Eigenvalues and eigenvectors of large matrices under random perturbations

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Eigenvalues and eigenvectors of large matrices under random perturbations

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Abstract

The present thesis is devoted to the study of the effect of a perturbation on the spectrum of a Hermitian matrix by a random matrix with small operator norm and whose entries in the eigenvector basis of the first one were independent, centered and with a variance profile. This is carried out through perturbative expansions of various types of spectral laws of the considered perturbed large matrices. First, we demonstrate different perturbative expansions of the empirical spectral measure in the cases of the perturbative regime and the semi-perturbative regime and highlight well known heuristic patterns in Physics, as the transition between semi-perturbative and perturbative regimes. Secondly, we provide a thorough study of the semi-perturbative regime and prove the new fact that this regime could be decomposed into infinitely many sub-regimes. Finally, we prove, through a perturbative expansion of spectral measures associated to the state defined by a given vector, a perturbative expansion of the coordinates of the eigenvectors of the perturbed matrices.

Résumé

La présente thèse est consacrée à l'étude de l'effet d'une perturbation sur le spectre d'une matrice hermitienne perturbée par une matrice aléatoire de petite norme opérateur et dont les entrées dans la base propre de la première matrice sont indépendantes, centrées et possèdent un profil de variance. Ceci est réalisé au travers de développements perturbatifs de divers types des lois spectrales des grandes matrices perturbées considérées. Dans un premier temps, nous démontrons différents développements perturbatifs de la mesure spectrale empirique dans les cas du régime perturbatif et du régime semi-perturbatif et mettons en évidence des modèles heuristiques bien connus en physique, comme la transition entre les régimes semi-perturbatifs et perturbatifs. Dans un deuxième temps, nous proposons une étude approfondie du régime semi-perturbatif et prouvons le fait nouveau que ce régime peut être décomposé en un nombre infini de sous-régimes. Enfin, nous démontrons, au travers d'un développement perturbatif des mesures spectrales associées à un vecteur donné, un développement perturbatif des coordonnées des vecteurs propres des matrices perturbées que nous considérons.

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Chapter 1

Introduction

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Numerous mathematical problems naturally lead us to fill a matrix with random coefficients and to study its spectral properties. Thus, in the 1920s, the statistician Wishart initiated the random matrix theory by studying in [Wis28] the first ever model of a random matrix: a random covariance matrix. Given n centered i.i.d. random vectors, $X_1, \ldots, X_n$ in $\mathbb{R}^p$ modeling a sample of a given population, assume that we want to estimate the covariance matrix Σ of their common distribution. We know from the law of large numbers that the random matrix $S_n = \frac{1}{n}(X_1, \ldots, X_n)(X_1, \ldots, X_n)^T$ is a good estimator of Σ, as long as p remains small compared to n. But one of the reasons we may be interested in the spectrum of these matrices, S_n, now identified as *Wishart matrices*, is that in the case where p has an order close to that of n, to obtain satisfying information about Σ, we need to use other statistical methods, such as a principal component analysis, which involve the largest eigenvalues of S_n and their associated eigenvectors.

However, it was not until the 1950s that a concrete mathematical theory of the spectrum of random matrices emerged through the work [Wig55] of Nobel Prize winner Eugene Wigner. In the context of his physical studies of heavy nuclei, the behavior of a nucleus is determined by a Hamiltonian operator. Wigner's idea was then to approximate this Hamiltonian operator by a Hermitian random matrix of high dimension. The interest for the spectrum of these matrices followed the postulate that the spacings between the lines in the spectrum of a heavy atom nucleus should resemble the spacings between their eigenvalues. Indeed, during this last study Wigner then conjectured a phenomenon of universality, namely that the asymptotic spectral properties of random matrices are described by universal laws.

This latter conjecture, which will be illustrated in Section 1.1, as well as the work of Dyson and Mehta, marked the beginning of the interest in random matrices as pure mathematical objects. Indeed, many important tools aiding the analysis of the properties of the spectrum of random matrices were then developed, for example, the method of orthogonal polynomials by Mehta and Gaudin in [MG60]. In addition, two years later in [Dys62], Dyson classified Hamiltonians into several categories based on their symmetry properties, implying the existence of three major sets of matrices: the Gaussian orthogonal ensemble (GOE), the Gaussian unitary ensemble (GUE) and the Gaussian symplectic ensemble (GSE). Those three ensembles of Gaussian random matrices, which are respectively real symmetric, complex Hermitian, and quaternionic self-dual, rapidly became fundamental in various areas of theoretical physics, mesoscopic physics, disordered electron systems, and in the field of quantum chaos.

Until the late 1980s, research on the properties of random matrices in the mathematical community was apparently not as intensive as it was in theoretical physics. Among the few purely mathematical articles of the two decades after the 1960s, we can cite in particular three works that greatly influenced the theory of random matrices: the work [MP67] of Marcenko and Pastur describing the spectrum of random covariance matrices, the work [FK81] of Furedi and Komlos on the distribution of eigenvalues in ensembles of random matrices with independent entries and finally Montgomery's work [Mon73], which provided an important link between the theory of random matrices and deep-rooted problems of number theory. Indeed, he revealed a certain similarity between the eigenvalues of matrices of the GUE and the zeros of the Riemann zeta function.

The situation changed considerably after the 1990s, thanks, in particular, to the works [Voi91, DVDN92] of Voiculescu *et al*, who developed the free probability theory. Since then, random matrices have been increasingly studied. Today, random matrix theory is a very large subject with applications in many disciplines of natural science, engineering and finance.

In the following sections of this introduction, we present some fundamental results of random matrix theory that will help us progressively introduce the main ideas and tools that we will use in this thesis, as well as presenting the historical context in which this thesis is ingrained. Finally, the last subsection presents the precise subject of this thesis.

1.1 Universality: from square matrices to circular distributions

The first, and perhaps the most fundamental, mathematical object used to study the eigenvalues of a random matrix was introduced by Eugen Wigner and corresponds to the following probability measure, commonly known as *empirical spectral measure*.

Definition 1 (Empirical spectral measure)**.** For any $n \times n$ square matrix A, the probability measure, μ_n, which puts equal mass on each eigenvalue of A is called the empirical spectral measure (or distribution) of A.

For example, in the specific case where we consider an $n \times n$ Hermitian (or real symmetric) matrix A, as the eigenvalues $\lambda_1, \dots, \lambda_n$ of A are all real, its empirical spectral measure is a real probability measure that could be expressed as

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}.$$

Moreover, one can note that in this last case for any Borel set $B \subseteq \mathbb{R}$,

$$\mu_n(B) = \frac{1}{n} \, \#\{\lambda_i \in B \mid i \in [\![1, n]\!]\}$$

is the proportion of eigenvalues of A contained in the set B. For this reason, many studies have investigated the empirical spectral measure of various models of random matrices in order to study the distribution of their eigenvalues. As we will see in the next fundamental examples presented in this introductory chapter, many studies have focused on the limiting spectrum of an $n \times n$ matrix A_n when its dimension, n, tends to infinity.

1.1.1 Wigner's semi-circle law

The matrix model initially introduced by Eugen Wigner in 1955 was a specific model of a random Hermitian matrix. He introduced it to study the eigenvalues of such a matrix when its dimension tends to infinity in order to analyze by approximation a self-adjoint operator that interested him in his work in the field of physics of heavy nuclei. This model is given as follows.

Definition 2 (Wigner matrix). A Wigner matrix is a random Hermitian matrix, $W_n = (W_n(i,j))_{i,j=1}^n$ such that the entries above its diagonal $\{W_n(i,j)\ ,\ 1 \leq i < j \leq n\}$ are real or complex valued i.i.d. random variables with zero mean and unit variance, and the diagonal entries $\{W_n(i,i)\ ,\ 1 \leq i \leq n\}$ are i.i.d. centered real-valued random variables with bounded mean and variance and are independent from the upper-triangular entries.

Remark 1. This definition may vary; rather than considering that the upper-triangular entries have a second moment equal to 1, other authors consider that the entries have just a bounded moment of order 2, or bounded moments of any order.

By studying the eigenvalues of this last model, Wigner demonstrated the following theorem which can be considered as the starting point of random matrix theory.

Theorem 3 (Wigner's semi-circle law). *The empirical spectral measure, $\mu_{\frac{1}{\sqrt{n}}W_n}$, of an $n \times n$ normalized Wigner matrix, $\frac{1}{\sqrt{n}}W_n$, converges almost surely as n tends to infinity to the Wigner semi-circle distribution with parameter 2:*

$$\mu_{\frac{1}{\sqrt{n}}W_n} \xrightarrow[n\to\infty]{a.s.} \mu_{sc}$$

with

$$\mu_{sc} := \frac{1}{2\pi}\sqrt{4-x^2}\ \mathbb{1}_{x\in[-2,2]}\ \mathrm{d}x.$$

In other words, the eigenvalues of a normalized large Wigner matrix will be approximately distributed in a semi-circle of radius 2. This phenomenon is illustrated in Figure 1.1, where we can observe the convergence to the semi-circle distribution of the eigenvalues as the dimension, n, grows.

The reason we have to normalize the matrix W_n by $\frac{1}{\sqrt{n}}$ comes from the fact that we want its spectrum to be bounded. Indeed, the magnitude of the eigenvalues of W_n is of order $O(\sqrt{n})$. Various proofs of this order of magnitude can be found in the literature. We can cite for example the so-called *Bai-Yin Theorem* which was proved in [BY86] and states that, in the case where the entries of W_n have finite fourth moment, almost surely:

$$\lim_{n\to\infty} \frac{\|W_n\|_{\mathrm{op}}}{\sqrt{n}} = 2.$$

Another noteworthy fact of Wigner's theorem is that since the eigenvalues of a random matrix are random variables, the involved empirical spectral measure is a random measure. It is therefore a convergence theorem of a random probability measure to a deterministic probability measure.

Finally, as for the classical central limit theorem, this last result shows a limiting distribution which does not depend on the initial laws of the entries of the considered matrix. In random matrix theory this phenomenon is called *universality*. In the next subsections, we present three other historically important results of universality for different classes of random matrices.

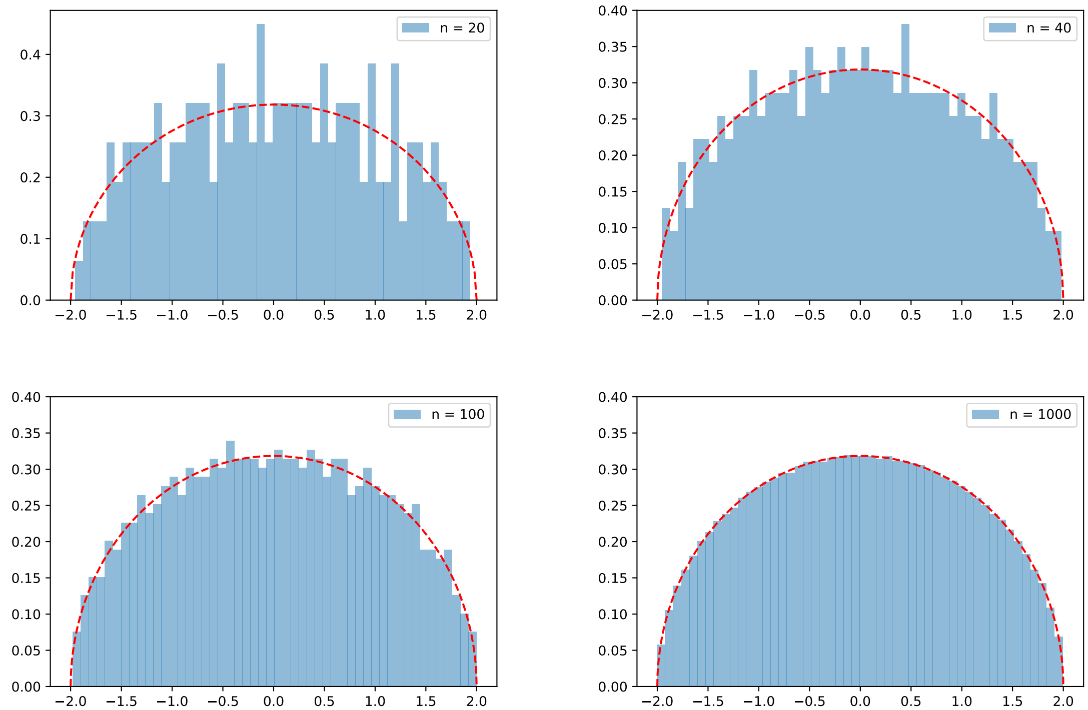

Figure 1.1: **Illustration of the Wigner's semi-circle law.** The red curve represents the density $x \mapsto \frac{1}{2\pi}\sqrt{4-x^2}\ \mathbb{1}_{x\in[-2,2]}$ of the semi-circle distribution with parameter 2 and the blue histogram is that of the eigenvalues of an $n \times n$ Gaussian Wigner Matrix $\frac{1}{\sqrt{n}}W_n$ for different values of n.

1.1.2 Girko's circle law

In 1965 the French mathematical physicist, Jean Ginibre, noticed and demonstrated in [Gin65] that the limiting spectral distribution of an $n \times n$ random matrix which lacks any condition of symmetry and whose entries are all i.i.d. centered Gaussian random variables, all with variance equal to $1/n$, is the uniform distribution over the unit disc.

Since this study, the ensemble of square matrices with Gaussian distribution of entries is called the *Ginibre ensemble* and has been very successful in describing various physical phenomena.

Despite the fact that, in 1984, Vyacheslav Girko introduced in [Gir85], an approach that allowed for the establishment of the circular law for more general distributions, it was not until 45 years after the initial result of Ginibre that, in 2010, Tao and Vu proved in [TV10] the universality of the so-called *Girko's circle law* under the minimum assumptions set out below.

Theorem 4 (Girko's circle law)**.** *Let $(G_n)_{n\in\mathbb{N}^*}$ be a sequence of $n \times n$ matrix ensembles whose entries are i.i.d. copies of a complex random variable with mean 0 and variance 1. Then, the empirical spectral measure of $\frac{1}{\sqrt{n}}G_n$ converges almost surely to the uniform measure on the unit disk as n tends to infinity:*

$$\mu_{\frac{1}{\sqrt{n}}G_n} \xrightarrow[n\to\infty]{a.s.} \mu_{disk}$$

with

$$\mu_{disk} := \frac{1}{\pi}\mathbb{1}_{\{x^2+y^2\leq 1\}}(x,y)\ \mathrm{d}x\mathrm{d}y$$

The eigenvalues, which are in this case complex random variables, will be approximately distributed in a disk of the complex plane of radius 1. This phenomenon is illustrated in Figure 1.2, where we can observe the convergence to the uniform distribution on the unit disk of the eigenvalues as the dimension n grows.

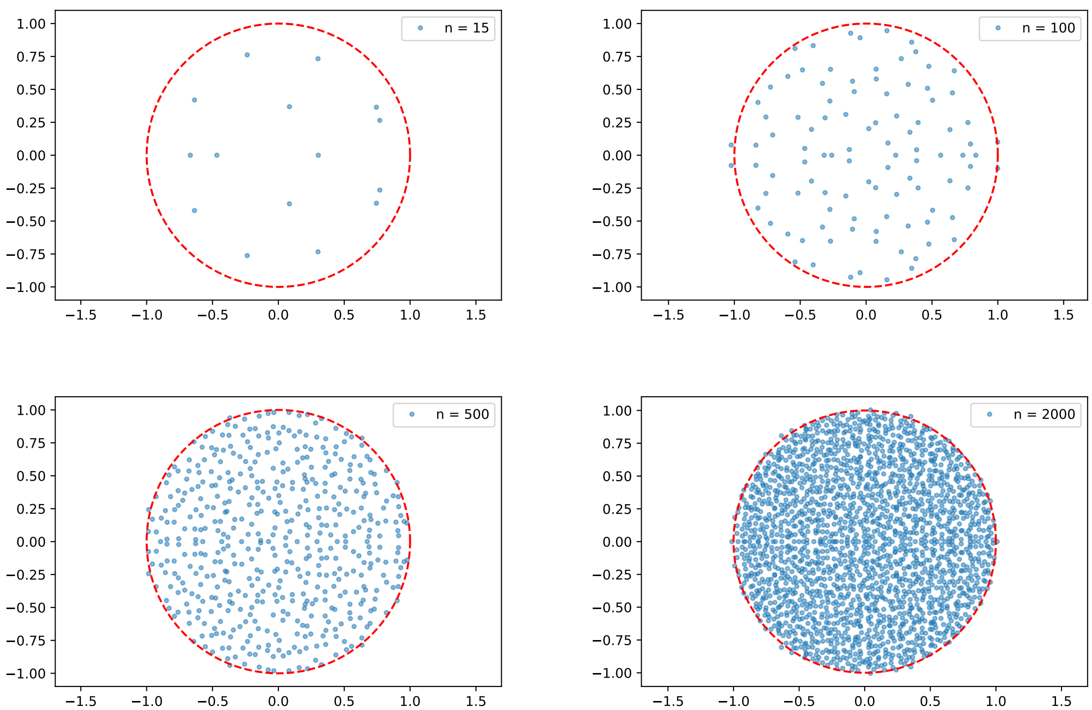

Figure 1.2: **Illustration of the Girko's circle law.** The blue dots represent the eigenvalues in the complex plane of a real Ginibre matrix $\frac{1}{\sqrt{n}}G_n$ for different values of n.

1.1.3 Marchenko-Pastur's quarter circle law

In 1967, two years after the first result of Jean Ginibre on the eigenvalues of Gaussian matrices, the two Ukrainian mathematicians Vladimir Marchenko and Leonid Pastur proved in [MP67] that the limiting distribution of the singular values of a large rectangular random matrix is a probability distribution, hence named the *Marchenko-Pastur distribution*.

We recall that the singular values of a matrix M are the eigenvalues of the matrix $\sqrt{MM^*}$. Thus, in the case where M is a rectangular matrix and therefore does not have eigenvalues, its singular values are helpful to compute its pseudoinverse, to determine its rank, its range, its operator norm, etc.

We present here a simpler, reformulated version, of the initial Marchenko-Pastur theorem which illustrates the convergence of the empirical spectral measure to the quarter circle distribution.

Theorem 5 (Marchenko-Pastur's quarter circle law)**.** *If X_n is an $n \times n$ random matrix such that its entries are independent, centered and have variance 1, then the empirical spectral distribution, $\mu_{\frac{1}{\sqrt{n}}\sqrt{MM^*}}$, of the $n \times n$ random matrix $\frac{1}{n}\sqrt{MM^*}$ converges to the Marchenko-Pastur's quarter circle distribution:*

$$\mu_{\frac{1}{\sqrt{n}}\sqrt{MM^*}} \xrightarrow[n\to\infty]{a.s.} \mu_{MP}$$

with

$$\mu_{MP} := \frac{1}{\pi}\sqrt{4-x^2}\ \mathbb{1}_{[0,2]}(x)\ \mathrm{d}x$$

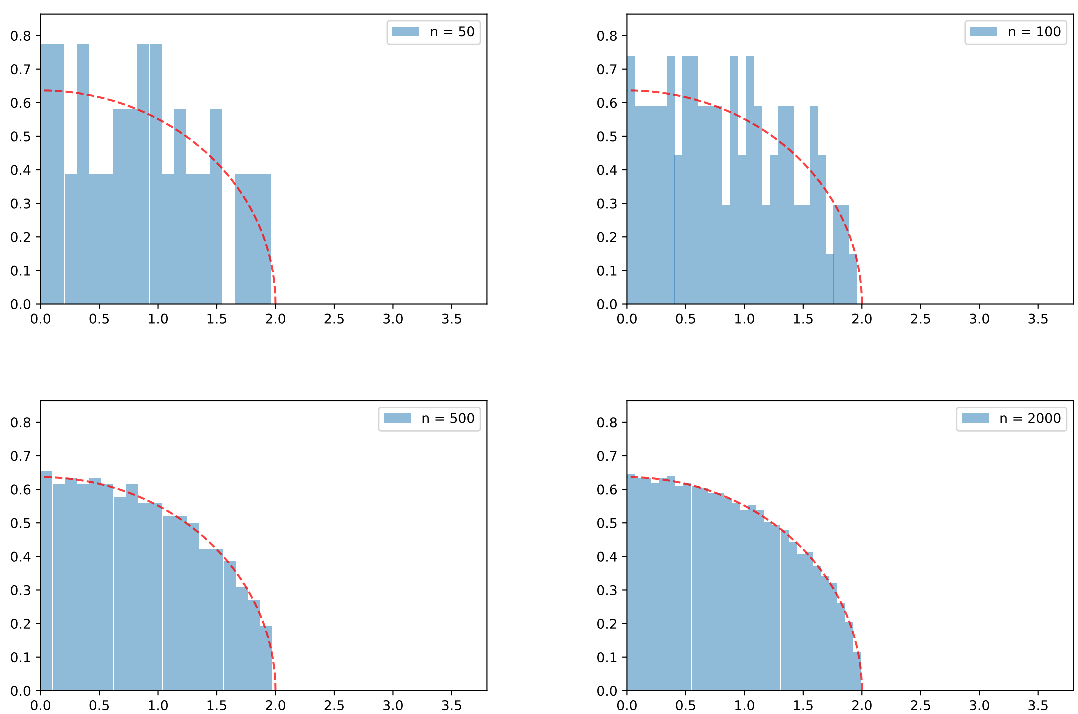

Figure 1.3: **Illustration of the Marchenko-Pastur's quarter circle law.** The red curve represents the density $x \mapsto \frac{1}{\pi}\sqrt{4-x^2}\ \mathbb{1}_{[0,2]}(x)$ of the quarter circle distribution and the blue histogram is that of the singular values of an $n \times n$ random matrix such that its entries are Gaussian random variables which are independent, centered and with unit variance, for different values of n.

1.1.4 Single Ring Theorem

A more recent result of universality that we can cite in this introduction is that of Guionnet, Krishnapur and Zeitouni. In 2011, they demonstrated, in an article entitled *Single Ring Theorem*, a new universality result for another class of matrices. For a fixed n-tuple $(s_1, \dots, s_n)$ of non-negative reals, these matrices could be defined as the matrices which are uniformly distributed among the matrices with singular values $(s_1, \dots, s_n)$. More explicitly, any matrix of this class can be written under the form,

$$A_n = U_n D_n V_n$$

for U_n and V_n two independent Haar distributed random matrices, that is to say two matrices uniformly distributed among the ensemble of unitary matrices, and $D_n = \mathrm{diag}(s_1, \dots, s_n)$ a non-negative diagonal matrix independent of U_n and V_n. This result [GKZ11], was initially partially demonstrated by the physicits Feinberg and Zee in [FZ97], and its assumptions have since been amended. Specifically, in [RV14], Rudelson and Vershynin showed that a hard-to-check assumption on the smallest singular value of $(A_n - z.I_n)$ was unnecessary and in [BD$^+$13], Basak and Dembo weakened the hypotheses so that the empirical spectral measure of D_n allows for the existence of some atoms. We state here a shortened version with optimal assumptions of the *Single Ring Theorem* borrowed from the very recent work [BES16] of Bao, Erdös and Schnelli which applies, like the first paper [BG$^+$17] of Benaych-Georges, to study the *Local Single Ring Theorem*, that is to say the *Single Ring Theorem* whithin the bulk regime.

Theorem 6 (Single Ring Theorem). *Let $(A_n)_{n\in\mathbb{N}^*}$ be a sequence of $n \times n$ matrices defined as previously. If the empirical spectral measure, μ_{D_n}, of D_n converges weakly to a measure μ whose support contains more than one point and the sequence of matrices $(D_n)_{n\in\mathbb{N}^*}$ is uniformly bounded, then then the empirical spectral measure of A_n converges in probability to a deterministic probability measure ν which possesses a radially-symmetric density, $\rho(z)$, with respect to the Lebesgue measure on $\mathbb{C}$ depending only on μ and supported on the single ring $\{z \in \mathbb{C}, r \leq z \leq R\}$ for*

$$r = \frac{1}{\sqrt{\int x^{-2}\mu(\mathrm{d}x)}} \qquad \textit{and} \qquad R = \sqrt{\int x^2 \mu(\mathrm{d}x)}$$

To put it another way, the distinctive property of matrices (A_n) is that their eigenvalues tend to spread over a single annulus centered in the origin. For example, if the eigenvalues of the matrix D_n are distributed such that the empirical spectral measure of D_n tends to the uniform measure on an interval $[a, b]$, then the eigenvalues of A_n will spread as n tends to infinity over the single ring $\{z \in \mathbb{C}, r = \sqrt{ab} \leq z \leq R = \sqrt{\frac{a^2+ab+b^2}{3}}\}$. We illustrate this phenomenon in Figure 1.4 in the case where μ is the uniform measure on the interval $[1, 6]$.

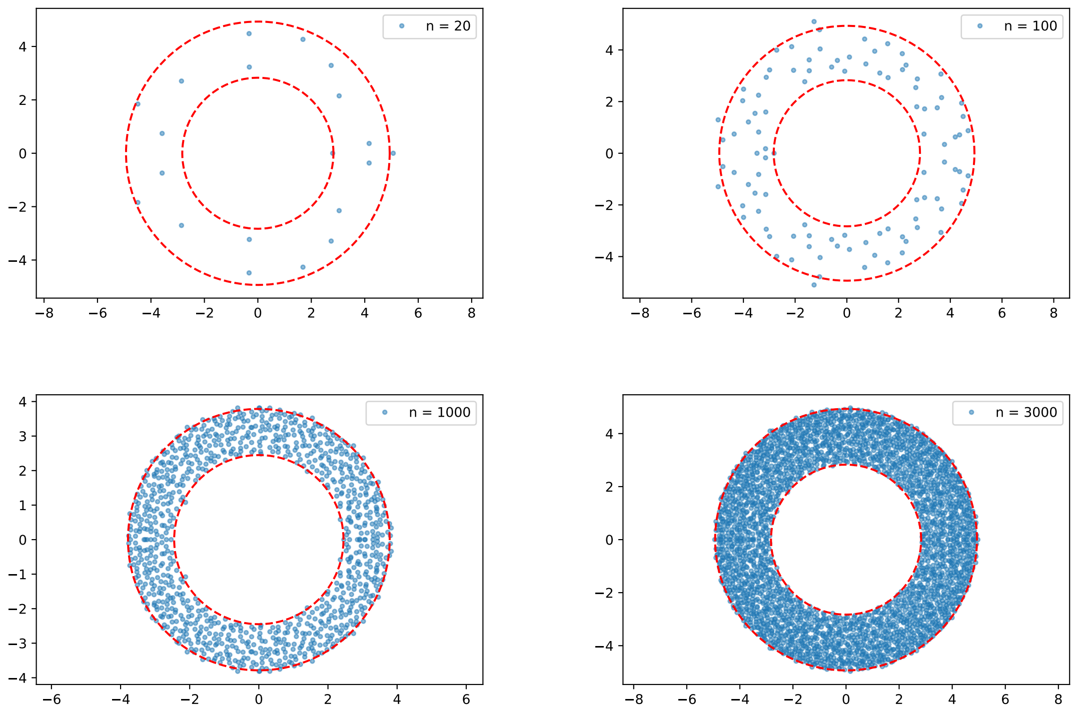

Figure 1.4: **Illustration of the Single Ring Theorem.** The blue dots represent the eigenvalues in the complex plane of the matrix $A_n = U_n D_n V_n$, for different values of n and for U_n and V_n two $n \times n$ Haar distributed random matrices and D_n an $n \times n$ random diagonal matrix such that its empirical spectral measure converges to the uniform measure on $[1, 6]$. The smaller circle has radius $r = \sqrt{6} \approx 2.45$ and the bigger one has radius $R = \sqrt{\frac{43}{3}} \approx 3.78$.

Furthermore, one can note that if the measure μ verifies $\sqrt{\int x^{-2}\mu(\mathrm{d}x)} = \infty$, then $r = 0$ and the support of the limiting measure ν is a disk, leading to the previously presented Ginibre case. Moreover, in the case where μ is a Dirac measure, $r = R$ so that the support of ν is simply a circle. These relations are illustrated in Figure 1.5.

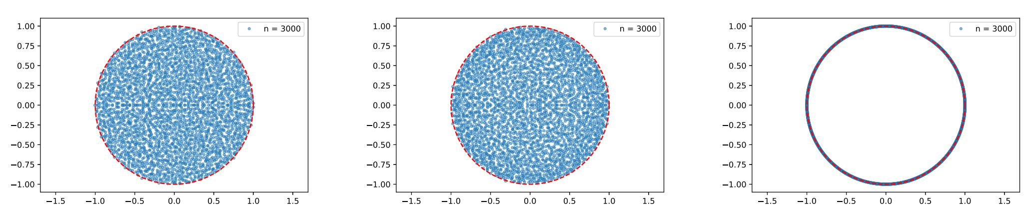

Figure 1.5: **Relation of the Single Ring Theorem to other results.** In the three figures above, the blue dots represent the eigenvalues in the complex plane respectively from left to right of a real Ginibre matrix $\frac{1}{\sqrt{n}} G_n$, of a matrix $A_n = U_n D_n V_n$ such that μ is the uniform measure on $[0, \sqrt{3}]$ and of a matrix $A_n = U_n D_n V_n$ such that $\mu = \delta_1$.

1.2 Eigenvectors in random matrix theory

Although the earliest foundational works of random matrix theory were concerned with eigenvalues, many studies were subsequently focused on eigenvectors. Indeed, many problems, particularly related to graph theory, naturally lead to questions about the eigenvectors of random matrices. We can mention for example problems related to spectral clustering or even the famous PageRank algorithm used by Google to sort its search results. Nevertheless, the eigenvectors literature is currently narrower than the eigenvalues one. Nevertheless, the literature concerning eigenvectors is currently scarce compared to that relating to eigenvalues.

This could be explained in particular by the fact that many ensembles of random matrices, such as those of the GOE or the GUE, are rotationally invariant in law which makes the study of their eigenvectors pointless as they are uniformly distributed on the unit sphere.

1.2.1 Spectral measure

In this section dedicated to eigenvectors, we introduce one of the tools that will be later addressed in this thesis: the spectral measure.

Definition 7 (Spectral measure over a vector)**.** For an $n \times n$ hermitian matrix, H_n, with eigenvalues $\lambda_1, \ldots, \lambda_n$ and associated eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$, we define the spectral measure of H_n over a vector $\mathbf{v}$, $\mu_{H_n,\mathbf{v}}$, by

$$\mu_{H,\mathbf{v}} = \sum_{j=1}^{n} |\mathbf{v} \cdot \mathbf{u}_j|^2 \delta_{\lambda_j}$$

In contrast with the empirical spectral measure, the spectral measure over a vector $\mathbf{v}$ does not weigh all the eigenvalues of a matrix H_n equally; the weighting depends here on the eigenvectors of H_n. Thus, the spectral measure gives information on the eigenvector basis of H_n. Moreover, this measure is often applied over specific vectors as vectors $(\mathbf{e}_i)_{i=1}^n$ of the canonical base of $\mathbb{R}^n$, as in this case it satisfies, for any function ϕ defined on the spectrum of H_n, the identity

$$(\phi(H_n))_{i,i} = \sum_{j=1}^{n} |\mathbf{e}_i \cdot \mathbf{u}_j|^2 \phi(\lambda_j) = \int \phi(x) \, \mathrm{d}\mu_{H_n,\mathbf{e}_i}$$

which follows directly from the spectral theorem.

For this reason as well as others, this measure is often used in studies on the eigenvectors of random matrices. For example, many estimates on eigenvectors of a random matrix, H_n, are obtained from its resolvent matrix, $(H_n - zI_n)^{-1}$ and using the previous formula with $\phi : x \mapsto \frac{1}{x-z}$ could be helpful in the sense that the entries of the resolvent matrix are easy to compute.

1.2.2 Anderson's transition

One of the sizeable current issues concerning the eigenvectors of random matrices is the so-called Anderson's transition. It concerns matrices called *band matrices*, that is to say Hermitian matrices whose entries are null beyond a band located around their diagonal, and which represent physical systems with local interactions. The question is then to know to which bandwidth, W, the spectral properties remain local, that is to say close to those of a diagonal matrix with independent diagonal entries. Roughly speaking, we say that an eigenvector is localized if it is essentially carried by few of its coordinates (see e.g. [Cha10]).

Although this topic is not addressed in this thesis, it may be interesting to give some fundamental results, because as we will later see we will approach it by studying the spectrum of a diagonal matrix summed with a random band matrix.

Despite the fact that this has not yet been formally demonstrated, a phase transition between the localized regime and the delocalized regime of the eigenvectors seems to occur when the bandwidth of an $n \times n$ random matrix is of order $\sqrt{n}$. Another way of formulating this latter conjecture could be the following: the average localization length of the eigenvectors of an $n \times n$ band matrix, of bandwidth W, is of the order of W^2. Figure 1.6 illustrates this transition for a 4000×4000 band matrix, as $\sqrt{4000} \approx 63$.

In [Sch09], Jeffrey Schenker shows that the localization length of the eigenvectors of a band random matrix whose bandwidth is equal to W, is lower than W^8. Similarly, in 2012, Erdös, Yau and Yin demonstrated, in [EYY12a] that, under certain assumptions, the localization length of the eigenvectors of a band matrix is greater than W. We can also mention the work [EKYY13] of Erdös, Knowles, Yau and Yin in which they prove that the eigenvectors of an $n \times n$ random band matrix are delocalized when the width of the strip W is of order greater that $n^{4/5}$. More generally, all the results currently in existence regarding this subject are in agreement with the conjecture previously mentioned.

