Flux vacua and compactification on smooth compact toric varieties

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Vides avec flux et compactification sur des variétés toriques compactes

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Résumé

L'étude des vides avec flux est une étape primordiale afin de mieux comprendre la compactification en théorie des cordes ainsi que ses conséquences phénoménologiques. En présence de flux, l'espace interne ne peut plus être Calabi-Yau, mais admet tout de même une structure SU(3) qui devient un outil privilégié. Après une introduction aux notions géométriques nécessaires, cette thèse examine le rôle des flux dans la compactification supersymétrique sous différents angles. Nous considérons tout d'abord des troncations cohérentes de la supergravité IIA. Nous montrons alors que des condensats fermioniques peuvent aider à supporter des flux et générer une contribution positive à la constante cosmologique. Ces troncations admettent donc des vides de Sitter qu'il serait autrement très difficile d'obtenir, si ce n'est impossible. L'argument est tout d'abord employé avec des condensats de dilatini puis amélioré en suggérant un mécanisme pour générer des condensats de gravitini à partir d'instantons gravitationnels. Ensuite l'attention se tourne sur les branes et leur comportement sous T-dualité non abélienne. Nous calculons les configurations duales à certaines solutions avec D branes de la supergravité de type II, et examinons les flux ainsi que leurs charges afin d'identifier les branes après dualité. La solution supersymétrique avec brane D2 est étudiée plus en détails en vérifiant explicitement les équations sur les spineurs généralisés, puis en discutant de la possibilité d'une déformation massive. Le dernier chapitre fournit une construction systématique de structures SU(3) sur une large classe de variétés toriques compactes. Cette construction définit un fibré en sphère au-dessus d'une variété torique 2d quelconque, mais fonctionne tout aussi bien sur une base Kähler-Einstein.

Abstract

The study of flux vacua is a primordial step in the understanding of string compactifications and their phenomenological properties. In presence of flux the internal manifold ceases to be Calabi-Yau, but still admits an $SU(3)$ structure which becomes thus the preferred framework. After introducing the relevant geometrical notions this thesis explores the role that fluxes play in supersymmetric compactification through several approaches. At first consistent truncations of type IIA supergravity are considered. It is shown that fermionic condensates can help support fluxes and generate a positive contribution to the cosmological constant. These truncations thus admit de Sitter vacua which are otherwise extremely difficult to get, if not impossible. The argument is initially performed with dilatini condensates and then improved by suggesting a mechanism to generate gravitini condensates from gravitational instantons. Then the focus shifts towards branes and their behavior under non abelian T-duality. The duals of several D-brane solutions of type II supergravity are computed and the branes are tracked down by investigating the fluxes and the charges they carry. The supersymmetric D2 brane is further studied by checking explicitly the generalized spinor equations and discussing the possibility of a massive deformation. The last chapter gives a systematic construction of $SU(3)$ structures on a wide class of compact toric varieties. The construction defines a sphere bundle on an arbitrary two-dimensional toric variety but also works when the base is Kähler-Einstein.

Publications

This thesis is based on the following publications:

- [1] R. Terrisse and D. Tsimpis, "SU(3) structures on S^2 bundles over four-manifolds," JHEP **1709**, 133 (2017)
- [2] R. Terrisse and D. Tsimpis, "Consistent truncation with dilatino condensation on nearly Kähler and Calabi-Yau manifolds," JHEP **1902**, 088 (2019),
- [3] R. Terrisse and D. Tsimpis, "Consistent truncation and de Sitter space from gravitational instantons," JHEP **1907**, 034 (2019)
- [4] R. Terrisse, D. Tsimpis and C. A. Whiting, "D-branes and non-Abelian T-duality," Nucl. Phys. B **947** (2019)

Introduction

Modern physics models are extremely successful in describing the universe. Virtually all physical phenomena observable on earth can be explained at a fundamental level by two theoretical frameworks.

The first one is the standard model (SM) of particle physics. It describes the basic constituents of matter as well as their interactions. The SM is a quantum field theory, its starting point is an action functional describing the dynamics of classical fields. The theory is then quantized to get the spectrum, *ie* the different states that are available in the theory. As a quantum theory the SM model predicts the probability of transition between states. When the interactions (parametrized by couplings in the Lagrangian) are weak, a state can be interpreted as a combination of particles propagating in space-time. If the particles come close to each other they can interact and the result of their interaction can be computed with a series expansion in the couplings. Such predictions are then tested in particle accelerators and actually lead to the most precise predictions ever achieved in physics.

The second one is general relativity (GR), which describes the gravitational interaction, not present in the standard model. In fact GR is much more than a mere theory of gravitation. In GR space-time is modeled by a manifold equipped with a Lorentzian metric which determines its "shape". Any physical object is then bound to follow geodesics in this geometric space. Reciprocally any energy density sources the curvature of the metric through Einstein's equations, and thus modifies locally the geometry of space-time. GR can be described by a classical field theory defined on the space-time manifold, in which the metric itself is a dynamical field.

Of course both frameworks are not used in all physical applications, as they are supposed to describe reality at its fundamental level. Newtonian approximation of GR is more than sufficient to get accurate models in most situations. On the other hand the SM describes accurately the interactions of elementary particles while most relevant situations involve an enormous number of those particles. Then a complete treatment would be impossible and actually useless. One uses instead effective models that are all, at least in principle, compatible with the standard model. Most scientific research seeks to resolve the tremendous complexity of the universe rather than its fundamental rules.

Still the SM and GR are two apparently unrelated frameworks, and thus do not qualify for a fundamental theory describing the real world. One then expects that a complete description of reality should involve a quantum version of GR that is compatible with the SM. However this is conceptually difficult to achieve. Indeed one of the most important properties of GR is that it does not depend on a choice a coordinate system. Space-time is a set of points, and the metric defines the distances between them. But this intrinsic description is not manageable in practice and one needs to introduce coordinates so that physical objects can be attributed numerical values. Eventually the result should not depend on this arbitrary choice. This is translated in the theory as an invariance under diffeomorphism of space-time. On the other hand quantum field theory is defined on a Minkowski space-time with a flat metric, which is of course a coordinate dependent statement. This is actually crucial as quantum fields need a background on which they propagate. If the metric is quantized, hence space-time itself, it becomes unclear on what background the theory lives.

One could of course ignore this conceptual barrier and consider the metric as a regular field by expanding it around the flat metric. The procedure fails drastically as the computation leads to inevitable divergences. The SM and GR seem by nature incompatible. Then how come they both lead to such accurate predictions ? The main reason is that, comparatively to the fundamental interactions of the SM, gravity is extremely weak. For example at the classical level, for two electrons the gravitational interaction is 10^{42} times weaker than electromagnetism. Thus gravitation is completely negligible in the interaction of elementary particles. When the number of particles increases gravitation becomes stronger and stronger as it is universally attractive. On the contrary quantum effects will be smoothed out by decoherence and as bound states are formed, the fundamental interactions are screened. When gravitational effects become relevant, quantum effects are negligible and the system can be described in classical GR.

The situation then looks surprisingly satisfying. When one framework is needed, the other one is negligible: one could argue that this is an acceptable answer. This is not the end of the story though: despite the formidable predictive power of GR and the SM some phenomena are still not understood at the fundamental level[1]

- The exact nature of dark matter is still not known. Moreover current cosmological models need a strictly positive cosmological constant (though extremely small) to explain the acceleration of the expansion of the universe, usually referred to as dark energy. In Planck units its value is of the order of 10^{-122}.
- A few marginal experimental results do not fit in the SM. One important example is the non vanishing mass of the neutrinos.
- The mathematical foundations of the SM model are not completely clear. The computations need to be regularized through a process called renormalization to give finite results. Moreover the expansion in the couplings is now known to be divergent. Also in a regime of strong interactions perturbative techniques cannot be used anymore and very few tools are available to make sensible predictions. For all these reasons, the SM should probably be considered as an effective theory.
- In regions with extremely high energy density, gravitational effects can become significant while quantum effects are still relevant. Even though such extreme conditions are unreachable on Earth they actually exist in our universe: during the Big Bang, at the

[1] If we look back at the historical development of modern theoretical physics, those "small" imperfections could most likely lead to major upheaval in our understanding of the universe.

core of neutron stars, or at the horizon of black holes. A fundamental description of reality should account for these effects.

Thus a quantum theory of gravitation seems necessary and may hopefully resolve some of the issues raised here. Such theory should also be unified with the standard model at some point. String theory stands as the most serious candidate.

In string theory the fundamental elements are not particles but extended one dimensional objects. A quantum particle can be described by a quantum field theory on a one dimensional space, its wordline, whose fields are its coordinates in space-time. In the same way a quantum string would corresponds to a two dimensional field theory on its worldsheet. However the one dimensional theory is rather trivial. Its only degree of freedom is its momentum and it describes merely a free propagating particle. In order to get something interesting one needs to combine different types of particles and add interactions. This is exactly what the standard model does.

