Reactive imcompressible flow with interfaces : macroscopic models and applications to self-healing composite materials

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TITRE

Écoulements incompressibles réactifs avec interfaces : modèles macroscopiques et applications aux matériaux composites auto-cicatrisants

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Résumé

Dans ce manuscrit, nous parlons des matériaux composites à matrice céramique (CMCs) qui sont envisagés pour intégrer les chambres de combustion de futurs moteurs aéronautiques civils. Pour faire face à des conditions extrêmes, ces matériaux possèdent la particularité de s'auto-protéger vis-à-vis de l'oxydation par la formation d'un oxyde passivant qui limite la diffusion des espèces oxydantes au sein des fissures matricielles. Nous modélisons l'écoulement d'un oxyde dans une fissure par l'équation de Navier-Stokes, puis la mettons sous forme non dimensionnelle, et les derivations de deux types de modèles sont intéressantes : les modèles de Saint-Venant et les modèles de lubrification. Ensuite nous nous engageons à chercher l'existence de solution faible de l'approximation de lubrification d'ordre 4 obtenue précédente dans le cas uni-dimensionnel. Enfin nous précisons la limite entre les équations de Saint-Venant et l'équation de lubrification.

Mots clés : Equation de Navier-Stokes, Modèles asymptotiques, Modèles de Saint-Venant, Modèles de lubrification.

Abstract

In this work, we are interested in the ceramic matrix composite materials(CMCs) which will be used to integrate the combustion chambers of future civil aeronautical engines. When facing extreme conditions, these materials possess the peculiarity to auto-protect itself towards the oxidation by the formation of an oxide passivate which limits the distribution of the oxidizing species within the matrix cracks. We model the flow of an oxide in a crack by the Navier-Stokes equation, then by asymptotic analysis we establish two types of asymptotic models : models of Saint-Venant (shallow water model) and lubrication models. Next we are interested in looking for the existence of weak solutions to the one-dimensional approximated lubrication equation of order 4 obtained before. Finally we talk about the limit between the Saint-Venant equations and the lubrication equation.

Keywords : Navier-Stokes equation, shallow water model, lubrication model

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Chapitre 1

Introduction

1.1 Introduction générale de la thèse

Les composites à matrice céramique (CMC) sont envisagés pour intégrer les chambres de combustion de futurs moteurs aéronautiques civils. Dans ces moteurs, ils atteignent leurs conditions limites d'utilisation à cause de la conjonction de fortes sollicitations thermiques et mécaniques et d'un environnement très oxydant. Pour faire face à ces conditions, des CMC à matrice auto-cicatrisante ont été développés. Ils présentent des durées de vie extrêmement longue même sous sollicitations mécaniques importantes dans des environnements oxydants. Ces matériaux possèdent la particularité de s'auto-protéger vis-à-vis de l'oxydation par la formation d'un oxyde passivant qui limite la diffusion des espèces oxydantes au sein des fissures matricielles.

L'objectif principal de cette thèse est d'introduire et d'étudier de nouveaux modèles asymptotiques qui décrivent la propagation d'un oxyde à travers une fissure transverse. Ce travail fait suite à la thèse de G. Perrot dans laquelle était introduit un nouveau modèle physico-chimique décrivant le processus d'auto-cicatrisation de ces matériaux composites. Dans un premier temps, nous allons enrichir la description de ce phénomène de bouchage en présentant une hiérarchie de modèles asymptotiques décrivant la propagation d'un fluide dans une fissure et obtenus à partir des équations de Navier-Stokes. Notons que cette thèse s'inscrit dans un projet ambitieux regroupant l'équipe INRIA CARDAMOM, le laboratoire LCTS de l'Université de Bordeaux et le laboratoire LAMA de l'Université de Chambéry visant la création d'une véritable plate-forme de calcul capable de certifier ces matériaux du futur. Le but ultime serait de pouvoir coupler des modèles fluides décrivant la propagation de l'oxyde passivant à travers les fissures avec un code de calcul mécanique afin de prévoir la durée de vie de ces matériaux. La simulation complète d'un vrai matériau 3D serait bien sûr très coûteuse en terme de temps de calcul. Dans cette optique, l'introduction de modèles asymptotiques s'avère indispensable. Dans ce manuscrit, nous introduisons deux familles d'équations. La première famille est constituée par un système d'équations de type Saint-Venant alors que la seconde fait appel à la théorie de la lubrification. Différents types de conditions aux bords permettront de complexifier ces modèles et de développer la hiérarchie évoquée ci-dessus.

Dans un second temps, nous nous intéresserons aux modèles de type lubrification. En effet, ces modèles semblent les plus appropriés pour décrire la propagation d'un oxyde, au vu des faibles nombre de Reynolds mis en jeu dans ce contexte. Nous établirons un résultat d'existence de solutions faibles. L'originalité et la difficulté principal de ces équations réside dans le fait qu'elle contienne des opérateurs différentiels d'ordre 4 en espace, rendant par exemple l'utilisation d'outils comme le principe du maximum impossible. Au passage, nous en profiterons pour montrer un résultat de convergence asymptotique des équations de Saint-Venant vers les équations de la lubrification, faisant le lien ainsi entre les deux familles de modèles.

1.2 Modélisation physique et mathématique

Nous commençons ce chapitre par une brève description des matériaux composites auto-cicatrisants (CMCs). Ces matériaux composites à matrice céramique sont composés d'un assemblage (sous forme de tissage) de fibres carbonés plongés dans une matrice céramique. La Figure 1.1 représente une coupe transversale d'un tel matériau : on peut voir que chaque fibre est entourée d'une matrice que l'on va décrire ci-après.

Les fibres sont le plus souvent constituées de carbone ou de carbure de silicium (SiC). Le SiC peut être considéré comme chimiquement inerte pour des températures inférieures à 1000 °C. Par contre en présence d'oxygène, les fibres s'oxydent provoquant jusqu'à leur rupture, affaiblissant ainsi le matériau. L'objectif d'un matériau auto-cicatrisant est de protéger les fibres en ralentissant la propagation de l'oxygène dans la fissure par le processus suivant. Chaque fibre est entourée d'une matrice composée de couches successives de carbure de silicium et de carbure de bore B_4C, en alternance. Le bore est un élément

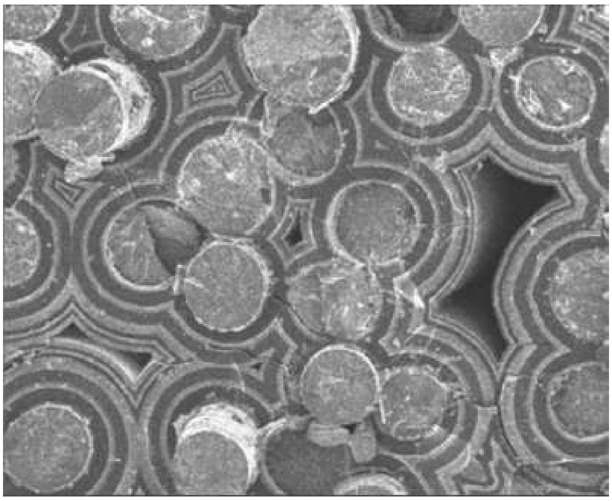

FIGURE 1.1 – Matériaux composites, vue des différentes couches de matrice autour des fibres, par V. Dréan et al. [31].

essentiel dans la protection des fibres car en présence d'oxygène (apporté par exemple lorsque le matériau se fissure), une réaction chimique s'initie et produit un oxyde passivant d'oxyde de Bore qui va diffuser dans la fissure jusqu'à la boucher. La réaction s'initie dès que la température atteint environ 450 °C. Le coefficient de diffusion de l'oxygène dans l'oxyde est environ 10 fois moins grand que le coefficient de diffusion dans l'air : le rôle de l'oxyde est ainsi de ralentir la diffusion de l'oxygène dans la fissure afin de préserver les fibres de l'oxydation et d'augmenter la durée de vie du matériau. Voir par exemple : une interphase d'épaisseur faible autour de la fibre se forme afin de mieux la protéger contre les dégradations mécaniques dues aux fissures [56], et cette configuration offre une meilleure protection au matériau [51].

1.2.1 Le modèle mathématique.

Pour décrire l'évolution d'un fluide visco-élastique, incompressible, irrotationnel à surface libre, il existe de nombreux modèles asymptotiques : systèmes de type Boussinesq (équations de Nwogu, équations de Madsen-Sorensen, équations de Peregrine,..), équations de Serre-Green-Naghdi, ... Ces modèles sont en général obtenus à partir de développements asymptotiques effectués sur les équations d'Euler ou de Navier-Stokes à surface libre. On commence par effectuer une analyse dimensionnelle du système en fonction du phénomène physique que l'on souhaite étudier. Cette analyse permet en général d'exhiber un ou plusieurs paramètres sans dimension, comme par exemple le paramètre de nonlinéarité ε et la paramètre de dispersion σ^2. Ces paramètres font intervenir les grandeurs caractéristiques de l'écoulement et permettent, sous certaines hypothèses de taille, de dériver des modèles asymptotiques contenant autant de phénomènes physiques que l'asymptotique est poussée loin. Dans le cadre de cette thèse, nous traiterons essentiellement le cas d'une fissure transverse idéalisée.

Nous supposons ici que la propagation de l'oxyde à l'intérieur d'une fissure transverse est gouvernée par les équations de Navier-Stokes à surface libre. L'originilaté provient du fait que le fond, constitué d'une succession de matrices actives ou passives, est variable au cours du temps et décrit le processus chimique de création de l'oxyde. Il y a donc gain ou perte de masse d'oxyde au cours du temps. Nous nous intéressons donc aux équations de Navier-Stokes bi-dimensionnelles à surface libre suivantes

$$\begin{cases} \mathrm{div}\mathbf{U} = 0, \\ \partial_t \mathbf{U} + \mathrm{div}(\mathbf{U} \otimes \mathbf{U}) = \mathrm{div}\sigma(\mathbf{U}) - \nabla p + \mathbf{g}, \end{cases} \tag{1.1}$$

où

- $\mathbf{U} = (u(t,x,z), w(t,x,z))$ représente la vitesse du fluide ;
- $p(t,x)$ est la pression du fluide ;
- $\xi(t,x)$ est la hauteur de la surface libre, $b(t,x)$ la bathymétrie du canal, $h = \xi - b$ la hauteur totale du fluide ;
- $\mathbf{g} = (0, -g)$ est la force de la gravité ;
- $\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{xz} \end{pmatrix}$ représente le tenseur de déformation : $\sigma(\mathbf{U}) = 2\mu D(\mathbf{U})$, où μ est la viscosité dynamique et $D(\mathbf{U})$ est donné par

$$D(\mathbf{U}) = \frac{1}{2}(\nabla \mathbf{U} + \nabla \mathbf{U}^T) = \begin{pmatrix} \partial_x u & (\partial_z u + \partial_x w)/2 \\ (\partial_z u + \partial_x w)/2 & \partial_z w \end{pmatrix}.$$

Toutes ces notations sont résumées sur la figure 1.3. Remarquons que le domaine $\Omega =$

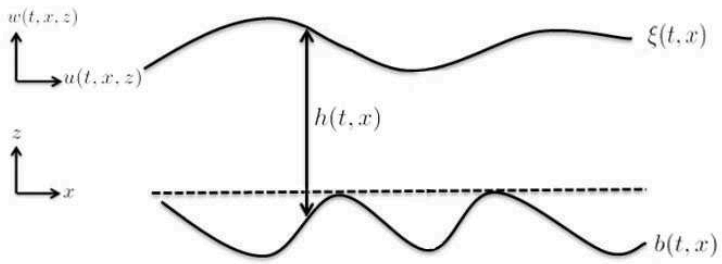

FIGURE 1.2 – Présentation d'une fissure transverse.

$\Omega(t,x)$ sur lequel sont posées les équations (1.1) est variable

$$\Omega(t,x) = \{(t,x,z); t \geq 0,\ x \in [0,L],\ b(t,x) \leq z \leq \xi(t,x)\},$$

où $b(t,x)$ désigne la bathymétrie (voir Figure 1.2). Pour décrire proprement le domaine $\Omega(t,x)$, nous allons utiliser une idée développée dans [43] en introduisant la fonction indicatrice suivante

$$\varphi(t,x,z) = \begin{cases} 1, & b(t,x) \leq z \leq \xi(t,x), \\ 0, & \text{sinon.} \end{cases}$$

Pour tenir compte de la réaction chimique entre les zones réactives du fond et l'oxygène qui se propage dans la fissure, nous introduisons aussi **la fonction indicatrice** ϕ_r caractéristique du fond, qui vaut 1 sur la zone réactive et qui est nulle partout ailleurs, ainsi que les vitesses $\mathbf{U}_i = (u_i, w_i)$ d'injection de l'oxyde passivant et $\mathbf{U}_r = (u_r, w_r)$ de recul du fond. La fonction φ est advectée par le flow et vérifie l'équation classique de convection

$$\partial_t \varphi + \Big(\mathbf{U} - \phi_r(\mathbf{U}_i - \mathbf{U}_r)\Big) \cdot \nabla \varphi = 0,$$

ou encore, en utilisant la condition d'incompressibilité

$$\partial_t \varphi + \nabla \cdot (\mathbf{U}\varphi) = \phi_r(\mathbf{U}_i - \mathbf{U}_r) \cdot \nabla \varphi. \tag{1.2}$$

Remarquons que les équations de Navier-Stokes (1.1) peuvent s'écrire sous la forme éclatée :

$$\begin{cases} \partial_x u + \partial_z w = 0, \\ \partial_t u + \partial_x(u^2) + \partial_z(uw) = \partial_x \sigma_{xx} + \partial_z \sigma_{xz} - \partial_x p, \\ \partial_t w + \partial_x(uw) + \partial_z(w^2) = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - \partial_z p - g. \end{cases} \tag{1.3}$$

Notons enfin que, puisque l'on va travailler sur un domaine fermé, il faut ajouter des conditions aux bords à ce système d'équations : d'une part une équation sur la surface libre (équation cinématique) et d'autre part une équation pour prescrire la bathymétrie $b(t, x)$ qui encodera la chimie du problème, ce qui constituera la principale originalité de ce travail.

1.2.2 Description des conditions aux bords.

Comme décrit dans la Figure 2.4, le domaine de calcul est délimité en haut par une surface libre et en bas par une bathymétrie. Il faut donc ajouter des conditions sur chacune de ces parties pour fermer le système (1.3).

Conditions à la surface libre : Nous utilisons la condition cinématique classique qui exprime le fait que la vitesse d'une particule à la surface est égale à la vitesse du fluide

$$\partial_t \xi + u_{|z=\xi} \partial_x \xi - w_{|z=\xi} = 0. \tag{1.4}$$

De plus, nous utilisons que l'écoulement en surface est soumis aux forces de capillarité, ce qui fournit la relation suivante sur le tenseur de contraintes total au niveau de la surface libre

$$(\sigma - p\mathrm{Id}) \cdot \mathbf{n}_\xi = -\gamma \kappa \mathbf{n}_\xi, \tag{1.5}$$

où $\gamma > 0$ est le coefficient de capillarité de la surface et $\kappa(x)$ la courbure moyenne de la surface en $z = \xi$, définie par

$$\kappa(x) = \frac{\partial_x^2 \xi}{(1 + (\partial_x \xi)^2)^{3/2}},$$

et $\mathbf{n}_\xi$ est le vecteur normal à la surface libre

$$\mathbf{n}_\xi(x) = \frac{1}{\sqrt{1 + (\partial_x \xi)^2}} \begin{pmatrix} -\partial_x \xi \\ 1 \end{pmatrix}.$$

Sur le fond : Nous considérons une condition de pénétration avec une zone chimiquement active caractérisée par une fonction indicatrice, notée ϕ_r, qui vaut 1 sur la zone réactive et qui est nulle partout ailleurs. On rappelle que $\mathbf{U}_i = (u_i, w_i)$ représente la vitesse d'injection du fluide à travers l'interface (formation de l'oxyde) et que $\mathbf{U}_r = (u_r, w_r)$ est la vitesse de recul de la surface dans les zones réactives, dûe essentiellement à la consommation du B_4C, (voir Figure 1.3). Les vitesses d'injection et de recul de l'oxyde créé par

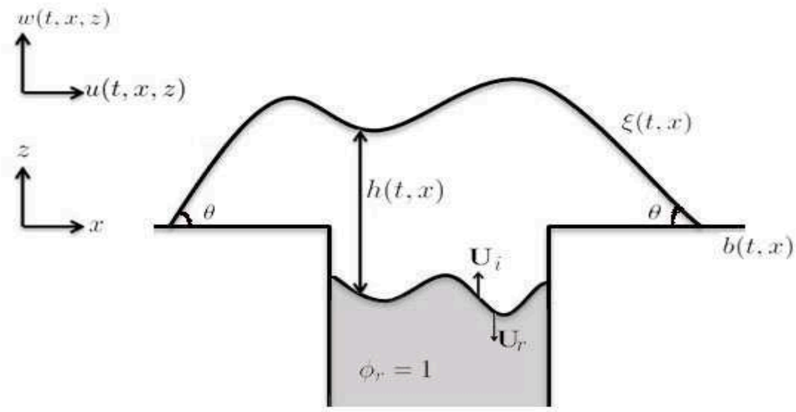

FIGURE 1.3

réaction chimique $\mathbf{U}_i$ et $\mathbf{U}_r$ à travers les interfaces sont supposées connues. Notons que la quantité importante est ici la vitesse relative définie par $\mathbf{U}_i - \mathbf{U}_r$.

Les conditions sur le fond sont les conditions de friction du fluide sur la paroi, de mouillage du fluide conditionnant son avancée/recul, et une équation caractérisant le recul de la surface réactive, représentant l'oxydation de la surface :

$$\begin{cases} \partial_t b = [\mathbf{U} - \phi_r(\mathbf{U}_i - \mathbf{U}_r)] \cdot \mathbf{n}_b, \\ (\sigma \cdot \mathbf{n}_b) \cdot \mathbf{t}_b = \alpha(\mathbf{U} - \phi_r(\mathbf{U}_i - \mathbf{U}_r)) \cdot \mathbf{t}_b + \beta \nabla\varphi \cdot \mathbf{t}_b(\cos\theta_{eq} - \cos\theta), \end{cases} \tag{1.6}$$

où α est le coefficient de friction entre les phases liquide et solide et β est le coefficient de capillarité. Nous renvoyons au Chapitre 2.1 pour une description complète et la dérivation de ces équations relatives à l'evolution du fond. De plus, il y a deux points triples sur les trois phases (gas, liquide et solide) qui, pour raison de symétrie, serons supposés identiques et dont l'angle de contact statique est défini par

$$\theta_{eq} = \frac{\gamma_{SG} - \gamma_{SL}}{\gamma_{LG}}.$$

où $\gamma_{SG}, \gamma_{SL}, \gamma_{LG}$ sont les tensions de surfaces respectivement entre les phases solide (S), gas (G) et liquide (L). L'angle de contact dynamique θ commun aux deux points triples est défini par

$$\cos\theta = \frac{1}{\sqrt{1 + (\partial_x \xi)^2}}.$$

Les vectors $\mathbf{n}_b$ and $\mathbf{t}_b$ sont respectivement le vecteur normal et le vecteur tangentiel à la surface solide

$$\mathbf{n}_b(t,x) = \frac{1}{\sqrt{1 + (\partial_x b)^2}} \begin{pmatrix} -\partial_x b \\ 1 \end{pmatrix},$$

$$\mathbf{t}_b(t,x) = \frac{1}{\sqrt{1 + (\partial_x b)^2}} \begin{pmatrix} 1 \\ \partial_x b \end{pmatrix}.$$

Pour l'étude d'asymptotique suivante, l'équation du fluide en dimension 1 se pose

$$\begin{cases} \partial_x u + \partial_z w = 0, \\ \partial_t u + \partial_x(u^2) + \partial_z(uw) = \partial_x \sigma_{xx} + \partial_z \sigma_{xz} - \partial_x p, \\ \partial_t w + \partial_x(uw) + \partial_z(w^2) = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - \partial_z p - g. \end{cases}$$

1.3 Deux types de modèles asymptotiques

Dans cette partie, nous procédons à une analyse dimensionnelle sur le système de départ en vue de faire apparaître les petits paramètres utiles pour développer une analyse asymptotique sur les équations de Navier-Stokes.

Notre but est ici de créer une hiérarchie de modèles, du plus simple au plus complexe. Cette hiérarchie sera construite en enrichissant au fur et à mesure les conditions aux bords. Au bas de l'échelle, nous étudierons une configuration avec un fond plat et une surface libre ne prenant pas en compte les effets de capillarité. Le modèle final prendra en compte les effets de capillarité ainsi que les conditions aux bords (1.6). Dans cette introduction nous ne présentons que le modèle final.

Dans un schéma simplifié, les écoulements des fluides sont gouvernés, entre autres, par les grandeurs physiques qui leur sont associées. Dans le cadre de l'écoulement d'un oxyde dans une fissure longitudinale, nous sommes amenés à introduire les grandeurs caractéristiques en espace : L suivant l'axe x et H suivant l'axe z. Nous introduisons aussi celles associées aux vitesses u et w que nous noterons respectivement U et W et celles correspondant à la pression p et le temps t : P et T. Dans ce contexte, adimensionner le système consiste à introduire les nouvelles variables suivantes

$$\tilde{x} = x/L, \quad \tilde{z} = z/H, \quad \tilde{\xi} = \xi/H, \quad \tilde{b} = b/H$$

$$\tilde{u} = u/U, \quad \tilde{w} = w/W, \quad \tilde{t} = t/T, \quad \tilde{p} = p/P,$$

et à réécrire les équations (1.3), (1.4), (1.5), (1.6) and (2.3) en utilisant ces nouvelles variables. Les fissures dans les matériaux auto-cicatrisants se caractérisent par une épaisseur très petite devant leur longueur. Ces considérations géométriques nous conduisent à introduire le paramètre suivant

$$\varepsilon = \frac{H}{L},$$

et à supposer $\varepsilon \ll 1$. Ce paramètre apparait naturellement lorsque l'on adimensionne les équations citées ci-dessus et permet de dériver différents modèles basés sur les développements asymptotiques de u, w et p en fonction de ε. Notons, dans un premier temps, que ce processus appliqué à la condition d'incompressibilité $\partial_x u + \partial_z w = 0$ conduit à l'équation suivante

$$\partial_{\tilde{x}} \tilde{u} + \frac{WL}{UH} \partial_{\tilde{z}} \tilde{w} = \partial_{\tilde{x}} \tilde{u} + \frac{W}{\varepsilon U} \partial_{\tilde{z}} \tilde{w} = 0. \tag{1.7}$$

Ainsi, pour ne pas considérer, au premier ordre en ε, des vitesses w constantes, nous sommes amenés à considérer l'hypothèse fondamentale suivante

$$\frac{H}{L} = \frac{W}{U} = \varepsilon, \tag{1.8}$$

qui transforme (1.7) en

$$\partial_{\tilde{x}} \tilde{u} + \partial_{\tilde{z}} \tilde{w} = 0.$$

Dans le même ordre d'idée, l'équation de conservation du moment conduit à l'équation suivante

$$\frac{U}{T}\partial_{\tilde{t}}\tilde{u} + \frac{U^2}{L}\partial_{\tilde{x}}(\tilde{u}^2) + \frac{UW}{H}\partial_{\tilde{z}}(\tilde{u}\tilde{w}) = \frac{U^2}{L}\partial_{\tilde{x}}\tilde{\sigma}_{xx} + \frac{U^2}{H}\partial_{\tilde{z}}\tilde{\sigma}_{xz} - \frac{P}{L}\partial_{\tilde{x}}\tilde{p}.$$

ce qui nous invite à prendre en considération la valeur de T suivante

$$T = \frac{L}{U}.$$

Dans la suite, nous introduirons aussi la viscosité μ de l'oxyde, le nombre de Reynolds

$$Re = \frac{UL}{\mu}$$

ainsi que le paramètre $\nu = 1/Re$. Le choix de l'adimmensionnement de la pression s'avère plus délicat et détermine le type de modèles asymptotiques que l'on peut obtenir. Dans ce manuscrit de thèse, nous allons considérer deux valeurs de P différentes. La première, définie par

$$P = U^2, \tag{1.9}$$

permet d'écrire des modèles de type Saint-Venant alors que le choix

$$P = \nu\frac{U^2}{\varepsilon^2} \tag{1.10}$$

nous conduira à des modèles qui s'inscrivent dans un régime de lubrification.
La première étape consiste à intégrer l'équation(1.2) sur ϕ entre $b(t,x)$ et la surface libre $\xi(t,x)$ afin d'obtenir l'équation qui décrit la conservation de la masse

$$\partial_t h + \partial_x(h\overline{u}) = \int_b^{\xi}\Big(\phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi\Big)dz + \phi_r\big(\frac{\partial b}{\partial x}(u_i - u_r) - (w_i - w_r)\big),$$

où $h(t,x) = b(t,x) - \xi(t,x)$ et $\overline{u}$ représente la vitesse moyennée suivant la verticale

$$\overline{u} = \frac{1}{h(t,x)}\int_{b(t,x)}^{\xi(t,x)} u(t,x,z)dz.$$

La suite consiste, pour fermer le système, à obtenir une description asymptotique de la vitesse $\overline{u}$ en utilisant les équations de conservation de mouvement et les conditions aux bords, dans chacun des deux régimes cités ci-dessus : le régime Saint-Venant et la théorie de la lubrification. Notons enfin que dans chacun de ces deux régimes considérés, plusieurs conditions aux bords seront envisagées sur le fond $b(t,x)$, du fond plat correspondant à $b(t,x) = 0$ à une situation plus complexe du point de vue physique incluant un fond variable, une condition de pénétration et la prise en compte de la réaction chimique de création de l'oxyde. Ces considérations nous amènerons, dans chacun des régimes, à écrire une hiérarchie de 5 modèles, du plus simple au plus complet (voir le Chapitre 2 pour plus de détail).

Pour finir, nous présentons les 2 modèles les plus complets que l'on obtient dans ce manuscrit :

• pour le régime Saint-Venant

$$
(\mathbf{SW5}) \quad \begin{cases}
\partial_t h + \partial_x(h\overline{u}) = \displaystyle\int_b^{\xi} \Big(\phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi\Big)dz + \phi_r\big(\partial_x b(u_i - u_r) - (w_i - w_r)\big) \\
\partial_t(h\overline{u}) + \partial_x(h\overline{u}^2) + \overline{u}(u_i - u_r)_{|z=b}\partial_x b - \overline{u}(w_i - w_r)_{|z=b} + \gamma_1 h\partial_{xxx}(h+b) \\
\quad - 4\nu\partial_x(h\partial_x\overline{u}) + \dfrac{g\partial_x h^2}{2} + gh\partial_x b + \alpha_1\overline{u} + \beta_1\partial_x(\varphi(t,x,b))(\cos\theta_{eq} - \cos\theta) = 0, \\
\partial_t\varphi + \partial_x(\overline{u}\varphi) + \partial_z\Big(\big(\partial_t b(t,x) + \partial_x b(t,x)\overline{u}(t,x) - \phi_r\big(\partial_x(u_i - u_r) - (w_i - w_r)\big) \\
\quad - \partial_x\overline{u}(t,x)(z-b)\big)\varphi\Big) = \phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi.
\end{cases}
$$

Notons que ce système comprend un terme de dérivée d'ordre 3 en h qui correspond à la prise en compte des effets de capillarité.

• pour la théorie de la lubrification

$$
(\mathbf{TF5})\quad
\begin{cases}
\partial_t h + \partial_x(R_5(h)) = \displaystyle\int_b^\xi \Big(\phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi\Big)dz + \phi_r\big(\frac{\partial b}{\partial x}(u_i - u_r) - (w_i - w_r)\big),\\
R_5(h) = -(\frac{h^3}{3} + \frac{h^2}{\alpha})\big(\beta_1\partial_{xxx}(h+b) + g\partial_x(h+b)\big) - \frac{\beta_2}{\alpha}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)h,\\
\partial_t\varphi + \partial(u\varphi) + \partial_z(w\varphi) = 0,\\
u = \frac{\partial_x p}{2}\big((h+b-z)^2 - h^2 + \frac{h}{\alpha_2}\big) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta),\\
w = w_{|z=b} + \partial_x^2 p\big(\frac{z^3-b^3}{6}\big) - \partial_x\big(\xi\partial_x p\big)\big(\frac{z^2-b^2}{2}\big) + \partial_x\big(\partial_x p(t,x)\big(\frac{h}{\alpha_2} - \frac{h^2-\xi^2}{2}\big)\big)(z-b)\\
\quad + \frac{\beta_2}{\alpha_2}\partial_x^2\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)(z-b),\\
w_{|z=b} = \partial_t b + u_{|z=b}\partial_x b - \phi_r\big(\partial_x b(u_i - u_r) - (w_i - w_r)\big),\\
u_{|z=b} = \frac{\partial_x p}{2}\big(\frac{h}{\alpha_2}\big) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta).
\end{cases}
\tag{1.11}
$$

Du point de vue structure, l'équation obtenue est une équation parabolique dégénérée d'ordre 4. Généralement, ce type de modèle sert à décrire des écoulements de fluides visqueux entre deux parois solides très poches en mouvement. Cependant dans le cadre de cette thèse, les parois ne sont plus entièrement solides, dû à la présence de couches chimiquement actives.

1.4 Résultats d'existence de solutions pour l'équations de lubrification

Dans cette partie nous considérons un modèle de type lubrification dérivé de l'équation de Navier-Stokes précédente,dans sa version la plus simple : celle donnée par l'équation (2.96) i.e. (TF1) . Cette équation de lubrification est une équation d'ordre 4 non-linéaire parabolique sur le domain $\Omega = (0, L)$, elle est L-périodique en espace :

$$
\begin{cases}
\partial_t h + \frac{1}{\alpha\mathrm{We}}\partial_x(F(h)\partial_x^3 h) - \frac{1}{\alpha\mathrm{Fr}^2}\partial_x(F(h)\partial_x h) = 0,\\
\partial_x^{(i)} h|_{x=0} = \partial_x^{(i)} h|_{x=L},\ \text{for } i = 0, ..., 4,\\
h|_{t=0} = h_0,
\end{cases}
\tag{1.12}
$$

avec g et β_1 deux coefficients strictement positifs, $F(s) = s^3/3 + s^2/\alpha$ avec $\alpha > 0$ et la fonction initiale h_0 satisfait

$$h_0 \in H^1(\Omega) \qquad \text{avec} \qquad h_0 \geq 0. \tag{1.13}$$

Il y a plusieurs papiers concernant le problème de lubrification sous la forme

$$\partial_t h + \partial_x(F(h)\partial_x^3 h) - \partial_x(G(h)\partial_x h) = 0. \tag{1.14}$$

Pour $F(h) = |h|^n$, $G(h) = 0$ en 1D, l'existence d'une solution non-negative faible de (1.14) est étudiée par F. Bernis and A. Friedman [5] pour $n > 1$; des résultats plus avancés sont obtenus ensuite par E. Beretta, M. Bertsch et R. Dal Passo [2] et A. Bertozzi et M. Pugh [8]. En particulier, dans [8] les auteurs ont classifié ce problème en deux types : $0 < n < 3$ avec fonction initiale $h_0 \geq 0$; $0 < n < +\infty$ avec fonction initiale $h_0 \geq m > 0$. D'ailleurs des résultats en dimension supérieure sont obtenus dans [42, 28].