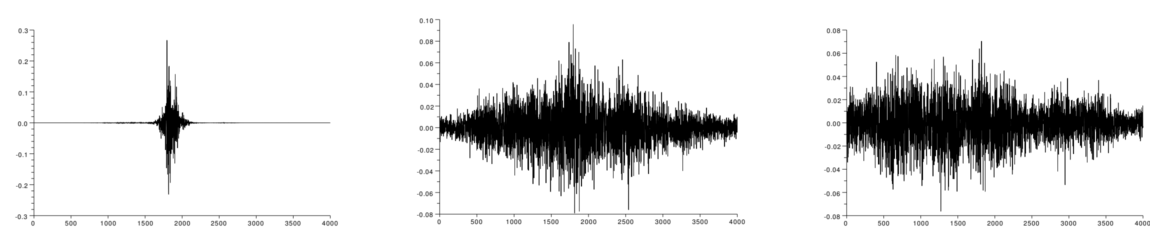

Figure 1.6: **Illustration of Anderson's transition.** For an $n \times n$ random band matrix whose entries are standard Gaussian variables, we have represented one of its eigenvectors for a bandwith equal to $W = 30$ (left), 63 (center) and 70 (right). Here $n = 4000$.

1.3 Perturbation theory

A natural and central question, in Mathematics and Physics, is to understand how the spectral properties of an operator are altered when the operator is subject to a small perturbation. This question is at the center of *Perturbation Theory* and was more precisely introduced by Rayleigh and Schrödinger as follows.

Let us consider a Hamiltonian operator, H, whose eigenvectors and eigenvalues are known and a second operator, X, whose entries are small compared to those of H. Thus, if we look at the operator $H+X$, the effect of X is to modify the spectrum of H. This can model, physically speaking, the perturbation by a weak magnetic or electric field represented by X of an atomic nucleus represented by H. The purpose of Perturbation Theory is then to give an approximate expression of the eigenvalues and eigenvectors of $H+X$ in terms of those of H. We refer the reader to Kato's book [Kat13] for a thorough account on this subject which has been studied in many different contexts.

1.3.1 The problem of Weyl

In 1912, the German mathematician, theoretical physicist and philosopher Hermann Weyl stated in [Wey12] the following problem: if A and B are two Hermitian matrices, how are the eigenvalues of the matrix $A+B$ related to those of A and B ?

This problem, which is more general than that of a perturbation, because here no hypothesis is made on the "smallness" of the perturbing matrix, currently does not have so many answers. For instance, by using the notation $\lambda_1(M) \leq \cdots \leq \lambda_n(M)$ to denote the eigenvalues of an $n \times n$ Hermitian matrix M, it is easy to see, from linearity of the trace, that

$$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B) \Longleftrightarrow \sum_{i=1}^{n} \lambda_i(A+B) = \sum_{i=1}^{n} \lambda_i(A) + \sum_{i=1}^{n} \lambda_i(B),$$

or that if A and B commute, so that A and B are diagonalizable in a same eigenbasis, then for any $i \in [\![1,n]\!]$,

$$\lambda_i(A+B) = \lambda_i(A) + \lambda_i(B),$$

but overall precise answers under broad hypotheses are scarce. Nevertheless, Weyl's answer to this problem has remained one of the most satisfactory thus far.

Theorem 8 (Weyl's inequality). *Let A and B be two $n \times n$ Hermitian matrices, then for any $i \in [\![1,n]\!]$, $i \in [\![0,n-i]\!]$,*

$$\lambda_i(A+B) \leq \lambda_{i+j}(A) + \lambda_{n-j}(B)$$

Moreover, if A and B have no common eigenvector, then every inequality is strict.

From this first theorem many other inequalities have been deduced, for instance,

Corollary 9. *Let A and B be two $n \times n$ Hermitian matrices, then for any $i \in [\![1, n]\!]$,*

$$\lambda_i(A) + \lambda_1(B) \leq \lambda_i(A + B) \leq \lambda_i(A) + \lambda_n(B)$$

or even,

Corollary 10. *Let A and B be two $n \times n$ Hermitian matrices, then for any $(i, j) \in [\![1, n]\!]^2$ such that $i + j - 1 \leq n$,*

$$\lambda_{i+j-1}(A + B) \leq \lambda_i(A) + \lambda_j(B)$$

All these inequalities, known today as Weyl's inequalities, are very useful in estimating perturbations in the spectrum of Hermitian matrices. In fact, in the case of a perturbation, $H + P$, if we have a bound on the perturbing matrix, P, in the sense that we know that its operator norm satisfies $\|P\|_{\text{op}} \leq \varepsilon$, then it follows, by definition of the operator norm, that all the eigenvalues of P are bounded in absolute value by ε. Applying Weyl's inequalities, it follows that the spectra of $H + P$ and H are close in the sense that for all $i \in [\![1, n]\!]$,

$$|\lambda_i(H + P) - \lambda_i(H)| \leq \varepsilon.$$

In other words, these inequalities guarantee that the spectrum of a Hermitian matrix is stable with respect to small perturbations, and are therefore a good starting point for any study of a perturbed system.

1.3.2 Eigenvalues of a perturbed matrix: bulk and outliers

Questions about perturbed matrices are as diverse (global, local, bulk, edge...) as questions about non perturbed matrices in random matrix theory. Namely, if we consider a deterministic $n \times n$ Hermitian matrix, H_n, perturbed by a random matrix X_n, as various new asymptotic behaviors of the spectrum of $H_n + X_n$ can appear, we can study, for instance, questions relating to the extreme eigenvalues of $H_n + X_n$ or the typical spacings between two consecutive eigenvalues of $H_n + X_n$.

We can distinguish two categories of perturbations; those which are small in terms of rank and those which are small in terms of operator norm.

Those of the first category are often referred to as *low rank perturbations*. They consist in perturbing a matrix H_n by a matrix X_n whose rank is a fixed integer which does not vary with n or by a matrix X_n whose rank is small compared to its dimension (as for example $\log(n)$). Weyl's inequalities guarantee that such perturbations do not influence the global statistics of the eigenvalues as the dimension n tends to infinity. Thus, the empirical spectral measures of $H_n + X_n$ and H_n have the same asymptotics and are governed by the universal law associated to the matrix ensemble to which H_n belongs. However, at a

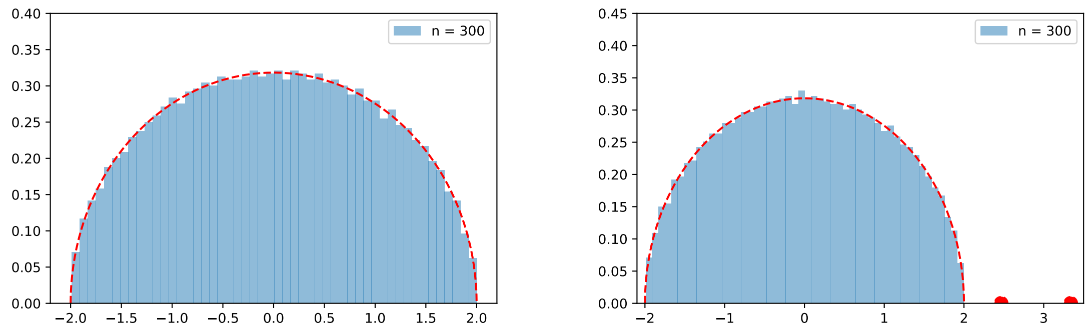

Figure 1.7: **Illustration of a finite rank perturbation.** In both figures, the red curve represents the density $x \mapsto \frac{1}{2\pi}\sqrt{4-x^2}\ \mathbb{1}_{[-2,2]}(x)$ of the semi circle distribution. The blue histogram of the left figure is that of the eigenvalues of an $n \times n$ Wigner matrix, $\frac{1}{\sqrt{n}}W_n$, whose entries are standard Gaussian variables. The blue histogram of the right figure is that of the eigenvalues of the same Wigner matrix but perturbed this time by a random diagonal matrix of rank 3, two outliers represented here by red dots are exhibited out of the bulk spectrum.

microscopic scale, the behavior of individual eigenvalues may change dramatically under such a perturbation. Indeed, the perturbed matrix may exhibit *outliers*, that is to say eigenvalues detached from the bulk spectrum.

Regarding the outliers of a matrix perturbed by a low rank matrix, there is a phase transition called the *BBP phase transition*, from the names of the authors Baik, Ben Arous and Péché who have highlighted it in [BAP+05] in the case of covariance matrices, after the initial work [Joh01] of Johnstone. The general principle of this transition is that if the amplitude of the perturbation remains below a certain threshold, the largest eigenvalues of the perturbed system do not move significantly, whereas beyond this threshold, they are at a macroscopic distance from the bulk spectrum. Studies on other ensembles of matrices, such as the Wigner ensemble, have shown similar phase transitions. We can cite for example the pioneering work [FK81] of Füredi and Komlós about the outliers of perturbed Wigner matrices, and the works [BGN11] or [BGGM+11] of Benaych-Georges, Guionnet, Maida and Nadakuditi that have generalized this transition to larger ensembles.

In this thesis we are interested in perturbations belonging to the second category: those which are small in terms of operator norm. This second kind of perturbation has, unlike the first, a tendency to macroscopically perturb the bulk spectrum of the considered matrix, and also has a phase transition phenomenon.

More precisely, in the typical case where the spacing between two consecutive eigenvalues of the matrix, H_n, is of order n^{-1} the phase transition takes place at $\|X_n\|_{\mathrm{op}} \sim n^{-1}$. The regime $\|X_n\|_{\mathrm{op}} \ll n^{-1}$ is called the *perturbative regime* and the regime $n^{-1} \ll \|X_n\|_{\mathrm{op}} \ll 1$ is often called the *semi-perturbative regime*, as the more the operator norm of a perturbation approaches the order 1, the more the considered regime begins to resemble that of a simple matrix addition. We illustrate this phase transition in Figure 1.8, where the distinction between the spectra of H_n and $H_n + X_n$ begins to be observable effectively as soon as the operator norm of the perturbative matrix, X_n, begins to be of order greater than n^{-1}.

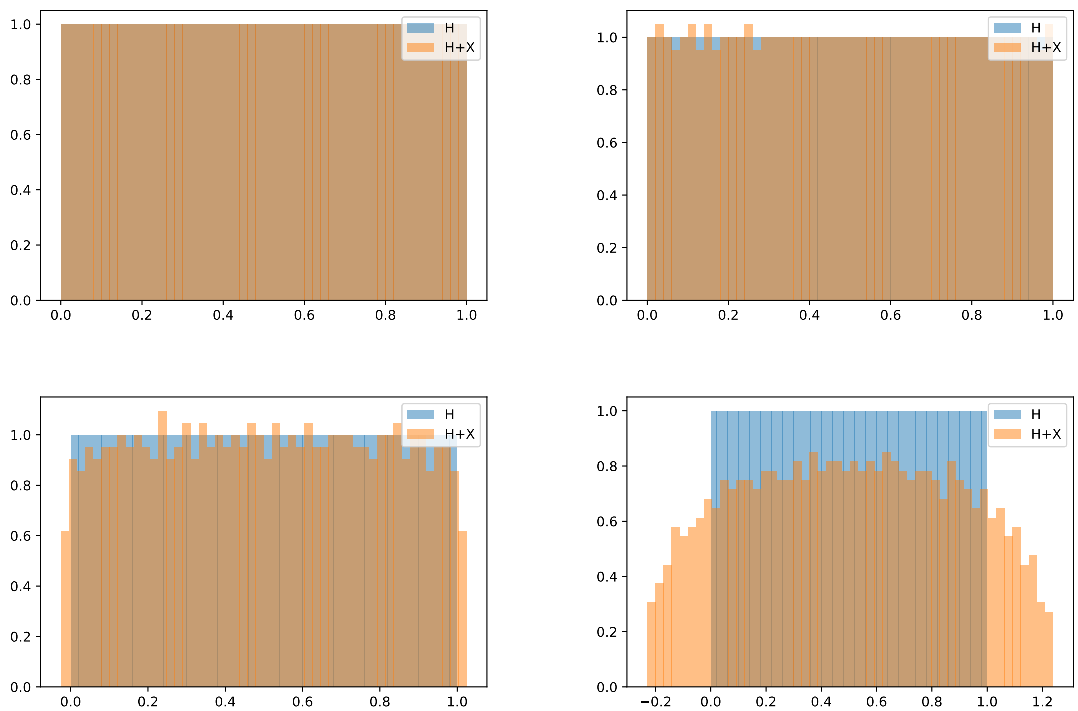

Figure 1.8: **Illustration of the phase transition phenomenon.** The blue histogram is that of the eigenvalues of the diagonal matrix $H_n = \mathrm{diag}(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n})$ and the beige histogram is that of the eigenvalues of $H_n + X_n$ where X_n is a Gaussian Wigner Matrix. Here $\|X_n\|_{\mathrm{op}} \sim n^{-1.2}$ (top left), n^{-1} (top right), $n^{-0.5}$ (bottom left) and $n^{-0.2}$ (bottom right).

In general, and as it is very well explained in [Fer00], the study of the perturbative regime mainly concerns applications in quantum mechanics, while the semi-perturbative regime deals with applications in the context of covariance matrices, as in [LP11], or applications to finance, as in [AB12].

1.3.3 Eigenvectors of a perturbed matrix

Currently, the existing results about the spectral properties of perturbed systems mostly concern the eigenvalues, and we have very little knowledge about eigenvectors. We can distinguish two major themes among the works focused on the study of the eigenvectors of a perturbed Hermitian matrix; those that provide bounds on the deviations of these eigenvectors under perturbation, as [OVW16, OVW17, vSW17, Zho17], and those, as [AB12, AB14, ABB14, Ben17], that provide explicit perturbative expansions.

For example, in [AB12], Allez and Bouchaud investigate the stability under a small perturbation of the subspace spanned by some consecutive eigenvectors of a generic symmetric matrix, and in [AB14] they consider the perturbation of a matricial stochastic process and they study the effect of this perturbation on the dynamics of a given eigenvector. In more recent work [Zho17], Zhong gives bounds of deviations of the eigenvector associated to the maximal eigenvalue of the perturbed matrix.

Let us suppose for exemple H_n to be diagonal. In this sense, it is possible to understand the current problem of perturbation as that of a perturbation of the canonical basis. Figure 1.9 illustrates this interpretation of the problem.

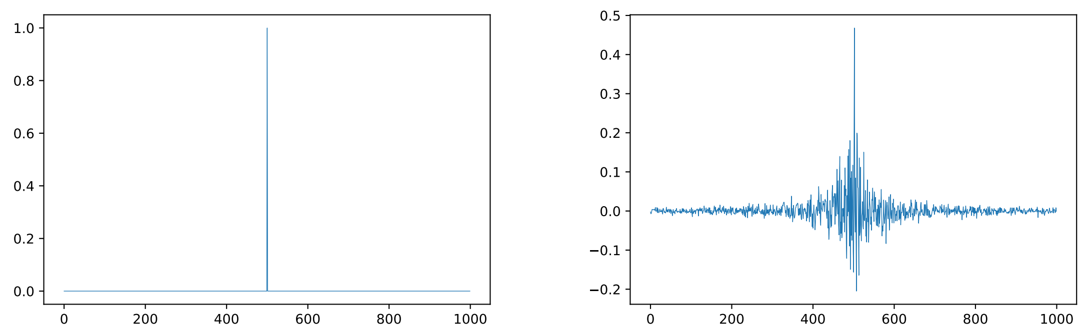

Figure 1.9: **Effect of a perturbation on an eigenvector.** For a diagonal matrix $H_n = \mathrm{diag}(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n})$ and a Gaussian Wigner matrix, X_n, which operator norm is of order $n^{-0.4}$, one can visualize in the left figure the $\frac{n}{2}$-th eigenvector of the original matrix, H_n, and in the right figure the $\frac{n}{2}$-th eigenvector of the perturbed matrix, $H_n + X_n$. Here $n = 1000$.

Major technical difficulties remain to approximate precisely the eigenvectors of a perturbed matrix. However, it is possible to find in the recent literature, some advances on this specific subject. We can cite for example the paper [Ben17] of Benigni, in which he shows that in the case of a perturbation of a diagonal matrix, H_n, by a Wigner matrix, X_n, with operator norm of order $n^{-1/2} \ll \varepsilon \ll 1$, we have for m specifically selected eigenvectors, $(\mathbf{u}_{k_i})_{i=1}^m$, of $H_n + X_n$ and any unit vector $\mathbf{q}$, the convergence in moments of the quantities $(\langle \mathbf{u}_{k_i}, \mathbf{q} \rangle)_{i=1}^m$ to those of a Gaussian vector if X_n is symmetric or to those of a complex Gaussian vector if X_n is Hermitian. By choosing, in particular, a unit vector $\mathbf{q}$ of the canonical basis this result could then specify an average on some coordinates of an eigenvector of the perturbed matrix, $H_n + X_n$.

1.4 Purpose of this thesis

The present thesis is devoted to the study of the sensitivity of the eigenvalues and the eigenvectors of a given operator under small perturbations, and more precisely to the understanding of the effect of a perturbation on the spectrum of a Hermitian matrix by a random matrix with small operator norm and whose entries in the eigenvector basis of the first one were independent, centered and with a variance profile.

Throughout this thesis we will consider the following model:

We perturb an $n \times n$ Hermitian matrix, D_n, by a random Hermitian matrix with small operator norm. Moreover, as we can diagonalize D_n up to a change of basis, we suppose it is diagonal and consider that the entries of the pertubative matrix, X_n, are independent in the eigenvector basis of D_n with a variance profile[1]. Furthermore, to better describe the magnitude of the perturbations, we suppose that the operator norm of X_n is of order 1, introduce a real sequence ε_n that tends to zero, and consider the perturbed matrix

$$D_n^\varepsilon := D_n + \varepsilon_n X_n$$

for different rates of convergence of ε_n to zero. In the following we prove that, depending on the order of magnitude of the perturbation, different regimes appear.

1.4.1 Organization of the thesis

This thesis is organized as follows:

- In Chapter 2, we introduce some useful tools and results which will be used throughout the thesis.

- In Chapter 3, we are interested in the empirical spectral distribution μ_n^ε of the perturbed matrix, $D_n^\varepsilon = D_n + \varepsilon_n X_n$, in the regime where the matrix size n tends to infinity and ε_n tends to 0. We shall prove the following perturbative expansion of μ_n^ε,

$$\begin{aligned}
\mu_n^\varepsilon &\approx \mu_n + \frac{\varepsilon_n}{n}\mathrm{d}Z & &\text{if } \varepsilon_n \ll n^{-1} \\
\mu_n^\varepsilon &\approx \mu_n + \frac{\varepsilon_n}{n}\left(c\mathrm{d}F + \mathrm{d}Z\right) & &\text{if } \varepsilon_n \sim \frac{c}{n} \\
\mu_n^\varepsilon &\approx \mu_n + \varepsilon_n^2\mathrm{d}F & &\text{if } n^{-1} \ll \varepsilon_n \ll 1
\end{aligned}$$

for μ_n the empirical spectral measure of D_n, $\mathrm{d}Z$ a Gaussian random linear form related to the so-called one-dimensional Gaussian free field, and $\mathrm{d}F$ a deterministic

[1] Note that if the perturbing matrix belongs to the GOE or GUE, then its law is invariant under this change of basis

linear form related to free probability theory. We will prove, moreover, that if $n^{-1} \ll \varepsilon_n \ll n^{-1/3}$, then the last of those three convergences can be refined as follows:

$$\mu_n^\varepsilon \;\approx\; \mu_n + \varepsilon_n^2 \mathrm{d}F + \frac{\varepsilon_n}{n}\mathrm{d}Z.$$

- The aim of Chapter 4 is to provide a thorough study of the semi-perturbative regime $n^{-1} \ll \varepsilon_n \ll 1$. We will see that this regime could be decomposed into infinitely many sub-regimes and that for all positive integer p, when $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$, the perturbative expansion of μ_n^ε could be decomposed in the following manner,

$$\mu_n^\varepsilon \;\approx\; \mu_n + \frac{\varepsilon_n}{n}\mathrm{d}Z + \varepsilon_n^2 \mathrm{d}C_2 + \varepsilon_n^4 \mathrm{d}C_4 + \cdots + \varepsilon_n^{2p}\mathrm{d}C_{2p}$$

  for some deterministic linear forms $(\mathrm{d}C_{2k})$. In this chapter we will explain how we should reinforce the hypothesis of our model each time we choose a slower rate of convergence to zero for ε_n and how the deterministic terms appear then.

- Finally, in Chapter 5, we will provide a perturbative expansion of the coordinates of the eigenvectors of the perturbated matrix D_n^ε. This will be done through a perturbative expansion of spectral measures associated to the state defined by a given vector.

Chapters 3 and 5 correspond respectively to the publications [BGEM17] and [BGEM18] done in collaboration with Florent Benaych-Georges and Nathanaël Enriquez. Chapter 4 corresponds to work still undergoing study and unpublished as of yet. Some of the concepts mentioned in this introduction have not been removed from these chapters so that they can be read and understood independently of the rest of this thesis.

Chapter 2

Useful tools and theorems

Contents

2.1 Stieltjes transform 31
2.2 Hellfer-Sjöstrand formula 32
2.3 CLT extension lemma 33
2.4 A functional density lemma 34

The tools and results we use along this thesis, namely the Stieltjes transform, the Helffer-Sjöstrand formula, the CLT extension lemma of Shcherbina and Tirozzi and a functional density lemma with its proof, can be found by the reader in the following sections.

2.1 Stieltjes transform

In this section we present one of the fundamental tools of random matrix theory called Stieltjes transform and which is very useful for proving the convergence of a sequence of measures.

Definition 11 (Stieltjes transform)**.** Given a probability measure μ on $\mathbb{R}$ we define, for any z outside of its support, its Stieltjes transform by

$$s_\mu(z) := \int_{\mathbb{R}} \frac{1}{x-z}\, d\mu(x)$$

One of the main advantages of the Stieltjes transform is that, as it is invertible, we can use the following theorem to study the convergence of a sequence of measures.

Theorem 12. *If $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of measures on $\mathbb{R}$ and μ another measure on $\mathbb{R}$, then μ_n converges almost surely to μ as n tends to infinity if and only if $s_{\mu_n}(z)$ converges almost surely as n tends to infinity to $s_\mu(z)$ for any $z \in \mathbb{C} \setminus \mathbb{R}$.*

Moreover, in the special case of the study of the empirical spectral measure, $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$ of a matrix M_n, this theorem allows us to resume our study to the convergence of the normalized trace of its resolvent matrix,

$$s_{\mu_n}(z) = \frac{1}{n}\sum_{i=1}^n \frac{1}{\lambda_i - z} = \frac{1}{n}\operatorname{Tr}\left(M_n - zI_n\right)^{-1}.$$

Finally, we will use in the next three chapters the following identity,

Lemma 13 (Resolvent expansion)**.** *For two hermitian matrices A and B, and for any z that is not in the spectrum of A, B and $A+B$, we have*

$$\frac{1}{z-(A+B)} = \frac{1}{z-A} + \frac{1}{z-A}B\frac{1}{z-(A+B)}.$$

2.2 Hellfer-Sjöstrand formula

The Hellfer-Sjöstrand formula expresses a regular function ϕ on $\mathbb{R}$ as an integral against functions $x \mapsto \frac{1}{x-z}$ and is very useful to extend to any $\mathcal{C}^p$ compactly supported function on $\mathbb{R}$ a convergence which has been previously proven for test functions of the type $x \mapsto \frac{1}{x-z}$. In the next chapters, this formula will allow us to extend the convergence results initially demonstrated for the resolvent of the Hermitian matrix D_n^ε to its empirical spectral measure against regular functions.

Proposition 14 (Hellfer-Sjöstrand formula)**.** *Let $n \in \mathbb{N}$ and $\phi \in \mathcal{C}^{p+1}(\mathbb{R})$. We define the* almost analytic extension of ϕ of degree p *through*

$$\widetilde{\phi}_p(x+\mathrm{i}y) \ := \ \sum_{k=0}^p \frac{1}{k!}(\mathrm{i}y)^k \phi^{(k)}(x)\,.$$

Let $\chi \in \mathcal{C}_c^\infty(\mathbb{C};[0,1])$ be a smooth cutoff function. Then for any $\lambda \in \mathbb{R}$ satisfying $\chi(\lambda) = 1$ we have

$$\phi(\lambda) \ = \ \frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{\partial}(\widetilde{\phi}_p(z)\chi(z))}{\lambda - z}\,\mathrm{d}^2 z\,,$$

where $\mathrm{d}^2 z$ denotes the Lebesgue measure on $\mathbb{C}$ and $\bar{\partial} := \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$ is the antiholomorphic derivative.

The proof of this formula can be found, e.g. in [BGK16].

2.3 CLT extension lemma

The following CLT extension lemma is borrowed from the paper of Shcherbina and Tirozzi [ST10]. We state here the version that can be found in the Appendix of [BGGM14].

Lemma 15. *Let $(\mathcal{L}, \|\cdot\|)$ be a normed space with a dense subspace $\mathcal{L}_1$ and, for each $n \geq 1$, $(N_n(\phi))_{\phi\in\mathcal{L}}$ a collection of real random variables such that:*

- *for each n , $\phi \longmapsto N_n(\phi)$ is linear,*
- *for each n and each $\phi \in \mathcal{L}$, $\mathbb{E}[N_n(\phi)] = 0$,*
- *there is a constant C such that for each n and each $\phi \in \mathcal{L}$, $\mathrm{Var}(N_n(\phi)) \leq C\|\phi\|^2$,*
- *there is a quadratic form $V : \mathcal{L}_1 \to \mathbb{R}_+$ such that for any $\phi \in \mathcal{L}_1$, we have the convergence in distribution $N_n(\phi) \underset{n\to\infty}{\longrightarrow} \mathcal{N}(0, V(\phi))$.*

Then V is continuous on $\mathcal{L}_1$, can (uniquely) be continuously extended to $\mathcal{L}$ and for any $\phi \in \mathcal{L}$, we have the convergence in distribution $N_n(\phi) \underset{n\to\infty}{\longrightarrow} \mathcal{N}(0, V(\phi))$.

One of the assumptions of previous the lemma concerns a variance domination. The next proposition provides a tool in order to check it. Let us first remind the definition of the Sobolev space $\mathcal{H}_s$.

For $\phi \in L^1(\mathbb{R}, \mathrm{d}x)$, we define

$$\widehat{\phi}(k) := \int \mathrm{e}^{\mathrm{i}kx}\phi(x)\mathrm{d}x \qquad\qquad (k \in \mathbb{R})$$

and, for $s > 0$,

$$\|\phi\|_{\mathcal{H}_s} := \|k \longmapsto (1 + 2|k|)^s\, \widehat{\phi}(k)\|_{L^2}.$$

We define the Sobolev space $\mathcal{H}_s$ as the set of functions with finite $\|\cdot\|_{\mathcal{H}_s}$ norm. Let us now state Proposition 2 of the paper [ST10] of Shcherbina and Tirozzi.

Proposition 16. *For any $s > 0$, there is a constant $C = C(s)$ such that for any n, any $n \times n$ Hermitian random matrix M, and any $\phi \in \mathcal{H}_s$, we have*

$$\mathrm{Var}(\mathrm{Tr}\,\phi(M)) \;\leq\; C\|\phi\|^2_{\mathcal{H}_s} \int_{y=0}^{\infty} y^{2s-1}\mathrm{e}^{-y} \int_{x\in\mathbb{R}} \mathrm{Var}(\mathrm{Tr}((x + \mathrm{i}y - M)^{-1}))\mathrm{d}x\mathrm{d}y.$$

2.4 A functional density lemma

We did not find Lemma 18 in the literature, so we provide its proof. Subsequently, the latter will be used with the CLT extension Lemma 15. Recall that for any $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\varphi_z(x) = \frac{1}{z-x}.$$

Lemma 17. *For any $z \in \mathbb{C}\backslash\mathbb{R}$, we have, in the L^2 sense,*

$$\widehat{\varphi_z} \;=\; (t \longmapsto -\operatorname{sgn}(\mathfrak{Im} z) 2\pi \mathrm{i} \mathbb{1}_{\mathfrak{Im}(z)t>0} \mathrm{e}^{\mathrm{i}tz}) \tag{2.1}$$

and φ_z belongs to each $\mathcal{H}_s$ for any $s \in \mathbb{R}$.

Proof. It is well known that if $\mathfrak{Re} z > 0$, then $\displaystyle\frac{1}{z} = \int_{t=0}^{+\infty} \mathrm{e}^{-tz}\mathrm{d}t$.

Let $z = E + \mathrm{i}\eta$, $E \in \mathbb{R}, \eta > 0$. For any $\xi \in \mathbb{R}$, we have

$$\varphi_z(\xi) = \frac{-\mathrm{i}}{\mathrm{i}(\xi - z)} = -\mathrm{i}\int_{t=0}^{+\infty} \mathrm{e}^{-\mathrm{i}t(\xi - z)}\mathrm{d}t = -\mathrm{i}\int_{t=0}^{+\infty} \mathrm{e}^{-\mathrm{i}t\xi}\mathrm{e}^{\mathrm{i}tz}\mathrm{d}t.$$

We deduce (2.1) for $\mathfrak{Im} z > 0$.
The general result can be deduced by complex conjugation. □

Lemma 18. *Let $\mathcal{L}_1$ denote the linear span of the functions $\varphi_z(x) := \frac{1}{z-x}$, for $z \in \mathbb{C}\backslash\mathbb{R}$. Then the space $\mathcal{L}_1$ is dense in $\mathcal{H}_s$ for any $s \in \mathbb{R}$.*

Proof. We know, by Lemma 17, that $\mathcal{L}_1 \subset \mathcal{H}_s$. Recall first the definition of the Poisson kernel, for $E \in \mathbb{R}$ and $\eta > 0$,

$$P_\eta(E) = \frac{1}{\pi}\frac{\eta}{E^2 + \eta^2} = \frac{1}{2\mathrm{i}\pi}\left(\varphi_{\mathrm{i}\eta}(E) - \varphi_{-\mathrm{i}\eta}(E)\right)$$

and that, by Lemma 17,

$$\widehat{P_\eta}(t) = \mathrm{e}^{-\eta|t|}.$$

Hence for any $f \in \mathcal{H}_s$, we have

$$\|f - P_\eta * f\|^2_{\mathcal{H}_s} = \int (1 + 2|x|)^{2s}|\widehat{f}(x)|^2(1 - \mathrm{e}^{-\eta|x|})^2\mathrm{d}x,$$

so that, by dominated convergence, $P_\eta * f \longrightarrow f$ in $\mathcal{H}_s$ as $\eta \to 0$.

To prove Lemma 18, it suffices to prove that any smooth compactly supported function can be approximated, in $\mathcal{H}_s$, by functions of $\mathcal{L}_1$. So let f be a smooth compactly supported function. By what precedes, it suffices to prove that for any fixed $\eta > 0$, $P_\eta * f$ can be approximated, in $\mathcal{H}_s$, by functions of $\mathcal{L}_1$. For $x \in \mathbb{R}$,

$$\begin{aligned} P_\eta * f(x) &= \frac{1}{\pi}\int f(t)\frac{\eta}{\eta^2+(x-t)^2}\mathrm{d}t \\ &= -\frac{1}{\pi}\int f(t)\mathfrak{Im}(\varphi_{t+\mathrm{i}\eta}(x))\mathrm{d}t \\ &= \frac{1}{2\pi\mathrm{i}}\int f(t)(\varphi_{t-\mathrm{i}\eta}(x)-\varphi_{t+\mathrm{i}\eta}(x))\mathrm{d}t. \end{aligned}$$

Without loss of generality, one can suppose that the support of f is contained in $[0,1]$. Then, for any $n \geq 1$,

$$P_\eta * f(x) = \frac{1}{2n\pi\mathrm{i}}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\varphi_{\frac{k}{n}-\mathrm{i}\eta}(x)-\varphi_{\frac{k}{n}+\mathrm{i}\eta}(x)\right)+R_n(x) \tag{2.2}$$

where for $[t]_n := \lceil nt\rceil/n$,

$$R_n(x)=\frac{1}{2\pi\mathrm{i}}\int f(t)\left(\varphi_{t-\mathrm{i}\eta}(x)-\varphi_{t+\mathrm{i}\eta}(x)\right)-f([t]_n)\left(\varphi_{[t]_n-\mathrm{i}\eta}(x)-\varphi_{[t]_n+\mathrm{i}\eta}(x)\right)\mathrm{d}t.$$

The error term $R_n(x)$ rewrites

$$\begin{aligned} R_n(x) &= \frac{1}{2\pi\mathrm{i}}\int(f(t)-f([t]_n))(\varphi_{t-\mathrm{i}\eta}-\varphi_{t+\mathrm{i}\eta})(x)\mathrm{d}t \\ &\quad+\frac{1}{2\pi\mathrm{i}}\int f([t]_n)(\varphi_{t-\mathrm{i}\eta}-\varphi_{[t]_n-\mathrm{i}\eta}+\varphi_{t+\mathrm{i}\eta}-\varphi_{[t]_n+\mathrm{i}\eta})(x)\mathrm{d}t. \end{aligned}$$

Now, note that for any $t \in \mathbb{R}$ and $\eta \in \mathbb{R}\backslash\{0\}$, we have by Lemma 17,

$$\widehat{\varphi_{t+i\eta}} = (x \mapsto -\operatorname{sgn}(\eta)2\pi\mathrm{i}\,\mathbb{1}_{\eta x>0}\,\mathrm{e}^{\mathrm{i}xz}),$$

so that when, for example, $\eta > 0$, for any $t \in \mathbb{R}$,

$$\|\varphi_{t+\mathrm{i}\eta}\|^2_{\mathcal{H}_s} = 4\pi^2\int_0^\infty(1+2|x|)^{2s}\mathrm{e}^{-2\eta x}\mathrm{d}x$$

does not depend on t and for any $t, t' \in \mathbb{R}$,

$$\begin{aligned} \|\varphi_{t+\mathrm{i}\eta}-\varphi_{t'+\mathrm{i}\eta}\|^2_{\mathcal{H}_s} &= 4\pi^2\int_0^\infty(1+2|x|)^{2s}|\mathrm{e}^{\mathrm{i}tx}-\mathrm{e}^{\mathrm{i}t'x}|^2\mathrm{e}^{-2\eta x}\mathrm{d}x \\ &= 4\pi^2\int_0^\infty(1+2|x|)^{2s}|\mathrm{e}^{\mathrm{i}(t-t')x}-1|^2\mathrm{e}^{-2\eta x}\mathrm{d}x \end{aligned}$$

depends only on $t'-t$ end tends to zero (by dominated convergence) when $t'-t \to 0$.