On the other hand the spectrum of the string is much richer, as it also contains the excitation modes of the string. From the space-time point of view this gives a tower of states with increasing masses. This would then give a quantum field theory with an infinite number of fields but the full theory is not known. From the worldsheet point of view the quantum string is a conformal two dimensional theory and amplitudes can be computed exactly. Moreover the massless spectrum contains a traceless symmetric tensor, which can be shown to behave like a graviton. Indeed at low energy, *ie* ignoring the tower of massive sates and higher derivative corrections, the effective action contains the Einstein-Hilbert term of GR. At this stage this is very promising: string theory naturally includes quantum gravity without suffering from the usual divergences. Moreover there still remains many degrees of freedom that could, hopefully, include the SM. However:

- String theory predicts a 26 dimensional space-time.
- The spectrum contains only bosonic states. It cannot then reproduce the fermions of the SM.
- The spectrum contains a tachyon, rendering the theory unstable.

Let us see how these problems can be avoided.

Compactification If extra dimensions seem to be inconsistent with our reality, they can be accommodated though a process called compactification. In fact extra dimensions should be considered as extra degrees of freedom (and can actually be beneficial). Consider a d-dimensional space-time M_d with $d > 4$. Now decompose $M_d = M_4 \times M_{d-4}$ where M_4 represents our four dimensional world and M_{d-4} is called the internal space and embodies the extra dimensions. From the four dimensional point of view the components of tensors along the internal space can be interpreted as extra fields. All field components are then functions of $x \in M_4$ and $y \in M_{d-4}$. If the internal manifold is compact of characteristic length l_i the y dependence can be decomposed into eigenmodes of the Laplacian[2]. In the four dimensional theory this leads to a tower a states whose first mass are of the order of $\frac{1}{l_i}$. This extra mass correspond to momentum in the internal directions. At low energies

[2]These are Fourier modes when M_{d-4} is a torus.

compared to $\frac{1}{l_i}$ these states cannot be generated and can thus be discarded from the effective theory.

Superstrings A natural way to have fermions in a theory is to include supersymmetry. The supersymmetry transformations have fermionic parameters and thus mix bosons and fermions. As a consequence a supersymmetric theory necessarily has the same number of bosonic and fermionic degrees of freedom. Moreover the successive action of two supersymmetry transformations generates translations, so supersymmetry naturally extends the Poincaré algebra. Superstring theory is then the supersymmetric version of string theory. Of course the spectrum now contains fermions, but another consequence is that supersymmetry removes the tachyon. The graviton is still present but now the critical dimension is ten. The low energy effective theory is now ten dimensional supergravity. Thus string theory is exactly what we were looking for: a consistent quantum theory of gravitation that potentially includes the SM.

If superstring theory were to actually describe our world, our energy scales would most likely be low compared to the string scale. Thus supergravity is expected to play an important role in phenomenological considerations. An effective description of these low energy processes should be determined by fluctuations around a supergravity solution, thus called in this context string vacua. Note that by itself, ten dimensional supergravity is not a theory of quantum gravity as it is not renormalizable. It should be considered only as a low energy approximation: at higher energies the approximation fails and stringy effects should be taken into account. Now in order to get to four dimensions, we will need at some point to proceed to a compactification. Eventually the four dimensional theory will highly depend on the choice of vacua, and thus on the reduction ansatz.

The search for string vacua then becomes an intensive research subject. At first reductions to a Minkowski space-time were considered. In that case the internal space should be a Calabi-Yau manifold, and some bosonic fields of supergravity called fluxes need to be switched off. Such compactifications lead to great progress in our understanding of string vacua. However along the reduction a high number of scalar field are generated in the four dimensional theory, coming for example from the components of the graviton. The modes with lowest mass are merely the constant modes in the internal directions and thus no mass is generated from the reduction. In fact these fields end up being massless. Such fields are called moduli and would not lead to a realistic phenomenological model. In the case of Calabi-Yau compactification, very few options are left to stabilize these moduli and all rely on quantum corrections.

Flux vacua This suggests that the initial ansatz may be too restrictive. Actually the masses of the moduli depend on the value of the background flux. Keeping the fluxes in the ansatz should then help stabilize the moduli. Moreover fluxes are genuine ingredients of supergravity and string theory, so keeping them is rather natural. Solutions with non vanishing fluxes are called flux vacua and are the main subject of this thesis. Still, turning on fluxes is not a trivial step. The internal manifold is not Calabi-Yau so that the associated mathematical machinery is no longer available. The geometrical conditions can be rephrased in the language of G-structures. More precisely the internal manifold must have $SU(3)$-structure (note that Calabi-Yau manifolds can be defined as manifolds with a very specific

$SU(3)$-structure). In the most general case, flux vacua should be studied in the framework of generalized complex geometry. Then the relevant structure group is $SU(3) \times SU(3)$.

This thesis is the compilation of four papers that were written during my Ph.D [1, 2, 3, 4]. They have been slightly adapted to better fit the structure of the thesis and to avoid unnecessary repetitions. Two chapters have been added to introduce the frameworks that are used throughout the papers.

The first chapter is devoted to mathematical preliminaries, namely the definition of G-structures. The emphasis is put on $SU(3)$-structures and their relations with spinors, which are essential in flux compactification. A minimal introduction to generalized complex geometry is given in order to properly define $SU(3) \times SU(3)$-structures. Chapter 2 then explains why those structures are relevant in flux compactification. This is also a good opportunity to discuss supergravity further, its relation with string theory and branes. Original content starts in chapter 3 which studies consistent truncations of type IIA supergravity in the presence of fermionic condensates. At first dilatini condensates are considered with a Nearly-Kähler internal space[2]. A second truncation is presented, with a Calabi-Yau internal manifold and gravitini condensates [3]. It is suggested that the latter can be generated from gravitational instantons. Chapter 4 is focused on the interplay between branes and non abelian T-duality [4]. Several brane configurations are considered together with their dual where the branes are studied by computing the flux charges. The D2 supersymmetric solution is further studied through its $SU(3)\times SU(3)$ structure. Chapter 5 presents a systematic construction of $SU(3)$-structures on a class of toric varieties after a brief introduction to the toric formalism [1]. The purpose is to investigate if such spaces could be suitable for flux compactification.

Contents

1 Geometrical tools

Throughout the thesis I will assume that the reader is familiar with standard tools from differential geometry, among which are the very definition of a smooth manifold, vector and tensor fields, differential forms (and their cohomology), Riemannian geometry. Unless otherwise stated all objects will be taken in the smooth category, since this is the most relevant case for our physical applications.

This chapter's purpose is to introduce further tools that will be necessary (or at least interesting) in the context of compactification. Being myself fond of mathematical nitpicking, I will try to focus on the point and avoid (too many) unnecessary digressions. The ultimate goals are the notions of $SU(3)$-structure and $SU(3)\times SU(3)$-structure which are the relevant structures for the internal six dimensional space in type II compactification. This will require a general discussion on G-structures on vector and principal bundles.

1.1 Fiber bundles in differential geometry

1.1.1 Fiber bundles

The easiest way to build higher dimensional manifolds is to merely "add" dimensions. Given two manifolds B and F of dimensions n and k, the Cartesian product $E = B \times F$ is a manifold of dimension $n+k$. Since each point of $y \in E$ can be written uniquely in the form $y = (x, \xi)$ where $x \in B$ and $\xi \in F$ we can define projections:

$$\begin{array}{rcl} \pi_B : B\times F & \to & B \\ (x,\xi) & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_F : B\times F & \to & F \\ (x,\xi) & \mapsto & \xi \end{array} \tag{1.1}$$

Fiber bundles are a generalization of this.

Fiber bundle A fiber bundle is given by three manifolds E, B and F together with a projection $\pi : E \to B$ such that: for each $x \in B$ there is an open neighborhood $U \subset B$ of x and a diffeomorphism $\varphi : \pi^{-1}U \to U \times F$ satisfying $\pi_U \circ \varphi = \pi$.

The data (U, φ) are called local trivializations of the bundle E which is a manifold of dimension $n+k$. Note that the trivialization condition implies that $E_x := \pi^{-1}\{x\} \simeq F$. E_x is called the fiber over x and the total space E can be seen as the disjoint union, over the base B, of all the fibers. The trivializations ensure that the union is locally well behaved, and looks like a Cartesian product. If the trivialization can be defined globally, *ie* $U = E$, then the bundle is trivial and the total space is merely a Cartesian product. In general, this will not be the case.

Similarly to the Cartesian product, an element $y \in E$ can be written $y = (x, \xi)$ with $x \in B$ and $\xi \in E_x$. However the role of B and F are not symmetric anymore since we have to choose a base point x before knowing in which fiber ξ lives. π naturally projects onto the base so that $x = \pi(y)$ but there is *a priori* no canonical projection onto the fiber.

