UNIVERSITÉ PARIS 13

Doctoral School **Galilée**

University Department **LAGA-Institut Galilée-Université Paris** 13

Thesis defended by **Tom DUTILLEUL**

Defended on **12th November, 2019**

In order to become Doctor from Université Paris 13

Academic Field **Mathematics**

Speciality **Dynamical systems**

Chaotic dynamics of spatially homogeneous spacetimes

Thesis supervised by François BÉGUIN

Committee members

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	Hans RINGSTRÖM	Professor at KTH in Stockholm
Examiners	Julien BARRAL	Professor at Université Paris 13
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Dynamique chaotique des espaces-temps spatialement homogènes

Thèse dirigée par François BÉGUIN

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A mes parents, Doumé et Marie,
pour leur amour
et leur soutien sans faille.
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Chaotic dynamics of spatially homogeneous spacetimes

Abstract

In 1963, Belinsky, Khalatnikov and Lifshitz have proposed a conjectural description of the asymptotic geometry of cosmological models in the vicinity of their initial singularity. In particular, it is believed that the asymptotic geometry of generic spatially homogeneous spacetimes should display an oscillatory chaotic behaviour modeled on a discrete map's dynamics (the so-called Kasner map). We prove that this conjecture holds true, if not for generic spacetimes, at least for a positive Lebesgue measure set of spacetimes.
In the context of spatially homogeneous spacetimes, the Einstein field equations can be reduced to a system of differential equations on a finite dimensional phase space: the Wainwright-Hsu equations. The dynamics of these equations encodes the evolution of the geometry of spacelike slices in spatially homogeneous spacetimes. Our proof is based on the non-uniform hyperbolicity of the Wainwright-Hsu equations. Indeed, we consider the return map of the solutions of these equations on a transverse section and prove that it is a non-uniformly hyperbolic map with singularities. This allows us to construct some local stable manifolds *à la Pesin* for this map and to prove that the union of the orbits starting in these local stable manifolds cover a positive Lebesgue measure set in the phase space. The chaotic oscillatory behaviour of the corresponding spacetimes follows.
The Wainwright-Hsu equations turn out to be quite interesting and challenging from a purely dynamical system viewpoint. In order to understand the asymptotic behaviour of (many of) the solutions of these equations, we will in particular be led to:

- carry a detailed analysis of the local dynamics of a vector field in the neighborhood of degenerate non-linearizable partially hyperbolic singularities,
- deal with non-uniformly hyperbolic maps with singularities for which the usual theory (due to Pesin and Katok-Strelcyn) is not relevant due to the poor regularity of the maps,
- consider some unusual arithmetic conditions expressed in terms of continued fractions and use some rather sophisticated ergodic properties of the Gauss map to prove that these properties are generic.

Keywords: non-uniformly hyperbolic dynamical systems, general relativity, cosmological models, ordinary differential equations, lorentzian geometry, continued fractions

Dynamique chaotique des espaces-temps spatialement homogènes

Résumé

En 1963, Belinsky, Khalatnikov et Lifshitz ont proposé une description conjecturale de la géométrie asymptotique des modèles cosmologiques au voisinage de leur singularité initiale. En particulier, il y est avancé que la géométrie asymptotique des espaces-temps spatialement homogènes « génériques » devrait avoir un comportement oscillatoire chaotique modelé sur la dynamique d'une application discrète : l'application de Kasner. Nous démontrons que cette conjecture est vraie au moins pour un ensemble d'espaces-temps de mesure de Lebesgue strictement positive.
Dans le contexte des espaces-temps spatialement homogènes, l'équation d'Einstein de la relativité générale se réduit à un système d'équations différentielles sur un espace des phases de dimension finie : les équations de Wainwright-Hsu. La dynamique de ces équations encode l'évolution de la géométrie des hypersurfaces spatiales dans les espaces-temps spatialement homogènes. Notre preuve est basée sur l'hyperbolicité non-uniforme des équations de Wainwright-Hsu. Nous considérons l'application de Poincaré associée aux solutions de ces équations sur une section transverse au flot et nous démontrons qu'il s'agit d'une application non-uniformément hyperbolique avec singularités. Ceci nous permet de construire des variétés stables locales « à la Pesin » pour cette application et de montrer que la réunion des orbites passant par ces variétés stables locales recouvre une partie de l'espace des phases de mesure de Lebesgue strictement positive. Le comportement oscillatoire chaotique des espaces-temps correspondant à ces orbites est une conséquence de cette construction.
Du point de vue des systèmes dynamiques, les équations de Wainwright-Hsu se révèlent être très riches et posent un certain nombre de défis. Pour comprendre le comportement asymptotique d'un nombre conséquent de solutions de ces équations, nous serons amenés à :

- faire une analyse fine de la dynamique locale d'un champ de vecteurs au voisinage d'une singularité partiellement hyperbolique dégénérée et non linéarisable,
- travailler avec des applications non-uniformément hyperboliques ayant des singularités, pour lesquelles la théorie usuelle (due à Pesin et Katok-Strelcyn) ne s'applique pas à cause de la faible régularité de ces applications,
- considérer des conditions arithmétiques exotiques exprimées en termes de fractions continues et utiliser des propriétés ergodiques quelque peu sophistiquées de l'application de Gauss pour montrer que ces propriétés sont génériques, *etc.*

Mots clés : systèmes dynamiques non uniformément hyperboliques, relativité générale, modèles cosmologiques, équations différentielles ordinaires, géométrie lorentzienne, fractions continues

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Contents

Chapter 1

Introduction

1.1 The BKL conjecture for Bianchi spacetimes

1.1.1 Bianchi spacetimes

In classical General Relativity, *spacetime* is modeled as a smooth 4-dimensional Lorentz manifold (M, g) verifying the *Einstein field equations*

$$\mathrm{Ric}_g + \left(\Lambda - \frac{1}{2}\,\mathrm{Scal}_g\right) g = T \tag{1.1}$$

where Ric_g is the Ricci curvature tensor, Scal_g is the scalar curvature, Λ is the *cosmological constant* and T is the *stress-energy tensor*, which encodes the presence of matter, radiation and non-gravitational force fields. Assuming that the gravitational force field only self-interacts and Λ is zero, (1.1) reduces to the *vacuum Einstein field equations*

$$\mathrm{Ric}_g = 0 \tag{1.2}$$

Informally, a *Bianchi spacetime* (also called *Bianchi cosmological model*) is a spacetime which is *spatially homogeneous*. We will work with the following formal definition: a Bianchi spacetime is a Lorentzian manifold of the form $(M, g) = (I \times G, -\mathrm{d}s^2 + h_s)$ where I is an interval of the real line, G is a simply-connected 3-dimensional real Lie group, s is a coordinate on I and h_s is a left-invariant Riemannian metric on $\{s\} \times G \simeq G$ for every $s \in I$. If the Lie group G is unimodular[1], then the Bianchi spacetime is said to be of *class A*, otherwise it is said to be of *class B*. We say that a Bianchi spacetime is *maximal* if it cannot be embedded isometrically as a strict submanifold of another Bianchi spacetime. In this work, we will restrict our attention to maximal vacuum (with zero cosmological constant) class A Bianchi spacetimes[2], that is, maximal class A Bianchi spacetimes solution to the vacuum Einstein field equations (1.2). It is well known (see *e.g.* [CE79]) that, up to a change of time orientation, every maximal vacuum class A Bianchi spacetime admits an initial singularity[3] (often called *Big-Bang*). We are mostly interested in the description of the past-asymptotic geometry of maximal vacuum class A Bianchi spacetimes, *i.e.*, in the description of their behaviour near their initial singularity.

1.1.2 BKL picture

In a series of papers, Belinskii, Khalatnikov and Lifshitz (see [BKL82] and [BKL70]) explained with heuristic arguments that general singularities should have the following properties:

1. As a first order approximation, the behaviour of the curvature of a spacetime near its initial singularity is dominated by the behaviour of its "spatially homogeneous part".

[1] A Lie group is called unimodular if its left invariant Haar measure is also right invariant.

[2] For some literature on class B Bianchi spacetimes, we refer to [HW93], [HHW03] and [Rad16].

[3] Precisely, we say that a maximal vacuum class A Bianchi spacetime $(M, g) = (I \times G, -\mathrm{d}s^2 + h_s)$ admits an initial singularity if $I =]s_-, s_+[$ with $s_- > -\infty$. Moreover, if this is the case, the curvature blows up when the time tends to s_- (see [Rin00]).

2. Solutions of the Einstein field equations with matter are well approximated, in the vicinity of their initial singularity, by solutions of the vacuum Einstein field equations. As the saying goes, near the initial singularity, "matter does not matter".

3. The geometry of the spatial hypersurfaces "oscillates" in a chaotic manner at the approach of the initial singularity.

What precedes is often referred to as the *BKL picture* or the *BKL conjecture*.

1.1.3 Wainwright-Hsu equations

The Einstein field equations are, in local coordinates, a system of non linear partial differential equations of order 2 about the coefficients of the metric g. A Bianchi spacetime can be seen as a family $(h_s)_{s\in I}$ of left-invariant Riemannian metrics on a simply-connected 3-dimensional real Lie group G. Since the space of left-invariant Riemannian metrics on a given Lie group is finite-dimensional, the vacuum Einstein field equations restricted to Bianchi spacetimes should translate as a system of ordinary differential equations (abbreviated as ODEs) on a finite-dimensional *phase space* $\mathscr{B}$. Adopting the view-point of vector fields, this allows one to study the vacuum Einstein field equations restricted to Bianchi spacetimes with dynamical systems methods. The first step to explicit the vacuum Einstein field equations is to choose a particular frame field or, equivalently, a coordinates system. One of the first successful attempts to do so has been made by Bogoyavlenski (see [Bog85]). Later on, Ellis and MacCallum (see [EM69]) and then Wainwright and Hsu (see [WH89]) introduced useful coordinates using the so-called *orthonormal frame method*.

In this work, we will use a *Hubble-renormalized* system of variables $(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3)$ closely related to the one used by Wainwright and Hsu. These variables are dimensionless, which means that they will not change if the spacetime metric is rescaled. Since these variables do not see the rescaling operation, one can hope that they remain bounded in the vicinity of the singularity. Such a system has been used by Heinzle and Uggla in [HU09] and Béguin in [Bég10]. We also choose a dimensionless time variable t and an "anti-physical" time orientation[4], which means that the initial singularities are located in $t = +\infty$.

Before we give more details about these variables, let us recall that the 3-dimensional real Lie algebras have been classified by Luigi Bianchi in 1898. This is the reason why the Bianchi spacetimes are called that way and why it is now standard to classify them according to their "Bianchi type" (see table 1.1 and, *e.g.*, [EM69],[HU09] and [Mil76]).

The numbers $N_1(t), N_2(t), N_3(t)$ describe the intrinsic curvature of the spacelike hypersurface $\{t\}\times G$ (that is, the curvature of the left-invariant riemannian metric h_t) and its Bianchi type. Actually, these three numbers are, up to a renormalization, the structure constants of the Lie algebra of G in a special basis (which depends on the metric h_t). The numbers $\Sigma_1(t), \Sigma_2(t), \Sigma_3(t)$ describe the extrinsic curvature of the spacelike hypersurface $\{t\} \times G$. These numbers verify two constraint equations:

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = 0 \tag{1.3}$$

(this relation comes from the fact that the numbers $\Sigma_1(t), \Sigma_2(t), \Sigma_3(t)$ are the diagonal coefficients of the trace-free part of the second fundamental form of the spacelike hypersurface $\{t\} \times G$) and

$$6 - \left(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2\right) - \frac{1}{2}\left(N_1^2 + N_2^2 + N_3^2\right) + (N_1N_2 + N_2N_3 + N_3N_1) = 0 \tag{1.4}$$

(this relation comes from the Gauss formula, which connects the intrinsic and the extrinsic curvatures of a given hypersurface to the curvature of the ambiant space, and the fact that the scalar curvature of the spacetime (M, g) is null). The left-hand side of (1.4) can be thought as the renormalized density parameter, which is null in the context of vacuum spacetimes. We will denote by $\mathscr{B}$ the *phase space*, defined as the set of points $(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathbb{R}^6$ verifying (1.3) and (1.4). In particular, it is a non-singular and non-compact 4-dimensional quadric.

When the vacuum Einstein field equations are written in this system of variables, it gives rise to an

[4] It is denoted by $-\tau$ in [HU09] (they choose to respect the "physical" time-orientation.)

autonomous system of six differential equations called the *Wainwright-Hsu equations*:

$$\begin{cases} N_1' &= -(q+2\Sigma_1)N_1 \\ N_2' &= -(q+2\Sigma_2)N_2 \\ N_3' &= -(q+2\Sigma_3)N_3 \\ \Sigma_1' &= (2-q)\Sigma_1 + S_1 \\ \Sigma_2' &= (2-q)\Sigma_2 + S_2 \\ \Sigma_3' &= (2-q)\Sigma_3 + S_3 \end{cases} \tag{1.5}$$

where

$$q \stackrel{\text{def}}{=} \frac{1}{3}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2)$$

and

$$S_i \stackrel{\text{def}}{=} \frac{1}{3}(2N_i^2 - N_j^2 - N_k^2 + 2N_jN_k - N_iN_j - N_iN_k), \qquad \{i,j,k\} = \{1,2,3\}$$

The numbers $S_1(t), S_2(t), S_3(t)$ are, up to renormalization, the components of the traceless Ricci tensor of the metric h_t and q is called the *deceleration parameter*.

The corresponding vector field is called the *Wainwright-Hsu vector field* and is denoted by $\mathscr{X}$. The first thing to remark is the fact that the Wainwright-Hsu equations (1.5) respect the constraint equations (1.3) and (1.4), *i.e.* the quadric $\mathscr{B}$ is invariant under the action of the flow of the Wainwright-Hsu vector field. The correspondence between maximal solutions of the Wainwright-Hsu equations contained in the phase space $\mathscr{B}$ and maximal vacuum class A Bianchi spacetimes will be discussed in Chapter 2.

Within the Wainwright-Hsu presentation, we have the major advantage of being able to study all the vacuum class A Bianchi spacetimes with the same equations (1.5) and in the same phase space $\mathscr{B} \subset \mathbb{R}^6$. It means that we can "compare" two different vacuum class A Bianchi spacetimes (even if these spacetimes are of different Bianchi types) using the metric of our choice in $\mathbb{R}^6$ and this approach has proved to be successful in the past (see *e.g.* [WH89], [Rin01],[Lie+11], [Bég10] and [Bre16]). Recall that with our choice of an anti-physical time orientation, describing the past-asymptotic states of a vacuum class A Bianchi spacetime amounts to describe the future-asymptotic states (that is, the ω-limit set[5]) of the corresponding orbit of the Wainwright-Hsu vector field.

1.1.4 Stratification of the phase-space

The classification of 3-dimensional Lie algebras induces a stratification of the phase space $\mathscr{B}$ in six strata invariant under the flow of the Wainwright-Hsu vector field $\mathscr{X}$. This invariant stratification is nothing more than the formalization of a simple fact: the signs of the variables N_i define a stratification and, according to the Wainwright-Hsu equations (1.5), the signs of the variables N_i are invariant along the orbits of the Wainwright-Hsu vector field. The different strata each correspond to a certain Bianchi type and will be called *Bianchi type* I *(resp.* II*,* VI_0*,* VII_0*,* VIII *and* IX*) stratum*. The orbits of the Wainwright-Hsu vector field $\mathscr{X}$ contained in the Bianchi type I (resp. II, VI_0,...) stratum will be called *type* I *(resp.* II*,* VI_0*,...) orbits*. The Bianchi type I stratum is an Euclidean circle, called the Kasner circle, and is denoted by $\mathscr{K}$. There are three particular ellipsoids intersecting along their common equator, which happens to be the Kasner circle $\mathscr{K}$. The Bianchi type II stratum is the union of these three ellipsoids minus the Kasner circle. Each one of these ellipsoids (minus the Kasner circle) is contained in a subset of the form $N_i \neq 0, N_j = 0, N_k = 0$, where $\{i,j,k\} = \{1,2,3\}$. These two strata are respectively of codimension three and two in the phase space $\mathscr{B}$. The Bianchi type VI_0 and VII_0 strata are both of codimension one while the Bianchi type VIII and IX strata are both open Zariski subsets of $\mathscr{B}$. We refer to section 3.2 for further details. Table 1.1 summarizes the preceding description.

[5]Precisely, the ω-limit set of an orbit $\mathscr{O}(t)$ is defined as the set $\omega(\mathscr{O}) \stackrel{\text{def}}{=} \cap_{s\geq 0}\overline{\{\mathscr{O}(t) \mid t \geq s\}}$. If $\mathscr{O}$ converges to a point x in the future, then $\omega(\mathscr{O}) = \{x\}$ and we say that x is the ω-limit point of $\mathscr{O}$.

Bianchi type	Name of the stratum	Dimension of the stratum	Signs of N_1, N_2, N_3 modulo permutation of the indices	Corresponding Lie algebra up to isomorphism
I	$\mathscr{K}$ or $\mathscr{B}_{\mathrm{I}}$	1	$0, 0, 0$	$\mathbb{R}^3$
II	$\mathscr{B}_{\mathrm{II}}$	2	$+, 0, 0$ or $-, 0, 0$	Heisenberg's algebra
VI_0 VII_0	$\mathscr{B}_{\mathrm{VI}_0}$ $\mathscr{B}_{\mathrm{VII}_0}$	3	$+, -, 0$ $+, +, 0$ or $-, -, 0$	$\mathfrak{isom}(\mathrm{Min}_2)$ $\mathfrak{isom}(\mathbb{R}^2)$
VIII IX	$\mathscr{B}_{\mathrm{VIII}}$ $\mathscr{B}_{\mathrm{IX}}$	4	$+, +, -$ or $-, -, +$ $+, +, +$ or $-, -, -$	$\mathfrak{sl}(2, \mathbb{R})$ $\mathfrak{su}(2)$

Table 1.1 – Stratification of the phase space.

1.1.5 Mixmaster attractor and past-asymptotic dynamics of Bianchi spacetimes

The union of the Kasner circle and the Bianchi type II stratum is called the Mixmaster attractor and is denoted by $\mathscr{A}$. Geometrically, it is the union of three ellipsoids intersecting along their common equator. The Mixmaster attractor is invariant under the flow of the Wainwright-Hsu vector field. The importance of this particular subset is expressed by the following theorem (see [Rin01] and [Bre16] for further details).

Theorem 1.1 (Ringström 2001, Brehm 2016)**.** *For Lebesgue almost all point q in the phase space $\mathscr{B}$, the distance between the orbit of the Wainwright-Hsu vector field with initial condition q and the Mixmaster attractor $\mathscr{A}$ converges to 0 in the future. For such an orbit, it means that its ω-limit set is included in $\mathscr{A}$.*

Given such an orbit converging to the Mixmaster attractor, one may ask the following question: is its future-asymptotic dynamics related to the global dynamics of the Wainwright-Hsu vector field restricted to the Mixmaster attractor? This question will be precised in the next paragraph and should be perceived as the point 3 of the BKL picture.

1.1.6 Restriction of the phase space

From now on, we will restrict ourselves to the part of the phase space characterized by

$$N_1 \geq 0, N_2 \geq 0, N_3 \geq 0$$

In particular, we will only state results for orbits that are contained in this subpart of the phase space, denoted by $\mathscr{B}^+$. Remark that

- $\mathscr{B}^+$ is invariant under the flow of the Wainwright-Hsu vector field.
- Generic orbits of $\mathscr{B}^+$ are type IX orbits.

This restriction will greatly simplify the presentation of the main result of this thesis. In particular it allows us to use simplified notations. We refer to the appendix A for a description of the results in the full phase space $\mathscr{B}$.

1.1.7 Basic facts about the dynamics of Bianchi spacetimes

We now state some well-known facts about the dynamics of the Wainwright-Hsu vector field in low dimensional stratas, in particular in the Mixmaster attractor. Any point of the Kasner circle $\mathscr{K} = \mathscr{B}_{\mathrm{I}}$ is a critical point of the Wainwright-Hsu vector field $\mathscr{X}$. This means that type I orbits are reduced to stationnary points and correspond to self-similar spacetimes (see [WH89] and [Ear74]). More precisely they correspond to Kasner spacetimes. There are three special points in the Kasner circle, called the *Taub points* and denoted by T_1, T_2, T_3, which will play a crucial role in the understanding of the behaviour of the solutions of the Wainwright-Hsu equations.

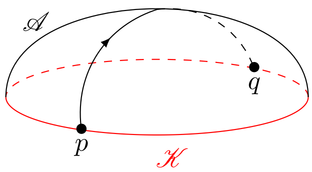

Figure 1.1 – A type II orbit connecting two points of the Kasner circle $\mathscr{K}$.

Any type II orbit is a heteroclinic connexion between two points of the Kasner circle. This means that any type II orbit converges in the future to a point $q \in \mathscr{K}$ and in the past to a point $p \in \mathscr{K}$. We will say that such an orbit *starts* at p and *arrives* at q. See figure 1.1. Of course, one should recall that type II orbits *never* reach the Kasner circle, so it is an abuse of terminology. Type II orbits are explicitly known. In particular, for every point p of the Kasner circle that is not a Taub point, there is exactly one type II orbit starting at p in $\mathscr{B}^+$. We refer to section 3.4 for further details.

The future-asymptotic behaviour of type VI_0 or VII_0 orbits is well-understood. Given such an orbit, its ω-limit set is either a single point of the Kasner circle or a flat point of type VII_0, the latter being only possible if the orbit is constant. We refer to [Ren97] for further details.

1.1.8 Kasner map, heteroclinic chains and shadowing

The fundamental tool to describe the dynamics of the Wainwright-Hsu vector field restricted to the Mixmaster attractor is the Kasner map. It is a map from the Kasner circle $\mathscr{K}$ to itself defined in such a way that it encodes the dynamics of type II orbits. More precisely, it is defined as follows. Let p be a point of the Kasner circle that is not a Taub point. The type II orbit starting at p converges to another point of the Kasner circle, denoted by $\mathscr{F}(p)$. We will denote this type II orbit by $\mathscr{O}_{p\to\mathscr{F}(p)}$. If p is a Taub point, set $\mathscr{F}(p) := p$. This defines a continuous map

$$\mathscr{F} : \mathscr{K} \to \mathscr{K}$$

called the *Kasner map*, whose dynamics is well understood:

- The Kasner map is topologically conjugated to $\theta \mapsto -2\theta$ on the cirle $\mathbb{R}/\mathbb{Z}$ (see [Bég10]). In particular, its dynamics is chaotic.
- There is an explicit “conjugation” between the Kasner map and an avatar of the Gauss transformation on the continued fractions (see section 1.2.1 below).
- The Kasner map admits a very simple geometric construction (see section 3.5).

We refer to sections 3.5 and 3.7 for further details on the Kasner map. One may rephrase the question asked in a preceding paragraph as follows: is the future-asymptotic dynamics of a generic type IX orbit in $\mathscr{B}^+$ “driven” by the Kasner map? We now introduce two concepts to make the preceding question rigorous: *heteroclinic chains* and *shadowing*.

Let p be a point of the Kasner circle (such that, for every $k \geq 0$, $\mathscr{F}^k(p)$ is not a Taub point). The *heteroclinic chain starting at* p is the concatenation of the unique type II orbit starting at p and arriving at $\mathscr{F}(p)$, then the unique type II orbit starting at $\mathscr{F}(p)$ and arriving at $\mathscr{F}^2(p)$, etc. Formally, this is the sequence

$$\left(\mathscr{O}_{p\to\mathscr{F}(p)}, \mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \mathscr{O}_{\mathscr{F}^2(p)\to\mathscr{F}^3(p)}, \dots\right) \tag{1.6}$$

Let $t \mapsto \mathscr{O}(t)$ be a type IX orbit in $\mathscr{B}^+$ converging to the Mixmaster attractor, p be a point of the Kasner circle (such that, for every $k \geq 0$, $\mathscr{F}^k(p)$ is not a Taub point) and $\mathscr{H}$ be the heteroclinic chain (1.6) starting at p.

Definition 1.2 (Shadowing)**.** We say that $\mathscr{O}$ *shadows* $\mathscr{H}$ (or $\mathscr{H}$ attracts $\mathscr{O}$) if there exists a strictly increasing sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that

1. $d(\mathscr{O}(t_n), \mathscr{F}^n(p)) \xrightarrow[n\to+\infty]{} 0$.

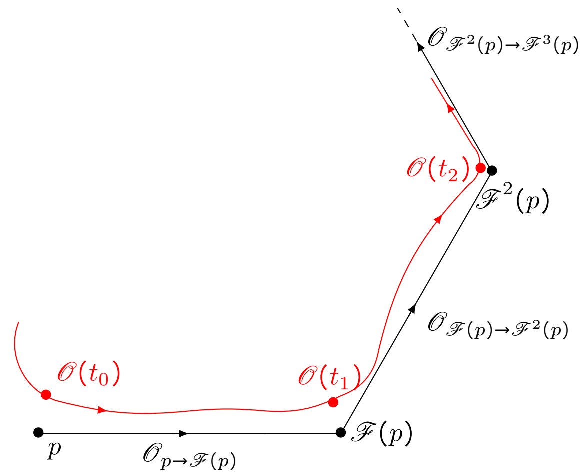

Figure 1.2 – Schematical representation of the first part of a heteroclinic chain shadowed by a type IX orbit in $\mathscr{B}^+$.

2. The Hausdorff distance between the orbit interval $\{\mathscr{O}(t) \mid t_n < t < t_{n+1}\}$ and the type II orbit $\mathscr{O}_{\mathscr{F}^n(p)\to\mathscr{F}^{n+1}(p)}$ tends to 0 when $n \to +\infty$.

See figure 1.2 for a schematical representation of the shadowing.

Given a type IX orbit in $\mathscr{B}^+$, the concept of shadowing formalizes the idea that its future-asymptotic dynamics is "driven" by the Kasner map. We can now refine our preceding questions: given a point p of the Kasner circle (such that, for every $k \geq 0$, $\mathscr{F}^k(p)$ is not a Taub point) and the heteroclinic chain $\mathscr{H}$ starting at p, what is the geometrical structure of the union of all the type IX orbits in $\mathscr{B}^+$ shadowing the heteroclinic chain $\mathscr{H}$? Are "typical" orbits driven by the Kasner map? More precisely, does the union of all the type IX orbits in $\mathscr{B}^+$ shadowing some heteroclinic chain has full Lebesgue measure in the phase space $\mathscr{B}^+$? If not, is it a set of positive measure?

1.1.9 Possible formalization of the BKL conjecture for Bianchi spacetimes

Using the preceding definitions, we propose[6] the following rewording of the part 3 of the BKL picture:

1. Almost every heteroclinic chain is shadowed by some type IX orbits in $\mathscr{B}^+$.

2. The union of all the type IX orbits in $\mathscr{B}^+$ shadowing some heteroclinic chain has full Lebesgue measure in the phase space $\mathscr{B}^+$.

1.2 Statement of the results

In this work, we intend to give a proof of item 1 and a partial proof of item 2 above. Our results can be stated in the following terms:

Theorem A. *For Lebesgue almost every point p of the Kasner circle, if $\mathscr{H}$ denotes the heteroclinic chain starting at p, then the union of all the type* IX *orbits shadowing $\mathscr{H}$ contains a* 3*-dimensional Lipschitz immersed submanifold. Moreover, the union of all the type* IX *orbits shadowing some heteroclinic chain has positive Lebesgue measure. More precisely, for all subset $\mathscr{E}$ of the Kasner circle with positive* 1*-dimensional Lebesgue measure, the union of all the type* IX *orbits shadowing some heteroclinic chain starting at a point of $\mathscr{E}$ has positive* 4*-dimensional Lebesgue measure.*

Remark 1.3. Informally, this means that if one picks randomly a spatially homogeneous spacetime, then this spacetime has a chaotic oscillatory past-asymptotic behaviour with nonzero probability.

[6] This formulation is classic and is based on the work of Beliinski, Khalatnikov and Lifschitz on one hand and Misner on the other hand.

The first part of Theorem A is a refinement of the work done by Reiterer & Trubowitz in [RT10]. To our knowledge, the second part of Theorem A is entirely new. It should be considered as the main result of this thesis.

The purpose of the next two subsections is to explain what are the heteroclinic chains that we manage to shadow with a sufficiently big set of type IX orbits. Let us say that a point p belonging to the Kasner circle is *admissible for the shadowing* if the union of all the type IX orbits shadowing the heteroclinic chain starting at p contains a 3-dimensional Lipschitz immersed submanifold. Let p be a point of the Kasner circle. Roughly speaking, our proof of Theorem A shows that if the orbit of p under the Kasner map "does not come too fast too close to the Taub points", then p is admissible for the shadowing . We are now going to introduce some tools to make this statement more precise.

1.2.1 Kasner parameter and Gauss transformation

The Kasner parameter $\omega : \mathscr{K}/\mathfrak{S}_3 \to [1, +\infty]$, where $\mathfrak{S}_3$ is the group of permutations of $\{1, 2, 3\}$, is a bijective parametrization of $\mathscr{K}/\mathfrak{S}_3$ by $[1, +\infty]$ satisfying the relation $\omega(T_i) = +\infty$, for any Taub point T_i. In this parametrization, the Kasner map $\mathscr{F}$ becomes an avatar of the Gauss transformation on the continued fractions. More precisely, let us define

$$f(\omega) = \begin{cases} \omega - 1 & \text{if } \omega \geq 2 \\ \frac{1}{\omega - 1} & \text{if } 1 < \omega \leq 2 \\ +\infty & \text{if } \omega = 1 \text{ ou } \omega = +\infty \end{cases}$$

The Kasner parameter is a C^0-conjuguacy from $(\mathscr{K}/\mathfrak{S}_3, \mathscr{F})$ to $([1, +\infty], f)$. It means that, for any given point p of the Kasner circle, the dynamical behaviour of its orbit under the Kasner map $\mathscr{F}$ depends on the continued fraction expansion of its Kasner parameter

$$\omega(p) = [k_0; k_1, k_2, k_3, \dots] = k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \dots}}}$$

We refer to section 3.7 for further details, see also [BCJ07].

1.2.2 Rephrasing of the results

Let p be a point of the Kasner circle and $\omega(p) = [k_0; k_1, k_2, k_3, \dots]$ be its Kasner parameter. According to the preceding paragraph, p is "close" to a Taub point if and only if k_0 is "large". Adopting the view-point of the continued fractions, we can say that, roughly speaking, a point p is admissible for the shadowing if the partial quotients k_i of the continued fraction expansion of its Kasner parameter $\omega(p)$ do not blow up "too fast". A precise meaning is given by the following definition.

Definition 1.4 (Moderate growth condition)**.** Let $\omega = [k_0; k_1, k_2, \dots] \in]1, +\infty[\setminus \mathbb{Q}$ be a continued fraction. We say that ω verifies the *moderate growth condition* if

$$k_{n+4}^4 = o_{n\to+\infty}\left(\sum_{i=1}^{n} k_i^5\right) \tag{MG}$$

Next lemma shows that the moderate growth condition is not too restrictive. A proof can be found in Appendix B. Define

$$\mathscr{K}_{(MG)} = \{p \in \mathscr{K} \mid \omega(p) \text{ verifies (MG)}\}$$

Lemma 1.5. *The set $\mathscr{K}_{(MG)}$ is a full Lebesgue measure subset of $\mathscr{K}$.*

We are now able to give a more precise statement of Theorem A.

Theorem B. *Let p be a point of the Kasner circle. If $\omega(p)$ verifies the moderate growth condition* (MG)*, then the union of all the type* IX *orbits shadowing the heteroclinic chain starting at p contains a 3-dimensional ball $D(p)$ Lipschitz embedded in the phase space $\mathscr{B}^+$. Moreover, for any $\mathscr{E} \subset \mathscr{K}_{(MG)}$ of positive 1-dimensional Lebesgue measure, the union of all the balls $D(p)$ for $p \in \mathscr{E}$ has positive 4-dimensional Lebesgue measure.*

We do not know whether the union of the type IX orbits intersecting a ball $D(p)$ for some p has full Lebesgue measure in the phase space. Hence, the following questions remains open:

Question 1. *Is the union of all type* IX *orbits shadowing a heteroclinic chain of type* II *orbits a full Lebesgue measure subset of the phase space $\mathscr{B}^+$?*

1.2.3 Examples of dynamical and geometrical consequences

In order to know the past-asymptotic behaviour of a maximal vacuum class A Bianchi spacetime, it is of prime interest to describe the ω-limit set of the corresponding orbit of the Wainwright-Hsu vector field. Knowing that for almost all point p of the Kasner circle (with respect to Lebesgue measure), the heteroclinic chain starting at p (seen as a subset of the phase space) is dense in the Mixmaster attractor $\mathscr{A}$, one gets the following result as a direct consequence of Theorem B.

Corollary 1.6. *Let q be a point of the phase space $\mathscr{B}^+$. With positive probability on q, the ω-limit set of the orbit of the Wainwright-Hsu vector field with initial condition q is the entire Mixmaster attractor $\mathscr{A}$.*

Theorem B says in particular that, with positive probability, a maximal vacuum class A Bianchi spacetime will have an oscillatory past-asymptotic behaviour. However, oscillatory has multiple meanings and they are not all equivalent. Corollary 1.7 below shows that the spacelike hypersurfaces (when following time towards the initial singularity) alternate indefinitely between periods where they are curved in a single direction and periods where they are curved in two or three directions.

Corollary 1.7. *Let q be a point of the phase space $\mathscr{B}^+$ and $(M,g) = \left(]s_-, s_+[\times G, -\mathrm{d}s^2 + h_s\right)$ be a maximal vacuum class A Bianchi spacetime corresponding to the orbit of the Wainwright-Hsu vector field with initial condition q, with $s_- > -\infty$. Denote by $\theta_{max}(t), \theta_{mid}(t), \theta_{min}(t)$ the three principal curvatures of the second fundamental form of the spacelike hypersurface $\{t\} \times G$ in M, with the convention $|\theta_{max}(t)| \geq |\theta_{mid}(t)| \geq |\theta_{min}(t)|$. With positive probability on q, there exists a sequence (s_n) strictly decreasing and converging to s_- such that*

1. $\lim_{n\to+\infty} \frac{|\theta_{max}(s_{2n})|}{|\theta_{mid}(s_{2n})|} = +\infty$.

2. *For every* $n \geq 0$, $\frac{|\theta_{max}(s_{2n+1})|}{|\theta_{min}(s_{2n+1})|} \leq 3$

1.2.4 Comparison with previous results

It was already known that some heteroclinic chains attract an injectively immersed manifold of codimension one. In [Lie+11], Liebscher & al prove this result for a periodic heteroclinic chain. Their method extend, with some technical work, to arbitrary heteroclinic chains bounded away from the Taub points. F. Béguin proved a similar result for aperiodic heteroclinic chains in [Bég10]. One should note that in both these papers, the set of heteroclinic chains that are shown to attract some type VIII or IX orbits correspond to a null measure subset of the Kasner circle. In the preprint [RT10], Reiterer & Trubowitz show that the set of points p for which the heteroclinic chains attract some type VIII or IX orbits is a Lebesgue full measure subset of the Kasner circle. However, their result, while showing that the union of all the type VIII or IX orbits shadowing a generic heteroclinic chain is in some sense "3-dimensional", does not describe its geometry as precisely as in [Lie+11] and [Bég10]. This is mainly due to the lack of information near the Taub points, or at least, the difficulty to extract this information.

The first part of Theorem B is essentially equivalent to the theorem proved by Reiterer & Trubowitz. There are three main differences between these two results:

- We do not work with the same equations. Indeed, while we use the *orthonormal frame method*, they use the *orthogonal frame method*. It means that their variables are the diagonal coefficients of the metric h_t and the diagonal coefficients of the second fundamental form of the spacelike hypersurface $\{t\} \times G$, while with the *orthonormal frame method*, as its name seems to indicate, the metric h_t is normalized (its diagonal coefficients are equal to 1).

- We do not obtain the same subsets of the Kasner circle. Indeed, the result of Reiterer & Trubowitz applies to any point p of the Kasner circle such that the sequence (k_i) of the partial quotients of the continued fraction expansion of the Kasner parameter of p grows at most polynomially, that is, such that the sequence (k_i) satisfies the *subpolynomial growth condition*

$$\text{there exists } P \in \mathbb{R}[X] \text{ such that for all } n \in \mathbb{N}, k_n \leq P(n) \tag{sPG}$$

  One can remark that between the two conditions (MG) and (sPG), neither is stronger than the other one.

- We obtain a slightly finer description of the geometry of the union of all the type IX orbits shadowing a generic heteroclinic chain. In our result, this set is proved to contain a Lipschitz manifold immersed in the phase space $\mathscr{B}^+$. In Reiterer & Trubowitz's work, it is not clear if the set they obtain is Lipschitz regular.

Moreover, our posture is quite different from Reiterer-Trubowitz's. Their goal is to provide a proof of their main theorem as concise as possible. On the contrary, our choice was to carry a rather complete and systematic investigation of the properties of the Wainwright-Hsu vector field from the viewpoint of non-uniformly hyperbolic systems theory. Theorem B appears as a kind of corollary of this investigation.

On the other hand, the second part of Theorem B is entirely new and relies on the precise geometrical description of the shadowing sets (*i.e.* the sets of orbits shadowing a heteroclinic chain).

As we already stated in Theorem 1.1, it was already known that the ω-limit set of almost all the orbits of the Wainwright-Hsu vector field is contained in the Mixmaster attractor $\mathscr{A}$. Moreover, Ringström ([Rin01]) and Brehm ([Bre16]) proved that the ω-limit set of a generic orbit is not reduced to a Taub point. This implies that almost all the orbits of the Wainwright-Hsu vector field have an oscillatory future-asymptotic behaviour (a generic orbit has at least three different ω-limit points in the Kasner circle), but this result does not give precise information about the oscillatory behaviour. In particular, using only this result, we do not know if these generic orbits shadow some heteroclinic chains or not.

Hence, we still do not know if corollary 1.6 holds for generic orbits of the Wainwright-Hsu vector field. The question whether or not it is true is of particular importance, so let us state this open question here.

Question 2. *Is it true that for a generic point q of the phase space $\mathscr{B}^+$ (with respect to Lebesgue measure), the ω-limit set of the orbit of the Wainwright-Hsu vector field with initial condition q coincides with the Mixmaster attractor $\mathscr{A}$?*

1.3 Some interesting dynamical features of the Wainwright-Hsu vector field

Even if one forgets its physical origin, the Wainwright-Hsu vector field appears to be very interesting from a purely dynamical systems viewpoint.

A catalogue of classical examples of dynamical systems. First of all, it is quite amusing that the Wainwright-Hsu equations somehow gathers in a single vector field several of the most classical examples of chaotic dynamical systems that are presented in most introductory courses.

- The behaviour of the type II orbits is described by the so-called Kasner map, which is an avatar of the most basic example of chaotic map: the *angle-doubling map* of the circle. More precisely, being a (non-uniformly) expanding map of degree -2 of the circle, the Kasner map is topologically conjugate to the map $\theta \mapsto -2\theta$ on $\mathbb{R}/\mathbb{Z}$.

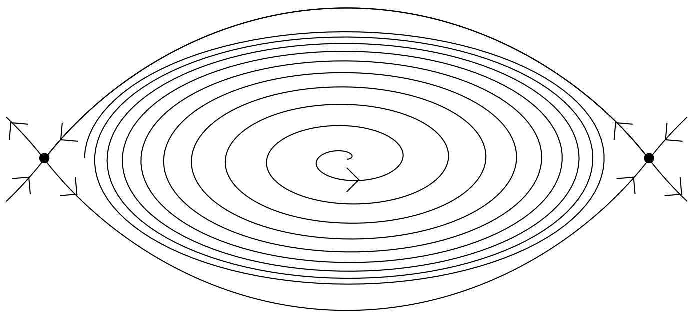

Figure 1.3 – Bowen's eye attractor.

- As explained in the previous pages, the Kasner parameter conjugate the Kasner map (modulo a finite quotient) to an avatar of the famous Gauss map $x \mapsto \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$. As an immediate consequence, the behaviour of the orbit of a point p under the Kasner map depends on the continued fraction development of the Kasner parameter of p. Some properties of the Gauss map will indeed play a crucial role in the proof of our main theorems (see Appendix B).
- Recall that the classical *Bowen's eye-attractor* is obtained by considering a vector field in the plane with an attracting cycle made of two heteroclinic orbits connecting two hyperbolic saddle-type singularities (see figure 1.3). This example is well-known because it has a very bad statistical behaviour: the Birkhoff sums along any orbit in the interior eye do not converge. The reason is that such an orbit will spend some time close to the left corner of the eye, then a much longer time close to the right corner of the eye, then a much much longer time close to the left corner of the eye, etc. This behaviour forces the Birkhoff sums to oscillate. Now consider a periodic chain of type II orbits in the Mixmaster attractor. It is nothing but a cycle of heteroclinic orbits connecting (partially) hyperbolic saddle-type singularities. It was proved by Georgi, Härterich, Liebscher and Webster that there is a three-dimensional set of type VIII or IX orbits that are attracted by this cycle (see [Lie+11]). The same arguments as for the classical Bowen's eye-attractor show that the Birkhoff sum along these orbits do not converge. Hence, every periodic chain of type II orbits can indeed be considered as a "generalized Bowen's eye-attractor". Therefore, the Mixmaster attractor somehow contains a "bunch of infinitely many interlaced (generalized) Bowen's eye-attractors".
- Yet another classical system "hidden" in the Wainwright-Hsu vector field ! In some variables that we will not use in this thesis, the flow of the restriction of the Wainwright-Hsu vector field to the Mixmaster attractor becomes a billiard in a ideal hyperbolic triangle, the so-called *cosmological billiard* (see *e.g.* [Dam] and [HU09]).

Non-linearizable degenerate partially hyperbolic singularities. When one tries to analyze in details the behaviour of the Wainwright-Hsu vector field, one realizes that this vector field presents some unusually complicated dynamical features.

For example, the starting point of the proof of our main theorems is the analysis of the dynamics of the Wainwright-Hsu vector field $\mathscr{X}$ in the neighbourhood of a point p of the Kasner circle. Recall that every such point p is a singularity of $\mathscr{X}$. The eigenvalues of $D\mathscr{X}(p)$ vary with p, and there often appears some resonance between them. As a consequence, there is a dense set of points p in the neighbourhood of which the Wainwright-Hsu vector field is not linearizable. As a further consequence, we are forced to study the local dynamics of $\mathscr{X}$ in the neighbourhood of such points p by very basic methods (which roughly consist in using repeatedly Grönwall's lemma to bound the effect of the non-linear terms). Note that the situation we face (the local dynamics of a non-linear vector field in the neighbourhood of a partially hyperbolic singularity in dimension 4, with arbitrarily bad resonances, the vector field being C^∞ flat in the central direction) seems to be more degenerate than what has been studied by experts.

Remark 1.8. A result of F. Takens [Tak71] allows to linearize the dynamics in the neighbourhood of a point p of the Kasner circle that is not pre-periodic for the Kasner map (these are exactly the points whose eigenvalues are non-resonant). But this result does not provide any lower bound on the size of the linearization neighbourhood, nor any upper bound on the derivatives of the linearizing coordinates. As a consequence, this result can only be used in order to build some local stable manifolds for chains of type II orbits that do not accumulate on a periodic orbit of the Kasner map (this has been done by F. Béguin in [Bég10]). Such chains are very rare: their union has zero Lebesgue measure in the Mixmaster attractor.

A non-uniformly hyperbolic return map with poor regularity. The proof of our main theorems relies on the non-uniformly hyperbolic behaviour of the Wainwright-Hsu vector field. In practice, we will consider the second iterate of the Poincaré return map $\hat{\Phi}$ of the orbits of the Wainwright-Hsu vector field on a transverse section S. We will prove some uniformly hyperbolic properties for this return map $\hat{\Phi}$: for every point p in the intersection of the section S with the Mixmaster attractor $\mathscr{A}$, if the return map $\hat{\Phi}$ is defined at p, then it contracts uniformly the direction transverse to $\mathscr{A}$ at p, and expands uniformly the direction tangent to $\mathscr{A}$. We insist on the fact that the contraction and expansion constants are independent of the point p. Moreover, the contraction in the direction transverse to $\mathscr{A}$ happens to be super-linear. Nevertheless, the map $\hat{\Phi}$ should be considered as a *non-uniformly* hyperbolic map. Indeed, the size of the neighbourhood of the point p on which one can prove some contraction/expansion properties is not bounded from below uniformly in p. This is due to:

- the presence of *singularities*: the return map $\hat{\Phi}$ is not defined everywhere (roughly speaking, an orbit which falls on a Taub point never comes back in the section);
- the lack of regularity of the return map $\hat{\Phi}$: we are only able to prove that $\hat{\Phi}$ is Lipschitz. Actually, $\hat{\Phi}$ might be C^1, but some evidence indicate that the derivative of $\hat{\Phi}$, if it happens to exist, cannot be α-Hölder for some uniform $\alpha > 0$.

As a consequence of this non-uniformity:

- we will be able to prove the existence of local stable manifolds for almost every orbit of $\hat{\Phi}$, but not for all orbits,
- the size of these stable manifolds will depend on the orbit, and will not be uniformly bounded from below.

Although we will prove some non-uniformly hyperbolic properties for the return map $\hat{\Phi}$, the classical Pesin's theory of non-uniformly hyperbolic maps (see *e.g.* [BP13]) does not apply to $\hat{\Phi}$. The theory of non-uniformly hyperbolic maps with singularities, as developed by A. Katok and J.-M. Strelcyn (see [KS86] or [Sat92]) does not apply directly either. The reason is once again the lack of regularity of $\hat{\Phi}$. Indeed, the above-mentioned theories concern maps whose derivatives may explode when one approaches some singular set, but which are quite regular (at least C^2) far from the singular set. This is not the case of $\hat{\Phi}$: as explained above, we are not able to prove that $\hat{\Phi}$ is differentiable. The hardest task in the proof of our main theorems is to obtain some hyperbolicity estimates for $\hat{\Phi}$, with some explicit controls of the size of the neighbourhoods where these estimates hold. It will cover chapters 4 to 9. Once we will have these estimates, we will need to « redo » Katok-Strelcyn's work in our specific context, using some Lipschitz estimates instead of the classical bounds on the first and second derivatives. Appart from the low regularity of our map, there is another important difference between Katok-Strelcyn's setting and ours:

- roughly speaking, Katok-Strelcyn's hypotheses are chosen so that the size of the neighbourhoods on which one gets various types of estimates is always polynomial with respect to the distance to the singularity;
- in our situation, we will often be forced to consider neighbourhoods with exponentially small size ...
- ... but the extremely small size of the neighbourhood on which we can prove interesting estimates will be balanced by the super-linear contraction in the direction transverse to the Mixmaster attractor.

Remark 1.9. Note that one really needs to use some specific properties of the Poincaré map $\hat{\Phi}$ to compensate its poor regularity. Indeed, C. Bonatti, S. Crovisier and K. Shinohara have proved that generic C^1 non-uniformly hyperbolic diffeomorphisms (such diffeomorphisms are not $C^{1+\alpha}$ for any $\alpha > 0$) do not admit non-trivial local stable manifolds (see [BCS14]).

Some unusual arithmetic conditions. In a non-uniformly hyperbolic system with singularities, it is not possible to construct non-trivial Pesin stable manifolds at every point p. A necessary condition (among others) is that the orbit of p should wait a long time before coming very close to the singularities. For the Wainwright-Hsu vector field, this means that we have to focus on points p of the Kasner circle whose orbits under the Kasner map will wait a long time before coming very close to the Taub points. Since the Kasner parameter turns the Kasner map into an avatar of the Gauss map, this naturally translates as a condition on the continued fraction development of the Kasner parameter of the point p. In other words, we will only be able to deal with points p whose Kasner parameter satisfies a certain arithmetic condition.

Arithmetic conditions appear in various areas of dynamical systems. They are usually of one of the following two types:

- Either one needs to consider real numbers that are badly approximated by rational numbers (so-called *Diophantine numbers* and their generalizations). This is typically the case when one wants to prove KAM-type results, solve cohomological equations, prove the convergence of a renormalization scheme, etc. The terms $(k_n)_{n\geq 0}$ of the continued fraction development of such numbers grow slowly with respect to n.

- Or one needs to consider real numbers that are very well-approximated by rational numbers (so-called *Liouville* or *super-Liouville* numbers). This is typically the case when one wants to construct exotic examples of elliptic dynamical systems as limits of periodic systems (for example, by using the so-called *Anosov-Katok method*). The terms $(k_n)_{n\geq 0}$ of the continued fraction development of such numbers grow very fast with respect to n.

The arithmetic condition (MG) we need to consider in our proof (which we call *moderate growth condition*) is of neither of the two above types. The integers $(k_n)_{n\geq 0}$ that appear in a continued fraction development satisfying this condition might grow either slowly or very fast with respect to n. What is important is that the size of k_n should be balanced by the size of $k_1, \dots, k_{n-1}$. This is due to the competition between two phenomena. Consider a chain of type II orbits starting at some point p of the Kasner circle, a type IX orbit whose initial condition is at distance $\epsilon \ll 1$ of p and denote by $(k_n)_{n\geq 0}$ the terms of the continued fraction development of the Kasner parameter of p.

- On the one hand, the contraction rate of the flow in the direction transversal to the Mixmaster attractor between a small transverse section close to p and a small transverse section close to $\mathscr{F}^{k_1+\cdots+k_{n-1}}(p)$ depends on $k_1 + \cdots + k_{n-1}$.

- On the other hand, the size of the neighbourhood of $\mathscr{F}^{k_1+\cdots+k_{n-1}}(p)$ where we have a good control of the behaviour of the orbits depends of k_n.

So, very roughly speaking, the orbits starting at distance 1 of the Mixmaster attractor will hit the neighbourhood of $\mathscr{F}^{k_1+\cdots+k_{n-1}}(p)$ where we can control their behaviour provided that k_n is small compared to $k_1 + \cdots + k_{n-1}$ (of course, we are oversimplifying). This is the reason why the moderate growth condition (MG) comes into the game.

Remark 1.10. Proving that Lebesgue almost every real number satisfies the moderate growth condition (MG) (see Lemma 1.5) is not that easy. The argument that was suggested to us by S. Gouëzel uses some rather sophisticated properties of the Gauss map (namely, the existence of a spectral gap for the transfer operator associated with the Gauss map, acting on the space of L^∞ functions with bounded essential variation).

A complicated statistical behaviour. We have explained above that a periodic chain of type II orbits of the Wainwright-Hsu vector field can be thought as a generalized Bowen's eye-attractor. But the global statistical behaviour of the Wainwright-Hsu vector field is certainly much more complicated than those of a Bowen's eye-attractor.

Indeed, for a classical Bowen's eye-attractor, the set of all the limit points (in the space of probability measures) of the Birkhoff sums is rather small: it is exactly the affine segment whose ends are the Dirac masses supported by the two eye corners. Now consider a non-periodic chain of type II orbits in the Mixmaster attractor. Such a chain will almost surely be dense in the Mixmaster attractor, *i.e.* the corners of the chain will almost surely be dense in the Kasner circle. Moreover, Theorem B shows that such a chain will almost surely be shadowed by a three-dimensional set of type IX orbits of the Wainwright-Hsu vector field. We are not able to compute exactly the set of limit points of the Birkhoff sums along such orbits (in the general case). But some informal arguments show that this set should typically be infinite dimensional. In any case, it is clear that the behaviour of the Birkhoff sums along most orbits of the Wainwright-Hsu vector field must be very wild.

P. Berger has introduced a quantity which quantifies the statistical complexity of a dynamical system: the *emergence* of the system. Roughly speaking, it measures the growth rate, as ϵ goes to 0, of the number of probability measures that are necessary to ϵ-approximate the set of all limit points of the Birkhoff sums along almost all the orbits (see [Ber16] for a precise definition). It is known that there exists systems with arbitrarily large emergence (such systems are actually locally generic). But the constructions rely on Baire arguments, and do not yield explicit examples. We guess that the Wainwright-Hsu vector field might be an explicit example of a dynamical system with very large emergence. So we conclude this section by the following problem:

Question 3. *Is it possible to compute the emergence of the Wainwright-Hsu vector field? Is it exponential?*

A high emergence rate for the Wainwright-Hsu vector field would bring another evidence that explicit models of physical systems might display a very wild dynamical behaviour ...

1.4 Heuristic arguments underlying the proof of the main theorem

The proof of Theorem B is based on the following heuristic argumentation, which can be attributed to Belinskii, Khalatnikov and Lifshitz (except for the very last part concerning the moderate growth condition).

Consider a point p of the Kasner circle, so that p is not one of the Taub points. The point $\mathscr{F}(p)$ (the image of p under the Kasner map) is a partially hyperbolic singularity of the Wainwright-Hsu vector field $\mathscr{X}$. More precisely, the linear part of $\mathscr{X}$ at $\mathscr{F}(p)$ has two negative eigenvalues $-\mu_{s_1}$ and $-\mu_{s_2}$ (with $\mu_{s_2} \geq \mu_{s_1}$), one zero eigenvalue (corresponding to the direction tangent to the Kasner circle), and one positive eigenvalue μ_u. The eigendirections associated with the two stable eigenvalues, $-\mu_{s_1}$ and $-\mu_{s_2}$, are tangent to the two type II orbits arriving at $\mathscr{F}(p)$ (hence, one of them, say the one associated with $-\mu_{s_1}$, is tangent to the type II orbit $\mathscr{O}_{p\to\mathscr{F}(p)}$) The eigendirection associated with the unstable eigenvalue μ_u is tangent to the type II orbit $\mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}$ going from $\mathscr{F}(p)$ to $\mathscr{F}^2(p)$.

Consider a type IX orbit $\mathscr{O}$ traveling very close to the type II orbit $\mathscr{O}_{p\to\mathscr{F}(p)}$. After some time, it will enter a neighbourhood B_1 of $\mathscr{F}(p)$. Let d_1 be the distance between the orbits $\mathscr{O}$ and $\mathscr{O}_{p\to\mathscr{F}(p)}$ when they enter in B_1. The orbit $\mathscr{O}$ will continue to follow $\mathscr{O}_{p\to\mathscr{F}(p)}$ until it comes very close to the point $\mathscr{F}(p)$ (going slower and slower since $\mathscr{F}(p)$ is a singularity). Then it will start to follow the unstable manifold of $\mathscr{F}(p)$, that is, to follow the type II orbit $\mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}$. Now, suppose for some moment that one could neglect the non-linear part of $\mathscr{X}$. Then we can compute explicitly the flow of $\mathscr{X}$, and we see that the orbit $\mathscr{O}$ will exit B_1 roughly at distance $d_1^{\mu_{s_1}/\mu_u}$ from the orbit $\mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}$. The crucial point is that the stable eigenvalues of the point of the Kasner circle are “stronger” than the unstable one. In other words, μ_{s_1}/μ_u is greater than 1 and therefore $d_1^{\mu_{s_1}/\mu_u}$ is much smaller than d_1.

Now, the orbit $\mathscr{O}$ will travel side to side with the type II orbit $\mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}$ until entering a small neighbourhood B_2 of the point $\mathscr{F}^2(p)$. It is impossible to control precisely the distance between $\mathscr{O}$ and $\mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}$ during this travel: we face the global behaviour of a non-linear vector field. But in any case, the travel from B_1 to B_2 will take a finite time T, and therefore the distance will grow at most linearly, the dilatation factor λ being the upper bound of the derivative of the time T map of the flow. As a consequence, the orbit $\mathscr{O}$ should enter the neighbourhood B_2 roughly at distance $d_2 := \lambda d_1^{\mu_{s_1}/\mu_u}$ of the orbit $\mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}$, which is much smaller than d_1 (if d_1 is small enough). See figure 1.4.

Iterating the argument, the orbit $\mathscr{O}$ should go through the small neighbourhood B_2 of $\mathscr{F}^2(p)$, follow the type II orbit $\mathscr{O}_{\mathscr{F}^2(p)\to\mathscr{F}^3(p)}$, and enter in a neighbourhood of $\mathscr{F}^3(p)$ at a distance $d_3 \ll d_2$, go through the small neighbourhood of $\mathscr{F}^3(p)$, follow the type II orbit $\mathscr{O}_{\mathscr{F}^3(p)\to\mathscr{F}^4(p)}$, and enter in a neighbourhood of $\mathscr{F}^4(p)$ at a distance $d_4 \ll d_3$, ... So we can hope to keep some control of the behaviour of $\mathscr{O}$ forever and prove that it shadows the heteroclinic chain $\left(\mathscr{O}_{p\to\mathscr{F}(p)}, \mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \dots\right)$.

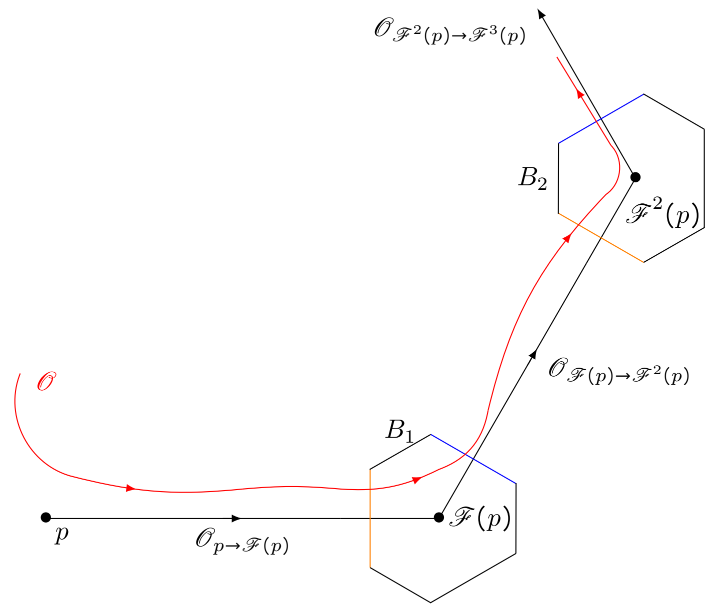

Figure 1.4 – The orbit $\mathscr{O}$ successively enters the neighbourhoods B_1, B_2,... Each time it passes inside one of these neighbourhoods, it gets much closer to the heteroclinic chain starting at p, due to the super-linear contraction.

Of course, this very rough heuristic argument dramatically oversymplifies the situation (otherwise the proof of Theorem B would not fill more than 150 pages of this thesis !). Yet it will serve us as a guideline, and our task will be to turn it into a rigorous proof.

The main difficulties that we will face are the following. When we analyze the behaviour of the orbit $\mathscr{O}$ inside a neighbourhood of $\mathscr{F}^\ell(p)$, we need to take into account the effect of the non-linear part of $\mathscr{X}$. These non-linear terms will in particular induce a drift in the central direction, *i.e.* in the direction of the Kasner circle. So the orbit $\mathscr{O}$ will deviate from the heteroclinic chain of type II orbits, and we shall need to control this deviation, and prove that it is somehow balanced by the very strong contraction due to the linear part of the vector field. We also have to take into account the fact that the stable and unstable eigenvalues $-\mu_{s_1}$, $-\mu_{s_2}$ and μ_u at the point $\mathscr{F}^\ell(p)$ critically depend on the position of this point on the Kasner circle: $-\mu_{s_1}$ and μ_u tend to zero as the point $\mathscr{F}^\ell(p)$ approaches one of the Taub points. This means that, when $\mathscr{F}^\ell(p)$ is very close to a Taub point, the hyperbolicity of the linear part of $\mathscr{X}$ at $\mathscr{F}^\ell(p)$ is very weak, and therefore can only compensate the effect of the non-linear part in an extremely small neighbourhood B_ℓ of $\mathscr{F}^\ell(p)$.

So there will be a competition. On the one hand, if the orbit $\mathscr{O}$ falls successively in the neighbourhooods $B_1, B_2, \dots, B_n$ of the points $\mathscr{F}^1(p), \mathscr{F}^2(p), \dots, \mathscr{F}^n(p)$, then the distance between $\mathscr{O}$ and the heteroclinic chain of type II orbits $\left(\mathscr{O}_{p\to\mathscr{F}(p)}, \mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \dots\right)$ will undergo a very strong contraction. Therefore the orbit $\mathscr{O}$ will have more chance to enter the neighbourhood B_{n+1} of the point $\mathscr{F}^{n+1}(p)$. On the other hand, if the point $\mathscr{F}^{n+1}(p)$ happens to be very close to one of the Taub points, then the neighbourhood B_{n+1} will be extremely small and it is quite likely that the orbit $\mathscr{O}$ will fail to enter this neighbourhood, in which case the future behaviour of $\mathscr{O}$ will get out of control. This is the reason why we will not always be able to prove the existence of type IX orbits shadowing the heteroclinic chain $\left(\mathscr{O}_{p\to\mathscr{F}(p)}, \mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \dots\right)$. Roughly speaking, we will need this heteroclinic chain to "wait enough time before going close to the Taub points".

In order to be more quantitative, let us consider the continued fraction expansion $[1; k_1, k_2, \dots]$ of the Kasner parameter of the point p. On the one hand, if the orbit $\mathscr{O}$ falls in the neighbourhoods $B_1, B_2, \dots, B_n$, then the contraction of the distance between $\mathscr{O}$ and the heteroclinic chain will roughly be controlled by $k_1^5 + \dots + k_n^5$. On the other hand, the size of the neighbourhood B_{n+1} will roughly be controlled by k_{n+4}^4. So we will be able to keep some control on the behaviour of the orbit $\mathscr{O}$ if and only if the continued fraction expansion satisfies the moderate growth condition (MG). Once again, we are oversymplifying, but this is indeed the origin of the moderate growth condition.

1.5 Strategy of the proof of the main theorem and organization of the memoir

In the next few pages, we will describe the content of the different chapters of the memoir. We hope that the strategy of the proof of our main theorems will arise from this description.

The vacuum Einstein field equations for Bianchi spacetimes. The purpose of Chapter 2 is to explain briefly how the vacuum Einstein field equations translate into the Wainwright-Hsu equations in the context of Bianchi spacetimes.

The Wainwright-Hsu vector field and the Mixmaster attractor. In Chapter 3, we describe the dynamics of the Wainwright-Hsu vector field $\mathscr{X}$ in restriction to the Mixmaster attractor (linear part of $\mathscr{X}$ at points of the Kasner circle, explicit expression of the type II orbits, Kasner map, Kasner parameter, etc.). This dynamics is well-known. The only original part of Chapter 3 is the description of a finite quotient of the classical phase space in which we shall work.

Local expression of the Wainwright-Hsu vector field in the neighbourhood of a point of the Kasner circle. As explained above in heuristic terms, the proof of Theorem B is based on the analysis of the local dynamics of the Wainwright-Hsu vector field $\mathscr{X}$ in the neighbourhood of a point p of the Kasner circle. To carry this analysis, we use a quite standard strategy: we first construct a coordinates system in which the vector field $\mathscr{X}$ has the simplest possible expression, and then, we use this expression to control the deviation of the true orbits of $\mathscr{X}$ from those of the linear part $D\mathscr{X}(p)$ of $\mathscr{X}$.

Hence, our first task is to find a "nice" local coordinate system in the neighbourhood of a point p of the Kasner circle (which is not one of the Taub points). Actually, the only property we need for this coordinates system is that it straightens the stable, central and unstable manifold of $\mathscr{X}$ at the point p. So the coordinates system will be provided by the stable manifold theorem. Yet we need a quite precise version of this result: in particular, we need some lower bounds on the size of the neighbourhoods on which the straightening coordinates are defined, and some upper bounds on the norm of the derivative of these coordinates, with some explicit dependance on a parameter. We could not find the appropriate statement in the literature, so we had to prove it (using some rather standard techniques); this is done in Appendix C. Once we have the suitable statement of the stable manifold theorem, we apply it three times (together with some other easy coordinate change) to get a local coordinate system straightening the strong stable, weak stable, central and strong unstable manifolds of p. Then we write the local expression of $\mathscr{X}$ in this "nice" local coordinate system, providing some upper bounds on the non-linear terms showing up in this expression. This is done in Chapter 4.

Local sections and transition maps. In Chapter 5, we define some sections transverse to the Wainwright-Hsu vector field $\mathscr{X}$. For every point p in the Kasner circle (which is not one of the Taub points), we define a local section S_p^s that will be intersected by the orbits of $\mathscr{X}$ when they arrive in a small neighbourhood of p. Similarly, we define a local section S_p^u that will be intersected by the orbits of $\mathscr{X}$ when they get out from a small neighbourhood of p. The size of these sections (in the different directions), as well as their distance to the point p, depend on several parameters. We also define a global section S which is intersected by all the type IX orbits that could possibly shadow some heteroclinic chain of type II orbits. Moreover, in order to understand the dynamics of the orbits traveling between two sections, we are led to define some transition maps. The transition map from a section S_1 to a section S_2 encodes, for an orbit $\mathscr{O}$ of $\mathscr{X}$ starting in S_1, the first intersection point of $\mathscr{O}$ with the section S_2.

Local dynamics in the neighbourhood of a point of the Kasner circle. In Chapter 6, we use the local expression of the Wainwright-Hsu vector field $\mathscr{X}$ in order to study the local dynamics of $\mathscr{X}$ in the neighbourhood of a point p of the Kasner circle. More precisely, we want to understand the transition map Υ_p of the orbits of $\mathscr{X}$ from a local section S_p^s at the entrance of a neighbourhood of p to a local section S_p^u at the exit of the same neighbourhood. The task consists in controlling the effect of the non-linear terms in the local expression of $\mathscr{X}$. The size of the neighbourhood of p, the size of the local sections S_p^s and S_p^u, and their distance to the point p, depend on the Kasner parameter of p. The outcome of the chapter is roughly the following: when the orbits of $\mathscr{X}$ cross a small neighbourhood of the point p, the distance from these orbits to the Mixmaster attractor undergoes a super-linear contraction, whereas the drift of the orbits in the direction tangent to the Mixmaster attractor is extremely small. In other words, the transition map Υ_p is strongly contracting in the direction transverse to the Mixmaster attractor (the contraction is super-linear), and almost isometric in the direction tangent to the Mixmaster attractor. An important point is that the dependence of the contraction (resp. drift) rate with respect to the Kasner parameter of p is explicit. Note that to get this explicit dependence, we extend the methods employed in [Lie+11].

Dynamics in the neighbourhood of a type II **orbit.** Consider again a point p on the Kasner circle. The purpose of Chapter 7 is to control the behaviour of a type IX orbit traveling very close to the type II orbit $\mathscr{O}_{p\to\mathscr{F}(p)}$. More precisely, we want to control the transition map Ψ_p of the orbits of $\mathscr{X}$ from a local section S_p^u at the exit of a neighbourhood of p to a local section $S^s_{\mathscr{F}(p)}$ at the entrance of a neighbourhood of the point $\mathscr{F}(p)$. The estimates we obtain are very loose, since we are considering the long range behaviour of a non-linear vector field. The only thing we can do is to:

- find an upper bound of the travel time of the orbits between the sections S_p^u and $S^s_{\mathscr{F}(p)}$,
- apply Grönwall's lemma to obtain some (very) rough control during this travel.

Dynamics along an epoch. Given a point p of the Kasner circle, the *epoch transition map* Φ_p is the transition map of the orbits of $\mathscr{X}$ from a section S_p^s at the entrance of a neighbourhood of the point p to a section $S^s_{\mathscr{F}(p)}$ at the entrance of a neighbourhood of the point $\mathscr{F}(p)$. Observe that Φ_p is nothing else than the composition of the maps Υ_p and Ψ_p considered in Chapters 6 and 7. So, we will only need to concatenate the estimates proven for the maps Υ_p and Ψ_p to obtain some estimates on Φ_p. The only difficulty is to find some size of the sections S_p^s and $S^s_{\mathscr{F}(p)}$ so that the map Φ_p is well-defined. This is done in Chapter 8. Once we know that Φ_p is well-defined and is the composition of Υ_p and Ψ_p, we easily obtain some partial hyperbolicity properties for Φ_p: it is super-contracting in the direction transverse to the Mixmaster attractor, and almost not contracting in the direction tangent to the Mixmaster attractor (this direction may be expanded, or very weakly contracted).

Dynamics along an era. Consider the region $\mathscr{K}_{]1,2]}$ of the Kasner circle where the Kasner parameter ranges between 1 and 2 (roughly speaking, this is the region of the Kasner circle which is far from the Taub points). Let p be a point in $\mathscr{K}_{]1,2]}$, and denote by k_1 the first term in the continued fraction expansion of the Kasner parameter $\omega(p)$. The heteroclinic chain of type II orbits starting at p first goes close (roughly at distance $\frac{1}{k_1}$) to one of the Taub points, say T_3, then bounces $k_1 - 1$ times from one side of T_3 to the other, slowly escaping from the vicinity of T_3, until it comes back in $\mathscr{K}_{]1,2]}$. An *era* is such a piece of heteroclinic chain, made of the concatenation of k_1 type II orbits, which starts and ends up in $\mathscr{K}_{]1,2]}$. See figure 1.5. The purpose of Chapter 9 is to study the behaviour of the orbits of the Wainwright-Hsu vector field $\mathscr{X}$ along such an era. More precisely, we want to study the *era transition map*, *i.e.* the transition map $\bar{\Phi}_p$ of the orbits of $\mathscr{X}$ from a local section S_p^s at the entrance of a neighbourhood of p to a local section $S^s_{\mathscr{F}^{k_1}(p)}$ at the entrance of a neighbourhood of the point $\mathscr{F}^{k_1}(p)$. This map can be seen as the composition of the k_1 epoch transition maps $\Phi_p, \Phi_{\mathscr{F}(p)}, \dots, \Phi_{\mathscr{F}^{k_1-1}(p)}$ provided that we can find some size of sections so that this composition is well-defined. We indeed manage to set up an induction scheme, based on the estimates of Chapter 8, showing that the composition of the epoch transitions maps $\Phi_p, \Phi_{\mathscr{F}(p)}, \dots, \Phi_{\mathscr{F}^{k_1-1}(p)}$ is well-defined on a tiny local section close to p.

It is natural to expect some uniform hyperbolicity properties for $\bar{\Phi}_p$. Yet a minor (but quite annoying) technical difficulty shows up. One soon realizes that the map $\bar{\Phi}_p$ cannot be uniformly expanding in the direction tangent to the Mixmaster attractor. This can be easily overcome by replacing $\bar{\Phi}_p$ by the "double era transition map" $\hat{\Phi}_p$, which describes the behaviour of the orbits during two eras instead of a single one. We are indeed able to prove some hyperbolicity properties for this map: it contracts uniformly the direction transverse to the Mixmaster attractor and expands uniformly the direction tangent to this attractor.

Moreover, the $\hat{\Phi}_p$'s can be glued together, in order to get a global Lipschitz map $\hat{\Phi}$. This map is the second iterate of the Poincaré's return map of the orbits of the Wainwright-Hsu vector field $\mathscr{X}$ on a global section S. We call it the *double era return map*. The section S is intersected by all the

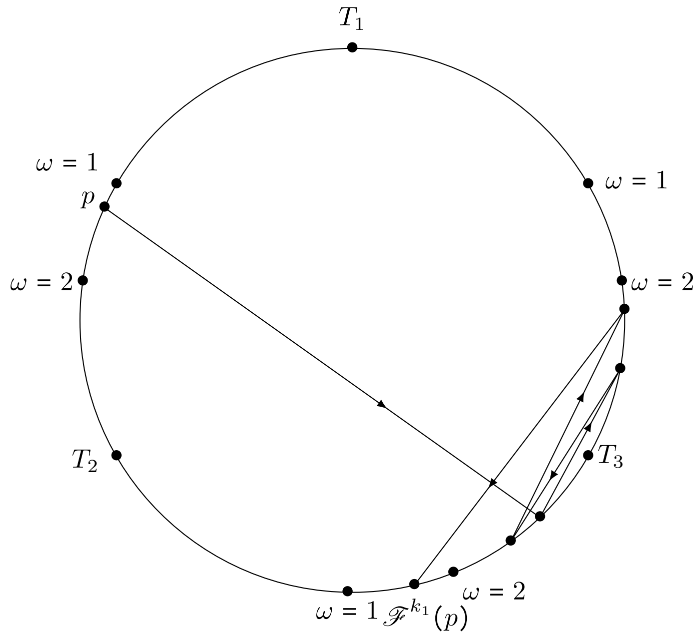

Figure 1.5 – First era of the heteroclinic chain starting at p, represented in projection on the plane containing the Kasner circle.

orbits that could potentially shadow some heteroclinic chain. Yet, it is important to note that $\hat{\Phi}$ is not well-defined on the whole section S. It is defined on a kind of hedge with variable height over the interval $]1, 2]$: the height of the hedge over the point $\omega \in]1, 2]$ depends on the four first terms of the continued fraction development of ω, and is equal to zero at certain points. The map $\hat{\Phi}$ is uniformly hyperbolic on this hedge-shaped domain.

Construction of local stable manifolds for the double era return map. In Chapter 10, we use the hyperbolicity of the double era return map $\hat{\Phi}$, together with the usual graph transform mapping technique, in order to construct some local stable manifolds for $\hat{\Phi}$. The main difficulty is to find some domains where the graph transform mapping can be iterated (recall that the map $\hat{\Phi}$ is not defined on the whole section S). This is where the moderate growth condition (MG) shows up. Roughly speaking, we can iterate the graph transform mapping over the orbit of a point $p \in \mathscr{K}_{]1,2]}$ if and only if the Kasner parameter of p satisfies the moderate growth condition. For such a point p, we obtain a non-trivial two-dimensional local stable manifold $W^s(p, \hat{\Phi})$. The size of this local stable manifold depends on p. In particular, it depends on the time n_0 one has to wait in order to "see" the domination of k_{n+4}^4 by the sum $\sum_{i=1}^n k_i^5$ for all $n \geq n_0$.

Shadowing of heteroclinic chains. Consider a point p of the Kasner circle having a non-trivial stable manifold $W^s(p, \hat{\Phi})$ for the double era return map $\hat{\Phi}$, and a point $q \in W^s(p, \hat{\Phi})$. In Chapter 11, we prove that the forward orbit of q (for the Wainwright-Hsu vector field) shadows (in the sense of definition 1.2) the heteroclinic chain $\left(\mathscr{O}_{p\to\mathscr{F}(p)}, \mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \dots\right)$. This easily follows from what has been done earlier. Thanks to some estimates proven in Chapters 6, 7 and 9, we know that, since q is close to p, the forward orbit of q will stay very close to the heteroclinic chain $\left(\mathscr{O}_{p\to\mathscr{F}(p)}, \mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \dots\right)$ during two complete eras. But since q is in the stable manifold $W^s(p, \hat{\Phi})$, this orbit hits the section S very close to $\hat{\mathscr{F}}(p) := \mathscr{F}^{k_1+k_2}(p)$ (we call $\hat{\mathscr{F}}$ the *double era Kasner map*). So, using again the estimates of Chapters 6, 7 and 9, we obtain that the forward orbit of q stays very close to the heteroclinic chain during two more eras. Then it hits the section S even closer to $\hat{\mathscr{F}}^2(p) = \mathscr{F}^{k_1+k_2+k_3+k_4}(p)$. Iterating this argument, we obtain that the forward orbit of q shadows the entire heteroclinic chain $\left(\mathscr{O}_{p\to\mathscr{F}(p)}, \mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \dots\right)$.

At this point, we have proved the first part of Theorem B, *i.e.* we have constructed a three-dimensional set of type IX orbits that shadow the heteroclinic chain $\left(\mathscr{O}_{p\to\mathscr{F}(p)}, \mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \dots\right)$ for every point p whose Kasner parameter satisfies the moderate growth condition (MG).

Absolute continuity of the local stable manifolds foliation. The second part of Theorem B is proven in Chapter 12. Namely, we consider a set $\mathscr{E}$ of positive one-dimensional Lebesgue measure in the Kasner circle, and we prove that the union of the type IX orbits shadowing a heteroclinic chain $\left(\mathscr{O}_{p\to\mathscr{F}(p)}, \mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \dots\right)$ with $p \in \mathscr{E}$ has positive four-dimensional Lebesgue measure in the phase space. Without loss of generality, one can assume that $\mathscr{E} \subset \mathscr{K}_{]1,2]}$. According to what has been explained above, it is enough to prove that the union of the local stable manifolds $W^s(p, \hat{\Phi})$ when p ranges over $\mathscr{E}$ has positive three-dimensional Lebesgue measure in the transverse section S.

Remark 1.11. Readers that are not familiar with non-uniformly hyperbolic dynamics might think that this is a straightforward consequence of the $W^s(p, \hat{\Phi})$'s being two-dimensional submanifolds which depend continuously on p. However, a continuous dependance is not sufficient to apply a Fubini type argument (recall that a continuous map might send positive Lebesgue measure sets to zero Lebesgue measure sets). Examples of non-uniformly hyperbolic dynamical systems with pathological local stable manifold “foliations” do exist.

We use a well-known strategy due to Pesin, which consists in considering the holonomy along the “foliation” in local stable manifolds, and proving that this holonomy is made of absolutely continuous maps. The absolute continuity of the holonomy maps follows from estimates on the action of these maps on the volume of discs transverse to the “foliation” in local stable manifolds. Such estimates are trivial for “big” discs. The trick is to turn small discs into big ones using the map $\hat{\Phi}$. Indeed the discs are transversal to the stable manifold, hence can be considered as unstable disc, and therefore, their images under $\hat{\Phi}^n$ become larger and larger as n goes to infinity.

Our setting is easier than the usual general setting because the “foliation" in stable manifolds is transversally one-dimensional, and therefore the discs transverse to the foliation are just arcs, whose volume can be computed easily (in particular, it roughly coincide with the diameter of these arcs). On the other hand, our setting is also more tricky because we have to work with a map $\hat{\Phi}$ which is not defined everywhere, so we have to be very careful when we consider large iterates of $\hat{\Phi}$ to make the discs grow.

Remark 1.12. One could be worried since it is well-known that Pesin's absolute continuity techniques only work for $C^{1+\alpha}$ maps, and since we have explained previously that our map $\hat{\Phi}$ is only Lipschitz. Actually, the $C^{1+\alpha}$-regularity is used for two purposes in Pesin's proof. First, to find some lower bounds for the size of the neighbourhoods of the points of the attractor where certain hyperbolicity estimates hold. We already have computed such sizes in the previous chapters. Second, to get some Hölder regularity on the unstable direction (tangent space to the attractor). In our case, this regularity is for free, since we know explicitly the attractor, and since the intersection of this attractor with the section S is extremely simple and regular: this is an affine interval in our local coordinates (which are at least C^4). Hence the low regularity of the map $\hat{\Phi}$ is not a true problem for this precise proof.

Statement of the main theorem in the full phase space. For sake of simplicity, we have stated Theorem B in the restricted phase space $\mathcal{B}^+$ (in particular, we have restricted ourselves to heteroclinic chains that can be shadowed by type IX orbits). Nevertheless, our proof also works in the full phase space, provided that we introduce a natural notion of *coherent heteroclinic chain*. The generalization of Theorem B to the full phase space $\mathscr{B}$ is stated in Appendix A.

Continued fractions. Some classical material about continued fractions and the Gauss map is gathered in Appendix B. This is also the place where we prove that the moderate growth condition is generic in the measure-theoretical sense.

A stable manifold theorem with parameters. As explained above, in order to construct a nice coordinates system in the neighbourhood of a point of the Kasner circle (Chapter 4), we need a version of the stable manifold theorem with some explicit estimates. This version is stated and proved in Appendix C.

Chapter 2

The vacuum Einstein field equations for Bianchi spacetimes

The purpose of this section is to explain how the Wainwright-Hsu equations (1.5) can be derived from the vacuum Einstein field equations (1.2) for class A Bianchi spacetimes. Moreover, we want to describe the correspondence between vacuum class A Bianchi spacetimes and solutions of (1.5). This chapter follows mostly [Rin13], hence we refer to this book for more details.

First, let us recall the definitions we are using in this work.

Definition 2.1 (Vacuum Bianchi spacetime)**.** A *vacuum Bianchi spacetime* is a Lorentzian manifold of the form $(M, g) = (I \times G, -ds^2 + h_s)$ satisfying the vacuum Einstein field equations

$$\mathrm{Ric}_g = 0$$

where I is an interval of the real line, G is a simply-connected 3-dimensional real Lie group, s is a coordinate on I and h_s is a left-invariant Riemannian metric on $\{s\} \times G \simeq G$ for every $s \in I$, such that

Definition 2.2. A Lie group is said to be *unimodular* if its left invariant Haar measure is also right-invariant. A Lie algebra $\mathfrak{g}$ is said to be *unimodular* if all the simply-connected Lie groups associated with $\mathfrak{g}$ are unimodular.

Definition 2.3 (Vacuum class A Bianchi spacetime)**.** A vacuum Bianchi spacetime

$$(M, g) = (I \times G, -\mathrm{d}s^2 + h_s)$$

is said to be *of class A* if the Lie group G is unimodular.

Definition 2.4 (Maximal vacuum class A Bianchi spacetime)**.** A vacuum class A Bianchi spacetime is *maximal* if it cannot be embedded isometrically as a strict submanifold of another vacuum class A Bianchi spacetime.

Throughout this work, we will focus on maximal vacuum class A Bianchi spacetimes, in the sense of the preceding definitions.

This section is divided in three parts. Firstly, we recall some general results on 3-dimensional Lie groups and their algebras. Secondly, we show that every vacuum class A Bianchi spacetime admits a particular frame field, called an *orthonormal-parallel frame*. Lastly, we explicit the coordinates for which the vacuum Einstein field equations (1.2) transcribe precisely as the Wainwright-Hsu equations (1.5).

2.1 3-dimensional unimodular real Lie groups and algebras

Proposition 2.5. *Let G be a connected Lie group whose Lie algebra is denoted by $\mathfrak{g}$. The Lie group G is unimodular if and only if, for all $x \in \mathfrak{g}$, the linear transformation $\mathrm{ad}(x) : y \mapsto [x, y]$ has trace zero.*

Proof. It is easy, see for example section 6 of Milnor's article [Mil76]. □

Definition 2.6 (Canonical orthonormal frame)**.** Let $\mathfrak{g}$ be a 3-dimensional unimodular real Lie algebra, h be a positive-definite quadratic form on $\mathfrak{g}$, $\langle\cdot,\cdot\rangle$ be the inner product associated with h and $B = (e_1, e_2, e_3)$ be an orthonormal frame of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. We say that B is a *canonical orthonormal frame* for $(\mathfrak{g}, h)$ if there exists a triple (n_1, n_2, n_3) of real numbers verifying

$$[e_1, e_2] = n_3 e_3, \quad [e_2, e_3] = n_1 e_1, \quad [e_3, e_1] = n_2 e_2$$

If this is the case, we will call the numbers n_1, n_2, n_3 (in that order) the *structure constants* for the frame B in the Lie algebra $\mathfrak{g}$.

Next proposition shows that any 3-dimensional unimodular real Lie algebra equipped with a positive-definite quadratic form admits a canonical frame that will greatly simplify computations with the Lie bracket.

Proposition 2.7. *If $\mathfrak{g}$ is a 3-dimensional unimodular real Lie algebra and if h is a positive-definite quadratic form on $\mathfrak{g}$, then there exists a canonical orthonormal frame B for $(\mathfrak{g}, h)$.*

Proof. We fix an orientation on $\mathfrak{g}$. We denote by $\langle\cdot,\cdot\rangle$ the inner product associated with h. Since $\mathfrak{g}$ is 3-dimensional and oriented, we can consider the cross product $(u, v) \mapsto u \times v$ on $\mathfrak{g}$ associated with the inner product $\langle\cdot,\cdot\rangle$. Let us fix (f_1, f_2, f_3) a direct orthonormal frame of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ and let us define the linear transformation $L : \mathfrak{g} \to \mathfrak{g}$ by $L(f_i) = [f_{i+1}, f_{i+2}]$ (where the index i is taken modulo 3). Since $f_i = f_{i+1} \times f_{i+2}$, we get that $L(u \times v) = [u, v]$ for all $u, v \in \mathfrak{g}$. Let B be an orthonormal frame of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. According to what precedes, B is a canonical orthonormal frame for $(\mathfrak{g}, h)$ if and only if B diagonalizes L. To conclude, we use Proposition 2.5 to get that $\mathrm{tr}(\mathrm{ad}(f_i)) = 0$ for all i. This implies that L is self-adjoint for the inner product $\langle\cdot,\cdot\rangle$. Hence, L is diagonalizable in an orthonormal frame for $\langle\cdot,\cdot\rangle$. □

Now let us discuss the "uniqueness" of the canonical orthonormal frame and the structure constants. In the preceding proof, we saw that the canonical orthonormal frames for $(\mathfrak{g}, h)$ are exactly the orthonormal frames which diagonalize the map L. Remark that changing the orientation amounts to consider the map $-L$ instead of L, hence we did not loose any canonical orthonormal frame by fixing the orientation in the preceding proof.

Let $B = (e_1, e_2, e_3)$ be an orthonormal frame which diagonalizes the map L. If the eigenvalues of L are pairwise distinct, any other orthonormal frame diagonalizing L can be constructed from B via the two following transformations:

- a permutation of the vectors e_1, e_2 and e_3.
- taking the opposite of some vectors from e_1, e_2 and e_3.

If some eigenvalues of L are equal, we can also allow the transformation of B by any isometry stabilizing the eigenspaces of L.

As a consequence, it is easy to check that the structure constants do not depend on the canonical orthonormal frame chosen, up to order and up to a sign reversal of the three n_i's.

According to the preceding discussion, one can choose B such that at most one of the structure constants for B is strictly negative. The signs of the real numbers n_1, n_2, n_3, with this convention and considered modulo permutation of the indices, are an invariant of the 3-dimensional unimodular real Lie algebras modulo isomorphism. This classification is summarized in the table 2.1.

Definition 2.8 (Canonical orthonormal frame field)**.** Let h be a left-invariant riemannian metric on a 3-dimensional unimodular real Lie group G, of Lie algebra $\mathfrak{g}$. This metric induces a positive-definite quadratic form on $\mathfrak{g}$, that we also denote by h. Let (e_1, e_2, e_3) be a left-invariant orthonormal frame field on (G, h). We say that (e_1, e_2, e_3) is a *canonical orthonormal frame field* for (G, h) if $(e_1, e_2, e_3)(\mathrm{Id})$ is a canonical orthonormal frame for $(\mathfrak{g}, h)$ (in the sense of definition 2.6).

Remark 2.9. Any canonical orthonormal frame for $(\mathfrak{g}, h)$ induces a canonical orthonormal frame field for (G, h).

Bianchi type	Signs of n_1, n_2, n_3 modulo permutation of the indices	Corresponding Lie algebra up to isomorphism
I	$0, 0, 0$	$\mathbb{R}^3$
II	$+, 0, 0$	Heisenberg's algebra
VI_0	$+, -, 0$	$\mathfrak{isom}(\mathrm{Min}_2)$
VII_0	$+, +, 0$	$\mathfrak{isom}(\mathbb{R}^2)$
VIII	$+, +, -$	$\mathfrak{sl}(2, \mathbb{R})$
IX	$+, +, +$	$\mathfrak{su}(2)$

Table 2.1 – Classification of 3-dimensional unimodular Lie algebras.

Remark 2.10. If (e_1, e_2, e_3) is a canonical orthonormal frame field for (G, h), then for all $i \in \{1, 2, 3\}$,

$$[e_i, e_{i+1}] = n_{i+2} e_{i+2} \qquad \text{on } G$$

where n_1, n_2, n_3 are the structure constants for the frame $(e_1, e_2, e_3)(\mathrm{Id})$ in the algebra $\mathfrak{g}$. We will say that n_1, n_2, n_3 (in that order) are the structure constants for the frame field (e_1, e_2, e_3).

Next proposition shows how the Ricci curvature can be expressed with the structure constants in a canonical orthonormal frame field.

Proposition 2.11. *Let h be a left-invariant riemannian metric on a 3-dimensional unimodular real Lie group G. Let (e_1, e_2, e_3) be a canonical orthonormal frame field for (G, h). The Ricci curvature and the scalar curvature read as follows:*

$$\mathrm{Ric}_h(e_i, e_j) = 0 \qquad \textit{if } i \neq j \tag{2.1a}$$

$$\mathrm{Ric}_h(e_i, e_i) = \frac{1}{2}(n_i^2 - n_j^2 - n_k^2) + n_j n_k \qquad \textit{where } \{i, j, k\} = \{1, 2, 3\} \tag{2.1b}$$

$$\mathrm{Scal}_h = -\frac{1}{2}(n_1^2 + n_2^2 + n_3^2) + n_1 n_2 + n_2 n_3 + n_3 n_1 \tag{2.1c}$$

where n_1, n_2, n_3 are the structure constants for the frame B.

Proof of Proposition 2.11. Let $\langle \cdot, \cdot \rangle$ be the inner product associated with h. Let x, y, z be three left-invariant vector fields on G. When used with left-invariant vector fields, Koszul's formula on the Levi-Civita connexion ∇ simplifies as follows:

$$\langle \nabla_x y, z \rangle = \frac{1}{2}\left(-\langle x, [y, z]\rangle - \langle y, [x, z]\rangle + \langle z, [x, y]\rangle\right) \tag{2.2}$$

Coming back to the canonical orthonormal frame (e_1, e_2, e_3), we deduce from this formula that

$$\nabla_{e_1} e_2 = \alpha_1 e_3, \qquad \nabla_{e_2} e_3 = \alpha_2 e_1, \qquad \nabla_{e_3} e_1 = \alpha_3 e_2 \tag{2.3}$$

where $\alpha_i := \frac{1}{2}(n_1 + n_2 + n_3) - n_i$. Since ∇ is torsion-free, ∇ is entirely determined by (2.3). Moreover, the curvature tensor is entirely determined by the connexion, hence formulas (2.1) are easily obtained from (2.3). □

2.2 Canonical orthonormal-parallel frame fields on vacuum class A Bianchi spacetimes

Let $(M, g) = (I \times G, -\mathrm{d}s^2 + h_s)$ be a Lorentzian manifold where I is an interval of the real line, G is a simply-connected 3-dimensional real Lie group of class A, s is a coordinate on I and h_s is a left-invariant Riemannian metric on $\{s\} \times G \simeq G$ for every $s \in I$. Let $\langle \cdot, \cdot \rangle$ be the bilinear form associated with g and ∇ be the Levi-Civita connection of g. We define the second fundamental form of the spacelike hypersurface $\{s\} \times G$ by $\mathrm{II}_s(x, y) := \langle \nabla_x \frac{\partial}{\partial s}, y \rangle$. As we already stated, we are going to choose a coordinates system on $M = I \times G$ and use it to explicit the vacuum Einstein field equations $\mathrm{Ric}_g = 0$. This amounts to choose a frame field.

Definition 2.12 (Canonical orthonormal-parallel frame field)**.** We call *canonical orthonormal-parallel frame field* on $(I \times G, -\mathrm{d}s^2 + h_s)$ any frame field (e_0, e_1, e_2, e_3) satisfying the following properties:

1. The vector field e_0 is equal to $\frac{\partial}{\partial s}$. The vector fields e_1, e_2, e_3 are tangent to the spacelike hypersurface $\{s\} \times G$ for all $s \in I$, and are left-invariant.
2. In restriction to $\{s\} \times G$, (e_1, e_2, e_3) is a canonical orthonormal frame field for (G, h_s) (in the sense of definition 2.8).
3. For all $i \in \{1, 2, 3\}$, we have $\nabla_{e_0} e_i = 0$.
4. The second fundamental form II_s of the spacelike hypersurface $\{s\} \times G$ is diagonalized in the frame $(e_1, e_2, e_3)(s)$ for all $s \in I$. We denote by $\theta_1(s), \theta_2(s), \theta_3(s)$ the diagonal coefficients of II_s.

Remark 2.13. Any canonical orthonormal-parallel frame field (e_1, e_2, e_3) carries six variables: the structure constants $n_1(s), n_2(s), n_3(s)$ and the coefficients of the second fundamental form $\theta_1(s), \theta_2(s), \theta_3(s)$. Moreover, in this frame, the second fundamental form depends only on the θ_i. We will say that (n_i, θ_i) are the variables associated with the frame field (e_1, e_2, e_3).

Next proposition gives a necessary and sufficient condition for a Lorentzian manifold of the form $(I \times G, -\mathrm{d}s^2 + h_s)$ to be a vacuum class A Bianchi spacetime, using the notion of canonical orthonormal-parallel frame field.

Proposition 2.14. *Let $(M, g) = (I \times G, -\mathrm{d}s^2 + h_s)$ be a Lorentzian manifold where I is an interval of the real line, G is a simply-connected 3-dimensional real Lie group of class A, s is a coordinate on I and h_s is a left-invariant Riemannian metric on $\{s\} \times G \simeq G$ for every $s \in I$.*

If (M, g) is a vacuum class A Bianchi spacetime, then there exists a canonical orthonormal-parallel frame field (e_0, e_1, e_2, e_3) on (M, g) and the variables (n_i, θ_i) associated with this frame field satisfy the three equations

$$\frac{\mathrm{d}n_i}{\mathrm{d}s} = (\theta_i - \theta_j - \theta_k) n_i \tag{2.4a}$$

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}s} = -\frac{1}{2}(n_i^2 - n_j^2 - n_k^2) - n_j n_k - \theta_i \theta \tag{2.4b}$$

$$0 = \frac{1}{2}(n_1^2 + n_2^2 + n_3^2) - (n_1 n_2 + n_1 n_3 + n_2 n_3) - \theta^2 + \theta_1^2 + \theta_2^2 + \theta_3^2 \tag{2.4c}$$

where $\theta = \theta_1 + \theta_2 + \theta_3$ and $\{i, j, k\} = \{1, 2, 3\}$. Moreover, if $(\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ is another canonical orthonormal-parallel frame field, the variables $(\tilde{n}_i, \tilde{\theta}_i)$ associated with this frame satisfy

$$\tilde{n}_i = \epsilon n_{\sigma(i)}, \qquad \tilde{\theta}_i = \theta_{\sigma(i)}, \qquad i \in \{1, 2, 3\} \tag{2.5}$$

for some $\sigma \in \mathfrak{S}_3$ and some $\epsilon \in \{1, -1\}$.

Conversely, if there exists a canonical orthonormal-parallel frame field (e_0, e_1, e_2, e_3) on (M, g) such that the variables (n_i, θ_i) associated with this frame field satisfy the system (2.4)*, then (M, g) is a vacuum class A Bianchi spacetime.*

Remark 2.15. The variables (n_i, θ_i) are called the Ellis-MacCallum coordinates.

Remark 2.16. If there exist $s_0 \in I$ and $i \neq j$ such that $n_i(s_0) = n_j(s_0)$ and $\theta_i(s_0) = \theta_j(s_0)$ then $n_i(s) = n_j(s)$ and $\theta_i(s) = \theta_j(s)$ for all $s \in I$. In this case, we say that (M, g) is a *locally rotationally symmetric* (LRS) Bianchi spacetime. Any rotation in the plane spanned by $e_i(s_0)$ and $e_j(s_0)$ gives rise to another canonical orthonormal-parallel frame field. If this is not the case, the canonical orthonormal-parallel frame field is essentially unique, as discussed after Proposition 2.7. Indeed, remark that a canonical orthonormal-parallel frame field is entirely defined by the choice of one canonical orthonormal frame for $(\mathfrak{g}, h_{s_0})$ where $s_0 \in I$ is fixed.

Remark 2.17. One can check that the constraint equation (2.4c) is invariant under the flow of the differential equations (2.4a) and (2.4b).

Remark 2.18. Equation (2.4a) implies that the signs of the variables n_i are constant. This is of course coherent with the fact that these signs (modulo permutation and up to a simultaneous sign reversal of the three n_i's) are invariant under Lie algebra's isomorphisms, as discussed after Proposition 2.7 (see table 2.1).

We stress that in this proposition, we will be using the vacuum Einstein field equations $\mathrm{Ric}_g = 0$ to construct the canonical orthonormal-parallel frame field.

Proof of Proposition 2.14. Let $e_0 = \frac{\partial}{\partial s}$ and fix $s_0 \in I$. We will begin with the construction of the vector fields e_1, e_2, e_3 in restriction to the spacelike hypersurface $\{s_0\} \times G$. After that, we will extend these vector fields on $I \times G$ by parallel transport. Once this is done, we will only have to check that the frame field (e_0, e_1, e_2, e_3) is a canonical orthonormal-parallel frame field.

Step 1: construction of the vector fields e_1, e_2, e_3 in restriction to the spacelike hypersurface $\{s_0\} \times G$. We fix a left-invariant orthonormal frame field $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ on $(\{s_0\} \times G \simeq G, h_{s_0})$. We denote by $(\hat{\theta}_{i,j})$ the (symmetrical) matrix of the second fundamental form of $\{s_0\} \times G$ in this frame field, that is,

$$\hat{\theta}_{i,j} = \langle \nabla_{\hat{e}_i} e_0, \hat{e}_j \rangle$$

We introduce the (symmetrical) matrix $(\hat{n}_{i,j})$ defined by

$$\hat{n}_{i,j} \stackrel{\text{def}}{=} \sum_{k,\ell=1}^{3} \epsilon(j,k,\ell)\hat{C}^i_{k,\ell} + \epsilon(i,k,\ell)\hat{C}^j_{k,\ell} \tag{2.6}$$

where $\hat{C}^k_{i,j} := \langle [\hat{e}_i, \hat{e}_j], \hat{e}_k \rangle$ and $\epsilon(j,k,\ell)$ is the signature of the permutation $(1,2,3) \mapsto (j,k,\ell)$ if $\{j,k,\ell\} = \{1,2,3\}$ and is zero if two indices among j,k,ℓ coincide. Let us call $(\hat{n}_{i,j})$ the *commutator matrix* associated with the frame field $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$.

Claim 1. *The matrix $\hat{n} = (\hat{n}_{i,j})$ is the unique matrix such that*

$$\hat{C}^i_{k,\ell} = \frac{1}{4} \sum_{m=1}^{3} \epsilon(k,\ell,m)\hat{n}_{m,i} \tag{2.7}$$

for all $\{k,\ell,i\} = \{1,2,3\}$. Moreover, if $(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$ is another left-invariant orthonormal frame field on $(\{s_0\} \times G \simeq G, h_{s_0})$ and A is the orthogonal matrix such that

$$\hat{e}'_i = \sum_{m=1}^{3} A_{m,i}\hat{e}_m$$

then the commutator matrix $\hat{n}' = (\hat{n}'_{i,j})$ associated with the frame field $(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$ satisfies the relation

$$\hat{n} = (\det A)^{-1}\, {}^tA\, \hat{n}' A \tag{2.8}$$

Proof of claim 1. See [Rin13], Lemma 19.3, p. 206 and Lemma 19.6, p. 207. The main ingredient for (2.7) is the fact that G is unimodular. It is used in the form given by Proposition 2.5: $\mathrm{tr}(\mathrm{ad}(\hat{e}_i)) = 0$ for all i. □

Claim 2. *The matrix $(\hat{n}_{i,j})$ is diagonal if and only if $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is a canonical orthonormal frame field for (G, h_{s_0}).*

Proof of claim 2. Assume that $(\hat{n}_{i,j})$ is diagonal. Using (2.7), one can see that $\hat{C}^i_{k,\ell} = 0$ whenever $i = k$ or $i = \ell$. Hence, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is a canonical orthonormal frame field for (G, h_{s_0}). Now assume that $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is a canonical orthonormal frame field for (G, h_{s_0}). If $i \neq j$, either $\epsilon(j,k,\ell)$ or $\hat{C}^i_{k,\ell}$ vanishes, hence $\hat{n}_{i,j} = 0$ by (2.6). □

Claim 3. *The matrices $(\hat{n}_{i,j})$ and $(\hat{\theta}_{i,j})$ commute.*

Proof of claim 3. This is a consequence of the fact that $\mathrm{Ric}_g(e_0, \hat{e}_l) = 0$ for all $l \in \{1,2,3\}$. Let us develop the main arguments. As in the proof of Proposition 2.11, Koszul's formula (2.2) gives

$$2\langle \nabla_{\hat{e}_l} \hat{e}_m, \hat{e}_k \rangle = \hat{C}^k_{l,m} - \hat{C}^l_{m,k} - \hat{C}^m_{l,k} \tag{2.9}$$

Moreover, by developing the relation

$$\mathrm{Ric}_g(e_0, \hat{e}_l) = -\langle R^g_{e_0,e_0} \hat{e}_l, e_0 \rangle + \sum_{k=1}^{3} \langle R^g_{\hat{e}_k,e_0} \hat{e}_l, \hat{e}_k \rangle$$

while using the fact that $\nabla_{e_0} e_0 = 0$ and the definition of the second fundamental form, we find that

$$\mathrm{Ric}_g(e_0, \hat{e}_l) = \sum_{k=1}^{3} \sum_{m=1}^{3} \left(-\theta_{k,m} \langle \nabla_{\hat{e}_l} \hat{e}_m, \hat{e}_k \rangle + \theta_{l,m} \langle \nabla_{\hat{e}_k} \hat{e}_m, \hat{e}_k \rangle + \hat{C}^m_{l,k} \theta_{m,k} \right) \tag{2.10}$$

Plugging (2.9) into (2.10) and using the definition of $(\hat{n}_{i,j})$, we can see that $\mathrm{Ric}_g(e_0, \hat{e}_l) = 0$ translates as the commutation of the matrices $(\hat{n}_{i,j})$ and $(\hat{\theta}_{i,j})$. □

As a consequence, one can diagonalize the matrices $(\hat{n}_{i,j})$ and $(\hat{\theta}_{i,j})$ in the same left-invariant orthonormal frame field (e_1, e_2, e_3). According to (2.8), the commutator matrix associated with the frame field (e_1, e_2, e_3) is also diagonal. Hence, claim 2 implies that (e_1, e_2, e_3) is a canonical orthonormal frame field for (G, h_{s_0}).

Step 2: extension of the frame field (e_1, e_2, e_3) to $I \times G$. We extend this frame field on $M = I \times G$ by parallel transport along the orbits of the vector field e_0. By definition, the frame field (e_0, e_1, e_2, e_3) satisfies items 1 and 3 of definition 2.12 on (M, g). It also satisfies items 2 and 4 in restriction to the spacelike hypersurface $\{s_0\} \times G$. Moreover, recall that the inner product is invariant under parallel transport. Hence, for all $s \in I$, $(e_1, e_2, e_3)(s)$ is a left-invariant orthonormal frame field on $(\{s\} \times G, h_s)$.

Step 3: the frame (e_0, e_1, e_2, e_3) satisfies items 3 and 4 of definition 2.12. We generalize the precedent matrices. Let us denote by $(\theta_{i,j}(s))$ the matrix of the second fundamental form of the spacelike hypersurface $\{s\} \times G$ in the frame $(e_1, e_2, e_3)(s)$. Consider the matrix $(n_{i,j}(s))$ defined by

$$n_{i,j}(s) \overset{\mathrm{def}}{=} \sum_{k,\ell=1}^{3} \epsilon(j,k,\ell) C^i_{k,\ell}(s) + \epsilon(i,k,\ell) C^j_{k,\ell}(s)$$

where $C^k_{i,j}(s) := \langle [e_i, e_j], e_k \rangle|_{\{s\}\times G}$.

Claim 4. *For all $i, j \in \{1,2,3\}$, the variable $n_{i,j}$ satisfies the differential equation*

$$\frac{\mathrm{d}n_{i,j}}{\mathrm{d}s} = -\theta n_{i,j} + \sum_{l=1}^{3} \theta_{i,l} n_{l,j} + \theta_{l,j} n_{i,l} \tag{2.11}$$

Proof of claim 4. Using $\nabla_{e_0} e_i = 0$, we get that

$$\frac{\mathrm{d}C^i_{k,l}}{\mathrm{d}s} = \langle \nabla_{e_0} [e_k, e_l], e_i \rangle$$

Using the fact that ∇ is torsion-free, the Jacobi identity, the definition of the second fundamental form and once again the fact that the vector fields e_i are parallel transported along the orbits of the vector field e_0, we obtain

$$\frac{\mathrm{d}C^i_{k,l}}{\mathrm{d}s} = \sum_{m=1}^{3} \left(C^m_{k,l} \theta_{m,i} + C^i_{l,m} \theta_{k,m} - C^i_{k,m} \theta_{l,m} \right)$$

Using the definition of $n_{i,j}$ and some painful algebraic manipulations, we find (2.11). □

Claim 5. *For all $i, j \in \{1,2,3\}$, the variable $\theta_{i,j}$ satisfies the differential equation*

$$\frac{\mathrm{d}\theta_{i,j}}{\mathrm{d}s} = -\operatorname{Ric}_{h_s}(e_i, e_j) - \theta_{i,j}\theta \tag{2.12}$$

where $\theta = \operatorname{tr}(\mathrm{II}_s) = \theta_{1,1} + \theta_{2,2} + \theta_{3,3}$.

Proof of claim 5. Using $\nabla_{e_0} e_i = 0$ (for $i \in \{1,2,3,4\}$), the definition of the curvature tensor and the fact that ∇ is torsion-free, we get

$$\frac{\mathrm{d}\theta_{i,j}}{\mathrm{d}s} = \langle R^g_{e_0,e_i} e_0, e_j\rangle - \sum_{m=1}^{3} \theta_{i,m}\theta_{m,j}$$

The equation $\operatorname{Ric}_g(e_i, e_j) = 0$ allows one to express $\langle R^g_{e_0,e_i} e_0, e_j\rangle$ independently of e_0:

$$\langle R^g_{e_0,e_i} e_0, e_j\rangle = -\sum_{k=1}^{3} \langle R^g_{e_k,e_i} e_j, e_k\rangle$$

Moreover, Gauss formula about the second fundamental form allows one to express the curvature tensor R^g with the curvature tensor of the spacelike hypersurface R^{h_s} and its second fundamental form:

$$\forall k \geq 1, \qquad \langle R^g_{e_k,e_i} e_j, e_k\rangle = \langle R^{h_s}_{e_k,e_i} e_j, e_k\rangle - \theta_{k,j}\theta_{i,k} + \theta_{k,k}\theta_{i,j}$$

With these relations, it is easy to finish the computation of $\frac{\mathrm{d}\theta_{i,j}}{\mathrm{d}s}$ and to find (2.12). □

Recall that $n_{i,j}(s_0) = 0 = \theta_{i,j}(s_0)$ for all $i \neq j$. Moreover, $\operatorname{Ric}_{h_s}(e_i, e_j)$ admits an expression in function of the coefficients of the matrix $(n_{i,j})$ and vanishes when this matrix is diagonal (see [Rin13], Lemma 19.11, p. 209). Equations (2.11) and (2.12) form an ODE system of order 1. Since

$$\forall i \neq j, n_{i,j} \equiv \theta_{i,j} \equiv 0$$

is a solution of this ODE system, Cauchy-Lipschitz theorem implies that for all $i \neq j$ and all $s \in I$, $n_{i,j}(s) = 0 = \theta_{i,j}(s)$. As a consequence, items 2 and 4 of definition 2.12 hold true (using claim 2).

Step 4: equations satisfied by the variables (n_i, θ_i). According to (2.7), $n_{i,i} = 4n_i$. Recall that $n_{i,j} = 0$ and $\theta_{i,j} = 0$ whenever $i \neq j$. Hence, (2.11) implies that

$$\frac{\mathrm{d}n_i}{\mathrm{d}s} = -\theta n_i + 2\theta_i n_i = (\theta_i - \theta_j - \theta_k)n_i$$

i.e. (2.4a) holds true. By definition, $\theta_i = \theta_{i,i}$. Hence, equation (2.4b) is a direct consequence of (2.12) and the expression of the Ricci curvature (2.1b).

According to Gauss formula about the second fundamental form, the equation $\operatorname{Ric}_g(e_0, e_0) = 0$ translates as

$$\operatorname{Scal}_{h_s} + (\operatorname{tr}\mathrm{II}_s)^2 - \operatorname{tr}(\mathrm{II}_s^2) = 0 \tag{2.13}$$

where II_s^2 is an abuse of notation for the quadratic form whose matrix is the square of the matrix of the quadratic form II_s in the frame $(e_1, e_2, e_3)(s)$. Plugging the expression of the scalar curvature (2.1c) into (2.13), we get (2.4c).

Step 5: change of canonical orthonormal-parallel frame field. Formula (2.5) describing how the variables (n_i, θ_i) change if one changes the canonical orthonormal-parallel frame field follows from the discussion about the "uniqueness" of a canonical orthonormal frame for $(\mathfrak{g}, h_{s_0})$.

Step 6: proof of the converse statement. Assume that there exists a canonical orthonormal-parallel frame field (e_0, e_1, e_2, e_3) on (M, g) such that the variables (n_i, θ_i) associated with this frame field satisfy the system (2.4). The goal is to prove that $\operatorname{Ric}_g = 0$. This is done in three computations:

- The matrices $(n_{i,j})$ and $(\theta_{i,j})$ are diagonal, hence they commute. Careful reading of the computations done in the proof of claim 3 show that $\operatorname{Ric}_g(e_0, e_l) = 0$ for all $l \in \{1,2,3\}$.

- According to (2.1a), $\mathrm{Ric}_{h_s}(e_i, e_j) = 0$ for all $i \neq j$. Moreover, the matrices $(n_{i,j})$ and $(\theta_{i,j})$ are diagonal, hence $n_{i,j} = 0 = \theta_{i,j}$ for all $i \neq j$. As a consequence, equation (2.12) holds for $i \neq j$. According to (2.1b) and (2.4b), equation (2.12) also hold for $i = j$. Careful reading of the computations done in the proof of claim 5 show that $\mathrm{Ric}_g(e_i, e_j) = 0$ for all $i, j \in \{1, 2, 3\}$.
- Using the expression of the scalar curvature (2.1c), we get that $\mathrm{Ric}_g(e_0, e_0) = 0$ is equivalent to the constraint equation (2.4c).

□

Proposition 2.14 shows that we can associate to any vacuum class A Bianchi spacetime a solution of the system (2.4), well defined modulo the symmetry given by (2.5). Now we want to explain how we can do the converse, that is, to construct a vacuum class A Bianchi spacetime from a solution of the system (2.4). This will imply that there is a one-to-one correspondence between maximal vacuum class A Bianchi spacetimes and maximal solutions of the system (2.4), modulo some simple symmetries.

Definition 2.19 (Isomorphism of maximal vacuum class A Bianchi spacetimes)**.** Let

$$(M = I \times G, g = -\mathrm{d}s^2 + h_s), \qquad (\tilde{M} = \tilde{I} \times \tilde{G}, \tilde{g} = -\mathrm{d}\tilde{s}^2 + \tilde{h}_s)$$

be two maximal vacuum class A Bianchi spacetimes. We say that (M, g) and $(\tilde{M}, \tilde{g})$ are *isomorphic* if

- $I = \tilde{I}$ and $s = \tilde{s}$.
- There exists a global isometry $\psi : (M, g) \to (\tilde{M}, \tilde{g})$ (*i.e.* a diffeomorphism such that $\psi_* g = \tilde{g}$) such that for all $(s, u) \in M = I \times G$, $\psi(s, u) = (s, \varphi(u))$ where $\varphi : G \to \tilde{G}$ is a Lie group isomorphism.

If this is the case, we say that ψ is an *isomorphism* between (M, g) and $(\tilde{M}, \tilde{g})$.

Remark 2.20. For "generic" maximal vacuum class A Bianchi spacetimes (without too many symmetries), any global isometry $\psi : (M, g) \to (\tilde{M}, \tilde{g})$ should be of the form described in definition 2.19, up to a translation in time.

Proposition 2.21. *Let $s \mapsto (n_i(s), \theta_i(s))$ be a maximal solution of the system* (2.4)*, defined on an interval I. There exists a maximal vacuum class A Bianchi spacetime $(M = I \times G, g)$ and a canonical orthonormal-parallel frame field (e_0, e_1, e_2, e_3) on (M, g) such that the variables associated with this frame are exactly the functions n_i and θ_i. Moreover, this maximal vacuum class A Bianchi spacetime is unique up to isomorphism (in the sense of definition 2.19).*

Proof. Let $s \mapsto (n_i(s), \theta_i(s))$ be a maximal solution of the system (2.4), defined on an interval I. Fix $s_0 \in I$. We construct a Lie bracket on $\mathbb{R}^3$ as follows: denote by (f_1, f_2, f_3) the canonical frame of $\mathbb{R}^3$ and define $[f_i, f_{i+1}] = n_{i+2}(s_0) f_{i+2}$ for $i \in \{1, 2, 3\}$ (the indices $i+1$ and $i+2$ are taken modulo 3). This defines a real Lie algebra structure on $\mathbb{R}^3$. According to Lie's third theorem, there exists a 3-dimensional simply-connected real Lie group G whose Lie algebra is $(\mathbb{R}^3, [.,.])$ defined above. The frame (f_1, f_2, f_3) induces a left-invariant frame field on G, that we still denote by (f_1, f_2, f_3). Let $M = I \times G$ and equip M with the frame field $(\frac{\partial}{\partial s}, f_1, f_2, f_3)$, *i.e.* we choose a frame field which does not depend on s.

For $i \in \{1, 2, 3\}$, we define the function $a_i : I \to \mathbb{R}$ by the formula

$$a_i(s) = e^{\int_{s_0}^{s} \theta_i(u)\,\mathrm{d}u}$$

Using the equation (2.4a) governing the evolution of the variables n_i, we easily obtain the relation

$$\frac{a_k(s)}{a_i(s) a_j(s)} n_k(s_0) = n_k(s) \tag{2.14}$$

for $\{i, j, k\} = \{1, 2, 3\}$ and $s \in I$. Now consider the metric

$$g = -\mathrm{d}s^2 + \sum_{i=1}^{3} a_i(s)^2 \mathrm{d}f_i^* \otimes \mathrm{d}f_i^* = -\mathrm{d}s^2 + h_s$$

where (f_1^*, f_2^*, f_3^*) is the dual frame of (f_1, f_2, f_3). Remark that g is well defined even if some of the variables n_i vanish. This is the reason why we used the coefficient $a_i(s)^2$ instead of $\frac{n_j(s_0)n_k(s_0)}{n_j(s)n_k(s)}$. The family (f_1, f_2, f_3) is orthogonal for this metric and even orthonormal in the spacelike hypersurface $\{s_0\} \times G$.