Pour $F(h) = |h|^n$, $G(h) = h^m$, une variété de résultats sont obtenus par A. L.Bertozzi et M. C. Pugh dans [7, 10, 9]. En comparant avec [8], l'équation (1.14) dans [7] est étudiée en ajoutant un 'porous media' terme d'ordre 2 $G(h)\partial_x h = \nabla h^m$. En particulier, il existe des solutions faibles non-negatives pour $0 < n < 3$, mais ça ne marche pas pour $n \geq 3$.

En particulier, pour $F(h) = G(h) = h^3$, l'équation (1.14) est considérée par by C. Imbert et A. Mellet [46] mais avec un terme $\lambda I(h)$ qui modélise l'effet de champs d'électricité, où l'opérateur $I(h)$ est un opérateur elliptique non-local d'ordre 1.

Pour $F(h) = h^3 + \lambda h^p$ avec $0 < p < 3$ et $\lambda > 0$, dans [49] A. A. Lacey a considéré le cas $G(h) = 0$ qui représente le mouvement glissant d'une goutte mince et visqueuse sur une surface solide. De plus il y a une instabilité des ondes longues (Ehrhard [34]).

Dans notre travail, nous nous intéressons à $F(h) = G(h) = h^3/3 + h^2/\alpha$ avec un constant $\alpha > 0$. L'objectif de cette partie est la suivante : étudier le problème de Cauchy (1.12) et montrer l'existence de solutions faibles non-negatives , et étudier la limite quand $\beta_1 \to 0$ avec g fixé et la limite $g \to 0$ avec β_1 fixé.

Puisque nous ne pouvons pas utiliser le principe du maximum sur les équations d'ordre 4, la difficulté est d'obtenir la positivité sans utiliser le principe du maximum.

1.4.1 Definitions et théorème d'existence

Ici nous ne parlons que de la définition de solutions faibles pour le système de lubrification et du résultat principal.

Définition 1. *Etant donné h_0 satisfaisant* (1.13). *Nous appelons h une* **solution faible** *pour équation* (1.12) *si*

$$h \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \qquad \sqrt{F(h)}\,\partial_x^3 h \in L^2(0,T;L^2(\Omega))$$

avec $h \geq 0$ et pour $\psi \in \mathcal{C}^\infty_\#([0,T]\times\Omega)$ telle que $\psi|_{t=T} = 0$

$$\begin{aligned}\int_\Omega h_0\psi(0)dx + \int_0^T\int_\Omega h\partial_t\psi\,dxdt \\ - \frac{1}{\alpha\mathrm{We}}\int_0^T\int_\Omega\Big[F'(h)\partial_x^2 h\partial_x h\partial_x\psi + F(h)\partial_x^2 h\partial_x^2\psi\Big]\,dxdt \\ - \frac{1}{\alpha\mathrm{Fr}^2}\int_0^T\int_\Omega F(h)\partial_x h\partial_x\psi\,dxdt = 0\end{aligned} \tag{1.15}$$

où $\mathcal{C}^\infty_\#([0,T]\times\Omega)$ est une fonction périodique en espace dans $\mathcal{C}^\infty([0,T]\times\Omega)$. L'objectif de cette partie est de montrer l'existence d'une solution faible globale pour le système (1.12) de différentes manières.

Théorème 1. *Etant donnée une condition initiale h_0 satisfaisant $h_0 \geq 0$ avec $h_0 \in H^1(\mathbb{T})$ et $-(1+h_0)\log(h_0/(1+h_0)) \in L^1(\mathbb{T})$ alors il existe une solution faible pour le système* (1.12) *dans le sens de définition 1 construite par une manière directe ou par la solution faible globale pour equation de type Saint-Venant viscosité avec la capillarité.*

D'après les travaux de A.L. Bertozzi et Pugh, l'idée pour montrer l'existence de (1.12) est de justifier que les solutions faibles peuvent être approchée respectivement comme les limites de h_η, sont solutions des systèmes suivants :

$$\begin{aligned} \partial_t h_\eta + \frac{1}{\alpha \mathrm{We}}\partial_x(\tilde{f}_\eta(h_\eta)\partial_x^3 h_\eta) - \frac{1}{\alpha \mathrm{Fr}^2}\partial_x(\tilde{f}_\eta(h_\eta)\partial_x h_\eta) = 0,& \\ \partial_x^{(i)} h_\eta|_{x=0} = \partial_x^{(i)} h_\eta|_{x=L} \qquad \text{for } i = 0, ..., 4,& \\ h_\eta|_{t=0} = h_0 + \eta.& \end{aligned} \tag{1.16}$$

où nous remplaçons $F(y)$ par,

$$\tilde{f}_\eta(y) = \frac{y^4|F(y)|}{\delta(\eta)|F(y)| + y^4}$$

avec $\delta(\eta) \in (0, 1/4)$ qui tend vers 0 lentement quand η tend vers 0.
Comparé à $f_\eta(y)$, $\tilde{f}_\eta(y)$ est positive ou nulle pour tout y, ce qui pourra nous aider à améliorer la borne uniforme et obtenir la positivité de h_η. Ainsi nous réécrivons le problème d'approximation afin de chercher une solution L-périodique pour le système (1.16).
Ensuite nous parlons de quelques estimations a priori :

Proposition 2. *Soit $h_\eta(t, x)$ une solution régulière pour l'équation* (1.16)*, alors*

$$\begin{aligned} \sup_{t\in(0,T}\int_{\mathbb{T}}(\frac{1}{\alpha \mathrm{Fr}^2}h_\eta^2 + \frac{1}{\alpha \mathrm{We}}(\partial_x h_\eta)^2)dx& \\ + \int_0^T\int_{\mathbb{T}}\frac{F(h_\eta)}{h_\eta^2}\Big|\frac{1}{\alpha \mathrm{We}}h_\eta\partial_x^3 h_\eta - \frac{1}{\alpha \mathrm{Fr}^2}h_\eta\partial_x h_\eta\Big|^2& \\ \le \int_{\mathbb{T}}(\frac{1}{\alpha \mathrm{Fr}^2}(h_\eta)^2(0) + \frac{1}{\alpha \mathrm{We}}(\partial_x h_\eta)^2(0))dx.& \end{aligned} \tag{1.17}$$

Remarque. Cette proposition nous offre une borne uniforme sur h_η dans $L^\infty(0, T; H^1(\Omega))$ si la condition initiale h_0 est dans $H^1(\Omega)$.

Proposition 3. *Soit $h_\eta(t, x)$ une solution régulière pour équation* (1.16)*, alors*

$$\int_\Omega G_\eta(h_\eta)dx + \int_0^t\int_\Omega \frac{1}{\alpha \mathrm{We}}(\partial_{xx}h_\eta)^2 dx + \frac{1}{\alpha \mathrm{Fr}^2}(\partial_x h_\eta)^2 dx \le \int_\Omega G_\eta(h_0)dx \tag{1.18}$$

Remarque. Cette proposition nous offre une borne uniforme sur h_η dans $L^2(0, T; H^2(\Omega))$ pour une hauteur initiale $h_{\eta 0} = h_0 + \eta$ et $G(h_0) \in L^1(\mathbb{T})$. (Quant à la définition de la fonction $G(\cdot)$, nous pouvons utiliser la définition (11) dans le chapitre 3.)

Proposition 4. *Soit $h_\eta(t,x)$ une solution régulière pour l'équation* (1.16) *avec $h_{\eta 0} = h_0 + \eta$, alors $h_\eta(t,x)$ est bornée uniformément par une M constante, et est strictement positif dans $[0,T] \times \Omega$.*

Proposition 5. *Soit $h_\eta(t,x)$ une solution régulière pour l'équation* (1.16) *avec $h_{\eta 0}$ satisfaisant* (1.13)*, alors il existe une D constante indépendente de η, telle que nous ayons l'estimation suivante*

$$\int_0^T \int_{\mathbb{T}} f_\eta(h_\eta)(\partial_x^3 h_\eta)^2 dx \leq D.$$

Par conséquent, $\partial_x^3 h_\eta$ est borné dans $L^2([0,T], L^2(\mathbb{T}))$ mais sa borne dépend de η.

Proposition 6. *Soit $h_\eta(t,x)$ une solution régulière pour l'équation* (1.16) *avec h_0 satisfaisant* (1.13)*, alors $\partial_t h_\eta$ est dans $L^2(0,T; H^{-1}(\mathbb{T}))$.*

Théorème 2. *Etant donnée une condition initiale h_0 satisfaisant* (1.13)*, il existe une solution faible positive ou nulle pour l'équation régularisée* (1.16)*, qui satisfait la formulation faible* (3.13) *dans $[0,T] \times \Omega$.*

Remarque : L'existence de solution faible de (1.16) peut être obtenue par la méthode de Galerkin. Ainsi nous avons besoin de chercher une base de fonction test $\phi_n(x)$ bien régulière et périodique sur $\partial\Omega$, pour cela nous considerons le problème de Dirichlet stationnaire d'ordre 4 :

$$\begin{cases} \phi^{(4)}(x) = \lambda \phi(x); \\ \phi^{(i)}(0) = \phi^{(i)}(L) \text{ for } i = 0, \cdots, 4. \end{cases} \tag{1.19}$$

Alors nous construisons la fonction semi-discrète suivante par la méthode de Galerkin

$$u^N(t,x) = \sum_{j=1}^{N} \alpha_j(t)\phi_j(x) \tag{1.20}$$

Ensuite nous pouvons trouver les coefficients $\alpha_j(t)$, en testant les équations (1.16). Puis nous pouvons montrer qu'il existe une unique suite $(\alpha_j)_{j=1,\cdots,N}$ qui nous mène à une unique forme pour $u^N(t,x)$ (voir la proposition 19). Enfin de la méthode de Galerkin, nous montrons les convergences pour faire écho aux remarques ci-dessus.

Théorème 3. *Soit $\{u^N\}_{N\in\mathbb{N}}$ une suite construite par la précédent processus (défini par* (3.17)*), alors*

i) il exist une suite $\{u^N\}_{N\in\mathbb{N}}$ *qui converge fortement dans* $L^2([0,T],H^1(\Omega))$ *;*
ii) il exist une suite $\{u^N\}_{N\in\mathbb{N}}$ *qui converge fortement dans* $L^2([0,T],H^2(\Omega))$ *;*
iii) il exist une suite $\{u^N\}_{N\in\mathbb{N}}$ *qui converge faiblement dans* $L^2([0,T],H^3(\Omega))$ *;*

1.5 Modèle de Saint-Venant et modèle de lubrification

Dans cette partie nous voulons préciser la limite entre l'équation de lubrification et l'équation de Saint-Venant précédente, plus de details de ces modèles sont présentés dans les articles [43], [12] (ainsi que les travaux [6], [10]).

Tout d'abord, nous pouvons construire une convergence faible en utilisant la borne uniforme pour une solution faible globale pour l'équation de lubrification. Ensuite nous montrons la convergence forte pour une solution de cette équation en utilisant une inégalité dans [24]–[25] pour le système de Navier-Stokes-Korteweg avec condition de compatiblité entre le terme de dispersion et le terme de diffusion. Sous la condition compressible, la densité dépendant de la viscosité s'annule si la densité s'annule.

Nous considérons, sur un domaine périodique $\Omega = \mathbb{T}$, l'équation de Saint-Venant avec tension de surface

$$\begin{aligned} &\partial_t h_\varepsilon + \partial_x(h_\varepsilon \overline{u}_\varepsilon) = 0, \\ &\partial_t(h_\varepsilon \overline{u}_\varepsilon) + \partial_x\left(h_\varepsilon \overline{u}_\varepsilon^2 + \frac{(h_\varepsilon)^2}{2\mathrm{Fr}^2}\right) = \frac{4}{\mathrm{Re}}\partial_x(h_\varepsilon \partial_x \overline{u}_\varepsilon) + \frac{1}{\mathrm{We}} h_\varepsilon \partial_x^3 h_\varepsilon - \alpha \overline{u}_\varepsilon, \end{aligned} \tag{1.21}$$

avec $\alpha > 0$ où Re le nombre de Reynolds, We le nombre de Weber et Fr le nombre de Froude. Notons que les termes à droite représentent respectivement le terme de viscosité, le terme de capillarité et un terme linéaire. Dans le cas unidimensionnel en espace, l'existence de solutions faibles globales de ce système est obtenue dans [18] où la BD-entropie a été introduite.

Nous considérons le fonction initiale

$$h_\varepsilon|_{t=0} = h_0^\varepsilon, \qquad (h_\varepsilon u_\varepsilon)|_{t=0} = m_0^\varepsilon.$$

Nous considérons la limite de lubrification ($\varepsilon \ll 1$) avec des nombres non-dimensionnels sous la forme

$$\mathrm{We} := \varepsilon W_e, \quad \mathrm{Fr}^2 := \varepsilon F^2, \quad \alpha := \frac{\overline{\alpha}}{\varepsilon}$$

et les autres nombres dimensionnels indénpendants de ε. Pour la limite $\varepsilon \to 0$, nous supposons des bornes uniformes pour toutes les dérivées ; formellement

$$\overline{\alpha}\,\overline{u} = \frac{1}{W_e} h \partial_x^3 h - \frac{h \partial_x h}{F^2} \tag{1.22}$$

et

$$\partial_t h + \partial_x (h \overline{u}) = 0. \tag{1.23}$$

Combinant les équations (1.22) avec (1.23), nous obtenons une équation de lubrification

$$\partial_t h + \partial_x \left(\frac{1}{\overline{\alpha} W_e} h^2 \partial_x^3 h - \frac{1}{\overline{\alpha} F^2} h^2 \partial_x h \right) = 0. \tag{1.24}$$

Estimation d'énergie :

$$\frac{d}{dt} \left(\int_{\mathbb{T}} \varepsilon \frac{h_\varepsilon \overline{u}_\varepsilon^2}{2} + \frac{h_\varepsilon^2}{2F^2} + \frac{(\partial_x h_\varepsilon)^2}{2W_e} \right) + \int_{\mathbb{T}} \frac{4\varepsilon}{R_e} h_\varepsilon (\partial_x \overline{u}_\varepsilon)^2 + \overline{\alpha}\,\overline{u}_\varepsilon^2 \le 0. \tag{1.25}$$

Cet estimation d'énergie est obtenue en multipliant l'équation du mouvement par $\overline{u}_\varepsilon$ et en ajoutant ce résultat à l'équation suivante

$$\frac{1}{2} [\partial_t h_\varepsilon^2 + \partial_x (h_\varepsilon^2 \overline{u}_\varepsilon) + h_\varepsilon^2 \partial_x \overline{u}_\varepsilon] = 0, \tag{1.26}$$

et ensuite nous l'intégrons en espace. Cette équation (1.26) est obtenue à partir de l'équation de la masse, en la multipliant par h_ε.

L'estimation de BD entropy est

$$\varepsilon\frac{d}{dt}\int_{\mathbb{T}}\frac{h_\varepsilon}{2}(\overline{u}_\varepsilon+4(R_e)^{-1}\frac{\partial_x h_\varepsilon}{h_\varepsilon})^2+\frac{d}{dt}\int_{\mathbb{T}}\Big(\frac{h_\varepsilon^2}{2F^2}+\frac{(\partial_x h_\varepsilon)^2}{2W_e}-\frac{4\overline{\alpha}}{R_e}\log_- h_\varepsilon\Big).$$
$$+\frac{4}{R_e}\int_{\mathbb{T}}\frac{(\partial_x h_\varepsilon)^2}{F^2}+\frac{(\partial_x^2 h_\varepsilon)^2}{W_e}+\int_{\mathbb{T}}\overline{\alpha}\,\overline{u}_\varepsilon^2\leq 0 \tag{1.27}$$

Définition : Nous appelons $(h_\varepsilon, u_\varepsilon)$ **solution faible globale** pour l'équation (1.21) si elle satisfait (1.25) et (1.27)

$$\int_0^\infty\int_{\mathbb{T}}h_\varepsilon\partial_t\psi+\int_{\mathbb{T}}h_0^\varepsilon\psi(\cdot,0)\,dx=-\int_0^\infty\int_{\mathbb{T}}h_\varepsilon\overline{u}_\varepsilon\partial_x\psi\,dxdt \tag{1.28}$$

et

$$\varepsilon\Big(\int_0^\infty\int_{\mathbb{T}}h_\varepsilon\overline{u}_\varepsilon\partial_t\phi+\int_{\mathbb{T}}m_0^\varepsilon\phi(\cdot,0)\,dx+\int_0^\infty\int_{\mathbb{T}}h_\varepsilon\overline{u}_\varepsilon^2\partial_x\phi\,dxdt\Big) \tag{1.29}$$
$$-\frac{4\varepsilon}{R_e}\int_0^\infty\int_{\mathbb{T}}h_\varepsilon\partial_x\overline{u}_\varepsilon\partial_x\phi\Big)-\frac{1}{W_e}\int_0^\infty\int_{\mathbb{T}}\partial_x h_\varepsilon\partial_x^2 h_\varepsilon\phi\,dxdt$$
$$-\frac{1}{W_e}\int_0^\infty\int_{\mathbb{T}}h_\varepsilon\partial_x^2 h_\varepsilon\partial_x\phi\,dxdt+\frac{1}{F^2}\int_0^\infty\int_{\mathbb{T}}h_\varepsilon^2\partial_x\phi\,dxdt-\overline{\alpha}\int_0^\infty\int_{\mathbb{T}}\overline{u}_\varepsilon\phi\,dxdt=0$$

pour tout $\psi\in\mathcal{C}_0^\infty(\mathbb{T}\times[0,\infty))$ et $\phi\in\mathcal{C}_0^\infty(\mathbb{T}\times[0,\infty))$.

Nous rappelons qu'il y a un résultat d'existence dans [16].

Théorème 4. *Soit* $(h_0^\varepsilon, m_0^\varepsilon)$ *est un couple tel que* $h_0^\varepsilon\geq 0$ *et*

$$h_0^\varepsilon\in H^1(\Omega),\quad \varepsilon|m_0^\varepsilon|^2/h_0^\varepsilon\in L^1(\Omega),\quad \sqrt{\varepsilon}\partial_x\sqrt{h_0^\varepsilon}\in L^2(\Omega),\quad -\log_- h_0^\varepsilon\in L^1(\Omega)$$

où $\log_-\cdot=\log\min(\cdot,1)$. *Alors il existe une solution faible globale pour l'équation* (1.21) *au sens de la definition* (1.28)–(1.29).

Théorème 5. *Soit* $(h_\varepsilon, u_\varepsilon)$ *une solution faible globale de* (3.25) *avec la condition initiale satisfait les bornes uniformes. Alors il existe une sous-suite de* $(h_\varepsilon,\overline{u}_\varepsilon)$, *notée par* $(h_\varepsilon,\overline{u}_\varepsilon)$, *qui converge vers* $(h,\overline{u})$ *la solution faible globale du système de lubrification* (4.2)–(4.3) *satisfaisant la condition initiale* $h|_{t=0}=h_0$ *avec* h_0 *la limite faible dans* $H^1(\mathbb{T})$ *de* h_0^ε.

Chapitre 2

Derivation of asymptotique models from Navier-Stokes equation

As a composite material accompanied with excellent properties, Ceramic-matrix composites (CMCs) contain a ceramic matrix reinforced by a refractory fiber, such as silicon carbide (SiC) fiber. Because of their high strength and low weight, CMCs are the focus of active research, for aerospace and energy applications involving high temperatures, either military or civil referring to [63], where CMCs are envisioned as lightweight replacements for metallic superalloys. These composites are no longer brittle in spite of being based on brittle ceramic components, owing to the use of a fiber/matrix interphase that manages to preserve the fibers from cracks appearing in the matrix, specifically, in protecting themselves against oxidation by the formation of a sealing oxide which fills the matrix cracks. Due to the self-healing process, CMCs have extremely long lifetimes even under severe mechanical and chemical solicitations. Recent developments aim at implementing in civil aero engines a specific class of CMCs which was already successfully tested in military engines that show a self-healing behavior.

Indeed, the lifetime-determining part of the material is the fibers, which are sensitive to oxidation ; when the composite is in use, it may appear cracks that provide a path for oxidation. Self-healing consists in filling these cracks with an oxide formed in-situ by the oxidation of parts of the matrix components surrounding the fibers. The oxide corks limit oxygen diffusion toward fibers, thus delaying fiber oxidation and failure. The obtained lifetimes can be of the order of hundreds of thousands of hours. These time spans make most full-scale experimental investigations impractical. Laboratory tests have necessarily to be replaced by predictions based on numerical models.

Remarkable preliminary results have already been obtained by Dréan, Perrot, Couégnat, Ricchiuto and Vignoles on the 2D numerical modeling of oxide formation in self-healing CMCs in [31], also by Caty and Rebillat on X-ray CMT characterization in [26]. This work is a continuation of that of G. Perrot [60], with the aim of deriving asymptotic models to describe the self-healing process and to develop mathematical tools in view of performing realistic numerical experiments.

In this paper, we introduce new models which describe both the self-healing behavior and oxygen diffusion towards fibers. In general, the evolution of an incompressible fluid can be described by the Navier-Stokes equations. However, we observe that a direct numerical method applied to these equations will induce a significant computational cost, especially in our case, due to the long lifespan of the material. Thus, we shall derive several asymptotic models from the Navier-Stokes equations with boundary conditions in order to reduce this cost, for example, shallow water models and thin film models. The idea is inspired by the method proposed for the one-dimensional case by Gerbeau and Perthame in [43], in which they recall the classical derivation of Saint-Venant system from the Navier-Stokes system for incompressible flows with a free moving boundary. Precisely, we transform bottom conditions related to the chemical reaction and normal-stress continuity conditions on the free surface. Moreover, the gravity, surface tension effects and capillary effects will be taken into consideration in our models.

On account of these complex boundary conditions, asymptotic models are not readily obtained, for this reason we set about boundary conditions from the simple to the complex. More specifically, we start from pure slip bottom without capillary penetrations as well as free surface without capillary effects, afterwards we add other conditions to current model stepwise. From dimensionless analysis implemented by some suitable scalings on Navier-Stokes equations with free surface boundary conditions and general boundary conditions at bottom (from adherence to pure slip), we derive two kinds of models, namely shallow water models and thin film models.

More precisely, solutions are expanded with respect to the geometrical ratio $\varepsilon = H/L$ between the classical defined characteristic fluid height H and the characteristic wavelength L of the flow. The main difference between these two types of models springs from the definition of rescaled full stress tensor, substituting the asymptotic expansions into the depth averaged continuity and momentum equation : we obtain shallow water models by taking $P = U^2$ and thin film models by taking $P = \nu U^2/\varepsilon^2$. Similar strategies are used in Boutounet, Monnier and Vila [8], related to thin free-surface flows for quasi-Newtonian fluids (see also Bresch and Noble [22] concerning the derivation of shallow water model assuming non zero surface tension coefficient).

2.1 Navier-Stokes equations and boundary conditions

In order to describe the cicatrisation process of a self-healing material, we assume that the oxyde created by the chemical reaction

$$B_4C + 0.6C + 4.6O_2 \longrightarrow 2B_2O_3 + 1.6CO_2$$

ou

$$B_4C + 4O_2 \longrightarrow 2B_2O_3 + CO_2$$

is a free-surface, isotropic, incompressible and irrotational fluid. In the context of a 2D simplified crack (see Figure 1.3), the propagation of the fluid is described by the 2D Navier-Stokes equations

$$\begin{cases} \mathrm{div}\mathbf{U} = 0, \\ \partial_t \mathbf{U} + \mathrm{div}(\mathbf{U} \otimes \mathbf{U}) = \mathrm{div}\sigma(\mathbf{U}) - \nabla p + \mathbf{g}, \end{cases} \tag{2.1}$$

where $\mathbf{U} = (u, w)$ is the velocity, $\sigma(u) = 2\mu D(\mathbf{U})$ is the deformation tensor with

$$D(\mathbf{U}) = \frac{\nabla\mathbf{U} + \nabla\mathbf{U}^T}{2},$$

μ is the viscosity coefficient, p is the local pressure in the fluid and g denotes the acceleration of gravity. Note that Equations (2.1) can be rewritten into the form

$$\begin{cases} \partial_x u + \partial_z w = 0, & (2.2a) \\ \partial_t u + \partial_x(u^2) + \partial_z(uw) = \partial_x\sigma_{xx} + \partial_z\sigma_{xz} - \partial_x p, & (2.2b) \\ \partial_t w + \partial_x(uw) + \partial_z(w^2) = \partial_x\sigma_{xz} + \partial_z\sigma_{zz} - \partial_z p - g. & (2.2c) \end{cases}$$

Let us mention that the fluid domain depends on time t and on the horizontal space variable x

$$\Omega(t, x) = \{(t, x, z); t \geq 0,\ x \in [0, L],\ b(t, x) \leq z \leq \xi(t, x)\},$$

where $b(t, x)$ denotes the bathymetry and $\xi(t, x)$ represents the surface elevation. The domain $\Omega(t, x)$ will be described by a level set formulation (see [65], [58]). In order to describe properly $\Omega(t, x)$, we use an idea of [43] by introducing an indicator function φ defined by

$$\varphi(t, x, z) = \begin{cases} 1, & \text{for } b(t, x) \leq z \leq \xi(t, x), \\ 0, & \text{otherwise.} \end{cases}$$

Note that, since at the bottom we have to take into account a chemical reaction zone we use a second indicator function ϕ_r which describes this chemical reactive zone. We introduce also $\mathbf{U_i} = (u_i, w_i)$, the velocity of fluid injection through the surface and $\mathbf{U_r} = (u_r, w_r)$ the velocity corresponding to the decline of the reactive zones (see Figure 1.3). The function φ is advected by the flow and then satisfies the usual equation

$$\partial_t\varphi + \Big(\mathbf{U} - \phi_r(\mathbf{U}_i - \mathbf{U}_r)\Big) \cdot \nabla\varphi = 0,$$

which can be rewritten using the incompressibility condition

$$\partial_t\varphi + \nabla \cdot (\mathbf{U}\varphi) = \phi_r(\mathbf{U}_i - \mathbf{U}_r) \cdot \nabla\varphi. \tag{2.3}$$

In order to close the system, we now need appropriate boundary conditions for respectively the free surface and the bottom of the geometry.

• at the bottom. The bottom is treated as a free surface and for sake of completeness, we recall here how to obtain the equation describing the evolution of this kind of surface. Consider a fixed time t_0 and a point $(x_0, b(t_0, x_0))$ on the interface. Consider a semi-Lagrangian surface $\omega(t)$ delimited by 4 points M(t), N(t), P, Q (see Figure 2.1). We are interested by following the deformation in the z-direction. Introduce two infinitesimal length Δx and $\Delta z(t_0)$ which are supposed to go to 0 later on. We set

$$M(t) = (x_0 - \frac{\Delta x}{2}, b(t, x_0 - \frac{\Delta x}{2})),$$
$$N(t) = (x_0 + \frac{\Delta x}{2}, b(t, x_0 + \frac{\Delta x}{2})),$$
$$P = (x_0 + \frac{\Delta x}{2}, b(t, x_0 + \frac{\Delta x}{2}) + \Delta z(t_0),$$
$$Q = (x_0 - \frac{\Delta x}{2}, b(t, x_0 - \frac{\Delta x}{2}) + \Delta z(t_0)).$$

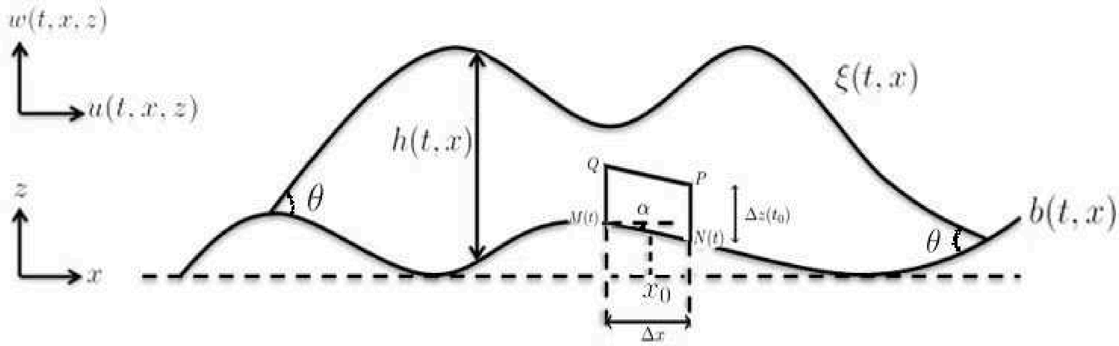

FIGURE 2.1 – .