Sections A section of a bundle is a function $\sigma : B \to E$ such that $\pi(\sigma(x)) = x$ for $x \in B$. Such a function do not necessarily exist for a general fiber bundle. We thus need to also consider local sections, namely sections of the bundle $E_{|U} = \pi^{-1}U$ over U. Local sections always exist on a local trivialization, since they can be identified as functions $\sigma_U : U \to F$.

Transition functions and structure group Consider now an atlas of E, *ie* a set of local trivializations (U_i, φ_i) covering the whole space: $B = \bigcup_i U_i$. On an intersection $U_i \cap U_j$, both φ_i and φ_j can be used and $\varphi_i \circ \varphi_j^{-1}$ is a function $U_i \cap U_j \times F \to U_i \cap U_j \times F$. Since both leave the base point unchanged we can write for $x \in B$, $\xi \in F$:

$$\varphi_i \circ \varphi_j^{-1}(x, \xi) = (x, t_{ij}(x)(\xi)) \tag{1.2}$$

which defines the transition functions $t_{ij} : U_i \cap U_j \to G$ where G is a group of transformations acting on F, called the structure group of the bundle. They satisfy several compatibility conditions:

$$\begin{aligned} t_{ii} &= \mathrm{id} \\ t_{ji} &= t_{ij}^{-1} \\ t_{ij}t_{jk}t_{ki} &= \mathrm{id} \end{aligned} \tag{1.3}$$

The last equation in (1.3) is the cocycle condition and is defined on triple intersection $U_i \cap U_j \cap U_k$. In fact the transition functions tell us how the different patches U_i are glued together and how the fibers are twisted when we move along the base. They give enough information to build the bundle, provided they satisfy (1.3).

In practice the structure group is usually defined as part as the data required for the bundle together with its action on the fiber and is chosen to be a nice group (*eg* a Lie group). Which structure group can be chosen for a specific bundle is a question that will keep us busy for the remaining of the chapter.

1.1.2 Vector bundles and principal bundles

I have up to now only discussed general fiber bundles where the fiber F is an arbitrary manifold. When some structure is further imposed on F the bundle will also inherit from it in a certain way. This is the case for vector bundles (the fiber is a vector space) and principal bundles (the fiber is a Lie group).

Vector Bundle A rank k vector bundle is a fiber bundle $\pi : E \to M$ with $F = \mathbb{R}^k$. For each $x \in M$, the fiber E_x is a vector space and the trivialization (U, φ) induce an isomorphism

$$\begin{aligned} \mathbb{R}^k &\to E_x \\ v &\mapsto \varphi^{-1}(x, v) \end{aligned} \tag{1.4}$$

A few remarks can readily be made from this definition:

- the same definition and all the following remarks would hold also for a complex vector bundle with obvious substitutions.

- on a double intersection $U_i \cap U_j$ the transition function t_{ij} are automorphisms of $\mathbb{R}^n$, *ie* matrices of $\mathrm{GL}_k(\mathbb{R})$. So the structure group is naturally $\mathrm{GL}_k(\mathbb{R})$
- E_x is a vector space and has its own zero vector. Thus E has a canonical global zero section given by $\sigma(x) := (x, 0_{E_x})$ for $x \in M$. The space of sections $\Gamma(E)$ is thus a vector space, more specifically this is a $C^\infty(M)$ module since we can define $f \cdot u\ (x) := f(x)u(x)$ for $u \in \Gamma(M)$.

Moreover operations on vector spaces can be extended to operations on vector bundles. Consider two vector bundles E and F over the same base M, we can define the following bundles

- Direct sum: $E \oplus F$ whose fiber over x is $E_x \oplus F_x$
- Tensor product: $E \otimes F$ whose fiber is $E_x \otimes F_x$
- Dual: E^* whose fiber is E_x^*

Principal bundles A G-principal bundle is a fiber bundle $\pi : P \to M$ with $F = G$ a Lie Group. There is a right action of G on P such that: for any $x \in M$, G preserves P_x and its action on P_x is free and transitive.

Note that the fiber P_x over each point is diffeomorphic to the group G but is not a group (in a relevant way). The obstruction is in fact the lack of an identity element. Suppose now that there is a global section $\sigma : M \to P$. Then for $p \in P$ and $x = \pi(p)$, p and $\sigma(x)$ belong in P_x. Since the action of G on P_x is free and transitive there is a unique $g \in G$ such that $p = \sigma(x) \cdot g$. This gives a diffeomorphism

$$\begin{array}{rcl} \Phi : M \times G & \to & P \\ (x, g) & \mapsto & \sigma(x)g \end{array} \tag{1.5}$$

Thus a principal bundle is trivial if and only if it admits a global section. This is drastically different from the previous case of vector bundle which always admits global sections. Despite their different properties, vector and principal bundles are related in a natural way.

Frame bundle Consider a rank k vector bundle $\pi : E \to M$. For $x \in M$ the fiber E_x is a finite dimensional vector space and thus admits bases. Take F_x the set of ordered bases for E_x: GL_k acts on F_x by a change of basis. Any two basis are related by a unique transformation of GL_k so the action is free and transitive. Then $F(E) := \bigcup_{x \in M} F_x$ is a GL_k-principal bundle called the frame bundle, whose fiber at $x \in M$ is exactly F_x. Note that a global section on $F(E)$ would define a canonical basis for any F_x and thus would trivialize $F(E)$: a vector bundle is trivial if and only if its frame bundle is trivial.

This shows that any vector bundle can be associated a principal bundle. It is also possible to go the other way around.

Associated bundle Consider a G-principal bundle $\pi : P \to M$. Let $\rho : G \to \mathrm{GL}(V)$ a k-dimensional representation of the Lie group G. Define now an action of G on $P \times V$: $g \cdot (p, v) := (p\, g, \rho(g)^{-1} v)$. Then $E = P \times V/G$ is a rank k vector bundle on M, called the associated bundle to P with fiber V.

On way to see this is that $P \times V$ is trivially a vector bundle over P. The group action relation will remove the degrees of freedom of the fibers P_x, but at the same time twist the fiber V. The twist of the bundle P will be shifted to E via ρ. If P is a trivial bundle, then E will also be trivial (but the converse is not true). Unsurprisingly any vector bundle is the associated bundle of its frame bundle. However a frame bundle is always a GL_k-principal bundle so that principal bundles are not necessarily frame bundles.

On a manifold M of dimension n the tangent bundle TM is a rank n vector bundle and its frame bundle FM is a GL_n-principal bundle. Both will play an especially important role in the study of properties of M. TM and its dual T^*M are used to build vector bundles through tensor products and direct sums whose sections are usual objects in physics:

- Vector fields are sections of TM
- 1-forms are sections of T^*M
- Higher order differential forms are sections of the antisymmetric products of T^*M: $\Omega M = \Lambda(T^*M)$
- Metrics are non-degenerate (to be understood fiberwise) sections of the symmetric part of $T^*M \otimes T^*M$.

1.2 G-structures

As we have seen, the structure group of a bundle restricts the transition functions, which in turn define how the bundle is twisted. This means that the more restricted the group, the less twisted the bundle. However looking back at the definitions, the structure group seems a bit arbitrary. For example for another group H such that G is a subgroup of H, it would be as valid as G for a structure group. This can also be seen for the trivialization functions which are nothing but a choice of coordinates. It is possible to add new charts with arbitrary complicated coordinates without actually changing the bundle, but enlarging the structure group. There can be in fact a lot a redundancy in the local trivializations and one would like to have some control on the twisting of the bundle. The question at stake here is given a fiber bundle, which groups could be chosen as its group structure, providing we made the right choice in defining the local trivializations ?

1.2.1 Reduction of the structure group

Consider first $\pi : P \to M$ a G-principal bundle and another group H with a homomorphism $\phi : G \to H$. It is then possible to build bundle associated to P with fiber H in a similar fashion as for vector bundles. Define an action of G on $P \times H$: $g \cdot (p, h) := (pg, \phi(g)^{-}1h)$. Then $Q := (P \times H)/G$ is a H-principal bundle over M where $h' \in H$ acts on Q by $h' \cdot (p, h) := (p, h\,h')$. This action commutes with the action of G and thus descends to the quotient. The bundle structure of Q comes from P and thus do not carry more information.

Considering the other direction, one can wonder whether the G-principal bundle P comes from a H-principal bundle Q:

Reduction of the structure group Let H be a group with a homomorphism $\phi : H \to G$. A reduction of the structure group of P from G to H is a H-principal bundle Q such that the bundle associated to Q is isomorphic to P.

For a rank k vector bundle E its frame bundle $F(E)$ is a GL_k-principal bundle, this leads to G-structure.

G-structure Consider a homomorphism $\phi : G \to \mathrm{GL}_k$. A G-structure on E is a reduction of the structure group of $F(E)$ to G.