We deduce that $\|R_n\|_{\mathcal{H}_s} \longrightarrow 0$ as $n \to \infty$, which closes the proof, by (2.2). $\square$

Chapter 3

Empirical spectral distribution of a matrix under perturbation

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3.1 Introduction

In this chapter, we provide a perturbative expansion for the empirical spectral distribution of a Hermitian matrix with large size perturbed by a random matrix with small operator norm whose entries in the eigenvector basis of the first one are independent with a variance profile. More explicitly, let D_n be an $n \times n$ Hermitian matrix, that, up to a change of basis, we suppose diagonal[1]. We denote by μ_n the empirical spectral distribution of D_n. This matrix is additively perturbed by a random Hermitian matrix $\varepsilon_n X_n$ whose entries are chosen at random independently and scaled so that the operator norm of X_n has order one. We are interested in the empirical spectral distribution μ_n^ε of

$$D_n^\varepsilon \ := \ D_n + \varepsilon_n X_n$$

in the regime where the matrix size n tends to infinity and ε_n tends to 0. We shall prove that, depending on the order of magnitude of the perturbation, several regimes can appear. We suppose that μ_n converges to a limiting measure $\rho(\lambda)\mathrm{d}\lambda$ and that the variance profile of the entries of X_n has a macroscopic limit σ_d on the diagonal and σ elsewhere. We then prove that there is a deterministic function F and a Gaussian random linear form $\mathrm{d}Z$ on the space of $\mathcal{C}^6$ functions on $\mathbb{R}$, both depending only on the limit parameters of the model ρ, σ and σ_d such that if one defines the distribution $\mathrm{d}F : \phi \longmapsto -\int \phi'(s)F(s)\mathrm{d}s$, then, for large n:

$$\mu_n^\varepsilon \ \approx \ \mu_n + \frac{\varepsilon_n}{n}\mathrm{d}Z \qquad \text{if } \varepsilon_n \ll n^{-1} \tag{3.1}$$

$$\mu_n^\varepsilon \ \approx \ \mu_n + \frac{\varepsilon_n}{n}\left(c\mathrm{d}F + \mathrm{d}Z\right) \qquad \text{if } \varepsilon_n \sim \frac{c}{n} \tag{3.2}$$

$$\mu_n^\varepsilon \ \approx \ \mu_n + \varepsilon_n^2\mathrm{d}F \qquad \text{if } n^{-1} \ll \varepsilon_n \ll 1 \tag{3.3}$$

and if, moreover, $n^{-1} \ll \varepsilon_n \ll n^{-1/3}$, then convergence (3.3) can be refined as follows:

$$\mu_n^\varepsilon \ \approx \ \mu_n + \varepsilon_n^2\mathrm{d}F + \frac{\varepsilon_n}{n}\mathrm{d}Z. \tag{3.4}$$

In Section 3.3 several figures show a very good matching of random simulations with these theoretical results. The definitions of the function F and of the process Z are given below in (3.6) and (3.7). In many cases, the linear form $\mathrm{d}F$ can be interpreted as the integration with respect to the signed measure $F'(x)\mathrm{d}x$. The function F is related to free probability theory, as explained in Section 3.4 below, whereas the linear form $\mathrm{d}Z$ is related to the so-called one-dimensional Gaussian free field defined, for instance, at [Dub09, Sect. 4.2].

[1] If the perturbing matrix belongs to the GOE or GUE, then its law is invariant under this change of basis, hence our results in fact apply to any self-adjoint matrix D_n.

If the variance profile of X_n is constant, then it is precisely the Laplacian of the Gaussian free field, defined in the sense of distributions.

The transition at $\varepsilon_n \sim n^{-1}$ is the well-known transition, in quantum mechanics, where the *perturbative regime* ends. Indeed, one can distinguish the two following regimes:

- The regime $\varepsilon_n \ll n^{-1}$, called the *perturbative regime* (see [Fer00]): the size of the perturbation (i.e. its operator norm) is much smaller than the typical spacing between two consecutive eigenvalues (level spacing), which is of order n^{-1} in our setting.

- The regime $n^{-1} \ll \varepsilon_n \ll 1$, sometimes called the *semi-perturbative regime*, where the size of the perturbation is not small compared to the level spacing. This regime concerns many applications [LP11, AB12] in the context of covariance matrices and applications to finance.

A surprising fact discovered during this study is that the semi-perturbative regime $n^{-1} \ll \varepsilon_n \ll 1$ decomposes into infinitely many sub-regimes. In the case $n^{-1} \ll \varepsilon_n \ll n^{-1/3}$, the expansion of $\mu_n^\varepsilon - \mu_n$ contains a single deterministic term before the random term $\frac{\varepsilon_n}{n}\mathrm{d}Z$. In the case $n^{-1/3} \ll \varepsilon_n \ll n^{-1/5}$, the expansion of $\mu_n^\varepsilon - \mu_n$ contains two of them. More generally, for all positive integer p, when $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$, the expansion contains p of them. For computational complexity reasons, the only case we state explicitly is the first one. We refer the reader to Section 3.6.5 for a discussion around this point.

In the papers [WW95, AB12, AB14, BABP16, ABB14], Wilkinson, Walker, Allez, Bouchaud *et al* have investigated some problems related to this one. Some of these works were motivated by the estimation of a matrix out of the observation of its noisy version. The present work differs from these ones mainly by the facts that firstly, we are interested in the perturbations of the *global* empirical distribution of the eigenvalues and not of a single one, and secondly, we push our expansion up to the random term, which does not appear in these papers. Besides, the noises they consider have constant variance profiles (either a Wigner-Dyson noise in the four first cited papers or a rotationally invariant noise in the fifth one). The transition at $\varepsilon_n \sim n^{-1}$ between the perturbative and the semi-perturbative regimes is already present in these texts. They also consider the transition between the perturbative regime $\varepsilon_n \ll 1$ and the *non perturbative* regime $\varepsilon_n \asymp 1$. As explained above, we exhibit the existence of an infinity of sub-regimes in this transition and focus on $\varepsilon_n \ll 1$ for the first order of the expansion and to $\varepsilon_n \ll n^{-1/3}$ for the second (and last) order. The study of other sub-regimes is postponed to forthcoming papers.

This chapter is organized as follows. Results, examples and comments are given in Sections 3.2 to 3.4, while the rest of the chapter is devoted to the proofs, except for Section 3.6.5, where we discuss the sub-regimes mentioned above.

Notations. For a_n, b_n some real sequences, $a_n \ll b_n$ (resp. $a_n \sim b_n$) means that a_n/b_n tends to 0 (resp. to 1). Also, $\xrightarrow{P}$ and $\xrightarrow{\text{dist.}}$ stand respectively for convergence in probability and convergence in distribution for all finite marginals.

3.2 Main result

3.2.1 Definition of the model and assumptions

For all positive integer n, we consider a real diagonal matrix $D_n = \operatorname{diag}(\lambda_n(1), \ldots, \lambda_n(n))$, as well as a Hermitian random matrix

$$X_n = \frac{1}{\sqrt{n}}[x_{i,j}^n]_{1\leq i,j\leq n}$$

and a positive number ε_n. The normalizing factor $n^{-1/2}$ and our hypotheses below ensure that the operator norm of X_n is of order one. We then define, for all n,

$$D_n^\varepsilon := D_n + \varepsilon_n X_n.$$

We now introduce the probability measures μ_n and μ_n^ε as the respective uniform distributions on the eigenvalues (with multiplicity) of D_n and D_n^ε. Our aim is to give a perturbative expansion of μ_n^ε around μ_n.

We make the following hypotheses:

(a) the entries $x_{i,j}^n$ of $\sqrt{n}X_n$ are independent (up to symmetry) random variables, centered, with variance denoted by $\sigma_n^2(i,j)$, such that $\mathbb{E}|x_{i,j}^n|^8$ is bounded uniformly on n, i, j,

(b) there are $f, \sigma_{\mathrm{d}}, \sigma$ real functions defined respectively on $[0,1]$, $[0,1]$ and $[0,1]^2$ such that, for each $x \in [0,1]$,

$$\lambda_n(\lfloor nx \rfloor) \underset{n\to\infty}{\longrightarrow} f(x) \quad \text{and} \quad \sigma_n^2(\lfloor nx \rfloor, \lfloor nx \rfloor) \underset{n\to\infty}{\longrightarrow} \sigma_{\mathrm{d}}(x)^2$$

and for each $x \neq y \in [0,1]$,

$$\sigma_n^2(\lfloor nx \rfloor, \lfloor ny \rfloor) \underset{n\to\infty}{\longrightarrow} \sigma^2(x,y).$$

We make the following hypothesis about the rate of convergence:

$$\eta_n := \max\{n\varepsilon_n, 1\} \times \sup_{1\leq i\neq j\leq n} \left(|\sigma_n^2(i,j) - \sigma^2(i/n, j/n)| + |\lambda_n(i) - f(i/n)|\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Let us now make some assumptions on the limiting functions σ and f:

(c) the function f is bounded and the push-forward of the uniform measure on $[0,1]$ by the function f has a density ρ with respect to the Lebesgue measure on $\mathbb{R}$ and a compact support denoted by $\mathcal{S}$,

(d) the variance of the entries of X_n essentially depends on the eigenspaces of D_n, namely, there exists a symmetric function $\tau(\,\cdot\,,\,\cdot\,)$ on $\mathbb{R}^2$ such that for all $x \neq y$, $\sigma^2(x,y) = \tau(f(x), f(y))$,

(e) the following regularity property holds: there exist $\eta_0 > 0, \alpha > 0$ and $C < \infty$ such that for almost all $s \in \mathbb{R}$, for all $t \in [s-\eta_0, s+\eta_0]$, $\quad |\tau(s,t)\rho(t) - \tau(s,s)\rho(s)| \leq C|t-s|^\alpha$.

We add a last assumption which strengthens assumption (c) and makes it possible to include the case where the set of eigenvalues of D_n contains some outliers:

(f) there is a real compact set $\widetilde{\mathcal{S}}$ such that

$$\max_{1\leq i\leq n} \operatorname{dist}(\lambda_n(i), \widetilde{\mathcal{S}}) \underset{n\to\infty}{\longrightarrow} 0.$$

Remark 1 (About the hypothesis that D_n is diagonal)**.** (i) If the perturbing matrix X_n belongs to the GOE (resp. to the GUE), then its law is invariant under conjugation by any orthogonal (resp. unitary) matrix. It follows that in this case, our results apply to any real symmetric (resp. Hermitian) matrix D_n with eigenvalues $\lambda_n(i)$ satisfying the above hypotheses.

(ii) As explained after Proposition 20 below, we conjecture that when the variance profile of X_n is constant, for $\varepsilon_n \gg n^{-1}$, we do not need the hypothesis that D_n is diagonal neither. However, if the perturbing matrix does not have a constant variance profile, then for a non-diagonal D_n and $\varepsilon \gg n^{-1}$, the spectrum of D_n^ε should depend heavily on the relation between the eigenvectors of D_n and the variance profile, which implies that our results should not remain true.

(iii) At last, it is easy to see that the random process (Z_ϕ) introduced at (3.7) satisfies, for any test function ϕ,

$$\frac{1}{\varepsilon_n}\sum_{i=1}^n \left(\phi(\lambda_n(i) + \frac{\varepsilon_n}{\sqrt{n}} x_{ii}) - \phi(\lambda_n(i))\right) \quad \underset{n\to\infty}{\overset{\text{dist.}}{\longrightarrow}} \quad Z_\phi.$$

Thus, regardless to the variance profile, the convergence of (3.8) rewrites, informally,

$$\mu_n^\varepsilon = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_n(i) + (\varepsilon_n/\sqrt{n})x_{ii}} + o(\varepsilon_n/n). \tag{3.5}$$

A so simple expression, up to a $o(\varepsilon_n/n)$ error, of the empirical spectral distribution of D_n^ε, with some independent translations $\frac{\varepsilon}{\sqrt{n}} x_{ii}$, should not remain true without the hypothesis that D_n is diagonal or that the distribution of X_n is invariant under conjugation.

3.2.2 Main result

Recall that the *Hilbert transform*, denoted by $H[u]$, of a function u, is the function

$$H[u](s) := \mathrm{p.\,v.} \int_{t\in\mathbb{R}} \frac{u(t)}{s-t}\mathrm{d}t$$

and define the function

$$F(s) = -\rho(s)H[\tau(s,\cdot)\rho(\cdot)](s). \tag{3.6}$$

Note that, by assumptions (c) and (e), F is well defined and supported by $\mathcal{S}$. Besides, for any ϕ supported on an interval where F is $\mathcal{C}^1$,

$$-\int \phi'(s)F(s)\mathrm{d}s = \int \phi(s)\mathrm{d}F(s),$$

where $\mathrm{d}F(s)$ denotes the measure $F'(s)\mathrm{d}s$.

We also introduce the centered Gaussian field, $(Z_\phi)_{\phi\in\mathcal{C}^6}$, indexed by the set of $\mathcal{C}^6$ complex functions on $\mathbb{R}$, with covariance defined by

$$\mathbb{E}Z_\phi Z_\psi \;=\; \int_0^1 \sigma_{\mathrm{d}}(t)^2\phi'(f(t))\psi'(f(t))\mathrm{d}t \qquad \text{and} \qquad \overline{Z_\psi} \;=\; Z_{\overline{\psi}}. \tag{3.7}$$

Note that the process $(Z_\phi)_{\phi\in\mathcal{C}^6}$ can be represented, for (B_t) is the standard one-dimensional Brownian motion, as

$$Z_\phi = \int_0^1 \sigma_{\mathrm{d}}(t)\phi'(f(t))\mathrm{d}B_t.$$

Theorem 19. *For all compactly supported $\mathcal{C}^6$ function ϕ on $\mathbb{R}$, the following convergences hold:*

- **Perturbative regime:** *if $\varepsilon_n \ll n^{-1}$, then,*

$$n\varepsilon_n^{-1}(\mu_n^\varepsilon - \mu_n)(\phi) \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad Z_\phi. \tag{3.8}$$

- **Critical regime:** *if $\varepsilon_n \sim c/n$, with c constant, then,*

$$n\varepsilon_n^{-1}(\mu_n^\varepsilon - \mu_n)(\phi) \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad -c\int \phi'(s)F(s)\mathrm{d}s + Z_\phi. \tag{3.9}$$

- **Semi-perturbative regime:** *if $n^{-1} \ll \varepsilon_n \ll 1$, then,*

$$\varepsilon_n^{-2}(\mu_n^\varepsilon - \mu_n)(\phi) \quad \xrightarrow[n\to\infty]{P} \quad -\int \phi'(s)F(s)\mathrm{d}s, \tag{3.10}$$

and if, moreover, $n^{-1} \ll \varepsilon_n \ll n^{-1/3}$, then,

$$n\varepsilon_n^{-1}\left((\mu_n^\varepsilon - \mu_n)(\phi) + \varepsilon_n^2\int \phi'(s)F(s)\mathrm{d}s\right) \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad Z_\phi. \tag{3.11}$$

Remark 2 (Sub-regimes for $n^{-1/3} \ll \varepsilon_n \ll 1$)**.** In the semi-perturbative regime, the reason why we provide an expansion up to a random term, only for $\varepsilon_n \ll n^{-1/3}$, is that the study of the regime $n^{-1/3} \ll \varepsilon_n \ll 1$ up to such a precision, requires further terms in the expansion of the resolvent of D_n^ε that make appear, beside $\mathrm{d}F$, additional determistic terms of smaller order, which are much larger than the probabilistic term containing Z_ϕ. The computation becomes rather intricate without any clear recursive formula. As we will see in Section 3.6.5, there are infinitely many regimes. Precisely, for any positive integer p, when $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$, there are p deterministic terms in the expansion before the term in Z_ϕ.

Remark 3 (Local law)**.** The approximation

$$\mu_n^\varepsilon(I) \;\approx\; \mu_n(I) + \varepsilon_n^2 \int_I \mathrm{d}F$$

of (3.10) should stay true even for intervals I with size tending to 0 as the dimension n grows, as long as the size of I stays much larger than the right-hand side term of (3.30), as can be seen from Proposition 23.

Remark 4. The second part of Hypothesis (b), concerning the speed of convergence of the profile of the spectrum of D_n as well as of the variance of its perturbation, is needed in order to express the expansion of $\mu_n^\varepsilon - \mu_n$ in terms of limit parameters of the model σ and ρ. We can remove this hypothesis and get analogous expansions where the terms $\mathrm{d}F$ and $\mathrm{d}Z$ are replaced by their discrete counterparts $\mathrm{d}F_n$ and $\mathrm{d}Z_n$, defined thanks to the "finite n" empirical versions of the limit parameters σ and ρ.

3.3 Examples

3.3.1 Uniform measure perturbation by a band matrix

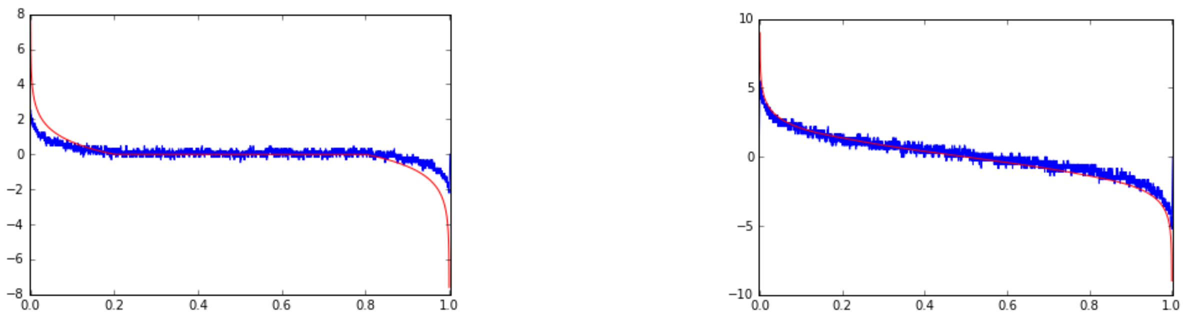
Here, we consider the case where $f(x) = x$, $\sigma_{\mathrm{d}}(x) \equiv m$ and $\sigma(x,y) = \mathbb{1}_{|y-x|\leq \ell}$, for some constants $m \geq 0$ and $\ell \in [0,1]$ (the relative width of the band). In this case, $\tau(\,\cdot\,,\,\cdot\,) = \sigma(\,\cdot\,,\,\cdot\,)^2$, hence

$$F(s) \;=\; \mathbb{1}_{(0,1)}(s)\,\mathrm{p.\,v.}\int_t \frac{\tau(s,t)}{s-t}\mathrm{d}t \;=\; -\mathbb{1}_{(0,1)}(s)\log\frac{\ell\wedge(1-s)}{\ell\wedge s} \tag{3.12}$$

and $(Z_\phi)_{\phi\in\mathcal{C}^6}$ is the centered complex Gaussian process with covariance defined by

$$\mathbb{E}Z_\phi\overline{Z_\psi} \;=\; m^2\int_0^1 \phi'(t)\,\overline{\psi'(t)}\,\mathrm{d}t \qquad \text{and} \qquad \overline{Z_\psi} \;=\; Z_{\overline{\psi}}.$$

Theorem 19 is then illustrated by Figure 3.1, where we ploted the cumulative distribution functions.



(a) $n = 10^4$, $\varepsilon_n = n^{-0.4}$, $\ell = 0.2$ (b) $n = 10^4$, $\varepsilon_n = n^{-0.4}$, $\ell = 0.8$

Figure 3.1: **Deforming the uniform distribution by a band matrix.** Cumulative distribution function of $\varepsilon_n^{-2}(\mu_n^\varepsilon - \mu_n)$ (in blue) and function $F(\,\cdot\,)$ of (3.12) (in red). The non smoothness of the blue curves results of the noise term Z_ϕ in Theorem 19. Each graphic is realized thanks to one single matrix (no averaging) perturbed by a real Gaussian band matrix.

3.3.2 Triangular pulse perturbation by a Wigner matrix

Here, we consider the case where $\rho(x) = (1 - |x|)\mathbb{1}_{[-1,1]}(x)$, $\sigma_d \equiv m$, for some real constant m, and $\sigma \equiv 1$ (what follows can be adapted to the case $\sigma(x, y) = \mathbb{1}_{|y-x|\le\ell}$, with a bit longer formulas). In this case, thanks to the formula (9.6) of $H[\rho(\,\cdot\,)]$ given p. 509 of [Kin09], we get

$$F(s) \;=\; (1 - |s|)\mathbb{1}_{[-1,1]}(s)\,\{(1 - s)\log(1 - s) - (1 + s)\log(1 + s) + 2s\log|s|\}\,. \tag{3.13}$$

and the covariance of $(Z_\phi)_{\phi\in\mathcal{C}^6}$ is given by

$$\mathbb{E}Z_\phi\overline{Z_\psi} \;=\; m^2\int_{-1}^{1}(1 - |t|)\;\phi'(t)\;\overline{\psi'(t)}\;\mathrm{d}t \qquad \text{and} \qquad \overline{Z_\psi} \;=\; Z_{\overline{\psi}}.$$

Theorem 19 is then illustrated by Figure 3.2 in the case where $\varepsilon_n \gg n^{-1/2}$. In Figure 3.2, we implicitly use some test functions of the type $\phi(x) = \mathbb{1}_{x\in I}$ for some intervals I. These functions are not $\mathcal{C}^6$, and one can easily see that for $\varepsilon_n \ll n^{-1/2}$, Theorem 19 cannot work for such functions. However, considering imaginary parts of Stietljes transforms, i.e. test functions

$$\phi(x) = \frac{1}{\pi}\frac{\eta}{(x - E)^2 + \eta^2} \qquad (E \in \mathbb{R},\ \eta > 0)$$

gives a perfect matching between the predictions from Theorem 19 and numerical simulations, also for $\varepsilon_n \ll n^{-1/2}$ (see Figure 3.3, where we use Proposition 22 and (3.17) to compute the theoretical limit).

3.3.3 Parabolic pulse perturbation by a Wigner matrix

Here, we consider the case where $\rho(x) = \frac{3}{4}(1 - x^2)\mathbb{1}_{[-1,1]}(x)$, $\sigma_d \equiv m$, for some real constant m, and $\sigma \equiv 1$ (again, this can be adapted to the case $\sigma(x, y) = \mathbb{1}_{|y-x|\le\ell}$). Theorem 19 is

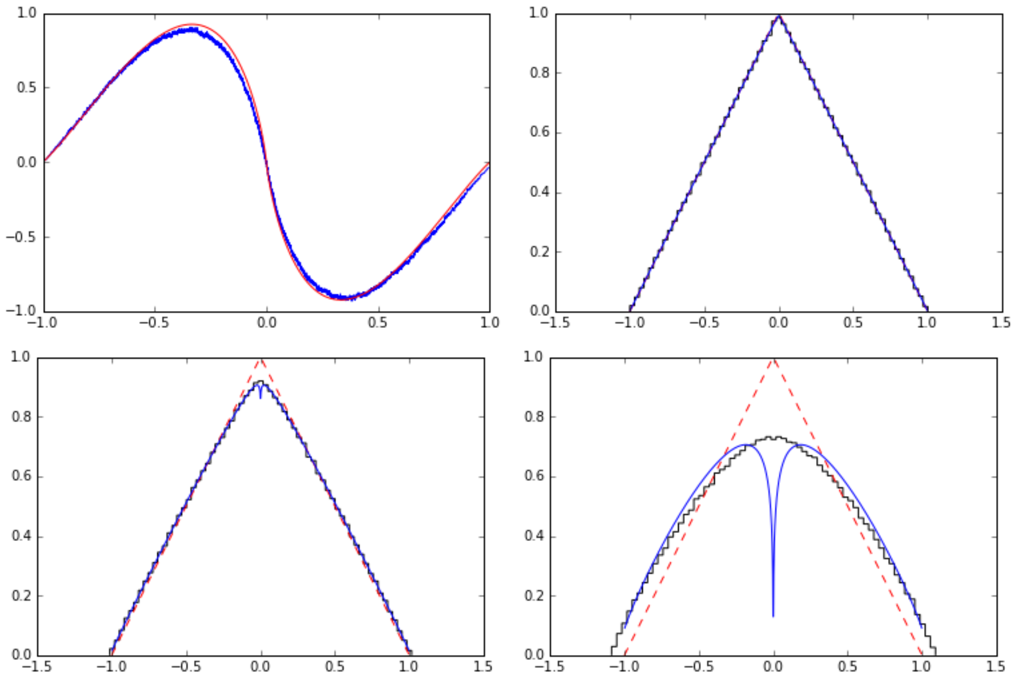


Figure 3.2: **Triangular pulse perturbation by a Wigner matrix: density and cumulative distribution function.** *Top left:* Cumulative distribution function of $\varepsilon_n^{-2}(\mu_n^\varepsilon - \mu_n)$(in blue) and function $F(\,\cdot\,)$ of (3.13) (in red). *Top right and bottom:* Density ρ (red dashed line), histogram of the eigenvalues of D_n^ε (in black) and theoretical density $\rho + \varepsilon_n^2 F'(s)$ of the eigenvalues of D_n^ε as predicted by Theorem 19 (in blue). Here, $n = 10^4$ and $\varepsilon_n = n^{-\alpha}$, with $\alpha = 0.25$ (up left), $\alpha = 0.4$ (up right), 0.25 (bottom left) and 0.1 (bottom right).

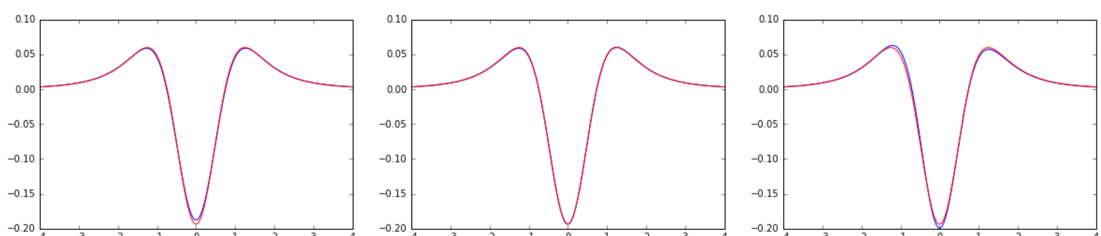


Figure 3.3: **Triangular pulse perturbation by a Wigner matrix: Stieltjes transform.** Imaginary part of the Stieltjes transform of $\varepsilon_n^{-2}(\mu_n^\varepsilon - \mu_n)$ (in blue) and of the measure dF (in red) at $z = E + \mathrm{i}$ as a function of the real part E for different values of ε_n. Here, $n = 10^4$ and $\varepsilon_n = n^{-\alpha}$, with $\alpha = 0.2$, 0.5 and 0.8 (from left to right).

then illustrated by Figure 3.4. In this case, thanks to the formula (9.10) of $H[\rho(\,\cdot\,)]$ given p. 509 of [Kin09], we get

$$F(s) \;=\; -\frac{9}{16}(1-s^2)\mathbb{1}_{[-1,1]}(s)\left\{2s - (1-s^2)\ln\left|\frac{s-1}{s+1}\right|\right\} \tag{3.14}$$

and the covariance of $(Z_\phi)_{\phi\in\mathcal{C}^6}$ is given by

$$\mathbb{E}Z_\phi\overline{Z_\psi} \;=\; \frac{3m^2}{4}\int_{-1}^{1}(1-t^2)\;\phi'(t)\;\overline{\psi'(t)}\;\mathrm{d}t \qquad \text{and} \qquad \overline{Z_\psi} \;=\; Z_{\overline{\psi}}.$$

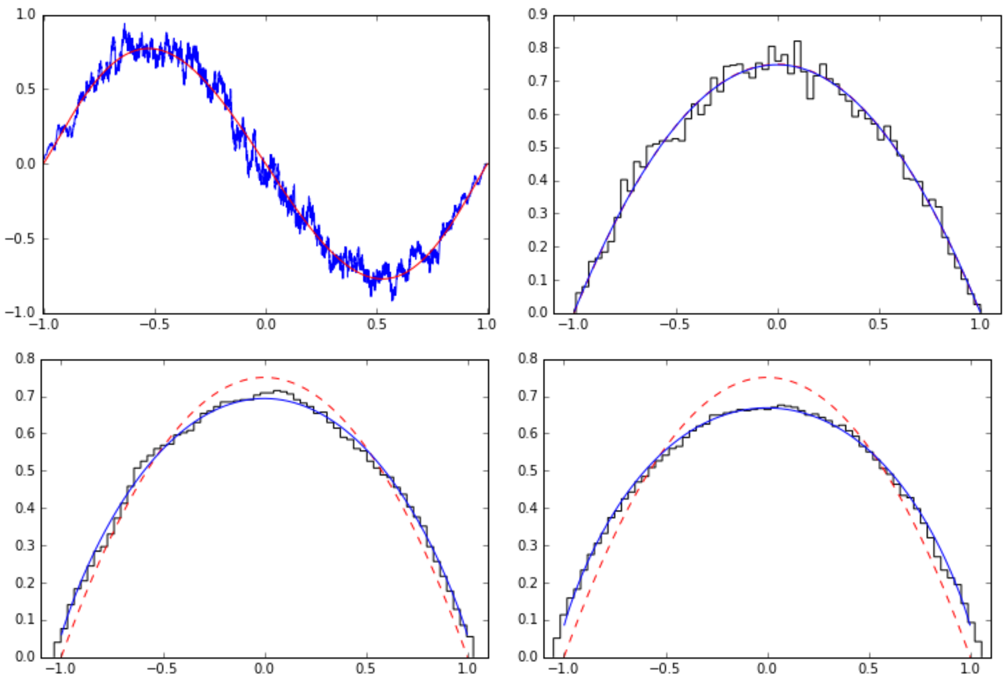


Figure 3.4: **Parabolic pulse perturbation by a Wigner matrix.** *Top left:* Cumulative distribution function of $\varepsilon_n^{-2}(\mu_n^\varepsilon - \mu_n)$(in blue) and function $F(\,\cdot\,)$ of (3.14) (in red). *Top right and bottom:* Density ρ (red dashed line), histogram of the eigenvalues of D_n^ε (in black) and theoretical density $\rho+\varepsilon_n^2 F'(s)$ of the eigenvalues of D_n^ε as predicted by Theorem 19 (in blue). Here, $n = 10^4$ and $\varepsilon_n = n^{-\alpha}$, with $\alpha = 0.25$ (up left), $\alpha = 0.4$ (up right), 0.2 (bottom left) and 0.18 (bottom right).

3.4 Relation to free probability theory

Let us now explain how this work is related to free probability theory. If, instead of letting ε_n tend to zero, one considers the model

$$D_n^t \;:=\; D_n + \sqrt{t}X_n$$

for a fixed $t > 0$, then, by [CG93a, CG93b, Shl96, AZ06], the empirical eigenvalue distribution of D_n^t has a limit as $n \to \infty$, that we shall denote here by μ_t. The law μ_t can be interpreted as the law of the sum of two elements in a non-commutative probability space which are free with an amalgamation over a certain sub-algebra (see [Shl96] for more details). The following proposition relates the function F from (3.6) to the first order expansion of μ_t around $t = 0$.

Proposition 20. *For any $z \in \mathbb{C}\backslash\mathbb{R}$, we have*

$$\frac{\partial}{\partial t}_{|t=0} \int \frac{\mathrm{d}\mu_t(\lambda)}{z-\lambda} \;=\; -\int \frac{F(\lambda)}{(z-\lambda)^2}\mathrm{d}\lambda \;=\; -\int F(\lambda)\frac{\partial}{\partial\lambda}\left(\frac{1}{z-\lambda}\right)\mathrm{d}\lambda.$$

This is related to the fact that in Equations (3.1)–(3.4), for ε_n large enough, the term $\varepsilon_n^2 \mathrm{d}F$ is the leading term.

In the particular case where X_n is a Wigner matrix, μ_t is the free convolution of the measure $\rho(\lambda)\mathrm{d}\lambda$ with a semicircle distribution and admits a density ρ_t, by [Bia97b, Cor. 2]. Then, Theorem 19 makes it possible to formally recover the *free Fokker-Planck equation with null potential*:

$$\begin{cases} \frac{\partial}{\partial t}\rho_t(s) + \frac{\partial}{\partial s}\{\rho_t(s)H[\rho_t](s)\} = 0, \\ \rho_0(s) = \rho(s), \end{cases}$$

where $H[\rho_t]$ denotes the Hilbert transform of ρ_t. This equation is also called *McKean-Vlasov* (or *Fokker-Planck*) *equation with logarithmic interaction* (see [Bia97a, BS98, BS01]).

Note also that when X_n is a Wigner matrix, the hypothesis that D_n is diagonal is not required to have the convergence of the empirical eigenvalue distribution of D_n^t to μ_t as $n \to \infty$. This suggests that, even for non diagonal D_n, the convergence of (3.10) still holds when X_n is a Wigner matrix.