We construct a new frame $(e_0 = \frac{\partial}{\partial s}, e_1, e_2, e_3)$ on M as follows. On the spacelike hypersurface $\{s\} \times G$, we set $e_i = a_i(s)^{-1} f_i$. By construction, (e_0, e_1, e_2, e_3) is an orthonormal frame on (M, g). Using equation (2.14), it is easy to check that in restriction to the spacelike hypersurface $\{s\} \times G$, $[e_i, e_{i+1}] = n_{i+2}(s) e_{i+2}$, hence (e_1, e_2, e_3) is a canonical orthonormal frame field for (G, h_s). Using Koszul's formula, we get that the second fundamental form II_s is diagonal in the frame (e_1, e_2, e_3), with diagonal coefficients being exactly the numbers $\theta_1(s)$, $\theta_2(s)$ and $\theta_3(s)$. Using the fact that the Levi-Civita connexion is torsion-free, we get that

$$\nabla_{f_i} e_0 = \nabla_{e_0} f_i = \theta_i a_i e_i + a_i \nabla_{e_0} e_i$$

According to Koszul's formula,

$$\langle \nabla_{f_i} e_0, f_i \rangle = \theta_i a_i^2$$

and if $i \neq j$, then $\langle \nabla_{f_i} e_0, f_j \rangle = 0$. It follows that $\nabla_{f_i} e_0 = \theta_i f_i = \theta_i a_i e_i$. Hence, $\nabla_{e_0} e_i = 0$, *i.e.* the frame field (e_1, e_2, e_3) is parallel. As a consequence, (e_0, e_1, e_2, e_3) is a canonical orthonormal-parallel frame field on (M, g) and the variables associated with this frame field are exactly the (n_i, θ_i) which satisfy the system (2.4) by definition. We can use Proposition 2.14 to conclude that $\mathrm{Ric}_g = 0$. Hence, (M, g) is a vacuum class A Bianchi spacetime and (e_0, e_1, e_2, e_3) is a canonical orthonormal-parallel frame field on (M, g) such that the variables associated with this frame are exactly the functions n_i and θ_i.

It is easy to see that the solution (M, g) is maximal using the maximality of the solution

$$s \mapsto (n_i(s), \theta_i(s))$$

Assume that $(\tilde{M} = I \times \tilde{G}, \tilde{g} = -\mathrm{d}s^2 + \tilde{h}_s)$ is a maximal vacuum class A Bianchi spacetime equipped with a canonical orthonormal-parallel frame field $(e_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ such that the variables associated with this frame are exactly the functions n_i and θ_i. Define the Lie algebra isomorphism $\phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ by $\phi(e_i(s_0)) = \tilde{e}_i(s_0)$. This Lie algebra isomorphism arises from a Lie group isomorphism $\varphi : G \to \tilde{G}$ (see *e.g.* [Lee03], Theorem 20.15, p. 532). Define $\psi : M \to \tilde{M}$ by the formula $\psi(s, u) = (s, \varphi(u))$. The map ψ is the desired isomorphism if and only if $\phi(e_i(s)) = \tilde{e}_i(s)$ for all $s \in I$. Recall that by definition, $e_i(s) = a_i(s)^{-1} e_i(s_0)$. Hence, we only need to prove that $\tilde{e}_i$ satisfies the same relation. Let $\tilde{f}_i(s) := a_i(s)\tilde{e}_i(s)$. Using the fact that ∇ is torsion-free, the relation $\nabla_{e_0} \tilde{e}_i = 0$ and the relation $\nabla_{\tilde{e}_i} e_0 = \theta_i \tilde{e}_i$, we get that

$$[e_0, \tilde{f}_i] = 0$$

Hence, $\tilde{f}_i$ does not depend on s: for all $s \in I$, $\tilde{f}_i(s) = \tilde{f}_i(s_0) = a_i(s_0)\tilde{e}_i(s_0) = \tilde{e}_i(s_0)$. This concludes the proof of Proposition 2.21. □

Let us summarize the results of this section.

Proposition 2.22. *There is a one-to-one correspondence between*

- *Maximal vacuum class A Bianchi spacetimes modulo isomorphism (in the sense of definition 2.19).*
- *Maximal solutions (n_i, θ_i) of the system of equations* (2.4) *modulo permutation of the indices and a simultaneous sign reversal of the three n_i's (see* (2.5)*).*

Remark 2.23. A translation in time for a maximal solution of the system (2.4) amounts to the same translation in time for the corresponding maximal vacuum class A Bianchi spacetime.

2.3 Some properties of the mean curvature

Let (M, g) be a maximal vacuum class A Bianchi spacetime. Let (e_0, e_1, e_2, e_3) be a canonical orthonormal-parallel frame field on (M, g). The *mean curvature* of the spacelike hypersurface $\{s\} \times G$ is $\frac{1}{3}\theta(s)$ where

$$\theta(s) = \mathrm{tr}(\mathrm{II}_s) = \theta_1(s) + \theta_2(s) + \theta_3(s)$$

Remark that θ is independent of the choice of the canonical orthonormal-parallel frame field. Let us consider three different properties for θ:

(Z_∞) For all $s \in I$, $\theta(s) = 0$.

(Z_1) There exists a unique $s_0 \in I$ such that $\theta(s_0) = 0$.

(Z_0) For all $s \in I$, $\theta(s) \neq 0$.

Next proposition shows that the set of vacuum class A Bianchi spacetimes is (non trivially) partitioned by these three properties.

Proposition 2.24. *Let $(M =]s_-, s_+[\times G, g)$ be a maximal vacuum class A Bianchi spacetime. The mean curvature must satisfy one of the three properties (Z_∞), (Z_1), (Z_0). More precisely,*

- *The mean curvature satisfies (Z_∞) if and only if (M, g) is the Minkowski spacetime.*
- *The mean curvature satisfies (Z_1) if and only if the Lie algebra of G is of type* IX. *In that case, there exists a unique $s_0 \in I$ such that $\theta > 0$ in $]s_-, s_0[$ and $\theta < 0$ in $]s_0, s_+[$. Moreover, θ is decreasing, $s_- > -\infty$, $s_+ < +\infty$ and*

$$\lim_{s \to s_-} \theta(s) = +\infty, \qquad \lim_{s \to s_+} \theta(s) = -\infty$$

- *The mean curvature satisfies (Z_0) if and only if (M, g) is not the Minkowski spacetime and the Lie algebra of G is not of type* IX. *If the time orientation is chosen so that $\theta < 0$ (anti-physical time orientation), then $s_- = -\infty$ and $s_+ < +\infty$. Moreover, θ is decreasing and*

$$\lim_{s \to s_-} \theta(s) = 0, \qquad \lim_{s \to s_+} \theta(s) = -\infty$$

Remark 2.25. Changing the time orientation amounts to consider the spacetime

$$(\tilde{M} = \tilde{I} \times G, \tilde{g} = -\mathrm{d}\tilde{s}^2 + \tilde{h}_{\tilde{s}})$$

where $\tilde{I} = -I$, $\tilde{s}$ is the coordinate $-s$ on $\tilde{I}$ and $\tilde{h}_{\tilde{s}} = h_{-\tilde{s}}$ for all $\tilde{s} \in \tilde{I}$.

Proof. See [Rin13], Lemma 20.6, p. 218 and Lemma 20.9, p. 220. For the type IX case, see [LW90]. □

Definition 2.26. Let $(M =]s_-, s_+[\times G, g)$ be a maximal vacuum class A Bianchi spacetime of type IX (*i.e.* the Lie algebra of G is of type IX). Let s_0 be the unique time such that $\theta > 0$ in $]s_-, s_0[$ and $\theta < 0$ in $]s_0, s_+[$. We say that $\left(M_- =]s_-, s_0[\times G, g_{|M_-}\right)$ and $\left(M_+ =]s_0, s_+[\times G, g_{|M_+}\right)$ are *maximal half vacuum class A Bianchi spacetimes of type* IX.

Remark 2.27. The set of maximal half vacuum class A Bianchi spacetimes of type IX is invariant under metric rescaling, time orientation reversal, time translation and isomorphism.

2.4 Wainwright-Hsu coordinates and Wainwright-Hsu equations

In the preceding section, we described a correspondence between maximal vacuum class A Bianchi spacetimes and maximal solutions (n_i, θ_i) of the system (2.4). In other words, the vacuum Einstein field equations $\mathrm{Ric}_g = 0$ is equivalent to the ODE system (2.4).

We are now going to introduce some new variables to replace the variables (n_i, θ_i). For some physical reasons, it is convenient to construct a new time variable using the mean curvature $\frac{1}{3}\theta$ of the spacelike hypersurfaces and to divide the variables n_i and θ_i by this mean curvature (see *e.g.* [WH89]).

Let us denote by $\sigma_1, \sigma_2, \sigma_3$ the diagonal coefficients of the trace-less second fundamental form, *i.e.*

$$\sigma_i = \theta_i - \frac{1}{3}\theta$$

where $\theta = \theta_1 + \theta_2 + \theta_3$. By definition, $\sigma_1 + \sigma_2 + \sigma_3 = 0$. Analogously, let us denote by s_1, s_2, s_3 the coefficients of the trace-less Ricci tensor, *i.e.*

$$s_i = \mathrm{Ric}_{h_s}(e_i, e_i) - \frac{1}{3}\,\mathrm{Scal}_{h_s}$$

According to Proposition 2.11, we have

$$s_i = \frac{1}{3}\left(2n_i^2 - n_j^2 - n_k^2 + 2n_j n_k - n_i n_j - n_i n_k\right)$$

Recall that $\frac{1}{3}\theta(s)$ is the mean curvature of the spacelike hypersurface $\{s\} \times G$. Define

$$N_i = \frac{3n_i}{\theta}, \qquad \Sigma_i = \frac{3\sigma_i}{\theta}, \qquad S_i = \frac{9s_i}{\theta^2}$$

The variables (N_i, Σ_i) are called the Wainwright-Hsu coordinates. Remark that

$$S_i = \frac{1}{3}\left(2N_i^2 - N_j^2 - N_k^2 + 2N_j N_k - N_i N_j - N_i N_k\right)$$

Now we consider the new time variable t defined by

$$\frac{\mathrm{d}t}{\mathrm{d}s}(s) = -\frac{\theta(s)}{3}$$

The mean curvature satisfies the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = (1+q)\theta$$

where

$$q = \frac{1}{3}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) \tag{2.15}$$

is sometimes called the *deceleration parameter*. Rewriting (2.4a) and (2.4b) with the normalized variables N_i, Σ_i and with the time variable t, we obtain the Wainwright-Hsu equations

$$\begin{cases} N_1' &= -(q + 2\Sigma_1)N_1 \\ N_2' &= -(q + 2\Sigma_2)N_2 \\ N_3' &= -(q + 2\Sigma_3)N_3 \\ \Sigma_1' &= (2-q)\Sigma_1 + S_1 \\ \Sigma_2' &= (2-q)\Sigma_2 + S_2 \\ \Sigma_3' &= (2-q)\Sigma_3 + S_3 \end{cases} \tag{2.16a}$$

where the notation $'$ refers to differentiation with respect to the time variable t. The constraint equation (2.4c) becomes

$$6 - \left(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2\right) - \frac{1}{2}\left(N_1^2 + N_2^2 + N_3^2\right) + (N_1N_2 + N_2N_3 + N_3N_1) = 0 \tag{2.16b}$$

The left-hand side of (2.16b) can be thought as a normalized density parameter, which is why it is null in the context of vacuum class A Bianchi spacetimes. Also, remark that we have another constraint equation:

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = 0 \tag{2.16c}$$

This is a direct consequence of the fact that the variables Σ_i are the renormalized coefficients of the trace-less second fundamental form.

Remark 2.28. The choice to explicit the vacuum Einstein field equations in a canonical orthonormal-parallel frame field (e_0, e_1, e_2, e_3) implies in particular that the coefficients of the metric g are constant, hence they do not appear in the differential system (2.16a). The numbers $N_i(t)$ are, up to renormalization, the structure constants of the Lie group G for the frame $(e_1(t), e_2(t), e_3(t))$ while the numbers $\Sigma_i(t)$ are, up to renormalization, the diagonal coefficients of the second fundamental form of the spacelike hypersurface $\{t\} \times G$ in the frame $(e_1(t), e_2(t), e_3(t))$.

Remark 2.29 (On the choice of the time variable). The variable θ is called the *Hubble's variable*. This variable naturally appears when one studies the Einstein field equations for spatially homogeneous and isotropic spacetimes (the so-called Friedmann–Lemaître–Robertson–Walker models). In this case, the Einstein field equations becomes essentially a differential equation of order 1 on θ. If one thinks about spatially homogeneous spacetimes as perturbations of the isotropic and spatially homogeneous model, then it is natural to use θ to construct the time variable since θ possesses the desired properties in the isotropic and spatially homogeneous case.

Remark 2.30 (On the orientation of time). We could choose $\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\theta}{3}$ instead of $\frac{\mathrm{d}t}{\mathrm{d}s} = -\frac{\theta}{3}$. This corresponds to a choice of time orientation for Bianchi spacetimes. Our choice is anti-physical: oriented solutions of the Wainwright-Hsu equations (2.16a) correspond to spacetimes in contraction, which will end in a Big-Crunch but which do not have any Big-Bang in the past (this is of course the opposite of our physical universe). As a consequence of this choice, the Wainwright-Hsu system (2.16a) will present an attractor. Most authors do the opposite choice.

Remark 2.31 (On the Wainwright-Hsu variables). By dividing the variables n_i and σ_i by the mean curvature θ and by deciding to use a time variable t such that $\mathrm{d}t/\mathrm{d}s = -\theta/3$, we loose some information. More precisely, we loose during this process the spacelike hypersurfaces $\{s\}\times G$ of the Bianchi spacetime $(I \times G, -\mathrm{d}s^2 + h_s)$ whose mean curvature vanish. According to Proposition 2.24, this can only appear in two cases:

- The Minkowski spacetime: all the spacelike hypersurface $\{s\} \times G$ have zero mean curvature.
- For any maximal vacuum class A Bianchi spacetime $(I \times G, -\mathrm{d}s^2 + h_s)$ of type IX (*i.e.* whose Lie algebra is $\mathfrak{g} = \mathfrak{su}(2)$), there exists a unique real $s_0 \in I$ such that the mean curvature of the spacelike hypersurface $\{s\} \times G$ vanish. Hence, the spacelike hypersurface $\{s_0\} \times G$ "split" $(I \times G, -\mathrm{d}s^2 + h_s)$ into two vacuum class A Bianchi spacetimes, one where the mean curvature is strictly negative and one where the mean curvature is strictly positive.

Moreover, the Wainwright-Hsu variables are dimensionless, which means that they will not change if the spacetime metric is rescaled (see [WH89]).

Remark 2.32 (Lie algebra type and signs of the variables N_i). It is clear from the Wainwright-Hsu equations that the signs of the variables N_i are constant. By construction, these signs are exactly the signs of the variables n_i. Hence, they encode the Lie algebra of G modulo isomorphism. This remark allows us to transfer the vocabulary introduced for Lie algebras in the table 2.1 to solutions of the Wainwright-Hsu equations. For example, we will say that a solution is of type IX if

$$N_1 > 0, N_2 > 0, N_3 > 0 \qquad \text{or} \qquad N_1 < 0, N_2 < 0, N_3 < 0$$

To have a clear picture of how the solutions of the Wainwright-Hsu equations are related to vacuum class A Bianchi spacetimes, let us describe the relation betwenn the time coordinate s for vacuum class A Bianchi spacetimes and the time coordinate t for the solutions of the Wainwright-Hsu equations.

Proposition 2.33. *Let $(M =]s_-, s_+[\times G, g)$ be a maximal vacuum class A Bianchi spacetime which is not the Minkowski spacetime and which is not of type* IX. *Choose the orientation of time such that $\theta < 0$. The corresponding solution of the Wainwright-Hsu equations* (2.16) *is defined on $\mathbb{R}$ (hence it is maximal) and*

$$\lim_{s \to s_-} t(s) = -\infty, \qquad \lim_{s \to s_+} t(s) = +\infty$$

Proof. See [Rin13], Lemma 22.2, p. 234. □

Proposition 2.34. *Let $(M =]s_-, s_+[\times G, g)$ be a vacuum class A Bianchi spacetime of type* IX. *Denote by s_0 the unique real number such that $\theta > 0$ in $I_- =]s_-, s_0[$ and $\theta < 0$ in $I_+ =]s_0, s_+[$. The solution of the Wainwright-Hsu equations corresponding to the interval I_- is defined on $]t_-, +\infty[$ with $t_- > -\infty$ and*

$$\lim_{s \to s_-} t(s) = +\infty$$

The solution of the Wainwright-Hsu equations corresponding to the interval I_+ *is defined on* $]t_+, +\infty[$ *with* $t_+ > -\infty$ *and*

$$\lim_{s \to s_+} t(s) = +\infty$$

Proof. See [Rin13], Lemma 22.3, p. 234. □

To conclude, we describe the correspondence between vacuum class A Bianchi spacetimes and solutions of the system (2.16) formed by the Wainwright-Hsu equations and the two constraint equations.

Proposition 2.35. *The Minkowski spacetime does not correspond to any solution of the system* (2.16)*. There is a one-to-one correspondence between*

- *Maximal vacuum class A Bianchi spacetimes which are not of type* IX *(minus the Minkowski spacetime), modulo isomorphism, metric rescaling, time orientation reversal and time translation.*
- *Maximal solutions of the system* (2.16) *which are not of type* IX*, modulo permutation of the indices, simultaneous sign reversal of the three* N_i*'s and time translation.*

There is a one-to-one correspondence between

- *Maximal half vacuum class A Bianchi spacetimes of type* IX*, modulo isomorphism, metric rescaling, time orientation reversal and time translation.*
- *Maximal solutions of type* IX *of the system* (2.16)*, modulo permutation of the indices, simultaneous sign reversal of the three* N_i*'s and time translation.*

Chapter 3

The Wainwright-Hsu vector field and the Mixmaster attractor

In this section, we will recall a number of well known facts about the Wainwright-Hsu vector field $\mathscr{X}$, its dynamics in restriction to the Mixmaster attractor, the Kasner map and the Kasner parameter. A good reference for these facts is [HU09]. However, there is something "new" in addition to what is presented in [HU09]: we will define the quotient phase space and the induced Wainwright-Hsu vector field $\mathcal{X}$ on that space (see section 3.6), as they will be more convenient to work with in what follows.

3.1 The Wainwright-Hsu vector field $\mathscr{X}$

Phase space. Recall from Proposition 2.35 that Bianchi spacetimes will be described as solutions of the system of equations (2.16). These solutions can be seen as the orbits of a certain flow on the *phase space*

$$\mathscr{B} = \left\{(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathbb{R}^6 \quad \text{satisfying (2.16c) and (2.16b)}\right\}$$

Observe that $\mathscr{B}$ is a non-singular and non-compact 4-dimensional quadric in $\mathbb{R}^6$.

Wainwright-Hsu vector field The Wainwright-Hsu vector field, denoted by $\mathscr{X}$, is defined as the vector field on $\mathscr{B}$ associated with the Wainwright-Hsu equations (2.16a), that is,

$$\mathscr{X}(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) = \begin{pmatrix} -(q+2\Sigma_1)N_1 \\ -(q+2\Sigma_2)N_2 \\ -(q+2\Sigma_3)N_3 \\ (2-q)\Sigma_1 + S_1 \\ (2-q)\Sigma_2 + S_2 \\ (2-q)\Sigma_3 + S_3 \end{pmatrix} \tag{3.1}$$

Symmetries. The Wainwright-Hsu vector field is equivariant for the action of the permutations group $\mathfrak{S}_3$:

$$\sigma.(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) = (N_{\sigma(1)}, N_{\sigma(2)}, N_{\sigma(3)}, \Sigma_{\sigma(1)}, \Sigma_{\sigma(2)}, \Sigma_{\sigma(3)}) \tag{3.2}$$

and for the action of the group $\mathbb{Z}/2\mathbb{Z} = \{\mathrm{Id}, \epsilon\}$ given by:

$$\epsilon.(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) = (-N_1, -N_2, -N_3, \Sigma_1, \Sigma_2, \Sigma_3)$$

One should remark that this implies that $\mathfrak{S}_3$ is acting on the space of the orbits of the Wainwright-Hsu vector field. Later on, to simplify the presentation, we will work in the quotient phase space $\mathscr{B}/\mathfrak{S}_3$ (see section 3.6).

3.2 Stratification of the phase space $\mathscr{B}$

Stratification of the phase space. According to (2.16a), the signs (positive, negative or null) of the variables N_i are invariant along the orbits of the Wainwright-Hsu vector field. This fact leads to a stratification of the phase space $\mathscr{B}$ in six subsets which are invariant under the flow of the Wainwright-Hsu vector field $\mathscr{X}$ (see table 3.1). Recall that the variables N_i are closely related to the structure constants of the Bianchi spacetime represented by the orbit. So this stratification is no more than a reinterpretation of the classification of the 3-dimensional unimodular Lie algebras. This stratification plays an important role in the study of the dynamics of the Wainwright-Hsu vector field $\mathscr{X}$. This is thanks to the following facts: the dynamics on the low dimensional strata (1 and 2) can be described entirely explicitly and the reunion of these low dimensional strata forms an attractor on which almost every orbit of the Wainwright-Hsu vector field accumulate.

Bianchi type	Name of the stratum	Dimension of the stratum	Signs of N_1, N_2, N_3 modulo permutation of the indices	Corresponding Lie algebra up to isomorphism
I	$\mathscr{K}$ or $\mathscr{B}_{\mathrm{I}}$	1	$0,0,0$	$\mathbb{R}^3$
II	$\mathscr{B}_{\mathrm{II}}$	2	$+,0,0$ or $-,0,0$	Heisenberg algebra
VI_0	$\mathscr{B}_{\mathrm{VI}_0}$	3	$+,-,0$	$\mathfrak{isom}(\mathrm{Min}_2)$
VII_0	$\mathscr{B}_{\mathrm{VII}_0}$	3	$+,+,0$ or $-,-,0$	$\mathfrak{isom}(\mathbb{R}^2)$
VIII	$\mathscr{B}_{\mathrm{VIII}}$	4	$+,+,-$ or $-,-,+$	$\mathfrak{sl}(2,\mathbb{R})$
IX	$\mathscr{B}_{\mathrm{IX}}$	4	$+,+,+$ or $-,-,-$	$\mathfrak{su}(2)$

Table 3.1 – Stratification of the phase space.

Restriction to the "positive" part of the phase space. As stated in the introduction, to avoid clutter with notations and to simplify the presentation, we will restrict our attention to the dynamics of the Wainwright-Hsu vector field in

$$\mathscr{B}^+ \stackrel{\text{def}}{=} \{(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathscr{B} \mid N_1 \geq 0, N_2 \geq 0, N_3 \geq 0\}$$

Recall that

- $\mathscr{B}^+$ is invariant under the flow of the Wainwright-Hsu vector field.
- Generic orbits of $\mathscr{B}^+$ are type IX orbits.

We will denote $\mathscr{B}^+_{\mathrm{II}} := \mathscr{B}_{\mathrm{II}} \cap \mathscr{B}^+$ and analogously for other stratas. Also, we will implicitly restrict the Wainwright-Hsu vector field to $\mathscr{B}^+$ from now on.

The Kasner circle. The stratum corresponding to abelian Lie algebra is one-dimensional. It is an euclidean circle denoted by $\mathscr{K}$ and called the *Kasner circle* (because its points correspond to Kasner spacetimes, see [WH89]):

$$\begin{aligned}\mathscr{K} &= \{(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathscr{B} \mid N_1 = N_2 = N_3 = 0\} \\ &= \left\{(0,0,0,\Sigma_1, \Sigma_2, \Sigma_3) \in \mathbb{R}^6 \mid \Sigma_1 + \Sigma_2 + \Sigma_3 = 0, \Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 = 6\right\}\end{aligned} \tag{3.3}$$

The stratum $\mathscr{B}^+_{\mathrm{II}}$. The stratum $\mathscr{B}^+_{\mathrm{II}}$ corresponding to Heisenberg Lie algebras is two-dimensional. It is the reunion of three open hemiellipsoids (see later figure 3.2), each having the Kasner circle as boundary:

$$\mathscr{B}^+_{\mathrm{II}} = \mathscr{B}^1_{\mathrm{II}} \sqcup \mathscr{B}^2_{\mathrm{II}} \sqcup \mathscr{B}^3_{\mathrm{II}}$$

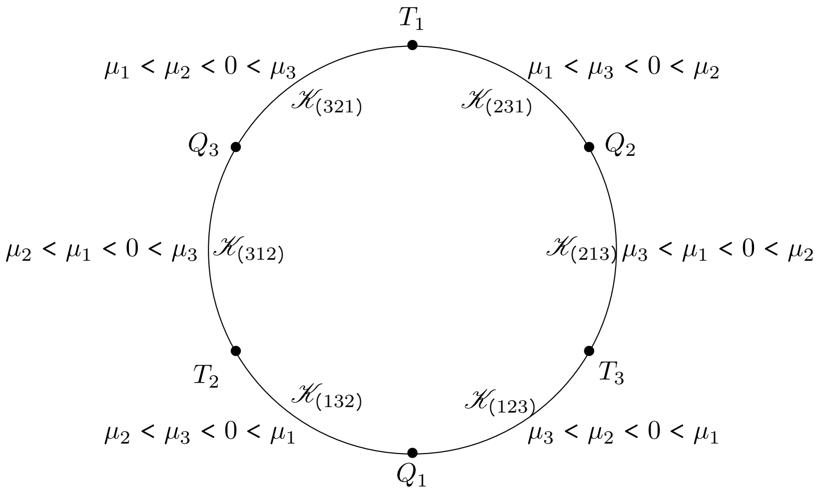

Figure 3.1 – Order of the eigenvalues.

where

$$\begin{aligned}\mathscr{B}^1_{\text{II}} &= \{(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathscr{B} \mid N_1 > 0, N_2 = N_3 = 0\} \\ &= \left\{(N_1, 0, 0, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathbb{R}^6 \mid N_1 > 0, \Sigma_1 + \Sigma_2 + \Sigma_3 = 0, \Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 + \frac{1}{2}N_1^2 = 6\right\} \end{aligned} \tag{3.4}$$

The hemiellipsoids $\mathscr{B}^2_{\text{II}}$ and $\mathscr{B}^3_{\text{II}}$ are defined analogously.

The Mixmaster attractor. The reunion of the Kasner circle $\mathscr{K}$ and the stratum $\mathscr{B}_{\text{II}}$ is called the *Mixmaster attractor* and is denoted by $\mathscr{A}$. We denote by $\mathscr{A}^+ := \mathscr{K} \cup \mathscr{B}^+_{\text{II}}$ the *positive part* of the Mixmaster attractor.

Generic orbits The strata $\mathscr{B}^+_{\text{IX}}$ corresponding to semi-simple Lie algebras is open and dense in $\mathscr{B}^+$. Generic orbits of the Wainwright-Hsu vector field $\mathscr{X}$ are contained in $\mathscr{B}^+_{\text{IX}}$.

3.3 Linearization of the Wainwright-Hsu vector field along the Kasner circle

Critical points. The critical points of the Wainwright-Hsu vector field correspond to self-similarly expanding spacetimes (see [WH89]). Using (2.16a) and (3.3), one can see that any point of the Kasner circle $\mathscr{K}$ is a critical point. The goal of this section is to describe the eigenvalues of $D\mathscr{X}(p)$ for any point p of the Kasner circle.

Notations (see figure 3.1). There are three particular points in the Kasner circle called the *Taub points*:

$$\begin{aligned} T_1 &= (0,0,0,2,-1,-1) \\ T_2 &= (0,0,0,-1,2,-1) \\ T_3 &= (0,0,0,-1,-1,2) \end{aligned}$$

These points split the Kasner circle in three open arcs $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$ defined as following: $\mathscr{K}_i$ is the connected component of $\mathscr{K} \setminus \{T_1, T_2, T_3\}$ admitting T_j and T_k as end points, where $\{i,j,k\} = \{1,2,3\}$. Alternatively, one can define $\mathscr{K}_i$ as the subset of $\mathscr{K}$ where

$$-2 < \Sigma_i < -1$$

We denote by Q_1, Q_2, Q_3 the diametrically opposite points of the Taub points in the Kasner circle, that is,

$$\begin{aligned} Q_1 &= (0,0,0,-2,1,1) \\ Q_2 &= (0,0,0,1,-2,1) \\ Q_3 &= (0,0,0,1,1,-2) \end{aligned}$$

One can remark that Q_i is the middle of the arc $\mathscr{K}_i$. As such, Q_i divides $\mathscr{K}_i$ in two open arcs $\mathscr{K}_{(ijk)}$ and $\mathscr{K}_{(ikj)}$ with respective end points Q_i, T_k and Q_i, T_j. Alternatively, one can define $\mathscr{K}_{(ijk)}$ as the subset of $\mathscr{K}$ where

$$\Sigma_i < \Sigma_j < \Sigma_k$$

Eigenvalues of the linearized vector field at points of the Kasner circle. In chapter 6, we will study the behaviour of the orbits passing close to a point p of the Kasner circle. The first step is to linearize the Wainwright-Hsu vector field at the points of the Kasner circle. Indeed, the local behaviour of the orbits is determined, at the first order, by the linear part of the vector field.

Let $p = (0,0,0,\Sigma_1,\Sigma_2,\Sigma_3)$ be a point of the Kasner circle $\mathscr{K}$. One can remark that

$$(\Sigma_3 - \Sigma_2)\partial_{\Sigma_1} + (\Sigma_1 - \Sigma_3)\partial_{\Sigma_2} + (\Sigma_2 - \Sigma_1)\partial_{\Sigma_3}$$

is tangent to $\mathscr{K}$ at p and that

$$(\partial_{N_1}, \partial_{N_2}, \partial_{N_3}, (\Sigma_3 - \Sigma_2)\partial_{\Sigma_1} + (\Sigma_1 - \Sigma_3)\partial_{\Sigma_2} + (\Sigma_2 - \Sigma_1)\partial_{\Sigma_3})$$

is a basis of $T_p\mathscr{B}$. In this basis, the matrix of $D\mathscr{X}(p)$ is

$$\begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.5}$$

where

$$\mu_i = -(2 + 2\Sigma_i) \tag{3.6}$$

We summarize the main properties of the eigenvalues μ_i in Proposition 3.1 and figure 3.1.

Proposition 3.1. *If $\{i,j,k\} = \{1,2,3\}$, then*

1. *On $\mathscr{K}_{(ijk)}$, we have $\mu_k < \mu_j < 0 < \mu_i$. Moreover, the unstable eigenvalue μ_i is "weaker" than the stable eigenvalues μ_j and μ_k: $\mu_i < |\mu_j|$ and $\mu_i < |\mu_k|$.*

2. *At the point T_i, we have $\mu_i < 0$ and $\mu_j = \mu_k = 0$.*

3. *At the point Q_i, we have $\mu_i > 0$ and $\mu_j = \mu_k < 0$.*

Proof. This is a straightforward consequence of (3.6). □

Remark 3.2. According to Proposition 3.1, the unstable direction and the weak stable direction swap at the Taub points while the weak stable direction and the strong stable direction swap at the points Q_i, $i = 1,2,3$.

Moreover, each Taub point has a one dimensional stable manifold and a three dimensional central manifold. Every other point of the Kasner circle has a two dimensional stable manifold, a one dimensional unstable manifold and a one dimensional central manifold.

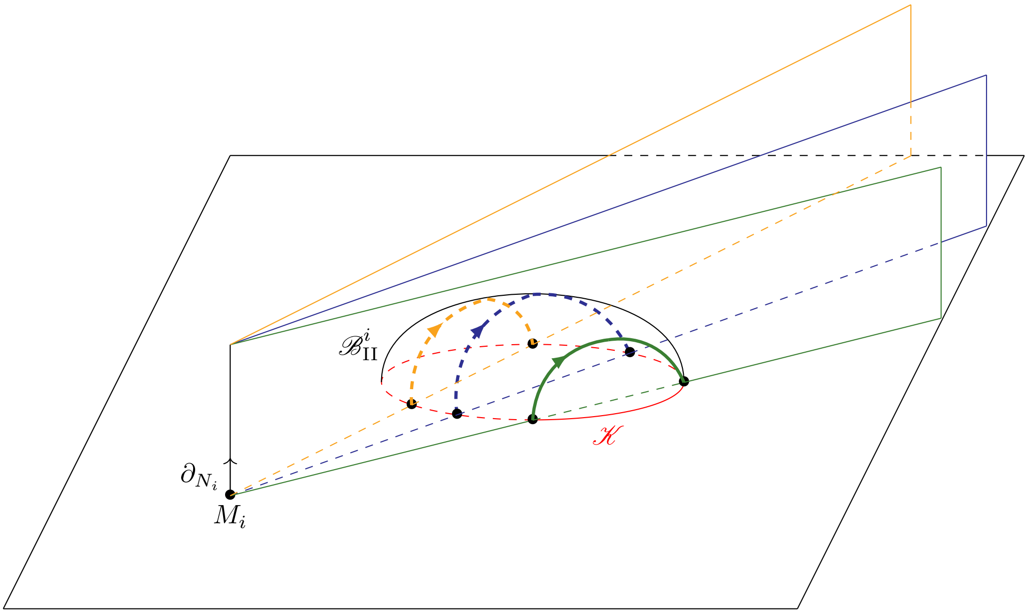

Figure 3.2 – Type II orbits contained in $\mathscr{B}^i_{\mathrm{II}}$.

3.4 Type II orbits

The orbits contained in the stratum $\mathscr{B}^+_{\mathrm{II}}$ are called *type* II *orbits*. These orbits can be explicitly described in an easy manner. Let

$$\begin{aligned} M_1 &= (0,0,0,-4,2,2) \\ M_2 &= (0,0,0,2,-4,2) \\ M_3 &= (0,0,0,2,2,-4) \end{aligned}$$

For $i \in \{1,2,3\}$, let P_i^2 be the sheaf of two-dimensional affine planes passing through M_i and whose direction contains ∂_{N_i}. Type II orbits contained in $\mathscr{B}^i_{\mathrm{II}}$ (see (3.4)) are exactly the intersection of $\mathscr{B}^i_{\mathrm{II}}$ with planes of the sheaf P_i^2 (see figure 3.2). As a consequence, any type II orbit is a *heteroclinic connexion*[1] between two points of the Kasner circle. One easy way to see this is to remark that, for a type II orbit contained in $\mathscr{B}^i_{\mathrm{II}}$, the Wainwright-Hsu equations (2.16a) lead to the conservation of the quantity $\frac{\Sigma_j-2}{\Sigma_k-2}$ along the orbit (see [Rin01] for more details).

Local view-point. Let $p \in \mathscr{K}_{(ijk)}$. There are exactly three type II orbits which establish a heteroclinic connexion between p and another point of the Kasner circle. We are now going to determine the "time direction" of these orbits, that is, to determine whether they admit p as an ω-limit point or an α-limit point. Recall that the Wainwright-Hsu vector field admits three non trivial eigenvalues $\mu_k < \mu_j < 0 < \mu_i$ at the point p. It follows that:

- The type II orbit contained in $\mathscr{B}^i_{\mathrm{II}}$, denoted by $\mathscr{O}^u_p$, admits the point p as its α-limit point. We will say that this orbit *starts* at p.
- The type II orbit contained in $\mathscr{B}^j_{\mathrm{II}}$ (resp. $\mathscr{B}^k_{\mathrm{II}}$), denoted by $\mathscr{O}^{s_1}_p$ (resp. $\mathscr{O}^{s_2}_p$), admits the point p as its ω-limit point. We will say that these orbits *arrive* at p.

Global view-point. $\mathscr{B}^i_{\mathrm{II}}$ is foliated by type II orbits in a very specific way. Any type II orbit contained in $\mathscr{B}^i_{\mathrm{II}}$ starts in $\mathscr{K}_i$ and arrives in $\mathscr{K}_j \cup \{T_i\} \cup \mathscr{K}_k$. More precisely, those starting in $\mathscr{K}_{(ijk)}$

[1] A heteroclinic connexion is an orbit "joining two different points". More precisely it is an orbit $t \mapsto \mathscr{O}(t)$ such that there exists two distinct points p and q verifying $\lim_{t\to+\infty} \mathscr{O}(t) = q$ and $\lim_{t\to-\infty} \mathscr{O}(t) = p$.

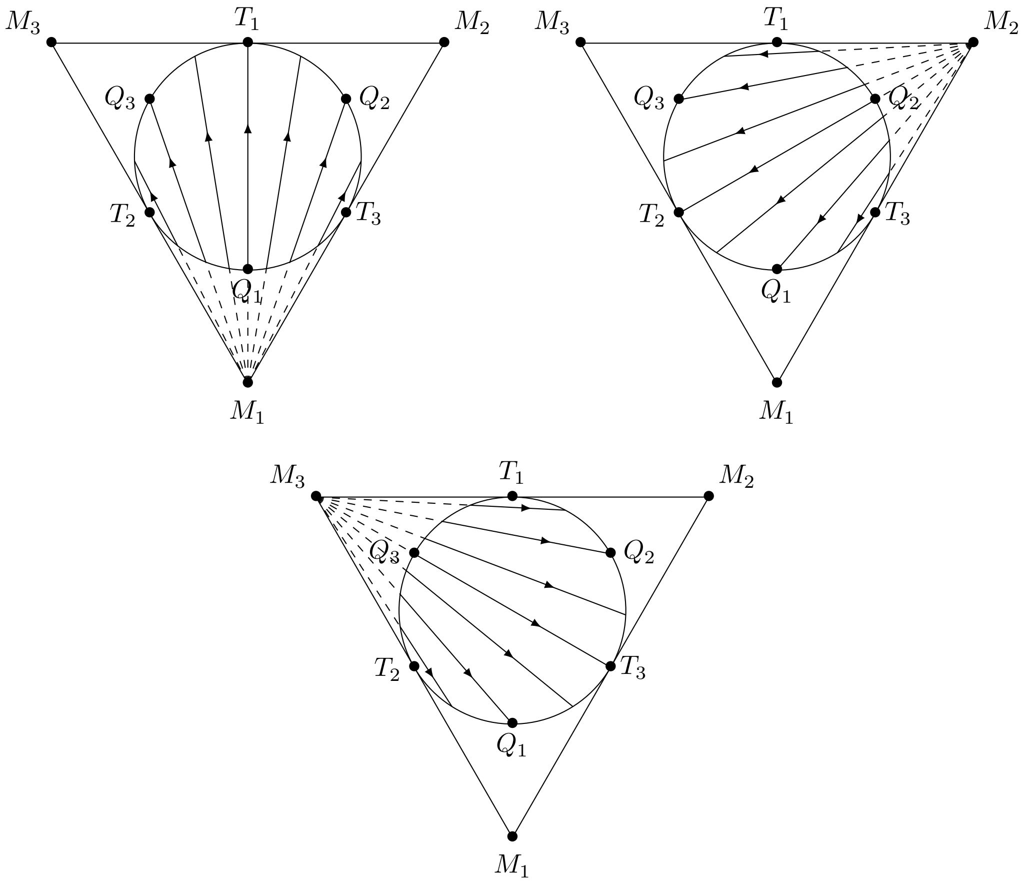

Figure 3.3 – Projections on the $(\Sigma_1, \Sigma_2, \Sigma_3)$-plane of type II orbits contained in $\mathscr{B}^1_{\mathrm{II}}$, $\mathscr{B}^2_{\mathrm{II}}$ and $\mathscr{B}^3_{\mathrm{II}}$ (left to right, top to bottom).

arrive in $\mathscr{K}_j$ and the one starting at Q_i arrives at T_i. There is no type II orbit starting from a Taub point.

Projection view-point. Another way to describe the type II orbits is to give their projection on the $(\Sigma_1, \Sigma_2, \Sigma_3)$-plane, that is, the two-dimensional plane containing the Kasner circle. Let P^1_i be the sheaf of one-dimensional affine lines passing through M_i and contained in the $(\Sigma_1, \Sigma_2, \Sigma_3)$-plane. According to what precedes, the projections of the type II orbits contained in $\mathscr{B}^i_{\mathrm{II}}$ are exactly the intersections of the open disc delimited by the Kasner circle and the lines of the sheaf P^1_i (see figure 3.3).

3.5 The Kasner map $\mathscr{F}$

Definition 3.3 (Kasner map)**.** Let $p \in \mathscr{K} \setminus \{T_1, T_2, T_3\}$. The type II orbit $\mathscr{O}^u_p$ starting at p converges (in the future) to a point of the Kasner circle denoted by $\mathscr{F}(p)$. Set $\mathscr{F}(T_i) = T_i$ for all $i \in \{1, 2, 3\}$. This defines a continuous map $\mathscr{F} : \mathscr{K} \to \mathscr{K}$, called the *Kasner map* (sometimes also called the *BKL map*). The orbit $\mathscr{O}^u_p$ will also be denoted by $\mathscr{O}_{p \to \mathscr{F}(p)}$.

Geometrical construction of $\mathscr{F}(p)$. Let $p \in \mathscr{K}_i$. The line $(M_i p)$ intersects the Kasner circle at two points. The closest to M_i is p while the farthest is $\mathscr{F}(p)$ (see figure 3.4). One can remark that $\mathscr{F}(Q_i) = T_i$ for $i \in \{1, 2, 3\}$.

Dynamics of the Kasner map. The dynamics of the Kasner map on the circle is chaotic (see figure 3.5). One can verify that the Kasner map is C^∞ and of degree -2, is non uniformly expanding

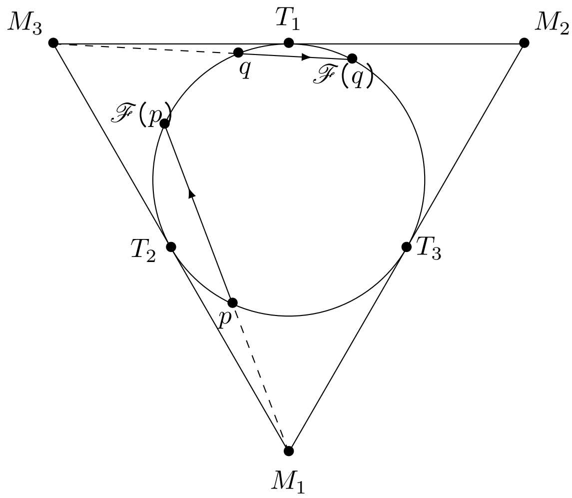

Figure 3.4 – The Kasner map.

(the derivative is, in absolute value, strictly superior to 1 except at the Taub points where its absolute value is equal to 1). By a classical argument (see [KH97]), the Kasner map is topologically conjugate to $\theta \mapsto -2\theta$ (on the circle $\mathbb{R}/\mathbb{Z}$) which has a well understood dynamics. In particular, $\mathscr{F}$ has the following properties:

- Periodic points of $\mathscr{F}$ are dense in $\mathscr{K}$.
- There exists points in $\mathscr{K}$ whose forward orbit under $\mathscr{F}$ are dense in $\mathscr{K}$. The set of all such points is a G_δ dense.
- For every point p of the Kasner circle, the complete backward orbit of p under $\mathscr{F}$ is dense in $\mathscr{K}$.
- The topological entropy of the Kasner map is positive (it is equal to $\log(2)$).
- $\mathscr{F}$ possesses an invariant measure (of infinite mass) absolutely continuous with respect to Lebesgue measure.

We will come back to the dynamics of the Kasner map in section 3.7, after introducing the Kasner parameter and reducing the dynamics modulo symmetries.

Generalized heteroclinic chains. Let p be a point of the Kasner circle. One can consider the orbit of p under the Kasner map $\left(p, \mathscr{F}(p), \mathscr{F}^2(p), \mathscr{F}^3(p), \dots\right)$ and a chain of heteroclinic connexions between the consecutive iterates of this sequence. This forms a continuous curve in the phase space $\mathscr{B}$. At every step, the heteroclinic connexion is the type II orbit $\mathscr{O}_{\mathscr{F}^n(p)\to\mathscr{F}^{n+1}(p)}$.

The following notion of heteroclinic chain is standard.

Definition 3.4 (Heteroclinic chain)**.** Let p be a point of the Kasner circle which is not one of the Taub points. The *heteroclinic chain* starting at p is the sequence

$$\mathscr{H}(p) \overset{\text{def}}{=} \left(\mathscr{O}_{p\to\mathscr{F}(p)}, \mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \mathscr{O}_{\mathscr{F}^2(p)\to\mathscr{F}^3(p)}, \dots\right) \tag{3.7}$$

If there exists $n \in \mathbb{N}^*$ such that $\mathscr{F}^n(p)$ is a Taub point, then the heteroclinic chain starting at p ends at that point.

To simplify the definition of some transition maps that we will use later on, we extend the above definition.

Definition 3.5 (Generalized heteroclinic chain)**.** Let $q \in \mathscr{B}^+ \setminus \mathscr{K}$. Denote by $\mathscr{O}(q)$ the forward $\mathscr{X}$-orbit of q. The *heteroclinic chain* $\mathscr{H}(q)$ starting at q is defined as follows:

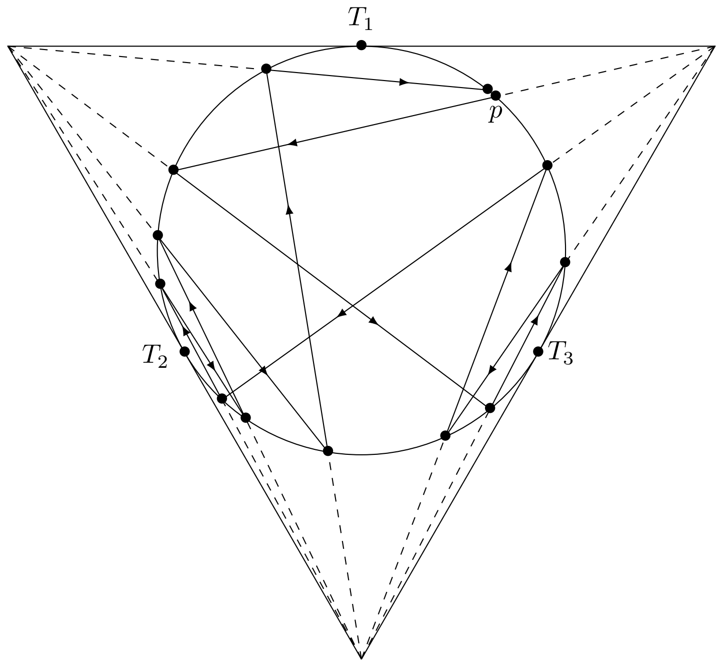

Figure 3.5 – The Kasner map is chaotic.

- If $\mathscr{O}(q)$ converges to a point p of the Kasner circle which is not a Taub point, then $\mathscr{H}(q)$ is the concatenation of $\mathscr{O}(q)$ with $\mathscr{H}(p)$:

$$\mathscr{H}(q) \stackrel{\text{def}}{=} \left(\mathscr{O}(q), \mathscr{O}_{p\to\mathscr{F}(p)}, \mathscr{O}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \dots\right) \tag{3.8}$$

- Otherwise, $\mathscr{H}(q)$ is simply the orbit $\mathscr{O}(q)$.

It is well known that type IX orbits cannot converge to a Taub point. Hence, if $q \in \mathscr{B}^+_{\text{IX}}$, the heteroclinic chain starting at q is nothing but the forward $\mathscr{X}$-orbit of q. Recall that we want to describe the heteroclinic chains starting at points of the Kasner circle which are shadowed by some type IX orbits (see definition 1.2).

3.6 Quotient phase space $\mathcal{B}$

Recall that $\mathfrak{S}_3$ acts on $\mathscr{B}$ by permutation of the indices $1, 2, 3$ (see (3.2)). From now on, we will make a systematic use of these symmetries. Let us define the *quotient phase space* and its positive part

$$\mathcal{B} \stackrel{\text{def}}{=} \mathscr{B}/\mathfrak{S}_3, \qquad \mathcal{B}^+ \stackrel{\text{def}}{=} \mathscr{B}^+/\mathfrak{S}_3$$

as well as the natural *projection map*

$$\pi : \mathscr{B} \to \mathcal{B} \tag{3.9}$$

Many results have a natural presentation in the quotient phase space $\mathcal{B}$. The only case where it is better to work in the phase space $\mathscr{B}$ is when one needs to use precisely the Wainwright-Hsu equations. This will not happen often in our work. We will mainly use the properties described in sections 3.3 and 3.4: behaviour of type II orbits and eigenvalues of $D\mathscr{X}(p)$ for $p \in \mathscr{K}$.

In order to have a better understanding of the quotient, one needs to describe the orbits under the action of $\mathfrak{S}_3$. Before going into the details in low dimensional strata, one can notice that $\mathfrak{S}_3$ acts freely and properly on $\mathscr{B} \setminus \mathscr{S}$ where $\mathscr{S}$ is the singular set defined by

$$\mathscr{S} \stackrel{\text{def}}{=} \left\{(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathscr{B} \mid \exists i \neq j, N_i = N_j \text{ and } \Sigma_i = \Sigma_j\right\}$$

Hence, $\mathcal{B}$ is a 4-dimensional orbifold with singular locus $\pi(\mathscr{S})$. The fact that $\mathcal{B}$ is singular is not a huge issue. Indeed, we will only work with heteroclinic chains which do not cross the singular locus. The orbits shadowing such heteroclinic chains will not cross the singular locus either (at least after a certain time).

Let us define the *regular part of the quotient phase space* by

$$\mathcal{B}_{\text{reg}} \overset{\text{def}}{=} \mathcal{B} \setminus \pi(\mathscr{S})$$

It will be convenient to work on a smaller part of the quotient phase space, so we define

$$\mathscr{B}_0 \overset{\text{def}}{=} \{(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathscr{B} \mid \forall i \neq j, \Sigma_i \neq \Sigma_j\} \tag{3.10a}$$

$$\mathscr{B}_0^+ \overset{\text{def}}{=} \left\{(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathscr{B}^+ \mid \forall i \neq j, \Sigma_i \neq \Sigma_j\right\} \tag{3.10b}$$

$$\mathcal{B}_0 \overset{\text{def}}{=} \pi(\mathscr{B}_0) \tag{3.10c}$$

$$\mathcal{B}_0^+ \overset{\text{def}}{=} \pi\left(\mathscr{B}_0^+\right) \tag{3.10d}$$

Observe that $\mathcal{B}_0$ is an open subset of $\mathcal{B}_{\text{reg}}$. Let

$$\mathscr{B}_{(123)} \overset{\text{def}}{=} \mathscr{B} \cap \{\Sigma_1 < \Sigma_2 < \Sigma_3\}$$

Proposition 3.6. *The projection map π restricted to $\mathscr{B}_{(123)}$ is a C^∞-diffeomorphism from $\mathscr{B}_{(123)}$ to $\mathcal{B}_0$. In particular, $\mathscr{B}_{(123)}$ is a fundamental domain of*

$$\pi : \mathscr{B}_0 \to \mathcal{B}_0$$

Proof. π restricted to $\mathscr{B}_{(123)}$ is injective and π is a local C^∞-diffeomorphism everywhere on $\mathscr{B}_{(123)}$ by definition of the quotient manifold structure so the result follows immediately. ☐

Orbits under the action of $\mathfrak{S}_3$ on the Kasner circle. Let $p \in \mathscr{K}$. If p is one of the three exceptional points T_1, T_2, T_3 (resp. Q_1, Q_2, Q_3), then the orbit of p under the action of $\mathfrak{S}_3$ is $\{T_1, T_2, T_3\}$ (resp. $\{Q_1, Q_2, Q_3\}$). On the other hand, if p is not one of the above points, then the orbit of p under the action of $\mathfrak{S}_3$ contains six points, one in each sixth of the Kasner circle $\mathscr{K}_{(ijk)}$.

General orbits under the action of $\mathfrak{S}_3$. Let $p \in \mathscr{B}$. Similarly to the previous case, if $p \in \mathscr{S}$, then its orbit under the action of $\mathfrak{S}_3$ contains three points. On the other hand, if $p \notin \mathscr{S}$, then its orbit under the action of $\mathfrak{S}_3$ contains six points.

Stratification of the quotient phase space. The stratification of the phase space $\mathscr{B}$ induces a stratification of the quotient phase space $\mathcal{B}$ (see table 3.2).

Bianchi type	Name of the stratum	Dimension of the stratum
I	$\mathcal{K}$	1
II	$\mathcal{B}_{\text{II}}$	2
VI_0	$\mathcal{B}_{\text{VI}_0}$	3
VII_0	$\mathcal{B}_{\text{VII}_0}$	3
VIII	$\mathcal{B}_{\text{VIII}}$	4
IX	$\mathcal{B}_{\text{IX}}$	4

Table 3.2 – Stratification of the quotient phase space.

Induced Kasner segment According to what precedes, the projection in $\mathcal{B}$ of the Kasner circle $\mathscr{K}$

$$\mathcal{K} \stackrel{\text{def}}{=} \mathscr{K} / \mathfrak{S}_3$$

is in fact a topological segment (hence we will speak of the *Kasner segment* $\mathcal{K}$). The end points of this segment, denoted by T and Q, are respectively the projection of the Taub points and the projection of the points Q_i in the quotient phase space $\mathcal{B}$. Let

$$\mathcal{K}_0 \stackrel{\text{def}}{=} \mathcal{K} \setminus \{T, Q\}$$

be the *(induced) Kasner interval*. Any point $p \in \mathcal{K}_0$ possesses a fiber containing six points, one in each sixth of the Kasner circle $\mathscr{K}_{(ijk)}$. Observe that $\mathcal{B}_0$ is an open neighbourhood of $\mathcal{K}_0$.

Distance on the quotient phase space. The Euclidean distance d_E on $\mathbb{R}^6$ induces a distance $d_{\mathcal{B}}$ on $\mathcal{B}$:

$$d_{\mathcal{B}}(\pi(p), \pi(q)) \stackrel{\text{def}}{=} \inf_{\sigma \in \mathfrak{S}_3} d_E(p, \sigma.q) \tag{3.11}$$

which we will always use to measure the radius of balls in $\mathcal{B}$.

Induced coordinate functions. The coordinates $N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3$ on $\mathscr{B}$ induce a set of smooth coordinates functions $N_u, N_{s_1}, N_{s_2}, \Sigma_u, \Sigma_{s_1}, \Sigma_{s_2}$ on $\mathcal{B}_0$ (here smooth stands for C^∞). They are defined as following: let $x \in \mathcal{B}_0$ and choose a point $y = (N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathscr{B}_0$ in the fiber over x. Let $\{i, j, k\} = \{1, 2, 3\}$ such that $\Sigma_i < \Sigma_j < \Sigma_k$, then we define

$$N_u \stackrel{\text{def}}{=} N_i, \quad N_{s_1} \stackrel{\text{def}}{=} N_j, \quad N_{s_2} \stackrel{\text{def}}{=} N_k$$
$$\Sigma_u \stackrel{\text{def}}{=} \Sigma_i, \quad \Sigma_{s_1} \stackrel{\text{def}}{=} \Sigma_j, \quad \Sigma_{s_2} \stackrel{\text{def}}{=} \Sigma_k$$

This definition does not depend on the choice of y in the fiber of x, hence N_u, N_{s_1}, N_{s_2}, Σ_u, Σ_{s_1}, Σ_{s_2} are well defined on $\mathcal{B}_0$. One cannot extend them by continuity on $\mathcal{B}$. In particular, beware of the fact that induced type II orbits in $\mathcal{B}$ are not contained in $\mathcal{B}_0$, this implies that N_u, N_{s_1}, N_{s_2}, Σ_u, Σ_{s_1}, Σ_{s_2} are not continued functions along type II orbits in the quotient phase space (one cannot extend them by continuity when the orbit cross $\mathcal{B} \setminus \mathcal{B}_0$).

Note that the map

$$x \mapsto \left(N_u(x), N_{s_1}(x), N_{s_2}(x), \Sigma_u(x), \Sigma_{s_1}(x), \Sigma_{s_2}(x)\right)$$

is a diffeomorphism from $\mathcal{B}_0$ to B_0 where

$$\begin{aligned} B_0 \stackrel{\text{def}}{=} \{N_u, N_{s_1}, N_{s_2}, \Sigma_u, \Sigma_{s_1}, \Sigma_{s_2} \in \mathbb{R}^6 \mid \Sigma_u + \Sigma_{s_1} + \Sigma_{s_2} = 0,& \\ 6 - \left(\Sigma_u^2 + \Sigma_{s_1}^2 + \Sigma_{s_2}^2\right) - \frac{1}{2}\left(N_u^2 + N_{s_1}^2 + N_{s_2}^2\right) + \left(N_u N_{s_1} + N_{s_1} N_{s_2} + N_{s_2} N_u\right) = 0,& \\ \Sigma_u < \Sigma_{s_1} < \Sigma_{s_2}\}& \end{aligned}$$

Induced Wainwright-Hsu vector field. The Wainwright-Hsu vector field $\mathscr{X}$ on $\mathscr{B}$ is equivariant under the action of $\mathfrak{S}_3$ and therefore induces a vector field $\mathcal{X}$ on $\mathcal{B}$. Let $p \in \mathcal{K}_0$. According to the discussion about the eigenvalues of the Wainwright-Hsu vector field $\mathscr{X}$ (see section 3.3), $D\mathcal{X}(p)$ is diagonalizable. More precisely, ∂_{N_u}, $\partial_{N_{s_1}}$, $\partial_{N_{s_2}}$ and the direction tangent to $\mathscr{K}$ at p are four eigendirections of $D\mathcal{X}(p)$ associated with the eigenvalues

$$\mu_u(p) \stackrel{\text{def}}{=} -(2 + 2\Sigma_u(p)), \quad -\mu_{s_1}(p) \stackrel{\text{def}}{=} -(2 + 2\Sigma_{s_1}(p)), \quad -\mu_{s_2}(p) \stackrel{\text{def}}{=} -(2 + 2\Sigma_{s_2}(p)) \quad \text{and} \quad 0$$

Beware of the fact that μ_{s_1} and μ_{s_2} denote the *modulus* of the stable eigenvalues. As a consequence of Proposition 3.1, we have

$$0 < \mu_u < \mu_{s_1} < \mu_{s_2} \quad \text{in } \mathcal{K}_0$$

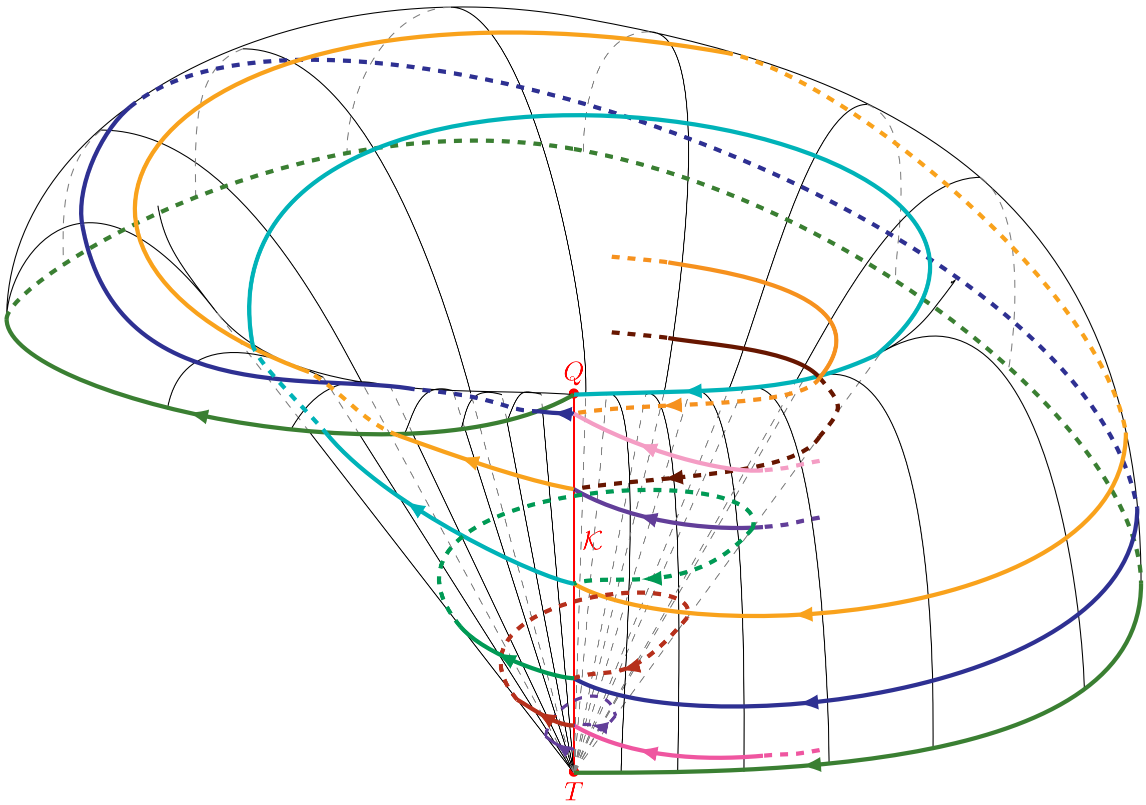

Figure 3.6 – Half of the quotient Mixmaster attractor and some type II orbits.

Induced Kasner map. The Kasner map is equivariant under the action of $\mathfrak{S}_3$ and therefore induces a map

$$\mathcal{F} : \mathcal{K} \to \mathcal{K}$$

called the *(induced) Kasner map*. We have in particular $\mathcal{F}(T) = \mathcal{F}(Q) = T$.

Quotient Mixmaster attractor. Let us denote by

$$\mathcal{A} \overset{\text{def}}{=} \mathscr{A}/\mathfrak{S}_3, \qquad \mathcal{A}^+ \overset{\text{def}}{=} \mathscr{A}^+/\mathfrak{S}_3$$

the quotient Mixmaster attractor and its "positive" part.

Induced type II orbits. One can remark that type II orbits in $\mathscr{A}^+$ which do not arrive at some Taub point do not cross the singular set $\mathscr{S}$. Hence they induce orbits of $\mathcal{X}$ in $\mathcal{A}^+$. We will use the following notations, where $p \in \mathcal{K}_0$ and q is a lift of p:

$$\mathcal{O}_{p\to\mathcal{F}(p)} \overset{\text{def}}{=} \pi\left(\mathscr{O}_{q\to\mathscr{F}(q)}\right)$$
$$\mathcal{O}_p^* \overset{\text{def}}{=} \pi\left(\mathscr{O}_q^*\right)$$

for $* \in \{u, s_1, s_2\}$. In the positive part of the quotient Mixmaster attractor, type II orbits look like a "loop" (see figure 3.6).

Induced heteroclinic chains.

Definition 3.7 (Induced heteroclinic chain)**.** Let $p \in \mathcal{B}^+ \setminus \{T\}$ and $q \in \mathscr{B}^+$ be a lift of p. The *heteroclinic chain* $\mathcal{H}(p)$ starting at p is the projection of $\mathscr{H}(q)$ by π.

For example, if p is a point of the Kasner interval $\mathcal{K}_0$ such that, for every $n \in \mathbb{N}$, $\mathcal{F}^n(p)$ is not a Taub point, then

$$\mathcal{H}(p) \stackrel{\text{def}}{=} \left(\mathcal{O}_{p \to \mathcal{F}(p)}, \mathcal{O}_{\mathcal{F}(p) \to \mathcal{F}^2(p)}, \mathcal{O}_{\mathcal{F}^2(p) \to \mathcal{F}^3(p)}, \dots\right) \tag{3.12}$$

3.7 Kasner parameter and Kasner map

The Kasner parameter. The main tool to study the dynamics of the Kasner map is the Kasner parameter. The Kasner parameter is a bijection $\mathcal{K} \to [1, +\infty]$ which conjugates the (induced) Kasner map $\mathcal{F}$ with the Gauss transformation on continued fractions (defined precisely in the next paragraph).

Definition 3.8 (Kasner parameter)**.** For every $p \in \mathcal{K}$, the *Kasner parameter* associated with p is defined by

$$\begin{aligned} \omega(p) &= \frac{\mu_{s_2}(p)}{\mu_{s_1}(p)} \in]1, +\infty[\quad \text{if } p \neq T,\, p \neq Q \qquad (3.13)\\ \omega(Q) &= 1 \\ \omega(T) &= +\infty \end{aligned}$$

This formula defines a bijection $\omega : \mathcal{K} \to [1, +\infty]$.

Let us also denote by μ_* $(* \in \{u, s_1, s_2\})$ the eigenvalue μ_* as a function of the Kasner parameter so that, for every $p \in \mathcal{K}$, $\mu_*(\omega(p)) := \mu_*(p)$. Formally this is an abuse of notations, but it will not give rise to confusion. A simple computation shows that, for every $\omega \in [1, +\infty]$,

$$\mu_u(\omega) = \frac{6\omega}{1 + \omega + \omega^2} \tag{3.14a}$$

$$\mu_{s_1}(\omega) = \frac{6(1+\omega)}{1 + \omega + \omega^2} \tag{3.14b}$$

$$\mu_{s_2}(\omega) = \frac{6\omega(1+\omega)}{1 + \omega + \omega^2} \tag{3.14c}$$

We refer to [HU09] for more details.

Conjugacy between the Kasner map and the Gauss transformation. As stated earlier, the Kasner parameter $\omega : \mathcal{K} \to [1, +\infty]$ conjugates the Kasner map $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ to the Gauss map $f : [1, +\infty] \to [1, +\infty]$ defined by

$$f(\omega) = \begin{cases} \omega - 1 & \text{if } \omega \geq 2 \\ \frac{1}{\omega - 1} & \text{if } 1 < \omega \leq 2 \\ +\infty & \text{if } \omega = 1 \text{ or } \omega = +\infty \end{cases} \tag{3.15}$$

We refer to [HU09] for more details. This conjugacy can be represented by the commutative diagram

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\mathcal{F}} & \mathcal{K} \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ [1, +\infty] & \xrightarrow{f} & [1, +\infty] \end{array}$$

We will also call f the Kasner map.

The era Kasner map. Let us define the *era Kasner map* $\bar{f} :]1, 2[\to [1, 2[$ by the formula

$$\bar{f}(\omega) = f^{r(\omega)}(\omega) \tag{3.16}$$

where $r(\omega) = \left\lfloor \frac{1}{\omega - 1} \right\rfloor$ (here, $\lfloor . \rfloor$ is the floor function).

Interpretation of the dynamics of the Kasner map in terms of continued fractions. Let $\omega_0 \in]1, +\infty[$. Let $[k_0; k_1, k_2, k_3, \dots]$ be the continued fraction expansion associated with ω_0, that is, the only (finite or infinite) sequence of integers such that

$$\omega_0 = k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \cdots}} \stackrel{\text{def}}{=} [k_0; k_1, k_2, k_3, \dots]$$

In terms of continued fractions, we have

$$f([k_0; k_1, k_2, k_3, \dots]) = \begin{cases} [k_0 - 1; k_1, k_2, k_3, \dots] & \text{if } k_0 \geq 2 \\ [k_1; k_2, k_3, \dots] & \text{if } k_0 = 1 \end{cases}$$

and if $1 < \omega_0 < 2$ (*i.e.* $k_0 = 1$), we have

$$\bar{f}([1; k_1, k_2, k_3, \dots]) = [1; k_2, k_3, \dots]$$

In other words, the era Kasner map $\bar{f}$ is a left-shift on the continued fractions.

In what follows, we assume that $\omega_0 \notin \mathbb{Q}$, so the continued fraction expansion associated with ω_0 is infinite. Let $(\omega_n)_{n\geq 0} \in [1, +\infty[^{\mathbb{N}}$ be the sequence generated by the Kasner map f from ω_0, *i.e.* $\omega_{n+1} = f(\omega_n)$ for every $n \geq 0$. Every term of this sequence is called an *epoch*. It is quite natural to consider the subsequence $(\hat{\omega}_n)_{n\geq 1}$ defined by

$$\hat{\omega}_n \stackrel{\text{def}}{=} \omega_{k_0 + k_1 + \cdots + k_{n-1} - 1} = [1; k_n, k_{n+1}, \dots]$$

This subsequence divides the sequence $(\omega_n)_{n\geq 0}$ in *eras* of the form:

$$\begin{aligned} \big(f(\hat{\omega}_n) = &[k_n; k_{n+1}, k_{n+2}, \dots], \\ &[k_n - 1; k_{n+1}, k_{n+2}, \dots], \\ &\dots, \\ &[1; k_{n+1}, k_{n+2}, \dots] = \hat{\omega}_{n+1}\big) \end{aligned}$$

On each era, $(\omega_n)_{n\geq 0}$ is decreasing. Moreover, one can remark that $\bar{f}(\hat{\omega}_n) = \hat{\omega}_{n+1}$ for any $n \geq 1$.

Chapter 4

Local expression of the Wainwright-Hsu vector field near the Kasner circle

In chapter 6, we will study the dynamics of the Wainwright-Hsu vector field $\mathcal{X}$ in the neighbourhood of a point of the Kasner interval $\mathcal{K}_0$. The aim of the present section is to describe a "nice" system of local coordinates ξ in the neighbourhood of $\mathcal{K}_0$ and to write a workable local form of $\mathcal{X}$ in these "nice" coordinates. The key property of these coordinates is the fact that they straighten the stable and the unstable manifolds of the points belonging to $\mathcal{K}_0$ for $\mathcal{X}$. We now proceed to define those stable and unstable manifolds.

For any $\omega \in]1, +\infty[$, let us denote by $\mathcal{P}_\omega$ the unique point of $\mathcal{K}_0$ whose Kasner parameter is ω. Recall that for every $\omega \in]1, +\infty[$, there exist:

- one type II orbit, denoted by $\mathcal{O}^u_\omega$, which converges to the point $\mathcal{P}_\omega$ as time goes to $-\infty$. This orbit is asymptotically tangent to the direction ∂_{N_u};
- two type II orbits, denoted by $\mathcal{O}^{s_1}_\omega$ and $\mathcal{O}^{s_2}_\omega$, which converge to the point $\mathcal{P}_\omega$ as time goes to $+\infty$. They are respectively asymptotically tangent to the directions $\partial_{N_{s_1}}$ and $\partial_{N_{s_2}}$.

Let us denote by

$$W^u(\mathcal{P}_\omega, \mathcal{X}) \stackrel{\text{def}}{=} \{\mathcal{P}_\omega\} \cup \mathcal{O}^u_\omega$$

the reunion of the point $\mathcal{P}_\omega$ with the type II orbit which converges to the point $\mathcal{P}_\omega$ as time goes to $-\infty$. This is a 1-dimensional smooth[1] embedded submanifold of $\mathcal{B}^+$ tangent to the direction ∂_{N_u} at the point $\mathcal{P}_\omega$. The notation $W^u(\mathcal{P}_\omega, \mathcal{X})$ comes from the fact that it is the unstable manifold of the point $\mathcal{P}_\omega$ for the vector field $\mathcal{X}$. Indeed, it follows from the stable manifold theorem that the unstable manifold of the point $\mathcal{P}_\omega$ for the vector field $\mathcal{X}$ is 1-dimensional. Moreover, from what precedes, we get that $W^u(\mathcal{P}_\omega, \mathcal{X})$ is included in the stable manifold of the point $\mathcal{P}_\omega$ for the vector field $\mathcal{X}$. By dimension, this inclusion must be an equality. In other words,

$$W^u(\mathcal{P}_\omega, \mathcal{X}) = \left\{x \in \mathcal{B}^+ \mid \mathcal{X}^t(x) \xrightarrow[t \to -\infty]{} \mathcal{P}_\omega\right\}$$

Analogously, let

$$W^{s_1}(\mathcal{P}_\omega, \mathcal{X}) \stackrel{\text{def}}{=} \{\mathcal{P}_\omega\} \cup \mathcal{O}^{s_1}_\omega$$
$$W^{s_2}(\mathcal{P}_\omega, \mathcal{X}) \stackrel{\text{def}}{=} \{\mathcal{P}_\omega\} \cup \mathcal{O}^{s_2}_\omega$$

$W^{s_1}(\mathcal{P}_\omega, \mathcal{X})$ (resp. $W^{s_2}(\mathcal{P}_\omega, \mathcal{X})$) is a 1-dimensional smooth embedded submanifold of $\mathcal{B}^+$ tangent to the direction $\partial_{N_{s_1}}$ (resp. $\partial_{N_{s_2}}$) at the point $\mathcal{P}_\omega$, called the "weak stable manifold" (resp. the "strong stable manifold"). Note that $W^{s_1}(\mathcal{P}_\omega, \mathcal{X})$ cannot be characterized as a stable manifold. Indeed,

[1]The word *smooth* will always stand for C^∞ in this work.

$W^{s_1}(\mathcal{P}_\omega, \mathcal{X})$ and $W^{s_2}(\mathcal{P}_\omega, \mathcal{X})$ are both included in the stable manifold of the point $\mathcal{P}_\omega$ for the vector field $\mathcal{X}$:

$$W^s(\mathcal{P}_\omega, \mathcal{X}) \stackrel{\text{def}}{=} \left\{ x \in \mathcal{B}^+ \mid \mathcal{X}^t(x) \xrightarrow[t \to +\infty]{} \mathcal{P}_\omega \right\}$$

It follows from the stable manifold theorem that the stable manifold of the point $\mathcal{P}_\omega$ for the vector field $\mathcal{X}$ is a 2-dimensional smooth embedded submanifold of $\mathcal{B}^+$.

The submanifolds $\{W^u(\mathcal{P}_\omega, \mathcal{X})\}_{\omega \in]1,+\infty[}$ foliate the 2-dimensional submanifold

$$\begin{aligned} W^u(\mathcal{K}_0, \mathcal{X}) &\stackrel{\text{def}}{=} \bigsqcup_{\omega \in]1,+\infty[} W^u(\mathcal{P}_\omega, \mathcal{X}) \\ &= \{N_{s_1} = N_{s_2} = 0, N_u \geq 0\} \\ &= \left\{ x \in \mathcal{B}^+ \mid \exists p \in \mathcal{K}_0, \mathcal{X}^t(x) \xrightarrow[t \to -\infty]{} p \right\} \end{aligned}$$

Analogously, the submanifolds $\{W^s(\mathcal{P}_\omega, \mathcal{X})\}_{\omega \in]1,+\infty[}$ foliate the 2-dimensional submanifold

$$\begin{aligned} W^s(\mathcal{K}_0, \mathcal{X}) &\stackrel{\text{def}}{=} \bigsqcup_{\omega \in]1,+\infty[} W^s(\mathcal{P}_\omega, \mathcal{X}) \\ &= \{N_u = 0, N_{s_1} \geq 0, N_{s_2} \geq 0\} \\ &= \left\{ x \in \mathcal{B}^+ \mid \exists p \in \mathcal{K}_0, \mathcal{X}^t(x) \xrightarrow[t \to +\infty]{} p \right\} \end{aligned}$$

Indeed, $\{N_u = 0, N_{s_1} \geq 0, N_{s_2} \geq 0\}$ is a 3-dimensional submanifold of $\mathcal{B}^+$ containing $W^s(\mathcal{K}_0, \mathcal{X})$ which is also 3-dimensional (being continuously foliated by 2-dimensional submanifolds).

Definition 4.1. For any $\omega \in \,]1, +\infty[$, $C > 0$ and $n \in \mathbb{N}$, let us denote by

$$\mathcal{B}_{\omega,C,n} \stackrel{\text{def}}{=} B\left(\mathcal{P}_\omega, \frac{1}{C\omega^n}\right) \subset \mathcal{B}_0^+$$

the ball of center $\mathcal{P}_\omega$ and radius $\frac{1}{C\omega^n}$ in the phase space $\mathcal{B}_0^+$ (for the distance $d_{\mathcal{B}}$, see (3.11)) and by

$$B_{\omega,C,n} \stackrel{\text{def}}{=} \left\{ (x_u, x_{s_1}, x_{s_2}, x_c) \in \left(\mathbb{R}^+\right)^3 \times \,]1, +\infty[\; \mid \max(x_u, x_{s_1}, x_{s_2}, |x_c - \omega|) \leq \frac{1}{C\omega^n} \right\}$$

the ball of center $(0, 0, 0, \omega)$ and radius $\frac{1}{C\omega^n}$ in $\left(\mathbb{R}^+\right)^3 \times \,]1, +\infty[$ (for the sup-norm).

We now proceed to give formal statements of the main results of this section. We delay the proofs until the following sections.

Proposition 4.2 (System of local coordinates)**.** *There exist two constants $C > 0$ and $n \in \mathbb{N}$, an open neighbourhood $\mathcal{U}_\xi$ of $\mathcal{K}_0$ in $\mathcal{B}_0^+$ (see (3.10c)), an open neighbourhood U_ξ of $\{0_{\mathbb{R}^3}\} \times \,]1, +\infty[$ in $\left(\mathbb{R}^+\right)^3 \times \,]1, +\infty[$ and a smooth system of local coordinates*

$$\xi = (x_u, x_{s_1}, x_{s_2}, x_c) : \mathcal{U}_\xi \to U_\xi$$

with the following properties:

1. *We have*

$$(x_u, x_{s_1}, x_{s_2}) = (N_u, N_{s_1}, N_{s_2}) \tag{4.1a}$$

and

$$x_c = \omega \quad \text{in restriction to } \mathcal{K}_0 \tag{4.1b}$$

In particular, ξ maps the Kasner interval $\mathcal{K}_0$ to $\{x_u = x_{s_1} = x_{s_2} = 0\} \cap U_\xi$:

$$\begin{aligned} \xi(\mathcal{K}_0) &= \{0_{\mathbb{R}^3}\} \times \,]1, +\infty[\\ &= \{x_u = x_{s_1} = x_{s_2} = 0\} \cap U_\xi \end{aligned} \tag{4.2}$$

2. *The chart ξ straightens the stable and unstable manifolds foliations along the Kasner interval $\mathcal{K}_0$. More precisely, for any $\omega \in]1,+\infty[$, we have*

$$\xi\left(W^u_{loc}(\mathcal{P}_\omega,\mathcal{X})\cap\mathcal{U}_\xi\right) = \{x_{s_1} = x_{s_2} = 0, x_c = \omega\}\cap U_\xi \tag{4.3a}$$
$$\xi\left(W^s_{loc}(\mathcal{P}_\omega,\mathcal{X})\cap\mathcal{U}_\xi\right) = \{x_u = 0, x_c = \omega\}\cap U_\xi \tag{4.3b}$$
$$\xi\left(W^{s_1}_{loc}(\mathcal{P}_\omega,\mathcal{X})\cap\mathcal{U}_\xi\right) = \{x_u = x_{s_2} = 0, x_c = \omega\}\cap U_\xi \tag{4.3c}$$
$$\xi\left(W^{s_2}_{loc}(\mathcal{P}_\omega,\mathcal{X})\cap\mathcal{U}_\xi\right) = \{x_u = x_{s_1} = 0, x_c = \omega\}\cap U_\xi \tag{4.3d}$$

3. *The open sets $\mathcal{U}_\xi$ and U_ξ are "big enough": for any $\omega \in]1,+\infty[$,*

$$\mathcal{B}_{\omega,C,n}\subset\mathcal{U}_\xi \quad \textit{and} \quad B_{\omega,C,n}\subset U_\xi \tag{4.4}$$

4. *The C^6-norm of ξ restricted to a neighbourhood of $\mathcal{P}_\omega$ admits an upper bound which is polynomial in ω, and similarly for ξ^{-1}. More precisely, for any $\omega\in]1,+\infty[$,*

$$\|\xi\|_{C^6}\le C\omega^n \quad \textit{in restriction to } \mathcal{B}_{\omega,C,n} \tag{4.5a}$$
$$\left\|\xi^{-1}\right\|_{C^6}\le C\omega^n \quad \textit{in restriction to } B_{\omega,C,n} \tag{4.5b}$$

From now on the system of local coordinates ξ given by Proposition 4.2 is fixed. We will use roman letters for objects viewed in the system of local coordinates ξ. For example, we will denote by X the vector field $\xi_*\mathcal{X}$. The Wainwright-Hsu vector field $\mathcal{X}$ has a "nice" expression in the local coordinates ξ:

- The fact that ξ straightens the stable and the unstable manifolds of $\mathcal{X}$ implies that a lot of non linear terms vanish in the development of X.
- The estimates on the C^6 norm of ξ and ξ^{-1} allow one to get analogous estimates on the C^3 norm of the non linear terms appearing in the development of X. These estimates will eventually lead to a C^1 control of the non linear terms appearing in the development of the renormalized vector field X_ω (see Proposition 4.8).

Proposition 4.3 (Local expression of X)**.** *There exist two constants $C>0$ and $n\in\mathbb{N}$ such that the local vector field X admits the following expression on the open set $\cup_{\omega\in]1,+\infty[}B_{\omega,C_0,n_0}\subset U_\xi$:*

$$X(x) = \begin{pmatrix}\mu_u(x_c) & 0 & 0 & 0\\ 0 & -\mu_{s_1}(x_c) & 0 & 0\\ 0 & 0 & -\mu_{s_2}(x_c) & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_u\\ x_{s_1}\\ x_{s_2}\\ x_c\end{pmatrix} + \begin{pmatrix}\bar X^{u,u}_u(x)x_u^2 + \bar X^{u,s_1}_u(x)x_ux_{s_1} + \bar X^{u,s_2}_u(x)x_ux_{s_2}\\ \bar X^{u,s_1}_{s_1}(x)x_ux_{s_1} + \bar X^{s_1,s_1}_{s_1}(x)x_{s_1}^2 + \bar X^{s_1,s_2}_{s_1}(x)x_{s_1}x_{s_2}\\ \bar X^{u,s_2}_{s_2}(x)x_ux_{s_2} + \bar X^{s_1,s_2}_{s_2}(x)x_{s_1}x_{s_2} + \bar X^{s_2,s_2}_{s_2}(x)x_{s_2}^2\\ \bar X^{u,s_1}_c(x)x_ux_{s_1} + \bar X^{u,s_2}_c(x)x_ux_{s_2}\end{pmatrix} \tag{4.6}$$

Moreover, for every $\omega\in]1,+\infty[$, the functions $\bar X^{*,*}_*$ (where $*\in\{u,s_1,s_2,c\}$ and different occurrences of $*$ are independent) appearing in the non linear part of (4.6) *satisfy*

$$\|\bar X^{*,*}_*\|_{C^3}\le C\omega^n \quad \textit{on } B_{\omega,C,n} \tag{4.7}$$

Remark 4.4. $\mu_u(x_c)$, $\mu_{s_1}(x_c)$ and $\mu_{s_2}(x_c)$ defined in (3.14) are the nonzero eigenvalues of the derivative $DX(0,0,0,x_c)$.

To further simplify the computations, we will renormalize the local vector field X (by multiplying it by a positive function γ_ω) in order to linearize the dynamics in the unstable direction. This trick will allow us to compute explicit travel time between two local sections.

Let $\omega\in]1,+\infty[$. We define the *renormalization function* γ_ω in the neighbourhood of $(0,0,0,\omega)$ by the formula

$$\gamma_\omega(x) = \frac{\mu_u(\omega)}{\mu_u(x_c) + \bar X^{u,u}_u(x)x_u + \bar X^{u,s_1}_u(x)x_{s_1} + \bar X^{u,s_2}_u(x)x_{s_2}} \tag{4.8}$$

The renormalization function γ_ω is chosen so that, according to (4.6), the coordinate of $(\gamma_\omega.X)$ in the direction ∂_{x_u} is $\mu_u(\omega)x_u$. In other words, $(\gamma_\omega.X)$ is "linear" in the direction ∂_{x_u}.

Lemma 4.5 (Domain of γ_ω). *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in]1,+\infty[$, for every $x \in B_{\omega,C,n}$, we have*

$$\mu_u(x_c) + \bar{X}_u^{u,u}(x)x_u + \bar{X}_u^{u,s_1}(x)x_{s_1} + \bar{X}_u^{u,s_2}(x)x_{s_2} > 0 \tag{4.9}$$

In particular, γ_ω is well defined and positive on $B_{\omega,C,n}$.

Definition 4.6. We define the *local vector field*

$$X_\omega \overset{\text{def}}{=} \gamma_\omega.X \tag{4.10}$$

on $B_{\omega,C,n}$ for C, n large enough so that, for every $\omega \in]1,+\infty[$, the conclusion of Lemma 4.5 is satisfied.

Remark 4.7. In restriction to $B_{\omega,C,n}$, the orbits of X_ω are the same than the one of X, up to a time reparametrization.

Proposition 4.8 (Local expression of X_ω). *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in]1,+\infty[$, the local vector field X_ω admits the following expression on the open ball $B_{\omega,C,n}$:*

$$X_\omega(x) = \begin{pmatrix} \mu_u(\omega) & 0 & 0 & 0 \\ 0 & -\tilde{\mu}_{\omega,s_1}(x_c) & 0 & 0 \\ 0 & 0 & -\tilde{\mu}_{\omega,s_2}(x_c) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_u \\ x_{s_1} \\ x_{s_2} \\ x_c \end{pmatrix} + \begin{pmatrix} 0 \\ X_{\omega,s_1}^{u,s_1}(x)x_u x_{s_1} + X_{\omega,s_1}^{s_1,s_1}(x)x_{s_1}^2 + X_{\omega,s_1}^{s_2,s_1}(x)x_{s_2}x_{s_1} \\ X_{\omega,s_2}^{u,s_2}(x)x_u x_{s_2} + X_{\omega,s_2}^{s_1,s_2}(x)x_{s_1}x_{s_2} + X_{\omega,s_2}^{s_2,s_2}(x)x_{s_2}^2 \\ X_{\omega,c}^{u,s_1}(x)x_u x_{s_1} + X_{\omega,c}^{u,s_2}(x)x_u x_{s_2} \end{pmatrix} \tag{4.11}$$

where

$$\tilde{\mu}_{\omega,s_i}(x_c) \overset{def}{=} \frac{\mu_u(\omega)}{\mu_u(x_c)}\mu_{s_i}(x_c) \tag{4.12}$$

Moreover, the functions $X_{\omega,*}^{*;*}$ (where $* \in \{u,s_1,s_2,c\}$ and different occurrences of $*$ are independent) appearing in the non linear part of (4.11) *satisfy*

$$\|X_{\omega,*}^{*;*}\|_{C^1} \leq C\omega^n \quad \text{on } B_{\omega,C,n} \tag{4.13}$$

Remark 4.9. $\mu_u(\omega)$, $-\tilde{\mu}_{\omega,s_1}(x_c)$, $-\tilde{\mu}_{\omega,s_2}(x_c)$ are the nonzero eigenvalues of $DX_\omega(0,0,0,x_c)$.

4.1 A straightening theorem for a stable manifold foliation

In this subsection, we present a general result on vector fields, Theorem 4.10 and its addendum, that will be used to construct the system of local coordinates ξ given by Proposition 4.2. This result is a reformulation and a simplification of Theorem C.5 and its addendum in our current context. We refer to appendix C for an independent and complete proof of these general theorems.

The context is as follows. Let Ω be an open set of $\mathbb{R}^n$, G a linear subspace of $\mathbb{R}^n$ and $Y : \Omega \to \mathbb{R}^n$ be a smooth vector field vanishing on $\Omega_0 := \Omega \cap G \neq \varnothing$. Assume that there exists a complement F of G such that for every $\omega \in \Omega_0$, F is stabilized and contracted by $DY(\omega)$ (Ω_0 is said to be "normally contracted"). Recall that we denote by $W^s(\omega, Y)$ the stable set of ω for Y, that is, the union of all the orbits of Y which converge to the point ω as time goes to $+\infty$. According to the standard stable manifold theorem (see *e.g.* [KH97], [Irw01], [Rue89], [Rob99], [BS02] and [HPS06]), $W^s(\omega, Y)$ is a smooth embedded submanifold passing through ω and the family of stable manifolds $(W^s(\omega,Y))_{\omega\in\Omega_0}$ is a smooth foliation of a small neighbourhood Ω of Ω_0. Moreover, the stable foliation $(W^s(\omega,Y))_{\omega\in\Omega_0}$ can be straightened using smooth local charts.

The standard result explained above can be stated as following:

Theorem 4.10 (Straightening of a stable foliation)**.** *Let Ω be an open set of $\mathbb{R}^n$, G be a linear subspace of $\mathbb{R}^n$ and $Y : \Omega \to \mathbb{R}^n$ be a smooth vector field such that*

1. *Y vanishes on $\Omega_0 \overset{def}{=} \Omega \cap G \neq \varnothing$;*
2. *There exists a complement F of G such that for every $\omega \in \Omega_0$, the derivative*

$$DY(\omega) : T_\omega\mathbb{R}^n \simeq \mathbb{R}^n \to T_\omega\mathbb{R}^n \simeq \mathbb{R}^n$$

stabilizes F and

$$\mu_{max}\Big((DY(\omega))_{|F}\Big) < 0$$

where $\mu_{max}\Big((DY(\omega))_{|F}\Big)$ denotes the maximum of the real parts of the eigenvalues of $(DY(\omega))_{|F}$.

Let $\omega_0 \in \Omega_0$. Then there exists a smooth local coordinate system ξ defined on a ball $B := B_{\mathbb{R}^n}(\omega_0, R)$ such that the family of stable manifolds $(W^s(\omega, Y))_{\omega\in\Omega_0\cap B}$ foliates B and is straightened by ξ: for every $\omega \in \Omega_0 \cap B$,

$$\xi\left(W^s(\omega, Y) \cap B\right) = (\omega + F) \cap \xi(B)$$

We emphasize the fact that Theorem 4.10 is a straightforward consequence of the stable manifold theorem. The point of appendix C is to prove the following addendum, which provides some explicit estimates on the radius R and on the derivatives of all orders of ξ and ξ^{-1}:

Addendum 4.11. *For every $r > 0$ such that $\overline{B_{\mathbb{R}^n}(\omega_0, r)} \subset \Omega$, one can find a radius R and a local coordinate system ξ on $B_{\mathbb{R}^n}(\omega_0, R)$ as above satisfying the following properties:*

1. *The radius R admits a lower bound which is*
   - *linear in r,*
   - *polynomial in the spectral gap $\left|\mu_{max}\Big((DY(\omega_0))_{|F}\Big)\right|$,*
   - *inversely linear in the norm of the second derivative of Y on the closed ball $\overline{B_{\mathbb{R}^n}(\omega_0, r)}$,*
   - *inversely polynomial in*
     - *the norm of $DY(\omega_0)$,*
     - *the angle between the generalized eigenspaces of $DY(\omega_0)$.*

   This lower bound depends only on the parameters cited above.
2. *For every $\epsilon > 0$, ξ restricted to $B_{\mathbb{R}^n}(\omega_0, \epsilon R)$ is ϵ-close to the identity in C^1-norm.*
3. *The norms of the k-th derivatives of ξ and ξ^{-1} admit an upper bound which is*
   - *polynomial in*
     - *the norm of $DY(\omega_0)$,*
     - *the angle between the generalized eigenspaces of $DY(\omega_0)$,*
     - *the norms of the $(k+1)$ first derivatives of Y on the closed ball $\overline{B_{\mathbb{R}^n}(\omega_0, r)}$*
   - *inversely polynomial in*
     - *the spectral gap $\left|\mu_{max}\Big((DY(\omega_0))_{|F}\Big)\right|$*
     - *r*

   This upper bound depends only on the parameters cited previously.

   Moreover, identifying $\mathbb{R}^n = F \oplus G$ with $F \times G$ the local coordinate system ξ has the following form:

$$\xi(x, y) = (x, y + \tilde{\xi}(x, y))$$

where $\tilde{\xi}(0, y) \equiv 0$.

Finally, the different charts are compatible in the following sense: for any two charts ξ and ξ' defined respectively on B and B', we have $\xi = \xi'$ in restriction to $B \cap B'$.

Remark 4.12. In order to get such estimates on R and ξ, one must choose a compact ball $\overline{B(\omega_0, r)} \subset \Omega$ on which one controls the derivatives of all orders of Y. There is no canonical choice and one can use the parameter r to make a choice depending on its needs.

4.2 System of local coordinates ξ

The existence of the system of local coordinates ξ which straightens the stable and the unstable foliations of $\mathcal{X}$ (see Proposition 4.2) is a consequence of Theorem 4.10 and Addendum 4.11. The proof will be divided in several steps. We first construct a chart which straightens $\mathcal{K}_0$, $W^s(\mathcal{K}_0,\mathcal{X})$ and $W^u(\mathcal{K}_0,\mathcal{X})$. This is done by using the Kasner parameter and the radial projection on the Kasner circle. We then apply Theorem 4.10 and Addendum 4.11 twice: in $W^s(\mathcal{K}_0,\mathcal{X})$ to straighten the foliation $\{W^s(\mathcal{P}_\omega,\mathcal{X})\}_{\omega\in]1,+\infty[}$ and in $W^u(\mathcal{K}_0,\mathcal{X})$ to straighten the foliation $\{W^u(\mathcal{P}_\omega,\mathcal{X})\}_{\omega\in]1,+\infty[}$. Finally, we merge the two families of charts (of lower dimension since we restricted ourselves to submanifolds) obtained above into a unique chart straightening both the stable foliation and the unstable foliation.

Proof of Proposition 4.2. It will be convenient for this proof to work in $\mathcal{B}_0$ instead of $\mathcal{B}_0^+$. The reason is technical: the Kasner circle is in the boundary of $\mathcal{B}_0^+$ and Theorem 4.10 and Addendum 4.11 apply to a vector field defined on an open set of a vector space. Taking a chart of $\mathcal{B}_0^+$ in the neighbourhood of the Kasner circle, we cannot apply Theorem 4.10 and Addendum 4.11 to the push forward of $\mathcal{X}$ by this chart.

Step 1: Straightening of $\mathcal{K}_0$, $W^s(\mathcal{K}_0,\mathcal{X})$ and $W^u(\mathcal{K}_0,\mathcal{X})$. Recall from section 3.6 that the map

$$\xi_0 : y \mapsto \left(N_u(y), N_{s_1}(y), N_{s_2}(y), \Sigma_u(y), \Sigma_{s_1}(y), \Sigma_{s_2}(y)\right)$$

is a diffeomorphism from $\mathcal{B}_0$ (see (3.10c)) to B_0 where

$$\begin{aligned} B_0 \overset{\text{def}}{=} \{(N_u, N_{s_1}, N_{s_2}, \Sigma_u, \Sigma_{s_1}, \Sigma_{s_2}) \in \mathbb{R}^6 \mid \Sigma_u + \Sigma_{s_1} + \Sigma_{s_2} = 0,& \\ 6 - \left(\Sigma_u^2 + \Sigma_{s_1}^2 + \Sigma_{s_2}^2\right) - \frac{1}{2}\left(N_u^2 + N_{s_1}^2 + N_{s_2}^2\right) + \left(N_u N_{s_1} + N_{s_1} N_{s_2} + N_{s_2} N_u\right) = 0,& \\ \Sigma_u < \Sigma_{s_1} < \Sigma_{s_2}\}& \end{aligned}$$

Let us identify the Kasner interval $\mathcal{K}_0$ with the set

$$\left\{(\Sigma_u, \Sigma_{s_1}, \Sigma_{s_2}) \in \mathbb{R}^3 \mid \Sigma_u + \Sigma_{s_1} + \Sigma_{s_2} = 0, \Sigma_u^2 + \Sigma_{s_1}^2 + \Sigma_{s_2}^2 = 6, \Sigma_u < \Sigma_{s_1} < \Sigma_{s_2}\right\}$$

The idea is to "straighten" B_0 into a subset of the product $\mathbb{R}^3 \times \mathcal{K}_0$. To do this, we use the radial projection from the sixth of the $(\Sigma_u, \Sigma_{s_1}, \Sigma_{s_2})$-plane

$$\left\{(\Sigma_u, \Sigma_{s_1}, \Sigma_{s_2}) \in \mathbb{R}^3 \mid \Sigma_u + \Sigma_{s_1} + \Sigma_{s_2} = 0, \Sigma_u < \Sigma_{s_1} < \Sigma_{s_2}\right\}$$

on the Kasner interval $\mathcal{K}_0$. In other words, we consider the chart

$$\xi_1 : \left| \begin{array}{rcl} B_0 & \to & \xi_1(B_0) \subset \mathbb{R}^3 \times \mathcal{K}_0 \\ (N_u, N_{s_1}, N_{s_2}, \Sigma_u, \Sigma_{s_1}, \Sigma_{s_2}) & \mapsto & \left(N_u, N_{s_1}, N_{s_2}, \frac{\Sigma_u}{\sqrt{\frac{q}{2}}}, \frac{\Sigma_{s1}}{\sqrt{\frac{q}{2}}}, \frac{\Sigma_{s2}}{\sqrt{\frac{q}{2}}}\right) \end{array} \right.$$

where q is the deceleration parameter (see (2.15)) and more explicitly

$$\sqrt{\frac{q}{2}} = \sqrt{\frac{1}{6}\left(\Sigma_u^2 + \Sigma_{s_1}^2 + \Sigma_{s_2}^2\right)}$$

Note that ξ_1 is well defined because $q \neq 0$ on B_0. Moreover, the equality

$$\sqrt{\frac{q}{2}} = \sqrt{1 - \frac{1}{12}\left(N_u^2 + N_{s_1}^2 + N_{s_2}^2\right) + \frac{1}{6}\left(N_u N_{s_1} + N_{s_1} N_{s_2} + N_{s_2} N_u\right)}$$

which holds true on B_0 shows that $\sqrt{\frac{q}{2}}$ is entirely determined by N_u, N_{s_1} and N_{s_2}. It follows that ξ_1 is invertible and its inverse is

$$\xi_1^{-1} : \left| \begin{array}{rcl} \xi_1(B_0) & \to & B_0 \\ (N_u, N_{s_1}, N_{s_2}, \Sigma_u, \Sigma_{s_1}, \Sigma_{s_2}) & \mapsto & \left(N_u, N_{s_1}, N_{s_2}, \sqrt{\frac{q}{2}}\Sigma_u, \sqrt{\frac{q}{2}}\Sigma_{s_1}, \sqrt{\frac{q}{2}}\Sigma_{s_2}\right) \end{array} \right.$$

Now recall that the Kasner parameter (defined in (3.13)) is a diffeomorphism from $\mathcal{K}_0$ to $]1,+\infty[$. The composition of the Kasner parameter with the charts ξ_1 and ξ_0 leads to a smooth chart

$$\xi_2 = (N_u, N_{s_1}, N_{s_2}, \omega) \colon \left| \begin{array}{ccc} \mathcal{B}_0 & \to & \xi_2(\mathcal{B}_0) \subset \mathbb{R}^3 \times]1,+\infty[\\ y & \mapsto & \big(N_u(y), N_{s_1}(y), N_{s_2}(y), \omega(\pi_2 \circ \xi_1 \circ \xi_0(y))\big) \end{array} \right. \tag{4.14}$$

where π_2 is the projection $\mathbb{R}^3 \times \mathcal{K}_0 \to \mathcal{K}_0$.

The chart ξ_2 straightens $\mathcal{K}_0$, $W^s(\mathcal{K}_0, \mathcal{X})$ and $W^u(\mathcal{K}_0, \mathcal{X})$. Remark that $\xi_2(\mathcal{B}_0)$ contains the open set $B_{\mathbb{R}^3}(0,1/2) \times]1,+\infty[$. Moreover, by a straightforward computation, there exist $C_2 > 0$ and $n_2 \in \mathbb{N}$ such that for every $\omega \in]1,+\infty[$, we have

$$\|\xi_2\|_{C^6} \le C_2 \omega^{n_2} \quad \text{in restriction to } \mathcal{B}_{\omega, C_2, n_2} \tag{4.15a}$$

$$\left\|\xi_2^{-1}\right\|_{C^6} \le C_2 \omega^{n_2} \quad \text{in restriction to } B_{\omega, C_2, n_2} \tag{4.15b}$$

Step 2: Straightening of the stable foliation of $(\xi_2)_* \mathcal{X}$. Let Y^s be the restriction of $(\xi_2)_* \mathcal{X}$ to $\Omega^s := \xi_2(\mathcal{B}_0) \cap \{N_u = 0\}$, that is, the stable manifold of $\{0_{\mathbb{R}^3}\} \times]1,+\infty[$ for $(\xi_2)_* \mathcal{X}$. Identifying $\mathbb{R}^4 \cap \{N_u = 0\}$ with $\mathbb{R}^3$ endowed with the coordinates N_{s_1}, N_{s_2} and ω, Ω^s is an open set of $\mathbb{R}^3$. Since

$$\Omega^s \cap \{N_{s_1} = N_{s_2} = 0\}$$

is the image of $\mathcal{K}_0$ by ξ_2, Y^s vanishes on $\Omega^s \cap \{N_{s_1} = N_{s_2} = 0\}$. Let $F := \mathbb{R}\partial_{N_{s_1}} \oplus \mathbb{R}\partial_{N_{s_2}}$ and $G := \mathbb{R}\partial_\omega$. According to (3.5) and (4.14), for every $\omega \in]1,+\infty[$, the decomposition $F \oplus G = \mathbb{R}^3$ is stabilized by $DY^s(0,0,\omega)$ and the eigenvalues μ_{s_1} and μ_{s_2} of $\big(DY^s(0,0,\omega)\big)_{|F}$ are both (strictly) negative.

For $\omega \in]1,+\infty[$, let

$$r(\omega) \stackrel{\text{def}}{=} \min\left(\frac{1}{4}, \frac{\omega - 1}{2}\right)$$

Observe that $B_{\mathbb{R}^3}((0,0,\omega), r(\omega)) \subset \Omega^s$.

According to Theorem 4.10 and Addendum 4.11, there exist two constants $C_s > 0$ and $n_s \in \mathbb{N}$ such that, for any $\omega_0 \in]1,+\infty[$, there exist an open set $V^s_{\omega_0} \subset \Omega^s$ and a smooth chart

$$\xi^s_{3,\omega_0} \colon \left| \begin{array}{ccc} V^s_{\omega_0} & \to & \xi^s_{3,\omega_0}\left(V^s_{\omega_0}\right) \subset \mathbb{R}^3 \\ (N_{s_1}, N_{s_2}, \omega) & \mapsto & (N_{s_1}, N_{s_2}, \omega + \tilde{\xi}^s_{2,\omega_0}(N_{s_1}, N_{s_2}, \omega)) \end{array} \right.$$

where $\tilde{\xi}^s_{3,\omega_0}(0,0,\omega) \equiv 0$, such that ξ^s_{3,ω_0} straightens the stable foliation of Y^s in $V^s_{\omega_0}$:

$$\xi^s_{3,\omega_0}\left(W^s((0,0,\bar{\omega}), Y^s) \cap V^s_{\omega_0}\right) = \left\{(N_{s_1}, N_{s_2}, \omega) \mid \omega = \bar{\omega}\right\} \cap \xi^s_{3,\omega_0}\left(V^s_{\omega_0}\right) \tag{4.16}$$

Moreover, $V^s_{\omega_0}$ and $\xi^s_{3,\omega_0}\left(V^s_{\omega_0}\right)$ both contain the open set[2]

$$B^s_{\omega_0} \stackrel{\text{def}}{=} B_{\mathbb{R}^2}\left(0, R^s_{\omega_0}\right) \times \left]\omega_0 - \min\left(R^s_{\omega_0}, \frac{\omega_0 - 1}{2}\right), \omega_0 + \min\left(R^s_{\omega_0}, \frac{\omega_0 - 1}{2}\right)\right[$$

where

$$R^s_{\omega_0} := \frac{1}{C_s \omega_0^{n_s}},$$

and

$$\left\|\xi^s_{3,\omega_0}\right\|_{C^6}, \left\|\left(\xi^s_{3,\omega_0}\right)^{-1}\right\|_{C^6} \le C_s \omega_0^{n_s} \tag{4.17}$$

Remark that the particular form of ξ^s_{3,ω_0} assures that the invariant manifolds $W^{s_1}(\mathcal{P}_\omega, \mathcal{X})$ and $W^{s_2}(\mathcal{P}_\omega, \mathcal{X})$ are both straightened automatically by the "composition" of ξ^s_{3,ω_0} with ξ_2:

$$\xi^s_{3,\omega_0}\left(\xi_2\left(W^{s_1}(\mathcal{P}_{\bar{\omega}}, \mathcal{X})\right) \cap V^s_{\omega_0}\right) = \left\{(N_{s_1}, N_{s_2}, \omega) \mid N_{s_2} = 0, \omega = \bar{\omega}\right\} \cap \xi^s_{3,\omega_0}\left(V^s_{\omega_0}\right) \tag{4.18a}$$

$$\xi^s_{3,\omega_0}\left(\xi_2\left(W^{s_2}(\mathcal{P}_{\bar{\omega}}, \mathcal{X})\right) \cap V^s_{\omega_0}\right) = \left\{(N_{s_1}, N_{s_2}, \omega) \mid N_{s_1} = 0, \omega = \bar{\omega}\right\} \cap \xi^s_{3,\omega_0}\left(V^s_{\omega_0}\right) \tag{4.18b}$$

[2]The result of the addendum gives a ball. Here, we adapt the result to our current context: the frontier $\omega = 1$ is only technical and is not the manifestation of some estimates that will degenerate when $\omega \to 1$, unlike the frontier $\omega = +\infty$. One can use some plateau map to obtain the desired result from the addendum.

Step 3: Straightening of the unstable foliation of $(\xi_2)_* \mathcal{X}$. This step will be treated analogously to step 2. Let Y^u be the restriction of $(\xi_2)_* \mathcal{X}$ to $\Omega^u := \xi_2(\mathcal{B}_0) \cap \{N_{s_1} = N_{s_2} = 0\}$, that is, the unstable manifold of $\{0_{\mathbb{R}^3}\} \times]1,+\infty[$ for $(\xi_2)_* \mathcal{X}$. Remark that the unstable foliation of Y^u is exactly the stable foliation of $-Y^u$. Identifying $\mathbb{R}^4 \cap \{N_{s_1} = N_{s_2} = 0\}$ with $\mathbb{R}^2$ endowed with the coordinates N_u and ω, Ω^u is an open set of $\mathbb{R}^2$. Since $\Omega^u \cap \{N_u = 0\}$ is the image of $\mathcal{K}_0$ by ξ_2, Y^u vanishes on $\Omega^u \cap \{N_u = 0\}$. Let $F := \mathbb{R}\partial_{N_u}$ and $G := \mathbb{R}\partial_\omega$. According to (3.5) and (4.14), for every $\omega \in]1,+\infty[$, the decomposition $F \oplus G = \mathbb{R}^2$ is stabilized by $D(-Y^u)(0,\omega)$ and the eigenvalue μ_u of $\left(D(-Y^u)(0,\omega)\right)_{|F}$ is (strictly) negative.

According to Theorem 4.10 and Addendum 4.11, there exist two constants $C_u > 0$ and $n_u \in \mathbb{N}$ such that, for any $\omega_0 \in]1,+\infty[$, there exist an open set $V^u_{\omega_0} \subset \Omega^u$ and a smooth chart

$$\xi^u_{3,\omega_0} : \left| \begin{array}{ccc} V^u_{\omega_0} & \to & \xi^u_{3,\omega_0}\left(V^u_{\omega_0}\right) \subset \mathbb{R}^2 \\ (N_u,\omega) & \mapsto & (N_u, \omega + \tilde{\xi}^u_{3,\omega_0}(N_u,\omega)) \end{array} \right.$$

where $\tilde{\xi}^u_{3,\omega_0}(0,\omega) \equiv 0$, such that ξ^u_{3,ω_0} straightens the stable foliation of $-Y^u$ in $V^u_{\omega_0}$:

$$\xi^u_{3,\omega_0}\left(W^s((0,\bar{\omega}),-Y^u) \cap V^u_{\omega_0}\right) = \{(N_u,\omega) \mid \omega = \bar{\omega}\} \cap \xi^u_{3,\omega_0}\left(V^u_{\omega_0}\right) \tag{4.19}$$

Moreover, $V^u_{\omega_0}$ and $\xi^u_{3,\omega_0}\left(V^u_{\omega_0}\right)$ both contain the open set

$$B^u_{\omega_0} \stackrel{\text{def}}{=}]-R^u_{\omega_0}, R^u_{\omega_0}[\times \left]\omega_0 - \min\left(R^u_{\omega_0}, \frac{\omega_0-1}{2}\right), \omega_0 + \min\left(R^u_{\omega_0}, \frac{\omega_0-1}{2}\right)\right[$$

where

$$R^u_{\omega_0} := \frac{1}{C_u \omega_0^{n_u}},$$

and

$$\left\|\xi^u_{3,\omega_0}\right\|_{C^6}, \left\|\left(\xi^u_{3,\omega_0}\right)^{-1}\right\|_{C^6} \le C_u \omega_0^{n_u} \tag{4.20}$$

Since a stable manifold of $-Y^u$ is an unstable manifold of $(\xi_2)_* \mathcal{X}$, it follows that ξ^u_{3,ω_0} straightens the unstable foliation of $(\xi_2)_* \mathcal{X}$ restricted to Ω^u.

Step 4: Straightening of both the stable and the unstable foliation of $(\xi_2)_* \mathcal{X}$. Let $\omega_0 \in]1,+\infty[$. Let

$$V_{\omega_0} = \left\{(N_u, N_{s_1}, N_{s_2}, \omega) \mid (N_{s_1}, N_{s_2}, \omega) \in V^s_{\omega_0}, (N_u,\omega) \in V^u_{\omega_0}\right\}$$

and let

$$\xi_{3,\omega_0} : \left| \begin{array}{ccc} V_{\omega_0} & \to & \mathbb{R}^4 \\ (N_u, N_{s_1}, N_{s_2}, \omega) & \mapsto & (N_u, N_{s_1}, N_{s_2}, \omega + \tilde{\xi}^s_{3,\omega_0}(N_{s_1}, N_{s_2}, \omega) + \tilde{\xi}^u_{3,\omega_0}(N_u,\omega)) \end{array} \right. \tag{4.21}$$

According to Addendum 4.11, the map $\tilde{\xi}^s_{3,\omega_0}$ (resp. $\tilde{\xi}^u_{3,\omega_0}$) restricted to $B^s_{\omega_0}$ (resp. $B^u_{\omega_0}$) where $R^s_{\omega_0}$ (resp. $R^u_{\omega_0}$) is replaced by $\epsilon R^s_{\omega_0}$ (resp. $\epsilon R^u_{\omega_0}$) is ϵ-close to 0 with respect to the C^1-norm. It follows that there exist two constants $C_3 \ge \max(C_s, C_u)$ and $n_3 \ge \max(n_s, n_u)$ such that for every $\omega_0 \in]1,+\infty[$, ξ_{3,ω_0} is invertible on

$$B_{\omega_0} := B_{\mathbb{R}^3}(0, R_{\omega_0}) \times \left]\omega_0 - \min\left(R_{\omega_0}, \frac{\omega_0-1}{2}\right), \omega_0 + \min\left(R_{\omega_0}, \frac{\omega_0-1}{2}\right)\right[$$

where

$$R_{\omega_0} := \frac{1}{C_3 \omega_0^{n_3}}$$

From now on, we make the abuse of notation to consider that ξ_{3,ω_0} is restricted to B_{ω_0}. Using (4.17) and (4.20), we get

$$\left\|\xi_{3,\omega_0}\right\|_{C^6}, \left\|\left(\xi_{3,\omega_0}\right)^{-1}\right\|_{C^6} \le C_3 \omega_0^{n_3} \tag{4.22}$$

By local uniqueness (see Addendum 4.11) of the charts ξ^s_{3,ω_0} and ξ^u_{3,ω_0}, the charts $\{\xi_{3,\omega_0}\}_{\omega_0 \in]1,+\infty[}$

glue together and induce a global chart ξ_3 on the neighbourhood

$$V := \bigcup_{\omega_0 \in]1,+\infty[} B_{\omega_0} \subset \mathbb{R}^3 \times]1,+\infty[\tag{4.23}$$

One can remark that V contains the open set $B_{\mathbb{R}^3}\left(0, \frac{1}{C_3 2^{n_3}}\right) \times]1,2[$: the size of V does not shrink when $\omega \to 1$, it only shrinks when $\omega \to +\infty$.

According to (4.16) and (4.19), ξ_3 straightens the stable and the unstable foliations of $(\xi_2)_* \mathcal{X}$:

$$\xi_3 \left(W^s((0,0,0,\bar{\omega}), (\xi_2)_* \mathcal{X}) \cap V\right) = \left\{(N_u, N_{s_1}, N_{s_2}, \omega) \mid N_u = 0, \omega = \bar{\omega}\right\} \cap \xi_3(V) \tag{4.24a}$$

$$\xi_3 \left(W^u((0,0,0,\bar{\omega}), (\xi_2)_* \mathcal{X}) \cap V\right) = \left\{(N_u, N_{s_1}, N_{s_2}, \omega) \mid N_{s_1} = N_{s_2} = 0, \omega = \bar{\omega}\right\} \cap \xi_3(V) \tag{4.24b}$$

Step 5:Straightening of both the stable and the unstable foliation of $\mathcal{X}$. Let us define

$$\xi = (x_u, x_{s_1}, x_{s_2}, x_c) \stackrel{\text{def}}{=} \xi_3 \circ \xi_2$$

The chart ξ is well defined on the open set $\mathcal{U}_\xi := \xi_2^{-1}(V) \subset \mathcal{B}_0$. Let $U_\xi := \xi(\mathcal{U}_\xi)$. We now proceed to check all the properties of ξ announced in Proposition 4.2.

Properties (4.1) and (4.2) follow immediately from (4.14) and (4.21).

Properties (4.3) follow immediately from from (4.18) and (4.24).

The fact that there exist two constants $C > 0$ and $n \in \mathbb{N}$ such that estimates (4.4) and (4.5) hold true for any $\omega \in]1,+\infty[$ is an immediate consequence of (4.15), (4.22) and (4.23). □

Remark 4.13. In step 3, we used the same argument as in step 2. Nevertheless, one do not need Theorem 4.10 and Addendum 4.11 to straighten the unstable foliation of $(\xi_2)_* \mathcal{X}$. Indeed, the leaves of this foliation are all type II orbits explicitly known: all the computations could be done explicitly without the help of a general result.

4.3 Proofs of the main results on the local expression of the Wainwright-Hsu vector field

Fix $C_0 > 0$ and $n_0 \in \mathbb{N}$ such that Proposition 4.2 holds true with these constants. We begin this subsection with a proof of Proposition 4.3.

Proof of Proposition 4.3. Denote by $\bar{X}_u$, $\bar{X}_{s_1}$, $\bar{X}_{s_2}$, $\bar{X}_c$ the coordinates of X and let

$$U_X \stackrel{\text{def}}{=} \bigcup_{\omega \in]1,+\infty[} B_{\omega, C_0, n_0}$$

According to Proposition 4.2, ξ is smooth. Since $\mathcal{X}$ is also smooth, it follows that X is smooth. Using the invariance of the set $\{N_u = 0\}$ by the flow of $\mathcal{X}$ and (4.1a), we get that the set $\{x_u = 0\}$ is invariant by the flow of X. Using the standard Hadamard's lemma in differential calculus, we get the existence of some smooth functions $\bar{X}_u^{u,u}$, $\bar{X}_u^{u,s_1}$ and $\bar{X}_u^{u,s_2}$ defined on the open set U_X such that

$$\bar{X}_u(x) = \mu_u(x_c) x_u + \bar{X}_u^{u,u}(x) x_u^2 + \bar{X}_u^{u,s_1}(x) x_u x_{s_1} + \bar{X}_u^{u,s_2}(x) x_u x_{s_2}$$

Analogously, the sets $\{x_{s_1} = 0\}$ and $\{x_{s_2} = 0\}$ are invariant so (4.6) holds true for the first three coordinates. For any $x_c \in]1,+\infty[$, the stable manifold of $(0,0,0,x_c)$ for X is invariant by the flow of X. Using (4.3b), it follows that $\bar{X}_c(0, x_{s_1}, x_{s_2}, x_c) \equiv 0$ and we get the existence of a smooth function $\bar{X}_c^u$ defined on U_X such that

$$\bar{X}_c(x) = x_u \bar{X}_c^u(x)$$

The unstable manifold of $(0,0,0,x_c)$ for X being also invariant, it follows by (4.3a) that there exist two smooth functions $\bar{X}_c^{u,s_1}$ and $\bar{X}_c^{u,s_2}$ defined on U_X such that

$$\bar{X}_c^u(x) = \bar{X}_c^{u,s_1}(x) x_{s_1} + \bar{X}_c^{u,s_2}(x) x_{s_2}$$

We can conclude that (4.6) holds true on U_X.