The relation with the structure group is what we could expect: a vector bundle E admits a G structure if and only if its structure group can be chosen to be G. Once again this does not mean that this condition is indeed fulfilled, but only that it is possible to find an atlas for which the transition functions all belong in G.

Before we get to examples a few remarks are in order:

- In the following we will mainly look at reductions of the frame bundle of a n-dimensional manifold M, so a G-structure on M will refer to its tangent bundle.
- In most relevant cases ϕ is an inclusion map, hence the name *reduction*, but this is not mandatory. When this is the case ϕ induces an inclusion $\tilde{\phi} : Q \to P$ so that the reduction Q is in fact a subbundle of P. For us G will most likely be a subgroup of GL_m, and the reduction will correspond to a subset of local bases for TM.
- The existence of a G-structure for a given group G is a topological property. However a G-structure itself is neither unique nor canonical.
- As we have seen the frame bundle FE of an associated bundle E to P is not in general the same thing as P. However P will give a reduction of the structure group of FE, for the morphism given by the representation.

1.2.2 Examples

This definition of G-structures is rather straightforward and elegant, however it is basically useless for computations. A G-structure can rather usually be defined by a specific G-invariant non degenerate tensor T. Provided that the defining properties of T are well chosen, it can be identified with the G-structure. It is then much easier (at least for our purposes) to work with this tensor than with the more abstract notion of principal bundle. It is also important to notice that this alternate definition enables to translate many common geometrical properties in the framework of G-structures. Rather than giving a general proof for this heuristic (and hazy) statement I will show how this works out on a case by case basis.

O(n) Consider a reduction P of the structure group of M to $O(n)$. Thus P is a subbundle of FM and gives for $x \in M$ a set of bases of T_xM related to each other by an $O(n)$-transformation. A basis defines canonically a metric g on T_xM, which does not depend on the choice of basis in P_x. This metric is well defined everywhere and furnishes M with a Riemannian structure. Note that a local section of P, which is also a local trivialization of TM, is a vielbein for g.

Conversely consider a Riemannian manifold M with metric g. The bundle of orthonormal bases for g is a $O(n)$-bundle and thus gives a reduction of FM.

An $O(n)$-structure on M is a Riemannian metric. Since it is possible to define a metric on any manifold, note that there is no obstruction to the reduction to $O(n)$. The same construction can be done with metrics of signature (p, q) and $O(p, q)$, but then there can be topological obstructions.

SO(n) This case is similar as the previous one but now the basis can be attributed an orientation since the $SO(n)$-transformations will preserve it. This will define an orientation on the whole space. Conversely an oriented Riemannian manifold enables to define oriented orthonormal bases and thus a $SO(n)$ subbundle of FM.

A manifold admits a $SO(n)$-structure if and only if it is orientable. Strictly speaking the orientation is defined by a Gl_n^+ structure but the reduction to $O(n)$ is automatic.

SO(n − 1) Consider an oriented Riemannian manifold with an $SO(n-1)$-structure. For our purposes we will use the following inclusion:

$$\begin{array}{rcl} \phi : SO(n-1) & \to & SO(n) \\ g & \mapsto & \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \end{array}$$

On a trivialization (U, φ) define the vector field $V(x) := \varphi^{-1}(x, e_1)$ for $x \in U$ and $e_n = (1, 0, \ldots, 0)$. Since e_1 is invariant under $SO(n-1)$, this definition is compatible with the transition functions and V can be extended to M. Note that the norm of V is 1 every where, and thus V does not vanish.

Conversely consider an oriented Riemannian manifold M with a non vanishing vector field V. For $x \in T_xM$, take a basis $\mathcal{B}$ of the orthogonal complement of $V(x)$. Then the bundle of bases of the form $(V(x), \mathcal{B})$ is a $(SO(n-1))$-subbundle of FM.

An oriented manifold admits a $SO(n-1)$-structure if and only if it has a non-vanishing vector field.

Spin(n) Consider an oriented Riemmanian manifold M with a $\mathrm{Spin}(n)$-structure. Here $\phi : \mathrm{Spin}(n) \to SO(n)$ is a double cover. This is a typical case where the structure is not reduced to a subgroup (and the only relevant one that I am aware of). Call P the reduction of the frame bundle. Take S_0 a spinorial representation of $\mathrm{Spin}(n)$ (it does not matter yet whether we take a Dirac, Weyl or Majorana representation). The associated bundle S of P with fiber S_0 is a spinor bundle and thus M admits spinors.

A spin manifold if an oriented Riemannian manifold with a $\mathrm{Spin}(n)$*-structure.* Note that the spin structure of M is defined relatively to a specific metric and orientation, and that it is in general not unique. M admits a spin structure if it is possible to lift the transition functions to the spin group, the topological obstruction is given by the second Stiefel-Whitney class.

1.2.3 Torsion

Some geometrical properties are stronger than the mere existence of a G-structure, and require its so-called integrability. These conditions are expressed in terms of the torsion classes of the G-structure. I will not delve into too much details and will rather define the torsion classes for a specific structure when needed. Here is a quick insight about this.

As we have seen a G-structure on a manifold M is a reduction P of its frame bundle. Since P is a principal bundle we can define a connection on it. This gives rise to a connection on TM with values in the Lie algebra of G. The torsion of the G-structure is related to the torsion of this connection and can be defined independently of the choice of connection.

For an $O(n)$-structure, a connection on P is a connection that is compatible with the metric. It is always possible to choose a connection without torsion: the Levi-Civita connection.

This means that the torsion of an $O(n)$-structure is trivially zero. If the structure group is further reduced to a group G, look at the class of $O(n)$-connections that are still compatible with the G-structure. The Levi-Civita connection needs not be in this class, in which case the torsion of the G-connection cannot be torsion free: the torsion of a G-structure gives obstruction to the compatibility of the Levi-Civita connection.

1.3 Complex structures and integrability

1.3.1 Complex manifolds

A complex manifold is defined in a similar way as a real one by replacing real smooth function by complex holomorphic ones. This leads to a more restrictive structure: a complex n dimensional manifold M is also a real $2n$ dimensional manifold but the converse is not true.

Tangent bundles As a manifold, M naturally comes equipped with a tangent bundle TM. However its structure as a complex manifold also provides M with a holomorphic tangent bundle $T^{(1,0)}M$. Both bundles are different but can be unified in a larger bundle, the complexified tangent bundle:

$$T_{\mathbb{C},z}M := T_{\mathbb{R},z}M \otimes \mathbb{C} \tag{1.6}$$

In fact $T^{(1,0)}M$ can be embedded in $T_{\mathbb{C}}M$. The embedding is given in local coordinates $z^k = x^k + iy^k$, $k = 1, \dots, n$ by $\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right)$. The tangent space to M is then the real vector space generated by the vector fields $\frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^k}$, while the holomorphic tangent space is the complex vector space generated by the holomorphic vector fields $\frac{\partial}{\partial z^k}$. This gives a decomposition of the complexified tangent bundle:

$$T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M \tag{1.7}$$

where $T^{(0,1)}M = T^{(1,\bar{0})}M$ is the anti-holomorphic tangent bundle. Let us resume here the rank of the different bundles:

$$\begin{aligned} \dim_{\mathbb{R}} TM &= 2n \\ \dim_{\mathbb{C}} T_{\mathbb{C}}M &= 2n \\ \dim_{\mathbb{C}} T^{(1,0)}M &= n \end{aligned}$$

Differential forms The decomposition 1.7 can also be extended to any complex tensor, hence for differential forms. This leads to the following decomposition for complex k-forms into (p, q)-forms:

$$\Omega^k M = \bigoplus_{p+q=k} \Omega^{p,q} M \tag{1.8}$$

where $\Omega^{p,q}M = \Lambda^p\left(T^{(1,0)}M\right)^* \otimes \Lambda^q\left(T^{(0,1)}M\right)^*$. In local coordinates this corresponds to a decomposition in wedge products of $\mathrm{d}z^k, \mathrm{d}\bar{z}^k$ whose pairing with vectors fields gives:

$$\begin{aligned} \mathrm{d}z^i \cdot \frac{\partial}{\partial z^j} &= \mathrm{d}\bar{z}^i \cdot \frac{\partial}{\partial \bar{z}^j} = \delta^i_j \\ \mathrm{d}z^i \cdot \frac{\partial}{\partial \bar{z}^j} &= \mathrm{d}\bar{z}^i \cdot \frac{\partial}{\partial z^j} = 0 \end{aligned} \tag{1.9}$$

Since the holomorphic and anti-holomorphic tangent spaces both have dimension n, p and q are limited to n: $\Omega^{p,q}M = \{0\}$ if $p > n$ or $q > n$. The exterior derivative can also be decomposed along these subspaces. For a complex function f on M:

$$\mathrm{d}f = \frac{\partial f}{\partial z^k}\mathrm{d}z^k + \frac{\partial f}{\partial \bar{z}^k}\mathrm{d}\bar{z}^k \tag{1.10}$$

This can be extended to higher forms so that $\mathrm{d} = \partial + \bar{\partial} : \Omega^{p,q}M \to \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M$.

