le l \le k$, and then

$$2\omega_l|\langle\nabla_x^{l-1}\nabla_v h, \nabla_x^l h\rangle| \le \frac{1}{2}\big(\omega_{l,l-1}\|\nabla_x^{l-1}\nabla_v h\|^2 + \omega_{l,l}\|\nabla_x^l h\|^2\big),$$

which follows that

$$((h,h))_{\dot{\mathrm{H}}^l} \le \frac{3}{2}\max\{\omega_{l,l-1},\omega_{l,l}\}\|h\|_{\dot{\mathrm{H}}^l}^2.$$

For $l=1$, by Poincaré inequality, we have seen in the previous section that

$$((h-\int h d\mu, h-\int h d\mu))_{\mathrm{H}^1} \le (\frac{5}{3}a+1)\|\nabla_v h\|^2 + (\frac{5}{3}c+\kappa)\|\nabla_x h\|^2$$

with the constants a,b,c given in Proposition 7.9, and κ given in the Poincaré inequality in the assumption.

Gathering the above inequalities, we obtain that

$$((h-\int h d\mu, h-\int h d\mu))_{\mathrm{H}^k} \le \mathrm{C}(\mathrm{Z}^2 + \mathrm{W}_x^2)$$

with $\mathrm{C} = \max\{\frac{5}{3}a+1, \frac{5}{3}c+\kappa, \frac{3}{2}\omega_{l,l-1}, \frac{3}{2}\omega_{l,l} \,|\, 2 \le l \le k\}$. Finally we conclude by (7.4.1) that

$$((h,\mathrm{L}h))_{\mathrm{H}^k} \ge \frac{\lambda_{k,0}}{\mathrm{C}}((h-\int h d\mu, h-\int h d\mu))_{\mathrm{H}^k}$$

which completes the proof (7.4.2).

□

Remark 7.13. In this remark, we comment on possible refinements on the constants and thus the rate of convergence in the proof of Proposition 7.10. The first possibility is that, just as the inequality (7.4.23) we have proved for Q(W), we can also find coefficients $\omega_k, \omega_{k,i}$ $(0 \le i \le k)$ such that

$$((h,\mathrm{L}h))_{\dot{\mathrm{H}}^k} \ge \delta\big(\omega_k \mathrm{W}_x^2 + \textstyle\sum_{i=0}^k \omega_{k,i}\mathrm{W}_i^2\big)$$

holds for some $\delta > 0$ and for all $h \in \mathscr{S}(\mathbb{R}^{2d})$.

Another possibility is to refine the lower bounds in terms of Z in the proof, for instance, we may distinguish $\|\nabla_v h\|$, $\|\nabla_v h\|$, $\|\nabla_v^2 h\|$, $\cdots$, $\|\nabla_v^k h\|$, $\|\nabla_v^{k-1}\nabla_x h\|$, $\cdots$, $\|\nabla^v\nabla_x^{k-1} h\|$ (i.e. roughly all the terms appearing in Z). It may result in a matrix of very large size. But it is still possible (although technically more complicated) to find coefficients such that a coercivity estimate holds.

Proof of Theorem 7.3. It follows from the coercivity estimate in Proposition 7.10. The proof goes in the very same way as the proof of theorem 7.1. □

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