The functions $\bar{X}_*^{*,*}$ depend on the second derivatives of X. A C^3 control of these functions involves a C^5 control of $\mathcal{X}$ and a C^6 control of ξ. A C^6 control of ξ is given by (4.5) while a C^5 control of $\mathcal{X}$ is trivial: there exists a constant $C_1 > 0$ such that $\|\mathcal{X}\|_{C^5} \le C_1$ on U_X. The conjunction of these two controls implies that (4.7) holds true for some $C > 0$, $n \in \mathbb{N}$ large enough. □

We now give a proof of Lemma 4.5.

Proof of Lemma 4.5. For every $C > 0$ and $n \in \mathbb{N}$, let us denote by $E_{C,n}$ the set of all (ω, x) such that $\omega \in]1, +\infty[$ and $x \in B_{\omega,C,n} \subset U_\xi$. Recall that

$$B_{\omega,C,n} = \left\{(x_u, x_{s_1}, x_{s_2}, x_c) \in \left(\mathbb{R}^+\right)^3 \times]1, +\infty[\mid \max(x_u, x_{s_1}, x_{s_2}, |x_c - \omega|) \le \frac{1}{C\omega^n}\right\}$$

According to 3.14a, for every $\omega \in]1, +\infty[$ and for every $x \in U_\xi$ such that $x_c \le 2\omega$, we have

$$\mu_u(x_c) \ge \frac{1}{\omega} \tag{4.25}$$

According to (4.7), there exist $C_1 \ge C_0$ and $n_1 \ge n_0$ such that for every $(\omega, x) \in E_{C_1,n_1}$, we have

$$\left|\bar{X}_u^{u,u}(x)x_u + \bar{X}_u^{u,s_1}(x)x_{s_1} + \bar{X}_u^{u,s_2}(x)x_{s_2}\right| \le C_1\omega^{n_1} \max(x_u, x_{s_1}, x_{s_2}) \tag{4.26}$$

Inequalities (4.25) and (4.26) imply that for every $(\omega, x) \in E_{2C_1,n_1+1}$, we have

$$\left|\bar{X}_u^{u,u}(x)x_u + \bar{X}_u^{u,s_1}(x)x_{s_1} + \bar{X}_u^{u,s_2}(x)x_{s_2}\right| < \mu_u(x_c)$$

which concludes the proof. □

Next lemma gives estimates on γ_ω and its derivatives, which will be useful to obtain estimates on X_ω later on.

Lemma 4.14 (Control of γ_ω and its derivatives)**.** *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in]1, +\infty[$, for every $x \in B_{\omega,C,n}$, we have*

$$\frac{1}{2} \le \gamma_\omega(x) \le \frac{3}{2} \tag{4.27}$$

and, for every $1 \le k \le 3$,

$$\left\|D^k\gamma_\omega(x)\right\| \le C\omega^N \tag{4.28}$$

Proof of Lemma 4.14. For every $C > 0$ and $n \in \mathbb{N}$, let us denote by $E_{C,n}$ the set of all (ω, x) such that $\omega \in]1, +\infty[$ and $x \in B_{\omega,C,n} \subset U_\xi$. Fix $3/4 < \alpha < 1$.

Proof of (4.27). Let $\bar{X}_u(x) := \bar{X}_u^{u,u}(x)x_u + \bar{X}_u^{u,s_1}(x)x_{s_1} + \bar{X}_u^{u,s_2}(x)x_{s_2}$. According to Lemma 4.5, there exist $C_0 > 0$ and $n_0 \in \mathbb{N}$ such that for every $C \ge C_0$, every $n \ge n_0$ and every $(\omega, x) \in E_{C,n}$, (4.9) holds true and it follows that

$$\begin{aligned}\frac{1}{2} \le \gamma_\omega(x) \le \frac{3}{2} &\iff \mu_u(x_c) + \bar{X}_u(x) \le 2\mu_u(\omega) \le 3(\mu_u(x_c) + \bar{X}_u(x)) \\ &\iff \begin{cases}\bar{X}_u(x) \le 2\mu_u(\omega) - \mu_u(x_c) \\ 2\mu_u(\omega) - 3\mu_u(x_c) \le 3\bar{X}_u(x)\end{cases}\end{aligned}$$

According to 3.14a and the mean value theorem, for every $\omega \in]1, +\infty[$ and for every $x \in U_\xi$ such that $|x_c - \omega| \le \frac{1}{6\omega}$, we have

$$|\mu_u(x_c) - \mu_u(\omega)| \le \frac{1}{\omega}$$

and it follows that

$$2\mu_u(\omega) - \mu_u(x_c) \ge \frac{1}{\omega} \tag{4.29}$$

Analogously, for every $\omega \in]1,+\infty[$ and for every $x \in U_\xi$ such that $|x_c - \omega| \leq \frac{1}{24\omega}$, we have

$$|\mu_u(x_c) - \mu_u(\omega)| \leq \frac{1}{4\omega}$$

and it follows that

$$2\mu_u(\omega) - 3\mu_u(x_c) \leq -\frac{1}{2\omega} \tag{4.30}$$

According to (4.7), (4.29) and (4.30), there exist $C_1 \geq C_0$ and $n_1 \geq n_0$ such that for every $C \geq C_1$, every $n \geq n_1$ and every $(\omega, x) \in E_{C,n}$, we have $\bar{X}_u(x) \leq 2\mu_u(\omega) - \mu_u(x_c)$ and $2\mu_u(\omega) - 3\mu_u(x_c) \leq 3\bar{X}_u(x)$ so (4.27) holds true.

Proof of (4.28). Recall that

$$\gamma_\omega(x) = \frac{\mu_u(\omega)}{\mu_u(x_c) + \bar{X}_u(x)}$$

and

$$D\gamma_\omega(x) = \frac{-\gamma_\omega(x)^2}{\mu_u(\omega)}(D\mu_u(x_c) + D\bar{X}_u(x)) \tag{4.31}$$

According to (3.14a) and (4.7), there exist $C_2 \geq C_1$ and $n_2 \geq n_1$ such that for every $C \geq C_2$, every $n \geq n_2$, every $(\omega, x) \in E_{C,n}$ and every $1 \leq k \leq 3$, we have

$$\left\| D^k \mu_u(x_c) \right\| \leq C_2 \omega^{n_2} \tag{4.32}$$

and

$$\left\| D^k \bar{X}_u(x) \right\| \leq C_2 \omega^{n_2} \tag{4.33}$$

Using (4.27), (4.31), (4.32), (4.33) and the inequality $\mu_u(\omega) \geq \frac{2}{\omega}$, we get that there exist $C_3 \geq C_2$ and $n_3 \geq n_2$ such that for every $C \geq C_3$, every $n \geq n_3$, every $(\omega, x) \in E_{C,n}$ and every $1 \leq k \leq 3$, we have

$$\left\| D^k \gamma_\omega(x) \right\| \leq C_3 \omega^{n_3}$$

so (4.28) holds true. □

We now have everything to prove the main result on X_ω.

Proof of Proposition 4.8. Expression (4.11) follows from (4.8) and analogous computations to the ones detailed for Proposition 4.3.

Estimate (4.13) follows from (4.5), Lemma 4.14, a C^5 control of $\mathcal{X}$ on an arbitrary compact neighbourhood of $\mathcal{K}$ and analogous computations to the ones detailed for Proposition 4.3. □

Chapter 5

Local sections and transition maps

The purpose of this chapter is to define some sections for the Wainwright-Hsu vector field, together with some transitions maps describing how the orbits of the flow travel from one section to another. All these sections will be located in the vicinity of the Kasner circle and will be defined in the local coordinate system ξ constructed in the previous chapter. The transition maps between the sections will play a central role in our investigation of the dynamics of the Wainwright-Hsu vector field.

We will first recall some properties of the local coordinate system ξ (section 5.1). Then we will define a "global section" S_h (section 5.2). The dynamics of the Wainwright-Hsu vector field is almost completely captured by the return map $\bar{\Phi}$ of the orbits of this vector field on the global section S_h. Therefore, understanding the dynamical properties of $\bar{\Phi}$ will be our long-term goal. But, since this goal cannot be achieved directly, it is necessary to decompose $\bar{\Phi}$ as a product of a large number of "local transitions maps". This will lead us to introduce some local sections (section 5.3), and some transitions maps describing how the orbits move from one local section to another (section 5.4).

5.1 Reminder on the local coordinate system ξ. The pseudo-norms $\|.\|_{\perp}$, $\|.\|_{/\!/}$ and the projection $\mathrm{Proj_A}$

In order to define the global and local sections, we first need to recall a few facts on the "nice" local coordinate system $\xi = (x_u, x_{s_1}, x_{s_2}, x_c)$ constructed in the previous chapter, and to introduce some pseudo-norms and projections related to this coordinate system.

Recall that the local coordinate system $\xi = (x_u, x_{s_1}, x_{s_2}, x_c)$ is defined on a neighbourhood $\mathcal{U}_\xi$ of the Kasner interval $\mathcal{K}_0$ in the quotient phase space $\mathcal{B}^+$. The range of ξ, denoted by U_ξ, is a neighbourhood of $\{0_{\mathbb{R}^3}\} \times]1, +\infty[$ in $\left(\mathbb{R}^+\right)^3 \times]1, +\infty[$.

The local coordinate system ξ maps the Kasner interval $\mathcal{K}_0$ to the interval $\{0_{\mathbb{R}^3}\} \times]1, +\infty[$. Moreover, in restriction to $\mathcal{K}_0$, the coordinate x_c is nothing but the Kasner parameter. In other words, the coordinates of the point $\mathcal{P}_\omega \in \mathcal{K}_0$ are $(x_u, x_{s_1}, x_{s_2}, x_c) = (0, 0, 0, \omega)$.

For $\omega \in]1, +\infty[$, there is one type II orbit, denoted by $\mathcal{O}^u_\omega$, starting at $\mathcal{P}_\omega$ and two type II orbits, denoted by $\mathcal{O}^{s_1}_\omega$ and $\mathcal{O}^{s_2}_\omega$, arriving at $\mathcal{P}_\omega$. These orbits are mapped by ξ to the straight lines

$$O^u_\omega \stackrel{\text{def}}{=} \{x_u > 0, x_{s_1} = x_{s_2} = 0, x_c = \omega\} \tag{5.1a}$$

$$O^{s_1}_\omega \stackrel{\text{def}}{=} \{x_{s_1} > 0, x_u = x_{s_2} = 0, x_c = \omega\} \tag{5.1b}$$

$$O^{s_2}_\omega \stackrel{\text{def}}{=} \{x_{s_2} > 0, x_u = x_{s_1} = 0, x_c = \omega\} \tag{5.1c}$$

More precisely, the connected component of $\mathcal{O}^u_\omega \cap \mathcal{U}_\xi$ starting[1] at $\mathcal{P}_\omega$ is mapped to $O^u_\omega \cap U_\xi$, and similarly for the two other type II orbits. We will abusively call the sets O^u_ω, $O^{s_1}_\omega$ and $O^{s_2}_\omega$ "type II orbits".

[1] Any connected component of $\mathcal{O}^u_\omega \cap \mathcal{U}_\xi$ is oriented by the flow of the Wainwright-Hsu vector field $\mathcal{X}$.

The Mixmaster attractor $\mathcal{A}$ is mapped by ξ to the set

$$A \overset{\text{def}}{=} \{x_u = x_{s_1} = 0\} \cup \{x_u = x_{s_2} = 0\} \cup \{x_{s_1} = x_{s_2} = 0\} \tag{5.2}$$

that is, $\xi(\mathcal{A} \cap \mathcal{U}_\xi) = A \cap U_\xi$.

Recall that our goal is to compare the behaviour of the type IX orbits winding around the Mixmaster attractor with the dynamics on the Mixmaster attractor itself. In view to that goal, it will be convenient to project the type IX orbits (or at least some of their points) on the Mixmaster attractor.

Definition 5.1 (Projection on the Mixmaster attractor)**.** Let us denote by Δ the set of all $x = (x_u, x_{s_1}, x_{s_2}, x_c)$ such that two of the three coordinates x_u, x_{s_1}, and x_{s_2} are equal and larger than the third one, that is, the set

$$\{(x_u, x_{s_1}, x_{s_2}, x_c) \mid x_u = x_{s_1} \geq x_{s_2} \text{ or } x_u = x_{s_2} \geq x_{s_1} \text{ or } x_{s_1} = x_{s_2} \geq x_u\}$$

We define a projection $\mathrm{Proj_A} : (\mathbb{R}^+)^3 \times]1, +\infty[\setminus \Delta \to A$ by the formula

$$\mathrm{Proj_A}(x_u, x_{s_1}, x_{s_2}, x_c) \overset{\text{def}}{=} \begin{cases} (x_u, 0, 0, x_c) & \text{if } x_u > \max(x_{s_1}, x_{s_2}) \\ (0, x_{s_1}, 0, x_c) & \text{if } x_{s_1} > \max(x_u, x_{s_2}) \\ (0, 0, x_{s_2}, x_c) & \text{if } x_{s_2} > \max(x_u, x_{s_1}) \end{cases} \tag{5.3}$$

Remark 5.2. According to the equation of the Mixmaster attractor in local coordinates (5.2), one can see that $\mathrm{Proj_A}(x)$ is the closest point to x (both for the Euclidean standard norm and the sup-norm) belonging to the Mixmaster attractor A. This is why we say that $\mathrm{Proj_A}(x)$ is the projection of x on the Mixmaster attractor.

For $x = (x_u, x_{s_1}, x_{s_2}, x_c) \in \mathbb{R}^4$, we denote

$$\|x\|_\infty = \max(|x_u|, |x_{s_1}|, |x_{s_2}|, |x_c|)$$

It will be convenient to discriminate the direction ∂_{x_c} from the other directions, for dynamical reasons. This leads us to introduce two pseudo-norms.

Definition 5.3 (Pseudo-norms)**.** For any $x = (x_u, x_{s_1}, x_{s_2}, x_c) \in \mathbb{R}^4$, we define

$$\|x\|_\perp \overset{\text{def}}{=} \max(|x_u|, |x_{s_1}|, |x_{s_2}|)$$
$$\|x\|_{/\!/} \overset{\text{def}}{=} |x_c|$$

Remark 5.4. For any $x \in \mathbb{R}^4$, $\|x\|_\infty = \max(\|x\|_\perp, \|x\|_{/\!/})$.

Remark 5.5. If the projection $\mathrm{Proj_A}(x)$ of x on the Mixmaster attractor is well defined (see definition 5.1), then $\|x - \mathrm{Proj_A}(x)\|_\perp = \|x - \mathrm{Proj_A}(x)\|_\infty$ is the distance between x and the Mixmaster attractor A.

5.2 The global section S_h, the era return map $\bar{\Phi}_h$ and the double era return map $\hat{\Phi}_h$

Definition 5.6 (Global section)**.** For $h > 0$, we define the *global section* $S_h := S_h^{s_1} \cup S_h^{s_2}$ where

$$S_h^{s_1} \overset{\text{def}}{=} \{x = (x_u, x_{s_1}, x_{s_2}, x_c) \mid x_{s_1} = h, \quad 0 \leq x_u \leq h, \quad 0 \leq x_{s_2} \leq h, \quad 1 < x_c < 2\} \tag{5.4}$$

and analogously for $S_h^{s_2}$. If h is small enough, the global section is included in the range U_ξ of the local coordinate system ξ. In this case, we consider the geometric global section

$$\mathcal{S}_h \overset{\text{def}}{=} \xi^{-1}(S_h)$$

Suppose that h is small enough, so that the geometric global section S_h is well-defined. On the one hand, for every $\omega \in]1, 2[$, the two type II orbits $O_\omega^{s_1}$, $O_\omega^{s_2}$ intersect the global section S_h. On the other hand, formula (3.15) shows that, for every $\omega \in]1,+\infty[\setminus\mathbb{Q}$, the forward orbit of ω under the Kasner map f passes infinitely many times in the interval $]1,2[$. It follows that every heteroclinic chain of type II orbits either converges to a Taub point, or crosses infinitely many times the global section S_h. Hence all type IX orbits that possibly shadow a heteroclinic chain of type II orbits must cross infinitely many times the global section S_h. This is the reason why we say that S_h is a *global* section.

The above discussion shows that our main Theorem B can be proved by investigating the dynamical properties of the return map of the orbits on the global section S_h, called the *era return map*. We will now proceed to the formal definition of this map. This definition is not completely straightforward for two reasons:

- the global section S_h was defined in the local coordinate system ξ but the segments orbits (or heteroclinic chains) travelling from S_h to S_h do not remain inside the open set where this local coordinate system is defined,
- we want to consider not only the returns of orbits, but also the return of heteroclinic chains.

Recall that, for every point $q \in \mathcal{B}^+$ which is not a Taub point, we have defined a generalized heteroclinic chain $\mathcal{H}(q)$ starting at q (see definitions 3.5 and 3.7). In particular, $\mathcal{H}(q)$ is nothing but the forward orbit of q when $q \in \mathcal{B}^+_{\mathrm{IX}}$.

Definition 5.7. For $x = (x_u, x_{s_1}, x_{s_2}, x_c) \in U_\xi$, we will denote by $\mathcal{H}(x)$ the heteroclinic chain starting at the point in $\mathcal{B}^+$ of coordinates x.

Definition 5.8 (Era return map)**.** Let $h > 0$ be small enough so that the global section S_h is included in the range U_ξ of the local coordinate system ξ. We define the *era return map*

$$\bar{\Phi}_h : S_h \to S_h$$

as follows. Let $x \in S_h$. If the heteroclinic chain $\mathcal{H}(x)$ intersects the section $\mathcal{S}_h$, then $\bar{\Phi}_h(x)$ is the 4-tuple of coordinates of the first intersection point of $\mathcal{H}(x)$ with $\mathcal{S}_h$. Otherwise, $\bar{\Phi}_h$ is not defined at the point x.

If $x_u, x_{s_1}, x_{s_2} > 0$, then the heteroclinic chain $\mathcal{H}(x)$ is nothing but the forward $\mathcal{X}$-orbit of the point $\xi^{-1}(x)$. As a consequence, in restriction to $\{x_u, x_{s_1}, x_{s_2} > 0\}$, the era return map $\bar{\Phi}_h$ is nothing but the first return map of the orbits of the Wainwright-Hsu vector field on the global section $\mathcal{S}_h$, expressed in local coordinates.

For technical reasons (namely, because we will discover that $\bar{\Phi}_h$ fails to be uniformly expanding in the direction parallel to the Kasner interval), we will be led to replace $\bar{\Phi}_h$ by its square. This motivates the following definition.

Definition 5.9 (Double era transition map)**.** Let $h > 0$ be small enough so that the global section S_h is included in the range U_ξ of the local coordinate system ξ. We define the *double era return map* $\hat{\Phi}_h : S_h \to S_h$ by the formula

$$\hat{\Phi}_h \stackrel{\text{def}}{=} \bar{\Phi}_h \circ \bar{\Phi}_h \tag{5.5}$$

The goal of the remainder of the memoir is to find a subset of S_h where the double era return map $\hat{\Phi}_h$ is well-defined, to prove that $\hat{\Phi}_h$ has nice hyperbolicity properties on this subset, to construct some local stable manifolds for the map $\hat{\Phi}_h$ and finally to prove that the union of these local stable manifold cover a subset of positive Lebesgue measure in S_h. Our main Theorem B will follow easily.

5.3 The local sections $S^u_{\omega,\mathbf{h}}$, $S^{s_1}_{\omega,\mathbf{h}}$ and $S^{s_2}_{\omega,\mathbf{h}}$

Definition 5.10. Let $\omega \in]1,+\infty[$ and $h > 0$. We denote respectively by $P^u_{\omega,h}, P^{s_1}_{\omega,h}, P^{s_2}_{\omega,h}$ the points on the type II orbits O^u_ω, $O^{s_1}_\omega$ and $O^{s_2}_\omega$ that are at distance h from the Kasner interval, that is,

$$P^u_{\omega,h} \stackrel{\text{def}}{=} (h,0,0,\omega)\,, \quad P^{s_1}_{\omega,h} \stackrel{\text{def}}{=} (0,h,0,\omega)\,, \quad P^{s_2}_{\omega,h} \stackrel{\text{def}}{=} (0,0,h,\omega)$$

If h is small enough, the points $P^u_{\omega,h}$, $P^{s_1}_{\omega,h}$ and $P^{s_2}_{\omega,h}$ are in the range U_ξ of the local coordinate system ξ. In this case, we denote $\mathcal{P}^u_{\omega,h} := \xi^{-1}(P^u_{\omega,h})$, $\mathcal{P}^{s_1}_{\omega,h} := \xi^{-1}(P^{s_1}_{\omega,h})$ and $\mathcal{P}^{s_2}_{\omega,h} := \xi^{-1}(P^{s_2}_{\omega,h})$.

We now define three local sections that intersect respectively the type II orbits O^u_ω, $O^{s_1}_\omega$ and $O^{s_2}_\omega$. These local sections will be crossed by the orbits traveling close to the heteroclinic chain passing through the point P_ω. They will serve as gates controlling the entrance in (resp. the exit from) a neighbourhood of the point P_ω.

Definition 5.11 (Local sections)**.** Let $\omega \in \,]1,+\infty[$ and $\mathbf{h} = \left(h, h_\perp, h_{/\!/}\right)$ where h, $h_\perp$ and $h_{/\!/}$ are some positive numbers. We consider the local sections

$$\begin{aligned} S^u_{\omega,\mathbf{h}} &\overset{\text{def}}{=} \left\{x = (x_u, x_{s_1}, x_{s_2}, x_c) \mid x_u = h, \quad \left\|x - P^u_{\omega,h}\right\|_\perp \le h_\perp, \quad \left\|x - P^u_{\omega,h}\right\|_{/\!/} \le h_{/\!/}\right\} \\ &= \left\{x = (x_u, x_{s_1}, x_{s_2}, x_c) \mid x_u = h, \quad 0 \le x_{s_1} \le h_\perp, \quad 0 \le x_{s_2} \le h_\perp, \quad \omega - h_{/\!/} \le x_c \le \omega + h_{/\!/}\right\} \end{aligned}$$

and

$$\begin{aligned} S^{s_1}_{\omega,\mathbf{h}} &\overset{\text{def}}{=} \left\{x = (x_u, x_{s_1}, x_{s_2}, x_c) \mid x_{s_1} = h, \quad \left\|x - P^{s_1}_{\omega,h}\right\|_\perp \le h_\perp, \quad \left\|x - P^{s_1}_{\omega,h}\right\|_{/\!/} \le h_{/\!/}\right\} \\ &= \left\{x = (x_u, x_{s_1}, x_{s_2}, x_c) \mid x_{s_i} = h, \quad 0 \le x_u \le h_\perp, \quad 0 \le x_{s_j} \le h_\perp, \quad \omega - h_{/\!/} \le x_c \le \omega + h_{/\!/}\right\} \end{aligned}$$

The local section $S^{s_2}_{\omega,\mathbf{h}}$ is defined analogously, permuting the roles of s_1 and s_2. See figure 5.1 for a representation of $S^{s_1}_{\omega,\mathbf{h}}$ and $S^u_{\omega,\mathbf{h}}$. Finally, set

$$S^s_{\omega,\mathbf{h}} \overset{\text{def}}{=} S^{s_1}_{\omega,\mathbf{h}} \sqcup S^{s_2}_{\omega,\mathbf{h}}$$

If the sections $S^u_{\omega,\mathbf{h}}$, $S^{s_i}_{\omega,\mathbf{h}}$ are included in the range U_ξ of the local coordinate system ξ, then we define the geometric local sections

$$\mathcal{S}^u_{\omega,\mathbf{h}} \overset{\text{def}}{=} \xi^{-1}\left(S^u_{\omega,\mathbf{h}}\right) \subset \mathcal{B}^+, \qquad \mathcal{S}^{s_i}_{\omega,\mathbf{h}} \overset{\text{def}}{=} \xi^{-1}\left(S^{s_i}_{\omega,\mathbf{h}}\right) \subset \mathcal{B}^+$$

Remark 5.12. The local sections $S^u_{\omega,\mathbf{h}}$, $S^{s_1}_{\omega,\mathbf{h}}$ and $S^{s_2}_{\omega,\mathbf{h}}$ are included in the range U_ξ of the local coordinate system ξ as soon as the parameters h, $h_\perp$ and $h_{/\!/}$ are chosen small enough. More precisely, there exists $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in \,]1,+\infty[$ and every $\mathbf{h} = \left(h, h_\perp, h_{/\!/}\right)$, if $\max\left(h, h_\perp, h_{/\!/}\right) \le (C\omega^n)^{-1}$, then the local sections $S^u_{\omega,\mathbf{h}}$, $S^{s_1}_{\omega,\mathbf{h}}$ and $S^{s_2}_{\omega,\mathbf{h}}$ are included in U_ξ. This is a direct consequence of Proposition 4.2 on the local coordinate system ξ.

Remark 5.13. Let $\omega \in \,]1,+\infty[$ and $\mathbf{h} = \left(h, h_\perp, h_{/\!/}\right)$ where h, $h_\perp$ and $h_{/\!/}$ are positive. Let $x \in S^{s_1}_{\omega,\mathbf{h}}$.

- Assume that $0 < h_\perp < \min(h, h_{/\!/})$. The (in)equalities

  $$\max(x_u, x_{s_2}) = \left\|x - P^{s_1}_{\omega,h}\right\|_\perp \le h_\perp < h = x_{s_1}$$

  show that the projection $\mathrm{Proj}_{\mathrm{A}}(x)$ of x on the Mixmaster attractor is well defined, and $\left\|x - P^{s_1}_{\omega,h}\right\|_\perp = \|x - \mathrm{Proj}_{\mathrm{A}}(x)\|_\perp$. In other words, $\left\|x - P^{s_1}_{\omega,h}\right\|_\perp$ is the distance from x to the Mixmaster attractor. Hence, $h_\perp$ can be seen as the size of the section in the direction transversal to the Mixmaster attractor A.

- Assume again that $0 < h_\perp < \min(h, h_{/\!/})$. The section $S^{s_1}_{\omega,\mathbf{h}}$ is a 3-dimensional "rectangle" in $\left(\mathbb{R}^+\right)^3 \times \,]1,+\infty[$. Using the fact that the Kasner interval corresponds to $x_u = x_{s_1} = x_{s_2} = 0$ and the preceding item, we see that

  - h is the distance from the section $S^{s_1}_{\omega,\mathbf{h}}$ to the Kasner interval $\mathcal{K}_0$.
  - $h_\perp$ is the size of $S^{s_1}_{\omega,\mathbf{h}}$ in the direction transversal to the Mixmaster attractor.
  - $h_{/\!/}$ is the size of $S^{s_1}_{\omega,\mathbf{h}}$ in the direction parallel to the Kasner interval $\mathcal{K}_0$.

  The section $S^{s_1}_{\omega,\mathbf{h}}$ cuts the type II orbit $O^{s_1}_\omega$ at the point $P^{s_1}_{\omega,h}$. Its intersection with the Mixmaster attractor A is the segment $\{0\} \times \{h\} \times \{0\} \times \left[\omega - h_{/\!/}, \omega + h_{/\!/}\right]$. See figure 5.1.

- Moreover, one can interpret the terms $\left\|x - P^{s_1}_{\omega,h}\right\|_\perp$ and $\left\|x - P^{s_1}_{\omega,h}\right\|_{/\!/}$ as follows: $\left\|x - P^{s_1}_{\omega,h}\right\|_\perp$ is the distance between x and the type II orbit $O^{s_1}_\omega$ in the direction transverse to the Mixmaster

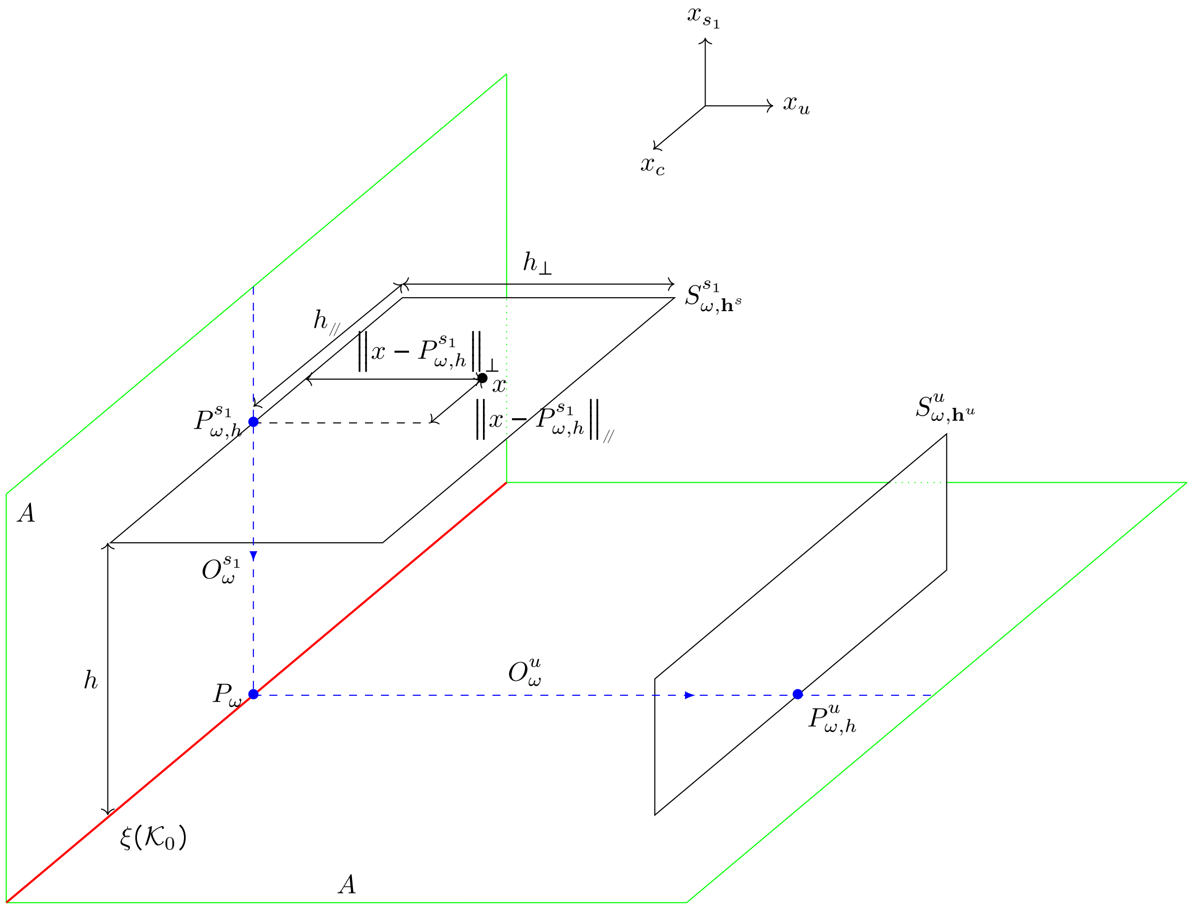

Figure 5.1 – Sections $S^{s_1}_{\omega,\mathbf{h}}$ and $S^u_{\omega,\mathbf{h}}$. We omit the x_{s_2}-direction since we cannot draw in four dimensions. The Mixmaster attractor A is represented in green. The Kasner interval $\xi(\mathcal{K}_0)$ is represented in red. The type II orbits arriving and starting at P_ω are represented in blue.

attractor while $\left\|x - P^{s_1}_{\omega,h}\right\|_{/\!/}$ is the distance between x and the type II orbit $O^{s_1}_{\omega}$ in the direction tangent to the Mixmaster attractor.

Similar remarks hold for the sections $S^{u}_{\omega,\mathbf{h}}$ and $S^{s_2}_{\omega,\mathbf{h}}$ as well.

5.4 Transition maps

We will now construct some transition maps between the local sections that were defined in section 5.3. These maps describe the behaviour of the orbits of the Wainwright-Hsu vector field in some specific regions (in the neighbourhood of a point of the Kasner circle, in the neighbourhood of a type II orbit, etc). Our goal is to decompose the era return map $\bar{\Phi}_h$ as a product of elementary transition maps, that are easier to understand than $\bar{\Phi}_h$ itself.

5.4.1 The era transition map $\bar{\Phi}_{\omega,\mathbf{h}}$ and the double era transition map $\hat{\Phi}_{\omega,\mathbf{h}}$

The orbits of the Wainwright-Hsu vector field can follow very different routes between two intersections with the global section S_h. For example, some orbits come back rather quickly to the global section S_h, whereas some others will spend a very long time oscillating in the vicinity of the Taub point T before coming back to S_h. For that reason, we cannot study the era return map $\bar{\Phi}_h$ *globally*: we need to define some localized version of $\bar{\Phi}_h$ (and $\hat{\Phi}_h$).

Definition 5.14 (Era transition map $\bar{\Phi}_{\omega,\mathbf{h}}$ and double era transition map $\hat{\Phi}_{\omega,\mathbf{h}}$)**.** Let $\omega \in \,]1,2[$ and $\mathbf{h} = (h, h_\perp, h_{/\!/})$ so that the sections $S^{s}_{\omega,\mathbf{h}}$ and S_h are included in the range U_ξ of the local coordinate system ξ. We define the *era transition map*

$$\bar{\Phi}_{\omega,\mathbf{h}} : S^{s}_{\omega,\mathbf{h}} \cap S_h \to S_h$$

as the restriction of $\bar{\Phi}_h$ to the section $S^{s}_{\omega,\mathbf{h}}$. See figure 5.2. Analogously, we define the *double era transition map*

$$\hat{\Phi}_{\omega,\mathbf{h}} : S^{s}_{\omega,\mathbf{h}} \cap S_h \to S_h$$

as the restriction of $\hat{\Phi}_h$ to the section $S^{s}_{\omega,\mathbf{h}}$.

Definition 5.15 (Maps $\bar{\Phi}^{A}_{\omega,\mathbf{h}}$ and $\hat{\Phi}^{A}_{\omega,\mathbf{h}}$)**.** Let $\omega \in \,]1,2[$ and $\mathbf{h} = (h, h_\perp, h_{/\!/})$ so that the sections $S^{s}_{\omega,\mathbf{h}}$ and S_h are included in U_ξ and so that the projection $\mathrm{Proj}_{\mathrm{A}}$ is well defined on the section $S^{s}_{\omega,\mathbf{h}}$. We define the map $\bar{\Phi}^{A}_{\omega,\mathbf{h}} : S^{s}_{\omega,\mathbf{h}} \cap S_h \to S_h$ by the formula

$$\bar{\Phi}^{A}_{\omega,\mathbf{h}} \stackrel{\text{def}}{=} \bar{\Phi}_{\omega,\mathbf{h}} \circ \mathrm{Proj}_{\mathrm{A}}$$

Analogously, we define the map $\hat{\Phi}^{A}_{\omega,\mathbf{h}} : S^{s}_{\omega,\mathbf{h}} \cap S_h \to S_h$ by the formula

$$\hat{\Phi}^{A}_{\omega,\mathbf{h}} \stackrel{\text{def}}{=} \hat{\Phi}_{\omega,\mathbf{h}} \circ \mathrm{Proj}_{\mathrm{A}}$$

Consider a point $x \in S^{s}_{\omega,\mathbf{h}} \cap S_h \cap B^{+}_{\mathrm{IX}}$. On the one hand, $\bar{\Phi}_h(x)$ is the first intersection point of the forward orbit of the point x with the section S_h. On the other hand, $\bar{\Phi}^{A}_{\omega,\mathbf{h}}(x)$ is the first intersection point of the heteroclinic chain $\mathcal{H}(\mathrm{Proj}_{\mathrm{A}}(x))$ with the section S_h. Since the point $\mathrm{Proj}_{\mathrm{A}}(x)$ belongs to the Mixmaster attractor, $\mathcal{H}(\mathrm{Proj}_{\mathrm{A}}(x))$ is a heteroclinic chain of type II orbits. As a consequence, the comparison between the maps $\bar{\Phi}_{\omega,\mathbf{h}}(x)$ and $\bar{\Phi}^{A}_{\omega,\mathbf{h}}(x)$ will allow us to understand whether the type IX orbits follow (or deviate from) the heteroclinic chains of type II orbits.

Remark 5.16. The map $\bar{\Phi}^{A}_{\omega,\mathbf{h}}$ admits an explicit expression. Recall that the era Kasner map is defined by

$$\bar{f}([1; k_1, k_2, \dots]) = [1; k_2, k_3, \dots]$$

and see (3.16) for more details. Both $\bar{f}$ and $\bar{\Phi}^{A}_{\omega,\mathbf{h}}$ encode the behaviour of heteroclinic chains of type II

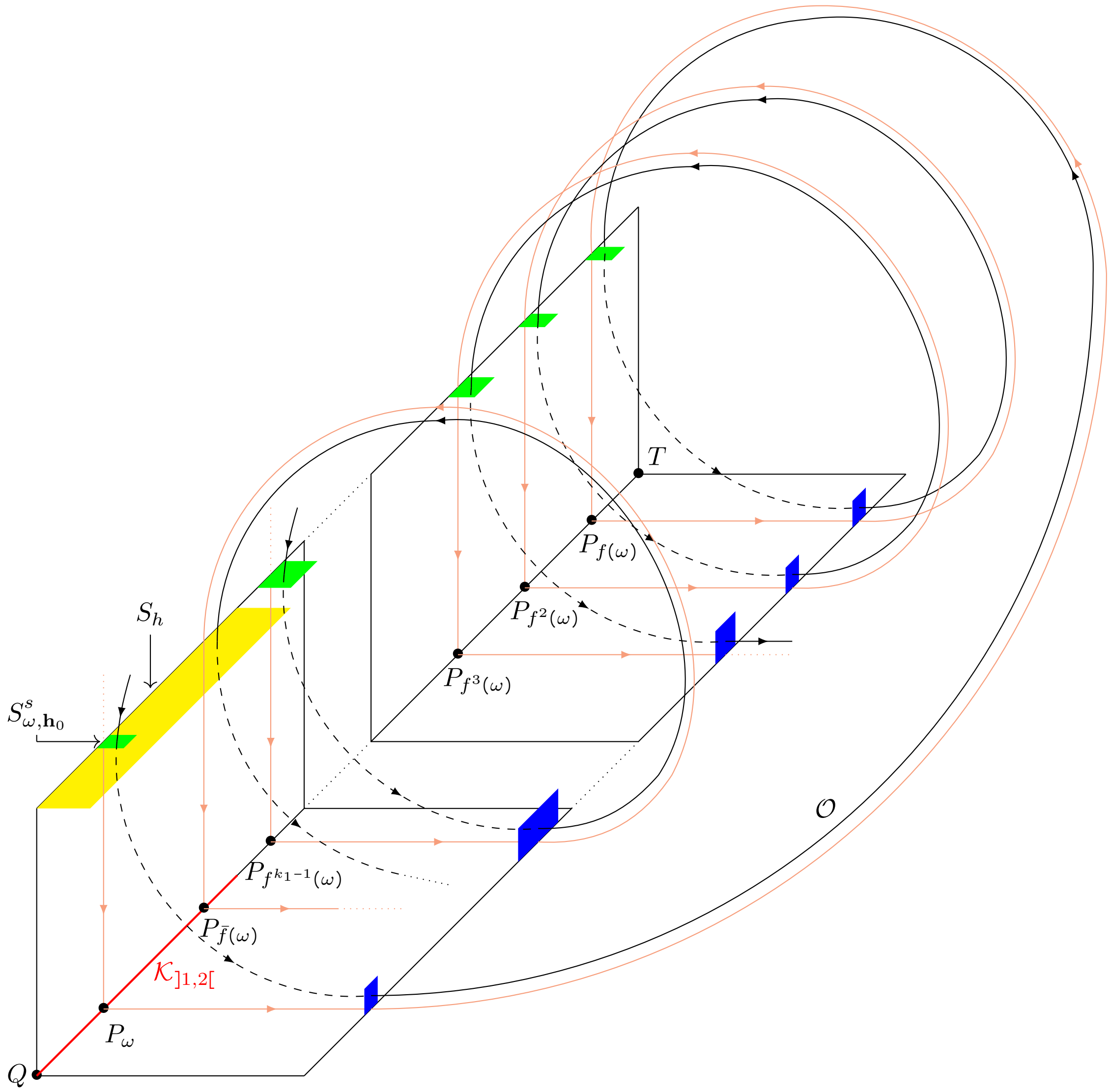

Figure 5.2 – A type IX orbit $\mathcal{O}$ (in black) traveling close to the first era of the heteroclinic chain starting at P_ω (in melon), where $\omega = [1; k_1, k_2, \ldots]$. The local sections $S^s_{\omega,\mathbf{h}_0}$, $S^s_{f(\omega),\mathbf{h}_1}$, ..., $S^s_{\bar{f}(\omega),\mathbf{h}_{k_1}}$ are represented in green, while the local sections $S^u_{\omega,\mathbf{h}'_0}$, $S^u_{f(\omega),\mathbf{h}'_1}$, ..., $S^u_{\bar{f}(\omega),\mathbf{h}'_{k_1}}$ are represented in blue. The era transition map $\bar{\Phi}_{\omega,\mathbf{h}_0}$ encodes the travel of the orbit $\mathcal{O}$ between the local section $S^s_{\omega,\mathbf{h}_0}$ and the global section S_h (in yellow). One can decompose the era transition map into several epoch transition maps, encoding the travel of the orbit $\mathcal{O}$ between two consecutive green sections.

orbits. Hence, they are naturally related. More precisely, one easily checks that:

$$\bar{\Phi}^A_{\omega,\mathbf{h}}(x) = \begin{cases} (0,h,0,\bar{f}(x_c)) & \text{if } k_1(x_c) \geq 2 \\ (0,0,h,\bar{f}(x_c)) & \text{if } k_1(x_c) = 1 \end{cases} \qquad \text{where } x_c = [1; k_1(x_c), k_2(x_c), \dots] \tag{5.6}$$

Analogously,

$$\hat{\Phi}^A_{\omega,\mathbf{h}}(x) = \begin{cases} (0,h,0,\hat{f}(x_c)) & \text{if } k_2(x_c) \geq 2 \\ (0,0,h,\hat{f}(x_c)) & \text{if } k_2(x_c) = 1 \end{cases} \qquad \text{where } x_c = [1; k_1(x_c), k_2(x_c), \dots] \tag{5.7}$$

where $\hat{f}$ is the *double era Kasner map* defined by

$$\hat{f}(\omega) \stackrel{\text{def}}{=} \bar{f} \circ \bar{f}(\omega)$$

5.4.2 The epoch transition map $\Phi_{\omega,\mathbf{h},\mathbf{h}'}$

Consider a type IX orbit orbit traveling between $\mathcal{S}^s_{\omega,\mathbf{h}} \cap \mathcal{S}_h$ and $\mathcal{S}_h$. Typically, this orbit stays close to the piece of heteroclinic chain connecting successively the points $\mathcal{P}_\omega$, $\mathcal{P}_{f(\omega)}$, …, $\mathcal{P}_{\bar{f}(\omega)}$. Since the global behaviour of this orbit is rather complex, it is a good idea to focus on a smaller part of this travel, namely a transition between a neighbourhood of a point $\mathcal{P}_{f^j(\omega)}$ and a neighbourhood of the point $\mathcal{P}_{f^{j+1}(\omega)}$. This leads to the definition of the epoch transition map.

Since the travel of the orbits of the Wainwright-Hsu vector field between $\mathcal{S}^s_{\omega,\mathbf{h}} \cap S_h$ and S_h is complex, we can study it piece by piece. During this travel, an orbit stays close to the piece of heteroclinic chain connecting successively the points $\mathcal{P}_\omega$, $\mathcal{P}_{f(\omega)}$, …, $\mathcal{P}_{\bar{f}(\omega)}$. Hence, it is a good idea to focus on a transition between a neighbourhood of a point $\mathcal{P}_{f^j(\omega)}$ and a neighbourhood of the point $\mathcal{P}_{f^{j+1}(\omega)}$. This leads to the definition of the epoch transition map.

Definition 5.17 (Epoch transition map $\Phi_{\omega,\mathbf{h},\mathbf{h}'}$)**.** Let $\omega \in]1,+\infty[\setminus \{2\}$, $\mathbf{h} = (h, h_\perp, h_{/\!/})$ and $\mathbf{h}' = (h', h'_\perp, h'_{/\!/})$ so that the sections $S^s_{\omega,\mathbf{h}}$ and $S^s_{f(\omega),\mathbf{h}'}$ are included in U_ξ. We define the *epoch transition map*

$$\Phi_{\omega,\mathbf{h},\mathbf{h}'} : S^s_{\omega,\mathbf{h}} \to S^s_{f(\omega),\mathbf{h}'}$$

as usual: if the heteroclinic chain $\mathcal{H}(x)$ intersects the section $\mathcal{S}^s_{f(\omega),\mathbf{h}'}$, then $\Phi_{\omega,\mathbf{h},\mathbf{h}'}(x)$ is the 4-tuple of coordinates of the first intersection point of $\mathcal{H}(x)$ with $\mathcal{S}^s_{f(\omega),\mathbf{h}'}$, otherwise $\Phi_{\omega,\mathbf{h},\mathbf{h}'}$ is not defined at the point x. See figures 5.2 and 5.3.

In restriction to the set $\{x_u > 0\}$, the epoch transition map $\Phi_{\omega,\mathbf{h},\mathbf{h}'}$ is simply the transition map of the orbits of the Wainwright-Hsu vector field $\mathcal{X}$ between the section $\mathcal{S}^s_{\omega,\mathbf{h}}$ and the section $\mathcal{S}^s_{f(\omega),\mathbf{h}'}$, expressed in local coordinates.

If $x_u = 0$, then x is contained in the stable manifold of the point $P_{x_c} = (0,0,0,x_c)$. So the heteroclinic chain $\mathcal{H}(x)$ is the concatenation of the orbit of x, the type II orbit $\mathcal{O}_{\mathcal{P}_{x_c} \to \mathcal{P}_{f(x_c)}}$, the type II orbit $\mathcal{O}_{\mathcal{P}_{f(x_c)} \to \mathcal{P}_{f^2(x_c)}}$, etc. Hence, for any reasonable choice of the parameters $\mathbf{h}$ and $\mathbf{h}'$, the point $\Phi_{\omega,\mathbf{h},\mathbf{h}'}(x)$ is the first intersection point of the type II orbit $\mathcal{O}_{\mathcal{P}_{x_c} \to \mathcal{P}_{f(x_c)}}$ with the section $\mathcal{S}^s_{f(\omega),\mathbf{h}'}$. That is,

$$\Phi_{\omega,\mathbf{h},\mathbf{h}'}(0, x_{s_1}, x_{s_2}, x_c) = \begin{cases} (0, h', 0, f(x_c)) & \text{if } \omega > 2 \\ (0,0,h', f(x_c)) & \text{if } 1 < \omega < 2 \end{cases} \tag{5.8}$$

This formula deserves some explanations. If $\omega > 2$, then $\mathcal{O}^u_{x_c} = \mathcal{O}^{s_1}_{f(x_c)}$ and in that case, the first intersection point of $\mathcal{O}^u_{x_c}$ with the section $\mathcal{S}^s_{f(\omega),\mathbf{h}'}$ is in $\mathcal{S}^{s_1}_{f(\omega),\mathbf{h}'}$, otherwise it is in $\mathcal{S}^{s_2}_{f(\omega),\mathbf{h}'}$.

Definition 5.18 (Map $\Phi^A_{\omega,\mathbf{h},\mathbf{h}'}$)**.** Let $\omega \in]1,+\infty[\setminus \{2\}$, $\mathbf{h} = (h, h_\perp, h_{/\!/})$ and $\mathbf{h}' = (h', h'_\perp, h'_{/\!/})$ so that the sections $S^s_{\omega,\mathbf{h}}$ and $S^s_{f(\omega),\mathbf{h}'}$ are included in U_ξ and so that Proj_A is well defined on the section $S^s_{\omega,\mathbf{h}}$. We define the map $\Phi^A_{\omega,\mathbf{h},\mathbf{h}'} : S^s_{\omega,\mathbf{h}} \to S^s_{f(\omega),\mathbf{h}'}$ by the formula

$$\Phi^A_{\omega,\mathbf{h},\mathbf{h}'}(x) \stackrel{\text{def}}{=} \Phi_{\omega,\mathbf{h},\mathbf{h}'} \circ \mathrm{Proj}_\mathrm{A}(x)$$

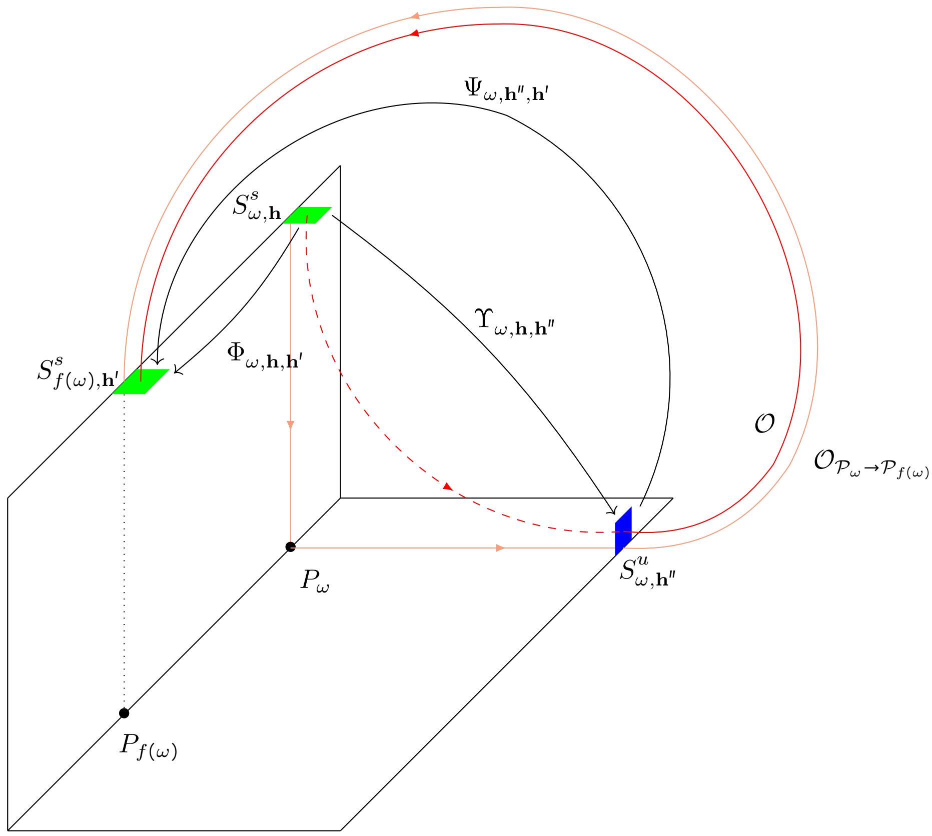

Figure 5.3 – A type IX orbit $\mathcal{O}$ (in red) traveling between the section $S^s_{\omega,\mathbf{h}}$ and the section $S^s_{f(\omega),\mathbf{h}'}$. During this travel, the orbit $\mathcal{O}$ stays close to a piece of heteroclinic chain (in melon) passing through the point $\mathcal{P}_\omega$. For a good choice of parameters $\mathbf{h}$, $\mathbf{h}'$ and $\mathbf{h}''$, the epoch transition map $\Phi_{\omega,\mathbf{h},\mathbf{h}'}$ is equal to the composition of $\Psi_{\omega,\mathbf{h}'',\mathbf{h}'}$ with $\Upsilon_{\omega,\mathbf{h},\mathbf{h}''}$, *i.e.* the orbit $\mathcal{O}$ does not intersect the section $S^s_{f(\omega),\mathbf{h}'}$ before it intersects the section $S^u_{\omega,\mathbf{h}''}$.

Remark 5.19. According to (5.8), the map $\Phi^A_{\omega,\mathbf{h},\mathbf{h}'}$ admits an explicit expression. For any reasonable choice of the parameters $\mathbf{h}$ and $\mathbf{h}'$,

$$\Phi^A_{\omega,\mathbf{h},\mathbf{h}'}(x) = \begin{cases} (0, h', 0, f(x_c)) & \text{if } \omega > 2 \\ (0, 0, h', f(x_c)) & \text{if } 1 < \omega < 2 \end{cases} \tag{5.9}$$

Recall that $\mathcal{S}^s_{\omega,\mathbf{h}}$ is the "entry gate" (for the orbits of the Wainwright-Hsu vector field) of a neighbourhood of the point $\mathcal{P}_\omega$ while $\mathcal{S}^s_{f(\omega),\mathbf{h}'}$ is the "entry gate" of a neighbourhood of the point $\mathcal{P}_{f(\omega)}$. As a consequence, when an orbit travels between $\mathcal{S}^s_{\omega,\mathbf{h}}$ and $\mathcal{S}^s_{f(\omega),\mathbf{h}'}$, there is a first phase where it is close to the point $\mathcal{P}_\omega$ (and, *a fortiori*, it is close to the Kasner interval) and a second phase where it is far away from the Kasner interval but close to the type II orbit $\mathcal{O}_{\mathcal{P}_\omega \to \mathcal{P}_{f(\omega)}}$. This leads us to introduce two more transition maps, $\Upsilon_{\omega,\mathbf{h},\mathbf{h}''}$ and $\Psi_{\omega,\mathbf{h}'',\mathbf{h}'}$, such that $\Phi_{\omega,\mathbf{h},\mathbf{h}'} = \Psi_{\omega,\mathbf{h}'',\mathbf{h}'} \circ \Upsilon_{\omega,\mathbf{h},\mathbf{h}'}$. Each one of these maps depicts the behaviour of the orbits during one of the two phases described above.

Note that until the end of this chapter, we will assume that all the local sections considered are included in U_ξ. We will also implicitly assume that $\mathrm{Proj_A}$ is well defined on these local sections. This is to avoid a lot of repetition in the following definitions, as they are all modeled on definitions 5.17 and 5.18.

5.4.3 The transition map $\Upsilon_{\omega,\mathbf{h},\mathbf{h}'}$

We start with the transition map $\Upsilon_{\omega,\mathbf{h},\mathbf{h}'}$ capturing the behaviour of the orbits in the neighbourhood of the point $\mathcal{P}_\omega$.

Definition 5.20 (Transition map $\Upsilon_{\omega,\mathbf{h},\mathbf{h}'}$). Let $\omega \in \,]1,+\infty[$, $\mathbf{h} = (h, h_\perp, h_{/\!/})$ and $\mathbf{h}' = (h', h'_\perp, h'_{/\!/})$. We define the transition map

$$\Upsilon_{\omega,\mathbf{h},\mathbf{h}'} : S^s_{\omega,\mathbf{h}} \to S^u_{\omega,\mathbf{h}'}$$

as usual: if the heteroclinic chain $\mathcal{H}(x)$ intersects the section $\mathcal{S}^u_{\omega,\mathbf{h}'}$, then $\Upsilon_{\omega,\mathbf{h},\mathbf{h}'}(x)$ is the 4-tuple of coordinates of the first intersection point of $\mathcal{H}(x)$ with $\mathcal{S}^u_{\omega,\mathbf{h}'}$, otherwise $\Upsilon_{\omega,\mathbf{h},\mathbf{h}'}$ is not defined at the point x. See figure 5.3.

Definition 5.21 (Map $\Upsilon^A_{\omega,\mathbf{h},\mathbf{h}'}$). Let $\omega \in \,]1,+\infty[$, $\mathbf{h} = (h, h_\perp, h_{/\!/})$ and $\mathbf{h}' = (h', h'_\perp, h'_{/\!/})$. We define the map $\Upsilon^A_{\omega,\mathbf{h},\mathbf{h}'} : S^s_{\omega,\mathbf{h}} \to S^u_{\omega,\mathbf{h}'}$ by the formula

$$\Upsilon^A_{\omega,\mathbf{h},\mathbf{h}'}(x) \stackrel{\text{def}}{=} \Upsilon_{\omega,\mathbf{h},\mathbf{h}'} \circ \mathrm{Proj_A}(x)$$

Remark 5.22. We can do the same remarks as for the epoch transition map $\Phi_{\omega,\mathbf{h},\mathbf{h}'}$. In particular, if $h'_{/\!/} \geq h_{/\!/}$, then for every $x \in S^s_{\omega,\mathbf{h}}$ such that $x_u = 0$, we have

$$\Upsilon_{\omega,\mathbf{h},\mathbf{h}'}(x) = (h', 0, 0, x_c) \tag{5.10}$$

In other words, $\Upsilon_{\omega,\mathbf{h},\mathbf{h}'}(x)$ is the unique intersection point of the type II orbit $O^u_{x_c}$ with $S^u_{\omega,\mathbf{h}'}$. As a consequence, the map $\Upsilon^A_{\omega,\mathbf{h},\mathbf{h}'}$ admits an explicit expression: if $h'_{/\!/} \geq h_{/\!/}$, then

$$\Upsilon^A_{\omega,\mathbf{h},\mathbf{h}'}(x) = (h', 0, 0, x_c) \tag{5.11}$$

5.4.4 The transition map $\Psi_{\omega,\mathbf{h},\mathbf{h}'}$

We conclude with the transition map $\Psi_{\omega,\mathbf{h},\mathbf{h}'}$ capturing the behaviour of the orbits in the neighbourhood of the type II orbit $\mathcal{O}_{\mathcal{P}_\omega \to \mathcal{P}_{f(\omega)}}$.

Definition 5.23 (Transition map $\Psi_{\omega,\mathbf{h},\mathbf{h}'}$). Let $\omega \in \,]1,+\infty[\setminus\{2\}$, $\mathbf{h} = (h, h_\perp, h_{/\!/})$ and $\mathbf{h}' = (h', h'_\perp, h'_{/\!/})$. We define the transition map

$$\Psi_{\omega,\mathbf{h},\mathbf{h}'} : S^u_{\omega,\mathbf{h}} \to S^s_{f(\omega),\mathbf{h}'}$$

as usual: if the heteroclinic chain $\mathcal{H}(x)$ intersects the section $\mathcal{S}^s_{f(\omega),\mathbf{h}'}$, then $\Psi_{\omega,\mathbf{h},\mathbf{h}'}(x)$ is the 4-tuple of coordinates of the first intersection point of $\mathcal{H}(x)$ with $\mathcal{S}^s_{f(\omega),\mathbf{h}'}$, otherwise $\Psi_{\omega,\mathbf{h},\mathbf{h}'}$ is not defined at the point x. See figure 5.3.

Definition 5.24 (Map $\Psi^A_{\omega,\mathbf{h},\mathbf{h}'}$)**.** Let $\omega \in]1,+\infty[$, $\mathbf{h} = (h, h_\perp, h_{/\!/})$ and $\mathbf{h}' = (h', h'_\perp, h'_{/\!/})$. We define the map $\Psi^A_{\omega,\mathbf{h},\mathbf{h}'} : S^u_{\omega,\mathbf{h}} \to S^s_{f(\omega),\mathbf{h}'}$ by the formula

$$\Psi^A_{\omega,\mathbf{h},\mathbf{h}'}(x) \stackrel{\text{def}}{=} \Psi_{\omega,\mathbf{h},\mathbf{h}'} \circ \mathrm{Proj}_{\mathrm{A}}(x)$$

Remark 5.25. The map $\Psi^A_{\omega,\mathbf{h},\mathbf{h}'}$ admits an explicit expression. More precisely, if $h'_{/\!/}$ is large enough compared to $h_{/\!/}$, then for every $x \in S^u_{\omega,\mathbf{h}}$,

$$\Psi^A_{\omega,\mathbf{h},\mathbf{h}'}(x) = \begin{cases} (0, h', 0, f(x_c)) & \text{if } \omega > 2 \\ (0, 0, h', f(x_c)) & \text{if } 1 < \omega < 2 \end{cases} \tag{5.12}$$

The explanation is the same than for formula (5.8).

In the next four chapters, we will study these transition maps. More precisely, we will study the transition map $\Upsilon_{\omega,\mathbf{h},\mathbf{h}'}$ in Chapter 6, then the transition map $\Psi_{\omega,\mathbf{h},\mathbf{h}'}$ in Chapter 7, then the epoch transition map $\Phi_{\omega,\mathbf{h},\mathbf{h}'}$ in Chapter 8 and finally the era transition map $\bar{\Phi}_{\omega,\mathbf{h}}$ and the double era transition map $\hat{\Phi}_{\omega,\mathbf{h}}$ in Chapter 9.

Chapter 6

Dynamics in the neighbourhood of a point of the Kasner circle

The goal of this section is to give some sharp estimates on the transition map $\Upsilon_{\omega,\mathbf{h}^s,\mathbf{h}^u}$ (see definition 5.20). Recall that $\Upsilon_{\omega,\mathbf{h}^s,\mathbf{h}^u}$ describes the transition of the orbits of the Wainwright-Hsu vector field from the section $S^s_{\omega,\mathbf{h}^s}$ to the section $S^u_{\omega,\mathbf{h}^u}$. Both sections $S^s_{\omega,\mathbf{h}^s}$ and $S^u_{\omega,\mathbf{h}^u}$ are close to the point $P_\omega := (0,0,0,\omega)$ belonging to the Kasner circle in local coordinates. Actually, $S^s_{\omega,\mathbf{h}^s}$ should be thought as the "entrance gate" to the neighbourhood of P_ω for the orbits, whereas $S^u_{\omega,\mathbf{h}^u}$ should be thought as the "exit gate". We will choose the parameters $\mathbf{h}^s$ and $\mathbf{h}^u$ so that the orbits starting in the section $S^s_{\omega,\mathbf{h}^s}$ hit the section $S^u_{\omega,\mathbf{h}^u}$ before they exit a small neighbourhood of P_ω where the local vector field X_ω is defined. Recall that the orbits of X_ω are the same as those of X. Hence, we are left to investigate, for any $\omega \in]1,+\infty[$, the dynamics generated by the local vector field X_ω near the point P_ω. The methods we use are generalizations and refinements of those used in the work of Liebscher & al. [Lie+11].

The following proposition is the main result of this section. For a technical reason explained below, we will often encounter the quantity $\frac{\omega-1}{4}$ in the estimates. Hence, we introduce the notation

$$d(\omega) \stackrel{\text{def}}{=} \frac{\omega-1}{4}$$

Proposition 6.1 (Control of the transition maps $\Upsilon_{\omega,\mathbf{h}^s,\mathbf{h}^u}$)**.** *There exist two constants $C>0$ and $n \in \mathbb{N}$ such that for every $\omega \in]1,+\infty[$, every $0 < h \le (C\omega^n)^{-1}$, every $0 < h_\perp < \min(h, d(\omega))$, for $\mathbf{h}^s = (h, h_\perp, \min(h, d(\omega)))$ and $\mathbf{h}^u = (h,h,2h)$, the transition map*

$$\Upsilon_{\omega,\mathbf{h}^s,\mathbf{h}^u} : S^s_{\omega,\mathbf{h}^s} \to S^u_{\omega,\mathbf{h}^u}$$

is well defined. Moreover, for every $x, \tilde{x} \in S^s_{\omega,\mathbf{h}^s}$, we have the following estimates where we denote $\Upsilon := \Upsilon_{\omega,\mathbf{h}^s,\mathbf{h}^u}$ and $\Upsilon^A := \Upsilon^A_{\omega,\mathbf{h}^s,\mathbf{h}^u}$:

(Distance to the Mixmaster attractor)

$$\operatorname{dist}_\infty(\Upsilon(x), A) = \left\|\Upsilon(x) - \Upsilon^A(x)\right\|_\perp \le h_\perp^{\frac{\omega+2}{\omega+1}} h^{-1} \tag{6.1}$$

(Drift in the direction tangent to the Mixmaster attractor)

$$\left\|\Upsilon(x) - \Upsilon^A(x)\right\|_{/\!/} \le h_\perp h C \omega^n \tag{6.2}$$

(Lipschitz control in the direction transverse to the Mixmaster attractor)

$$\left\|(\Upsilon(x) - \Upsilon(\tilde{x})) - (\Upsilon^A(x) - \Upsilon^A(\tilde{x}))\right\|_\perp \le h_\perp^{\frac{1}{\omega+1}} h^{-1} \|x - \tilde{x}\|_\infty \tag{6.3}$$

(Lipschitz control in the direction tangent to the Mixmaster attractor)

$$\left\|(\Upsilon(x)-\Upsilon(\tilde{x}))-(\Upsilon^A(x)-\Upsilon^A(\tilde{x}))\right\|_{/\!/} \leq C\omega^n h\,\|x-\tilde{x}\|_\perp + C\omega^n h_\perp\,\|x-\tilde{x}\|_{/\!/} \tag{6.4}$$

Remark 6.2 (Purpose of Proposition 6.1). Recall that Υ describes the behaviour of all the orbits of the local vector field X_ω near $\mathcal{K}$ and $\mathcal{A}$ while Υ^A describes the behaviour of the heteroclinic chains in $\mathcal{A}$. The purpose of Proposition 6.1 is to compare the dynamics of Υ to the one of Υ^A.

Remark 6.3 (Explanation of the different estimates). To fix the ideas, consider an orbit $\tau \mapsto x(\tau)$ of the local vector field X_ω traveling between the sections $S^{s_1}_{\omega,\mathbf{h}^s}$ and $S^u_{\omega,\mathbf{h}^u}$. Denote by $x^{\text{in}} = (x_u^{\text{in}}, h, x_{s_2}^{\text{in}}, x_c^{\text{in}}) \in S^s_{\omega,\mathbf{h}^s}$ its initial condition. Estimate (6.1) means that the distance between the orbit x and the Mixmaster attractor is contracted during its travel. Moreover, it shows that this contraction degenerates when $\omega \to +\infty$, that is, when x travels very close to a Taub point. Estimate (6.2) means that the more x^{in} is close to the Mixmaster attractor, the more the orbit x does not deviate, in the direction tangent to the Mixmaster attractor, from the type II orbit passing through the point $(0,h,0,x_c^{\text{in}})$. Estimates (6.3) and (6.4) prove that $\Upsilon - \Upsilon^A$ is Lipschitz and explicit a Lipschitz constant for this map. As a summary, one can remember the following fact. There is a competition between two factors for the above estimates:

- The more the points are close to the Mixmaster attractor, the more the estimates are precise.
- The more the points are close to a Taub point, the less the estimates are precise.

Remark 6.4. Estimate (6.2) could be rewritten in a simpler way as $|\Upsilon(x)_c - x_c| \leq h_\perp h C\omega^n$. We did not make this choice to make it clear here that we compare Υ and Υ^A. Same remark goes for (6.4).

Remark 6.5. If $x \in S^u_{\omega,\mathbf{h}^u}$, then for all $y \in S^u_{\omega,\mathbf{h}^u} \cap A$

$$\text{dist}_\infty(x, A) = \|x-y\|_\perp$$

Now remark that if $x \in S^s_{\omega,\mathbf{h}^s}$, then $\Upsilon^A(x) \in S^u_{\omega,\mathbf{h}^u} \cap A$. This is the reason why

$$\text{dist}_\infty(\Upsilon(x), A) = \left\|\Upsilon(x) - \Upsilon^A(x)\right\|_\perp$$

Remark 6.6 (Technical detail). The quantity $\frac{\omega-1}{4}$ in the upper bound on the size of the sections is purely technical. It is closely related to the fact that the coordinates are not defined for $x_c = 1$. Basically, we need to make sure that the orbits do not start too close to this frontier so that they intersect the section $S^u_{\omega,\mathbf{h}^u}$ before they possibly encounter this frontier and cease to exist.

To prove Proposition 6.1, we divide the study in two parts. In section 6.1, we study the behaviour of one orbit of X_ω. This will lead to estimates (6.1) and (6.2). In section 6.2, we compare the behaviour of two orbits. This will lead to estimates (6.3) and (6.4).

Following the notations of Proposition 4.2, we will denote by $x = (x_u, x_{s_1}, x_{s_2}, x_c)$ the coordinates of any point $x \in U_\xi \subset \mathbb{R}^4$.

Recall from Proposition 4.8 that the differential equations associated with X_ω have the following form

$$\begin{cases} x_u' &= \mu_u(\omega)x_u \\ x_{s_1}' &= -\tilde{\mu}_{\omega,s_1}(x_c)\,x_{s_1} + X^{u,s_1}_{\omega,s_1}(x)x_u x_{s_1} + X^{s_1,s_1}_{\omega,s_1}(x)x_{s_1}^2 + X^{s_2,s_1}_{\omega,s_1}(x)x_{s_2}x_{s_1} \\ x_{s_2}' &= -\tilde{\mu}_{\omega,s_2}(x_c)\,x_{s_2} + X^{u,s_2}_{\omega,s_2}(x)x_u x_{s_2} + X^{s_1,s_2}_{\omega,s_2}(x)x_{s_1}x_{s_2} + X^{s_2,s_2}_{\omega,s_2}(x)x_{s_2}^2 \\ x_c' &= X^{u,s_1}_{\omega,c}(x)x_u x_{s_1} + X^{u,s_2}_{\omega,c}(x)x_u x_{s_2} \end{cases} \tag{6.5}$$

Remark 6.7. Let $\omega \in \,]1,+\infty[$ and $\tau \mapsto x(\tau)$ be an orbit of the local vector field X_ω. Denote by x^{in} its initial condition. Using (6.5), one can see that the coordinate x_u is strictly increasing. It follows that the section $S^u_{\omega,\mathbf{h}^u}$ is intersected by the orbit x at most once. If this is the case, the time of first (and unique) intersection of the orbit x with the section $S^u_{\omega,\mathbf{h}^u}$ is

$$\tau^{\text{out}}(x,\omega,h) \overset{\text{def}}{=} \frac{1}{\mu_u(\omega)} \ln \frac{h}{x_u^{\text{in}}} \tag{6.6}$$

where $\mathbf{h}^u = (h, h_\perp, h_{/\!/})$.

We should insist on the fact that we are talking about the local vector field X_ω and its orbits. The "real" orbits for the Wainwright-Hsu vector field $\mathcal{X}$ can intersect the section $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ infinitely many times. The local vector field X_ω being a renormalization of $\mathcal{X}$ in a neighbourhood $\mathcal{U}$ of the point $\mathcal{P}_\omega$ on the Kasner circle, an orbit of X_ω is exactly a connected component of an orbit of $\mathcal{X}$ intersected with $\mathcal{U}$ (modulo the local coordinate system ξ).

To conclude this introduction, we state the first ingredient for the shadowing theorem (see chapter 11).

Proposition 6.8 (Shadowing near the Kasner circle)**.** *There exist two constants $C \geq 1$ and $n \in \mathbb{N}$ such that the properties below hold for every $\omega \in]1,+\infty[$, every $i \in \{1,2\}$, every $0 < h \leq (C\omega^n)^{-1}$, every $0 < h_\perp \leq \min(\frac{1}{3}h, d(\omega))$, for $\mathbf{h}^s = (h, h_\perp, h_\perp)$ and for $\mathbf{h}^u = (h, h, 2h)$. Let $\tau \mapsto x(\tau)$ be an orbit of the local vector field X_ω whose initial condition $x^{in} := x(0)$ belongs to the section $S^s_{\omega,\mathbf{h}^s}$ and such that $x^{in}_u \neq 0$. See figure 6.1. Set*

$$\tau^* = \tau^*_{\omega,h}(x^{in}) \stackrel{def}{=} \frac{1}{\mu_u(\omega) + \frac{4}{5}\mu_{s_1}(\omega)} \ln \frac{4}{5}\frac{\mu_{s_1}(\omega)h}{\mu_u(\omega)x^{in}_u}$$

Then

1. *$0 \leq \tau^* \leq \tau^{out}$ where $\tau^{out} = \tau^{out}(x,\omega,h)$.*

2. *The point $x(\tau^*)$ is very close to the point P_ω on the Kasner circle. More precisely,*

$$\left\|x(\tau^*) - P_\omega\right\|_\infty \leq 8h_\perp^{\frac{1}{3}} \tag{6.7}$$

3. *The orbit segment joining x^{in} to $x(\tau^*)$ is very close to the type* II *orbit interval joining $P^{s_i}_{\omega,h} \in S^{s_i}_{\omega,\mathbf{h}^s}$ to P_ω with respect to the Hausdorff distance. More precisely,*

$$d_{\mathcal{H}}\left(x\left([0,\tau^*]\right), \left[P^{s_i}_{\omega,h}, P_\omega\right[\right) \leq 8h_\perp^{\frac{1}{3}} \tag{6.8a}$$

4. *The orbit segment joining $x(\tau^*)$ to x^{out} is very close to the type* II *orbit interval joining P_ω to $P^u_{\omega,h} \in S^u_{\omega,\mathbf{h}^u}$ with respect to the Hausdorff distance. More precisely,*

$$d_{\mathcal{H}}\left(x\left(\left[\tau^*, \tau^{out}\right]\right), \left]P_\omega, P^u_{\omega,h}\right]\right) \leq 8h_\perp^{\frac{1}{3}} \tag{6.8b}$$

6.1 Control of one orbit

Our goal is, for any given $\omega \in]1,+\infty[$, to study the behaviour of the orbits of the local vector field X_ω which travel between two given sections $S^s_{\omega,\mathbf{h}^s}$ and $S^u_{\omega,\mathbf{h}^u}$.

We first study the behaviour of the orbits that are assumed to stay in a small neighbourhood of P_ω, where the dynamics is "almost linear". Lemma 6.14 shows that, for such an orbit $\tau \mapsto x(\tau)$, the coordinate x_u is exponentially increasing (if it is not identically zero), the coordinates x_{s_1} and x_{s_2} are exponentially decreasing (if they are not identically zero) and the variation of the coordinate x_c is small. Then, we prove that for $\mathbf{h}^s$ and $\mathbf{h}^u$ well chosen, any orbit crossing $S^s_{\omega,\mathbf{h}^s}$ will eventually cross $S^u_{\omega,\mathbf{h}^u}$ and stays in a small neighbourhood of P_ω during its travel between those two sections. To do this, we need two preliminary tools:

- A control of the eigenvalues $-\tilde{\mu}_{\omega,s_1}(x_c)$ and $-\tilde{\mu}_{\omega,s_2}(x_c)$ of $DX_\omega(0,0,0,x_c)$. This is done in corollary 6.11.

- An estimate on the quantity

$$\chi = x_u \left\|x_{s_1,s_2}\right\|_1 = x_u(x_{s_1} + x_{s_2})$$

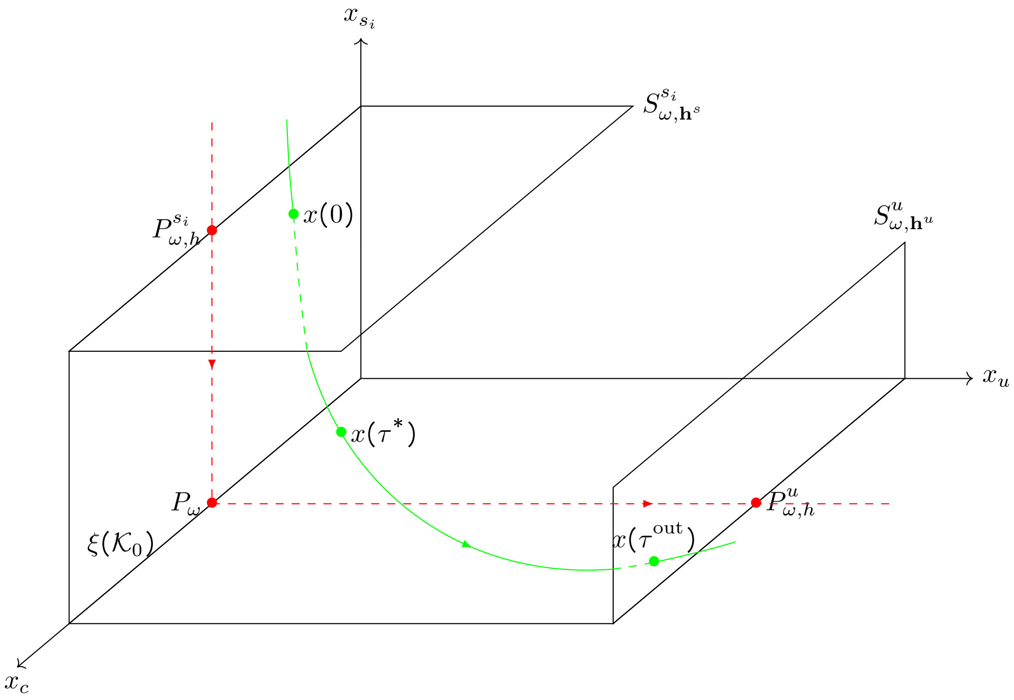

Figure 6.1 – Notations for local shadowing.

used to bound the variation of the coordinate x_c. Lemma 6.12 shows that this map is exponentially decreasing along the orbits of the local vector field X_ω which travel sufficiently close to the point P_ω.

Lemma 6.9 (Control of the eigenvalues μ_u, μ_{s_1} et μ_{s_2}). *Let $\omega \in \,]1,+\infty[$ and $\alpha \in]0,1[$. For every $\omega' \in \,]1,+\infty[$ such that*

$$\left|\omega' - \omega\right| \leq \frac{1-\alpha}{3\omega}$$

we have

$$\alpha \leq \frac{\mu_u(\omega')}{\mu_u(\omega)}, \frac{\mu_{s_1}(\omega')}{\mu_{s_1}(\omega)}, \frac{\mu_{s_2}(\omega')}{\mu_{s_2}(\omega)} \leq \alpha^{-1} \tag{6.9}$$

Proof. According to (3.14a),

$$\left|\frac{\mathrm{d}\mu_u}{\mathrm{d}\omega}(\omega')\right| = 6\left|\frac{1-\omega'^2}{\left(1+\omega'+\omega'^2\right)^2}\right| \leq 6$$

Let $\omega' \in \,]1,+\infty[$ such that

$$\left|\omega' - \omega\right| \leq \frac{1-\alpha}{3\omega}$$

According to the mean value theorem,

$$\left|\mu_u(\omega') - \mu_u(\omega)\right| \leq 6\frac{1-\alpha}{3\omega} \leq 6\frac{(1-\alpha)\omega}{1+\omega+\omega^2} = (1-\alpha)\mu_u(\omega)$$

Observe that $1-\alpha \leq \alpha^{-1} - 1$. Hence, (6.9) holds true for μ_u. The computations are analogous for μ_{s_1} and μ_{s_2}. □

Recall from Proposition 4.8 that

$$\tilde{\mu}_{\omega,s_i}\left(\omega'\right) = \frac{\mu_u(\omega)}{\mu_u(\omega')}\mu_{s_i}(\omega')$$

is an eigenvalue of $DX_\omega(0,0,0,\omega')$.

Proposition 6.10 (Control of the eigenvalue $\tilde{\mu}_{\omega,s_i}$)**.** *Let* $\omega \in\,]1,+\infty[$ *and* $\alpha \in]0,1[$. *For any* $i \in \{1,2\}$ *and any* $\omega' \in\,]1,+\infty[$ *such that*

$$\left|\omega' - \omega\right| \leq \frac{1-\alpha}{6\omega},$$

the eigenvalue $\tilde{\mu}_{\omega,s_i}\left(\omega'\right)$ *of* $DX_\omega(0,0,0,\omega')$ *satisfies*

$$\alpha \leq \frac{\tilde{\mu}_{\omega,s_i}\left(\omega'\right)}{\mu_{s_i}(\omega)} \leq \alpha^{-1} \tag{6.10}$$

Proof. Using the formula

$$\frac{\tilde{\mu}_{\omega,s_i}\left(\omega'\right)}{\mu_{s_i}(\omega)} = \frac{\mu_u(\omega)}{\mu_u(\omega')}\frac{\mu_{s_i}(\omega')}{\mu_{s_i}(\omega)}$$

and the straightforward inequality

$$\frac{1-\alpha}{6} \leq \frac{1-\sqrt{\alpha}}{3}$$

we get the estimate (6.10) from Lemma 6.9 applied twice with $\sqrt{\alpha}$ instead of α. □

Corollary 6.11 below is a refinement of Proposition 6.10 easier to use in the proof of Lemma 6.12.

Corollary 6.11 (Control of the eigenvalue $\tilde{\mu}_{\omega,s_i}$, second version)**.** *Let* $\omega \in\,]1,+\infty[$ *and* $\alpha \in]0,1[$. *For every* $\omega' \in\,]1,+\infty[$ *such that*

$$\left|\omega' - \omega\right| \leq \frac{1-\alpha}{24\omega^2},$$

for $i = 1,2$, *the eigenvalue* $\tilde{\mu}_{\omega,s_i}\left(\omega'\right)$ *of* $DX_\omega(0,0,0,\omega')$ *satisfies*

$$\alpha' \leq \frac{\tilde{\mu}_{\omega,s_i}\left(\omega'\right)}{\mu_{s_i}(\omega)} \leq \alpha'^{-1} \tag{6.11}$$

where

$$\alpha' = \frac{\frac{1-\alpha}{2}\mu_u(\omega) + \frac{1+\alpha}{2}\mu_{s_1}(\omega)}{\mu_{s_1}(\omega)}$$

Proof. Using (3.14), it is straightforward to check that

$$\frac{1-\alpha}{24\omega^2} \leq \frac{1-\alpha}{12(1+\omega)\omega} = \frac{1-\alpha'}{6\omega} \quad \text{for every } \omega \in\,]1,+\infty[\text{ and every } \alpha \in]0,1[.$$

Hence, (6.11) follows immediately from Proposition 6.10. □

Lemma 6.12 (Control of χ)**.** *There exist two constants* $C > 0$ *and* $n \in \mathbb{N}$ *such that for every* $\omega \in\,]1,+\infty[$, *every* $3/4 < \alpha < 1$, *every* $t \geq 0$ *and every orbit* $\tau \mapsto x(\tau)$ *of the local vector field* X_ω *satisfying*

$$\max\left(x_u(\tau), x_{s_1}(\tau), x_{s_2}(\tau), |x_c(\tau)-\omega|\right) \leq \frac{1-\alpha}{C\omega^n} \quad \textit{for every } \tau \in [0,t]$$

the function $\chi(\tau) = x_u(\tau)(x_{s_1}(\tau)+x_{s_2}(\tau))$ *satisfies*

$$\chi(t) \leq e^{-\alpha(\mu_{s_1}(\omega)-\mu_u(\omega))t}\chi(0) \tag{6.12}$$

Remark 6.13. Recall that $\mu_{s_1}(\omega) \geq \mu_u(\omega)$. Hence, χ is exponentially decreasing along the orbits of the local vector field X_ω.

Proof. Basically, the proof amounts to obtain a differential inequation on χ and then to use a Gronwall estimate. For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $E_{C,n}$ the set of all (ω,α,t,x) such that

$\omega \in]1, +\infty[$, $3/4 < \alpha < 1$, $t \geq 0$ and $\tau \mapsto x(\tau)$ is an orbit of the local vector field X_ω satisfying

$$\max(x_u(\tau), x_{s_1}(\tau), x_{s_2}(\tau), |x_c(\tau) - \omega|) \leq \frac{1-\alpha}{C\omega^n} \quad \text{for every } \tau \in [0, t] \tag{6.13}$$

Let $C > 0$ and $n \in \mathbb{N}$ be large enough such that for every $\omega \in]1, +\infty[$, X_ω is well defined on the open ball $B_{\omega,C,n}$ (see definition 4.6). Let $(\omega, \alpha, t, x) \in E_{C,n}$. We compute the derivative of $\chi : \tau \mapsto x_u(\tau)(x_{s_1}(\tau) + x_{s_2}(\tau))$ by replacing the derivatives of x_u, x_{s_1} and x_{s_2} by their respective expressions according to (6.5). The time $\tau \in [0, t]$ is implicit in the following computations.

$$\begin{aligned}
\chi' = {}& \mu_u(\omega) x_u (x_{s_1} + x_{s_2}) \\
& + x_u \left(-\tilde{\mu}_{\omega,s_1}(x_c)\, x_{s_1} + X^{u,s_1}_{\omega,s_1}(x) x_u x_{s_1} + X^{s_1,s_1}_{\omega,s_1}(x) x_{s_1}^2 + X^{s_2,s_1}_{\omega,s_1}(x) x_{s_2} x_{s_1} \right) \\
& \quad + x_u \left(-\tilde{\mu}_{\omega,s_2}(x_c)\, x_{s_2} + X^{s_1,s_2}_{\omega,s_2}(x) x_{s_1} x_{s_2} + X^{u,s_2}_{\omega,s_2}(x) x_u x_{s_2} + X^{s_2,s_2}_{\omega,s_2}(x) x_{s_2}^2 \right)
\end{aligned}$$

According to the estimate on the non-linear terms (4.13), the inequality (6.13) and the fact that $\tilde{\mu}_{\omega,s_1} \leq \tilde{\mu}_{\omega,s_2}$, there exist $C_0 \geq 24$ and $n_0 \geq 2$ such that for every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, \alpha, t, x) \in E_{C,n}$, we have

$$\chi' \leq \left[(\mu_u(\omega) - \tilde{\mu}_{\omega,s_1}(x_c)) + C_0 \omega^{n_0} \max(x_u, x_{s_1}, x_{s_2}) \right] \chi \tag{6.14}$$

According to the estimate (6.11) on the eigenvalues and the inequality (6.13), for every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, \alpha, t, x) \in E_{C,n}$, we have the following control on $\tilde{\mu}_{\omega,s_1}$:

$$-\tilde{\mu}_{\omega,s_1}(x_c) \leq -\alpha' \mu_{s_1}(\omega) \tag{6.15}$$

where

$$\alpha' = \frac{(1 - \frac{1+\alpha}{2}) \mu_u(\omega) + \frac{1+\alpha}{2} \mu_{s_1}(\omega)}{\mu_{s_1}(\omega)}$$

Plugging (6.15) into (6.14), it follows that for every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, \alpha, t, x) \in E_{C,n}$, we have

$$\begin{aligned}
\chi' &\leq \left[(\mu_u(\omega) - \alpha' \mu_{s_1}(\omega)) + C_0 \omega^{n_0} \max(x_u, x_{s_1}, x_{s_2}) \right] \chi \\
&\leq \left[\frac{1+\alpha}{2} (\mu_u(\omega) - \mu_{s_1}(\omega)) + C_0 \omega^{n_0} \max(x_u, x_{s_1}, x_{s_2}) \right] \chi
\end{aligned} \tag{6.16}$$

According to (3.14),

$$\mu_{s_1}(\omega) - \mu_u(\omega) \geq \frac{2}{\omega^2} \tag{6.17}$$

It follows by (6.16) and (6.17) that for every $C \geq C_0$, every $n \geq n_0 + 2$ and every $(\omega, \alpha, t, x) \in E_{C,n}$, we have

$$\chi' \leq \alpha(\mu_u(\omega) - \mu_{s_1}(\omega)) \chi$$

which is the desired differential inequation. Indeed, one just needs to apply the standard Gronwall's lemma to obtain (6.12). □

Lemma 6.14 (Control of one orbit close to P_ω). *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in]1, +\infty[$, every $3/4 < \alpha < 1$, every time $t \geq 0$ and every orbit $\tau \mapsto x(\tau)$ of the local vector field X_ω satisfying*

$$\max(x_u(\tau), x_{s_1}(\tau), x_{s_2}(\tau), |x_c(\tau) - \omega|) \leq \frac{1-\alpha}{C\omega^n} \quad \textit{for every } \tau \in [0, t]$$

we have the following controls:

(Control in the unstable direction)

$$x_u(t) = e^{\mu_u(\omega) t} x_u(0) \tag{6.18a}$$

(Control in the stable direction) *For every $i \in \{1, 2\}$,*

$$x_{s_i}(t) \le e^{-\alpha\mu_{s_1}(\omega)t} x_{s_i}(0) \tag{6.18b}$$

(Control in the central direction)

$$|x_c(t) - x_c(0)| \le C\omega^n x_u(0)(x_{s_1}(0) + x_{s_2}(0)) \left(1 - e^{-\alpha(\mu_{s_1}(\omega) - \mu_u(\omega))t}\right) \tag{6.18c}$$

Proof. The three controls are proven independently. The first one is a direct consequence of the evolution equation of x_u in (6.5).

For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $E_{C,n}$ the set of all (ω, α, t, x) such that $\omega \in]1, +\infty[$, $3/4 < \alpha < 1$, $t \ge 0$ and $\tau \mapsto x(\tau)$ is an orbit of the local vector field X_ω satisfying

$$\max(x_u(\tau), x_{s_1}(\tau), x_{s_2}(\tau), |x_c(\tau) - \omega|) \le \frac{1-\alpha}{C\omega^n} \quad \text{for every } \tau \in [0, t] \tag{6.19}$$

Control of the coordinate x_{s_1} (the case of x_{s_2} being analogous). We compute the derivative of x_{s_1} using (6.5). The time τ is implicit in the following computations. According to (4.13) and (6.19), there exist $C_0 \ge 24$ and $n_0 \ge 2$ such that for every $C \ge C_0$, every $n \ge n_0$ and every $(\omega, \alpha, t, x) \in E_{C,n}$, we have

$$x_{s_1}' \le \left(-\tilde{\mu}_{\omega,s_1}(x_c) + C_0\omega^{n_0} \max(x_u, x_{s_1}, x_{s_2})\right) x_{s_1} \tag{6.20}$$

According to the estimate (6.11) on the eigenvalues and the inequality (6.19), for every $C \ge C_0$, every $n \ge n_0$ and every $(\omega, \alpha, t, x) \in E_{C,n}$, we have the following control on $\tilde{\mu}_{\omega,s_1}$:

$$-\tilde{\mu}_{\omega,s_1}(x_c) \le -\alpha'\mu_{s_1}(\omega) \le -\frac{1+\alpha}{2}\mu_{s_1}(\omega) \tag{6.21}$$

where

$$\alpha' = \frac{(1 - \frac{1+\alpha}{2})\mu_u(\omega) + \frac{1+\alpha}{2}\mu_{s_1}(\omega)}{\mu_{s_1}(\omega)}$$

It follows from (6.20) and (6.21) that, for every $C \ge C_0$, every $n \ge n_0$ and every $(\omega, \alpha, t, x) \in E_{C,n}$, we have

$$x_{s_1}' \le \left(-\frac{1+\alpha}{2}\mu_{s_1}(\omega) + C_0\omega^{n_0} \max(x_u, x_{s_1}, x_{s_2})\right) x_{s_1} \tag{6.22}$$

Using (3.14), we get

$$\mu_{s_1}(\omega) \ge \frac{4}{\omega^2} \tag{6.23}$$

According to (6.22) and (6.23), for every $C \ge C_0$, every $n \ge n_0 + 2$ and every $(\omega, \alpha, t, x) \in E_{C,n}$, we have

$$x_{s_1}' \le -\alpha\mu_{s_1}(\omega)x_{s_1}$$

Hence, Gronwall's lemma gives the desired control (6.18b) on $x_{s_1}(t)$.

Control of the coordinate x_c. We compute the derivative of x_c using (6.5). The time τ is implicit in the following computations. According to (4.13) and (6.19), there exist $C_1 \ge C_0$ and $n_1 \ge n_0$ such that for every $C \ge C_1$, every $n \ge n_1$ and every $(\omega, \alpha, t, x) \in E_{C,n}$, we have

$$\left|x_c'\right| \le C_1\omega^{n_1}\chi \tag{6.24}$$

where $\chi(\tau) = x_u(\tau)(x_{s_1}(\tau) + x_{s_2}(\tau))$. According to (6.24) and Lemma 6.12, there exist $C_2 \ge C_1$ and $n_2 \ge n_1$ such that for every $C \ge C_2$, every $n \ge n_2$ and every $(\omega, \alpha, t, x) \in E_{C,n}$, we have

$$|x_c(t) - x_c(0)| \le C_1\omega^{n_1} \int_0^t e^{-\alpha(\mu_{s_1}(\omega) - \mu_u(\omega))\tau} \chi(0)\, \mathrm{d}\tau$$

so

$$|x_c(t) - x_c(0)| \le C_1\omega^{n_1}\chi(0) \frac{1 - e^{-\alpha(\mu_{s_1}(\omega) - \mu_u(\omega))t}}{\alpha(\mu_{s_1}(\omega) - \mu_u(\omega))} \tag{6.25}$$

Using (3.14), we get

$$\mu_{s_1}(\omega) - \mu_u(\omega) \geq \frac{2}{\omega^2} \tag{6.26}$$

Recall that $\alpha > 3/4$. It follows by (6.25) and (6.26) that for every $C \geq \max(C_2, \frac{8}{3}C_1)$, every $n \geq \max(n_2, n_1 + 2)$ and every $(\omega, \alpha, t, x) \in E_{C,n}$, we have

$$|x_c(t) - x_c(0)| \leq C\omega^n \chi(0) \left(1 - e^{-\alpha(\mu_{s_1}(\omega) - \mu_u(\omega))t}\right)$$

which concludes the proof. □

We are now ready to formulate, for any given $\omega \in]1, +\infty[$, two statements about the orbits $\tau \mapsto x(\tau)$ of the local vector field X_ω starting in the section $S^s_{\omega,\mathbf{h}^s}$. Proposition 6.15 deals with the generic orbits whose initial condition x^{in} verifies $x_u^{\text{in}} \neq 0$ (generic case) while Proposition 6.18 deals with the exceptional orbits for which $x_u^{\text{in}} = 0$.

More precisely, Proposition 6.15 below gives an explicit interval where the orbit x is well defined. On this interval, x will satisfy the controls of Lemma 6.14. Recall that $d(\omega) = \frac{\omega - 1}{4}$ and $\tau^{\text{out}}(x, \omega, h) = \frac{1}{\mu_u(\omega)} \ln \frac{h}{x_u^{\text{in}}}$ (see remark 6.7).

Proposition 6.15 (Behaviour of generic orbits). *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in]1, +\infty[$, every $0 < h \leq (C\omega^n)^{-1}$ and for $\mathbf{h}^s = (h, \min(h, d(\omega)), \min(h, d(\omega)))$, the following properties hold true. Let $\tau \mapsto x(\tau)$ be an orbit of the local vector field X_ω. If its initial condition $x^{in} = (x_u^{in}, x_{s_1}^{in}, x_{s_2}^{in}, x_c^{in}) := x(0)$ belongs to the section $S^s_{\omega,\mathbf{h}^s}$ and satisfies $x_u^{in} \neq 0$ then x is defined (at least) on $\left[0, \tau^{out}(x, \omega, h)\right]$. Moreover, if $3/4 < \alpha < 1$ and $h \leq (1 - \alpha)(C\omega^n)^{-1}$, then for every $\tau \in [0, \tau^{out}(x, \omega, h)]$, we have the following controls:*

(Control in the unstable direction)

$$x_u(\tau) = e^{\mu_u(\omega)\tau} x_u^{in} \tag{6.27a}$$

(Control in the stable direction) *For $i \in \{1, 2\}$,*

$$x_{s_i}(\tau) \leq e^{-\alpha\mu_{s_1}(\omega)\tau} x_{s_i}^{in} \tag{6.27b}$$

(Control in the central direction)

$$\left|x_c(\tau) - x_c^{in}\right| \leq C\omega^n x_u^{in} \max\left(x_{s_1}^{in}, x_{s_2}^{in}\right)\left(1 - e^{-\alpha(\mu_{s_1}(\omega) - \mu_u(\omega))\tau}\right) \tag{6.27c}$$

Proof. For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $E_{C,n}$ the set of all (ω, α, h, x) such that $\omega \in]1, +\infty[$, $3/4 < \alpha < 1$, $0 < h \leq (1 - \alpha)(C\omega^n)^{-1}$ and $\tau \mapsto x(\tau)$ is an orbit of the local vector field X_ω such that $x^{\text{in}} := x(0) \in S^s_{\omega,\mathbf{h}^s}$ where $\mathbf{h}^s = (h, \min(h, d(\omega)), \min(h, d(\omega)))$ and $x_u^{\text{in}} \neq 0$ (hence $x_u^{\text{in}} > 0$). Let $C_0 > 0$ and $n \in \mathbb{N}$ be large enough such that we can apply Lemma 6.14 with these two constants and such that for any $\omega \in]1, +\infty[$ the local vector field X_ω is well defined on the open ball $B_{\omega,C_0,n}$ in $(\mathbb{R}^+)^3 \times]1, +\infty[$ (see definition 4.1 and Proposition 4.8). Let $C_1 = 100C_0$ and $(\omega, \alpha, h, x) \in E_{C_1,n}$. We are going to show that the orbit x is well defined on $[0, \tau^{\text{out}}(x, \omega, h)]$ and that for every $\tau \in [0, \tau^{\text{out}}(x, \omega, h)]$, we have $\max(x_u, x_{s_1}, x_{s_2}, |x_c - \omega|) \leq \frac{1-\alpha}{C_0\omega^n}$.

Let us denote by $]\tau_-, \tau_+[$ the maximal existence interval of x (with $\tau_- < 0$ and $\tau_+ > 0$). For every $\tau \in]\tau_-, \tau_+[$, let

$$N_\perp(\tau) \stackrel{\text{def}}{=} \max(x_u(\tau), x_{s_1}(\tau), x_{s_2}(\tau))$$
$$N_{/\!/}(\tau) \stackrel{\text{def}}{=} |x_c(\tau) - \omega|$$

Remark that $N_\perp(0) \leq \min(h, d(\omega)) < \frac{1-\alpha}{C_0\omega^n}$ and $N_{/\!/}(0) \leq \min(h, d(\omega)) < \min(\frac{1-\alpha}{C_0\omega^n}, \frac{\omega-1}{2})$. Let us denote by $\tau_{\max}$ the supremum of all time $t_0 \in [0, \tau_+[$ such that for every $\tau \in [0, t_0]$, $N_\perp(\tau) \leq \frac{1-\alpha}{C_0\omega^n}$

and $N_{/\!/}(\tau) \leq \min(\frac{1-\alpha}{C_0\omega^n}, \frac{\omega-1}{2})$. By definition, for every $\tau \in [0, \tau_{\max}[$, we have $N_\perp(\tau) \leq \frac{1-\alpha}{C_0\omega^n}$ and $N_{/\!/}(\tau) \leq \min(\frac{1-\alpha}{C_0\omega^n}, \frac{\omega-1}{2})$. It follows that x must be defined at $\tau = \tau_{\max}$ since it cannot blow up. Now assume that $\tau_{\max} \leq \tau^{\text{out}}(x, \omega, h)$. By continuity, for every $\tau \in [0, \tau_{\max}]$, we have $N_\perp(\tau) \leq \frac{1-\alpha}{C_0\omega^n}$ and $N_{/\!/}(\tau) \leq \min(\frac{1-\alpha}{C_0\omega^n}, \frac{\omega-1}{2})$. As a consequence, we can use Lemma 6.14 on $[0, \tau_{\max}]$ to get

$$x_u(\tau_{\max}) \leq e^{\mu_u(\omega)\tau^{\text{out}}(x,\omega,h)} x_u^{\text{in}} = h \leq \frac{1-\alpha}{C_1\omega^n} < \frac{1-\alpha}{C_0\omega^n},$$

$$x_{s_1}(\tau_{\max}), x_{s_2}(\tau_{\max}) \leq h < \frac{1-\alpha}{C_0\omega^n},$$

and

$$\begin{aligned}
|x_c(\tau_{\max}) - \omega| &\leq \left|x_c(\tau_{\max}) - x_c^{\text{in}}\right| + \left|x_c^{\text{in}} - \omega\right| \\
&\leq 2C_0\omega^n h \min(h, d(\omega)) + \min(h, d(\omega)) \\
&< \min\left(\frac{1-\alpha}{C_0\omega^n}, \frac{\omega-1}{2}\right)
\end{aligned}$$

which contradicts the maximality of $\tau_{\max}$ by continuity. Hence, $\tau_{\max} > \tau^{\text{out}}(x, \omega, h)$. This proves that the orbit x is well defined on $[0, \tau^{\text{out}}(x, \omega, h)]$ and that for every $\tau \in [0, \tau^{\text{out}}(x, \omega, h)]$, $N_\perp(\tau) \leq \frac{1-\alpha}{C_0\omega^n}$ and $N_{/\!/}(\tau) \leq \min(\frac{1-\alpha}{C_0\omega^n}, \frac{\omega-1}{2})$. As a consequence, we can use Lemma 6.14 on $[0, \tau^{\text{out}}(x, \omega, h)]$, which proves (6.27a), (6.27b) and (6.27c).

Now remark that x is well defined on the interval $[0, \tau^{\text{out}}(x, \omega, h)]$, which is independant of α. Hence, we should find a condition that is also independant of α, as stated in Proposition 6.15. Let $C = 5C_1$. If $\omega \in]1, +\infty[$, $0 < h \leq (C\omega^n)^{-1}$ and x is an orbit of the local vector field X_ω such that $x^{\text{in}} := x(0) \in S^s_{\omega,\mathbf{h}^s}$ where $\mathbf{h}^s = (h, \min(h, d(\omega)), \min(h, d(\omega)))$ and $x_u^{\text{in}} \neq 0$, then $(\omega, 4/5, h, x) \in E_{C_1,n}$ and we can apply the above reasoning to x. This proves that x is well defined on $\left[0, \tau^{\text{out}}(x, \omega, h)\right]$ and this concludes the proof. □

Corollary 6.16 below complements Proposition 6.15. It shows that if $S^s_{\omega,\mathbf{h}^s}$ and $S^u_{\omega,\mathbf{h}^u}$ are two sections "close enough" to P_ω and if $S^s_{\omega,\mathbf{h}^s}$ is "sufficiently small", then any generic orbit of the local vector field X_ω starting in $S^s_{\omega,\mathbf{h}^s}$ will eventually pass through $S^u_{\omega,\mathbf{h}^u}$ before leaving the neighbourhood of P_ω where X_ω is well defined. Moreover, it gives precise estimates about the position of the orbit in the section $S^u_{\omega,\mathbf{h}^u}$.

Corollary 6.16 (Estimates in the section $S^u_{\omega,\mathbf{h}^u}$). *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in]1, +\infty[$, every $0 < h \leq (C\omega^n)^{-1}$, every $0 < h_\perp < \min(h, d(\omega))$, for $\mathbf{h}^s = (h, h_\perp, \min(h, d(\omega)))$ and $\mathbf{h}^u = (h, h, 2h)$, the following properties hold true. Let $\tau \mapsto x(\tau)$ be an orbit of the local vector field X_ω. If its initial condition $x^{in} := x(0)$ belongs to the section $S^s_{\omega,\mathbf{h}^s}$ and satisfies $x_u^{in} \neq 0$, then x intersects the section $S^u_{\omega,\mathbf{h}^u}$ exactly at the time $\tau = \tau^{out}(x, \omega, h)$. Moreover, if $3/4 < \alpha < 1$ and $h \leq (1-\alpha)(C\omega^n)^{-1}$, then*

(Distance to the Mixmaster attractor)

$$\left\|\Upsilon(x^{in}) - \Upsilon^A(x^{in})\right\|_\perp \leq (h_\perp)^{1+\frac{\alpha}{\omega}} h^{-\frac{\alpha}{\omega}} \tag{6.28a}$$

(Drift in the direction tangent to the Mixmaster attractor)

$$\left\|\Upsilon(x^{in}) - \Upsilon^A(x^{in})\right\|_{/\!/} \leq h_\perp h C\omega^n \tag{6.28b}$$

Remark 6.17. Choosing $\alpha = \max\left(\frac{\omega}{\omega+1}, \frac{4}{5}\right)$, estimate (6.28a) will lead to estimate (6.1) and estimate (6.28b) will lead to estimate (6.2).

Proof. Let $C_0 > 0$ and $n_0 \in \mathbb{N}$ be large enough such that we can apply Proposition 6.15 with these two constants. Let $C_1 = 2C_0$ and $n_1 = n_0 + 1$. Fix $\omega \in]1, +\infty[$, $3/4 < \alpha < 1$, $0 < h \leq (1-\alpha)(C_1\omega^{n_1})^{-1}$,

$0 < h_\perp < \min(h, d(\omega))$ and an orbit $\tau \mapsto x(\tau)$ of the local vector field X_ω whose initial condition $x^{\text{in}} := x(0)$ belongs to the section $S^s_{\omega,\mathbf{h}^s}$ where $\mathbf{h}^s = (h, h_\perp, \min(h, d(\omega)))$ and such that $x_u^{\text{in}} \neq 0$. According to Proposition 6.15, x is well defined on $\left[0, \tau^{\text{out}}(x, \omega, h)\right]$.

Using (6.27a), we get

$$x_u(\tau^{\text{out}}(x,\omega,h)) = e^{\mu_u(\omega)\tau^{\text{out}}(x,\omega,h)} x_u^{\text{in}} = h$$

According to (6.27b),

$$x_{s_1}(\tau^{\text{out}}(x,\omega,h)), x_{s_2}(\tau^{\text{out}}(x,\omega,h)) \leq h$$

and according to (6.27c),

$$\left| x_c(\tau^{\text{out}}(x,\omega,h)) - \omega \right| \leq 2h$$

It follows that $x(\tau^{\text{out}}(x,\omega,h)) \in S^u_{\omega,\mathbf{h}^u}$ where $\mathbf{h}^u = (h, h, 2h)$. Recall from remark 6.7 that $\tau^{\text{out}}(x,\omega,h)$ is the unique time of intersection. Hence, $\Upsilon(x^{\text{in}}) = x(\tau^{\text{out}}(x,\omega,h))$ is well defined.

Remark that $x_u^{\text{in}} \leq h_\perp$, $\max\left(x_{s_1}^{\text{in}}, x_{s_2}^{\text{in}}\right) \leq h$ and $\left(\Upsilon^A(x^{\text{in}})\right)_c = x_c^{\text{in}}$ so (6.28b) is a direct consequence of (6.27c) applied with $\tau = \tau^{\text{out}}(x,\omega,h)$.

Let $\alpha' = \frac{(1-\alpha)\mu_u(\omega) + \alpha\mu_{s_1}(\omega)}{\mu_{s_1}(\omega)}$. Using (3.14), one can remark that

$$\alpha' \frac{\mu_{s_1}(\omega)}{\mu_u(\omega)} - 1 = \alpha \frac{\mu_{s_1}(\omega) - \mu_u(\omega)}{\mu_u(\omega)} = \frac{\alpha}{\omega}$$

and

$$1 - \alpha' = (1-\alpha)\frac{\mu_{s_1}(\omega) - \mu_u(\omega)}{\mu_{s_1}(\omega)} = \frac{1-\alpha}{1+\omega} \geq \frac{1-\alpha}{2\omega}$$

Hence,

$$h \leq \frac{1-\alpha}{C_1\omega^{n_1}} = \frac{1-\alpha}{2\omega}\frac{1}{C_0\omega^{n_0}} \leq \frac{1-\alpha'}{C_0\omega^{n_0}}$$

Since

$$\left\| \Upsilon(x^{\text{in}}) - \Upsilon^A(x^{\text{in}}) \right\|_\perp = \max\left(x_{s_1}(\tau^{\text{out}}(x,\omega,h)), x_{s_2}(\tau^{\text{out}}(x,\omega,h)) \right),$$

(6.28a) follows from (6.27b) applied with $\tau^{\text{out}}(x,\omega,h)$ instead of τ and with α' instead of α.

Now remark that the fact that x intersects the section $S^u_{\omega,\mathbf{h}^u}$ is independant of α. Hence, we should find a condition that is also independant of α, as stated in corollary 6.16. Let $C = 5C_1$, $\omega \in]1, +\infty[$, $0 < h \leq (C\omega^{n_1})^{-1}$ and $0 < h_\perp \leq \min(h, d(\omega))$. Let x be an orbit of the local vector field X_ω whose initial condition $x^{\text{in}} := x(0)$ belongs to the section $S^s_{\omega,\mathbf{h}^s}$ where $\mathbf{h}^s = (h, h_\perp, \min(h, d(\omega)))$ and such that $x_u^{\text{in}} \neq 0$. Remark that $h \leq 1/5(C_1\omega^{n_1})^{-1}$. Hence, we can apply the above reasoning to x with $\alpha = 4/5$. This proves that x intersects the section $S^u_{\omega,\mathbf{h}^u}$ (where $\mathbf{h}^u = (h, h, 2h)$) and concludes the proof. □

We now deal with the exceptional orbits $\tau \mapsto x(\tau)$ whose initial condition x^{in} verifies $x_u^{\text{in}} = 0$. Recall that the local coordinate system ξ is constructed in such a way that the stable manifold of a point $(0,0,0,\bar{x}_c)$ for the local vector field X_ω has for equation "$x_u = 0, x_c = \bar{x}_c$" (see (4.3b)). Hence, any exceptional orbit converges to a point of $\{0_{\mathbb{R}^3}\} \times]1, +\infty[$ (which is the Kasner interval in local coordinates).

Proposition 6.18 (Behaviour of exceptional orbits)**.** *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in]1, +\infty[$, every $3/4 < \alpha < 1$, every $0 < h \leq (1-\alpha)(C\omega^n)^{-1}$, for $\mathbf{h}^s = (h, h, h)$ and for every orbit $\tau \mapsto x(\tau)$ of the local vector field X_ω whose initial condition $x^{in} := x(0)$ belongs to the section $S^s_{\omega,\mathbf{h}^s}$ and such that $x_u^{in} = 0$, x is well defined on $[0, +\infty[$ and stays forever in the stable manifold of the point $(0,0,0,x_c^{in})$,* i.e. *for every $\tau \geq 0$, $x_u(\tau) = 0$ and $x_c(\tau) = x_c^{in}$.*

Moreover, the orbit x converges exponentially fast to $(0,0,0,x_c^{in})$. More precisely, for $i = 1, 2$ and for every $\tau \geq 0$,

$$x_{s_i}(\tau) \leq e^{-\alpha\mu_{s_1}(\omega)\tau} x_{s_i}^{in} \tag{6.29}$$

Proof. Using the equations (6.5), this is a straightforward consequence of Lemma 6.14. □

Next corollary shows that if the section $S^s_{\omega,\mathbf{h}^s}$ is small enough, any orbit of the local vector field X_ω will intersect it at most once. This is useful for two reasons. Firstly, it allows us to define a time of intersection without ambiguity (see definition 6.20). Secondly, it implies that the time length between two consecutive intersections of an orbit $t \mapsto x(t)$ of the Wainwright-Hsu vector field $\mathcal{X}$ with the section $\mathcal{S}^s_{\omega,\mathbf{h}^s}$ cannot be arbitrary small, *i.e.* admits a uniform positive lower bound (see Lemma 7.10).

Again, we insist on the fact that corollary 6.19 below is about the local vector field X_ω and its orbits. The "real" orbits for the Wainwright-Hsu vector field $\mathcal{X}$ can intersect the section $\mathcal{S}^s_{\omega,\mathbf{h}^s}$ infinitely many times. The local vector field X_ω being a renormalization of $\mathcal{X}$ in the neighbourhood of the point $\mathcal{P}_\omega$ on the Kasner circle, an orbit of X_ω is exactly a connected component of an orbit of $\mathcal{X}$ (modulo the local coordinate system ξ).

Corollary 6.19 (Unique intersection with $S^s_{\omega,\mathbf{h}^s}$)**.** *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in \,]1,+\infty[$, every $0 < h \leq (C\omega^n)^{-1}$, for $\mathbf{h}^s = (h, \min(h, d(\omega)), \min(h, d(\omega)))$ and for every orbit $\tau \mapsto x(\tau)$ of the local vector field X_ω whose initial condition $x(0)$ belongs to the section $S^s_{\omega,\mathbf{h}^s}$, x does not intersect $S^s_{\omega,\mathbf{h}^s}$ again in the future nor in the past.*

Proof. Let $C_0 > 0$ and $n \in \mathbb{N}$ be large enough such that we can apply Proposition 6.15 and Proposition 6.18 with these two constants. Let $C = 10C_0$ and $\alpha = \frac{4}{5}$. Let $\omega \in \,]1,+\infty[$, $0 < h \leq (C\omega^n)^{-1}$, $\mathbf{h}^s = (h, \min(h, d(\omega)), \min(h, d(\omega)))$ and $\tau \mapsto x(\tau)$ be an orbit of the local vector field X_ω whose initial condition $x^{\text{in}} := x(0)$ belongs to the section $S^s_{\omega,\mathbf{h}^s}$. Let us denote by $]\tau_-, \tau_+[$ the maximal existence interval of x. By symmetry, it is enough to prove that x does not intersect again $S^s_{\omega,\mathbf{h}^s}$ in the future.

First, assume that $x^{\text{in}}_u \neq 0$. Using Proposition 6.15 with C_0 and α, we can apply (6.27b) to get that for every $\tau \in \left]0, \tau^{\text{out}}(x,\omega,h)\right]$ and every $i \in \{1,2\}$, $x_{s_i}(\tau) < h$ so $x(\tau) \notin S^s_{\omega,\mathbf{h}^s}$. Analogously, we can apply (6.27a) to get that for every $\tau \in \left]\tau^{\text{out}}(x,\omega,h), \tau_+\right[$, $x_u(\tau) > h$ so $x(\tau) \notin S^s_{\omega,\mathbf{h}^s}$.

We are left to deal with the case where $x^{\text{in}}_u = 0$. Using (6.29), we get that for every $\tau > 0$ and every $i \in \{1,2\}$, $x_{s_i}(\tau) < h$. Hence, $x(\tau) \notin S^s_{\omega,\mathbf{h}^s}$. This concludes the proof. □

We can now give a proof of the proposition on the shadowing of a heteroclinic chain, stated in the introduction of the present section.

Proof of Proposition 6.8. To simplify the proof, let us treat the case $i = 1$. Let $C_0 \geq 1$ and $n \in \mathbb{N}$ be large enough such that we can apply Proposition 6.15 and corollary 6.16 with these two constants. Let $C_1 = 5C_0$ and $\alpha = \frac{4}{5}$. Remark that $C_1^{-1} = (1-\alpha)C_0^{-1}$. Let $\omega \in \,]1,+\infty[$, $0 < h \leq (C_1\omega^n)^{-1}$, $0 < h_\perp \leq \min(\frac{1}{3}h, d(\omega))$ and $\tau \mapsto x(\tau)$ be an orbit of the local vector field X_ω whose initial condition $x^{\text{in}} := x(0)$ belongs to the section $S^s_{\omega,\mathbf{h}^s}$ (where $\mathbf{h}^s = (h, h_\perp, h_\perp)$) and satisfies $x^{\text{in}}_u \neq 0$ (hence $x^{\text{in}}_u > 0$). Let $\mathbf{h}^u = (h, h, 2h)$. Using (3.14a), (3.14b), (6.6) and the fact that $x^{\text{in}}_u \leq \frac{1}{3}h$, it is straightforward to check that $0 \leq \tau^* \leq \tau^{\text{out}}$ where $\tau^{\text{out}} = \tau^{\text{out}}(x,\omega,h)$. According to corollary 6.16, x is well defined on $\left[0, \tau^{\text{out}}(x,\omega,h)\right]$. According to (6.27a), (6.27b) and (6.27c), we have

$$\left\|x(\tau^*) - P_\omega\right\|_\infty \leq e^{\mu_u(\omega)\tau^*} x^{\text{in}}_u + e^{-\alpha\mu_{s_1}(\omega)\tau^*} h + 2h_\perp \tag{6.30}$$

Using (3.14a), (3.14b), we get that

$$\begin{aligned} e^{\mu_u(\omega)\tau^*} &\leq e^{\frac{2}{3}\ln\frac{\alpha\mu_{s_1}(\omega)h}{\mu_u(\omega)x^{\text{in}}_u}} \\ &\leq 4h^{\frac{2}{3}}(x^{\text{in}}_u)^{-\frac{2}{3}} \end{aligned} \tag{6.31}$$

and

$$\begin{aligned} e^{-\alpha\mu_{s_1}(\omega)\tau^*} &\leq e^{-\frac{1}{3}\ln\frac{\alpha\mu_{s_1}(\omega)h}{\mu_u(\omega)x^{\text{in}}_u}} \\ &\leq 2h^{-\frac{1}{3}}(x^{\text{in}}_u)^{\frac{1}{3}} \end{aligned} \tag{6.32}$$

Plugging (6.31) and (6.32) into (6.30), we get that (6.7) holds true.