1.3.2 Almost complex structures

Let us first recall a few algebraic properties. An n-dimensional complex vector space is trivially also a $2n$-dimensional vector space. The converse is not trivial. A real vector space V does not come equipped with a complex multiplication, we thus need to define it by hand. Consider $I \in \mathcal{L}(V)$ such that $I^2 = -1$, then define for $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ and $v \in V$:

$$\lambda\, v := (\lambda_1 + \lambda_2 I)\; v$$

With this multiplication V is a complex vector space, hence I is called a complex structure on V. Such I always exists but the choice of I is in general not canonical. Now complex endomorphisms are endomorphisms that commute with complex multiplication, *ie* with I:

$$\mathcal{L}(V, \mathbb{C}) = \{A \in \mathcal{L}(V) \mid AI = IA\}$$

For $V = \mathbb{R}^{2n}$ we define a canonical complex structure:

$$I_{2n} = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} \tag{1.11}$$

This gives a natural inclusion $\phi : M_n(\mathbb{C}) \to M_{2n}$.

$$\phi(A) = \begin{pmatrix} \operatorname{Re} A & \operatorname{Im} A \\ -\operatorname{Im} A & \operatorname{Re} A \end{pmatrix} \tag{1.12}$$

Let us come back now to the case of a complex manifold and its tangent bundle. The decomposition 1.7 will define a complex structure on each fiber. Define an operator I on $T_{\mathbb{C}}M$ by its action on each subspace. For $u, v \in T^{(1,0)}M$:

$$I(u + \bar{v}) = iu - i\bar{v} \tag{1.13}$$

This leads to $I^2 = -1$. Moreover each real vector field of TM can be written uniquely as $u + \bar{u}$ for $u \in\in T^{(1,0)}M$ so that $I(u + \bar{u}) = (iu) + \overline{(iu)}$ is still a real vector field. Thus I can be seen as an endomorphism on TM.

Conversely suppose that there exists an endomorphism I on TM such that $I^2 = -1$. Then I can be diagonalized on $T_{\mathbb{C}}M$ with eigenvalues $\pm i$. Define $T^{(1,0)}M$ (resp. $T^{(0,1)}M$) the eigenspaces of I for the eigenvalue i (resp. $-i$). For I to be real both should be conjugate to each other and thus have same dimension. This leads to a decomposition similar to 1.7. As a side note this imposes the dimension of M to be even.

An almost complex structure on an even dimensional manifold is an endomorphism I of TM such that $I^2 = -1$. As we have just seen, a complex manifold has a canonical almost complex structure. The question that naturally arises now is whether an almost complex structure comes from a complex manifold. The answer is already given in the word *almost*: there are additional conditions. Indeed the almost complex structure gives a decomposition of the tangent bundle. Locally this is just an algebraic condition on the tangent spaces but says nothing about the manifold itself. A complex manifold would require complex coordinates that are compatible with this decomposition. The Newlander-Nirenberg theorem states that this is possible if and only if the Nijenhuis tensor vanishes. It is a 2-form with values in vector fields given by:

$$N(u, v) = [u, v] + I[Iu, v] + I[u, Iv] - [Iu, Iv] \tag{1.14}$$

for $u, v \in \Gamma(TM)$. In local coordinates this becomes:

$$N_{ij}^k = -I_i^l\, \partial_l I_j^k + I_j^l\, \partial_l I_j^k + I_l^k\, (\partial_i I_j^l - \partial_j I_i^l) \tag{1.15}$$

When this is the case the almost complex structure is said integrable and can be promoted to a complex structure, which enables to define complex coordinates on the manifold. When they exist complex structures are very numerous compared to smooth structures and generally span continuous degree(s) of freedom.

$Gl_n(\mathbb{C}$-structures The previous results concerning almost complex structure can all be translated in the formalism of G-structures, in a similar fashion as was done in the examples of 1.2.2. Consider a manifold M of dimension $2n$ with almost complex structure I. The complex frame bundle of $T^{(1,0)}M$ is a $Gl_n(\mathbb{C})$-principal bundle and gives a reduction of FM. Indeed if $f_1, \ldots, f_n$ is a basis for $T^{(1,0)}M$, then we can define a basis $(e_1, \ldots, e_{2n})$ of TM by $e_i = \operatorname{Re} f_i, e_{i+n} = \operatorname{Im} f_i$ for $i = 1 \ldots n$.

Conversely suppose the structure group of TM is reduced to $Gl_n(\mathbb{C})$, for the inclusion ϕ. I_{2n} acts naturally on each local trivialization of TM and commutes with the transition functions, and thus can be globally extended to an almost complex structure I.

An almost complex structure is a $Gl_n(\mathbb{C})$-structure.

Here the G-structure only corresponds to the *almost* case. In fact the integrability condition can be expressed as a condition on the torsion G-structure: the Nijenhuis tensor is exactly the torsion of the $Gl_n(\mathbb{C})$-structure.

I will end this section with a remark about orientation. Consider a matrix $A \in Gl_n(\mathbb{C})$. Then it is easy to show that

$$\det \phi(A) = |\det A|^2 \tag{1.16}$$

This means that when considered as a real matrix, the determinant of A is positive. So that the inclusion is in fact $Gl_n(\mathbb{C}) \subset Gl_{2n}^+$. Ultimately any $Gl_n(\mathbb{C})$-structure is also a Gl_{2n}^+-structure and an (almost) complex manifold is canonically oriented.

1.3.3 Hermitian metrics

In the same way that real vector bundles can be equipped with a metric, complex vector bundles can be equipped with an hermitian metric: *ie* an hermitian form on each fiber. For a bundle E, an hermitian metric h is a section of $E^* \otimes \bar{E}^*$. Focusing on an (almost) complex manifold M we take $E = T^{(1,0)}M$ so that h is a section of $T^{(1,0)}M^* \otimes T^{(0,1)}M^*$. It is crucial to notice that h defines an hermitian form on the holomorphic tangent bundle only, and not on the full complexified tangent space. h can be understood equivalently as a bilinear form on $T^{(1,0)}M \times T^{(0,1)}M$ or as a sesquilinear form on $T^{(1,0)}M \times T^{(1,0)}M$.

$U(n)$-structure On a complex vector space the group of transformation that preserve an hermitian form is precisely the unitary group $U(n)$. Thus by an argument nearly identical to the $O(n)$-case:

An $U(n)$-structure on M is an hermitian metric. In this case the reduction of the frame bundle provides local bases (f_i), $i = 1, \dots, n$ for $T^{1,0}M^*$ that are complex vielbeins for h:

$$h = f_i \otimes \bar{f}_i \tag{1.17}$$

Decompose h into real and imaginary parts:

$$h = g - iJ \quad , \text{where} \quad \left\{ \begin{array}{lcl} g & = & \frac{1}{2}(f_i \otimes \bar{f}_i + \bar{f}_i \otimes f_i) = f_i \, \bar{f}_i \\ J & = & \frac{1}{2}(f_i \otimes \bar{f}_i - \bar{f}_i \otimes f_i) = \frac{i}{2} f_i \wedge \bar{f}_i \end{array} \right. \tag{1.18}$$

Thus h is written in terms of two sections of $T^*M \otimes T^*M$: g is symmetric, and J is antisymmetric and thus lves in $\Omega^{1,1}M$. This means that an hermitian metric comes together with a riemannian metric and a symplectic form. They are related by the almost complex structure:

$$J(u, v) = g(Iu, v) \quad \text{or in local coordinates:} \quad J_{mn} = I_m^{\ p} g_{pn} \tag{1.19}$$

Conversely consider a Riemannian manifold M with metric g and an almost complex structure I. Then define a symplectic form J and an hermitian metric h using 1.19 then 1.17. For this construction to work, g and I need to be compatible, namely J should be antisymmetric. This leads to the condition:

$$I_m^{\ p} I_n^{\ q} g_{pq} = g_{mn} \tag{1.20}$$

which merely means that I is orthogonal towards g. This is not really restrictive since it is always possible to define a corrected metric such that 1.20 holds:

$$g'_{mn} := \frac{1}{2}(g_{mn} + I_m^{\ p} I_n^{\ q} g_{pq}) \tag{1.21}$$

Any almost complex manifold admits an $U(n)$-structure.