The volume of $\omega(t)$ is $|\omega(t)| = \Delta x \Delta z(t)$. As a consequence, one has

$$\frac{\partial}{\partial t}|\omega(t)| = \Delta x \frac{\partial \Delta z(t)}{\partial t} = \Delta x \frac{\partial b}{\partial t}(t, x_0).$$

Let α be the angle of the film surface with respect to the horizontal variable, so that

$$\tan(\alpha) = \frac{\partial b}{\partial x}(t_0, x_0),$$

and consider a long-wave regime $\alpha << 1$, that is

$$\tan(\alpha) \sim \sin(\alpha) \sim \frac{\partial b}{\partial x}(t_0, x_0), \quad \cos(\alpha) \sim 1. \tag{2.4}$$

By conservation of mass, one can write that the variation of the volume of $\omega(t)$ is equal to the mass fluxes on each part of the boundary of $\omega(t)$. Thus

$$\begin{aligned}\frac{\partial}{\partial t}|\omega(t)| = \int_{[MQ]} udz - \int_{[NP]} udz + \int_{[PQ]} (-\sin(\alpha)u + \cos(\alpha)w)d\gamma \\ + \int_{[MN]} (\sin(\alpha)\phi_r(u_i - u_r) - \cos(\alpha)\phi_r(w_i - w_r))d\gamma.\end{aligned}$$

We assume furthermore that all the velocities involved are continuous around (t_0, x_0), which provides

$$\begin{aligned}
u(x,z) &= u(x_0, b(t_0,x_0)) + O(\Delta x, \Delta z(t_0)),\\
w(x,z) &= w(x_0, b(t_0,x_0)) + O(\Delta x, \Delta z(t_0)),\\
u_i(x,z) &= u_i(x_0, b(t_0,x_0)) + O(\Delta x, \Delta z(t_0)),\\
w_i(x,z) &= w_i(x_0, b(t_0,x_0)) + O(\Delta x, \Delta z(t_0)),\\
u_r(x,z) &= u_r(x_0, b(t_0,x_0)) + O(\Delta x, \Delta z(t_0)),\\
w_r(x,z) &= w_r(x_0, b(t_0,x_0)) + O(\Delta x, \Delta z(t_0)),
\end{aligned}$$

and furnishes

$$\int_{[MQ]} udz - \int_{[NP]} udz = \Delta z(t_0) O(\Delta x, \Delta z(t_0)),$$

and

$$\begin{aligned}&\int_{[PQ]} (-\sin(\alpha)u) + \cos(\alpha)w)d\gamma \\ &= \frac{\Delta x}{\cos(\alpha)}(-\sin(\alpha)u(x_0, b(t_0,x_0) + \cos(\alpha)w(x_0, b(t_0,x_0))) + O(\Delta x, \Delta z(t_0)),\end{aligned}$$

$$\begin{aligned}
&\int_{[MN]} (\sin(\alpha)\phi_r(u_i - u_r) - \cos(\alpha)\phi_r(w_i - w_r))d\gamma \\
&= \frac{\Delta x}{\cos(\alpha)}\Big(\sin(\alpha)\phi_r(u_i(x_0, b(t_0, x_0)) - u_r(x_0, b(t_0, x_0))) \\
&- \cos(\alpha)\phi_r(w_i(x_0, b(t_0, x_0)) - w_r(x_0, b(t_0, x_0)))\Big), \\
&+ O(\Delta x, \Delta z(t_0)).
\end{aligned}$$

Then we deduce

$$\begin{aligned}
\Delta x \frac{\partial b}{\partial t}(t_0, x_0) =& \Delta z(t_0) O(\Delta x, \Delta z(t_0)) + \frac{\Delta x}{\cos(\alpha)}(-\sin(\alpha)u(x_0, b(t_0, x_0) + \cos(\alpha)w(x_0, b(t_0, x_0))) \\
&+ \frac{\Delta x}{\cos(\alpha)}\Big(\sin(\alpha)\phi_r(u_i(x_0, b(t_0, x_0)) - u_r(x_0, b(t_0, x_0))) \\
&- \cos(\alpha)\phi_r(w_i(x_0, b(t_0, x_0)) - w_r(x_0, b(t_0, x_0)))\Big) + \Delta x O(\Delta x, \Delta z(t_0)).
\end{aligned}$$

Using (2.4) and sending Δx and $\Delta z(t_0)$ to 0, one obtains

$$\frac{\partial b}{\partial t} + u_{|z=b}\frac{\partial b}{\partial x} - w_{|z=b} = \phi_r\big(\frac{\partial b}{\partial x}(u_i - u_r) - (w_i - w_r)\big). \tag{2.5}$$

it can be also written as

$$\partial_t b = [\mathbf{U} - \phi_r(\mathbf{U}_i - \mathbf{U}_r)] \cdot \mathbf{n}_b$$

To complete the equation (2.5), we also use the so-called Generalized Navier Boundary Condition (see [62]), which reads in this context

$$(\sigma \cdot \mathbf{n}_b) \cdot \mathbf{t}_b = \alpha \mathbf{U} \cdot \mathbf{t}_b + \beta \nabla\varphi \cdot \mathbf{t}_b(\cos\theta_{eq} - \cos\theta), \tag{2.6}$$

where α is the liquid-solid friction coefficient and β the capillary coefficient on the solid-fluid surface. We remark that there appears two triple points of three phases (gas, liquid, and solid), at which the equilibrium contact angle is defined as $\theta_{eq} = \frac{\gamma_{SG}-\gamma_{SL}}{\gamma_{LG}}$, where γ is the surface tension between phases gas (G), liquid (L) and solid (S). The contact angles θ are supposed to be equal by symmetry arguments and are defined by $\cos\theta = \frac{1}{\sqrt{1+(\partial_x \xi)^2}}$.

Vectors $\mathbf{n}_b$ and $\mathbf{t}_b$ are respectively the outward normal and the tangent to the bottom :

$$\mathbf{n}_b(t,x) = \frac{1}{\sqrt{1+(\partial_x b)^2}} \begin{pmatrix} -\partial_x b \\ 1 \end{pmatrix},$$

$$\mathbf{t}_b(t,x) = \frac{1}{\sqrt{1+(\partial_x b)^2}} \begin{pmatrix} 1 \\ \partial_x b \end{pmatrix}.$$

At the free surface. We use the well-known kinematic condition, which expresses the continuity of the velocity at the free surface and that can be obtained with the same tools used to derive (2.5),

$$\partial_t \xi + u_{|z=\xi} \partial_x \xi - w_{|z=\xi} = 0. \tag{2.7}$$

In addition, we express the continuity of the fluid stress tensor

$$(\sigma - p\mathrm{Id}) \cdot \mathbf{n}_\xi = -\gamma \kappa \mathbf{n}_\xi, \tag{2.8}$$

where $\gamma > 0$ is a capillary coefficient on free surface, $\kappa(x)$ the mean curvature of the surface at point x, defined by

$$\kappa(x) = \frac{\partial_x^2 \xi}{(1+(\partial_x \xi)^2)^{3/2}},$$

and $\mathbf{n}_\xi$ is the outward normal to the free surface, defined by

$$\mathbf{n}_\xi(x) = \frac{1}{\sqrt{1+(\partial_x \xi)^2}} \begin{pmatrix} -\partial_x \xi \\ 1 \end{pmatrix}.$$

Finally, we give now the complete system describing the propagation of the oxyde inside the crack :

$$\begin{cases} \mathrm{div}\mathbf{U} = 0, \text{ on } \Omega(t,x), \\ \partial_t \mathbf{U} + \mathrm{div}(\mathbf{U}\otimes\mathbf{U}) = \mathrm{div}D(\mathbf{U}) - \nabla p + \mathbf{g}, \text{ on } \Omega(t,x), \\ \partial_t\varphi + \nabla\cdot(\mathbf{U}\varphi) = \phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi, \text{ on } \Omega(t,x), \\ \dfrac{\partial b}{\partial t} + u_{|z=b}\dfrac{\partial b}{\partial x} - w_{|z=b} = \phi_r\big(\dfrac{\partial b}{\partial x}(u_i - u_r) - (w_i - w_r)\big), \text{ at } z = b(t,x), \\ \partial_t\xi + u_{|z=\xi}\partial_x\xi - w_{|z=\xi} = 0, \text{ at } z = \xi(t,x), \\ (\sigma\cdot\mathbf{n}_b)\cdot\mathbf{t}_b = \alpha\mathbf{U}\cdot\mathbf{t}_b + \beta\nabla\varphi\cdot\mathbf{t}_b(\cos\theta_{eq} - \cos\theta), \text{ on } z = b(t,x), \\ (\sigma - p\mathrm{Id})\cdot\mathbf{n}_\xi = -\gamma\kappa\mathbf{n}_\xi, \text{ at } z = \xi(t,x). \end{cases} \tag{2.9}$$

2.2 Different kind of Models

In order to create a hierarchy of models, we consider in the sequel different kind of boundary conditions, from the simplest to the more complex ones. The idea is to write first simple models which can be used in first approximation and to construct step by step more complicated models which contains more physics. We then modified the boundary conditions on the bottom. In this context two family of asymptotic models will be derived : the first one belongs to the shallow water types models and the second one come from the lubrication theory. Note that we still deal here with an ideal situation where the fluid is symmetric in the crack, meaning that the two triple points have the same dynamical angle.

At the bottom, five situations are investigated from a flat bottom to a bumpy bottom, with, for the last one, the inclusion of a reactive chemical zone.

• **case 1** : we consider a flat bottom without capillary penetration (adherence) (see Figure 2.2), which leads to the following boundary conditions :

$$\begin{cases} b = 0, & (2.10a) \\ (\sigma\cdot\mathbf{n}_b)\cdot\mathbf{t}_b = \alpha\mathbf{U}\cdot\mathbf{t}_b. & (2.10b) \end{cases}$$

Note that Equation of (2.10b) expands

$$\mu(\partial_z u + \partial_x w) = \alpha u. \tag{2.11}$$

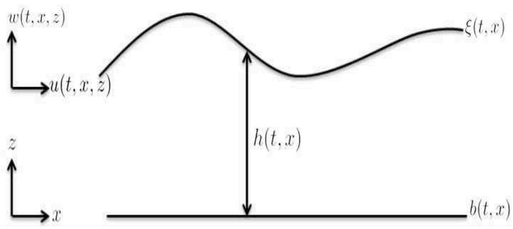

FIGURE 2.2 – Boundary conditions corresponding to case 1.

• **case 2** : we still consider a flat bottom but we assume that capillary penetration occurs (see Figure 2.3). This furnishes

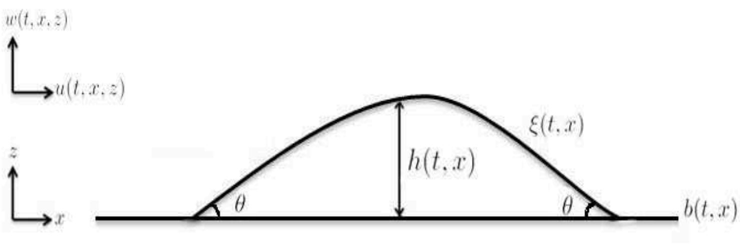

FIGURE 2.3 – Boundary conditions corresponding to case 2.

$$\begin{cases} b = 0, & \text{(2.12a)} \\ (\sigma \cdot \mathbf{n}_b) \cdot \mathbf{t}_b = \alpha \mathbf{U} \cdot \mathbf{t}_b + \beta \nabla \varphi \cdot \mathbf{t}_b (\cos \theta_{eq} - \cos \theta). & \text{(2.12b)} \end{cases}$$

In this case, Equation of (2.12b) expands

$$\mu(\partial_z u + \partial_x w) = \alpha u + \beta \partial_x \varphi (\cos \theta_{eq} - \cos \theta), \tag{2.13}$$

and is completed by the equation on ϕ

$$\partial_t \phi + \nabla \cdot \big(\phi \mathbf{U}\big) = 0. \tag{2.14}$$

• **case 3** : we come back to a no penetration condition associated to a bumpy bottom (see Figure 2.4). The boundary conditions can be written into

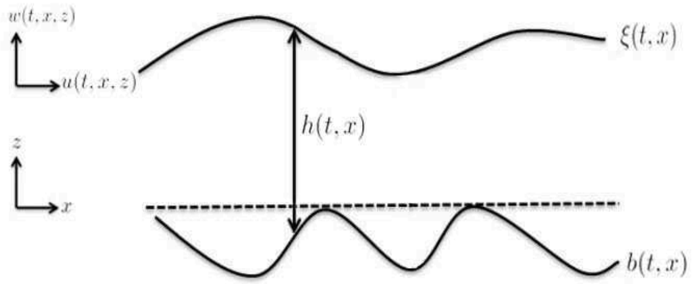

FIGURE 2.4 – Boundary conditions corresponding to case 3.

$$\begin{cases} \partial_t b + u\partial_x b = w, & (2.15a) \\ (\sigma \cdot \mathbf{n}_b) \cdot \mathbf{t}_b = \alpha \mathbf{U} \cdot \mathbf{t}_b, & (2.15b) \end{cases}$$

Equation (2.15b) expands

$$\begin{aligned} -2\mu\partial_x u\partial_x b + \mu(\partial_z u + \partial_x w) - \mu(\partial_z u + \partial_x w)(\partial_x b)^2 + 2\mu\partial_z w\partial_x b \\ = \alpha(u + w\partial_x b)\sqrt{1 + (\partial_x b)^2}. \end{aligned} \quad (2.16)$$

• **case 4** : this situation mixes a penetration condition on a bumpy bottom (see Figure (2.5)). The boundary conditions are

$$\begin{cases} \partial_t b + u\partial_x b = w, & (2.17a) \\ (\sigma \cdot \mathbf{n}_b) \cdot \mathbf{t}_b = \alpha \mathbf{U} \cdot \mathbf{t}_b + \beta\nabla\varphi \cdot \mathbf{t}_b(\cos\theta_{eq} - \cos\theta), & (2.17b) \end{cases}$$

where φ solves Equation (2.14).

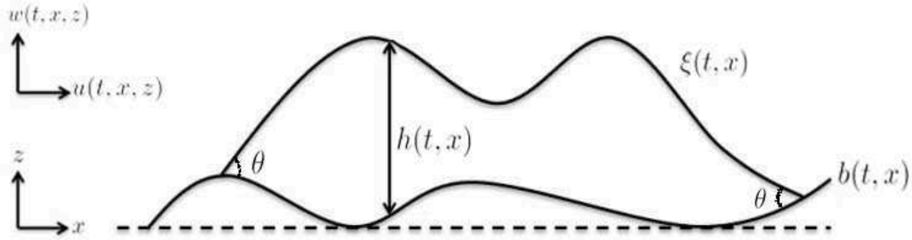

FIGURE 2.5 – Boundary conditions corresponding to case 4.

Equation (2.17b) becomes

$$\begin{aligned} &- 2\mu\partial_x u\partial_x b + \mu(\partial_z u + \partial_x w) - \mu(\partial_z u + \partial_x w)(\partial_x b)^2 + 2\mu\partial_z w\partial_x b \\ &= \Big(\alpha(u + w\partial_x b) + \beta(\partial_x\varphi + \partial_z\varphi\partial_x b)(\cos\theta_{eq} - \cos\theta)\Big)\sqrt{1 + (\partial_x b)^2}. \end{aligned} \tag{2.18}$$

• **case 5** : this is the most complete physical situation that we treat here : a penetration condition is studied with a chemical reactive zone (see Figure 1.3). This leads to

$$\begin{cases} \dfrac{\partial b}{\partial t} + u\dfrac{\partial b}{\partial x} - w = \phi_r\big(\dfrac{\partial b}{\partial x}(u_i - u_r) - (w_i - w_r)\big), & (2.19a) \\ (\sigma \cdot \mathbf{n}_b) \cdot \mathbf{t}_b = \alpha\mathbf{U} \cdot \mathbf{t}_b + \beta\nabla\varphi \cdot \mathbf{t}_b(\cos\theta_{eq} - \cos\theta), & (2.19b) \end{cases}$$

where φ solves Equation (2.3)

$$\partial_t\varphi + \nabla \cdot (\mathbf{U}\varphi) = \phi_r(\mathbf{U}_i - \mathbf{U}_r) \cdot \nabla\varphi.$$

2.3 Shallow Water models

This section is devoted to the obtention of asymptotics models of Shallow water types. These models are obtained by combining an appropriate scaling and the different physical situations presented in the previous section.

2.3.1 The nondimensionalized system

The aim of this section is to introduce a non-dimensionalized version of system (2.9). To this end, let us introduce the characteristic values of the flow : H in the direction x, L in the direction z, U is the order of magnitude of the velocity u and W is the one for the velocity w. The mass equation suggests to take $W = UH/L$. We then introduce some characteristic dimensions for the time t, $T = L/U$ and for the pressure p, $P = U^2$. The dimensionless variables can be defined as :

$$\tilde{x} = x/L,\ \tilde{z} = z/H,\ \tilde{\xi} = \xi/H,\ \tilde{b} = b/H,$$

$$\tilde{u} = u/U,\ \tilde{w} = w/W,\ \tilde{t} = t/T,\ \tilde{p} = p/P,$$

$$\tilde{u_i} = u_i/U,\ \tilde{u_r} = u_r/U,\ \tilde{w_i} = w_i/W,\ \tilde{w_r} = W_r/U,$$

$$\tilde{\alpha} = \frac{\alpha}{U},\ \tilde{\beta} = \frac{\beta}{LU^2},\ \tilde{\gamma} = \frac{\gamma}{LU^2}.$$

The geometric scaling parameter is defined by :

$$\varepsilon = \frac{H}{L} = \frac{W}{U}.$$

In the shallow water asymptotic regime considered here, note that one can assume $\varepsilon << 1$. This fundamental hypothesis will of course play a crucial role in the sequel. The free surface $z = \xi$ becomes $\tilde{z} = \tilde{\xi}$ after scaling. Some dimensionless numbers are also introduced : the Reynolds number $Re = UL/\mu$ and the inverse of the Reynolds number $\nu = \mu/(UL)$. Moreover, an easy computation shows that, formally $\partial_x = \varepsilon\partial_{\tilde{x}}$, so that

$$\mathbf{n}_b(t,x) = \frac{1}{\sqrt{1+(\partial_x b)^2}}\begin{pmatrix} -\partial_x b \\ 1 \end{pmatrix} = \frac{1}{\sqrt{1+\varepsilon^2(\partial_{\tilde{x}}\tilde{b})^2}}\begin{pmatrix} -\varepsilon\partial_{\tilde{x}}\tilde{b} \\ 1 \end{pmatrix} = \begin{pmatrix} -\varepsilon\partial_{\tilde{x}}\tilde{b} \\ 1 \end{pmatrix} + \mathcal{O}(\varepsilon^2)$$

$$\mathbf{n}_\xi(t,x) = \frac{1}{\sqrt{1+(\partial_x \xi)^2}}\begin{pmatrix} -\partial_x \xi \\ 1 \end{pmatrix} = \frac{1}{\sqrt{1+\varepsilon^2(\partial_{\tilde{x}}\tilde{\xi})^2}}\begin{pmatrix} -\varepsilon\partial_{\tilde{x}}\tilde{\xi} \\ 1 \end{pmatrix} = \begin{pmatrix} -\varepsilon\partial_{\tilde{x}}\tilde{\xi} \\ 1 \end{pmatrix} + \mathcal{O}(\varepsilon^2)$$

We also introduced the rescaled indicator function as $\varphi(t,x,z) = \tilde{\varphi}(\tilde{t},\tilde{x},\tilde{z})$ and the rescaled deviatoric stress tensor $\tilde{\sigma}$ is

$$\tilde{\sigma} = \frac{1}{U^2}\sigma = \begin{pmatrix} 2\nu\partial_{\tilde{x}}\tilde{u} & \frac{\nu}{\varepsilon}\partial_{\tilde{z}}\tilde{u} + \nu\varepsilon\partial_{\tilde{x}}\tilde{w} \\ \frac{\nu}{\varepsilon}\partial_{\tilde{z}}\tilde{u} + \nu\varepsilon\partial_{\tilde{x}}\tilde{w} & 2\nu\partial_{\tilde{z}}\tilde{w} \end{pmatrix}.$$

We rescale also the gravity $\tilde{g} = gH/U^2$. Dropping the " $\tilde{}$ " for the sake of simplicity, one can write the nondimensionalized version of (1.3) as follows

$$\begin{cases} \partial_x u + \partial_z w = 0, & \text{(2.20a)} \\ \partial_t u + \partial_x(u^2) + \partial_z(uw) = -\partial_x p + 2\nu\partial_x^2 u + \dfrac{\nu}{\varepsilon^2}\partial_z^2 u + \nu\partial_{xz}^2 w, & \text{(2.20b)} \\ \varepsilon^2(\partial_t w + \partial_x(uw) + \partial_z(w^2)) = -\partial_z p + \nu\partial_{xz}^2 u + \nu\varepsilon^2\partial_x^2 w + 2\nu\partial_z^2 w - g. & \text{(2.20c)} \end{cases}$$

Observe that our change of variables leave the equation on ϕ unchanged, that is

$$\partial_t\varphi + \nabla\cdot(\mathbf{U}\varphi) = \phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi. \tag{2.21}$$

The kinematic condition (2.7) can be rewritten into

$$\partial_t\xi + u\partial_x\xi - w_{|z=\xi} = 0, \tag{2.22}$$

while the the continuity of the fluid stress tensor at the free surface becomes

$$\begin{cases} (-2\nu\partial_x u + p)\partial_x\xi + \dfrac{\nu}{\varepsilon^2}\partial_z u + \nu\partial_x w = \gamma\varepsilon\dfrac{\partial_x^2\xi}{(1+\varepsilon^2(\partial_x\xi)^2)^{\frac{3}{2}}}\partial_x\xi, & \text{(2.23a)} \\ -(\nu\partial_z u + \varepsilon^2\nu\partial_x w)\partial_x\xi + 2\nu\partial_z w - p = -\gamma\varepsilon\dfrac{\partial_x^2\xi}{(1+\varepsilon^2(\partial_x\xi)^2)^{\frac{3}{2}}}. & \text{(2.23b)} \end{cases}$$

Again, in this context, the equation (2.5) describing the evolution of the bottom in the most general situation (that is by considering a chemical reactive zone) is not affected by the use of dimensionless variables and can be recalled here

$$\frac{\partial b}{\partial t} + u_{|z=b}\frac{\partial b}{\partial x} - w_{|z=b} = \phi_r\big(\frac{\partial b}{\partial x}(u_i - u_r) - (w_i - w_r)\big). \tag{2.24}$$

This is not the case for the Generalized Navier Boundary Condition (2.6) which becomes,

$$\begin{aligned}&-4\varepsilon\nu\partial_x u\partial_x b+\frac{\nu}{\varepsilon}\big(1-\varepsilon^2(\partial_x b)^2\big)\partial_z u+\nu\varepsilon\big(1-\varepsilon^2(\partial_x b)^2\big)\partial_x w\\&=\sqrt{1+\varepsilon^2(\partial_x b)^2}\Big(\alpha(u+\varepsilon^2\partial_x b)+\beta(\partial_x\phi+\partial_z\phi\partial_x b)(\cos(\theta_{eq})-\cos(\theta))\Big).\end{aligned} \tag{2.25}$$

2.3.2 The vertically averaged nondimensionalized system

In view of obtaining simple asymptotics models, we first derive in this section an intermediate system for quantities that have been vertically averaged. In order to be more precise, we first introduce some notations. For a generic function f depending on (t, x, z), we denote by $\bar{f}$ its vertical average

$$\bar{f}(t,x)=\frac{1}{h(t,x)}\int_b^\xi f(t,x,\eta)d\eta,$$

where $h(t, x) = \xi(t, x) - b(t, x)$. We integrate the mass conservation equation (2.21) from $b(t, x)$ to $\xi(t, x)$ to obtain

$$\int_b^\xi\Big(\partial_t\varphi+\nabla\cdot(\varphi\mathbf{U})\Big)dz=\int_b^\xi\Big(\phi_r(\mathbf{U}_i-\mathbf{U}_r)\cdot\nabla\varphi\Big)dz.$$

Using the Leibniz rule, we then derive

$$\partial_t h+\partial_x(h\overline{u})-(\partial_t\xi+u\partial_x\xi-w)_{|z=\xi}+(\partial_t b+u\partial_x b-w)_{|z=b}=\int_b^\xi\Big(\phi_r(\mathbf{U}_i-\mathbf{U}_r)\cdot\nabla\varphi\Big)dz$$

Using Equation (2.22) and (2.24), one can write

$$\partial_t h+\partial_x(h\overline{u})=\int_b^\xi\Big(\phi_r(\mathbf{U}_i-\mathbf{U}_r)\cdot\nabla\varphi\Big)dz+\phi_r\big(\frac{\partial b}{\partial x}(u_i-u_r)-(w_i-w_r)\big) \tag{2.26}$$

Remark that in the cases 1, 2, 3 and 4, that is without taking into account a reactive chemical zone, equation (2.26) becomes

$$\partial_t h+\partial_x(h\overline{u})=0. \tag{2.27}$$

Applying the same procedure to (2.20b), one gets

$$\begin{aligned} &\partial_t \int_b^\xi u dz + \partial_x \int_b^\xi u^2 dz - (u\partial_t \xi + u^2 \partial_x \xi - uw)_{|z=\xi} \\ +&(u\partial_t b + u^2\partial_x b - uw)_{|z=b} = -\partial_x \int_b^\xi p dz + 2\nu \partial_x \int_b^\xi \partial_x u dz \\ &-(-p\partial_x \xi + 2\nu\partial_x u \partial_x \xi - \frac{\nu}{\varepsilon^2}\partial_z u - \nu \partial_x w)_{|z=\xi} \\ &+(-p\partial_x b + 2\nu \partial_x u \partial_x b - \frac{\nu}{\varepsilon^2}\partial_z u - \nu\partial_x w)_{|z=b}. \end{aligned} \tag{2.28}$$

We now derive some asymptotics expansions on the velocities u and w. Observe first that from (2.20b) and (2.20c), one obtains directly

$$\begin{cases} \partial_z^2 u = \mathcal{O}(\varepsilon^2), & (2.29a) \\ \partial_z p = \nu \partial_{xz}^2 u + 2\nu \partial_z^2 w - g + \mathcal{O}(\varepsilon^2). & (2.29b) \end{cases}$$

From (2.23a), one has

$$\partial_z u(t,x,\xi) = \mathcal{O}(\varepsilon^2), \tag{2.30}$$

from which we deduce that

$$\partial_z u(t,x,z) = \partial_z u(t,x,\xi) - \int_z^\xi \partial_z^2 u(t,x,y) dy = \mathcal{O}(\varepsilon^2). \tag{2.31}$$

Integrating (2.31) from respectively b to z and z to ξ, one derives

$$\begin{cases} u(t,x,z) = u(t,x,b) + \mathcal{O}(\varepsilon^2), & (2.32a) \\ u(t,x,z) = u(t,x,\xi) + \mathcal{O}(\varepsilon^2)., & (2.32b) \end{cases}$$

which provides

$$\begin{cases} u(t,x,b) = \overline{u}(t,x) + \mathcal{O}(\varepsilon^2), & (2.33a) \\ u(t,x,\xi) = \overline{u}(t,x) + \mathcal{O}(\varepsilon^2), & (2.33b) \\ u(t,x,z) = \overline{u}(t,x) + \mathcal{O}(\varepsilon^2). & (2.33c) \end{cases}$$

In the next lemma, we make a link between the derivative of u with respect to x and its mean value.

Lemme 7. *The following expansions with respect to ε hold true*

$$\partial_x u(t,x,z) = \overline{\partial_x u}(t,x) + \mathcal{O}(\varepsilon^2), \tag{2.34}$$

$$\overline{\partial_x u}(t,x) = \partial_x \overline{u}(t,x) + \mathcal{O}(\varepsilon^2), \tag{2.35}$$

$$\overline{u^2}(t,x) = \overline{u}^2(t,x) + \mathcal{O}(\varepsilon^2), \tag{2.36}$$

$$w(t,x,z) = w(t,x,b) - \partial_x \overline{u}(t,x)(z-b) + \mathcal{O}(\varepsilon^2). \tag{2.37}$$

Proof. We first apply ∂_x on Equation (2.20b), considering that u and w are regular, to obtain

$$\partial_{tx} u + \partial_x^2(u^2) + \partial_{zx}^2(uw) = -\partial_x^2 p + 2\nu\partial_x^3 u + \frac{\nu}{\varepsilon^2}\partial_z^2(\partial_x u) + \nu\partial_x^2(\partial_z w),$$

from which we deduces that

$$\partial_z^2(\partial_x u) = \mathcal{O}(\varepsilon^2). \tag{2.38}$$

Applying ∂_x on Equation (2.23a), we have

$$(-\partial_x p + 2\nu\partial_x^2 u)_{|z=\xi}(-\partial_x \xi) + (-p + 2\nu\partial_x u)_{|z=\xi}(-\partial_x^2 \xi) + \frac{\nu}{\varepsilon^2}\partial_{xz}^2 u_{|z=\xi} + \nu\partial_x^2 w_{|z=\xi} = \mathcal{O}(\varepsilon^2)$$

from which we deduce

$$\partial_{xz}^2 u(t,x,\xi) = \mathcal{O}(\varepsilon^2). \tag{2.39}$$

Gathering relations (2.38) and (2.39), one derives

$$\partial_{xz}^2 u(t,x,z) = \partial_{xz}^2 u(t,x,\xi) - \int_z^\xi \partial_z^2(\partial_x u)(t,x,z')dz') = \mathcal{O}(\varepsilon^2),$$

and

$$\partial_x u(t,x,z) = \partial_x u(t,x,\xi) - \int_z^\xi \partial_{xz}^2 u(t,x,z')dz' = \partial_x u(t,x,\xi) + \mathcal{O}(\varepsilon^2).$$

As a consequence, one can compute

$$\begin{aligned}\overline{\partial_x u}(t,x) &= \frac{1}{h(t,x)}\int_b^\xi \partial_x u(t,x,z)dz = \frac{1}{h(t,x)}\int_b^\xi \partial_x u(t,x,\xi)dz + \mathcal{O}(\varepsilon^2) \\ &= \partial_x u(t,x,\xi) + \mathcal{O}(\varepsilon^2) = \partial_x u(t,x,z) + \mathcal{O}(\varepsilon^2),\end{aligned}$$

which proves (2.34).

Moreover, by Leibniz's rule,

$$\begin{aligned}
\overline{\partial_x u}(t,x) &= \frac{1}{h(t,x)}\int_b^\xi \partial_x u(t,x,z)dz \\
&= \frac{1}{h(t,x)}\Big(\partial_x \int_b^\xi u(t,x,z)dz - (\partial_x \xi)u(t,x,\xi) + (\partial_x b)u(t,x,b)\Big) \\
&= \frac{1}{h(t,x)}\Big(\partial_x (h\overline{u})(t,x) - \overline{u}(t,x,z)\partial_x h\Big) + \mathcal{O}(\varepsilon^2), \text{ using (2.32a) and (2.32b)} \\
&= \partial_x \overline{u} + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

which proves (2.35). Furthermore, using (2.32a) and (2.33a), one has

$$\frac{1}{h(t,x)}\int_b^\xi u^2(t,x,z')dz' = \frac{1}{h(t,x)}\overline{u}(t,x)\int_b^\xi u(t,x,z')dz' + \mathcal{O}(\varepsilon^2) = \overline{u}^2(t,x) + \mathcal{O}(\varepsilon^2).$$

Finally, from (2.20a), one can write

$$\begin{aligned}
w(t,x,z) &= w(t,x,z) + \int_b^z \partial_z w(t,x,z')dz' \\
&= w(t,x,b) - \int_b^z \partial_u(t,x,z')dz' = w(t,x,b) - \partial_x \overline{u}(t,x)(z-b) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

by (2.34), which ends the proof of Lemma 7. □

2.3.3 Presentation of the Shallow water models

In this section, we derive the different Shallow water models corresponding to case 1 up to case 5 (see Section 2.2 for a complete description). For the sake of clearness, we begin with case 1 and we refer to [22] and [43] and for the explanation of the usual asymtptotic method in this framework. Observe first that equation (2.27) is the first equation of the system we derive in cases 1, 2, 3 and 4.