Recall that

$$\left[P^{s_1}_{\omega,h}, P_\omega\right[= \{(0,z,0,\omega) \mid 0 < z \leq h\}$$

According to (6.27a), (6.27b), (6.27c) and (6.31), we have, for every $\tau \in [0, \tau^*]$, $0 \leq x_{s_1}(\tau) \leq h$ and

$$\left\|x(\tau) - (0, x_{s_1}(\tau), 0, \omega)\right\|_\infty \leq 4h^{\frac{2}{3}} h_\perp^{\frac{1}{3}} + 3h_\perp \leq 7h_\perp^{\frac{1}{3}} \tag{6.33}$$

Moreover, using (6.7), we have, for every $z \in \left]0, x_{s_1}(\tau^*)\right]$,

$$\left\|x(\tau^*) - (0, z, 0, \omega)\right\|_\infty \leq \left\|x(\tau^*) - P_\omega\right\|_\infty \leq 6h^{\frac{2}{3}} h_\perp^{\frac{1}{3}} + 2h_\perp \leq 8h_\perp^{\frac{1}{3}} \tag{6.34}$$

Finally, for every $z \in [x_{s_1}(\tau^*), h = x_{s_1}(0)]$, there exists $\tau \in [0, \tau^*]$ such that $z = x_{s_1}(\tau)$ and we can use (6.33) and (6.34) to conclude that

$$d_{\mathcal{H}}\left(x\left([0, \tau^*]\right), \left[P^{s_1}_{\omega,h}, P_\omega\right[\right) \leq 8h_\perp^{\frac{1}{3}}$$

Analogously, we recall that

$$\left]P_\omega, P^u_{\omega,h}\right] = \{(z, 0, 0, \omega) \mid 0 < z \leq h\}$$

and we get by a straightforward computation that

$$d_{\mathcal{H}}\left(x\left(\left[\tau^*, \tau^{\text{out}}\right]\right), \left]P_\omega, P^u_{\omega,h}\right]\right) \leq 8h_\perp^{\frac{1}{3}}$$

Hence, (6.8a) and (6.8b) hold true. □

6.2 Comparison of two orbits

In this section, we are going to precisely compare two orbits of the local vector field X_ω which simultaneously intersect a section $S^u_{\omega,\mathbf{h}^u}$. This will lead to the Lipschitz estimates (6.3) and (6.4) on Υ.

Until the end of this section, we fix $C_0 \geq 2$ and $n_0 \in \mathbb{N}$ large enough such that we can apply Proposition 6.15, corollary 6.16 and corollary 6.19 with these two constants. In particular, for every $C \geq C_0 > 0$, every $n \geq n_0 \in \mathbb{N}$, every $0 < h \leq (C\omega^n)^{-1}$, for $\mathbf{h}^s = (h, \min(h, d(\omega)), \min(h, d(\omega)))$, for $\mathbf{h}^u = (h, h, 2h)$ and for any orbit $\tau \mapsto x(\tau)$ of the local vector field X_ω, if x intersects $S^s_{\omega,\mathbf{h}^s}$ at least once, then it intersects $S^s_{\omega,\mathbf{h}^s}$ and $S^u_{\omega,\mathbf{h}^u}$ exactly once.

Definition 6.20 (Time of intersection with $S^s_{\omega,\mathbf{h}^s}$)**.** With the same notations as above, if x intersects $S^s_{\omega,\mathbf{h}^s}$ then we denote by $\tau^{\text{in}}(x, \omega, \mathbf{h}^s)$ the unique time $\tau \in \mathbb{R}$ such that $x(\tau) \in S^s_{\omega,\mathbf{h}^s}$.

Definition 6.21 (Pair of synchronized orbits)**.** Let $\omega \in \left]1, +\infty\right[$, $0 < h \leq (C\omega^n)^{-1}$, $\mathbf{h}^s = (h, \min(h, d(\omega)), \min(h, d(\omega)))$ and $\mathbf{h}^u = (h, h, 2h)$. Let $\tau \mapsto x(\tau)$ and $\tau \mapsto \tilde{x}(\tau)$ be two orbits of the local vector field X_ω. We say that $(x, \tilde{x})$ is a pair of $\left(S^s_{\omega,\mathbf{h}^s}, S^u_{\omega,\mathbf{h}^u}\right)$-synchronized orbits if

1. $x(0) \in S^s_{\omega,\mathbf{h}^s}$.

2. $\tilde{x}$ intersects $S^s_{\omega,\mathbf{h}^s}$ before x, *i.e* $\tau^{\text{in}}(\tilde{x}, \omega, \mathbf{h}^s) < 0$.

3. x and $\tilde{x}$ intersect $S^u_{\omega,\mathbf{h}^u}$ at the same time, *i.e.* $\tau^{\text{out}}(x, \omega, h) = \tau^{\text{out}}(\tilde{x}, \omega, h)$.

If this is the case, we define

$$\begin{aligned}
x^{\text{in}} &\stackrel{\text{def}}{=} x(\tau^{\text{in}}(x, \omega, \mathbf{h}^s)) \in S^s_{\omega,\mathbf{h}^s} \\
\tilde{x}^{\text{in}} &\stackrel{\text{def}}{=} \tilde{x}(\tau^{\text{in}}(\tilde{x}, \omega, \mathbf{h}^s)) \in S^s_{\omega,\mathbf{h}^s} \\
x^{\text{out}} &\stackrel{\text{def}}{=} x(\tau^{\text{out}}(x, \omega, h)) \in S^u_{\omega,\mathbf{h}^u} \\
\tilde{x}^{\text{out}} &\stackrel{\text{def}}{=} \tilde{x}(\tau^{\text{out}}(\tilde{x}, \omega, h)) \in S^u_{\omega,\mathbf{h}^u}
\end{aligned}$$

See also figure 6.2.

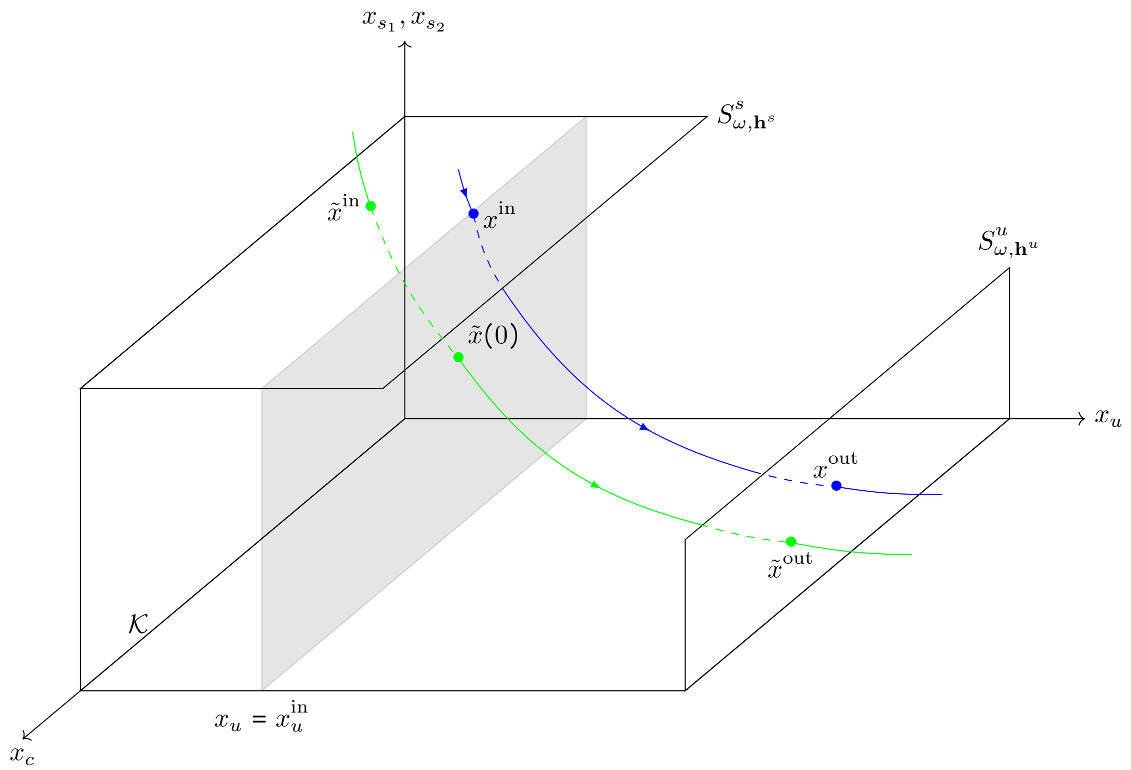

Figure 6.2 – A pair of $(S^s_{\omega,\mathbf{h}^s}, S^u_{\omega,\mathbf{h}^u})$-synchronized orbits.

Remark 6.22. Let $(x,\tilde{x})$ be a pair of $(S^s_{\omega,\mathbf{h}^s}, S^u_{\omega,\mathbf{h}^u})$-synchronized orbits. Since x and $\tilde{x}$ both intersect $S^u_{\omega,\mathbf{h}^u}$, it follows that $x_u^{\text{in}} > 0$ and $\tilde{x}_u^{\text{in}} > 0$. With these notations, it is straightforward to check that

$$\begin{aligned}
\tau^{\text{in}}(x,\omega,\mathbf{h}^s) &= 0\\
\tau^{\text{in}}(\tilde{x},\omega,\mathbf{h}^s) &= -\frac{1}{\mu_u(\omega)}\ln\frac{x_u^{\text{in}}}{\tilde{x}_u^{\text{in}}} \leq 0\\
\tau^{\text{out}}(x,\omega,h) = \tau^{\text{out}}(\tilde{x},\omega,h) &= \frac{1}{\mu_u(\omega)}\ln\frac{h}{x_u^{\text{in}}}
\end{aligned}$$

Remark 6.23. If $(x,\tilde{x})$ is a pair of $(S^s_{\omega,\mathbf{h}^s}, S^u_{\omega,\mathbf{h}^u})$-synchronized orbits, then for every $\tau \in \mathbb{R}$ such that $x(\tau)$ and $\tilde{x}(\tau)$ are well defined, we have $x_u(\tau) = \tilde{x}_u(\tau)$ (see figure 6.2).

Remark 6.24. Up to reparametrization, any pair of orbits which both intersect the section $S^s_{\omega,\mathbf{h}^s}$ is a pair of $(S^s_{\omega,\mathbf{h}^s}, S^u_{\omega,\mathbf{h}^u})$-synchronized orbits. More precisely, let $\tau \mapsto x(\tau)$ and $\tau \mapsto \tilde{x}(\tau)$ be two orbits of the local vector field X_ω which both intersect the section $S^s_{\omega,\mathbf{h}^s}$. Up to a translation in time of x, one can assume that $x(0) \in S^s_{\omega,\mathbf{h}^s}$. According to corollary 6.16, x and $\tilde{x}$ both intersect the section $S^u_{\omega,\mathbf{h}^u}$. Up to a translation in time of $\tilde{x}$, one can assume that x and $\tilde{x}$ intersect simultaneously the section $S^u_{\omega,\mathbf{h}^u}$. Up to symmetry, one can assume that $\tilde{x}$ intersects the section $S^s_{\omega,\mathbf{h}^s}$ before x. With these conventions, $(x,\tilde{x})$ is a pair of $(S^s_{\omega,\mathbf{h}^s}, S^u_{\omega,\mathbf{h}^u})$-synchronized orbits.

Given a pair $(x,\tilde{x})$ of $(S^s_{\omega,\mathbf{h}^s}, S^u_{\omega,\mathbf{h}^u})$-synchronized orbits, the following lemma provides some controls about the orbit $\tilde{x}$ at the time $t = 0$.

Lemma 6.25. *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in \,]1,+\infty[$, every $0 < h \leq (C\omega^n)^{-1}$, for $\mathbf{h}^s = (h, \min(h,d(\omega)), \min(h,d(\omega)))$, $\mathbf{h}^u = (h,h,2h)$ and for every pair $(x,\tilde{x})$ of $(S^s_{\omega,\mathbf{h}^s}, S^u_{\omega,\mathbf{h}^u})$-synchronized orbits, we have the following estimates:*

(Control in the stable direction) *For every $i \in \{1,2\}$,*

$$\left|\tilde{x}_{s_i}(0) - \tilde{x}_{s_i}^{in}\right| \leq C\omega h \frac{1}{x_u^{in}} \left|x_u^{in} - \tilde{x}_u^{in}\right| \tag{6.36}$$

(Control in the central direction)

$$\left|\tilde{x}_c(0) - \tilde{x}_c^{in}\right| \leq C\omega^n h \frac{\tilde{x}_u^{in}}{x_u^{in}} \left|x_u^{in} - \tilde{x}_u^{in}\right| \tag{6.37}$$

Proof. The two controls are proven independantly.

For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $E_{C,n}$ the set of all $(\omega, h, x, \tilde{x})$ such that $\omega \in]1, +\infty[$, $0 < h \leq (C\omega^n)^{-1}$ and $(x, \tilde{x})$ is a pair of synchronized orbits of the local vector field X_ω (with respect to the sections $S^s_{\omega,\mathbf{h}^s}$ and $S^u_{\omega,\mathbf{h}^u}$, where $\mathbf{h}^s = (h, \min(h, d(\omega)), \min(h, d(\omega)))$ and $\mathbf{h}^u = (h, h, 2h)$). Fix $3/4 < \alpha < 1$.

Control of the coordinate $\tilde{x}_c$. Recall that $\tau^{\text{in}}(\tilde{x}, \omega, \mathbf{h}^s) = -\frac{1}{\mu_u(\omega)} \ln \frac{x_u^{\text{in}}}{\tilde{x}_u^{\text{in}}}$. Applying (6.27c) on the time interval $\left[\tau^{\text{in}}(\tilde{x}, \omega, \mathbf{h}^s), 0\right]$ (using a translation in time), we get that for every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\begin{aligned}\left|\tilde{x}_c(0) - \tilde{x}_c^{\text{in}}\right| &= \left|\tilde{x}_c(0) - \tilde{x}_c\left(-\frac{1}{\mu_u(\omega)} \ln \frac{x_u^{\text{in}}}{\tilde{x}_u^{\text{in}}}\right)\right| \\ &\leq C_0 \omega^{n_0} \tilde{x}_u^{\text{in}} h \left(1 - \left(\frac{\tilde{x}_u^{\text{in}}}{x_u^{\text{in}}}\right)^{\alpha \frac{\mu_{s_1}(\omega) - \mu_u(\omega)}{\mu_u(\omega)}}\right)\end{aligned}$$

Moreover, using (3.14), we get that $\alpha \frac{\mu_{s_1}(\omega) - \mu_u(\omega)}{\mu_u(\omega)} = \frac{\alpha}{\omega} < 1$. Recall that $0 < \frac{\tilde{x}_u^{\text{in}}}{x_u^{\text{in}}} \leq 1$. Hence, estimate (6.37) is a consequence of the above inequality.

Control of the coordinate $\tilde{x}_{s_i}$ ($i \in \{1,2\}$). Let $C \geq C_0$, $n \geq n_0$ and $(\omega, h, x, \tilde{x}) \in E_{C,n}$. According to (6.5), $\tilde{x}_{s_i}$ is a solution of the following first order linear differential equation of variable y:

$$y' = -\tilde{\mu}_{s_i}(\tilde{x}_c) y + X_{s_i}(\tilde{x}) \tilde{x}_{s_i}$$

where

$$X_{s_i}(\tilde{x}) \stackrel{\text{def}}{=} X_{s_i}^{u,s_i}(\tilde{x}) \tilde{x}_u + X_{s_i}^{s_1,s_i}(\tilde{x}) \tilde{x}_{s_1} + X_{s_i}^{s_2,s_i}(\tilde{x}) \tilde{x}_{s_2}$$

Using variation of parameters, we get an implicit expression of $\tilde{x}_{s_i}(0)$, which can be written as follows:

$$\tilde{x}_{s_i}(0) - \tilde{x}_{s_i}^{\text{in}} = A_1 + A_2$$

where

$$A_1 = \left(e^{\int_{\tau^{\text{in}}(\tilde{x},\omega,\mathbf{h}^s)}^0 -\tilde{\mu}_{s_i}(\tilde{x}_c(w))\,\mathrm{d}w} - 1\right) \tilde{x}_{s_i}^{\text{in}}$$

and

$$A_2 = \int_{\tau^{\text{in}}(\tilde{x},\omega,\mathbf{h}^s)}^0 e^{\int_w^0 -\tilde{\mu}_{s_i}(\tilde{x}_c(\sigma))\,\mathrm{d}\sigma} X_{s_i}(\tilde{x}(w)) \tilde{x}_{s_i}(w)\,\mathrm{d}w$$

Control of $|A_1|$. Applying (6.27c) once again on the time interval $\left[\tau^{\text{in}}(\tilde{x}, \omega, \mathbf{h}^s), 0\right]$, we get that for every $\tau \in \left[\tau^{\text{in}}(\tilde{x}, \omega, \mathbf{h}^s), 0\right]$,

$$|\tilde{x}_c(\tau) - \omega| \leq \left|\tilde{x}_c(\tau) - \tilde{x}_c^{\text{in}}\right| + h \leq 2h \tag{6.38}$$

There exist $C_1 \geq C_0$ and $n_1 \geq n_0$ such that for every $C \geq C_1$, every $n \geq n_1$ and every $0 < h \leq (C\omega^n)^{-1}$, we have $2h \leq \frac{1-\alpha}{6\omega}$. It follows from (6.38) and (6.10) that for every $C \geq C_1$, every $n \geq n_1$ and every

$(\omega, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\left| e^{\int_{\tau^{\mathrm{in}}(\tilde{x},\omega,\mathbf{h}^s)}^{0} -\tilde{\mu}_{s_i}(\tilde{x}_c(w))\,\mathrm{d}w} - 1 \right| \le 1 - e^{-\alpha\mu_{s_2}(\omega)\tau^{\mathrm{in}}(\tilde{x},\omega,\mathbf{h}^s)} \le 1 - \left(\frac{\tilde{x}_u^{\mathrm{in}}}{x_u^{\mathrm{in}}} \right)^{\alpha \frac{\mu_{s_2}(\omega)}{\mu_u(\omega)}}$$

Hence,

$$|A_1| \le h \left(1 - \left(\frac{\tilde{x}_u^{\mathrm{in}}}{x_u^{\mathrm{in}}} \right)^{\alpha \frac{\mu_{s_2}(\omega)}{\mu_u(\omega)}} \right)$$

Moreover, we have the elementary fact

$$\text{for every } 0 < z < 1 \text{ and every } v > 0, \quad 1 - z^v \le \max(1, v)(1 - z) \tag{6.39}$$

Let us apply (6.39) with $z = \frac{\tilde{x}_u^{\mathrm{in}}}{x_u^{\mathrm{in}}}$ and $v = \frac{\alpha\mu_{s_2}(\omega)}{\mu_u(\omega)} = \alpha(1 + \omega) \ge 1$. It gives:

$$|A_1| \le h\alpha(1 + \omega) \left(1 - \frac{\tilde{x}_u^{\mathrm{in}}}{x_u^{\mathrm{in}}} \right) \le 2\omega h \frac{1}{x_u^{\mathrm{in}}} \left| x_u^{\mathrm{in}} - \tilde{x}_u^{\mathrm{in}} \right| \tag{6.40}$$

Control of $|A_2|$. According to (6.10), for every $C \ge C_1$, every $n \ge n_1$ and every $(\omega, h, x, \tilde{x}) \in E_{C,n}$, we have

$$e^{\int_w^0 -\tilde{\mu}_{s_i}(\tilde{x}_c(\sigma))\,\mathrm{d}\sigma} \le e^{\alpha\mu_{s_1}(\omega)w}$$

Hence, according to (4.13) and Proposition 6.15, there exist $C_2 \ge C_1$ and $n_2 \ge n_1$ such that for every $C \ge C_2$, every $n \ge n_2$ and every $(\omega, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\begin{aligned} |A_2| &\le C_2\omega^{n_2}h^2 \int_{\tau^{\mathrm{in}}(\tilde{x},\omega,\mathbf{h}^s)}^{0} e^{\alpha\mu_{s_1}(\omega)w}\,\mathrm{d}w \\ &\le \frac{C_2\omega^{n_2}h^2}{\alpha\mu_{s_1}(\omega)} \left(1 - \left(\frac{\tilde{x}_u^{\mathrm{in}}}{x_u^{\mathrm{in}}} \right)^{\frac{\alpha\mu_{s_1}(\omega)}{\mu_u(\omega)}} \right) \end{aligned}$$

Let us apply (6.39) with $z = \frac{\tilde{x}_u^{\mathrm{in}}}{x_u^{\mathrm{in}}}$ and $v = \frac{\alpha\mu_{s_1}(\omega)}{\mu_u(\omega)} = \alpha\frac{1+\omega}{\omega}$. One can remark that $v < 1$ for every ω large enough and $\mu_{s_1}(\omega) \sim_{\omega\to+\infty} \frac{6}{\omega}$. It follows that there exist $C_3 \ge C_2$ such that for every $C \ge C_3$, every $n \ge n_3 := n_2 + 1$ and every $(\omega, h, x, \tilde{x}) \in E_{C,n}$, we have

$$|A_2| \le C_3\omega^{n_3}h^2 \frac{1}{x_u^{\mathrm{in}}} \left| x_u^{\mathrm{in}} - \tilde{x}_u^{\mathrm{in}} \right| \le h \frac{1}{x_u^{\mathrm{in}}} \left| x_u^{\mathrm{in}} - \tilde{x}_u^{\mathrm{in}} \right| \tag{6.41}$$

It follows from (6.40) and (6.41) that for every $C \ge C_3$, every $n \ge n_3$ and every $(\omega, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\left| \tilde{x}_{s_i}(0) - \tilde{x}_{s_i}^{\mathrm{in}} \right| \le 3\omega h \frac{1}{x_u^{\mathrm{in}}} \left| x_u^{\mathrm{in}} - \tilde{x}_u^{\mathrm{in}} \right|$$

so estimate (6.36) holds true with $C = \max(C_3, 3)$ and $n = n_3$. □

We are now going to prove the main technical result of this section. The following proposition gives Lipschitz estimates on the distance between two synchronized orbits when intersecting the section $S^u_{\omega,\mathbf{h}^u}$. We prove that the Lipschitz constant mostly depends on the distance between their initial conditions in the section $S^s_{\omega,\mathbf{h}^s}$ and the Mixmaster attractor.

Proposition 6.26 (Lipschitz estimates in the section $S^u_{\omega,\mathbf{h}^u}$)**.** *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in]1, +\infty[$, every $3/4 < \alpha < 1$, every $0 < h \le (1 - \alpha)(C\omega^n)^{-1}$, every $0 < h_\perp \le \min(h, d(\omega))$, for $\mathbf{h}^s = (h, h_\perp, \min(h, d(\omega)))$, $\mathbf{h}^u = (h, h, 2h)$ and for every pair $(x, \tilde{x})$ of $(S^s_{\omega,\mathbf{h}^s}, S^u_{\omega,\mathbf{h}^u})$-synchronized orbits, we have the following estimates:*

(Lipschitz estimate in the direction transverse to the Mixmaster attractor)

$$\left\|x^{out}-\tilde{x}^{out}\right\|_{\perp} \leq (h_{\perp})^{\frac{\alpha}{\omega}} C\omega^n h^{-\frac{\alpha}{\omega}} \left\|x^{in}-\tilde{x}^{in}\right\|_{\infty} \tag{6.42}$$

(Lipschitz estimate in the direction tangent to the Mixmaster attractor)

$$\left\|\left(x^{out}-\tilde{x}^{out}\right)-\left(x^{in}-\tilde{x}^{in}\right)\right\|_{/\!/} \leq C\omega^n h \left\|x^{in}-\tilde{x}^{in}\right\|_{\perp} + C\omega^n h_{\perp} \left\|x^{in}-\tilde{x}^{in}\right\|_{/\!/} \tag{6.43}$$

Proof. We can assume that $x^{\mathrm{in}} \neq \tilde{x}^{\mathrm{in}}$. Otherwise, the orbits $\tau \mapsto x(\tau)$ and $\tau \mapsto \tilde{x}(\tau)$ coincide and the estimates in Proposition 6.26 are trivial.

Notations. For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $E_{C,n}$ the set of all $(\omega, \alpha, h, h_\perp, x, \tilde{x})$ such that $\omega \in \,]1, +\infty[$, $3/4 < \alpha < 1$, $0 < h \leq (1-\alpha)(C\omega^n)^{-1}$, $0 < h_\perp \leq \min(h, d(\omega))$ and $(x, \tilde{x})$ is a pair of $\left(S^s_{\omega,\mathbf{h}^s}, S^u_{\omega,\mathbf{h}^u}\right)$-synchronized orbits, where $\mathbf{h}^s = (h, h_\perp, \min(h, d(\omega)))$ and $\mathbf{h}^u = (h, h, 2h))$. Let $C_0 > 0$ and $n_0 \in \mathbb{N}$ be large enough such that we can apply Proposition 4.8 and Lemma 6.25 with these two constants. For every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, let

$$\begin{aligned}
\tau^{\mathrm{out}} &\stackrel{\mathrm{def}}{=} \frac{1}{\mu_u(\omega)} \ln \frac{h}{x_u^{\mathrm{in}}} = \tau^{\mathrm{out}}(x, \omega, h) = \tau^{\mathrm{out}}(\tilde{x}, \omega, h) \\
d_c(\tau) &\stackrel{\mathrm{def}}{=} \int_0^\tau \left| \frac{\mathrm{d}x_c}{\mathrm{d}\tau}(z) - \frac{\mathrm{d}\tilde{x}_c}{\mathrm{d}\tau}(z) \right| \mathrm{d}z && \text{for every } \tau \in \left[0, \tau^{\mathrm{out}}\right] \\
d_s(\tau) &\stackrel{\mathrm{def}}{=} \left|x_{s_1}(\tau) - \tilde{x}_{s_1}(\tau)\right| + \left|x_{s_2}(\tau) - \tilde{x}_{s_2}(\tau)\right| && \text{for every } \tau \in \left[0, \tau^{\mathrm{out}}\right] \\
\alpha_0 &\stackrel{\mathrm{def}}{=} \frac{(1-\alpha)\mu_u(\omega) + \alpha\mu_{s_1}(\omega)}{\mu_{s_1}(\omega)} \\
\alpha_0' &\stackrel{\mathrm{def}}{=} \frac{\frac{1-\alpha_0}{2}\mu_u(\omega) + \frac{1+\alpha_0}{2}\mu_{s_1}(\omega)}{\mu_{s_1}(\omega)}
\end{aligned}$$

First, remark that $\left\|x^{\mathrm{out}} - \tilde{x}^{\mathrm{out}}\right\|_\perp \leq d_s(\tau^{\mathrm{out}})$. Secondly, remark that

$$1 - \alpha_0 = \frac{1-\alpha}{1+\omega} \tag{6.44}$$

and

$$\alpha_0 \frac{\mu_{s_1}(\omega)}{\mu_u(\omega)} - 1 = \alpha\left(\frac{\mu_{s_1}(\omega)}{\mu_u(\omega)} - 1\right) = \frac{\alpha}{\omega} \tag{6.45}$$

Idea of the proof. We are looking for upper bounds of $d_c(\tau^{\mathrm{out}})$ and $d_s(\tau^{\mathrm{out}})$. The idea is to obtain cross estimates on both $d_c(\tau)$ and $d_s(\tau)$, and then to progress step by step to some controls that are independant from each other by using Gronwall estimates.

Step 1: control of $d_s(0)$. According to (6.36), for every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$d_s(0) \leq 2C_0\omega h \frac{1}{x_u^{\mathrm{in}}} \left|x_u^{\mathrm{in}} - \tilde{x}_u^{\mathrm{in}}\right| + \left|x_{s_1}^{\mathrm{in}} - \tilde{x}_{s_1}^{\mathrm{in}}\right| + \left|x_{s_2}^{\mathrm{in}} - \tilde{x}_{s_2}^{\mathrm{in}}\right| \tag{6.46}$$

Step 2: control of $|x_c(\tau) - \tilde{x}_c(\tau)|$ for $\tau \in [0, \tau^{out}]$. According to (6.37), for every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\begin{aligned}
|x_c(\tau) - \tilde{x}_c(\tau)| &\leq d_c(\tau) + |x_c(0) - \tilde{x}_c(0)| \\
&\leq d_c(\tau) + \left|\tilde{x}_c(0) - \tilde{x}_c^{\mathrm{in}}\right| + \left|x_c^{\mathrm{in}} - \tilde{x}_c^{\mathrm{in}}\right| \\
&\leq d_c(\tau) + C_0\omega^{n_0} h \frac{\tilde{x}_u^{\mathrm{in}}}{x_u^{\mathrm{in}}} \left|x_u^{\mathrm{in}} - \tilde{x}_u^{\mathrm{in}}\right| + \left|x_c^{\mathrm{in}} - \tilde{x}_c^{\mathrm{in}}\right|
\end{aligned}$$

Moreover, $C_0\omega^{n_0}h \leq 1$ and $\frac{\tilde{x}_u^{\text{in}}}{x_u^{\text{in}}} \leq 1$ so

$$|x_c(\tau) - \tilde{x}_c(\tau)| \leq d_c(\tau) + \left\|x^{\text{in}} - \tilde{x}^{\text{in}}\right\|_{\perp} + \left\|x^{\text{in}} - \tilde{x}^{\text{in}}\right\|_{/\!/} \tag{6.47}$$

Step 3: a control of $d_s(\tau)$ depending on $d_c(\tau)$. From now on, τ will often be implicit in the estimates. Those estimates are valid for every $\tau \in [0, \tau^{\text{out}}]$. By definition, we have

$$\frac{\mathrm{d}d_s}{\mathrm{d}\tau} = \frac{\tilde{x}_{s_1} - x_{s_1}}{|x_{s_1} - \tilde{x}_{s_1}|}\left(\frac{\mathrm{d}\tilde{x}_{s_1}}{\mathrm{d}\tau} - \frac{\mathrm{d}x_{s_1}}{\mathrm{d}\tau}\right) + \frac{\tilde{x}_{s_2} - x_{s_2}}{|x_{s_2} - \tilde{x}_{s_2}|}\left(\frac{\mathrm{d}\tilde{x}_{s_2}}{\mathrm{d}\tau} - \frac{\mathrm{d}x_{s_2}}{\mathrm{d}\tau}\right)$$

According to (6.5),

$$\begin{aligned}\frac{\mathrm{d}\tilde{x}_{s_1}}{\mathrm{d}\tau} - \frac{\mathrm{d}x_{s_1}}{\mathrm{d}\tau} = &-\tilde{\mu}_{\omega,s_1}(\tilde{x}_c)(\tilde{x}_{s_1} - x_{s_1}) + (\tilde{\mu}_{\omega,s_1}(x_c) - \tilde{\mu}_{\omega,s_1}(\tilde{x}_c))x_{s_1} \\ &+ X_{\omega,s_1}(\tilde{x})(\tilde{x}_{s_1} - x_{s_1}) + (X_{\omega,s_1}(\tilde{x}) - X_{\omega,s_1}(x))x_{s_1}\end{aligned} \tag{6.48}$$

where $X_{\omega,s_1}(x) = X_{\omega,s_1}^{u,s_1}(x)x_u + X_{\omega,s_1}^{s_1,s_1}(x)x_{s_1} + X_{\omega,s_1}^{s_2,s_1}(x)x_{s_2}$. According to (6.27c), there exist $C_1 \geq C_0$ and $n_1 \geq n_0$ such that for every $C \geq C_1$, every $n \geq n_1$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$|\tilde{x}_c - \omega| \leq \frac{1-\alpha}{24(1+\omega)\omega^2} \tag{6.49}$$

Using (6.49) with (6.44) and (6.11), we get

$$-\tilde{\mu}_{\omega,s_1}(\tilde{x}_c) \leq -\alpha_0' \mu_{s_1}(\omega) \leq -\frac{1+\alpha_0}{2}\mu_{s_1}(\omega) \tag{6.50}$$

According to the expression of $\tilde{\mu}_{\omega,s_1}$ (see (4.12)) and formulas (3.14), for every $C \geq C_1$, every $n \geq n_1$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$|\tilde{\mu}_{\omega,s_1}(x_c) - \tilde{\mu}_{\omega,s_1}(\tilde{x}_c)| \leq 6\,|x_c - \tilde{x}_c| \tag{6.51}$$

According to (4.13), for every $C \geq C_1$, every $n \geq n_1$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\begin{aligned}|X_{\omega,s_1}(\tilde{x})|\,|x_{s_1} - \tilde{x}_{s_1}| &\leq C_0\omega^{n_0}h\,|x_{s_1} - \tilde{x}_{s_1}| \\ |X_{\omega,s_1}(\tilde{x}) - X_{\omega,s_1}(x)|\,x_{s_1} &\leq C_0\omega^{n_0}\left(\|x - \tilde{x}\|_{\perp} + |x_c - \tilde{x}_c|\right)x_{s_1}\end{aligned} \tag{6.52}$$

We can control in the same way the terms that appear in the expression of $\frac{\mathrm{d}\tilde{x}_{s_2}}{\mathrm{d}\tau} - \frac{\mathrm{d}x_{s_2}}{\mathrm{d}\tau}$. It follows from (6.48), (6.50), (6.51) and (6.52) (and similar estimates for s_2 instead of s_1) that there exist $C_2 \geq C_1$ and $n_2 \geq n_1$ such that for every $C \geq C_2$, every $n \geq n_2$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\begin{aligned}\frac{\mathrm{d}d_s}{\mathrm{d}\tau} \leq -\frac{1+\alpha_0}{2}\mu_{s_1}(\omega)d_s &+ C_2\omega^{n_2}\,|x_c - \tilde{x}_c|\,(x_{s_1} + x_{s_2}) + C_2\omega^{n_2}hd_s \\ &+ C_2\omega^{n_2}\left(\|x - \tilde{x}\|_{\perp} + |x_c - \tilde{x}_c|\right)(x_{s_1} + x_{s_2})\end{aligned}$$

For every $\tau \in [0, \tau^{\text{out}}]$, we have $x_u(\tau) = \tilde{x}_u(\tau)$. Hence, $\|x - \tilde{x}\|_{\perp} \leq d_s$. Using $(x_{s_1} + x_{s_2}) \leq 2h$, it follows that for every $C \geq C_2$, every $n \geq n_2$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\frac{\mathrm{d}d_s}{\mathrm{d}\tau} \leq \left(-\frac{1+\alpha_0}{2}\mu_{s_1}(\omega) + 3C_2\omega^{n_2}h\right)d_s + 2C_2\omega^{n_2}\,|x_c - \tilde{x}_c|\,(x_{s_1} + x_{s_2})$$

Using (3.14), we get

$$\mu_{s_1}(\omega) \geq \frac{2}{\omega} \tag{6.53}$$

According to (6.44) and (6.53), there exist $C_3 \geq 2C_2$ and $n_3 \geq n_2$ such that for every $C \geq C_3$, every

$n \geq n_3$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$3C_2\omega^{n_2}h \leq \frac{1-\alpha_0}{2}\mu_{s_1}(\omega)$$

Hence, for every $C \geq C_3$, every $n \geq n_3$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\frac{\mathrm{d}d_s}{\mathrm{d}\tau} \leq -\alpha_0\mu_{s_1}(\omega)d_s + C_3\omega^{n_3}\left|x_c - \tilde{x}_c\right|(x_{s_1} + x_{s_2}) \tag{6.54}$$

According to (6.44), there exist $C_4 \geq 2C_3$ and $n_4 \geq n_3$ such that for every $C \geq C_4$, every $n \geq n_4$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we can apply the control (6.27b) to (ω, α_0, h, x) and obtain the following control on $(x_{s_1} + x_{s_2})$:

$$(x_{s_1}(\tau) + x_{s_2}(\tau)) \leq e^{-\alpha_0\mu_{s_1}(\omega)\tau}(x_{s_1}(0) + x_{s_2}(0)) \tag{6.55}$$

Plugging (6.47) and (6.55) into (6.54) and using the fact that $(x_{s_1}(0) + x_{s_2}(0)) \leq 2h$, we get that for every $C \geq C_4$, every $n \geq n_4$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$,

$$\frac{\mathrm{d}d_s}{\mathrm{d}\tau}(\tau) \leq -\alpha_0\mu_{s_1}(\omega)d_s(\tau) + C_4\omega^{n_3}he^{-\alpha_0\mu_{s_1}(\omega)\tau}(d_c(\tau) + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_\perp + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_{/\!/})$$

which can be rewritten in the form

$$e^{\alpha_0\mu_{s_1}(\omega)\tau}\frac{\mathrm{d}d_s}{\mathrm{d}\tau}(\tau) + \alpha_0\mu_{s_1}(\omega)e^{\alpha_0\mu_{s_1}(\omega)\tau}d_s(\tau) \leq C_4\omega^{n_3}h\left(d_c(\tau) + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_\perp + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_{/\!/}\right)$$

We recognize the derivative of $e^{\alpha_0\mu_{s_1}(\omega)\tau}d_s(\tau)$ in the left side of the above inequality. By integrating between 0 and τ, we find:

$$\begin{aligned} d_s(\tau) \leq\ & e^{-\alpha_0\mu_{s_1}(\omega)\tau}d_s(0) \\ & + C_4\omega^{n_3}he^{-\alpha_0\mu_{s_1}(\omega)\tau}\left(\int_0^\tau d_c(z)\,\mathrm{d}z + \tau\left(\left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_\perp + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_{/\!/}\right)\right) \end{aligned} \tag{6.56}$$

Step 4: an integral inequation for $d_c(\tau)$. According to (6.5) and the fact that $x_u(\tau) = \tilde{x}_u(\tau)$ for every $\tau \in [0, \tau^{\mathrm{out}}]$, we have

$$\begin{aligned} \frac{\mathrm{d}d_c}{\mathrm{d}\tau} = x_u \big| & X_c^{u,s_1}(\tilde{x})(\tilde{x}_{s_1} - x_{s_1}) + (X_c^{u,s_1}(\tilde{x}) - X_c^{u,s_1}(x))x_{s_1} \\ & + X_c^{u,s_2}(\tilde{x})(\tilde{x}_{s_2} - x_{s_2}) + (X_c^{u,s_2}(\tilde{x}) - X_c^{u,s_2}(x))x_{s_2}\big| \end{aligned}$$

According to the estimate (4.13) on the non linear terms, there exist $C_5 \geq C_4$ and $n_5 \geq n_4$ such that for every $C \geq C_5$, every $n \geq n_5$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\frac{\mathrm{d}d_c}{\mathrm{d}\tau} \leq C_5\omega^{n_5}x_u(d_s + \left|x_c - \tilde{x}_c\right|(x_{s_1} + x_{s_2})) \tag{6.57}$$

Plugging (6.47) and (6.55) into (6.57), using the formula $x_u(\tau) = e^{\mu_u(\omega)\tau}x_u^{\mathrm{in}}$ and the estimate

$$(x_{s_1}(0) + x_{s_2}(0)) \leq 2h$$

we get that, for every $C \geq C_5$, every $n \geq n_5$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$:

$$\begin{aligned} \frac{\mathrm{d}d_c}{\mathrm{d}\tau}(\tau) \leq\ & C_5\omega^{n_5}e^{\mu_u(\omega)\tau}x_u^{\mathrm{in}}d_s(\tau) \\ & + 2C_5\omega^{n_5}he^{(\mu_u(\omega)-\alpha_0\mu_{s_1}(\omega))\tau}x_u^{\mathrm{in}}\left(d_c(\tau) + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_\perp + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_{/\!/}\right) \end{aligned} \tag{6.58}$$

Plugging (6.56) into (6.58), there exist $C_6 \geq C_5$ and $n_6 \geq n_5$ such that for every $C \geq C_6$, every $n \geq n_6$

and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\frac{\mathrm{d}d_c}{\mathrm{d}\tau}(\tau) \leq C_6 \omega^{n_6} e^{(\mu_u(\omega)-\alpha_0\mu_{s_1}(\omega))\tau} x_u^{\mathrm{in}} \Big[d_s(0) + h\left(d_c(\tau) + \int_0^\tau d_c(z)\,\mathrm{d}z + (1+\tau)\left(\left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_\perp + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_{/\!/} \right)\right)\Big]$$

For any $C \geq C_6$, any $n \geq n_6$ and any $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, let

$$T(x,\tilde{x}) \overset{\mathrm{def}}{=} \sup\left\{ t \geq 0 \mid \forall \tau \in [0,t], d_c(\tau) \leq \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_\perp + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_{/\!/} \right\} > 0$$
$$\hat{T}(x,\tilde{x}) \overset{\mathrm{def}}{=} \min\left(T(x,\tilde{x}), \tau^{\mathrm{out}} \right)$$

By definition of $T(x,\tilde{x})$, for every $C \geq C_6$, every $n \geq n_6$, every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$ and every $\tau \in [0, \hat{T}(x,\tilde{x})]$, we have

$$\begin{aligned}\frac{\mathrm{d}d_c}{\mathrm{d}\tau}(\tau) \leq{}& C_6 \omega^{n_6} h e^{(\mu_u(\omega)-\alpha_0\mu_{s_1}(\omega))\tau} x_u^{\mathrm{in}} d_c(\tau) \\ &+ 2C_6\omega^{n_6} e^{(\mu_u(\omega)-\alpha_0\mu_{s_1}(\omega))\tau} x_u^{\mathrm{in}} \left(d_s(0) + h(1+\tau)\left(\left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_\perp + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_{/\!/} \right)\right)\end{aligned}$$

and by using formula (6.45) under the form $\mu_u(\omega) - \alpha_0\mu_{s_1}(\omega) = -\alpha(\mu_{s_1}(\omega) - \mu_u(\omega))$, we get

$$\begin{aligned}\frac{\mathrm{d}d_c}{\mathrm{d}\tau}(\tau) \leq{}& C_6 \omega^{n_6} h e^{-\alpha(\mu_{s_1}(\omega)-\mu_u(\omega))\tau} x_u^{\mathrm{in}} d_c(\tau) \\ &+ 2C_6\omega^{n_6} e^{-\alpha(\mu_{s_1}(\omega)-\mu_u(\omega))\tau} x_u^{\mathrm{in}} \left(d_s(0) + h(1+\tau)\left(\left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_\perp + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_{/\!/} \right)\right)\end{aligned} \tag{6.59}$$

Using the fact that $d_c(0) = 0$, integration of the inequality (6.59) between 0 and τ gives

$$d_c(\tau) \leq \gamma_1(\tau) + \int_0^\tau \gamma_2(z) d_c(z)\,\mathrm{d}z$$

where

$$\begin{aligned}\gamma_1(\tau) ={}& 2C_6\omega^{n_6} x_u^{\mathrm{in}} \left(d_s(0) + h\left(\left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_\perp + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_{/\!/} \right)\right) \int_0^\tau e^{-\alpha(\mu_{s_1}(\omega)-\mu_u(\omega))z}\,\mathrm{d}z \\ &+ 2C_6\omega^{n_6} x_u^{\mathrm{in}} h \left(\left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_\perp + \left\|x^{\mathrm{in}} - \tilde{x}^{\mathrm{in}}\right\|_{/\!/} \right) \int_0^\tau z e^{-\alpha(\mu_{s_1}(\omega)-\mu_u(\omega))z}\,\mathrm{d}z\end{aligned}$$

and

$$\gamma_2(\tau) = C_6\omega^{n_6} h e^{-\alpha(\mu_{s_1}(\omega)-\mu_u(\omega))\tau} x_u^{\mathrm{in}}$$

Using Gronwall's lemma, we obtain the following control of $d_c(\tau)$ for $\tau \in [0, \hat{T}(x,\tilde{x})]$:

$$d_c(\tau) \leq \gamma_1(\tau) + \int_0^\tau \gamma_1(z)\gamma_2(z) e^{\int_z^\tau \gamma_2(\sigma)\,\mathrm{d}\sigma}\,\mathrm{d}z \tag{6.60}$$

Here, the proof is essentially complete. Indeed, (6.60) is an explicit estimate on $d_c(\tau)$ and by plugging it in (6.56), we obtain an explicit estimate on $d_s(\tau)$. We are left to find upper bounds on the explicit functions γ_1 and γ_2.

Step 5: controls of γ_1 *and* γ_2 *on* $[0, \hat{T}(x,\tilde{x})]$. Using (3.14), we get

$$\mu_{s_1}(\omega) - \mu_u(\omega) \geq \frac{2}{\omega^2}$$

Hence, for every $C \geq C_6$, every $n \geq n_6$, every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$ and every $\tau \in [0, \hat{T}(x, \tilde{x})]$, we have

$$\begin{aligned} \int_0^\tau e^{-\alpha(\mu_{s_1}(\omega)-\mu_u(\omega))z}\,\mathrm{d}z &\leq \frac{1}{\alpha(\mu_{s_1}(\omega)-\mu_u(\omega))} \leq \frac{2}{3}\omega^2 \\ \int_0^\tau z e^{-\alpha(\mu_{s_1}(\omega)-\mu_u(\omega))z}\,\mathrm{d}z &\leq \frac{1}{\alpha^2(\mu_{s_1}(\omega)-\mu_u(\omega))^2} \leq \frac{4}{9}\omega^4 \end{aligned} \tag{6.61}$$

According to (6.61) and using the fact that $x_u^{\text{in}} \leq h$, there exist $C_7 \geq C_6$ and $n_7 \geq n_6$ such that for every $C \geq C_7$, every $n \geq n_7$, every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, every $\tau \in [0, \hat{T}(x, \tilde{x})]$ and every $0 \leq z \leq \tau$, we have

$$\int_z^\tau \gamma_2(\sigma)\,\mathrm{d}\sigma \leq C_7 \omega^{n_7} h^2 \tag{6.62}$$

According to (6.61), there exist $C_8 \geq C_7$ and $n_8 \geq n_7$ such that for every $C \geq C_8$, every $n \geq n_8$, every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$ and every $\tau \in [0, \hat{T}(x, \tilde{x})]$, we have

$$\gamma_1(\tau) \leq C_8 \omega^{n_8} x_u^{\text{in}} \left(d_s(0) + h \left(\left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_\perp + \left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_{/\!/} \right) \right) \tag{6.63}$$

Plugging (6.46) into (6.63), it follows that there exist $C_9 \geq C_8$ and $n_9 \geq n_8$ such that for every $C \geq C_9$, every $n \geq n_9$, every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$ and every $\tau \in [0, \hat{T}(x, \tilde{x})]$, we have

$$\gamma_1(\tau) \leq C_9 \omega^{n_9} h \left| x_u^{\text{in}} - \tilde{x}_u^{\text{in}} \right| + C_9 \omega^{n_9} x_u^{\text{in}} \left(\left| x_{s_1}^{\text{in}} - \tilde{x}_{s_1}^{\text{in}} \right| + \left| x_{s_2}^{\text{in}} - \tilde{x}_{s_2}^{\text{in}} \right| + \left| x_c^{\text{in}} - \tilde{x}_c^{\text{in}} \right| \right) \tag{6.64}$$

Step 6: control of d_c and proof of (6.43). Plugging the estimates obtained in the preceding step into (6.60), it follows that there exist $C_{10} \geq C_9$ and $n_{10} \geq n_9$ such that for every $C \geq C_{10}$, every $n \geq n_{10}$, every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$ and every $\tau \in [0, \hat{T}(x, \tilde{x})]$, we have

$$d_c(\tau) \leq C_{10} \omega^{n_{10}} h \left| x_u^{\text{in}} - \tilde{x}_u^{\text{in}} \right| + C_{10} \omega^{n_{10}} x_u^{\text{in}} \left(\left| x_{s_1}^{\text{in}} - \tilde{x}_{s_1}^{\text{in}} \right| + \left| x_{s_2}^{\text{in}} - \tilde{x}_{s_2}^{\text{in}} \right| + \left| x_c^{\text{in}} - \tilde{x}_c^{\text{in}} \right| \right) \tag{6.65}$$

Let $C_{11} = 2C_{10}$ and $n_{11} = n_{10}$. For every $C \geq C_{11}$, every $n \geq n_{11}$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$C_{10} \omega^{n_{10}} h \leq \frac{1}{6} \tag{6.66}$$

Let $C \geq C_{11}$, $n \geq n_{11}$ and $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$. Assume that $T(x, \tilde{x}) < \tau^{\text{out}}$. Using (6.65) and (6.66), we have,

$$d_c(T(x, \tilde{x})) \leq \frac{1}{2} \left(\left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_\perp + \left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_{/\!/} \right)$$

Since d_c is well defined and continuous (at least) on $\left[0, \tau^{\text{out}}\right]$, the above inequality contradicts the maximality of $T(x, \tilde{x})$. It follows that

$$T(x, \tilde{x}) \geq \tau^{\text{out}} \tag{6.67}$$

By definition of d_c, we have

$$\left| (\tilde{x}_c^{\text{out}} - x_c^{\text{out}}) - (\tilde{x}_c^{\text{in}} - x_c^{\text{in}}) \right| \leq d_c(\tau^{\text{out}}) + \left| \tilde{x}_c(0) - \tilde{x}_c^{\text{in}} \right| \tag{6.68}$$

Plugging (6.65) into (6.68) and using (6.37), it follows that there exists $C_{12} \geq C_{11}$ and $n_{12} \geq n_{11}$ such that for every $C \geq C_{12}$, every $n \geq n_{12}$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\left\| \left(x^{\text{out}} - \tilde{x}^{\text{out}} \right) - \left(x^{\text{in}} - \tilde{x}^{\text{in}} \right) \right\|_{/\!/} \leq C_{12} \omega^{n_{12}} h \left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_\perp + C_{12} \omega^{n_{12}} h_\perp \left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_{/\!/}$$

Hence, (6.43) holds true.

Step 7: control of d_s and proof of (6.42). According to (6.67) and (6.56), for every $C \geq C_{12}$, every $n \geq n_{12}$, every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$ and every $\tau \in [0, \tau^{\text{out}}]$, we have

$$d_s(\tau) \leq e^{-\alpha_0 \mu_{s_1}(\omega)\tau} d_s(0) + 2C_4 \omega^{n_3} h \tau e^{-\alpha_0 \mu_{s_1}(\omega)\tau} \left(\left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_\perp + \left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_{/\!/} \right) \tag{6.69}$$

Plugging (6.46) into (6.69), it follows that there exist $C_{13} \geq C_{12}$ and $n_{13} \geq n_{12}$ such that for every

$C \geq C_{13}$, every $n \geq n_{13}$, every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$ and every $\tau \in [0, \tau^{\text{out}}]$, we have

$$d_s(\tau) \leq C_{13}\omega^{n_{13}} h e^{-\alpha_0 \mu_{s_1}(\omega)\tau} \left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_\infty \left(\frac{1}{x_u^{\text{in}}} + \tau \right) \tag{6.70}$$

It remains to evaluate this inequality for $t = \tau^{\text{out}}$. According to formula (6.45), we have

$$\begin{aligned} \left\| x^{\text{out}} - \tilde{x}^{\text{out}} \right\|_\perp &\leq C_{13}\omega^{n_{13}} h \left(\frac{x_u^{\text{in}}}{h} \right)^{\alpha_0 \frac{\mu_{s_1}(\omega)}{\mu_u(\omega)}} \left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_\infty \left(\frac{1}{x_u^{\text{in}}} + \frac{1}{\mu_u(\omega)} \ln \frac{h}{x_u^{\text{in}}} \right) \\ &\leq C_{13}\omega^{n_{13}} \left(\frac{x_u^{\text{in}}}{h} \right)^{\frac{\alpha}{\omega}} \left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_\infty \left(1 - \frac{h}{\mu_u(\omega)} \frac{x_u^{\text{in}}}{h} \ln \frac{x_u^{\text{in}}}{h} \right) \end{aligned}$$

Moreover, $z \mapsto z \ln z$ is bounded on $[0, 1]$ and $\mu_u(\omega) \sim_{\omega \to +\infty} 6/\omega$ (see (3.14)) so there exist $C_{14} \geq C_{13}$ and $n_{14} \geq n_{13}$ such that for every $C \geq C_{14}$, every $n \geq n_{14}$ and every $(\omega, \alpha, h, x, \tilde{x}) \in E_{C,n}$, we have

$$\left\| x^{\text{out}} - \tilde{x}^{\text{out}} \right\|_\perp \leq (h_\perp)^{\frac{\alpha}{\omega}} C_{14}\omega^{n_{14}} h^{-\frac{\alpha}{\omega}} \left\| x^{\text{in}} - \tilde{x}^{\text{in}} \right\|_\infty \tag{6.71}$$

Hence, (6.42) holds true. □

6.3 Control of the transition maps $\Upsilon_{\omega,\mathbf{h}^s,\mathbf{h}^u}$

Recall that

$$d(\omega) = \frac{\omega - 1}{4}$$

Proof of Proposition 6.1. Let $C_0 > 0$ and $n_0 \in \mathbb{N}$ be large enough such that we can apply corollary 6.16 and Proposition 6.26 with these two constants.

Proof of (6.1) *and* (6.2). There is nothing to prove when $x_u = 0$ since in that case the left hands of the inequalities vanish (see (5.10)). For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $E_{C,n}$ the set of all $(\omega, h, h_\perp, x)$ such that $\omega \in]1, +\infty[$, $0 < h \leq (C\omega^n)^{-1}$, $0 < h_\perp < \min(h, d(\omega))$ and x is an orbit of the local vector field X_ω whose initial condition $x^{\text{in}} := x(0)$ belongs to the section $S^s_{\omega,\mathbf{h}^s}$ where $\mathbf{h}^s = (h, h_\perp, \min(h, d(\omega)))$ and such that $x_u^{\text{in}} \neq 0$. For every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, h, h_\perp, x) \in E_{C,n}$, we denote $\mathbf{h}^s := (h, h_\perp, \min(h, d(\omega)))$ and $\mathbf{h}^u := (h, h, 2h)$ and $\Upsilon := \Upsilon_{\omega,\mathbf{h}^s,\mathbf{h}^u}$. For every $\omega \in]1, +\infty[$, let

$$\alpha(\omega) \stackrel{\text{def}}{=} \max\left(\frac{\omega}{\omega + 1}, \frac{4}{5} \right)$$

Observe that

$$1 - \alpha(\omega) = \min\left(\frac{1}{\omega + 1}, \frac{1}{5} \right) \geq \frac{1}{5\omega}$$

Set $C_1 = 5C_0$ and $n_1 = n_0 + 1$. Let $C \geq C_1$, $n \geq n_1$ and $(\omega, h, h_\perp, x) \in E_{C,n}$. Observe that

$$h \leq \frac{1}{C\omega^n} \leq \frac{1 - \alpha(\omega)}{C_0\omega^{n_0}}$$

It follows that we can apply corollary 6.16 to $(\omega, \alpha(\omega), h, h_\perp, x)$. This yields

$$\begin{aligned} \left\| \Upsilon(x^{\text{in}}) - \Upsilon^A(x^{\text{in}}) \right\|_\perp &\leq (h_\perp)^{1 + \frac{\alpha(\omega)}{\omega}} h^{-\frac{\alpha(\omega)}{\omega}} \\ \left\| \Upsilon(x^{\text{in}}) - \Upsilon^A(x^{\text{in}}) \right\|_{/\!/} &\leq h_\perp h C_0 \omega^{n_0} \end{aligned}$$

Moreover, we have

$$\frac{\alpha(\omega)}{\omega} < 1,$$
$$1 + \frac{\alpha(\omega)}{\omega} \geq \frac{\omega+2}{\omega+1},$$

and $0 < h_\perp < 1$. Hence,

$$(h_\perp)^{1+\frac{\alpha(\omega)}{\omega}} h^{-\frac{\alpha(\omega)}{\omega}} \leq h_\perp^{\frac{\omega+2}{\omega+1}} h^{-1}$$

This concludes the proof of (6.1) and (6.2).

Continuity of Υ. Recall that for every $z \in S^s_{\omega,\mathbf{h}^s}$ such that $z_u = 0$, we have $\Upsilon(z) = (h, 0, 0, z_c)$ (see (5.10)). According to (6.1) and (6.2), for every $z \in S^s_{\omega,\mathbf{h}^s}$ such that $z_u = 0$, we have

$$\lim_{x\to z} \Upsilon(x) = (h, 0, 0, z_c) = \Upsilon(z)$$

so Υ is continuous at z.

Proof of (6.3) *and* (6.4). By symmetry, we can reduce the problem to the case where $\tilde{x}_u \leq x_u$. By continuity of the map Υ at points z such that $z_u = 0$, we can reduce the problem to the case where $0 < \tilde{x}_u \leq x_u$. For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $F_{C,N}$ the set of all $(\omega, h, h_\perp, x, \tilde{x})$ such that $\omega \in]1, +\infty[$, $0 < h \leq (C\omega^n)^{-1}$, $0 < h_\perp < \min(h, d(\omega))$ and $(x, \tilde{x})$ is a pair of $\left(S^s_{\omega,\mathbf{h}^s}, S^u_{\omega,\mathbf{h}^u}\right)$-synchronized orbits, where $\mathbf{h}^s := (h, h_\perp, \min(h, d(\omega)))$ and $\mathbf{h}^u := (h, h, 2h)$. For every $C \geq C_1$, every $n \geq n_1$ and every $(\omega, h, h_\perp, x, \tilde{x}) \in F_{C,N}$, we have

$$h \leq (C\omega^n)^{-1} \leq \frac{1-\alpha(\omega)}{C_0\omega^{n_0}}$$

Hence, we can apply Proposition 6.26 to $(\omega, \alpha(\omega), h, h_\perp, x)$, which yields

$$\left\|\Upsilon(x^{\text{in}}) - \Upsilon(\tilde{x}^{\text{in}})\right\|_\perp \leq (h_\perp)^{\frac{\alpha(\omega)}{\omega}} C_0\omega^{n_0} h^{-\frac{\alpha(\omega)}{\omega}} \left\|x^{\text{in}} - \tilde{x}^{\text{in}}\right\|_\infty$$
$$\left\|\left(\Upsilon(x^{\text{in}}) - \Upsilon(\tilde{x}^{\text{in}})\right) - (x^{\text{in}} - \tilde{x}^{\text{in}})\right\|_{/\!/} \leq C_0\omega^{n_0} h \left\|x^{\text{in}} - \tilde{x}^{\text{in}}\right\|_\perp + C_0\omega^{n_0} h_\perp \left\|x^{\text{in}} - \tilde{x}^{\text{in}}\right\|_{/\!/}$$

One can remark that there exists $4/5 < d < 1$ such that for every $\omega \in]1, +\infty[$, $\frac{\alpha(\omega)}{\omega} \leq d$. Moreover, for every $C_2 \geq C_1$ and every $n_2 \geq n_1$ such that

$$C_2^{1-d} \geq C_0 \quad \text{and} \quad n_2(1-d) \geq n_0$$

every $\omega \in]1, +\infty[$ and every $0 < h \leq (C_2\omega^{n_2})^{-1}$, we have $C_0\omega^{n_0} h^{-\frac{\alpha(\omega)}{\omega}} \leq h^{-1}$. Hence, for every $C \geq C_2$, every $n \geq n_2$ and every $(\omega, h, h_\perp, x, \tilde{x}) \in F_{C,N}$, we have

$$\left\|\Upsilon(x^{\text{in}}) - \Upsilon(\tilde{x}^{\text{in}})\right\|_\perp \leq h_\perp^{\frac{1}{\omega+1}} h^{-1} \|x - \tilde{x}\|_\infty$$
$$\left\|\left(\Upsilon(x^{\text{in}}) - \Upsilon(\tilde{x}^{\text{in}})\right) - (x^{\text{in}} - \tilde{x}^{\text{in}})\right\|_{/\!/} \leq C_0\omega^{n_0} h \left\|x^{\text{in}} - \tilde{x}^{\text{in}}\right\|_\perp + C_0\omega^{n_0} h_\perp \left\|x^{\text{in}} - \tilde{x}^{\text{in}}\right\|_{/\!/}$$

Since

$$\left\|\Upsilon(x^{\text{in}}) - \Upsilon(\tilde{x}^{\text{in}})\right\|_\perp = \left\|\left(\Upsilon(x^{\text{in}}) - \Upsilon(\tilde{x}^{\text{in}})\right) - \left(\Upsilon^A(x^{\text{in}}) - \Upsilon^A(\tilde{x}^{\text{in}})\right)\right\|_\perp$$
$$\left\|\left(\Upsilon(x^{\text{in}}) - \Upsilon(\tilde{x}^{\text{in}})\right) - (x^{\text{in}} - \tilde{x}^{\text{in}})\right\|_{/\!/} = \left\|\left(\Upsilon(x^{\text{in}}) - \Upsilon(\tilde{x}^{\text{in}})\right) - \left(\Upsilon^A(x^{\text{in}}) - \Upsilon^A(\tilde{x}^{\text{in}})\right)\right\|_{/\!/}$$

this concludes the proof of (6.3) and (6.4). □

Chapter 7

Dynamics in the neighbourhood of a type II orbit

The goal of this section is to give some estimates on the transition map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ (see definition 5.23). We will show that this map is "very close" to the Kasner map f. Recall that $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ describes the behaviour of the orbits of the Wainwright-Hsu vector field $\mathcal{X}$ in the neighbourhood of the type II orbit $\mathcal{O}_{\mathcal{P}_\omega\to\mathcal{P}_{f(\omega)}}$. More precisely, $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ is the transition map from the section $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ (which intersects $\mathcal{O}_{\mathcal{P}_\omega\to\mathcal{P}_{f(\omega)}}$ close to its "initial point" $\mathcal{P}_\omega$) to the section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$ (which intersects $\mathcal{O}_{\mathcal{P}_\omega\to\mathcal{P}_{f(\omega)}}$ close to its "final point" $\mathcal{P}_{f(\omega)}$) Observe that the situation is quite different from those of chapter 6. We are no more studying the local dynamics of a vector field in the vicinity of a singular point, but rather the large scale dynamics of a non-linear vector field. As a consequence, the estimates proven here for the map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ will be far less precise than the ones obtained in Proposition 6.1 for the map $\Upsilon_{\omega,\mathbf{h}^s,\mathbf{h}^u}$.

Define, for any $\omega\in\,]1,+\infty[\,\setminus\{2\}$,

$$i(\omega)\overset{\text{def}}{=}\begin{cases}1 & \text{if } \omega>2\\ 2 & \text{if } 1<\omega<2\end{cases}\tag{7.1}$$

Recall that for any $\omega\in\,]1,+\infty[\,\setminus\{2\}$, the type II orbit $\mathcal{O}_{\mathcal{P}_\omega\to\mathcal{P}_{f(\omega)}}$ is tangent to the direction $\partial_{x_{s_{i(\omega)}}}$ at the final point $\mathcal{P}_{f(\omega)}$. As a consequence, by continuity of the flow, if the section $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ is sufficiently small, the orbits starting in $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ will intersect the section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$ for the first time in $\mathcal{S}^{s_{i(\omega)}}_{f(\omega),\mathbf{h}^s}$.

For a technical reason explained below, we will often encounter the quantity $\min\left(1,(\omega-2)^2\right)$ in the estimates. Hence, we introduce the notation

$$m(\omega)\overset{\text{def}}{=}\min\left(1,(\omega-2)^2\right)$$

Recall that $\mathrm{Proj}_{\mathrm{A}}$ is the projection on the Mixmaster attractor (see definition 5.1) and recall that $\Psi^A_{\omega,\mathbf{h}^u,\mathbf{h}^s}=\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}\circ\mathrm{Proj}_{\mathrm{A}}$. Moreover, the map $\Psi^A_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ admits an explicit expression (see (5.12)). We can now give a formal statement of the main results of this section.

Proposition 7.1 (Control of the transition map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$). *There exist two constants $\tilde{C}_1\geq 1$ and $\tilde{n}_1\in\mathbb{N}$ such that the properties below hold for $\omega\in\,]1,+\infty[\,\setminus\{2\}$, $0<h^u\leq(\tilde{C}_1\omega^{\tilde{n}_1})^{-1}$, $0<h^s\leq(\tilde{C}_1 f(\omega)^{\tilde{n}_1})^{-1}$, $h=\min\left(h^u,h^s\right)$, $0<h_\perp\leq h^{\tilde{C}_1\omega}$, $\mathbf{h}^u=\left(h^u,h_\perp m(\omega),h^{\tilde{C}_1\omega}m(\omega)\right)$ and $\mathbf{h}^s=\left(h^s,h^s,h^s\right)$. The transition map*

$$\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}:S^u_{\omega,\mathbf{h}^u}\to S^s_{f(\omega),\mathbf{h}^s}$$

is well defined and takes its values in $S^{s_{i(\omega)}}_{f(\omega),\mathbf{h}^s}$. Moreover, for every $y,\tilde{y}\in S^u_{\omega,\mathbf{h}^u}$ we have the following estimates, where $\Psi:=\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ and $\Psi^A:=\Psi^A_{\omega,\mathbf{h}^u,\mathbf{h}^s}$:

(Control of the distance to the Mixmaster attractor)

$$\mathrm{dist}_\infty\left(\Psi(y),A\right)=\left\|\Psi(y)-\Psi^A(y)\right\|_\perp\leq h_\perp h^{-\tilde{C}_1\omega}\tag{7.2}$$

(Control of the drift tangential to the Mixmaster attractor)

$$\left\|\Psi(y) - \Psi^A(y)\right\|_{/\!/} \leq h_\perp h^{-\tilde{C}_1\omega} \tag{7.3}$$

(Lipschitz control in the direction transverse to the Mixmaster attractor)

$$\left\|(\Psi(y) - \Psi(\tilde{y})) - \left(\Psi^A(y) - \Psi^A(\tilde{y})\right)\right\|_\perp \leq \left(\|y - \tilde{y}\|_\perp + h_\perp \|y - \tilde{y}\|_{/\!/}\right) h^{-\tilde{C}_1\omega} \tag{7.4}$$

(Lipschitz control in the direction tangent to the Mixmaster attractor)

$$\left\|(\Psi(y) - \Psi(\tilde{y})) - \left(\Psi^A(y) - \Psi^A(\tilde{y})\right)\right\|_{/\!/} \leq \left(\|y - \tilde{y}\|_\perp + h_\perp \|y - \tilde{y}\|_{/\!/}\right) h^{-\tilde{C}_1\omega} \tag{7.5}$$

Remark 7.2. Proposition 7.1 describes the behaviour of the orbits of the Wainwright-Hsu vector field $\mathcal{X}$ traveling from a section $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ to a section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$. The vector field is non-linear and the traveling time is very long (it tends to infinity as $h \to 0$ or $\omega \to +\infty$). As a consequence, to ensure that an orbit starting in $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ will cut the section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$, the size of the section $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ must be very small. This is why, in Proposition 7.1, the size

$$h^{\tilde{C}_1\omega} m(\omega)$$

of the section $S^u_{\omega,\mathbf{h}^u}$ is "extremely small" compared to the parameters h^u and h^s, especially when ω is very large, *i.e.* when the type II orbit $\mathcal{O}_{\mathcal{P}_\omega \to \mathcal{P}_{f(\omega)}}$ is "close" to the Taub point.

Remark 7.3 (Technical detail). The quantity $m(\omega)$ appear in the upper bound of the size of the section $S^u_{\omega,\mathbf{h}^u}$ for some purely technical reasons. If $\omega = 2$, the type II orbit $\mathcal{O}^u_\omega$ arrives at the point $\mathcal{P}_{f(\omega)}$ of Kasner parameter $f(\omega) = 1$. However, the local coordinate system $\xi = (x_u, x_{s_1}, x_{s_2}, x_c)$ is not defined in the neighbourhood of this point. For this reason, we do not want the section $S^u_{\omega,\mathbf{h}^u}$ to cross the hyperplane $\omega = 2$.

The second result of this section will be used in Chapter 11 to prove that certain orbits shadow a heteroclinic chain.

Proposition 7.4 (Shadowing of a type II orbit)**.** *For every $\epsilon > 0$, there exists $\eta > 0$ such that for $\omega \in \,]1, +\infty[\, \setminus \{2\}$, $0 < h^u \leq (\tilde{C}_1\omega^{\tilde{n}_1})^{-1}$, $0 < h^s \leq (\tilde{C}_1 f(\omega)^{\tilde{n}_1})^{-1}$, $h = \min(h^u, h^s)$, $\mathbf{h}^u = \left(h^u, \eta h^{\tilde{C}_1\omega} m(\omega), \eta h^{\tilde{C}_1\omega} m(\omega)\right)$, $\mathbf{h}^s = (h^s, h^s, h^s)$ and $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$, the Hausdorff distance between two (minimal) orbit segments joining the section $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ and the section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$ (in that order) is less than ϵ.*

We now define a hitting time with the section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$ for the orbits in $\mathcal{B}^+$.

Definition 7.5 (Hitting time)**.** Let $\omega \in \,]1, +\infty[\, \setminus \{2\}$, $h^s > 0$ and $\mathbf{h}^s = (h^s, h^s, h^s)$. Assume that $S^s_{f(\omega),\mathbf{h}^s}$ is included in the range of the local coordinates ξ, so that the geometrical section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$ is well defined. For every $q \in \mathcal{B}^+$, we define

$$\tau_{\omega,h^s}(q) \stackrel{\text{def}}{=} \inf\left\{t > 0 \mid \mathcal{X}^t(q) \in \mathcal{S}^s_{f(\omega),\mathbf{h}^s}\right\} \in \,]0, +\infty]$$

Remark 7.6. With the notations of Proposition 7.1, for $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$, $\tau_{\omega,h^s}(q)$ is the traveling time between q and its image by the transition map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$. In particular, $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$ is the traveling time of the type II orbit $\mathcal{O}_{\mathcal{P}_\omega \to \mathcal{P}_{f(\omega)}}$ between the sections $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ and $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$.

Organization of the proof of Proposition 7.1. The main difficulty is to find some estimates on the traveling time τ_{ω,h^s}. Once we will have proven these estimates on τ_{ω,h^s}, we will easily deduce the estimates on the transition map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ using Gronwall's lemma. To study τ_{ω,h^s}, we proceed as follows:

1. We first obtain an estimate on $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$ using directly the Wainwright-Hsu equations (2.16a). This is possible because $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$ is the traveling time of the type II orbit $\mathcal{O}_{\mathcal{P}_\omega \to \mathcal{P}_{f(\omega)}}$ between the sections $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ and $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$ and this orbit is explicit.

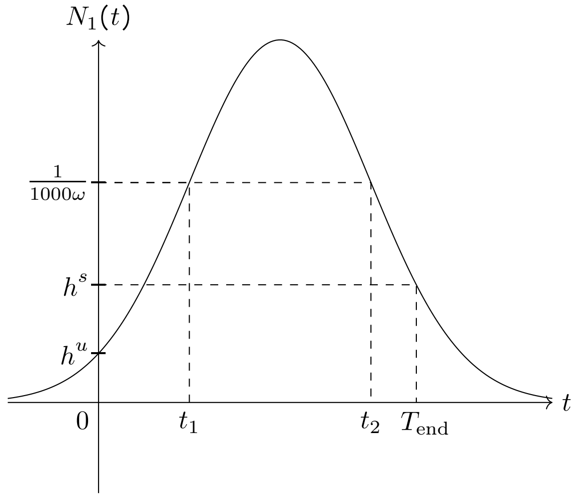

Figure 7.1 – Graph of $t \mapsto N_1(t)$. $T_{\text{end}} = \tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$.

2. Then we construct a flow box in the neighbourhood of the point $\mathcal{P}^{s_{i(\omega)}}_{f(\omega),h^s}$ and we bound the flow box coordinates. Recall that $\mathcal{P}^{s_{i(\omega)}}_{f(\omega),h^s} = \mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(\mathcal{P}^u_{\omega,h^u}) = \xi^{-1} \circ \Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s} \circ \xi(\mathcal{P}^u_{\omega,h^u})$.
3. Finally, we use a formula for $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ depending on $\mathcal{X}$, the traveling time $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$ and the flow box to get the desired estimates on $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$.

7.1 Traveling time of type II orbits

Recall that $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$ is the traveling time of the type II orbit $\mathcal{O}_{\mathcal{P}_\omega \to \mathcal{P}_{f(\omega)}}$ from the section $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ to the section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$.

Proposition 7.7 (Estimates on the traveling time of type II orbits). *There exist two constants $\tilde{C}_2 > 0$ and $\tilde{n}_2 \in \mathbb{N}$ such that for every $\omega \in]1,+\infty[$, every $0 < h^u \le (\tilde{C}_2 \omega^{\tilde{n}_2})^{-1}$, every $0 < h^s \le (\tilde{C}_2 f(\omega)^{\tilde{n}_2})^{-1}$ and for $h = \min\left(h^u, h^s\right)$, the traveling time satisfies*

$$\frac{\omega}{\tilde{C}_2} \le \tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) \le \tilde{C}_2 \omega \ln\left(\frac{1}{h}\right) \tag{7.6}$$

Proof. According to Proposition 4.2, there exist $C_0 > 0$ and $n_0 \ge 1$ such that for any $\omega \in]1,+\infty[$, the range U_ξ of the local coordinate system contains the ball B_{ω,C_0,n_0}. We can and we will assume that $C_0 \ge 2000$. Let $\omega \in]1,+\infty[$, $0 < h^u \le (C_0 \omega^{n_0})^{-1}$, $0 < h^s \le (C_0 f(\omega)^{n_0})^{-1}$ and $h = \min\left(h^u, h^s\right)$. To control the traveling time $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$, we can lift the type II orbit $\mathcal{O}_{\mathcal{P}_\omega \to \mathcal{P}_{f(\omega)}}$ into $\mathscr{B}^+$. Recall that $\mathcal{O}_{\mathcal{P}_\omega \to \mathcal{P}_{f(\omega)}}$ has six lifts in $\mathscr{B}^+$. Two of these lifts are such that $N_1 > 0$, $N_2 = 0$ and $N_3 = 0$. We choose one, denoted by

$$t \mapsto \mathscr{O}(t) = (N_1(t), 0, 0, \Sigma_1(t), \Sigma_2(t), \Sigma_3(t))$$

Using a time translation, we can and we will assume that $\mathscr{O}(0)$ is a lift of $\mathcal{P}^u_{\omega,h^u}$. This property is equivalent to $N_1(0) = h^u$ and $N_1'(0) > 0$. With this parametrization, $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$ is the unique time T verifying $N_1(T) = h^s$ and $N_1'(T) < 0$. Moreover, $\mathscr{O}(\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}))$ is a lift of $\mathcal{P}^{s_{i(\omega)}}_{f(\omega),h^s}$. See figure 7.1. Denote by $\mathscr{P}_\omega$ the lift of $\mathcal{P}_\omega$ such that $\mathscr{O}$ starts at $\mathscr{P}_\omega$, *i.e.* $\lim_{t\to-\infty} \mathscr{O}(t) = \mathscr{P}_\omega$. Recall that near the point $\mathscr{P}_\omega$, $N_1 = x_u$, while near the point $\mathscr{F}(\mathscr{P}_\omega)$, $N_1 = x_{s_{i(\omega)}}$.

Recall the evolution equations

$$N_1' = -(q + 2\Sigma_1) N_1 \tag{7.7}$$

$$\Sigma_1' = \frac{1}{6} N_1^2 (\Sigma_1 + 4) \tag{7.8}$$

where $q = \frac{1}{3}\left(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2\right)$. To control the traveling time, one must control the quantities $q + 2\Sigma_1$ and N_1^2. Next lemma shows that these two quantities cannot be simultaneously "too small".

Claim 1. *For every point* $(N_1, 0, 0, \Sigma_1, \Sigma_2, \Sigma_3)$ *in the type* II *orbit* $\mathscr{O}$, *either* $N_1 > \frac{1}{1000\omega}$ *or* $|q + 2\Sigma_1| \geq \frac{1}{\omega}$.

Proof of claim 1. Let $(N_1, 0, 0, \Sigma_1, \Sigma_2, \Sigma_3)$ be a point in the type II orbit $\mathscr{O}$. Let $M = (0, 0, 0, \Sigma_1, \Sigma_2, \Sigma_3)$ be its projection onto the $(\Sigma_1, \Sigma_2, \Sigma_3)$-plane. Denote by d the Euclidean distance on $\mathbb{R}^6$. The proof essentially follows from the formula

$$q + 2\Sigma_1 = \frac{1}{3}d(M, Q_1)^2 - 2 \tag{7.9}$$

which proves that $q + 2\Sigma_1$ varies as a squared distance. Using (3.14) and the fact that at the point $\mathscr{P}_\omega$, the quantity $-(q + 2\Sigma_1)$ coincide with the unstable eigenvalue of the Wainwright-Hsu vector field $-(2 + 2\Sigma_1)$, it follows from (7.9) that

$$\left|\frac{1}{3}d(\mathscr{P}_\omega, Q_1)^2 - 2\right| \geq \frac{2}{\omega} \tag{7.10}$$

Analogously, we have

$$\left|\frac{1}{3}d(\mathscr{F}(\mathscr{P}_\omega), Q_1)^2 - 2\right| \geq \frac{2}{\omega} \tag{7.11}$$

Recall the constraint equation (3.4):

$$6 - 3q = \frac{1}{2}N_1^2 \tag{7.12}$$

and observe that $3q$ is the square of the distance between the point M and the center of the Kasner circle and 6 is the square of the radius of the Kasner circle. The constraint equation (7.12) implies that, if N_1 is small, then M is very close to the Kasner circle. Since M belongs to the projection of the type II orbit $\mathscr{O}$, M must be close to one of the two end points $\mathscr{P}_\omega$ and $\mathscr{F}(\mathscr{P}_\omega)$. More precisely, one easily checks that if $N_1 \leq \frac{1}{1000\omega}$, then

$$\min\left(d\left(M, \mathscr{P}_\omega\right), d\left(M, \mathscr{F}(\mathscr{P}_\omega)\right)\right) \leq \frac{1}{100\omega} \tag{7.13}$$

Using (7.10), (7.11) and (7.13), we get that if $N_1 \leq \frac{1}{1000\omega}$, then $\left|\frac{1}{3}d(M, Q_1)^2 - 2\right| \geq \frac{1}{\omega}$. The claim follows from (7.9). ☐

We know that N_1 is increasing and then decreasing along the type II orbit. Moreover, recall that $N_1(0) = h^u$ and $N_1(\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})) = h^s$ (see figure 7.1). Hence, $|N_1(0)| < \frac{1}{1000\omega}$ and $\left|N_1(\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}))\right| < \frac{1}{1000\omega}$. It follows that there exist $0 < t_1 < t_2 < \tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$ such that

1. On $[0, t_1]$, N_1 is increasing and $N_1(t) \leq \frac{1}{1000\omega}$.
2. On $]t_1, t_2[$, $N_1(t) > \frac{1}{1000\omega}$.
3. On $\left[t_2, \tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})\right]$, N_1 is decreasing and $N_1(t) \leq \frac{1}{1000\omega}$.

Upper bound for t_1 *and* $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) - t_2$. Using the evolution equation (7.7) and claim 1 on $[0, t_1]$, we get that for every $t \in [0, t_1]$, $\left|N_1'(t)\right| \geq \frac{1}{\omega}N_1(t)$. By integrating this inequality between 0 and t_1, we get

$$t_1 \leq \omega \ln \frac{1}{h^u} \leq \omega \ln \frac{1}{h} \tag{7.14}$$

By an analogous reasoning on $\left[t_2, \tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})\right]$, we get

$$\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) - t_2 \leq \omega \ln \frac{1}{h^s} \leq \omega \ln \frac{1}{h} \tag{7.15}$$

Lower and upper bounds for $t_2 - t_1$.

Claim 2. *For every point* $(N_1, 0, 0, \Sigma_1, \Sigma_2, \Sigma_3)$ *in the type* II *orbit* $\mathcal{O}$, *we have*

$$N_1 \leq \frac{100}{\omega} \tag{7.16}$$

Proof of claim 2. Let $(N_1, 0, 0, \Sigma_1, \Sigma_2, \Sigma_3)$ be a point in the type II orbit $\mathcal{O}$. Let $M = (0, 0, 0, \Sigma_1, \Sigma_2, \Sigma_3)$ be its projection onto the $(\Sigma_1, \Sigma_2, \Sigma_3)$-plane. The projection of $\mathcal{O}$ onto the $(\Sigma_1, \Sigma_2, \Sigma_3)$-plane is explicitly known: it is the chord whose end points are $\mathscr{P}_\omega$ and $\mathscr{F}(\mathscr{P}_\omega)$. Using the coordinates of $\mathscr{P}_\omega$ and $\mathscr{F}(\mathscr{P}_\omega)$, one can get that $d(\mathscr{P}_\omega, \mathscr{F}(\mathscr{P}_\omega)) \leq \frac{18\sqrt{2}}{\omega}$. Hence, $d(M, \mathscr{K}) \leq \frac{100}{\omega^2}$. Recall that $3q$ is the square of the distance between the point M and the center of the Kasner circle and 6 is the square of the radius of the Kasner circle. It follows that $3q \geq 6 - \frac{1000}{\omega^2}$ and, using the constraint equation (7.12), we get $N_1 \leq \frac{100}{\omega}$. This concludes the proof of claim 2. □

We are left to find some lower and upper bounds for the variation of Σ_1 on $]t_1, t_2[$. According to the constraint equation (7.12), $q(t_1) = q(t_2)$. According to claim 1, $(q + 2\Sigma_1)(t_2) \geq \frac{1}{\omega}$ and $(q + 2\Sigma_1)(t_1) \leq -\frac{1}{\omega}$. Hence,

$$\Sigma_1(t_2) - \Sigma_1(t_1) \geq \frac{1}{\omega} \tag{7.17}$$

Moreover, Σ_1 is increasing along the type II orbit and its variation $\Sigma_1(t_2) - \Sigma_1(t_1)$ is smaller than its variation between $\mathscr{P}_\omega$ and $\mathscr{F}(\mathscr{P}_\omega)$. Using (3.14), we get

$$\Sigma_1(t_2) - \Sigma_1(t_1) \leq \frac{12}{\omega} \tag{7.18}$$

Using the estimate (7.16), the fact that $2 \leq \Sigma_1 + 4 \leq 6$ and the evolution equation (7.8), we get that for every $t \in]t_1, t_2[$,

$$\frac{1}{10^7 \omega^2} \leq \Sigma_1'(t) \leq \frac{10^4}{\omega^2} \tag{7.19}$$

Integrating (7.19) between t_1 and t_2, estimates (7.17) and (7.18) give

$$\frac{\omega}{10^4} \leq t_2 - t_1 \leq 10^9 \omega \tag{7.20}$$

Estimates (7.14), (7.15) and (7.20) give the desired control on $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$. This concludes the proof with $\tilde{C}_2 := \max\left(C_0, 10^{10}\right)$ and $\tilde{n}_2 := n_0$. □

7.2 Construction of a flow box

Given $\omega \in]1, +\infty[$, $i \in \{1, 2\}$ and a small constant $h > 0$, we are going to construct a flow box in a neighbourhood of the point $\mathcal{P}^{s_i}_{\omega,h}$. The usual flow box theorem states that, since $\mathcal{P}^{s_i}_{\omega,h}$ is a non singular point for $\mathcal{X}$, there exists a neighbourhood of $\mathcal{P}^{s_i}_{\omega,h}$ (called a "flow box") and a local coordinate system on this neighbourhood such that the integral curves of the vector field $\mathcal{X}$ are parallel straight lines in this local coordinate system.

The following lemma, in addition to give a precise statement of the flow box theorem in our context, gives estimates about the size of the flow box and the C^2-norm of the local coordinate system.

To study the map map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$, we will apply this lemma at $f(\omega)$ instead of ω.

Lemma 7.8 (Construction of a flow box)**.** *There exist two constants* $\tilde{C}_3 \geq \tilde{C}_2$ *and* $\tilde{n}_3 \geq \tilde{n}_2$ *such that for every* $\omega \in]1, +\infty[$, *every* $i \in \{1, 2\}$, *every* $0 < h \leq (\tilde{C}_3 \omega^{\tilde{n}_3})^{-1}$, *for*

$$r^{box} = \min\left(h^2 (\tilde{C}_3 \omega^{\tilde{n}_3})^{-1}, \frac{\omega - 1}{2}\right)$$
$$\mathbf{h}^{box} = (h, r^{box}, r^{box})$$

there exist a neighbourhood $\mathcal{V}_{\omega,h}$ *of* $\mathcal{P}^{s_i}_{\omega,h}$ *in* $\mathcal{B}^+$ *and a* C^2*-diffeomorphism*

$$\theta_{\omega,h} : \mathcal{V}_{\omega,h} \to \left[-r^{box}, r^{box}\right] \times \left[0, r^{box}\right]^2 \times \left[-r^{box}, r^{box}\right] \subset \mathbb{R}^4$$

with the following properties. If we denote by (x_1, x_2, x_3, x_4) *the coordinates on the space* $\mathbb{R}^4$ *where* $\theta_{\omega,h}$ *takes its values, then*

1. $\theta_{\omega,h}\left(\mathcal{P}^{s_i}_{\omega,h}\right) = (0,0,0,0)$.

2. $\mathcal{V}_{\omega,h}$ *contains the section* $\mathcal{S}^{s_i}_{\omega,\mathbf{h}^{box}}$ *and* $\theta_{\omega,h}$ *maps* $\mathcal{S}^{s_i}_{\omega,\mathbf{h}^{box}}$ *to* $\{0\} \times \left[0, r^{box}\right]^2 \times \left[-r^{box}, r^{box}\right]$.

3. $\theta_{\omega,h}$ *straightens the vector field* $\mathcal{X}$ *onto the vector field* $\frac{\partial}{\partial x_1}$.

4. *The* C^2*-norm of* $\theta_{\omega,h}$ *admits an upper bound which is polynomial in* $\frac{\omega}{h}$. *More precisely:*

$$\begin{aligned} \left\|D\theta_{\omega,h}\right\|_\infty &\le \frac{\tilde{C}_3 \omega^{\tilde{n}_3}}{h^2} \\ \left\|D^2\theta_{\omega,h}\right\|_\infty &\le \frac{\tilde{C}_3 \omega^{\tilde{n}_3}}{h^6} \end{aligned} \tag{7.21}$$

5. *The* C^2*-norm of* $\theta^{-1}_{\omega,h}$ *admits an upper bound which is polynomial in* ω. *More precisely:*

$$\left\|\theta^{-1}_{\omega,h}\right\|_{C^2} \le \tilde{C}_3 \omega^{\tilde{n}_3} \tag{7.22}$$

6. *For every* $0 < r' \le r^{box}$, $\theta^{-1}_{\omega,h}\left(\left[-r', r'\right] \times \left[0, r'\right]^2 \times \left[-r', r'\right]\right)$ *contains the ball* $B\left(\mathcal{P}^{s_i}_{\omega,h}, r^{box} r'\right)$ *open in* $\mathcal{U}_\xi$ *and*

$$\theta^{-1}_{\omega,h}\left(\{0\} \times \left[0, r'\right]^2 \times \left[-r', r'\right]\right) = \mathcal{S}^{s_i}_{\omega,(h,r',r')} \subset \mathcal{V}_{\omega,h} \tag{7.23}$$

Remark 7.9. Items 2 and 3 imply that for every $y \in \mathcal{V}_{\omega,h}$, $-x_1(\theta_{\omega,h}(y))$ is the unique time $t \in \left[-r^{\text{box}}, r^{\text{box}}\right]$ such that $\mathcal{X}^t(y) \in \mathcal{S}^{s_i}_{\omega,\mathbf{h}^{\text{box}}}$. In particular, for a flow box around the point $\mathcal{P}^{s_{i(\omega)}}_{f(\omega),h}$, if $-x_1(\theta_{f(\omega),h}(y)) > 0$, then $\tau_{\omega,h}(y) = -x_1(\theta_{f(\omega),h}(y))$.

In order to make the proof of Lemma 7.8 easier to read, we extract here an independant result that will be used in the course of the proof. Roughly speaking, this result states that the orbits of the Wainwright-Hsu vector field $\mathcal{X}$ crossing a section $\mathcal{S}^{s_i}_{\omega,\mathbf{h}}$ do not cross it again "too fast".

Lemma 7.10 (No loop in small time)**.** *There exist two constants* $C > 0$ *and* $n \in \mathbb{N}$ *such that the following property holds for* $\omega \in \left]1, +\infty\right[$ *and* $0 < h \le (C\omega^n)^{-1}$. *Let* $t \mapsto q(t)$ *be an orbit of the Wainwright-Hsu vector field* $\mathcal{X}$ *whose initial condition* $q(0)$ *belongs to the section* $\mathcal{S}^s_{\omega,\mathbf{h}}$ *where* $\mathbf{h} = \left(h, \frac{h}{2}, \frac{h}{2}\right)$. *Then,* q *is well defined (at least) on the time interval* $\left[0, \frac{\ln 2}{12}\right]$ *and does not cross the section* $\mathcal{S}^s_{\omega,\mathbf{h}}$ *for* $t \in \left]0, \frac{\ln 2}{12}\right]$.

Proof. Let $C_0 > 0$ and $n_0 \in \mathbb{N}$ be large enough such that we can apply Proposition 6.15, Proposition 6.18 and corollary 6.19 with these two constants. Let $\omega \in \left]1, +\infty\right[$, $0 < h \le (C_0 \omega^{n_0})^{-1}$ and $t \mapsto q(t)$ be an orbit of the vector field $\mathcal{X}$ whose initial condition $q(0)$ belongs to the section $\mathcal{S}^s_{\omega,\mathbf{h}}$ where $\mathbf{h} = \left(h, \frac{h}{2}, \frac{h}{2}\right)$. Let $y^{\text{in}} := \xi(q(0)) \in S^s_{\omega,\mathbf{h}}$ and denote by $t \mapsto y(t)$ the orbit of the vector field $X = \xi_* \mathcal{X}$ with initial condition $y(0) = y^{\text{in}}$. Remark that $y = \xi \circ q$ whenever y is well defined.

Case $y^{in}_u = 0$. In that case, the orbit y converges exponentially fast to the point $(0,0,0,y_c)$ and according to (6.29), for every $t > 0$ and every $i \in \{1,2\}$, $y_i(t) < h/2$. This implies that for every $t > 0$, $y(t) \notin S^s_{\omega,\mathbf{h}}$. Hence, q does not cross the section $\mathcal{S}^s_{\omega,\mathbf{h}}$ for $t > 0$.

Case $y^{in}_u > 0$. Denote by $t \mapsto x(t)$ the orbit of the renormalized local vector field $X_\omega = \gamma_\omega.X$ (see definition 4.10) with initial condition $x(0) = y^{\text{in}}$. Remark that x is a reparametrization of the orbit y. According to Proposition 6.15, x is at least defined for $t \in [0, \tau^{\text{loc}}]$ where

$$\tau^{\text{loc}} = \frac{1}{\mu_u(\omega)} \ln \frac{h}{y^{\text{in}}_u} \ge \frac{\ln 2}{6}$$

Using the estimate (4.27) about the renormalization function γ_ω, we get that the orbit y is at least defined for $t \in \left[0, \frac{\ln 2}{12}\right]$ and there exists a C^1-map $s : \left[0, \frac{\ln 2}{12}\right] \to \left[0, \frac{\ln 2}{6}\right]$ such that $s(0) = 0$ and for every $t \in [0, \frac{\ln 2}{12}]$, $y(t) = x(s(t))$. Moreover, x intersects the section $S^s_{\omega,\mathbf{h}}$ at most one time (see corollary 6.19) so y intersects the section $S^s_{\omega,\mathbf{h}}$ at most one time on $\left[0, \frac{\ln 2}{12}\right]$. It follows that q intersects the section $\mathcal{S}^s_{\omega,\mathbf{h}}$ at most one time on $\left[0, \frac{\ln 2}{12}\right]$. This concludes the proof. □

Proof of Lemma 7.8. To fix the ideas, we will only treat the case $i = 2$, that is, we will construct a flow box around the point $\mathcal{P}^{s_2}_{\omega,h} = (0, 0, h, \omega)$.

For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $E_{C,n}$ the set of all (ω, h) such that $\omega \in]1, +\infty[$ and $0 < h \leq (C\omega^n)^{-1}$. Let $C_0 \geq 100$ and $n_0 \in \mathbb{N}$ be large enough such that we can apply Proposition 4.2 and Lemma 7.10 with these two constants.

We will use several times in this proof that the vector field $\mathcal{X}$ is bounded on every compact subset of $\mathcal{B}$ for the C^2-norm. In particular, even if it means taking C_0 larger, we can assume that $\|\mathcal{X}\|_{C^2} \leq C_0$ on a compact set containing all the orbits playing a role in this proof.

For every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, h) \in E_{C,n}$, let

$$r \overset{\text{def}}{=} \min\left(h^2(C\omega^n)^{-1}, \frac{\omega - 1}{2}\right)$$
$$\mathbf{h} \overset{\text{def}}{=} (h, r, r)$$
$$D \overset{\text{def}}{=} [0, r]^2 \times [-r, r]$$

let

$$\chi\colon \left| \begin{array}{ccc} D & \to & \mathcal{S}^{s_2}_{\omega,\mathbf{h}} \\ (x_u, x_{s_1}, x_c) & \mapsto & \xi^{-1}(x_u, x_{s_1}, h, x_c + \omega) \end{array} \right.$$

and let

$$\varphi\colon \left| \begin{array}{ccc} [-r, r] \times D & \to & \mathcal{U}_\xi \\ (t, z) & \mapsto & \mathcal{X}^t(\chi(z)) \end{array} \right.$$

where $\mathcal{X}^t$ denotes the flow of the Wainwright-Hsu vector field $\mathcal{X}$. The map χ is a bijective C^2 parametrization of the section $\mathcal{S}^{s_2}_{\omega,\mathbf{h}}$ such that $\chi(0) = \mathcal{P}^{s_2}_{\omega,h}$. The map φ is a C^2 map such that, for every $z \in D$, $t \mapsto \varphi(t, z)$ is a (local) parametrization of the orbit of the Wainwright-Hsu vector field $\mathcal{X}$ passing through the point $\chi(z) \in \mathcal{S}^{s_2}_{\omega,\mathbf{h}}$ at $t = 0$. Note that the domain of φ depends on C, n and (ω, h). Roughly speaking, the map $\theta_{\omega,h}$ will be obtained as the inverse of φ.

Claim 1. *For every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, h) \in E_{C,n}$, φ is injective on $[-r, r] \times D$.*

Proof of claim 1. Let $C \geq C_0$, $n \geq n_0$ and $(\omega, h) \in E_{C,n}$. Let $(t, z), (t', z') \in [-r, r] \times D$ and assume that $\varphi(t, z) = \varphi(t', z')$. By symmetry, one can assume that $t \leq t'$. We have $\chi(z) = \mathcal{X}^{t'-t}(\chi(z'))$ and since $r < \frac{\ln 2}{24}$, we have $0 \leq t' - t \leq \frac{\ln 2}{12}$. According to Lemma 7.10, we necessarily have $t = t'$. It follows that $\chi(z) = \chi(z')$ and since χ is injective, we have $z = z'$. It follows that φ is injective. □

Claim 2. *There exist $C_1 \geq C_0$ and $n_1 \geq n_0$ such that for every $C \geq C_1$, every $n \geq n_1$ and every $(\omega, h) \in E_{C,n}$, $\|\varphi\|_{C^2} \leq C_1\omega^{n_1}$.*

Proof of claim 2. Let $C \geq C_0$, $n \geq n_0$ and $(\omega, h) \in E_{C,n}$. The first and second derivatives of the flow $(t, y) \mapsto \mathcal{X}^t(y)$ are controlled by the C^2-norm of $\mathcal{X}$ (which is bounded by C_0) and the size of the time interval on which we study the flow. This time interval is $[-r, r]$ so its size is bounded independantly of (ω, h). Moreover, according to the estimate (4.5) about the adapted system of local coordinates ξ, we have $\|\chi\|_{C^2} \leq C_0\omega^{n_0}$. Since $\varphi(t, z) = \mathcal{X}^t(\chi(z))$, this leads to the desired result. □

Claim 3. *There exist $C_2 \geq C_1$ and $n_2 \geq n_1$ such that for every $C \geq C_2$, every $n \geq n_2$ and every $(\omega, h) \in E_{C,n}$, the derivative $D\varphi(0)$ is invertible and $\left\|(D\varphi(0))^{-1}\right\| \leq \frac{C_2\omega^{n_2}}{h^2}$*

Proof of claim 3. Let $C \geq C_1$, $n \geq n_1$ and $(\omega, h) \in E_{C,n}$. Observe that

$$D\varphi(0) = \left(\ \mathcal{X}(\mathcal{P}^{s_2}_{\omega,h}) \ \ \middle| \ \ \frac{\partial \xi^{-1}}{\partial x_u}\left(P^{s_2}_{\omega,h}\right) \ \ \middle| \ \ \frac{\partial \xi^{-1}}{\partial x_{s_1}}\left(P^{s_2}_{\omega,h}\right) \ \ \middle| \ \ \frac{\partial \xi^{-1}}{\partial x_c}\left(P^{s_2}_{\omega,h}\right) \ \right)$$

and

$$\mathcal{X}(\mathcal{P}^{s_2}_{\omega,h}) = D\xi^{-1}(P^{s_2}_{\omega,h}).X\left(P^{s_2}_{\omega,h}\right)$$

Recall from the formula (4.6) that $X\left(P^{s_2}_{\omega,h}\right)$ is collinear to the vector $\frac{\partial}{\partial x_{s_2}}$. It follows that

$$\mathcal{X}(\mathcal{P}^{s_2}_{\omega,h}) = a\frac{\partial \xi^{-1}}{\partial x_{s_2}}\left(P^{s_2}_{\omega,h}\right) \tag{7.24}$$

for a certain $a \in \mathbb{R}$ and

$$|\det D\varphi(0)| = |a| \left|\det D\xi^{-1}(P^{s_2}_{\omega,h})\right| \tag{7.25}$$

According to (4.5), there exist $C_1' \geq C_1$ and $n_1' \geq n_1$ such that for every $C \geq C_1', n \geq n_1'$, for every $(\omega, h) \in E_{C,n}$,

$$\left|\det D\xi^{-1}(P^{s_2}_{\omega,h})\right| \geq \frac{1}{C_1'\omega^{n_1'}} \tag{7.26}$$

and

$$\left\|\frac{\partial \xi^{-1}}{\partial x_{s_2}}\left(P^{s_2}_{\omega,h}\right)\right\| \leq C_1'\omega^{n_1'} \tag{7.27}$$

According to (4.1a) and the expression of the vector field $\mathcal{X}$ induced by (2.16a), we have

$$\begin{aligned}\Sigma_{s_2}\left(\mathcal{X}(\mathcal{P}^{s_2}_{\omega,h})\right) &= \frac{1}{6}N_{s_2}\left(\mathcal{P}^{s_2}_{\omega,h}\right)^2\left(\Sigma_{s_2}\left(\mathcal{P}^{s_2}_{\omega,h}\right)+4\right)\\ &= \frac{1}{6}h^2\left(\Sigma_{s_2}\left(\mathcal{P}^{s_2}_{\omega,h}\right)+4\right)\\ &\geq \frac{1}{3}h^2\end{aligned}$$

so

$$\left\|\mathcal{X}(\mathcal{P}^{s_2}_{\omega,h})\right\| \geq \frac{h^2}{3} \tag{7.28}$$

Using (7.24), (7.25), (7.26), (7.27) and (7.28), we find that for every $C \geq C_1'$, every $n \geq n_1'$ and every $(\omega, h) \in E_{C,n}$, we have

$$|\det D\varphi(0)| \geq \frac{\left\|\mathcal{X}(\mathcal{P}^{s_2}_{\omega,h})\right\|}{\left\|\frac{\partial \xi^{-1}}{\partial x_{s_2}}\left(P^{s_2}_{\omega,h}\right)\right\|}\left|\det D\xi^{-1}(P^{s_2}_{\omega,h})\right| \geq \frac{h^2}{3\left(C_1'\omega^{n_1'}\right)^2} \tag{7.29}$$

In particular, $D\varphi(0)$ is invertible. Denote by ${}^{\mathrm{t}}\mathrm{Co}(A)$ the adjugate of a square matrix A. Using (7.29), the standard formula

$$(D\varphi(0))^{-1} = \frac{1}{\det D\varphi(0)}\,{}^{\mathrm{t}}\mathrm{Co}(D\varphi(0))$$

and claim 2, it follows that there exist $C_2 \geq C_1$ and $n_2 \geq n_1$ such that for every $C \geq C_2$, every $n \geq n_2$ and every $(\omega, h) \in E_{C,n}$, $D\varphi(0)$ is invertible and $\left\|(D\varphi(0))^{-1}\right\| \leq \frac{C_2\omega^{n2}}{h^2}$. □

Next claim relies on a standard argument for the local inversion theorem. Denote

$$\mathcal{V}_{\omega,h} \overset{\text{def}}{=} \varphi\left([-r,r]\times D\right)$$

Claim 4. *There exist $C_3 \geq C_2$ and $n_3 \geq n_2$ such that for every $C \geq C_3$, every $n \geq n_3$ and every $(\omega, h) \in E_{C,n}$, φ is a C^2-diffeomorphism from $[-r,r]\times D$ onto $\mathcal{V}_{\omega,h}$ and $\left\|D\varphi^{-1}\right\|_\infty \leq \frac{C_3\omega^{n3}}{h^2}$.*

Proof of claim 4. Let $C \geq C_2$, $n \geq n_2$ and $(\omega, h) \in E_{C,n}$. Let $u = D\varphi(0)$ and $\eta = D\varphi(0) - D\varphi$. We have $D\varphi = u(\mathrm{Id} - u^{-1}\eta)$. According to claim 2 and the mean value theorem, $\|\eta\|_\infty \leq C_1\omega^{n_1}r$. According to claim 3, $\left\|u^{-1}\right\| \leq \frac{C_2\omega^{n2}}{h^2}$. It follows that, for every $C \geq C_3 := 2C_1C_2$, every $n \geq n_3 := n_1 + n_2$ and every $(\omega, h) \in E_{C,n}$, $\left\|u^{-1}\eta\right\|_\infty \leq \frac{1}{2}$. Hence, for every $C \geq C_3$, every $n \geq n_3$ and every $(\omega, h) \in E_{C,n}$, $D\varphi$ is invertible on $[-r,r]\times D$ and $\left\|D\varphi^{-1}\right\|_\infty \leq \frac{2C_2\omega^{n2}}{h^2} \leq \frac{C_3\omega^{n3}}{h^2}$. Recall from claim 1 that φ is injective. So, according to the global inversion theorem, φ is a C^2-diffeomorphism from $[-r,r]\times D$ to $\mathcal{V}_{\omega,h}$. □

Let us denote by θ the inverse of φ. By construction, it is clear that $\theta\left(\mathcal{P}^{s_2}_{\omega,h}\right) = (0,0,0,0)$. Next claim is also a standard computation for the local inversion theorem.

Claim 5. *There exist $C_4 \geq C_3$ and $n_4 \geq n_3$ such that for every $C \geq C_4$, every $n \geq n_4$ and every $(\omega,h) \in E_{C,n}$, $\|D\theta\|_\infty \leq \frac{C_4\omega^{n_4}}{h^2}$ and $\left\|D^2\theta\right\|_\infty \leq \frac{C_4\omega^{n_4}}{h^6}$.*

Proof of claim 5. Let $C \geq C_3$, $n \geq n_3$ and $(\omega,h) \in E_{C,n}$. Let us denote by $I : M \mapsto M^{-1}$ the inversion in $\mathrm{GL}(\mathbb{R}^4)$. We have $D\theta = I \circ D\varphi \circ \theta$ and $D^2\theta = DI(D\varphi\circ\theta)D^2\varphi(\theta)D\theta$. According to claims 2 and 4 and the inequality $\|DI(D\varphi\circ\theta)\|_\infty \leq \|D\theta\|^2_\infty$, we get the desired result. □

Next claim is a double statement. First part is a standard consequence of the mean value theorem. Second part is a direct consequence of the definition of φ and $\mathcal{S}^{s_2}_{\omega,(h,r',r')}$.

Claim 6. *For every $C \geq C_4$, every $n \geq n_4$, every $(\omega,h) \in E_{C,n}$ and every $0 < r' \leq r$, $\theta^{-1}\left(\left[-r',r'\right]\times\left[0,r'\right]^2\times\left[-r',r'\right]\right)$ contains the open ball $B\left(\mathcal{P}^{s_2}_{\omega,h}, rr'\right)$ in $\mathcal{U}_\xi$ and*

$$\theta^{-1}_{\omega,h}\left(\{0\}\times\left[0,r'\right]^2\times\left[-r',r'\right]\right) = \mathcal{S}^{s_2}_{\omega,(h,r',r')} \subset \mathcal{V}_{\omega,h}$$

Proof of claim 6. Let $C \geq C_4$, $n \geq n_4$, $(\omega,h)\in E_{C,n}$ and $0 < r' \leq r$. Let us denote by R the supremum of every $\delta > 0$ such that

$$B(\mathcal{P}^{s_2}_{\omega,h},\delta) \subset \theta^{-1}\left(\left[-r',r'\right]\times\left[0,r'\right]^2\times\left[-r',r'\right]\right)$$

Recall that $\theta\left(\mathcal{P}^{s_2}_{\omega,h}\right) = (0,0,0,0)$. Using the mean value theorem and claim 5, we get that

$$r' \leq \|D\theta\|_\infty R \leq \frac{C_4\omega^{n_4}}{h^2}R$$

Hence,

$$R \geq r'\frac{h^2}{C_4\omega^{n_4}} \geq r'r$$

Moreover,

$$\begin{aligned}\theta^{-1}_{\omega,h}\left(\{0\}\times\left[0,r'\right]^2\times\left[-r',r'\right]\right) &= \varphi\left(\{0\}\times\left[0,r'\right]^2\times\left[-r',r'\right]\right)\\ &= \chi\left(\left[0,r'\right]^2\times\left[-r',r'\right]\right)\\ &= \xi^{-1}\left(\left[0,r'\right]^2\times\{h\}\times\left[-r',r'\right]\right)\\ &= \mathcal{S}^{s_2}_{\omega,(h,r',r')} \subset \mathcal{V}_{\omega,h}\end{aligned}$$

This concludes the proof of claim 6. □

As a particular case with $r' = r$, it follows from claim 6 that $\theta_{\omega,h}$ maps the section $\mathcal{S}^{s_2}_{\omega,\mathbf{h}}$ to $\{0\}\times[0,r]^2\times[-r,r]$. Moreover, by definition of φ, $\frac{\partial\varphi}{\partial t}(t,z) = \mathcal{X}(\varphi(t,z))$ so $D\theta(\varphi(t,z))\mathcal{X}(\varphi(t,z)) = \frac{\partial}{\partial x_1}$. Hence, $\theta_{\omega,h}$ straightens the vector field $\mathcal{X}$ onto the vector field $\frac{\partial}{\partial x_1}$.

This shows that Lemma 7.8 holds with $\tilde{C}_3 := C_4$ and $\tilde{n}_3 := n_4$. □

7.3 Hitting time

Lemma 7.11 (Hitting time)**.** *There exist two constants $\tilde{C}_4 \geq \tilde{C}_3$ and $\tilde{n}_4 \geq \tilde{n}_3$ such that the properties below hold for $\omega \in \,]1,+\infty[\,\setminus\{2\}$, $0 < h^u \leq (\tilde{C}_4\omega^{\tilde{n}_4})^{-1}$, $0 < h^s \leq (\tilde{C}_4 f(\omega)^{\tilde{n}_4})^{-1}$, $0 < \eta \leq 1$, $h = \min\left(h^u,h^s\right)$, $\mathbf{h}^u = \left(h^u, \eta h^{\tilde{C}_4\omega}m(\omega), \eta h^{\tilde{C}_4\omega}m(\omega)\right)$ and $\mathbf{h}^s = \left(h^s,h^s,h^s\right)$.*

1. *For every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$, the forward $\mathcal{X}$-orbit of q intersects the section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$ and its first intersection point belongs to $\mathcal{S}^{s_{i(\omega)}}_{f(\omega),\mathbf{h}^s}$. Moreover,*

$$\tau_{\omega,h^s}(q) = \tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) - x_1\left(\theta_{f(\omega),h^s}\left(\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q)\right)\right) \tag{7.30}$$

2. *For every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$ and every $t \in [0, 2\tau_{\omega,h^s}(q)]$, we have*

$$d_{\mathcal{B}}\left(\mathcal{X}^t(q), \mathcal{X}^t(\mathcal{P}^u_{\omega,h^u})\right) \le \eta \tag{7.31}$$

Proof. Setting. For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $E_{C,n}$ the set of all (ω, h^u, h^s, η) such that $\omega \in \,]1,+\infty[\, \setminus \{2\}$, $0 < h^u \le (C\omega^n)^{-1}$, $0 < h^s \le (Cf(\omega)^n)^{-1}$ and $0 < \eta \le 1$. Let $C_0 \ge \tilde{C}_3$ and $n_0 \ge \tilde{n}_3$ be large enough such that we can apply Proposition 4.2, Proposition 7.7 and Lemma 7.8 with these two constants.

For every $C \ge C_0$, every $n \ge n_0$ and every $(\omega, h^u, h^s, \eta) \in E_{C,n}$, let $h = \min\left(h^u, h^s\right)$, $\mathbf{h}^u = \left(h^u, \eta h^{C\omega} m(\omega), \eta h^{C\omega} m(\omega)\right)$, $\mathbf{h}^s = \left(h^s, h^s, h^s\right)$ and define the map $g : \mathcal{S}^u_{\omega,\mathbf{h}^u} \to \mathbb{R}$ by the formula

$$g(q) = \tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) - x_1\left(\theta_{f(\omega),h^s}\left(\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q)\right)\right)$$

Remark that $g(q)$ is well defined if and only if $\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q)$ belongs to the flow box $\mathcal{V}_{f(\omega),h^s}$. According to remark 7.9, if $g(q)$ is well defined then $\mathcal{X}^{g(q)}(q) \in \mathcal{S}^s_{f(\omega),\mathbf{h}^{\mathrm{box}}} \subset \mathcal{S}^s_{f(\omega),\mathbf{h}^s}$ where

$$\begin{aligned} r^{\mathrm{box}} &= \min\left(\frac{(h^s)^2}{\tilde{C}_3 f(\omega)^{\tilde{n}_3}}, \frac{f(\omega)-1}{2}\right) \\ \mathbf{h}^{\mathrm{box}} &= (h^s, r^{\mathrm{box}}, r^{\mathrm{box}}) \end{aligned}$$

We are going to prove that

1. If C and n are large enough, then, for every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$, $g(q)$ is well defined and $g(q) > 0$ (claim 1).

2. If C and n are large enough, then, for every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$, $g(q)$ is the first time such that the forward $\mathcal{X}$-orbit of q intersects the section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$ (claims 2 and 3). More precisely, first we prove that $g(q)$ is the first time such that the forward $\mathcal{X}$-orbit of q intersects a small section $\mathcal{S}^{s_{i(\omega)}}_{f(\omega),\tilde{\mathbf{h}}}$ (defined below) and then we extend this result to our initial section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$.

As an immediate consequence of these results, we will get that $g = \tau_{\omega,h^s}$ on $\mathcal{S}^u_{\omega,\mathbf{h}^u}$. Inequality (7.31) will be proved along the way. The main arguments are the logarithmic upper bound (7.6) of $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$, Gronwall's lemma, and the lower bound on the size of the flow box $\mathcal{V}_{f(\omega),h^s}$.