The relation between g, J and I can be understood by looking at G-structures on a $2n$-dimensional manifold M. As we have seen g and I are respectively equivalent to $O(2n)$- and $Gl_n(\mathbb{C})$-structures. In fact J can also be identified with a $Sp(2n, \mathbb{R})$-structure: the symplectic group is indeed the group of transformation that preserve a symplectic form. Here is a reminder of what these matrix groups look like (here g is the identity matrix so that the matrix representation of I and J are both I_{2n} even though they are still different tensors):

$$\begin{array}{rcl} O(2n) & = & \{A \in Gl_{2n} \mid {}^t A = A^{-1}\} \\ Gl_n(\mathbb{C}) & = & \{A \in Gl_{2n} \mid A^{-1} \, I_{2n} \, A = I_{2n}\} \\ Sp(2n, \mathbb{R}) & = & \{A \in Gl_{2n} \mid {}^t A \, I_{2n} \, A = I_{2n}\} \end{array}$$

Notice also that the inclusion $\phi : M_n(\mathbb{C}) \to M_{2n}$ verifies:

$$\phi(A^\dagger) = {}^t\phi(A) \tag{1.22}$$

so that:

$$U(n) = O(2n) \cap Gl_n(\mathbb{C}) = O(2n) \cap Sp(2n, \mathbb{R}) = Gl_n(\mathbb{C}) \cap Sp(2n, \mathbb{R})$$

This translates the fact that two among g, I, J are sufficient to define the third one (provided they are compatible). Note also that 1.16 implies that $U(n) \subset SO(2n)$. Thus the hermitian metric defines an orientation. Since J is non-degenerate, $\frac{1}{n!}J^n$ defines a volume form on M.

1.4 $SU(3)$-structures

We now have all the necessary tools to define $SU(3)$-structures. We first consider the general $SU(n)$-case before specializing to $n = 3$ for specific properties.

1.4.1 $SU(n)$-structures

Consider a $2n$ dimensional manifold M with $U(n)$-structure defined by a hermitian metric h. In order to reduce to $SU(n)$ we need to ensure the determinant of the transition functions to be 1. The determinant here refers to the determinant as a complex matrix, since the real determinant is already 1 (recall that M is already oriented). The ideal solution would be to find a tensor which is affected only by the determinant of $U(n)$-transformations, *ie* a complex equivalent to a top form. Such an element is given by the decomposition 1.8: we are interested in $\Omega^{n,0}M$:

Suppose the structure group if further reduced to $SU(n)$ and take a (local) vielbein f_i, $i = 1, \dots n$ in the reduced frame bundle. Since we are reducing from $U(n)$, 1.17 is still satisfied. Define locally a $(n, 0)$ form:

$$\Omega = \bigwedge_{i=1}^{n} f_i \tag{1.23}$$

This definition is trivially compatible with the transition functions and thus leads to a global non vanishing form $\Omega \in \Omega^{n,0}M$. Using the local expressions 1.18 and 1.23 it is easy to find the following relations with J:

$$\begin{aligned} J \wedge \Omega &= 0 \\ \Omega \wedge \bar{\Omega} &= (-1)^{\frac{1}{2}n(n+1)} \frac{(2i)^n}{n!} J^n \end{aligned} \tag{1.24}$$

These are fundamental for the $SU(n)$-structure. The first equation comes from the fact that $J \in \Omega^{1,1}M$ and $\Omega \in \Omega^{n,0}M$. The second on translates the non-degeneracy of Ω.

Conversely suppose now that there exists on M a global non vanishing $(n, 0)$-form Ω. Without further restriction Ω can be normalized so that 1.24 is satisfied. In a local vielbein f_i for $U(n)$, Ω is thus:

$$\Omega = e^{i\varphi} \bigwedge_{i=1}^{n} f_i \tag{1.25}$$

Let us keep only the bases for which $e^{i\varphi} = 1$: this defines a reduction of the frame bundle to $SU(n)$. Thus Ω defines a $SU(n)$-structure on M.

In fact the hermitian structure is not necessary as it can be retrieved from J, Ω only. For this to work, Ω needs to be complex decomposable and non vanishing. This means that there exits local one-forms $\eta_i, i = 1 \ldots n$ such that $\Omega = \bigwedge_{i=1}^n \eta_i$. The complex subspace generated by η_i then defines an almost complex structure. This can also be seen by defining the endomorphism I explicitly. For example, when $n = 3$:

$$\tilde{I}_j^k = \epsilon^{kpqrst} \operatorname{Im} \Omega_{j[pq} \operatorname{Im} \Omega_{rst]} \tag{1.26}$$

Then normalize:

$$I = \frac{1}{\sqrt{-\frac{1}{6} \operatorname{tr} \tilde{I}^2}} \tilde{I} \tag{1.27}$$

Then I is an almost complex structure towards which Ω is a $(3,0)$-form. Now using J, relation 1.19 gives a metric. However this construction does not give any control on the signature of the metric, so in order to really get an $SU(3)$-structure (and not an $SU(p,q)$), the positivity of the metric should be imposed by hand. In practice this will not be a problem here since a Riemannian metric will always be defined upstream. The goal will then be to find compatible J and Ω.

1.4.2 Relations with pure spinors

A primordial aspect of $SU(n)$-structures is that they can be equivalently defined by non vanishing pure spinors. We will specify here to $n = 3$ but the argument can be extended for general n.

Consider a 6-dimensional spin manifold M. This means that M has a Spin(6)-structure (call P the reduction) and thus an $SO(6)$: M is oriented and Riemannian with metric g. In $6d$ the irreducible spinorial representations are Weyl spinors. Denote S^+ and S^- the bundle of spinor of positive and negative chirality, of complex rank 4. The fibers correspond in fact, through the accidental isomorphism Spin(6) $\sim SU(4)$, to the vector representations of $SU(4)$ and are conjugate to each other. Now suppose that there exists on M a non vanishing spinor $\eta \in \Gamma(S^+)$ of positive chirality. By an argument similar to what was done for $SO(n-1)$, this gives a reduction of P to $SU(3)$. Since TM is also associated to P, we can conclude that η defines an $SU(3)$-structure on M.

The structure can be explicitly constructed using spinor bilinears in η:

$$\begin{aligned} J_{ij} &= i\,\eta^\dagger \gamma_{ij} \eta = -i\,\tilde{\eta}\gamma_{ij}\eta^c \\ \Omega_{ijk} &= \eta^\dagger \gamma_{ijk} \eta^c \\ \Omega^*_{ijk} &= -\tilde{\eta}\gamma_{ijk}\eta \end{aligned} \tag{1.28}$$

The choices here are made to be compatible with (1.24) and (1.19), and this can be checked by Fierzing, using:

$$\begin{aligned} \eta\eta^\dagger &= \tfrac{1}{4}(1 + \tfrac{i}{2} J_{ij}\gamma^{ij})\, P^+ \\ \eta\tilde{\eta} &= \tfrac{1}{48}\bar{\Omega}_{ijk}\, P^+ \gamma^{ijk} \\ \eta^c\eta^\dagger &= -\tfrac{1}{48}\Omega_{ijk}\, P^- \gamma^{ijk} \end{aligned} \tag{1.29}$$

This result can be extended to other dimensions. But without the accidental isomorphism, the conclusion is not as straightforward and an additional constraint has to be imposed: the spinor needs to be pure. [1] Note that the holomorphic tangent space can also be defined directly from the pure spinor. First let us recall that a vector V acts on spinors η as a gamma matrix:

$$V \cdot \eta := V^i \gamma_i \eta \tag{1.30}$$

Now a pure spinor is defined as a non vanishing spinor that is annihilated by a subspace $L \subset T_{\mathbb{C}}M$ of maximal dimension, namely n. This gives a splitting of the tangent space similar to (1.7):

$$T_{\mathbb{C}}M = L + \bar{L} \tag{1.31}$$

and thus defines an almost complex structure, whose holomorphic tangent space is L (or $\bar{L}$ depending on the choice of convention).

The formulation in terms of spinors is what makes $SU(3)$-structures so important in the context of supersymmetric compactification, as we will see in the next chapter.

1.4.3 Torsion classes

As we have seen in 1.2.3 any G-structure can be attributed a torsion, which provides additional information about the structure. For $SU(3)$-structure the torsion can be retrieved by decomposing $\mathrm{d}J$ and $\mathrm{d}\Omega$ along irreducible representations of $SU(3)$ (this is not a proof). The almost complex structure splits the tangent space into holomorphic and anti-holomorphic parts and enables to decompose differential form along 1.8. However the almost complex structure need not be integrable in general, and there is no reason for the exterior derivative to be compatible with the decomposition (this is in fact a sufficient condition for integrability).