Case 1 : flat bottom without capillary penetration. In order to close the system, we have to derive a second equation using Equation (2.28) which becomes, in the context of case

1, recalling that $b = 0$,

$$\begin{aligned}\partial_t \int_b^\xi u dz + \partial_x \int_b^\xi u^2 dz = &- \partial_x \int_b^\xi p dz + 2\nu\partial_x \int_b^\xi \partial_x u dz \\ &- (-p\partial_x\xi + 2\nu\partial_x u\partial_x\xi - \frac{\nu}{\varepsilon^2}\partial_z u - \nu\partial_x w)_{|z=\xi} \\ &+ (-\frac{\nu}{\varepsilon^2}\partial_z u - \nu\partial_x w)_{|z=b}. \end{aligned} \tag{2.40}$$

Now, we recall the dimensionless boundary conditions (2.23a)-(2.23b) on the free surface $z = \xi(t, x)$

$$\begin{cases} (p - 2\nu\partial_x u)(\partial_x\xi) + \frac{\nu}{\varepsilon^2}\partial_z u + \nu\partial_x w = \gamma\varepsilon\partial_x^2\xi\partial_x\xi + \mathcal{O}(\varepsilon^2), \\ -(\nu\partial_z u + \varepsilon^2\nu\partial_x w)\partial_x\xi + (-p + 2\nu\partial_z w) = -\gamma\varepsilon\partial_x^2\xi + \mathcal{O}(\varepsilon^2). \end{cases}$$

As a direct consequence, (2.41) becomes, taking into account (2.25)

$$\partial_t \int_b^\xi u dz + \partial_x \int_b^\xi u^2 dz = -\partial_x \int_b^\xi p dz + 2\nu\partial_x \int_b^\xi \partial_x u dz + \gamma\varepsilon\partial_x^2\xi\partial_x\xi - \frac{\alpha}{\varepsilon}u_{|z=b} + \mathcal{O}(\varepsilon^2). \tag{2.41}$$

It remains to derive an asymptotic expansion on the pressure p. By the incompressibility condition (2.20a), one has $\partial_x u = -\partial_z w$ and using (2.30), one derives an asymptotic value for the pressure p on the top of the fluid

$$p(t, x, \xi) = -2\nu\partial_x u(t, x, \xi) + \gamma\varepsilon\partial_x^2\xi + \mathcal{O}(\varepsilon^2). \tag{2.42}$$

As a consequence, we obtain the following expression for the pressure p at any altitude z

$$\begin{aligned} p(t, x, z) &= p(t, x, \xi) - \int_z^\xi \partial_z p(t, x, z')dz' \\ &= -2\nu\partial_x u(t, x, \xi) + \gamma\varepsilon\partial_x^2\xi - \int_z^\xi \big(\nu\partial_{zx}^2 u + 2\nu\partial_z^2 w - g\big)(t, x, z')dz' + \mathcal{O}(\varepsilon^2) \text{ by (2.29b)} \\ &= -3\nu\partial_x u(t, x, \xi) + \gamma\varepsilon\partial_x^2\xi + \nu\partial_x u(t, x, z) - 2\nu\big(\partial_z w(t, x, \xi) - \partial_z w(t, x, z)\big) \\ &\quad + g(\xi - z) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{2.43}$$

Again using (2.20a), one obtains from Lemma 7

$$\begin{aligned}\partial_z w(t,x,\xi) - \partial_z w(t,x,z) &= -\partial_x u(t,x,\xi) + \partial_x u(t,x,z)\\ &= \overline{\partial_x u}(t,x) - \overline{\partial_x u}(t,x) + \mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^2),\end{aligned} \tag{2.44}$$

from which we deduce

$$p(t,x,z) = -3\nu\partial_x u(t,x,\xi) + \gamma\varepsilon\partial_x^2\xi + \nu\partial_x u(t,x,z) + g(\xi - z) + \mathcal{O}(\varepsilon^2). \tag{2.45}$$

We are now able to compute

$$\begin{aligned}\int_b^\xi p(t,x,z')dz &= \int_b^\xi \big(-3\nu\partial_x u(t,x,\xi) + \gamma\varepsilon\partial_x^2\xi + \nu\partial_x u(t,x,z') + g(\xi - z')\big)dz' + \mathcal{O}(\varepsilon^2)\\ &= -3\nu h\partial_x u(t,x,\xi) + \gamma\varepsilon h\partial_x^2\xi + \nu h\overline{\partial_x u}(t,x) + \frac{gh^2}{2} + \mathcal{O}(\varepsilon^2)\\ &= -2\nu h\partial_x\overline{u} + \frac{gh^2}{2} + \gamma\varepsilon h\partial_x^2\xi + \mathcal{O}(\varepsilon^2) \text{ by Lemma 7,}\end{aligned} \tag{2.46}$$

which provides

$$\partial_x \int_b^\xi p dz = -2\nu\partial_x(h\partial_x\overline{u}) + \gamma\varepsilon\partial_x(h\partial_x^2\xi) + \frac{g\partial_x h^2}{2} + \mathcal{O}(\varepsilon^2).$$

Introduce $\alpha_1 = \alpha/(\varepsilon)$ and $\gamma_1 = \gamma\varepsilon$, Equation (2.41) becomes, using (2.33a) and (2.36),

$$\partial_t(h\overline{u}) + \partial_x(h\overline{u}^2) + \gamma_1 h\partial_x^3\xi - 4\nu\partial_x(h\partial_x\overline{u}) + \frac{g\partial_x h^2}{2} + \alpha_1\overline{u} = \mathcal{O}(\varepsilon^2) \tag{2.47}$$

As the Shallow water regime corresponds to an approximation up to ε^2, one finally obtains the first asymptotic model, fitting the boundary conditions of case 1

$$(\mathbf{SW1}) \quad \begin{cases}\partial_t h + \partial_x(h\overline{u}) = 0,\\ \partial_t(h\overline{u}) + \partial_x(h\overline{u}^2) + \gamma_1 h\partial_x^3\xi - 4\nu\partial_x(h\partial_x\overline{u}) + \dfrac{g\partial_x h^2}{2} + \alpha_1\overline{u} = 0.\end{cases} \tag{2.48}$$

Case 2 : flat bottom with capillary penetration. In this case, Conditions (2.24) and

(2.25) at the bottom become

$$w(t,x,b) = 0, \tag{2.49}$$

$$\tfrac{\nu}{\varepsilon}\partial_z u_{|z=b} + \nu\varepsilon\partial_x w_{|z=b} = \alpha u_{|z=b} + \beta\partial_x\phi(\cos(\theta_{eq}) - \cos(\theta)), \tag{2.50}$$

where φ solves Equation (2.14). The continuity of the fluid stress tensor on the free surface $z = \xi(t,x)$ (2.23a)-(2.23b) can be rewritten as

$$\begin{cases} (-2\nu\partial_x u + p)\partial_x\xi + \dfrac{\nu}{\varepsilon^2}\partial_z u + \nu\partial_x w = \gamma\varepsilon\partial_x^2\xi\partial_x\xi + \mathcal{O}(\varepsilon^2) & \text{(2.51a)} \\ -(\nu\partial_z u + \varepsilon^2\nu\partial_x w)\partial_x\xi + 2\nu\partial_z w - p = -\gamma\varepsilon\partial_x^2\xi + \mathcal{O}(\varepsilon^2), & \text{(2.51b)} \end{cases}$$

from which we deduce that, using (2.20a)

$$\begin{cases} \partial_z u(t,x,\xi) = \mathcal{O}(\varepsilon^2), & \text{(2.52a)} \\ p(t,x,\xi) = -2\nu\partial_x u(t,x,\xi) + \gamma\varepsilon\partial_x^2\xi(t,x) + \mathcal{O}(\varepsilon^2) & \text{(2.52b)} \end{cases}$$

In this situation, Equation (2.28) becomes

$$\begin{aligned} \partial_t \int_b^\xi u dz + \partial_x \int_b^\xi u^2 dz = & - \partial_x \int_b^\xi p dz + 2\nu\partial_x \int_b^\xi \partial_x u dz + \gamma\varepsilon\partial_x^2\xi\partial_x\xi \\ & - (\alpha u + \beta\partial_x\phi(\cos(\theta_{eq}) - \cos(\theta))_{|z=b} + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{2.53}$$

We now derive the asymptotic expansion on p, using (2.52b) and (2.29b), as follows

$$\begin{aligned} p(t,x,z) &= p(t,x,\xi) - \int_z^\xi \partial_z p(t,x,z')dz' \\ &= -2\nu\partial_x u(t,x,\xi) + \gamma\varepsilon\partial_x^2\xi(t,x) - \int_z^\xi \big(\nu\partial_{zx}^2 u + 2\nu\partial_z^2 w - g\big)(t,x,z')dz' + \mathcal{O}(\varepsilon^2) \\ &= -3\nu\partial_x u(t,x,\xi) + \nu\partial_x u(t,x,z) - 2\nu\big(\partial_z w(t,x,\xi) - \partial_z w(t,x,z)\big) \\ &\quad + g(\xi - z) + \gamma\varepsilon\partial_x^2\xi(t,x) + \mathcal{O}(\varepsilon^2), \qquad\qquad \text{(2.54)} \\ &= -3\nu\partial_x u(t,x,\xi) + \nu\partial_x u(t,x,z) + g(\xi - z) + \gamma\varepsilon\partial_x^2\xi(t,x) + \mathcal{O}(\varepsilon^2), \qquad \text{(2.55)} \end{aligned}$$

by (2.44). We deduce that

$$\begin{aligned}\int_b^\xi p(t,x,z')dz &= \int_b^\xi \big(-3\nu\partial_x u(t,x,\xi) + \nu\partial_x u(t,x,z') + g(\xi - z') + \gamma\varepsilon\partial_x^2\xi(t,x)\big)dz' + \mathcal{O}(\varepsilon^2)\\ &= -3\nu h\partial_x u(t,x,\xi) + \nu h\overline{\partial_x u}(t,x) + \frac{gh^2}{2} + \gamma\varepsilon h\partial_x^2\xi(t,x) + \mathcal{O}(\varepsilon^2)\\ &= -2\nu h\partial_x\overline{u} + \frac{gh^2}{2} + \gamma\varepsilon h\partial_x^2\xi(t,x) + \mathcal{O}(\varepsilon^2) \text{ by Lemma 7,}\end{aligned} \tag{2.56}$$

and

$$\partial_x\int_b^\xi pdz = -2\nu\partial_x(h\partial_x\overline{u}) + \frac{g\partial_x h^2}{2} + \gamma\varepsilon\partial_x\big(h\partial_x^2\xi(t,x)\big) + \mathcal{O}(\varepsilon^2). \tag{2.57}$$

Furthermore, since $b = 0$, one can notice that $\partial_x h = \partial_x\xi$. Then, collecting (2.53) and (2.57), we obtain

$$\begin{aligned}\partial_t\int_b^\xi udz + \partial_x\int_b^\xi u^2dz = {}& 2\nu\partial_x(h\partial_x\overline{u}) - \frac{g\partial_x h^2}{2} - \gamma\varepsilon\partial_x\big(h\partial_x^2\xi(t,x)\big) + \gamma\varepsilon\partial_x^2\xi\partial_x\xi\\ &+ 2\nu\partial_x\int_b^\xi \partial_x udz - (\alpha u + \beta\partial_x\phi(\cos(\theta_{eq}) - \cos(\theta))_{|z=b}\\ &+ \mathcal{O}(\varepsilon^2),\\ = {}& 4\nu\partial_x(h\partial_x\overline{u}) - \frac{g\partial_x h^2}{2} - (\alpha u + \beta\partial_x\phi(\cos(\theta_{eq}) - \cos(\theta))_{|z=b}\\ &- \gamma\varepsilon h\partial_x^3 h + \mathcal{O}(\varepsilon^2)\end{aligned} \tag{2.58}$$

Recalling that $\gamma_1 = \gamma\varepsilon$ and using (2.33a), we conclude that $\overline{u}$ satisfies

$$\partial_t(h\overline{u}) + \partial_x(h\overline{u}^2) + \gamma_1 h\partial_x^3 h - 4\nu\partial_x(h\partial_x\overline{u}) + \frac{g\partial_x h^2}{2} + \alpha_1\overline{u} \tag{2.59}$$

$$= -\beta\partial_x\phi(t,x,b)\big(\cos(\theta_{eq}) - \cos(\theta)\big) + \mathcal{O}(\varepsilon^2) \tag{2.60}$$

We finally obtain, since $w(t,x,b) = 0$ and using (2.21), (2.33c), (2.36) and (2.37) the following system

$$(\mathbf{SW2}) \quad \begin{cases} \partial_t h + \partial_x(h\overline{u}) = 0, \\ \partial_t(h\overline{u}) + \partial_x(h\overline{u}^2) + \gamma_1 h\partial_{xxx}h - 4\nu\partial_x(h\partial_x\overline{u}) + \dfrac{g\partial_x h^2}{2} + \alpha\overline{u} \\ \quad + \beta\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta) = 0, \\ \partial_t\varphi + \partial_x(\overline{u}\varphi) - \partial_z\big((z-b)\partial_x\overline{u}\varphi\big) = 0. \end{cases} \tag{2.61}$$

Case 3 : bumpy bottom without capillary penetration. In this case, equation (2.24) becomes

$$\partial_t b + u_{|z=b}\partial_x b - w_{|z=b} = 0, \tag{2.62}$$

while (2.25) is, taking into account (2.20a) and (2.30),

$$-4\nu\partial_x u_{|z=b}\partial_x b + \frac{\nu}{\varepsilon^2}\partial_z u_{|z=b} + \nu\partial_x w_{|z=b} + = \alpha_1 u_{|z=b}, \tag{2.63}$$

where α_1 is a rescaled coefficient and is equal to

$$\alpha_1 = \frac{\alpha}{\varepsilon}.$$

Equation (2.28) can be written, using (2.22), (2.62) and (2.63), into the form

$$\begin{aligned} \partial_t \int_b^\xi u dz + \partial_x \int_b^\xi u^2 dz = -\partial_x \int_b^\xi p dz + 2\nu\partial_x \int_b^\xi \partial_x u dz \\ +\gamma_1\partial_x^2\xi\partial_x\xi - (p\partial_x b - 2\nu\partial_x u\partial_x b - \alpha_1 u)_{|z=b}, \end{aligned} \tag{2.64}$$

where we recall that $\gamma_1 = \gamma\varepsilon$. By (2.45), one has

$$p(t,x,b) = -3\nu\partial_x u(t,x,\xi) + \nu\partial_x u(t,x,b) + gh + \mathcal{O}(\varepsilon^2),$$

from which we deduce that

$$p\partial_x b - 2\nu\partial_x u\partial_x b = \big(-3\nu\partial_x u(t,x,\xi) + 3\nu\partial_x u(t,x,b)\big)\partial_x b + gh\partial_x b + \mathcal{O}(\varepsilon^2).$$

Using Lemma 7, one derives,

$$p\partial_x b - 2\nu\partial_x u\partial_x b = gh\partial_x b + \mathcal{O}(\varepsilon^2).$$

As a consequence, one can rewrite (2.64) in the following way

$$\partial_t \int_b^\xi udz + \partial_x \int_b^\xi u^2 dz = -\partial_x \int_b^\xi pdz + 2\nu\partial_x \int_b^\xi \partial_x udz \\ + \gamma_1 \partial_x^2 \xi \partial_x \xi - gh\partial_x b - \alpha_1 u + \mathcal{O}(\varepsilon^2). \tag{2.65}$$

Collecting (2.57) and (2.65), one obtains, since $\xi = h + b$,

$$\partial_t(h\overline{u}) + \partial_x(h\overline{u}^2) + \gamma_1 h\partial_x^3(h+b) - 4\nu\partial_x(h\partial_x\overline{u}) + \frac{g\partial_x h^2}{2} + gh\partial_x b\alpha_1\overline{u} = \mathcal{O}(\varepsilon^2). \tag{2.66}$$

Thus, the final model is, collecting (2.27) and (2.66),

$$(\mathbf{SW3}) \quad \begin{cases} \partial_t h + \partial_x(h\overline{u}) = 0, \\ \partial_t(h\overline{u}) + \partial_x(h\overline{u}^2) + \gamma_1 h\partial_x^3(h+b) - 4\nu\partial_x(h\partial_x\overline{u}) + \dfrac{g\partial_x h^2}{2} + gh\partial_x b + \alpha_1\overline{u} = 0, \end{cases} \tag{2.67}$$

Case 4 : bumpy bottom with capillary penetration. Equations (2.5) and (2.25) are now

$$\partial_t b + u_{|z=b}\partial_x b - w_{|z=b} = 0, \tag{2.68}$$

$$-4\nu\partial_x u_{|z=b}\partial_x b + \tfrac{\nu}{\varepsilon^2}\partial_z u_{|z=b} + \nu\partial_x w_{|z=b} \\ = \alpha_1 u_{|z=b} + \beta(\partial_x\phi + \partial_z\varphi\partial_x b)_{|z=b}(\cos(\theta_{eq}) - \cos(\theta)), \tag{2.69}$$

while Equation (2.28) takes the form

$$\partial_t \int_b^\xi udz + \partial_x \int_b^\xi u^2 dz = -\partial_x \int_b^\xi pdz + 2\nu\partial_x \int_b^\xi \partial_x udz + \gamma_1\partial_x^2\xi\partial_x\xi \\ -\big(p\partial_x b - 2\nu\partial_x u\partial_x b - \alpha_1 u - \beta_1(\partial_x\phi + \partial_z\varphi\partial_x b)(\cos(\theta_{eq}) - \cos(\theta))\big)_{|z=b}, \tag{2.70}$$

where β_1 is a rescaled coefficient

$$\beta_1 = \frac{\beta}{\varepsilon}.$$

We mix the procedure used for case 2 and case 3. However, one has to pay attention to

(2.37) since in this situation, the property (2.49) is no longer valid and has to be replaced by (2.68). More precisely, one has, by (2.33a) and (2.37)

$$\begin{aligned} w(t,x,z) &= w(t,x,b) - \partial_x \overline{u}(t,x)(z-b) + \mathcal{O}(\varepsilon^2) \\ &= \partial_t b(t,x) + \partial_x b(t,x)\overline{u}(t,x) - \partial_x \overline{u}(t,x)(z-b) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

from which we deduce that

$$(\mathbf{SW4}) \quad \begin{cases} \partial_t h + \partial_x(h\overline{u}) = 0, \\ \partial_t(h\overline{u}) + \partial_x(h\overline{u}^2) + \gamma_1 h\partial_{xxx}(h+b) - 4\nu\partial_x(h\partial_x\overline{u}) + \dfrac{g\partial_x h^2}{2} \\ + gh\partial_x b + \alpha_1\overline{u} + \beta_1(\partial_x\varphi(t,x,b) + \partial_z\varphi(t,x,b)\partial_x b)(\cos\theta_{eq} - \cos\theta) = 0, \\ \partial_t\varphi + \partial_x(\overline{u}\varphi) + \partial_z\big((\partial_t b(t,x) + \partial_x b(t,x)\overline{u}(t,x) - \partial_x\overline{u}(t,x)(z-b))\varphi\big) = 0. \end{cases} \tag{2.71}$$

Case 5 : bumpy bottom with capillary penetration and a chemical reactive zone. This is the most complete situation treated in this paper, that is we have to use the full boundary conditions (2.24) and (2.25) on the bottom that we recall here

$$\begin{aligned} &\frac{\partial b}{\partial t} + u_{|z=b}\frac{\partial b}{\partial x} - w_{|z=b} = \phi_r\big(\partial_x b(u_i - u_r) - (w_i - w_r)\big), \\ &- 4\nu\partial_x u\partial_x b + \frac{\nu}{\varepsilon^2}\partial_z u + \nu\partial_x w = \alpha_1 u + \beta_1(\partial_x\phi + \partial_z\phi\partial_x b)(\cos(\theta_{eq}) - \cos(\theta)). \end{aligned} \tag{2.72}$$

Let us mentionned that the evolution of φ is governed by Equation (2.21) and the conservation of mass leads to Equation (2.26), which is the first equation of our system

$$\partial_t h + \partial_x(h\overline{u}) = \int_b^\xi \Big(\phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi\Big)dz + \phi_r\big(\partial_x b(u_i - u_r) - (w_i - w_r)\big).$$

Equation (2.28) becomes in this context

$$\partial_t \int_b^\xi u dz + \partial_x \int_b^\xi u^2 dz + \phi_r u_{|z=b}\big(\partial_x b(u_i - u_r) - (w_i - w_r)\big) \tag{2.73}$$

$$= -\partial_x \int_b^\xi p dz + 2\nu\partial_x \int_b^\xi \partial_x u dz + \gamma_1\partial_x^2\xi\partial_x\xi \tag{2.74}$$

$$- \big(p\partial_x b - 2\nu\partial_x u\partial_x b - \alpha_1 u - \beta_1(\partial_x\phi + \partial_z\varphi\partial_x b)(\cos(\theta_{eq}) - \cos(\theta))\big)_{|z=b}. \tag{2.75}$$

Since $\phi_r = 1$ on $z = b(t,x)$ and by (2.33a), $u(t,x,b) = \overline{u}(t,x) + \mathcal{O}(\varepsilon^2)$, one derives from (2.73),

$$\partial_t(h\overline{u}) + \partial_x(h\overline{u}^2) + \overline{u}(u_i - u_r)_{|z=b}\partial_x b - \overline{u}(w_i - w_r)_{|z=b} + \gamma_1 h\partial_{xxx}(h+b) - 4\nu\partial_x(h\partial_x\overline{u}) + \frac{g\partial_x h^2}{2} + gh\partial_x b + \alpha_1\overline{u} + \beta_1\partial_x(\varphi(t,x,b))(\cos\theta_{eq} - \cos\theta) = \mathcal{O}(\varepsilon^2).$$

Moreover, taking into account (2.24), one has

$$\begin{aligned} w(t,x,z) &= w(t,x,b) - \partial_x\overline{u}(t,x)(z-b) + \mathcal{O}(\varepsilon^2) \\ &= \partial_t b(t,x) + \partial_x b(t,x)\overline{u}(t,x) - \phi_r\big(\partial_x(u_i - u_r) - (w_i - w_r)\big) \\ &\quad - \partial_x\overline{u}(t,x)(z-b) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

which leads to the final system

$$(\mathbf{SW5}) \quad \begin{cases} \partial_t h + \partial_x(h\overline{u}) = \displaystyle\int_b^\xi \Big(\phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi\Big)dz + \phi_r\big(\partial_x b(u_i - u_r) - (w_i - w_r)\big) \\ \partial_t(h\overline{u}) + \partial_x(h\overline{u}^2) + \overline{u}(u_i - u_r)_{|z=b}\partial_x b - \overline{u}(w_i - w_r)_{|z=b} + \gamma_1 h\partial_{xxx}(h+b) \\ -4\nu\partial_x(h\partial_x\overline{u}) + \dfrac{g\partial_x h^2}{2} + gh\partial_x b + \alpha_1\overline{u} + \beta_1\partial_x(\varphi(t,x,b))(\cos\theta_{eq} - \cos\theta) = 0, \\ \partial_t\varphi + \partial_x(\overline{u}\varphi) + \partial_z\Big(\big(\partial_t b(t,x) + \partial_x b(t,x)\overline{u}(t,x) - \phi_r\big(\partial_x(u_i - u_r) - (w_i - w_r)\big) \\ - \partial_x\overline{u}(t,x)(z-b)\big)\varphi\Big) = \phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi. \end{cases} \tag{2.76}$$

2.4 Thin Film models

This section is devoted to the obtention of thin film models in the context discussed in Section 2.2. Compare to Section 2.3 we introduce a different scaling on the pressure p and on the coefficients α and β of the boundary conditions of the Navier-Stokes system (2.9).

2.4.1 The nondimensionalized system and asymptotic description of the velocity and the pressure.

In view of obtaining thin film models, we consider, in this section, the following scaling on the variables of System (2.9). For the sake of completeness, we recall here the scaling

$$\tilde{x} = x/L,\ \tilde{z} = z/H,\ \tilde{\xi} = \xi/H,\ \tilde{b} = b/H$$

$$\tilde{u} = u/U,\ \tilde{w} = w/W,\ \tilde{t} = t/T,\ \tilde{p} = p/P$$

$$\tilde{u_i} = u_i/U,\ \tilde{u_r} = u_r/U,\ \tilde{w_i} = w_i/W,\ \tilde{w_r} = W_r/U$$

$$\tilde{\alpha} = \frac{\alpha}{U},\ \tilde{\beta} = \frac{\beta}{LU^2},\ \tilde{\gamma} = \frac{\gamma}{LU^2}.$$

The geometric scaling parameter is still defined by :

$$\varepsilon = \frac{H}{L} = \frac{W}{U},$$

and we assume that $\varepsilon << 1$. In this section, the scaling parameter P of the pressure p is

$$P = \frac{\nu U^2}{\varepsilon^2},$$

while the gravity is rescaled as

$$\tilde{g} = \frac{H\varepsilon^2}{\nu U^2} g.$$

The rescaled indicator function is $\varphi(t, x, z) = \tilde{\varphi}(\tilde{t}, \tilde{x}, \tilde{z})$ and the deviatoric stress tensor $\tilde{\sigma}$ can be written into

$$\tilde{\sigma} = \frac{1}{U^2}\sigma = \begin{pmatrix} 2\nu\partial_{\tilde{x}}\tilde{u} & \frac{\nu}{\varepsilon}\partial_{\tilde{z}}\tilde{u} + \nu\varepsilon\partial_{\tilde{x}}\tilde{w} \\ \frac{\nu}{\varepsilon}\partial_{\tilde{z}}\tilde{u} + \nu\varepsilon\partial_{\tilde{x}}\tilde{w} & 2\nu\partial_{\tilde{z}}\tilde{w} \end{pmatrix}.$$

Dropping the "~" terms, the nondimensionalized equation (1.3) becomes :

$$\begin{cases} \partial_x u + \partial_z w = 0, & (2.77a) \\ \partial_t u + \partial_x(u^2) + \partial_z(uw) = -\frac{\nu}{\varepsilon^2}\partial_x p + 2\nu\partial_x^2 u + \frac{\nu}{\varepsilon^2}\partial_z^2 u + \nu\partial_{xz}^2 w, & (2.77b) \\ \varepsilon^2(\partial_t w + \partial_x(uw) + \partial_z(w^2)) = -\frac{\nu}{\varepsilon^2}\partial_z p + \nu\partial_{xz}^2 u + \nu\varepsilon^2\partial_x^2 w + 2\nu\partial_z^2 w - \frac{\nu}{\varepsilon^2}g. & (2.77c) \end{cases}$$

The function ϕ is still solution to (2.21)

$$\partial_t\varphi + \nabla\cdot(\mathbf{U}\varphi) = \phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi.$$

The kinematic condition (2.7) can be rewritten into

$$\partial_t\xi + u\partial_x\xi - w_{|z=\xi} = 0, \tag{2.78}$$

while the the continuity of the fluid stress tensor at the free surface becomes

$$\begin{cases} (-2\nu\partial_x u + \frac{\nu}{\varepsilon^2}p)\partial_x\xi + \frac{\nu}{\varepsilon^2}\partial_z u + \nu\partial_x w = \gamma\varepsilon\frac{\partial_x^2\xi}{(1+\varepsilon^2(\partial_x\xi)^2)^{\frac{3}{2}}}\partial_x\xi & (2.79a) \\ -(\nu\partial_z u + \varepsilon^2\nu\partial_x w)\partial_x\xi + 2\nu\partial_z w - \frac{\nu}{\varepsilon^2}p = -\gamma\varepsilon\frac{\partial_x^2\xi}{(1+\varepsilon^2(\partial_x\xi)^2)^{\frac{3}{2}}} & (2.79b) \end{cases}$$

The equation (2.5) describing the evolution of the bottom in the most general situation (that is by considering a chemical reactive zone) is not affected pas the use of dimensionless variables and can be recall here

$$\frac{\partial b}{\partial t} + u_{|z=b}\frac{\partial b}{\partial x} - w_{|z=b} = \phi_r\big(\frac{\partial b}{\partial x}(u_i - u_r) - (w_i - w_r)\big). \tag{2.80}$$

The Generalized Navier Boundary Condition (2.6) looks like

$$\begin{aligned} &-4\varepsilon\nu\partial_x u\partial_x b + \frac{\nu}{\varepsilon}\big(1-\varepsilon^2(\partial_x b)^2\big)\partial_z u + \nu\varepsilon\big(1-\varepsilon^2(\partial_x b)^2\big)\partial_x w \\ &= \sqrt{1+\varepsilon^2(\partial_x b)^2}\Big(\alpha(u+\varepsilon^2\partial_x b) + \beta(\partial_x\phi + \partial_z\phi\partial_x b)(\cos(\theta_{eq}) - \cos(\theta))\Big). \end{aligned} \tag{2.81}$$

We now have to observe that Equation (2.26), obtained with an averaged process, is still valid with our new nondimensionalized variables

$$\partial_t h + \partial_x(h\overline{u}) = \int_b^\xi \Big(\phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi\Big)dz + \phi_r\big(\frac{\partial b}{\partial x}(u_i - u_r) - (w_i - w_r)\big).$$

Remark that (2.26) is reduced to (2.27) in the cases 1,2,3 and 4

$$\partial_t h + \partial_x(h\overline{u}) = 0.$$

In lubrication framework, in order to close Equation (2.26), one has to express the velocity u in terms of $h(t,x)(t,x,z)$. For that purpose, we have to describe the asymptotic behaviour of u and p with respect to ε. First observe that, from (2.77b) and (2.77c), one obtains directly

$$\begin{cases} \partial_x p = \partial_z^2 u + \mathcal{O}(\varepsilon^2), & (2.82a)\\ \partial_z p = -g + \mathcal{O}(\varepsilon^2). & (2.82b)\end{cases}$$

Let us introduced the following new scaling on γ

$$\gamma_2 = \frac{\varepsilon^3}{\nu}\gamma,$$

so that (2.79a)-(2.79b) can be rewritten into

$$\begin{cases} (-2\nu\partial_x u + \frac{\nu}{\varepsilon^2}p)\partial_x\xi + \frac{\nu}{\varepsilon^2}\partial_z u + \nu\partial_x w = \frac{\nu}{\varepsilon^2}\gamma_2\frac{\partial_x^2\xi}{(1+\varepsilon^2(\partial_x\xi)^2)^{\frac{3}{2}}}\partial_x\xi, & (2.83a)\\ -(\nu\partial_z u + \varepsilon^2\nu\partial_x w)\partial_x\xi + 2\nu\partial_z w - \frac{\nu}{\varepsilon^2}p = -\frac{\nu}{\varepsilon^2}\gamma_2\frac{\partial_x^2\xi}{(1+\varepsilon^2(\partial_x\xi)^2)^{\frac{3}{2}}}. & (2.83b)\end{cases}$$

By (2.83b), one has

$$p_{|z=\xi} = \gamma_2\partial_x^2\xi + \mathcal{O}(\varepsilon^2), \tag{2.84}$$

from which we deduce by (2.81) that

$$\partial_z u_{|z=\xi} = \big(-p_{|z=\xi} + \gamma_2\partial_x^2\xi\big) + \mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^2). \tag{2.85}$$

Integrating (2.82b) from z to ξ, one derives

$$\begin{aligned} p(t,x,z) &= p(t,x,\xi) - \int_z^{\xi} \partial_z p(t,x,z')dz' \\ &= \gamma_2 \partial_x^2 \xi - g(\xi - z) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{2.86}$$

from which we obtain, by applying ∂_x on (2.86)

$$\partial_x p(t,x,z) = \gamma_2 \partial_x^3 \xi(t,x) - g\partial_x \xi(t,x) + \mathcal{O}(\varepsilon^2). \tag{2.87}$$

On the other hand, by (2.82a),

$$\begin{aligned} \partial_z^2 u(t,x,z) &= \partial_x p(t,x) + \mathcal{O}(\varepsilon^2) \\ &= \gamma_2 \partial_x^3 \xi(t,x) - g\partial_x \xi(t,x) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{2.88}$$

Since the right-hand side of (2.88) is independent of f, one obtains by a direct integration, that there exists two functions $A(t,x)$ and $B(t,x)$ independent of z such that

$$u(t,x,z) = \frac{\partial_x p(t,x)}{2} z^2 + A(t,x)z + B(t,x) + \mathcal{O}(\varepsilon^2). \tag{2.89}$$

Obviously, from (2.89), one has

$$\partial_z u(t,x,z) = \partial_x p(t,x) z + A(t,x) + \mathcal{O}(\varepsilon^2),$$

which can be combined with (2.85) to obtain

$$A(t,x) = -\partial_x p(t,x)\xi(t,x) + \mathcal{O}(\varepsilon^2).$$

Thus

$$\begin{aligned} u(t,x,z) &= \frac{\partial_x p(t,x)}{2} z^2 - \partial_x p \xi(t,x) z + B(t,x) + \mathcal{O}(\varepsilon^2) \\ &= \partial_x p(t,x) \frac{(\xi(t,x)-z)^2}{2} - \partial_x p \frac{\xi(t,x)^2}{2} + B(t,x) + \mathcal{O}(\varepsilon^2), \\ &= \big(\gamma_2 \partial_x^3 \xi(t,x) - g\partial_x \xi(t,x)\big)\big(\frac{(\xi(t,x)-z)^2}{2} - \frac{\xi^2(t,x)}{2}\big) + B(t,x) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{2.90}$$

The function $B(t,x)$ will be determined by using the different boundary conditions on the bottom listed in Section 2.2. This is the goal of the next section. Moreover, by (2.77a), one can obtained an asymptotic description of the velocity w. Indeed,

$$\partial_z w = -\partial_x u = -\partial_x^2 p\big(\frac{z^2}{2} - \zeta(t,x)z\big) - \partial_x B(t,x) + \mathcal{O}(\varepsilon^2),$$

from which we derive

$$\begin{aligned} w(t,x,z) &= w(t,x,b) + \int_b^z \partial_z w(t,x,z')dz' \\ &= w(t,x,b) - \int_b^z \Big(\partial_x^2 p\big(\frac{z^2}{2} - \zeta(t,x)z\big) + \partial_x B(t,x)\Big)dz' \\ &= w(t,x,b) - \partial_x^2 p\big(\frac{z^3}{6} - \frac{b^3}{6} - \xi(t,x)(\frac{z^2}{2} - \frac{b^2}{2})\big) - \partial_x B(t,x)(z-b). \end{aligned} \tag{2.91}$$

2.4.2 Thin film models

In this section, we derive the thin film models corresponding to the different boundary conditions introduced in Section 2.2.