Using (3.15), it is straightforward to check that for every $\omega \in \,]1,+\infty[$, we have

$$f(\omega) - 1 \ge |\omega - 2|$$

Hence, there exist $C_1 \ge C_0$ and $n_1 \ge n_0$ such that for every $C \ge C_1$, every $n \ge n_1$ and every $(\omega, h^u, h^s, \eta) \in E_{C,n}$, we have

$$\left(C_0 f(\omega)^{n_0}\right)^2 \frac{(h^s)^4}{C_1 f(\omega)^{n_1}} m(\omega) < \frac{1}{2}\left(r^{\mathrm{box}}\right)^2 \tag{7.32}$$

For every $C \ge C_1$, every $n \ge n_1$ and every $(\omega, h^u, h^s, \eta) \in E_{C,n}$, let

$$\begin{aligned} \tilde{r} &= \frac{(h^s)^4}{C_1 f(\omega)^{n_1}} m(\omega) \\ \tilde{\mathbf{h}} &= \left(h^s, \tilde{r}, \tilde{r}\right) \end{aligned}$$

Claim 1. *There exist $\tilde{C}_4 \geq C_1$ and $\tilde{n}_4 \geq n_1$ such that for all $C \geq \tilde{C}_4$, $n \geq \tilde{n}_4$, $(\omega, h^u, h^s, 1) \in E_{C,n}$ and $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$, $g(q)$ is well defined, $g(q) > 0$ and $\mathcal{X}^{g(q)}(q) \in \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\tilde{\mathbf{h}}} \subset \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\mathbf{h}^s}$.*

Proof of claim 1. Since $\mathcal{X}$ is bounded on every compact, Gronwall's lemma implies that there exist $C_2 \geq C_1$, $n_2 \geq n_1$ and $A > 0$ such that for every $C \geq C_2$, every $n \geq n_2$, every $(\omega, h^u, h^s, \eta) \in E_{C,n}$, every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$ and every $t \in [0, 4\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})]$, we have

$$d_{\mathcal{B}}\left(\mathcal{X}^t(q), \mathcal{X}^t(\mathcal{P}^u_{\omega,h^u})\right) \leq e^{4A\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})} d_{\mathcal{B}}\left(q, \mathcal{P}^u_{\omega,h^u}\right) \tag{7.33}$$

By definition of the size of the section $\mathcal{S}^u_{\omega,\mathbf{h}^u}$ in the direction transverse to the Mixmaster attractor, the distance between q and the Mixmaster attractor in local coordinates is less than $\eta h^{C\omega} m(\omega)$. Hence, according to the estimate (4.5b) on the derivative of the local coordinate system and the mean value theorem,

$$d_{\mathcal{B}}\left(q, \mathcal{P}^u_{\omega,h^u}\right) \leq C_0 \omega^{n_0} \eta h^{C\omega} m(\omega) \tag{7.34}$$

Recall from (7.6) that

$$\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) \leq \tilde{C}_2 \omega \ln\left(\frac{1}{h}\right)$$

Take $C_3 \geq C_2$ and $n_3 \geq n_2$ such that for every $C \geq C_3$, every $n \geq n_3$ and every $(\omega, h^u, h^s, \eta) \in E_{C,n}$, we have

$$h^{-4AC_0\omega} C_0 \omega^{n_0} h^{C\omega} \leq 1 \tag{7.35}$$

It follows from (7.33), (7.34), (7.35) and (7.6) that for every $C \geq C_3$, every $n \geq n_3$, every $(\omega, h^u, h^s, \eta) \in E_{C,n}$, every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$ and every $t \in [0, 4\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})]$, we have

$$d_{\mathcal{B}}\left(\mathcal{X}^t(q), \mathcal{X}^t(\mathcal{P}^u_{\omega,h^u})\right) \leq \eta m(\omega) \tag{7.36}$$

Using (7.36) with $\eta = h^{(C-C_3)\omega}$, we get that there exists $\tilde{C}_4 \geq C_3$ such that for every $C \geq \tilde{C}_4$, every $n \geq \tilde{n}_4 := n_3$, every $(\omega, h^u, h^s, 1) \in E_{C,n}$, every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$ and every $t \in [0, 4\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})]$, we have

$$d_{\mathcal{B}}\left(\mathcal{X}^t(q), \mathcal{X}^t(\mathcal{P}^u_{\omega,h^u})\right) < \frac{1}{2} \frac{\tilde{r}r^{\text{box}}}{C_0 f(\omega)^{n_0}} \tag{7.37}$$

In particular, for $t = \tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$, we obtain

$$d_{\mathcal{B}}\left(\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q), \mathcal{P}^{s_{i(\omega)}}_{f(\omega),h^s}\right) = d_{\mathcal{B}}\left(\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q), \mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(\mathcal{P}^u_{\omega,h^u})\right) < \tilde{r}r^{\text{box}}$$

Using point 6 of Lemma 7.8, it follows from the above inequality that for every $C \geq \tilde{C}_4$, every $n \geq \tilde{n}_4$, every $(\omega, h^u, h^s, 1) \in E_{C,n}$ and every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$ we have

$$\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q) \in \theta^{-1}_{f(\omega),h^s}\left([-\tilde{r}, \tilde{r}] \times [0, \tilde{r}]^2 \times [-\tilde{r}, \tilde{r}]\right) \subset \mathcal{V}_{f(\omega),h^s}$$

Hence, g is well defined on $\mathcal{S}^u_{\omega,\mathbf{h}^u}$. Moreover, using (7.23), we get that $\mathcal{X}^{g(q)}(q) \in \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\tilde{\mathbf{h}}} \subset \mathcal{S}^s_{f(\omega),\mathbf{h}^s}$. Now, remark that according to the lower bound (7.6) on $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$,

$$\left|x_1\left(\theta_{f(\omega),h^s}\left(\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q)\right)\right)\right| < \tilde{r} < \frac{1}{2}r^{\text{box}} \leq \frac{1}{2\tilde{C}_3} \leq \frac{1}{2\tilde{C}_2} \leq \frac{\omega}{2\tilde{C}_2} \leq \frac{1}{2}\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) \tag{7.38}$$

It follows that $g(q) > 0$. Hence, the forward $\mathcal{X}$-orbit of q intersects the section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$. This concludes the proof of claim 1. □

Let us fix $C \geq \tilde{C}_4$, $n \geq \tilde{n}_4$, $(\omega, h^u, h^s, 1) \in E_{C,n}$ and $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$ until the end of this proof.

Claim 2. *$g(q)$ is the time of first intersection of the forward $\mathcal{X}$-orbit of q with the section $\mathcal{S}^{s_{i(\omega)}}_{f(\omega),\tilde{\mathbf{h}}}$.*

Proof of claim 2. Let us denote by $t_{\min} \in [0, g(q)]$ the time of first intersection of the forward $\mathcal{X}$-orbit of q with the section $\mathcal{S}^{s_{i(\omega)}}_{f(\omega),\tilde{\mathbf{h}}}$. We have $t_{\min} = g(q)$ if and only if

$$t_{\min} - \tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) = -x_1\left(\theta_{f(\omega),h^s}\left(\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q)\right)\right)$$

Moreover

$$\mathcal{X}^{t_{\min}-\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q)) = \mathcal{X}^{t_{\min}}(q) \in \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\tilde{\mathbf{h}}} \subset \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\mathbf{h}^{\mathrm{box}}}$$

and $-x_1\left(\theta_{f(\omega),h^s}\left(\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q)\right)\right)$ is, according to remark 7.9, the unique time $t \in \left[-r^{\mathrm{box}}, r^{\mathrm{box}}\right]$ such that $\mathcal{X}^t(\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q)) \in \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\mathbf{h}^{\mathrm{box}}}$. Hence, it is sufficient to prove that

$$\left|t_{\min} - \tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})\right| < r^{\mathrm{box}} \tag{7.39}$$

According to (7.38), we have

$$t_{\min} \leq g(q) \leq \tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) + \left|x_1\left(\theta_{f(\omega),h^s}\left(\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(q)\right)\right)\right| \leq 2\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) \tag{7.40}$$

Using (7.37), (7.32), the estimate (4.5b) on the local coordinate system ξ and the mean value theorem, we get

$$d_{\mathcal{B}}\left(\mathcal{X}^{t_{\min}}(\mathcal{P}^u_{\omega,h^u}), \mathcal{P}^{s_{i(\omega)}}_{f(\omega),h^s}\right) \leq d_{\mathcal{B}}\left(\mathcal{X}^{t_{\min}}(q), \mathcal{P}^{s_{i(\omega)}}_{f(\omega),h^s}\right) + d_{\mathcal{B}}\left(\mathcal{X}^{t_{\min}}(q), \mathcal{X}^{t_{\min}}(\mathcal{P}^u_{\omega,h^u})\right) < \left(r^{\mathrm{box}}\right)^2$$

Hence, using point 6 of Lemma 7.8, $\mathcal{X}^{t_{\min}}(\mathcal{P}^u_{\omega,h^u}) \in \mathcal{V}_{f(\omega),h^s}$. Moreover, the type II orbit $\mathcal{O}_{\mathcal{P}_\omega \to \mathcal{P}_{f(\omega)}}$ passes through the section $\mathcal{S}^{s_{i(\omega)}}_{f(\omega),\tilde{\mathbf{h}}}$ exactly one time so, according to remark 7.9,

$$-x_1\left(\theta_{f(\omega),h^s}\left(\mathcal{X}^{t_{\min}}(\mathcal{P}^u_{\omega,h^u})\right)\right)$$

is the unique time $t \in \mathbb{R}$ such that $\mathcal{X}^t(\mathcal{X}^{t_{\min}}(\mathcal{P}^u_{\omega,h^u})) \in \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\tilde{\mathbf{h}}}$ and it satisfies

$$\left|x_1\left(\theta_{f(\omega),h^s}\left(\mathcal{X}^{t_{\min}}(\mathcal{P}^u_{\omega,h^u})\right)\right)\right| < r^{\mathrm{box}} \tag{7.41}$$

Since

$$\mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})-t_{\min}}(\mathcal{X}^{t_{\min}}(\mathcal{P}^u_{\omega,h^u})) = \mathcal{X}^{\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})}(\mathcal{P}^u_{\omega,h^u})) = \mathcal{P}^{s_{i(\omega)}}_{f(\omega),h^s} \in \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\tilde{\mathbf{h}}}$$

it follows that

$$\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) - t_{\min} = -x_1\left(\theta_{f(\omega),h^s}\left(\mathcal{X}^{t_{\min}}(\mathcal{P}^u_{\omega,h^u})\right)\right) \tag{7.42}$$

Hence, (7.39) is a consequence of (7.41) and (7.42). This concludes the proof of claim 2. □

We now extend claim 2 to the full section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$.

Claim 3. *$g(q)$ is the time of first intersection of the forward $\mathcal{X}$-orbit of q with the section $\mathcal{S}^s_{f(\omega),\mathbf{h}^s}$.*

Proof of claim 3. Let $j(\omega) = 2$ if $i(\omega) = 1$ and $j(\omega) = 1$ if $i(\omega) = 2$. By definition, $\tau_{\omega,h^s}(q) \leq g(q)$. Assume that $\tau_{\omega,h^s}(q) < g(q)$. This implies that either $\mathcal{X}^{\tau_{\omega,h^s}(q)}(q) \in \mathcal{S}^{s_{j(\omega)}}_{f(\omega),\mathbf{h}^s}$ or $\mathcal{X}^{\tau_{\omega,h^s}(q)}(q) \in \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\mathbf{h}^s} \setminus \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\tilde{\mathbf{h}}}$, otherwise it would contradict claim 2. According to (7.40), we can use (7.37) to get

$$d_{\mathcal{B}}\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(q), \mathcal{X}^{\tau_{\omega,h^s}(q)}(\mathcal{P}^u_{\omega,h^u})\right) < \frac{1}{2}\frac{\tilde{r}r^{\mathrm{box}}}{C_0 f(\omega)^{n_0}} \tag{7.43}$$

According to the estimate (4.5b) on the local coordinate system ξ and the mean value theorem, we have

$$\left\|\xi\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(q)\right) - \xi\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(\mathcal{P}^u_{\omega,h^u})\right)\right\|_\infty \leq C_0 f(\omega)^{n_0} d_{\mathcal{B}}\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(q), \mathcal{X}^{\tau_{\omega,h^s}(q)}(\mathcal{P}^u_{\omega,h^u})\right) \tag{7.44}$$

We are now going to treat the two cases differently.

Case $\mathcal{X}^{\tau_{\omega,h^s}(q)}(q) \in \mathcal{S}^{s_{j(\omega)}}_{f(\omega),\mathbf{h}^s}$. Remark that the orbit of the Wainwright-Hsu vector field $\mathcal{X}$ starting at $\mathcal{P}^u_{\omega,h^u}$ is a type II orbit passing through the section $\mathcal{S}^{s_{i(\omega)}}_{f(\omega),\mathbf{h}^s}$. Hence,

$$h^s \leq \left| x_{s_{j(\omega)}}\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(q)\right) - x_{s_{j(\omega)}}\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(\mathcal{P}^u_{\omega,h^u})\right)\right| \leq \left\|\xi\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(q)\right) - \xi\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(\mathcal{P}^u_{\omega,h^u})\right)\right\|_\infty \tag{7.45}$$

It follows from (7.43), (7.44) and (7.45) that $h^s \leq \frac{1}{2}\tilde{r}r^{\mathrm{box}}$, which is absurd.

Case $\mathcal{X}^{\tau_{\omega,h^s}(q)}(q) \in \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\mathbf{h}^s} \setminus \mathcal{S}^{s_{i(\omega)}}_{f(\omega),\tilde{\mathbf{h}}}$. In that case, we have

$$\left\|\xi\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(q)\right) - P^{s_{i(\omega)}}_{f(\omega),h^s}\right\|_\infty > \tilde{r}$$

and

$$x_{s_{i(\omega)}}\left(\xi\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(q)\right)\right) = x_{s_{i(\omega)}}\left(P^{s_{i(\omega)}}_{f(\omega),h^s}\right) = h^s$$

Moreover, since the orbit of the Wainwright-Hsu vector field $\mathcal{X}$ starting at $\mathcal{P}^u_{\omega,h^u}$ is a type II orbit passing through the section $\mathcal{S}^{s_{i(\omega)}}_{f(\omega),h^s}$ at the point $\mathcal{P}^{s_{i(\omega)}}_{f(\omega),h^s}$, it follows that $\xi\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(\mathcal{P}^u_{\omega,h^u})\right)$ and $P^{s_{i(\omega)}}_{f(\omega),h^s}$ have the same coordinates except for the coordinate $x_{s_{i(\omega)}}$. Hence,

$$\left\|\xi\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(q)\right) - \xi\left(\mathcal{X}^{\tau_{\omega,h^s}(q)}(\mathcal{P}^u_{\omega,h^u})\right)\right\|_\infty > \tilde{r} \tag{7.46}$$

It follows from (7.43), (7.44) and (7.46) that $\tilde{r} \leq \frac{1}{2}\tilde{r}r^{\mathrm{box}}$, which is absurd. This concludes the proof of claim 3. ☐

It follows that $\tau_{\omega,h^s}(q) = g(q)$. To finish the proof, remark that (7.31) is a consequence of estimates (7.36) and (7.38). ☐

7.4 Control of the transition map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$

With the context and notations of Lemma 7.11, item 1 of Lemma 7.11 implies that the map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ is well defined at every point of the section $S^u_{\omega,\mathbf{h}^u}$ and is C^2. Recall that for every $y \in S^u_{\omega,\mathbf{h}^u}$,

$$\Psi^A_{\omega,\mathbf{h}^u,\mathbf{h}^s}(y_u, y_{s_1}, y_{s_2}, y_c) = \Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s} \circ \mathrm{Proj}_{\mathrm{A}}(y_u, y_{s_1}, y_{s_2}, y_c) = \Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}(y_u, 0, 0, y_c)$$

Using standard Hadamard's lemma, we get that there exists a C^1 map $\Psi^\Delta_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ from $S^u_{\omega,\mathbf{h}^u}$ into the space of (4×2) real valued matrices such that for every $y \in S^u_{\omega,\mathbf{h}^u}$,

$$\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}(y_u, y_{s_1}, y_{s_2}, y_c) = \Psi^A_{\omega,\mathbf{h}^u,\mathbf{h}^s}(y_u, y_{s_1}, y_{s_2}, y_c) + \Psi^\Delta_{\omega,\mathbf{h}^u,\mathbf{h}^s}(y_u, y_{s_1}, y_{s_2}, y_c).(y_{s_1}, y_{s_2}) \tag{7.47}$$

One can think about the map $\Psi^\Delta_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ as a tool to measure the "deviation" of the transition map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ from the map $\Psi^A_{\omega,\mathbf{h}^u,\mathbf{h}^s}$. Since the map $\Psi^A_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ is essentially the Kasner map f, it amounts to study the deviation of generic orbits from type II orbits. Next lemma gives some estimates on $\Psi^\Delta_{\omega,\mathbf{h}^u,\mathbf{h}^s}$.

Lemma 7.12 (Control of $\Psi^\Delta_{\omega,\mathbf{h}^u,\mathbf{h}^s}$)**.** *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that for every $\omega \in \,]1, +\infty[\, \setminus \{2\}$, every $0 < h^u \leq (C\omega^n)^{-1}$, every $0 < h^s \leq (Cf(\omega)^n)^{-1}$, for $h = \min(h^u, h^s)$, $\mathbf{h}^u = \left(h^u, h^{C\omega}m(\omega), h^{C\omega}m(\omega)\right)$ and $\mathbf{h}^s = (h^s, h^s, h^s)$, we have*

$$\left\|\Psi^\Delta_{\omega,\mathbf{h}^u,\mathbf{h}^s}\right\|_{C^1} \leq h^{-C\omega} \tag{7.48}$$

Proof. For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $E_{C,n}$ the set of all (ω, h^u, h^s) such that $\omega \in \,]1, +\infty[\, \setminus \{2\}$, $0 < h^u \leq (C\omega^n)^{-1}$ and $0 < h^s \leq (Cf(\omega)^n)^{-1}$. For every $C \geq \tilde{C}_4$, every $n \geq \tilde{n}_4$ and every $(\omega, h^u, h^s) \in E_{C,n}$, define h, $\mathbf{h}^u$ and $\mathbf{h}^s$ as in Lemma 7.12.

According to the standard Hadamard's lemma, estimates on the k-th derivative of $\Psi^\Delta_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ follow from estimates on the $(k+1)$-th derivative of $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$. By definition of the transition map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$

and the hitting time τ_{ω,h^s}, for every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$,

$$\xi^{-1} \circ \Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s} \circ \xi(q) = \mathcal{X}^{\tau_{\omega,h^s}(q)}(q)$$

Hence, estimates on $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ are consequences of estimates on the local coordinate system ξ, the flow of the Wainwright-Hsu vector field and the hitting time τ_{ω,h^s}.

According to Proposition 7.7, $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u}) \leq -\tilde{C}_2\omega \ln h$. Moreover, $\mathcal{X}$ is bounded on every compact. Hence, Gronwall's lemma implies that there exist $C_1 \geq C_0$ and $n_1 \geq n_0$ such that for every $C \geq C_1$, every $n \geq n_1$, every $(\omega, h^u, h^s) \in E_{C,n}$, every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$ and every $t \in [0, 2\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})]$, we have

$$\left\| D_{t,q}\mathcal{X}^t(q) \right\| \leq h^{-C\omega} \quad \text{and} \quad \left\| D^2_{t,q}\mathcal{X}^t(q) \right\| \leq h^{-C\omega} \tag{7.49}$$

According to the expression of the hitting time (7.30), the estimate (7.21) on the derivative of the flow box coordinates and the preceding control on the flow of the Wainwright-Hsu vector field $\mathcal{X}$, there exist $C_2 \geq C_1$ and $n_2 \geq n_1$ such that for every $C \geq C_2$, every $n \geq n_2$, every $(\omega, h^u, h^s) \in E_{C,n}$ and every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$, we have

$$\left\| D\tau_{\omega,h^s}(q) \right\| \leq h^{-C\omega} \quad \text{and} \quad \left\| D^2\tau_{\omega,h^s}(q) \right\| \leq h^{-C\omega} \tag{7.50}$$

Using (7.49) and (7.50), we get some estimates on the first and second derivatives of $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$: there exist $C_3 \geq C_2$ and $n_3 \geq n_2$ such that for every $C \geq C_3$, every $n \geq n_3$, every $(\omega, h^u, h^s) \in E_{C,n}$ and every $q \in \mathcal{S}^u_{\omega,\mathbf{h}^u}$, we have

$$\left\| D(\xi^{-1} \circ \Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s} \circ \xi)(q) \right\| \leq h^{-C\omega} \quad \text{and} \quad \left\| D^2(\xi^{-1} \circ \Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s} \circ \xi)(q) \right\| \leq h^{-C\omega} \tag{7.51}$$

Estimates (7.51) together with estimates (4.5) on the local coordinate system ξ yield some estimates on the first and second derivatives of $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$. These estimates give the desired estimates on $\Psi^\Delta_{\omega,\mathbf{h}^u,\mathbf{h}^s}$. □

At this point, Proposition 7.1 on the transition map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ must be seen as a straightforward consequence of Lemma 7.12.

Proof of Proposition 7.1. Let $C_0 \geq \tilde{C}_4$ and $n_0 \geq \tilde{n}_4$ be large enough such that we can apply Lemma 7.12 with these two constants. For every $C > 0$ and $n \in \mathbb{N}$, we denote by $E_{C,N}$ the set of all $(\omega, h^u, h^s, h_\perp, y, \tilde{y})$ such that $\omega \in]1, +\infty[\setminus \{2\}$, $0 < h^u \leq (C\omega^n)^{-1}$, $0 < h^s \leq (Cf(\omega)^n)^{-1}$, $0 < h_\perp \leq h^{C\omega}$ and $y, \tilde{y} \in S^u_{\omega,\mathbf{h}^u}$ where $h = \min(h^u, h^s)$ and $\mathbf{h}^u = (h^u, h_\perp m(\omega), h^{C\omega} m(\omega))$. For every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, h^u, h^s, h_\perp, y, \tilde{y}) \in E_{C,N}$, we use the notations $\mathbf{h}^s = (h^s, h^s, h^s)$, $\Psi = \Psi_{\omega,\mathbf{h}^u,\mathbf{h}^s}$, $\Psi^A = \Psi^A_{\omega,\mathbf{h}^u,\mathbf{h}^s}$ and $\Psi^\Delta = \Psi^\Delta_{\omega,\mathbf{h}^u,\mathbf{h}^s}$.

Let $C \geq C_0$, $n \geq n_0$ and $(\omega, h^u, h^s, h_\perp, y, \tilde{y}) \in E_{C,N}$. According to Lemma 7.11, Ψ is well defined. According to (7.47), we have

$$\Psi(y) - \Psi^A(y) = \Psi^\Delta(y) y_{s_1,s_2}$$

where $y_{s_1,s_2} = (y_{s_1}, y_{s_2})$. Hence, using (7.48), we get

$$\left\| \Psi(y) - \Psi^A(y) \right\|_{/\!/} \leq \left\| \Psi^\Delta(y) y_{s_1,s_2} \right\|_{/\!/} \leq \left\| \Psi^\Delta \right\|_{C^1} \left\| y_{s_1,s_2} \right\|_\infty \leq h^{-C\omega} h_\perp$$

This proves estimate (7.3). Estimate (7.2) is proven analogously. According to (7.47), we have

$$(\Psi(y) - \Psi(\tilde{y})) - \left(\Psi^A(y) - \Psi^A(\tilde{y})\right) = \left(\Psi^\Delta(y) - \Psi^\Delta(\tilde{y})\right) y_{s_1,s_2} + \Psi^\Delta(\tilde{y}) \left(y_{s_1,s_2} - \tilde{y}_{s_1,s_2}\right)$$

Moreover,

$$\begin{aligned}
\left\| \left(\Psi^\Delta(y) - \Psi^\Delta(\tilde{y})\right) y_{s_1,s_2} + \Psi^\Delta(\tilde{y}) \left(y_{s_1,s_2} - \tilde{y}_{s_1,s_2}\right) \right\|_\infty &\leq \left\| \Psi^\Delta \right\|_{C^1} \left(\|y - \tilde{y}\|_\infty h_\perp + \left\| y_{s_1,s_2} - \tilde{y}_{s_1,s_2} \right\|_\infty \right) \\
&\leq \left\| \Psi^\Delta \right\|_{C^1} \left(\|y - \tilde{y}\|_\infty h_\perp + \|y - \tilde{y}\|_\perp \right) \\
&\leq 2h^{-C\omega} \left(\|y - \tilde{y}\|_{/\!/} h_\perp + \|y - \tilde{y}\|_\perp \right)
\end{aligned}$$

using (7.48). There exist $C_1 \geq C_0$ and $n_1 \geq n_0$ such that for every $(\omega, h^u, h^s, h_\perp, y, \tilde{y}) \in E_{C,N}$, we have

$$2h^{-C_0\omega} \leq h^{-C_1\omega}$$

This proves estimates (7.4) and (7.5). This shows that Proposition 7.1 holds true with $\tilde{C}_1 := C_1$ and $\tilde{n}_1 := n_1$. ☐

We finish this section with a short proof of Proposition 7.4.

Proof of Proposition 7.4. Using the notations of Proposition 7.4, this is a straightforward consequence of the Gronwall's estimate (7.31) and the fact that $\tau_{\omega,h^s}(q)$ is uniformly arbitrary close to $\tau_{\omega,h^s}(\mathcal{P}^u_{\omega,h^u})$ when η is taken small enough. ☐

Chapter 8

Dynamics along an epoch

The goal of this chapter is to give some estimates on the epoch transition map $\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}}$ (see definition 5.17). Recall that this map describes the behaviour of the orbits of the Wainwright-Hsu vector field between the sections $\mathcal{S}^s_{\omega,\mathbf{h}_\omega}$ and $\mathcal{S}^s_{f(\omega),\mathbf{h}_{f(\omega)}}$. In other words, it describes the behaviour of the orbits between the moment they arrive in the neighbourhood of the point $\mathcal{P}_\omega$ and the moment they arrive in the neighbourhood of the point $\mathcal{P}_{f(\omega)}$.

Our first task will be to prove that we can write the epoch transition map $\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}}$ as a composition

$$\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}} = \Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}} \circ \Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u} \tag{8.1}$$

of the transition maps $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}}$ and $\Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}$ studied in the two preceding chapters. This amounts to prove that, for $\mathbf{h}_\omega$, $\mathbf{h}^u$ and $\mathbf{h}_{f(\omega)}$ well chosen, any orbit starting in the section $S^s_{\omega,\mathbf{h}_\omega}$ will pass through the section $S^u_{\omega,\mathbf{h}^u}$ before hitting the section $S^s_{f(\omega),\mathbf{h}_{f(\omega)}}$.

Once the relation (8.1) will be proven, we will be able to combine the estimates proven in the two preceding chapters and deduce from them some estimates on the map $\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}}$. More precisely, we will show that this map is a strong contraction in the direction transversal to the Mixmaster attractor while it is very close to the Kasner map f in the direction tangential to the Mixmaster attractor. The key point is the fact that the super-linear contraction of $\Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}$ in the direction transversal to the Mixmaster attractor dominates everything else.

From now on, we will systematically use the continued fraction expansion of the Kasner parameter ω. This will make our results easier to formulate and to read. Recall that we denote by $[k_0; k_1, k_2, k_3, \dots]$ the unique (infinite) continued fraction

$$k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \dots}}}$$

Moreover, we denote by $[k_0(\omega); k_1(\omega), k_2(\omega), \dots]$ the continued fraction expansion of a real number $\omega \in\,]0, +\infty[\, \setminus \mathbb{Q}$. Also, recall that

$$m(\omega) = \min\left(1, (\omega-2)^2\right), \quad i(\omega) = \begin{cases} 1 & \text{if } \omega > 2 \\ 2 & \text{if } 1 < \omega < 2 \end{cases}$$

Remind that Proj_A is the projection on the Mixmaster attractor (see definition 5.1) and $\Phi^A_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}} = \Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}} \circ \mathrm{Proj}_\mathrm{A}$.

Now, let us introduce some constants that will be used to quantify the dilatation properties of the Kasner map. Define, for $\omega \in\,]1, +\infty[\, \setminus \mathbb{Q}$,

$$K_f(\omega) \stackrel{\text{def}}{=} \begin{cases} \frac{36}{25} & \text{if } 1 < \omega < \frac{5}{3} \\ 1 & \text{if } \omega > \frac{5}{3} \end{cases} \tag{8.2}$$

$$\mathrm{Lip}_f(\omega) \stackrel{\text{def}}{=} \begin{cases} 16k_1(\omega)^2 & \text{if } 1<\omega<2 \\ 1 & \text{if } \omega>2 \end{cases} \tag{8.3}$$

and

$$\mathrm{Lip}_{f'}(\omega) \stackrel{\text{def}}{=} \begin{cases} 128k_1(\omega)^3 & \text{if } 1<\omega<2 \\ 0 & \text{if } \omega>2 \end{cases} \tag{8.4}$$

We will prove that, on the one hand, $K_f(\omega)$ is a local expansion constant for the Kasner map and, on the other hand, $\mathrm{Lip}_f(\omega)$ and $\mathrm{Lip}_{f'}(\omega)$ are some local Lipschitz constants for the Kasner map and its derivative in the neighbourhood of ω.

Proposition 8.1 is the main result of this chapter, it shows that the decisive parameter to control the epoch transition map is the size $h_\perp$ of the section $S^s_{\omega,\mathbf{h}_\omega}$ in the direction transverse to the Mixmaster attractor. Its proof does not require new ideas, it is just the concatenation of Proposition 6.1 and Proposition 7.1.

Proposition 8.1 (Control of the epoch transition map)**.** *There exist two constants* $\tilde{C}_5 \geq 1$ *and* $\tilde{n}_5 \in \mathbb{N}$ *such that the properties below hold for* $\omega \in]1,+\infty[\setminus \mathbb{Q}$, $0 < h_\omega \leq (\tilde{C}_5\omega^{\tilde{n}_5})^{-1}$, $0 < h_{f(\omega)} \leq (\tilde{C}_5 f(\omega)^{\tilde{n}_5})^{-1}$, $h = \min\left(h_\omega, h_{f(\omega)}\right)$, $0 < h_\perp \leq h^{\tilde{C}_5 k_0(\omega)^3} m(\omega)$, $\mathbf{h}_\omega = \left(h_\omega, h_\perp, h^{\tilde{C}_5 k_0(\omega)} m(\omega)\right)$ *and* $\mathbf{h}_{f(\omega)} = (h_{f(\omega)}, h_{f(\omega)}, h_{f(\omega)})$. *If* $k_0(\omega) = k_1(\omega) = 1$, *assume that* $h_\omega = h_{f(\omega)}$. *The epoch transition map*

$$\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}} : S^s_{\omega,\mathbf{h}_\omega} \to S^s_{f(\omega),\mathbf{h}_{f(\omega)}}$$

is well defined and takes its values in $S^{s_i(\omega)}_{f(\omega),\mathbf{h}_{f(\omega)}}$. *Moreover, for every* $x, \tilde{x} \in S^s_{\omega,\mathbf{h}_\omega}$, *we have the following estimates, where* $\Phi := \Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}}$ *and* $\Phi^A := \Phi^A_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}}$*:*

(Control of the distance to the Mixmaster attractor)

$$\mathrm{dist}_\infty\left(\Phi(x), A\right) = \left\|\Phi(x) - \Phi^A(x)\right\|_\perp \leq h_\perp^{\frac{k_0(\omega)+4}{k_0(\omega)+3}} \tag{8.5}$$

(Control of the drift tangential to the Mixmaster attractor)

$$\left\|\Phi(x) - \Phi^A(x)\right\|_{/\!/} \leq 2h_\perp \,\mathrm{Lip}_f(\omega) \tag{8.6}$$

(Contraction in the direction transverse to the Mixmaster attractor)

$$\|\Phi(x) - \Phi(\tilde{x})\|_\perp \leq h_\perp^{\frac{1}{k_0(\omega)+3}} \|x - \tilde{x}\|_\infty \tag{8.7}$$

(Lipschitz control in the direction tangential to the Mixmaster attractor)

$$\left\|(\Phi(x) - \Phi(\tilde{x})) - (\Phi^A(x) - \Phi^A(\tilde{x}))\right\|_{/\!/} \leq h_\perp^{\frac{1}{k_0(\omega)+3}} \|x - \tilde{x}\|_\infty + \mathrm{Lip}_f(\omega)\|x - \tilde{x}\|_\perp \tag{8.8}$$

(Expansion in the direction tangential to the Mixmaster attractor)

$$\|\Phi(x) - \Phi(\tilde{x})\|_{/\!/} \geq K_f(\omega)\|x - \tilde{x}\|_{/\!/} - h_\perp^{\frac{1}{k_0(\omega)+3}}\|x - \tilde{x}\|_\infty - \tilde{C}_5 k_0(\omega)^{\tilde{n}_5} h_\omega \|x - \tilde{x}\|_\perp \tag{8.9}$$

(Global lipschitz constant)

$$\|\Phi(x) - \Phi(\tilde{x})\|_\infty \leq 4\,\mathrm{Lip}_f(\omega)\|x - \tilde{x}\|_\infty \tag{8.10}$$

8.1 Some estimates about the Kasner map

In this section, we explore two properties of the Kasner map f: the fact that it is locally expansive and the fact that it is locally Lipschitz. Those properties are direct consequences of the explicit formula (3.15). We detail them because we need a precise local control of the Kasner map. Next proposition shows that $K_f(\omega)$ (defined by formula (8.2)) is a local expansion constant for f in the neighbourhood of ω.

Proposition 8.2 (Local expansion constant for f)**.** *For $\omega \in]1,+\infty[\setminus \mathbb{Q}$, the Kasner map f is $K_f(\omega)$-expansive on the interval $]\omega-\eta,\omega+\eta[$ where $\eta = \min\left(\frac{\omega-1}{2}, \frac{|\omega-2|}{2}\right)$.*

Proof. We divide the proof in three cases: $\omega > 2$, $\frac{5}{3} < \omega < 2$ and $1 < \omega < \frac{5}{3}$. If $\omega > 2$, then $f = \mathrm{Id}$ on $]\omega-\eta,\omega+\eta[$ according to (3.15). Using (3.15), remark that for every $x \in]1,2[$,

$$f'(x) = -\frac{1}{(x-1)^2}$$

and f is monotonous on $]1,2[$. Let $y, \tilde{y} \in]\omega-\eta,\omega+\eta[$. If $\frac{5}{3} < \omega < 2$, then

$$|f(y) - f(\tilde{y})| \geq \min_{x\in]\omega-\eta,\omega+\eta[} \left|f'(x)\right| |y-\tilde{y}| \geq \left|f'(2)\right| |y-\tilde{y}| \geq K_f(\omega)\,|y-\tilde{y}|$$

If $1 < \omega < \frac{5}{3}$, then

$$|f(y) - f(\tilde{y})| \geq \min_{x\in]\omega-\eta,\omega+\eta[} \left|f'(x)\right| |y-\tilde{y}| \geq \left|f'\left(\frac{11}{6}\right)\right| |y-\tilde{y}| \geq K_f(\omega)\,|y-\tilde{y}|$$

Hence, Proposition 8.2 has been proved for all ω. □

Next proposition shows that $\mathrm{Lip}_f(\omega)$ and $\mathrm{Lip}_{f'}(\omega)$ (see (8.3) and (8.4)) are some local Lipschitz constants for f and its derivative f' in the neighbourhood of ω.

Proposition 8.3 (Local lipschitz constant for f and f')**.** *For $\omega \in]1,+\infty[\setminus \mathbb{Q}$, the Kasner map f is $\mathrm{Lip}_f(\omega)$-Lipschitz and its derivative f' is $\mathrm{Lip}_{f'}(\omega)$-Lipschitz on the interval $]\omega-\eta,\omega+\eta[$ where $\eta = \min\left(\frac{\omega-1}{2}, \frac{|\omega-2|}{2}\right)$.*

Proof. If $\omega > 2$, then $f = \mathrm{Id}$ on $]\omega-\eta,\omega+\eta[$ according to (3.15). If $1 < \omega < 2$, then (3.15) implies that

$$\max_{x\in]\omega-\eta,\omega+\eta[} \left|f'(x)\right| \leq \left|f'\left(\frac{\omega+1}{2}\right)\right| \leq \frac{4}{(\omega-1)^2} \leq 16k_1^2$$

and

$$\max_{x\in]\omega-\eta,\omega+\eta[} \left|f''(x)\right| \leq \left|f''\left(\frac{\omega+1}{2}\right)\right| \leq \frac{16}{(\omega-1)^3} \leq 128k_1^3$$

The statement follows immediately from these inequalities and the mean value theorem. □

8.2 Travels along an epoch

In this section, we state a proposition that gives some conditions under which we can write $\Phi_\omega = \Psi_\omega \circ \Upsilon_\omega$. Equivalently, we give some conditions on $\mathbf{h}_\omega$, $\mathbf{h}^u$ and $\mathbf{h}_{f(\omega)}$ under which every orbit starting in the section $S^s_{\omega,\mathbf{h}_\omega}$ will pass through the section $S^u_{\omega,\mathbf{h}^u}$ before hitting the section $S^s_{f(\omega),\mathbf{h}_{f(\omega)}}$. There are essentially two cases depending on $\omega = [k_0; k_1, k_2, \ldots] \in]1,+\infty[\setminus \mathbb{Q}$: the first case is when $k_0 = k_1 = 1$ and the second case is when either $k_0 \geq 2$ or ($k_0 = 1$ and $k_1 \geq 2$). For the first case, we use the contraction in the direction transversal to the Mixmaster attractor. For the second case, we use the gap between the sections S^s_ω and $S^s_{f(\omega)}$ in the direction tangential to the Mixmaster attractor. The first case is special, in the sense that we need to choose more carefully the parameters for the sections than in the second case.

Lemma 8.4. *There exist two constants $C > 0$ and $n \in \mathbb{N}$ such that the properties below hold true for $\omega \in]2,+\infty[\setminus \mathbb{Q}$, $0 < h_\omega \leq (C\omega^n)^{-1}$, $0 < h_{f(\omega)} \leq (Cf(\omega)^n)^{-1}$, $h = \min(h_\omega, h_{f(\omega)})$, $\mathbf{h}_\omega = \left(h_\omega, h^{Ck_0(\omega)}m(\omega), h^{Ck_0(\omega)}m(\omega)\right)$, $\mathbf{h}^u = \left(h_\omega, h^{\tilde{C}_1\omega}m(\omega), h^{\tilde{C}_1\omega}m(\omega)\right)$ and $\mathbf{h}_{f(\omega)} = (h_{f(\omega)}, h_{f(\omega)}, h_{f(\omega)})$. If $k_0(\omega) = k_1(\omega) = 1$, assume that $h_\omega = h_{f(\omega)}$. The epoch transition map*

$$\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}} : S^s_{\omega,\mathbf{h}_\omega} \to S^s_{f(\omega),\mathbf{h}_{f(\omega)}}$$

is well defined and takes its values in $S^{s_{i(\omega)}}_{f(\omega),\mathbf{h}_{f(\omega)}}$*. The map* $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}} \circ \Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}$ *is also well defined on the section* $S^s_{\omega,\mathbf{h}_\omega}$*. Moreover,*

$$\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}} = \Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}} \circ \Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}$$

Proof. For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $E_{C,n}$ the set of all $(\omega, h_\omega, h_{f(\omega)})$ such that $\omega = [k_0; k_1, k_2, \dots] \in]1,+\infty[\setminus \mathbb{Q}$, $0 < h_\omega \leq (C\omega^n)^{-1}$, $0 < h_{f(\omega)} \leq (Cf(\omega)^n)^{-1}$ such that $h_\omega = h_{f(\omega)}$ if $k_0(\omega) = k_1(\omega) = 1$. We also define h, $\mathbf{h}_\omega$, $\mathbf{h}^u$, $\mathbf{h}_{f(\omega)}$ as in Lemma 8.4. Let $C_0 \geq 100$ and $n_0 \in \mathbb{N}$ be large enough such that we can apply Proposition 6.1, Proposition 6.15 and Proposition 7.1 with these two constants. Take $C_1 \geq C_0$ such that for every $C \geq C_1$, every $n \geq n_0$ and every $(\omega, h_\omega, h_{f(\omega)}) \in E_{C,n}$, we have

$$\left(h_\omega^{Ck_0} m(\omega)\right)^{\frac{\omega+2}{\omega+1}} h_\omega^{-1} \leq h_\omega^{\tilde{C}_1\omega} m(\omega) \tag{8.11a}$$

$$h_\omega^{Ck_0} m(\omega) \left(h_\omega C_0 \omega^{n_0} + 1\right) \leq h_\omega^{\tilde{C}_1\omega} m(\omega) \tag{8.11b}$$

Let $C \geq C_0$, $n \geq n_0$ and $(\omega, h_\omega, h_{f(\omega)}) \in E_{C,n}$. According to Proposition 7.1, $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}}$ is well defined on $S^u_{\omega,\mathbf{h}^u}$ and takes its values in $S^{s_{i(\omega)}}_{f(\omega),\mathbf{h}_{f(\omega)}}$. According to Proposition 6.1, we know that $\Upsilon_{\omega,\mathbf{h}_\omega,(h_\omega,h_\omega 2h_\omega)}$ is well defined so $\Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}$ is well defined if

$$\Upsilon_{\omega,\mathbf{h}_\omega,(h_\omega,h_\omega 2h_\omega)}\left(S^s_{\omega,\mathbf{h}_\omega}\right) \subset S^u_{\omega,\mathbf{h}^u} \tag{8.12}$$

Let $x \in S^s_{\omega,\mathbf{h}_\omega}$. According to (6.1) and (8.11a), we have

$$\left\|\Upsilon_{\omega,\mathbf{h}_\omega,(h_\omega,h_\omega 2h_\omega)}(x) - \Upsilon^A_{\omega,\mathbf{h}_\omega,(h_\omega,h_\omega 2h_\omega)}(x)\right\|_\perp \leq \left(h_\omega^{Ck_0} m(\omega)\right)^{\frac{\omega+2}{\omega+1}} h_\omega^{-1} \leq h_\omega^{\tilde{C}_1\omega} m(\omega)$$

According to (6.2), we have

$$\left\|\Upsilon_{\omega,\mathbf{h}_\omega,(h_\omega,h_\omega 2h_\omega)}(x) - x\right\|_{/\!/} \leq h_\omega^{Ck_0} m(\omega) h_\omega C_0 \omega^{n_0}$$

so, using (8.11b), we get

$$\begin{aligned}\left\|\Upsilon_{\omega,\mathbf{h}_\omega,(h_\omega,h_\omega 2h_\omega)}(x) - P^u_{\omega,h_\omega}\right\|_{/\!/} &\leq \left\|\Upsilon_{\omega,\mathbf{h}_\omega,(h_\omega,h_\omega 2h_\omega)}(x) - x\right\|_{/\!/} + \left\|x - P^u_{\omega,h_\omega}\right\|_{/\!/} \\ &\leq h_\omega^{Ck_0} m(\omega) h_\omega C_0 \omega^{n_0} + h_\omega^{Ck_0} m(\omega) \leq h_\omega^{\tilde{C}_1\omega} m(\omega)\end{aligned}$$

It follows that for every $C \geq C_1$, every $n \geq n_0$ and every $(\omega, h_\omega, h_{f(\omega)}) \in E_{C,n}$, (8.12) holds true. Hence, the maps $\Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}$ and $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}} \circ \Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}$ are well defined on the section $S^s_{\omega,\mathbf{h}_\omega}$. Moreover, the map $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}} \circ \Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}$ takes its values in $S^{s_{i(\omega)}}_{f(\omega),\mathbf{h}_{f(\omega)}}$. This implies that the epoch transition map $\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}}$ is well defined. We are left to prove that $\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}} = \Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}} \circ \Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}$. Let $x \in S^s_{\omega,\mathbf{h}_\omega}$.

First case: $x_u = 0$. According to (5.8), (5.10) and (5.12),

$$\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}}(x) = \Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}} \circ \Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}(x)$$

Second case: $x_u \neq 0$. By definition of $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}}$, we only need to prove that the orbit $t \mapsto y(t)$ of the locally renormalized Wainwright-Hsu vector field X_ω starting from x does not intersect the section $S^s_{f(\omega),\mathbf{h}_{f(\omega)}}$ before it intersects the section $S^u_{\omega,\mathbf{h}^u}$.

Assume that $k_0 = k_1 = 1$. It follows from (6.27b) that during its travel between $S^s_{\omega,\mathbf{h}_\omega}$ and $S^u_{\omega,\mathbf{h}^u}$, the orbit y satisfies $y_{s_1}(t) < h_\omega$ and $y_{s_2}(t) < h_\omega$. Since $h_\omega = h_{f(\omega)}$, $y(t)$ does not belong to the section $S^s_{f(\omega),\mathbf{h}_{f(\omega)}}$.

Assume that $k_0 = 1$ and $k_1 \geq 2$. It follows from (6.27c) that during its travel between $S^s_{\omega,\mathbf{h}_\omega}$ and $S^u_{\omega,\mathbf{h}^u}$, the orbit y satisfies $|y_c(t) - \omega| \leq 2h^C_{f(\omega)} \leq \frac{1}{8}$. Hence, $y_c(t) \leq \frac{13}{8}$. Moreover, $h_{f(\omega)} \leq \frac{1}{8}$ and $f(\omega) \geq 2$ so any point z belonging to the section $S^s_{f(\omega),\mathbf{h}_{f(\omega)}}$ must satisfy $z_c \geq 2 - \frac{1}{8} = \frac{15}{8}$. Hence, $y(t)$ does not belong to the section $S^s_{f(\omega),\mathbf{h}_{f(\omega)}}$.

Assume that $k_0 \geq 2$. Having in mind that in this case, $f(\omega) = \omega - 1$, one can repeat the above argument.

This shows that $\Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}} \circ \Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}(x)$ is the first intersection point of y with the section $S^s_{f(\omega),\mathbf{h}_{f(\omega)}}$. Hence $\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}}(x) = \Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}} \circ \Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}(x)$. □

8.3 Control of the epoch transition map $\Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}}$

In this section, we prove Proposition 8.1 using the decomposition $\Phi_\omega = \Psi_\omega \circ \Upsilon_\omega$ (see Lemma 8.4), the estimates on Υ_ω proven in Chapter 6 (see Proposition 6.1) and the estimates on Ψ_ω proven in Chapter 7 (see Proposition 7.1).

Proof of Proposition 8.1. For every $C > 0$ and every $n \in \mathbb{N}$, we denote by $E_{C,n}$ the set of all $(\omega, h_\omega, h_{f(\omega)}, h_\perp, x, \tilde{x})$ such that $\omega = [k_0; k_1, k_2, \dots] \in\ ]1, +\infty[\ \setminus \mathbb{Q}$, $0 < h_\omega \leq (C\omega^n)^{-1}$, $0 < h_{f(\omega)} \leq (Cf(\omega)^n)^{-1}$ such that $h_\omega = h_{f(\omega)}$ if $k_0(\omega) = k_1(\omega) = 1$, $0 < h_\perp \leq h^{Ck_0^3} m(\omega)$ where $h = \min\big(h_\omega, h_{f(\omega)}\big)$ and $x, \tilde{x} \in S^s_{\omega,\mathbf{h}_\omega}$ where $\mathbf{h}_\omega = (h_\omega, h_\perp, h^{Ck_0} m(\omega))$. Let $C_0 > 0$ and $n_0 \in \mathbb{N}$ be large enough such that we can apply Proposition 6.1, Proposition 7.1 and Lemma 8.4 with these two constants. For every $C \geq C_0$, every $n \geq n_0$ and every $(\omega, h_\omega, h_{f(\omega)}, h_\perp, x, \tilde{x}) \in E_{C,n}$, define $\mathbf{h}^u$ and $\mathbf{h}_{f(\omega)}$ as in Proposition 8.1 and let $\Upsilon := \Upsilon_{\omega,\mathbf{h}_\omega,\mathbf{h}^u}$, $\Psi := \Psi_{\omega,\mathbf{h}^u,\mathbf{h}_{f(\omega)}}$ and $\Phi := \Phi_{\omega,\mathbf{h}_\omega,\mathbf{h}_{f(\omega)}}$.

Step 1: estimates (8.5) *and* (8.6).

Let $C \geq C_0$, $n \geq n_0$ and $(\omega, h_\omega, h_{f(\omega)}, h_\perp, x, \tilde{x}) \in E_{C,n}$. Recall that

$$\left\|\Phi(x) - \Phi^A(x)\right\|_\perp = \mathrm{dist}_\infty\left(\Phi(x), A\right) = \|\Phi(x) - y\|_\perp$$

for any $y \in S^{s_{i(\omega)}}_{f(\omega),\mathbf{h}_{f(\omega)}} \cap A$. Hence,

$$\begin{aligned}\left\|\Phi(x) - \Phi^A(x)\right\|_\perp &= \left\|\Psi \circ \Upsilon(x) - \Psi \circ \Upsilon^A(x)\right\|_\perp && \text{using Lemma 8.4}\\ &= \left\|\Psi \circ \Upsilon(x) - \Psi^A \circ \Upsilon(x)\right\|_\perp\end{aligned}$$

It follows that

$$\begin{aligned}\left\|\Phi(x) - \Phi^A(x)\right\|_\perp &\leq \left\|\Upsilon(x) - \Upsilon^A(x)\right\|_\perp h^{-\tilde{C}_1\omega} && \text{using (7.2)}\\ &\leq h_\perp^{\frac{\omega+2}{\omega+1}} h_\omega^{-1} h^{-\tilde{C}_1\omega} && \text{using (6.1)}\end{aligned} \tag{8.13}$$

To simplify the estimate found above, let us fix $C_1 \geq C_0$ such that for every $C \geq C_1$, every $n \geq n_0$ and every $(\omega, h_\omega, h_{f(\omega)}, h_\perp, x, \tilde{x}) \in E_{C,n}$, we have

$$h_\perp^{\frac{\omega+2}{\omega+1}} h_\omega^{-1} h^{-\tilde{C}_1\omega} \leq h_\perp^{\frac{k_0+4}{k_0+3}} \tag{8.14}$$

Plugging (8.14) into (8.13), we get that estimate (8.5) holds true.

According to (6.2), there exists $C_2 \geq C_1$ such that for every $C \geq C_2$, every $n \geq n_0$ and every $(\omega, h_\omega, h_{f(\omega)}, h_\perp, x, \tilde{x}) \in E_{C,n}$, we have

$$|\Upsilon(x)_c - \omega| \leq \min\left(\frac{\omega - 1}{4}, \frac{|\omega - 2|}{2}\right) \tag{8.15}$$

Let $C \geq C_2$, $n \geq n_0$ and $(\omega, h_\omega, h_{f(\omega)}, h_\perp, x, \tilde{x}) \in E_{C,n}$. We have

$$\begin{aligned}
&\left\|\Phi(x) - \Phi^A(x)\right\|_{/\!/} \\
=&\left\|\Psi \circ \Upsilon(x) - \Psi^A \circ \Upsilon^A(x)\right\|_{/\!/} \qquad \text{using Lemma 8.4} \\
\leq&\left\|\Psi \circ \Upsilon(x) - \Psi^A \circ \Upsilon(x)\right\|_{/\!/} + \left\|\Psi^A \circ \Upsilon(x) - \Psi^A \circ \Upsilon^A(x)\right\|_{/\!/} \\
\leq&\left\|\Upsilon(x) - \Upsilon^A(x)\right\|_\perp h^{-\tilde{C}_1\omega} + \mathrm{Lip}_f(\omega)\left\|\Upsilon(x) - \Upsilon^A(x)\right\|_{/\!/} \\
&\text{using (7.3), (8.15) and proposition 8.3} \\
\leq& h_\perp^{\frac{\omega+2}{\omega+1}} h_\omega^{-1} h^{-\tilde{C}_1\omega} + \mathrm{Lip}_f(\omega) h_\perp h_\omega C_0 \omega^{n_0} \qquad \text{using (6.1) and (6.2)} \\
\leq& h_\perp^{\frac{k_0+4}{k_0+3}} + h_\perp \,\mathrm{Lip}_f(\omega) \qquad \text{using (8.14)} \\
\leq& 2h_\perp \,\mathrm{Lip}_f(\omega)
\end{aligned}$$

Hence, estimate (8.6) holds true.

Step 2: estimates (8.7)*,* (8.9) *and* (8.10). Using estimates (6.1), (6.3) and (6.4) and taking C_3 large enough, we get that

$$\begin{aligned}
&\left(\|\Upsilon(x) - \Upsilon(\tilde{x})\|_\perp + \left\|\Upsilon(x) - \Upsilon^A(x)\right\|_\perp \|\Upsilon(x) - \Upsilon(\tilde{x})\|_{/\!/}\right) h^{-\tilde{C}_1\omega} \\
\leq&\left(h_\perp^{\frac{1}{\omega+1}} h_\omega^{-1} + h_\perp^{\frac{\omega+2}{\omega+1}} h_\omega^{-1}\left(1 + C_0\omega^{n_0} h_\omega + C_0\omega^{n_0} h_\perp\right)\right) h^{-\tilde{C}_1\omega} \|x - \tilde{x}\|_\infty \qquad (8.16) \\
\leq& h_\perp^{\frac{1}{k_0+2}} h^{-C_3k_0} \|x - \tilde{x}\|_\infty
\end{aligned}$$

Plugging (8.16) into (7.4), we get that there exists $C_4 \geq C_3$ such that for every $C \geq C_4$, every $n \geq n_0$ and every $(\omega, h_\omega, h_{f(\omega)}, h_\perp, x, \tilde{x}) \in E_{C,n}$, we have

$$\|\Phi(x) - \Phi(\tilde{x})\|_\perp \leq h_\perp^{\frac{1}{k_0+2}} h^{-C_3k_0} \|x - \tilde{x}\|_\infty \leq h_\perp^{\frac{1}{k_0+3}} \|x - \tilde{x}\|_\infty$$

Hence, estimate (8.7) holds true.

Plugging (8.16) into (7.5), we get that

$$\left\|(\Phi(x) - \Phi(\tilde{x})) - \left(\Psi^A \circ \Upsilon(x) - \Psi^A \circ \Upsilon(\tilde{x})\right)\right\|_{/\!/} \leq h_\perp^{\frac{1}{k_0+2}} h^{-C_3k_0} \|x - \tilde{x}\|_\infty \qquad (8.17)$$

Recall that Ψ^A is essentially the Kasner map (see remark 5.25), hence

$$\begin{aligned}
&\|\Phi(x) - \Phi(\tilde{x})\|_{/\!/} \\
\geq&\left\|\Psi^A \circ \Upsilon(x) - \Psi^A \circ \Upsilon(\tilde{x})\right\|_{/\!/} - \left\|(\Phi(x) - \Phi(\tilde{x})) - \left(\Psi^A \circ \Upsilon(x) - \Psi^A \circ \Upsilon(\tilde{x})\right)\right\|_{/\!/} \\
\geq& K_f(\omega)\|\Upsilon(x) - \Upsilon(\tilde{x})\|_{/\!/} - h_\perp^{\frac{1}{k_0+2}} h^{-C_3k_0} \|x - \tilde{x}\|_\infty \qquad \text{using (8.15), proposition 8.2 and (8.17)}
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|\Phi(x) - \Phi(\tilde{x})\|_{/\!/} \\
\geq& K_f(\omega)\|x - \tilde{x}\|_{/\!/} - K_f(\omega) C_0\omega^{n_0} h_\omega \|x - \tilde{x}\|_\perp - K_f(\omega) C_0\omega^{n_0} h_\perp \|x - \tilde{x}\|_{/\!/} \\
&- h_\perp^{\frac{1}{k_0+2}} h^{-C_3k_0} \|x - \tilde{x}\|_\infty \qquad \text{using (6.4)} \\
\geq& K_f(\omega)\|x - \tilde{x}\|_{/\!/} - C_5 k_0^{n_0} h_\omega \|x - \tilde{x}\|_\perp - h_\perp^{\frac{1}{k_0+2}} h^{-C_5k_0} \|x - \tilde{x}\|_\infty \qquad \text{for } C_5 \text{ large enough}
\end{aligned}$$

According to the above inequality, there exists $C_6 \geq \max(C_4, C_5)$ such that for every $C \geq C_6$, every

$n \geq n_0$ and every $(\omega, h_\omega, h_{f(\omega)}, h_\perp, x, \tilde{x}) \in E_{C,n}$, we have

$$\|\Phi(x) - \Phi(\tilde{x})\|_{/\!/} \geq K_f(\omega)\, \|x - \tilde{x}\|_{/\!/} - C_5 k_0^{n_0} h_\omega \, \|x - \tilde{x}\|_\perp - h_\perp^{\frac{1}{k_0+3}} \, \|x - \tilde{x}\|_\infty$$

Hence, estimate (8.9) holds true.

We have

$$\begin{aligned}
&\|\Phi(x) - \Phi(\tilde{x})\|_{/\!/} \\
\leq &\left\|\Psi^A \circ \Upsilon(x) - \Psi^A \circ \Upsilon(\tilde{x})\right\|_{/\!/} + \left\|(\Phi(x) - \Phi(\tilde{x})) - \left(\Psi^A \circ \Upsilon(x) - \Psi^A \circ \Upsilon(\tilde{x})\right)\right\|_{/\!/} \\
\leq &\operatorname{Lip}_f(\omega)\, \|\Upsilon(x) - \Upsilon(\tilde{x})\|_{/\!/} + h_\perp^{\frac{1}{k_0+2}} h^{-C_3 k_0} \, \|x - \tilde{x}\|_\infty \\
&\text{using (8.15), Proposition 8.3 on the Kasner map and (8.17)} \\
\leq &\left(h_\perp^{\frac{1}{k_0+2}} h^{-C_3 k_0} + 3 \operatorname{Lip}_f(\omega)\right) \|x - \tilde{x}\|_\infty \qquad \text{using (6.4)} \\
\leq &4 \operatorname{Lip}_f(\omega)\, \|x - \tilde{x}\|_\infty \qquad \text{for } C \geq C_4
\end{aligned}$$

It follows from the above inequality and (8.7) that estimate (8.10) holds true.

Step 3: estimate (8.8). Let $C \geq C_6$, $n \geq n_0$ and $(\omega, h_\omega, h_{f(\omega)}, h_\perp, x, \tilde{x}) \in E_{C,n}$. We have

$$\begin{aligned}
\left\|(\Phi(x) - \Phi(\tilde{x})) - (\Phi^A(x) - \Phi^A(\tilde{x}))\right\|_{/\!/} \leq \left\|(\Phi(x) - \Phi(\tilde{x})) - \left(\Psi^A \circ \Upsilon(x) - \Psi^A \circ \Upsilon(\tilde{x})\right)\right\|_{/\!/} + \\
\left\|\left(\Psi^A \circ \Upsilon(x) - \Psi^A \circ \Upsilon(\tilde{x})\right) - \left(\Psi \circ \Upsilon^A(x) - \Psi \circ \Upsilon^A(\tilde{x})\right)\right\|_{/\!/} \quad (8.18)
\end{aligned}$$

The first term of the right hand side of (8.18) is controlled by (8.17). To control the second term of the right hand side of (8.18), let us define the map

$$\lambda : x \mapsto \Psi \circ \Upsilon^u(x) - \Psi \circ \Upsilon^A(x)$$

where $\Upsilon^u = \mathrm{Proj}_{\mathrm{A}} \circ \Upsilon$. Remark that the second term is equal to $\|\lambda(x) - \lambda(\tilde{x})\|_{/\!/}$ so we are left to apply the mean value theorem to λ. Remark that λ is continuous on $S^s_{\omega,\mathbf{h}_\omega}$ and smooth on $\operatorname{Int} S^s_{\omega,\mathbf{h}_\omega}$ (we do not know if it is smooth on the hyperplane $\{x_u = 0\}$). Let us identify the tangent space $\mathrm{T}_x\, S^{s_1}_{\omega,\mathbf{h}_\omega} = \operatorname{Vect} \frac{\partial}{\partial x_u} \oplus \operatorname{Vect} \frac{\partial}{\partial x_{s_2}} \oplus \operatorname{Vect} \frac{\partial}{\partial x_c}$ with $\mathbb{R}^3$ (and analogously for $\mathrm{T}_x\, S^{s_2}_{\omega,\mathbf{h}_\omega}$, permuting the roles of s_1 and s_2). Assume that $x, \tilde{x} \in \operatorname{Int} S^s_{\omega,\mathbf{h}_\omega}$ and $x \neq \tilde{x}$. We will only prove estimate (8.8) in the case where $\|x - \tilde{x}\|_\perp \leq \|x - \tilde{x}\|_{/\!/}$ (this is the only case useful later on and the other case is similar). Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ such that $\|(v_1, v_2)\|_\infty \leq \alpha\, |v_3|$, where $\alpha = \frac{\|x-\tilde{x}\|_\perp}{\|x-\tilde{x}\|_{/\!/}}$. We have

$$D\lambda(x).v = \left(D\Psi(\Upsilon^u(x)) - D\Psi(\Upsilon^A(x))\right) D\Upsilon^u(x).v + D\Psi(\Upsilon^A(x)) \left(D\Upsilon^u(x).v - D\Upsilon^A(x).v\right)$$

Recall that

$$\Upsilon^u(x) = (h_\omega, 0, 0, \Upsilon(x)_c)$$

Using (5.12), (6.4) and Proposition 8.3 (with (8.15)), we get

$$\begin{aligned}
\left\|D\Psi(\Upsilon^A(x)) \left(D\Upsilon^u(x).v - D\Upsilon^A(x).v\right)\right\|_{/\!/} &\leq \operatorname{Lip}_f(\omega) \left\|D\Upsilon^u(x).v - D\Upsilon^A(x).v\right\|_{/\!/} \\
&\leq \operatorname{Lip}_f(\omega) \left(\alpha + C_0 \omega^{n_0} h_\perp\right) \|v\|_\infty
\end{aligned}$$

Using (5.12), (6.2), (6.4) and Proposition 8.3 (with (8.15)), we get

$$\begin{aligned}
\left\|\left(D\Psi(\Upsilon^u(x)) - D\Psi(\Upsilon^A(x))\right) D\Upsilon^u(x).v\right\|_{/\!/} &\leq \operatorname{Lip}_{f'}(\omega)\, \|\Upsilon(x) - x\|_{/\!/} \, \|D\Upsilon^u(x).v\|_{/\!/} \\
&\leq \operatorname{Lip}_{f'}(\omega) h_\perp h_\omega C_0 \omega^{n_0} \left(1 + C_0 \omega^{n_0} (h_\omega \alpha + h_\perp)\right) \|v\|_\infty
\end{aligned}$$

There exist $C_8 \geq C_7 \geq C_6$ such that for every $C \geq C_8$, every $n \geq n_0$ and every $(\omega, h_\omega, h_{f(\omega)}, h_\perp, x, \tilde{x}) \in$

$E_{C,n}$, we have

$$\begin{aligned} \mathrm{Lip}_f(\omega) C_0 \omega^{n_0} &\leq h^{-C_7 k_0} \\ \mathrm{Lip}_{f'}(\omega) h_\omega C_0 \omega^{n_0} \left(1 + C_0 \omega^{n_0} (h_\omega \alpha + h_\perp)\right) &\leq h^{-C_7 k_0} \\ 3 h_\perp^{\frac{1}{k_0+2}} h^{-C_7 k_0} &\leq h_\perp^{\frac{1}{k_0+3}} \end{aligned}$$

Applying the mean value theorem to the last coordinate of λ, it follows that

$$\begin{aligned} &\left\| (\Phi(x) - \Phi(\tilde{x})) - (\Phi^A(x) - \Phi^A(\tilde{x})) \right\|_{/\!/} \\ \leq & h_\perp^{\frac{1}{k_0+2}} h^{-C_3 k_0} \|x - \tilde{x}\|_\infty + \mathrm{Lip}_f(\omega) \|x - \tilde{x}\|_\perp + 2 h_\perp h^{-C_7 k_0} \|x - \tilde{x}\|_\infty \\ \leq & h_\perp^{\frac{1}{k_0+3}} \|x - \tilde{x}\|_\infty + \mathrm{Lip}_f(\omega) \|x - \tilde{x}\|_\perp \end{aligned}$$

Hence, estimate (8.8) holds true on $\mathrm{Int}\, S^s_{\omega,\mathbf{h}_\omega}$ and then on $S^s_{\omega,\mathbf{h}_\omega}$ by continuity. To conclude, Proposition 8.1 holds true with $\tilde{C}_5 = C_8$ and $\tilde{n}_5 = n_0$. □

Chapter 9

Dynamics along an era

The goal of this section is to give some estimates on the era return map $\bar{\Phi}_h : S_h \to S_h$ and the double era return map $\hat{\Phi}_h : S_h \to S_h$ (see definitions 5.8 and 5.9). Recall that $\bar{\Phi}_h : S_h \to S_h$ is essentially the first return map of the orbits of the Wainwright-Hsu vector field on the global section S_h and that $\hat{\Phi}_h : S_h \to S_h$ is just the square of $\bar{\Phi}_h$.

Our first task will be to prove that, for any $\omega \in \,]1,2[\, \setminus \mathbb{Q}$, we can write the era transition map $\bar{\Phi}_{\omega,\mathbf{h}}$ as a composition of $k_1(\omega)$ epoch transition maps,

$$\bar{\Phi}_{\omega,\mathbf{h}} = \Phi_{k_1(\omega)-1} \circ \cdots \circ \Phi_0 \tag{9.1}$$

where Φ_j is the epoch transition map from a section $S^s_{f^j(\omega),\mathbf{h}_j}$ at the entrance of a neighbourhood of $P_{f^j(\omega)}$ to a section $S^s_{f^{j+1}(\omega),\mathbf{h}_{j+1}}$ at the entrance of a neighbourhood of $P_{f^{j+1}(\omega)}$.

Once the relation (9.1) will be proven, we will be able to use the estimates proven in the preceding section on the epoch transition maps (see Proposition 8.1) to get some estimates on the era transition map $\bar{\Phi}_{\omega,\mathbf{h}}$ (see Proposition 9.4). The main technical difficulty will be to set up an induction on the length of the era. Analogously to the epoch transition maps, we will show that the era transition map $\bar{\Phi}_{\omega,\mathbf{h}}$ is a strong contraction in the direction transversal to the Mixmaster attractor while it is very close to the era Kasner map $\bar{f}$ (see (3.16)) in the direction tangential to the Mixmaster attractor.

We want to prove that the era transition map $\bar{\Phi}_{\omega,\mathbf{h}}$ admits some hyperbolic properties. Unfortunately, it does not expand enough in the direction tangential to the Mixmaster attractor. Indeed, the era transition map is "close" to the era Kasner map $\bar{f}$ in the direction tangential to the Mixmaster attractor and $\bar{f}$ does not expand uniformly in the neighbourhood of $\omega = 2$. Nevertheless, since $\bar{f}$ expands uniformly on every interval $]1, 2-\epsilon[$ and since $\bar{f}\left(]2-\epsilon, 2[\right) \subset \,]1, 2-\epsilon[$ for ϵ small enough, it follows that $\hat{f} := \bar{f} \circ \bar{f}$ expands uniformly on $]1,2[$. This is the reason why we introduce the double era transition map $\hat{\Phi}_{\omega,\mathbf{h}}$.

Before we give the estimates on the double era transition map $\hat{\Phi}_{\omega,\mathbf{h}}$, we need some definitions. Let $\hat{K}_{\hat{f}} := \frac{36}{25}$. Let us explain why $\hat{K}_{\hat{f}}$ is a local expansion constant for the double era Kasner map $\hat{f}$. Recall that we denote by $K_f(\omega)$ a local expansion constant for the Kasner map f in the neighbourhood of ω (see Proposition 8.2). Let $\omega = [1; k_1, k_2, \dots] \in \,]1,2[\, \setminus \mathbb{Q}$ and $1 \le j \le k_1 - 1$. Using formula (8.2), we get that

$$K_f(f^j(\omega)) = 1$$

Hence,

$$\prod_{j=0}^{k_1-1} K_f(f^j(\omega)) = K_f(\omega)$$

As a consequence, $K_f(\omega)$ is also a local expansion constant for the era Kasner map $\bar{f}$ in the neighbourhood of ω. As a consequence, $K_f(\bar{f}(\omega))K_f(\omega)$ is a local expansion constant for the double era Kasner map $\hat{f}$ in the neighbourhood of ω. Using formula (8.2), it is easy to check that

$$K_f(\bar{f}(\omega))K_f(\omega) \ge \hat{K}_{\hat{f}}$$

We are going to prove that the double era transition map is, as the double era Kasner map, expansive in the direction tangent to the Mixmaster attractor with a slightly lesser constant, say

$$K_c \stackrel{\text{def}}{=} \frac{1 + \hat{K}_{\hat{f}}}{2} \tag{9.2}$$

Later on (see chapter 10), we will show that there exists a cone field invariant by the double era transition map, say of width $\hat{\sigma}$. This invariant cone field will allow us to define a graph transformation that maps $\hat{\sigma}$-Lipschitz graphs to $\hat{\sigma}$-Lipschitz graphs. We will construct a local stable manifold for the double era transition map as an invariant graph for the graph transformation. The condition for this transformation graph to be a contraction mapping is

$$K_c \left(1 - \hat{\sigma}^2\right) > 1 \tag{9.3}$$

Hence, we fix now a positive constant $\hat{\sigma}$ satisfying (9.3) and we will prove an expansion estimate for the double era transition map that is adapted to this particular constant. Remark that $\hat{\sigma} < 1/2$.

Definition 9.1. For any $\omega \in \,]1, 2[\, \setminus \mathbb{Q}$, let us define

$$s_2(\omega) \stackrel{\text{def}}{=} k_1(\omega)^2 + k_2(\omega)^2 + k_3(\omega)^2 + k_4(\omega)^2$$
$$s_4(\omega) \stackrel{\text{def}}{=} k_1(\omega)^4 + k_2(\omega)^4 + k_3(\omega)^4 + k_4(\omega)^4$$

Define, for any $\omega \in \,]1, 2[\, \setminus \mathbb{Q}$,

$$\hat{\imath}(\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } k_2(\omega) \geq 2 \\ 2 & \text{if } k_2(\omega) = 1 \end{cases} \tag{9.4}$$

Proposition 9.2 (Double era transition map). *There exist two constants $\tilde{C}_8 \geq 1$ and $\hat{h} > 0$ such that the properties below hold for every $\omega \in \,]1, 2[\, \setminus \mathbb{Q}$ and $0 < h_\perp \leq e^{-\tilde{C}_8 s_4(\omega)}$. Let $\mathbf{h} = \left(\hat{h}, h_\perp, e^{-\tilde{C}_8 s_2(\omega)}\right)$. The double era transition map*

$$\hat{\Phi}_{\omega,\mathbf{h}} : S^s_{\omega,\mathbf{h}} \subset S_{\hat{h}} \to S_{\hat{h}}$$

is well defined and takes its values in $S_{\hat{h}}^{s_{\hat{\imath}(\omega)}}$. Moreover, for every $x, \tilde{x} \in S^s_{\omega,\mathbf{h}}$, we have the following estimates, where $\hat{\Phi} := \hat{\Phi}_{\omega,\mathbf{h}}$ and $\hat{\Phi}^A := \hat{\Phi}^A_{\omega,\mathbf{h}}$:

(Control of the distance to the Mixmaster attractor)

$$\text{dist}_\infty\left(\hat{\Phi}(x), A\right) = \left\|\hat{\Phi}(x) - \hat{\Phi}^A(x)\right\|_\perp \leq h_\perp^{1 + \frac{k_1(\omega)}{4} + \frac{k_2(\omega)}{4}} \tag{9.5}$$

(Control of the drift tangential to the Mixmaster attractor)

$$\left\|\hat{\Phi}(x) - \hat{\Phi}^A(x)\right\|_{/\!/} \leq 306 h_\perp k_1(\omega)^2 k_2(\omega)^2 \tag{9.6}$$

(Contraction in the direction transverse to the Mixmaster attractor)

$$\left\|\hat{\Phi}(x) - \hat{\Phi}(\tilde{x})\right\|_\perp \leq h_\perp^{\frac{k_1(\omega)}{100} + \frac{k_2(\omega)}{100}} \|x - \tilde{x}\|_\infty \tag{9.7}$$

(Lipschitz control in the direction tangential to the Mixmaster attractor)

$$\left\|\left((\hat{\Phi}(x) - \hat{\Phi}(\tilde{x})\right) - \left(\hat{\Phi}^A(x) - \hat{\Phi}^A(\tilde{x})\right)\right\|_{/\!/} \leq$$
$$\left(h_\perp^{\frac{1}{26 k_1(\omega)}} + h_\perp^{\frac{1}{26 k_2(\omega)}}\right) \|x - \tilde{x}\|_\infty + 16^2 k_1(\omega)^2 k_2(\omega)^2 \|x - \tilde{x}\|_\perp \tag{9.8}$$

(Expansion in the direction tangent to the Mixmaster attractor)

$$\left\|\hat{\Phi}(x) - \hat{\Phi}(\tilde{x})\right\|_{/\!/} \geq \hat{K}_{\hat{f}} \left\|x - \tilde{x}\right\|_{/\!/} - \frac{\hat{K}_{\hat{f}} - K_c}{1 + \frac{1}{\hat{\sigma}}} \left\|x - \tilde{x}\right\|_{\infty} \tag{9.9}$$

9.1 Control of the era transition map $\bar{\Phi}_{\omega,\mathbf{h}}$

The purpose of this section is to give some estimates on the era transition map $\bar{\Phi}_{\omega,\mathbf{h}}$. Proposition 9.4 below shows that the decisive parameter to control the era transition map is the size $h_\perp$ of the section $S^s_{\omega,\mathbf{h}}$ in the direction transverse to the Mixmaster attractor.

Recall that for all $\omega \in]1,+\infty[\setminus \mathbb{Q}$,

$$m(\omega) = \min\left(1, (\omega - 2)^2\right)$$

and define for all $\omega \in]1,2[\setminus \mathbb{Q}$,

$$\bar{m}(\omega) \stackrel{\text{def}}{=} \min_{0 \leq j \leq k_1(\omega)} m(f^j(\omega))$$

Lemma 9.3. *For all $\omega \in]1,+\infty[\setminus \mathbb{Q}$,*

$$m(\omega) \geq \begin{cases} \frac{1}{(4k_2(\omega))^2} & \text{if } k_0(\omega) = 1 \\ \frac{1}{(2k_1(\omega))^2} & \text{if } k_0(\omega) = 2 \\ 1 & \text{if } k_0(\omega) \geq 3 \end{cases} \tag{9.10}$$

For all $\omega \in]1,2[\setminus \mathbb{Q}$,

$$\bar{m}(\omega) \geq \frac{1}{(4k_2(\omega)k_3(\omega))^2} \tag{9.11}$$

Proof. Estimate (9.10) is straightforward. Let $\omega \in]1,2[\setminus \mathbb{Q}$. Remark that $k_0(f^j(\omega)) \geq 3$ for $1 \leq j \leq k_1 - 2$ and

$$k_0(f^{k_1-1}(\omega)) = 2 \qquad k_1(f^{k_1-1}(\omega)) = k_2$$
$$k_0(f^{k_1}(\omega)) = 1 \qquad k_2(f^{k_1}(\omega)) = k_3$$

Hence, (9.11) follows from (9.10). □

Recall that $\tilde{n}_5$ is a constant fixed in Proposition 8.1.

Proposition 9.4 (Era transition map). *There exists a constant $\tilde{C}_6 \geq \tilde{C}_5$ such that the properties below hold for $\omega \in]1,2[\setminus \mathbb{Q}$, $0 < h \leq \tilde{C}_6^{-1}$, $\tilde{h} = hk_1(\omega)^{-\tilde{n}_5}$, $0 < h_\perp \leq \tilde{h}^{\tilde{C}_6(k_1(\omega)+1)^3}\bar{m}(\omega)^2$. Let $\mathbf{h} = \left(h, h_\perp, \tilde{h}^{\tilde{C}_6(k_1(\omega)+1)}\bar{m}(\omega)^2\right)$. The era transition map*

$$\bar{\Phi}_{\omega,\mathbf{h}} : S^s_{\omega,\mathbf{h}} \subset S_{\hat{h}} \to S_{\hat{h}}$$

is well defined and takes its values in $S^{s_{\bar{i}(\omega)}}_{\hat{h}}$. Moreover, for every $x, \tilde{x} \in S^s_{\omega,\mathbf{h}}$, we have the following estimates, where $\bar{\Phi} = \bar{\Phi}_{\omega,\mathbf{h}}$ and $\bar{\Phi}^A = \bar{\Phi}^A_{\omega,\mathbf{h}}$:

(Control of the distance to the Mixmaster attractor)

$$\mathrm{dist}_\infty\left(\bar{\Phi}(x), A\right) = \left\|\bar{\Phi}(x) - \bar{\Phi}^A(x)\right\|_\perp \leq h_\perp^{1+\frac{k_1(\omega)}{4}} \tag{9.12}$$

(Control of the drift tangential to the Mixmaster attractor)

$$\left\|\bar{\Phi}(x) - \bar{\Phi}^A(x)\right\|_{/\!/} \leq 34 h_\perp k_1(\omega)^2 \tag{9.13}$$

(Contraction in the direction transverse to the Mixmaster attractor)

$$\left\|(\bar{\Phi}(x) - \bar{\Phi}(\tilde{x})\right\|_{\perp} \leq h_{\perp}^{\frac{k_1(\omega)}{25}} \left\|x - \tilde{x}\right\|_{\infty} \tag{9.14}$$

(Lipschitz control in the direction tangential to the Mixmaster attractor)

$$\left\|\left((\bar{\Phi}(x) - \bar{\Phi}(\tilde{x})\right) - \left(\bar{\Phi}^A(x) - \bar{\Phi}^A(\tilde{x})\right)\right\|_{/\!/} \leq h_{\perp}^{\frac{1}{k_1(\omega)+4}} \left\|x - \tilde{x}\right\|_{\infty} + 16k_1(\omega)^2 \left\|x - \tilde{x}\right\|_{\perp} \tag{9.15}$$

(Control of the expansion in the direction tangent to the Mixmaster attractor)

$$\left\|\bar{\Phi}(x) - \bar{\Phi}(\tilde{x})\right\|_{/\!/} \geq K_f(\omega) \left\|x - \tilde{x}\right\|_{/\!/} - h_{\perp}^{\frac{1}{k_1(\omega)+4}} \left\|x - \tilde{x}\right\|_{\infty} - h\tilde{C}_6 \left\|x - \tilde{x}\right\|_{\perp} \tag{9.16}$$

(Global lipschitz constant)

$$\left\|\bar{\Phi}(x) - \bar{\Phi}(\tilde{x})\right\|_{\infty} \leq 4^{k_1(\omega)+2} k_1(\omega)^2 \left\|x - \tilde{x}\right\|_{\infty} \tag{9.17}$$

First, we will show that the map $\bar{\Phi}_{\omega,\mathbf{h}}$ can be expressed as a composition of several epoch transition maps. Once this is done, we will be left to apply recursively Proposition 8.1 to obtain the estimates on the era transition map.

From now on, assume that $\tilde{C}_4 \geq 1$, $\tilde{C}_5 \geq 1000\tilde{C}_4$ and $\tilde{n}_5 \geq 1000$. Fix $\omega \in]1, 2[\setminus \mathbb{Q}$ and $0 < h \leq \tilde{C}_5^{-1}$. For the remainder of this section, We now proceed to define the epoch transition maps that we will be using to decompose the era transition map. Let $\tilde{h} := hk_1(\omega)^{-\tilde{n}_5}$ and

$$h_j \stackrel{\text{def}}{=} \begin{cases} h & \text{if } 1 \leq j \leq k_1(\omega) - 1 \\ \tilde{h} & \text{if } j = 0 \text{ or } j = k_1(\omega) \end{cases}$$

Define, for $0 \leq j \leq k_1(\omega)$, the section parameters

$$h_{j,\perp} \stackrel{\text{def}}{=} \tilde{h}^{\tilde{C}_5(k_1(\omega)-j+1)^3} \bar{m}(\omega)^2, \qquad h_{j,/\!/} \stackrel{\text{def}}{=} \tilde{h}^{\tilde{C}_5(k_1(\omega)-j+1)} \bar{m}(\omega)^2$$

and

$$\mathbf{h}_j \stackrel{\text{def}}{=} \left(h_j, h_{j,\perp}, h_{j,/\!/}\right), \qquad \mathbf{h}'_j \stackrel{\text{def}}{=} \left(h_j, h_j, h_j\right)$$

We will use the epoch transition maps

$$\Phi_j \stackrel{\text{def}}{=} \Phi_{f^j(\omega),\mathbf{h}_j,\mathbf{h}'_{j+1}} : S^s_{f^j(\omega),\mathbf{h}_j} \to S^s_{f^{j+1}(\omega),\mathbf{h}'_{j+1}}, \qquad 0 \leq j \leq k_1(\omega) - 1$$

Define, for $0 \leq j \leq k_1(\omega) - 1$,

$$\Phi^*_j \stackrel{\text{def}}{=} \Phi_j \circ \cdots \circ \Phi_0, \qquad \Phi^{*A}_j \stackrel{\text{def}}{=} \Phi^*_j \circ \mathrm{Proj_A}$$

Our goal is to prove that

$$\bar{\Phi}_{\omega,\mathbf{h}_0} = \Phi^*_{k_1(\omega)-1}$$

To simplify the notations, let

$$S^s_j \stackrel{\text{def}}{=} S^s_{f^j(\omega),\mathbf{h}_j}, \qquad 0 \leq j \leq k_1(\omega)$$

Remark that the sections we consider become larger as j increases from 0 to $k_1(\omega)$. The departure section S^s_0 and the arrival section $S^s_{k_1(\omega)}$ are at distance h from the Kasner circle. The intermediate sections S^s_j $(0 < j < k_1(\omega))$ are chosen much closer to the Kasner circle, at distance $\tilde{h} \ll h$.

Lemma 9.5. *For every $0 \leq j \leq k_1(\omega) - 1$, the epoch transition map Φ_j is well defined on the section S^s_j.*

Proof. Remark that

$$k_0(f^j(\omega)) = k_1(\omega) - j + 1, \qquad \text{for all } 1 \leq j \leq k_1(\omega)$$

and $\min(h, \tilde{h}) = \tilde{h}$. Moreover, recall that

$$\bar{m}(\omega) \leq m(f^j(\omega)), \qquad \text{for all } 0 \leq j \leq k_1(\omega)$$

Hence, Lemma 9.5 is a direct consequence of Proposition 8.1. □

Lemma 9.6. *For every* $0 \leq j \leq k_1(\omega) - 1$, $\Phi_j\left(S_j^s\right) \subset S_{j+1}^s$.

Proof. Let $x \in S_0^s$. According to (8.5),

$$\mathrm{dist}_\infty\left(\Phi_0(x), A\right) = \left\|\Phi_0(x) - \Phi_0^A(x)\right\|_\perp \leq h_{0,\perp}^{\frac{5}{4}} \leq h_{0,\perp} \leq h_{1,\perp}$$

According to (8.6) and Proposition 8.3 on the local Lipschitz constant for the Kasner map,

$$\begin{aligned}
\left\|\Phi_0(x) - P^{s_2}_{f(\omega),\tilde{h}}\right\|_{/\!/} &\leq \left\|\Phi_0(x) - \Phi_0^A(x)\right\|_{/\!/} + \left\|\Phi_0^A(x) - P^{s_{i(\omega)}}_{f(\omega),\tilde{h}}\right\|_{/\!/} \\
&\leq 2h_{0,\perp}\,\mathrm{Lip}_f(\omega) + \mathrm{Lip}_f(\omega) h_{0,/\!/} \\
&\leq 16k_1(\omega)^2\left(2\tilde{h}^{\tilde{C}_5(k_1(\omega)+1)^3} + \tilde{h}^{\tilde{C}_5(k_1(\omega)+1)}\right)\bar{m}(\omega)^2 \quad \text{using } \mathrm{Lip}_f(\omega) = 16k_1(\omega)^2 \\
&\leq \tilde{h}^{\tilde{C}_5 k_1(\omega)}\bar{m}(\omega)^2 \times 48k_1(\omega)^2\tilde{h}^{\tilde{C}_5}
\end{aligned}$$

Since $\tilde{C}_5 \geq 1000$ and $\tilde{n}_5 \geq 1000$, one can check that $48k_1(\omega)^2\tilde{h}^{\tilde{C}_5} \leq 1$. Hence,

$$\left\|\Phi_0(x) - P^{s_2}_{f(\omega),\tilde{h}}\right\|_{/\!/} \leq h_{1,/\!/}$$

and we can conclude that $\Phi_0\left(S_0^s\right) \subset S_1^s$. Now, fix $1 \leq j \leq k_1(\omega) - 1$ and $x \in S_j^s$. According to (8.5),

$$\mathrm{dist}_\infty\left(\Phi_j(x), A\right) = \left\|\Phi_j(x) - \Phi_j^A(x)\right\|_\perp \leq h_{j,\perp}^{\frac{k_1(\omega)-j+5}{k_1(\omega)-j+4}} \leq h_{j,\perp} \leq h_{j+1,\perp}$$

According to (8.6) and Proposition 8.3 on the local Lipschitz constant for the Kasner map,

$$\begin{aligned}
\left\|\Phi_j(x) - P^{s_1}_{f^j(\omega),\tilde{h}}\right\|_{/\!/} &\leq \left\|\Phi_j(x) - \Phi_j^A(x)\right\|_{/\!/} + \left\|\Phi_j^A(x) - P^{s_1}_{f^j(\omega),\tilde{h}}\right\|_{/\!/} \\
&\leq 2h_{j,\perp}\,\mathrm{Lip}_f(f^j(\omega)) + \mathrm{Lip}_f(f^j(\omega)) h_{j,/\!/} \\
&\leq \left(2\tilde{h}^{\tilde{C}_5(k_1(\omega)-j+1)^3} + \tilde{h}^{\tilde{C}_5(k_1(\omega)-j+1)}\right)\bar{m}(\omega)^2 \qquad \text{using } \mathrm{Lip}_f(f^j(\omega)) = 1 \\
&\leq h_{j+1,/\!/} \times 3\tilde{h}^{\tilde{C}_5} \\
&\leq h_{j+1,/\!/}
\end{aligned}$$

Hence, $\Phi_j\left(S_j^s\right) \subset S_{j+1}^s$. This concludes the proof of Lemma 9.6. □

Define, for any $\omega \in]1,2[\setminus \mathbb{Q}$,

$$\bar{i}(\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } k_1(\omega) \geq 2 \\ 2 & \text{if } k_1(\omega) = 1 \end{cases} \tag{9.18}$$

Corollary 9.7. *The map* $\Phi^*_{k_1(\omega)-1}$ *is well defined on the whole section* S_0^s *and takes its values in the global section* $S_h^{s_{\bar{i}(\omega)}}$.

Proof. Recall that $S_h = S_h^{s_1} \cup S_h^{s_2}$ where

$$S_h^{s_1} \stackrel{\text{def}}{=} \left\{x = (x_u, x_{s_1}, x_{s_2}, x_c) \mid x_{s_1} = h, \quad 0 \leq x_u \leq h, \quad 0 \leq x_{s_2} \leq h, \quad 1 < x_c < 2\right\}$$

and analogously for $S_h^{s_2}$. The fact that $\Phi^*_{k_1(\omega)-1}$ is well defined on the section S_0^s is a direct consequence of Lemma 9.6. Moreover, Lemma 9.6 informs us that $\Phi^*_{k_1(\omega)-1}$ takes its values in the section $S^s_{k_1(\omega)}$. Hence, for any $x \in S_0^s$,

$$\left\|\Phi^*_{k_1(\omega)-1}(x) - \Phi^{*A}_{k_1(\omega)-1}(x)\right\|_\perp \leq h_{k_1(\omega),\perp} \leq h$$

and

$$\left\|\Phi^*_{k_1(\omega)-1}(x) - P^{s_{\bar{i}(\omega)}}_{\bar{f}(\omega),h}\right\|_{/\!/} \leq h_{k_1(\omega),/\!/} \leq \frac{1}{2}\bar{m}(\omega) \leq \min\left(\frac{\bar{f}(\omega)-1}{2}, \frac{2-\bar{f}(\omega)}{2}\right)$$

Since $1 < \bar{f}(\omega) < 2$ and $x_c\left(P^{s_{\bar{i}(\omega)}}_{\bar{f}(\omega),h}\right) = \bar{f}(\omega)$, the above inequality implies that

$$1 < x_c\left(\Phi^*_{k_1(\omega)-1}(x)\right) < 2$$

Hence, $\Phi^*_{k_1(\omega)-1}(x) \in S_h$. More precisely, Proposition 8.1 implies that $\Phi^*_{k_1(\omega)-1}$ takes its values in $S_h^{s_{\bar{i}(\omega)}}$. □

Lemma 9.8. *The era transition map $\bar{\Phi}_{\omega,\mathbf{h}_0}$ is well defined on the whole section S_0^s and takes its values in $S_h^{s_{\bar{i}(\omega)}}$. Moreover,*

$$\bar{\Phi}_{\omega,\mathbf{h}_0} = \Phi^*_{k_1(\omega)-1} = \Phi_{k_1(\omega)-1} \circ \cdots \circ \Phi_0 \tag{9.19}$$

Proof. In this proof, we will denote $k_1 = k_1(\omega)$ and we will assume that $k_1 \geq 2$. Indeed, if $k_1 = 1$, the era transition map coincide with the epoch transition map and Lemma 9.8 is a straightforward consequence of Proposition 8.1.

Claim 1. *The era transition map $\bar{\Phi}_{\omega,\mathbf{h}_0}$ is well defined on $S_0^s \cap B_{\mathrm{IX}}$ and $\bar{\Phi}_{\omega,\mathbf{h}_0}(x) = \Phi^*_{k_1-1}(x)$ for every $x \in S_0^s \cap B_{\mathrm{IX}}$.*

Proof of claim 1. Let $x \in S_0^s \cap B_{\mathrm{IX}}$ and q be the orbit of the Wainwright-Hsu vector field $\mathcal{X}$ with initial condition $q(0) = \xi^{-1}(x)$. Let $q^0 = \xi^{-1}(x)$ and $q^j = \xi^{-1}(\Phi^*_{j-1}(x))$ for any $1 \leq j \leq k_1$. Since $x \in B_{\mathrm{IX}}$, we have $x_u \neq 0$, $x_{s_1} \neq 0$ and $x_{s_2} \neq 0$. It follows by induction that for any $0 \leq j \leq k_1 - 1$, we have $q_u^j \neq 0$ and q^{j+1} is the first intersection point of the orbit of the Wainwright-Hsu vector field $\mathcal{X}$ starting at q^j with the section $\mathcal{S}^s_{j+1} := \xi\left(S^s_{j+1}\right)$. Hence, q^{k_1} is a point belonging both to the orbit starting at $q^0 = \xi^{-1}(x)$ and to the global section $\mathcal{S}_h$. This proves that the era transition map $\bar{\Phi}_{\omega,\mathbf{h}_0}$ is well defined on $S_0^s \cap B_{\mathrm{IX}}$.

We are now going to prove that $\bar{\Phi}_{\omega,\mathbf{h}_0}(x) = \Phi^*_{k_1-1}(x)$, *i.e.* that $\xi^{-1}(\Phi^*_{k_1-1}(x))$ is the first intersection point of the orbit q with the section $\mathcal{S}_h$. Let $t_0^s = 0$ and let t_0^u be the first time $t > 0$ such that $q(t) \in \mathcal{S}^u_{\omega,\mathbf{h}'_0}$. By induction, define, for every $1 \leq j \leq k_1 - 1$,

$$\begin{aligned} t_j^s &= \min\left\{t > t^u_{j-1} \mid q(t) \in \mathcal{S}^s_j\right\} \\ t_j^u &= \min\left\{t > t^s_j \mid q(t) \in \mathcal{S}^u_{f^j(\omega),\mathbf{h}'_j}\right\} \\ t^s_{k_1} &= \min\left\{t > t^u_{k_1-1} \mid q(t) = \xi^{-1}(\Phi^*_{k_1-1}(x))\right\} \end{aligned}$$

With these notations, we are left to prove that for any $t \in \left]0, t^s_{k_1}\right[$, $q(t)$ does not belong to $\mathcal{S}_h$. The general idea is simple: either $q(t)$ is close to a type II orbit that is far away from the section $\mathcal{S}_h$ or $q(t)$ is close to the Kasner circle and we can use the local estimates of chapter 6.

Case $t \in \left]t_j^s, t_j^u\right]$, $0 \leq j \leq k_1 - 1$. According to Proposition 6.15 and corollary 6.16, q_{s_1} and q_{s_2} are exponentially decreasing. Hence, $q_{s_1}(t) < q_{s_1}(t_j^s) \leq h$ and $q_{s_2}(t) < q_{s_1}(t_j^s) \leq h$. This implies that $q(t)$ does not belong to the section $\mathcal{S}_h$.

Case $t \in \left]t_0^u, t_1^s\right]$. Recall that the local coordinates $(x_u, x_{s_1}, x_{s_2}, x_c)$ are defined on the open ball $B_{\omega,\tilde{C}_5,\tilde{n}_5}$, *i.e.* $B_{\omega,\tilde{C}_5,\tilde{n}_5} \subset U_\xi$ (see definition 4.1). Let t_0^{out} be the first time $t > t_0^u$ when the orbit $\xi \circ q$ leaves the open ball $B_{\omega,2\tilde{C}_5,\tilde{n}_5}$. On $\left]t_0^u, t_0^{\mathrm{out}}\right]$, q_u is strictly increasing and $q_u(t_0^u) = h$ so $q(t)$ does not belong to the section $\mathcal{S}_h$.

Let us denote by p the (type II) orbit with initial condition $p(t_0^u) = \mathcal{P}^u_{\omega,h}$. For every $t \in \left]t_0^u, t_1^s\right]$, we have

$$\begin{aligned}
d_{\mathcal{B}}\left(q(t), p(t)\right) &\leq \frac{\left\|\xi(q(t_0^u)) - P^u_{\omega,h}\right\|_\perp}{\tilde{h}^{\tilde{C}_4\omega} m(\omega)} && \text{using (7.31)} \\
&\leq \frac{(h_{0,\perp})^{\frac{\omega+2}{\omega+1}} h^{-1}}{\tilde{h}^{\tilde{C}_4\omega} m(\omega)} && \text{using (6.1)} \\
&\leq \frac{\tilde{h}^{\tilde{C}_5(k_1+1)^3} \bar{m}(\omega)^2}{\tilde{h}^{\tilde{C}_4\omega} m(\omega)} \\
&\leq \tilde{h}^{100} \bar{m}(\omega) \\
&\leq \frac{1}{\tilde{C}_5 f(\omega)^{\tilde{n}_5} \omega^{\tilde{n}_5}} \frac{h}{1000} \bar{m}(\omega) && (9.20)
\end{aligned}$$

According to (9.20) and the control (4.5b) on the local coordinate system ξ, for every $t \in \left]t_0^u, t_0^{\text{out}}\right]$, we have $\xi \circ p(t) \in B_{\omega,\tilde{C}_5,\tilde{n}_5} \subset U_\xi$. It follows from the evolution equations (6.5) that p_u is increasing on $\left]t_0^u, t_0^{\text{out}}\right]$. Since p is a type II orbit, $p(t) = (p_u(t), 0, 0, \omega)$ on $\left]t_0^u, t_0^{\text{out}}\right]$. As a consequence, estimate (9.20) together with the control (4.5b) on the local coordinate system ξ imply that

$$q_{s_1}(t_0^{\text{out}}) < \frac{1}{2\tilde{C}_5\omega^{\tilde{n}_5}}, \quad q_{s_2}(t_0^{\text{out}}) < \frac{1}{2\tilde{C}_5\omega^{\tilde{n}_5}}, \quad \left|q_c(t_0^{\text{out}}) - \omega\right| < \min\left(\frac{1}{2\tilde{C}_5\omega^{\tilde{n}_5}}, \frac{\omega-1}{2}\right)$$

Recall that the orbit $\xi \circ q$ leaves the open ball $B_{\omega,2\tilde{C}_5,\tilde{n}_5}$ at time $t = t_0^{\text{out}}$. Hence, $q_u(t_0^{\text{out}}) = \frac{1}{2\tilde{C}_5\omega^{\tilde{n}_5}} \geq 3h$. Using (9.20) together with (4.5b) once again, we get that $p_u(t_0^{\text{out}}) \geq 2h$. It follows that for every $t \in \left[t_0^{\text{out}}, t_1^s\right]$, one of the three following properties hold:

1. $p_u(t) \geq 2h$ (roughly, before p leaves $\mathcal{U}_\xi$)

2. $p(t) \notin \mathcal{U}_\xi$

3. $p_c(t) = f(\omega)$ (roughly, after p re-enters $\mathcal{U}_\xi$)

Let $t \in \left[t_0^{\text{out}}, t_1^s\right]$ and assume that $q(t) \in \mathcal{S}_h$. Estimate (9.20) implies that $p(t)$ belongs to the domain $\mathcal{U}_\xi$ of the local coordinates system ξ. If $p_u(t) \geq 2h$, then we get $q_u(t) > h$ using (9.20) with (4.5b). If $p_c(t) = f(\omega)$, then we get

$$|q_c(t) - f(\omega)| \leq \frac{f(\omega) - 2}{2}$$

using (9.20) with (4.5b) once again. Since $f(\omega) > 2$, it follows that $q_c(t) > 2$. In both cases, $q(t) \notin \mathcal{S}_h$ so this is absurd.

Case $t \in \left[t_j^u, t_{j+1}^s\right]$, $1 \leq j \leq k_1 - 2$. Let us denote by p the (type II) orbit with initial condition $p(t_j^u) = \mathcal{P}^u_{f^j(\omega),\tilde{h}}$. Recall that

$$k_1 - j + 1 \leq f^j(\omega) = [k_1 - j + 1; k_2, \dots] \leq k_1 - j + 2$$

For every $t \in \left[t_j^u, t_{j+1}^s\right]$, we have

$$
\begin{aligned}
d_{\mathcal{B}}\left(q(t), p(t)\right) &\le \frac{\left\|\xi(q(t_j^u)) - \mathcal{P}^u_{f^j(\omega),\tilde{h}}\right\|_\perp}{\tilde{h}^{\tilde{C}_4 f^j(\omega)} m(f^j(\omega))} && \text{using (7.31)} \\
&\le \frac{(h_{j,\perp})^{\frac{f^j(\omega)+2}{f^j(\omega)+1}}\tilde{h}^{-1}}{\tilde{h}^{\tilde{C}_4 f^j(\omega)} m(f^j(\omega))} && \text{using (6.1)} \\
&\le \frac{\tilde{h}^{\tilde{C}_5 (k_1 - j + 1)^3}\bar{m}(\omega)^2}{\tilde{h}^{\tilde{C}_4 f^j(\omega)} m(f^j(\omega))} \\
&\le \frac{1}{\tilde{C}_5 f^j(\omega)^{\tilde{n}_5} f^{j+1}(\omega)^{\tilde{n}_5}} \frac{h}{1000}\bar{m}(\omega) && (9.21)
\end{aligned}
$$

Let $t \in \left[t_j^u, t_{j+1}^s\right]$ and assume that $q(t) \in \mathcal{S}_h$. It follows from (9.21) and the control (4.5b) on the local coordinate system ξ that $p(t) \in \mathcal{U}_\xi$. Since p is a type II orbit, we have either $p_c(t) = f^j(\omega)$ or $p_c(t) = f^{j+1}(\omega)$. Moreover, estimate (9.21) together with the control (4.5b) on the local coordinate system implies that

$$
|q_c(t) - p_c(t)| \le \frac{f^{j+1}(\omega) - 2}{2}
$$

Since $2 < f^{j+1}(\omega) < f^j(\omega)$, it follows that $q_c(t) > 2$. This contradicts the fact that $q(t)$ belongs to the section $\mathcal{S}_h$.

Case $t \in \left[t^u_{k_1-1}, t^s_{k_1}\right]$. Assume that there exists $t \in \left[t^u_{k_1-1}, t^s_{k_1}\right[$ such that $q(t) \in \mathcal{S}_h$. Then it must satisfy $q(t) \in \mathcal{S}_h \setminus \mathcal{S}^s_{\bar{f}(\omega),\mathbf{h}'_{k_1(\omega)}}$ because $\xi^{-1}(\Phi^*_{k_1-1}(x))$ is the first intersection point of the orbit starting at $q(t^u_{k_1-1})$ with the section $\mathcal{S}^s_{\bar{f}(\omega),\mathbf{h}'_{k_1(\omega)}}$. But estimate (9.21) with $j = k_1 - 1$ is also valid on $\left[t^u_{k_1-1}, t^s_{k_1}\right]$ and implies that $q(t) \in \mathcal{S}^s_{\bar{f}(\omega),\mathbf{h}'_{k_1(\omega)}}$. This is absurd.

This concludes the proof of claim 1. □

Claim 2. *The era transition map $\bar{\Phi}_{\omega,\mathbf{h}_0}$ is well defined at every point of $S_0^s \cap B_{\mathrm{II}}$ and $\bar{\Phi}_{\omega,\mathbf{h}_0}(x) = \Phi^*_{k_1-1}(x)$ for every $x \in S_0^s \cap B_{\mathrm{II}}$.*

Proof of claim 2. Let $x \in S_0^s \cap B_{\mathrm{II}}$. In particular, $x_u = 0$. Iteration of formula (5.8) gives

$$
\Phi^*_{k_1-1}(x) = \begin{cases} (0, h, 0, f^{k_1}(x_c)) & \text{if } k_1 \ge 2 \\ (0, 0, h, f^{k_1}(x_c)) & \text{if } k_1 = 1 \end{cases}
$$

Morevoer, if we denote by $[1; k_1(x_c), k_2(x_c), \dots]$ the continued fraction associated with x_c, formula (5.6) can be rewritten as follows:

$$
\bar{\Phi}_{\omega,\mathbf{h}}(x) = \begin{cases} (0, h, 0, f^{k_1(x_c)}(x_c)) & \text{if } k_1(x_c) \ge 2 \\ (0, 0, h, f^{k_1(x_c)}(x_c)) & \text{if } k_1(x_c) = 1 \end{cases}
$$

We are left to prove that $k_1(x_c) = k_1$. This is a consequence of Proposition B.6 together with the fact that $|x_c - \omega| \le h_{0,/\!/} < (10 k_1(\omega)^2 k_2(\omega) k_3(\omega))^{-1}$. This concludes the proof of claim 2. □

Claim 3. *The era transition map $\bar{\Phi}_{\omega,\mathbf{h}_0}$ is well defined at every point of $S_0^s \cap B_{\mathrm{VII}_0}$ and $\bar{\Phi}_{\omega,\mathbf{h}_0}(x) = \Phi^*_{k_1-1}(x)$ for every $x \in S_0^s \cap B_{\mathrm{VII}_0}$.*

Proof of claim 3. This is a mix of claim 1 (before the orbit starting at x converges to a point of the Kasner circle) and claim 2 (after the orbit starting at x converges to a point of the Kasner circle). □

This concludes the proof of Lemma 9.8, since $S_0^s = (S_0^s \cap B_{\mathrm{II}}) \sqcup (S_0^s \cap B_{\mathrm{VII}_0}) \sqcup (S_0^s \cap B_{\mathrm{IX}})$. □

Proof of Proposition 9.4. As in the preceding proof, we will denote $k_1 = k_1(\omega)$ and we will assume that $k_1 \ge 2$. The proof of Proposition 9.4 relies on the decomposition (9.19) of the era transition map $\bar{\Phi}_{\omega,\mathbf{h}_0}$ as a product of k_1 epoch transition maps, together with the estimates on these epoch transition maps stated in Proposition 8.1. In other words, estimates (9.12), …, (9.16) will be obtained by applying k_1 times the corresponding estimates of Proposition 8.1. More precisely, we are going to prove Proposition 9.4 for a restriction of the map $\bar{\Phi}_{\omega,\mathbf{h}_0}$. Let $C \geq \tilde{C}_5$ and define

$$\tilde{h}_{0,\perp} \stackrel{\text{def}}{=} \tilde{h}^{C(k_1+1)^3}\bar{m}(\omega)^2, \qquad h_{0,/\!/} \stackrel{\text{def}}{=} \tilde{h}^{\tilde{C}_5(k_1(\omega)+1)}\bar{m}(\omega)^2$$

Let $0 < h_\perp \leq \tilde{h}_{0,\perp}$, $\mathbf{h} = (h, h_\perp, h_{0,/\!/})$, $x, \tilde{x} \in S^s_{\omega,\mathbf{h}}$, $\bar{\Phi} = \bar{\Phi}_{\omega,\mathbf{h}}$ and $\bar{\Phi}^A = \bar{\Phi}^A_{\omega,\mathbf{h}}$. We are left to prove the following statement: provided that C is large enough, estimates (9.12), …, (9.16) hold true. From now on, we will use the notation $\Phi_0 := \Phi_{\omega,\mathbf{h},\mathbf{h}_1'}$. Beware of the fact that this is the restriction of the former epoch transition map Φ_0 to the smaller section $S^s_{\omega,\mathbf{h}}$.

Proof of estimate (9.12). Define $a_0 = h_\perp$ and

$$a_j = \sup_{x \in S^s_{\omega,\mathbf{h}}} \left\| \Phi^*_{j-1}(x) - \Phi^{*A}_{j-1}(x) \right\|_\perp, \qquad 1 \leq j \leq k_1$$

Claim 1. *For all* $1 \leq j \leq k_1$,

$$a_j \leq h_\perp^{\frac{5}{4}\frac{k_1+4}{k_1-j+5}} \tag{9.22}$$

Proof of claim 1. Recall that $k_0(\omega) = 1$. Applying (8.5) to the epoch transition map Φ_0, we get that $a_1 \leq h_\perp^{\frac{5}{4}}$. Assume that $a_j \leq h_\perp^{\frac{5}{4}\frac{k_1+4}{k_1-j+5}}$ for some $1 \leq j \leq k_1 - 1$. We are now going to apply Proposition 8.1 with $h_\perp = a_j$. More precisely, we apply (8.5) to the epoch transition map Φ_j restricted to the section $S^s_{f^j(\omega),(h_j,a_j,h_{j,/\!/})}$. We get that

$$a_{j+1} \leq a_j^{\frac{k_0(f^j(\omega))+4}{k_0(f^j(\omega))+3}} = a_j^{\frac{k_1-j+5}{k_1-j+4}} \leq h_\perp^{\frac{5}{4}\frac{k_1+4}{k_1-j+4}}$$

By induction on j, claim 1 holds true. □

Since $\bar{\Phi} = \Phi^*_{k_1-1}$, estimate (9.12) is a direct consequence of estimate (9.22) with $j = k_1$.

Proof of estimate (9.13). Define

$$b_j = \sup_{x \in S^s_{\omega,\mathbf{h}}} \left\| \Phi^*_{j-1}(x) - \Phi^{*A}_{j-1}(x) \right\|_{/\!/}, \qquad 1 \leq j \leq k_1$$

Using (8.6) and the fact that $\mathrm{Lip}_f(\omega) = 16k_1^2$ (see the explicit formula (8.3)), we get

$$b_1 \leq 2h_\perp \,\mathrm{Lip}_f(\omega) \leq 32k_1^2 h_\perp \tag{9.23}$$

We are now going to find a relation between b_{j+1} and b_j. Let $1 \leq j \leq k_1 - 1$. Remark that

$$\left\|\Phi^*_j(x) - \Phi^{*A}_j(x)\right\|_{/\!/} \leq \left\|\Phi_j\left(\Phi^*_{j-1}(x)\right) - \Phi^A_j\left(\Phi^*_{j-1}(x)\right)\right\|_{/\!/} + \left\|\Phi^A_j\left(\Phi^*_{j-1}(x)\right) - \Phi^A_j\left(\Phi^{*A}_{j-1}(x)\right)\right\|_{/\!/} \tag{9.24}$$

As a direct consequence of estimate (8.6) applied to the epoch transition map Φ_j restricted to the section $S^s_{f^j(\omega),(\tilde{h},a_j,h_{j,/\!/})}$, we get that

$$\left\|\Phi_j\left(\Phi^*_{j-1}(x)\right) - \Phi^A_j\left(\Phi^*_{j-1}(x)\right)\right\|_{/\!/} \leq 2a_j \,\mathrm{Lip}_f(f^j(\omega)) \leq 2a_j \tag{9.25}$$

since $f^j(\omega) > 2$ implies that $\mathrm{Lip}_f(f^j(\omega)) = 1$ by the explicit formula (8.3). Moreover, recall that the x_c-coordinate of Φ^A_j is essentially the Kasner map (see (5.9)). Hence, Proposition 8.3 implies that

$$\left\|\Phi^A_j\left(\Phi^*_{j-1}(x)\right) - \Phi^A_j\left(\Phi^{*A}_{j-1}(x)\right)\right\|_{/\!/} \leq \mathrm{Lip}_f(f^j(\omega)) \left\|\Phi^*_{j-1}(x) - \Phi^{*A}_{j-1}(x)\right\|_{/\!/} \leq b_j \tag{9.26}$$

Pluging (9.25) and (9.26) in (9.24), we get that $b_{j+1} \leq 2a_j + b_j$. Hence, using (9.22) and (9.23), it follows that for all $1 \le j \le k_1$,

$$b_j \le 2\sum_{r=1}^{j-1} a_r + b_1 \le 2k_1 h_\perp + 32k_1^2 h_\perp \le 34k_1^2 h_\perp \tag{9.27}$$

Since $\bar{\Phi} = \Phi^*_{k_1-1}$, estimate (9.13) is a direct consequence of estimate (9.27) with $j = k_1$.

Proof of estimate (9.17). Let $0 \le l \le k_1 - 1$. Recall that Φ^*_l is defined as the product of $l+1$ epoch transition maps. Using $l+1$ times inequality (8.10), we obtain the following Lipschitz estimate for Φ^*_l:

$$\left\|\Phi^*_l(x) - \Phi^*_l(\tilde{x})\right\|_\infty \le \left(\prod_{j=0}^{l} 4\,\mathrm{Lip}_f(f^j(\omega))\right)\|x-\tilde{x}\|_\infty$$

Recall that $\mathrm{Lip}_f(.)$ is defined by the explicit formula (8.3) which yields

$$\left(\prod_{j=0}^{l} 4\,\mathrm{Lip}_f(f^j(\omega))\right) = 4^{l+1}16k_1^2$$

since $f^j(\omega) > 2$ for $1 \le j \le k_1 - 1$. Hence, we obtain

$$\left\|\Phi^*_l(x) - \Phi^*_l(\tilde{x})\right\|_\infty \le 4^{l+3}k_1^2\,\|x-\tilde{x}\|_\infty \tag{9.28}$$

Taking $l = k_1 - 1$, we obtain the desired estimate (9.17) for the era transition map $\bar{\Phi} = \Phi^*_{k_1-1}$.

Proof of estimate (9.14). Let us turn to the contraction estimate (9.14). The idea is to decompose the era transition map $\bar{\Phi} = \Phi^*_{k_1-1}$ as $\Phi_{k_1-1} \circ \Phi^*_{k_1-2}$ and to use the contraction estimate (8.7) for the epoch transition map Φ_{k_1-1} restricted to the section $S^s_{f^{k_1-1}(\omega),(\tilde{h},a_{k_1-1},h_{k_1-1,/\!/})}$ as well as the Lipschitz estimate (9.28) for $\Phi^*_{k_1-2}$. For $1 \le l \le k_1 - 1$,

$$\begin{aligned}
&\left\|\Phi^*_l(x) - \Phi^*_l(\tilde{x})\right\|_\perp \\
=&\left\|\Phi_l(\Phi^*_{l-1}(x)) - \Phi_l(\Phi^*_{l-1}(\tilde{x}))\right\|_\perp \\
\le& a_l^{\frac{1}{k_0(f^l(\omega))+3}}\left\|\Phi^*_{l-1}(x) - \Phi^*_{l-1}(\tilde{x})\right\|_\infty \qquad \text{using (8.7) for } \Phi_l \text{ restricted to } S^s_{f^l(\omega),(\tilde{h},a_l,h_{l,/\!/})} \\
\le& a_l^{\frac{1}{k_1-l+4}}4^{l+2}k_1^2\,\|x-\tilde{x}\|_\infty \qquad \text{using (9.28) and } k_0(f^l(\omega)) = k_1 - l + 1
\end{aligned} \tag{9.29}$$

Taking $l = k_1 - 1$, we get

$$\begin{aligned}
\left\|\Phi^*_{k_1-1}(x) - \Phi^*_{k_1-1}(\tilde{x})\right\|_\perp &\le a_{k_1-1}^{\frac{1}{5}}4^{k_1+1}k_1^2\,\|x-\tilde{x}\|_\infty \\
&\le h_\perp^{\frac{k_1}{24}}4^{k_1+1}k_1^2\,\|x-\tilde{x}\|_\infty && \text{using (9.22)} \\
&\le h_\perp^{\frac{k_1}{25}}\,\|x-\tilde{x}\|_\infty && \text{provided that } C \text{ is large enough}
\end{aligned}$$

Since $\bar{\Phi} = \Phi^*_{k_1-1}$, estimate (9.14) is a direct consequence of the above inequality.

Proof of estimate (9.16). The idea is to decompose the era transition map $\bar{\Phi}$ as the product of the k_1 epoch transition maps Φ_j and then to apply recursively the expansion estimate (8.9) to these epoch transition maps. Define,

$$\begin{aligned}
\Lambda_1 &= 4^3k_1^2a_1^{\frac{1}{k_1+3}} + \tilde{C}_5k_1^{\tilde{n}_5}\tilde{h}h_\perp^{\frac{1}{4}} \\
\Lambda_j &= 4^{j+1}k_1^2\left(4a_j^{\frac{1}{k_1-j+4}} + \tilde{C}_5(k_1-j+1)^{\tilde{n}_5}\tilde{h}a_{j-1}^{\frac{1}{k_1-j+5}}\right) && 2 \le j \le k_1 - 1
\end{aligned}$$

According to (8.9), we have

$$\|\Phi_0(x)-\Phi_0(\tilde{x})\|_{/\!/} \geq K_f(\omega)\,\|x-\tilde{x}\|_{/\!/} - h_\perp^{\frac{1}{4}}\,\|x-\tilde{x}\|_\infty - \tilde{C}_5 h\,\|x-\tilde{x}\|_\perp$$

Let $1 \leq j \leq k_1-1$. Remark once again that

$$\left\|\Phi_j^*(x)-\Phi_j^*(\tilde{x})\right\|_{/\!/} = \left\|\Phi_j\left(\Phi_{j-1}^*(x)\right)-\Phi_j\left(\Phi_{j-1}^*(\tilde{x})\right)\right\|_{/\!/}$$

Hence, expansion estimate (8.9) applied to the epoch transition map Φ_j restricted to the section $S^s_{f^j(\omega),(\tilde{h},a_j,h_{j,/\!/})}$ gives

$$\begin{aligned}\left\|\Phi_j^*(x)-\Phi_j^*(\tilde{x})\right\|_{/\!/} &\geq K_f(f^j(\omega))\left\|\Phi_{j-1}^*(x)-\Phi_{j-1}^*(\tilde{x})\right\|_{/\!/}\\ &\quad - a_j^{\frac{1}{k_1-j+4}}\left\|\Phi_{j-1}^*(x)-\Phi_{j-1}^*(\tilde{x})\right\|_\infty - \tilde{C}_5(k_1-j+1)^{\tilde{n}_5}\tilde{h}\left\|\Phi_{j-1}^*(x)-\Phi_{j-1}^*(\tilde{x})\right\|_\perp\end{aligned}$$

Remark that $f^j(\omega) > \frac{5}{3}$, hence $K_f(f^j(\omega)) = 1$ by the explicit formula (8.2). If $j=1$, use (9.28) to estimate the term $\left\|\Phi_{j-1}^*(x)-\Phi_{j-1}^*(\tilde{x})\right\|_\infty$ and (8.7) to estimate the term $\left\|\Phi_{j-1}^*(x)-\Phi_{j-1}^*(\tilde{x})\right\|_\perp$. This gives

$$\left\|\Phi_1^*(x)-\Phi_1^*(\tilde{x})\right\|_{/\!/} \geq \left\|\Phi_0^*(x)-\Phi_0^*(\tilde{x})\right\|_{/\!/} - \Lambda_1\,\|x-\tilde{x}\|_\infty$$

If $2 \leq j \leq k_1-1$, use (9.28) to estimate the term $\left\|\Phi_{j-1}^*(x)-\Phi_{j-1}^*(\tilde{x})\right\|_\infty$ and (9.29) to estimate the term $\left\|\Phi_{j-1}^*(x)-\Phi_{j-1}^*(\tilde{x})\right\|_\perp$. This gives

$$\left\|\Phi_j^*(x)-\Phi_j^*(\tilde{x})\right\|_{/\!/} \geq \left\|\Phi_{j-1}^*(x)-\Phi_{j-1}^*(\tilde{x})\right\|_{/\!/} - \Lambda_j\,\|x-\tilde{x}\|_\infty$$

By induction on j, it follows that

$$\left\|\Phi_{k_1-1}^*(x)-\Phi_{k_1-1}^*(\tilde{x})\right\|_{/\!/} \geq K_f(\omega)\,\|x-\tilde{x}\|_{/\!/} - \left(h_\perp^{\frac{1}{4}} + \sum_{j=1}^{k_1-1}\Lambda_j\right)\|x-\tilde{x}\|_\infty - \tilde{C}_5 h\,\|x-\tilde{x}\|_\perp$$

Using (9.22), one can see that if C is large enough, then

$$h_\perp^{\frac{1}{4}} + \sum_{j=1}^{k_1-1}\Lambda_j \leq h_\perp^{\frac{1}{k_1+4}}$$

Hence, if C is large enough, then

$$\left\|\Phi_{k_1-1}^*(x)-\Phi_{k_1-1}^*(\tilde{x})\right\|_{/\!/} \geq K_f(\omega)\,\|x-\tilde{x}\|_{/\!/} - h_\perp^{\frac{1}{k_1+4}}\,\|x-\tilde{x}\|_\infty - \tilde{C}_5 h\,\|x-\tilde{x}\|_\perp$$

which is precisely the desired estimate (9.16).