Thus a holomorphic 1-form $\eta \in \Omega^1 M$ can "loose its holomorphicity" after the action of d: $\mathrm{d}\eta \in \Omega^{2,0}M \oplus \Omega^{1,1}M \oplus \Omega^{0,2}$. Now according to Leibniz rule d acts separately on each factor of a wedge product, so that in the worst case a higher degree form will lose 1 holomorphic degree and gains 2 anti-holomorphic degrees:

$$\mathrm{d}\Omega^{p,q} \subset \Omega^{p+2,q-1} \oplus \Omega^{p+1,q} \oplus \Omega^{p,q+1} \oplus \Omega^{p-1,q+2} \tag{1.32}$$

Applying this to J and Ω implies that $\mathrm{d}J \in \Omega^{3,0} \oplus \Omega^{2,1} \oplus \Omega^{1,2} \oplus \Omega^{0,3}$ and $\Omega \in \Omega^{3,1} \oplus \Omega^{2,2}$. Since J is real the $(3,0)$ and $(2,1)$ parts are respectively conjugate to the $(0,3)$ and $(1,2)$. As $SU(3)$ acts trivially on J, the symplectic structure enables to further decompose by extracting terms proportional to J. This leads to a decomposition of $\mathrm{d}J, \mathrm{d}\Omega$ into representations of $SU(3)$:

$$\begin{aligned} \mathrm{d}J &= \tfrac{2}{3}\,\mathrm{Im}\, W_1^* \Omega + W_3 + W_4 \wedge J \\ \mathrm{d}\Omega &= W_1\, J^2 + W_2 \wedge J + W_5^* \wedge \Omega \end{aligned} \tag{1.33}$$

where the W_i are called the torsion classes of the $SU(3)$-structure.

- W_1 is a complex scalar
- $W_2 \in \Omega^{1,1}M$ is a primitive 2-form
- $W_3 \in \Omega^{2,1} \oplus \Omega^{1,2}$ is a real primitive 3-form

[1] In 6-dimension any Weyl spinor is pure and this is why this is not an issue here.

- $W_4 \in \Omega^{1,0} \oplus \Omega^{0,1}$ is a real 1-form
- $W_5 \in \Omega^{1,0}$ is a holomorphic 1-form

Primitive refers to the fact that the component along J has been projected out: the scalar $J \cdot W_2$ and the 1-form $J \cdot W_3$ vanish, where the dot represents contraction with the metric (this constraint is similar to a traceless condition). Also, the double occurrence of W_1 results from the constraint $J \wedge \Omega = 0$.

If the $SU(3)$ structure is defined using a pure spinor η the torsion classes can also be found in the covariant derivative of η:

$$\begin{aligned}\nabla_m \eta = & \frac{1}{2}\left(W_{4m}^{(1,0)} + W_{5m} - \text{c.c}\right)\eta \\ & + \frac{1}{16}\left(4W_1 g_{mn} - 2W_4^p \Omega_{pmn} + 4iW_{2mn} - iW_{3mpq}\Omega^{pq}{}_n\right)\gamma^n \eta^c \ .\end{aligned} \tag{1.34}$$

This relation can be translated to (1.33) by using (1.28,1.29).

The structure can then be classified by the vanishing of certain classes. This in fact corresponds to integrability condition of "almost structures" that live along the $SU(3)$.

Complex: If the manifold is complex then the exterior derivative is given by 1.10 so that $W_1 = W_2 = 0$. It can be shown that the converse is in fact true.

Symplectic: For a symplectic manifold the symplectic form J should be closed. Here this is equivalent to $W_1 = W_3 = W_4 = 0$

Kähler: A Kähler manifold is a complex manifold with a hermitian metric such that the associated symplectic form is closed *ie* a symplectic complex manifold: $W_1 = W_2 = W_3 = W_4 = 0$. More details are given in appendix B.

Calabi-Yau: A Calabi-Yau manifold is a Kähler manifold with a holomorphic 3-form. Even if $\Omega \in \Omega^{3,0}M$ lives in the correct subspace, it is *a priori* only a smooth section and is not a holomorphic form (where holomorphic refers also to the smoothness). Holomorphicity of Ω amounts to taking $W_5 = 0$, and thus all torsion classes vanish. Then a Calabi-Yau manifold can be equivalently defined as a manifold with a torsion-free $SU(3)$-structure, or also an integrable $SU(3)$-structure.

Nearly-Kähler: A nearly-Kähler manifold is a manifold with $SU(3)$ structure for which $W_2 = W_3 = W_4 = W_5 = 0$. W_1 is taken imaginary $W_1 = 4i\omega$, and relations (1.33,1.34) then simplify to:

$$\begin{aligned}\nabla_m \eta &= i\omega\gamma_m \eta^c \\ \mathrm{d}J &= -6\omega \mathrm{Re}\Omega \\ \mathrm{dIm}\Omega &= 4\omega J \wedge J \ .\end{aligned} \tag{1.35}$$

1.5 Generalized complex geometry

For most of our purposes, $SU(3)$ structures will be sufficient. However understanding supersymmetric flux compactification in its full generality requires Generalized complex geometry (GCG). GCG was introduced first in [5, 6] and quickly became the natural framework to study flux compactification [7, 8, 9, 10]. This section will settle for the few details that will be needed in the following, and thus will head straight to the definition of $SU(3) \times SU(3)$ structures. For more details on the interplay between GCG and compactification, see [11] for a review or [12] for a shorter course.

1.5.1 Generalized tangent bundle

The starting point of GCG is the generalized tangent bundle. For a manifold M of dimension n, define:

$$\mathcal{T}M = TM \oplus T^*M \tag{1.36}$$

At first glance, $\mathcal{T}M$ and TM will basically carry the same information. However the range of structures for the generalized tangent bundle is much richer than for the usual tangent bundle. In fact many apparently unrelated geometric notions will be unified in the language of GCG.

Thus let us now focus on the structure group of $\mathcal{T}M$. As a consequence of the decomposition 1.36, $\mathcal{T}M$ admits a metric $\mathcal{G}$ that represents the natural pairing of vectors and 1-forms:

$$\mathcal{G}(X+\xi, Y+\eta) = \xi(Y) + \eta(X) \tag{1.37}$$

for $X, Y \in TM$ and $\xi, \eta \in T^*M$. This metric is split, *ie* has signature (n, n) and thus reduces the structure group to $O(n, n)$. It is also possible to define a volume form

$$v_{\mathcal{T}} = \frac{\partial}{\partial x^1} \wedge \cdots \frac{\partial}{\partial x^n} \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \quad , \tag{1.38}$$

which does not depend on the choice of coordinates, and defines a canonical orientation on $\mathcal{T}M$. This further reduces to $SO(d, d)$. This is in fact not surprising, as we already now that the structure group of TM is Gl_n in general, which is a subgroup of $SO(d, d)$ for the inclusion:

$$\begin{array}{rcl} \iota : Gl_n & \rightarrow & SO(d,d) \\ A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \end{array} \tag{1.39}$$

All of the structures discussed in the following will need to be compatible with $SO(n, n)$-structure as we will look for further reduction of this structure group. As was the case for ordinary G-structures, the generalized structure can also be subject to integrability conditions, which will not be considered here.

Generalized spinors

As we have seen earlier, it may then be possible to lift the structure group to $\mathrm{Spin}(n, n)$. For a general orthogonal group this is not trivial but there is in fact no obstruction in the case of a split signature. This can be shown by explicitly constructing a spinor bundle.

Define $\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$ the set of polyforms on M. An element $\omega \in \Omega(M)$ is merely a sum of forms of different ranks. Vectors and 1-forms naturally act on ω respectively by interior and exterior product. This defines an action of the generalized tangent space on $\Omega(M)$:

$$(X + \xi) \cdot \omega = (\iota_X + \xi\wedge)\, \omega \tag{1.40}$$

Then a simple computation gives:

$$\{X + \xi, Y + \eta\} \cdot \omega = (\xi(Y) + \eta(X))\, \omega = \mathcal{G}(X + \xi, Y + \eta)\, \omega \tag{1.41}$$

This means that the action of $\mathcal{T}M$ on $\Omega(M)$ respects the Clifford relation, hence $\Omega(M)$ is a representation of the Clifford bundle of $\mathcal{G}$. We will thus identify generalized spinors and polyforms in the following. Note that this identification is slightly abusive: the correct action of $\mathrm{Spin}(n, n)$ on $\Omega(M)$ depends on a choice of volume form. Since in practice M will always be equipped with a metric and associated volume form, I will not discuss this issue any further.

The action (1.40) is actually real so that polyform are in fact Majorana spinors. Moreover (1.40) either lowers or raises the rank by 1. Thus even elements of the Clifford algebra will not change the rank parity of forms. It is then possible to decompose polyforms into even and odds polyforms:

$$\Omega(M) = \Omega^+(M) \oplus \Omega^-(M) \tag{1.42}$$

$\Omega^{\pm}(M)$ will then be Majoran-Weyl spinors of positive/negative chirality, as expected for a split signature (n, n). Let us now define the charge conjugation operator C. This can be seen as a bilinear form on spinors by identifying the notations:

$$C(\omega_1, \omega_2) \leftrightarrow \tilde{\omega}_1 \omega_2 \ . \tag{1.43}$$

For polyforms, this is given by the Mukai pairing:

$$\langle \omega_1, \omega_2 \rangle = \omega_1 \wedge \lambda(\omega_2)|_{top} \tag{1.44}$$

where we keep only the top part of the polyform, and λ reverses the order of the wedge products: for a p form ω_p, $\lambda(\omega_p) = (-1)^{p\frac{p-1}{2}}$. The symmetry properties then depend on the dimension of M:

$$\langle \omega_2, \omega_1 \rangle = (-1)^{n\frac{n-1}{2}} \langle \omega_1, \omega_2 \rangle \tag{1.45}$$

At last note that the Mukai pairing takes values in top forms, if we really want a bilinear form, we need to divide by the volume form of M.