Case 1 : flat bottom without capillary penetration. First recall that in this situation, $b = 0$ and $\xi(t,x) = h(t,x)$. We rescale the coefficient α in (2.81) by introducing

$$\alpha_2 = \frac{\varepsilon}{\nu}\alpha,$$

so that (2.81) becomes

$$\partial_z u_{|z=b} = \alpha_2 u_{|z=b} + \mathcal{O}(\varepsilon^2). \tag{2.92}$$

Collecting (2.90) and (2.92), one obtains directly

$$B(t,x) = -\frac{\partial_x p(t,x)}{\alpha_2} h(t,x) + \mathcal{O}(\varepsilon^2),$$

which furnishes the following asymptotic expansion on u

$$u(t,x,z) = \partial_x p \frac{(h-z)^2}{2} - \partial_x p \frac{h^2}{2} - \frac{\partial_x p h}{\alpha_2} + \mathcal{O}(\varepsilon^2). \tag{2.93}$$

Integrating (2.93) between $z = 0$ and $z = h(t,x)$, we obtain

$$\int_0^h u(t,x,z)dz = -\frac{h^3}{3}\partial_x p - \frac{h^2}{\alpha_2}\partial_x p + \mathcal{O}(\varepsilon^2). \tag{2.94}$$

Replacing $\partial_x p = \beta_1 \partial_{xxx} h - g\partial_x h$ into (2.94), we get

$$\begin{aligned} \int_0^h u(t,x,z)dz &= -\frac{h^3}{3}(\beta_1 \partial_{xxx} h - g\partial_x h) - \frac{h^2}{\alpha}(\beta_1 \partial_{xxx} h - g\partial_x h) + \mathcal{O}(\varepsilon^2) \\ &= -(\frac{h^3}{3} + \frac{h^2}{\alpha})(\beta_1 \partial_{xxx} h - g\partial_x h) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{2.95}$$

Plugging (2.95) into (2.27), up to order ε^2, one can write the following first thin film model composed by a single equation on the surface elevation $h(t,x)$

$$(\mathbf{TF1}) \quad \begin{cases} \partial_t h + \partial_x (R_1(h)) = 0, \\ R_1(h) = -(\frac{h^3}{3} + \frac{h^2}{\alpha})(\beta_1 \partial_{xxx} h - g\partial_x h). \end{cases} \tag{2.96}$$

Case 2 : flat bottom with penetration condition. We still considerer that $b = 0$, and we rescale equation (2.81) with

$$\beta_2 = \frac{\varepsilon}{\nu}\beta,$$

to obtain

$$\partial_z u_{|z=b} = \alpha_2 u_{|z=b} + \beta_2 \partial_x \varphi(t,x,b)(\cos\theta_{eq} - \cos\theta) + \mathcal{O}(\varepsilon^2). \tag{2.97}$$

Plugging (2.97) into (2.90), one obtains the following value of B

$$B(t,x) = -\frac{\partial_x p(t,x) h(t,x)}{\alpha_2} - \frac{\beta_2}{\alpha_2}\partial_x \varphi(t,x,b)(\cos\theta_{eq} - \cos\theta) + \mathcal{O}(\varepsilon^2).$$

The asymptotic expansion of u is then

$$u(t,x,z) = \partial_x p \frac{(h-z)^2}{2} - \partial_x p \frac{h^2}{2} - \frac{\partial_x p h}{\alpha_2} - \frac{\beta_2}{\alpha_2}\partial_x \varphi(t,x,b)(\cos\theta_{eq} - \cos\theta) + \mathcal{O}(\varepsilon^2), \quad (2.98)$$

while the one for w is (see (2.91))

$$w(t,x,z) = -\partial_x^2 p(\frac{z^3}{6} - \frac{hz}{\alpha_2}) + \partial_x p \partial_x h\big(\frac{z^2}{2} + \frac{z}{\alpha_2}\big) + \frac{\beta_2}{\alpha_2}\partial_x^2 \varphi(t,x,b)(\cos(\theta_{eq}) - \cos(\theta))\big)z + \mathcal{O}(\varepsilon^2). \quad (2.99)$$

Integrating (2.98) between $z = 0$ and $z = h(t,x)$ and using (2.87), we obtain

$$\begin{aligned}
\int_0^h u(t,x,z)dz &= -\frac{h^3}{3}\partial_x p - \frac{h^2}{\alpha}\partial_x p - \frac{\beta_2}{\alpha}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)h + \mathcal{O}(\varepsilon^2) \\
&= -\frac{h^3}{3}(\beta_1\partial_{xxx}h - g\partial_x h) - \frac{h^2}{\alpha_2}(\beta_1\partial_{xxx}h - g\partial_x h) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)h + \mathcal{O}(\varepsilon^2) \\
&= -(\frac{h^3}{3} + \frac{h^2}{\alpha_2})(\beta_1\partial_{xxx}h - g\partial_x h) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)h + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Plugging (2.100) into (2.27), one obtains the following system

$$(\mathbf{TF2}) \quad \begin{cases}
\partial_t h + \partial_x(R_2(h)) = 0, \\
R_2(h) = -(\frac{h^3}{3} + \frac{h^2}{\alpha_2})(\beta_1\partial_{xxx}h - g\partial_x h) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)h, \\
\partial_t\varphi + \partial(u\varphi) + \partial_z(w\varphi) = 0, \\
u = \partial_x p\frac{(h-z)^2}{2} - \partial_x p\frac{h^2}{2} - \frac{\partial_x p h}{\alpha_2} - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta), \\
w = -\partial_x^2 p(\frac{z^3}{6} - \frac{hz}{\alpha_2}) + \partial_x p\partial_x h\big(\frac{z^2}{2} + \frac{z}{\alpha_2}\big) + \frac{\beta_2}{\alpha_2}\partial_x^2\varphi(t,x,b)(\cos(\theta_{eq}) - \cos(\theta))\big)z.
\end{cases} \quad (2.100)$$

Case 3 : moving bottom without capillary penetration. In this case, $b \neq 0$, a direct computation similar to case 1 furnishes, using (2.92),

$$B(t,x) = -\partial_x p(t,x)\big(\frac{h}{\alpha_2} + \frac{h^2 - \xi^2}{2}\big) + \mathcal{O}(\varepsilon^2),$$

and

$$u(t,x,z) = \frac{\partial_x p}{2}(\xi - z)^2 - \frac{\partial_x p}{2}\xi^2(t,x) - \partial_x p\big(\frac{h}{\alpha_2} + \frac{h^2-\xi^2}{2}\big) + \mathcal{O}(\varepsilon^2),$$
$$= \frac{\partial_x p}{2}\big((\xi - z)^2 - h^2 + \frac{h}{\alpha_2}\big) + \mathcal{O}(\varepsilon^2). \quad (2.101)$$

Furthermore, we integrate (2.101) from b to ξ to obtain, using (2.87) and recalling that $h(t,x) = \xi(t,x) - b(t,x)$,

$$\int_b^\xi u(t,x,z)dz = -\frac{h^3}{3}\partial_x p - \frac{h^2}{\alpha_2}\partial_x p + \mathcal{O}(\varepsilon^2)$$
$$= -\big(\frac{h^3}{3} + \frac{h^2}{\alpha_2}\big)(\beta_1\partial_{xxx}\xi + g\partial_x\xi) + \mathcal{O}(\varepsilon^2);$$
$$= -\big(\frac{h^3}{3} + \frac{h^2}{\alpha_2}\big)(\beta_1\partial_{xxx}(h+b) - g\partial_x(h+b)) + \mathcal{O}(\varepsilon^2).$$

Finally, the final lubrication model system corresponding to case 3 is

$$(\mathbf{TF3}) \quad \begin{cases} \partial_t h + \partial_x(R_3(h)) = 0, \\ R_3(h) = -\big(\frac{h^3}{3} + \frac{h^2}{\alpha_2}\big)(\beta_1\partial_{xxx}(h+b) - g\partial_x(h+b)). \end{cases} \quad (2.102)$$

Case 4 : moving bottom with penetration condition. We mix the computations of Case 2 and Case 3 to obtain

$$B(t,x) = -\partial_x p(t,x)\big(\frac{h}{\alpha_2} + \frac{h^2-\xi^2}{2}\big) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta),$$

$$u(t,x,z) = \frac{\partial_x p}{2}\big((\xi - z)^2 - h^2 + \frac{h}{\alpha_2}\big) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta) + \mathcal{O}(\varepsilon^2),$$

and

$$w(t,x,z) = w(t,x,b) + \partial_{xp}^2 p\big(\frac{z^3-b^3}{6}\big) - \partial_x\big(\xi\partial_x p\big)\big(\frac{z^2-b^2}{2}\big)$$
$$+ \partial_x\big(\partial_x p(t,x)\big(\frac{h}{\alpha_2} - \frac{h^2-\xi^2}{2}\big)\big)(z-b)$$
$$+ \frac{\beta_2}{\alpha_2}\partial_x^2\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)(z-b) + \mathcal{O}(\varepsilon^2). \quad (2.103)$$

Consequently,

$$\begin{aligned}&\int_b^{\xi} u(t,x,z)dz\\ &= -\frac{h^3}{3}\partial_x p - \frac{h^2}{\alpha}\partial_x p - \frac{\beta_2}{\alpha}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)h + \mathcal{O}(\varepsilon^2),\\ &= -(\frac{h^3}{3} + \frac{h^2}{\alpha})(\beta_1\partial_{xxx}(h+b) + g\partial_x(h+b)) - \frac{\beta_2}{\alpha}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)h + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{2.104}$$

which provides the following model

$$(\mathbf{TF4})\quad \begin{cases}\partial_t h + \partial_x(R_4(h)) = 0,\\ R_4(h) = -(\frac{h^3}{3} + \frac{h^2}{\alpha})\big(\beta_1\partial_{xxx}(h+b) + g\partial_x(h+b)\big) - \frac{\beta_2}{\alpha}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)h,\\ \partial_t\varphi + \partial(u\varphi) + \partial_z(w\varphi) = 0,\\ u = \frac{\partial_x p}{2}\big((h+b-z)^2 - h^2 + \frac{h}{\alpha_2}\big) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta),\\ w = w_{|z=b} + \partial_x^2 p(\frac{z^3 - b^3}{6}) - \partial_x\big(\xi\partial_x p\big)\big(\frac{z^2 - b^2}{2}\big) + \partial_x\big(\partial_x p(t,x)\big(\frac{h}{\alpha_2} - \frac{h^2 - \xi^2}{2}\big)\big)(z-b)\\ + \frac{\beta_2}{\alpha_2}\partial_x^2\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)(z-b),\\ w_{|z=b} = \partial_t b + u_{|z=b}\partial_x b,\\ u_{|z=b} = \frac{\partial_x p}{2}\big(\frac{h}{\alpha_2}\big) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta).\end{cases} \tag{2.105}$$

Case 5 : moving bottom with reactive chemical zone and capillary penetration. Concerning the velocity u, this situation is similar to case 4 and we obtain again

$$B(t,x) = -\partial_x p(t,x)\big(\frac{h}{\alpha_2} + \frac{h^2 - \xi^2}{2}\big) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta),$$

$$u(t,x,z) = \frac{\partial_x p}{2}\big((\xi - z)^2 - h^2 + \frac{h}{\alpha_2}\big) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta) + \mathcal{O}(\varepsilon^2).$$

We can also still write the expression (2.103) of w but one has to be careful with the value of $w_{|z=b}$, since one has to take into account the full expression given by (2.80).

Remarking that (2.104) does not change in this context, we derive, using (2.26),

$$
(\mathbf{TF5}) \quad \begin{cases}
\partial_t h + \partial_x(R_5(h)) = \displaystyle\int_b^{\xi} \Big(\phi_r(\mathbf{U}_i - \mathbf{U}_r)\cdot\nabla\varphi\Big)dz + \phi_r\big(\frac{\partial b}{\partial x}(u_i - u_r) - (w_i - w_r)\big),\\
R_5(h) = -(\frac{h^3}{3} + \frac{h^2}{\alpha})\big(\beta_1\partial_{xxx}(h+b) + g\partial_x(h+b)\big) - \frac{\beta_2}{\alpha}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)h,\\
\partial_t\varphi + \partial(u\varphi) + \partial_z(w\varphi) = 0,\\
u = \frac{\partial_x p}{2}\big((h+b-z)^2 - h^2 + \frac{h}{\alpha_2}\big) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta),\\
w = w_{|z=b} + \partial_x^2 p\big(\frac{z^3 - b^3}{6}\big) - \partial_x\big(\xi\partial_x p\big)\big(\frac{z^2 - b^2}{2}\big) + \partial_x\big(\partial_x p(t,x)\big(\frac{h}{\alpha_2} - \frac{h^2 - \xi^2}{2}\big)\big)(z-b)\\
\quad + \frac{\beta_2}{\alpha_2}\partial_x^2\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta)(z-b),\\
w_{|z=b} = \partial_t b + u_{|z=b}\partial_x b - \phi_r\big(\partial_x b(u_i - u_r) - (w_i - w_r)\big),\\
u_{|z=b} = \frac{\partial_x p}{2}\big(\frac{h}{\alpha_2}\big) - \frac{\beta_2}{\alpha_2}\partial_x\varphi(t,x,b)(\cos\theta_{eq} - \cos\theta).
\end{cases}
\tag{2.106}
$$

Chapitre 3

Global Existence for a Lubrication System from Viscous Shallow-Water Equations with Drag Term and Capillarity Forces

This chapter concerns an alternative proof of existence of global weak solutions of a lubrication system rather than the one introduced for instance by F. Bernis–A. Friedman or A. Bertozzi–M. Pugh. More precisely we start from global weak solutions of viscous shallow-water equations with capillary forces and an appropriate drag term to derive global weak solutions of the lubrication systems. By this way, we make the link between the dissipative entropy introduced by F. Bernis–A. Friedman for the lubrication equation and the BD entropy introduced by D. Bresch and B. Desjardins. The method may be used for more general lubrication systems from the compressible Navier-Stokes equations with density dependent viscosities to investigate more qualitative properties : some remarks are given in this direction in the end of the chapter.

3.1 Introduction

In this chapter we consider a lubrication model derived from Navier-Stokes equation in [13] where surface tension and gravity effects are taken into account. Such derivation occurs when considering self-healing composite material comprise a ceramic matrix called ceramic-matrix composites (CMCs), where the self-healing process runs owing to the use of a fiber/matrix interphase that manages to preserve the fibers from cracks appearing in the matrix, specifically, in protecting themselves against oxidation by the formation of a sealing oxide which fills the matrix cracks. A simple model is obtained when considering the crack as a fixed strip through a given function with a liquid filling it through a free surface process managed by the Navier-Stokes equations.

In [13], taking into account the thin thickness of the crack, we derived a 1D lubrication equation by dimensionless analysis from the 2D Navier-Stokes equation with gravity force and free surface boundary condition including surface tension. More precisely, the lubrication model is a fourth order nonlinear degenerate diffusion equation in a domain $\Omega = (0, L)$ governing a L-periodic in space height function h through the following system :

$$\begin{cases} \partial_t h + \dfrac{1}{\alpha \mathrm{We}} \partial_x (F(h)\partial_x^3 h) - \dfrac{1}{\alpha \mathrm{Fr}^2} \partial_x (F(h)\partial_x h) = 0, \\ \partial_x^{(i)} h|_{x=0} = \partial_x^{(i)} h|_{x=L}, \text{ for } i = 0, ..., 4, \\ h|_{t=0} = h_0, \end{cases} \tag{3.1}$$

where α, We, Fr are three positive constants and $F(s) = s^3 + s^2$ where the initial function h_0 satisfies

$$h_0 \in H^1(\Omega) \qquad \text{with} \qquad h_0 \geq 0. \tag{3.2}$$

There are many papers devoted to the study of lubrication problems under the form

$$\partial_t h + \partial_x (F(h)\partial_x^3 h) - \partial_x (G(h)\partial_x h) = 0. \tag{3.3}$$

We will give comments in the last section to consider such general equation.

Let us present for reader's convenience, some results that are already known on such equation as mentioned in [46]. For $F(h) = |h|^n$, $G(h) = 0$ in one space dimension, the existence of non-negative weak solutions for (3.3) was first established by F. Bernis and A. Friedman [5] for $n > 1$. Further results ($n > 0$) were later obtained by E. Beretta, M. Bertsch and R. Dal Passo [2] and A. Bertozzi and M. Pugh [8]. In particular in [8] authors classified the problem in two main cases : $0 < n < 3$ with initial data $h_0 \geq 0$; $0 < n < +\infty$ with initial data $h_0 \geq m > 0$. Results in higher dimension were obtained in

[42, 28]. For $F(h) = |h|^n$, $G(h) = h^m$, various results have been obtained mainly by A. L. Bertozzi and M. C. Pugh in [7, 10, 9]. Compared to [8], Equation (3.3) in [7] was studied by adding a second order 'porous media' term $G(h)\partial_x h = \nabla h^m$. In particular it is shown that there is existence of nonnegative weak solutions with increasing support $0 < n < 3$ but the techniques failed for $n \geq 3$. Next in [10] they showed that the large-y behavior of $G(y)/F(y)$ determines the presence or absence of a finite-time blow-up, thus it is proved that there is no blow-up for $m < n + 2$. Nearly in [9] for $F(h) = h$, $G(h) = h^m$ they proved that there is blow-up for $m \geq 3$ and initial data in $H^1(\mathbb{R})$ with negative "energy". For $F(h) = G(h) = h$, equation (3.3) models a thin neck of fluid in the Hele-Shaw cell [27, 32, 33]. This model was also widely disscussed in many papers, for instance R. E. Goldstein, A. I. Pesci and M. J. Shelley [39, 40], etc.. Note that for $F(h) = G(h) = h^3$, equation (3.3) was considered by C. Imbert and A. Mellet [46] with adding a extra term $\lambda I(h)$ which models the effects of the electric field on the thin film, where the operator $I(h)$ is a nonlocal elliptic operator of order 1. Also in [68], equation (3.3) is studied with adding a Hilbert transform operator. For $F(h) = h^3 + \lambda h^p$ with $0 < p < 3$ and $\lambda > 0$. In [49] A.A.Lacey considered $G(h) = 0$ the motion with slip of a thin viscous droplet over a solid surface. Moreover there is a long-wave instability if $G(h) \geq 0$, in the gravity-destabilized thin film problem, $G(h) \sim h^3$ has been shown by Ehrhard [34] concerning long-wave instabilities. In addition there are some other papers concerning similar problems in form (3.3), for instance [41, 70] authors considered $F(h) = G(h)$ non negative diffusion function in $C^1(\mathbb{R}) \cap H^{1,\infty}(\mathbb{R})$, with the second order term being $A(h)$ instead of $G(h)h_x$ where A a C^α function with $0 < \alpha < 1$.

Turning to our work, we are interested in $F(h) = G(h) = h^3 + h^2$. We investigate the Cauchy problem (3.1) and show the existence of its non-negative weak solution. In our study, our main concern is to show that the strategy introduced in the work of D. Bresch and B. Desjardins can be adapted to our situation.

the main objective is to show that the strategy developed in the lubrication works mentioned before may be deduced from the method introduced by D.Bresch and B. Desjardins on the shallow-water equation based on the BD entropy. This makes the link between the dissipative entropy introduced by F. Bernis–A. Friedman and the BD entropy introduced by D. Bresch and B. Desjardins. Note [48] and [14] which consider some lubrication equations from shallow-water equations : the first paper focuses on the existence and limit with singular pressure law to the intermediate-slip model and the second one concerns the existence and convergence with a monotone pressure law with a linear drag term. This chapter is organized as follows : in the second section, we revisit the usual proof for the lubrication equation. For that, we consider the approximated lubrication model (3.1) for its energy estimations, entropy, positivity and some other properties in order to get the convergence results. We recall how to obtain weak solutions to equation (3.1) by Galerkin's method. In the third part, we explain how to get the global existence of weak solutions of the lubrication equation from the viscous shallow-water equations. This is in this section that we make the link between the dissipative entropy introduced by F. Bernis–A. Friedman and the BD-entropy introduced by D. Bresch and B. Desjardins. In a last section, we present remarks to show that our method may be used for more general lubrication systems from the compressible Navier-Stokes-Korteweg equations with drag terms.

Let us recall for the reader's convenience the definition of weak solutions for the lubrication system and the main result.

Definition 8. *We call h a* **weak solution** *to equation* (3.1) *if*

$$h \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \qquad \sqrt{F(h)}\,\partial_x^3 h \in L^2(0,T;L^2(\Omega))$$

with $h \geq 0$ and it satisfies for $\psi \in \mathcal{C}^\infty_\#([0,T]\times\Omega)$ such that $\psi|_{t=T} = 0$

$$\begin{aligned}\int_\Omega h_0\psi(0)dx + \int_0^T\int_\Omega h\partial_t\psi\,dxdt \\ -\frac{1}{\alpha \mathrm{We}}\int_0^T\int_\Omega \Big[F'(h)\partial_x^2 h\partial_x h\partial_x\psi + F(h)\partial_x^2 h\partial_x^2\psi\Big]\,dxdt \\ -\frac{1}{\alpha \mathrm{Fr}^2}\int_0^T\int_\Omega F(h)\partial_x h\partial_x\psi\,dxdt = 0\end{aligned} \tag{3.4}$$

where $\mathcal{C}^\infty_\#([0,T]\times\Omega)$ denotes the $\mathcal{C}^\infty([0,T]\times\Omega)$ function periodic in space. The main objective of the paper is to show existence of global weak solutions to System (3.1) in a different manner than the usual trick. More precisely we want to prove the following result

Theorem 9. *Given an initial condition h_0 satisfying $h_0 \geq 0$ with $h_0 \in H^1(\mathbb{T})$ and $-(1+h_0)\log(h_0/(1+h_0)) \in L^1(\mathbb{T})$ then there exists a* **weak** *solution to system* (3.1) *in the sense of definition 8 constructed in a direct way or from global weak solution of a viscous shallow-water equations with capillarity effect and an appropriate drag term.*

To prove the theorem written above, we recall and revisit in a first part of the chapter the method developed these last years in lubrication theory and in a second part of the paper, we present a method based on the viscous shallow-water equation. This last method is really flexible and could be used for more general lubrication system than the one considered in this chapter : this will be discussed in the last section.

3.2 Usual proof for the lubrication equation

As introduced in the papers discussed in the introduction, in order to obtain non-negative solution the function F is changed by

$$\tilde{f}_\eta(y) = \frac{y^4|F(y)|}{\delta(\eta)|F(y)| + y^4} \tag{3.5}$$

for $f_\eta(y)$. Compared to $f_\eta(y)$, $\tilde{f}_\eta(y)$ is non-negative for all y which will help to use uniform the bounds and conclude on the positivy ot h_η. We rewrite the approximate problem as looking at L-periodic solutions to the following system :

$$\begin{aligned} \partial_t h_\eta + \frac{1}{\alpha \mathrm{We}}\partial_x(\tilde{f}_\eta(h_\eta)\partial_x^3 h_\eta) - \frac{1}{\alpha \mathrm{Fr}^2}\partial_x(\tilde{f}_\eta(h_\eta)\partial_x h_\eta) = 0, \\ \partial_x^{(i)} h_\eta|_{x=0} = \partial_x^{(i)} h_\eta|_{x=L} \qquad \text{for } i = 0, ..., 4, \\ h_\eta|_{t=0} = h_0 + \eta. \end{aligned} \tag{3.6}$$

We will start with *a-priori* estimates, dissipative entropy and bounds uniform with respect to η. Then we show the weak-stability on such *a-priori* smooth sequence. The construction of such approximate solutions will be done through a Galerkin method later-on. This is the method developed in papers related to lubrication equations mentioned for instance in the introduction.

3.2.1 Energy estimate

In this section we will derive the following energy estimate of h_η in proposition 10.

Proposition 10. *Let $h_\eta(t,x)$ be a regular solution to* (3.6)*, then*

$$\sup_{t\in(0,T)} \int_{\mathbb{T}} (\frac{1}{\alpha \mathrm{Fr}^2} h_\eta^2 + \frac{1}{\alpha \mathrm{We}} (\partial_x h_\eta)^2) dx \\ + \int_0^T \int_{\mathbb{T}} \frac{F(h_\eta)}{h_\eta^2} \Big| \frac{1}{\alpha \mathrm{We}} h_\eta \partial_x^3 h_\eta - \frac{1}{\alpha \mathrm{Fr}^2} h_\eta \partial_x h_\eta \Big|^2 \\ \leq \int_{\mathbb{T}} (\frac{1}{\alpha \mathrm{Fr}^2} (h_\eta)^2(0) + \frac{1}{\alpha \mathrm{We}} (\partial_x h_\eta)^2(0)) dx. \tag{3.7}$$

Remark. The proposition above provides a uniform bound on h_η in $L^\infty(0,T;H^1(\Omega))$ if initially the height $h_0 \in H^1(\Omega)$.

Proof : This estimate is well known in previous works but let us provide a more simple proof. It suffices to observe that the lubrication equation may be written under the form

$$\partial_t h_\eta + \partial_x (h_\eta v_\eta) = 0,$$

and

$$\frac{h_\eta^2 v_\eta}{F(h_\eta)} = \frac{1}{\alpha \mathrm{We}} h_\eta \partial_x^3 h_\eta - \frac{1}{\alpha \mathrm{Fr}^2} h_\eta \partial_x h_\eta.$$

Multiplying the second equation by v_η and using the first equation, we get after integration in space

$$\int_{\mathbb{T}} \frac{h_\eta^2 |v_\eta|^2}{F(h_\eta)} = \frac{1}{\alpha \mathrm{We}} \int_{\mathbb{T}} \partial_x^2 h_\eta \partial_t h_\eta - \frac{1}{\alpha \mathrm{Fr}^2} \int_{\mathbb{T}} h_\eta \partial_t h_\eta$$

Using the expression of v_η and integrating in time, we get the result. □

3.2.2 Dissipative entropy

In this section we present the entropy $\int_\Omega G_\eta(h_\eta) dx$ introduced by F. Bernis–A. Friedman, then prove it dissipates allowing to control vacuum. We start from defining the function $G_\eta(y)$ below.

Definition and Lemma 11. *Assuming $\tilde{f}_\eta(y)$ defined in* (3.5)*, there exists a piecewise $\mathcal{C}^2(\mathbb{R}\backslash\{0,-1\})$ differentiable function $G_\eta(y)$ such that it satisfies following properties*

i) $G''_\eta(y) = 1/\tilde{f}(y)$ *for* $y \in \mathbb{R}\backslash\{0, -1\}$,

ii) $G_\eta(y)$ *is uniformly bounded for* $\eta_0 \leq y \leq M$,

iii) $G_\eta(y) - \delta(\eta)/6y^2$ *has an uniform lower bound* c_4 *for* $|y| \leq 2/3$,

iv) $G_\eta(y) \geq 0$ *for* $y \in \mathbb{R}\backslash\{0, -1\}$.