Proof of Proposition 20. By [Shl96, Th. 4.3], we have

$$\int \frac{\mathrm{d}\mu_t(\lambda)}{z-\lambda} \;=\; \int_{x=0}^{1} C_t(x,z)\mathrm{d}x, \tag{3.15}$$

where $C_t(x,z)$ is bounded by $|\mathfrak{Im} z|^{-1}$ and satisfies the fixed-point equation

$$C_t(x,z) = \frac{1}{z - f(x) - t\int_{y=0}^{1}\sigma^2(x,y)C_t(y,z)\mathrm{d}y}.$$

Hence as $t \to 0$, $C_t(x,z) \longrightarrow \frac{1}{z-f(x)}$ uniformly in x. Thus

$$\begin{aligned} C_t(x,z) - \frac{1}{z-f(x)} &= \frac{t\int_{y=0}^{1}\sigma^2(x,y)C_t(y,z)\mathrm{d}y}{(z-f(x)-t\int_{y=0}^{1}\sigma^2(x,y)C_t(y,z)\mathrm{d}y)(z-f(x))} \\ &= t\frac{1}{(z-f(x))^2}\int_{y=0}^{1}\sigma^2(x,y)C_t(y,z)\mathrm{d}y + o(t) \\ &= t\frac{1}{(z-f(x))^2}\int_{y=0}^{1}\frac{\sigma^2(x,y)}{z-f(y)}\mathrm{d}y + o(t) \end{aligned}$$

where each $o(t)$ is uniform in $x \in [0,1]$. Then, by (3.15), we deduce that

$$\frac{\partial}{\partial t}_{|t=0} \int \frac{\mathrm{d}\mu_t(\lambda)}{z-\lambda} = \int_{(x,y)\in[0,1]^2} \frac{\sigma^2(x,y)}{(z-f(x))^2(z-f(y))} \mathrm{d}x\mathrm{d}y.$$

The right-hand side term of the previous equation is precisely the number $B(z)$ introduced at (3.17) below. Then, one concludes using Proposition 22 from Section 3.6.1. □

3.5 Strategy of the proof

We shall first prove the convergence results of Theorem 19 for test functions ϕ of the form $\varphi_z(x) := \frac{1}{z-x}$. This is done in Section 3.6 by writing an expansion of the resolvent of D_n^ε.

Once we have proved that the convergences hold for the resolvent of D_n^ε, we can extend them to the larger class of compactly supported $\mathcal{C}^6$ functions on $\mathbb{R}$.

In Section 3.7, we use the Helffer-Sjöstrand formula to extend the convergence in probability in the semi-pertubative regime (3.10) to the case of compactly supported $\mathcal{C}^6$ functions on $\mathbb{R}$.

In Section 3.8, the convergences in distribution (3.8), (3.9) and (3.11) are proved in two steps. The overall strategy is to apply an extension lemma of Shcherbina and Tirozzi which states that a CLT that applies to a sequence of *centered* random linear forms on some space can be extended, by density, to a larger space, as long as the variance of the image of these random linear forms by a function ϕ of the larger space is uniformly bounded by the norm of ϕ. Therefore, our task is twofold. We need first to prove that the sequences of variables involved in the convergences (3.8), (3.9) and (3.11) can be replaced by their *centered* counterparts $n\varepsilon_n^{-1}(\mu_n^\varepsilon(\phi) - \mathbb{E}[\mu_n^\varepsilon(\phi)])$ (i.e. they differ by $o(1)$). In a second step, we dominate the variance of these latter variables, in order to apply the extension lemma which is precisely stated in Chapter 2 as Lemma 15.

3.6 Stieltjes transforms convergence

As announced in the previous section, we start with the proof of Theorem 19 in the special case of test functions of the type $\varphi_z := \frac{1}{z-x}$. We decompose it into two propositions. Their statement and proof are the purpose of the three following subsections. The two last subsections 3.6.4 and 3.6.5 are devoted respectively to a local type convergence result and to a discussion about the possibility of an extension of the expansion result to a wider range of rate of convergence of ε_n, namely beyond $n^{-1/3}$.

3.6.1 Two statements

Let denote, for $z \in \mathbb{C}\backslash\mathbb{R}$,

$$Z(z) := Z_{\varphi_z} \quad \text{for} \quad \varphi_z(x) := \frac{1}{z-x} \tag{3.16}$$

where $(Z_\phi)_{\phi\in\mathcal{C}^6}$ is the Gaussian field with covariance defined by (3.7). We also introduce, for $z \in \mathbb{C}\backslash\mathbb{R}$,

$$B(z) := \int_{(s,t)\in[0,1]^2} \frac{\sigma^2(s,t)}{(z-f(s))^2(z-f(t))}\mathrm{d}s\mathrm{d}t \tag{3.17}$$

and

$$\Delta \mathrm{G}_n(z) \ := \ (\mu_n^\varepsilon - \mu_n)(\varphi_z) \ = \ \frac{1}{n}\operatorname{Tr}\frac{1}{z-D_n^\varepsilon} - \frac{1}{n}\operatorname{Tr}\frac{1}{z-D_n}. \tag{3.18}$$

Proposition 21. *Under Hypotheses (a), (b), (f),*

- *if $\varepsilon_n \ll n^{-1}$, then for all $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$n\varepsilon_n^{-1}\Delta \mathrm{G}_n(z) \quad \underset{n\to\infty}{\overset{\text{dist.}}{\longrightarrow}} \quad Z(z) \tag{3.19}$$

- *if $\varepsilon_n \sim c/n$, with c constant, then for all $z \in \mathbb{C}\backslash\mathbb{R}$*

$$n\varepsilon_n^{-1}\Delta \mathrm{G}_n(z) \quad \underset{n\to\infty}{\overset{\text{dist.}}{\longrightarrow}} \quad cB(z) + Z(z)\,, \tag{3.20}$$

- *if $n^{-1} \ll \varepsilon_n \ll n^{-1/3}$, then for all $z \in \mathbb{C}\backslash\mathbb{R}$*

$$n\varepsilon_n^{-1}\left(\Delta \mathrm{G}_n(z) - \varepsilon_n^2 B(z)\right) \quad \underset{n\to\infty}{\overset{\text{dist.}}{\longrightarrow}} \quad Z(z)\,. \tag{3.21}$$

- *if $n^{-1} \ll \varepsilon_n \ll 1$, then for all $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$\varepsilon_n^{-2}\Delta \mathrm{G}_n(z) - B(z) \quad \underset{n\to\infty}{\overset{P}{\longrightarrow}} \quad 0\,. \tag{3.22}$$

Remark. Note that (3.20) is merely an extension of (3.21) in the critical regime.

The following statement expresses $B(z)$ as the image of a φ_z by a linear form. So, in the expansion of the previous proposition, both quantities $Z(z)$ and $B(z)$ depend linearly on φ_z. Note that as F vanishes at $\pm\infty$, Proposition 22 does not contradicts the fact that as $|z|$ gets large, $B(z) = O(|z|^{-3})$.

Proposition 22. *Under Hypotheses (c), (d), (e), for any $z \in \mathbb{C}\backslash\mathcal{S}$, for F defined by* (3.6),

$$B(z) \ = \ -\int \frac{F(s)}{(z-s)^2}\mathrm{d}s \ = \ -\int \varphi_z'(s)F(s)\mathrm{d}s.$$

3.6.2 Proof of Proposition 21

The proof is based on a perturbative expansion of the resolvent $\frac{1}{n}\operatorname{Tr}\frac{1}{z-D_n^\varepsilon}$. To make notations lighter, we shall sometimes suppress the subscripts and superscripts n, so that D_n^ε, D_n, X_n and $x_{i,j}^n$ will be respectively denoted by D^ε, D, X and $x_{i,j}$. Let us fix $z \in \mathbb{C}\backslash\widetilde{\mathcal{S}}$. We can deduce from the expansion of the resolvent of D^ε:

$$\Delta \mathrm{G}_n(z) = A_n(z) + B_n(z) + C_n(z) + R_n^\varepsilon(z),$$

with

$$\begin{aligned}
A_n(z) &:= \frac{\varepsilon_n}{n}\operatorname{Tr}\frac{1}{z-D}X\frac{1}{z-D} = \frac{\varepsilon_n}{n}\frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{x_{i,i}}{(z-\lambda_n(i))^2} \\
B_n(z) &:= \frac{\varepsilon_n^2}{n}\operatorname{Tr}\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D} = \frac{\varepsilon_n^2}{n^2}\sum_{i,j}\frac{|x_{i,j}|^2}{(z-\lambda_n(i))^2(z-\lambda_n(j))} \\
C_n(z) &:= \frac{\varepsilon_n^3}{n}\operatorname{Tr}\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D} \\
&= \frac{\varepsilon_n^3}{n^{5/2}}\sum_{i,j,k=1}^n \frac{x_{i,j}\ x_{j,k}\ x_{k,i}}{(z-\lambda_n(i))^2\ (z-\lambda_n(j))\ (z-\lambda_n(k))} \\
R_n^\varepsilon(z) &:= \frac{\varepsilon_n^4}{n}\operatorname{Tr}\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D^\varepsilon}.
\end{aligned}$$

The purpose of the four following claims is to describe the asymptotic behavior of each of these four terms.

Claim 1. *The finite dimension marginals of the centered process*

$$\left(n\varepsilon_n^{-1}A_n(z)\right)_{z\in\mathbb{C}\backslash\widetilde{\mathcal{S}}}$$

converge in distribution to those of the centered Gaussian process $(Z(z))_{z\in\mathbb{C}\backslash\widetilde{\mathcal{S}}}$. *Besides, there is* $C > 0$ *such that for any* $z \in \mathbb{C}\backslash\widetilde{\mathcal{S}}$*,*

$$\mathbb{E}[|n\varepsilon_n^{-1}A_n(z)|^2] \ \le\ \frac{C}{\operatorname{dist}(z,\widetilde{\mathcal{S}})^4}. \tag{3.23}$$

Proof. Estimate (3.23) follows from

$$\mathbb{E}[|A_n(z)|^2] \ =\ \frac{\varepsilon_n^2}{n^3}\sum_{i=1}^n \frac{\mathbb{E}\left[|x_{i,i}|^2\right]}{|z-\lambda_n(i)|^4} \ \le\ \frac{\varepsilon_n^2}{n^3}\sum_{i=1}^n \frac{\sigma_n^2(i,i)}{\operatorname{dist}(z,\widetilde{\mathcal{S}})^4}$$

and from the existence of a uniform upper bound for $\sigma_n^2(i,i)$ which comes from Hypothesis (a) which stipulates that the 8-th moments of the entries $x_{i,j}$ are uniformly bounded.

We turn now to the proof of the convergence in distribution of $n\varepsilon_n^{-1}A_n(z)$ which actually does not depend on the sequence (ε_n). For all $\alpha_1, \beta_1, \ldots, \alpha_p, \beta_p \in \mathbb{C}$ and for all $z_1, \ldots, z_p \in \mathbb{C}\backslash\widetilde{\mathcal{S}}$,

$$\sum_{i=1}^{p} \alpha_i \left(n\varepsilon_n^{-1}A_n(z_i)\right) + \beta_i \overline{(n\varepsilon_n^{-1}A_n(z_i))} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_{j,j} \left(\sum_{i=1}^{p} \xi_n(i,j)\right)$$

for $\xi_n(i,j) = \dfrac{\alpha_i}{(z_i - \lambda_n(j))^2} + \dfrac{\beta_i}{(\overline{z_i} - \lambda_n(j))^2}$.

On one hand, by dominated convergence, the covariance matrix of the above two dimensional random vector converges.

On the other hand, $\mathbb{E}|x_{i,j}|^4$ is uniformly bounded in i, j and n, by Hypothesis (a). Moreover, for n large enough, for all i, j,

$$|\xi_n(i,j)| \le 2 \max_{1\le i\le p}(|\alpha_i| + |\beta_i|) \times (\min_{1\le i\le p} \operatorname{dist}(z_i, \mathcal{S}))^{-1}.$$

Hence, the conditions of Lindeberg Central Limit Theorem are satisfied and the finite dimension marginals of the process $(n\varepsilon_n^{-1}A_n(z))_{z\in\mathbb{C}\backslash\widetilde{\mathcal{S}}}$ converge in distribution to those of the centered Gaussian process $(Z_z)_{z\in\mathbb{C}\backslash\widetilde{\mathcal{S}}}$ defined by its covariance structure

$$\begin{aligned}
\mathbb{E}\left(Z(z)\overline{Z(z')}\right) &= \lim_{n\to\infty} \mathbb{E}\left[\left(n\varepsilon_n^{-1}A_n(z)\right) . \left(n\varepsilon_n^{-1}\overline{A_n(z')}\right)\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i,j=1}^{n} \frac{\mathbb{E}\left[x_{i,i}\ \overline{x_{j,j}}\right]}{(z-\lambda_n(i))^2\ (\overline{z'}-\lambda_n(j))^2} \\
&= \int_0^1 \frac{\sigma_{\mathrm{d}}(t)^2}{(z-f(t))^2\ (\overline{z'}-f(t))^2} \mathrm{d}t
\end{aligned}$$

and by the fact that $\overline{Z(z)} = Z(\overline{z})$ which comes from $\overline{A_n(z)} = A_n(\overline{z})$. □

Claim 2. *There is a constant C such that, for η_n as in Hypotheses (b),*

- *if $\varepsilon_n \ll n^{-1}$, then*

$$\mathbb{E}[|n\varepsilon_n^{-1}B_n(z)|^2] \ \le\ \frac{C(n\varepsilon_n)^2}{\operatorname{dist}(z,\widetilde{\mathcal{S}})^6} + \frac{C\eta_n^2}{\operatorname{dist}(z,\widetilde{\mathcal{S}})^8},$$

- *if $\varepsilon_n \sim c/n$ or if $n^{-1} \ll \varepsilon_n \ll 1$, then*

$$\mathbb{E}[|n\varepsilon_n^{-1}(B_n(z) - \varepsilon_n^2 B(z))|^2] \ \le\ \frac{C\varepsilon_n^2}{\operatorname{dist}(z,\widetilde{\mathcal{S}})^6} + \frac{C\eta_n^2}{\operatorname{dist}(z,\widetilde{\mathcal{S}})^8}.$$

Proof. Remind that,

$$B_n(z) = \frac{\varepsilon_n^2}{n^2} \sum_{i,j} \frac{|x_{i,j}|^2}{(z-\lambda_n(i))^2(z-\lambda_n(j))}.$$

Introduce the variable $b_n^\circ(z)$ obtained by centering the variable $n\varepsilon_n^{-2}B_n(z)$:

$$b_n^\circ(z) := n\varepsilon_n^{-2}(B_n(z) - \mathbb{E}B_n(z)) = \frac{1}{n}\sum_{i,j}\frac{|x_{i,j}|^2 - \sigma_n^2(i,j)}{(z-\lambda_n(i))^2(z-\lambda_n(j))}$$

and the defect variable

$$\begin{aligned}\delta_n(z) &:= \varepsilon_n^{-2}\mathbb{E}B_n(z) - B(z)\\ &= \frac{1}{n^2}\sum_{i,j}\frac{\sigma_n^2(i,j)}{(z-\lambda_n(i))^2(z-\lambda_n(j))} - \int_{(s,t)\in[0,1]^2}\frac{\sigma^2(s,t)}{(z-f(s))^2(z-f(t))}\mathrm{d}s\mathrm{d}t.\end{aligned}$$

In the two regimes $\varepsilon_n \ll n^{-1}$ and $\varepsilon_n \geq c/n$, we want to dominate the L^2 norms respectively of

$$n\varepsilon_n^{-1}B_n(z) = \varepsilon_n b_n^\circ(z) + n\varepsilon_n(\delta_n(z)+B(z)) \quad \text{and} \quad n\varepsilon_n^{-1}(B_n(z) - \varepsilon_n^2 B(z)) = \varepsilon_n b_n^\circ + n\varepsilon_n\delta_n(z).$$

For this purpose, we successively dominate b_n°, $\delta_n(z)$ and $B(z)$.

Using the independence of the $x_{i,j}$'s, the fact that they are bounded in L^4 and the fact that z stays at a macroscopic distance of the $\lambda_n(i)$'s, we can write for all $z \in \mathbb{C}\backslash\widetilde{\mathcal{S}}$

$$\begin{aligned}\mathbb{E}[|b_n^\circ(z)|^2] &= \frac{1}{n^2}\operatorname{Var}\left(\sum_{i\leq j}\left(x_{i,j}^2 + \mathbb{1}_{i\neq j}\overline{x_{i,j}}^2\right)\frac{1}{(z-\lambda_n(i))^2(z-\lambda_n(j))}\right)\\ &= \frac{1}{n^2}\sum_{i\leq j}\operatorname{Var}\left(\left(x_{i,j}^2 + \mathbb{1}_{i\neq j}\overline{x_{i,j}}^2\right)\frac{1}{(z-\lambda_n(i))^2(z-\lambda_n(j))}\right)\\ &\leq C\operatorname{dist}(z,\widetilde{\mathcal{S}})^{-6}.\end{aligned} \tag{3.24}$$

Now, the term $\delta_n(z)$ rewrites

$$\begin{aligned}\delta_n(z) &= O(n^{-1})\\ &\quad + \int_{(s,t)\in[0,1]^2}\mathbb{1}_{\lfloor ns\rfloor\neq\lfloor nt\rfloor}\left(\frac{\sigma_n^2(\lfloor ns\rfloor,\lfloor nt\rfloor)}{(z-\lambda_n(\lfloor ns\rfloor))^2(z-\lambda_n(\lfloor nt\rfloor))} - \frac{\sigma^2(s,t)}{(z-f(s))^2(z-f(t))}\right)\mathrm{d}s\mathrm{d}t.\end{aligned}$$

Since, for $M_\sigma := \sup_{0\leq x\neq y\leq 1}\sigma(x,y)^2$ and for any fixed $z \notin \widetilde{\mathcal{S}}$, the function

$$\psi_z : (s,\lambda,\lambda') \in [0, M_\sigma + 1] \times \{x \in \mathbb{R}\,;\, \operatorname{dist}(x,\widetilde{\mathcal{S}}) \leq \operatorname{dist}(z,\widetilde{\mathcal{S}})/2\}^2 \longmapsto \frac{s}{(z-\lambda)^2(z-\lambda')}$$

is $C\operatorname{dist}(z,\widetilde{\mathcal{S}})^{-4}$-Lipschitz, for C a universal constant, by Hypothesis (b),

$$\delta_n(z) = O(n^{-1}) + \frac{O(\eta_n)}{\max\{n\varepsilon_n, 1\}\operatorname{dist}(z,\widetilde{\mathcal{S}})^4}. \tag{3.25}$$

Finally, the expression of $B(z)$ given in (3.17) implies,

$$B(z) \leq \frac{C}{\operatorname{dist}(z, \widetilde{\mathcal{S}})^3} \tag{3.26}$$

Collecting estimations (3.24), (3.25) and (3.26), we conclude. □

Claim 3. *There is a constant C such that for any $z \in \mathbb{C} \backslash \widetilde{\mathcal{S}}$,*

$$\mathbb{E}[|n\varepsilon_n^{-1} C_n(z)|^2] \ \leq \ \frac{C\varepsilon_n^4}{\operatorname{dist}(z, \widetilde{\mathcal{S}})^8}.$$

Proof. We start by writing for all $z \in \mathbb{C} \backslash \widetilde{\mathcal{S}}$

$$\mathbb{E}[|n\varepsilon_n^{-1} C_n(z)|^2] = \frac{\varepsilon_n^4}{n^3} \mathbb{E}\left[\left|\sum_{i,j,k=1}^{n} \frac{x_{i,j}\ x_{j,k}\ x_{k,i}}{(z-\lambda_n(i))^2\ (z-\lambda_n(j))\ (z-\lambda_n(k))}\right|^2\right]$$

$$= \frac{\varepsilon_n^4}{n^3} \sum_{i,j,k,l,m,p=1}^{n} \frac{\mathbb{E}\left(x_{i,j}\ x_{j,k}\ x_{k,i}\ \overline{x_{l,m}\ x_{m,p}\ x_{p,l}}\right)}{(z-\lambda_n(i))^2\ (z-\lambda_n(j))\ (z-\lambda_n(k))\ (\overline{z}-\lambda_n(l))^2\ (\overline{z}-\lambda_n(m))\ (\overline{z}-\lambda_n(p))}.$$

Generically, the set of "edges" $\{(l,m),(m,p),(p,l)\}$ must be equal to the set $\{(i,j),(j,k),(k,i)\}$ in order to get a non zero term. Therefore, the complexity of the previous sum is $O(n^3)$. Note that other non zero terms involving third or fourth moments are much less numerous. Hence,

$$\mathbb{E}[|n\varepsilon_n^{-1} C_n(z)|^2] \leq \frac{\varepsilon_n^4}{n^3} \times \frac{O(n^3)}{\operatorname{dist}(z, \widetilde{\mathcal{S}})^8} \ \leq \ \frac{C\varepsilon_n^4}{\operatorname{dist}(z, \widetilde{\mathcal{S}})^8}$$

□

Claim 4. *There is a constant C such that for any $z \in \mathbb{C} \backslash \mathbb{R}$,*

$$\mathbb{E}[|n\varepsilon_n^{-1} R_n^{\varepsilon}(z)|^2] \ \leq \ \frac{O(n^2\varepsilon_n^6)}{|\mathfrak{Im}(z)|^2 \operatorname{dist}(z, \widetilde{\mathcal{S}})^8}.$$

Proof. Remind that,

$$R_n^{\varepsilon}(z) := \frac{\varepsilon_n^4}{n} \operatorname{Tr} \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D^{\varepsilon}}.$$

Hence,

$$
\begin{aligned}
\mathbb{E}[|n\varepsilon_n^{-1}R_n^\varepsilon(z)|^2] &\leq \varepsilon_n^6\ \mathbb{E}\left[\left|\operatorname{Tr}\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D^\varepsilon}\right|^2\right] \\
&\leq \varepsilon_n^6\ \mathbb{E}\left[\operatorname{Tr}\left|\left(\frac{1}{z-D}X\right)^4\right|^2 \times \operatorname{Tr}\left|\frac{1}{z-D^\varepsilon}\right|^2\right] \\
&\leq \varepsilon_n^6\ \mathbb{E}\left[\operatorname{Tr}\left(\left(\frac{1}{z-D}X\right)^4\left(\overline{\frac{1}{z-D}X}\right)^4\right)\frac{n}{|\operatorname{Im}(z)|^2}\right] \\
&\leq \frac{n\varepsilon_n^6}{|\operatorname{Im}(z)|^2}\ \mathbb{E}\left[\operatorname{Tr}\left(\left(\frac{1}{z-D}X\right)^4\left(\overline{\frac{1}{z-D}X}\right)^4\right)\right] \\
&\leq \frac{n\varepsilon_n^6}{|\operatorname{Im}(z)|^2}\frac{O(n^5)}{n^4\ \operatorname{dist}(z,\widetilde{\mathcal{S}})^8}\ \leq\ \frac{O(n^2\varepsilon_n^6)}{|\operatorname{Im}(z)|^2\operatorname{dist}(z,\widetilde{\mathcal{S}})^8}.
\end{aligned}
$$

The inequality of the last line takes into account that

- the L^8 norm of the entries of $\sqrt{n}X$ is uniformly bounded
- the norm of the entries of X is of order $n^{-1/2}$
- the norm of the coefficients of $(z-D)^{-1}$ is smaller than $\operatorname{dist}(z,\widetilde{\mathcal{S}})^{-1}$
- the complexity of the sum defining the trace is of order $O(n^5)$ since its non-null terms are encoded by four edges trees which have therefore five vertices.

□

We gather now the results of the previous claims.

For any rate of convergence of ε_n, Claim 1 proves that the process $n\varepsilon_n^{-1}A_n(z)$ converges in distribution to the centered Gaussian variable $Z(z)$. Moreover,

- if $\varepsilon_n \ll n^{-1}$, then as Claims 2, 3 and 4 imply that the processes $n\varepsilon_n^{-1}B_n(z)$, $n\varepsilon_n^{-1}C_n(z)$ and $n\varepsilon_n^{-1}R_n^\varepsilon(z)$ converge to 0 in probability, we can conclude, by Slutsky's theorem, that for any $z \in \mathbb{C}\setminus\mathbb{R}$:
$$n\varepsilon_n^{-1}\Delta \mathrm{G}_n(z)\ \xrightarrow[n\to\infty]{\text{dist}}\ Z(z)$$
- if $\varepsilon_n \sim \frac{c}{n}$, then, as Claims 2, 3 and 4 imply that the processes $n\varepsilon_n^{-1}B_n(z)$, $n\varepsilon_n^{-1}C_n(z)$ and $n\varepsilon_n^{-1}R_n^\varepsilon(z)$ converge respectively to $cB(z)$, 0 and 0 in probability, we can conclude, by Slutsky's theorem, that for any $z \in \mathbb{C}\setminus\mathbb{R}$:
$$n\varepsilon_n^{-1}\Delta \mathrm{G}_n(z)\ \xrightarrow[n\to\infty]{\text{dist}}\ Z(z)+cB(z)$$

- if $n^{-1} \ll \varepsilon_n \ll n^{-1/3}$, then, as Claims 2, 3 and 4 imply that the three processes $n\varepsilon_n^{-1}(B_n(z) - \varepsilon_n^2 B(z))$, $n\varepsilon_n^{-1}C_n(z)$ and $n\varepsilon_n^{-1}R_n^\varepsilon(z)$ converge to 0 in probability, we can conclude, by Slutsky's theorem, that for any $z \in \mathbb{C} \setminus \mathbb{R}$:

$$n\varepsilon_n^{-1}\left(\Delta \mathrm{G}_n(z) - \varepsilon_n^2 B(z)\right) \quad \underset{n\to\infty}{\xrightarrow{\text{dist.}}} \quad Z(z)$$

Regarding the convergence in probability (3.22), in the case $n^{-1} \ll \varepsilon_n \ll 1$, Claims 1, 2, 3 and 4 imply that the processes $\varepsilon_n^{-2}A_n(z)$, $\varepsilon_n^{-2}B_n(z) - B(z)$, $\varepsilon_n^{-2}C_n(z)$ and $\varepsilon_n^{-2}R_n^\varepsilon(z)$ converge to 0.

This finishes the proof of the convergences of Proposition 21. $\square$

3.6.3 Proof of Proposition 22

Recall that

$$B(z) = \int_{(s,t)\in[0,1]^2} \frac{\sigma^2(s,t)}{(z-f(s))^2(z-f(t))} \mathrm{d}s\mathrm{d}t.$$

Recall that ρ is the density of the push-forward of the uniform measure on $[0,1]$ by the map f.
Let τ be as in Hypotheis (d). We have

$$B(z) = \int_{\mathbb{R}^2} \frac{\tau(s,t)\ \rho(s)\ \rho(t)}{(z-s)^2\ (z-t)} \mathrm{d}s\mathrm{d}t.$$

By a partial fraction decomposition we have for all $a \neq b$

$$\frac{1}{(z-a)^2(z-b)} = \frac{1}{(b-a)^2}\left(\frac{1}{z-b} - \frac{1}{z-a} - \frac{b-a}{(z-a)^2}\right).$$

Thus, as the Lebesgue measure of the set $\{(y_1,y_2) \in [0,1]^2 \ ; \ y_1 = y_2\}$ is null, we have

$$B(z) = \int_{\mathbb{R}^2} \frac{\tau(s,t)\ \rho(s)\ \rho(t)}{(t-s)^2}\left(\frac{1}{z-t} - \frac{1}{z-s} - \frac{t-s}{(z-s)^2}\right) \mathrm{d}s\mathrm{d}t.$$

Moreover, for φ_z the function $\varphi_z : x \longmapsto \frac{1}{z-x}$, we obtain

$$B(z) = \int_{\mathbb{R}^2} \frac{\tau(s,t)\ \rho(s)\ \rho(t)}{(t-s)^2}\left(\varphi_z(t) - \varphi_z(s) - (t-s)\varphi_z'(s)\right) \mathrm{d}s\mathrm{d}t.$$

Now, we want to prove that $B(z) = -\displaystyle\int_{\mathbb{R}^2} \frac{\tau(s,t)\ \rho(s)\ \rho(t)}{t-s}\ \varphi_z'(s)\ \mathrm{d}s\mathrm{d}t.$

To do this, we will use a symmetry argument: in fact both terms in $\varphi_z(t)$ and $\varphi_z(s)$ neutralize each other, and it remains only to prove, that we did not remove ∞ to ∞ and that the remaining term has the desired form.

Let us define

$$B^\eta(z) := \int_{|s-t|>\eta} \frac{\tau(s,t)\ \rho(s)\ \rho(t)}{(t-s)^2} \left(\varphi_z(t) - \varphi_z(s) - (t-s)\varphi_z'(s)\right) \mathrm{d}s\mathrm{d}t.$$

By the Taylor-Lagrange inequality we obtain:

$$\left| \frac{\tau(s,t)\ \rho(s)\ \rho(t)}{(t-s)^2} \left(\varphi_z(t) - \varphi_z(s) - (t-s)\varphi_z'(s)\right) \right| \leq \frac{\rho(s)\ \rho(t)\ \|\tau(\cdot,\cdot)\|_{L^\infty}\ \|\varphi_z''\|_{L^\infty}}{2}.$$

So that, since ρ is a density, by dominated convergence, we have

$$\lim_{\eta\to 0} B^\eta(z) = B(z).$$

Moreover, by symmetry, for any η,

$$B^\eta(z) = \int_{|s-t|>\eta} \frac{\tau(s,t)\ \rho(s)\ \rho(t)}{t-s}(-\varphi_z'(s))\mathrm{d}s\mathrm{d}t.$$

So

$$\begin{aligned} B(z) &= \lim_{\eta\to 0} \int_{|s-t|>\eta} \frac{\tau(s,t)\ \rho(s)\ \rho(t)}{t-s}(-\varphi_z'(s))\mathrm{d}t\mathrm{d}s \\ &= -\lim_{\eta\to 0} \int_{s\in\mathbb{R}} F_\eta(s)\varphi_z'(s)\mathrm{d}s \end{aligned} \tag{3.27}$$

where for $\eta > 0$ and $s \in \mathbb{R}$, we define

$$F_\eta(s) := \rho(s) \int_{t\in\mathbb{R}\setminus[s-\eta,s+\eta]} \frac{\tau(s,t)\ \rho(t)}{t-s}\mathrm{d}t.$$

Note that that by definition of the function F given at (3.6), for any s, we have

$$F(s) = \lim_{\eta\to 0} F_\eta(s). \tag{3.28}$$

Thus by (3.27) and (3.28), to conclude the proof of Proposition 22, by dominated convergence, one needs only to state that F_η is dominated, uniformly in η, by an integrable function. This follows from the following computation.

Note first that by symmetry, we have

$$F_\eta(s) = \rho(s) \int_{t\in\mathbb{R}\setminus[s-\eta,s+\eta]} \frac{\tau(s,t)\ \rho(t) - \tau(s,s)\ \rho(s)}{t-s}\mathrm{d}t. \tag{3.29}$$

Let $M > 0$ such that the support of the function ρ is contained in $[-M, M]$. Then, for η_0, α, C as in Hypothesis (e), using the expression of $F_\eta(s)$ given at (3.29), we have

$$\begin{aligned}|F_\eta(s)| \leq \quad & 2C\rho(s)\int_{t=s}^{s+\eta_0} |t-s|^{\alpha-1}\mathrm{d}t + \int_{t\in[s-2M,s-\eta_0]\cup[s+\eta_0,s+2M]} \left|\frac{\tau(s,t)\rho(s)\rho(t)}{t-s}\right| \mathrm{d}t \\ \leq \quad & \frac{2C\rho(s)}{\alpha}\eta_0^\alpha + \frac{1}{\eta_0}\int_{t\in\mathbb{R}} |\tau(s,t)\rho(s)\rho(t)|\,\mathrm{d}t \\ \leq \quad & \frac{2C\rho(s)}{\alpha}\eta_0^\alpha + \frac{\|\tau(\cdot,\cdot)\|_{L^\infty}}{\eta_0}\rho(s).\end{aligned}$$

□

3.6.4 A local type convergence result

One can precise the convergence (3.22) by replacing the complex variable z by a complex sequence (z_n) which converges slowly enough to the real axis. This convergence won't be used in the sequel. As it is discussed in [BGK16], this type of result is a first step towards a local result for the empirical distribution.

Proposition 23. *Under Hypotheses (a), (b), (f), if $n^{-1} \ll \varepsilon_n \ll 1$, then for any nonreal complex sequence (z_n), such that*

$$\mathfrak{Im}(z_n) \gg \max\left\{(n\varepsilon_n)^{-1/2}\ ,\ \left(\frac{\eta_n}{n\varepsilon_n}\right)^{1/4}\ ,\ \varepsilon_n^{2/5}\right\} \tag{3.30}$$

the following convergence holds

$$\varepsilon_n^{-2}\Delta\mathrm{G}_n(z_n) - B(z_n) \quad \xrightarrow[n\to\infty]{P} \quad 0\,.$$

Remark. In the classical case where $\dfrac{\eta_n}{n\varepsilon_n} = \sup\limits_{i\neq j}(|\sigma_n^2(i,j) - \sigma^2(i/n, j/n)| + |\lambda_n(i) - f(i/n)|)$ is of order $\dfrac{1}{n}$, the above assumption boils down to $\mathfrak{Im}(z_n) \gg \max\left\{(n\varepsilon_n)^{-1/2}\ ,\ \varepsilon_n^{2/5}\right\}$.