Proof of estimate (9.15). Define, for $0 \leq j \leq k_1-1$,

$$N_{j,/\!/} = \left\|\left(\Phi_j^*(x)-\Phi_j^*(\tilde{x})\right)-\left(\Phi_j^{*A}(x)-\Phi_j^{*A}(\tilde{x})\right)\right\|_{/\!/}$$

Let $1 \leq j \leq k_1-1$. Decompose Φ_j^* as $\Phi_j \circ \Phi_{j-1}^*$ and Φ_j^{*A} as $\Phi_j \circ \Phi_{j-1}^{*A}$. Using standard triangle inequality, we get

$$\begin{aligned}N_{j,/\!/} \leq &\left\|\left(\Phi_j(\Phi_{j-1}^*(x))-\Phi_j(\Phi_{j-1}^*(\tilde{x}))\right)-\left(\Phi_j^A(\Phi_{j-1}^*(x))-\Phi_j^A(\Phi_{j-1}^*(\tilde{x}))\right)\right\|_{/\!/}\\ &+\left\|\left(\Phi_j^A(\Phi_{j-1}^*(x))-\Phi_j^A(\Phi_{j-1}^*(\tilde{x}))\right)-\left(\Phi_j^A(\Phi_{j-1}^{*A}(x))-\Phi_j^A(\Phi_{j-1}^{*A}(\tilde{x}))\right)\right\|_{/\!/}\end{aligned} \tag{9.30}$$

Let us begin with the second term. Recall that the x_c-coordinate of Φ_j^A is essentially the Kasner map f (see (5.9)). Moreover, $f(u) = u-1$ for all $u \geq 2$ and the x_c-coordinates of the four points $\Phi_{j-1}^*(x)$,

$\Phi^*_{j-1}(\tilde{x})$, $\Phi^{*A}_{j-1}(x)$ and $\Phi^{*A}_{j-1}(\tilde{x})$ are all greater than 2. Hence,

$$\begin{aligned}\Big\|\Big(\Phi^A_j(\Phi^*_{j-1}(x)) - \Phi^A_j(\Phi^*_{j-1}(\tilde{x}))\Big) - \Big(\Phi^A_j(\Phi^{*A}_{j-1}(x)) - \Phi^A_j(\Phi^{*A}_{j-1}(\tilde{x}))\Big)\Big\|_{/\!/} \\ = \Big\|\Big(\Phi^*_{j-1}(x) - \Phi^*_{j-1}(\tilde{x})\Big) - \Big(\Phi^{*A}_{j-1}(x) - \Phi^{*A}_{j-1}(\tilde{x})\Big)\Big\|_{/\!/} = N_{j-1,/\!/} \end{aligned} \quad (9.31)$$

Let us now turn to the first term. Using the Lipschitz estimate (8.8) with the epoch transition map Φ_j restricted to the section $S^s_{f^j(\omega),(\tilde{h},a_j,h_{j,/\!/})}$, we get

$$\begin{aligned}\Big\|\Big(\Phi_j(\Phi^*_{j-1}(x)) - \Phi_j(\Phi^*_{j-1}(\tilde{x}))\Big) - \Big(\Phi^A_j(\Phi^*_{j-1}(x)) - \Phi^A_j(\Phi^*_{j-1}(\tilde{x}))\Big)\Big\|_{/\!/} \leq \\ a_j^{\frac{1}{k_1-j+4}} \left\|\Phi^*_{j-1}(x) - \Phi^*_{j-1}(\tilde{x})\right\|_\infty + \mathrm{Lip}_f(f^j(\omega)) \left\|\Phi^*_{j-1}(x) - \Phi^*_{j-1}(\tilde{x})\right\|_\perp\end{aligned}$$

Since $f^j(\omega) < 2$, explicit formula (8.3) implies that $\mathrm{Lip}_f(f^j(\omega)) = 1$. Now, use (9.22) to estimate $a_j^{\frac{1}{k_1-j+4}}$ and (9.28) to estimate $\left\|\Phi^*_{j-1}(x) - \Phi^*_{j-1}(\tilde{x})\right\|_\infty$. We obtain

$$\begin{aligned}\Big\|\Big(\Phi_j(\Phi^*_{j-1}(x)) - \Phi_j(\Phi^*_{j-1}(\tilde{x}))\Big) - \Big(\Phi^A_j(\Phi^*_{j-1}(x)) - \Phi^A_j(\Phi^*_{j-1}(\tilde{x}))\Big)\Big\|_{/\!/} \leq \\ h_\perp^{\frac{5}{4}\frac{1}{k_1-j+4}} 4^{j+2} k_1^2 \left\|x - \tilde{x}\right\|_\infty + \left\|\Phi^*_{j-1}(x) - \Phi^*_{j-1}(\tilde{x})\right\|_\perp\end{aligned}$$

If $j = 1$, use (8.7) to estimate the term $\left\|\Phi^*_{j-1}(x) - \Phi^*_{j-1}(\tilde{x})\right\|_\perp$. This gives

$$\begin{aligned}\Big\|\Big(\Phi_1(\Phi^*_0(x)) - \Phi_1(\Phi^*_0(\tilde{x}))\Big) - \Big(\Phi^A_1(\Phi^*_0(x)) - \Phi^A_1(\Phi^*_0(\tilde{x}))\Big)\Big\|_{/\!/} \\ \leq \left(h_\perp^{\frac{5}{4}\frac{1}{k_1+3}} 4^3 k_1^2 + h_\perp^{\frac{1}{4}}\right) \left\|x - \tilde{x}\right\|_\infty \\ \leq h_\perp^{\frac{5}{4}\frac{1}{k_1+3}} 4^4 k_1^2 \left\|x - \tilde{x}\right\|_\infty\end{aligned} \quad (9.32)$$

If $2 \leq j \leq k_1 - 1$, use (9.29) to estimate the term $\left\|\Phi^*_{j-1}(x) - \Phi^*_{j-1}(\tilde{x})\right\|_\perp$. This gives

$$\begin{aligned}\Big\|\Big(\Phi_j(\Phi^*_{j-1}(x)) - \Phi_j(\Phi^*_{j-1}(\tilde{x}))\Big) - \Big(\Phi^A_j(\Phi^*_{j-1}(x)) - \Phi^A_j(\Phi^*_{j-1}(\tilde{x}))\Big)\Big\|_{/\!/} \\ \leq \left(h_\perp^{\frac{5}{4}\frac{1}{k_1-j+4}} 4^{j+2} k_1^2 + a_{j-1}^{\frac{1}{k_1-(j-1)+4}} 4^{j+1} k_1^2\right) \left\|x - \tilde{x}\right\|_\infty \\ \leq h_\perp^{\frac{5}{4}\frac{1}{k_1+3}} 4^{j+3} k_1^2 \left\|x - \tilde{x}\right\|_\infty\end{aligned} \quad (9.33)$$

Plugging (9.31), (9.32) and (9.33) into (9.30), we get

$$N_{j,/\!/} \leq h_\perp^{\frac{5}{4}\frac{1}{k_1+3}} 4^{j+3} k_1^2 \left\|x - \tilde{x}\right\|_\infty + N_{j-1,/\!/}$$

By induction on j, it follows that

$$N_{k_1-1,/\!/} \leq h_\perp^{\frac{5}{4}\frac{1}{k_1+3}} 4^{k_1+3} k_1^3 \left\|x - \tilde{x}\right\|_\infty + N_{0,/\!/}$$

As a direct consequence of the Lipschitz estimate (8.8) applied to the epoch transition map Φ_0, we get

$$N_{0,/\!/} \leq h_\perp^{\frac{1}{4}} \left\|x - \tilde{x}\right\|_\infty + 16 k_1^2 \left\|x - \tilde{x}\right\|_\perp$$

Hence, provided that C is large enough,

$$N_{k_1-1,/\!/} \leq h_\perp^{\frac{1}{k_1+4}} \left\|x - \tilde{x}\right\|_\infty + 16 k_1^2 \left\|x - \tilde{x}\right\|_\perp$$

which is the desired estimate (9.15).

This concludes the proof of Proposition 9.4. □

9.2 Control of the double era transition map $\hat{\Phi}_{\omega,\mathbf{h}}$

Proof of Proposition 9.2. Let us fix $\hat{h} > 0$ small enough such that

$$\hat{h}\tilde{C}_6 \leq \frac{1}{4}\frac{\hat{K}_{\hat{f}} - K_c}{1 + \frac{1}{\hat{\sigma}}} \tag{9.34}$$

Let $\omega = [1 : k_1, k_2, \dots] \in]1, 2[\setminus \mathbb{Q}$. Let $\mathbf{h} = \left(\hat{h}, h_\perp, e^{-Cs_2(\omega)}\right)$, where $0 < h_\perp \leq e^{-Cs_4(\omega)}$ and C is some large constant. The idea is to decompose the double era transition map $\hat{\Phi}_{\omega,\mathbf{h}}$ as

$$\hat{\Phi}_{\omega,\mathbf{h}} = \bar{\Phi}_{\bar{f}(\omega),\mathbf{h}_{\bar{f}(\omega)}} \circ \bar{\Phi}_{\omega,\mathbf{h}} \tag{9.35}$$

where

$$\mathbf{h}_{\bar{f}(\omega)} = \left(\hat{h}, \left(\hat{h}k_2^{-\tilde{n}_5}\right)^{\tilde{C}_6(k_2+1)^3} \bar{m}(\bar{f}(\omega))^2, \left(\hat{h}k_2^{-\tilde{n}_5}\right)^{\tilde{C}_6(k_2+1)} \bar{m}(\bar{f}(\omega))^2\right)$$

One should remark that the section $S^s_{\omega,\mathbf{h}}$ is much more smaller than what we need. Indeed, recall from Lemma 9.3 that $\bar{m}(\omega) \geq \frac{1}{(4k_2k_3)^2}$ and $\bar{m}(\bar{f}(\omega)) \geq \frac{1}{(4k_3k_4)^2}$. Hence, according to (9.12) and (9.13), it would be enough to take $\mathbf{h}$ equal to

$$\left(\hat{h}, \left(\hat{h}k_1^{-\tilde{n}_5}\right)^{C(k_1+1)^3} \left(\hat{h}k_2^{-\tilde{n}_5}\right)^{C(k_2+1)^3} \bar{m}(\omega)^2\bar{m}(\bar{f}(\omega))^2, \left(\hat{h}k_1^{-\tilde{n}_5}\right)^{C(k_1+1)} \left(\hat{h}k_2^{-\tilde{n}_5}\right)^{C(k_2+1)} \bar{m}(\omega)^2\bar{m}(\bar{f}(\omega))^2\right)$$

with C large enough for (9.35) to hold true. Nevertheless, computations will be much more easier with our choice of size for the section $S^s_{\omega,\mathbf{h}}$. While being smaller than needed, it will still be large enough for the graph transformation to be implemented.

Since, for C large enough, the map $\bar{\Phi}_{\omega,\mathbf{h}}$ maps the section $S^s_{\omega,\mathbf{h}}$ into the section $S^s_{\bar{f}(\omega),\mathbf{h}'}$ where

$$\mathbf{h}' \stackrel{\text{def}}{=} \left(\hat{h}, h_\perp^{1+\frac{k_1}{4}}, \left(\hat{h}k_2^{-\tilde{n}_5}\right)^{\tilde{C}_6(k_2+1)} \bar{m}(\bar{f}(\omega))^2\right)$$

one can apply Proposition 9.4 twice to get Proposition 9.2: once with $\bar{\Phi}_{\bar{f}(\omega),\mathbf{h}_{\bar{f}(\omega)}}$ restricted to the section $S^s_{\bar{f}(\omega),\mathbf{h}'}$ and once with $\bar{\Phi}_{\omega,\mathbf{h}}$. □

9.3 Shadowing of a heteroclinic chain along an era

Let $\omega \in]1, 2[\setminus \mathbb{Q}$. Let us call the point $P^{s_i}_{\omega,\hat{h}}$ the *center* of the section $S^{s_i}_{\omega,\mathbf{h}}$. The following proposition states that if $x \in S_{\hat{h}}$ is close to the center $P^{s_i}_{\omega,\hat{h}}$ of the section $S^{s_i}_{\omega,\mathbf{h}}$, then the orbit segment $[x, \bar{\Phi}_{\omega,\mathbf{h}}(x)]$ passes through the sections $S^{s_2}_1$, $S^{s_1}_2$, $\dots$, $S^{s_1}_{k_1(\omega)-1}$ close to their respective centers, *i.e.* the intersection points of $[x, \bar{\Phi}_{\omega,\mathbf{h}}(x)]$ with the sections $S^{s_{l(j)}}_j$ (where $l(j) = 2$ if $j = 1$ and $l(j) = 1$ if $j \geq 2$) and the intersection points of the heteroclinic chain starting at $P^{s_i}_{\omega,\hat{h}}$ with the sections $S^{s_{l(j)}}_j$ are respectively close to each other. Moreover, the estimates can be made uniform on ω. Recall that $\Phi^*_j = \Phi_j \circ \cdots \circ \Phi_0$. Recall that $\tilde{n}_5$ is a constant fixed in Proposition 8.1. Recall that $s_2(\omega)$ and $s_4(\omega)$ are defined by (9.1).

Proposition 9.9 (Shadowing along an era)**.** *There exists a constant $\tilde{C}_9 \geq \tilde{C}_8$ such that the property below holds for $\omega \in]1, 2[\setminus \mathbb{Q}$, $i \in \{1, 2\}$ and $0 < \epsilon \leq 1$. Let $x \in S^{s_i}_{\omega,\mathbf{h}}$ where $\mathbf{h} = \left(\hat{h}, e^{-\tilde{C}_8 s_4(\omega)}, e^{-\tilde{C}_8 s_2(\omega)}\right)$. If*

$$\left\|x - P^{s_i}_{\omega,\hat{h}}\right\|_\infty \leq \epsilon e^{-\tilde{C}_9 s_4(\omega)}$$

then, for any $1 \leq j \leq k_1(\omega) - 1$,

$$\left\|\Phi^*_{j-1}(x) - \Phi^*_{j-1}\left(P^{s_i}_{\omega,\hat{h}}\right)\right\|_\infty \leq \epsilon\left(\hat{h}k_1^{-\tilde{n}_5}\right)^{\tilde{C}_5 f^j(\omega)} m(f^j(\omega)) \tag{9.36}$$

Proof. There exists a constant $\tilde{C}_9 \geq \tilde{C}_8$ such that for every $\omega = [1; k_1, k_2, \dots] \in]1,2[\setminus \mathbb{Q}$ and every $1 \leq j \leq k_1 - 1$, we have

$$50k_1^2 e^{-\tilde{C}_9 s_4(\omega)} \leq \left(\hat{h} k_1^{-\tilde{n}_5}\right)^{\tilde{C}_5 f^j(\omega)} m(f^j(\omega)) \tag{9.37}$$

Let $0 < \epsilon \leq 1$, $\omega = [1; k_1, k_2, \dots] \in]1,2[\setminus \mathbb{Q}$ and $x \in S^{s_i}_{\omega,\mathbf{h}}$ where $\mathbf{h} = \left(\hat{h}, e^{-\tilde{C}_8 s_4(\omega)}, e^{-\tilde{C}_8 s_2(\omega)}\right)$. Assume that

$$\frac{\left\| x - P^{s_i}_{\omega,\hat{h}} \right\|_\infty}{e^{-\tilde{C}_9 s_4(\omega)}} \leq \epsilon$$

Let $1 \leq j \leq k_1 - 1$. Using estimate (9.22) with $h_\perp = \left\| x - P^{s_i}_{\omega,\hat{h}} \right\|_\perp$, we get

$$\begin{aligned}
\left\| \Phi^*_{j-1}(x) - \Phi^*_{j-1}\left(P^{s_i}_{\omega,\hat{h}}\right) \right\|_\perp &\leq \left\| x - P^{s_i}_{\omega,\hat{h}} \right\|_\perp^{\frac{5}{4}} \\
&\leq \epsilon e^{-\tilde{C}_9 s_4(\omega)} \\
&\leq \epsilon \left(\hat{h} k_1^{-\tilde{n}_5}\right)^{\tilde{C}_5 f^j(\omega)} m(f^j(\omega)) \qquad \text{using (9.37)}
\end{aligned}$$

Remark that

$$\left\| \Phi^*_{j-1}(x) - \Phi^*_{j-1}\left(P^{s_i}_{\omega,\hat{h}}\right) \right\|_{/\!/} \leq \left\| \Phi^*_{j-1}(x) - \Phi^{*A}_{j-1}(x) \right\|_{/\!/} + \left\| \Phi^{*A}_{j-1}(x) - \Phi^*_{j-1}\left(P^{s_i}_{\omega,\hat{h}}\right) \right\|_{/\!/}$$

Using estimate (9.27) with $h_\perp = \left\| x - P^{s_i}_{\omega,\hat{h}} \right\|_\perp$, we get

$$\left\| \Phi^*_{j-1}(x) - \Phi^{*A}_{j-1}(x) \right\|_{/\!/} \leq 34k_1^2 \left\| x - P^{s_i}_{\omega,\hat{h}} \right\|_\perp \leq 34k_1^2 \epsilon e^{-\tilde{C}_9 s_4(\omega)}$$

Using Proposition 8.3 on the local Lipschitz constant for the Kasner map, we get

$$\left\| \Phi^{*A}_{j-1}(x) - \Phi^*_{j-1}\left(P^{s_i}_{\omega,\hat{h}}\right) \right\|_{/\!/} \leq \left(\prod_{l=0}^{j-1} \mathrm{Lip}_f(f^l(\omega)) \right) \left\| x - P^{s_i}_{\omega,\hat{h}} \right\|_{/\!/} \leq 16k_1^2 \epsilon e^{-\tilde{C}_9 s_4(\omega)}$$

Hence, using (9.37), we get

$$\left\| \Phi^*_{j-1}(x) - \Phi^*_{j-1}\left(P^{s_i}_{\omega,\hat{h}}\right) \right\|_{/\!/} \leq 50k_1^2 \epsilon e^{-\tilde{C}_9 s_4(\omega)} \leq \epsilon \left(\hat{h} k_1^{-\tilde{n}_5}\right)^{\tilde{C}_5 f^j(\omega)} m(f^j(\omega))$$

This concludes the proof of Proposition 9.9. □

Next proposition allows one to use Proposition 9.9 twice during a double era.

Proposition 9.10 (Shadowing along two eras)**.** *There exists a constant $\tilde{C}_{10} \geq \tilde{C}_9$ such that the property below holds for $\omega \in]1,2[\setminus \mathbb{Q}$, $i \in \{1,2\}$ and $0 < \epsilon \leq 1$. Let $x \in S^{s_i}_{\omega,\mathbf{h}}$ where $\mathbf{h} = \left(\hat{h}, e^{-\tilde{C}_8 s_4(\omega)}, e^{-\tilde{C}_8 s_2(\omega)}\right)$. Let $\bar{\Phi} = \bar{\Phi}_{\omega,\mathbf{h}}$. If*

$$\left\| x - P^{s_i}_{\omega,\hat{h}} \right\|_\infty \leq \epsilon e^{-\tilde{C}_{10} s_4(\omega)}$$

then,

$$\left\| \bar{\Phi}(x) - \bar{\Phi}\left(P^{s_i}_{\omega,\hat{h}}\right) \right\|_\infty \leq \epsilon e^{-\tilde{C}_9 s_4(\bar{\Phi}(\omega))} \tag{9.38}$$

Proof. This is a straightforward consequence of (9.12) and (9.13). Computations are similar to the ones in the proof of Proposition 9.9.

□

Chapter 10

Local stable manifolds of the double era return map

The purpose of this chapter is to construct some local stable manifolds for the double era return map $\hat{\Phi} := \hat{\Phi}_{\hat{h}}$. These local stable manifolds play a central role in the proof of our main Theorem. In Chapter 11, we will prove that any type IX orbit whose starting point lies in the local stable manifold of a point p will shadow the heteroclinic chain starting at p. In Chapter 12, we will prove that the union of these local stable manifolds over a positive 1-dimensional Lebesgue measure subset of the Kasner circle has positive 3-dimensional Lebesgue measure. Their construction rely on the estimates proven in Chapter 9.

Recall that for any $\omega \in]1,2[\setminus \mathbb{Q}$,

$$s_2(\omega) = k_1(\omega)^2 + k_2(\omega)^2 + k_3(\omega)^2 + k_4(\omega)^2$$
$$s_4(\omega) = k_1(\omega)^4 + k_2(\omega)^4 + k_3(\omega)^4 + k_4(\omega)^4$$

Recall that we have fixed a constant $\hat{h}$ in the preceding section (see (9.34) and Proposition 9.2). According to Proposition 9.2, the double era return map $\hat{\Phi}$ is well defined on the set

$$\bigcup_{\omega \in]1,2[\setminus\mathbb{Q}} S^s_{\omega,\mathbf{h}_\omega} \subset S_{\hat{h}}$$

where $\mathbf{h}_\omega = \left(\hat{h}, e^{-\tilde{C}_8 s_4(\omega)}, e^{-\tilde{C}_8 s_2(\omega)}\right)$.

Recall that $P_\omega = (0,0,0,\omega)$ denotes the point (in local coordinates) of Kasner parameter ω on the Kasner interval $\mathcal{K}_0$. Moreover, $P^{s_1}_{\omega,\hat{h}} = (0,\hat{h},0,\omega)$ and $P^{s_2}_{\omega,\hat{h}} = (0,0,\hat{h},\omega)$ denote the intersection points of the two type II orbits arriving at P_ω with the global section $S_{\hat{h}}$ (see definition 5.10).

Definition 10.1 (Local stable set)**.** Let $\omega \in]1,2[\setminus \mathbb{Q}$, $i \in \{1,2\}$ and $\eta > 0$. We call *local stable set of* $P^{s_i}_{\omega,\hat{h}}$ *of size* η and we denote by $W^s_\eta\left(P^{s_i}_{\omega,\hat{h}}, \hat{\Phi}\right)$ the set of all $x \in S_{\hat{h}}$ such that for every $n \geq 0$, $\hat{\Phi}^n(x)$ is well defined and satisfies

$$\left\|\hat{\Phi}^n(x) - \hat{\Phi}^n\left(P^{s_i}_{\omega,\hat{h}}\right)\right\|_\infty \leq \eta$$

We want to prove that for Lebesgue almost all $\omega \in]1,2[$ and for η small enough, the local stable set $W^s_\eta\left(P^{s_i}_{\omega,\hat{h}}, \hat{\Phi}\right)$ contains a Lipschitz graph.

Definition 10.2 (Rooted graph)**.** Let $\omega \in]1,2[$ and $0 < a \leq \hat{h}$. A *graph of size* a *rooted at* $P^{s_1}_{\omega,\hat{h}}$ is a set $\gamma \subset S^{s_1}_{\hat{h}}$ of the form

$$\gamma = \mathrm{Graph}^{s_1}(\zeta) \stackrel{\text{def}}{=} \left\{\left(x_u, \hat{h}, x_{s_2}, \zeta(x_u, x_{s_2})\right) \mid (x_u, x_{s_2}) \in [0,a]^2\right\}$$

where $\zeta : [0,a]^2 \to \mathbb{R}$ is a map such that $\zeta(0, x_{s_2}) = \omega$ for all $x_{s_2} \in [0,a]$. We define analogously the graphs rooted at $P^{s_2}_{\omega,\hat{h}}$.

Remark 10.3. We say that $\gamma = \mathrm{Graph}^{s_1}(\zeta)$ is a graph rooted at $P^{s_1}_{\omega,\hat{h}}$ because $P^{s_1}_{\omega,\hat{h}} = \left(0, \hat{h}, 0, \zeta(0,0)\right)$.

Remark 10.4. Recall from Proposition 4.2 that the local coordinates x_u, x_{s_1} and x_{s_2} are positive. This is why the map ζ is defined on $[0,a]^2$.

The map ζ is entirely determined by $\gamma = \mathrm{Graph}^{s_1}(\zeta)$. Recall that $\hat{\sigma}$ is a constant that has been fixed in the preceding section (see (9.3)). We say that a graph γ is $\hat{\sigma}$-Lipschitz if it is associated to a $\hat{\sigma}$-Lipschitz map ζ (for the infinite norm).

We can now state the main theorem of this chapter, which describes the local stable manifolds of the double era return map $\hat{\Phi}$. We refer to Theorem 10.20 for a version that characterizes the size of the local stable manifolds. Recall that $\tilde{C}_8$ is the constant fixed in Proposition 9.2 on the double era transition map.

Theorem 10.5 (Local stable manifolds of the double era return map)**.** *There exists a full Lebesgue measure set $\Omega_{graph} \subset \,]1,2[$ with the following properties. For all $\omega \in \Omega_{graph}$ and all $i \in \{1,2\}$, for η small enough, the local stable set $W^s_\eta\left(P^{s_i}_{\omega,\hat{h}}, \hat{\Phi}\right)$ of $P^{s_i}_{\omega,\hat{h}}$ of size η contains a $\hat{\sigma}$-Lipschitz graph of size η rooted at $P^{s_i}_{\omega,\hat{h}}$. Moreover, for all x belonging to this graph and all $n \geq 0$,*

$$\left\|\hat{\Phi}^n(x) - \hat{\Phi}^n\left(P^{s_i}_{\omega,\hat{h}}\right)\right\|_\infty \leq \left\|x - P^{s_i}_{\omega,\hat{h}}\right\|_\perp e^{-\tilde{C}_8 \sum_{j=1}^{2n} k_j(\omega)^5} \tag{10.1}$$

Remark 10.6. Note that $e^{-\tilde{C}_8 \sum_{j=1}^{2n} k_j(\omega)^5} \ll e^{-2\tilde{C}_8 n}$. Hence, we have a super exponential convergence to the Mixmaster attractor for the orbits starting in those graphs.

We are going to prove Theorem 10.5 using the so-called Hadamard graph transformation method. Let us describe informally our strategy, which may not be the most standard one.

The first step is to show that the double era return map $\hat{\Phi}$ satisfies some hyperbolic properties. We prove the existence of two invariant cone fields, namely: the unstable cone field containing the direction tangent to the Mixmaster attractor and the stable cone field containing the direction transverse to the Mixmaster attractor. The unstable cone field is forward invariant while the stable cone field is backward invariant. Moreover, the map $\hat{\Phi}$ expands the length of the vectors in the unstable cone field and contracts the length of the vectors in the stable cone field. See Proposition 10.8.

Once we know that the double era return map $\hat{\Phi}$ satisfies some hyperbolic properties, we can show that the preimage of a $\hat{\sigma}$-Lipschitz graph rooted at $P^{s_{\hat{\imath}(\omega)}}_{\hat{f}(\omega),\hat{h}}$ by the double era transition map $\hat{\Phi}_{|S^{s_i}_{\omega,\mathbf{h}}}$ is a $\hat{\sigma}$-Lipschitz graph rooted at $P^{s_i}_{\omega,\hat{h}}$. To make this statement correct, one must carefully choose the size of the graph rooted at $P^{s_{\hat{\imath}(\omega)}}_{\hat{f}(\omega),\hat{h}}$ and the size of the section $S^{s_i}_{\omega,\mathbf{h}}$. See Lemma 10.11.

Next step consists in constructing a space Γ of families of $\hat{\sigma}$-Lipschitz graphs invariant by the preimage procedure described in the above paragraph. For $\Omega_{\mathrm{graph}} \subset \,]1,2[\, \setminus \mathbb{Q}$ and $\hat{h}_\perp : \Omega_{\mathrm{graph}} \to \,]0,+\infty[$ fixed, define

$$\Gamma \stackrel{\text{def}}{=} \left\{\gamma = \left(\gamma_{\omega,i}\right)_{\omega \in \Omega_{\mathrm{graph}}, i \in \{1,2\}} \mid \gamma_{\omega,i} \text{ is a } \hat{\sigma}\text{-Lipschitz graph of size } \hat{h}_\perp(\omega) \text{ rooted at } P^{s_i}_{\omega,\hat{h}}\right\}$$

Roughly speaking, the graph transformation $\hat{\Phi}^* : \Gamma \to \Gamma$ is defined as follows. For $\gamma \in \Gamma$, $\left(\hat{\Phi}^*\gamma\right)_{\omega,i}$ is the preimage of $\gamma_{\hat{f}(\omega),\hat{\imath}(\omega)}$ by a suitable restriction of the double era return map $\hat{\Phi}_{|S^{s_i}_{\omega,\mathbf{h}}}$ (see definition 10.17). Our goal is to find a full Lebesgue measure set $\Omega_{\mathrm{graph}} \subset \,]1,2[$ invariant by the double era Kasner map $\hat{f}$ and a function $\hat{h}_\perp : \Omega_{\mathrm{graph}} \to \,]0,+\infty[$ such that the graph transformation $\hat{\Phi}^*$ defined above is well defined. See definition 10.14 and Proposition 10.15.

Using the hyperbolic properties of $\hat{\Phi}$, the graph transformation $\hat{\Phi}^*$ will be proved to be a contraction mapping. The standard contraction mapping theorem will provide a fixed point $\hat{\gamma} = \left(\hat{\gamma}_{\omega,i}\right) \in \Gamma$.

Final step consists in checking that $\hat{\gamma}_{\omega,i}$ is contained in the local stable set of the point $P^{s_i}_{\omega,\hat{h}}$ for the double era return map $\hat{\Phi}$ for all ω and i.

10.1 Cone field invariant by the double era return map

Recall that $\hat{\sigma}$ has been fixed in the preceding section (see (9.3)). This parameter will serve as the angle of the invariant cone field.

Definition 10.7 (Cones)**.** Let $\omega \in]1,2[\setminus \mathbb{Q}$, $\mathbf{h} = (\hat{h}, h_\perp, h_{/\!/})$ with $\min(h_\perp, h_{/\!/}) > 0$, $i \in \{1,2\}$ and $x \in S^{s_i}_{\omega,\mathbf{h}}$. We define the *tangential cone* at x as

$$V^{/\!/}_{\omega,\mathbf{h},i}(x) \stackrel{\text{def}}{=} \left\{\tilde{x} \in S^{s_i}_{\omega,\mathbf{h}} \mid \|x - \tilde{x}\|_\perp \leq \hat{\sigma}\, \|x - \tilde{x}\|_{/\!/}\right\}$$

As usual in hyperbolic dynamical system theory, we define the *interior* of $V^{/\!/}_{\omega,\mathbf{h},i}(x)$ as

$$\operatorname{Int} V^{/\!/}_{\omega,\mathbf{h},i}(x) \stackrel{\text{def}}{=} \{x\} \cup \left\{\tilde{x} \in S^{s_i}_{\omega,\mathbf{h}} \mid \|x - \tilde{x}\|_\perp < \hat{\sigma}\, \|x - \tilde{x}\|_{/\!/}\right\}$$

In other words, $\operatorname{Int} V^{/\!/}_{\omega,\mathbf{h},i}(x)$ is the union of the topological interior of $V^{/\!/}_{\omega,\mathbf{h},i}(x)$ and its vertex $\{x\}$. Analogously, we define the *transverse cone* at x as

$$V^{\perp}_{\omega,\mathbf{h},i}(x) \stackrel{\text{def}}{=} \left\{\tilde{x} \in S^{s_i}_{\omega,\mathbf{h}} \mid \|x - \tilde{x}\|_{/\!/} \leq \hat{\sigma}\, \|x - \tilde{x}\|_\perp\right\}$$

and the *interior* of $V^{\perp}_{\omega,\mathbf{h},i}(x)$ as

$$\operatorname{Int} V^{\perp}_{\omega,\mathbf{h},i}(x) \stackrel{\text{def}}{=} \{x\} \cup \left\{\tilde{x} \in S^{s_i}_{\omega,\mathbf{h}} \mid \|x - \tilde{x}\|_{/\!/} < \hat{\sigma}\, \|x - \tilde{x}\|_\perp\right\}$$

Recall that $K_c > 1$ is an explicit constant fixed in the preceding section (see (9.2)).

Proposition 10.8 (Hyperbolic properties of the double era return map)**.** *There exists a constant $\tilde{C}_{11} \geq \tilde{C}_{10}$ such that the properties below hold for $\omega \in]1,2[\setminus \mathbb{Q}$, $i \in \{1,2\}$, $\mathbf{h} = \left(\hat{h}, e^{-\tilde{C}_{11} s_4(\omega)}, e^{-\tilde{C}_8 s_2(\omega)}\right)$ and $\mathbf{h}' = \left(\hat{h}, \hat{h}, \hat{h}\right)$.*

(Forward invariance of the tangential cone field) *For all $x \in S^{s_i}_{\omega,\mathbf{h}}$,*

$$\hat{\Phi}\left(V^{/\!/}_{\omega,\mathbf{h},i}(x)\right) \subset \operatorname{Int} V^{/\!/}_{\hat{f}(\omega),\mathbf{h}',\hat{\imath}(\omega)}\left(\hat{\Phi}(x)\right) \tag{10.2}$$

(Backward invariance of the transverse cone field) *For all $x \in S^{s_i}_{\omega,\mathbf{h}}$,*

$$\left(\hat{\Phi}_{|S^{s_i}_{\omega,\mathbf{h}}}\right)^{-1}\left(V^{\perp}_{\hat{f}(\omega),\mathbf{h}',\hat{\imath}(\omega)}\left(\hat{\Phi}(x)\right)\right) \subset \operatorname{Int} V^{\perp}_{\omega,\mathbf{h},i}(x) \tag{10.3}$$

(Expansion in the tangential cone field) *For every $x, \tilde{x} \in S^{s_i}_{\omega,\mathbf{h}}$, if $\tilde{x} \in V^{/\!/}_{\omega,\mathbf{h},i}(x)$, then*

$$\left\|\hat{\Phi}(x) - \hat{\Phi}(\tilde{x})\right\|_{/\!/} \geq K_c\, \|x - \tilde{x}\|_{/\!/} \tag{10.4}$$

(Contraction in the transverse cone field) *For every $x, \tilde{x} \in S^{s_i}_{\omega,\mathbf{h}}$, if $\hat{\Phi}(\tilde{x}) \in V^{\perp}_{\hat{f}(\omega),\mathbf{h}',\hat{\imath}(\omega)}\left(\hat{\Phi}(x)\right)$, then*

$$\left\|\hat{\Phi}(x) - \hat{\Phi}(\tilde{x})\right\|_\perp \leq e^{-\tilde{C}_8\left(k_1(\omega)^5 + k_2(\omega)^5\right)}\, \|x - \tilde{x}\|_\perp \tag{10.5}$$

See figures 10.1 and 10.2.

Proof. Let $C \geq 100\tilde{C}_8$ such that

$$e^{-\frac{C}{100}}\left(1 + \frac{1}{\hat{\sigma}}\right)K_c^{-1} < \hat{\sigma} \tag{10.6}$$

Fix $\omega \in]1,2[\setminus \mathbb{Q}$ and $x, \tilde{x} \in S^{s_i}_{\omega,\mathbf{h}}$ ($i \in \{1,2\}$). Let $\mathbf{h} = \left(\hat{h}, e^{-C s_4(\omega)}, e^{-\tilde{C}_8 s_2(\omega)}\right)$ and $\mathbf{h}' = \left(\hat{h}, \hat{h}, \hat{h}\right)$.

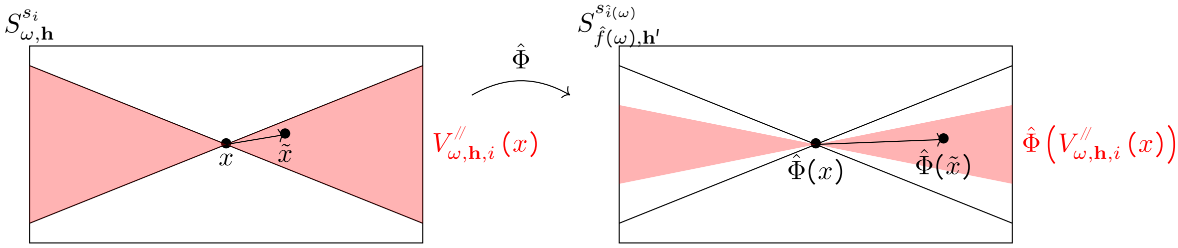

Figure 10.1 – Forward invariance of the tangential cone field.

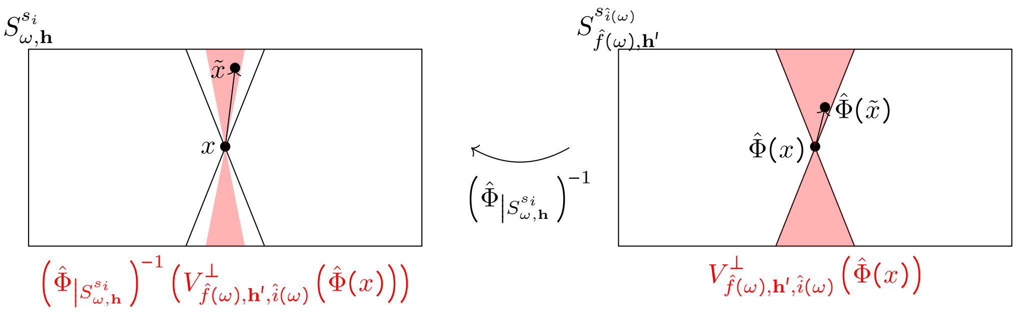

Figure 10.2 – Backward invariance of the transverse cone field.

Recall that $\hat{\sigma} < 1$. Hence,

$$\tilde{x} \in V^{/\!/}_{\omega,\mathbf{h},i}(x) \implies \|x - \tilde{x}\|_\infty = \|x - \tilde{x}\|_{/\!/} \tag{10.7a}$$

$$\tilde{x} \in V^{\perp}_{\omega,\mathbf{h},i}(x) \implies \|x - \tilde{x}\|_\infty = \|x - \tilde{x}\|_{\perp} \tag{10.7b}$$

Expansion estimate (10.4). According to (10.7a), if $\tilde{x} \in V^{/\!/}_{\omega,\mathbf{h},i}(x)$, then

$$\|x - \tilde{x}\|_\infty = \|x - \tilde{x}\|_{/\!/} \leq \left(1 + \frac{1}{\hat{\sigma}}\right) \|x - \tilde{x}\|_{/\!/}$$

Hence, expansion estimate (9.9) implies that the expansion estimate (10.4) holds true.

Forward invariance of the tangential cone field (10.2). If $\tilde{x} \in V^{/\!/}_{\omega,\mathbf{h},i}(x)$, then

$$\begin{aligned}
\left\|\hat{\Phi}(x) - \hat{\Phi}(\tilde{x})\right\|_{\perp} &\leq \left(e^{-Cs_4(\omega)}\right)^{\left(\frac{k_1}{100}+\frac{k_2}{100}\right)} \|x - \tilde{x}\|_\infty && \text{using (9.7)}\\
&\leq e^{-\frac{C}{100}} \|x - \tilde{x}\|_{/\!/} && \text{using (10.7a)}\\
&\leq e^{-\frac{C}{100}} K_c^{-1} \left\|\hat{\Phi}(x) - \hat{\Phi}(\tilde{x})\right\|_{/\!/} && \text{using (10.4)}\\
&< \hat{\sigma} \left\|\hat{\Phi}(x) - \hat{\Phi}(\tilde{x})\right\|_{/\!/} && \text{using (10.6)}
\end{aligned}$$

Hence, $\hat{\Phi}(\tilde{x}) \in \operatorname{Int} V^{/\!/}_{\hat{f}(\omega),\mathbf{h}',\hat{\imath}(\omega)}\left(\hat{\Phi}(x)\right)$.

Backward invariance of the transverse cone field (10.3). Assume that $\|x - \tilde{x}\|_{\perp} \leq \frac{1}{\hat{\sigma}} \|x - \tilde{x}\|_{/\!/}$. Hence,

$$\|x - \tilde{x}\|_\infty \leq \left(1 + \frac{1}{\hat{\sigma}}\right) \|x - \tilde{x}\|_{/\!/} \tag{10.8}$$

As a consequence, the expansion estimate (10.4) remains true in that case:

$$\left\|\hat{\Phi}(x) - \hat{\Phi}(\tilde{x})\right\|_{/\!/} \geq K_c \|x - \tilde{x}\|_{/\!/} \tag{10.9}$$

Hence,

$$\begin{aligned}\left\|\hat{\Phi}(x)-\hat{\Phi}(\tilde{x})\right\|_{\perp} &\leq \left(e^{-Cs_4(\omega)}\right)^{\left(\frac{k_1}{100}+\frac{k_2}{100}\right)}\|x-\tilde{x}\|_{\infty} && \text{using (9.7)}\\ &\leq e^{-\frac{C}{100}}\left(1+\frac{1}{\hat{\sigma}}\right)\|x-\tilde{x}\|_{/\!/} && \text{using (10.8)}\\ &\leq e^{-\frac{C}{100}}\left(1+\frac{1}{\hat{\sigma}}\right)K_c^{-1}\left\|\hat{\Phi}(x)-\hat{\Phi}(\tilde{x})\right\|_{/\!/} && \text{using (10.9)}\\ &< \frac{1}{\hat{\sigma}}\left\|\hat{\Phi}(x)-\hat{\Phi}(\tilde{x})\right\|_{/\!/} && \text{using (10.6) and } \hat{\sigma}\leq\frac{1}{\hat{\sigma}} \qquad (10.10)\end{aligned}$$

Backward invariance of the transverse cone field is a straightforward consequence of (10.10) by contraposition, *i.e.* (10.3) holds true.

Contraction estimate (10.5). Assume that $\hat{\Phi}(\tilde{x}) \in V^{\perp}_{\hat{f}(\omega),\mathbf{h}',\hat{\imath}(\omega)}\left(\hat{\Phi}(x)\right)$. By backward invariance of the trasnverse cone field, we have $\tilde{x} \in V^{\perp}_{\omega,\mathbf{h},i}(x)$. Hence,

$$\begin{aligned}\left\|\hat{\Phi}(x)-\hat{\Phi}(\tilde{x})\right\|_{\perp} &\leq \left(e^{-Cs_4(\omega)}\right)^{\left(\frac{k_1}{100}+\frac{k_2}{100}\right)}\|x-\tilde{x}\|_{\infty} && \text{using (9.7)}\\ &\leq e^{-\frac{C}{100}(k_1^5+k_2^5)}\|x-\tilde{x}\|_{\perp} && \text{using (10.7b)}\\ &\leq e^{-\tilde{C}_8(k_1^5+k_2^5)}\|x-\tilde{x}\|_{\perp} && \text{using } C\geq 100\tilde{C}_8\end{aligned}$$

which is the desired estimate (10.5). □

Remark 10.9. Let us describe precisely how we will use Proposition 10.8. The forward invariance of the tangential cone field (10.2) together with the expansion estimate (10.4) are used two times:

1. To show that the preimage of a $\hat{\sigma}$-Lipschitz graph is a graph.
2. To show that the graph transformation is a contraction mapping.

Knowing that the preimage of a $\hat{\sigma}$-Lipschitz graph rooted at $P^{s_{\hat{\imath}(\omega)}}_{\hat{f}(\omega),\hat{h}}$ by the double era transition map $\hat{\Phi}|_{S^{s_i}_{\omega,\mathbf{h}}}$ is a graph rooted at $P^{s_i}_{\omega,\hat{h}}$, the backward invariance of the transverse cone field (10.3) implies that this graph is also $\hat{\sigma}$-Lipschitz. This property is essential to show that the set of $\hat{\sigma}$-Lipschitz graphs families is invariant by the graph transformation.

Finally, the contraction estimate in the transverse cone field (10.5) is used to prove the exponential convergence (10.1) for a point in a graph constructed as the fixed point of the graph transformation. It also proves that this graph is contained in the local stable set of some point for the double era return map.

10.2 Local graph transformation

Our next task is to understand the preimage of a $\hat{\sigma}$-Lipschitz graph by the double era return map $\hat{\Phi}$. This is the purpose of Lemma 10.11 below. To make the computations in coordinates easier to follow, let us identify the section $S^{s_1}_{\hat{h}}$ with a subset of $\mathbb{R}^3$, forgetting the coordinate x_{s_1} which is constant equal to $\hat{h}$ on $S^{s_1}_{\hat{h}}$. More precisely, we identify the point $(x_u, \hat{h}, x_{s_2}, x_c) \in S^{s_1}_{\hat{h}}$ with $(x_\perp, x_{/\!/}) \in \mathbb{R}^2 \times \mathbb{R}$ where $x_\perp = (x_u, x_{s_2})$ and $x_{/\!/} = x_c$. We will use the same notation in $S^{s_2}_{\hat{h}}$, letting $x_\perp = (x_u, x_{s_1})$. We will not work in both sections at the same time, hence this notation will not be ambiguous. Remark that with these coordinates, a graph of size a rooted at $P^{s_i}_{\omega,\hat{h}}$ is a subset γ of $\mathbb{R}^2 \times \mathbb{R}$ of the form

$$\mathrm{Graph}(\zeta) = \left\{(x_\perp, \zeta(x_\perp)) \mid x_\perp \in [0,a]^2\right\}$$

where $\zeta : [0,a]^2 \to \mathbb{R}$ satisfies $\zeta(0,z) = \omega$ for all $z \in [0,a]$.

Remark 10.10. Recall from Proposition 4.2 that the local coordinates x_u, x_{s_1} and x_{s_2} are positive. This is why the map ζ is defined on $[0,a]^2$.

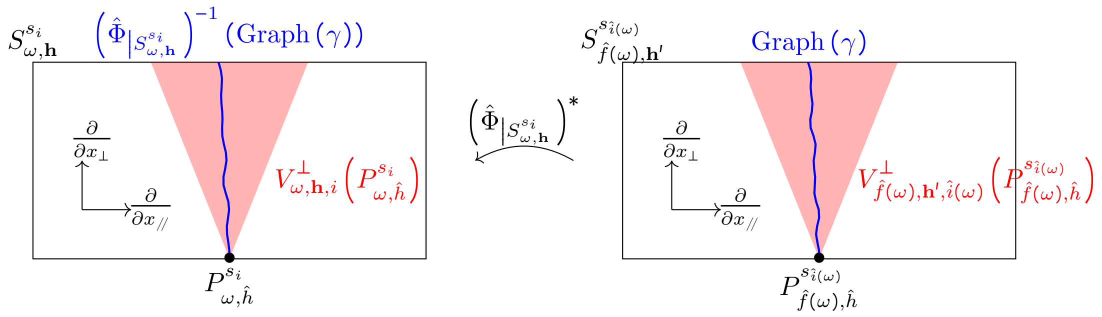

Figure 10.3 – The graph transformation.

Lemma 10.11 (Graph transformation over one point)**.** *There exists a constant $\tilde{C}_{12} > \tilde{C}_{11}$ such that the property below holds for $\omega \in \,]1,2[\, \setminus \mathbb{Q}$, $i \in \{1,2\}$, $0 < a \leq e^{-\tilde{C}_{12}s_4(\omega)}$, $a' \geq ae^{-\tilde{C}_8\left(k_1(\omega)^5+k_2(\omega)^5\right)}$ and $\mathbf{h} = \left(\hat{h}, a, e^{-\tilde{C}_8 s_2(\omega)}\right)$. If γ is a $\hat{\sigma}$-Lipschitz graph of size a' rooted at $P^{s_{\hat{\imath}(\omega)}}_{\hat{f}(\omega),\hat{h}}$, then its preimage by the double era transition map $\hat{\Phi}_{|S^{s_i}_{\omega,\mathbf{h}}}$ is a $\hat{\sigma}$-Lipschitz graph of size a rooted at $P^{s_i}_{\omega,\hat{h}}$. See figure 10.3.*

Remark 10.12. Beware of the fact that, in Lemma 10.11, γ denotes a single graph and not a family of graphs as in the definition of Γ.

Before we prove Lemma 10.11, let us state a simple property of the $\hat{\sigma}$-Lipschitz graphs.

Lemma 10.13. *Let $\omega \in \,]1,2[\, \setminus \mathbb{Q}$, $i \in \{1,2\}$, $\mathbf{h} = \left(\hat{h}, e^{-\tilde{C}_8 s_4(\omega)}, e^{-\tilde{C}_8 s_2(\omega)}\right)$, $0 < a \leq e^{-\tilde{C}_8 s_4(\omega)}$ and γ be a graph of size a rooted at $P^{s_i}_{\omega,\hat{h}}$. The graph γ is $\hat{\sigma}$-Lipschitz if and only if for all $x \in \gamma$,*

$$\gamma \subset V^\perp_{\omega,\mathbf{h},i}(x)$$

Proof of Lemma 10.13. This is a straightforward consequence of the definition of the cone $V^\perp_{\omega,\mathbf{h},i}(x)$. ∎

Proof of Lemma 10.11. Take $\tilde{C}_{12} \geq 4\tilde{C}_{11}$ large enough so that for every $\omega \in \,]1,2[\, \setminus \mathbb{Q}$,

$$\hat{\sigma}\left(e^{-\tilde{C}_{12}s_4(\omega)}\right)^{1+\frac{k_1}{4}+\frac{k_2}{4}} + 306e^{-\tilde{C}_{12}s_4(\omega)}k_1(\omega)^2 k_2(\omega)^2 \leq 4e^{-\tilde{C}_8 s_2(\omega)} \tag{10.11}$$

Fix $\omega = [1; k_1, k_2, \dots]$, i, a, a' and γ as in the statement. There exists a unique map $\zeta : [0,a]^2 \to \mathbb{R}$ such that $\gamma = \text{Graph}^{s_{\hat{\imath}(\omega)}}(\zeta)$. Let $\mathbf{h} = \left(\hat{h}, a, e^{-\tilde{C}_8 s_2(\omega)}\right)$, $\mathbf{h}' = \left(\hat{h}, \hat{h}, \hat{h}\right)$, $\hat{\Phi}^{s_i} = \hat{\Phi}_{|S^{s_i}_{\omega,\mathbf{h}}}$ and $\hat{\Phi}^{A,s_i} = \hat{\Phi}^A_{|S^{s_i}_{\omega,\mathbf{h}}}$. To prove that $(\hat{\Phi}^{s_i})^{-1}(\gamma)$ is a graph of size a rooted at $P^{s_i}_{\omega,\hat{h}}$, we first need a technical claim. Define $I_\omega := \left[\omega - e^{-\tilde{C}_8 s_2(\omega)}, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right]$.

Claim 1. *Fix $x_\perp \in [0,a]^2$. Define a map $\lambda : I_\omega \to \mathbb{R}$ by the formula*

$$\lambda(x_{/\!/}) \overset{def}{=} \left(\hat{\Phi}^{s_i}(x_\perp, x_{/\!/})\right)_{/\!/} - \zeta\left(\left(\hat{\Phi}^{s_i}(x_\perp, x_{/\!/})\right)_\perp\right)$$

The map λ is well defined. Moreover, $\lambda\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right) \geq 0$ and $\lambda\left(\omega - e^{-\tilde{C}_8 s_2(\omega)}\right) \leq 0$. See figure 10.4.

Proof of claim 1. Recall that $\tilde{C}_{12} \geq 4\tilde{C}_{11} \geq 4\tilde{C}_8$. According to (9.5), for every $x_{/\!/} \in I_\omega$, we have

$$\left\|\hat{\Phi}^{s_i}(x_\perp, x_{/\!/}) - \hat{\Phi}^{A,s_i}(x_\perp, x_{/\!/})\right\|_\perp \leq a^{1+\frac{k_1+k_2}{4}} \leq ae^{-\tilde{C}_8\left(k_1^5+k_2^5\right)} \leq a'$$

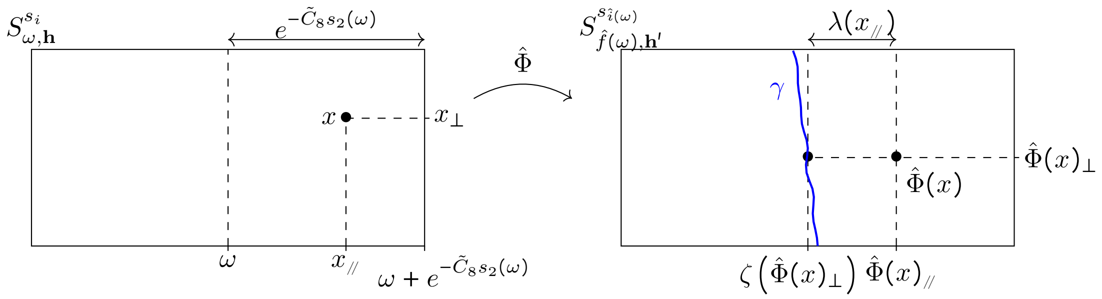

Figure 10.4 – Interpretation of the map λ. If $x_{/\!/} = \omega + e^{-\tilde{C}_8 s_2(\omega)}$, then $\hat{\Phi}(\tilde{x})$ is on the right side of γ. If $x_{/\!/} = \omega - e^{-\tilde{C}_8 s_2(\omega)}$, then $\hat{\Phi}(\tilde{x})$ is on the left side of γ.

This means that $\left(\hat{\Phi}^{s_i}(x_\perp, x_{/\!/})\right)_\perp \in \left[0, a'\right]^2$. Since γ is a graph of size a', the map λ is well defined. Roughly speaking, since $x_\perp$ is "small", $\left(\hat{\Phi}^{s_i}(x_\perp, x_{/\!/})\right)_{/\!/}$ is close to $\hat{f}(x_{/\!/})$ and $\left(\hat{\Phi}^{s_i}(x_\perp, x_{/\!/})\right)_\perp$ is close to $(0,0)$. Using this approximation and the fact that γ is a graph rooted at $P^{s_{\hat{\imath}(\omega)}}_{\hat{f}(\omega),\hat{h}}$, we get

$$\lambda\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right) \simeq \hat{f}\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right) - \hat{f}(\omega)$$

Using the fact that $\hat{f}$ is increasing on I_ω (see Proposition B.7), we get

$$\lambda\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right) \geq 0$$

We are now going to make rigorous this computation. Remark that

$$\begin{aligned}\lambda\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right) = &\left[\left(\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right)_{/\!/} - \hat{f}\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right] \\ &+ \left[\hat{f}\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right) - \hat{f}(\omega)\right] + \left[\hat{f}(\omega) - \zeta\left(\left(\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right)_\perp\right)\right]\end{aligned}$$

We will compare the three terms in the right-hand side of the above equation. Recall from (5.7) that

$$\hat{\Phi}^{A,s_i}(x) = \begin{cases}(0, \hat{h}, 0, \hat{f}(x_c)) & \text{if } k_2(x_c) \geq 2 \\ (0, 0, \hat{h}, \hat{f}(x_c)) & \text{if } k_2(x_c) = 1\end{cases} \qquad \text{where } x_c = [1; k_1(x_c), k_2(x_c), \dots]$$

Hence, estimate (9.6) gives

$$\left|\left(\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right)_{/\!/} - \hat{f}\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right| \leq 306 a k_1(\omega)^2 k_2(\omega)^2 \tag{10.12}$$

According to Proposition B.7 about the Gauss transformation, we have the expansion estimate

$$\left|\hat{f}\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right) - \hat{f}(\omega)\right| \geq 4 e^{-\tilde{C}_8 s_2(\omega)} \tag{10.13}$$

Moreover, using the fact that γ is a graph rooted at $P^{s_{\hat{\imath}(\omega)}}_{\hat{f}(\omega),\hat{h}}$, we get

$$\left|\hat{f}(\omega) - \zeta\left(\left(\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right)_\perp\right)\right| = \left|\zeta(0,0) - \zeta\left(\left(\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right)_\perp\right)\right| \tag{10.14}$$

Since γ is $\hat{\sigma}$-Lipschitz, it follows that

$$\begin{aligned}\left|\zeta(0,0) - \zeta\left(\left(\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right)_\perp\right)\right| &\leq \hat{\sigma}\left\|\left(\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right)_\perp\right\|_\infty \\ &\leq \hat{\sigma}\left\|\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right) - \hat{\Phi}^{A,s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right\|_\perp\end{aligned} \tag{10.15}$$

According to the estimate (9.5), we have

$$\left\|\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right) - \hat{\Phi}^{A,s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right\|_\perp \leq a^{1+\frac{k_1}{4}+\frac{k_2}{4}} \tag{10.16}$$

Putting together (10.14), (10.15) and (10.16), we get

$$\left|\hat{f}(\omega) - \zeta\left(\left(\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right)_\perp\right)\right| \leq \hat{\sigma} a^{1+\frac{k_1}{4}+\frac{k_2}{4}} \tag{10.17}$$

It follows from (10.12), (10.13), (10.17) and (10.11) that

$$\begin{aligned}\left|\left(\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right)_{/\!/} - \hat{f}\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right| + \left|\hat{f}(\omega) - \zeta\left(\left(\hat{\Phi}^{s_i}\left(x_\perp, \omega + e^{-\tilde{C}_8 s_2(\omega)}\right)\right)_\perp\right)\right| \\ \leq \left|\hat{f}\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right) - \hat{f}(\omega)\right|\end{aligned}$$

Hence, $\lambda\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right)$ and $\left(\hat{f}\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right) - \hat{f}(\omega)\right)$ have the same sign. Since $\hat{f}$ is increasing on I_ω, we get that $\lambda\left(\omega + e^{-\tilde{C}_8 s_2(\omega)}\right) \geq 0$. The arguments are analogous for $\lambda\left(\omega - e^{-\tilde{C}_8 s_2(\omega)}\right)$ □

Claim 2. $(\hat{\Phi}^{s_i})^{-1}(\gamma)$ *is a graph of size a rooted at* $P^{s_i}_{\omega,\hat{h}}$.

Proof of claim 2. Fix $x_\perp \in [0,a]^2$. First, let us prove that there exists a unique $x_{/\!/} \in I_\omega$, denoted by $(\hat{\Phi}^{s_i})^*\zeta(x_\perp)$, such that $\hat{\Phi}^{s_i}(x_\perp, x_{/\!/}) \in \gamma$. This will show that

$$\left(\hat{\Phi}^{s_i}\right)^{-1}(\gamma) = \left\{\left(x_\perp, (\hat{\Phi}^{s_i})^*\zeta(x_\perp)\right) \mid x_\perp \in [0,a]^2\right\} = \mathrm{Graph}^{s_i}\left((\hat{\Phi}^{s_i})^*\zeta\right)$$

Remark that for $x = (x_\perp, x_{/\!/})$ with $x_{/\!/} \in I_\omega$,

$$\hat{\Phi}^{s_i}(x) \in \gamma \iff \left(\hat{\Phi}^{s_i}(x)\right)_{/\!/} = \zeta\left(\left(\hat{\Phi}^{s_i}(x)\right)_\perp\right) \iff \lambda(x_{/\!/}) = 0$$

Since ζ is $\hat{\sigma}$-Lipschitz and $\hat{\Phi}^{s_i}$ is continuous, λ is continuous on I_ω. According to claim 1 and the intermediate value theorem, there exists $x_{/\!/} \in I_\omega$ such that $\lambda(x_{/\!/}) = 0$.

Let $x_{/\!/}, \tilde{x}_{/\!/} \in I_\omega$ such that $\lambda(x_{/\!/}) = \lambda(\tilde{x}_{/\!/}) = 0$. Set $x = (x_\perp, x_{/\!/}) \in S^{s_i}_{\omega,\mathbf{h}}$ and $\tilde{x} = (x_\perp, \tilde{x}_{/\!/}) \in S^{s_i}_{\omega,\mathbf{h}}$. By definition of the tangential cone, we have

$$\tilde{x} \in V^{/\!/}_{\omega,\mathbf{h},j}(x)$$

and by forward invariance of the tangential cone field (see (10.2)), we get

$$\hat{\Phi}^{s_i}(\tilde{x}) \in V^{/\!/}_{\hat{f}(\omega),\mathbf{h}',\hat{\imath}(\omega)}\left(\hat{\Phi}^{s_i}(x)\right)$$

Moreover, $\hat{\Phi}^{s_i}(x)$ and $\hat{\Phi}^{s_i}(\tilde{x})$ both belong to γ which is a $\hat{\sigma}$-Lipschitz graph so Lemma 10.13 implies that

$$\hat{\Phi}^{s_i}(\tilde{x}) \in V^{\perp}_{\hat{f}(\omega),\mathbf{h}',\hat{\imath}(\omega)}\left(\hat{\Phi}^{s_i}(x)\right)$$

It follows that

$$\hat{\Phi}^{s_i}(\tilde{x}) = \hat{\Phi}^{s_i}(x)$$

Using the expansion estimate (10.4) in the direction tangent to the Mixmaster attractor, we get $x_{/\!/} = \tilde{x}_{/\!/}$.

Let $z \in [0, a]$. We have

$$\zeta\left(\left(\hat{\Phi}^{s_i}((0,z),\omega)\right)_{\perp}\right) = \zeta(0,0) = \hat{f}(\omega) = \left(\hat{\Phi}^{s_i}((0,z),\omega)\right)_{/\!/}$$

Hence, $\hat{\Phi}^{s_i}((0,z),\omega) \in \gamma$. By uniqueness, we get that $(\hat{\Phi}^{s_i})^* \zeta(0,z) = \omega$. This concludes the proof of claim 2. □

Claim 3. $(\hat{\Phi}^{s_i})^{-1}(\gamma)$ *is a $\hat{\sigma}$-Lipschitz graph.*

Proof of claim 3. Let $x_\perp, \tilde{x}_\perp \in [0,a]^2$. Set

$$x = \left(x_\perp, (\hat{\Phi}^{s_i})^* \zeta(x_\perp)\right) \in (\hat{\Phi}^{s_i})^{-1}(\gamma) \qquad \text{and} \qquad \tilde{x} = \left(\tilde{x}_\perp, (\hat{\Phi}^{s_i})^* \zeta(\tilde{x}_\perp)\right) \in (\hat{\Phi}^{s_i})^{-1}(\gamma)$$

The graph γ is $\hat{\sigma}$-Lipschitz so Lemma 10.13 implies that

$$\hat{\Phi}^{s_i}(\tilde{x}) \in V^{\perp}_{\hat{f}(\omega),\mathbf{h}',\hat{\imath}(\omega)}\left(\hat{\Phi}^{s_i}(x)\right)$$

and by backward invariance of the transverse cone field (see (10.3)), we get

$$\tilde{x} \in V^{\perp}_{\omega,\mathbf{h},i}(x)$$

Using Lemma 10.13 once again, we get that $(\hat{\Phi}^{s_i})^{-1}(\gamma) = \mathrm{Graph}^{s_i}\left((\hat{\Phi}^{s_i})^* \zeta\right)$ is $\hat{\sigma}$-Lipschitz. □

This concludes the proof of Lemma 10.11. □

10.3 The set of admissible points for the graph transformation

According to Lemma 10.11, we have a graph transformation over *one* point $\omega \in]1,2[\setminus \mathbb{Q}$, pulling-back a graph rooted at $P^{s_{\hat{\imath}(\omega)}}_{\hat{f}(\omega),\hat{h}}$. Recall that our goal is to define a graph transformation over a *full-measure set* $\Omega_{\text{graph}} \subset]1,2[\setminus \mathbb{Q}$, called the set of *admissible points for the graph transformation*. To do this, we need to *iterate* the procedure described in Lemma 10.11. This means that the set Ω_{graph} must be invariant under the Kasner double era map $\hat{f}$. Hence, it is a reunion of orbits $(\omega, \hat{f}(\omega), \hat{f}^2(\omega), \dots)$.

Roughly speaking, the idea is to attach a graph to each point $\omega \in \Omega_{\text{graph}}$ and to replace the graph rooted at $P^{s_i}_{\omega,\hat{h}}$ by the preimage of the graph rooted at $P^{s_{\hat{\imath}(\omega)}}_{\hat{f}(\omega),\hat{h}}$ by the double era return map $\hat{\Phi}$. If we go into the technical details, there are two sections $S^{s_1}_\omega$ and $S^{s_2}_\omega$ above each point $\omega \in \Omega_{\text{graph}}$ so we need to consider two graphs rooted at each point. Let us temporarily simplify the discussion by acting as if there were only one section, say S^s_ω.

The graph transformation acts above the orbit $(\omega, \hat{f}(\omega), \hat{f}^2(\omega), \dots)$ as follows: for all $n \geq 0$, the graph rooted at $P^s_{\hat{f}^n(\omega),\hat{h}}$ is replaced by the preimage of the graph rooted at $P^s_{\hat{f}^{n+1}(\omega),\hat{h}}$ by the double era return map. Informally, the graph transformation is well defined above the orbit $(\omega, \hat{f}(\omega), \hat{f}^2(\omega), \dots)$ if there exists a sequence $(a_n)_{n\geq 0}$ of positive real numbers such that for any family $(\gamma_n)_{n\geq 0}$ where γ_n is a $\hat{\sigma}$-Lipschitz graph of size a_n rooted at $P^s_{\hat{f}^n(\omega),\hat{h}}$, the following property holds: for all $n \geq 0$, the preimage of the graph γ_{n+1} by the double era return map defines a $\hat{\sigma}$-Lipschitz graph of size a_n rooted at $P^s_{\hat{f}^n(\omega),\hat{h}}$. Remark that for all $n \geq 0$, $k_1(\hat{f}^n(\omega)) = k_{2n+1}(\omega)$, $k_2(\hat{f}^n(\omega)) = k_{2n+2}(\omega)$, etc. Hence, Lemma 10.11 gives a sufficient condition: if there exists a sequence $(a_n)_{n\geq 0}$ of positive real numbers such that for every $n \geq 0$,

$$a_n \leq e^{-\tilde{C}_{12} s_4(\hat{f}^n(\omega))} = e^{-\tilde{C}_{12}\left(k_{2n+1}(\omega)^4 + k_{2n+2}(\omega)^4 + k_{2n+3}(\omega)^4 + k_{2n+4}(\omega)^4\right)}$$

and

$$a_n e^{-\tilde{C}_8\left(k_{2n+1}(\omega)^5 + k_{2n+2}(\omega)^5\right)} = a_n e^{-\tilde{C}_8\left(k_1(\hat{f}^n(\omega))^5 + k_2(\hat{f}^n(\omega))^5\right)} \leq a_{n+1}$$

then the graph transformation is well defined above the orbit $(\omega, \hat{f}(\omega), \hat{f}^2(\omega), \dots)$. This leads to the following definition.

Definition 10.14 (Admissible points for the graph transformation)**.** Let $\omega \in \,]1,2[\, \setminus \mathbb{Q}$ and $h_\perp > 0$. We associate with ω and $h_\perp$ a sequence $(a_n(\omega, h_\perp))_{n\geq 0}$ defined by

$$\begin{cases} a_0(\omega, h_\perp) = h_\perp \\ a_{n+1}(\omega, h_\perp) = a_n(\omega, h_\perp) e^{-\tilde{C}_8\left(k_{2n+1}(\omega)^5 + k_{2n+2}(\omega)^5\right)} \end{cases}$$

We say that ω is *admissible* (for the graph transformation) if there exists $h_\perp > 0$ such that for every $n \geq 0$, $a_n(\omega, h_\perp) \leq e^{-\tilde{C}_{12} s_4(\hat{f}^n(\omega))}$. If ω is admissible, we define

$$\hat{h}_\perp(\omega) \stackrel{\text{def}}{=} \sup\left\{h_\perp > 0 \mid \forall n \geq 0,\ a_n(\omega, h_\perp) \leq e^{-\tilde{C}_{12} s_4(\hat{f}^n(\omega))}\right\} \tag{10.18}$$

We denote by Ω_{graph} the set of all admissible points in $]1,2[\, \setminus \mathbb{Q}$.

Proposition 10.15. *The set of the admissible points is invariant in the future and the past by the Kasner double era map,* i.e. $\hat{f}^{-1}(\Omega_{graph}) = \Omega_{graph}$.

Proof. One can remark that for every $\omega \in \,]1,2[\, \setminus \mathbb{Q}$, for every $h_\perp > 0$ and for every $n \geq 0$,

$$a_n\left(\hat{f}(\omega), a_1(\omega, h_\perp)\right) = a_{n+1}(\omega, h_\perp) \tag{10.19}$$

Proposition 10.15 is a straightforward consequence of formula (10.19). □

Recall that $\omega \in \,]1,2[\, \setminus \mathbb{Q}$ is said to satisfy the moderate growth condition if

$$k_{n+4}(\omega)^4 = o_{n\to+\infty}\left(\sum_{i=1}^{n} k_i(\omega)^5\right) \tag{MG}$$

Also, recall that the moderate growth condition is Lebesgue generic (see Lemma B.1).

Proposition 10.16 (Genericity of the admissible points)**.** *Any point $\omega \in \,]1,2[\, \setminus \mathbb{Q}$ satisfying the moderate growth condition* (MG) *is admissible. In particular, Ω_{graph} is a Lebesgue full measure subset of* $]1,2[\, \setminus \mathbb{Q}$.

Proof. Let $\omega \in \,]1,2[\, \setminus \mathbb{Q}$. Observe that

$$a_n(\omega, h_\perp) = h_\perp e^{-\tilde{C}_8 \sum_{i=1}^{2n} k_i(\omega)^5},$$

As a consequence, $\omega \in \Omega_{\text{graph}}$ as soon as

$$s_4(\hat{f}^n(\omega)) = o_{n\to+\infty}\left(\sum_{i=1}^{2n} k_i(\omega)^5\right) \tag{10.20}$$

On the other hand, (MG) clearly implies (10.20). □

10.4 Global graph transformation $\hat{\Phi}^*$

Now that the set Ω_{graph} and the function $\hat{h}_\perp : \Omega_{\text{graph}} \to \,]0,+\infty[$ are defined, recall that

$$\Gamma \stackrel{\text{def}}{=} \left\{\gamma = \left(\gamma_{\omega,i}\right)_{\omega\in\Omega_{\text{graph}}, i\in\{1,2\}} \mid \gamma_{\omega,i} \text{ is a } \hat{\sigma}\text{-Lipschitz graph of size } \hat{h}_\perp(\omega) \text{ rooted at } P^{s_i}_{\omega,\hat{h}}\right\}$$

Beware of the fact that in this definition, γ is not a graph but a family of graphs.

For $\omega \in \,]1,2[\, \setminus \mathbb{Q}$, define a "canonical" triplet of parameters

$$\hat{\mathbf{h}}_\omega \stackrel{\text{def}}{=} \left(\hat{h}, \hat{h}_\perp(\omega), e^{-\tilde{C}_8 s_2(\omega)}\right)$$

The double era return map defines a natural transformation $\hat{\Phi}^* : \Gamma \to \Gamma$.

Definition 10.17 (Graph transformation)**.** The *graph transformation* $\hat{\Phi}^* : \Gamma \to \Gamma$ is defined by the formula

$$\left(\hat{\Phi}^* \gamma\right)_{\omega,i} \stackrel{\text{def}}{=} \left(\hat{\Phi}_{|S^{s_i}_{\omega,\hat{\mathbf{h}}_\omega}}\right)^{-1} \left(\gamma_{\hat{f}(\omega),\hat{\imath}(\omega)}\right)$$

for all $\gamma \in \Gamma$, all $\omega \in \Omega_{\text{graph}}$ and all $i \in \{1,2\}$.

Proposition 10.18. *The graph transformation $\hat{\Phi}^* : \Gamma \to \Gamma$ is well defined.*

Proof. This is a straightforward consequence of Lemma 10.11 and Proposition 10.15. □

10.5 Local stable manifolds of the double era return map

For $\gamma \in \Gamma$, there exists a unique family of maps $\zeta = \left(\zeta_{\omega,i}\right)$ where $\zeta_{\omega,i}$ is a map from $\left[0, \hat{h}_\perp(\omega)\right]^2$ to $\mathbb{R}$ and $\gamma_{\omega,i} = \text{Graph}^{s_i}\left(\zeta_{\omega,i}\right)$. We will denote $\gamma = \text{Graph}\left(\zeta\right)$.

We endow Γ with the distance

$$d_{\text{graph}}\left(\text{Graph}\left(\zeta\right), \text{Graph}\left(\tilde{\zeta}\right)\right) \stackrel{\text{def}}{=} \sup_{\omega\in\Omega_{\text{graph}}, i\in\{1,2\}} \left\|\zeta_{\omega,i} - \tilde{\zeta}_{\omega,i}\right\|_{\infty,\left[0,\hat{h}_\perp(\omega)\right]^2}$$

where

$$\left\|\zeta_{\omega,i} - \tilde{\zeta}_{\omega,i}\right\|_{\infty,\left[0,\hat{h}_\perp(\omega)\right]^2} \stackrel{\text{def}}{=} \sup_{x_\perp\in\left[0,\hat{h}_\perp(\omega)\right]^2} \left\|\zeta_{\omega,i}(x_\perp) - \tilde{\zeta}_{\omega,i}(x_\perp)\right\|_\infty$$

Remark that $\left(\Gamma, d_{\text{graph}}\right)$ is a complete space.

Lemma 10.19 (Fixed point of the graph transformation)**.** *The graph transformation $\hat{\Phi}^*$ is a contraction mapping of the complete metric space $\left(\Gamma, d_{graph}\right)$ with*

$$\text{Lip}\,\hat{\Phi}^* \leq \frac{1}{K_c(1-\hat{\sigma}^2)}$$

As a consequence, $\hat{\Phi}^*$ admits a unique fixed point in Γ, denoted by $\hat{\gamma} = \text{Graph}\left(\hat{\zeta}\right)$.

Proof. Let $\gamma = \text{Graph}\left(\zeta\right), \tilde{\gamma} = \text{Graph}\left(\tilde{\zeta}\right) \in \Gamma$. We are going to prove that

$$d_{\text{graph}}\left(\hat{\Phi}^*\gamma, \hat{\Phi}^*\tilde{\gamma}\right) \leq \frac{1}{K_c(1-\hat{\sigma}^2)} d_{\text{graph}}\left(\gamma, \tilde{\gamma}\right)$$

If we denote $\text{Graph}\left(\hat{\Phi}^*\zeta\right) = \hat{\Phi}^*\gamma$ and $\text{Graph}\left(\hat{\Phi}^*\tilde{\zeta}\right) = \hat{\Phi}^*\tilde{\gamma}$, then it is enough to prove that for all $\omega \in \Omega_{\text{graph}}$ and all $i \in \{1,2\}$,

$$\left\|\left(\hat{\Phi}^*\zeta\right)_{\omega,i} - \left(\hat{\Phi}^*\tilde{\zeta}\right)_{\omega,i}\right\|_{\infty,\left[0,\hat{h}_\perp(\omega)\right]^2} \leq \frac{1}{K_c(1-\hat{\sigma}^2)} \left\|\zeta_{\hat{f}(\omega),\hat{\imath}(\omega)} - \tilde{\zeta}_{\hat{f}(\omega),\hat{\imath}(\omega)}\right\|_{\infty,\left[0,\hat{h}_\perp(\hat{f}(\omega))\right]^2} \tag{10.21}$$

Let $\omega \in \Omega_{\text{graph}}$, $i \in \{1,2\}$ and $y \in [0, \hat{h}_\perp(\omega)]^2$. Consider two points with the same first coordinate:

$$x = \left(y, \left(\hat{\Phi}^*\zeta\right)_{\omega,i}(y)\right) \in \left(\hat{\Phi}^*\gamma\right)_{\omega,i} \qquad z = \hat{\Phi}(x) \in \gamma_{\hat{f}(\omega),\hat{\imath}(\omega)}$$
$$\tilde{x} = \left(y, \left(\hat{\Phi}^*\tilde{\zeta}\right)_{\omega,i}(y)\right) \in \left(\hat{\Phi}^*\tilde{\gamma}\right)_{\omega,i} \qquad \tilde{z} = \hat{\Phi}(\tilde{x}) \in \tilde{\gamma}_{\hat{f}(\omega),\hat{\imath}(\omega)}$$

Since $x_\perp = \tilde{x}_\perp$, we have $\|x - \tilde{x}\|_\perp = 0$. It follows that $\tilde{x} \in V^{/\!/}_{\omega,\hat{\mathbf{h}}_\omega,i}(x)$. By forward invariance of the tangential cone field (see (10.2)), we have $\tilde{z} \in V^{/\!/}_{\hat{f}(\omega),\hat{\mathbf{h}}_{\hat{f}(\omega)},\hat{\imath}(\omega)}(z)$ which means

$$\|z - \tilde{z}\|_\perp \leq \hat{\sigma}\,\|z - \tilde{z}\|_{/\!/} \tag{10.22}$$

Hence,

$$\begin{aligned}
\|z-\tilde{z}\|_{/\!/} &= \left|\zeta_{\hat{f}(\omega),\hat{\imath}(\omega)}(z_\perp) - \tilde{\zeta}_{\hat{f}(\omega),\hat{\imath}(\omega)}(\tilde{z}_\perp)\right| \\
&\le \left|\zeta_{\hat{f}(\omega),\hat{\imath}(\omega)}(z_\perp) - \zeta_{\hat{f}(\omega),\hat{\imath}(\omega)}(\tilde{z}_\perp)\right| + \left\|\zeta_{\hat{f}(\omega),\hat{\imath}(\omega)} - \tilde{\zeta}_{\hat{f}(\omega),\hat{\imath}(\omega)}\right\|_{\infty,\left[0,\hat{h}_\perp(\hat{f}(\omega))\right]^2} \\
&\le \hat{\sigma}\,\|z-\tilde{z}\|_\perp + \left\|\zeta_{\hat{f}(\omega),\hat{\imath}(\omega)} - \tilde{\zeta}_{\hat{f}(\omega),\hat{\imath}(\omega)}\right\|_{\infty,\left[0,\hat{h}_\perp(\hat{f}(\omega))\right]^2} \\
&\le \hat{\sigma}^2\,\|z-\tilde{z}\|_{/\!/} + \left\|\zeta_{\hat{f}(\omega),\hat{\imath}(\omega)} - \tilde{\zeta}_{\hat{f}(\omega),\hat{\imath}(\omega)}\right\|_{\infty,\left[0,\hat{h}_\perp(\hat{f}(\omega))\right]^2} \qquad \text{using (10.22)}
\end{aligned}$$

or equivalently

$$\|z-\tilde{z}\|_{/\!/} \le \frac{1}{1-\hat{\sigma}^2}\left\|\zeta_{\hat{f}(\omega),\hat{\imath}(\omega)} - \tilde{\zeta}_{\hat{f}(\omega),\hat{\imath}(\omega)}\right\|_{\infty,\left[0,\hat{h}_\perp(\hat{f}(\omega))\right]^2} \tag{10.23}$$

Recall that $\tilde{x} \in V^{/\!/}_{\omega,\hat{\mathbf{h}}_\omega,i}(x)$. By expansion in the tangential cone field (see (10.4)), we have

$$\|z-\tilde{z}\|_{/\!/} \ge K_c\,\|x-\tilde{x}\|_{/\!/} = K_c\left|\left(\hat{\Phi}^*\zeta\right)_{\omega,i}(x_\perp) - \left(\hat{\Phi}^*\tilde{\zeta}\right)_{\omega,i}(x_\perp)\right| \tag{10.24}$$

Using (10.23) and (10.24), we get

$$\left|\left(\hat{\Phi}^*\zeta\right)_{\omega,i}(x_\perp) - \left(\hat{\Phi}^*\tilde{\zeta}\right)_{\omega,i}(x_\perp)\right| \le \frac{1}{K_c(1-\hat{\sigma}^2)}\left\|\zeta_{\hat{f}(\omega),\hat{\imath}(\omega)} - \tilde{\zeta}_{\hat{f}(\omega),\hat{\imath}(\omega)}\right\|_{\infty,\left[0,\hat{h}_\perp(\hat{f}(\omega))\right]^2}$$

Hence, (10.21) holds true. According to (9.3), $K_c\left(1-\hat{\sigma}^2\right) > 1$ so $\hat{\Phi}^*$ is a contraction mapping. Using the standard contraction mapping theorem, we get that $\hat{\Phi}^*$ admits a unique fixed point in Γ. This concludes the proof. □

Theorem 10.20 (Local stable manifolds of the double era return map)**.** *For every $\omega \in \Omega_{graph}$ and every $i \in \{1,2\}$, the local stable set of $P^{s_i}_{\omega,\hat{h}}$ of size $\hat{h}_\perp(\omega)$ contains a Lipschitz submanifold of dimension 2. More precisely,*

$$\hat{\gamma}_{\omega,i} \subset W^s_{\hat{h}_\perp(\omega)}\left(P^{s_i}_{\omega,\hat{h}},\hat{\Phi}\right)$$

Moreover, the convergence is exponential in the graph: for every $x \in \hat{\gamma}_{\omega,i}$ and every $n \ge 0$,

$$\left\|\hat{\Phi}^n(x) - \hat{\Phi}^n\left(P^{s_i}_{\omega,\hat{h}}\right)\right\|_\infty \le \left\|x - P^{s_i}_{\omega,\hat{h}}\right\|_\perp e^{-\tilde{C}_8\sum_{i=1}^{2n} k_i(\omega)^5} \tag{10.25}$$

Proof. Let $\omega = [1;k_1,k_2,\dots] \in \Omega_{\text{graph}}$, $i \in \{1,2\}$ and $x \in \hat{\gamma}_{\omega,i} = \text{Graph}^{s_i}\left(\hat{\zeta}_{\omega,i}\right)$. By definition, $\hat{\gamma} = \hat{\Phi}^*\hat{\gamma}$. Hence, for every $n \ge 0$ and every $j \in \{1,2\}$,

$$\hat{\Phi}\left(\hat{\gamma}_{\hat{f}^n(\omega),j}\right) \subset \hat{\gamma}_{\hat{f}^{n+1}(\omega),\hat{\imath}(\hat{f}^n(\omega))}$$

Using the fact that $\hat{\zeta}_{\hat{f}^{n+1}(\omega),\hat{\imath}(\hat{f}^n(\omega))}$ is $\hat{\sigma}$-Lipschitz with $\hat{\sigma} \le 1$, we get that for every $n \ge 0$,

$$\hat{\gamma}_{\hat{f}^{n+1}(\omega),\hat{\imath}(\hat{f}^n(\omega))} \subset S^s_{\hat{f}^{n+1}(\omega),\hat{\mathbf{h}}_{\hat{f}^{n+1}(\omega)}}$$

By induction, we get that for every $n \ge 0$, $\hat{\Phi}^n(x)$ is well defined and belongs to $S^s_{\hat{f}^n(\omega),\hat{\mathbf{h}}_{\hat{f}^n(\omega)}}$.

Let $n \ge 0$. Since $\hat{\Phi}^{n+1}(x)$ and $\hat{\Phi}^{n+1}\left(P^{s_i}_{\omega,\hat{h}}\right)$ both belong to $\hat{\gamma}_{\hat{f}^{n+1}(\omega),\hat{\imath}(\hat{f}^n(\omega))}$, it follows from Lemma 10.13 that

$$\hat{\Phi}^{n+1}(x) \in V^\perp_{\hat{f}^{n+1}(\omega),\mathbf{h}',\hat{\imath}(\hat{f}^n(\omega))}\left(\hat{\Phi}^{n+1}\left(P^{s_i}_{\omega,\hat{h}}\right)\right)$$

where $\mathbf{h}' = \left(\hat{h},\hat{h},\hat{h}\right)$. Hence, the contraction estimate (10.5) in the transverse cone gives

$$\left\|\hat{\Phi}^{n+1}(x) - \hat{\Phi}^{n+1}\left(P^{s_i}_{\omega,\hat{h}}\right)\right\|_\perp \le \left\|\hat{\Phi}^n(x) - \hat{\Phi}^n\left(P^{s_i}_{\omega,\hat{h}}\right)\right\|_\perp e^{-\tilde{C}_8\left(k^5_{2n+1}+k^5_{2n+2}\right)} \tag{10.26}$$

Moreover,

$$\left\|\hat{\Phi}^{n+1}(x)-\hat{\Phi}^{n+1}\left(P^{s_i}_{\omega,\hat{h}}\right)\right\|_{/\!/}=\left\|\hat{\zeta}_{\hat{f}^{n+1}(\omega),\hat{i}(\hat{f}^{n}(\omega))}\left(\left(\hat{\Phi}^{n+1}(x)\right)_{\perp}\right)-\hat{\zeta}_{\hat{f}^{n+1}(\omega),\hat{i}(\hat{f}^{n}(\omega))}\left(\left(\hat{\Phi}^{n+1}\left(P^{s_i}_{\omega,\hat{h}}\right)\right)_{\perp}\right)\right\|$$
$$\leq \hat{\sigma}\left\|\hat{\Phi}^{n+1}(x)-\hat{\Phi}^{n+1}\left(P^{s_i}_{\omega,\hat{h}}\right)\right\|_{\perp} \tag{10.27}$$

Using (10.26), (10.27) and the fact that $\hat{\sigma}\leq 1$, we get that for every $n\geq 0$,

$$\left\|\hat{\Phi}^{n+1}(x)-\hat{\Phi}^{n+1}\left(P^{s_i}_{\omega,\hat{h}}\right)\right\|_{\infty}\leq e^{-\tilde{C}_8\left(k^5_{2n+1}+k^5_{2n+2}\right)}\left\|\hat{\Phi}^{n}(x)-\hat{\Phi}^{n}\left(P^{s_i}_{\omega,\hat{h}}\right)\right\|_{\perp}$$

By induction, we get that for every $n\geq 0$,

$$\left\|\hat{\Phi}^{n}(x)-\hat{\Phi}^{n}\left(P^{s_i}_{\omega,\hat{h}}\right)\right\|_{\infty}\leq \left\|x-P^{s_i}_{\omega,\hat{h}}\right\|_{\perp}e^{-\tilde{C}_8\sum_{i=1}^{2n}k_i(\omega)^5} \tag{10.28}$$

Hence, $x\in W^s_{\hat{h}_\perp(\omega)}\left(P^{s_i}_{\omega,\hat{h}},\hat{\Phi}\right)$ and the convergence is exponential in the graph. This concludes the proof of Theorem 10.20.

□

10.6 Continuity of the local stable manifolds

We want to show that the graphs $\hat{\gamma}_{\omega,i}$ depend continuously on $\omega\in\Omega_{\text{graph}}$. Equivalently, we can show that the maps $\hat{\zeta}_{\omega,i}$ depend continuously on $\omega\in\Omega_{\text{graph}}$. Now remark that if $P^{s_i}_{\omega,\hat{h}}$ and $P^{s_j}_{\tilde{\omega},\hat{h}}$ are close to each other, then $i=j$. Hence, we can fix $i=1$ and discuss the regularity of the map $\hat{\zeta}_1:\omega\in\Omega_{\text{graph}}\mapsto\hat{\zeta}_{\omega,1}$.

Recall that for all $\omega\in\Omega_{\text{graph}}$, $\hat{\zeta}_{\omega,1}:\left[0,\hat{h}_\perp(\omega)\right]^2\to\mathbb{R}$ is a $\hat{\sigma}$-Lipschitz map such that $\hat{\zeta}_{\omega,1}(0,z)=\omega$ for all $z\in\left[0,\hat{h}_\perp(\omega)\right]$. We want to compare two different maps $\hat{\zeta}_{\omega,1}$ and $\hat{\zeta}_{\tilde{\omega},1}$ when ω and $\tilde{\omega}$ are close together. The most natural way to compare $\hat{\zeta}_{\omega,1}$ and $\hat{\zeta}_{\tilde{\omega},1}$ is to restrict them to $\left[0,\min\left(\hat{h}_\perp(\omega),\hat{h}_\perp(\tilde{\omega})\right)\right]^2$ and then to use the sup-norm. We do not want the function $\min\left(\hat{h}_\perp(\omega),\hat{h}_\perp(\tilde{\omega})\right)$ to collapse to 0 while $\tilde{\omega}$ tends to ω so we will restrict ourselves to points ω that satisfy $\hat{h}_\perp(\omega)\geq h_\perp$ where $h_\perp>0$ is an arbitrary fixed number. This leads us to define the following subset of Ω_{graph}:

$$\Omega_{\text{graph}}\left(h_\perp\right)=\left\{\omega\in\Omega_{\text{graph}}\mid\hat{h}_\perp(\omega)\geq h_\perp\right\}$$

One should note that $\Omega_{\text{graph}}=\bigcup_{n\geq 1}\Omega_{\text{graph}}\left(\frac{1}{n}\right)$. According to Proposition 10.16, for $h_\perp$ small enough, $\Omega_{\text{graph}}\left(h_\perp\right)$ has positive Lebesgue measure. In the following proposition, $\text{Lip}_{h_\perp}$ denotes the set of all real valued $\hat{\sigma}$-Lipschitz map defined on $[0,h_\perp]^2$.

Proposition 10.21. *For every $h_\perp>0$, the map $\hat{\zeta}_{1,h_\perp}:\omega\in\Omega_{graph}\left(h_\perp\right)\mapsto\left(\hat{\zeta}_{\omega,1}\right)_{|[0,h_\perp]^2}\in\text{Lip}_{h_\perp}$ is continous for the sup-norm topology on $\text{Lip}_{h_\perp}$.*

Proof. Let $\epsilon>0$, $h_\perp>0$ and $Kparam=[1;k_1,k_2,\dots]\in\Omega_{\text{graph}}\left(h_\perp\right)$. We are going to show that there exists $\eta>0$ (depending only on ϵ and ω) such that for all $\tilde{\omega}\in\Omega_{\text{graph}}\left(h_\perp\right)$, if $|\omega-\tilde{\omega}|\leq\eta$, then $\left\|\hat{\zeta}_{\omega,1}-\hat{\zeta}_{\tilde{\omega},1}\right\|_{\infty}\leq\epsilon$ (where the sup-norm is to be understood over $[0,h_\perp]^2$). Let $\gamma=\text{Graph}\left(\zeta\right)\in\Gamma$ be the "constant" graph family, defined by $\zeta_{z,i}\equiv z$ for all $z\in\Omega_{\text{graph}}$ and all $i\in\{1,2\}$. Since $\hat{\Phi}^*$ is a contraction mapping (see Lemma 10.19), there exists an integer n such that

$$d_{\text{graph}}\left(\hat{\gamma},\left(\hat{\Phi}^*\right)^n\gamma\right)\leq\epsilon$$

From now on, we fix such a n. Denote $\left(\hat{\Phi}^*\right)^n \gamma = \text{Graph}\left(\left(\hat{\Phi}^*\right)^n \zeta\right)$. We then have

$$
\begin{aligned}
&\left\|\hat{\zeta}_{\omega,1} - \hat{\zeta}_{\tilde{\omega},1}\right\|_\infty \\
&\leq \left\|\hat{\zeta}_{\omega,1} - \left(\left(\hat{\Phi}^*\right)^n \zeta\right)_{\omega,1}\right\|_\infty + \left\|\left(\left(\hat{\Phi}^*\right)^n \zeta\right)_{\omega,1} - \left(\left(\hat{\Phi}^*\right)^n \zeta\right)_{\tilde{\omega},1}\right\|_\infty + \left\|\left(\left(\hat{\Phi}^*\right)^n \zeta\right)_{\tilde{\omega},1} - \hat{\zeta}_{\tilde{\omega},1}\right\|_\infty \\
&\leq \left\|\left(\left(\hat{\Phi}^*\right)^n \zeta\right)_{\omega,1} - \left(\left(\hat{\Phi}^*\right)^n \zeta\right)_{\tilde{\omega},1}\right\|_\infty + 2\epsilon
\end{aligned}
$$

One can remark that for every $\text{Graph}\left(\Lambda\right) \in \Gamma$ and every $z, \tilde{z} \in \Omega_{\text{graph}}\left(h_\perp\right)$ close enough, we have

$$
\left\|\left(\hat{\Phi}^*\Lambda\right)_{z,1} - \left(\hat{\Phi}^*\Lambda\right)_{\tilde{z},1}\right\|_{\infty,[0,h_\perp]^2} \leq \lambda \left\|\Lambda_{\hat{f}(z),1} - \Lambda_{\hat{f}(\tilde{z}),1}\right\|_{\infty,\left[0,h_\perp e^{-\tilde{C}_8\left(k_1(z)^5+k_2(z)^5\right)}\right]^2} \tag{10.29}
$$

where $\lambda = \frac{1}{K_c(1-\hat{\sigma}^2)}$. This inequality follows from the very same argument as in Lemma 10.19. One just needs to check that if $\tilde{z}$ is close enough to z, we can indeed use the invariant cone field from Proposition 10.8. Now remark that if $\tilde{\omega}$ is close enough to ω, then for every $0 \leq j \leq n$, $\hat{f}^j(\tilde{\omega})$ is close enough to $\hat{f}^j(\omega)$ so that estimate (10.29) holds true with $z = \hat{f}^j(\omega)$ and $\tilde{z} = \hat{f}^j(\tilde{\omega})$. By induction, we get

$$
\begin{aligned}
\left\|\left(\left(\hat{\Phi}^*\right)^n \zeta\right)_{\omega,1} - \left(\left(\hat{\Phi}^*\right)^n \zeta\right)_{\tilde{\omega},1}\right\|_{\infty,[0,h_\perp]^2} &\leq \lambda^n \left\|\zeta_{\hat{f}^n(\omega),\hat{\imath}(\hat{f}^{n-1}(\omega))} - \zeta_{\hat{f}^n(\tilde{\omega}),\hat{\imath}(\hat{f}^{n-1}(\tilde{\omega}))}\right\|_{\infty,\left[0,h_\perp e^{-\tilde{C}_8 \sum_{i=1}^{2n} k_i^5}\right]^2} \\
&\leq \lambda^n \left|\hat{f}^n(\omega) - \hat{f}^n(\tilde{\omega})\right|
\end{aligned}
$$

Moreover, if $\tilde{\omega}$ is close enough to ω, then

$$
\hat{f}^n(\omega) = f^{k_1+\cdots+k_{2n}}(y_c) \quad \text{and} \quad \hat{f}^n(\tilde{\omega}) = f^{k_1+\cdots+k_{2n}}(\tilde{\omega})
$$

Hence, using Proposition 8.3 on the Kasner map, we get that

$$
\left\|\left(\left(\hat{\Phi}^*\right)^n \zeta\right)_{\omega,1} - \left(\left(\hat{\Phi}^*\right)^n \zeta\right)_{\tilde{\omega},1}\right\|_{\infty,[0,h_\perp]^2} \leq \left(\lambda^n \prod_{i=1}^{2n} 16k_i^2\right) |\omega - \tilde{\omega}|
$$

Take $\eta > 0$ such that $\eta\left(\lambda^n \prod_{i=1}^{2n} 16k_i^2\right) \leq \epsilon$. If $|\omega - \tilde{\omega}| \leq \eta$, then

$$
\left\|\hat{\zeta}_{\omega,1} - \hat{\zeta}_{\tilde{\omega},1}\right\|_\infty \leq 3\epsilon
$$

which concludes the proof. □

Chapter 11

Shadowing of heteroclinic chains

If p is a point of the Kasner circle $\mathscr{K}$, let us denote by $\mathrm{Shad}(p)$ the union of all the type IX orbits in $\mathscr{B}^+$ shadowing the heteroclinic chain $\mathscr{H}(p)$ starting at p. Recall that $\omega \in]1, +\infty[\setminus \mathbb{Q}$ satisfies the moderate growth condition if

$$k_{n+4}(\omega)^4 = o_{n\to+\infty}\left(\sum_{i=1}^{n} k_i(\omega)^5\right) \tag{MG}$$

We are now ready to prove the first part of Theorem B stated in the introduction. Let us recall the statement.

Theorem 11.1 (Theorem B, first part)**.** *Let p be a point of the Kasner circle. If $\omega(p)$ verifies the moderate growth condition* (MG)*, then* $\mathrm{Shad}(p)$ *contains a 3-dimensional ball Lipschitz embedded in the phase space $\mathscr{B}^+$.*

We will reduce Theorem 11.1 to a more technical statement, see Theorem 11.4 below. Let us recall some notations. For any $\omega \in]1, 2[\setminus \mathbb{Q}$, we denote by $\mathcal{P}_\omega$ the unique point belonging to the Kasner interval $\mathcal{K}_0$ whose Kasner parameter is ω and by $\mathcal{H}(\omega)$ the heteroclinic chain starting at $\mathcal{P}_\omega$ (see definition 3.7). Recall that $\hat{\gamma} = (\hat{\gamma}_{\omega,i})$ denotes the fixed point of the graph transformation, that is, the graph family invariant by the double era return map $\hat{\Phi}$ constructed in Chapter 10. Roughly speaking, we will prove that the orbits starting in $\hat{\gamma}_{\omega,i}$ will shadow the heteroclinic chain $\mathcal{H}(\omega)$ (see definition 1.2). In practice, we need to impose a stronger condition on ω : the moderate growth condition (MG).

Definition 11.2. We denote by Ω_{shad} the set of all the points $\omega \in]1, 2[\setminus \mathbb{Q}$ satisfying the moderate growth condition (MG).

Proposition 11.3. *$\Omega_{shad} \subset \Omega_{graph}$ (see definition 10.14) and Ω_{shad} is a Lebesgue full measure subset of $]1, 2[\setminus \mathbb{Q}$. Moreover, if $\omega \in \Omega_{shad}$, then*

$$k_{2n+1}(\omega)^4 + k_{2n+2}(\omega)^4 + k_{2n+3}(\omega)^4 + k_{2n+4}(\omega)^4 = o_{n\to+\infty}\left(\sum_{i=1}^{2n} k_i(\omega)^5\right) \tag{11.1}$$

Proof. The first part of Proposition 11.3 is a direct consequence of Proposition 10.16 and Lemma B.1. The fact that (MG) implies (11.1) is straightforward. □

11.1 Shadowing theorem

Recall that $\hat{\gamma} = (\hat{\gamma}_{\omega,i})$ denotes the fixed point of the graph transformation. Recall that the type IX points are those satisfying, in local coordinates, the condition

$$x_u > 0, \qquad x_{s_1} > 0, \qquad x_{s_2} > 0$$

In particular, any point in the interior of $\hat{\gamma}_{\omega,i}$ is of type IX. Recall that $\mathrm{Shad}(\omega)$ is the union of all the type IX orbits in $\mathcal{B}^+$ shadowing $\mathcal{H}(\omega)$.

Theorem 11.4 (Partial description of the shadowing sets). *For every $\omega \in \Omega_{shad}$, every $i \in \{1,2\}$ and every point $q_0 \in \xi^{-1}(\hat{\gamma}_{\omega,i})$ of type* IX*, the orbit of the Wainwright-Hsu vector field starting at q_0 shadows the heteroclinic chain $\mathcal{H}(\omega)$. In particular, the shadowing set* Shad(ω) *contains a* 3*-dimensional injectively immersed Lipschitz manifold, namely the set*

$$\bigcup_{t\in\mathbb{R}} \mathcal{X}^t\left(\xi^{-1}(\hat{\gamma}_{\omega,i}) \cap \mathcal{B}_{\mathrm{IX}}\right)$$

where $\mathcal{X}^t$ is the flow of the induced Wainwright-Hsu vector field $\mathcal{X}$.

Proof of Theorem 11.4. The proof relies on the following ingredients: Theorem 10.20, Proposition 9.10, Proposition 9.9, Proposition 7.4 and Proposition 6.8. To make the proof easier to read, we will sometimes identify a point in $\mathcal{U}_\xi \subset \mathcal{B}^+$ with its image by the local coordinate system ξ. Let $\omega = [1;k_1,k_2,\dots] \in \Omega_{\mathrm{shad}}$, $i \in \{1,2\}$, $q_0 \in \xi^{-1}(\hat{\gamma}_{\omega,i})$ be a type IX point and $q : t \mapsto q(t)$ be the forward $\mathcal{X}$-orbit of q_0.

Our goal is to prove that $q : t \mapsto q(t)$ shadows the heteroclinic chain $\mathcal{H}(\omega)$. Recall that $\mathcal{H}(\omega)$ is the concatenation of the type II orbit $\mathcal{O}_{\mathcal{P}_\omega \to \mathcal{P}_{f(\omega)}}$ with $\mathcal{O}_{\mathcal{P}_{f(\omega)} \to \mathcal{P}_{f^2(\omega)}}$ and so on. Hence, the orbit $(\omega_n)_{n\geq 0} = \left(\omega, f(\omega), f^2(\omega), \dots\right)$ will play a fundamental role. It will be convenient to gather the terms of this sequence by eras, that is, to look at it as the double sequence

$$\begin{aligned}(\omega_{j,l})_{(j,l)\in E_\omega} = (\omega_{0,0} = \omega, \omega_{0,1} = f(\omega), \dots, \omega_{0,k_1-1} = f^{k_1-1}(\omega),\\ \omega_{1,0} = \bar{f}(\omega), \omega_{1,1} = f(\bar{f}(\omega)), \dots, \omega_{1,k_2-1} = f^{k_2-1}(\bar{f}(\omega)),\\ \omega_{2,0} = \bar{f}^2(\omega), \omega_{2,1} = f(\bar{f}^2(\omega)), \dots)\end{aligned}$$

where

$$E_\omega = \left\{(j,l) \in \mathbb{N}^2 \mid 0 \leq l \leq k_{j+1} - 1\right\}$$

is endowed with the lexicographical order. We will alternate between those two points of view, using the increasing bijection $\varphi : E_\omega \to \mathbb{N}$ defined by

$$\varphi(j,l) = l + \sum_{m=1}^{j} k_m$$

In other words, we associate with any formal sequence $(a_n)_{n\in\mathbb{N}}$ a sequence $(a_{j,l})_{(j,l)\in E_\omega}$ where $a_{j,l} := a_{\varphi(j,l)}$ and conversely.

According to Theorem 10.20, q_0 belongs to the local stable manifold of $P^{s_i}_{\omega,\hat{h}}$ of size $\hat{h}_\perp(\omega)$. In particular, $\hat{\Phi}^j(q_0)$ is well defined for all $j \geq 0$ and, *a fortiori*, $\bar{\Phi}^j_{\hat{h}}(q_0)$ is also well defined for all $j \geq 0$. Let $T_{0,0} = 0$, $T_{1,0}$, $T_{2,0}$, ... be the successive times when the orbit q intersects the section $S_{\hat{h}}$. For $j \geq 0$, define $h_{\perp,2j,0} = e^{-\tilde{C}_{10} s_4(\hat{f}^j(\omega))}$, $\mathbf{h}_{2j,0} = \left(\hat{h}, h_{\perp,2j,0}, e^{-\tilde{C}_8 s_2(\hat{f}^j(\omega))}\right)$ and $S_{2j,0} = S^s_{\hat{f}^j(\omega),\mathbf{h}_{2j,0}}$.

Claim 1. *For all $j \geq 0$, $\mathcal{X}^{T_{2j,0}}(q_0) \in S_{2j,0}$.*

Proof of claim 1. Recall that $\hat{h}_\perp(\omega) \leq h_{\perp,0,0}$ and $\hat{\zeta}_{\omega,i}$ is $\hat{\sigma}$-Lipschitz, hence the claim is trivial for $j = 0$. For $j \geq 1$, remark that

$$\mathcal{X}^{T_{2j,0}}(q_0) = \hat{\Phi}^j(q_0) \in \mathrm{Graph}^{s_{\hat{\imath}(\hat{f}^{j-1}(\omega))}}\left(\hat{\zeta}_{\hat{f}^j(\omega),\hat{\imath}(\hat{f}^{j-1}(\omega))}\right) \subset S^s_{\hat{f}^j(\omega),\hat{\mathbf{h}}_{\hat{f}^j(\omega)}} \subset S_{2j,0}$$

□

For $j \geq 0$, define $h_{\perp,2j+1,0} := e^{-\tilde{C}_9 s_4(\bar{f}^{2j+1}(\omega))}$, $\mathbf{h}_{2j+1,0} := \left(\hat{h}, h_{\perp,2j+1,0}, e^{-\tilde{C}_8 s_2(\bar{f}^{2j+1}(\omega))}\right)$ and $S_{2j+1,0} := S^s_{\bar{f}^{2j+1}(\omega),\mathbf{h}_{2j+1,0}}$.

Claim 2. *For all $j \geq 0$, $\mathcal{X}^{T_{2j+1,0}}(q_0) \in S_{2j+1,0}$.*

Proof of claim 2. This is an immediate consequence of claim 1 and Proposition 9.10. □

For $j \geq 0$ and $1 \leq l \leq k_{j+1}-1$, define $h_{\perp,j,l} = \left(\hat{h}k_{j+1}^{-\tilde{n}_5}\right)^{\tilde{C}_5 f^l(\bar{f}^j(\omega))} m(f^l(\bar{f}^j(\omega)))$, $\mathbf{h}_{j,l} = \left(\hat{h}k_{j+1}^{-\tilde{n}_5}, h_{\perp,j,l}, h_{\perp,j,l}\right)$ and $S_{j,l} = S^s_{f^l(\bar{f}^j(\omega)),\mathbf{h}_{j,l}}$. Let $j \geq 0$. According to Lemma 9.8 and Proposition 9.9, the orbit segment $\left[\mathcal{X}^{T_{j,0}}(q_0), \mathcal{X}^{T_{j+1,0}}(q_0)\right]$ passes through all the sections $S_{j,1}$, $S_{j,2}$, …, $S_{j,k_{j+1}-1}$ in that order. We denote by

$$T_{j,0} < T_{j,1} < \cdots < T_{j,k_{j+1}-1} < T_{j+1,0}$$

the successive first times $T_{j,l}$ such that $\mathcal{X}^{T_{j,l}}(q_0) \in S_{j,l}$ for all $1 \leq l \leq k_{j+1}-1$. More precisely, $T_{j,1} < \cdots < T_{j,k_{j+1}-1}$ are defined recursively as follows

$$\begin{aligned} T_{j,1} &= \min\left\{t > T_{j,0} \mid \mathcal{X}^t(q_0) \in S_{j,1}\right\} \\ T_{j,2} &= \min\left\{t > T_{j,1} \mid \mathcal{X}^t(q_0) \in S_{j,2}\right\} \\ &\dots \\ T_{j,k_{j+1}-1} &= \min\left\{t > T_{j,k_{j+1}-2} \mid \mathcal{X}^t(q_0) \in S_{j,k_{j+1}-1}\right\} \end{aligned}$$

Let $(P_n)_{n\geq 0}$ be the sequence of the successive intersection points of the heteroclinic chain $\mathcal{H}\left(P^{s_i}_{\omega,\hat{h}}\right)$ with the sections S_0, S_1, etc. According to Proposition 9.10, if $\mathcal{X}^{T_{2j,0}}(q_0)$ is close to $P_{2j,0}$ (relatively to the size $h_{\perp,2j,0}$ of the section $S_{2j,0}$ in the direction transverse to the Mixmaster attractor), then $\mathcal{X}^{T_{2j+1,0}}(q_0)$ is close to $P_{2j+1,0}$ (relatively to the size $h_{\perp,2j+1,0}$). More precisely, if

$$\frac{\left\|\mathcal{X}^{T_{2j,0}}(q_0) - P_{2j,0}\right\|_\infty}{h_{\perp,2j,0}} \leq \epsilon$$

with $0 < \epsilon \leq 1$, then

$$\frac{\left\|\mathcal{X}^{T_{2j+1,0}}(q_0) - P_{2j+1,0}\right\|_\infty}{h_{\perp,2j+1,0}} \leq \epsilon$$

According to Proposition 9.9, if $\mathcal{X}^{T_{j,0}}(q_0)$ is close to $P_{j,0}$ (relatively to the size $h_{\perp,j,0}$), then, for every $1 \leq l \leq k_{j+1}-1$, $\mathcal{X}^{T_{j,l}}(q_0)$ is close to $P_{j,l}$ (relatively to the size $h_{\perp,j,l}$). For $(j,l) \in E_\omega$, define

$$h_{j,l} = \begin{cases} \hat{h} & \text{if } l = 0 \\ \hat{h}k_{j+1}^{-\tilde{n}_5} & \text{if } l \geq 1 \end{cases}$$

and

$$t_n = T_n + \tau^*_{f^n(\omega),h_n}\left(\mathcal{X}^{T_n}(q_0)\right)$$

where τ^* is defined in Proposition 6.8. According to Proposition 6.8 together with Proposition 6.1 and Proposition 7.4, if $\mathcal{X}^{T_n}(q_0)$ is close to P_n (relatively to the size $h_{\perp,n}$), then

1. $\mathcal{X}^{t_n}(q_0)$ is close to $\mathcal{P}_{f^n(\omega)}$.

2. The orbit segment $\left[\mathcal{X}^{T_n}(q_0), \mathcal{X}^{T_{n+1}}(q_0)\right]$ is close to the heteroclinic chain segment $[P_n, P_{n+1}]$ for the Hausdorff distance.

Hence, we are left to prove that the ratio between $\left\|\mathcal{X}^{T_{2j,0}}(q_0) - P_{2j,0}\right\|_\infty$ and $h_{\perp,2j,0}$ tends to 0 as j tends to $+\infty$. One can rewrite (10.25) as follows:

$$\left\|\mathcal{X}^{T_{2j,0}}(q_0) - P_{2j,0}\right\|_\infty \leq \|q_0 - P_0\|_\perp \, e^{-\tilde{C}_8 \sum_{m=1}^{2j} k_m^5}$$

Hence,

$$\frac{\left\|\mathcal{X}^{T_{2j,0}}(q_0) - P_{2j,0}\right\|_\infty}{h_{\perp,2j,0}} \leq \|q_0 - P_0\|_\perp \, e^{\tilde{C}_{10} s_4(\hat{f}^j(\omega)) - \tilde{C}_8 \sum_{m=1}^{2j} k_m^5}$$

To conclude, recall that $\omega \in \Omega_{\text{shad}}$. Equation (11.1) implies that

$$\lim_{j\to+\infty} e^{\tilde{C}_{10} s_4(\hat{f}^j(\omega)) - \tilde{C}_8 \sum_{m=1}^{2j} k_m^5} = 0$$

Hence, the orbit $q : t \mapsto q(t)$ shadows the heteroclinic chain $\mathcal{H}(\omega)$. This concludes the proof since $\text{Shad}(\omega)$ is clearly invariant by the flow of the Wainwright-Hsu vector field $\mathcal{X}$. □

Proof of Theorem 11.1. Let p be a point of the Kasner circle such that $\omega(p)$ verifies the moderate growth condition (MG). One can find an iterate $\mathscr{F}^j(p)$ such that $\omega(\mathscr{F}^j(p)) \in \,]1,2[\setminus \mathbb{Q}$. Moreover, $\text{Shad}(p) = \text{Shad}(\mathscr{F}^j(p))$ and $\omega(\mathscr{F}^j(p))$ verifies the moderate growth condition (MG). Hence, one can assume that $\omega = \omega(p) \in \,]1,2[\setminus \mathbb{Q}$ without loss of generality. According to Proposition 11.3, $\omega \in \Omega_{\text{shad}}$. Remark that $\hat{\gamma}_{\omega,i} \cap B_{\text{IX}}$ is a 2-dimensional Lipschitz manifold. Since the local coordinate system ξ is a diffeomorphism, it follows that

$$\xi^{-1}\left(\hat{\gamma}_{\omega,i}\right) \cap \mathcal{B}_{\text{IX}}$$

is a 2-dimensional Lipschitz manifold as well. According to Theorem 11.4, for all point $q_0 \in \xi^{-1}\left(\hat{\gamma}_{\omega,i}\right)$ of type IX, the orbit of the Wainwright-Hsu vector field starting at q_0 shadows the heteroclinic chain $\mathcal{H}(\omega)$. In other words

$$\bigcup_{t\in\mathbb{R}} \mathcal{X}^t\left(\xi^{-1}\left(\hat{\gamma}_{\omega,i}\right) \cap \mathcal{B}_{\text{IX}}\right) \subset \text{Shad}(\omega)$$

Moreover, for $\epsilon > 0$ small enough, the set

$$\bigcup_{t\in]-\epsilon,\epsilon[} \left(\xi^{-1}\left(\hat{\gamma}_{\omega,i}\right) \cap \mathcal{B}_{\text{IX}}\right)$$

is a 3-dimensional ball Lipschitz embedded in the phase space $\mathcal{B}^+$. Hence, the shadowing set $\text{Shad}(\omega)$ contains a 3-dimensional ball Lipschitz embedded in the phase space $\mathcal{B}^+$. Recall that $\mathcal{B}^+$ is a quotient of $\mathscr{B}^+$ (see section 3.6). As a consequence, $\text{Shad}(p)$ contains a 3-dimensional ball Lipschitz embedded in the phase space $\mathscr{B}^+$. This concludes the proof of Theorem 11.1. □

Chapter 12

Absolute continuity of the stable manifolds foliation

If p is a point of the Kasner circle $\mathscr{K}$, recall that we denote by $\mathrm{Shad}(p)$ the reunion of all the type IX orbits in $\mathscr{B}^+$ shadowing the heteroclinic chain starting at p. The purpose of this last chapter is to prove the second part of Theorem B stated in the introduction. Let us recall the statement.

Theorem 12.1 (Theorem B, second part)**.** *If $\mathscr{E} \subset \mathscr{K}$ has positive 1-dimensional Lebesgue measure, then $\bigcup_{p\in\mathscr{E}} \mathrm{Shad}(p)$ has positive 4-dimensional Lebesgue measure in the phase space $\mathscr{B}^+$.*

We can reduce Theorem 12.1 to the following proposition. Recall that $\Omega_{\mathrm{shad}} \subset \Omega_{\mathrm{graph}} \subset]1,2[$. In Chapter 10, we constructed a graph included in the local stable set of $P^{s_i}_{\omega,\hat{h}}$ for all $\omega \in \Omega_{\mathrm{graph}}$, denoted by $\hat{\gamma}_{\omega,i}$ (see Theorem 10.20). For $F \subset \Omega_{\mathrm{graph}}$ and $i \in \{1,2\}$, let

$$W^{s_i}\left(F,\hat{\Phi}\right) \stackrel{\mathrm{def}}{=} \bigsqcup_{\omega\in F} \hat{\gamma}_{\omega,i}$$

Proposition 12.2. *If $E \subset \Omega_{shad}$ has positive 1-dimensional Lebesgue measure, then $W^{s_1}\left(E,\hat{\Phi}\right)$ has positive 3-dimensional Lebesgue measure. The same result holds true with $W^{s_2}\left(E,\hat{\Phi}\right)$.*

Proof of Theorem 12.1 using Proposition 12.2. Assume that Proposition 12.2 holds true. Fix $\mathscr{E} \subset \mathscr{K}$ of positive 1-dimensional Lebesgue measure. Let $\omega(\mathscr{E}) := \{\omega(p) \mid p \in \mathscr{E}\}$. Since the map $\omega \mapsto \mathcal{P}_\omega$ is absolutely continuous, we get that $\omega(\mathscr{E})$ has positive 1-dimensional Lebesgue measure. As in the proof of Theorem 11.1, one can assume that $\omega(\mathscr{E}) \cap]1,2[$ has positive 1-dimensional Lebesgue measure without loss of generality. Recall from Proposition 11.3 that Ω_{shad} is a Lebesgue full measure subset of $]1,2[$. Hence, $E := \Omega_{\mathrm{shad}} \cap \omega(\mathscr{E})$ has positive 1-dimensional Lebesgue measure. Now apply Proposition 12.2 with the set E. We get that

$$W^{s_1}\left(E,\hat{\Phi}\right) \subset S^{s_1}_{\hat{h}}$$

has positive 3-dimensional Lebesgue measure. Since the local coordinate system ξ is a diffeomorphism, it follows that

$$\xi^{-1}\left(W^{s_1}\left(E,\hat{\Phi}\right)\right) \subset \mathcal{S}^{s_1}_{\hat{h}}$$

has positive 3-dimensional Lebesgue measure as well. Hence, the set

$$\bigcup_{t\in\mathbb{R}} \mathcal{X}^t\left(\xi^{-1}\left(W^{s_1}\left(E,\hat{\Phi}\right)\right)\right)$$

has positive 4-dimensional Lebesgue measure in $\mathcal{B}^+$. Moreover, according to Theorem 11.4, $\bigcup_{\omega\in E} \mathrm{Shad}(\omega)$ contains the above set. Recall that $\mathcal{B}^+$ is a finite quotient of $\mathscr{B}^+$ (see chapter 3.6). As a consequence, $\bigcup_{p\in\mathscr{E}} \mathrm{Shad}(p)$ contains a 4-dimensional Lebesgue measure set. Hence, Theorem 12.1 holds true. □

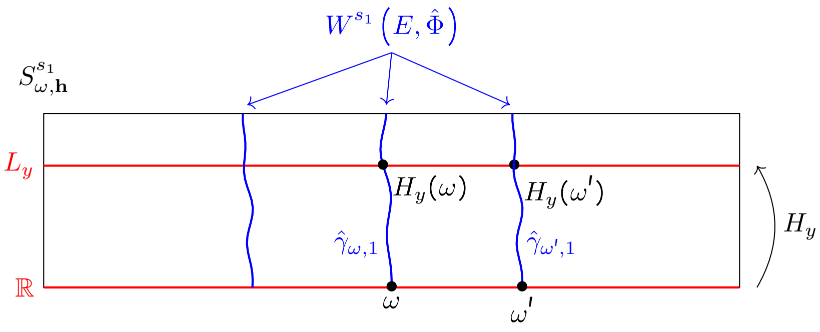

Figure 12.1 – The map H_y.