Proof : We start from finding an indefinite primitive function of $G''_\eta(y) = 1/\tilde{f}(y)$, secondly we will prove ii) and iii) hold, finally we will choose proper parameters to satisfy property iv). In polynomial form $G''_\eta(y)$ reads

$$G''_\eta(y) = \frac{\delta(\eta)}{y^4} + \frac{1}{|F(y)|} = \frac{\delta(\eta)}{y^4} + \frac{1}{y^2|1+y|}.$$

The absolute value sign brings us two expressions

$$\text{if } y > -1 \text{ and } y \neq 0 \qquad \text{then} \qquad G''_\eta(y) = \frac{\delta(\eta)}{y^4} + \frac{1}{y^2} - \frac{1}{y} + \frac{1}{1+y}$$

$$\text{if } y < -1 \qquad \text{then} \qquad G''_\eta(y) = \frac{\delta(\eta)}{y^4} - \frac{1}{y^2} + \frac{1}{y} - \frac{1}{1+y}.$$

They have primitive functions

$$\begin{aligned} G'_\eta(y) &= -\frac{\delta(\eta)}{3y^3} - \frac{1}{y} - \ln|y| + \ln|1+y| + c_1 \qquad \text{when} \qquad y > -1 \text{ and } y \neq 0; \\ G'_\eta(y) &= -\frac{\delta(\eta)}{3y^3} + \frac{1}{y} + \ln|y| - \ln|1+y| - c_1 \qquad \text{when} \qquad y < -1. \end{aligned}$$

where c_1 is an arbitrary constant. We recall the following antiderivatives

$$\int \ln|y| = y\ln|y| - y \qquad \text{and} \qquad \int \ln|1+y| = (1+y)\ln|1+y| - y.$$

An indefinite integration for $G'_\eta(y)$ yields the primitive function $G_\eta(y)$

$$G_\eta(y) = \frac{\delta(\eta)}{6y^2} \mp \ln|y| \mp y\ln|y| \pm (1+y)\ln|1+y| \pm c_1 y \pm c_2, \tag{3.8}$$

with c_2 is an arbitrary constant, where signs "$\pm$" or "$\mp$" correspond respectively to "$y > -1$, $y \neq 0$" and "$y < -1$". It is evident to prove ii) for any c_1 or c_2 because G_η is continuous on $\mathbb{R}^+$.

Remark. Pay attention that signs "±" or "∓" correspond to two cases "$y > -1$, $y \neq 0$" and "$y < -1$" but not correspond to $y \in \mathbb{R}^+$ or $\mathbb{R}^-$.

Proof of iii) : When $y \to 0$, without loss of generality, we assume $|y| < 2/3$ (domain included in case $y > -1$, $y \neq 0$) then $G_\eta(y)$ reads

$$G_\eta(y) = \frac{\delta(\eta)}{6y^2} - \ln|y| - y\ln|y| + (1+y)\ln|1+y| + c_1 y + c_2.$$

In this situation, we find that $-y\ln|y| + \alpha(1+y)\ln|1+y| + c_1 y + c_2$ is a continuous positive function in $\mathbb{R}$ so it admits a minimum value in $[-\frac{2}{3}, \frac{2}{3}]$ that we denote c_3. Then we have the following inequality

$$G_\eta(y) - \frac{\delta(\eta)}{6y^2} \geq -\ln\frac{2}{3} + c_3,$$

for simplicity we denote $c_4 = -\ln\frac{2}{3} + c_3$ in next.

Proof of iv) : Let's choose parameters c_1 and c_2 to make iv) true. We first consider behaviors of G_η to $\pm\infty$ by combing related terms, $G_\eta(y)$ can also be rewritten as

$$\begin{aligned} G_\eta(y) &= \frac{\delta(\eta)}{6y^2} \pm (1+y)\ln\left|\frac{1}{y}+1\right| \pm c_1 y + c_2 \\ &= \frac{\delta(\eta)}{6y^2} \pm (1+y)\ln\left|1+\frac{1}{y}\right| \pm c_1 y + c_2 \end{aligned} \tag{3.9}$$

We give an exact value to constant $c_1 = 0$, $G_\eta(y)$ reads

$$G_\eta(y) = \frac{\delta(\eta)}{6y^2} \pm (1+y)\ln\left|1+\frac{1}{y}\right| + c_2. \tag{3.10}$$

Since we know that

$$\lim_{y\to\pm\infty} y\ln\left|1+\frac{1}{y}\right| = 1,$$

it is easy to show that

$$\lim_{y\to\pm\infty} (1+y)\ln\left|1+\frac{1}{y}\right| = 1.$$

In brief, when $y \to \pm\infty$, $G_\eta(y) \to \pm 1 + c_2$. On the other hand, $G_\eta(y)$ has two discontinuous points. We find from (3.8) that the point $y = -1$ is a jump discontinuity and the point $y = 0$ is an essential discontinuity, precisely $G_\eta(y)$ tends to $+\infty$ as y approaches 0^- or 0^+. Therefore we conclude that $G_\eta(y)$ has a uniform lower bound on $\mathbb{R}\backslash\{0,-1\}$. We just need to choose a sufficient large $c_2 > 0$ such that $G_\eta(y)$ can be defined positive for all y in $\mathbb{R}\backslash\{0,-1\}$, which ends the proof of lemma. □

Proposition 12. *Let $h_\eta(t,x)$ be a regular solution to* (3.6)*, then*

$$\int_\Omega G_\eta(h_\eta)dx + \int_0^t \int_\Omega \frac{1}{\alpha \mathrm{We}}(\partial_{xx}h_\eta)^2 dx + \frac{1}{\alpha \mathrm{Fr}^2}(\partial_x h_\eta)^2 dx \leq \int_\Omega G_\eta(h_0)dx. \tag{3.11}$$

Remark. The proposition above provides a uniform bound on h_η in $L^2(0,T;H^2(\Omega))$ if initially the height $h_{\eta 0} = h_0 + \eta$ and $G(h_0) \in L^1(\mathbb{T})$.

Proof : Assuming h_η be a smooth solution to (3.6), we calculate the time derivative on $\int_\Omega G_\eta(h_\eta)dx$, using two integrations by parts

$$\begin{aligned}
\frac{d}{dt}\int_\Omega G_\eta(h_\eta)dx &= \int_\Omega G'_\eta(h_\eta)\partial_t h_\eta dx \\
&= -\int_\Omega G'_\eta(h_\eta)[\frac{1}{\alpha \mathrm{We}}\partial_x(\tilde{f}_\eta(h_\eta)\partial_x^3 h_\eta) - \frac{1}{\alpha \mathrm{Fr}^2}\partial_x(\tilde{f}_\eta(h_\eta)\partial_x h_\eta)]dx \\
&= \int_\Omega G''_\eta(h_\eta)\partial_x h_\eta \Big[\frac{1}{\alpha \mathrm{We}}\tilde{f}_\eta(h_\eta)\partial_x^3 h_\eta - \frac{1}{\alpha \mathrm{Fr}^2}\tilde{f}_\eta(h_\eta)\partial_x h_\eta\Big]dx \\
&= \frac{1}{\alpha \mathrm{We}}\int_\Omega \partial_x h_\eta \partial_x^3 h_\eta dx - \frac{1}{\alpha \mathrm{Fr}^2}\int_\Omega (\partial_x h_\eta)^2 dx \\
&= -\frac{1}{\alpha \mathrm{We}}\int_\Omega (\partial_{xx}h_\eta)^2 dx - \frac{1}{\alpha \mathrm{Fr}^2}\int_\Omega (\partial_x h_\eta)^2 dx \leq 0
\end{aligned}$$

Boundary terms equal to zero due to periodic boundary conditions in (3.6). We derive (3.11) by integrating the resulting inequality on time. Note that $G_\eta(h_0)$ has an uniform bound due to definition and lemma 11, therefore its integral on such a bounded domain Ω is uniformly bounded, that we denote the bound C_3. As a consequence, h_η is uniformly bounded in $L^2(0,T;H^2(\Omega))$. □

3.2.3 A priori constraints of the solution to (3.6).

In this section we can get help from the dissipative entropy in proposition 12 and Cauchy-Schwarz inequality with $H^1(\Omega)$ bound of h_η to prove the a priori positivity. The main idea is to lead a contradiction by explosion in bounded entropy. We first need a continuous lemma on h_η.

Lemme 13. *Let $h_\eta(t,x)$ be a solution to* (3.6)*, for all $t \in [0,T]$ and $x \in \Omega$, h_η is uniformly $\frac{1}{2}-$Hölder continuous in space and $\frac{1}{8}-$Hölder continuous in time.*

Proof. Follow [5] for detailed proof. □

Since h_η is Hölder continuous, once we proved that it could not cross 0, the positivity of initial condition and Hölder continuous property can guarantee the a priori positivity.

Proposition 14. *Let $h_\eta(t,x)$ be a solution to* (3.6) *with $h_{\eta 0} = h_0 + \eta$, then $h_\eta(t,x)$ is uniformly bounded by a constant M, and it is positive on $[0,T] \times \Omega$.*

Proof : It is easy to show $h_\eta(t,x)$ being uniformly bounded by a constant M, due to its uniform bound in $L^\infty(0,T;H^1(\Omega))$. For positivity we assume that there exists a $x_0 \in \Omega$ such that $h_\eta(x_0) = 0$, from proposition 10 we have obtained that h_η is bounded in $H^1(\Omega)$, we can get following argument by the Cauchy-Schwarz inequality, for x near to x_0 :

$$|h_\eta(x)| = |h_\eta(x) - h_\eta(x_0)| \leq \left(\int_{x_0}^{x} (\partial_x h_\eta)^2 \right)^{\frac{1}{2}} |x - x_0|^{\frac{1}{2}} \leq \sqrt{C_1 |x - x_0|}$$

obviously

$$\frac{1}{h_\eta(x)^2} \geq \frac{1}{C_1} \frac{1}{|x - x_0|}.$$

On the other hand, we get from definition and lemma 11 that $G_\eta(y) - \delta(\eta)/6y^2$ has an uniform lower bound for $|y| \leq 2/3$ which reads

$$G(y) \geq \frac{\delta(\eta)}{6y^2} + c_4.$$

In order to apply this inequality to h_η, we just have to bound h_η by 2/3. Indeed, since x near to x_0, without loss of generality we assume x in a ball $B(x_0, r)$ where we choose $r > 0$ such that $C_1 r \leq 4/\alpha^2$, therefore in this situation $|h_\eta(x)| \leq \sqrt{C_1 r} \leq 2/3$:

$$G(h_\eta) \geq \frac{\delta(\eta)}{6h_\eta^2} + c_4.$$

From proposition 12, we have a control of entropy $\int_\Omega G_\eta(h_\eta)$ which reads :

$$\begin{aligned} C_3 \geq \int_\Omega G(h_\eta)dx \geq& \int_\Omega \left(\frac{\delta(\eta)}{6h_\eta(x)^2} + c_4\right)dx \\ \geq& \frac{\delta(\eta)}{6}\int_{x_0}^{x} \frac{1}{C_1|x-x_0|}dx + \int_\Omega c_4 dx \\ =& +\infty \end{aligned} \tag{3.12}$$

It leads to a contradiction. As a result, such a x_0 does not exist, that is to say, $h_\eta(x)$ can not cross 0, it should be either positive or negative. Because the initial height $h_0 \geq 0$ and so $h_\eta(x) > 0$, which ends the proof of proposition. □

Besides, we can derive refined constraints on h_η.

Proposition 15. *Let h_η be a regular solution to* (3.6) *with $h_{\eta 0} = h_0 + \eta$ and $\delta(\eta) = (\ln(1+\eta^{-1/2}))^{-1/2}$, then for η sufficient small, $h_\eta > \eta$.*

Proof : The idea of the proof is similar to the one that we proved the positivity in previous section. Because η will be send to 0 later, without loss of generality we suppose η small such that $\eta < \eta_0$. For all $t \in [0,T]$, we assume that there exists a x_2 such that $h_\eta(t,x_2) = \eta$. The Cauchy-Schwarz inequality guarantees for $x \in B(x_2,\epsilon)$

$$\begin{aligned} |h_\eta(t,x) - \eta| \leq& \left(\int_{B(x_2,\epsilon)} (\partial_x h_\eta)^2(t,x)dx\right)^{\frac{1}{2}} |x-x_2|^{\frac{1}{2}} \\ \leq& \sqrt{C_1|x-x_2|} \leq \sqrt{C_1\epsilon}. \end{aligned}$$

ϵ will be chosen later. Then we get

$$0 < h_\eta(t,x) \leq \eta + \sqrt{C_1|x-x_2|} \leq \eta + \sqrt{C_1\epsilon},$$

which is equivalent to

$$\frac{1}{h_\eta^2(t,x)} \geq \frac{1}{(\eta+\sqrt{C_1|x-x_2|})^2}.$$

Like we proved the positivity, to find a final contradiction, ϵ should be chosen such that

$\eta + \sqrt{C_1\epsilon}$ bounded by 2/3, then the control of entropy $\int_\Omega G_\eta(h_\eta)$ reads for all $t \in [0, T]$

$$C_3 \geq \int_\Omega G(h_\eta)dx \geq \int_\Omega \frac{\delta(\eta)}{6h_\eta^2}dx + c_4|\Omega| \geq \frac{\delta(\eta)}{6}\int_{x_2}^{x_2+\epsilon} \frac{1}{(\eta + \sqrt{C_1|z-x_2|})^2}dz + c_4|\Omega|.$$

In calculus, we make a substitution $u = \sqrt{C_1|z - x_2|}$, then we obtain $C_1 dz = 2udu$ and hence

$$\begin{aligned} C_3 - c_4|\Omega| &\geq \frac{\delta(\eta)}{3C_1}\int_0^{\sqrt{C_1\epsilon}} \frac{u}{(\eta+u)^2}du \\ &= \frac{\delta(\eta)}{3C_1}\left[\ln(\eta+u) + \frac{\eta}{\eta+u}\right]_0^{\sqrt{C_1\epsilon}} \\ &= \frac{\delta(\eta)}{3C_1}\left(\ln(1+\frac{\sqrt{C_1\epsilon}}{\eta}) + \frac{\eta}{\eta+\sqrt{C_1\epsilon}} - 1\right) \end{aligned}$$

Particularly, we choose $\epsilon = \eta/C_1$ hence $\sqrt{C_1\epsilon} = \eta^{1/2}$ and $\delta(\eta) = (\ln(1+\eta^{-1/2}))^{-1/2}$. It is easy to see that $\delta(\eta)$ approaches to 0 at a low speed. The above inequality gives

$$C_3 - c_4|\Omega| \geq \frac{1}{3C_1}\left[(\ln(1+\eta^{-1/2}))^{1/2} + (\frac{\eta}{\eta+\eta^{1/2}} - 1)(\ln(1+\eta^{-1/2}))^{-1/2}\right].$$

We find that the right hand side tends to $+\infty$ when η tends to 0, this leads to a contradiction. That is to say, for η sufficient small, such a x_2 does not exist, $h_\eta(t,x)$ can not cross η which ends the proof. □

3.2.4 Estimate of $\partial_x^3 h_\eta$

Proposition 16. *Let h_η be a solution to* (3.6) *with $h_{\eta 0}$ satisfying* (3.2)*, then there exists a constant D independent of η, such that we have the following uniform estimate*

$$\int_0^T \int_{\mathbb{T}} f_\eta(h_\eta)(\partial_x^3 h_\eta)^2 dx \leq D.$$

As a result, $\partial_x^3 h_\eta$ is bounded in $L^2([0,T], L^2(\mathbb{T}))$ but its bound depends on η.

Proof : It suffices to use the energy estimate which provides the bound

$$\int_0^T \int_{\mathbb{T}} \frac{F(h_\eta)}{h_\eta^2}\Big|\frac{1}{\alpha\mathrm{We}}h_\eta\partial_x^3 h_\eta - \frac{1}{\alpha\mathrm{Fr}^2}h_\eta\partial_x h_\eta|^2\Big| \leq C < +\infty$$

and the uniform bound of h_η in $L^\infty(0,T;L^\infty(\mathbb{T}))$ and the uniform bound of $\partial_x h_\eta$ in $L^2(0,T;L^2(\mathbb{T}))$. □

3.2.5 Convergence (the proof of theorem 9)

In this section we will pass η to 0 to prove theorem 9. Thanks to proposition (12) the uniform $L^2([0,T],H^2(\mathbb{T}))$ estimate on h_η, it shows that there exists a weak convergent subsequence in $L^2([0,T],H^2(\mathbb{T}))$, further by Aubin-Lions lemma [1, 66] one can extract a strong convergent subsequence in $L^2([0,T],L^2(\mathbb{T}))$ and $L^2([0,T],H^1(\mathbb{T}))$, where we denote these limit as h. More specifically to satisfy criteria for Aubin-Lions lemma, we just need to prove $\partial_t h_\eta$ in $L^2(0,T;H^{-1}(\mathbb{T}))$.

Proposition 17. *Let h_η be a solution to* (3.6) *with h_0 satisfying* (3.2)*, then $\partial_t h_\eta$ in $L^2(0,T;H^{-1}(\mathbb{T}))$.*

Proof. For all ψ in $L^2(0,T;H^1(\mathbb{T}))$, using integration by parts one has

$$
\begin{aligned}
&\Big|\int_0^T\int_{\mathbb{T}}\partial_t h_\eta\psi dxdt\Big|\\
=&\Big|\int_0^T\int_{\mathbb{T}}\big(-\frac{1}{\alpha\mathrm{We}}\partial_x(f(h_\eta)\partial_x^3 h_\eta)+\frac{1}{\alpha\mathrm{Fr}^2}\partial_x(f(h_\eta)\partial_x h_\eta)\big)\psi dxdt\Big|\\
=&\Big|\int_0^T\int_{\mathbb{T}}\frac{1}{\alpha\mathrm{We}}f(h_\eta)\partial_x^3 h_\eta\partial_x\psi dxdt\Big|+\Big|\int_0^T\int_{\mathbb{T}}\frac{1}{\alpha\mathrm{Fr}^2}f(h_\eta)\partial_x h_\eta\partial_x\psi dxdt\Big|.
\end{aligned}
$$

Then the Cauchy-Schwarz inequality provides

$$
\begin{aligned}
&\Big|\int_0^T\int_{\mathbb{T}}\frac{1}{\alpha\mathrm{We}}f(h_\eta)\partial_x^3 h_\eta\partial_x\psi dxdt\Big|\\
\le&\frac{1}{\alpha\mathrm{We}}\Big(\int_0^T\int_{\mathbb{T}}f(h_\eta)(\partial_x^3 h_\eta)^2\Big)^{1/2}\Big(\int_0^T\int_{\mathbb{T}}f(h_\eta)(\partial_x\psi)^2 dxdt\Big)^{1/2}\\
\le&C\sqrt{F(M)}\|\psi\|_{L^2(0,T;H^1(\mathbb{T}))},
\end{aligned}
$$

and

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{T}} gf(h_\eta)\partial_x h_\eta \partial_x \psi dxdt\right| \\
\leq & CF(M)\Big(\int_0^t \int_{\mathbb{T}} \frac{1}{\alpha \mathrm{We}}(\partial_x h_\eta)^2\Big)^{1/2}\Big(\int_0^t \int_{\mathbb{T}} (\partial_x \psi)^2 dxdt\Big)^{1/2} \\
\leq & CF(M)\sqrt{C_1}\|\psi\|_{L^2(0,T;H^1(\mathbb{T}))}.
\end{aligned}$$

In brief

$$\left|\int_0^T \int_{\mathbb{T}} \partial_t h_\eta \psi dxdt\right| \leq \Big(\frac{1}{\alpha \mathrm{We}}\sqrt{D}\sqrt{F(M)} + CF(M)\sqrt{C_1}\Big)\|\psi\|_{L^2(0,T;H^1(\mathbb{T}))}$$

then we finished the proof. □

Lemme 18. *(Uniform convergence of $f_\eta(y)$ and its derivatives.) Recall that*

$$f_\eta(y) = \frac{y^4 F(y)}{\eta F(y) + y^4}, \qquad F(y) = y^3 + y^2.$$

Then respectively, $f_\eta(y)$, $f'_\eta(y)$ and $f''_\eta(y)$ converge uniformly on $[0, M]$ to $F(y)$, $F'(y)$ and $F''(y)$ as η goes to 0

Proof : Refer to appendix in [8]. □

Proof of theorem 9 : Let h_η be a weak solution to (3.6) it reads

$$h_\eta \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \qquad \sqrt{F(h_\eta)}\,\partial_x^3 h_\eta \in L^2(0,T;L^2(\Omega))$$

with

$$0 < h_\eta \leq M$$

and it satisfies for $\psi \in \mathcal{C}^\infty_\#([0,T]\times\Omega)$ such that $\psi|_{t=T} = 0$

$$\begin{aligned} &\int_\Omega h_0\psi(0)dx + \int_0^T\int_\Omega h_\eta \partial_t\psi\, dxdt \\ &-\frac{1}{\alpha\mathrm{We}}\int_0^T\int_\Omega \Big[F'(h_\eta)\partial_x^2 h_\eta \partial_x\psi + F(h_\eta)\partial_x^2 h_\eta \partial_x^2\psi\Big]\, dxdt \\ &\qquad -\frac{1}{\alpha\mathrm{Fr}^2}\int_0^T\int_\Omega F(h_\eta)\partial_x h_\eta \partial_x\psi\, dxdt = 0. \end{aligned} \tag{3.13}$$

We can pass to the limit in a straightforward manner using the uniform bounds we get on h_η and get the result of the theorem namely the global existence of weak solutions.

3.2.6 Galerkin method (η fixed)

The aim of this section is to apply the usual Galerkin method to system (3.6) in order to obtain existence of its weak solution. As a preparation, we look for a base of test function $\phi_n(x)$, which are sufficiently smooth and periodic on $\partial\Omega$ (for one dimensional case, we suppose $\Omega = [0, L]$).

We discuss the fourth order stationary Dirichlet problem with periodic boundary conditions

$$\begin{cases} \phi^{(4)}(x) = \lambda\phi(x); \\ \phi^{(i)}(0) = \phi^{(i)}(L) \text{ for } i = 0, \cdots, 4. \end{cases} \tag{3.14}$$

For this problem, we find a set of solutions $\phi_n(x) = \sqrt{\frac{2}{L}}\cos(\lambda_n^{\frac{1}{4}}x)$ for $n \in \mathbb{N}$ with $\lambda_n = (\frac{2n\pi}{L})^4$. In this instance, we find that $\int_\Omega \phi_n^2(x) = 1$, $\phi_n'' = -\sqrt{\lambda_n}\phi_n$, $\phi_n^{(3)} = -\sqrt{\lambda_n}\phi_n'$, moreover, it also verifies the periodic boundary conditions. Then $\{\phi_n\}_{n=1}^N$ forms an orthonormal basis in $L^2(\Omega)$, we denote V_N the spanned space. For every given smooth function v on Ω periodic on $\partial\Omega$, its projection function $P_N(v)$ on V_N can be expressed by $P_N(v) = \sum_{j=1}^N P(v^j)\phi_j(x)$. On $L^2(\Omega)$ one can define a scalar product $\langle f(x), g(x)\rangle = \int_\Omega f(x)g(x)dx$. The semi-discrete Galerkin method consists of finding

$$u^N(t,x) = \sum_{j=1}^N \alpha_j(t)\phi_j(x), \tag{3.15}$$

with time-dependent coefficients $\alpha_j(t)$ such that for all $\varphi \in V_N$

$$\begin{cases} \langle (u^N)_t, \varphi\rangle + \dfrac{1}{\alpha \mathrm{We}}\langle (f(u^N)u^N_{xxx})_x, \varphi\rangle - \dfrac{1}{\alpha \mathrm{Fr}^2}\langle (f(u^N)u^N_x)_x, \varphi\rangle = 0, \\ \langle u^N(0,x), \varphi\rangle = \langle h_{\eta 0}, \varphi\rangle. \end{cases} \tag{3.16}$$

Proposition 19. *Let u^N a function of form* (3.15)*, then for all N there exists a unique sequence $(\alpha_j)_{j=1,\cdots,N}$ such that u^N satisfies system* (3.16)*.*

Testing (3.16) again $\varphi = \phi_j$, $j = 1, 2, ..., N$, leads to the following system of linear ordinary differential equations for α_j, $j = 1, ..., N$,

$$\begin{cases} \langle (u^N)_t, \phi_k\rangle + \dfrac{1}{\alpha \mathrm{We}}\langle (f(u^N)u^N_{xxx})_x, \phi_k\rangle + \dfrac{1}{\alpha \mathrm{Fr}^2}\langle (f(u^N)u^N_x)_x, \phi_k\rangle = 0, \text{ for } k = 1, ..., N, \\ \qquad \langle u^N(0,x), \phi_k\rangle = \langle h_{\eta 0}, \phi_k\rangle, \text{ for } k = 1, ..., N. \end{cases} \tag{3.17}$$

We expand the first equation of (3.17),

$$\begin{aligned} &\sum_{j=1}^N \alpha'(t)\langle \phi_j, \phi_k\rangle - \frac{1}{\alpha \mathrm{We}}\sum_{j=1}^N \langle f(\sum_{l=1}^N \alpha_l(t)\phi_l(x))\alpha_j(t)\phi_j^{(3)}, \phi_k'\rangle \\ &\qquad + \frac{1}{\alpha \mathrm{Fr}^2}\sum_{j=1}^N \langle f(\sum_{l=1}^N \alpha_l(t)\phi_l(x))\alpha_j(t)\phi_j', \phi_k'\rangle = 0, \end{aligned} \tag{3.18}$$

which is equivalent to ,

$$\begin{aligned} &\sum_{j=1}^N \alpha'(t)\langle \phi_j, \phi_k\rangle + \frac{1}{\alpha \mathrm{We}}\sum_{j=1}^N \alpha_j(t)\langle f(\sum_{l=1}^N \alpha_l(t)\phi_l(x))\sqrt{\lambda_j}\phi_j', \phi_k'\rangle \\ &\qquad + \frac{1}{\alpha \mathrm{Fr}^2}\sum_{j=1}^N \alpha_j(t)\langle f(\sum_{l=1}^N \alpha_l(t)\phi_l(x))\phi_j', \phi_k'\rangle = 0. \end{aligned} \tag{3.19}$$

We denote $\mathcal{B} = (b_{jk})$ the mass matrix with $b_{jk} = \langle \phi_j, \phi_k\rangle$, $\mathcal{A} = (a_{jk})$ with $a_{jk} = \langle f(\sum_{l=1}^N \alpha_l(t)\phi_l(x))\phi_j', \phi_k'\rangle$, $\mathcal{A}_\lambda = (a_{\lambda jk})$ with $a_{\lambda jk} = \langle f(\sum_{l=1}^N \alpha_l(t)\phi_l(x))\sqrt{\lambda_j}\phi_j', \phi_k'\rangle$, $\alpha(t$ the vector of unknowns $\alpha_j(t)$. The dimension of all these items equals to N, the dimension of V_N. Then (3.19) can be formulated as

$$\mathcal{B}\alpha'(t) + \frac{1}{\alpha \mathrm{We}}\mathcal{A}_\lambda \alpha(t) + g\mathcal{A}\alpha(t) = 0 \tag{3.20}$$

For all ξ_j,

$$\sum_{j,k=1}^{N} a_{j,k}\xi_j\xi_k = \sum_{j,k=1}^{N} \langle f(\sum_{l=1}^{N}\alpha_l(t)\phi_l(x))\phi'_j, \phi'_k\rangle \xi_j\xi_k = \int f(\sum_{l=1}^{N}\alpha_l(t)\phi_l(x))\sum_{j=1}^{N}(\phi'_j\xi_j)^2 \geq 0$$

then $\mathcal{A}$ is a positive-definite matrix and invertible, so does $\mathcal{B}$ and $\mathcal{A}_\lambda$. Therefore the discret problem (3.20) has a unique solution on $[0, T_N]$. We can easily extend the solution to long time since we obtained a uniform bound on u^N.

Semi-discrete energy estimate

The basic discrete energy for $u^N \in V_N$ is defined as follows :

$$E_1[u^N] = \int_\Omega (u^N)^2 = \int_\Omega (\sum_{j=1}^{N} u_j)^2 \tag{3.21}$$

with $u_j = \alpha_j(t)\phi_j(x)$, $u^N = \sum_{j=1}^{N} u_j$.

Proposition 20. *The semi-discrete energy is dissipative in the following form*

$$\frac{d}{dt}\int_\Omega \frac{1}{\alpha \mathrm{Fr}^2}(u^N)^2 + \frac{1}{\alpha \mathrm{We}}(u_x^N)^2 \leq 0,$$

and we deduce that $u^N \in H^1(\Omega)$.