Proof. Assume $n^{-1} \ll \varepsilon_n \ll 1$. One can directly obtain, for all non-real complex sequences (z_n), that

- by Claim 1, if $\mathrm{dist}(z_n, \widetilde{\mathcal{S}}) \gg (n\varepsilon_n)^{-1/2}$, then

$$\mathbb{E}[|\varepsilon_n^{-2}A_n(z_n)|^2] \ \leq\ \frac{C}{(n\varepsilon_n)^2\ \mathrm{dist}(z_n,\widetilde{\mathcal{S}})^4} \ \xrightarrow[n\to\infty]{}\ 0,$$

- by Claim 2, if $\mathrm{dist}(z_n, \widetilde{\mathcal{S}}) \gg \max\left\{n^{-1/3}\ ,\ (\eta_n/(n\varepsilon_n))^{1/4}\right\}$, then

$$\mathbb{E}[|\varepsilon_n^{-2}B_n(z_n) - B(z_n)|^2] \ \leq\ \frac{C}{n^2\,\mathrm{dist}(z_n,\widetilde{\mathcal{S}})^6} + \frac{C\eta_n^2}{(n\varepsilon_n)^2\ \mathrm{dist}(z_n,\widetilde{\mathcal{S}})^8} \quad \xrightarrow[n\to\infty]{} \quad 0,$$

- by Claim 3, if $\operatorname{dist}(z_n, \widetilde{\mathcal{S}}) \gg (\varepsilon_n/n)^{1/4}$, then

$$\mathbb{E}[|\varepsilon_n^{-2} C_n(z_n)|^2] \ \leq \ \frac{C\varepsilon_n^2}{n^2 \ \operatorname{dist}(z_n, \widetilde{\mathcal{S}})^8} \quad \underset{n\to\infty}{\longrightarrow} \quad 0,$$

- by Claim 4, if $|\mathfrak{Im}(z_n)| \operatorname{dist}(z_n, \widetilde{\mathcal{S}})^4 \gg \varepsilon_n^2$, then

$$\mathbb{E}[|\varepsilon_n^{-2} R_n^\varepsilon(z_n)|^2] \ \leq \ \frac{O(\varepsilon_n^4)}{|\mathfrak{Im}(z_n)|^2 \operatorname{dist}(z_n, \widetilde{\mathcal{S}})^8} \quad \underset{n\to\infty}{\longrightarrow} \quad 0.$$

Therefore, when

$$\operatorname{dist}(z_n, \widetilde{\mathcal{S}}) \gg \max\left\{ (n\varepsilon_n)^{-1/2} \ , \ n^{-1/3} \ , \ \left(\frac{\eta_n}{n\varepsilon_n}\right)^{1/4} \ , \ \left(\frac{\varepsilon_n}{n}\right)^{1/4} \right\} \ \text{ and } \ |\mathfrak{Im}(z_n)| \operatorname{dist}(z_n, \widetilde{\mathcal{S}})^4 \gg \varepsilon_n^2,$$

the four processes, $\varepsilon_n^{-2} A_n(z_n)$, $\varepsilon_n^{-2} B_n(z_n) - B(z_n)$, $\varepsilon_n^{-2} C_n(z_n)$ and $\varepsilon_n^{-2} R_n^\varepsilon(z_n)$ converge to 0 in probability. Since $\operatorname{dist}(z_n, \widetilde{\mathcal{S}}) \geq \mathfrak{Im}(z_n)$, the above condition is implied by

$$\mathfrak{Im}(z_n) \gg \max\left\{ (n\varepsilon_n)^{-1/2} \ , \ n^{-1/3} \ , \ \left(\frac{\eta_n}{n\varepsilon_n}\right)^{1/4} \ , \ \left(\frac{\varepsilon_n}{n}\right)^{1/4} \ , \ \varepsilon_n^{2/5} \right\}.$$

Observing finally that the two terms $n^{-1/3}$ and $\left(\frac{\varepsilon_n}{n}\right)^{1/4}$ are dominated by the maximum of the three other ones, we conclude the proof. □

3.6.5 Possible extensions to larger ε_n

The convergence in distribution result of Theorem 19 is valid for $\varepsilon_n \ll n^{-1/3}$ but fails above $n^{-1/3}$. Let us consider, for example, the case where $n^{-1/3} \ll \varepsilon_n \ll n^{-1/5}$. In this case, the contribution of the first term $A_n(z)$ in the expansion of $\Delta \mathrm{G}_n(z)$ which yields the random limiting quantity, is dominated *not only* by the term $B_n(z)$ as it used to be previously. It is also dominated by a further and smaller term $D_n(z)$ of the expansion

$$\Delta \mathrm{G}_n(z) = A_n(z) + B_n(z) + C_n(z) + D_n(z) + E_n(z) + R_n^\varepsilon,$$

with:

$$\begin{aligned}
A_n(z) \ &:= \ \frac{\varepsilon_n}{n} \operatorname{Tr} \frac{1}{z-D} X \frac{1}{z-D} \\
&\ \vdots \\
E_n(z) \ &:= \ \frac{\varepsilon_n^5}{n} \operatorname{Tr} \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} \\
R_n^\varepsilon(z) \ &:= \ \frac{\varepsilon_n^6}{n} \operatorname{Tr} \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D} X \frac{1}{z-D^\varepsilon}.
\end{aligned}$$

In this case, the random term $Z(z)$ is still produced by $A_n(z)$ and has an order of magnitude of ε_n/n. Meanwhile, the term $D_n(z)$ writes

$$D_n(z) := \frac{\varepsilon_n^4}{n^3} \sum_{i,j,k,l=1}^{n} \frac{x_{i,j}\ x_{j,k}\ x_{k,l}\ x_{l,i}}{(z-\lambda_n(i))^2\ (z-\lambda_n(j))\ (z-\lambda_n(k))\ (z-\lambda_n(l))}.$$

All the indices satisfying $j = l$ contribute to the previous sum, since they produce a term in $|x_{i,l}|^2|x_{k,l}|^2$. Their cardinality is of order n^3. Therefore, the term $D_n(z)$ is of order ε_n^4 which prevails on the order ε_n/n of $A_n(z)$, as soon as $\varepsilon_n \gg n^{-1/3}$. One can also observe that the odd terms $C_n(z)$ and $E_n(z)$ in the expansion are negligible with respect to $A_n(z)$ due to the fact that the entries $x_{i,j}$ are centered. One can then state an analogous result to Proposition 21, but the deterministic limiting term $D(z)$ arising from $D_n(z)$ does not find a nice expression as the image of φ_z by a linear form as it was the case for $B(z)$ in Proposition 22. Therefore we did not state an extension of Theorem 19.

More generally, for all positive integer p, when $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$, the expansion will contain p deterministic terms, produced by the even variables, $B_n(z)$, $D_n(z)$, $F_n(z)$, $H_n(z)$ ... All the other odd terms, $C_n(z)$, $E_n(z)$, $G_n(z)$ ... being negligible due to the centering of the entries. The limits of the even terms $B_n(z)$, $D_n(z)$, $F_n(z)$, $H_n(z)$ ... can be expressed thanks to operator-valued free probability theory, using the results of [Shl96] (namely, Th. 4.1), but expressing these limits as the images of φ_z by linear forms is a quite involved combinatorial problem that we did not solve yet.

3.7 Convergence in probability in the semi-perturbative regime

Our goal now is to extend the convergence in probability result (3.22) of Proposition 21, proved for test functions $\varphi_z(x) := \frac{1}{z-x}$, to any $\mathcal{C}^6$ and compactly supported function on $\mathbb{R}$. We do it in the following lemma by using the Helffer-Sjöstrand formula which is stated in Proposition 14 of Chapter 2.

Lemma 24. *If $n^{-1} \ll \varepsilon_n \ll 1$, then, for any compactly supported $\mathcal{C}^6$ function ϕ on $\mathbb{R}$,*

$$\varepsilon_n^{-2}(\mu_n^\varepsilon - \mu_n)(\phi) \xrightarrow[n\to\infty]{P} -\int \phi'(s)F(s)\ \mathrm{d}s\,.$$

Proof. Let us introduce the Banach space $\mathcal{C}^1_{\mathrm{b,b}}$ of bounded $\mathcal{C}^1$ functions on $\mathbb{R}$ with bounded derivative, endowed with the norm $\|\phi\|_{\mathcal{C}^1_{\mathrm{b,b}}} := \|\phi\|_\infty + \|\phi'\|_\infty$.

On this space, let us define the random continuous linear form

$$\Pi_n(\phi) := \varepsilon_n^{-2}(\mu_n^\varepsilon - \mu_n)(\phi) + \int \phi'(s)F(s)\ \mathrm{d}s.$$

Convergence (3.22) of Proposition 21 can now be formulated as

$$\forall z \in \mathbb{C} \setminus \mathbb{R}, \qquad \Pi_n(\varphi_z) \xrightarrow[n\to\infty]{P} 0.$$

Actually, we can be more precise by adding the upper bounds of Claims 1, 2, 3 and 4, and obtain, uniformly in z,

$$\begin{aligned}\mathbb{E}[|\Pi_n(\varphi_z)|^2] &= \mathbb{E}[|\varepsilon_n^{-2}\Delta \mathrm{G}_n(z) - B(z)|^2] \\ &\le \frac{(n\varepsilon_n)^{-2}}{\min\left(\operatorname{dist}(z,\widetilde{\mathcal{S}})^4\ ,\ \operatorname{dist}(z,\widetilde{\mathcal{S}})^8\ ,\ |\mathfrak{Im}(z)|^2 \operatorname{dist}(z,\widetilde{\mathcal{S}})^8\right)}.\end{aligned} \tag{3.31}$$

Now, let ϕ be a compactly supported $\mathcal{C}^6$ function on $\mathbb{R}$ and let us introduce the almost analytic extension of degree 5 of ϕ defined by

$$\forall z = x + \mathrm{i}y \in \mathbb{C}, \qquad \widetilde{\phi}_5(z) := \sum_{k=0}^{5} \frac{1}{k!}(\mathrm{i}y)^k \phi^{(k)}(x)\,.$$

An elementary computation gives, by successive cancellations, that

$$\bar{\partial}\widetilde{\phi}_5(z) = \frac{1}{2}\left(\partial_x + \mathrm{i}\partial_y\right)\widetilde{\phi}_5(x+\mathrm{i}y) = \frac{1}{2\times 5!}(\mathrm{i}y)^5\phi^{(6)}(x). \tag{3.32}$$

Furthermore, by Helffer-Sjöstrand formula (Proposition 14), for $\chi \in \mathcal{C}_c^\infty(\mathbb{C};[0,1])$ a smooth cutoff function with value one on the support of ϕ,

$$\phi(\cdot) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{\partial}(\widetilde{\phi}_5(z)\chi(z))}{y^5} y^5 \varphi_z(\cdot)\,\mathrm{d}^2 z$$

where $\mathrm{d}^2 z$ denotes the Lebesgue measure on $\mathbb{C}$.

Note that by (3.32), $z \mapsto \mathbb{1}_{y\neq 0}\frac{\bar{\partial}(\widetilde{\phi}_5(z)\chi(z))}{y^5}$ is a continuous compactly supported function and that $z \in \mathbb{C} \mapsto \mathbb{1}_{y\neq 0} y^5 \varphi_z \in \mathcal{C}^1_{\mathrm{b,b}}$ is continuous, hence,

$$\Pi_n(\phi) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{\partial}(\widetilde{\phi}_5(z)\chi(z))}{y^5}\ y^5 \Pi_n(\varphi_z)\,\mathrm{d}^2 z.$$

Therefore,

$$\begin{aligned}\mathbb{E}\left(|\Pi_n(\phi)|^2\right) &= \mathbb{E}\left(\left|\frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{\partial}(\widetilde{\phi}_5(z)\chi(z))}{y^5}\ y^5\Pi_n(\varphi_z)\,\mathrm{d}^2 z\right|^2\right) \\ &\le \mathbb{E}\left(\frac{1}{\pi^2}\int_{\mathbb{C}}\left|\frac{\bar{\partial}(\widetilde{\phi}_5(z)\chi(z))}{y^5}\ y^5\Pi_n(\varphi_z)\right|^2\ \mathrm{d}^2 z\right) \\ &= \frac{1}{\pi^2}\int_{\mathbb{C}}\left|\frac{\bar{\partial}(\widetilde{\phi}_5(z)\chi(z))}{y^5}\right|^2\ y^{10}\ \mathbb{E}\left(|\Pi_n(\varphi_z)|^2\right)\ \mathrm{d}^2 z\,.\end{aligned}$$

Since the function $\left|\frac{\bar\partial(\widetilde{\phi}_5(z)\chi(z))}{y^5}\right|^2$ is continuous and compactly supported and that, by (3.31), for $n^{-1} \ll \varepsilon_n \ll 1$, uniformly in z,

$$y^{10}\ \mathbb{E}\left(|\Pi_n(\varphi_z)|^2\right) \leq y^{10}\ \frac{o(1)}{\min(y^4,\ y^{10})} \quad \underset{n\to\infty}{\longrightarrow} \quad 0.$$

Thus, for any compactly supported $\mathcal{C}^6$ function on $\mathbb{R}$,

$$\mathbb{E}\left(|\Pi_n(\phi)|^2\right) \leq \frac{1}{\pi^2}\int_{\mathbb{C}}\left|\frac{\bar\partial(\widetilde{\phi}_5(z)\chi(z))}{y^5}\right|^2\ y^{10}\ \mathbb{E}\left(|\Pi_n(\varphi_z)|^2\right)\ \mathrm{d}^2 z \quad \underset{n\to\infty}{\longrightarrow} \quad 0$$

which implies that $\Pi_n(\phi)$ converges to 0 in probability. □

3.8 Convergence in distribution towards the Gaussian variable Z_ϕ

The purpose of this section is to extend the convergences in distribution of Proposition 21, from test functions of the type $\varphi_z := \frac{1}{z-x}$, to compactly supported $\mathcal{C}^6$ functions on $\mathbb{R}$. To do so, we will use an extension lemma of Shcherbina and Tirozzi, stated as Lemma 15 in Chapter 2, which concerns the convergence of a sequence of *centered* random fields with uniformly bounded variance. Hence, we need to show first that our non centered random sequence is not far from being centered, which is done in subsection 3.8.1 by using again the Helffer-Sjöstrand formula (14). In subsection 3.8.2, we dominate the variance of this centered random field thanks to another result of Shcherbina and Tirozzi stated in Proposition 16 of Chapter 2. Subsection 3.8.3 collects the preceding results to conclude the proof.

3.8.1 Coincidence of the expectation of μ_n^ε with its deterministic approximation

The asymptotic coincidence of the expectation of μ_n^ε with its deterministic approximation is the content of next lemma:

Lemma 25. *Let us define, for ϕ a $\mathcal{C}^1$ function on $\mathbb{R}$,*

$$\Lambda_n(\phi) := \begin{cases} n\varepsilon_n^{-1}\left(\mathbb{E}[\mu_n^\varepsilon(\phi)] - \mu_n(\phi)\right) & \text{if } \varepsilon_n \ll n^{-1}, \\ \\ n\varepsilon_n^{-1}\left(\mathbb{E}[\mu_n^\varepsilon(\phi)] - \mu_n(\phi) + \varepsilon_n^2\int \phi'(s)F(s)\mathrm{d}s\right) & \text{if } \varepsilon_n \sim c/n \text{ or } n^{-1} \ll \varepsilon_n \ll n^{-1/3}. \end{cases}$$

Then, as $n \to \infty$, for any compactly supported $\mathcal{C}^6$ function ϕ or any ϕ of the type $\varphi_z(x) = \frac{1}{z-x}$, $z \in \mathbb{C}\backslash\mathbb{R}$, we have

$$\Lambda_n(\phi) \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. First note that, as the variables $x_{i,j}$ are centered, $\mathbb{E}[A_n(z)] = 0$. Moreover, by adding the renormalized upper bounds of Claims 2, 3 and 4 one can directly obtain the two following inequalities for any $z \in \mathbb{C} \setminus \mathbb{R}$:

- If $\varepsilon_n \ll n^{-1}$, then

$$\begin{aligned}
|\Lambda_n(\varphi_z)| &= n\varepsilon_n^{-1}|\mathbb{E}[\Delta \mathrm{G}_n(z)]| \\
&\leq n\varepsilon_n^{-1}\left(|\mathbb{E}[A_n(z)]| + \mathbb{E}[|B_n(z)|] + \mathbb{E}[|C_n(z)|] + \mathbb{E}[|R_n^\varepsilon(z)|]\right) \\
&\leq \frac{C(n\varepsilon_n + \eta_n)}{\min\left\{\operatorname{dist}(z,\widetilde{\mathcal{S}})^3, \operatorname{dist}(z,\widetilde{\mathcal{S}})^4, |\mathfrak{Im}(z)| \operatorname{dist}(z,\widetilde{\mathcal{S}})^4\right\}} \quad \underset{n\to\infty}{\longrightarrow} \quad 0\,.
\end{aligned}$$

- If $\varepsilon_n \sim c/n$ or $n^{-1} \ll \varepsilon_n \ll n^{-1/3}$, then

$$\begin{aligned}
|\Lambda_n(\varphi_z)| &= n\varepsilon_n^{-1}|\mathbb{E}[\Delta \mathrm{G}_n(z) - \varepsilon_n^2 B(z)]| \\
&\leq n\varepsilon_n^{-1}\big(|\mathbb{E}[A_n(z)]| + \mathbb{E}[|B_n(z) - \varepsilon_n^2 B(z)|] + \mathbb{E}[|C_n(z)|] + \mathbb{E}[|R_n^\varepsilon(z)|]\big) \\
&\leq \frac{C(\varepsilon_n + \eta_n + n\varepsilon_n^3)}{\min\left\{\operatorname{dist}(z,\widetilde{\mathcal{S}})^3, \operatorname{dist}(z,\widetilde{\mathcal{S}})^4, |\mathfrak{Im}(z)| \operatorname{dist}(z,\widetilde{\mathcal{S}})^4\right\}} \quad \underset{n\to\infty}{\longrightarrow} \quad 0\,.
\end{aligned}$$

Hence, in all cases, $\Lambda_n(\varphi_z) \underset{n\to\infty}{\longrightarrow} 0$.

The extension of this result to compactly supported $\mathcal{C}^6$ test functions on $\mathbb{R}$ goes the same way as for Π_n in the proof of Lemma 24. □

3.8.2 Domination of the variance of μ_n^ε

The second ingredient goes through a domination of the variance of $\mu_n^\varepsilon(\phi)$:

Lemma 26. *Let $s > 5$. There is a constant C such that for each n and each $\phi \in \mathcal{H}_s$,*

$$\operatorname{Var}(n\varepsilon_n^{-1}\mu_n^\varepsilon(\phi)) \;\leq\; C\|\phi\|_{\mathcal{H}_s}^2\,.$$

Proof. By Proposition 16, it suffices to prove that

$$\int_{y=0}^{\infty} y^{2s-1}\mathrm{e}^{-y}\int_{x\in\mathbb{R}} \operatorname{Var}(\varepsilon_n^{-1}\operatorname{Tr}((x+\mathrm{i}y - D_n^\varepsilon)^{-1}))\mathrm{d}x\mathrm{d}y$$

are bounded independently of n.

Note that for $\Delta \mathrm{G}_n(z)$ defined in (3.18),

$$\operatorname{Var}(\varepsilon_n^{-1}\operatorname{Tr}((z-D_n^\varepsilon)^{-1})) = n^2\varepsilon_n^{-2}\operatorname{Var}(\Delta \mathrm{G}_n(z)).$$

Moreover, the sum of the inequalities of Claims 1, 2, 3 and 4 yields

$$\operatorname{Var}(n\varepsilon_n^{-1}\Delta \mathrm{G}_n(z)) \quad \leq \quad \frac{C}{\operatorname{dist}(z,\widetilde{\mathcal{S}})^4} + \frac{C}{|\mathfrak{Im}(z)|^2\operatorname{dist}(z,\widetilde{\mathcal{S}})^8}.$$

Let $M>0$ such that $\widetilde{\mathcal{S}}\subset[-M,M]$. Then

$$\operatorname{dist}(z,\widetilde{\mathcal{S}}) \geq \begin{cases} y & \text{if } |x|\leq M,\\ \sqrt{y^2+(|x|-M)^2} & \text{if } |x|>M.\end{cases}$$

Thus $\operatorname{dist}(z,\widetilde{\mathcal{S}})\geq y$ if $|x|\leq M$ and, for $|x|>M$,

$$\frac{1}{\operatorname{dist}(z,\widetilde{\mathcal{S}})} \leq \frac{y^{-1}}{\sqrt{1+((|x|-M)/y)^2}}$$

and for any $y>0$,

$$\begin{aligned}\int_{x\in\mathbb{R}}\operatorname{Var}(n\varepsilon_n^{-1}\Delta \mathrm{G}_n(x+\mathrm{i}y))\mathrm{d}x \;&\leq\; 2CM(y^{-10}+y^{-4}) + 2C\int_0^{+\infty}\frac{y^{-4}}{(1+(\frac{x}{y})^2)^2}+\frac{y^{-10}}{(1+(\frac{x}{y})^2)^4}\mathrm{d}x\\ &\leq\; 2CM(y^{-10}+y^{-4}) + C\left(\frac{\pi}{2}y^{-3}+\frac{5\pi}{16}y^{-9}\right)\\ &\leq\; k\left(y^{-10}+y^{-3}\right),\end{aligned}$$

for a suitable constant k.

We deduce that, as soon as $2s-10>0$, i.e. $s>5$,

$$\int_{y=0}^{\infty} y^{2s-1}\mathrm{e}^{-y}\int_{x\in\mathbb{R}}\operatorname{Var}(\varepsilon_n^{-1}\operatorname{Tr}((x+\mathrm{i}y-D_n^\varepsilon)^{-1}))\mathrm{d}x\mathrm{d}y \;\leq\; k\int_0^\infty y^{2s-1}e^{-y}(y^{-10}+y^{-3})\mathrm{d}y \;<\; \infty.$$

□

3.8.3 Proof of the convergences in distribution of Theorem 19

Since we have proved in Lemma 25 that for all compactly supported $\mathcal{C}^6$ function ϕ, the deterministic term $\mu_n(\phi)$ could be replaced by $\mathbb{E}[\mu_n^\varepsilon(\phi)]$, we only have to prove, that for all $\phi\in\mathcal{C}^6$,

$$n\varepsilon_n^{-1}(\mu_n^\varepsilon(\phi)-\mathbb{E}[\mu_n^\varepsilon(\phi)]) \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad Z_\phi.$$

For the time being, we know this result to be valid for functions ϕ belonging to the space $\mathcal{L}_1$, defined as the linear span of the family of functions $\varphi_z(x) := \frac{1}{z-x}$, $z \in \mathbb{C}\backslash\mathbb{R}$.

By applying Lemma 15 to the centered field $\mu_n^\varepsilon - \mathbb{E}[\mu_n^\varepsilon]$, we are going to extend the result from the space $\mathcal{L}_1$ to the Sobolev space $(\mathcal{H}_s, \|\cdot\|_{\mathcal{H}_s})$ with $s \in (5, 6)$. Note that, since $s < 6$, this latter space contains the space of $\mathcal{C}^6$ compactly supported functions (see [Hör03, Sec. 7.9]).

It remains to check the two hypotheses of Lemma 15. First, the subspace $\mathcal{L}_1$ is dense in every space $(\mathcal{H}_s, \|\cdot\|_{\mathcal{H}_s})$. This is the content of Lemma 18 of Chapter 2. Second, by Lemma 26, since $s > 5$, $\mathrm{Var}(n\varepsilon_n^{-1}\mu_n^\varepsilon(\phi)) \leq C\|\phi\|_{\mathcal{H}_s}^2$ for a certain constant C.

This concludes the proof.

Chapter 4

Expansions of the empirical spectral measure of a perturbed matrix

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4.1 Introduction

In the present chapter, we study this problem in the special case of Hermitian matrices. Namely, we perturb an $n \times n$ Hermitian matrix, D_n, by a random Hermitian matrix with small operator norm. Moreover, as up to a change of basis, we can diagonalize D_n, we suppose it diagonal and consider that the entries of the pertubative matrix, X_n, are independent in the eigenvector basis of D_n with a variance profile[1]. Furthermore, to better describe the magnitude of the perturbations, we suppose that the operator norm of X_n is of order 1 and introduce a real sequence ε_n that tends to zero, so as to interest us in the empirical spectral distribution, μ_n^ε, of the perturbed matrix

$$D_n^\varepsilon := D_n + \varepsilon_n X_n$$

for different rates of convergence of ε_n to zero. In the following we prove that, depending on the order of magnitude of the perturbation, an infinity of different regimes appear.

Previously, in [BGEM17] (Chapter 3), we have explicited the three first regimes, respectively called *pertubative, critical* and *semi-perturbative regimes*, and appear under the three following expressions:

$$\begin{aligned}
\mu_n^\varepsilon &\approx \mu_n + \frac{\varepsilon_n}{n}\mathrm{d}Z & &\text{if } \varepsilon_n \ll n^{-1} \\
\mu_n^\varepsilon &\approx \mu_n + \frac{\varepsilon_n}{n}\left(c\mathrm{d}C_2 + \mathrm{d}Z\right) & &\text{if } \varepsilon_n \sim \frac{c}{n} \\
\mu_n^\varepsilon &\approx \mu_n + \varepsilon_n^2 \mathrm{d}C_2 & &\text{if } n^{-1} \ll \varepsilon_n \ll 1
\end{aligned}$$

where, μ_n is the empirical spectral measure of D_n, $\mathrm{d}C_2$ a deterministic linear form depending of the limit parameters of the off-diagonal entries of X_n and $\mathrm{d}Z$ a random linear form on the space of $\mathcal{C}^1$ functions on $\mathbb{R}$ depending only on the limit parameters of the diagonal entries of X_n. This transition at $\varepsilon_n \sim n^{-1}$ is the well-known transition, in quantum mechanics, where the *perturbative regime* ends. A surprising fact discovered during this first study was that the semi-perturbative regime $n^{-1} \ll \varepsilon_n \ll 1$, which concerns many applications in the context of covariance matrices in finance (see, for example, [AB12] or [ABB14]), decomposes into infinitely many sub-regimes. In the case $n^{-1} \ll \varepsilon_n \ll n^{-1/3}$, the expansion of $\mu_n^\varepsilon - \mu_n$ contains a single deterministic term before the random term $\frac{\varepsilon_n}{n}\mathrm{d}Z$. In the case $n^{-1/3} \ll \varepsilon_n \ll n^{-1/5}$, the expansion of $\mu_n^\varepsilon - \mu_n$ contains two of them. And more generally, for all positive integer p, when $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$, the expansion contains p of them:

$$\mu_n^\varepsilon \approx \mu_n + \frac{\varepsilon_n}{n}\mathrm{d}Z + \varepsilon_n^2 \mathrm{d}C_2 + \varepsilon_n^4 \mathrm{d}C_4 + \cdots + \varepsilon_n^{2p}\mathrm{d}C_{2p}.$$

In this case, each linear form $\mathrm{d}C_{2k}$ may concern up to the k first derivatives of the test function ϕ on which we evaluate the measure μ_n^ε.

[1] Note that if the perturbing matrix belongs to the GOE or GUE, then its law is invariant under this change of basis

Thereby, the aim of this chapter is, now, to provide a thorough study of thoses sub-regimes of the semi-pertubative regime, explaining how we should reinforce the hypothesis of our model each time we choose a slower rate of convergence to zero for ε_n and how the deterministic terms appear then.

The perturbative expansion of the empirical spectral measure is, in this paper, done for real test-functions of the type $\varphi_z(x) = \frac{1}{z-x}$, for a non-real complex number z. This class of test functions, which plays a central role in random matrix theory, is very useful because it can be used to extend easily a convergence in law to larger classes of test functions, as it is discussed in Section 4.6.

In this case, the deterministic terms (C_k), which come from the linear forms $(\mathrm{d}C_k)$, of this expansion are related to some combinatorial objects as, for example, rooted planar trees and the random linear form $\mathrm{d}Z$ is related to the so-called one-dimensional Gaussian free field defined, for instance at [Dub09, Sect. 4.2] (if the variance profile of X_n is flat, then it is precisely the Laplacian of the Gaussian free field, defined in the sense of distributions).

This chapter is organized as follows; the main results, comments and examples are given in Sections 4.2 and 4.3. Sections 4.4 and 4.5 are devoted to the proofs. And finally, Section 4.6 is a discussion about possible extensions of the main results to larger classes of test functions.

Notations. For a_n, b_n some real sequences, $a_n \ll b_n$ (resp. $a_n \sim b_n$) means that a_n/b_n tends to 0 (resp. to 1). Also, $\xrightarrow{P}$ and $\xrightarrow{\text{dist.}}$ stand respectively for convergence in probability and convergence in distribution for all finite marginals.

4.2 Main results

4.2.1 Definition of the model and assumptions

For all positive integer n, we consider a real diagonal matrix $D_n = \mathrm{diag}(\lambda_n(1), \ldots, \lambda_n(n))$, as well as a Hermitian random matrix

$$X_n = \frac{1}{\sqrt{n}}[x_{i,j}^n]_{1\leq i,j\leq n}$$

and a positive number ε_n. The normalizing factor $n^{-1/2}$ and our hypotheses below ensure that the operator norm of X_n is of order one. We then define, for all n,

$$D_n^\varepsilon := D_n + \varepsilon_n X_n.$$

We now introduce the probability measures μ_n and μ_n^ε as the respective uniform distributions on the eigenvalues (with multiplicity) of D_n and D_n^ε. Our aim is to explicit all the different perturbative expansions of μ_n^ε around μ_n when $n^{-1} \ll \varepsilon_n \ll 1$.

We make the following general hypotheses:

(a) the entries $x^n_{i,j}$ of $\sqrt{n}X_n$ are independent (up to symmetry) random variables, centered, with variance denoted by $\sigma_n^2(i,j)$,

(b) there are $f, \sigma_\mathrm{d}, \sigma$ real functions defined respectively on $[0,1]$, $[0,1]$ and $[0,1]^2$ such that, for each $x \in [0,1]$,

$$\lambda_n(\lfloor nx \rfloor) \underset{n\to\infty}{\longrightarrow} f(x) \quad \text{and} \quad \sigma_n^2(\lfloor nx \rfloor, \lfloor nx \rfloor) \underset{n\to\infty}{\longrightarrow} \sigma_\mathrm{d}(x)^2$$

and for each $x \neq y \in [0,1]$,

$$\sigma_n^2(\lfloor nx \rfloor, \lfloor ny \rfloor) \underset{n\to\infty}{\longrightarrow} \sigma^2(x,y).$$

We make the following hypothesis about the rate of convergence:

$$\eta_n \;:=\; \max\{n\varepsilon_n, 1\} \times \sup_{1\le i\neq j\le n} \left(|\sigma_n^2(i,j) - \sigma^2(i/n, j/n)| + |\lambda_n(i) - f(i/n)|\right) \quad \underset{n\to\infty}{\longrightarrow} \quad 0.$$

Let us now make a last general assumption which makes it possible to include the case where the set of eigenvalues of D_n contains some outliers:

(c) there is a real compact set $\widetilde{\mathcal{S}}$ such that

$$\max_{1\le i\le n} \operatorname{dist}(\lambda_n(i), \widetilde{\mathcal{S}}) \underset{n\to\infty}{\longrightarrow} 0.$$

Furthermore, we add a last particular hypothese which depends on the rate of convergence of ε_n. In the case $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$:

(d) the moment $\mathbb{E}|x^n_{i,j}|^{4p+4}$ is bounded uniformly on n, i, j

Finally, we must introduce the last notion of rooted planar tree that is used in the two main results.

Definition 27 (Dyck path)**.** A *Dyck path* of length $k \in 2\mathbb{N}$, also called *non-negative Bernoulli walk* or *Bernoulli excursion*, is a function $\gamma : [\![0,k]\!] \mapsto \mathbb{N}$ satisfying

$$\gamma(0) = \gamma(k) = 0 \quad \text{and} \quad \forall i \in [\![1, k-1]\!],\ |\gamma(i+1) - \gamma(i)| = 1$$

Remark 5. The set of Dyck paths of length $k \in 2\mathbb{N}$ is in bijection with several other sets of combinatorial objects, as the set of rooted planar trees with $k/2$ edges, that is to say the set of trees which are embedded in the plane and are given one distinguished vertex called the root. One can find, for example, a proof of this well known result in the Subsection 1.1.2 of [Gui08]. Namely, the bijection consists in considering a clockwise path around the tree which begins and finishes at its root. This way, each time the walk around the tree meets a vertex for the first time then the Dyck path rises by one unit and each time the walk around the tree meets an already visited vertex the Dyck path descends from one unit, as it can be seen in Figure 4.1.

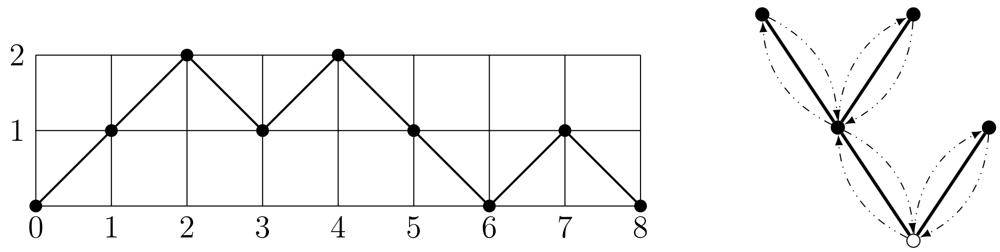


Figure 4.1: A Dyck path and its corresponding rooted planar tree (root is white)

Notation. In the following we formulate the main results in terms of rooted planar trees rather than in terms of Dyck paths. In order to be able to precisely designate the different vertices and edges of these rooted planar trees we labeled them in the following way: for a rooted planar tree with $k+1$ vertices, the root is labeled 1 and as the other vertices could be arbitrarly labeled from 2 to $k+1$ (only the structure of the tree matters), we choose the so-called *left-to-right depth-first labelling with source the root*, that is to say we label the vertices in the order that the clockwise walk around the tree begining at the root, induced by the associated Dyck path, discover them. This arbitrary labeling could be observed in Figure 4.2.

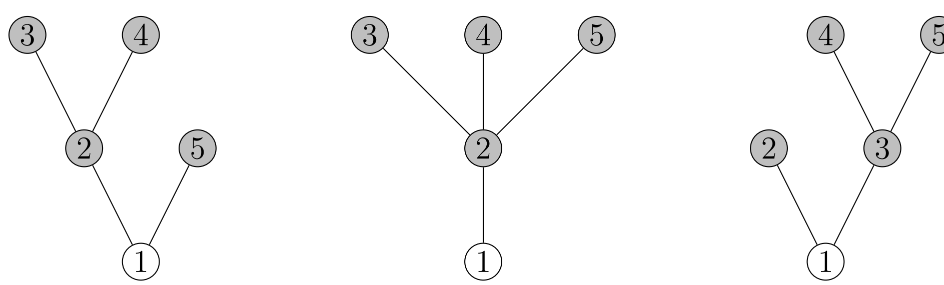


Figure 4.2: Three different labeled planar rooted trees

Moreover, we denote $\mathfrak{T}(k)$ the set of the rooted planar trees with k edges. Thereby, for any tree $T \in \mathfrak{T}(k)$, we denote by $\mathcal{E}_T$ the set of its (un-oriented) edges and by v_ℓ^T the degree of the vertex ℓ.