To prove Proposition 12.2, we will use a strategy due to Pesin, which consists in considering the holonomy along the "foliation" in local stable manifolds, and proving that this holonomy is made of absolutely continuous maps. This strategy is well-known in the context of non-uniformly hyperbolic maps. We call "foliation" in local stable manifolds the set

$$\left\{\hat{\gamma}_{\omega,i} \mid \omega \in E, i \in \{1,2\}\right\}$$

According to Proposition 10.21, the map $\hat{\zeta}_\omega$ depends continuously on ω. Now let us explain why Proposition 12.2 is not a direct consequence of this continuity. Consider a set $E \subset \Omega_{\text{shad}}$ of positive 1-dimensional Lebesgue measure. For $y \in \left[0,\hat{h}\right]^2$, introduce the *horizontal line*

$$L_y \stackrel{\text{def}}{=} \left\{(x_u,\hat{h},x_{s_2},x_c) \in S^{s_1}_{\hat{h}} \mid (x_u,x_{s_2}) = y\right\}$$

Recall that $\hat{h}_\perp(\omega)$ is the "size" of the graph $\hat{\gamma}_{\omega,i}$ (see definition (10.18) and Theorem 10.20). To simplify the discussion, assume that there exists $h_\perp$ such that for all $\omega \in E$, the graph $\hat{\gamma}_{\omega,1}$ has a size larger than $h_\perp$, *i.e.* $\hat{h}_\perp(\omega) \geq h_\perp$. According to Fubini's theorem, the set $W^{s_1}\left(E,\hat{\Phi}\right)$ has positive 3-dimensional Lebesgue measure if and only if there exists a set $Y \subset \left[0,\hat{h}\right]^2$ of positive 2-dimensional Lebesgue measure such that for all $y \in Y$, $L_y \cap W^{s_1}\left(E,\hat{\Phi}\right)$ has positive 1-dimensional Lebesgue measure. For $y \in [0,h_\perp]^2$, define the map $H_y : E \to L_y \cap W^{s_1}\left(E,\hat{\Phi}\right)$ by the formula

$$H_y(\omega) = \hat{\zeta}_{\omega,1}(y)$$

See figure 12.1. Remark that

$$H_y(E) = L_y \cap W^{s_1}\left(E,\hat{\Phi}\right)$$

If one wants to deduce the fact that $W^{s_1}\left(E,\hat{\Phi}\right)$ has positive 1-dimensional Lebesgue measure from the fact that E has positive 1-dimensional Lebesgue measure using the maps H_y, one needs to show that these maps send positive Lebesgue measure sets onto positive Lebesgue measure sets for all $y \in Y \subset [0,h_\perp]^2$ where Y has positive 2-dimensional Lebesgue measure. However, Proposition 10.21 only implies that H_y is a homeomorphism and it is well known that not all homeomorphisms send positive Lebesgue measure sets onto positive Lebesgue measure sets. In other words, some homeomorphisms send non-zero Lebesgue measure sets onto zero Lebesgue measure sets. Hence, Proposition 12.2 is not a straightforward consequence of Proposition 10.21. We must show that the maps H_y send positive Lebesgue measure sets onto positive Lebesgue measure sets using another method. Let us now describe this method, which is due to Pesin.

From now on and until the end of this section, we fix a set $E \subset \Omega_{\text{shad}}$ of positive 1-dimensional Lebesgue measure. We are going to replace E by a subset $\tilde{E}$ such that we have some uniform estimates on the continued fraction expansion of points of $\tilde{E}$ and such that $\tilde{E}$ still has positive 1-dimensional Lebesgue measure. Define

$$F^{s_1} \stackrel{\text{def}}{=} \left\{\hat{\gamma}_{\omega,1} \mid \omega \in \tilde{E}\right\}$$

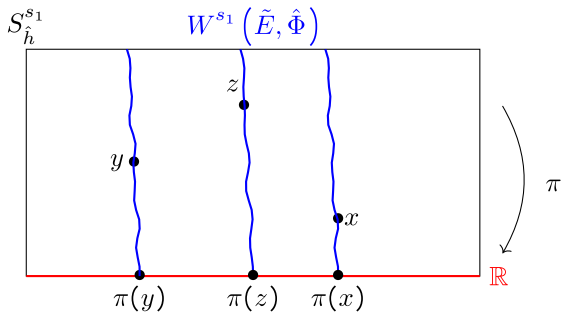

Figure 12.2 – The projection map π.

Remark that Ω_{shad} is totally disconnected, hence F^{s_1} is the family of connected components of $W^{s_1}\left(\tilde{E},\hat{\Phi}\right)$. Even if F^{s_1} is not a foliation of the section $S^{s_1}_{\hat{h}}$, we will call F^{s_1} the local stable manifolds "foliation" of the double era return map $\hat{\Phi}$. Remark that the F^{s_1} is leaf-invariant by $\hat{\Phi}$. The uniform estimates on points of $\tilde{E}$ will be crucial to prove that the local stable manifolds "foliation" of the double era return map $\hat{\Phi}$ is absolutely continuous. We now proceed to define $\tilde{E}$. According to Lemma B.1, there exists n_0 and l_0 such that the set

$$\tilde{E} \overset{\text{def}}{=} \left\{ \omega \in E \;\middle|\; \begin{array}{ll} \forall n \geq n_0, & \sum_{i=1}^{2n} k_i(\omega)^5 \geq n^{5-\frac{1}{10}} \\ \forall n \geq n_0, & k_{2n+1}(\omega)^4 + k_{2n+2}(\omega)^4 + k_{2n+3}(\omega)^4 + k_{2n+4}(\omega)^4 \leq n^{4+\frac{1}{10}} \\ \forall 1 \leq n \leq 2n_0, & k_n(\omega) \leq l_0 \end{array} \right\} \tag{12.1}$$

has positive 1-dimensional Lebesgue measure. Remark that the quantity

$$e^{\tilde{C}_{12} s_4(\hat{f}^n(\omega)) - \tilde{C}_8 \sum_{i=0}^{2n} k_i(\omega)^5}$$

is uniformly bounded from above for $\omega \in \tilde{E}$. Hence, according to the very definition of $\hat{h}_\perp(\omega)$ (see (10.18)),

$$\delta_{\perp,0} \overset{\text{def}}{=} \inf_{\omega \in \tilde{E}} \hat{h}_\perp(\omega) > 0 \tag{12.2}$$

In other words, the size of the graph $\hat{\gamma}_{\omega,1}$ is uniformly bounded from below by $\delta_{\perp,0}$ for $\omega \in \tilde{E}$.

In a second time, let us introduce a projection map π which is somehow the inverse of H_y. Roughly speaking, we will project points of $W^{s_1}\left(\tilde{E},\hat{\Phi}\right)$ onto the Mixmaster attractor along the foliation F^{s_1} and then project to the last coordinate. See figure 12.2.

Definition 12.3 (Projection map)**.** The projection map $\pi : W^{s_1}\left(\tilde{E},\hat{\Phi}\right) = \bigsqcup_{\omega \in \tilde{E}} \hat{\gamma}_{\omega,1} \to \mathbb{R}$ is defined by $\pi(x) = \omega$ for all $x \in \hat{\gamma}_{\omega,1}$.

Remark 12.4. The restriction $\pi_{|L_y}$ of the projection map is the inverse of H_y.

Remark 12.5. To make the reading easier, we will make the abuse of notation to write $\pi(G)$ instead of $\pi\left(G \cap W^{s_1}\left(\tilde{E},\hat{\Phi}\right)\right)$ for any set $G \subset S_{\hat{h}}$.

We denote by Leb_n the n-dimensional Lebesgue measure. Lemma 12.6 states precisely that the projection map π is absolutely continuous in restriction to horizontal lines. For $y \in \left[0,\hat{h}\right]^2$, let π_y be the restriction of π to the horizontal line L_y.

Lemma 12.6. *There exists $0 < h_\perp \leq \delta_{\perp,0}$ such that for all $y \in [0,h_\perp]^2$ and all $G \subset L_y \cap W^{s_1}\left(\tilde{E},\hat{\Phi}\right)$,*

$$\text{Leb}_1(G) = 0 \implies \text{Leb}_1\left(\pi(G)\right) = 0 \tag{12.3}$$

Proof of Proposition 12.2 using Lemma 12.6. Assume that Lemma 12.6 holds true. Take $0 < h_\perp \leq \delta_{\perp,0}$

as in the statement of Lemma 12.6. Assume that

$$\mathrm{Leb}_3\left(W^{s_1}\left(E,\hat{\Phi}\right)\right) = 0$$

This implies that

$$\mathrm{Leb}_3\left(W^{s_1}\left(\tilde{E},\hat{\Phi}\right)\right) = 0$$

Using Fubini's theorem, we get that for Lebesgue almost all $y \in [0,h_\perp]^2$,

$$\mathrm{Leb}_1\left(L_y \cap W^{s_1}\left(\tilde{E},\hat{\Phi}\right)\right) = 0 \tag{12.4}$$

Fix such a transversal L_y.

Claim 1. $\pi_y\left(W^{s_1}\left(\tilde{E},\hat{\Phi}\right)\right) = \tilde{E}$.

Proof of claim 1. The inclusion $\pi_y\left(W^{s_1}\left(\tilde{E},\hat{\Phi}\right)\right) \subset \tilde{E}$ is obvious by definition of π. Let $\omega \in \tilde{E}$. Since $h_\perp \leq \delta_{\perp,0}$, the size $\hat{h}_\perp(\omega)$ of the graph $\hat{\gamma}_{\omega,1}$ is larger than $h_\perp$ (see (12.2)). Hence, $\hat{\gamma}_{\omega,1}$ intersects the horizontal line L_y exactly one time, say at x. By definition, we have $\pi_y(x) = \omega$. This concludes the proof of claim 1. □

According to (12.3) and (12.4), we have

$$\mathrm{Leb}_1\left(\pi_y\left(W^{s_1}\left(\tilde{E},\hat{\Phi}\right)\right)\right) = 0$$

Using claim 1, we get that

$$\mathrm{Leb}_1\left(\tilde{E}\right) = 0$$

This is the desired contradiction. Hence,

$$\mathrm{Leb}_3\left(W^{s_1}\left(E,\hat{\Phi}\right)\right) > 0$$

and Proposition 12.2 holds true. □

We are left to prove Lemma 12.6. Let us explain the general strategy of the proof. Consider a set $G \subset L_y \cap W^{s_1}\left(\tilde{E},\hat{\Phi}\right)$ such that $\mathrm{Leb}_1(G) = 0$. We will cover G by a countable union of little horizontal segments. We need two definitions to make this idea precise.

Definition 12.7 (Diameter)**.** Let $G \subset S_{\hat{h}}$. We define the *diameter* (in the direction tangential to the Mixmaster attractor) of G by

$$|G| \stackrel{\text{def}}{=} \sup_{x,\tilde{x}\in G} \|x-\tilde{x}\|_{/\!/} = \sup_{x,\tilde{x}\in G} |x_c - \tilde{x}_c|$$

Definition 12.8 (Horizontal segment centered above ω)**.** Let $\omega \in \tilde{E}$ and $D \subset S_{\hat{h}}^{s_1}$. We say that D is an *horizontal segment* if there exists $y \in \left[0,\hat{h}\right]^2$ such that D is a compact and connected subset of L_y. If this is the case, there exists a unique pair $(x,\tilde{x}) \in L_y$ such that $|D| = \tilde{x}_c - x_c$. In other words,

$$D = \left\{z = (z_u,\hat{h},z_{s_2},z_c) \in S_{\hat{h}}^{s_1} \mid (z_u,z_{s_2}) = y, x_c \leq z_c \leq \tilde{x}_c\right\}$$

We call x and $\tilde{x}$ the *end points* of D. Moreover, we say that D is *centered above* ω if $y \in \left[0,\hat{h}_\perp(\omega)\right]^2$ and if the middle of the segment $[x_c,\tilde{x}_c]$ coincides with v_c, where v denotes the intersection point between L_y and $\hat{\gamma}_{\omega,1}$. See figure 12.3.

Since G has zero 1-dimensional Lebesgue measure, one can find a countable family $(D_i)_{i\in\mathbb{N}}$ of horizontal segments centered above points of $\tilde{E}$ covering G and satisfying

$$\sum_{i=0}^{+\infty} \mathrm{Leb}_1(D_i) \leq \epsilon$$

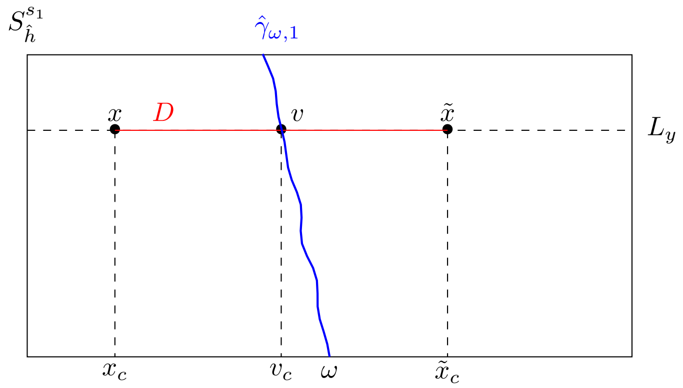

Figure 12.3 – Segment D centered above ω with its end points.

where ϵ is an arbitrary fixed positive number. Assume that there exists a constant M (independant of the choice of the segments D_i) such that for all $i \in \mathbb{N}$,

$$\mathrm{Leb}_1(\pi(D_i)) \leq M\,\mathrm{Leb}_1(D_i)$$

We get that $\pi(G)$ is covered by the countable union of sets $\pi(D_i)$ whose total measure is arbitrary small. As a consequence, $\pi(G)$ has zero 1-dimensional Lebesgue measure. Hence, we are left to control the projection of an horizontal segment D by the map π. Informally, we will prove the following statement.

Informal statement. *There exists a constant $M > 0$ with the following property. Take an horizontal segment D centered above a point $\omega \in \tilde{E}$. If D is sufficiently close to the Mixmaster attractor and has a sufficiently small diameter, then*

$$\mathrm{Leb}_1(\pi(D)) \leq M\,\mathrm{Leb}_1(D)$$

As stated earlier, the strategy used to prove the above statement is borrowed from Pesin's work on non-uniformly hyperbolic dynamical systems. First, remark that if D has a diameter $\delta_{/\!/}$ and is positioned at distance $\delta_\perp$ from the Mixmaster attractor with $\delta_\perp \leq \delta_{/\!/}$, then the above result is easy to prove. Indeed, recall that the graphs $\hat{\gamma}_{\omega,1}$ are all $\frac{1}{2}$-Lipschitz. Hence, the projection $\pi(D)$ has a diameter less than $\delta_\perp + \delta_{/\!/} \leq 2\delta_{/\!/}$. For the general case, one can try to "push by $\hat{\Phi}$" the horizontal segment D so that $\hat{\Phi}^n(D)$ is in the configuration of the previous situation. Indeed, recall that $\hat{\Phi}$ contracts the direction transverse to the Mixmaster attractor and expands the direction tangent to the Mixmaster attractor. Hence, for n large enough, $\hat{\Phi}^n(D)$ will have a "large" diameter and will be "close" to the Mixmaster attractor. As a consequence, the result should hold true if we replace D by $\hat{\Phi}^n(D)$. To conclude, one needs to tackle two difficulties.

The first one is the ability to "come back to D". In other words, we need to prove that if the result holds true for $\hat{\Phi}^n(D)$, then it holds true for D as well. This amounts to prove the following thing: $\hat{\Phi}^n$ expands D in the tangent direction to the Mixmaster attractor and $\hat{f}^n$ expands $\pi(D)$ with almost the same factor, independantly of n. This is the distorsion estimate proved in Proposition 12.10.

The second one is the fact that n must be "well chosen": large enough so that $\hat{\Phi}^n(D)$ has a "large" diameter and is "close" to the Mixmaster attractor but not too large because we need to ensure that for all $0 \leq j \leq n$, $\hat{\Phi}^j(D)$ is contained in a small section $S^s_{\hat{f}^j(\omega),\mathbf{h}_j(\omega)}$ where all the objects of interest are well defined and well controlled.

12.1 Distorsion estimate

Recall that we introduced a constant $\tilde{C}_{12}$ in Lemma 10.11, when we described the preimage of a $\hat{\sigma}$-Lipschitz graph by the double era return map $\hat{\Phi}$. Let us fix $\tilde{C}_{13} \geq \tilde{C}_{12}$ large enough so that for all

$a \geq 1$,

$$128^2 a^6 e^{-\tilde{C}_{13} a} \leq \frac{1}{8} \tag{12.5}$$

For $\omega \in]1,2[\setminus \mathbb{Q}$ and $j \geq 0$, define

$$m_j(\omega) \stackrel{\text{def}}{=} s_2(\hat{f}^j(\omega)) + \max_{1 \leq l \leq 2j} k_l(\omega),$$

$$\mathbf{h}_j(\omega) \stackrel{\text{def}}{=} \left(\hat{h}, e^{-\tilde{C}_{13} s_4(\hat{f}^j(\omega))}, e^{-\tilde{C}_{13} m_j(\omega)}\right)$$

and the interval

$$I_j(\omega) \stackrel{\text{def}}{=} \left[\hat{f}^j(\omega) - e^{-\tilde{C}_{13} m_j(\omega)}, \hat{f}^j(\omega) + e^{-\tilde{C}_{13} m_j(\omega)}\right]$$

Remark 12.9. $m_0(\omega) = s_2(\omega)$.

For $\omega \in]1,2[\setminus \mathbb{Q}$, D an horizontal segment and n a positive integer, define the property

$$(H_{\omega,D,n}) : \begin{cases} \forall 0 \leq j \leq n-1, & \hat{\Phi}^j(D) \subset S^s_{\hat{f}^j(\omega),\mathbf{h}_j(\omega)} \\ \forall 0 \leq j \leq n-1, & \hat{f}^j\left(\pi(D)\right) \subset I_j(\omega) \end{cases}$$

Property $(H_{\omega,D,n})$ implies that all the objects playing a role in the distorsion estimate are well defined and well controled for n iterates. Next proposition gives a precise statement about the distorsion estimate we need.

Proposition 12.10 (Distorsion estimate). *There exists a constant $\Delta \geq 1$ and a constant $\delta_{\perp,2} > 0$ such that the following property holds true for $\omega \in \tilde{E}$, $y \in \left[0, \delta_{\perp,2}\right]^2$ and $n \geq 0$. Let $D \subset L_y$ be an horizontal segment centered above ω. If $(H_{\omega,D,n})$ holds true, then*

$$\frac{\left|\hat{\Phi}^n(D)\right|}{\left|\hat{f}^n\left(\pi(D)\right)\right|} \leq \Delta \frac{|D|}{|\pi(D)|}$$

Roughly speaking, the distorsion estimate means that D and $\pi(D)$ are “similarly” expanded by $\hat{\Phi}^n$ and $\hat{f}^n$ respectively, uniformly with respect to n.

The first step to prove this distorsion estimate is to show that under the hypotheses of Proposition 12.10, the j-th iterate of the horizontal segment D by the double era return map is almost horizontal.

Recall that $\tilde{C}_8$ is the constant defined in Proposition 9.2 on the double era transition map. Remark that

$$\delta_{\perp,0} \leq \inf_{\omega \in \tilde{E}} e^{-\tilde{C}_8 s_4(\omega)}$$

For $\omega \in \tilde{E}$ and $n \geq 0$, let

$$\alpha_n(\omega) \stackrel{\text{def}}{=} \frac{e^{-\sqrt{n}}}{4 \times 16^2 k_{2n+1}(\omega)^2 k_{2n+2}(\omega)^2}$$

For $\omega \in]1,2[\setminus \mathbb{Q}$, let

$$\mathbf{h}_\omega \stackrel{\text{def}}{=} \left(\hat{h}, e^{-\tilde{C}_8 s_4(\omega)}, e^{-\tilde{C}_8 s_2(\omega)}\right)$$

Proposition 12.11 (Decreasing angle with the Mixmaster attractor). *There exists $0 < \delta_{\perp,1} \leq \delta_{\perp,0}$ such that the following property holds for $\omega \in \tilde{E}$, $y \in \left[0, \delta_{\perp,1}\right]^2$, $x, \tilde{x} \in L_y$ and $n \geq 0$. Suppose that both $\hat{\Phi}^j(x)$ and $\hat{\Phi}^j(\tilde{x})$ are well-defined and belong to the section $S^s_{\hat{f}^j(\omega),\mathbf{h}_{\hat{f}^j(\omega)}}$ for $0 \leq j \leq n-1$, then*

$$\left\|\hat{\Phi}^n(x) - \hat{\Phi}^n(\tilde{x})\right\|_\perp \leq \alpha_n(\omega) \left\|\hat{\Phi}^n(x) - \hat{\Phi}^n(\tilde{x})\right\|_{/\!/} \tag{12.6}$$

Remark 12.12. The ratio $\frac{\left\|\hat{\Phi}^n(x)-\hat{\Phi}^n(\tilde{x})\right\|_\perp}{\left\|\hat{\Phi}^n(x)-\hat{\Phi}^n(\tilde{x})\right\|_{/\!/}}$ measures the angle between the segment $\left[\hat{\Phi}^n(x), \hat{\Phi}^n(\tilde{x})\right]$ and the horizontal direction (the direction tangent to the Mixmaster attractor). Proposition 12.11 states that this angle decreases at a rate of a "stretched exponential".

Proof of Proposition 12.11. The proof relies on the fact that the double era return map $\hat{\Phi}$ contracts the direction transverse to the Mixmaster attractor and expands the direction tangent to the Mixmaster attractor. We begin with the definition of the size $\delta_{\perp,1}$. For any $\omega = [1; k_1, k_2, \dots] \in \tilde{E}$ and any $j \geq 0$, let

$$\tilde{h}_{\perp,j} \stackrel{\text{def}}{=} e^{-\frac{\tilde{C}_8}{4}\sum_{i=1}^{2j} k_i^5}$$

Using the uniform estimates (12.1) for points of $\tilde{E}$, we get that there exists $n_1 \geq n_0$ (depending only on n_0) such that for every $\omega = [1; k_1, k_2, \dots] \in \tilde{E}$ and every $j \geq n_1$, we have

$$\tilde{h}_{\perp,j}^{\frac{1}{26k_{2j+1}}} + \tilde{h}_{\perp,j}^{\frac{1}{26k_{2j+2}}} \leq \frac{1}{4}$$

and

$$\tilde{h}_{\perp,j}^{\frac{k_{2j+1}}{100}+\frac{k_{2j+2}}{100}} \leq \frac{1}{2}(4 \times 16^2 k_{2j+3}^2 k_{2j+4}^2)^{-1} e^{-\sqrt{j+1}} = \frac{1}{2}\alpha_{j+1}(\omega)$$

Since the coefficients $k_1(\omega), \dots, k_{2n_1}(\omega)$ of any point $\omega \in \tilde{E}$ admit a uniform upper bound depending only on n_0 and n_1 (see (12.1)), one can find a constant $0 < \delta_{\perp,1} \leq \delta_{\perp,0}$ such that for every $\omega = [1; k_1, k_2, \dots] \in \tilde{E}$ and every $j \geq 0$, we have

$$\left(\delta_{\perp,1}\tilde{h}_{\perp,j}\right)^{\frac{1}{26k_{2j+1}}} + \left(\delta_{\perp,1}\tilde{h}_{\perp,j}\right)^{\frac{1}{26k_{2j+2}}} \leq \frac{1}{4} \tag{12.7a}$$

and

$$\left(\delta_{\perp,1}\tilde{h}_{\perp,j}\right)^{\frac{k_{2j+1}}{100}+\frac{k_{2j+2}}{100}} \leq \frac{1}{2}\alpha_{j+1}(\omega) \tag{12.7b}$$

Let $\omega = [1; k_1, k_2, \dots] \in \tilde{E}$, y, x, $\tilde{x}$ as in the statement of Proposition 12.11. Assume that $x \neq \tilde{x}$. Let $n \geq 0$ such that for every $0 \leq j \leq n-1$, we have

$$\hat{\Phi}^j(x) \in S^s_{\hat{f}^j(\omega),\mathbf{h}_{\hat{f}^j(\omega)}} \qquad \text{and} \qquad \hat{\Phi}^j(\tilde{x}) \in S^s_{\hat{f}^j(\omega),\mathbf{h}_{\hat{f}^j(\omega)}}$$

For $0 \leq j \leq n$, let

$$\begin{aligned}
h_{\perp,j} &= \max\left(\left\|\hat{\Phi}^j(x) - (\hat{\Phi}^A)^j(x)\right\|_\perp, \left\|\hat{\Phi}^j(\tilde{x}) - (\hat{\Phi}^A)^j(\tilde{x})\right\|_\perp\right)\\
h_{/\!/,j} &= e^{-\tilde{C}_8 s_2(\hat{f}^j(\omega))}\\
\alpha_j &= \frac{\left\|\hat{\Phi}^j(x) - \hat{\Phi}^j(\tilde{x})\right\|_\perp}{\left\|\hat{\Phi}^j(x) - \hat{\Phi}^j(\tilde{x})\right\|_{/\!/}}
\end{aligned}$$

According to (9.5) and the fact that both $\hat{\Phi}^j(x)$ and $\hat{\Phi}^j(\tilde{x})$ belong to the section $S^s_{\hat{f}^j(\omega),\mathbf{h}_{\hat{f}^j(\omega)}}$ for $0 \leq j \leq n-1$, we get by induction on j that for every $0 \leq j \leq n$,

$$h_{\perp,j} \leq \delta_{\perp,1} e^{-\frac{\tilde{C}_8}{4}\sum_{i=1}^{2j} k_i^5} \tag{12.8}$$

Claim 1. *For all $0 \leq j \leq n$, $\alpha_j \leq \alpha_j(\omega)$.*

Proof of claim 1. By hypothesis, x and $\tilde{x}$ belong to a same horizontal line, hence $\|x - \tilde{x}\|_\perp = 0$. In other words, $\alpha_0 = 0$ so $\alpha_0 \leq \alpha_0(\omega)$ holds true. Fix $0 \leq j \leq n-1$ and assume that $\alpha_j \leq \alpha_j(\omega)$ holds true. We apply (9.8) to the map $\hat{\Phi}$ restricted to the section $S^s_{\hat{f}^j(\omega),(\hat{h},h_{\perp,j},h_{/\!/,j})}$:

$$\begin{aligned}
&\left\|\left(\hat{\Phi}\left(\hat{\Phi}^j(x)\right) - \hat{\Phi}\left(\hat{\Phi}^j(\tilde{x})\right)\right) - \left(\hat{\Phi}^A\left(\hat{\Phi}^j(x)\right) - \hat{\Phi}^A\left(\hat{\Phi}^j(\tilde{x})\right)\right)\right\|_{/\!/} \leq\\
&\qquad\left(h_{\perp,j}^{\frac{1}{26k_{2j+1}}} + h_{\perp,j}^{\frac{1}{26k_{2j+2}}} + 16^2 k_{2j+1}^2 k_{2j+2}^2 \alpha_j\right)\left\|\hat{\Phi}^j(x) - \hat{\Phi}^j(\tilde{x})\right\|_\infty
\end{aligned} \tag{12.9}$$

Plugging (12.8) into (12.7a), we get

$$h_{\perp,j}^{\frac{1}{26k_{2j+1}}} + h_{\perp,j}^{\frac{1}{26k_{2j+2}}} \leq \frac{1}{4} \tag{12.10}$$

Using the hypothesis $\alpha_j \leq \alpha_j(\omega)$, we get

$$16^2 k_{2j+1}^2 k_{2j+2}^2 \alpha_j \leq \frac{1}{4} \tag{12.11}$$

Plugging (12.10) and (12.11) into (12.9), we get

$$\left\| \left(\hat{\Phi}\left(\hat{\Phi}^j(x)\right) - \hat{\Phi}\left(\hat{\Phi}^j(\tilde{x})\right) \right) - \left(\hat{\Phi}^A\left(\hat{\Phi}^j(x)\right) - \hat{\Phi}^A\left(\hat{\Phi}^j(\tilde{x})\right) \right) \right\|_{/\!/} \leq \frac{1}{2} \left\| \hat{\Phi}^j(x) - \hat{\Phi}^j(\tilde{x}) \right\|_\infty \tag{12.12}$$

Moreover, the Kasner map being expansive (see Proposition 8.2), we have

$$\left\| \hat{\Phi}^A\left(\hat{\Phi}^j(x)\right) - \hat{\Phi}^A\left(\hat{\Phi}^j(\tilde{x})\right) \right\|_{/\!/} \geq \left\| \hat{\Phi}^j(x) - \hat{\Phi}^j(\tilde{x}) \right\|_{/\!/} = \left\| \hat{\Phi}^j(x) - \hat{\Phi}^j(\tilde{x}) \right\|_\infty \tag{12.13}$$

It follows from (12.12) and (12.13) that

$$\left\| \hat{\Phi}^{j+1}(x) - \hat{\Phi}^{j+1}(\tilde{x}) \right\|_{/\!/} \geq \frac{1}{2} \left\| \hat{\Phi}^j(x) - \hat{\Phi}^j(\tilde{x}) \right\|_\infty \tag{12.14}$$

Now we apply (9.7) to the map $\hat{\Phi}$ restricted to the section $S^s_{\hat{f}^j(\omega),\left(\hat{h},h_{\perp,j},h_{/\!/,j}\right)}$:

$$\left\| \hat{\Phi}\left(\hat{\Phi}^j(x)\right) - \hat{\Phi}\left(\hat{\Phi}^j(\tilde{x})\right) \right\|_\perp \leq h_{\perp,j}^{\frac{k_{2j+1}}{100}+\frac{k_{2j+2}}{100}} \left\| \hat{\Phi}^j(x) - \hat{\Phi}^j(\tilde{x}) \right\|_\infty \tag{12.15}$$

Plugging (12.8) into (12.7b), we get

$$\left(h_{\perp,j}\right)^{\frac{k_{2j+1}}{100}+\frac{k_{2j+2}}{100}} \leq \frac{1}{2}\alpha_{j+1}(\omega) \tag{12.16}$$

Plugging (12.16) into (12.15), we get

$$\left\| \hat{\Phi}^{j+1}(x) - \hat{\Phi}^{j+1}(\tilde{x}) \right\|_\perp \leq \frac{1}{2}\alpha_{j+1}(\omega) \left\| \hat{\Phi}^j(x) - \hat{\Phi}^j(\tilde{x}) \right\|_\infty \tag{12.17}$$

According to (12.14) and (12.17), we have

$$\alpha_{j+1} \leq \alpha_{j+1}(\omega)$$

This concludes the proof of claim 1. □

In particular, $\alpha_n \leq \alpha_n(\omega)$ holds true, which is the desired result. This concludes the proof of Proposition 12.11. □

Proof of Proposition 12.10. We begin with the definition of the size $\delta_{\perp,2}$. For $\omega = [1; k_1, k_2, \dots] \in \tilde{E}$ and $j \geq 0$, recall that

$$\tilde{h}_{\perp,j} = e^{-\frac{\tilde{C}_8}{4}\sum_{i=1}^{2j} k_i^5}$$

Using the uniform estimates (12.1) for points of $\tilde{E}$, we get that there exists $n_1 \geq n_0$ (depending only on n_0) such that for every $\omega = [1; k_1, k_2, \dots] \in \tilde{E}$ and every $j \geq n_1$, we have the following estimates

$$\tilde{h}_{\perp,j}^{\frac{1}{26k_{2j+1}}} + \tilde{h}_{\perp,j}^{\frac{1}{26k_{2j+2}}} \leq \frac{1}{4}e^{-\sqrt{j}}$$
$$\tilde{h}_{\perp,j} 128^2 k_{2j+1}^3 k_{2j+2}^3 \leq \frac{1}{4}e^{-\sqrt{j}}$$

Since the coefficients $k_1(\omega), \dots, k_{2n_1}(\omega)$ of any point $\omega \in \tilde{E}$ admit a uniform upper bound depending only on n_0 and n_1 (see (12.1)), one can find a constant $0 < \delta_{\perp,2} \leq \delta_{\perp,1}$ such that for every $\omega =$

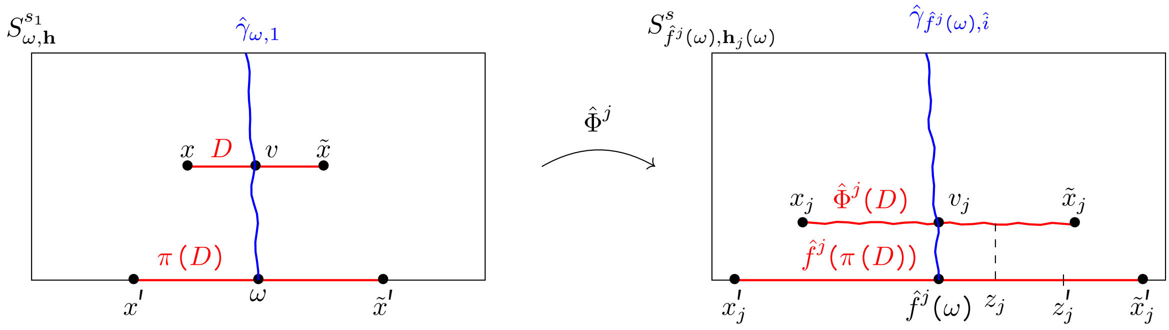

Figure 12.4 – Iteration of the two horizontal segments. To avoid clutter, we denote $\hat{\imath} = \hat{\imath}(\hat{f}^{j-1}(\omega))$.

$[1; k_1, k_2, \dots] \in \tilde{E}$ and every $j \geq 0$, we have

$$\left(\delta_{\perp,2}\tilde{h}_{\perp,j}\right)^{\frac{1}{26k_{2j+1}}} + \left(\delta_{\perp,2}\tilde{h}_{\perp,j}\right)^{\frac{1}{26k_{2j+2}}} \leq \frac{1}{4}e^{-\sqrt{j}} \tag{12.18a}$$

$$\delta_{\perp,2}\tilde{h}_{\perp,j}128^2 k^3_{2j+1}k^3_{2j+2} \leq \frac{1}{4}e^{-\sqrt{j}} \tag{12.18b}$$

Fix $\omega \in \tilde{E}$ and $y \in [0, \delta_{\perp,1}]^2$. Fix $D \subset L_y$ an horizontal segment centered above ω. Fix $n \geq 0$ such that $(H_{\omega,D,n})$ holds true. Denote by x and $\tilde{x}$ the end points of D. Analogously, let $x' = \inf \pi(D)$ and $\tilde{x}' = \sup \pi(D)$. The forward invariance of the tangential cone field and the expansion estimate (10.4) imply that, for all $0 \leq j \leq n$, $\hat{\Phi}^j(D)$ is an arc "almost horizontal" in the section $S^s_{\hat{f}^j(\omega),\mathbf{h}_j(\omega)}$. In particular, its diameter satisfies the relation

$$\left|\hat{\Phi}^j(D)\right| = \|x_j - \tilde{x}_j\|_{/\!/}$$

where $x_j := \hat{\Phi}^j(x)$ and $\tilde{x}_j := \hat{\Phi}^j(\tilde{x})$. Using the expansion of the double era Kasner map (see Proposition 8.2), one has an analogous result for $\pi(D)$, letting $x'_j := \hat{f}^j(x')$ and $\tilde{x}'_j := \hat{f}^j(\tilde{x}')$. See figure 12.4.

Using the points x_j, $\tilde{x}_j$, x'_j and $\tilde{x}'_j$, one can write

$$\begin{aligned}
\frac{\left|\hat{\Phi}^n(D)\right|}{\left|\hat{f}^n(\pi(D))\right|}\frac{|\pi(D)|}{|D|} &= \frac{\left\|\hat{\Phi}^n(x) - \hat{\Phi}^n(\tilde{x})\right\|_{/\!/}}{\|x - \tilde{x}\|_{/\!/}}\frac{|x' - \tilde{x}'|}{\left|\hat{f}^n(x') - \hat{f}^n(\tilde{x}')\right|} \\
&= \prod_{j=0}^{n-1}\frac{\left\|\hat{\Phi}(x_j) - \hat{\Phi}(\tilde{x}_j)\right\|_{/\!/}}{\|x_j - \tilde{x}_j\|_{/\!/}}\prod_{j=0}^{n-1}\frac{|x'_j - \tilde{x}'_j|}{\left|\hat{f}(x'_j) - \hat{f}(\tilde{x}'_j)\right|} \\
&= \prod_{j=0}^{n-1}\frac{\left\|\hat{\Phi}(x_j) - \hat{\Phi}(\tilde{x}_j)\right\|_{/\!/}}{\left\|\hat{\Phi}^A(x_j) - \hat{\Phi}^A(\tilde{x}_j)\right\|_{/\!/}}\prod_{j=0}^{n-1}\frac{\left\|\hat{\Phi}^A(x_j) - \hat{\Phi}^A(\tilde{x}_j)\right\|_{/\!/}}{\|x_j - \tilde{x}_j\|_{/\!/}}\frac{|x'_j - \tilde{x}'_j|}{\left|\hat{f}(x'_j) - \hat{f}(\tilde{x}'_j)\right|} \\
&= R_1R_2
\end{aligned} \tag{12.19}$$

where

$$R_1 \stackrel{\text{def}}{=} \prod_{j=0}^{n-1}\frac{\left\|\hat{\Phi}(x_j) - \hat{\Phi}(\tilde{x}_j)\right\|_{/\!/}}{\left\|\hat{\Phi}^A(x_j) - \hat{\Phi}^A(\tilde{x}_j)\right\|_{/\!/}} \quad \text{and} \quad R_2 \stackrel{\text{def}}{=} \prod_{j=0}^{n-1}\frac{\left\|\hat{\Phi}^A(x_j) - \hat{\Phi}^A(\tilde{x}_j)\right\|_{/\!/}}{\|x_j - \tilde{x}_j\|_{/\!/}}\frac{|x'_j - \tilde{x}'_j|}{\left|\hat{f}(x'_j) - \hat{f}(\tilde{x}'_j)\right|}$$

For $0 \le j \le n$, let

$$h_{\perp,j} = \max_{z\in\{x,\tilde{x}\}} \left\|\hat{\Phi}^j(z) - (\hat{\Phi}^A)^j(z)\right\|_\perp$$
$$h_{/\!/,j} = e^{-\tilde{C}_{13} m_j(\omega)}$$

According to (9.5), we get by induction on j that for every $0 \le j \le n$,

$$h_{\perp,j} \le \delta_{\perp,2} e^{-\frac{\tilde{C}_8}{4}\sum_{i=1}^{2j} k_i^5} \tag{12.20}$$

Let

$$C_1 \stackrel{\text{def}}{=} \frac{1}{2}\sum_{j=0}^{+\infty} e^{-\sqrt{j}}$$

Claim 1. $R_1 \le e^{C_1}$.

Proof of claim 1. Let $0 \le j \le n-1$. We apply (9.8) to the map $\hat{\Phi}$ restricted to the section $S^s_{\hat{f}^j(\omega),\left(\hat{h},h_{\perp,j},h_{/\!/,j}\right)}$:

$$\left\|\left(\hat{\Phi}(x_j) - \hat{\Phi}(\tilde{x}_j)\right) - \left(\hat{\Phi}^A(x_j) - \hat{\Phi}^A(\tilde{x}_j)\right)\right\|_{/\!/} \le \left(h_{\perp,j}^{\frac{1}{26k_{2j+1}}} + h_{\perp,j}^{\frac{1}{26k_{2j+2}}}\right)\left\|x_j - \tilde{x}_j\right\|_\infty + 16^2 k_{2j+1}^2 k_{2j+2}^2 \left\|x_j - \tilde{x}_j\right\|_\perp \tag{12.21}$$

Plugging (12.20) into (12.18a), we get

$$h_{\perp,j}^{\frac{1}{26k_{2j+1}}} + h_{\perp,j}^{\frac{1}{26k_{2j+2}}} \le \frac{1}{4} e^{-\sqrt{j}} \tag{12.22}$$

According to (12.6),

$$16^2 k_{2j+1}^2 k_{2j+2}^2 \left\|x_j - \tilde{x}_j\right\|_\perp \le 16^2 k_{2j+1}^2 k_{2j+2}^2 \alpha_j(\omega)\left\|x_j - \tilde{x}_j\right\|_{/\!/} \le \frac{1}{4}e^{-\sqrt{j}}\left\|x_j - \tilde{x}_j\right\|_\infty \tag{12.23}$$

where we used $\left\|x_j - \tilde{x}_j\right\|_{/\!/} = \left\|x_j - \tilde{x}_j\right\|_\infty$. Plugging (12.22) and (12.23) into (12.21), we get

$$\left\|\left(\hat{\Phi}(x_j) - \hat{\Phi}(\tilde{x}_j)\right) - \left(\hat{\Phi}^A(x_j) - \hat{\Phi}^A(\tilde{x}_j)\right)\right\|_{/\!/} \le \frac{1}{2}e^{-\sqrt{j}}\left\|x_j - \tilde{x}_j\right\|_\infty \tag{12.24}$$

Recall that the map $\hat{\Phi}^A$ is essentially the double era Kasner map (see (5.7)). Moreover, the Kasner map is expansive (see Proposition 8.2), hence

$$\left\|\hat{\Phi}^A(x_j) - \hat{\Phi}^A(\tilde{x}_j)\right\|_{/\!/} \ge \left\|x_j - \tilde{x}_j\right\|_{/\!/} = \left\|x_j - \tilde{x}_j\right\|_\infty \tag{12.25}$$

It follows from (12.24) and (12.25) that

$$\left|\frac{\left\|\hat{\Phi}(x_j) - \hat{\Phi}(\tilde{x}_j)\right\|_{/\!/}}{\left\|\hat{\Phi}^A(x_j) - \hat{\Phi}^A(\tilde{x}_j)\right\|_{/\!/}} - 1\right| \le \frac{1}{2}e^{-\sqrt{j}}$$

As a consequence of the above estimate, we get

$$\ln R_1 = \sum_{j=0}^{n-1} \ln \frac{\left\|\hat{\Phi}(x_j) - \hat{\Phi}(\tilde{x}_j)\right\|_{/\!/}}{\left\|\hat{\Phi}^A(x_j) - \hat{\Phi}^A(\tilde{x}_j)\right\|_{/\!/}} \le \sum_{j=0}^{n-1} \frac{1}{2}e^{-\sqrt{j}} \le C_1$$

This concludes the proof of claim 1. ∎

Recall that $K_c > 1$ is the expansivity constant in the tangential cone field (see (10.4)). Let

$$C_2 \stackrel{\text{def}}{=} \frac{1}{4}\left(\sum_{j=0}^{+\infty} K_c^{-j} + \sum_{j=0}^{+\infty} e^{-\sqrt{j}}\right)$$

Claim 2. $R_2 \leq e^{C_2}$.

Proof of claim 2. Recall that the last coordinate of the double era return map restricted to the Mixmaster attractor is exactly the double era Kasner map, hence

$$R_2 = \prod_{j=0}^{n-1} \frac{\left|\hat{f}\left((x_j)_c\right) - \hat{f}\left((\tilde{x}_j)_c\right)\right|}{\left|(x_j)_c - (\tilde{x}_j)_c\right|} \frac{\left|x_j' - \tilde{x}_j'\right|}{\left|\hat{f}(x_j') - \hat{f}(\tilde{x}_j')\right|}$$

Applying the mean value theorem to the function $\hat{f}$, we get that

$$R_2 = \prod_{j=0}^{n-1} \frac{\left|\hat{f}'(z_j)\right|}{\left|\hat{f}'(z_j')\right|}$$

where $z_j \in \left[(x_j)_c, (\tilde{x}_j)_c\right]$ and $z_j' \in \left[x_j', \tilde{x}_j'\right]$ (see figure 12.4). Let $0 \leq j \leq n-1$. According to Proposition 8.3,

$$\left|\hat{f}'(z_j) - \hat{f}'(z_j')\right| \leq 128^2 k_{2j+1}^3 k_{2j+2}^3 \left|z_j - z_j'\right|$$

Let us denote by v the intersection point of D with $\hat{\gamma}_{\omega,1}$. Let $v_j = \hat{\Phi}^j(v)$. Remark that $(v_j)_c \in \left[(x_j)_c, (\tilde{x}_j)_c\right]$ and $\hat{f}^j(\omega) \in \left[x_j', \tilde{x}_j'\right]$. According to the above estimate,

$$\begin{aligned}
\left|\hat{f}'(z_j) - \hat{f}'(z_j')\right| &\leq 128^2 k_{2j+1}^3 k_{2j+2}^3 \left(\left|z_j - (v_j)_c\right| + \left|(v_j)_c - \hat{f}^j(\omega)\right| + \left|\hat{f}^j(\omega) - z_j'\right|\right) \\
&\leq 128^2 k_{2j+1}^3 k_{2j+2}^3 \left(\left\|x_j - \tilde{x}_j\right\|_{/\!/} + \left|(v_j)_c - \hat{f}^j(\omega)\right| + \left|x_j' - \tilde{x}_j'\right|\right) \qquad (12.26)
\end{aligned}$$

According to the forward invariance of the tangential cone field and the expansion estimate (10.4),

$$\left\|x_j - \tilde{x}_j\right\|_{/\!/} \leq K_c^{j+1-n} \left\|x_{n-1} - \tilde{x}_{n-1}\right\|_{/\!/} \leq K_c^{j+1-n} e^{-\tilde{C}_{13} m_{n-1}(\omega)} \qquad (12.27)$$

One has an analogous estimate for $\left|x_j' - \tilde{x}_j'\right|$. Putting together (12.27) and (12.5) (with $a = \max_{1 \leq l \leq 2n} k_l$), we get

$$128^2 k_{2j+1}^3 k_{2j+2}^3 \left(\left\|x_j - \tilde{x}_j\right\|_{/\!/} + \left|x_j' - \tilde{x}_j'\right|\right) \leq \frac{1}{4} K_c^{j+1-n} \qquad (12.28)$$

Since v belongs to the graph $\hat{\gamma}_{\omega,1}$, (10.25) implies that

$$\left|(v_j)_c - \hat{f}^j(\omega)\right| \leq \delta_{\perp,2} e^{-\tilde{C}_8 \sum_{i=1}^{2j} k_i^5}$$

Using (12.18b), it follows from the above estimate that

$$128^2 k_{2j+1}^3 k_{2j+2}^3 \left|(v_j)_c - \hat{f}^j(\omega)\right| \leq \frac{1}{4} e^{-\sqrt{j}} \qquad (12.29)$$

Plugging (12.28) and (12.29) into (12.26), we get

$$\left|\hat{f}'(z_j) - \hat{f}'(z_j')\right| \leq \frac{1}{4} K_c^{j+1-n} + \frac{1}{4} e^{-\sqrt{j}}$$

Since $\hat{f}$ is expansive (see Proposition 8.2), it follows that $\left|\hat{f}'(z_j')\right| \geq 1$ and

$$\left|\frac{\left|\hat{f}'(z_j)\right|}{\left|\hat{f}'(z_j')\right|} - 1\right| \leq \frac{1}{4}K_c^{j+1-n} + \frac{1}{4}e^{-\sqrt{j}}$$

As in the proof of claim 1, we get as a consequence of the above estimate that

$$\ln R_2 \leq \frac{1}{4}\left(\sum_{j=0}^{n-1} K_c^{j+1-n} + \sum_{j=0}^{n-1} e^{-\sqrt{j}}\right) \leq \frac{1}{4}\left(\sum_{j=0}^{n-1} K_c^{-j} + \sum_{j=0}^{n-1} e^{-\sqrt{j}}\right) \leq C_2$$

This concludes the proof of claim 2. □

Using claim 1 and claim 2 together with (12.19), we get that

$$\frac{\left|\hat{\Phi}^n(D)\right|}{\left|\hat{f}^n(\pi(D))\right|}\frac{|\pi(D)|}{|D|} \leq e^{C_1+C_2}$$

Hence, Proposition 12.10 holds true with $\Delta := e^{C_1+C_2}$. □

12.2 Absolute continuity of the projection map π

Define

$$\delta_{/\!/} \stackrel{\text{def}}{=} \inf_{\omega\in\tilde{E}} \frac{e^{-\tilde{C}_{13}s_2(\omega)}}{2} > 0$$

For G a subset of $S_{\hat{h}}^{s_1}$, define the "maximal gap" between G and the Mixmaster attractor by

$$\operatorname{dist}_\perp(G,A) \stackrel{\text{def}}{=} \sup_{x\in G}\operatorname{dist}_\infty(x,A) = \sup_{x\in G}\max\left(x_u, x_{s_2}\right)$$

and analogously if G is a subset of $S_{\hat{h}}^{s_2}$.

Next lemma gives a sufficient condition so that we can control all the iterates $\hat{\Phi}^j(D)$, $0 \leq j \leq n-1$, for a time n sufficiently large so that the distance between the Mixmaster attractor and $\hat{\Phi}^n(D)$ is smaller than its diameter in the direction tangential to the Mixmaster attractor.

Lemma 12.13. *There exists a constant $0 < \delta_{\perp,3} \leq \delta_{\perp,2}$ such that the following property holds for $\omega \in \tilde{E}$ and $y \in [0,\delta_{\perp,3}]^2$. Let $D \subset L_y$ be an horizontal segment centered above ω. If*

$$|D| \leq \delta_{/\!/}$$

then there exists an integer $n(D) \geq 0$ such that the property $(H_{\omega,D,n(D)})$ holds true and such that

$$\operatorname{dist}_\perp\left(\hat{\Phi}^{n(D)}(D), A\right) \leq \left|\hat{\Phi}^{n(D)}(D)\right| \tag{12.30}$$

Proof. We begin with the definition of the size $\delta_{\perp,3}$. For $\omega = [1;k_1,k_2,\dots] \in \tilde{E}$ and $j \geq 0$, recall that

$$\tilde{h}_{\perp,j} = e^{-\frac{\tilde{C}_8}{4}\sum_{i=1}^{2j}k_i^5}$$

Using the uniform estimates (12.1) for points of $\tilde{E}$, we get that there exists $n_1 \geq n_0$ (depending only on n_0) such that for every $\omega = [1;k_1,k_2,\dots] \in \tilde{E}$ and every $j \geq n_1$, we have the following estimates

$$\tilde{h}_{\perp,j} \leq e^{-\tilde{C}_{13}s_4(\hat{f}^j(\omega))}$$

$$\tilde{h}_{\perp,j} \leq \frac{e^{-\tilde{C}_{13}m_j(\omega)}}{2}$$

Since the coefficients $k_1(\omega), \dots, k_{2n_1}(\omega)$ of any point $\omega \in \tilde{E}$ admit a uniform upper bound depending only on n_0 and n_1 (see (12.1)), one can find a constant $0 < \delta_{\perp,3} \leq \min\left(\delta_{/\!/}, \delta_{\perp,2}\right)$ such that for every $\omega = [1; k_1, k_2, \dots] \in \tilde{E}$ and every $j \geq 0$, we have

$$\delta_{\perp,3}\tilde{h}_{\perp,j} \leq e^{-\tilde{C}_{13}s_4(\hat{f}^j(\omega))} \tag{12.31a}$$

$$\delta_{\perp,3}\tilde{h}_{\perp,j} \leq \frac{e^{-\tilde{C}_{13}m_j(\omega)}}{2} \tag{12.31b}$$

Fix $\omega = [1; k_1, k_2, \dots] \in \tilde{E}$, $y \in \left[0, \delta_{\perp,3}\right]^2$ and $D \subset L_y$ an horizontal segment centered above ω. Assume that

$$|D| \leq \delta_{/\!/}$$

Recall that

$$I_j(\omega) = \left[\hat{f}^j(\omega) - e^{-\tilde{C}_{13}m_j(\omega)}, \hat{f}^j(\omega) + e^{-\tilde{C}_{13}m_j(\omega)}\right]$$
$$\mathbf{h}_j(\omega) = \left(\hat{h}, e^{-\tilde{C}_{13}s_4(\hat{f}^j(\omega))}, e^{-\tilde{C}_{13}m_j(\omega)}\right)$$

Define

$$n(D) \stackrel{\text{def}}{=} \max\left\{n \in \mathbb{N} \mid \forall 0 \leq j \leq n-1, \left|\hat{\Phi}^j(D)\right| \leq \frac{e^{-\tilde{C}_{13}m_j(\omega)}}{2}\right\}$$
$$N(D) \stackrel{\text{def}}{=} \max\left\{n \in \mathbb{N} \mid \forall 0 \leq j \leq n-1, \hat{\Phi}^j(D) \subset S^s_{\hat{f}^j(\omega),\mathbf{h}_j(\omega)}\right\}$$
$$N(\pi(D)) \stackrel{\text{def}}{=} \max\left\{n \in \mathbb{N} \mid \forall 0 \leq j \leq n-1, \hat{f}^j(\pi(D)) \subset I_j(\omega)\right\}$$

Saying that $(H_{\omega,D,n(D)})$ holds true amounts to saying that $N(D) \geq n(D)$ and $N(\pi(D)) \geq n(D)$.

According to (9.5), we get by induction on j that for every $0 \leq j \leq N(D)$,

$$\operatorname{dist}_\perp\left(\hat{\Phi}^j(D), A\right) \leq \delta_{\perp,3}e^{-\frac{\tilde{C}_8}{4}\sum_{i=1}^{2j}k_i^5} = \delta_{\perp,3}\tilde{h}_{\perp,j} \tag{12.32}$$

Claim 1. $N(D) \geq n(D)$.

Proof of claim 1. We need to prove that for all $0 \leq j \leq n(D) - 1$,

$$\hat{\Phi}^j(D) \subset S^s_{\hat{f}^j(\omega),\mathbf{h}_j(\omega)}$$

For $0 \leq j \leq n(D) - 1$, define the property

$$(P_j) : \hat{\Phi}^j(D) \subset S^s_{\hat{f}^j(\omega),\mathbf{h}_j(\omega)}$$

Recall that D is centered above ω. Hence, for all $x \in D$,

$$|x_c - \omega| \leq |x_c - v_c| + |v_c - \omega| \leq \delta_{/\!/} + \delta_{\perp,3} \leq 2\delta_{/\!/}$$

where v denotes the intersection point of D with $\hat{\gamma}_{\omega,1}$. Using (12.31a) and the definition of $\delta_{/\!/}$, we get that $D \subset S^s_{\omega,\mathbf{h}_0(\omega)}$. Hence, (P_0) holds true. Fix $0 \leq l \leq n(D) - 2$ and assume that for all $0 \leq j \leq l$, (P_j) holds true. It follows that $l + 1 \leq N(D)$. Let $x \in D$. Plugging (12.32) into (12.31a), we get

$$\left\|\hat{\Phi}^{l+1}(x) - (\hat{\Phi}^A)^{l+1}(x)\right\|_\perp \leq e^{-\tilde{C}_{13}s_4(\hat{f}^{l+1}(\omega))}$$

By standard triangle inequality, we get

$$\left\|\hat{\Phi}^{l+1}(x) - \hat{\Phi}^{l+1}\left(P^{s_1}_{\omega,\hat{h}}\right)\right\|_{/\!/} \leq \left\|\hat{\Phi}^{l+1}(x) - \hat{\Phi}^{l+1}(v)\right\|_{/\!/} + \left\|\hat{\Phi}^{l+1}(v) - \hat{\Phi}^{l+1}\left(P^{s_1}_{\omega,\hat{h}}\right)\right\|_{/\!/}$$

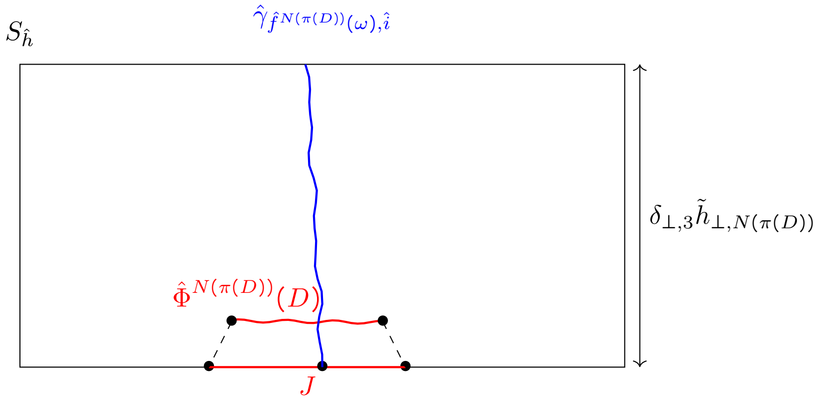

Figure 12.5 – $\pi\left(\hat{\Phi}^{N(\pi(D))}(D)\right) \subset J$ with $|J| \leq \left|\hat{\Phi}^{N(\pi(D))}(D)\right| + \delta_{\perp,3}\tilde{h}_{\perp,N(\pi(D))}$. To avoid clutter, we denote $\hat{\imath} = \hat{\imath}(\hat{f}^{N(\pi(D))-1}(\omega))$.

According to (10.25),

$$\begin{aligned}
\left\|\hat{\Phi}^{l+1}(v) - \hat{\Phi}^{l+1}\left(P^{s_1}_{\omega,\hat{h}}\right)\right\|_{/\!/} &\leq \left\|v - P^{s_1}_{\omega,\hat{h}}\right\|_{\perp} e^{-\tilde{C}_8 \sum_{i=1}^{2(l+1)} k_i^5} \\
&\leq \delta_{\perp,3} e^{-\tilde{C}_8 \sum_{i=1}^{2(l+1)} k_i^5} \\
&\leq \frac{e^{-\tilde{C}_{13} m_{l+1}(\omega)}}{2} && \text{using (12.31b)}
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left\|\hat{\Phi}^{l+1}(x) - \hat{\Phi}^{l+1}(v)\right\|_{/\!/} &\leq \left|\hat{\Phi}^{l+1}(D)\right| \\
&\leq \frac{e^{-\tilde{C}_{13} m_{l+1}(\omega)}}{2} && \text{using } l+1 \leq n(D)-1
\end{aligned}$$

It follows that

$$\left\|\hat{\Phi}^{l+1}(x) - \hat{\Phi}^{l+1}\left(P^{s_1}_{\omega,\hat{h}}\right)\right\|_{/\!/} \leq \frac{e^{-\tilde{C}_{13} m_{l+1}(\omega)}}{2} + \frac{e^{-\tilde{C}_{13} m_{l+1}(\omega)}}{2} \leq e^{-\tilde{C}_{13} m_{l+1}(\omega)}$$

Hence, $\hat{\Phi}^{l+1}(x) \in S^s_{\hat{f}^{l+1}(\omega),\mathbf{h}_{l+1}(\omega)}$ and (P_{l+1}) holds true. By induction, we get that for all $0 \leq j \leq n(D)-1$, (P_j) holds true. This concludes the proof of claim 1. □

Claim 2. $N(\pi(D)) \geq n(D)$.

Proof of claim 2. Assume that $N(\pi(D)) < n(D)$. Recall that the local stable manifolds "foliation" F^{s_1} of the double era return map $\hat{\Phi}$ is made of $\hat{\sigma}$-Lipschitz graphs with $\hat{\sigma} \leq \frac{1}{2}$ (see (9.3)). Using (12.32) with $j = N(\pi(D)) \leq N(D)$ (see claim 1), we get that

$$\left|\pi\left(\hat{\Phi}^{N(\pi(D))}(D)\right)\right| \leq \left|\hat{\Phi}^{N(\pi(D))}(D)\right| + \delta_{\perp,3}\tilde{h}_{\perp,N(\pi(D))}$$

See figure 12.5. Since $N(\pi(D)) < n(D)$,

$$\left|\hat{\Phi}^{N(\pi(D))}(D)\right| \leq \frac{e^{-\tilde{C}_{13} m_{N(\pi(D))}(\omega)}}{2}$$

According to (12.31b),

$$\delta_{\perp,3}\tilde{h}_{\perp,N(\pi(D))} \leq \frac{e^{-\tilde{C}_{13} m_{N(\pi(D))}(\omega)}}{2}$$

It follows that
$$\left|\pi\left(\hat{\Phi}^{N(\pi(D))}(D)\right)\right| \le e^{-\tilde{C}_{13}m_{N(\pi(D))}(\omega)}$$
Recall that the local stable manifolds "foliation" F^{s_1} is leaf-invariant by $\hat{\Phi}$. Hence, π semi-conjugate $\hat{\Phi}$ and $\hat{f}$:
$$\pi \circ \hat{\Phi} = \hat{f} \circ \pi$$
As a consequence,
$$\left|\hat{f}^{N(\pi(D))}(\pi(D))\right| \le e^{-\tilde{C}_{13}m_{N(\pi(D))}(\omega)}$$
Moreover, $\hat{f}^{N(\pi(D))}(\omega) \in \hat{f}^{N(\pi(D))}(\pi(D))$, hence
$$\hat{f}^{N(\pi(D))}(\pi(D)) \subset I_{N(\pi(D))}(\omega)$$
This contradicts the maximality of $N(\pi(D))$ and this concludes the proof of claim 2. □

It follows from claim 1 and claim 2 that $(H_{\omega,D,n(D)})$ holds true. Using (12.32) with $j = n(D) \le N(D)$ (see claim 1), we get that
$$\operatorname{dist}_\perp\left(\hat{\Phi}^{n(D)}(D), A\right) \le \delta_{\perp,3}\tilde{h}_{\perp,n(D)}$$
By definition of $n(D)$, we have
$$\left|\hat{\Phi}^{n(D)}(D)\right| > \frac{e^{-\tilde{C}_{13}m_{n(D)}(\omega)}}{2} \tag{12.33}$$
According to (12.31b) and (12.33), we have
$$\delta_{\perp,3}\tilde{h}_{\perp,n(D)} \le \left|\hat{\Phi}^{n(D)}(D)\right|$$
Hence,
$$\operatorname{dist}_\perp\left(\hat{\Phi}^{n(D)}(D), A\right) \le \left|\hat{\Phi}^{n(D)}(D)\right|$$
This concludes the proof of Lemma 12.13. □

Proposition 12.14 (Absolute continuity of the projection map). *Let $\omega \in \tilde{E}$, $y \in [0, \delta_{\perp,3}]^2$ and $D \subset L_y$ be an horizontal segment centered above ω. If*
$$|D| \le \delta_{/\!/}$$
then
$$\operatorname{Leb}_1(\pi(D)) \le 2\Delta \operatorname{Leb}_1(D) \tag{12.34}$$

Proof. According to Lemma 12.13, we can apply Proposition 12.10 to get
$$\frac{\left|\hat{\Phi}^{n(D)}(D)\right|}{\left|\hat{f}^{n(D)}(\pi(D))\right|} \le \Delta\frac{|D|}{|\pi(D)|}$$
Hence,
$$|\pi(D)| \le \Delta\frac{\left|\hat{f}^{n(D)}(\pi(D))\right|}{\left|\hat{\Phi}^{n(D)}(D)\right|}|D| \tag{12.35}$$

Claim 1. $\left|\hat{f}^{n(D)}(\pi(D))\right| \le 2\left|\hat{\Phi}^{n(D)}(D)\right|$.

Proof of claim 1. Recall that the local stable manifolds "foliation" F^{s_1} of the double era return map $\hat{\Phi}$ is made of $\hat{\sigma}$-Lipschitz graphs with $\hat{\sigma} \le \frac{1}{2}$ (see (9.3)). Hence, estimate (12.30) implies that
$$\left|\pi\left(\hat{\Phi}^{n(D)}(D)\right)\right| \le \left|\hat{\Phi}^{n(D)}(D)\right| + \operatorname{dist}_\perp\left(\hat{\Phi}^{n(D)}(D), A\right)$$

Now recall that the integer $n(D)$ was chosen so that $\mathrm{dist}_{\perp}\left(\hat{\Phi}^{n(D)}(D), A\right) \leq \left|\hat{\Phi}^{n(D)}(D)\right|$ (see Lemma 12.13) so we get

$$\left|\pi\left(\hat{\Phi}^{n(D)}(D)\right)\right| \leq 2\left|\hat{\Phi}^{n(D)}(D)\right|$$

The conjugacy relation $\pi \circ \hat{\Phi} = \hat{f} \circ \pi$ implies that $\pi\left(\hat{\Phi}^{n(D)}(D)\right) = \hat{f}^{n(D)}(\pi(D))$. This concludes the proof of claim 1. □

Claim 1 together with (12.35) gives

$$|\pi(D)| \leq 2\Delta |D|$$

Remark that $\mathrm{Leb}_1(\pi(D)) \leq |\pi(D)|$ and $\mathrm{Leb}_1(D) = |D|$. Hence

$$\mathrm{Leb}_1(\pi(D)) \leq 2\Delta\, \mathrm{Leb}_1(D)$$

which is the desired estimate. This concludes the proof of Proposition 12.14. □

Proof of Lemma 12.6. Let $y \in \left[0, \delta_{\perp,3}\right]^2$ and $G \subset L_y \cap W^{s_1}\left(\tilde{E}, \hat{\Phi}\right)$. Assume that $\mathrm{Leb}_1(G) = 0$. To show that $\mathrm{Leb}_1(\pi(G)) = 0$, cover G by a countable union of small horizontal segments and use the estimate (12.34). □

Appendix A

Statement of the main theorem in the entire phase space

In this appendix, we explain how to extend Theorem B to type VIII orbits. To this end, we show how some objects defined in the introduction (especially type II orbits, the Kasner map and heteroclinic chains) can be generalized to the entire phase space. A technical complication arises since most abstract heteroclinic chains cannot be shadowed by any type VIII or IX orbit for elementary reasons. This will lead us to introduce a notion of *coherent heteroclinic chain*.

Type II orbits. Recall that in $\mathscr{B}^+$, for every point p of the Kasner circle that is not a Taub point, there is exactly one type II orbit starting at p. When looking at the full phase space $\mathscr{B}$, we have the following result. For every point p of the Kasner circle that is not a Taub point, there are exactly two type II orbits starting at p. These two orbits are exchanged by the symmetry

$$(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \mapsto (-N_1, -N_2, -N_3, \Sigma_1, \Sigma_2, \Sigma_3)$$

fixing the points of the plane $(N_1 = N_2 = N_3 = 0)$ containing the Kasner circle. As an immediate consequence, these two type II orbits converge to the same point of $\mathscr{K}$ in the future.

Kasner map Let p be a point of the Kasner circle which is not a Taub point. When we restrict ourselves to $\mathscr{B}^+$, there is exactly one type II orbit starting at p and this orbit converges to a point denoted by $\mathscr{F}(p)$ (the image of p by the Kasner map). This is indeed how we defined the Kasner map (see section 3.5). As stated above, in $\mathscr{B}$, there are two (symmetrical) type II orbits starting at p. Since they are symmetrical, they both converge to the same point of the Kasner circle, that is, the point $\mathscr{F}(p)$. We will denote these two type II orbits by $\mathscr{O}^+_{p\to\mathscr{F}(p)}$ and $\mathscr{O}^-_{p\to\mathscr{F}(p)}$, $\mathscr{O}^+_{p\to\mathscr{F}(p)}$ being the one entirely contained in $\mathscr{B}^+$.

Coherent heteroclinic chains

Definition A.1 (Heteroclinic chains)**.** Let p be a point of the Kasner circle (such that, for every $k \geq 0$, $\mathscr{F}^k(p)$ is not a Taub point). A *heteroclinic chain* (starting at p) is a concatenation of one type II orbit starting at p and arriving at $\mathscr{F}(p)$, then one type II orbit starting at $\mathscr{F}(p)$ and arriving at $\mathscr{F}^2(p)$, etc. Formally, this is a sequence of the form

$$\left(\mathscr{O}^{\epsilon_0}_{p\to\mathscr{F}(p)}, \mathscr{O}^{\epsilon_1}_{\mathscr{F}(p)\to\mathscr{F}^2(p)}, \mathscr{O}^{\epsilon_2}_{\mathscr{F}^2(p)\to\mathscr{F}^3(p)}, \dots\right) \tag{A.1}$$

where $\epsilon_n \in \{\pm\}$ corresponds to a choice of one of the two symmetrical type II orbits starting at $\mathscr{F}^n(p)$.

As we will see, some heteroclinic chains cannot be shadowed by type VIII or type IX orbits. First, let us recall the definition of shadowing, generalized to the full phase space in a straightforward manner.

Definition A.2 (Shadowing)**.** Let $t \mapsto \mathscr{O}(t)$ be a type VIII or IX orbit in $\mathscr{B}$, p be a point of the Kasner circle (such that, for every $k \geq 0$, $\mathscr{F}^k(p)$ is not a Taub point) and $\mathscr{H}$ be a heteroclinic chain (A.1) starting at p. We say that $\mathscr{O}$ *shadows* $\mathscr{H}$ (or $\mathscr{H}$ attracts $\mathscr{O}$) if there exists a strictly increasing sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that

1. $d(\mathscr{O}(t_n), \mathscr{F}^n(p)) \xrightarrow[n\to+\infty]{} 0$.
2. The Hausdorff distance between the orbit interval $\{\mathscr{O}(t) \mid t_n < t < t_{n+1}\}$ and the type II orbit $\mathscr{O}^{\epsilon_n}_{\mathscr{F}^n(p)\to\mathscr{F}^{n+1}(p)}$ tends to 0 when $n \to +\infty$.

Recall that any type II orbit is contained in a subset of the phase space of the form

$$\{N_i > 0, N_j = 0, N_k = 0\} \qquad \text{or} \qquad \{N_i < 0, N_j = 0, N_k = 0\}$$

where $\{i,j,k\} = \{1,2,3\}$. Consider for example a heteroclinic chain made of an infinite number of type II orbits traveling in $\{N_1 > 0, N_2 = 0, N_3 = 0\}$ and an infinite number of type II orbits traveling in $\{N_1 < 0, N_2 = 0, N_3 = 0\}$. Let $t \mapsto \mathscr{O}(t) = (N_1(t), N_2(t), N_3(t), \Sigma_1(t), \Sigma_2(t), \Sigma_3(t))$ be a type VIII or IX orbit. Recall that the signs of the variables N_i are constant. Hence, it is obvious that $\mathscr{O}$ cannot shadow this heteroclinic chain, as it would violate the fact that the sign of N_1 is constant along $\mathscr{O}$. This means that any heteroclinic chain "alternating" between two signs as in the example above has zero chance to attract some type VIII or IX orbits.

This leads us to the definition of *coherent heteroclinic chains*. Recall that the Mixmaster attractor is the union of three ellipsoids and each of these ellipsoids is the union of two symmetrical hemiellipsoids (they correspond to opposite signs for one of the three variables N_i). In other words,

$$\mathscr{A} = \mathscr{B}_{\mathrm{II}}^{1,+} \cup \mathscr{B}_{\mathrm{II}}^{1,-} \cup \mathscr{B}_{\mathrm{II}}^{2,+} \cup \mathscr{B}_{\mathrm{II}}^{2,-} \cup \mathscr{B}_{\mathrm{II}}^{3,+} \cup \mathscr{B}_{\mathrm{II}}^{3,-}$$

where

$$\mathscr{B}_{\mathrm{II}}^{1,+} \stackrel{\text{def}}{=} \{(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathscr{B} \mid N_1 > 0, N_2 = N_3 = 0\}$$
$$\mathscr{B}_{\mathrm{II}}^{1,-} \stackrel{\text{def}}{=} \{(N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3) \in \mathscr{B} \mid N_1 < 0, N_2 = N_3 = 0\}$$

and analogously for the other hemiellipsoids.

Definition A.3 (Coherent heteroclinic chain)**.** A heteroclinic chain of type II orbits is *coherent* if it is included in the union of three hemiellipsoids (in three different directions) bounded by the Kasner $\mathscr{K}$, that is, if it is included in a set of the form

$$\mathscr{B}_{\mathrm{II}}^{1,\epsilon_1} \cup \mathscr{B}_{\mathrm{II}}^{2,\epsilon_2} \cup \mathscr{B}_{\mathrm{II}}^{3,\epsilon_3} \cup \mathscr{K}$$

For every point p of the Kasner circle (such that, for every $k \geq 0$, $\mathscr{F}^k(p)$ is not a Taub point), there are exactly eight coherent heteroclinic chains starting at p corresponding to the eight different choices of three hemiellipsoids (or, analogously, corresponding to the eight different choices of three signs for the variables N_i). One should remark that a type VIII orbit cannot shadow the same coherent heteroclinic chain than a type IX orbit. Among the eight coherent heteroclinic chains starting at p, six can be shadowed by type VIII orbits and two by type IX orbits.

Having this definition in mind, it is clear that Theorem B must be generalized by replacing the unique heteroclinic chain in $\mathscr{B}^+$ starting at p by one of the eight coherent heteroclinic chains in $\mathscr{B}$ starting at p. Recall that $\mathscr{K}_{(MG)}$ denotes the set of all the points $p \in \mathscr{K}$ such that $\omega(p)$ verifies the moderate growth condition (MG).

Theorem C. *Let p be a point of the Kasner circle and let $\mathscr{H}$ be a coherent heteroclinic chain starting at p. If $\omega(p)$ verifies the moderate growth condition* (MG)*, then the union of all the type* VIII *or* IX *orbits shadowing the heteroclinic chain $\mathscr{H}$ contains a* 3*-dimensional ball $D(p, \mathscr{H})$ Lipschitz embedded in the phase space $\mathscr{B}^+$. Moreover, for any $\mathscr{E} \subset \mathscr{K}_{(MG)}$ of positive* 1*-dimensional Lebesgue measure, the union of all the balls $D(p, \mathscr{H})$ for $p \in \mathscr{E}$ and $\mathscr{H}$ a coherent heteroclinic chain starting at p has positive* 4*-dimensional Lebesgue measure.*

Appendix B

Continued fractions

In this appendix, we gather the results about continued fractions that are used in the memoir. The main result is Lemma 1.5. We also prove a result on the expansivity of the Gauss transformation.

We first need to introduce some notations. Set $\Omega = [0,1] \setminus \mathbb{Q}$. For every $x \in \Omega$, there exists a unique sequence $(k_n(x))_{n\geq 1}$ of integers larger than 1 such that $x = \lim_{n\to+\infty}[k_1(x), \dots, k_n(x)]$ where

$$[k_1(x), \dots, k_n(x)] = \cfrac{1}{k_1(x) + \cfrac{1}{k_2(x) + \cfrac{1}{\dots + \cfrac{1}{k_n(x)}}}}$$

We use the notation

$$[k_1(x), k_2(x), \dots] \overset{\text{def}}{=} \lim_{n\to+\infty}[k_1(x), \dots, k_n(x)]$$

Lemma 1.5 is a straightforward consequence of the following lemma.

Lemma B.1. *For Lebesgue almost every $x \in \Omega$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,*

$$\sum_{i=1}^{2n} k_i(x)^5 \geq n^{5-\frac{1}{10}} \tag{B.1}$$

and

$$k_{2n+1}(x)^4 + k_{2n+2}(x)^4 + k_{2n+3}(x)^4 + k_{2n+4}(x)^4 \leq n^{4+\frac{1}{10}} \tag{B.2}$$

Inequality (B.1) is a consequence of a standard fact: for Lebesgue almost every point $x \in \Omega$, the sequence $(k_i(x))_{i\geq 0}$ of the partial quotients does not grow "too fast" (see corollary B.3). Inequality (B.2) is a consequence of a less standard result: for Lebesgue almost every point $x \in \Omega$ and for every $n \in \mathbb{N}$ large enough, there is at least one partial quotient among $k_1(x)$, ..., $k_n(x)$ that is "large" (see Proposition B.4). More precisely, the standard result can be rigorously stated as follows.

Proposition B.2. *Let $\varphi : \mathbb{N}^* \to \mathbb{R}_+^*$. Either the set*

$$E_x \overset{def}{=} \{n \in \mathbb{N}^* \mid k_n(x) \geq \varphi(n)\}$$

is finite for Lebesgue almost all $x \in \Omega$, or it is infinite for Lebesgue almost all $x \in \Omega$. More precisely, this dichotomy depends on φ as follows:

1. *If $\sum \frac{1}{\varphi(n)}$ is divergent, then for Lebesgue almost all $x \in \Omega$, there exists infinitely many $n \in \mathbb{N}^*$ such that $k_n(x) \geq \varphi(n)$.*
2. *If $\sum \frac{1}{\varphi(n)}$ is convergent, then for Lebesgue almost all $x \in \Omega$, there exists $n_0(x) \in \mathbb{N}^*$ such that for every $n \geq n_0(x)$, $k_n(x) < \varphi(n)$.*

Proof. See [Khi64]. □

Corollary B.3. *Let $\epsilon > 0$. For Lebesgue almost every point $x \in \Omega$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $k_n(x) \leq n^{1+\epsilon}$.*

Proof. For any $\epsilon > 0$, the serie $\sum n^{-1-\epsilon}$ is convergent. □

We now give a precise formulation of the second result needed to prove Lemma B.1.

Proposition B.4. *For Lebesgue almost all $x \in \Omega$, for every $\epsilon > 0$, there exists $n_0(x, \epsilon) \geq 1$ such that for every $n \geq n_0(x, \epsilon)$, there exists an integer $1 \leq j \leq n$ such that $k_j(x) \geq n^{1-\epsilon}$.*

Let us introduce some tools that will be needed to prove Proposition B.4. We denote by $\tau : \Omega \to \Omega$ the Gauss transformation defined by $\tau(x) = \left\{\frac{1}{x}\right\}$ where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. The very definition of τ implies that, for every continued fraction $[k_1, k_2, \dots]$,

$$\tau([k_1, k_2, \dots]) = [k_2, k_3, \dots]$$

In other words, τ is conjugated to the left shift on the space of sequences $(k_n)_{n \geq 1}$ of integers larger than 1.

Let us denote by γ_G the Gauss measure, defined by

$$\gamma_G(A) = \frac{1}{\ln 2} \int_A \frac{1}{x+1} \,\mathrm{d}\lambda(x) \quad \text{for every Borel set } A \text{ of } [0,1] \tag{B.3}$$

where λ denotes the Lebesgue measure. One can remark that the Gauss measure γ_G is equivalent to the Lebesgue measure λ on $[0,1]$. The fundamental fact is that γ_G is τ-invariant, *i.e.*

$$\gamma_G\left(\tau^{-1}(A)\right) = \gamma_G(A) \quad \text{for every Borel set } A \text{ of } [0,1].$$

For any map $f : [0,1] \to \mathbb{C}$, let

$$\operatorname{var} f \overset{\text{def}}{=} \sup \sum_{i=1}^{n-1} |f(t_{i+1}) - f(t_i)|$$

where the supremum is taken on all the finite sequences $0 \leq t_1 < \dots < t_n \leq 1$, $n \geq 2$. If $\operatorname{var} f < +\infty$, then we say that f is *of bounded variation*. For any map $f \in L^\infty_\lambda([0,1])$, we call *essential variation of* f and we denote by $v(f)$ the number $\inf \operatorname{var} \tilde{f}$ where the infimum is taken on all the maps $\tilde{f}$ equal to f mod 0. If $v(f) < +\infty$, then we say that f is *of bounded essential variation*. Let us denote by $\mathrm{BEV}([0,1])$ the set of all maps $f \in L^\infty_\lambda([0,1])$ such that $v(f) < +\infty$. Let us equip $\mathrm{BEV}([0,1])$ with the norm

$$\|f\|_{\mathrm{BEV}} = v(f) + \|f\|_1$$

We define the Perron-Frobenius operator U as the "dual" of the composition operator induced by τ. More precisely, U is defined as the unique bounded linear operator $L^1_\lambda([0,1]) \to L^1_\lambda([0,1])$ satisfying, for every $f \in L^1_\lambda([0,1])$ and for every $g \in L^\infty_\lambda([0,1])$,

$$\int_0^1 (g \circ \tau) \cdot f \,\mathrm{d}\gamma_G = \int_0^1 g \cdot Uf \,\mathrm{d}\gamma_G$$

Proposition B.5 (Spectral gap for the Perron-Frobenius operator)**.** *The Perron-Frobenius operator has a spectral gap: there exists $0 < \alpha < 1$ and $C > 0$ such that, for every $f \in \mathrm{BEV}([0,1])$,*

$$\left\| U^n f - \int_0^1 f \,\mathrm{d}\gamma_G \right\|_1 \leq C\alpha^n \|f\|_{\mathrm{BEV}}$$

Proof. See [IK13]. □

Proof of Proposition B.4. [1] Let us define, for $n \geq 2$ and $\epsilon > 0$ small:

$$X_{n,\epsilon} = \left\{x \in \Omega \mid k_1(x) < n^{1-\epsilon}\right\}$$
$$Y_{n,\epsilon} = \left\{x \in \Omega \mid \forall 1 \leq j \leq n, k_j(x) < n^{1-\epsilon}\right\}$$

According to the Borel-Cantelli lemma, it is enough to prove that

$$\sum_{n\geq 1} \gamma_G\left(Y_{n,\epsilon}\right) < +\infty \tag{B.4}$$

One can remark that

$$Y_{n,\epsilon} = \bigcap_{0\leq j\leq n-1} \tau^{-j}\left(X_{n,\epsilon}\right)$$

so

$$\gamma_G\left(Y_{n,\epsilon}\right) = \int_0^1 \prod_{j=0}^{n-1} \mathbf{1}_{X_{n,\epsilon}} \circ \tau^j(x)\,\mathrm{d}\gamma_G(x)$$

Let $c = \left\lfloor n^{\frac{\epsilon}{2}} \right\rfloor$ and $K = \left\lfloor \frac{n-1}{n^{\frac{\epsilon}{2}}} \right\rfloor$. We can estimate the above integral by keeping only the terms whose indices are multiples of c:

$$\begin{aligned}
\gamma_G\left(Y_{n,\epsilon}\right) &\leq \int_0^1 \prod_{j=0}^{K} \mathbf{1}_{X_{n,\epsilon}} \circ \tau^{jc}(x)\,\mathrm{d}\gamma_G(x)\\
&= \int_0^1 \mathbf{1}_{X_{n,\epsilon}}(x).\left(\left(\prod_{j=0}^{K-1} \mathbf{1}_{X_{n,\epsilon}} \circ \tau^{jc}\right) \circ \tau^c(x)\right) \mathrm{d}\gamma_G(x)\\
&= \int_0^1 \left(U^c \mathbf{1}_{X_{n,\epsilon}}(x)\right).\left(\prod_{j=0}^{K-1} \mathbf{1}_{X_{n,\epsilon}} \circ \tau^{jc}(x)\right) \mathrm{d}\gamma_G(x)
\end{aligned}$$

However, the family $\left(\mathbf{1}_{X_{n,\epsilon}}\right)_n$ is uniformly bounded by 2 in BEV$([0,1])$ and $\prod_{j=0}^{K-1} \mathbf{1}_{X_{n,\epsilon}} \circ \tau^{jc}$ is bounded by 1 in $L^\infty_\lambda([0,1])$ so according to the Proposition B.5,

$$\gamma_G\left(Y_{n,\epsilon}\right) \leq \gamma_G\left(X_{n,\epsilon}\right) \int_0^1 \prod_{j=0}^{K-1} \mathbf{1}_{X_{n,\epsilon}} \circ \tau^{jc}(x)\,\mathrm{d}\gamma_G(x) + O_{n\to\infty}\left(\alpha^c\right)$$

By induction, we get

$$\gamma_G\left(Y_{n,\epsilon}\right) \leq \gamma_G\left(X_{n,\epsilon}\right)^{K+1} + O_{n\to\infty}\left(K\alpha^c\right)$$

However, $X_{n,\epsilon} = \Omega \cap \left]\frac{1}{\lfloor n^{1-\epsilon}\rfloor+1}, 1\right[$ and using (B.3), we get that

$$\gamma_G\left(X_{n,\epsilon}\right) = \frac{1}{\ln 2} \int_{\frac{1}{\lfloor n^{1-\epsilon}\rfloor+1}}^1 \frac{1}{x+1}\,\mathrm{d}x = 1 - \frac{1}{n^{1-\epsilon}\ln 2} + O_{n\to+\infty}\left(\frac{1}{n^{2-2\epsilon}}\right)$$

Moreover,

$$\left(\left\lfloor \frac{n-1}{n^{\frac{\epsilon}{2}}} \right\rfloor + 1\right) \ln\left(1 - \frac{1}{n^{1-\epsilon}\ln 2} + O_{n\to+\infty}\left(\frac{1}{n^{2-2\epsilon}}\right)\right) = -\frac{1}{\ln 2} n^{\frac{\epsilon}{2}} + o_{n\to+\infty}(1)$$

Hence,

$$\gamma_G\left(X_{n,\epsilon}\right)^{K+1} \sim_{n\to+\infty} e^{-\frac{1}{\ln 2} n^{\frac{\epsilon}{2}}}$$

and $\gamma_G\left(X_{n,\epsilon}\right)^{K+1}$ is the general term of a convergent series. Analogously, $K\alpha^c$ is the general term of a convergent series. Hence, (B.4) holds true. This concludes the proof of Proposition B.4. □

Proof of Lemma B.1. Inequalities (B.1) and (B.2) are straightforward consequences of corollary B.3

[1] We would like to thank Sébastien Gouëzel for explaining to us how to use the Perron-Frobenius operator here.

and Proposition B.4 respectively, with $\epsilon = 10^{-2}$. $\square$

The following result provide some explicit conditions ensuring that the continued fraction expansion of two nearby real numbers start by the same integer. It is used to prove Lemma 9.8. In particular, it is useful to find a sufficiently small size for the section $S^s_{\omega,\mathbf{h}_\omega}$ so that all the points (in fact, their coordinate x_c) in $S^s_{\omega,\mathbf{h}_\omega}$ have the same first partial quotient.

Proposition B.6. *For $x, x' \in \Omega$, if*

$$\left|x - x'\right| < \frac{1}{10}\frac{1}{k_1(x)^2 k_2(x) k_3(x)}$$

then $k_1(x') = k_1(x)$.

Proof. Fix $x = [k_1, k_2, \dots] \in \Omega$. Let $x' = [k_1', k_2', \dots] \in \Omega$ such that

$$\left|x - x'\right| < \frac{1}{3}\frac{1}{k_1^2 k_2 k_3}$$

One can remark that

$$[k_1 + 1] < [k_1, k_2, k_3 + 1] < x < [k_1, k_2, k_3] < [k_1]$$

By a straightforward computation, one get

$$[k_1] - [k_1, k_2, k_3] \geq \frac{1}{3k_1^2 k_2 k_3}$$

and

$$[k_1, k_2, k_3 + 1] - [k_1 + 1] \geq \frac{1}{10 k_1^2 k_2 k_3}$$

It follows that

$$[k_1 + 1] < x' < [k_1]$$

Hence, $k_1' = k_1$. $\square$

The following result provide some explicit conditions ensuring that the continued fraction expansion of two nearby real numbers start by the same first two integers. Moreover, it shows that the double Gauss transformation τ^2 is expansive. It is particularly useful to prove Lemma 10.11.

Proposition B.7 (Expansivity of τ^2)**.** *For $x, x' \in \Omega$, if*

$$\left|x - x'\right| < \frac{1}{24}\frac{1}{k_1(x)^2 k_2(x)^2 k_3(x) k_4(x)}$$

then $k_1(x') = k_1(x)$, $k_2(x') = k_2(x)$ and

$$\left|\tau^2(x) - \tau^2(x')\right| \geq 4\left|x - x'\right|$$

Proof. Fix $x = [k_1, k_2, \dots] \in \Omega$. Let $x' = [k_1', k_2', \dots] \in \Omega$ such that

$$\left|x - x'\right| < \frac{1}{24}\frac{1}{k_1^2 k_2^2 k_3 k_4}$$

One can remark that

$$[k_1, k_2] < [k_1, k_2, k_3, k_4] < x < [k_1, k_2, k_3, k_4 + 1] < [k_1, k_2 + 1]$$

By a straightforward computation, one get

$$\begin{aligned}[k_1, k_2+1] - [k_1, k_2, k_3, k_4+1] &= \frac{1+(k_3-1)(k_4+1)}{(k_1(k_2+1)+1)\left((k_1k_2k_3+k_3+k_1)(k_4+1)+k_1k_2+1\right)} \\ &\geq \frac{1}{24}\frac{1}{k_1^2k_2^2k_3k_4}\end{aligned}$$

and

$$\begin{aligned}[k_1, k_2, k_3, k_4] - [k_1, k_2] &= \frac{k_4}{(k_1k_2+1)((k_1k_2+1)(k_3k_4+1)+k_1k_4)} \\ &\geq \frac{1}{10}\frac{1}{k_1^2k_2^2k_3}\end{aligned}$$

It follows that

$$[k_1, k_2] < x' < [k_1, k_2+1]$$

Hence, $k_1' = k_1$ and $k_2' = k_2$. Writing

$$x = \frac{1}{k_1 + \frac{1}{k_2+\tau^2(x)}}, \quad x' = \frac{1}{k_1 + \frac{1}{k_2+\tau^2(x')}}$$

leads to

$$x - x' = \frac{\tau^2(x) - \tau^2(x')}{(k_1k_2 + k_1\tau^2(x)+1)(k_1k_2+k_1\tau^2(x')+1)}$$

Since $k_1k_2 + 1 \geq 2$, we get

$$\left|\tau^2(x) - \tau^2(x')\right| \geq 4\left|x - x'\right|$$

$\square$

Appendix C

Stable manifold theorem with parameters

In this appendix, we investigate the standard stable manifold theorem in the context of a partially hyperbolic singularity of a vector field depending on a parameter. As stated in the introduction, we need some precisions on this theorem in order to construct the local coordinate system ξ used throughout this memoir. We will prove some estimates on the size of the neighbourhood where the local stable manifold is known to be the graph of a function, and some estimates about the derivatives of all orders of this function. We will make explicit the different constants arising and their dependance on the vector field. As an application, we consider the situation where a vector field vanishes on a submanifold N and contracts a direction transverse to N. We prove some estimates on the size of the neighbourhood of N where there are some charts straightening the stable foliation while giving some controls on the derivatives of all orders of the charts.

C.1 Introduction

Fix a smooth[1] vector field Y on a Riemannian manifold M and let x be a *singularity* of the vector field Y, that is, a point of M such that $Y(x) = 0$. For any $\gamma < 0$ and for any $\eta > 0$, the *local γ-stable set* $W_\eta^{s,\gamma}(x, Y)$ of x for Y is the set of points in M whose forward orbit under the flow of Y stay in the η-neighbourhood of x and converge to x faster than $e^{\gamma t}$ as $t \to +\infty$ (see (C.3d)). This is one of the most fundamental objects when one tries to understand the asymptotic dynamics of the flow of Y near x. Its geometry is very well understood in the context of a hyperbolic (or partially hyperbolic) singularity, as explained in what follows.

C.1.1 Stable manifold theorem

Assume that the singularity x is partially hyperbolic: up to replacing Y by $-Y$, this means that there exists a non trivial decomposition $T_xM = F \oplus G$ of the tangent space at x such that F and G are stabilized by $DY(x)$ and there exists a negative real γ such that the real parts of the eigenvalues of $DY(x)_{|F}$ are strictly less than γ and the real parts of the eigenvalues of $DY(x)_{|G}$ are strictly more than γ. In this context, the Stable Manifold Theorem asserts that for any positive η small enough, the local γ-stable set $W_\eta^{s,\gamma}(x, Y)$ is an embedded submanifold of M tangent to F at x, called the local γ-stable manifold. It can be seen as the graph of a smooth map $\phi : U \subset F \to V \subset G$, from a neighbourhood U of 0 in F to a neighbourhood V of 0 in G, satisfying $\phi(0) = 0$ and $D\phi(0) = 0$. Moreover, if Y depends smoothly on a parameter $\mu \in \mathbb{R}^s$, then this is also the case for the local γ-stable set $W_\eta^{s,\gamma}(x, Y)$, that is, $\phi_\mu(z) = \phi(z, \mu)$ is smooth as a map of the two variables $z \in U$, $\mu \in \mathbb{R}^s$.

Though this standard theorem has been presented and generalized in many articles (see *e.g.* [Irw70], [HP70]) and books (see *e.g.* [KH97], [Irw01], [Rue89], [Rob99], [BS02] for classical introductory readings and [HPS06] for a deeper treatment but a tougher reading), we have not found a version of this result that gives explicit estimates on the C^k-norms of $\phi(z, \mu)$ ($k \in \mathbb{N}^*$) and on the size of the neighbourhood where these estimates hold true. In most of the books, authors state that if Y is C^r, then ϕ_μ is also

[1] Recall that in this work, *smooth* stands for C^∞.

C^r and $\mu \mapsto \phi_\mu$ is a continuous map from $\mathbb{R}^s$ to the space of C^r maps equipped with the C^r topology, which is a weaker statement than saying that $\phi(z,\mu)$ is smooth. The closest result to what we were looking for has been found in [Chu+98] (chapter 5). In that book, it is proven that the map $\phi(z,\mu)$ is smooth but do not provide explicit estimates. This reference has been the main starting point of this appendix, whose purpose is to give such estimates. Since we are only interested by local estimates, we may (and do) assume that $M = \mathbb{R}^n$ and $x = 0$ (it suffices to work in a local chart and to multiply the vector field by a smooth plateau map in the neighbourhood of 0).

The classical stable manifold theorem (which can be found in the above references) can be stated as follows:

Theorem C.1 (Stable manifold theorem with parameters)**.** *Let $X = (X_\mu)_{\mu\in\mathbb{R}^s}$ be a smooth family of smooth vector fields on $\mathbb{R}^n$ such that*

1. *For every $\mu \in \mathbb{R}^s$, the origin of $\mathbb{R}^n$ is a singularity of X_μ,* i.e.
$$X_\mu(0) = 0$$
2. *The endomorphism $A := D_x X(0,0)$ admits a* partially hyperbolic splitting $R^n = F \oplus G$ *such that*
$$\lambda_{max}\left(A_{|F}\right) < \min\left(0, \lambda_{min}\left(A_{|G}\right)\right)$$

where $\lambda_{max}\left(A_{|F}\right)$ (resp. $\lambda_{min}\left(A_{|G}\right)$) denotes the maximum (resp. minimum) of the real parts of the eigenvalues of $A_{|F}$ (resp. $A_{|G}$). Let $\gamma \in \left]\lambda_{max}\left(A_{|F}\right), \min\left(0, \lambda_{min}\left(A_{|G}\right)\right)\right[$. Then there exists $\epsilon > 0$ and $\eta > 0$ such that for every $\mu \in B_{\mathbb{R}^s}(0,\epsilon)$, the local γ-stable set $W^{s,\gamma}_\eta(0, X_\mu)$ is the graph of a smooth function $\phi_\mu : F \to G$ intersected with the ball $B_{\mathbb{R}^n}(0,\eta)$. Moreover, the map

$$\phi : (z,\mu) \in F \times B_{\mathbb{R}^s}(0,\epsilon) \mapsto \phi_\mu(z) \in G$$

is smooth, for every $\mu \in B_{\mathbb{R}^s}(0,\epsilon)$, $\phi_\mu(0) = 0$ and $D\phi_0(0) = 0$.

As explained above, our goal is to supplement this result by providing explicit estimates on the constants ϵ and η and on the derivatives of all orders of ϕ. What we prove is summarized in the following addendum (for a precise version, see Theorem C.19):

Addendum C.2. *For every $r > 0$, one can find a radius ϵ, a size η and a map ϕ as above satisfying the following properties:*

- *the radius ϵ is linear in r, polynomial on the distance between γ and the real part of the spectrum of A, inversely linear on the norm of the second derivative of X on the closed ball $\overline{B_{\mathbb{R}^n\times\mathbb{R}^s}(0,r)}$ and inversely polynomial on the norm of A and the angle between the generalized eigenspaces of A.*
- *the size η is linear in r, polynomial on the spectral gap* $\min\left(0, \lambda_{min}\left(A_{|G}\right)\right) - \lambda_{max}\left(A_{|F}\right)$*, inversely linear on the norm of the second derivative of X on the closed ball $\overline{B_{\mathbb{R}^n\times\mathbb{R}^s}(0,r)}$ and inversely polynomial on the norm of A and the angle between the generalized eigenspaces of A.*
- *the norm of the k-th derivative of ϕ on $B_F(0,\eta)\times B_{\mathbb{R}^s}(0,\eta)$ is a polynomial function of degree $\simeq nk^2$ depending on the norm of A, the angle between the generalized eigenspaces of A, the inverse of r, the norms of the $(k+1)$ first derivatives of X on the closed ball $\overline{B_{\mathbb{R}^n\times\mathbb{R}^s}(0,r)}$ and the inverse of the spectral gap.*

Remark C.3. The parameter r describes quantitatively how the local γ-stable manifold is, indeed, a local object. It allows one to get some information on the size of the local γ-stable manifold when one is only using a control of X over the ball of radius r.

Remark C.4. The strategy used to prove Theorem C.1 is standard. We find the orbits contained in a stable manifold as the fixed points of an "integral" operator (depending on the parameter μ) on a suitable space of functions. The construction of the operator is natural and gives the desired description of the stable manifolds as graphs of some family of maps ϕ_μ. This is the technique used in [Chu+98], but with a major simplification. We directly prove that on the one hand the operator is smooth with respect to all variables including the parameters and on the other hand it is a contraction mapping with respect to the space of functions, thus we obtain that the family of graphs ϕ is smooth with respect to the variable in the phase space and the parameter, using a global version of the implicit function theorem (which can be seen as a contraction mapping theorem with parameters). This makes the proof easier and more natural compared to the one in [Chu+98]. Indeed, in this reference, the authors do not prove that the operator is smooth and thus need to use a family of truncated operators to obtain the smoothness of the fixed point.

C.1.2 Vector fields vanishing on submanifolds

Theorem C.1 allows us to describe the stable foliation associated with a normally contracted submanifold on which a vector field vanishes. The context is as follows. Let M be a smooth manifold of dimension n and let N be a smooth submanifold of M. Let Y be a smooth vector field on M vanishing on N such that for every point $x \in N$, there exists a direction transverse to T_xN which is stabilized and contracted by $DY(x)$. Recall that, given $x \in N$, the *stable set* $W^s(x,Y)$ of x for Y is the set of points in M whose forward orbit under the flow of Y converge to x. It is well known (this is an easy consequence of Theorem C.1) that the family of stable manifolds $(W^s(x,Y))_{x\in N}$ foliates a neighbourhood W of N and the stable foliation

$$\mathcal{F}^s \overset{\text{def}}{=} \{W^s(x,Y) \cap W \mid x \in N\}$$

can be locally smoothly straightened.

Fixing a point $x \in N$ and a local chart (independantly of Y) centered around x which straightens N, and looking at the situation in this chart, we "can assume that" M is an open set Ω of $\mathbb{R}^n$ and N is the set $\Omega_0 := \Omega \cap G \neq \varnothing$ where G is a linear subspace of $\mathbb{R}^n$. The standard result explained above can be stated as follows:

Theorem C.5 (Straightening of a stable foliation)**.** *Let Ω be an open set of $\mathbb{R}^n$, G be a linear subspace of $\mathbb{R}^n$ and $Y : \Omega \to \mathbb{R}^n$ be a smooth vector field such that*

1. *Y vanishes on $\Omega_0 := \Omega \cap G$;*
2. *For every $\mu \in \Omega_0$, there exists a decomposition $F_\mu \oplus G = \mathbb{R}^n$ stabilized by $A_\mu := DY(\mu)$ and such that*
$$\lambda_{max}\left((A_\mu)_{|F_\mu}\right) < 0$$

Let $\mu_0 \in \Omega_0$. Then there exists a smooth local coordinate system ξ defined on a ball $B := B_{\mathbb{R}^n}(\mu_0, R)$ such that the family of stable manifolds $(W^s(\mu,Y))_{\mu\in\Omega_0\cap B_{\mathbb{R}^n}(\mu_0,R)}$ foliates B and is straightened by ξ: for every $\mu \in \Omega_0 \cap B$,

$$\xi\left(W^s(\mu,Y) \cap B\right) = \left(\mu + F_{\mu_0}\right) \cap \xi(B)$$

We emphasize the fact that Theorem C.5 is a straightforward consequence of the stable manifold theorem. Once again, our goal is to provide some explicit estimates on the radius R and on the derivatives of all orders of ξ and ξ^{-1}. What we prove is summarized in the following addendum (for a precise version, see Theorem C.22):

Addendum C.6. *For every $r > 0$ such that $\overline{B_{\mathbb{R}^n}(\mu_0, r)} \subset \Omega$, one can find a radius R and a local coordinate system ξ on $B_{\mathbb{R}^n}(\mu_0, r)$ as above satisfying the following properties:*

- *The radius R admits a lower bound which is linear in r, polynomial in the spectral gap $\left|\lambda_{max}\left((A_{\mu_0})_{|F_{\mu_0}}\right)\right|$, inversely linear in the norm of the second derivative of Y on the closed ball $\overline{B_{\mathbb{R}^n}(\mu_0, r)}$, inversely polynomial in the norm of A_{μ_0} and the angle between the generalized eigenspaces of A_{μ_0}. This lower bound depends only on the parameters listed above.*
- *For every $\epsilon > 0$, ξ restricted to $B_{\mathbb{R}^n}(\mu_0, \epsilon R)$ is ϵ-close to the identity in C^1-norm.*
- *The norms of the k-th derivatives of ξ and ξ^{-1} admit an upper bound polynomial in the norm of A_{μ_0}, the angle between the generalized eigenspaces of A_{μ_0} and the norms of the $(k+1)$ first derivatives of Y on the closed ball $\overline{B_{\mathbb{R}^n}(\mu_0, r)}$ and inversely polynomial in the spectral gap and r. Moreover, this upper bound depends only on the parameters listed previously.*

Remark C.7. In order to get such estimates on R and ξ, one must choose a compact ball $\overline{B(\mu_0, r)} \subset \Omega$ on which one controls the derivatives of all orders of Y. There is no canonical choice and one can use the parameter r to make a choice depending on his/her needs.

This appendix is organized as follows. Section C.2 compiles some notations used throughout the appendix. In section C.3, we prove Theorem C.1. We first treat the global case (see Proposition C.8), which is the main technical result of this appendix, and then we apply it to the local case. In chapter C.4, we prove Theorem C.5 using Theorem C.1. Appendix C.5 recalls some well-known estimates of linear algebra that are extensively used throughout this appendix.

C.2 General notations.

For any $n \in \mathbb{N}$, we denote by $\|.\|$ the Euclidean norm on $\mathbb{R}^n$. For any family $(E_1, \|.\|_1), \dots, (E_r, \|.\|_r)$, $(F, \|.\|_F)$ of normed vector spaces (possibly of infinite dimension), for any continuous r-linear map $L : E_1 \times \cdots \times E_r \to F$, we will usually denote by $|||L|||$ its subordinate norm, that is,

$$|||L||| = \sup_{(x_1,\dots,x_r)\in\prod_{i=1}^r E_i} \frac{\|L(x_1,\dots,x_r)\|_F}{\prod_{i=1}^r \|x_i\|_i}$$

For any linear subspaces F, G of $\mathbb{R}^n$, let us recall that the *angle* between F and G, denoted by $\sphericalangle(F, G)$, is defined as the minimal (unsigned) angle between a vector in F and a vector in G. The angle between F and G is strictly positive if and only if $F \cap G = \{0\}$. If this is the case, let

$$m(F,G) \stackrel{\text{def}}{=} \left(\frac{2}{1-\cos\sphericalangle(F,G)}\right)^{\frac{1}{2}}$$

We generalise this notion by defining the angle between a finite family $E_1, \dots, E_r$ of linear subspaces of $\mathbb{R}^n$ as follows:

$$\sphericalangle(E_1,\dots,E_r) \stackrel{\text{def}}{=} \min_{1\le j\le r} \sphericalangle\left(E_j, \bigoplus_{i\ne j} E_i\right)$$

For $A \in \mathcal{M}_n(\mathbb{R})$, define

$$\lambda_{\max}(A) \stackrel{\text{def}}{=} \max_{\lambda\in \mathrm{Sp}_{\mathbb{C}}(A)} \mathrm{Re}(\lambda), \tag{C.1a}$$

$$\lambda_{\min}(A) \stackrel{\text{def}}{=} \min_{\lambda\in \mathrm{Sp}_{\mathbb{C}}(A)} \mathrm{Re}(\lambda), \tag{C.1b}$$

$$m(A) \stackrel{\text{def}}{=} \left(\frac{2}{1-\cos\sphericalangle(E_1,\dots,E_r)}\right)^{\frac{r-1}{2}} \tag{C.2a}$$

where $E_1, \dots, E_r$ are the generalized eigenspaces of A,

$$M(A) \stackrel{\text{def}}{=} \max(1, |||A|||)^{n-1}\, m(A) \tag{C.2b}$$

where $|||.|||$ is the subordinate norm with respect to the Euclidean norm, and

$$\hat{M}(A) \stackrel{\text{def}}{=} 2^{2n-2}(n-1)^{n-1} M(A) \tag{C.2c}$$

Given a Riemannian manifold M with distance d, a smooth vector field Y on M with flow Y^t and a singularity x of Y, we define the following *stable* sets:

- The *global stable set* $W^s(x, Y)$ of x for Y is the set of points in M whose forward orbit under the flow of Y converge to x, that is,

$$W^s(x,Y) = \left\{y \in M \mid \lim_{t\to+\infty} d\left(Y^t(y), x\right) = 0\right\} \tag{C.3a}$$

- For any $\gamma < 0$, the *global γ-stable set* $W^{s,\gamma}(x, Y)$ of x for Y is the set of points in M whose forward orbit under the flow of Y converge to x at least as fast as $e^{\gamma t}$, that is,

$$W^{s,\gamma}(x,Y) = \left\{ y \in M \mid d\left(Y^t(y), x\right) = O_{t\to+\infty}\left(e^{\gamma t}\right) \right\} \tag{C.3b}$$

- For any $\eta > 0$, the *local stable set* $W^s_\eta(x,Y)$ of x for Y is the set of points in $W^s(x,Y)$ whose forward orbit under the flow of Y stay in the η-neighbourhood of x, that is,

$$W^s_\eta(x,Y) = \left\{ y \in W^s(x,Y) \mid \forall t \geq 0,\ d\left(Y^t(y), x\right) < \eta \right\} \tag{C.3c}$$

- For any $\gamma < 0$, for any $\eta > 0$, the *local γ-stable set* $W^{s,\gamma}_\eta(x,Y)$ of x for Y is the set of points in $W^{s,\gamma}(x,Y)$ whose forward orbit under the flow of Y stay in the η-neighbourhood of x, that is,

$$W^{s,\gamma}_\eta(x,Y) = \left\{ y \in W^{s,\gamma}(x,Y) \mid \forall t \geq 0,\ d\left(Y^t(y), x\right) < \eta \right\} \tag{C.3d}$$

One can remark that if one chooses a distance d' equivalent to d, then the stable sets for d' coincide with the stable sets for d.

C.3 Estimates for the stable manifold theorem with parameters

C.3.1 Setup

Fix an integer $n \geq 2$ and an integer $s \in \mathbb{N}^*$. We define a *smooth family of vector fields* $(X_\mu)_{\mu\in\mathbb{R}^s}$ as a smooth map

$$X\colon \left| \begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^s & \to & \mathbb{R}^n \\ (x,\mu) & \mapsto & X_\mu(x) \end{array} \right.$$

where $\mathbb{R}^n$ is the phase space and $\mathbb{R}^s$ is the set of parameters. Given such a X, let us consider some hypotheses:

Hypothesis 1. *For every $\mu \in \mathbb{R}^s$, the origin is a singularity of X_μ,* i.e.

$$X_\mu(0) = 0$$

Hypothesis 2. *The endomorphism $A := D_x X(0,0)$ admits a* partially hyperbolic splitting (F,G), i.e. *there exists a non trivial decomposition $R^n = F \oplus G$ such that F and G are stabilized by A and*

$$\lambda_{max}\left(A_{|F}\right) < \min\left(0, \lambda_{min}\left(A_{|G}\right)\right)$$

Given such a partially hyperbolic splitting, we will consider the interval:

$$I_A \overset{\text{def}}{=} \left]\lambda_{\max}\left(A_{|F}\right), \min\left(0, \lambda_{\min}\left(A_{|G}\right)\right)\right[\tag{C.4}$$

and the "inverse of the spectral gap":

$$\sigma(A) \overset{\text{def}}{=} \min\left(1, \min\left(0, \lambda_{\min}\left(A_{|G}\right)\right) - \lambda_{\max}\left(A_{|F}\right)\right)^{-(n-1)} \tag{C.5}$$

Hypothesis 3. *Given a partially hyperbolic splitting (F,G), the first derivative of X satisfies*

$$\sup_{(x,\mu)\in\mathbb{R}^n\times\mathbb{R}^s} |||D_x X(x,\mu) - A||| \leq \left(2^{3n-1}(n-1)^{n-1}\sqrt{2}\, M(A)\sigma(A)\right)^{-1}$$

Hypothesis 4. *The derivatives of all orders of X are bounded,* i.e. *for every $k \geq 1$,*

$$\sup_{(x,\mu)\in\mathbb{R}^n\times\mathbb{R}^s} \left|\left|\left| D^k_{x,\mu} X(x,\mu) \right|\right|\right| < +\infty$$

In subsection C.3.2, we will assume that X satisfies all the above hypotheses and we will prove a global stable manifold theorem with global estimates while in section C.3.3, we will only assume that the first two hypotheses hold true and we will prove a local stable manifold theorem where we make explicit the local estimates and the size of the neighbourhood where these estimates hold true. The local theorem will be a consequence of the global one. The idea is to multiply the non linear part of X by a smooth plateau map on a small neighbourhood of $(0,0)$ such that the new X satisfies all the above hypotheses.

C.3.2 Global estimates

In this section, we state and prove a (global) stable manifold theorem with parameters for smooth families of vector fields $(X_\mu)_{\mu\in\mathbb{R}^s}$ satisfying the hypotheses 1, 2, 3 and 4. For such a X, let

$$M_1(X) \overset{\text{def}}{=} \sup_{(x,\mu)\in\mathbb{R}^n\times\mathbb{R}^s} |||D_x X(x,\mu) - A|||$$

where $A := D_x X(0,0)$, and for every integer $k \geq 2$, let

$$M_k(X) \overset{\text{def}}{=} \sup_{2\leq j\leq k} \sup_{(x,\mu)\in\mathbb{R}^n\times\mathbb{R}^s} \left|\left|\left| D^j_{x,\mu} X(x,\mu) \right|\right|\right|$$

and

$$\bar{M}_k(X) \overset{\text{def}}{=} \max(1, M_k(X))$$

Let us recall that in the current context, for any $\mu \in \mathbb{R}^s$ and $\gamma < 0$, the global γ-stable set of 0 for X_μ is

$$W^{s,\gamma}(0, X_\mu) = \left\{ x_0 \in \mathbb{R}^n \mid \left\| X^t_\mu(x_0) \right\| = O_{t\to+\infty}\left(e^{\gamma t}\right) \right\} \tag{C.6}$$

For any $\gamma < 0$, let

$$d_A(\gamma) \overset{\text{def}}{=} \min\bigl(1, d(\gamma, \mathrm{Re}(\mathrm{Sp}_{\mathbb{C}}(A)))\bigr)^{n-1}$$

where d is the usual distance on $\mathbb{R}$.

Proposition C.8 (Global estimates for the stable manifold theorem with parameters)**.** *There exists a positive constant C and a sequence of positive constants $(C_{1,k})_{k\in\mathbb{N}}$ (depending only on the dimension n of the phase space) such that for every smooth family of vector fields $(X_\mu)_{\mu\in\mathbb{R}^s}$ satisfying the hypotheses 1, 2, 3 and 4, and every partially hyperbolic splitting (F, G) of $A := D_x X(0,0)$, there exists a unique smooth map*

$$\phi : \left| \begin{array}{ccc} F\times\mathbb{R}^s & \to & G \\ (z,\mu) & \mapsto & \phi_\mu(z) \end{array} \right.$$

such that

1. *Graph structure of the global γ-stable set: for every $\mu \in \mathbb{R}^s$ and every $\gamma \in I_A$ (see (C.4)) satisfying*

$$M_1(X) \leq \frac{1}{C_1}\frac{d_A(\gamma)}{M(A)} \tag{C.7}$$

   where $C_1 = 2^{2n}(n-1)^{n-1}\sqrt{2}$, the stable set $W^{s,\gamma}(0, X_\mu)$ is exactly the graph of the map $\phi_\mu : F \to G$. In particular, $W^{s,\gamma}(0, X_\mu)$ does not depend on the choice of such a γ.

2. *Local γ-stable set: for every $\gamma \in I_A$ satisfying (C.7), every $\mu \in \mathbb{R}^s$, every $\eta > 0$ and every $0 < \delta \leq \frac{\eta}{CM(A)\sigma(A)}$,*

$$W^{s,\gamma}_\eta(0, X_\mu) \cap B_{\mathbb{R}^n}(0,\delta) = \mathrm{Graph}\left(\phi_\mu\right) \cap B_{\mathbb{R}^n}(0,\delta) \tag{C.8}$$

3. *Controls on ϕ: for every $(z,\mu) \in F\times\mathbb{R}^s$,*

$$\|\phi(z,\mu)\| \leq C_{1,0}\,\sigma(A)\,M(A) M_1(X)\,\|z\| \tag{C.9a}$$

$$|||D_z\phi(z,\mu)||| \leq C_{1,1}\,\sigma(A)\,M(A) M_1(X) \tag{C.9b}$$

$$|||D_\mu\phi(z,\mu)||| \leq C_{1,1}\,\sigma(A)\,M(A) M_2(X)\,\|z\| \tag{C.9c}$$

and more generally, using the norm $\|(z,\mu)\| = \|z\| + \|\mu\|$ *on* $F \times \mathbb{R}^s$*, we have, for all* $k \geq 2$*,*

$$\left|\left|\left|D^k\phi(z,\mu)\right|\right|\right| \leq C_{1,k}\left(\sigma(A)^2 M(A)^2 \bar{M}_{k+1}(X)\max(1,\|z\|)\right)^{2k-1} \tag{C.9d}$$

where $\sigma(A)$ *is defined by* (C.5).