Proof : We replace the test function φ in (3.16) with u^N or u^N_{xx} to obtain

$$\langle (u^N)_t, u^N\rangle + \frac{1}{\alpha \mathrm{We}}\langle (f(u^N)u^N_{xxx})_x, u^N\rangle - \frac{1}{\alpha \mathrm{Fr}^2}\langle (f(u^N)u^N_x)_x, u^N\rangle = 0$$

$$\langle (u^N)_t, u^N_{xx}\rangle + \frac{1}{\alpha \mathrm{We}}\langle (f(u^N)u^N_{xxx})_x, u^N_{xx}\rangle - \frac{1}{\alpha \mathrm{Fr}^2}\langle (f(u^N)u^N_x)_x, u^N_{xx}\rangle = 0$$

Here we have

$$\begin{aligned}
\langle (u^N)_t, u^N\rangle &= \langle \sum_{j=1}^N \alpha_j'\phi_j, \sum_{j=1}^N \alpha_j\phi_j\rangle = \int_\Omega \sum_{j=1}^N \alpha_j'\alpha_j\phi_j^2 = \frac{1}{2}\frac{d}{dt}\int_\Omega \sum_{j=1}^N \alpha_j^2\phi_j^2 \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega (\sum_{j=1}^N \alpha_j\phi_j)^2 = \frac{1}{2}\frac{d}{dt}\int_\Omega (u^N)^2 \\
\langle (u^N)_t, u^N_{xx}\rangle &= \langle \sum_{j=1}^N \alpha_j'\phi_j, \sum_{j=1}^N \alpha_j\phi_j''\rangle = \sum_{j=1}^N \alpha_j'\alpha_j\langle \phi_j, \phi_j''\rangle = -\sum_{j=1}^N \alpha_j'\alpha_j \int_\Omega (\phi_j')^2 \\
&= -\frac{1}{2}\frac{d}{dt}\int_\Omega (\sum \alpha_j\phi_j')^2 = -\frac{1}{2}\frac{d}{dt}\int_\Omega (u^N_x)^2
\end{aligned}$$

We denote $E_2[u^N] = \int_\Omega (u^N_x)^2 = \int_\Omega (\sum_{j=1}^N (u_j)_x)^2$, then we can find that

$$\frac{d}{dt}(\frac{1}{\alpha \mathrm{Fr}^2}E_1[u^N] + \frac{1}{\alpha \mathrm{We}}E_2[u^N]) \leq 0.$$

To see that, let us compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\frac{1}{\alpha \mathrm{Fr}^2}E_1[u^N] + \frac{1}{\alpha \mathrm{We}}E_2[u^N]) =& \frac{1}{2}\frac{d}{dt}(\frac{1}{\alpha \mathrm{Fr}^2}\int_\Omega (u^N)^2 + \frac{1}{\alpha \mathrm{We}}\int_\Omega (u^N_x)^2) \\
=& \frac{1}{\alpha \mathrm{Fr}^2}\langle (u^N)_t, u^N\rangle - \frac{1}{\alpha \mathrm{We}}\langle (u^N)_t, u^N_{xx}\rangle \\
=& -\frac{1}{\alpha \mathrm{Fr}^2}\frac{1}{\alpha \mathrm{We}}\langle (f(u^N)u^N_{xxx})_x, u^N\rangle + (\frac{1}{\alpha \mathrm{Fr}^2})^2\langle (f(u^N)u^N_x)_x, u^N\rangle \\
&+ (\frac{1}{\alpha \mathrm{We}})^2\langle (f(u^N)u^N_{xxx})_x, u^N_{xx}\rangle - \frac{1}{\alpha \mathrm{Fr}^2}\frac{1}{\alpha \mathrm{We}}\langle (f(u^N)u^N_x)_x, u^N_{xx}\rangle \\
=& \frac{1}{\alpha \mathrm{Fr}^2}\frac{1}{\alpha \mathrm{We}}\langle f(u^N)u^N_{xxx}, u^N_x\rangle - (\frac{1}{\alpha \mathrm{Fr}^2})^2\langle f(u^N)u^N_x, u^N_x\rangle \\
&- (\frac{1}{\alpha \mathrm{We}})^2\langle f(u^N)u^N_{xxx}, u^N_{xxx}\rangle + \frac{1}{\alpha \mathrm{Fr}^2}\frac{1}{\alpha \mathrm{We}}\langle f(u^N)u^N_x, u^N_{xxx}\rangle \\
\leq& 0.
\end{aligned} \tag{3.22}$$

Indeed, like we used Cauchy-Schwarz inequality in the continuous case,

$$\begin{aligned}
&\frac{1}{\alpha \mathrm{Fr}^2}\frac{1}{\alpha \mathrm{We}}\langle f(u^N)u^N_{xxx}, u^N_x\rangle + \frac{1}{\alpha \mathrm{Fr}^2}\frac{1}{\alpha \mathrm{We}}\langle f(u^N)u^N_x, u^N_{xxx}\rangle\\
=&2\frac{1}{\alpha \mathrm{Fr}^2}\frac{1}{\alpha \mathrm{We}}\sum_{j=1}^N\int_\Omega f(\sum_{k=1}^N u_k)(u_j)_x(u_j)_{xxx}\\
\le&(\frac{1}{\alpha \mathrm{Fr}^2})^2\sum_{j=1}^N\int_\Omega f(\sum_{k=1}^N u_k)(u_j)_x^2 + (\frac{1}{\alpha \mathrm{We}})^2\sum_{j=1}^N\int_\Omega f(\sum_{k=1}^N u_k)(u_j)_{xxx}^2\\
=&(\frac{1}{\alpha \mathrm{Fr}^2})^2\langle f(u^N)u^N_x, u^N_x\rangle + (\frac{1}{\alpha \mathrm{We}})^2\langle f(u^N)u^N_{xxx}, u^N_{xxx}\rangle
\end{aligned}$$

Integrating (3.22) on time from 0 to t to obtain

$$\frac{1}{\alpha \mathrm{Fr}^2}E_1[u^N] + \frac{1}{\alpha \mathrm{We}}E_2[u^N] \le \frac{1}{\alpha \mathrm{Fr}^2}E_1[u^N_0] + \frac{1}{\alpha \mathrm{We}}E_2[u^N_0] < +\infty.$$

□

Remarque 21. *i) $E_1[u^N_0]$ and $E_2[u^N_0]$ are all bounded respectively by $\int_\Omega h^2_{\eta 0}$ and $\int_\Omega \partial_x h^2_{\eta 0}$ which are bounded by C_1 in proposition 3.2.1, that is to say, u^N bounded in $H^1(\Omega)$.*
ii) There is no extra boundary values, because the basis function ϕ_j is periodic on boundary as well as their derivatives in (3.14)*.*

Semi-discrete entropy

The discret entropy for $u^N \in V_N$ is defined analogously to the continuous case, that is we define a function $G(y) > 0$ in (3.9) to satisfy $G''(y) = \frac{1}{f(y)}$, and we are interested in $\int_\Omega G(u^N)$.

Proposition 22. *Let $u^N(t,x)$ be a function satisfying* (3.17)*, then the semi-discret entropy $\int_\Omega G_\eta(u^N)dx$ dissipates in following form*

$$\frac{d}{dt}\int_\Omega G_\eta(u^N)dx \le 0. \tag{3.23}$$

as a consequence $u^N(t,x)$ is uniformly bounded in $L^2([0,T], H^1(\Omega))$.

Proof : We compute its time derivatives as we have done in the continuous case

$$\begin{aligned}
\frac{d}{dt}\int_\Omega G(u^N) &= \langle u_t^N, G'(u^N)\rangle \\
&= -\frac{1}{\alpha \mathrm{We}}\langle (f(u^N)u_{xxx}^N)_x, G'(u^N)\rangle + \frac{1}{\alpha \mathrm{Fr}^2}\langle (f(u^N)u_x^N)_x, G'(u^N)\rangle \\
&= \frac{1}{\alpha \mathrm{We}}\langle f(u^N)u_{xxx}^N, G''(u^N)u_x^N\rangle - \frac{1}{\alpha \mathrm{Fr}^2}\langle f(u^N)u_x^N, G''(u^N)u_x^N\rangle \\
&= \frac{1}{\alpha \mathrm{We}}\langle u_{xxx}^N, u_x^N\rangle - \frac{1}{\alpha \mathrm{Fr}^2}\langle u_x^N, u_x^N\rangle \\
&= -\frac{1}{\alpha \mathrm{We}}\langle u_{xx}^N, u_{xx}^N\rangle - \frac{1}{\alpha \mathrm{Fr}^2}\langle u_x^N, u_x^N\rangle \\
&\leq 0
\end{aligned} \tag{3.24}$$

According to (3.9) the definition of $G(y)$, $G(u^N)$ is defined positive, so we conclude that $\int_\Omega G(u^N)$, $\int_0^{T_N}\int_\Omega (u^N)_{xx}^2$ and $\int_0^{T_N}\int_\Omega (u^N)_x^2$ are all bounded by a constant independent of N. □

3.2.7 Estimate on u_{xxx}^N

We derive a same result $u^N > \eta$ by using same method to continuous case, and we can easily get a estimate for u_{xxx}^N like in continuous case,

Proposition 23. *Let $u^N(t,x)$ be a function satisfying (3.17), then there exists a constant D independent of N, such that the following uniform estimate holds*

$$\int_0^t \int_\Omega f_\eta(u^N)(u_{xxx}^N)^2 dx \leq D.$$

As a result, u_{xxx}^N is uniformly bounded in $L^2([0,T], L^2(\Omega))$

$$\int_0^t \int_\Omega (u_{xxx}^N)^2 dx \leq \frac{D}{f_\eta(\eta^\theta)}.$$

Proposition 24. *Let $u^N(t,x)$ be a function satisfying (3.17), then $u_{xxx}^N(t,x) \in L^2([0,T], H^{-1}(\Omega))$*

The proof is trivial, for all $\psi \in L^2(0,T;H^1(\Omega))$ one has

$$\int_0^t \langle u^N_{xxx}, \psi\rangle^2 dt = \int_0^t \langle u^N_{xx}, \psi_x\rangle^2 dt \leq \int_0^t \int_\Omega (u^N_{xx})^2 dxdt \int_0^t \int_\Omega (\psi_x)^2 dxdt < +\infty$$

Convergence

Theorem 25. *Let* $\{u^N\}_{N\in\mathbb{R}}$ *a sequence satisfying* (3.17)*, then*
i) there exists a subsequence of u^N*, strongly convergent in* $L^2([0,T],H^1(\Omega))$*,*
ii) there exists a subsequence of u^N*, strongly convergent in* $L^2([0,T],H^2(\Omega))$*,*
iii) there exists a subsequence of u^N*, weakly convergent in* $L^2([0,T],H^3(\Omega))$*.*

Proof : Up to this time, we know that for a fixed η, u^N is bounded uniformly with N in $H^1(\Omega)$, $L^2_tH^1(\Omega)$, $L^2_tH^2(\Omega)$ and $L^2_tH^3(\Omega)$. To pass to the limit, we want to use the Aubin-Lions lemma, which requires an estimate on u^N_t. For all $\psi \in L^2(0,T;H^1(\Omega))$ one has

$$\begin{aligned}
&\int_0^t \langle (u^N)_t, \psi\rangle^2 dt \\
=&\int_0^t \langle -\frac{1}{\alpha \mathrm{We}}(f(u^N)u^N_{xxx})_x + \frac{1}{\alpha \mathrm{Fr}^2}(f(u^N)u^N_x)_x, \psi\rangle^2 dt \\
\leq& 2\int_0^t \langle -\frac{1}{\alpha \mathrm{We}} f(u^N)u^N_{xxx}, \psi_x\rangle^2 + \langle \frac{1}{\alpha \mathrm{Fr}^2} f(u^N)u^N_x, \psi_x\rangle^2 dt \\
\leq& 2(\frac{1}{\alpha \mathrm{We}})^2 \max f_\eta(y) \int_0^t \int_\Omega f_\eta(u^N)(u^N_{xxx})^2 dxdt \int_0^t \int_\Omega \psi_x^2 dxdt \\
&+ 2(\frac{1}{\alpha \mathrm{Fr}^2})^2 \max f_\eta(y) \int_0^t \int_\Omega f_\eta(u^N)(u^N_x)^2 dxdt \int_0^t \int_\Omega \psi_x^2 dxdt \\
\leq& 2(\frac{1}{\alpha \mathrm{We}})^2 F(1)\|\psi\|^2_{L^2_tH^1_x} + 2(\frac{1}{\alpha \mathrm{Fr}^2})^2 (F(1))^2 C_1 \|\psi\|^2_{L^2_tH^1_x},
\end{aligned}$$

which implies $u^N_t \in L^2_tH^{-1}(\Omega)$. Now we can apply Aubin-Lions lemma in [1] and [66], note that $H^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ and $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ and that $L^2(\Omega)$ is continuously embedded in $H^{-1}(\Omega)$. Let

$$M = \Big\{ v \in L^2([0,T],H^1(\Omega)), \partial_t v \in L^2([0,T],H^{-1}(\Omega)) \Big\},$$

then the embedding of M into $L^2([0,T],L^2(\Omega))$ is compact, and then from the sequence $\{u^N\} \in M$, we can extract a convergent subsequence in $L^2([0,T],L^2(\Omega))$.
Furthermore, since $H^2(\Omega)$ is compactly embedded in $H^1(\Omega)$, let

$$M_2 = \Big\{ v \in L^2([0,T],H^2(\Omega)), \partial_t v \in L^2([0,T],H^{-1}(\Omega)) \Big\},$$

then the embedding of M_2 into $L^2([0,T],H^1(\Omega))$ is compact, and for sequence $\{u^N\} \in M_2$, we can extract a convergent subsequence in $L^2([0,T],H^1(\Omega))$. An easy induction gives that for sequence $\{u^N\}$ bounded in $L^2([0,T],H^3(\Omega))$, we can also extract a convergent subsequence in $L^2([0,T],H^2(\Omega))$, we denote its limit as u. Moreover we can easily prove that there exists subsequence u^N weakly convergent to u in $L^2([0,T],H^3(\Omega))$, which ends the proof. □

Theorem 26. *Given any positive initial condition h_0 satisfying* (3.2)*. There exists a non-negative* **weak** *solution to the regularized equation* (3.6) *satisfying the weak formulation (3.4) in* $[0,T]\times\Omega$ *.*

Proof : Indeed, $f_\eta(y)$ and $f'_\eta(y)$ are Lipschitz continuous functions, due to the uniform estimate on u^N in $L^2([0,T],H^1(\Omega))$ and $L^2([0,T],H^3(\Omega))$ of Theorem (25), we can show, by passing into the limit in a product of a weak-strong convergence, that in the sense of distribution u satisfies (3.4), that is a weak solution to (3.6). □

3.3 Existence from shallow-water equations

As preceded in the previous section, we aim to prove the existence of a global in time weak solution of the lubrication system, taking into consideration a new drag term relative to some fabricated function of weight F(h). This weight has been used by A.L.Bertozzi in the physical and mathematical justification of lubrication model.

Consider the Shallow Water system corresponding to the new drag term $h_\varepsilon^2 \bar{u}_\varepsilon / F(h_\varepsilon)$, defined in a periodic domain $\mathbb{T}$:

$$\begin{aligned}
&\partial_t h_\varepsilon + \partial_x (h_\varepsilon \bar{u}_\varepsilon) = 0, \\
&\varepsilon\left(\partial_t (h_\varepsilon \bar{u}_\varepsilon) + \partial_x (h_\varepsilon \bar{u}_\varepsilon^2)\right) + \frac{h_\varepsilon \partial_x (h_\varepsilon)}{\mathrm{Fr}^2} \\
&\qquad = \varepsilon\left(\frac{4}{\mathrm{R_e}} \partial_x (h_\varepsilon \partial_x \bar{u}_\varepsilon)\right) + \frac{1}{\mathrm{W_e}} h_\varepsilon \partial_x^3 h_\varepsilon - \alpha \frac{h_\varepsilon^2 \bar{u}_\varepsilon}{F(h_\varepsilon)}
\end{aligned} \tag{3.25}$$

where α is a positive constant ; $\mathrm{R_e}$, $\mathrm{W_e}$ and Fr are respectively the adimentional Reynolds, Weber and Froude numbers. The initial conditions are given by :

$$h_\varepsilon|_{t=0} = h_0^\varepsilon, \qquad (h_\varepsilon u_\varepsilon)|_{t=0} = m_0^\varepsilon.$$

Let's recall the definition of the weak formulation of system (3.25)

Definition 27. *A weak formulation of the shallow water model with the nonlinear drag term represented in system* (3.25) *is given by*

$$\int_0^\infty \int_{\mathbb{T}} h_\varepsilon \partial_t \psi \; dx \; dt + \int_{\mathbb{T}} h_0^\varepsilon \psi(\cdot, 0) \; dx = - \int_0^\infty \int_{\mathbb{T}} h_\varepsilon \bar{u}_\varepsilon \partial_x \psi \; dx \; dt, \tag{3.26}$$

and

$$\begin{aligned}
&\varepsilon\Big(\int_0^\infty \int_{\mathbb{T}} h_\varepsilon \bar{u}_\varepsilon \partial_t \phi \; dx \; dt + \int_{\mathbb{T}} m_0^\varepsilon \phi(\cdot, 0) \; dx + \int_0^\infty \int_{\mathbb{T}} h_\varepsilon \bar{u}_\varepsilon^2 \partial_x \phi \; dx \; dt\Big) \\
&\quad - \frac{4\varepsilon}{R_e} \int_0^\infty \int_{\mathbb{T}} h_\varepsilon \partial_x \bar{u}_\varepsilon \partial_x \phi \; dx \; dt - \frac{1}{W_e} \int_0^\infty \int_{\mathbb{T}} \partial_x h_\varepsilon \partial_x^2 h_\varepsilon \phi \; dx \; dt \\
&- \frac{1}{W_e} \int_0^\infty \int_{\mathbb{T}} h_\varepsilon \partial_x^2 h_\varepsilon \partial_x \phi \; dx \; dt + \frac{1}{Fr^2} \int_0^\infty \int_{\mathbb{T}} h_\varepsilon^2 \partial_x \phi \; dx \; dt - \alpha \int_0^\infty \int_{\mathbb{T}} \frac{h_\varepsilon^2 \bar{u}_\varepsilon}{F(h_\varepsilon)} \phi \; dx \; dt = 0,
\end{aligned} \tag{3.27}$$

for all $\psi \in \mathcal{C}^\infty(\mathbb{T} \times [0, T])$ *and* $\phi \in \mathcal{C}^\infty(\mathbb{T} \times [0, T])$ *with* $\psi(\cdot, T) = \phi(\cdot, T) = 0$.

Energy estimate. The energy equation corresponding for the viscous shallow-water equation is obtained by multiplying the momentum equation by u_ε. Integrating then in space and using the mass equation, we obtain

$$\frac{d}{dt}\left(\int_{\mathbb{T}} \varepsilon \frac{h_\varepsilon \bar{u}_\varepsilon^2}{2} + \frac{h_\varepsilon^2}{2Fr^2} + \frac{(\partial_x h_\varepsilon)^2}{2W_e}\right) dx$$
$$+ \int_{\mathbb{T}} \frac{4\varepsilon}{R_e} h_\varepsilon (\partial_x \bar{u}_\varepsilon)^2 + \alpha \frac{h_\varepsilon^2 \bar{u}_\varepsilon^2}{F(h_\varepsilon)} \, dx = 0. \tag{3.28}$$

BD entropy. As for the BD-entropy inequality, it reads (the reader can refer to appendix for its derivation)

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} h_\varepsilon v_\varepsilon^2 \, dx + \frac{1}{\varepsilon}\left[\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} \frac{h_\varepsilon^2}{Fr^2} + \frac{(\partial_x h_\varepsilon)^2}{W_e} \, dx + \alpha \int_{\mathbb{T}} \frac{h_\varepsilon^2 \bar{u}_\varepsilon^2}{F(h_\varepsilon)} \, dx\right]$$
$$+ \frac{4}{\varepsilon \mathrm{R_e}}\left[\int_{\mathbb{T}} \frac{\partial_x h_\varepsilon^2}{\mathrm{Fr}^2} \, dx + \int_{\mathbb{T}} \frac{\partial_{xx} h_\varepsilon^2}{\mathrm{W_e}} \, dx + \alpha \int_{\mathbb{T}} \frac{h_\varepsilon \bar{u}_\varepsilon}{F(h_\varepsilon)} \partial_x h_\varepsilon \, dx\right] = 0,$$

where $v_\varepsilon = \bar{u}_\varepsilon + \partial_x(\log h_\varepsilon)$. The last term can be rewritten as

$$\begin{aligned}
\int_{\mathbb{T}} \frac{h_\varepsilon \bar{u}_\varepsilon}{F(h_\varepsilon)} \partial_x h_\varepsilon &= \int_{\mathbb{T}} h_\varepsilon \bar{u}_\varepsilon \frac{\partial_x h_\varepsilon}{h_\varepsilon^2 + h_\varepsilon^3}, \\
&= \int_{\mathbb{T}} h_\varepsilon \bar{u}_\varepsilon \partial_x h_\varepsilon \left(\frac{1}{h_\varepsilon^2} + \frac{1}{1+h_\varepsilon} - \frac{1}{h_\varepsilon}\right), \\
&= \int_{\mathbb{T}} h_\varepsilon \bar{u}_\varepsilon \partial_x \left(\frac{-1}{h_\varepsilon} + \log(1+h_\varepsilon) - \log(h_\varepsilon)\right), \\
&= \frac{d}{dt}\int_{\mathbb{T}} \underbrace{-(1+h_\varepsilon)\log\left(\frac{h_\varepsilon}{1+h_\varepsilon}\right)}_{>0}.
\end{aligned}$$

Finally, using the fact that $\frac{h_\varepsilon}{1+h_\varepsilon} < 1$, the BD-entropy inequality reads

$$\frac{d}{dt}\int_{\mathbb{T}} \frac{\varepsilon}{2} h_\varepsilon v_\varepsilon^2 - \frac{4\alpha}{R_e}(1+h_\varepsilon)\log\left(\frac{h_\varepsilon}{1+h_\varepsilon}\right) + \frac{h_\varepsilon^2}{2Fr^2} + \frac{(\partial_x h_\varepsilon)^2}{2W_e} \, dx$$
$$+ \alpha \int_{\mathbb{T}} \frac{h_\varepsilon^2 \bar{u}_\varepsilon^2}{F(h_\varepsilon)} + \frac{4}{\mathrm{R_e}}\left[\int_{\mathbb{T}} \frac{\partial_x h_\varepsilon^2}{\mathrm{Fr}^2} \, dx + \frac{\partial_{xx} h_\varepsilon^2}{\mathrm{W_e}} \, dx\right] \leq 0. \tag{3.29}$$

As discussed before, the mathematical justification of this model relies solely on the energy and BD-entropy estimates (4) and (5). To proceed, we give the definition of a global weak solution of (3.25), as well as its existence result

Definition 28. *A couple (h,u) is said to be a global weak solution of* (3.25) *if it satisfies the weak formulations* (3.26)–(3.27) *as well as the energy and BD-entropy inequalities* (3.28)–(3.29)*.*

Theorem 29. *Let* $(h_0^\varepsilon, m_0^\varepsilon)$ *be such that* $h_0^\varepsilon > 0$ *and*

$$h_0^\varepsilon \in H^1(\mathbb{T}), \quad \varepsilon|m_0^\varepsilon|^2/h_0^\varepsilon \in L^1(\mathbb{T}), \quad \sqrt{\varepsilon}\partial_x\sqrt{h_0^\varepsilon} \in L^2(\mathbb{T}),$$

$$-(1+h_0^\varepsilon)\log(\frac{h_0^\varepsilon}{1+h_0^\varepsilon}) \in L^1(\mathbb{T}).$$

Then there exists a global weak solution of (3.25) *in the sense of definition* (3.26)–(3.27)*.*

Such theorem has been first obtain by D.Bresch and B. Desjardins in [16]. Performing the the lubrication limit ($\varepsilon \to 0$) in (1), we obtain the following lubrication system with the new nonlinear drag term

$$\partial_t h + \partial_x\big(\frac{1}{\alpha W_e}F(h)\partial_x^3 h - \frac{1}{\alpha Fr^2}F(h)\partial_x h\big) = 0. \tag{3.30}$$

Note that taking the new variable

$$hv = \frac{1}{\alpha W_e}F(h)\partial_x^3 h - \frac{1}{\alpha Fr^2}F(h)\partial_x h,$$

the lubrication model can be reformulated as a

$$\begin{aligned} &\partial_t h + \partial_x(hv) = 0, \\ &hv = \frac{1}{\alpha W_e}F(h)\partial_x^3 h - \frac{1}{\alpha Fr^2}F(h)\partial_x h. \end{aligned} \tag{3.31}$$

It is also important to note that the term coming from the drag term in the BD-entropy provides the control of the quantity

$$G(h) = -\int_{\mathbb{T}}(1+h_\varepsilon)\log(\frac{h_\varepsilon}{1+h_\varepsilon})$$

in $L^\infty(0,T;L^1(\Omega))$. This is exactly the dissipative entropy introduced by in [5]. Remarks will be done in the last section of the paper. Concerning our system, we are able to prove the following result

Theorem 30. *Given a sequence $(h_\varepsilon, \bar{u}_\varepsilon)_\varepsilon$ a global solution of (4.1), satisfying the initial conditions given in the previous theorem, then there exists a subsequence of $(h_\varepsilon, \bar{u}_\varepsilon)$ that converges to a couple (h,u), a global weak solution of the lubrication system satisfying the initial condition $h|_{t=0} = h_0$, where h_0 is the weak limit of h_0^ε in $H^1(\mathbb{T})$ such that $G(h_0^\varepsilon) \in L^1(\mathbb{T})$ uniformly with respect to ε.*

Proof : The proof starts from integrating the energy and BD-energy inequalities in the time interval (0,T). We obtain :

$$\begin{aligned} &\frac{1}{Fr^2}\int_{\mathbb{T}} h_\varepsilon^2 dx \quad + \frac{\varepsilon}{2}\int_{\mathbb{T}} h_\varepsilon^2 \bar{u}_\varepsilon^2 dx \quad + \frac{(\partial_x h_\varepsilon)^2}{W_e}\, dx \\ &+ \frac{4\varepsilon}{R_e}\int_0^T \int_{\mathbb{T}} h_\varepsilon (\partial_x \bar{u}_\varepsilon)^2 dx dt \quad + \alpha \int_0^T \int_{\mathbb{T}} \frac{h_\varepsilon^2 \bar{u}_\varepsilon^2}{F(h_\varepsilon)} dx dt = \int_{\mathbb{T}} h_0^2\, dx. \end{aligned} \tag{3.32}$$

and :

$$\begin{aligned} &\frac{\varepsilon}{2}\int_{\mathbb{T}} h_\varepsilon v_\varepsilon^2\, dx + \frac{1}{2}\int_{\mathbb{T}} \frac{h_\varepsilon^2}{Fr^2}\, dx + \frac{(\partial_x h_\varepsilon)^2}{W_e}\, dx - \frac{4\alpha}{R_e}\int_{\mathbb{T}} (1+h_\varepsilon)\log(\frac{h_\varepsilon}{1+h_\varepsilon})\, dx \\ &+ \alpha \int_0^T \int_{\mathbb{T}} \frac{h_\varepsilon^2 \bar{u}_\varepsilon^2}{F(h_\varepsilon)}\, dx\, dt + \frac{4}{R_e}\int_0^T \int_{\mathbb{T}} \frac{(\partial_x h_\varepsilon)^2}{Fr^2} + \frac{(\partial_{xx} h_\varepsilon)^2}{W_e^2} \\ &+ \alpha \int_0^T \|\frac{h_\varepsilon^2}{F(h_\varepsilon)}\|_{H^1(\mathbb{T})} \|\bar{u}_\varepsilon\|_{L^2(\mathbb{T})} \\ &\le \frac{\varepsilon}{2}\int_{\mathbb{T}} \frac{(m_0^\varepsilon)^2}{h_0^\varepsilon} + 2\frac{m_0^\varepsilon}{h_0^\varepsilon}\partial_x h_0^\varepsilon + \frac{(\partial_x h_0^\varepsilon)^2}{h_0^\varepsilon} + \frac{(h_0^\varepsilon)^2}{Fr^2} + \frac{\partial_x h_0^\varepsilon)^2}{W_e} - \frac{4\alpha}{R_e}\int_{\mathbb{T}} (1+h_0^\varepsilon)\log(\frac{h_0^\varepsilon}{1+h_0^\varepsilon})\, dx, \end{aligned} \tag{3.33}$$

where $v_\varepsilon = \bar{u}_\varepsilon + \partial_x(log(h_\varepsilon))$.

Using the uniform bounds on the initial data, and since h_0^ε is bounded, we obtain the following estimates from the energy equation

$$\|h_\varepsilon\|_{L^\infty(0,T;H^1(\mathbb{T}))} \le C, \qquad \sqrt{\varepsilon}\|\sqrt{h_\varepsilon}\bar{u}_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{T}))} \le C,$$

$$\sqrt{\varepsilon}\|\sqrt{h_\varepsilon}\partial_x \bar{u}_\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}))} \le C,$$

$$\|\bar{u}_\varepsilon\|_{L^2(0,T;L^2(\mathbb{T}))} \le C, \qquad (\text{ as } \frac{h_\varepsilon^2}{F(h_\varepsilon)} \ge 1).$$

As for the BD-energy, we have

$$h_\varepsilon v_\varepsilon^2 = h_\varepsilon \bar{u}_\varepsilon^2 + 2\bar{u}_\varepsilon \partial_x(h_\varepsilon) + 4(\partial_x(\sqrt{h_\varepsilon}))^2.$$

Hence we get the following additional information :

$$\sqrt{\varepsilon}\|\partial_x(\sqrt{h}_\varepsilon)|_{L^\infty(0,T;L^2(\mathbb{T}))} \leq C,$$

$$\|h_\varepsilon\|_{L^\infty(0,T;H^2(\mathbb{T}))} \leq C.$$

With C being a generic constant independent of time (it depends on the measure of $\mathbb{T}$).

As a result of the previous estimations, and writing $\partial_x(h_\varepsilon \bar{u}_\varepsilon) = \partial_x h_\varepsilon \bar{u}_\varepsilon + h_\varepsilon \partial_x \bar{u}_\varepsilon$, we get new estimation for $\sqrt{\varepsilon}(h_\varepsilon \bar{u}_\varepsilon)$:

$$\sqrt{\varepsilon}\|\partial_x(h_\varepsilon \bar{u}_\varepsilon)\|_{L^2(0,T;L^1(\mathbb{T}))} \leq C,$$

$$\sqrt{\varepsilon}\|h_\varepsilon \bar{u}_\varepsilon\|_{L^2(0,T;L^1(\mathbb{T}))} \leq C\sqrt{\varepsilon}\|\sqrt{h_\varepsilon}\bar{u}_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}))}\|h_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{T}))}.$$

Thus

$$\sqrt{\varepsilon}\|h_\varepsilon \bar{u}_\varepsilon\|_{L^2(0,T;W^{1,1}(\mathbb{T}))} \leq C,$$

Due to the uniform boundedness of h_ε and $\bar{u}_\varepsilon$ in $L^\infty(0,T;H^2(\mathbb{T}))$ and $L^2(0,T;L^2(\mathbb{T}))$ respectively, and using Rellich-Kandrachov compactness theory for h_ε, we get the existence of strongly convergent subsequence (denoted again (h_ε)) in $C([0,T];L^2(\mathbb{T}))$, and a weakly convergent subsequence (denoted again $(\bar{u}_\varepsilon)$) in $L^2(0,T;L^2(\mathbb{T}))$ that converges to h, u respectively.