4.2.2 Convergence in distribution of μ_n^ε

Let introduce, for $z \in \mathbb{C}\backslash\widetilde{\mathcal{S}}$, a centered Gaussian process, $(Z(z))_{z\in\mathbb{C}\backslash\widetilde{\mathcal{S}}}$, with distribution defined by its covariance structure

$$\mathbb{E}\left(Z(z)Z(z')\right) = \int_0^1 \frac{\sigma_{\mathrm{d}}(t)^2}{(z-f(t))^2\ (z'-f(t))^2}\mathrm{d}t$$

and by the fact that $\overline{Z(z)} = Z(\overline{z})$.

Note that the field $(Z(z))_{z\in\mathbb{C}\backslash\widetilde{\mathcal{S}}}$ can be represented as

$$Z(z) = \int_0^1 \frac{\sigma_d(t)}{(z-f(t))^2}\mathrm{d}B_t$$

where (B_t) is the standard one-dimensional Brownian motion.

We also introduce, for $z \in \mathbb{C}\backslash\widetilde{\mathcal{S}}$,

$$\Delta\mathrm{G}_n(z) \ := \ (\mu_n^\varepsilon - \mu_n)(\varphi_z) \ = \ \frac{1}{n}\operatorname{Tr}\frac{1}{z-D_n^\varepsilon} - \frac{1}{n}\operatorname{Tr}\frac{1}{z-D_n}.$$

Theorem 28. *Let $p \in \mathbb{N}^*$. Under the semi-perturbative regime $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$ and under Hypotheses (a), (b), (c), (d), the following convergence in distribution holds,*

$$n\varepsilon_n^{-1}\left((\mu_n^\varepsilon - \mu_n)(\varphi_z) - \sum_{\substack{k=2\\ k \text{ even}}}^{2p} \varepsilon_n^k C_k(z)\right) \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad Z(z)$$

where the terms $(C_k(z))$ are deterministic functions of z, defined by

$$C_k(z) = \sum_{T\in\mathfrak{T}\left(\frac{k}{2}\right)} \int_{[0,1]^{\frac{k}{2}+1}} \frac{\prod_{\{i,j\}\in\mathcal{E}_T}\sigma^2(x_i,x_j)}{(z-f(x_1))\prod_{i=1}^{k/2+1}(z-f(x_i))^{v_i^T}}\ \mathrm{d}x_1\ldots\mathrm{d}x_{\frac{k}{2}+1} \tag{4.1}$$

Remark 6. The second part of Hypothesis (b), concerning the speed of convergence of the profile of the spectrum of D_n as well as of the variance of its perturbation, is needed in order to express the expansion of $\mu_n^\varepsilon - \mu_n$ in terms of limit parameters of the model σ and ρ. We can remove this hypothesis and get analogous expansions where the terms $C_k(z)$ and $\mathrm{d}Z$ are replaced by their discrete counterparts $C_k^{(n)}$ and $\mathrm{d}Z_n$, defined thanks to the "finite n" empirical versions of the limit parameters σ and ρ.

4.2.3 A local type convergence result

One can precise the convergence of Theorem 28 by replacing the complex variable z by a complex sequence (z_n) which converges slowly enough to the real axis. As it is discussed in [BGK16], this type of result is a first step towards a local result for the empirical distribution.

Theorem 29. *Under Hypotheses (a), (b), (c), (d), and if $n^{-1} \ll \varepsilon_n \ll 1$, then for any nonreal complex sequence (z_n), such that*

$$\mathfrak{Im}(z_n) \gg \max\left\{(n\varepsilon_n^{p-1})^{-1/2}\ ,\ \left(\frac{\eta_n}{n\varepsilon_n^{p-1}}\right)^{1/4}\ ,\ \varepsilon_n^{\frac{2}{p+3}}\right\}.$$

and any even $p \geq 2$, the following convergence in probability holds

$$\varepsilon_n^{-p}(\mu_n^\varepsilon - \mu_n)(\varphi_{z_n}) - \sum_{\substack{k=0 \\ k \text{ even}}}^{p-2} \varepsilon_n^{-k} C_{p-k}(z_n) \quad \xrightarrow[n\to\infty]{P} \quad 0\,,$$

Remark. In the classical case where $\dfrac{\eta_n}{n\varepsilon_n} = \sup_{i\neq j}(|\sigma_n^2(i,j) - \sigma^2(\frac{i}{n},\frac{j}{n})| + |\lambda_n(i) - f(\frac{i}{n})|)$ is of order $\dfrac{1}{n}$, the above assumption boils down to $\mathfrak{Im}(z_n) \gg \max\left\{(n\varepsilon_n^{p-1})^{-1/2}\ ,\ \varepsilon_n^{\frac{2}{p+3}}\right\}$.

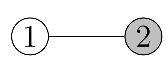
4.3 Examples and simulations

To illustrate Theorem 28 we explicit, in Subsection 4.3.1, the terms of the expansion of μ_n^ε until the semi-perturbative regime where $\varepsilon_n \sim n^{-1/7}$ and give some simulations based on them in Subsection 4.3.2.

4.3.1 Explanation of the first three deterministic terms of the expansion of μ_n^ε

In accordance with Theorem 28, the three first sub-regimes, $n^{-1} \ll \varepsilon_n \ll n^{-1/3}$, $n^{-1/3} \ll \varepsilon_n \ll n^{-1/5}$ and $n^{-1/5} \ll \varepsilon_n \ll n^{-1/7}$, of the semi-perturbative regime, involve the three first deterministic terms, $C_2(z), C_4(z)$ and $C_6(z)$, that we explicit in this subsection in order to re-use them in the next subsection which is dedicated to simulations.

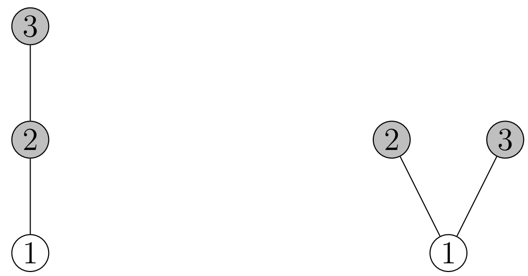
The term first deterministic term $C_2(z)$ is linked to the only planar rooted tree with a single double-edge of the set $\mathfrak{T}(1)$,



Thus, one can deduce directly deduce from formula (4.1) that

$$C_2(z) := \int_{(x_1,x_2)\in[0,1]^2} \frac{\sigma^2(x_1,x_2)}{(z-f(x_1))^2(z-f(x_2))} \mathrm{d}x_1 \mathrm{d}x_2.$$

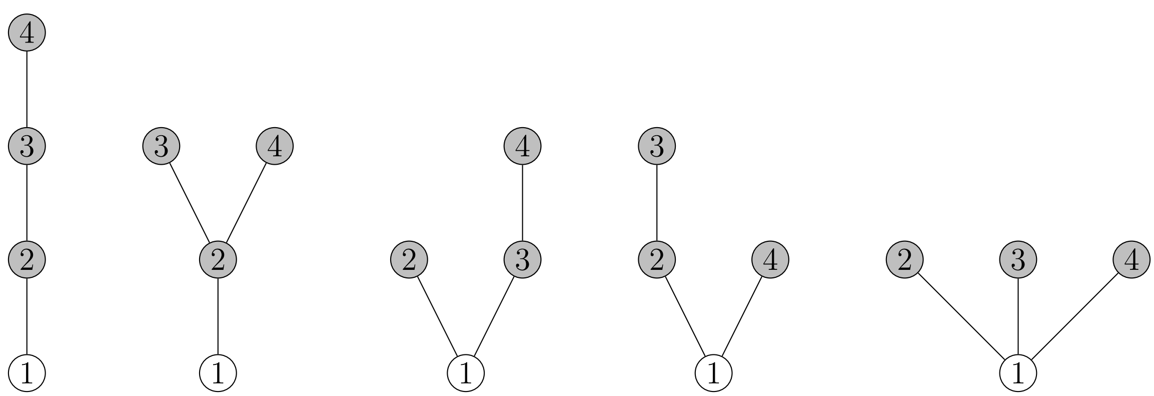
Similarly, as $C_4(z)$ is linked to the set, $\mathfrak{T}(2)$ containing the 2 planar rooted trees with 2 double-edges,



we deduce from formula (4.1) that,

$$C_4(z) = \int_{[0,1]^3} \frac{\sigma^2(x_1,x_2)\ \sigma^2(x_2,x_3)}{(z-x_1)^2\ (z-x_2)^2\ (z-x_3)} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 + \int_{[0,1]^3} \frac{\sigma^2(x_1,x_2)\ \sigma^2(x_1,x_3)}{(z-x_1)^3\ (z-x_2)\ (z-x_3)} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3.$$

And finally, as $C_6(z)$ is linked to the set $\mathfrak{T}(3)$ containing the 5 planar rooted trees with 4 double-edges,



one can deduce from formula (4.1) and by symmetry between two of those trees that,

$$\begin{aligned} C_6(z) &= \int_{[0,1]^4} \frac{\sigma^2(x_1,x_2)\ \sigma^2(x_2,x_3)\ \sigma^2(x_3,x_4)}{(z-f(x_1))^2\ (z-f(x_2))^2\ (z-f(x_3))^2\ (z-f(x_4))} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 \\ &+ \int_{[0,1]^4} \frac{\sigma^2(x_1,x_2)\ \sigma^2(x_2,x_3)\ \sigma^2(x_2,x_4)}{(z-f(x_1))^2\ (z-f(x_2))^3\ (z-f(x_3))\ (z-f(x_4))} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 \\ &+ 2\int_{[0,1]^4} \frac{\sigma^2(x_1,x_2)\ \sigma^2(x_1,x_3)\ \sigma^2(x_3,x_4)}{(z-f(x_1))^3\ (z-f(x_2))\ (z-f(x_3))^2\ (z-f(x_4))} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 \\ &+ \int_{[0,1]^4} \frac{\sigma^2(x_1,x_2)\ \sigma^2(x_1,x_3)\ \sigma^2(x_1,x_4)}{(z-f(x_1))^4\ (z-f(x_2))\ (z-f(x_3))\ (z-f(x_4))} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 \end{aligned}$$

4.3.2 Numerical simulations

Parabolic pulse pertubation by a Wigner matrix

We consider the case where
$\rho(x) = \frac{3}{4}(1-x^2)\mathbb{1}_{[-1,1]}(x), \sigma_d \equiv m$, for some real constant m, and $\sigma \equiv 1$ (this can be adapted to the case $\sigma(x,y) = \mathbb{1}_{|y-x|\leq \ell}$). Theorem 28 is then illustrated by Figure 4.3, where we have considered, for visual reasons, the imaginary parts of the studied Stieltjes transforms,

$$\frac{1}{\pi}\mathfrak{Im}(\mu_n^{\varepsilon}(\varphi_z) - \mu_n(\varphi_z)),$$

which boils down to replace, in Theorem 28, the function φ_z by the function

$$\phi(x) = \frac{1}{\pi}\frac{\eta}{(x-E)^2+\eta^2} \qquad (E \in \mathbb{R}, \eta > 0).$$

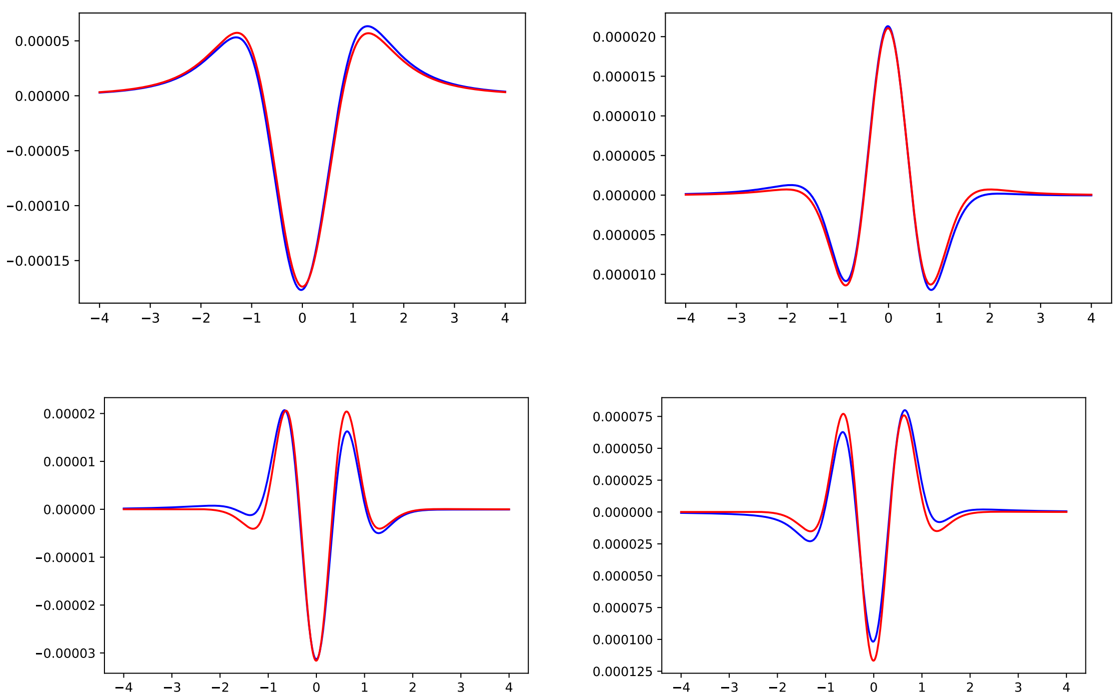


Figure 4.3: **Parabolic pulse perturbation by a Wigner matrix.** For different regimes of $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$ (from left to right and from top to bottom $\varepsilon_n = n^{-\alpha}$ for $\alpha =$ 0.5,0.25,0.17 and 0.143): imaginary part of the term $(\mu_n^{\varepsilon} - \mu_n)(\varphi_z) - \sum_{k=1}^{p-1} \varepsilon_n^k C_{2k}(z)$ (in blue) and imaginary part of the last deterministic term of the expansion $C_{2p}(z)$ (in red) at $z = E + i$ as a function of the real part E. In the four cases, $n = 10^4$.

As it can be seen on Figure 4.3, the predictions from Theorem 28 give a perfect matching with the numerical simulations for any $n^{-1/7} \ll \varepsilon_n \ll 1$.

Uniform measure perturbation by different band matrices

The deterministic terms, $C_k(z)$, of the perturbative expansion we provide are functions of the non-diagonal entries of the pertubative matrix X_n. Thus if X_n is a band matrix then a natural question might be that of the precision of the theoritical model, stated in Theorem 28, as a function of the width, ℓ, of the band. Indeed, the thinner the band, the fewer non-diagonal elements are taken into account in the terms $(C_k(z))$.

To study this question, we consider, here, the case where $f(x) = x$, $\sigma_d(x) \equiv m$ and $\sigma(x, y) = \mathbb{1}_{|y-x|\leq l}$, for some constants $m \geq 0$ and $\ell \in [0, 1]$, which is the relative width of the band.

Theorem 28 is then illustrated by Figure 4.4 where we ploted once again the imaginary parts of the Stieltjes transforms, for different band matrices.

Figure 4.4 shows that the predictions from Theorem 28 give a perfect matching with the numerical simulations and, finally, that the width of the band does not influence significantly the accuracy of the result.

4.4 Strategy of the proof

The proof of Theorems 28 and 29 is based on an iteration of the resolvent formula, which states that for two hermitian matrices A and B, and for any z that is not in the spectrum of A, B and $A + B$ the following egality holds;

$$\frac{1}{z - (A + B)} = \frac{1}{z - A} B \frac{1}{z - (A + B)}.$$

This formula permits to subdivise $(\mu_n^{\varepsilon} - \mu_n)(\varphi_z)$ into several terms that either converge in distribution to $Z(z)$, either converge in probability to the previously cited deterministic terms, or are negligible in probability compared to the other ones. This is the purpose of Subsection 4.5.3. Note that, the deterministic terms, $C_k(z)$, are to be related to planar rooted trees and Catalan numbers. The way that they are obtained is explicited in Subsection 4.5.2.

Once, those convergences are proved, then, thanks to Slutsky's theorem one can deduce both Theorems 28 and 29 as it is done respectively in Subsections 4.5.4 and 4.5.5.

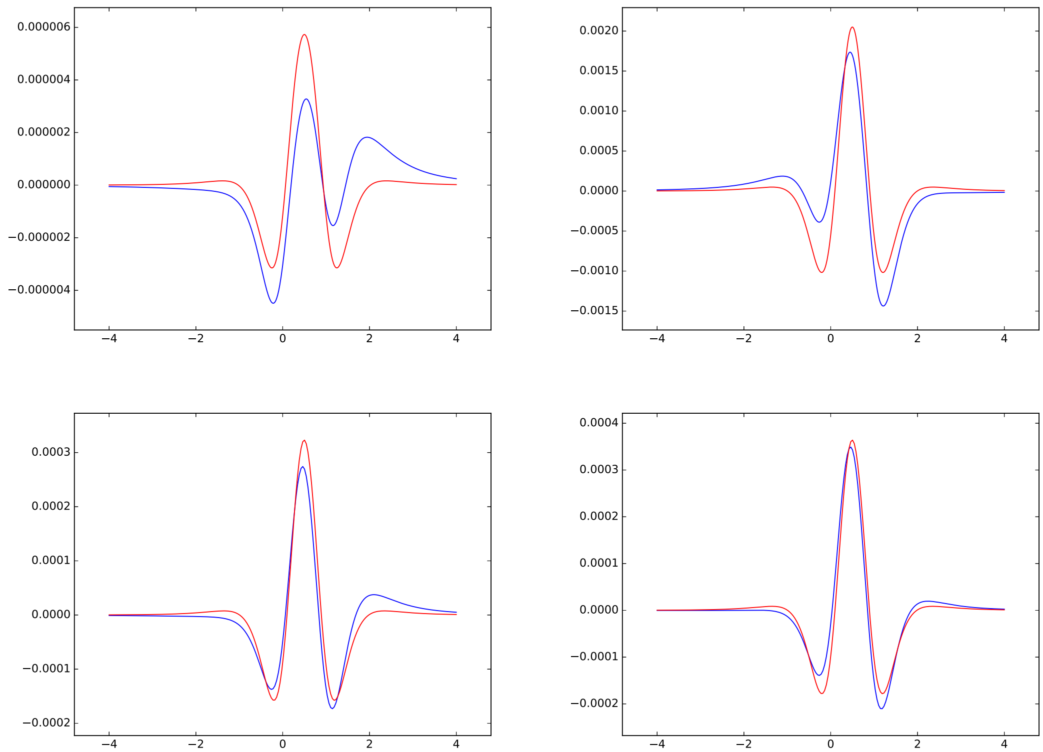


Figure 4.4: **Uniform measure perturbation by a Gaussian band matrix.** Imaginary part of the Stieltjes transform of $(\mu_n^\varepsilon - \mu_n)(\phi) - \varepsilon_n^2 C_2(z)$ (in blue) and of the imaginary part of the last deterministic term $\varepsilon_n^4 C_4(z)$ (in red) at $z = E + \mathrm{i}$ as a function of the real part E for different widths of the band of the perturbative matrix X_n. Here, $n = 10^4$, $\varepsilon_n = n^{-\frac{1}{4}}$ and the band's width take the following values, from left to right and from top to bottom, $\ell = 0.25$, $\ell = 0.5$, $\ell = 0.75$ and $\ell = 1$.

4.5 Proofs

4.5.1 Statements

Moreover, for $p \in \mathbb{N}^*$, let suppose that $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$. One can deduce from the expansion of the resolvent of D_n^ε that,

$$\Delta \mathrm{G}_n(z) = A_1^{(n)}(z) + A_2^{(n)}(z) + A_3^{(n)}(z) + \cdots + A_{2p+1}^{(n)}(z) + A_{2p+2}^{\varepsilon}(z), \tag{4.2}$$

for

$$
\begin{aligned}
A_1^{(n)}(z) &:= \frac{\varepsilon_n}{n}\operatorname{Tr}\frac{1}{z-D}X\frac{1}{z-D} = \frac{\varepsilon_n}{n}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{x_{i,i}}{(z-\lambda_n(i))^2}\\
A_2^{(n)}(z) &:= \frac{\varepsilon_n^2}{n}\operatorname{Tr}\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D} = \frac{\varepsilon_n^2}{n^2}\sum_{i,j}\frac{|x_{i,j}|^2}{(z-\lambda_n(i))^2(z-\lambda_n(j))}\\
&\vdots\\
A_k^{(n)}(z) &:= \frac{\varepsilon_n^k}{n}\operatorname{Tr}\underbrace{\frac{1}{z-D}X\frac{1}{z-D}X\dots\frac{1}{z-D}X}_{k \text{ times } \frac{1}{z-D}X}\frac{1}{z-D}\\
&= \frac{\varepsilon_n^k}{n^{\frac{k+2}{2}}}\sum_{i_1,\dots,i_k=1}^{n}\frac{x_{i_1,i_2}\ x_{i_2,i_3}\ \dots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}}{(z-\lambda_n(i_1))^2\ (z-\lambda_n(i_2))\ \dots\ (z-\lambda_n(i_k))}\\
&\vdots\\
A_{2p+1}^{(n)}(z) &:= \frac{\varepsilon_n^{2p+1}}{n}\operatorname{Tr}\underbrace{\frac{1}{z-D}X\frac{1}{z-D}X\dots\frac{1}{z-D}X}_{(2p+1) \text{ times } \frac{1}{z-D}X}\frac{1}{z-D}\\
A_{2p+2}^{\varepsilon}(z) &:= \frac{\varepsilon_n^{2p+2}}{n}\operatorname{Tr}\underbrace{\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D}X\frac{1}{z-D}X}_{(2p+2) \text{ times } \frac{1}{z-D}X}\frac{1}{z-D^{\varepsilon}}.
\end{aligned}
$$

Notations. To make notations lighter, we shall sometimes suppress the subscripts and superscripts n, so that D_n^{ε}, D_n, $A_k^{(n)}$, X_n and $x_{i,j}^n$ will be respectively denoted by D^{ε}, D, A_k, X and $x_{i,j}$.

4.5.2 How the deterministic terms $C_k(z)$ appear

In order to lighten the proof of Theorems 28 and 29, the way the deterministic terms $C_k(z)$ are obtained is explicited in the following lemma.

Lemma 30. *For any $k \in \mathbb{N}$ and any $z \in \mathbb{C}\backslash\widetilde{\mathcal{S}}$,*

$$\lim_{n\to\infty}\varepsilon_n^{-k}\ \mathbb{E}\left[A_k(z)\right] = \begin{cases} 0 & \text{if } k \text{ is odd}\\ C_k(z) & \text{if } k \text{ is even}\end{cases}.$$

Remark 7. The limits of the even terms $A_2(z), A_4(z), A_6(z), \dots$ could also be expressed thanks to operator-valued free probability theory, using the result [Shl96, Th. 4.1].

To prove Lemma 30, we shall provide first a lemma of graph theory which provides a link beetween the cardinality of the set of the vertices, V, and the cardinality of the set of the edges, E, of a graph G.

Lemma 31. *For any connected graph, $G = (V, E)$, the following inequality holds,*

$$|V| \leq |E| + 1.$$

Moreover, the equality holds if and only if G is a tree.

Proof. The proof is performed by induction over $|V|$.

If the graph is composed of only one vertex then the property is true.
Assume now that $|V| \geq 1$. Let's choose a vertex v of V and list $e_1, \ldots, e_l$ the l edges of G that contain v. One can split the graph G into m connected graphs $G_1, \ldots, G_m$ such that each of the edges $e_1, \ldots, e_l$ is contained in only one of those m graphs. Note that, $m \leq l$. By denoting, for $i \in \{1, \ldots, m\}$, $G_i := (V_i, E_i)$, by induction hypothesis, $|V_i| \leq |E_i| + 1$ and thus

$$|V| = 1 + \sum_{i=1}^{m} |V_i| = 1 + \sum_{i=1}^{m} (|E_i| + 1) = 1 + m + \sum_{i=1} |E_i| = 1 + m + |E| - l \leq 1 + |E|$$

In addition, for the case of equality, note that if $|V| = |E| + 1$ then we must have, for all i, $|V_i| = |E_i| + 1$ and $m = l$. But, if there is a loop in G, then one can find a vertex v such that $m < l$. So, by contradiction, the case of equality is verified only and only if G is a tree. $\square$

Proof. [of Lemma 4.1] Note that for any $k \in \mathbb{N}$ and any $z \in \mathbb{C} \setminus \widetilde{\mathcal{S}}$,

$$\varepsilon_n^{-k}\ \mathbb{E}\left[A_k(z)\right] = \frac{1}{n^{\frac{k}{2}+1}} \sum_{i_1,\ldots,i_k=1}^{n} \frac{\mathbb{E}\left[x_{i_1,i_2}\ x_{i_2,i_3}\ \ldots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}\right]}{(z-\lambda_n(i_1))^2\ (z-\lambda_n(i_2))\ \ldots\ (z-\lambda_n(i_k))}$$

- If k is odd, as the variables (x_{ij}) are centered, in order to get a non zero term in $\mathbb{E}\left[x_{i_1,i_2}\ x_{i_2,i_3}\ \ldots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}\right]$, we need to consider all the couplings such that no first order moment appear. In other words, one should consider all the sets of edges $\{(i_1, i_2), \ldots, (i_{k-1}, i_k), (i_k, i_1)\}$ verifying that every edge is equal at least to an other one. Thus, as we need to consider graphs with at most $\lfloor \frac{k}{2} \rfloor$ different edges, by Lemma 31, we know that those graphs have at most $\lfloor \frac{k}{2} \rfloor + 1$ vertices $i_1, \ldots, i_{\lfloor \frac{k}{2} \rfloor + 1}$.

Thus, since the indices $i_1, i_2, \ldots$ vary from 1 to n, there at most $n^{\lfloor \frac{k}{2} \rfloor + 1} = n^{\frac{k+1}{2}}$ indices contributing in the previous sum and finally,

$$\varepsilon_n^{-k}\ \mathbb{E}\left[A_k(z)\right] = \frac{1}{n^{\frac{k}{2}+1}} \underbrace{\sum_{i_1,\ldots,i_k=1}^{n} \frac{\mathbb{E}\left[x_{i_1,i_2}\ x_{i_2,i_3}\ \ldots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}\right]}{(z-\lambda_n(i_1))^2\ (z-\lambda_n(i_2))\ \ldots\ (z-\lambda_n(i_k))}}_{O\left(n^{\frac{k+1}{2}}\right)} \xrightarrow[n\to\infty]{} 0$$

- If k is even, as the variables $(x_{i,j})$ are centered, we need once again to consider all the sets of edges $\{(i_1, i_2), \ldots, (i_{k-1}, i_k), (i_k, i_1))\}$ that are, at least, equal two by two, in order to get a non zero term in $\mathbb{E}\left[x_{i_1,i_2}\ x_{i_2,i_3}\ \ldots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}\right]$. Namely, we have to consider all the graphs that have at most $\frac{k}{2}$ edges which are at least double.

From Lemma 31 one can deduce that among all these graphs, those whith a triple-edge or more have at most $\frac{k}{2}-1$ edges and so at most $\frac{k}{2}$ vertices. So the indices $i_1, i_2, \ldots$ which are associated to those last graphs contribute at most for $O(n^{k/2})$ in the previous sum.

Moreover, for the remaining case of graphs having exactly $\frac{k}{2}$ double-edges, note that a cyclic path of k edges, $\{i_1,i_2\},\{i_2,i_3\},\ldots,\{i_{k-1},i_k\},\{i_k,i_1\}$, such that each edge is repeated exactly twice, that is to say such that

$$\forall \ell \in [\![1,k-1]\!], \exists! \ell' \in [\![1,k-1]\!],\ \ell \neq \ell', \quad \{i_\ell, i_{\ell+1}\} = \{i_{\ell'}, i_{\ell'+1}\}$$

corresponds to a unique Dyck path, and so, to its associated rooted planar tree. Indeed, as a Dyck path of length k is non-negative and return at time k to 0, all the edges of its associated rooted planar tree form a cyclic path of $k/2$ edges, from the root to the root, which are visited exactly twice (see Remark 5). Thus, as the cyclic order of the edges, $\{(i_1,i_2),\ldots,(i_{k-1},i_k),(i_k,i_1)\}$, induces a unique orientation to those trees which make them planar with $\frac{k}{2}$ edges and with root i_1 (the only vertex which is "visited" twice), one can deduce from Lemma 31 that they are constituted by $\frac{k}{2}+1$ vertices. Thereby, the indices $i_1, i_2, \ldots$ which contribute the most to the previous sum, that is to say for $O(n^{\frac{k}{2}+1})$, are exactly those that "reconstitute" the planar rooted trees of the set $\mathfrak{T}(\frac{k}{2})$.

Therefore, as the number of rooted planar trees with $\frac{k}{2}$ edges is equal to the $(\frac{k}{2}+1)$-th Catalan number, that is to say to $\frac{2}{k+2}\binom{k+2}{\frac{k}{2}+1}$, one can conclude that the sum

$$\sum_{i_1,\ldots,i_k=1}^{n} \frac{\mathbb{E}\left[x_{i_1,i_2}\ x_{i_2,i_3}\ \ldots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}\right]}{(z-\lambda_n(i_1))^2\ (z-\lambda_n(i_2))\ \ldots\ (z-\lambda_n(i_k))}$$

could be decomposed into $\frac{2}{k+2}\binom{k+2}{\frac{k}{2}+1}$ sums related to rooted planar trees and which complexity is $O(n^{\frac{k}{2}+1})$, and other sums which complexity is at most $O(n^{\frac{k}{2}})$. In other words,

$$\varepsilon_n^{-k}\ \mathbb{E}[A_k(z)] = \frac{1}{n^{\frac{k}{2}+1}} \sum_{i_1,\ldots,i_{\frac{k}{2}+1}=1}^{n} \left(\sum_{T\in\mathfrak{T}\left(\frac{k}{2}\right)} \frac{\prod_{(i,j)\in\mathcal{E}_T} \mathbb{E}\left[x_i, x_j\right]}{(z-\lambda_n(x_1)) \prod_{i=1}^{k/2+1} (z-\lambda_n(x_i))^{v_i^T}} + O(n^{\frac{k}{2}}) \right)$$

Thus, by recognition of Riemann sums, as n tends to infinity, the term $\varepsilon_n^{-k}\ \mathbb{E}[A_k(z)]$ tends to the term $C_k(z)$ defined in (4.1). $\square$

4.5.3 Asymptotic behavior of the terms resulting from the expansion of the resolvent of D_n^ε.

The purpose of the four following claims is to describe the asymptotic behavior of each of the terms $(A_k(z))$. Namely, we prove in Claim 5 that the first term, $A_1(z)$, of the expansion converges in distribution to a Gaussian variable, in Claim 6 that all the even terms

$A_2(z), A_4(z), A_6(z)\ldots$ of the expansion converge in probability respectively to the deterministic terms $C_2(z), C_4(z), C_6(z)\ldots$, in Claim 7 that all the even terms $A_3(z), A_5(z), A_7(z)\ldots$ converge in probability to zero and in Claim 8 that the last remainder term, $A^{\varepsilon}_{2p+2}(z)$ converge in probability to zero and is negligible compared to the other terms.

Claim 5. *The finite dimension marginals of the centered process*

$$(n\varepsilon_n^{-1}A_1(z))_{z\in\mathbb{C}\backslash\widetilde{\mathcal{S}}}$$

converge in distribution to those of the centered Gaussian process $(Z(z))_{z\in\mathbb{C}\backslash\widetilde{\mathcal{S}}}$. *Besides, there is* $C>0$ *such that for any* $z\in\mathbb{C}\backslash\widetilde{\mathcal{S}}$,

$$\mathbb{E}[|n\varepsilon_n^{-1}A_1(z)|^2] \;\leq\; \frac{C}{\operatorname{dist}(z,\widetilde{\mathcal{S}})^4}. \tag{4.3}$$

Proof. Estimate (4.3) follows from

$$\mathbb{E}[|n\varepsilon_n^{-1}A_1(z)|^2] \;=\; \frac{1}{n}\sum_{i=1}^{n}\frac{\mathbb{E}\left[|x_{i,i}|^2\right]}{|z-\lambda_n(i)|^4} \;\leq\; \frac{1}{n}\sum_{i=1}^{n}\frac{\sigma_n^2(i,i)}{\operatorname{dist}(z,\widetilde{\mathcal{S}})^4}$$

and from the existence of a uniform upper bound for $\sigma_n^2(i,i)$ which comes from Hypothesis (f) which stipulates that the $(4p+4)$-th moments of the entries $x_{i,j}$ are uniformly bounded.

We turn now to the proof of the convergence in distribution of $n\varepsilon_n^{-1}A_1(z)$ which actually does not depend on the sequence (ε_n). For all $\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p\in\mathbb{C}$ and for all $z_1,\ldots,z_p\in\mathbb{C}\backslash\widetilde{\mathcal{S}}$,

$$\sum_{i=1}^{p}\alpha_i\left(n\varepsilon_n^{-1}A_1(z_i)\right)+\beta_i\overline{(n\varepsilon_n^{-1}A_1(z_i))}=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}x_{j,j}\left(\sum_{i=1}^{p}\xi_n(i,j)\right)$$

for $\xi_n(i,j)=\dfrac{\alpha_i}{(z_i-\lambda_n(j))^2}+\dfrac{\beta_i}{(\overline{z_i}-\lambda_n(j))^2}$.

On one hand, by dominated convergence, the covariance matrix of the above two dimensional random vector converges.