Remark C.9. If one is working with a different norm than the Euclidean one, one will have the same result but with different constants $C_1, C_{1,0}, C_{1,1}, \ldots$

Remark C.10. Hypothesis 3 is not fundamentally necessary for Proposition C.8 to be true. This hypothesis implies that there exists a $\gamma \in I_A$ satisfying (C.7) in item 1, so it is only a convenient and explicit sufficient condition for the proposition to not be empty. When proving the local version in section C.3.3, we will not check that hypothesis 3 holds true, we will directly work with a given γ and check that (C.7) holds true.

The proof of Proposition C.8 is heavily based on the contraction mapping theorem, applied in the Banach space introduced in definition C.12 below.

Definition C.11 (γ-norm)**.** For any $\gamma \in \mathbb{R}$ and $(z,v) : [0,+\infty[\to \mathbb{R}^p \times \mathbb{R}^q$, we define the γ-norm of (z,v) by

$$\|(z,v)\|_\gamma \stackrel{\text{def}}{=} \sup_{t\geq 0} \max(\|z(t)\|, \|v(t)\|)e^{-\gamma t} \in [0,+\infty]$$

Definition C.12 (Function space H^γ)**.** Let $\gamma \in \mathbb{R}$. Denote by H^γ the vector space of continuous maps $(z,v) : [0,+\infty[\to \mathbb{R}^p \times \mathbb{R}^q$ whose γ-norm are finite. The vector space H^γ endowed with the γ-norm is a Banach space.

Remark C.13. For any $\gamma < \gamma'$, $H^\gamma \subset H^{\gamma'}$ and for every $(z,v) \in H^\gamma$, $\|(z,v)\|_{\gamma'} \leq \|(z,v)\|_\gamma$.

Remark C.14. It will be useful to see $H_n^\gamma := H^\gamma$ as the cartesian product $H_p^\gamma \times H_q^\gamma$ when $n = p + q$.

Proof of Proposition C.8. Before we make explicit the strategy of the proof, we need some preparatory work. Fix a smooth family of vector fields $(X_\mu)_{\mu\in\mathbb{R}^s}$ satisfying the hypotheses 1, 2, 3 and 4 and a partially hyperbolic splitting (F,G) of $A := D_x X(0,0)$. Let $p = \dim F$ and $q = \dim G$. Fix $\gamma \in I_A$ satisfying

$$M_1(X) \leq \frac{1}{2^{2n}(n-1)^{n-1}\sqrt{2}} \frac{d_A(\gamma)}{M(A)} \tag{C.10}$$

Conjugation of X. We start by conjugating X in such a way that F and G become orthogonal linear subspaces of $\mathbb{R}^n$. For that purpose, let us fix an isomorphism $L : \mathbb{R}^n \to \mathbb{R}^n \simeq \mathbb{R}^p \times \mathbb{R}^q$ such that $L_{|F}$ (resp. $L_{|G}$) is an isometry from F (resp. G) to $\mathbb{R}^p \times \{0\} \simeq \mathbb{R}^p$ (resp. $\{0\} \times \mathbb{R}^q \simeq \mathbb{R}^q$). According to Lemma C.24,

$$\|L\| \leq m(F,G), \quad \left\|L^{-1}\right\| \leq \sqrt{2} \tag{C.11}$$

We now define

$$\tilde{X}: \left|\begin{array}{ccc} \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^s & \to & \mathbb{R}^p \times \mathbb{R}^q \\ (z,v,\mu) & \mapsto & L\big(X_\mu(L^{-1}(z,v))\big) \end{array}\right. \tag{C.12}$$

One can remark that for every $k \geq 1$,

$$M_k(\tilde{X}) \leq \sqrt{2}^k\, m(F,G)\, M_k(X) \tag{C.13}$$

Let

$$\tilde{A} \stackrel{\text{def}}{=} D_{z,v}\tilde{X}(0,0,0) = LAL^{-1} = \begin{pmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{pmatrix}$$

where $\tilde{A}_1 = \tilde{A}_{|\mathbb{R}^p}$ and $\tilde{A}_2 = \tilde{A}_{|\mathbb{R}^q}$, with respect to the canonical basis. Using the fact that L is an isometry in restriction to F and G, we have the following properties on $\tilde{A}_1$ and $\tilde{A}_2$:

$$\mathrm{Sp}_\mathbb{C}(\tilde{A}_1) = \mathrm{Sp}_\mathbb{C}(A_{|F}), \qquad \mathrm{Sp}_\mathbb{C}(\tilde{A}_2) = \mathrm{Sp}_\mathbb{C}(A_{|G}) \tag{C.14a}$$

$$m(\tilde{A}_1) = m(A_{|F}), \qquad m(\tilde{A}_2) = m(A_{|G}) \tag{C.14b}$$

$$M(\tilde{A}_1) = M(A_{|F}), \qquad M(\tilde{A}_2) = M(A_{|G}) \tag{C.14c}$$

Property (C.14a) implies that $d_{\tilde{A}}(\gamma) = d_A(\gamma)$.

Differential equation view-point. Let $\mu \in \mathbb{R}^s$. The differential equation associated with the vector field $\tilde{X}_\mu$ can be written in the following form

$$\begin{cases} z' &= \tilde{A}_1 z + f(z, v, \mu) \\ v' &= \tilde{A}_2 v + g(z, v, \mu) \end{cases} \tag{C.15}$$

where $(f, g) : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^s \to \mathbb{R}^p \times \mathbb{R}^q$ is a smooth map defined by

$$\begin{pmatrix} f(z, v, \mu) \\ g(z, v, \mu) \end{pmatrix} = \tilde{X}(z, v, \mu) - D_{z,v}\tilde{X}(0, 0).(z, v)$$

and satisfying

$$(f, g)(0, 0, \mu) = 0 \qquad \text{for all } \mu \in \mathbb{R}^s \tag{C.16a}$$

$$D(f, g)(0, 0, 0) = 0 \tag{C.16b}$$

$$\left|\left|\left| D_{z,v}(f, g)(z, v, \mu) \right|\right|\right| \leq M_1\left(\tilde{X}\right) \qquad \text{for all } (z, v, \mu) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^s \tag{C.16c}$$

$$\left|\left|\left| D^k(f, g)(z, v, \mu) \right|\right|\right| \leq M_N\left(\tilde{X}\right) \quad \text{for all } N \geq 2,\ 2 \leq k \leq N \text{ and } (z, v, \mu) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^s \tag{C.16d}$$

Property (C.16a) implies that

$$\forall \mu \in \mathbb{R}^s, \quad \forall k \in \mathbb{N}, \quad D^k_\mu(f, g)(0, 0, \mu) = 0 \tag{C.16e}$$

Estimates on exponential of matrices. We now state an estimate that will be used several times throughout this proof. Let

$$\alpha = \frac{\gamma + \lambda_{\max}\left(A_{|F}\right)}{2}, \quad \beta = \frac{\gamma + \lambda_{\min}\left(A_{|G}\right)}{2}$$

According to Lemma C.26 and (C.14c), we have, for every $s \geq 0$,

$$\begin{aligned} \left|\left|\left| e^{s\tilde{A}_1} \right|\right|\right| &\leq \frac{\hat{M}(A_{|F})}{d_A(\gamma)^{\frac{p-1}{n-1}}} e^{\alpha s} \\ \left|\left|\left| e^{-s\tilde{A}_2} \right|\right|\right| &\leq \frac{\hat{M}(A_{|G})}{d_A(\gamma)^{\frac{q-1}{n-1}}} e^{-\beta s} \end{aligned} \tag{C.17}$$

where $\hat{M}(.)$ is defined by (C.2c). Beware of the fact that the integer n must be replaced by p (resp. q) for $\hat{M}(A_{|F})$ (resp. $\hat{M}(A_{|G})$).

Main operator of the proof. Let us define the operator

$$\mathcal{O}^\gamma : \left| \begin{array}{ccc} H^\gamma \times \mathbb{R}^p \times \mathbb{R}^s & \to & H^\gamma \\ ((z, v), \omega, \mu) & \mapsto & \mathcal{O}^\gamma_{\omega,\mu}(z, v) \end{array} \right.$$

by the formula

$$\mathcal{O}^\gamma_{\omega,\mu}(z, v)(t) = \begin{pmatrix} e^{t\tilde{A}_1}\omega + \int_0^t e^{(t-s)\tilde{A}_1} f(z(s), v(s), \mu)\,\mathrm{d}s \\ -\int_t^{+\infty} e^{-(s-t)\tilde{A}_2} g(z(s), v(s), \mu)\,\mathrm{d}s \end{pmatrix}$$

where $\tilde{A}_1$, $\tilde{A}_2$, f and g are defined in (C.15).

Strategy of the proof. Fix $\mu \in \mathbb{R}^s$. We want to prove that the global γ-stable set $W^{s,\gamma}\left(0, \tilde{X}_\mu\right)$ is a graph over $\mathbb{R}^p$. This amounts to prove that for every $\omega \in \mathbb{R}^p$, there exists a unique $v_0 \in \mathbb{R}^q$ such that $(\omega, v_0) \in W^{s,\gamma}\left(0, \tilde{X}_\mu\right)$. This is also equivalent to say that, for every $\omega \in \mathbb{R}^p$, there exists a unique solution (z, v) of (C.15) such that $z(0) = \omega$ and $(z, v) \in H^\gamma$. We introduced the operator $\mathcal{O}^\gamma_{\omega,\mu}$ because its fixed points are exactly the solutions (z, v) of (C.15) such that $z(0) = \omega$ and $(z, v) \in H^\gamma$ (see Lemma C.18). Hence, it is enough to prove that $\mathcal{O}^\gamma_{\omega,\mu}$ admits a unique fixed point in H^γ, denoted

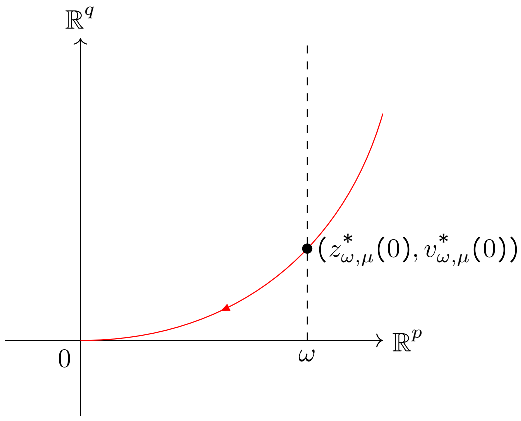

Figure C.1 – $(z^*_{\omega,\mu}, v^*_{\omega,\mu})$ is the unique orbit of $\tilde{X}_\mu$ contained in the global γ-stable set $W^{s,\gamma}\left(0, \tilde{X}_\mu\right)$ with initial condition of the form (ω, v_0), $v_0 \in \mathbb{R}^q$.

by $(z^*_{\omega,\mu}, v^*_{\omega,\mu})$ (see Lemma C.16). See figure C.1. The estimates on the graph follow from estimates on $v^*_{\omega,\mu}$ (see Lemma C.16) which themselves follow from estimates on $\mathcal{O}^\gamma$ (see Lemma C.15).

Technical details of the proof. We now state and prove three lemmas which constitute the main part of the proof.

For every $k \geq 0$, we denote by $\mathcal{L}_k\left(H^\gamma \times \mathbb{R}^s, H^\gamma\right)$ the space of k-linear maps from $\left(H^\gamma \times \mathbb{R}^s\right)^k$ to H^γ and we define the operator

$$\Lambda_k : H^\gamma \times \mathbb{R}^s \to \mathcal{L}_k\left(H^\gamma \times \mathbb{R}^s, H^\gamma\right)$$

by the following formula: for every $((z,v),\mu) \in H^\gamma \times \mathbb{R}^s$, $((z_i,v_i),\mu_i)_{1\leq i\leq k} \in \left(H^\gamma \times \mathbb{R}^s\right)^k$, $t \geq 0$,

$$\Lambda_k((z,v),\mu).((z_i,v_i),\mu_i)(t) = \begin{pmatrix} \int_0^t e^{(t-s)\tilde{A}_1} D^k f(z(s),v(s),\mu).((z_i(s),v_i(s)),\mu_i)\,\mathrm{d}s \\ -\int_t^{+\infty} e^{-(s-t)\tilde{A}_2} D^k g(z(s),v(s),\mu).((z_i(s),v_i(s)),\mu_i)\,\mathrm{d}s \end{pmatrix}$$

We also define the operator

$$\Gamma : \left| \begin{array}{ccc} \mathbb{R}^p & \to & H^\gamma \\ \omega & \mapsto & \left[t \mapsto \begin{pmatrix} e^{t\tilde{A}_1}\omega \\ 0 \end{pmatrix} \right] \end{array} \right.$$

In the next lemma, we will use the following norm on $H^\gamma \times \mathbb{R}^p \times \mathbb{R}^s$:

$$\|((z,v),\omega,\mu)\| = \|(z,v)\|_\gamma + \|\omega\| + \|\mu\|$$

Lemma C.15. *The operator $\mathcal{O}^\gamma$ is smooth. For all $k \geq 1$, $((z,v),\omega,\mu) \in H^\gamma \times \mathbb{R}^p \times \mathbb{R}^s$ and $((z_i,v_i),\omega_i,\mu_i)_{1\leq i\leq k} \in \left(H^\gamma \times \mathbb{R}^p \times \mathbb{R}^s\right)^k$,*

$$D^k\mathcal{O}^\gamma((z,v),\omega,\mu).((z_i,v_i),\omega_i,\mu_i) = \begin{cases} \Gamma(\omega_1) + \Lambda_1((z,v),\mu).((z_1,v_1),\mu_1) & \text{if } k = 1 \\ \Lambda_k((z,v),\mu).((z_i,v_i),\mu_i) & \text{if } k \geq 2 \end{cases} \tag{C.18}$$

Moreover, the derivatives of $\mathcal{O}^\gamma$ satisfy the following estimates: for every $((z,v),\omega,\mu) \in H^\gamma \times \mathbb{R}^p \times \mathbb{R}^s$,

$$\left|\left|\left|D_{z,v}\,\mathcal{O}^\gamma((z,v),\omega,\mu)\right|\right|\right|_\gamma \le \frac{1}{2} \tag{C.19a}$$

$$\left|\left|\left|D_{\omega}\,\mathcal{O}^\gamma((z,v),\omega,\mu)\right|\right|\right|_\gamma \le \frac{\hat{M}(A_{|F})}{d_A(\gamma)^{\frac{p-1}{n-1}}} \tag{C.19b}$$

$$\left|\left|\left|D_{\mu}\,\mathcal{O}^\gamma((z,v),\omega,\mu)\right|\right|\right|_\gamma \le \frac{2\max\left(\hat{M}(A_{|F}),\hat{M}(A_{|G})\right)}{d_A(\gamma)} M_2\left(\tilde{X}\right)\|(z,v)\|_\gamma \tag{C.19c}$$

where $|||.|||_\gamma$ *denotes the standard norm of continuous linear maps from* H^γ *(resp.* $\mathbb{R}^p$*, resp.* $\mathbb{R}^s$*) to* H^γ *and, for every* $k \ge 2$*,*

$$\left|\left|\left|D^k\,\mathcal{O}^\gamma((z,v),\omega,\mu)\right|\right|\right|_\gamma \le \frac{2\max\left(\hat{M}(A_{|F}),\hat{M}(A_{|G})\right)}{d_A(\gamma)}\left(M_k\left(\tilde{X}\right) + M_{k+1}\left(\tilde{X}\right)\|(z,v)\|_\gamma\right) \tag{C.19d}$$

where $|||.|||_\gamma$ *denotes the standard norm of continuous* k*-linear maps from* $\left(H^\gamma \times \mathbb{R}^p \times \mathbb{R}^s\right)^k$ *to* H^γ*.*

Proof of Lemma C.15. One can remark that $\mathcal{O}^\gamma$ is the sum of two operators, the first one being the linear map Γ and the second one being Λ_0. Since $\mathbb{R}^p$ is a finite dimensional vector space, Γ is smooth. It follows that we only need to prove that the operator $\Lambda_0 : H^\gamma \times \mathbb{R}^s \to H^\gamma$ is smooth to prove the first part of the lemma. Using the classical algebraic identification

$$\mathcal{L}_{k+1}\left(H^\gamma \times \mathbb{R}^s, H^\gamma\right) \simeq \mathcal{L}\left(H^\gamma \times \mathbb{R}^s, \mathcal{L}_k\left(H^\gamma \times \mathbb{R}^s, H^\gamma\right)\right)$$

we are going to prove that for every $k \ge 0$, Λ_k is differentiable and $D\,\Lambda_k = \Lambda_{k+1}$.

Step 1. For every $k \ge 0$*,* Λ_k *is well defined and for every* $k \ge 1$ *and* $((z,v),\mu) \in H^\gamma \times \mathbb{R}^s$*,* $\Lambda_k((z,v),\mu)$ *is a continuous* k*-linear map.* Let $k \ge 0$, $((z,v),\mu) \in H^\gamma \times \mathbb{R}^s$ and $((z_i,v_i),\mu_i)_{1\le i\le k} \in \left(H^\gamma \times \mathbb{R}^s\right)^k$. For every $s \ge 0$,

$$D^k(f,g)(z(s),v(s),\mu).((z_i(s),v_i(s)),\mu_i)_{1\le i\le k} = \sum_{\substack{0\le l\le k\\ \sigma\in\mathfrak{S}_k(l)}} D^l_{z,v}D^{k-l}_\mu(f,g)(z(s),v(s),\mu).\left(\sigma.((z_i,v_i),\mu_i)_{1\le i\le k}\right)$$

where

$$\sigma.((z_i,v_i),\mu_i)_{1\le i\le k} = \left(\left(z_{\sigma(1)}(s),v_{\sigma(1)}(s)\right),\dots,\left(z_{\sigma(l)}(s),v_{\sigma(l)}(s)\right),\mu_{\sigma(l+1)},\dots,\mu_{\sigma(k)}\right)$$

and $\mathfrak{S}_k(l)$ is the set of all permutations of $\{1,\dots,k\}$ which are increasing on both the integer intervals $[\![1,l]\!]$ and $[\![l+1,k]\!]$. Using estimates (C.16) and the mean value theorem, we obtain the following estimates. For all $s \ge 0$,

$$\|(f,g)(z(s),v(s),\mu)\| \le M_1\left(\tilde{X}\right)\|(z(s),v(s))\| \le e^{\gamma s}M_1\left(\tilde{X}\right)\|(z,v)\|_\gamma$$

$$\left\|D_{z,v}(f,g)(z(s),v(s),\mu).(z_1(s),v_1(s))\right\| \le M_1\left(\tilde{X}\right)\|(z_1(s),v_1(s))\| \le e^{\gamma s}M_1\left(\tilde{X}\right)\|(z_1,v_1)\|_\gamma$$

$$\left\|D_{\mu}(f,g)(z(s),v(s),\mu).\mu_1\right\| \le M_2\left(\tilde{X}\right)\|(z(s),v(s))\|\,\|\mu_1\| \le e^{\gamma s}M_2\left(\tilde{X}\right)\|(z,v)\|_\gamma\,\|\mu_1\|$$

For all $s \geq 0$, $0 \leq l \leq k$ and $\sigma \in \mathfrak{S}_k(l)$,

$$\begin{aligned}\left\| D^l_{z,v} D^{k-l}_\mu (f,g)(z(s),v(s),\mu).\left(\sigma.((z_i,v_i),\mu_i)_{1\leq i\leq k}\right)\right\| \\ \leq M_k\left(\tilde{X}\right)\prod_{i=1}^{l}\left\|\left(z_{\sigma(i)}(s),v_{\sigma(i)}(s)\right)\right\|\prod_{j=l+1}^{k}\left\|\mu_{\sigma(j)}\right\| \\ \leq e^{l\gamma s} M_k\left(\tilde{X}\right)\prod_{i=1}^{l}\left\|\left(z_{\sigma(i)},v_{\sigma(i)}\right)\right\|_\gamma\prod_{j=l+1}^{k}\left\|\mu_{\sigma(j)}\right\|\end{aligned}$$

When $l = 0$, the above estimate is not useful since there is no exponential decay, so we replace it with an estimate using $M_{k+1}\left(\tilde{X}\right)$ instead of $M_k\left(\tilde{X}\right)$:

$$\begin{aligned}\left\| D^k_\mu (f,g)(z(s),v(s),\mu).(\mu_i)_{1\leq i\leq k}\right\| &\leq M_{k+1}\left(\tilde{X}\right)\left\|(z(s),v(s))\right\|\prod_{i=1}^{k}\|\mu_i\| \\ &\leq e^{\gamma s} M_{k+1}\left(\tilde{X}\right)\|(z,v)\|_\gamma\prod_{i=1}^{k}\|\mu_i\|\end{aligned}$$

We now summarize the above (useful) estimates, using the inequality $e^{l\gamma s} \leq e^{\gamma s}$ for $l \neq 0$. For any $s \geq 0$, $0 \leq l \leq k$ and $\sigma \in \mathfrak{S}_k(l)$, we get

$$\left\| D^l_{z,v} D^{k-l}_\mu (f,g)(z(s),v(s),\mu).\left(\sigma.((z_i,v_i),\mu_i)_{1\leq i\leq k}\right)\right\| \leq \\ \begin{cases} e^{\gamma s} M_{k+1}\left(\tilde{X}\right)\prod_{i=1}^{k}\|\mu_i\|\,\|(z,v)\|_\gamma & \text{if } k\geq 0, l=0 \\ e^{\gamma s} M_k\left(\tilde{X}\right)\prod_{i=1}^{l}\left\|\left(z_{\sigma(i)},v_{\sigma(i)}\right)\right\|_\gamma\prod_{j=l+1}^{k}\left\|\mu_{\sigma(j)}\right\| & \text{if } k\geq 1, l\neq 0\end{cases} \tag{C.20}$$

It follows from (C.20) that the map $s \mapsto e^{-(s-t)\tilde{A}_2} D^k g(z(s),v(s),\mu).((z_i(s),v_i(s)),\mu_i)_{1\leq i\leq k}$ is integrable on $[t,+\infty[$, so Λ_k is well defined.

According to (C.17) and (C.20), and using the inequality $e^{l\gamma s} \leq e^{\gamma s}$, we have, for every $t \geq 0$,

$$\left\|\int_0^t e^{(t-s)\tilde{A}_1} D^k f(z(s),v(s),\mu).((z_i(s),v_i(s)),\mu_i)_{1\leq i\leq k}\,\mathrm{d}s\right\| \leq \\ \frac{1}{\gamma-\alpha}\frac{\hat{M}(A_{|F})}{d_A(\gamma)^{\frac{p-1}{n-1}}}e^{\gamma t}\times\begin{cases} M_1\left(\tilde{X}\right)\|(z,v)\|_\gamma & \text{if } k=0 \\ \left(M_k\left(\tilde{X}\right)+M_{k+1}\left(\tilde{X}\right)\|(z,v)\|_\gamma\right)\prod_{i=1}^{k}\|((z_i,v_i),\mu_i)\| & \text{if } k\geq 1\end{cases} \tag{C.21}$$

where we used the equality

$$\sum_{\substack{0\leq l\leq k\\ \sigma\in\mathfrak{S}_k(l)}}\prod_{i=1}^{l}\left\|\left(z_{\sigma(i)},v_{\sigma(i)}\right)\right\|_\gamma\prod_{j=l+1}^{k}\left\|\mu_{\sigma(j)}\right\| = \prod_{i=1}^{k}\left(\|(z_i,v_i)\|_\gamma+\|\mu_i\|\right) = \prod_{i=1}^{k}\|((z_i,v_i),\mu_i)\|$$

Of course, we have an analogous estimate for g: for every $t \geq 0$,

$$\left\|\int_t^{+\infty} e^{-(s-t)\tilde{A}_2} D^k g(z(s),v(s),\mu).((z_i(s),v_i(s)),\mu_i)_{1\leq i\leq k}\,\mathrm{d}s\right\| \leq \\ \frac{1}{\beta-\gamma}\frac{\hat{M}(A_{|G})}{d_A(\gamma)^{\frac{q-1}{n-1}}}e^{\gamma t}\times\begin{cases} M_1\left(\tilde{X}\right)\|(z,v)\|_\gamma & \text{if } k=0 \\ \left(M_k\left(\tilde{X}\right)+M_{k+1}\left(\tilde{X}\right)\|(z,v)\|_\gamma\right)\prod_{i=1}^{k}\|((z_i,v_i),\mu_i)\| & \text{if } k\geq 1\end{cases} \tag{C.22}$$

According to (C.21), (C.22) and the fact that $\max(p,q) \leq n-1$,

$$\|\Lambda_0((z,v),\mu)\|_\gamma \leq \frac{2\max\left(\hat{M}(A_{|F}),\hat{M}(A_{|G})\right)}{d_A(\gamma)} M_1\left(\tilde{X}\right)\|(z,v)\|_\gamma \tag{C.23a}$$

and for all $k \geq 1$,

$$\|\Lambda_k((z,v),\mu).((z_i,v_i),\mu_i)_{1\leq i\leq k}\|_\gamma \leq \frac{2\max\left(\hat{M}(A_{|F}),\hat{M}(A_{|G})\right)}{d_A(\gamma)}\left(M_k\left(\tilde{X}\right)+M_{k+1}\left(\tilde{X}\right)\|(z,v)\|_\gamma\right)\prod_{i=1}^{k}\|((z_i,v_i),\mu_i)\| \tag{C.23b}$$

According to (C.23b), for every $k \geq 1$, $\Lambda_k((z,v),\mu)$ is a continuous k-linear map whose subordinate norm satisfies

$$|||\Lambda_k((z,v),\mu)|||_\gamma \leq \frac{2\max\left(\hat{M}(A_{|F}),\hat{M}(A_{|G})\right)}{d_A(\gamma)}\left(M_k\left(\tilde{X}\right)+M_{k+1}\left(\tilde{X}\right)\|(z,v)\|_\gamma\right) \tag{C.24}$$

Step 2. For every $k \geq 0$, for every $((z,v),\mu) \in H^\gamma \times \mathbb{R}^s$, Λ_k is differentiable at the point $((z,v),\mu)$ and $D\,\Lambda_k((z,v),\mu) = \Lambda_{k+1}((z,v),\mu)$. Let $k \geq 0$, $((z,v),\mu), ((\Delta z,\Delta v),\Delta\mu) \in H^\gamma \times \mathbb{R}^s$ and $((z_i,v_i),\mu_i)_{1\leq i\leq k} \in \left(H^\gamma \times \mathbb{R}^s\right)^k$. According to Taylor-Lagrange formula, for every $s \geq 0$,

$$\begin{aligned}&\Big\|D^k(f,g)(z(s)+\Delta z(s),v(s)+\Delta v(s),\mu+\Delta\mu).((z_i(s),v_i(s)),\mu_i)_{1\leq i\leq k}-\\&\quad D^k(f,g)(z(s),v(s),\mu).\big(((z_i(s),v_i(s)),\mu_i)_{1\leq i\leq k}\big)-\\&\quad D^{k+1}(f,g)(z(s),v(s),\mu).\big(((z_i(s),v_i(s)),\mu_i)_{1\leq i\leq k},(\Delta z(s),\Delta v(s),\Delta\mu)\big)\Big\| \leq \\&\qquad \frac{1}{2}\sup_{w\in[0,1]}\left\|\Phi_s''(w)\right\|\end{aligned}$$

where $\Phi_s''(w)$ is the second derivative with respect to w of the real function $w \mapsto \Phi_s(w)$ defined by

$$\Phi_s(w) = D^k(f,g)((z(s),v(s),\mu)+w(\Delta z(s),\Delta v(s),\Delta\mu)).((z_i(s),v_i(s)),\mu_i)_{1\leq i\leq k}$$

By (C.20) and computations similar to the ones done in the preceding step,

$$\sup_{w\in[0,1]}\left\|\Phi_s''(w)\right\| \leq e^{\gamma s}O_{((\Delta z,\Delta v),\Delta\mu)\to 0}\left(\|((\Delta z,\Delta v),\Delta\mu)\|^2\right)\prod_{i=1}^{k}\|((z_i,v_i),\mu_i)\|$$

so

$$\begin{aligned}&|||\Lambda_k(((z,v),\mu)+((\Delta z,\Delta v),\Delta\mu))-\Lambda_k((z,v),\mu)-\Lambda_{k+1}((z,v),\mu).((\Delta z,\Delta v),\Delta\mu)|||_\gamma = \\&\qquad O_{((\Delta z,\Delta v),\Delta\mu)\to 0}\left(\|((\Delta z,\Delta v),\Delta\mu)\|^2\right)\end{aligned}$$

By a staightforward induction on k, this implies that Λ_0 is smooth and for every $k \geq 1$, $D^k\,\Lambda_0 = \Lambda_k$. As a further consequence, $\mathcal{O}^\gamma$ is smooth and formula (C.18) holds true.

Step 3. Proof of estimates (C.19). First, notice that estimate (C.19d) is a direct consequence of (C.24) and estimate (C.19b) is a direct consequence of (C.17) and (C.18). To prove (C.19a) and (C.19c), let $((z,v),\omega,\mu) \in H^\gamma \times \mathbb{R}^p \times \mathbb{R}^s$ and $((\Delta z,\Delta v),\Delta\mu) \in H^\gamma \times \mathbb{R}^s$. According to (C.18), we have

$$D_{z,v}\,\mathcal{O}^\gamma((z,v),\omega,\mu).(\Delta z,\Delta v) = \Lambda_1((z,v),\mu).((\Delta z,\Delta v),0)$$

By (C.20) and similar computations to the ones done in the first step,

$$\|\Lambda_1((z,v),\mu).((\Delta z,\Delta v),0)\|_\gamma \leq \frac{2\max\left(\hat{M}(A_{|F}),\hat{M}(A_{|G})\right)}{d_A(\gamma)}M_1\left(\tilde{X}\right)\|(\Delta z,\Delta v)\|_\gamma$$

so, by (C.10), (C.13), (C.2c) and the fact that $\max\left(M(A_{|F}),M(A_{|G})\right)m\left(F,G\right) \leq M(A)$, we have

$$\frac{2\max\left(\hat{M}(A_{|F}),\hat{M}(A_{|G})\right)}{d_A(\gamma)}M_1\left(\tilde{X}\right) \leq \frac{1}{2} \tag{C.25}$$

so estimate (C.19a) holds true. By similar computations, we obtain

$$\|\Lambda_1((z,v),\mu).((0,0),\Delta\mu)\|_\gamma \leq \frac{2\max\left(\hat{M}(A_{|F}),\hat{M}(A_{|G})\right)}{d_A(\gamma)} M_2\left(\tilde{X}\right)\|(z,v)\|_\gamma \|\Delta\mu\|$$

which implies the estimate (C.19c). This concludes the proof of Lemma C.15. □

Lemma C.16. *For every $(\omega,\mu)\in\mathbb{R}^p\times\mathbb{R}^s$, the operator $\mathcal{O}^\gamma_{\omega,\mu}$ admits a unique fixed point, which is independant of the choice of γ and is denoted by $(z^*_{\omega,\mu},v^*_{\omega,\mu})$. This fixed point satisifes the following inequality: for every $t\geq 0$,*

$$\left\|(z^*_{\omega,\mu}(t),v^*_{\omega,\mu}(t))\right\| \leq \frac{2\sqrt{2}\hat{M}(A_{|F})}{d_A(\gamma)^{\frac{p-1}{n-1}}}\left\|(z^*_{\omega,\mu}(0),v^*_{\omega,\mu}(0))\right\| \tag{C.26}$$

Moreover, the map

$$(z^*,v^*)\colon \left|\begin{array}{ccc} \mathbb{R}^p\times\mathbb{R}^s & \to & H^\gamma \\ (\omega,\mu) & \mapsto & (z^*_{\omega,\mu},v^*_{\omega,\mu}) \end{array}\right.$$

is smooth and, using the norm $\|(\omega,\mu)\| = \|\omega\|+\|\mu\|$ on $\mathbb{R}^p\times\mathbb{R}^s$, we have the following estimates: for every $(\omega,\mu)\in\mathbb{R}^p\times\mathbb{R}^s$,

$$\left\|v^*_{\omega,\mu}\right\|_\gamma \leq \frac{4\hat{M}(A_{|F})\hat{M}(A_{|G})}{d_A(\gamma)}M_1\left(\tilde{X}\right)\|\omega\| \tag{C.27a}$$

$$\left|\left|\left|D_\omega v^*_{\omega,\mu}\right|\right|\right|_\gamma \leq \frac{4\hat{M}(A_{|F})\hat{M}(A_{|G})}{d_A(\gamma)}M_1\left(\tilde{X}\right) \tag{C.27b}$$

$$\left|\left|\left|D_\mu v^*_{\omega,\mu}\right|\right|\right|_\gamma \leq \frac{8\hat{M}(A_{|F})\hat{M}(A_{|G})}{d_A(\gamma)}M_2\left(\tilde{X}\right)\|\omega\| \tag{C.27c}$$

and, more generally, for every $k\geq 2$,

$$\left|\left|\left|D^k v^*_{\omega,\mu}\right|\right|\right|_\gamma \leq a_k\left(\frac{\max\left(\hat{M}(A_{|F}),\hat{M}(A_{|G})\right)^2}{d_A(\gamma)^2}\bar{M}_{k+1}\left(\tilde{X}\right)\max\left(1,\|\omega\|\right)\right)^{2k-1} \tag{C.27d}$$

where a_k is a positive constant independant of X, (F,G), ω and μ.

Remark C.17. To conclude the proof of Proposition C.8, we only need estimates on v^*, this is why we did not give estimates on z^* in the above statement. Such estimates will be used in the following proof though.

Proof of Lemma C.16. According to (C.19a) and the contraction mapping theorem, for all $(\omega,\mu)\in\mathbb{R}^p\times\mathbb{R}^s$, $\mathcal{O}^\gamma_{\omega,\mu}$ admits a unique fixed point

$$(z^*_{\gamma,\omega,\mu},v^*_{\gamma,\omega,\mu})$$

Let $\gamma'\in]\lambda_{\max}\left(A_{|F}\right),\min(0,\lambda_{\min}\left(A_{|G}\right))[$ satisfying (C.10). According to remark C.13, we have

$$H^{\min(\gamma,\gamma')}\subset H^{\max(\gamma,\gamma')}$$

so the fixed point $\left(z^*_{\min(\gamma,\gamma'),\omega,\mu},v^*_{\min(\gamma,\gamma'),\omega,\mu}\right)$ is also a fixed point of $\mathcal{O}^{\max(\gamma,\gamma')}_{\omega,\mu}$. By uniqueness, this proves that the two fixed points coincide. Denote this unique fixed point by $(z^*_{\omega,\mu},v^*_{\omega,\mu})$.

To prove that the fixed point depends smoothly on (ω,μ), the idea is to apply the global inverse function theorem to the map

$$G^\gamma\colon \left|\begin{array}{ccc} H^\gamma\times\mathbb{R}^p\times\mathbb{R}^s & \to & H^\gamma\times\mathbb{R}^p\times\mathbb{R}^s \\ ((z,v),\omega,\mu) & \mapsto & \left(\mathcal{O}^\gamma_{\omega,\mu}(z,v)-(z,v),\omega,\mu\right) \end{array}\right.$$

Indeed, according to Lemma C.15, G^γ is smooth and according to (C.19a), G^γ is injective and its differential is at any point invertible. According to the global inverse function theorem,

$$V^\gamma \stackrel{\text{def}}{=} G^\gamma(H^\gamma \times \mathbb{R}^p \times \mathbb{R}^s)$$

is an open set of $H^\gamma \times \mathbb{R}^p \times \mathbb{R}^s$, the map $G^\gamma : H^\gamma \times \mathbb{R}^p \times \mathbb{R}^s \to V^\gamma$ is a diffeomorphism and its inverse is smooth. Denote by

$$\left(G^\gamma\right)_1^{-1} : V^\gamma \to H^\gamma$$

the first coordinate of $\left(G^\gamma\right)^{-1}$. By definition of (z^*, v^*), for every $(\omega, \mu) \in \mathbb{R}^p \times \mathbb{R}^s$,

$$G^\gamma\big((z^*_{\omega,\mu}, v^*_{\omega,\mu}), \omega, \mu\big) = \big((0,0), \omega, \mu\big)$$

so

$$(z^*_{\omega,\mu}, v^*_{\omega,\mu}) = \left(G^\gamma\right)_1^{-1}\left((0,0), \omega, \mu\right)$$

Since $\left(G^\gamma\right)^{-1}$ is smooth, this completes the first part of the proof of Lemma C.16.

Fix $(\omega, \mu) \in \mathbb{R}^p \times \mathbb{R}^s$. From the fixed point equation

$$(z^*_{\omega,\mu}, v^*_{\omega,\mu}) = \mathcal{O}^\gamma_{\omega,\mu}\left(z^*_{\omega,\mu}, v^*_{\omega,\mu}\right) \tag{C.28}$$

and (C.17), (C.23a), it follows that for all $t \geq 0$,

$$\left\|(z^*_{\omega,\mu}, v^*_{\omega,\mu})(t)\right\| \leq \frac{\hat{M}(A_{|F})}{d_A(\gamma)^{\frac{p-1}{n-1}}} e^{\alpha t} \|\omega\| + \frac{2\max\left(\hat{M}(A_{|F}), \hat{M}(A_{|G})\right)}{d_A(\gamma)} M_1\left(\tilde{X}\right) e^{\gamma t} \left\|(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right\|_\gamma$$

so, according to (C.25) and the inequality $e^{\alpha t} \leq e^{\gamma t}$,

$$\left\|(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right\|_\gamma \leq \frac{2\hat{M}(A_{|F})}{d_A(\gamma)^{\frac{p-1}{n-1}}} \|\omega\| \tag{C.29}$$

Plugging (C.22) (case $k = 0$) into (C.28), we obtain, for all $t \geq 0$,

$$\left\|v^*_{\omega,\mu}(t)\right\| \leq \frac{1}{\beta - \gamma} \frac{\hat{M}(A_{|G})}{d_A(\gamma)^{\frac{q-1}{n-1}}} e^{\gamma t} M_1\left(\tilde{X}\right) \left\|(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right\|_\gamma \tag{C.30}$$

Plugging (C.29) into (C.30) and using $p + q = n$, we obtain estimate (C.27a). Note that (C.26) follows from (C.29).

Taking the derivative of (C.28) with respect to the variable ω and using (C.19a) and (C.19b), we get

$$\left|\left|\left|D_\omega(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right|\right|\right|_\gamma \leq \frac{2\hat{M}(A_{|F})}{d_A(\gamma)^{\frac{p-1}{n-1}}} \tag{C.31}$$

Moreover, taking the derivative of (C.28) with respect to the variable ω, we obtain, for every $\omega_1 \in \mathbb{R}^p$ and every $t \geq 0$,

$$D_\omega v^*_{\omega,\mu}.\omega_1(t) = -\int_t^{+\infty} e^{-(s-t)\tilde{A}_2}\left(D_{z,v} g(z^*_{\omega,\mu}(s), v^*_{\omega,\mu}(s), \mu).D_\omega(z^*_{\omega,\mu}, v^*_{\omega,\mu}).\omega_1(s)\right) \mathrm{d}s$$

so using (C.17) and (C.20), we obtain

$$\left|\left|\left|D_\omega v^*_{\omega,\mu}\right|\right|\right|_\gamma \leq \frac{1}{\beta - \gamma} \frac{\hat{M}(A_{|G})}{d_A(\gamma)^{\frac{q-1}{n-1}}} M_1\left(\tilde{X}\right) \left|\left|\left|D_\omega(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right|\right|\right|_\gamma \tag{C.32}$$

Plugging (C.31) into (C.32) and using $p + q = n$, we get estimate (C.27b).

Taking the derivative of (C.28) with respect to the variable μ and using (C.19a) and (C.19c), we get

$$|||D_\mu(z^*_{\omega,\mu}, v^*_{\omega,\mu})|||_\gamma \leq \frac{4\max\left(\hat{M}(A_{|F}), \hat{M}(A_{|G})\right)}{d_A(\gamma)} M_2\left(\tilde{X}\right) \left\|(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right\|_\gamma \tag{C.33}$$

Moreover, taking the derivative of (C.28) with respect to the variable μ, we obtain, for every $\mu_1 \in \mathbb{R}^s$ and every $t \geq 0$,

$$D_\mu v^*_{\omega,\mu}.\mu_1(t) = -\int_t^{+\infty} e^{-(s-t)\tilde{A}_2}\left(D_{z,v}g(z^*_{\omega,\mu}(s), v^*_{\omega,\mu}(s), \mu).D_\mu(z^*_{\omega,\mu}, v^*_{\omega,\mu}).\mu_1(s)\right) \mathrm{d}s \\ - \int_t^{+\infty} e^{-(s-t)\tilde{A}_2}\left(D_\mu g(z^*_{\omega,\mu}(s), v^*_{\omega,\mu}(s), \mu).\mu_1\right) \mathrm{d}s$$

so using (C.17) and (C.20), we obtain

$$|||D_\mu v^*_{\omega,\mu}|||_\gamma \leq \frac{1}{\beta-\gamma}\frac{\hat{M}(A_{|G})}{d_A(\gamma)^{\frac{q-1}{n-1}}}\left(M_1\left(\tilde{X}\right)|||D_\mu(z^*_{\omega,\mu}, v^*_{\omega,\mu})|||_\gamma + M_2\left(\tilde{X}\right)\left\|(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right\|_\gamma\right) \tag{C.34}$$

Plugging (C.33) into (C.34), we get

$$|||D_\mu v^*_{\omega,\mu}|||_\gamma \leq \frac{1}{\beta-\gamma}\frac{\hat{M}(A_{|G})}{d_A(\gamma)^{\frac{q-1}{n-1}}} M_2\left(\tilde{X}\right)\left\|(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right\|_\gamma\left(1 + \frac{4\max\left(\hat{M}(A_{|F}), \hat{M}(A_{|G})\right)}{d_A(\gamma)} M_1\left(\tilde{X}\right)\right)$$

Using (C.2c), (C.10), (C.13), the inequality

$$\max\left(M(A_{|F}), M(A_{|G})\right) m\left(F, G\right) \leq M(A)$$

and the inequality $\max(p, q) < n$, we get

$$\frac{4\max\left(\hat{M}(A_{|F}), \hat{M}(A_{|G})\right)}{d_A(\gamma)} M_1\left(\tilde{X}\right) \leq 1$$

so

$$|||D_\mu v^*_{\omega,\mu}|||_\gamma \leq 2\frac{1}{\beta-\gamma}\frac{\hat{M}(A_{|G})}{d_A(\gamma)^{\frac{q-1}{n-1}}} M_2\left(\tilde{X}\right)\left\|(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right\|_\gamma \tag{C.35}$$

Plugging (C.29) into (C.35) and using $p + q = n$, we get estimate (C.27c).

We are now going to prove (C.27d). To avoid clutter with constants independant of X, (F, G), ω and μ in the following estimates, we introduce the following notation: for any real positive functions δ_1, δ_2 depending on (X, F, G, ω, μ) we define the order relation $\precsim$ by

$$\delta_1 \precsim \delta_2 \iff \exists C > 0, \quad \delta_1 \leq C\delta_2 \tag{C.36}$$

For every $k \geq 2$ and $(\omega, \mu) \in \mathbb{R}^p \times \mathbb{R}^s$, let

$$u_k \overset{\text{def}}{=} \frac{\max\left(\hat{M}(A_{|F}), \hat{M}(A_{|G})\right)}{d_A(\gamma)} \max\left(1, \left\|(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right\|_\gamma\right) \bar{M}_k\left(\tilde{X}\right) \tag{C.37}$$

Note that (u_k) is increasing. We are going to prove by induction on k that, for every $k \geq 1$,

$$\left|\left|\left|D^k(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right|\right|\right|_\gamma \precsim u_{k+1}^{2k-1} \tag{C.38}$$

Inequalities (C.31) and (C.33) yield

$$|||D(z^*_{\omega,\mu}, v^*_{\omega,\mu})|||_\gamma \precsim u_2$$

which proves (C.38) in the case $k = 1$. Let $k \geq 2$. Deriving (C.28), we get

$$D^k(z^*_{\omega,\mu}, v^*_{\omega,\mu}) = D_{z,v}\,\mathcal{O}^\gamma((z^*_{\omega,\mu}, v^*_{\omega,\mu}),\omega,\mu).D^k(z^*_{\omega,\mu}, v^*_{\omega,\mu}) + \sum_{j=2}^{k} \sum_{\substack{i_1,\dots,i_j\geq 1\\ i_1+\dots+i_j=k}} C_{i_1,\dots,i_j} D^j\,\mathcal{O}^\gamma((z^*_{\omega,\mu}, v^*_{\omega,\mu}),\omega,\mu). \left(D^{i_1}((z^*_{\omega,\mu}, v^*_{\omega,\mu}),\omega,\mu),\dots,D^{i_j}((z^*_{\omega,\mu}, v^*_{\omega,\mu}),\omega,\mu)\right) \quad \text{(C.39)}$$

where the $C_{i_1,\dots,i_j}$ are the constants appearing in the standard Faà di Bruno's formula. According to (C.19d), for all $j \geq 2$

$$\left|\left|\left|D^j\,\mathcal{O}^\gamma((z^*_{\omega,\mu}, v^*_{\omega,\mu}),\omega,\mu)\right|\right|\right|_\gamma \precsim u_{j+1} \quad \text{(C.40)}$$

Plugging estimates (C.19a) and (C.40) into (C.39) and using the induction hypothesis, we get

$$\begin{aligned}\left|\left|\left|D^k(z^*_{\omega,\mu}, v^*_{\omega,\mu})\right|\right|\right|_\gamma &\precsim \max_{\substack{2\leq j\leq k\\ i_1,\dots,i_j\geq 1\\ i_1+\dots+i_j=k}} u_{j+1}\prod_{l=1}^{j} u_{i_l+1}^{2i_l-1}\\ &\precsim u_{k+1}\max_{\substack{2\leq j\leq k\\ i_1,\dots,i_j\geq 1\\ i_1+\dots+i_j=k}} u_{k+1}^{\sum_{l=1}^{j}(2i_l-1)}\\ &\precsim u_{k+1}^{2k-1}\end{aligned}$$

which proves (C.38) for all $k \geq 1$ by induction. According to (C.29),

$$\max\left(1, \|(z^*_{\omega,\mu}, v^*_{\omega,\mu})\|_\gamma\right) \precsim \frac{\max\left(\hat{M}(A_{|F}), \hat{M}(A_{|G})\right)}{d_A(\gamma)} \max(1, \|\omega\|) \quad \text{(C.41)}$$

Plugging (C.41) into (C.38), we finally obtain estimate (C.27d). This concludes the proof of Lemma C.16. □

Let us define the map

$$\tilde{\phi}: \left| \begin{array}{ccc} \mathbb{R}^p\times\mathbb{R}^s & \to & \mathbb{R}^q \\ (\omega,\mu) & \mapsto & \tilde{\phi}_\mu(\omega) := v^*_{\omega,\mu}(0) \end{array}\right.$$

Lemma C.18. *For every $\mu \in \mathbb{R}^s$, the global γ-stable set $W^{s,\gamma}\left(0,\tilde{X}_\mu\right)$ is exactly the graph of the map $\tilde{\phi}_\mu : \mathbb{R}^p \to \mathbb{R}^q$. Moreover, $\tilde{\phi}$ is smooth and for every $k \geq 0$, every $(\omega,\mu) \in \mathbb{R}^p\times\mathbb{R}^s$ and every $(\omega_i,\mu_i)_{1\leq i\leq k} \in \left(\mathbb{R}^p\times\mathbb{R}^s\right)^k$, the following formula holds:*

$$D^k\tilde{\phi}(\omega,\mu).(\omega_i,\mu_i)_{1\leq i\leq k} = D^k v^*_{\omega,\mu}.(\omega_i,\mu_i)_{1\leq i\leq k}(0) \quad \text{(C.42)}$$

Proof of Lemma C.18. Fix $\mu \in \mathbb{R}^s$. Let $(\omega, v_0) \in \mathbb{R}^p\times\mathbb{R}^q$. We are going to prove the following equivalence:

$$(\omega, v_0) \in W^{s,\gamma}\left(0,\tilde{X}_\mu\right) \iff v_0 = \tilde{\phi}_\mu(\omega)$$

Let $(z,v) : [0,+\infty) \to \mathbb{R}^p\times\mathbb{R}^q$ be a continuous map such that $z(0) = \omega$. By a straighforward computation, (z,v) is an orbit of $\tilde{X}_\mu$ (that is, a solution of (C.15)) if and only if for every $0 \leq t \leq \tau$,

$$\begin{aligned} z(t) &= e^{t\tilde{A}_1}\omega + \int_0^t e^{(t-s)\tilde{A}_1} f(z(s),v(s),\mu)ds \\ v(t) &= e^{-(\tau-t)\tilde{A}_2}v(\tau) - \int_t^\tau e^{-(s-t)\tilde{A}_2} g(z(s),v(s),\mu)ds \end{aligned}$$

If we assume that $(z,v) \in H^\gamma$, then, according to (C.16a), (C.16c) and (C.17), the second integral above converges as τ goes to $+\infty$. Letting τ tend to $+\infty$, we get that $(z,v) \in H^\gamma$ and (z,v) is a solution of (C.15) if and only if $(z,v) \in H^\gamma$ and for every $t \geq 0$,

$$\begin{aligned} z(t) &= e^{t\tilde{A}_1}\omega + \int_0^t e^{(t-s)\tilde{A}_1} f(z(s),v(s),\mu)ds \\ v(t) &= -\int_t^{+\infty} e^{-(s-t)\tilde{A}_2} g(z(s),v(s),\mu)ds \end{aligned} \tag{C.43}$$

i.e. if and only if (z,v) is a fixed point of $\mathcal{O}^\gamma_{\omega,\mu}$.

From now on (z,v) denotes the orbit of $\tilde{X}_\mu$ starting from (ω, v_0) at $t=0$, that is, $(z(t),v(t)) = \tilde{X}^t_\mu(\omega,v_0)$. We have the following equivalences:

$$\begin{aligned}
&(\omega,v_0) \in W^{s,\gamma}\left(0,\tilde{X}_\mu\right) && \\
\iff &(z,v) \in H^\gamma && \text{by (C.6)} \\
\iff &(z,v) \text{ is a fixed point of } \mathcal{O}^\gamma_{\omega,\mu} && \text{by the above reasoning} \\
\iff &(z,v) = (z^*_{\omega,\mu}, v^*_{\omega,\mu}) && \text{by Lemma C.15} \\
\iff &(z(0),v(0)) = (z^*_{\omega,\mu}(0), v^*_{\omega,\mu}(0)) && \text{by uniqueness in Cauchy-Lipschitz theorem} \\
\iff &(\omega,v_0) = (\omega,\tilde{\phi}_\mu(\omega)) && \text{by definition of } \tilde{\phi}_\mu
\end{aligned}$$

which conclude the first part of the proof. Let E_0 be the “evaluation at time $t=0$” map

$$E_0\colon \left| \begin{array}{rcl} H^\gamma_q & \to & \mathbb{R}^q \\ v & \mapsto & v(0) \end{array}\right.$$

By definition of the γ-norm, E_0 is a linear continuous map (with $\|E_0\| \leq 1$) and as such is smooth. Since

$$\tilde{\phi} = E_0 \circ v^*$$

it follows from Lemma C.15 that $\tilde{\phi}$ is smooth and (C.42) holds true. This concludes the proof of Lemma C.18. ☐

Plugging estimates (C.27) into (C.42) and using (C.13), (C.2c) and the fact that

$$M(A_{|F})M(A_{|G})m(F,G) \leq M(A)$$

we have, for every $(\omega,\mu) \in \mathbb{R}^p \times \mathbb{R}^s$,

$$\left\|\tilde{\phi}(\omega,\mu)\right\| \precsim \frac{M(A)}{d_A(\gamma)} M_1(X)\|\omega\| \tag{C.44a}$$

$$\left\|\left|D_\omega\tilde{\phi}(\omega,\mu)\right|\right\| \precsim \frac{M(A)}{d_A(\gamma)} M_1(X) \tag{C.44b}$$

$$\left\|\left|D_\mu\tilde{\phi}(\omega,\mu)\right|\right\| \precsim \frac{M(A)}{d_A(\gamma)} M_2(X)\|\omega\| \tag{C.44c}$$

and, more generally, for every $k \geq 2$,

$$\left\|\left|D^k\tilde{\phi}(\omega,\mu)\right|\right\| \precsim \left(\frac{M(A)^2}{d_A(\gamma)^2}\bar{M}_{k+1}(X)\max(1,\|\omega\|)\right)^{2k-1} \tag{C.44d}$$

where $\precsim$ is defined by (C.36). Let us now define

$$\phi\colon \left| \begin{array}{rcl} F\times\mathbb{R}^s & \to & G \\ (\omega,\mu) & \mapsto & L^{-1}\left(\tilde{\phi}_\mu(L(\omega))\right) \end{array}\right.$$

One can remark that

$$L(\operatorname{Graph}\phi) = \operatorname{Graph}\tilde{\phi}$$

and, according to (C.12), we have, for every $\mu \in \mathbb{R}^s$,

$$L\left(W^{s,\gamma}\left(0, X_\mu\right)\right) = W^{s,\gamma}\left(0, \tilde{X}_\mu\right)$$

so, according to Lemma C.18, we get that item 1 of Proposition C.8 holds true.

We are now going to prove estimates (C.9). According to the fact that $L_{|F}$ and $(L^{-1})_{|\mathbb{R}^q}$ are isometries, it follows that estimates (C.44) hold true for ϕ instead of $\tilde{\phi}$, up to a formal replacement of $\omega \in \mathbb{R}^p$ by $z \in F$. To conclude, it suffices to remark that these estimates are valid for all $\gamma \in I_A$ satisfying (C.10). It is straightforward to check that the function $d_A(\gamma)$ defined for all $\gamma \in]\lambda_{\max}\left(A_{|F}\right), \min(0, \lambda_{\min}\left(A_{|G}\right))]$ satisfying (C.10) is maximal at the point $\min\left(0, \left(\lambda_{\max}\left(A_{|F}\right) + \lambda_{\min}\left(A_{|G}\right)\right)/2\right)$ and its maximum is more than $\left(2^{n-1}\sigma\left(A\right)\right)^{-1}$, where $\sigma\left(A\right)$ is defined by (C.5). Letting γ tend to $\min\left(0, \left(\lambda_{\max}\left(A_{|F}\right) + \lambda_{\min}\left(A_{|G}\right)\right)/2\right)$ in estimates (C.44), it follows that estimates (C.9) hold true for some constants $C_{1,0}, C_{1,1}, \ldots$ independant of X, (F, G), ω and μ.

It remains to prove item 2. Fix $\mu \in \mathbb{R}^s$. Let (z, v) be an orbit of X_μ. By definition of $\tilde{X}_\mu$, $L(z, v)$ is an orbit of $\tilde{X}_\mu$. According to (C.26), we have, for all $t \geq 0$,

$$\|L(z(t), v(t))\| \leq \frac{2\sqrt{2}\hat{M}(A_{|F})}{d_A(\gamma)^{\frac{p-1}{n-1}}} \|L(z(0), v(0))\|$$

so, using (C.11), we get

$$\|(z(t), v(t))\| \leq \frac{4\hat{M}(A_{|F})m\left(F, G\right)}{d_A(\gamma)^{\frac{p-1}{n-1}}} \|(z(0), v(0))\|$$

Letting γ tend to $\min\left(0, \left(\lambda_{\max}\left(A_{|F}\right) + \lambda_{\min}\left(A_{|G}\right)\right)/2\right)$ in the above estimate, there exists a positive constant C (independant of X, (F, G), ω, μ and (z, v)) such that for all $t \geq 0$,

$$\|(z(t), v(t))\| \leq CM(A)\sigma\left(A\right) \|(z(0), v(0))\|$$

The above estimate implies that for every $\eta > 0$ and every $0 < \delta \leq \frac{\eta}{CM(A)\sigma(A)}$,

$$W^{s,\gamma}\left(0, X_\mu\right) \cap B_{\mathbb{R}^n}(0, \delta) \subset W^{s,\gamma}_\eta(0, X_\mu) \cap B_{\mathbb{R}^n}(0, \delta)$$

The other inclusion being straightforward, item 2 follows. This concludes the proof of Proposition C.8. $\square$

C.3.3 Local estimates

In this section, we state and prove a precise version of the local stable manifold theorem. This version is given by Theorem C.19 below. Recall that we defined four hypotheses on smooth families of vector fields in subsection C.3.1. Given a parameter $r > 0$ and a smooth family of vector fields $(X_\mu)_{\mu \in \mathbb{R}^s}$ satisfying the hypotheses 1 and 2, let

$$M_1\left(X, r\right) \overset{\text{def}}{=} \sup_{(x,\mu)\in\overline{B((0,0),r)}} |||D_x X(x, \mu) - A|||$$

where $A := D_x X(0, 0)$, and for every integer $k \geq 2$, let

$$M_k\left(X, r\right) \overset{\text{def}}{=} \sup_{2\leq j\leq k} \sup_{(x,\mu)\in\overline{B((0,0),r)}} \left|\left|\left|D^j X(x, \mu)\right|\right|\right|$$

where $\overline{B((0,0), r)}$ is the closed ball in $\mathbb{R}^n \times \mathbb{R}^s$ of center $(0, 0)$ and radius r and let

$$\bar{M}_k\left(X, r\right) \overset{\text{def}}{=} \max\left(1, M_k\left(X, r\right)\right) \tag{C.45}$$

Theorem C.19 (Local estimates for the stable manifold theorem with parameters)**.** *There exists* *a positive constant $C_2 \geq 1$ and a sequence of positive constants $(C_{2,k})_{k\in\mathbb{N}}$ (both depending on the dimension n) such that for every smooth family of vector fields $(X_\mu)_{\mu\in\mathbb{R}^s}$ satisfying the hypotheses 1 and 2, every partially hyperbolic splitting (F,G) of $A := D_xX(0,0)$ and every $r > 0$,*

1. *Uniqueness of the stable sets: for every $\gamma, \gamma' \in I_A$ (see (C.4)) and every $\mu \in \mathbb{R}^s$ such that*

$$\|\mu\| \leq \frac{1}{C_2}\min\left(\frac{\min\left(d_A(\gamma), d_A(\gamma')\right)}{M(A)\bar{M}_2(X,r)}, r\right)$$

   one has

$$W^{s,\gamma}(0, X_\mu) = W^{s,\gamma'}(0, X_\mu) \tag{C.46}$$

2. *Graph structure: there exists a (non unique) smooth map*

$$\phi: \left| \begin{array}{ccc} F\times\mathbb{R}^s & \to & G \\ (z,\mu) & \mapsto & \phi_\mu(z) \end{array} \right.$$

   such that for every $\gamma \in I_A$, every $\mu \in \mathbb{R}^s$ such that

$$\|\mu\| \leq \frac{1}{C_2}\min\left(\frac{\min\left(d_A(\gamma), \sigma(A)^{-1}\right)}{M(A)\bar{M}_2(X,r)}, r\right)$$

   every $0 < \eta \leq \tilde{\eta}$ and every $0 < \delta \leq \frac{\eta}{CM(A)\sigma(A)}$, the following equality holds:

$$W^{s,\gamma}_\eta(0, X_\mu) \cap B_{\mathbb{R}^n}(0,\delta) = \mathrm{Graph}\left(\phi_\mu\right) \cap B_{\mathbb{R}^n}(0,\delta) \tag{C.47}$$

   where

$$\tilde{\eta} = \frac{1}{C_2}\min\left(\left(\sigma(A)\, M(A)\bar{M}_2(X,r)\right)^{-1}, r\right) \quad \textit{(see (C.2b) and (C.5))}$$

3. *Controls on ϕ: for every $(z,\mu) \in B_F\left(0,\tilde{\delta}\right) \times B_{\mathbb{R}^s}\left(0,\tilde{\delta}\right)$,*

$$\|\phi(z,\mu)\| \leq C_{2,0}\sigma(A)^2\, M(A)^2 M_2(X,r)\left(\|z\| + \|\mu\|\right)\|z\| \tag{C.48a}$$

$$|||D_z\phi(z,\mu)||| \leq C_{2,1}\sigma(A)^2\, M(A)^2 M_2(X,r)\left(\|z\| + \|\mu\|\right) \tag{C.48b}$$

$$\left|\left|\left|D_\mu\phi(z,\mu)\right|\right|\right| \leq C_{2,1}\sigma(A)\, M(A) M_2(X,r)\,\|z\| \tag{C.48c}$$

   where

$$\tilde{\delta} = \frac{1}{C_2\sigma(A)\, M(A)}\min\left(\left(\sigma(A)\, M(A)\bar{M}_2(X,r)\right)^{-1}, r\right)$$

   and more generally, using the norm $\|(z,\mu)\| = \|z\| + \|\mu\|$ on $F\times\mathbb{R}^s$, for all $k \geq 2$,

$$\left|\left|\left|D^k\phi(z,\mu)\right|\right|\right| \leq C_{2,k}\left(\sigma(A)^2\, M(A)^2 \max\left(\sigma(A)\, M(A)\bar{M}_2(X,r), r^{-1}\right)^{k-1}\bar{M}_{k+1}(X,r)\right)^{2k-1} \tag{C.48d}$$

Remark C.20. If the singularity is hyperbolic (*i.e.* $\lambda_{\min}\left(A_{|G}\right) > 0$), then the global γ-stable set $W^{s,\gamma}\left(0, X_\mu\right)$ coincide with the global stable set $W^s(0, X_\mu)$ (for μ sufficiently small).

Remark C.21. If one is working with a different norm than the Euclidean one, one will have the same result but with different constants $C_2, C_{2,0}, C_{2,1}, \ldots$

Proof of Theorem C.19. Fix a smooth family of vector fields $(X_\mu)_{\mu\in\mathbb{R}^s}$ satisfying the hypotheses 1 and 2, a partially hyperbolic splitting (F,G) of $A = D_xX(0,0)$ and $r > 0$. Fix a smooth plateau map $\chi : [0,+\infty] \to [0,1]$ such that

$$\chi(u) = \begin{cases} 1 & \text{if } 0 \leq u \leq 1 \\ 0 & \text{if } u \geq 2 \end{cases}$$

For every $k \geq 1$, let $a_k = \max\left(1, \sup_{u \geq 0} \left|\chi^{(k)}(u)\right|\right)$. For any $0 < \xi \leq 1$, let us define the "truncated" smooth family of vector fields $(X^\xi_\mu)_{\mu \in \mathbb{R}^s}$ by

$$\forall (x,\mu) \in \mathbb{R}^n \times \mathbb{R}^s, X^\xi(x,\mu) = Ax + \chi\left(\frac{\|(x,\mu)\|^2}{\xi^2}\right)\theta(x,\mu)$$

where $\theta(x,\mu) = X(x,\mu) - Ax$. We now state a claim about X^ξ.

Claim 1. *There exists a sequence of constants $(c_k)_{k \geq 1}$, $c_k \geq 1$, independant of X, (F,G) and r, such that for every $0 < \xi \leq \min(1, r/\sqrt{2})$,*

1. *X^ξ is a smooth family of vector fields satisfying the hypotheses 1, 2 and 4.*
2. *$D_x X^\xi(0,0) = A$.*
3. *The derivatives of X^ξ satisfy*

$$M_1\left(X^\xi\right) \leq c_1 \xi M_2(X,r) \tag{C.49a}$$

$$\forall k \geq 2, M_k\left(X^\xi\right) \leq c_k \xi^{2-k} M_k(X,r) \tag{C.49b}$$

Moreover, for every $\gamma \in I_A$, every $0 < \xi \leq \xi(\gamma)$ where

$$\xi(\gamma) \overset{def}{=} \min\left(\frac{1}{c_1 C_1}\frac{d_A(\gamma)}{M(A)\bar{M}_2(X,r)}, \frac{r}{\sqrt{2}}\right) \in \left]0, \min(1, r/\sqrt{2})\right] \tag{C.50}$$

every $\mu \in \mathbb{R}^s$ such that $\|\mu\| \leq \xi/2$, every $0 < \eta \leq \xi/2$ and every $0 < \delta \leq \frac{\eta}{CM(A)\sigma(A)}$, the following equality holds:

$$W^{s,\gamma}_\eta(0, X_\mu) \cap B_{\mathbb{R}^n}(0,\delta) = W^{s,\gamma}\left(0, X^\xi_\mu\right) \cap B_{\mathbb{R}^n}(0,\delta) \tag{C.51}$$

Proof of claim 1. Fix $0 < \xi \leq \min(1, r/\sqrt{2})$. Let $(x,\mu) \in \mathbb{R}^n \times \mathbb{R}^s$. By definition of χ,

$$X^\xi(x,\mu) = \begin{cases} X(x,\mu) & \text{if } \|(x,\mu)\| \leq \xi \\ Ax & \text{if } \|(x,\mu)\| \geq \xi\sqrt{2} \end{cases} \tag{C.52}$$

It follows from (C.52) that $D_x X^\xi(0,0) = A$ and X^ξ satisfies the hypotheses 1, 2 and 4.

We are now going to prove estimates (C.49). According to (C.52), we only need estimates on the derivatives of X^ξ on $B_{\mathbb{R}^n \times \mathbb{R}^s}(0, \xi\sqrt{2})$. As in the proof of Lemma C.16 (see (C.36)), we introduce a notation to avoid clutter with constants independant of X, ξ, r, x and μ in the following estimates: for any real positive functions δ_1, δ_2 depending on (X, ξ, r, x, μ) where $0 < \xi \leq \min(1, r/\sqrt{2})$ and $(x,\mu) \in B(0, \xi\sqrt{2})$, we define the order relation $\lesssim$ by

$$\delta_1 \lesssim \delta_2 \iff \exists C > 0, \quad \delta_1 \leq C\delta_2 \tag{C.53}$$

Using $\theta(0,0) = 0$, $D\theta(0,0) = 0$ and $D^k\theta = D^k X$ for all $k \geq 2$, if follows from the mean value theorem that

$$\begin{aligned} \|\theta(x,\mu)\| &\lesssim \xi^2 M_2(X,r) \\ |||D\theta(x,\mu)||| &\lesssim \xi M_2(X,r) \\ \forall k \geq 2, M_k(\theta) &= M_k(X,r) \end{aligned} \tag{C.54}$$

For every $(x,\mu) \in B(0, \xi\sqrt{2})$, let

$$N^\xi(x,\mu) = \frac{\|(x,\mu)\|^2}{\xi^2}$$

and let $\chi^\xi = \chi \circ N^\xi$. Using the standard Faà di Bruno's formula, we obtain, for all $j \geq 1$,

$$\left|\left|\left|D^j \chi^\xi(x,\mu)\right|\right|\right| \lesssim \xi^{-j} \tag{C.55}$$

Using estimates (C.54) and (C.55), we get

$$\left|\left|\left|DX^{\xi}(x,\mu) - A\right|\right|\right| \precsim \xi^{-1}\xi^{2}M_2\left(X,r\right) + \xi M_2\left(X,r\right) \precsim \xi M_2\left(X,r\right)$$

Since $A = D_x X^{\xi}(0,0)$, it follows that estimate (C.49a) holds true for some constant $c_1 \geq 1$ independant of X, (F,G), ξ and r. Using Leibniz formula and estimates (C.54), (C.55), we obtain, for all $k \geq 2$,

$$\left|\left|\left|D^k(\chi^{\xi}\theta)(x,\mu)\right|\right|\right| \precsim \xi^{2-k}M_k\left(X,r\right)$$

Since $D^k(\chi^{\xi}\theta) = D^k X^{\xi}$ for all $k \geq 2$, it follows that (C.49b) holds true for some constant $c_k \geq 1$ independant of X, (F,G), ξ and r.

Now, let us fix $\gamma \in I_A$. Let $0 < \xi \leq \xi(\gamma)$ (see (C.50)). According to (C.49a), condition (C.7) is satisfied for γ and X^{ξ} so according to item 2 of Proposition C.8, we obtain, for every $\mu \in \mathbb{R}^s$, every $\eta > 0$ and every $0 < \delta \leq \frac{\eta}{CM(A)\sigma(A)}$,

$$W_{\eta}^{s,\gamma}(0,X_{\mu}^{\xi}) \cap B_{\mathbb{R}^n}(0,\delta) = W^{s,\gamma}\left(0,X_{\mu}^{\xi}\right) \cap B_{\mathbb{R}^n}(0,\delta)$$

According to (C.52), for every $\mu \in \mathbb{R}^s$ such that $\|\mu\| \leq \xi/2$ and every $0 < \eta \leq \xi/2$,

$$W_{\eta}^{s,\gamma}(0,X_{\mu}) = W_{\eta}^{s,\gamma}(0,X_{\mu}^{\xi})$$

Hence, (C.51) holds true. This concludes the proof of claim 1. □

According to claim 1, for every $0 < \xi \leq \min(1, r/\sqrt{2})$, X^{ξ} is a smooth family of vector fields satisfying the hypotheses 1, 2 and 4 and (F,G) is a partially hyperbolic splitting of $D_x X^{\xi}(0,0) = A$. Denote by ϕ^{ξ} the smooth map associated with X^{ξ} and (F,G) by Proposition C.8 (well defined for all ξ small enough by (C.49a)).

Let $\gamma, \gamma' \in I_A$. Let $\xi = \min(\xi(\gamma), \xi(\gamma'))$ (see (C.50)). Estimate (C.49a) implies that γ and γ' satisfy (C.7) for X^{ξ}. In particular ϕ^{ξ} is well defined. Let $\mu \in \mathbb{R}^s$ such that $\|\mu\| \leq \xi/2$, $0 < \eta \leq \xi/2$ and $0 < \delta \leq \frac{\eta}{CM(A)\sigma(A)}$. We have

$$\begin{aligned} W_{\eta}^{s,\gamma}(0,X_{\mu}) \cap B_{\mathbb{R}^n}(0,\delta) &= W^{s,\gamma}\left(0,X_{\mu}^{\xi}\right) \cap B_{\mathbb{R}^n}(0,\delta) && \text{using (C.51)} \\ &= \operatorname{Graph}\left(\phi_{\mu}^{\xi}\right) \cap B_{\mathbb{R}^n}(0,\delta) && \text{using item 1 of Proposition C.8} \end{aligned}$$

and since the above computation holds true for γ' as well, it follows that

$$W_{\eta}^{s,\gamma}(0,X_{\mu}) \cap B_{\mathbb{R}^n}(0,\delta) = W_{\eta}^{s,\gamma'}(0,X_{\mu}) \cap B_{\mathbb{R}^n}(0,\delta)$$

and finally,

$$W^{s,\gamma}\left(0,X_{\mu}\right) = W^{s,\gamma'}\left(0,X_{\mu}\right) \tag{C.56}$$

It follows that item 1 of Theorem C.19 holds true.

Let

$$\tilde{\gamma} \stackrel{\text{def}}{=} \frac{\lambda_{\max}\left(A_{|F}\right) + \min\left(0, \lambda_{\min}\left(A_{|G}\right)\right)}{2}$$

One can remark that

$$d_A(\tilde{\gamma}) \geq \left(2^{n-1}\sigma\left(A\right)\right)^{-1}$$

Let

$$\tilde{\xi} \stackrel{\text{def}}{=} \min\left(\left(c_1 C_1 2^{n-1}\sigma\left(A\right)M(A)\bar{M}_2\left(X,r\right)\right)^{-1}, \frac{r}{\sqrt{2}}\right) \leq \xi(\tilde{\gamma}) \tag{C.57}$$

Let $\phi \stackrel{\text{def}}{=} \phi^{\tilde{\xi}}$. Estimate (C.49a) implies that $\tilde{\gamma}$ satisfies (C.7) for $X^{\tilde{\xi}}$ so ϕ is well defined. According to Proposition C.8 and claim 1, for every $\mu \in \mathbb{R}^s$ such that $\|\mu\| \leq \tilde{\xi}/2$, every $0 < \eta \leq \tilde{\xi}/2$ and every $0 < \delta \leq \frac{\eta}{CM(A)\sigma(A)}$,

$$W_{\eta}^{s,\tilde{\gamma}}(0,X_{\mu}) \cap B_{\mathbb{R}^n}(0,\delta) = \operatorname{Graph}\left(\phi_{\mu}\right) \cap B_{\mathbb{R}^n}(0,\delta)$$

According to (C.56), it follows that for every $\gamma \in I_A$, every $\mu \in \mathbb{R}^s$ such that $\|\mu\| \le \min(\tilde{\xi}, \xi(\gamma))/2$, every $0 < \eta \le \tilde{\xi}/2$ and every $0 < \delta \le \frac{\eta}{CM(A)\sigma(A)}$,

$$W_\eta^{s,\gamma}(0, X_\mu) \cap B_{\mathbb{R}^n}(0,\delta) = \mathrm{Graph}\left(\phi_\mu\right) \cap B_{\mathbb{R}^n}(0,\delta) \tag{C.58}$$

Hence, item 2 of Theorem C.19 holds true.

We are now going to prove estimates (C.48). Using (C.51), one can remark that for every $0 < \xi \le \tilde{\xi}$ and every $\|\mu\| \le \xi/2$,

$$\mathrm{Graph}\left(\phi_\mu\right) \cap B_{\mathbb{R}^n}(0,\delta(\xi)) = \mathrm{Graph}\left(\phi_\mu^\xi\right) \cap B_{\mathbb{R}^n}(0,\delta(\xi))$$

where

$$\delta(\xi) \stackrel{\text{def}}{=} \frac{\xi}{2CM(A)\sigma(A)}$$

It follows that for every $0 < \xi \le \tilde{\xi}$, every $\|\mu\| \le \xi/2$ and every $z \in F$ such that $\left\|z + \phi_\mu(z)\right\| < \delta(\xi)$,

$$\phi_\mu(z) = \phi_\mu^\xi(z)$$

In order to obtain the estimates about ϕ and its derivatives at a given point (z,μ), the idea is to remark that it will be the same estimates for ϕ^ξ for some well chosen $\xi = \xi(z,\mu)$. Plugging (C.49a) into (C.9a), we obtain, for every $(z,\mu) \in F \times \mathbb{R}^s$,

$$\|\phi(z,\mu)\| \le \frac{C_{1,0}}{2^{n-1}C_1}\|z\|$$

It follows from the previous estimate that for every $(z,\mu) \in F \times \mathbb{R}^s \setminus \{(0,0)\}$ such that

$$\begin{aligned}
&\|z\| < (CM(A)\sigma(A))^{-1}\min\left(\frac{1}{8c_1(C_1 + C_{1,0})2^{n-1}}\left(\sigma(A)M(A)\bar{M}_2(X,r)\right)^{-1}, \frac{r}{8\sqrt{2}\left(1 + \frac{C_{1,0}}{2^{n-1}C_1}\right)}\right)\\
&\|\mu\| < (CM(A)\sigma(A))^{-1}\min\left(\frac{1}{8c_1C_12^{n-1}}\left(\sigma(A)M(A)\bar{M}_2(X,r)\right)^{-1}, \frac{r}{8\sqrt{2}}\right)
\end{aligned} \tag{C.59}$$

the number

$$\xi(z,\mu) \stackrel{\text{def}}{=} 4CM(A)\sigma(A)\left(\left(1 + \frac{C_{1,0}}{2^{n-1}C_1}\right)\|z\| + \|\mu\|\right) \tag{C.60}$$

satisfies $0 < \xi(z,\mu) \le \tilde{\xi}$ and the following property: for every $(z',\mu') \in F \times \mathbb{R}^s$,

$$\left(\left\|z'\right\| < 2\|z\| \quad \text{and} \quad \left\|\mu'\right\| < 2\|\mu\|\right) \implies \left(\left\|z' + \phi_{\mu'}(z')\right\| < \delta(\xi(z,\mu)) \quad \text{and} \quad \left\|\mu'\right\| \le \xi(z,\mu)/2\right)$$

Let us now fix $(z,\mu) \in F \times \mathbb{R}^s \setminus \{(0,0)\}$ satisfying (C.59). According to the above arguments, the maps ϕ and $\phi^{\xi(z,\mu)}$ coincide on $B_F(0, 2\|z\|) \times B_{\mathbb{R}^s}(0, 2\|\mu\|)$, in particular all their derivatives at the point (z,μ) coincide. Hence, we get

$$\begin{aligned}
\|\phi(z,\mu)\| &= \left\|\phi^{\xi(z,\mu)}(z,\mu)\right\| \\
&\le C_{1,0}\sigma(A)M(A)M_1\left(X^{\xi(z,\mu)}\right)\|z\| && \text{using (C.9a)}\\
&\le c_1C_{1,0}\sigma(A)M(A)\xi(z,\mu)M_2(X,r)\|z\| && \text{using (C.49a)}\\
&\le 4c_1C_{1,0}C\left(1 + \frac{C_{1,0}}{2^{n-1}C_1}\right)\sigma(A)^2M(A)^2M_2(X,r)\|z\|(\|z\| + \|\mu\|) && \text{using (C.60)}
\end{aligned}$$

so estimate (C.48a) holds true (for some different constants). By the same arguments, we obtain estimates (C.48b) and (C.48c). Using (C.49b), we get, for all $k \geq 2$,

$$\begin{aligned}\bar{M}_{k+1}\left(X^{\tilde{\xi}}\right) &\leq \max\left(1, c_{k+1}\tilde{\xi}^{1-k} M_{k+1}(X)\right) \\ &\leq c_{k+1}\tilde{\xi}^{1-k}\bar{M}_{k+1}(X, r)\end{aligned}$$

Using (C.57), we get, for all $k \geq 2$,

$$\begin{aligned}\bar{M}_{k+1}\left(X^{\tilde{\xi}}\right) &\leq c_{k+1}\max\left(\left(c_1 C_1 2^{n-1}\sigma(A)\, M(A)\bar{M}_2(X, r)\right)^{k-1}, \left(\frac{\sqrt{2}}{r}\right)^{k-1}\right)\bar{M}_{k+1}(X, r) \\ &\leq c_{k+1}\max\left(\left(c_1 C_1 2^{n-1}\right)^{k-1}, \sqrt{2}^{k-1}\right)\max\left(\sigma(A)\, M(A)\bar{M}_2(X, r), r^{-1}\right)^{k-1}\bar{M}_{k+1}(X, r)\end{aligned}$$

Plugging this estimate into (C.9d) applied to $\phi^{\tilde{\xi}} = \phi$, it follows that (C.48d) holds true. This concludes the proof of Theorem C.19. □

C.4 Estimates for vector fields vanishing on submanifolds

Fix $n \in \mathbb{N}$ and a linear subspace G of $\mathbb{R}^n$. Denote by $\|.\|$ the Euclidean norm on $\mathbb{R}^n$. Let Ω be an open neighbourhood of 0 in $\mathbb{R}^n$. Fix a smooth vector field $Y : \Omega \to \mathbb{R}^n$. Assume that

1. Y vanishes on $\Omega_0 := \Omega \cap G$;
2. For every $\mu \in \Omega_0$, there exists a decomposition $F_\mu \oplus G = \mathbb{R}^n$ stabilized by $A_\mu := DY(\mu)$ and such that
$$\lambda_{\max}\left((A_\mu)_{|F_\mu}\right) < 0$$

   For every $\mu \in \Omega_0$, let
$$\beta(\mu) \stackrel{\text{def}}{=} \min\left(1, \left|\lambda_{\max}\left((A_\mu)_{|F_\mu}\right)\right|\right)^{n-1} \quad \text{(see (C.1a))} \tag{C.61}$$

Let $\mathcal{F}^s$ be the stable foliation associated with the contracted subspace G on which the vector field Y vanishes, that is, the partition

$$\mathcal{F}^s \stackrel{\text{def}}{=} \left\{W^s(\mu, Y) \mid \mu \in \Omega_0\right\}$$

where the stable manifolds $W^s(\mu, Y)$ are called the leaves of the foliation $\mathcal{F}^s$.

For every integer $k \geq 2$, every $\mu \in \Omega_0$ and every $r > 0$ such that $\overline{B_{\mathbb{R}^n}(\mu, r)} \subset \Omega$, let

$$M_k(Y, \mu, r) \stackrel{\text{def}}{=} \sup_{2 \leq j \leq k}\ \sup_{y \in \overline{B_{\mathbb{R}^n}(\mu, r)}} \left\|\left|D^j Y(y)\right|\right\|$$

and

$$\bar{M}_k(Y, \mu, r) \stackrel{\text{def}}{=} \max\left(1, M_k(Y, \mu, r)\right)$$

Our next theorem states that in this context, the foliation $\mathcal{F}^s$ can be locally smoothly straightened in the neighbourhood of any point $\mu \in \Omega_0$.

Theorem C.22 (Local straightening of the stable foliation of a vector field)**.** *There exists two positive constants $C_3 \geq C_3' \geq 1$, a sequence of positive constants $(C_{3,k})_{k\geq 2}$ and a sequence of integers $(N_k)_{k\geq 2}$ (all independant of Y) such that for every map $r : \Omega_0 \to]0, 1]$ satisfying*

$$\forall \mu \in \Omega_0, \quad \overline{B_{\mathbb{R}^n}(\mu, r(\mu))} \subset \Omega$$

there exists

- *two families $(U_\mu)_{\mu\in\Omega_0}$ and $(V_\mu)_{\mu\in\Omega_0}$ of open sets of $\mathbb{R}^n$;*

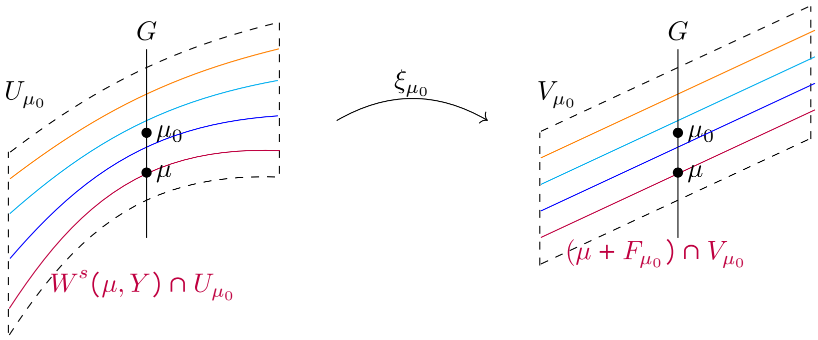

Figure C.2 – The local coordinate system ξ_{μ_0} straightens the stable foliation induced by $\mathcal{F}^s$ on U_{μ_0}.

- *a family of smooth diffeomorphisms*

$$(\xi_\mu : U_\mu \to V_\mu)_{\mu \in \Omega_0}$$

satisfying the following properties. Given $\mu_0 \in \Omega_0$:

1. *Both U_{μ_0} and V_{μ_0} are neighbourhoods of μ_0. More precisely, they both contain the open ball $B_{\mathbb{R}^n}\left(\mu_0, R_{\mu_0}\right)$ where*

$$R_{\mu_0} \stackrel{def}{=} \frac{\beta(\mu_0)}{C_3 m\left(F_{\mu_0}, G\right)^2 M(A_{\mu_0})} \min\left(\frac{\beta(\mu_0)}{M(A_{\mu_0})\bar{M}_2\left(Y, \mu_0, r(\mu_0)\right)}, r(\mu_0)\right) \tag{C.62}$$

and $M(A_{\mu_0})$ is defined by (C.2b)*.*

2. *$\mathcal{F}^s$ foliates U_{μ_0} and ξ_{μ_0} is a local coordinate system straightening the stable foliation $\mathcal{F}^s$ (see figure C.2). More precisely,*

$$U_{\mu_0} = \bigsqcup_{\mu \in \Omega_0 \cap U_{\mu_0}} W^s(\mu, Y) \cap U_{\mu_0}$$

and, for every $\mu \in \Omega_0 \cap U_{\mu_0}$,

$$\xi_{\mu_0}\left(W^s(\mu, Y) \cap U_{\mu_0}\right) = \left(\mu + F_{\mu_0}\right) \cap V_{\mu_0}$$

Moreover, for every $\mu \in \Omega_0 \cap U_{\mu_0}$,

$$W^s(\mu, Y) \cap U_{\mu_0} = W^{s,\gamma}_\eta(\mu, Y) \cap U_{\mu_0}$$

where

$$\gamma = -\frac{\left|\lambda_{max}\left((A_{\mu_0})_{|F_{\mu_0}}\right)\right|}{2}$$
$$\eta = \frac{1}{C_3'} \min\left(\frac{\beta(\mu_0)}{M(A_{\mu_0})\bar{M}_2\left(Y, \mu_0, r(\mu_0)\right)}, r(\mu_0)\right)$$

3. *Identifying $\mathbb{R}^n$ and $F_{\mu_0} \times G$, the local coordinate system has the following form:*

$$\xi_{\mu_0}(z, \mu) = (z, \mu) + \left(0, \tilde{\xi}_{\mu_0}(z, \mu)\right)$$

where $\tilde{\xi}_{\mu_0}(0, \mu) = 0$.

4. *For every $0 < \epsilon \leq 1$, ξ_{μ_0} restricted to $B_{\mathbb{R}^n}\left(\mu_0, \epsilon R_{\mu_0}\right)$ is ϵ-close to the identity with respect to the*

C^1-norm:

$$\left\|\xi_{\mu_0} - \mathrm{Id}\right\|_{C^1} \leq \epsilon \quad \textit{in restriction to } B_{\mathbb{R}^n}\left(\mu_0, \epsilon R_{\mu_0}\right)$$
$$\left\|\xi_{\mu_0}^{-1} - \mathrm{Id}\right\|_{C^1} \leq \epsilon \quad \textit{in restriction to } B_{\mathbb{R}^n}\left(\mu_0, \epsilon R_{\mu_0}\right)$$

5. *The C^k-norms have a sub-polynomial growth with respect to $\beta(\mu_0)^{-1}$: more precisely, for every $k \geq 2$,*

$$\left\|\xi_{\mu_0}\right\|_{C^k}, \left\|\xi_{\mu_0}^{-1}\right\|_{C^k} \leq C_{3,k}\left(\frac{M(A_{\mu_0})\bar{M}_{k+1}\left(Y, \mu_0, r(\mu_0)\right)}{\beta(\mu_0) r(\mu_0)}\right)^{N_k}$$

6. *For every $\mu_1 \in \Omega_0$, ξ_{μ_0} and ξ_{μ_1} "coincide" on $U_{\mu_0} \cap U_{\mu_1}$ modulo the choice of the direction on which the stable manifolds are projected. More precisely, if we denote by π_μ the linear projection along G onto F_μ for every $\mu \in \Omega_0$, we have*

$$\xi_{\mu_0} - \xi_{\mu_1} = \pi_{\mu_0} - \pi_{\mu_1} \quad \textit{in restriction to } U_{\mu_0} \cap U_{\mu_1}$$

Remark C.23. The charts $(\xi_{\mu_0})_{\mu_0 \in \Omega_0}$ do not form a foliation coordinate atlas because ξ_{μ_0} straightens the leaf $W^s(\mu, Y) \cap U_{\mu_0}$ onto the affine subspace $\mu + F_{\mu_0}$ which depends on μ_0. Nevertheless, identifying $\mathbb{R}^n$ and $F_{\mu_0} \times G$, one only needs to compose ξ_{μ_0} with $(\pi_{|F_{\mu_0}}, \mathrm{Id}_G)$ where π denotes a linear projection along G onto a fixed complement of G (for example $G^\perp$) to obtain a foliation coordinate atlas. This would change the estimates on the norms of the derivatives of $\xi_{\mu_0}^{-1}$ by a factor $m\left(F_{\mu_0}, G\right)$ and would make ξ_{μ_0} close to $(\pi_{|F_{\mu_0}}, \mathrm{Id}_G)$ in item 4. We did not make this choice for two reasons: there is no canonical complement of G and we want to obtain the fact that ξ_{μ_0} can be made arbitrarily close to Id with respect to the C^1-norm.

Proof. Presentation of the proof as a consequence of Theorem C.19. Fix a map $r : \Omega_0 \to]0, 1]$ satisfying

$$\forall \mu \in \Omega_0, \quad \overline{B(\mu, r(\mu))} \subset \Omega$$

Fix $\mu_0 \in \Omega_0$. Even if it means translating the vector field Y, one can assume that $\mu_0 = 0$. So, we will prove the desired result in the neighbourhood of 0. Recall that we want to straighten, for all $\mu \in G$ small enough, the local stable manifold $W^s_\eta(\mu, Y)$ for some η depending on μ. This leads us to define, for every $\mu \in G$ and $x \in \mathbb{R}^n$ such that $\mu + x \in \Omega$,

$$X(x, \mu) \stackrel{\text{def}}{=} X_\mu(x) = Y(\mu + x)$$

We will prove later on that the local stable manifolds of Y coincide with the local γ-stable manifolds for some $\gamma < 0$ well chosen (see (C.66)). Hence, we focus on the description of those local γ-stable manifolds. The local γ-stable manifold of $\mu \in \Omega_0$ for Y is exactly the translation of the local γ-stable manifold of 0 for X_μ by $t_\mu : x \mapsto \mu + x$. More precisely, for every $\mu \in \Omega_0$ and every $0 < \delta \leq r(\mu)$,

$$\mu + W^{s,\gamma}_\delta(0, X_\mu) = W^{s,\gamma}_\delta(\mu, Y) \tag{C.63}$$

According to the above equation, we are left to straighten the local γ-stable manifolds $W^{s,\gamma}_\delta(0, X_\mu)$ for μ small enough.

Construction of ξ_{μ_0}. We are now going to extend X so that we can apply Theorem C.19. One can remark that X is well defined on a neighbourhood of the closed ball $\overline{B_{\mathbb{R}^n \times G}((0,0), r(\mu_0)/2)}$. Multiplying X by a smooth plateau map equal to 1 on $\overline{B_{\mathbb{R}^n \times G}((0,0), r(\mu_0)/2)}$ and vanishing outside of a small neighbourhood of $\overline{B_{\mathbb{R}^n \times G}((0,0), r(\mu_0)/2)}$, we obtain a smooth family of vector fields (as defined in subsection C.3.1) defined on $\mathbb{R}^n \times G$, still denoted by X. With this new smooth family of vector fields, equation (C.63) implies: for every $\mu \in G$ such that $\|\mu\| \leq r(\mu_0)/4$ and every $0 < \delta \leq r(\mu_0)/4$, we have

$$\mu + W^{s,\gamma}_\delta(0, X_\mu) = W^{s,\gamma}_\delta(\mu, Y) \tag{C.64}$$

By hypothesis 1 on Y, for every $\mu \in G$, $X_\mu(0,0) = (0,0)$. By hypothesis 2 on Y, (F_{μ_0}, G) is a partially hyperbolic splitting of $A_{\mu_0} = D_{z,v}X(0,0,0)$ so $X := (X_\mu)_{\mu \in G}$ is a smooth family of vector fields satisfying the hypotheses 1 and 2. Using the estimate

$$\forall k \geq 2, \quad \bar{M}_k\left(X, r(\mu_0)/2\right) \leq \bar{M}_k\left(Y, \mu_0, r(\mu_0)\right) \quad \text{(see (C.45))}$$

it follows from Theorem C.19 applied to (X, F_{μ_0}, G) with $r = r(\mu_0)/2$ that there exists a smooth map

$$\phi\colon \left| \begin{array}{ccc} F_{\mu_0} \times G & \to & G \\ (z,\mu) & \mapsto & \phi_\mu(z) \end{array} \right.$$

such that for every $\mu \in B_G\left(0, \tilde{\eta}\right)$, every $0 < \eta \leq \tilde{\eta}$ and every $0 < \delta \leq \frac{\eta\beta(\mu_0)}{CM(A_{\mu_0})}$,

$$W_\eta^{s,\gamma}(0, X_\mu) \cap B_{\mathbb{R}^n}(0,\delta) = \operatorname{Graph}\left(\phi_\mu\right) \cap B_{\mathbb{R}^n}(0,\delta) \tag{C.65}$$

where

$$\gamma \stackrel{\text{def}}{=} -\frac{\left|\lambda_{\max}\left((A_{\mu_0})_{|F_{\mu_0}}\right)\right|}{2} \tag{C.66}$$

$$\tilde{\eta} \stackrel{\text{def}}{=} \frac{1}{4C_2} \min\left(\frac{\beta(\mu_0)}{M(A_{\mu_0})\bar{M}_2\left(Y, \mu_0, r(\mu_0)\right)}, r(\mu_0)\right) \quad \text{(see (C.2b))} \tag{C.67}$$

Moreover, for every $(z,\mu) \in B_{F_{\mu_0}}\left(0, \tilde{\delta}\right) \times B_G\left(0, \tilde{\delta}\right)$,

$$\|\phi(z,\mu)\| \leq C_{2,0}\beta(\mu_0)^{-2} M(A_{\mu_0})^2 \bar{M}_2\left(Y, \mu_0, r(\mu_0)\right)\left(\|z\| + \|\mu\|\right)\|z\| \tag{C.68a}$$

$$|||D_z\phi(z,\mu)||| \leq C_{2,1}\beta(\mu_0)^{-2} M(A_{\mu_0})^2 \bar{M}_2\left(Y, \mu_0, r(\mu_0)\right)\left(\|z\| + \|\mu\|\right) \tag{C.68b}$$

$$\left|\left|\left|D_\mu\phi(z,\mu)\right|\right|\right| \leq C_{2,1}\beta(\mu_0)^{-1} M(A_{\mu_0}) \bar{M}_2\left(Y, \mu_0, r(\mu_0)\right)\|z\| \tag{C.68c}$$

where

$$\tilde{\delta} \stackrel{\text{def}}{=} \frac{\beta(\mu_0)}{4C_2 M(A_{\mu_0})} \min\left(\frac{\beta(\mu_0)}{M(A_{\mu_0})\bar{M}_2\left(Y, \mu_0, r(\mu_0)\right)}, r(\mu_0)\right)$$

and more generally, using the norm $\|(z,\mu)\| = \|z\| + \|\mu\|$ on $F_{\mu_0} \times \mathbb{R}^s$, we have, for all $k \geq 2$,

$$\left|\left|\left|D^k\phi(z,\mu)\right|\right|\right| \leq C_{2,k}\left[\left(\frac{M(A_{\mu_0})}{\beta(\mu_0)}\right)^2 \times \max\left(\frac{M(A_{\mu_0})\bar{M}_2\left(Y, \mu_0, r(\mu_0)\right)}{\beta(\mu_0)}, \frac{2}{r(\mu_0)}\right)^{k-1} \bar{M}_{k+1}\left(Y, \mu_0, r(\mu_0)\right)\right]^{2k-1} \tag{C.68d}$$

Let

$$\psi\colon \left| \begin{array}{ccc} F_{\mu_0} \times G & \to & F_{\mu_0} \times G \\ (z,\mu) & \mapsto & (z, \mu + \phi(z,\mu)) \end{array} \right. \tag{C.69}$$

One can rewrite (C.69) as $\psi = \operatorname{Id} + h$ where $h(z,\mu) = (0, \phi(z,\mu))$. According to (C.68a), (C.68b), (C.68c) and the mean value theorem, there exists a constant $K \geq 1$ (independant of Y, r and μ_0) such that for every $0 < \epsilon \leq 1$ and every $(z,\mu) \in B_{F_{\mu_0}}\left(0, \frac{\epsilon\tilde{\delta}}{K}\right) \times B_G\left(0, \frac{\epsilon\tilde{\delta}}{K}\right)$, we have

$$|||Dh(z,\mu)||| \leq |||D\phi(z,\mu)||| \leq \frac{\epsilon}{2} \tag{C.70}$$

and

$$V_{\mu_0} \subset B_{\mathbb{R}^n}\left(0, \frac{\hat{\eta}}{2}\right) \tag{C.71a}$$

$$U_{\mu_0} \subset B_{\mathbb{R}^n}\left(0, \frac{\hat{\eta}}{2}\right) \tag{C.71b}$$

where

$$\begin{aligned}
V_{\mu_0} &\stackrel{\text{def}}{=} L\left(\tilde{V}_{\mu_0}\right)\\
\tilde{V}_{\mu_0} &\stackrel{\text{def}}{=} B_{F_{\mu_0}}\left(0, \frac{\tilde{\delta}}{K}\right) \times B_G\left(0, \frac{\tilde{\delta}}{K}\right)\\
U_{\mu_0} &\stackrel{\text{def}}{=} L\left(\tilde{U}_{\mu_0}\right)\\
\tilde{U}_{\mu_0} &\stackrel{\text{def}}{=} \psi\left(B_{F_{\mu_0}}\left(0, \frac{\tilde{\delta}}{K}\right) \times B_G\left(0, \frac{\tilde{\delta}}{K}\right)\right)\\
\hat{\eta} &\stackrel{\text{def}}{=} \frac{\tilde{\eta}\beta(\mu_0)}{CM(A_{\mu_0})}
\end{aligned}$$

and $L : F_{\mu_0} \times G \to \mathbb{R}^n$ is the canonical isomorphism $(z, \mu) \mapsto z + \mu$. According to Lemma C.24, we have

$$|||L||| \le 1, \quad \left|\left|\left|L^{-1}\right|\right|\right| \le m\left(F_{\mu_0}, G\right), \tag{C.72}$$

the linear spaces $\mathbb{R}^n$, F_{μ_0} and G being equipped with the Euclidean norm and the linear space $F_{\mu_0} \times G$ being equipped with $\|(z, \mu)\| = \|z\| + \|\mu\|$. Using (C.70) with $\epsilon = 1$, we get that ψ is injective on $\tilde{V}_{\mu_0}$ and according to the global inverse function theorem, ψ is invertible on $\tilde{V}_{\mu_0}$. Let us define $\tilde{\xi}_{\mu_0}$ as the local inverse of ψ:

$$\tilde{\xi}_{\mu_0} \stackrel{\text{def}}{=} \left(\psi_{|\tilde{V}_{\mu_0}}\right)^{-1} : \tilde{U}_{\mu_0} \to \tilde{V}_{\mu_0}$$

and let

$$\xi_{\mu_0} = L \circ \tilde{\xi}_{\mu_0} \circ L^{-1} : U_{\mu_0} \to V_{\mu_0} \tag{C.73}$$

Proof of item 2. The first thing to remark is the fact that ψ is constructed so that it maps straight lines to the graphs induced by ϕ: more precisely, we have, for every $\mu \in G$,

$$\psi\left(F_{\mu_0} \times \{\mu\}\right) = (0, \mu) + \operatorname{Graph} \phi_\mu \tag{C.74}$$

Identifying $\operatorname{Graph} \phi_\mu \subset F_{\mu_0} \times G$ with its image in $\mathbb{R}^n$, we have, for every $\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)$,

$$\begin{aligned}
\left(\mu + \operatorname{Graph} \phi_\mu\right) \cap U_{\mu_0} &= \mu + \left(\operatorname{Graph} \phi_\mu\right) \cap \underbrace{\left(U_{\mu_0} - \mu\right)}_{\subset B_{\mathbb{R}^n}(0, \hat{\eta})} && \text{using (C.71b)}\\
&= \left(\mu + W_{\tilde{\eta}}^{s,\gamma}(0, X_\mu)\right) \cap U_{\mu_0} && \text{using (C.65)}
\end{aligned}$$

so, using (C.64), we get that

$$\left(\mu + \operatorname{Graph} \phi_\mu\right) \cap U_{\mu_0} = W_{\tilde{\eta}}^{s,\gamma}(\mu, Y) \cap U_{\mu_0} \tag{C.75}$$

Since the family $(W_{\tilde{\eta}}^{s,\gamma}(\mu, Y))_{\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)}$ is pairwise disjoint, the family

$$\left(\left(\mu + \operatorname{Graph} \phi_\mu\right) \cap U_{\mu_0}\right)_{\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)}$$

is also pairwise disjoint. The preceding remark allows us to write

$$
\begin{aligned}
U_{\mu_0} &= \xi_{\mu_0}^{-1}(V_{\mu_0}) \\
&= \xi_{\mu_0}^{-1}\left(\bigsqcup_{\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)} (\mu + F_{\mu_0}) \cap V_{\mu_0} \right) \\
&= \bigsqcup_{\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)} \xi_{\mu_0}^{-1}\left((\mu + F_{\mu_0}) \cap V_{\mu_0} \right) && \text{by injectivity of } \xi_{\mu_0}^{-1} \\
&\subset \bigsqcup_{\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)} (\mu + \operatorname{Graph} \phi_\mu) \cap U_{\mu_0} && \text{using (C.74)} \\
&\subset \bigsqcup_{\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)} W_{\tilde{\eta}}^{s,\gamma}(\mu, Y) \cap U_{\mu_0} && \text{using (C.75)} \\
&\subset U_{\mu_0}
\end{aligned}
$$

where $\sqcup$ denotes a disjoint union. It follows that all the preceding inclusions must be equalities. As consequences, we get that the family $\left(W_{\tilde{\eta}}^{s,\gamma}(\mu, Y)\right)_{\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)}$ foliates U_{μ_0}:

$$
U_{\mu_0} = \bigsqcup_{\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)} W_{\tilde{\eta}}^{s,\gamma}(\mu, Y) \cap U_{\mu_0} \tag{C.76}
$$

and for every $\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)$,

$$
\xi_{\mu_0}^{-1}\left((\mu + F_{\mu_0}) \cap V_{\mu_0} \right) = W_{\tilde{\eta}}^{s,\gamma}(\mu, Y) \cap U_{\mu_0} \tag{C.77}
$$

Let $\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)$. Since any orbit contained in $W^s(\mu, Y)$ must eventually enter U_{μ_0}, it follows from (C.76) and the fact that such an orbit converges to μ, that

$$
W_{\tilde{\eta}}^{s,\gamma}(\mu, Y) \cap U_{\mu_0} = W^s(\mu, Y) \cap U_{\mu_0} \tag{C.78}
$$

According to (C.77) and (C.78), we have, for every $\mu \in B_G\left(0, \frac{\tilde{\delta}}{K}\right)$,

$$
\xi_{\mu_0}^{-1}\left((\mu + F_{\mu_0}) \cap V_{\mu_0} \right) = W^s(\mu, Y) \cap U_{\mu_0}
$$

so item 2 holds true.

Proof of item 3. This is a direct consequence of (C.69) and the fact that $\phi(0, \mu) = 0$ (see (C.68a)).

Proof of items 1 and 4. We have $D\tilde{\xi}_{\mu_0}^{-1} = \operatorname{Id} + Dh$ so

$$
D\tilde{\xi}_{\mu_0} = \operatorname{Id} + \sum_{k \geq 1} (-1)^k (Dh)^k \tag{C.79}
$$

Using (C.70), it follows that for every $0 < \epsilon \leq 1$ and every $(z, \mu) \in B_{F_{\mu_0}}\left(0, \frac{\epsilon\tilde{\delta}}{K}\right) \times B_G\left(0, \frac{\epsilon\tilde{\delta}}{K}\right)$,

$$
\left|\left|\left| D\tilde{\xi}_{\mu_0}(z, \mu) - \operatorname{Id} \right|\right|\right| \leq \epsilon \tag{C.80}
$$

According to (C.71b), (C.80), the mean value theorem and the fact that $\phi(0, 0) = (0, 0)$, there exists a contant $K' \geq K$ (independant of Y, r and μ_0)) such that

$$
B_{F_{\mu_0}}\left(0, \frac{\tilde{\delta}}{K'}\right) \times B_G\left(0, \frac{\tilde{\delta}}{K'}\right) \subset \tilde{U}_{\mu_0} \cap \tilde{V}_{\mu_0}
$$

Then, we use (C.72) to obtain that, for every $0 < \epsilon \leq 1$,

$$B_{\mathbb{R}^n}(0, \epsilon R_{\mu_0}) \subset L\left(B_{F_{\mu_0}}\left(0, \frac{\epsilon}{m\left(F_{\mu_0}, G\right)}\frac{\tilde{\delta}}{K'}\right) \times B_G\left(0, \frac{\epsilon}{m\left(F_{\mu_0}, G\right)}\frac{\tilde{\delta}}{K'}\right)\right) \subset U_{\mu_0} \cap V_{\mu_0}$$

where R_{μ_0} is defined by (C.62) for some constant C_3 large enough (independant of Y, r and μ_0). The above inclusion with $\epsilon = 1$ proves that item 1 holds true. Even if it means taking C_3 larger, item 4 holds true as well, using (C.70), (C.72), (C.80), the mean value theorem and the fact that $\phi(0,0) = (0,0)$.

Proof of item 5. This is a consequence of (C.79), (C.69), (C.68d), (C.72) and the fact that the sequence $(\bar{M}_k(Y, \mu_0, r(\mu_0)))_{k\geq 2}$ is increasing.

Proof of item 6. One can construct, for any $\mu \in \Omega_0$, a local coordinate system ξ_μ satisfying items 1-5 in the same way than ξ_{μ_0} (and with the same constants). Let $\mu_1 \in \Omega_0$. By construction of ξ_{μ_0} and ξ_{μ_1} (see (C.69) and (C.73)), for all $\mu \in \Omega_0 \cap U_{\mu_0} \cap U_{\mu_1}$ and for all $y \in W^{s,\gamma}_{\tilde{\eta}}(\mu, Y) \cap U_{\mu_0} \cap U_{\mu_1}$, we have

$$\begin{cases} \xi_{\mu_0}(y) & = \mu + \pi_{\mu_0}(y) \\ \xi_{\mu_1}(y) & = \mu + \pi_{\mu_1}(y) \end{cases}$$

where π_μ denotes the linear projection along G onto F_μ. It follows that for all $y \in U_{\mu_0} \cap U_{\mu_1}$, we have

$$\xi_{\mu_0}(y) - \xi_{\mu_1}(y) = \pi_{\mu_0}(y) - \pi_{\mu_1}(y)$$

so item 6 holds true. This completes the proof of Theorem C.22. □

C.5 Some linear algebra lemmas

We recall here some elementary facts of linear algebra used throughout this appendix. We refer to section C.2 for the notations.

Lemma C.24. *Let $n \in \mathbb{N}$ and let F, G be two linear subspaces of $\mathbb{R}^n$ such that $F \cap G = \{0\}$. For every $x = x_F + x_G \in F \oplus G$,*

$$\|x_F\|_2 + \|x_G\|_2 \leq m(F, G)\|x\|_2$$

Proof. Recall that $m(F,G) = \left(\frac{2}{1-\cos\sphericalangle(F,G)}\right)^{\frac{1}{2}}$. Let $x = x_F + x_G \in F \oplus G$. It is sufficient to prove the straightforward inequality

$$\frac{a^2 + b^2 + 2ab}{a^2 + b^2 - 2abc} \leq \frac{2}{1-c}$$

where $a = \|x_F\|_2^2$, $b = \|x_G\|_2^2$ and $c = \cos\sphericalangle(F,G) \in [0,1[$. □

Lemma C.25. *Let $n \in \mathbb{N}$ and $A \in \mathcal{M}_n(\mathbb{R})$. Let $\mathbb{R}^n = \oplus_{1\leq i\leq r} E_i$ be the decomposition of $\mathbb{R}^n$ as the direct sum of the generalized eigenspaces of A. Accordingly, for any $x \in \mathbb{R}^n$, we will use the decomposition $x = \sum_{i=1}^r x_i$ where $x_i \in E_i$. The following control holds true for every $x \in \mathbb{R}^n$:*

$$\sum_{i=1}^{r} \|x_i\|_2 \leq m(A)\|x\|_2$$

Proof. The proof is a straightforward induction on the number r of generalized eigenspaces of A, using Lemma C.24. □

Lemma C.26. *Let $n \in \mathbb{N}$ and $A \in \mathcal{M}_n(\mathbb{R})$. We have, for every $\alpha > \lambda_{max}(A)$ and every $s \geq 0$,*

$$\left|\left|\left|e^{sA}\right|\right|\right|_2 \leq 2^{n-1}(n-1)^{n-1}\frac{\max(1, |||A|||_2)^{n-1} m(A)}{\min(1, \alpha - \lambda_{max}(A))^{n-1}} e^{\alpha s}$$

Proof. Fix $\lambda_{\max}(A) < \alpha \leq \lambda_{\max}(A) + 1$ and $s \geq 0$. Let

$$\mathbb{R}^n = \oplus_{1\leq i\leq r} \operatorname{Ker}(A - \mu_i \operatorname{Id})^{d_i}$$

be the decomposition of $\mathbb{R}^n$ as the direct sum of the generalized eigenspaces of A. Fix $x = \sum_{i=1}^r x_i \in \mathbb{R}^n$, where $x_i \in \operatorname{Ker}(A - \mu_i \operatorname{Id})^{d_i}$. For every $1 \le i \le r$,

$$\begin{aligned}
\left\|e^{sA} x_i\right\|_2 &= \left\|e^{s\mu_i \operatorname{Id}} e^{s(A-\mu_i \operatorname{Id})} x_i\right\|_2 \\
&= \left\|e^{s\mu_i \operatorname{Id}} \sum_{j=0}^{d_i-1} \frac{s^j}{j!} (A - \mu_i \operatorname{Id})^j x_i\right\|_2 \\
&\le e^{s\operatorname{Re}(\mu_i)} \|x_i\|_2 \sum_{j=0}^{d_i-1} \frac{s^j}{j!} (2 \, |||A|||_2)^j \\
&\le e^{s\alpha} \|x_i\|_2 \, 2^{d_i-1} \max(1, |||A|||_2)^{d_i-1} e^{s(\operatorname{Re}(\mu_i)-\alpha)} (1+s)^{d_i-1}
\end{aligned}$$

where we used the fact that $|\mu_i| \le |||A|||_2$ by Browne theorem. By a straightforward computation, we obtain

$$\max_{t \ge 0} e^{t(\operatorname{Re}(\mu_i)-\alpha)} (1+t)^{d_i-1} \le \begin{cases} \frac{(d_i-1)^{d_i-1}}{(\alpha - \operatorname{Re}(\mu_i))^{d_i-1}} & \text{if } \alpha - \operatorname{Re}(\mu_i) \le d_i - 1 \\ 1 & \text{if } \alpha - \operatorname{Re}(\mu_i) > d_i - 1 \end{cases}$$

It follows that

$$\left\|e^{sA} x\right\|_2 \le e^{s\alpha} 2^{n-1} (n-1)^{n-1} \frac{\max(1, |||A|||_2)^{n-1}}{\min(1, \alpha - \lambda_{\max}(A))^{n-1}} \sum_{i=1}^r \|x_i\|_2$$

Using Lemma C.25, we obtain the desired inequality. $\square$

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Chaotic dynamics of spatially homogeneous spacetimes

Abstract

In 1963, Belinsky, Khalatnikov and Lifshitz have proposed a conjectural description of the asymptotic geometry of cosmological models in the vicinity of their initial singularity. In particular, it is believed that the asymptotic geometry of generic spatially homogeneous spacetimes should display an oscillatory chaotic behaviour modeled on a discrete map's dynamics (the so-called Kasner map). We prove that this conjecture holds true, if not for generic spacetimes, at least for a positive Lebesgue measure set of spacetimes.

In the context of spatially homogeneous spacetimes, the Einstein field equations can be reduced to a system of differential equations on a finite dimensional phase space: the Wainwright-Hsu equations. The dynamics of these equations encodes the evolution of the geometry of spacelike slices in spatially homogeneous spacetimes. Our proof is based on the non-uniform hyperbolicity of the Wainwright-Hsu equations. Indeed, we consider the return map of the solutions of these equations on a transverse section and prove that it is a non-uniformly hyperbolic map with singularities. This allows us to construct some local stable manifolds *à la Pesin* for this map and to prove that the union of the orbits starting in these local stable manifolds cover a positive Lebesgue measure set in the phase space. The chaotic oscillatory behaviour of the corresponding spacetimes follows.

The Wainwright-Hsu equations turn out to be quite interesting and challenging from a purely dynamical system viewpoint. In order to understand the asymptotic behaviour of (many of) the solutions of these equations, we will in particular be led to:

- carry a detailed analysis of the local dynamics of a vector field in the neighborhood of degenerate non-linearizable partially hyperbolic singularities,
- deal with non-uniformly hyperbolic maps with singularities for which the usual theory (due to Pesin and Katok-Strelcyn) is not relevant due to the poor regularity of the maps,
- consider some unusual arithmetic conditions expressed in terms of continued fractions and use some rather sophisticated ergodic properties of the Gauss map to prove that these properties are generic.

Keywords: non-uniformly hyperbolic dynamical systems, general relativity, cosmological models, ordinary differential equations, lorentzian geometry, continued fractions

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Dynamique chaotique des espaces-temps spatialement homogènes

Résumé

En 1963, Belinsky, Khalatnikov et Lifshitz ont proposé une description conjecturale de la géométrie asymptotique des modèles cosmologiques au voisinage de leur singularité initiale. En particulier, il y est avancé que la géométrie asymptotique des espaces-temps spatialement homogènes « génériques » devrait avoir un comportement oscillatoire chaotique modelé sur la dynamique d'une application discrète : l'application de Kasner. Nous démontrons que cette conjecture est vraie au moins pour un ensemble d'espaces-temps de mesure de Lebesgue strictement positive.

Dans le contexte des espaces-temps spatialement homogènes, l'équation d'Einstein de la relativité générale se réduit à un système d'équations différentielles sur un espace des phases de dimension finie : les équations de Wainwright-Hsu. La dynamique de ces équations encode l'évolution de la géométrie des hypersurfaces spatiales dans les espaces-temps spatialement homogènes. Notre preuve est basée sur l'hyperbolicité non-uniforme des équations de Wainwright-Hsu. Nous considérons l'application de Poincaré associée aux solutions de ces équations sur une section transverse au flot et nous démontrons qu'il s'agit d'une application non-uniformément hyperbolique avec singularités. Ceci nous permet de construire des variétés stables locales « à la Pesin » pour cette application et de montrer que la réunion des orbites passant par ces variétés stables locales recouvre une partie de l'espace des phases de mesure de Lebesgue strictement positive. Le comportement oscillatoire chaotique des espaces-temps correspondant à ces orbites est une conséquence de cette construction.

Du point de vue des systèmes dynamiques, les équations de Wainwright-Hsu se révèlent être très riches et posent un certain nombre de défis. Pour comprendre le comportement asymptotique d'un nombre conséquent de solutions de ces équations, nous serons amenés à :

- faire une analyse fine de la dynamique locale d'un champ de vecteurs au voisinage d'une singularité partiellement hyperbolique dégénérée et non linéarisable,
- travailler avec des applications non-uniformément hyperboliques ayant des singularités, pour lesquelles la théorie usuelle (due à Pesin et Katok-Strelcyn) ne s'applique pas à cause de la faible régularité de ces applications,
- considérer des conditions arithmétiques exotiques exprimées en termes de fractions continues et utiliser des propriétés ergodiques quelque peu sophistiquées de l'application de Gauss pour montrer que ces propriétés sont génériques, *etc.*

Mots clés : systèmes dynamiques non uniformément hyperboliques, relativité générale, modèles cosmologiques, équations différentielles ordinaires, géométrie lorentzienne, fractions continues

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