Passing to the limit in (3.26) using the fact that $\lim_{\varepsilon\to 0} h_0^\varepsilon := h_0 = h|_{t=0}$ in the weak sense, we get that (h,u) satisfies weak formulation of the mass equation of the lubrication system. In fact, we have

$$\int_0^T \int_{\mathbb{T}} h_\varepsilon \partial_t \psi \, dx \, dt \xrightarrow[\text{LDCT}]{\text{using}} \int_0^T \int_{\mathbb{T}} h \partial_t \psi \, dx \, dt.$$

$$\int_{\mathbb{T}} h_0^\varepsilon \psi(.,0)\ dx\ dt =< h_0^\varepsilon, \psi(.,0) >_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})}$$

$$\longrightarrow < h_0, \psi(.,0) >_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})} = \int_0^T \int_{\mathbb{T}} h_0 \psi(.,0)\ dx\ dt.$$

$$\int_0^T \int_{\mathbb{T}} h_\varepsilon \bar{u}_\varepsilon \partial_x \psi\ dx\ dt =< h_\varepsilon \bar{u}_\varepsilon, \partial_x \psi >_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})}$$

$$\longrightarrow < hu, \partial_x \psi >_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})} = \int_0^T \int_{\mathbb{T}} hu \partial_x \psi\ dx\ dt.$$

We follow now the same analysis for (3.27), the integrals multiplied by ε tend to zero upon the limit as they are bounded (due to energy estimates).
As the for fifth, sixth and seventh terms, we use the strong convergence of h_ε in $C([0,T];L^2(\mathbb{T}))$ and the weak $\star$ convergence of $\partial_x h_\varepsilon$ and $\partial_{xx} h_\varepsilon$ in $L^\infty(0,T;L^2(\mathbb{T}))$, i.e :

$$\int_0^T \int_{\mathbb{T}} h_\varepsilon^2 \partial_x \phi\ dx\ dt \xrightarrow[\text{LDCT}]{\text{using}} \int_0^T \int_{\mathbb{T}} h^2 \partial_x \phi\ dx\ dt.$$

(as h_ε converges strongly to h and thus pointwisely along a subsequence).

$$\int_0^T \int_{\mathbb{T}} h_\varepsilon \partial_{xx} h_\varepsilon \partial_x \phi\ dx\ dt =< h_\varepsilon \partial_{xx} h_\varepsilon, \partial_x \phi >_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})}$$

$$\xrightarrow[\varepsilon\to 0]{} < h \partial_{xx} h, \partial_x \phi >_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})} = \int_0^T \int_{\mathbb{T}} h \partial_{xx} h \partial_x \phi\ dx\ dt.$$

As for the last term, we have

$$\int_0^T \int_{\mathbb{T}} \frac{h_\varepsilon^2}{F(h_\varepsilon)} \bar{u}_\varepsilon \phi\ dx\ dt =< \frac{h_\varepsilon^2}{F(h_\varepsilon)} \bar{u}_\varepsilon, \phi >_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})}$$

$$\xrightarrow[\varepsilon\to 0]{} < \frac{h^2}{F(h)} u, \phi >_{H^{-1}(\mathbb{T}),H^1(\mathbb{T})} = \int_0^T \int_{\mathbb{T}} \frac{h^2}{F(h)} u \phi\ dx\ dt.$$

Hence, we get that (h,u) satisfies

$$\frac{1}{W_e} \int_0^T \int_{\mathbb{T}} h \partial_x^3 h \phi\ dx\ dt - \frac{1}{Fr^2} \int_0^T \int_{\mathbb{T}} h \partial_x h \phi\ dx\ dt - \alpha \int_0^T \int_{\mathbb{T}} \frac{h^2}{F(h)} u \phi\ dx\ dt = 0,$$

which is the weak formulation of the second equation of the lubrication system. □

3.4 The more general case revisited.

Let us consider the following lubrication equation

$$\partial_t h + \partial_x(F(h)\partial_x^3 h) - \partial_x(G(h)\partial_x h) = 0 \tag{3.34}$$

that has been studied in various papers. It may be written as

$$\partial_t h + \partial_x(hu) = 0$$

with

$$\alpha\frac{h^2 u}{F(h)} = \frac{1}{\mathrm{We}} h\partial_x^3 h - \frac{1}{\mathrm{Fr}^2}\frac{hG(h)}{F(h)}\partial_x h.$$

This system may be obtained as the limit of the compressible Navier-Stokes equations

$$\begin{aligned} &\partial_t h_\varepsilon + \partial_x(h_\varepsilon \bar{u}_\varepsilon) = 0,\\ &\varepsilon\left(\partial_t(h_\varepsilon\bar{u}_\varepsilon) + \partial_x(h_\varepsilon\bar{u}_\varepsilon^2)\right) + \frac{h_\varepsilon G(h_\varepsilon)}{F(h_\varepsilon)}\frac{\partial_x(h_\varepsilon)}{\mathrm{Fr}^2}\\ &\qquad = \varepsilon\left(\frac{4}{\mathrm{R_e}}\partial_x(h_\varepsilon\partial_x\bar{u}_\varepsilon)\right) + \frac{1}{\mathrm{W_e}}h_\varepsilon\partial_x^3 h_\varepsilon - \alpha\frac{h_\varepsilon^2\bar{u}_\varepsilon}{F(h_\varepsilon)}\ . \end{aligned} \tag{3.35}$$

The lubrication equation accept an energy estimate and a dissipative entropy estimate following the same line than the ones we give in the second section. Remark that the ratio $G(h)/F(h)$ is important in the pressure term. The energy and the BD entropy for the shallow-water equations are linked to the energy estimate and the dissipative entropy estimates for the lubrication equations. More general systems will be considered in future works by the authors.

Chapitre 4

Lubrication Theory and Viscous Shallow-Water Equations

4.1 Introduction

It is a real pleasure for us to write a proceeding in honor of Enrique Fernandez-Cara's birthday : D. Bresch is happy to have Enrique as friend and to have the opportunity in this special issue to precise that he started to work on compressible fluid system after discussion in Clermont-Ferrand with Enrique around 1997 in Jacques Simon' office and a first meet with Benoît Desjardins in 1998. The Navier-Stokes equations are considered as a basic mathematical model to describe the motion of a liquid. In his celebrated article [52] "Sur le mouvement d'un liquide visqueux emplissant l'espace" published in *Acta Mathematica* in 1934, Jean Leray (1906-1998) introduced the concept of global in time weak solution with a precise definition of what could be an irregular solution of the system, and he has proved the existence of such weak solutions in the homogeneous (constant density) incompressible setting. We now talk about "solutions à la Leray" these solutions of finite energy. Even if the global existence of weak solutions does not give too much information about the well posedness of the system, such analysis has a lot of practical interests. Beside the physical signification, because the assumed regularity on the data is minimal and strongly related to physical quantities well identified, the properties of stability of weak solutions on the continuous model help to better understand how to construct stable numerical schemes for which strong regularity estimates are not preserved.

The starting story correspond to the Leray solutions for the incompressible homogeneous Navier-Stokes equations in 1934, then results have been obtained for non-homogeneous incompressible Navier-Stokes equations by A. Kazhikhov (see [47] with initial density far from vacuum and bounded and with constant viscosity μ), J. Simon (see [67] with initial density with possible vacuum and with constant viscosity), Enrique Fernandez-Cara/Francisco Guillen-Gonzalez (see [37] with initial density with possible vacuum and strictly positive density dependent viscosity $\mu(\rho)$ and the interesting review paper [35] by Enrique Fernandez-Cara. See also [38] in unbounded domains) and P.–L. Lions (see [55] for a full picture for non-homogeneous incompressible Navier-Stokes equations) and also the recent interesting result in [29] by R. Danchin and P. Mucha concerning global strong solution in the two dimensional setting for bound initial density and H^1 initial velocity. The starting story, in the multi-dimensional setting, for the compressible Navier-Stokes equations concerns **constant viscosities** μ and λ by P.–L. Lions (see [55]), E. Feireisl *et al.* (see [35]) with pressure laws as $P(\rho) = a\rho^\gamma$ or Van der Vaals type laws (which are increasing after a certain fixed density value). Recently it has been possible to obtain a more general result by D.B. and J.-E. Jabin covering thermodynamically unstable pressure laws (no monotonicity assumption) and some anisotropy in the viscosities, see for instance [20], [21].

Remark that **density dependent viscosities**, in the compressible setting, has been firstly studied by D.B., B. Desjardins based on an observation with C.K. Lin for the Korteweg-Navier-Stokes system in [18] that in the case where $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$ we can control space derivative of the density if initial it is the case. It has been generalized in [17] to the case where $\mu(\rho)$ and $\lambda(\rho)$ are linked through the algebraic relation $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$. It envolves a new mathematical entropy called now BD-entropy helping to control the gradient of a function of the density if initially it is the case. The method of proving global existence of weak solution is completely different and may be seen as a dual one compared to the constant viscosities case. The role and difficulties between the density and the velocity field are completely exchanged : In the constant viscosities case, the difficulties occur for compactness on the density to pass to the limit in the non-linear pressure law $p(\rho)$. In the density dependent viscosities case, the difficulties occur for compactness on the velocity field to pass to the limit in the non-linear quadratic term $\rho u \otimes u$. In the first case, control on $L \log L$ on the density through renormalization technic on the mass equation allow to get such compactness if the pressure law is an increasing function at least after a fixed value. In the second case, control on $L \log L$ quantity on the modulus of the velocity through renormalization technic on the momentum equations allow to get such compactness. The interested reader is referred to [15] and to the recent Bourbaki paper written by [64] for more information around density dependent viscosities and compressible Navier-Stokes equations and to the recent papers [54] and [69].

In this paper, we want to precise the limit between the viscous shallow-water equations with capillarity and drag terms envolved in [43] to the lubrication equation related to the height studied in [12] (see also works by [6], [10]). Firstly, we get a weak convergence using the uniform bounds to a global weak solution of the lubrication system. Then we prove strong convergence to a strong solution of the lubrication system using a recent entropy inequality introduced in [24]–[25] (and extended in [19] for Navier-Stokes-Korteweg system with compatibility condition between dispersive term and diffusive term). It is interesting to note here that in the compressible setting, density dependent viscosities vanishes if the density vanishes. This kind of dependency is not actually allowed in the non-homogeneous incompressible setting since a strictly positive properties is asked in the viscosity : see the very interesting review paper by Enrique Fernandez-Cara in [36].

4.2 Derivation from shallow-water equations

Let us consider, in a periodic domain $\Omega = \mathbb{T}$, the shallow-water equations with linear drag term and surface tension :

$$\begin{aligned} &\partial_t h_\varepsilon + \partial_x(h_\varepsilon \overline{u}_\varepsilon) = 0, \\ &\partial_t(h_\varepsilon \overline{u}_\varepsilon) + \partial_x \left(h_\varepsilon \overline{u}_\varepsilon^2 + \frac{(h_\varepsilon)^2}{2\mathrm{Fr}^2} \right) = \frac{4}{\mathrm{Re}} \partial_x(h_\varepsilon \partial_x \overline{u}_\varepsilon) + \frac{1}{\mathrm{We}} h_\varepsilon \partial_x^3 h_\varepsilon - \alpha \overline{u}_\varepsilon, \end{aligned} \tag{4.1}$$

with $\alpha > 0$ where Re is the Reynolds number, We is the Weber number and Fr is the Froude number. Note that the terms in the right-hand side of the momentum equation represent respectively the viscous term, the capillarity term and the linear drag term. In the one-dimensional in space case, global existence of weak solutions of this system has been obtained by [18] where the BD-entropy has been firstly introduced in the simplified setting. The more general BD entropy relation may be found in [17]. We consider the initial data

$$h_\varepsilon|_{t=0} = h_0^\varepsilon, \qquad (h_\varepsilon u_\varepsilon)|_{t=0} = m_0^\varepsilon.$$

In this paper, we consider the lubrication limit ($\varepsilon \ll 1$) with adimensionalized numbers in [11, 22] under the form

$$\mathrm{We} := \varepsilon W_e, \quad \mathrm{Fr}^2 := \varepsilon F^2, \quad \alpha := \frac{\overline{\alpha}}{\varepsilon}$$

and the other dimensional numbers independent on ε. In the limit $\varepsilon \to 0$, on such system, assuming uniform bounds for all derivatives on the unknowns, we formally find

$$\overline{\alpha}\,\overline{u} = \frac{1}{W_e} h\partial_x^3 h - \frac{h\partial_x h}{F^2} \tag{4.2}$$

and

$$\partial_t h + \partial_x(h\overline{u}) = 0. \tag{4.3}$$

Combining the equation (4.2) with (4.3), we obtain a lubrication equation

$$\partial_t h + \partial_x\left(\frac{1}{\overline{\alpha}W_e} h^2\partial_x^3 h - \frac{1}{\overline{\alpha}F^2} h^2\partial_x h\right) = 0. \tag{4.4}$$

The mathematical justification of such derivation is linked to the energy estimates and a mathematical entropy arising for the degenerate viscous shallow-water system that has been discovered in its first form in [18] and in its general form in [17]. We will discuss this derivation in the first section. Note that a similar asymptotic study has been performed recently in [48] focusing on singular van-der-waals type pressure laws. Here we consider the standard shallow-water system occuring in geophysics, see for instance [43], [57] justified in [23].

Note that the energy estimate reads :

$$\frac{d}{dt}\left(\int_{\mathbb{T}} \varepsilon\frac{h_\varepsilon \overline{u}_\varepsilon^2}{2} + \frac{h_\varepsilon^2}{2F^2} + \frac{(\partial_x h_\varepsilon)^2}{2W_e}\right) + \int_{\mathbb{T}} \frac{4\varepsilon}{R_e} h_\varepsilon(\partial_x \overline{u}_\varepsilon)^2 + \overline{\alpha}\,\overline{u}_\varepsilon^2 \leq 0. \tag{4.5}$$

This energy estimate is obtained multiplying the momentum equation by $\overline{u}_\varepsilon$ and adding the result to the following equation

$$\frac{1}{2}[\partial_t h_\varepsilon^2 + \partial_x(h_\varepsilon^2 \overline{u}_\varepsilon) + h_\varepsilon^2 \partial_x \overline{u}_\varepsilon] = 0$$

and then integrating in space. This last equation is obtained from the mass equation formally multiplying by h_ε and rewriting it. The BD entropy estimate is given by (see recall after proof) :

$$\varepsilon \frac{d}{dt}\int_{\mathbb{T}} \frac{h_\varepsilon}{2}(\overline{u}_\varepsilon + 4(R_e)^{-1}\frac{\partial_x h_\varepsilon}{h_\varepsilon})^2 + \frac{d}{dt}\int_{\mathbb{T}}\Big(\frac{h_\varepsilon^2}{2F^2} + \frac{(\partial_x h_\varepsilon)^2}{2W_e} - \frac{4\overline{\alpha}}{R_e}\log_- h_\varepsilon\Big)$$
$$+\frac{4}{R_e}\int_{\mathbb{T}} \frac{(\partial_x h_\varepsilon)^2}{F^2} + \frac{(\partial_x^2 h_\varepsilon)^2}{W_e} + \int_{\mathbb{T}} \overline{\alpha}\,\overline{u}_\varepsilon^2 \leq 0 \tag{4.6}$$

Our result concerns weak solutions and is based on the following definition. The couple $(h_\varepsilon, u_\varepsilon)$ is called a global weak solutions of (4.1) if it satisfies (4.5)–(4.6) and

$$\int_0^\infty \int_{\mathbb{T}} h_\varepsilon \partial_t \psi + \int_{\mathbb{T}} h_0^\varepsilon \psi(\cdot, 0)\, dx = -\int_0^\infty \int_{\mathbb{T}} h_\varepsilon \overline{u}_\varepsilon \partial_x \psi \, dxdt \tag{4.7}$$

and

$$\varepsilon\Big(\int_0^\infty \int_{\mathbb{T}} h_\varepsilon \overline{u}_\varepsilon \partial_t \phi + \int_{\mathbb{T}} m_0^\varepsilon \phi(\cdot,0)\, dx + \int_0^\infty \int_{\mathbb{T}} h_\varepsilon \overline{u}_\varepsilon^2 \partial_x \phi \, dxdt\Big) \tag{4.8}$$
$$-\frac{4\varepsilon}{R_e}\int_0^\infty \int_{\mathbb{T}} h_\varepsilon \partial_x \overline{u}_\varepsilon \partial_x \phi\Big) - \frac{1}{W_e}\int_0^\infty \int_{\mathbb{T}} \partial_x h_\varepsilon \partial_x^2 h_\varepsilon \phi \, dxdt$$
$$-\frac{1}{W_e}\int_0^\infty \int_{\mathbb{T}} h_\varepsilon \partial_x^2 h_\varepsilon \partial_x \phi \, dxdt + \frac{1}{F^2}\int_0^\infty \int_{\mathbb{T}} h_\varepsilon^2 \partial_x \phi \, dxdt - \overline{\alpha}\int_0^\infty \int_{\mathbb{T}} \overline{u}_\varepsilon \phi \, dxdt = 0$$

for all $\psi \in \mathcal{C}_0^\infty(\mathbb{T}\times[0,\infty))$ and $\phi \in \mathcal{C}_0^\infty(\mathbb{T}\times[0,\infty))$.

Let us first recall an existence result which may be found in [16].

Theorem 31. *Let $(h_0^\varepsilon, m_0^\varepsilon)$ be such that $h_0^\varepsilon \geq 0$ and*

$$h_0^\varepsilon \in H^1(\Omega), \quad \varepsilon |m_0^\varepsilon|^2/h_0^\varepsilon \in L^1(\Omega), \quad \sqrt{\varepsilon}\partial_x \sqrt{h_0^\varepsilon} \in L^2(\Omega), \quad -\log_- h_0^\varepsilon \in L^1(\Omega)$$

where $\log_- \cdot = \log \min(\cdot, 1)$. Then there exists a global weak solution of (4.1) in the sense of definition (4.7)–(4.8).

Then we can give the following theorem which will be a straightforward application of bounds given by the energy and BD entropy. We will give the proof for reader's convenience.

Theorem 32. *Let* $(h_\varepsilon, u_\varepsilon)$ *be a global weak solution of* (4.1) *as given in Theorem 31 with initial data satisfying the bounds uniformly. Then there exists a subsequence of* $(h_\varepsilon, \overline{u}_\varepsilon)$*, already denoted by* $(h_\varepsilon, \overline{u}_\varepsilon)$*, which converges to* $(h, \overline{u})$ *global weak solution of the lubrication system* (4.2)–(4.3) *satisfying the initial condition* $h|_{t=0} = h_0$ *with* h_0 *the weak limit in* $H^1(\Omega)$ *(up to a subsequence) of* h_0^ε*.*

Démonstration. Due to the estimates, we have the following uniform bounds

$$\sqrt{\varepsilon}\|\sqrt{h_\varepsilon}\overline{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \le C, \qquad \|h_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \le C,$$

$$\sqrt{\varepsilon}\|\sqrt{h_\varepsilon}\partial_x\overline{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \le C, \qquad \|\overline{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \le C.$$

Using this bounds, due to the BD entropy, the following extra uniform bounds

$$\sqrt{\varepsilon}\|\partial_x\sqrt{h_\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} \le C, \quad \|h_\varepsilon\|_{L^2(0,T;H^2(\Omega))} \le C.$$

Remark that, using the uniform $L^\infty(0,T;H^1(\Omega)$ bound of h_ε, we get

$$\|h_\varepsilon\|_{L^\infty} \le C.$$

Using that $\partial_x(h_\varepsilon\overline{u}_\varepsilon) = h_\varepsilon\partial_x\overline{u}_\varepsilon + \overline{u}_\varepsilon\partial_x h_\varepsilon$ and the uniform bounds related to $\sqrt{h_\varepsilon}\partial_x\overline{u}_\varepsilon$ and h_ε and $\overline{u}_\varepsilon$, we get

$$\sqrt{\varepsilon}\|\partial_x(h_\varepsilon\overline{u}_\varepsilon)\|_{L^2(0,T;L^1(\Omega))} \le C$$

Thus

$$\sqrt{\varepsilon}\|h_\varepsilon\overline{u}_\varepsilon\|_{L^2(0,T;W^{1,1}(\Omega))} \le C.$$

Let us now pass to the limit in the weak formulation. In the mass equation, we use compactness on h_ε in $\mathcal{C}([0,T]\times\Omega)$ (due to bounds related to capillarity and estimates on $\partial_t h_\varepsilon$ looking at the mass equation in the distribution sense) and weak convergence in $L^2((0,T)\times\Omega)$ on $\overline{u}_\varepsilon$. Concerning the momentum equation, We easily pass to the limit in the third terms which will converge to 0 since they are multiplied by ε. The fourth one also converges to 0 since it concerns $\sqrt{h_\varepsilon}$ and $\sqrt{\varepsilon}\sqrt{h_\varepsilon}\partial_x\overline{u}_\varepsilon$. Concerning the terms involving h_ε, we use the strong convergence of h_ε in $\mathcal{C}([0,T];H^s(\Omega))$ for all $s<1$ and in $L^2(0,T;H^s(\Omega))$ for $s<2$ and weak convergence of $\partial_x^2 h_\varepsilon$ in $L^2((0,T)\times\Omega)$. The last term is easy using the L^2 uniform bound on $\overline{u}_\varepsilon$. □

Remark. This is interesting to note that the BD entropy degenerates to similar entropy involved in lubrication theory for instance described in [6] and [10].

Recall. Let us recall for reader's convenience how to derive the BD entropy. We differentiate the mass equation with respect to the space variable, it gives

$$\partial_t\partial_x h_\varepsilon + \partial_x(\overline{u}_\varepsilon\partial_x h_\varepsilon) + \partial_x(h_\varepsilon\partial_x\overline{u}_\varepsilon) = 0.$$

We remark that the last term is the same than the diffusive term in the momentum equation with an opposite sign if we multiply this equation by 4/Re. Thus multiplying by 4/Re and adding the resulting equation with the momentum equation, we get

$$\partial_t\Big(h_\varepsilon(\overline{u}_\varepsilon + \frac{4}{\mathrm{Re}}\partial_x \log h_\varepsilon)\Big) + \partial_x\Big(h_\varepsilon(\overline{u}_\varepsilon + \frac{4}{\mathrm{Re}}\partial_x \log h_\varepsilon)\Big) + h_\varepsilon\Big(\frac{h_\varepsilon}{\mathrm{Fr}^2}\frac{\partial_x^3 h_\varepsilon}{\mathrm{We}}\Big) + \alpha\overline{u}_\varepsilon = 0.$$

Multiplying this equation by $\overline{u}_\varepsilon + \frac{4}{\mathrm{Re}}\partial_x \log h_\varepsilon$ and using the mass equation $\partial_t h_\varepsilon + \partial_x(h_\varepsilon\overline{u}_\varepsilon) = 0$, we get integrating by parts the BD entropy.

Comment. Note that weak solutions to lubrication equations in the presence of strong slippage has been obtained in [48] from shallow-water equations. Strong slippage assumption with surface tension provides a height far from vanishing state (due to singular pressure laws and high derivative control of the height).

4.3 Relative entropy and strong convergences

Let us now explain how to get better convergence result namely strong convergence from viscous shallow-water system to lubrication equation. More precisely let us consider again the system (4.1)

$$\begin{aligned}
&\partial_t h_\varepsilon + \partial_x(h_\varepsilon\overline{u}_\varepsilon) = 0,\\
&\partial_t(h_\varepsilon\overline{u}_\varepsilon) + \partial_x\left(h_\varepsilon\overline{u}_\varepsilon^2 + \frac{h_\varepsilon^2}{2\mathrm{Fr}^2}\right) = \frac{4}{\mathrm{Re}}\partial_x(h_\varepsilon\partial_x\overline{u}_\varepsilon) - \alpha\overline{u}_\varepsilon + \frac{1}{\mathrm{We}}h_\varepsilon\partial_x^3 h_\varepsilon,
\end{aligned}$$

where Re is the Reynolds number, We is the Weber number and Fr is the Froude number. The coefficient α represents the friction due to the bottom and is assumed to be strictly positive. As explained before, such system has been studied initially in [18] and global existence of weak solutions has been proved in the one-dimensional setting using the linear drag term. Assuming the

$$\mathrm{We} := \varepsilon W_e, \qquad \mathrm{Fr}^2 := \varepsilon F^2, \qquad \alpha := \frac{\overline{\alpha}}{\varepsilon}$$

with W_e, F and $\overline{\alpha}$ fixed as explained in the introduction and looking at the limit $\varepsilon \to 0$, we can modulate the energy and BD-entropy in order to perform strong convergence between global weak solutions of the viscous shallow-water equation with damping and capillarity terms to strong solution $(\overline{h}, \overline{u})$ of the lubrication equations (4.4) using the fact that the height satisfies at the limit $h \geq c > 0$. Note that the limit quantity $(h, \overline{u})$ satisfies the following equations

$$\partial_t h + \partial_x (h\bar{u}) = 0. \tag{4.9}$$

Using (4.2), it also satisfies the momentum equation

$$\varepsilon\big(\partial_t(h\overline{u}) + \partial_x(h\overline{u}^2)\big) + \partial_x\Big(\frac{h^2}{2F^2}\Big) - \frac{4\varepsilon}{\mathrm{Re}}\partial_x(h\partial_x\overline{u}) - \frac{1}{We}h\partial_x^3 h + \overline{\alpha}\,\overline{u} = R_\varepsilon \tag{4.10}$$

where

$$R_\varepsilon = \varepsilon\big(\partial_t(h\overline{u}) + \partial_x(h\overline{u}^2)\big) - \frac{4\varepsilon}{\mathrm{Re}}\partial_x(h\partial_x\overline{u}).$$

Note that modulated technique for Navier-Stokes with density dependent viscosities has been recently developed in [24]–[25] (extending an initial study by B. Haspot in [44] where the density dependent pressure law is assumed to be proportional to the density dependent viscosity). Namely such technic has been developed for the following system composed of mass equation

$$\partial_t \rho + \partial_x(\rho u) = 0$$

and momentum equation

$$\partial_t(\rho u) + \partial_x(\rho u^2) - \nu \partial_x(\rho \partial_x u) + \partial_x p(\rho) = 0$$

where $p(\rho) = a\rho^\gamma$. A weak-strong uniqueness result is also performed using this well defined modulated energy (relative entropy) control. Compared to what has been done in [24]-[25], new terms here are therefore envolved namely the drag term $\overline{\alpha}\,\overline{u}$, the surface tension term $h\partial_x^3 h/We$ and the right-hand side R_ε.

Note that we also cannot use the recent work in [19] because in this paper capillarity coefficient and viscosity are assumed to be linked together in an appropriate way which is not satisfied by our model. To check if things work we need to look at new terms writing them in terms of the unknowns. Concerning R_ε, it is sufficient to assume that $R_\varepsilon \to 0$ when ε goes to zero in $L^1((0,T_\star);L^2(\mathbb{T}))$ where $T_\star$ is the existence time of strong solution of the lubrication equation where h is stricly positive. To control this rest this ask for regularity properties on $(h,\overline{u})$ and this justified the fact that we consider strong limit solution of lubrication equation. Concerning the surface tension term, it suffices to write it as $h\partial_x^3 h = h\partial_x^2(hv)$ with $v = \partial_x \log h$. Concerning the drag term, since it is a linear one, it does not provide any difficulties. If we consider the following relative entropy

$$\begin{aligned} E(h_\varepsilon, \overline{u}_\varepsilon, \partial_x h_\varepsilon \ \big| h, \overline{u}, \partial_x h) = & \\ \varepsilon \int_{\mathbb{T}} \frac{h_\varepsilon}{2} \Big(\big| \overline{u}_\varepsilon - \bar{u} + 4(R_e)^{-1} \big(\frac{\partial_x h_\varepsilon}{h_\varepsilon} - \frac{\partial_x h}{h} \big) \big|^2 + |\overline{u}_\varepsilon - \bar{u}|^2 \Big) & \\ + \int_{\mathbb{T}} \Big(\frac{|h_\varepsilon - h|^2}{F^2} + \frac{|\partial_x(h_\varepsilon - \overline{h})|^2}{2W_e} - \frac{4\overline{\alpha}}{R_e} \log_-(h_\varepsilon/h) \Big) & \end{aligned} \tag{4.11}$$

then by similar calculations than in [24] (we will not do such long but straightforward calculations again) we can get denoting the relative entropy $E_\varepsilon = E(h_\varepsilon, \overline{u}_\varepsilon, \partial_x h_\varepsilon \big| h, \overline{u}, \partial_x h)$ that

$$\begin{aligned} E_\varepsilon(t) \leq & \ E_\varepsilon(0) \exp[c(h) \int_0^t \|\partial_x u\|_{L^\infty} + \|\partial_x \log h\|_{L^\infty}^2 + \|\partial_x^2 \log h\|_{L^\infty(\Omega)}] \\ & + \int_0^t \exp[c(h) \int_s^t (\|\partial_x u\|_{L^\infty} + \|\partial_x \log h\|_{L^\infty}^2 + \|\partial_x^2 \log h\|_{L^\infty(\Omega)}) d\tau] \\ & \quad \int_\Omega h_\varepsilon \big(R_\varepsilon (\overline{u}_\varepsilon - \bar{u} + 4(R_e)^{-1} \big(\frac{\partial_x h_\varepsilon}{h_\varepsilon} - \frac{\partial_x h}{h})) + R_\varepsilon(\overline{u}_\varepsilon - \bar{u}) \big). \end{aligned}$$

Thus assuming that $E_\varepsilon(0)$ goes to zero when ε go to zero, we get the convergence result using the convergence of R_ε to zero in $L^1(0,T_\star;L^2(\mathbb{T}))$.

4.4 Change of time by Y. Brenier and X. Duan.

In this last section, let us precise for reader's convenience a very interesting result by Y. Brenier and X. Duan concerning an appropriate quadratic change of variable that will provide a derivation of lubrication type model from the viscous-shallow system without any drag term initially present in the system. Let us consider the viscous shallow water without drag term namely the equation

$$\begin{aligned} &\partial_t h_\varepsilon + \partial_x (h_\varepsilon \overline{u}_\varepsilon) = 0, \\ &\partial_t (h_\varepsilon \overline{u}_\varepsilon) + \partial_x \left(h_\varepsilon \overline{u}_\varepsilon^2 + \frac{h_\varepsilon^2}{2\mathrm{Fr}^2} \right) = \frac{4}{\mathrm{Re}} \partial_x (h_\varepsilon \partial_x \overline{u}_\varepsilon) + \frac{1}{\mathrm{We}} h_\varepsilon \partial_x^3 h_\varepsilon. \end{aligned} \tag{4.12}$$

Let set

$$t \to \theta = t^2/2, \qquad h_\varepsilon(t,x) \to h_\varepsilon(\theta,x), \qquad \bar{u}_\varepsilon(t,x) \to \bar{u}_\varepsilon(\theta,x)\frac{d\theta}{dt}.$$

the system reads

$$\begin{aligned} &\partial_\theta h_\varepsilon + \partial_x (h_\varepsilon \overline{u}_\varepsilon) = 0, \\ &h_\varepsilon \overline{u}_\varepsilon + 2\theta[\partial_\theta (h_\varepsilon \overline{u}_\varepsilon) + \partial_x \left(h_\varepsilon \overline{u}_\varepsilon^2 \right)] + \partial_x (\frac{h_\varepsilon^2}{2\mathrm{Fr}^2}) = \frac{4\sqrt{2\theta}}{\mathrm{Re}} \partial_x (h_\varepsilon \partial_x \overline{u}_\varepsilon) + \frac{1}{\mathrm{We}} h_\varepsilon \partial_x^3 h_\varepsilon. \end{aligned}$$

Thus letting formally θ goes to zero, we get the non-degenerate lubrication model

$$\partial_\theta h - \frac{1}{\mathrm{Fr}^2} \partial_x (h \partial_x h) + \frac{1}{We} \partial_x (h \partial_x^3 h) = 0.$$

Obviously more general lubrication may be obtained starting from general Euler-Korteweg type systems. This would be interested to justify such asymptotic using the Relative entropy framework developed in [19] similarly than what has been done by Y. Brenier and X. Duan on curve-shortening flow in [3].

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