On the other hand, $\mathbb{E}|x_{i,j}|^4$ is uniformly bounded in i, j and n, by Hypothesis (f). Moreover, for n large enough, for all i,j,

$$|\xi_n(i,j)|\leq 2\max_{1\leq i\leq p}(|\alpha_i|+|\beta_i|)\times(\min_{1\leq i\leq p}\operatorname{dist}(z_i,\mathcal{S}))^{-1}.$$

Hence, the conditions of Lindeberg Central Limit Theorem are satisfied and the finite dimension marginals of the process $(n\varepsilon_n^{-1}A_1(z))_{z\in\mathbb{C}\backslash\widetilde{\mathcal{S}}}$ converge in distribution to those of the centered Gaussian process $(Z_z)_{z\in\mathbb{C}\setminus\widetilde{\mathcal{S}}}$ defined by its covariance structure

$$\begin{aligned}\mathbb{E}\left(Z(z)\overline{Z(z')}\right) &= \lim_{n\to\infty}\mathbb{E}\left[\left(n\varepsilon_n^{-1}A_1(z)\right).\left(n\varepsilon_n^{-1}\overline{A_n(z')}\right)\right]\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i,j=1}^{n}\frac{\mathbb{E}\left[x_{i,i}\ \overline{x_{j,j}}\right]}{(z-\lambda_n(i))^2\ (\overline{z'}-\lambda_n(j))^2}\\ &= \int_0^1\frac{\sigma_{\mathrm{d}}(t)^2}{(z-f(t))^2\ (\overline{z'}-f(t))^2}\mathrm{d}t\end{aligned}$$

and by the fact that $\overline{Z(z)}=Z(\overline{z})$ which comes from $\overline{A_1(z)}=A_1(\overline{z})$. $\square$

Claim 6. *For any even $k\in\{2,\dots,2p+1\}$, there is a constant C such that for any $z\in\mathbb{C}\setminus\widetilde{\mathcal{S}}$,*

$$\mathbb{E}\left[\left|n\varepsilon_n^{-1}\left(A_k(z)-\varepsilon_n^kC_k(z)\right)\right|^2\right]\leq\frac{C\varepsilon_n^{2k-2}}{\mathrm{dist}(z,\widetilde{\mathcal{S}})^{2k+2}}+\frac{C\eta_n^2\varepsilon_n^{2k-4}}{\mathrm{dist}(z,\widetilde{\mathcal{S}})^{2k+4}}$$

Proof. Introduce the variable $A_k^o(z)$ obtained by centering the variable $n\varepsilon_n^{-k}A_k(z)$:

$$\begin{aligned}A_k^o(z) &:= n\varepsilon_n^{-k}(A_k(z)-\mathbb{E}A_k(z))\\ &= \frac{1}{n^{\frac{k}{2}}}\sum_{i_1,\dots,i_k}^{n}\frac{x_{i_1,i_2}\ x_{i_2,i_3}\ \dots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}-\mathbb{E}\left[x_{i_1,i_2}\ x_{i_2,i_3}\ \dots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}\right]}{(z-\lambda_n(i_1))^2\ (z-\lambda_n(i_2))\ \dots\ (z-\lambda_n(i_k))}\end{aligned}$$

and the defect variable

$$\begin{aligned}\delta_n(z) &:= \varepsilon_n^{-k}\mathbb{E}A_k(z)-C_k(z)\\ &= \frac{1}{n^{\frac{k+2}{2}}}\sum_{i_1,\dots,i_k=1}^{n}\frac{\mathbb{E}\left[x_{i_1,i_2}\ x_{i_2,i_3}\ \dots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}\right]}{(z-\lambda_n(i_1))^2\ (z-\lambda_n(i_2))\ \dots\ (z-\lambda_n(i_k))}\\ &\qquad -\lim_{n\to\infty}\frac{1}{n^{\frac{k+2}{2}}}\sum_{i_1,\dots,i_k=1}^{n}\frac{\mathbb{E}\left[x_{i_1,i_2}\ x_{i_2,i_3}\ \dots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}\right]}{(z-\lambda_n(i_1))^2\ (z-\lambda_n(i_2))\ \dots\ (z-\lambda_n(i_k))}\end{aligned}$$

we want to dominate the L^2 norm of $n\varepsilon_n^{-1}\left(A_k(z)-\varepsilon_n^kC_k(z)\right)=\varepsilon_n^{k-1}A_k^o(z)+n\varepsilon_n^{k-1}\delta_n(z)$. For this purpose, we successively dominate $A_k^o(z)$, $\delta_n(z)$ and $C_k(z)$.

Using the independence of the $x_{i,j}$'s, the fact that they are bounded in L^{4p+4} and the fact that z stays at a macroscopic distance of the $\lambda_n(i)$'s, we can write for all $z\in\mathbb{C}\setminus\widetilde{\mathcal{S}}$

$$\begin{aligned}\mathbb{E}[|A_k^o(z)|^2] &\leq \mathbb{E}[|n\varepsilon_n^{-k}A_k(z)|^2]\\ &\leq \mathbb{E}\left[\left|\frac{1}{n^{\frac{k}{2}}}\sum_{i_1,\dots,i_k=1}^{n}\frac{x_{i_1,i_2}\ x_{i_2,i_3}\ \dots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}}{(z-\lambda_n(i_1))^2\ (z-\lambda_n(i_2))\ \dots\ (z-\lambda_n(i_k))}\right|^2\right]\\ &\leq \frac{1}{n^k}\sum_{i_1,\dots,i_{2k}=1}^{n}\frac{\mathbb{E}\left[x_{i_1,i_2}\ \dots\ x_{i_k,i_1}\ \ x_{i_{k+1},i_{k+2}}\ \dots\ x_{i_{2k-1},i_{2k}}\right]}{\mathrm{dist}(z,\widetilde{\mathcal{S}})^{2k+2}}\end{aligned}$$

Generically, the set of "edges" $\{(i_1,i_2),(i_2,i_3),\dots,(i_{k-1},i_k)\}$ must be equal to the set $\{(i_{k+1},i_{k+2}),\dots,(i_{2k-1},i_{2k})\}$ in order to get a non zero term. Therefore the complexity of the previous sum is $O(n^k)$. Note that other non zero terms involving third or fourth moments are much less numerous. Hence,

$$\mathbb{E}[|A_k^o(z)|^2] \le \frac{O(n^k)}{n^k \mathrm{dist}(z,\widetilde{\mathcal{S}})^{2k+2}} \le \frac{C}{\mathrm{dist}(z,\widetilde{\mathcal{S}})^{2k+2}}. \tag{4.4}$$

Now, note that the expression of the finite and deterministic term of $C_k(z)$, defined in (4.1), implies that,

$$C_k(z) \le \frac{C}{\mathrm{dist}(z,\widetilde{\mathcal{S}})^{k+1}}. \tag{4.5}$$

Finally, the term $\delta_n(z)$ rewrites

$$\begin{aligned}\delta_n(z) \;=\; O(n^{-1}) + \sum_{T\in\mathfrak{T}\left(\frac{k}{2}\right)} \int_{[0,1]^{\frac{k}{2}+1}} \mathbb{1}_{\mathfrak{B}} \Bigg(&\frac{\prod_{\{i,j\}\in\mathcal{E}_T} \sigma^2(\lfloor nx_i\rfloor,\lfloor nx_j\rfloor)}{(z-\lambda_n(\lfloor nx_1\rfloor))\prod_{i=1}^{k/2+1}(z-\lambda_n(\lfloor nx_i\rfloor))^{v_i^T}} \\ &- \frac{\prod_{\{i,j\}\in\mathcal{E}_T}\sigma^2(x_i,x_j)}{(z-f(x_1))\prod_{i=1}^{k/2+1}(z-f(x_i))^{v_i^T}}\,\mathrm{d}x_2\dots\mathrm{d}x_{\frac{k}{2}+1}\Bigg)\,\mathrm{d}x_1\dots\mathrm{d}x_{\frac{k}{2}+1},\end{aligned}$$

where $\mathfrak{B} := \left\{(x_1,\dots,x_{\frac{k}{2}+1})\in[0,1]^{\frac{k}{2}+1}\;;\;\forall i\neq j, \lfloor nx_i\rfloor \neq \lfloor nx_j\rfloor\right\}$.

Since, for $M_\sigma := \sup_{0\le x\neq y\le 1}\sigma(x,y)^2$ and for any fixed $z\notin\widetilde{\mathcal{S}}$, the function

$$\psi_z(s,\lambda_1,\dots,\lambda_{\frac{k}{2}+1}) \mapsto \frac{s}{(z-\lambda_1))\prod_{i=1}^{k/2+1}(z-\lambda_i)^{v_i^T}}$$

defined for $(s,\lambda_1,\dots,\lambda_{\frac{k}{2}+1})\in[0,M_\sigma+1]\times\{x\in\mathbb{R}\,;\,\mathrm{dist}(x,\widetilde{\mathcal{S}})\le\mathrm{dist}(z,\widetilde{\mathcal{S}})/2\}^{\frac{k}{2}+1}$ is $C\,\mathrm{dist}(z,\widetilde{\mathcal{S}})^{-(k+2)}$-Lipschitz, for C a universal constant, by Hypothesis (b),

$$\delta_n(z) \;=\; O(n^{-1}) + \frac{O(\eta_n)}{\max\{n\varepsilon_n,1\}\,\mathrm{dist}(z,\widetilde{\mathcal{S}})^{k+2}}. \tag{4.6}$$

Collecting estimations (4.4), (4.5) and (4.6) we conclude. □

Claim 7. *For any odd $k\in\{3,\dots,2p+1\}$ there is a constant C such that for any $z\in\mathbb{C}\backslash\widetilde{\mathcal{S}}$,*

$$\mathbb{E}[|n\varepsilon_n^{-1}A_k(z)|^2] \;\le\; \frac{C\varepsilon_n^{2k-2}}{\mathrm{dist}(z,\widetilde{\mathcal{S}})^{2k+2}}.$$

Proof. We start by writing for all $z \in \mathbb{C}\backslash\widetilde{\mathcal{S}}$,

$$\mathbb{E}[|n\varepsilon_n^{-1}A_k(z)|^2] = \frac{\varepsilon_n^{2k-2}}{n^k}\mathbb{E}\left[\left|\sum_{i_1,\dots,i_k=1}^{n}\frac{x_{i_1,i_2}\ x_{i_2,i_3}\ \dots\ x_{i_{k-1},i_k}\ x_{i_k,i_1}}{(z-\lambda_n(i_1))^2\ (z-\lambda_n(i_2))\ \dots\ (z-\lambda_n(i_k))}\right|^2\right]$$

$$=\sum_{i_1,\dots,i_{2k}=1}^{n}\frac{\varepsilon_n^{2k-2}\ \mathbb{E}\left[x_{i_1,i_2}\ x_{i_2,i_3}\ \dots\ x_{i_{2k-1},i_{2k}}\right]}{n^k\ (z-\lambda_n(i_1))^2(z-\lambda_n(i_2))\dots(z-\lambda_n(i_k))(\overline{z}-\lambda_n(i_{k+1}))^2(\overline{z}-\lambda_n(i_{k+2}))\dots(\overline{z}-\lambda_n(i_{2k}))}.$$

Generically, the set of "edges" $\{(i_1,i_2),(i_2,i_3),\dots,(i_{k-1},i_k)\}$ must be equal to the set $\{(i_{k+1},i_{k+2}),\dots,(i_{2k-1},i_{2k})\}$ in order to get a non zero term. Therefore, as k is odd, the complexity of the previous sum is $O(n^k)$. Note that other non zero terms involving third or fourth moments are much less numerous. Hence,

$$\mathbb{E}[|n\varepsilon_n^{-1}A_k(z)|^2] \le \frac{\varepsilon_n^{2k-2}}{n^k}\times\frac{O(n^k)}{\operatorname{dist}(z,\widetilde{\mathcal{S}})^{2k+2}}\ \le\ \frac{C\varepsilon_n^{2k-2}}{\operatorname{dist}(z,\widetilde{\mathcal{S}})^{2k+2}}$$

□

Claim 8. *For any $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$\mathbb{E}[|n\varepsilon_n^{-1}A^{\varepsilon}_{2p+2}(z)|^2]\ \le\ \frac{O(n^2\varepsilon_n^{4p+2})}{|\mathfrak{Im}(z)|^2\operatorname{dist}(z,\widetilde{\mathcal{S}})^{4p+4}}.$$

Proof. Remind that,

$$A^{\varepsilon}_{2p+2}(z) := \frac{\varepsilon_n^{2p+2}}{n}\operatorname{Tr}\underbrace{\frac{1}{z-D}X\dots\frac{1}{z-D}X}_{(2p+2)\text{ times }\frac{1}{z-D}X}\frac{1}{z-D^{\varepsilon}}$$

Hence,

$$\begin{aligned}
\mathbb{E}[|n\varepsilon_n^{-1}A^{\varepsilon}_{2p+2}(z)|^2] &\le \varepsilon_n^{4p+2}\ \mathbb{E}\left[\left|\operatorname{Tr}\underbrace{\frac{1}{z-D}X\dots\frac{1}{z-D}X}_{(2p+2)\text{ times }\frac{1}{z-D}X}\frac{1}{z-D^{\varepsilon}}\right|^2\right]\\
&\le \varepsilon_n^{4p+2}\ \mathbb{E}\left[\operatorname{Tr}\left|\left(\frac{1}{z-D}X\right)^{2p+2}\right|^2\times\operatorname{Tr}\left|\frac{1}{z-D^{\varepsilon}}\right|^2\right]\\
&\le \varepsilon_n^{4p+2}\ \mathbb{E}\left[\operatorname{Tr}\left(\left(\frac{1}{z-D}X\right)^{2p+2}\left(\overline{\frac{1}{z-D}X}\right)^{2p+2}\right)\frac{n}{|\operatorname{Im}(z)|^2}\right]\\
&\le \frac{n\varepsilon_n^{4p+2}}{|\operatorname{Im}(z)|^2}\ \mathbb{E}\left[\operatorname{Tr}\left(\left(\frac{1}{z-D}X\right)^{2p+2}\left(\overline{\frac{1}{z-D}X}\right)^{2p+2}\right)\right]\\
&\le \frac{n\varepsilon_n^{4p+2}}{|\operatorname{Im}(z)|^2}\frac{O(n^{(2p+3)})}{n^{2p+2}\ \operatorname{dist}(z,\widetilde{\mathcal{S}})^{4p+4}}\ \le\ \frac{O(n^2\varepsilon_n^{4p+2})}{|\operatorname{Im}(z)|^2\operatorname{dist}(z,\widetilde{\mathcal{S}})^{4p+4}}.
\end{aligned}$$

The inequality of the last line takes into account that

- the L^{4p+4} norm of the entries of $\sqrt{n}X$ is uniformly bounded
- the norm of the entries of X is of order $n^{-1/2}$
- the norm of the coefficients of $(z-D)^{-1}$ is smaller than $\operatorname{dist}(z,\widetilde{\mathcal{S}})^{-1}$
- the complexity of the sum defining the trace is of order $O(n^{2p+3})$ since its non-null terms are encoded by $(2p+2)$ edges trees which have therefore $(2p+3)$ vertices.

Moreover, note that, as we have assumed that $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$, the term $n^2\varepsilon_n^{4p+2}$ is $o(1)$. □

4.5.4 Proof of Theorem 28

The proof of Theorem 28 is based on the perturbative expansion (4.2) of the resolvent $\frac{1}{n}\operatorname{Tr}\frac{1}{z-D_n^\varepsilon}$ and the four previous Claims.

Proof. We gather now the results of the previous claims. If $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$, then, for any $z \in \mathbb{C}\setminus\mathbb{R}$, we need to make the following expansion until the order $(2p+2)$,

$$n\varepsilon_n^{-1}\Delta \mathrm{G}_n(z) = n\varepsilon_n^{-1}\left(A_1(z)+A_2(z)+A_3(z)+\cdots+A_{2p+1}(z)+A_{2p+2}^\varepsilon(z)\right)$$

and

- Claim 5 proves that the process $n\varepsilon_n^{-1}A_1(z)$ converges in distribution to the centered Gaussian variable $Z(z)$,
- Claim 6 proves that for any even $k\in\{2,\dots,2p\}$ each term $n\varepsilon_n^{-1}A_k(z)$ converges in probability to the deterministic term $C_k(z)$,
- Claim 7 proves that for any odd $k\in\{3,\dots,2p+1\}$ each term $n\varepsilon_n^{-1}A_k(z)$ converges in probability to zero,
- and Claim 8 proves that the remainder term $n\varepsilon_n^{-1}A_{2p+2}^\varepsilon(z)$ converges in probability to zero.

Thus, by Slutsky's theorem, for any $z\in\mathbb{C}\setminus\mathbb{R}$:

$$n\varepsilon_n^{-1}\left(\Delta \mathrm{G}_n(z)-\varepsilon_n^2C_2(z)-\varepsilon_n^4C_4(z)\cdots-\varepsilon_n^{2p}C_{2p}(z)\right) \quad \underset{n\to\infty}{\xrightarrow{\text{dist.}}} \quad Z(z)$$

This finishes the proof of Theorem 28. □

4.5.5 Proof of the local type convergence result

Thanks to the perturbative expansion (4.2) of the resolvent $\frac{1}{n}\operatorname{Tr}\frac{1}{z-D_n^\varepsilon}$ and Claim 5, Claim 6, Claim 7 and Claim 8 we can also proceed to the proof of Theorem 29.

Proof. Assume $n^{-1} \ll \varepsilon_n \ll 1$ and note that $\varepsilon_n^{-p}\Delta \mathrm{G}_n(z_n) - \sum_{\substack{k=0 \\ k \text{ even}}}^{p-2} \varepsilon_n^k C_{p-k}(z_n)$ rewrites

$$\varepsilon_n^{-p}(A_1(z_n))+\varepsilon_n^{-p}\left(A_2(z_n)-\varepsilon_n^2 C_2(z_n)\right)+\varepsilon_n^{-p}A_3(z_n)+\varepsilon_n^{-p}\left(A_4(z_n)-\varepsilon_n^4 C_4(z_n)\right)+\cdots+\varepsilon_n^{-p}A_{p+2}^\varepsilon(z_n)$$

One can directly obtain, for all non-real complex sequences (z_n), that

- by Claim 5, if $\operatorname{dist}(z_n,\widetilde{\mathcal{S}}) \gg (n\varepsilon_n^{p-1})^{-1/2}$, then

$$\mathbb{E}[|\varepsilon_n^{-p}A_1(z_n)|^2] \le \frac{C}{(n\varepsilon_n^{p-1})^2 \operatorname{dist}(z_n,\widetilde{\mathcal{S}})^4} \underset{n\to\infty}{\longrightarrow} 0,$$

- by Claim 6, for any even $k \in \{2,p\}$, if $\operatorname{dist}(z_n,\widetilde{\mathcal{S}}) \gg \max\left\{\left(\frac{\varepsilon_n^{2-p}}{n}\right)^{1/3}, (\eta_n/(n\varepsilon_n^{p-1}))^{1/4}\right\}$, then

$$\mathbb{E}\left[\left|\varepsilon_n^{-p}\left(A_k(z_n)-\varepsilon_n^k C_k(z_n)\right)\right|^2\right] \le \frac{C\varepsilon_n^{2(k-p)}}{n^2\operatorname{dist}(z_n,\widetilde{\mathcal{S}})^{2k+2}} + \frac{C\eta_n^2\varepsilon_n^{-2p}}{n^2\operatorname{dist}(z_n,\widetilde{\mathcal{S}})^{2k+4}} \underset{n\to\infty}{\longrightarrow} 0$$

- by Claim 7, for any odd $k \in \{3,p+1\}$, if $\operatorname{dist}(z_n,\widetilde{\mathcal{S}}) \gg \left(\frac{\varepsilon_n^{3-p}}{n}\right)^{1/4}$, then

$$\mathbb{E}[|\varepsilon_n^{-p}C_k(z_n)|^2] \le \frac{C\varepsilon_n^{2(k-p)}}{n^2 \operatorname{dist}(z_n,\widetilde{\mathcal{S}})^{2k+2}} \underset{n\to\infty}{\longrightarrow} 0,$$

- by Claim 8, if $|\mathfrak{Im}(z_n)|\operatorname{dist}(z_n,\widetilde{\mathcal{S}})^{p+2} \gg \varepsilon_n^2$, then

$$\mathbb{E}[|\varepsilon_n^{-p}A_{p+2}^\varepsilon(z_n)|^2] \le \frac{O(\varepsilon_n^4)}{|\mathfrak{Im}(z_n)|^2\operatorname{dist}(z_n,\widetilde{\mathcal{S}})^{2p+4}} \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, when

$$\operatorname{dist}(z_n,\widetilde{\mathcal{S}}) \gg \max\left\{(n\varepsilon_n^{p-1})^{-1/2}, \left(\frac{\varepsilon_n^{2-p}}{n}\right)^{1/3}, \left(\frac{\eta_n}{n\varepsilon_n^{p-1}}\right)^{1/4}, \left(\frac{\varepsilon_n^{3-p}}{n}\right)^{1/4}\right\}$$

and

$$|\mathfrak{Im}(z_n)|\operatorname{dist}(z_n,\widetilde{\mathcal{S}})^{p+2} \gg \varepsilon_n^2,$$

all the processes that constitute the process $\varepsilon_n^{-p}\Delta \mathrm{G}_n(z_n) - \sum_{\substack{k=0 \\ k \text{ even}}}^{p-2} \varepsilon_n^{-k} C_{p-k}(z)$ converge to 0 in probability. Since $\mathrm{dist}(z_n, \widetilde{\mathcal{S}}) \geq \mathfrak{Im}(z_n)$, the above condition is implied by

$$\mathfrak{Im}(z_n) \gg \max\left\{ (n\varepsilon_n^{p-1})^{-1/2} \ , \ \left(\frac{\varepsilon_n^{2-p}}{n}\right)^{1/3} \ , \ \left(\frac{\eta_n}{n\varepsilon_n^{p-1}}\right)^{1/4} \ , \ \left(\frac{\varepsilon_n^{3-p}}{n}\right)^{1/4} \ , \ \varepsilon_n^{\frac{2}{p+3}} \right\}.$$

Observing finally that the two terms $\left(\frac{\varepsilon_n^{2-p}}{n}\right)^{1/3}$ and $\left(\frac{\varepsilon_n^{3-p}}{n}\right)^{1/4}$ are dominated by the maximum of the three other ones, we conclude the proof. $\square$

4.6 A possible extension to other classes of test functions

Theorems 28 and 29 could be extented from test functions of the type $\varphi_z(x) := \frac{1}{z-x}$ to larger classes of test functions. To make this extension possible it is necessary to succeed in expressing the terms $C_k(z)$, defined in (4.1), as linear expressions of the function $\varphi_z(x)$. Thereby, the expansions of $(\mu_n^\varepsilon - \mu_n)(\varphi_z)$ that we have previously considered, would be expressed entirely as linear forms of φ_z, which will allow to extend this convergence, by density of the linear span of the functions $\varphi_z(x)$ in the fractional Sobolev space $\mathcal{H}_s$, for any $s > 0$, to a larger class of regular functions through a CLT extension theorem. For example, one could use the CLT extension lemma of Scherbina and Tirozzi which could be found in [ST10].

This extension have been done in [BGEM17], for all the regimes where $\varepsilon_n \ll n^{-1/3}$, that is to say for all the regimes that involve at most one deterministic term, which comes from $C_2(z)$. In fact, in this paper the expression $C_2(z)$ have been expressed under the following form which is linear in $\varphi_z(x)$:

$$C_2(z) = \int_{[0,1]^2} \frac{\sigma^2(x_1, x_2)}{(z - f(x_1))^2\ (z - f(x_2))}\, \mathrm{d}x_1 \mathrm{d}x_2 = -\int_{\mathbb{R}} \varphi_z'(s) F(s) \mathrm{d}s$$

where $F(s) = \int_{\mathbb{R}} \frac{\tau(s,t)\rho(s)\rho(r)}{s-t} \mathrm{d}t$ for τ a function such that $\tau(s,t) = \sigma^2(f^{-1}(s), f^{-1}(t))$ and ρ the density of the push-forward of the uniform measure on $[0,1]$ by the function f.

But from the second deterministic term $C_4(z)$ it becomes much more difficult to give a linear expression in $\varphi_z(x)$ of $C_k(z)$ without making too restrictive assumptions on the diagonal entries of the matrix X_n.

If one succeeds in overcoming this technical limitation on the diagonal entries of X_n, then by a similar approach that in [BGEM17] a result, as for example the following that in addition to using the above-mentioned CLT extension lemma uses the Helffer-Sjőstrand formula (see [HS89]) and the fact that $\mathcal{C}^{2p+4}(\mathbb{R}) \subset \mathcal{H}_s$ for $s < 2p + 4$ (see [Hör03]), could be proved:

Let $p \in \mathbb{N}^*$. Under the semi-perturbative regime, if $n^{-1/(2p-1)} \ll \varepsilon_n \ll n^{-1/(2p+1)}$ then there exist p linear deterministic functions, $D_1, \dots, D_p$, defined on the space of compactly supported $\mathcal{C}^{2p+4}$ functions on $\mathbb{R}$ such that for any compactly supported $\mathcal{C}^{2p+4}$ function, ϕ, on $\mathbb{R}$, the following convergence holds :

$$n\varepsilon_n^{-1}\left((\mu_n^\varepsilon - \mu_n)(\phi) + D_1(\phi) + \cdots + D_p(\phi)\right) \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad Z_\phi.$$

where $(Z_\phi)_{\phi\in\mathcal{C}^1}$ is the centered Gaussian field indexed by the set of $\mathcal{C}^1$ complex functions on $\mathbb{R}$, with covariance structure defined by

$$\mathbb{E}Z_\phi Z_\psi \;=\; \int_0^1 \sigma_{\mathrm{d}}(t)^2\phi'(f(t))\psi'(f(t))\mathrm{d}t \qquad \text{and} \qquad \overline{Z_\psi} \;=\; Z_{\overline{\psi}}$$

and which can be represented as

$$Z_\phi = \int_0^1 \sigma_{\mathrm{d}}(t)\phi'(f(t))\mathrm{d}B_t$$

where (B_t) is the standard one-dimensional Brownian motion.

Note that the computation of the deterministic functions $D_1, \dots, D_p$ is rather intricate and without any clear recursive formula. Indeed, each term D_k comes from the deterministic term $C_{2k}(z)$ after some computations which have no clear reccurence relation such that partial fraction decompositions. Also, according to this link between D_k and $C_{2k}(z)$, all the determistic terms $D_1(\phi), \dots, D_p(\phi)$ have a much larger order than the probabilistic term containing Z_ϕ and, more precisely, for any $k \in \{2, \dots, p-1\}$ the deterministic term $D_k(\phi)$ has a smaller order than $D_{k-1}(\phi)$ and a larger than $D_{k+1}(\phi)$. Namely, $D_k(\phi)$ is of order $(\varepsilon_n)^k$.

Chapter 5

Eigenvectors of a matrix under random perturbation

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5.1 Introduction

This last chapter is devoted to the study of the sensitivity of the eigenvectors of a given operator under small perturbations. In the previous chapters we studied the effect of a perturbation on the spectrum of a Hermitian matrix by a random matrix with small operator norm and whose entries in the eigenvector basis of the first one were independent, centered, with a variance profile. We provided a perturbative expansion of the empirical spectral distribution, but did not consider the deformation of the eigenvectors basis with respect to the canonical basis. In the present paper, to complete this first study, we deal with the spectral measure of our matrix associated to the state defined by a given vector.

To define this measure, let us introduce some notations. We consider a real diagonal matrix $D_n = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ (the eigenvalue λ_i implicitly depends on n), as well as a Hermitian random matrix

$$X_n = \frac{1}{\sqrt{n}} \left[x^n_{i,j}\right]_{1\leq i,j\leq n}$$

such that the x_{ij} are independent (up to the symmetry), centered, with a variance profile. The normalizing factor $n^{-1/2}$ and our hypotheses below ensure that the operator norm of X_n is of order one. We then define, for $\varepsilon > 0$,

$$D_n^\varepsilon := D_n + \varepsilon X_n.$$

In contrast with [BGEM17], where we studied the empirical spectral measure μ_n^ε of the matrix D_n^ε, we consider here the spectral measure $\mu_{n,\mathbf{e}_i}^\varepsilon$ of D_n^ε over a vector $\mathbf{e}_i$ of the canonical basis, defined through an eigenvector basis $(\mathbf{u}_j^\varepsilon)_{j\in\{1,\dots,n\}}$ of D_n^ε and the related eigenvalues $(\lambda_j^\varepsilon)_{j\in\{1,\dots,n\}}$ by

$$\mu_{n,\mathbf{e}_i}^\varepsilon := \sum_{j=1}^n |\langle \mathbf{u}_j^\varepsilon, \mathbf{e}_i\rangle|^2 \delta_{\lambda_j^\varepsilon}.$$

The interest of these measures is that they give information on the eigenvector basis of D_n^ε, while being tractable since they satisfy, for any test function φ, the key identity

$$\int \varphi(x)\, \mathrm{d}\mu_{n,\mathbf{e}_i}^\varepsilon(x) = \sum_{j=1}^n |\langle \mathbf{u}_j^\varepsilon, \mathbf{e}_i\rangle|^2 \varphi(\lambda_j^\varepsilon) = \left(\varphi\left(D_n^\varepsilon\right)\right)_{i,i}. \tag{5.1}$$

Our main result, Theorem 32, gives a perturbative expansion of $\mu_{n,\mathbf{e}_i}^\varepsilon$. More precisely, using a resolvent expansion and the Helffer-Sjöstrand formula, we give an asymptotic expansion of

$$\int_{\mathbb{R}} \varphi(t) \mathrm{d}\mu_{n,\mathbf{e}_i}^\varepsilon(t)$$

for any $\mathcal{C}^5$ test function φ. From that, we deduce, at Equation (5.17), a heuristic first order estimation of the overlaps $|\langle \mathbf{u}_j^\varepsilon, \mathbf{e}_i\rangle|^2$ between the eigenvectors $\mathbf{u}_j^\varepsilon$ of D_n^ε and the ones $\mathbf{e}_i$ of D_n.

Some other works, on models closed to our one or contained in it, are devoted to the sensitivity to perturbations of the eigenvectors. Some of them, as [OVW16, OVW17, vSW17, Zho17], provide bounds on the deviations of these eigenvectors under perturbation, while some other, as [AB12, AB14, ABB14, Ben17], provide explicit perturbative expansions. As we explained in the previous paragraph, this is what we do here, and it allows us to recover (see (5.17)) the fact that the overlaps $|\langle \mathbf{u}_j^\varepsilon, \mathbf{e}_i\rangle|^2$ have order $\varepsilon^2(\lambda_j - \lambda_i)^2/n$. This cannot be proved for all i, j but only on average over more or less large windows. This is why estimates of this flavor have already appeared in various papers, as in [AB12, AB14, ABB14, Ben17] under various forms (for example, one of the differences between our result and [Ben17, Coro. 1.6] is that we average on j and not on i, as explained in Remark 5). In addition to the fact that it only relies on short and elementary computations, one of the interests of this paper is also to consider rather general perturbations, since we do not suppose that all entries of X_n have the same variance nor that they are Gaussian. Another interest is to provide, with the functional $\Xi_s(\varphi)$ from (5.5) and (5.9), an expression for the first order expansion of the measure $\mu_{n,\mathbf{e}_i}^\varepsilon$ from (5.1), which, up to our knowledge, did not appear so far.

The paper is organized as follows. Results and comments are given in Section 5.2, proofs in Section 5.3, while Section 5.4 is devoted to the heuristic derivation of the consequences of Theorem 32, and to some simulations which corroborate our predictions.

Notation. For $u = u_n$, $v = v_n$ some sequences, $u \ll v$ means that u_n/v_n tends to 0.

5.2 Main result

We consider a real diagonal matrix $D_n = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ (the eigenvalue λ_i implicitly depend on n), as well as a Hermitian random matrix

$$X_n = \frac{1}{\sqrt{n}} \left[x^n_{i,j}\right]_{1 \le i,j \le n}$$

and define, for $\varepsilon > 0$,

$$D_n^{\varepsilon} := D_n + \varepsilon X_n.$$

We make the following hypotheses:

(a) the entries $x^n_{i,j}$ of $\sqrt{n}X_n$ are independent (up to symmetry) random variables, centered, with variance denoted by $\sigma_n^2(i,j)$, such that $\mathbb{E}|x^n_{i,j}|^6$ is bounded uniformly on n, i, j,

(b) there are two bounded real functions, f and σ, defined respectively on $[0,1]$ and $[0,1]^2$ such that, denoting λ_i by $\lambda_{n,i}$ to emphasize the implicit dependence in n, the error bound

$$\eta_n \quad := \quad \sup_{x\in[0,1]} |\lambda_{n,\lfloor nx \rfloor} - f(x)| + \sup_{(x,y)\in[0,1]^2} |\sigma_n^2(\lfloor nx \rfloor, \lfloor ny \rfloor) - \sigma^2(x,y)| \tag{5.2}$$

satisfies

$$\eta_n \underset{n\to\infty}{\longrightarrow} 0.$$

Let us now make some assumptions on the limiting functions σ and f:

(c) the push-forward of the uniform measure on $[0,1]$ by the function f has a density ρ with respect to the Lebesgue measure on $\mathbb{R}$ and a compact support denoted by $\mathcal{S}$,

(d) the variances of the entries of X_n essentially depend on the eigenspaces of D_n, namely, there exists a symmetric function $\tau(\,\cdot\,,\,\cdot\,)$ on $\mathbb{R}^2$ such that for all $x \neq y$, $\sigma^2(x,y) = \tau(f(x), f(y))$.

Let $\mu^\varepsilon_{n,\mathbf{e}_i}$ denote the probability measure defined, for any test function φ, by

$$\int \varphi(t)\mathrm{d}\mu^\varepsilon_{n,\mathbf{e}_i}(t) := (\varphi(D^\varepsilon_n))_{ii}. \tag{5.3}$$

One can equivalently define $\mu^\varepsilon_{n,\mathbf{e}_i}$ by

$$\mu^\varepsilon_{n,\mathbf{e}_i} := \sum_{j=1}^n |\langle \mathbf{u}^\varepsilon_j, \mathbf{e}_i\rangle|^2 \delta_{\lambda^\varepsilon_j}, \tag{5.4}$$

where $\mathbf{e}_i$ denotes the i-th vector of the canonical basis, the λ^ε_j's denote the eigenvalues of D^ε_n and the $\mathbf{u}^\varepsilon_j$'s denote the associated eigenvectors.

We now introduce a functional which is central in the statement of our result. This functional admits another expression, given in Proposition 33 below.

Let, for $s \in \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{C}$ a $\mathcal{C}^2$ function,

$$\Xi_s(\varphi) := \int_{\mathbb{R}} \tau(s,t)\rho(t)\frac{(\varphi(t) - \varphi(s) - (t-s)\varphi'(s))}{(t-s)^2}\mathrm{d}t \tag{5.5}$$

Theorem 32. *Let us suppose that $\varepsilon \ll 1$. Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a compactly supported $\mathcal{C}^5$ function. For $x \in [0,1]$, set $i = i(n,x) = \lfloor nx \rfloor$. Then we have*

$$\int_{\mathbb{R}} \varphi(t)\mathrm{d}\mu^\varepsilon_{n,\mathbf{e}_i}(t) = \varphi\left(\lambda_i + \frac{\varepsilon}{\sqrt{n}}x_{ii}\right) + \varepsilon^2\Xi_{f(x)}(\varphi) + O_{L^2}\left(\varepsilon^2\|\varphi^{(5)}\|_\infty(\varepsilon + n^{-1/2} + \eta_n)\right)$$

for η_n as in (5.2).

Remark 2 (Leading order transition)**.** Note that for any $\mathcal{C}^1$ test function φ,

$$\varphi\left(\lambda_i + \frac{\varepsilon}{\sqrt{n}}x_{ii}\right) = \varphi(\lambda_i) + \frac{\varepsilon}{\sqrt{n}}x_{ii}\varphi'(\lambda_i) + O_{L^2}\left(\frac{\varepsilon^2}{n}\|\varphi''\|_\infty\right).$$

Thus the previous theorem allows to expand the measure $\mu^\varepsilon_{n,\mathbf{e}_i}$ around δ_{λ_i} as follows. With the notations and the hypothesis of the theorem,

$$\begin{aligned}\int \varphi(t)\mathrm{d}\mu^\varepsilon_{n,\mathbf{e}_i}(t) &= \varphi(\lambda_i) + \frac{\varepsilon}{\sqrt{n}}x_{ii}\varphi'(\lambda_i) + \varepsilon^2\Xi_s(\varphi) \\ &\quad + O_{L^2}\left(\varepsilon^2\|\varphi^{(5)}\|_\infty(\varepsilon + n^{-1/2} + \eta_n) + \frac{\varepsilon^2}{n}\|\varphi''\|_\infty\right).\end{aligned} \tag{5.6}$$

As far as the leading term of the expansion of $\mu^\varepsilon_{n,\mathbf{e}_i} - \delta_{\lambda_i}$ is concerned, we deduce the following transition: for η_n small enough, if $\varepsilon \ll n^{-1/2}$, then the leading term is $\frac{\varepsilon}{\sqrt{n}}x_{ii}\varphi'(\lambda_i)$, whereas if $n^{-1/2} \ll \varepsilon \ll 1$, the leading term is $\varepsilon^2\Xi_s(\varphi)$.

Remark 3. Strikingly, the image of a function φ by the operator $\Xi_{f(x)}$ is not changed if one adds an affine function to φ. This can be understood because the measure $\mu^{\varepsilon}_{n,\mathbf{e}_i} - \delta_{\lambda_i}$ is of null mass and with first moment of order $o(\varepsilon^2)$ since by (5.1),

$$\int_{\mathbb{R}} x \,\mathrm{d}\left(\mu^{\varepsilon}_{n,\mathbf{e}_i} - \delta_{\lambda_i}\right) = (D^{\varepsilon}_n)_{ii} - \lambda_i = \frac{\varepsilon}{\sqrt{n}} x_{ii} = o(\varepsilon^2).$$

Note that when both $\varphi(f(x))$ and $\varphi'(f(x))$ are null, the function $\Xi_{f(x)}(\varphi)$ boils down to the integral

$$\Xi_{f(x)}(\varphi) = \int_{\mathbb{R}} \tau(f(x),t)\rho(t)\frac{\varphi(t)}{(t-f(x))^2}\mathrm{d}t. \tag{5.7}$$

We will use this fact in Section 5.4 for test functions φ whose support does not contain $f(x)$.

Proposition 33. *Let us define, for any $s \in \mathbb{R}$, the function ζ_s defined on $\mathbb{R}$ by*

$$\zeta_s(y) := \int_1^{+\infty} \frac{r-1}{r^2}\tau(s, s+r(y-s))\rho(s+r(y-s))\mathrm{d}r. \tag{5.8}$$

Then for any $\mathcal{C}^2$ function φ and any $s \in \mathbb{R}$, the functional Ξ_s defined at (5.5) *rewrites*

$$\Xi_s(\varphi) \quad = \quad \int_{\mathbb{R}} \varphi''(y)\zeta_s(y)\mathrm{d}y. \tag{5.9}$$

Proof. Taylor formula yields

$$\varphi(t) - \varphi(s) - (t-s)\varphi'(s) = \int_s^t \varphi''(x)(t-x)\mathrm{d}x = (t-s)^2 \int_{u=0}^1 \varphi''(s+u(t-s))(1-u)\mathrm{d}u.$$

Hence,

$$\Xi_s(\varphi) = \int_{t\in\mathbb{R}} \int_{u=0}^1 \varphi''(s+u(t-s))(1-u)\mathrm{d}u \;\tau(s,t)\rho(t)\mathrm{d}t$$

We now perform the change of variable $(r,y) = \Psi_s(u,t)$ with

$$\Psi_s : (u,t) \in (0,1)\times\mathbb{R} \mapsto (r,y) = \left(\frac{1}{u}, u(t-s)+s\right) \in (1,\infty)\times\mathbb{R}$$

which gives the result $\square$

5.3 Proof (of Theorem 32)

The proof is divided into two parts. We shall first prove a convergence result for test functions φ of the type $\varphi_z := \frac{1}{z-x}$. This is the purpose of Subsection 5.3.1. It will be obtained by writing an expansion of the resolvent of D^{ε}_n.

Once we have proved that such a convergence holds for the resolvent of D^{ε}_n, we will be able to extend it to the class of compactly supported $\mathcal{C}^5$ functions on $\mathbb{R}$, by using the Helffer-Sjöstrand formula (see [HS89] or [BGK16]) which expresses a regular function φ on $\mathbb{R}$ as an integral against functions φ_z of the previous type. This is done in Subsection 5.3.2.

5.3.1 Stieltjes transform

Let us introduce the Banach space $\mathcal{C}_{\mathrm{b}}^2$ of bounded $\mathcal{C}^2$ functions on $\mathbb{R}$ with bounded first and second derivatives, endowed with the norm $\|\varphi\|_{\mathcal{C}_{\mathrm{b}}^2} := \|\varphi\|_\infty + \|\varphi'\|_\infty + \|\varphi''\|_\infty$.

On this space, let us define, for $x \in [0,1]$ and $i = \lfloor nx \rfloor$, the random continuous linear form

$$\Pi_n(\varphi) := \varepsilon^{-2}\left(\int \varphi(t)\mathrm{d}\mu^\varepsilon_{n,\mathbf{e}_i}(t) - \varphi(\lambda_i + \frac{\varepsilon}{\sqrt{n}}x_{ii})\right) - \Xi_{f(x)}(\varphi).$$

Lemma 34. *Uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\mathbb{E}[|\Pi_n(\varphi_z)|^2] \;=\; O\left(\left(\eta_n^2 + \frac{1}{n}\right)|\mathfrak{Im} z|^{-6}\right) + O(\varepsilon^2 |\mathfrak{Im} z|^{-8}). \tag{5.10}$$

Remark 4. This result implies that $\forall z \in \mathbb{C}\setminus\mathbb{R},\ \Pi_n(\varphi_z) \xrightarrow[n\to\infty]{P} 0.$

Let us prove the above lemma. We denote, for short, $x^n_{i,j}$ by x_{ij} and introduce the diagonal matrix

$$\widetilde{D}^\varepsilon_n := \operatorname{diag}\left(\left(\widetilde{\lambda}^\varepsilon_n(i) := \lambda_i + \frac{\varepsilon}{\sqrt{n}}x_{ii}\right)_{i=1,\ldots,n}\right) \tag{5.11}$$

which is the diagonal part of the matrix D^ε_n. Note that with this notation and by using identity (5.1), the quantity we are interested in can be written:

$$\Pi_n(\varphi_z) = \varepsilon^{-2}\left((z - D^\varepsilon_n)^{-1} - (z - \widetilde{D}^\varepsilon_n)^{-1}\right)_{ii} - \Xi_{f(x)}(z). \tag{5.12}$$

To deal with this quantity we introduce the null diagonal matrix

$$\widetilde{X}_n := \varepsilon^{-1}(D^\varepsilon_n - \widetilde{D}^\varepsilon_n) = X_n - n^{-1/2}\operatorname{diag}((x_{ii})_{i=1,\ldots,n})$$

obtained by vanishing the diagonal of the matrix X.

A perturbative expansion of the resolvent of $D^\varepsilon_n = \widetilde{D}^\varepsilon_n + \varepsilon\widetilde{X}_n$ yields

$$\begin{aligned}(z - D^\varepsilon_n)^{-1} - (z - \widetilde{D}^\varepsilon_n)^{-1} \;=\;& \varepsilon(z - \widetilde{D}^\varepsilon_n)^{-1}\widetilde{X}_n(z - \widetilde{D}^\varepsilon_n)^{-1} \\ &+\varepsilon^2(z - \widetilde{D}^\varepsilon_n)^{-1}\widetilde{X}_n(z - \widetilde{D}^\varepsilon_n)^{-1}\widetilde{X}_n(z - \widetilde{D}^\varepsilon_n)^{-1} \\ &+\varepsilon^3(z - \widetilde{D}^\varepsilon_n)^{-1}\widetilde{X}_n(z - \widetilde{D}^\varepsilon_n)^{-1}\widetilde{X}_n(z - \widetilde{D}^\varepsilon_n)^{-1}\widetilde{X}_n(z - D^\varepsilon_n)^{-1}.\end{aligned} \tag{5.13}$$

We now want to analyze the corresponding expansion of $\left((z - D^\varepsilon_n)^{-1} - (z - \widetilde{D}^\varepsilon_n)^{-1}\right)_{ii}$.

Claim 9. *For all $i \in [\![1,n]\!]$, $\left((z - \widetilde{D}^\varepsilon_n)^{-1}\widetilde{X}_n(z - \widetilde{D}^\varepsilon_n)^{-1}\right)_{ii} = 0$.*

Proof. This comes from the fact that the matrix $\widetilde{X}_n$ has a null diagonal. $\square$

Claim 10. *If, for all $i \in [\![1, n]\!]$, we denote*

$$B_n(z,i) := \left((z-\widetilde{D}_n^{\varepsilon})^{-1}\widetilde{X}_n(z-\widetilde{D}_n^{\varepsilon})^{-1}\widetilde{X}_n(z-\widetilde{D}_n^{\varepsilon})^{-1}\right)_{ii},$$

then, for all $x \in [0,1]$, $B_n(z, \lfloor nx \rfloor) - \Xi_{f(x)}(\varphi_z) = O_{L_2}\left(\left(\eta_n + \frac{1}{\sqrt{n}}\right)|\mathfrak{Im} z|^{-3}\right)$.

Proof. With the notations of (5.11), the term $B_n(z,i)$ writes

$$B_n(z,i) = \frac{1}{n}\sum_j \frac{|x_{ij}|^2}{\left(z-\widetilde{\lambda}_n^{\varepsilon}(i)\right)^2\left(z-\widetilde{\lambda}_n^{\varepsilon}(j)\right)},$$

and, for $x \in [0,1]$,

$$\begin{aligned}
\Xi_{f(x)}(\varphi_z) &= \int_{t\in\mathbb{R}} \frac{\tau(f(x),t)\rho(t)}{(t-f(x))^2}\left(\frac{1}{z-t} - \frac{1}{z-f(x)} + \frac{t-f(x)}{(z-f(x))^2}\right)\mathrm{d}t \\
&= \int_{t\in\mathbb{R}} \frac{\tau(f(x),t)}{(z-f(x))^2(z-t)}\rho(t)\mathrm{d}t \qquad (5.14)\\
&= \int_{y\in[0,1]} \frac{\sigma(x,y)^2}{(z-f(x))^2(z-f(y))}\mathrm{d}y. \qquad (5.15)
\end{aligned}$$

The difference of these quantities writes,

$$\begin{aligned}
&B_n(z,\lfloor nx\rfloor) - \Xi_{f(x)}(\varphi_z)\\
&= \frac{1}{n}\sum_{j=1}^n \frac{|x_{\lfloor nx\rfloor,j}|^2 - \sigma_n^2\left(\frac{\lfloor nx\rfloor}{n},\frac{j}{n}\right)}{\left(z-\widetilde{\lambda}_n^{\varepsilon}(\lfloor nx\rfloor)\right)^2\left(z-\widetilde{\lambda}_n^{\varepsilon}(j)\right)}\\
&\quad + \frac{1}{n}\sum_{j=1}^n \frac{\sigma_n^2\left(\frac{\lfloor nx\rfloor}{n},\frac{j}{n}\right)}{\left(z-\widetilde{\lambda}_n^{\varepsilon}(\lfloor nx\rfloor)\right)^2\left(z-\widetilde{\lambda}_n^{\varepsilon}(j)\right)} - \frac{\sigma_n^2\left(\frac{\lfloor nx\rfloor}{n},\frac{j}{n}\right)}{\left(z-\lambda_{\lfloor nx\rfloor}\right)^2(z-\lambda_j)}\\
&\quad + \int_{y\in[0,1]} \frac{\sigma_n^2\left(\frac{\lfloor nx\rfloor}{n},\frac{\lfloor ny\rfloor}{n}\right)}{(z-\lambda_{\lfloor nx\rfloor}^2)(z-\lambda_{\lfloor ny\rfloor})}\mathrm{d}y - \int_{y\in[0,1]} \frac{\sigma(x,y)^2}{(z-f(x))^2(z-f(y))}\mathrm{d}y.
\end{aligned}$$

The L_2 norm of the first line of the right hand side of the previous equality writes

$$\left\|\frac{1}{n}\sum_{j=1}^n \frac{|x_{ij}|^2-\sigma_n(i/n,j/n)^2}{(z-\lambda_i)^2(z-\lambda_j)}\right\|_{L^2} = \frac{1}{n}\left(\sum_{j=1}^n \frac{\mathbb{E}[(|x_{ij}|^2-\sigma_n(i/n,j/n)^2)^2]}{|z-\lambda_i|^4|z-\lambda_j|^2}\right)^{\frac{1}{2}} = O\left(\frac{1}{\sqrt{n}\,|\mathfrak{Im} z|^3}\right),$$

the L_1 norm of the second line is bounded, for $C > 0$, by

$$\frac{C}{n}\sum_{j=1}^{n}\mathbb{E}\left[\frac{\varepsilon}{\sqrt{n}}\frac{|x_{\lfloor nx\rfloor,\lfloor nx\rfloor}|+|x_{jj}|}{|\mathfrak{Im}(z)|^3}\right] = O\left(\frac{\varepsilon}{\sqrt{n}\ |\mathfrak{Im}(z)|^3}\right)$$

and, finally, from assumption (b), the third line is $O(\eta_n|\mathfrak{Im}z|^{-3})$. □

Claim 11. *For all $i \in [\![1, n]\!]$,*

$$\left((z-\widetilde{D}_n^{\varepsilon})^{-1}\widetilde{X}(z-\widetilde{D}_n^{\varepsilon})^{-1}\widetilde{X}(z-\widetilde{D}_n^{\varepsilon})^{-1}\widetilde{X}(z-D_n^{\varepsilon})^{-1}\right)_{ii} = O_{L^2}(|\mathfrak{Im}z|^{-4})$$

Proof. By taking into account that the L^6 norm of the entries of $\sqrt{n}X$ is finite, and that the norms of the coefficients of $(z-\widetilde{D}_n^{\varepsilon})^{-1}$ and of $(z-D_n^{\varepsilon})^{-1}$ are smaller than $|\mathfrak{Im}(z)|^{-1}$, we deduce that

$$\begin{aligned}
&\mathbb{E}\left[\left|\left((z-\widetilde{D}_n^{\varepsilon})^{-1}\widetilde{X}_n(z-\widetilde{D}_n^{\varepsilon})^{-1}\widetilde{X}_n(z-\widetilde{D}_n^{\varepsilon})^{-1}\widetilde{X}_n(z-D_n^{\varepsilon})^{-1}\right)_{ii}\right|^2\right]^{\frac{1}{2}}\\
&=\mathbb{E}\left[\left|\sum_{j,k,l=1}^{n}((z-\widetilde{D}_n^{\varepsilon})^{-1})_{i,i}(\widetilde{X}_n)_{i,j}((z-\widetilde{D}_n^{\varepsilon})^{-1})_{j,j}(\widetilde{X}_n)_{j,k}((z-\widetilde{D}_n^{\varepsilon})^{-1})_{k,k}(\widetilde{X}_n)_{k,l}((z-D_n^{\varepsilon})^{-1})_{l,i}\right|^2\right]^{\frac{1}{2}}\\
&\le\frac{1}{\mathfrak{Im}(z)^4}\mathbb{E}\left[\left|\sum_{j,k,l=1}^{n}(\widetilde{X}_n)_{i,j}(\widetilde{X}_n)_{j,k}(\widetilde{X}_n)_{k,l}\right|^2\right]^{\frac{1}{2}}\\
&\le\frac{1}{\mathfrak{Im}(z)^4\ n^{3/2}}\mathbb{E}\left[\left|\sum_{j,k,l=1}^{n}x_{i,j}\ x_{j,k}\ x_{k,l}\right|^2\right]^{\frac{1}{2}}\\
&\le\frac{1}{\mathfrak{Im}(z)^4\ n^{3/2}}\mathbb{E}\left[\sum_{j,k,l,m,p,q=1}^{n}x_{i,j}\ x_{j,k}\ x_{k,l}\ \overline{x_{i,m}}\ \overline{x_{m,p}}\ \overline{x_{p,q}}\right]^{\frac{1}{2}}.
\end{aligned}$$

Since the entries $(x_{i,j})$ are independent and centered, the set of "edges" $\{(i,m),(m,p),(p,q)\}$ must be equal to the set $\{(i,j),(j,k),(k,l)\}$ in order to get a non zero term. Therefore, the complexity of the previous sum is $O(n^3)$. Note that other non zero terms involving third or fourth moments are much less numerous.

Therefore,

$$\mathbb{E}\left[\left((z-\widetilde{D}_n^{\varepsilon})^{-1}\widetilde{X}_n(z-\widetilde{D}_n^{\varepsilon})^{-1}\widetilde{X}_n(z-\widetilde{D}_n^{\varepsilon})^{-1}\widetilde{X}_n(z-D_n^{\varepsilon})^{-1}\right)_{ii}^2\right]^{\frac{1}{2}} \le \frac{1}{\mathfrak{Im}(z)^4}$$

□

Gathering Formulas (5.12), (5.13) and Claims 1, 2 and 3, we prove Lemma 34.

5.3.2 From Stieltjes transform to $\mathcal{C}^5$ functions

Now, let φ be a $\mathcal{C}^5$ function on $\mathbb{R}$ with bounded fifth derivative and let us introduce the almost analytic extension of degree 5 of φ defined by

$$\forall z = x + \mathrm{i}y \in \mathbb{C}, \qquad \widetilde{\varphi}_4(z) \ := \ \sum_{k=0}^{4} \frac{1}{k!}(\mathrm{i}y)^k \varphi^{(k)}(x)\,.$$

An elementary computation gives, by successive cancellations, that

$$\bar{\partial}\widetilde{\varphi}_4(z) = \frac{1}{2}\left(\partial_x + \mathrm{i}\partial_y\right)\widetilde{\varphi}_4(x+\mathrm{i}y) = \frac{1}{2\times 4!}(\mathrm{i}y)^4\varphi^{(5)}(x). \tag{5.16}$$

Furthermore, by Helffer-Sjöstrand formula [BGEM17, Propo. 9], for $\chi \in \mathcal{C}_c^\infty(\mathbb{C};[0,1])$ a smooth cutoff function with value one on the support of φ,

$$\varphi(\cdot) \ = \ -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{\partial}(\widetilde{\varphi}_4(z)\chi(z))}{y^4} y^4 \varphi_z(\cdot)\,\mathrm{d}^2 z$$

where $\mathrm{d}^2 z$ denotes the Lebesgue measure on $\mathbb{C}$.

Note that by (5.16), $z \mapsto \mathbb{1}_{y\neq 0}\frac{\bar{\partial}(\widetilde{\varphi}_4(z)\chi(z))}{y^4}$ is a continuous compactly supported function and that $z \in \mathbb{C} \mapsto \mathbb{1}_{y\neq 0} y^4 \varphi_z \in \mathcal{C}^1_{\mathrm{b}}$ is continuous, hence,

$$\Pi_n(\varphi) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{\partial}(\widetilde{\varphi}_4(z)\chi(z))}{y^4}\ y^4 \Pi_n(\varphi_z)\,\mathrm{d}^2 z.$$

Therefore, using the Cauchy-Schwarz inequality and the fact that χ has compact support at the second step, for a certain constant C, we have

$$\begin{aligned}
\mathbb{E}\left(|\Pi_n(\varphi)|^2\right) &= \mathbb{E}\left(\left|\frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{\partial}(\widetilde{\varphi}_4(z)\chi(z))}{y^4}\ y^4\Pi_n(\varphi_z)\,\mathrm{d}^2 z\right|^2\right) \\
&\leq C\mathbb{E}\left(\int_{\mathbb{C}} \left|\frac{\bar{\partial}(\widetilde{\varphi}_4(z)\chi(z))}{y^4}\ y^4\Pi_n(\varphi_z)\right|^2 \mathrm{d}^2 z\right) \\
&= C\int_{\mathbb{C}} \left|\frac{\bar{\partial}(\widetilde{\varphi}_4(z)\chi(z))}{y^4}\right|^2\ y^8\ \mathbb{E}\left(|\Pi_n(\varphi_z)|^2\right)\ \mathrm{d}^2 z\,.
\end{aligned}$$

By (5.16), the function $\left|\frac{\bar{\partial}(\widetilde{\varphi}_5(z)\chi(z))}{y^5}\right|^2$ is continuous and compactly supported and bounded by $C\|\varphi^{(5)}\|_\infty^2$ for some constant C. Besides, by Lemma 34, uniformly in z,

$$y^8\ \mathbb{E}\left(|\Pi_n(\varphi_z)|^2\right) = O\left((1+y^2)\ \left(\eta_n^2 + \frac{1}{n} + \varepsilon^2\right)\right).$$

We deduce that

$$\mathbb{E}\left(|\Pi_n(\varphi)|^2\right) \leq C\int_{\mathbb{C}} \left|\frac{\bar{\partial}(\widetilde{\varphi}_4(z)\chi(z))}{y^4}\right|^2\ y^8\ \mathbb{E}\left(|\Pi_n(\varphi_z)|^2\right)\ \mathrm{d}^2 z \ = \ O\left(\|\varphi^{(5)}\|_\infty^2\ \left(\eta_n^2 + \frac{1}{n} + \varepsilon^2\right)\right),$$

which closes the proof of Theorem 32.

5.4 Consequence for the eigenvectors

In this section we want to present some concrete applications of our main result and some numerical simulations that comfort well our developments. Therefore, some of the following considerations will be at a heuristic level.

Let $x_0 \in [0,1]$ and $t \in f([0,1])$, with $t \neq f(x_0)$. Let us assume that the function f is invertible (which significantly lessens the generality of the model).

If we denote by μ_n^ε the empirical spectral measure of D_n^ε, namely

$$\mu_n^\varepsilon := \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j^\varepsilon},$$

the definition of $\mu_{n,\mathbf{e}_i}^\varepsilon$ implies, for $t \neq f(x_0)$,

$$\mu_{n,\mathbf{e}_i}^\varepsilon(\mathrm{d}t) \approx n|\langle \mathbf{u}^\varepsilon_{\lfloor nf^{-1}(t)\rfloor}, \mathbf{e}_{\lfloor nx_0\rfloor}\rangle|^2 \mu_n^\varepsilon(\mathrm{d}t).$$

In addition, Formulas (5.7) and (5.6) give, for $t \neq f(x_0)$, the following expansion

$$\mu_{n,\mathbf{e}_i}^\varepsilon \approx \varepsilon^2\, \frac{\tau(f(x_0),t)\rho(t)}{(t-f(x_0))^2}\mathrm{d}t.$$

Keeping in mind the convergence of $\mu_n^\varepsilon(t)$ towards the measure $\rho(t)\mathrm{d}t$, we conclude that we have the asymptotic equality, in the sense of distributions, between functions of t,

$$\varepsilon^{-2}n|\langle \mathbf{u}^\varepsilon_{\lfloor nf^{-1}(t)\rfloor}, \mathbf{e}_{\lfloor nx_0\rfloor}\rangle|^2 \;\approx\; \frac{\tau(f(x_0),t)}{(t-f(x_0))^2}. \tag{5.17}$$

We present now two simulations (displayed in Figures 5.1 and 5.2) which show a good matching with this theoretical prediction. First we consider the case where the deterministic matrix D_n is perturbed by a Gaussian Wigner matrix, X_n. More precisely, we take for D_n the diagonal matrix with $\frac{i}{n}$ as i^{th} entry, so that $f(x) = x$ and the density ρ is equal $x \mapsto \mathbb{1}_{[0,1]}(x)$. The entries of the perturbating matrix X_n are all Gaussian and independent with variance one. Then, we consider the case where the same matrix D_n is perturbed by a band matrix. In other words, we consider now that $\sigma(x,y) = \mathbb{1}_{|x-y|\leq\ell}$, where $\ell \in [0,1]$ is the relative width of the band. Note that in this second example, even though there is absolutely no deterministic reason why $\langle \mathbf{u}^\varepsilon_{\lfloor ny\rfloor}, \mathbf{e}_{\lfloor nx\rfloor}\rangle$ would vanish when $|y-x| > \ell$, we see that at first order, it is actually almost zero (Figure 5.2). This is related to the question of the localization of the eigenvectors of band random matrices (see e.g. [Cha10, EKYY13, EK11b, EK11a, EYY12b, EYY12a, FM91, Sch09]).

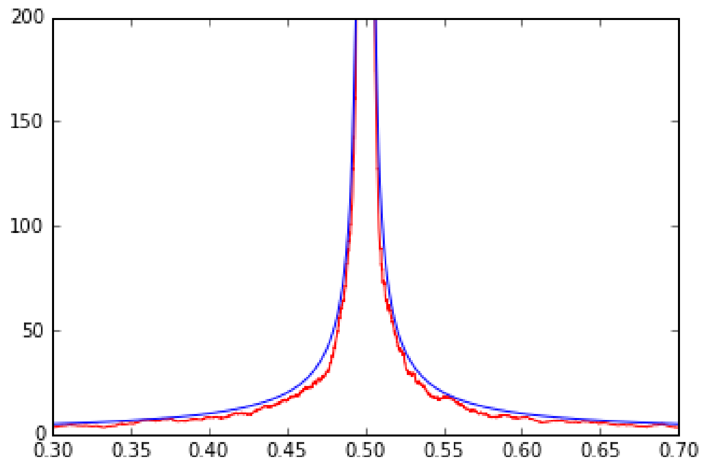

Figure 5.1: **Uniform measure perturbation by a Wigner matrix.** The red curve represents (part of) the function $t \in [0,1] \longmapsto \varepsilon^{-1}\sqrt{n}|\langle \mathbf{u}^{\varepsilon}_{\lfloor nf^{-1}(t)\rfloor}, \mathbf{e}_{\lfloor nx_0\rfloor}\rangle|$, which results from the numeric computation of the eigenvectors of D_n^{ε} and the blue curve represents our theoretical prediction $t \longmapsto |t-x_0|^{-1}$. As (5.17) is an equality between distributions, we have smoothed both curves thanks to a simple moving average with window length $\sqrt{n}$. Here $n = 10^4$, $\varepsilon = n^{-0.7}$ and $x_0 = \frac{1}{2}$.

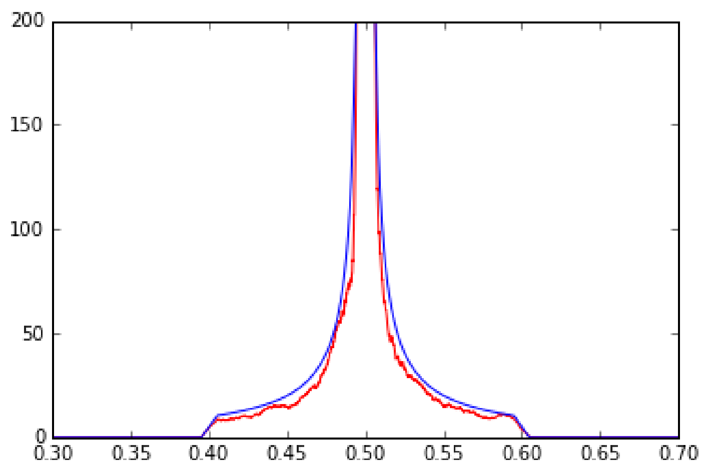

Figure 5.2: **Uniform measure perturbation by a Gaussian band matrix.** The red curve represents (part of) the function $t \in [0,1] \longmapsto \varepsilon^{-1}\sqrt{n}|\langle \mathbf{u}^{\varepsilon}_{\lfloor nf^{-1}(t)\rfloor}, \mathbf{e}_{\lfloor nx_0\rfloor}\rangle|$, which results from the numeric computation of the eigenvectors of D_n^{ε} and the blue curve represents our theoretical prediction $t \longmapsto \mathbb{1}_{|t-f(x_0)|\leq \ell}|t-x_0|^{-1}$. As (5.17) is an equality between distributions, we have smoothed both curves thanks to a simple moving average with window length $\sqrt{n}$. Here $n = 10^4$, $\ell = 0.1$, $\varepsilon = n^{-0.7}$ and $x_0 = \frac{1}{2}$.

Remark 5. It could be interesting to compare (5.17) with the recent work [Ben17] of Benigni. Corollary 1.6 of [Ben17] precisely implies the same estimate as the one we give in (5.17). More precisely, it says that if the distribution of the eigenvalues of D_n is regular enough (in our case, these conditions can be expressed as a bound on our error term η_n

of (5.2)), if X_n is a Wigner matrix (in our case, it amounts to suppose that $\tau \equiv 1$), if ε satisfies, roughly, $n^{-1} \ll \varepsilon \ll 1$ and if $\gamma_{j,\varepsilon}$ denotes the quantile of the density ρ^ε of the free convolution of the spectral distribution of D_n with a semicircle law with variance ε^2, i.e.

$$\int_{-\infty}^{\gamma_{j,\varepsilon}} \rho^\varepsilon(x)\mathrm{d}x \;=\; \frac{j}{n},$$

then the approximation

$$\varepsilon^{-2} n |\langle \mathbf{u}_{j_0}^\varepsilon, \mathbf{e}_i \rangle|^2 \;\approx\; \frac{1}{(\lambda_i - \gamma_{j_0,\varepsilon})^2 + O(\varepsilon^4)},$$

holds when empirical means on i in the neighborhood of a fixed j_0 have been taken. We recover the first order approximation from (5.17). Note however that here, empirical means are taken on i, whereas in our result, with these notations, they are taken on j_0.

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Abstract

The present thesis is devoted to the study of the effect of a perturbation on the spectrum of a Hermitian matrix by a random matrix with small operator norm and whose entries in the eigenvector basis of the first one were independent, centered and with a variance profile. This is carried out through perturbative expansions of various types of spectral laws of the considered perturbed large matrices. First, we demonstrate different perturbative expansions of the empirical spectral measure in the cases of the perturbative regime and the semi-perturbative regime and highlight well known heuristic patterns in Physics, as the transition between semi-perturbative and perturbative regimes. Secondly, we provide a thorough study of the semi-perturbative regime and prove the new fact that this regime could be decomposed into infinitely many sub-regimes. Finally, we prove, through a perturbative expansion of spectral measures associated to the state defined by a given vector, a perturbative expansion of the coordinates of the eigenvectors of the perturbed matrices.

Résumé

La présente thèse est consacrée à l'étude de l'effet d'une perturbation sur le spectre d'une matrice hermitienne perturbée par une matrice aléatoire de petite norme opérateur et dont les entrées dans la base propre de la première matrice sont indépendantes, centrées et possèdent un profil de variance. Ceci est réalisé au travers de développements perturbatifs de divers types des lois spectrales des grandes matrices perturbées considérées. Dans un premier temps, nous démontrons différents développements perturbatifs de la mesure spectrale empirique dans les cas du régime perturbatif et du régime semi-perturbatif et mettons en évidence des modèles heuristiques bien connus en physique, comme la transition entre les régimes semi-perturbatifs et perturbatifs. Dans un deuxième temps, nous proposons une étude approfondie du régime semi-perturbatif et prouvons le fait nouveau que ce régime peut être décomposé en un nombre infini de sous-régimes. Enfin, nous démontrons, au travers d'un développement perturbatif des mesures spectrales associées à un vecteur donné, un développement perturbatif des coordonnées des vecteurs propres des matrices perturbées que nous considérons.