Generalized Metric

The metric $\mathcal{G}$ is split and thus has nothing to do with an actual metric on M. Let us thus introduce generalized metrics.

A metric on $\mathcal{T}M$ can be written in a matrix form, as a morphism from $TM \oplus T^*M$ to $T^*M \oplus TM$. In this basis, the canonical metric becomes:

$$\mathcal{G} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \quad . \tag{1.46}$$

A generalized metric G is then a positive definite metric on $\mathcal{T}M$ satisfying the compatibility condition:

$$(\mathcal{G}^{-1}G)^2 = \mathbb{I} \tag{1.47}$$

As expected, a Riemannian metric g on M leads to a generalized metric:

$$G = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \tag{1.48}$$

The generic case contains a little more information. A bit of algebra shows that a generalized metric G defines a metric on M as well as a 2-form B, called the B-field, which is here written as a morphism $TM \to T^*M$ (and need not be invertible). In terms of these, G is written:

$$G = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -Bg^{-1} & g^{-1} \end{pmatrix} \tag{1.49}$$

Such a metric enables to split the structure group $SO(n,n)$ and thus reduce to $SO(n) \times SO(n)$.

1.5.2 Generalized almost complex structures

A generalized almost complex structure $\mathcal{I}$ on M is an almost complex structure on $\mathcal{T}M$ compatible with the metric $\mathcal{G}$. Thus $\mathcal{I}$ is an endomorphism of $\mathcal{T}M$ such that:

$$\begin{aligned} \mathcal{I}^2 &= -\mathbb{I} \\ \mathcal{G}(\mathcal{I}\mathcal{X}, \mathcal{I}\mathcal{Y}) &= \mathcal{G}(\mathcal{X}, \mathcal{Y}) \quad , \end{aligned} \tag{1.50}$$

for $\mathcal{X}, \mathcal{Y} \in \mathcal{T}M$. By similar arguments as for ordinary almost complex structures, $\mathcal{I}$ reduces the structure group from $SO(n,n)$ to $U(n/2, n/2)$, which can be achieved only in even dimension.

We can see directly that we are actually generalizing almost complex structures. Indeed, if I is an almost complex structure on M, define:

$$\mathcal{I}_I = \begin{pmatrix} I & 0 \\ 0 & -{}^tI \end{pmatrix} \quad , \tag{1.51}$$

which is indeed a generalized almost complex structure. But this is not the only possibility. Consider a symplectic form ω, and define:

$$\mathcal{I}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \quad , \tag{1.52}$$

which is also a generalized almost complex structure. In fact an arbitrary $\mathcal{I}$ will correspond to an intermediate situation between an almost complex structure and an almost symplectic one.

$U(n/2) \times U(n/2)$-structures

As was pointed out in 1.5.1, an $SO(n)$-structure on M enabled to split the structure group of $\mathcal{T}M$. We could then expect a similar behavior for $U(n/2)$-structures. In fact an $U(n/2)$ can be defined by an almost complex structure I and a symplectic 2-form ω. These in turn lead to two generalized almost complex structures $\mathcal{I}_I$ and $\mathcal{I}_\omega$.

This is exactly what we were looking for: an $U(n/2) \times U(n/2)$-structure is given by two compatible generalized almost complex structures $\mathcal{I}_1, \mathcal{I}_2$. The compatibility conditions are:

$$\begin{aligned} [\mathcal{I}_1, \mathcal{I}_2] =&0 \\ -\mathcal{G}\,\mathcal{I}_1\mathcal{I}_2 \quad & \text{is a generalized metric} \end{aligned} \tag{1.53}$$

1.5.3 $SU(n/2) \times SU(n/2)$-structures

The next step is to define an $SU(n/2, n/2)$-structure. Instead of looking for the generalization of the holomorphic top form Ω let us switch to the point of view of pure spinors: remarkably the results of section 1.4.2 can be readily generalized. An $SU(n/2, n/2)$-structure is indeed defined by a generalized pure spinor Ψ. Since Ψ is pure its annihilator space is an n dimensional subspace of $\mathcal{T}M$ and is actually the i eigenspace of the corresponding generalized almost complex structure. In order to further reduce to $SU(n/2) \times SU(n/2)$ we now need a second pure spinor:

An $SU(n/2) \times SU(n/2)$ is defined by two non vanishing pure spinors Ψ_1, Ψ_2, such that their generalized almost structures are compatible. They can be normalized to:

$$\langle \Psi_1, \bar{\Psi}_1 \rangle = \langle \Psi_2, \bar{\Psi}_2 \rangle = 8 v_6 \ . \tag{1.54}$$

In terms of the pure spinors the compatibility conditions say that they should share half of their annihilator space.

From pure spinors to $SU(3) \times SU(3)$-structures

Let us now specialize to our case of interest $n = 6$. $SU(3) \times SU(3)$-structures can be quite cumbersome to characterize but in the following we will only need the case where it actually comes from two $SU(3)$-structures. Suppose that we have not only one but two spinors η_1, η_2 of positive chirality on M, normalized so that $\eta_1^\dagger \eta_1 = \eta_2^\dagger \eta_2 = 1$. Each defines its own $SU(3)$- and almost complex structure, according to (1.28). They both combine to form an $SU(3) \times SU(3)$-structure on M, given by two generalized pure spinors:

$$\Psi_1 = 8 \eta_1 \eta_2^\dagger \quad , \quad \Psi_2 = 8 \eta_1 \tilde{\eta}_2 \ . \tag{1.55}$$

Note that, thanks to Fierz isomorphism, generalized spinors can be seen equivalently as bispinors or polyforms (in even dimension). We thus want to express Ψ_1, Ψ_2 as polyforms. It is also important to notice that η_1, η_2 are not necessarily independent. Since η_1 is pure, the space of spinors can be constructed from η_1 or η_1^c and (anti-)holomorphic gamma matrices. Note that we call a holomorphic gamma matrix the image of a holomorphic one-form in the Clifford algebra, with respect to the complex structure defined by η_1.

It follows that any normalized spinor of positive chirality, such as η_2 in particular, can be written as:

$$\eta_2 = e^{i\nu} \cos\varphi \, \eta_1 + \sin\varphi \, \chi \ . \tag{1.56}$$

Here $e^{i\nu}\cos\varphi = \eta_1^\dagger \eta_2$ and χ is a normalized spinor, orthogonal to η_1, and defined from an anti-holomorphic one-form $\bar{K}$ such that $\bar{K}\cdot K = 2$:

$$\chi = \frac{1}{2}\bar{K}_i\,\gamma^i\,\eta_1^c\ . \tag{1.57}$$

In this definition K (and thus χ) does not need to be globally well-defined, provided $\sin\varphi$ vanishes whenever the definition of K fails. Moreover, χ (and K) defines locally another $SU(3)$-structure, "orthogonal" to η_1's:

$$\begin{array}{rcl} J^\perp &=& \chi^\dagger\gamma_{(2)}\chi = iK\wedge\bar{K} - J \\ \Omega^\perp &=& \chi^\dagger\gamma_{(3)}\chi^c = \frac{1}{2}K\cdot\bar{\Omega}\wedge K\ . \end{array} \tag{1.58}$$

We can also define a local $SU(2)$-structure (j,ω):

$$\begin{array}{rcl} j &=& \frac{i}{2}(\eta_1^\dagger\gamma_{(2)}\eta_1 - \chi^\dagger\gamma_{(2)}\chi) = \frac{1}{2}(J - J^\perp) = J - \frac{i}{2}K\wedge\bar{K} \\ \omega &=& -\tilde{\chi}\gamma_{(2)}\eta_1^c = \frac{1}{2}\bar{K}\cdot\Omega\ . \end{array} \tag{1.59}$$

The two orthogonal $SU(3)$-structures can be reconstructed from the local $SU(2)$ and K:

$$\begin{array}{rcl} J &=& j + \frac{i}{2}K\wedge\bar{K} \\ J^\perp &=& -j + \frac{i}{2}K\wedge\bar{K} \\ \Omega &=& \omega\wedge K \\ \Omega^\perp &=& \bar{\omega}\wedge K\ . \end{array} \tag{1.60}$$

Now we can compute the generalized spinors using Fierz identities:

$$\begin{array}{rcl} \Psi_1 &=& e^{-\frac{1}{2}K\wedge\bar{K}}\,\left(e^{-i\nu}\cos\varphi\,e^{i\,j} + \sin\varphi\,\bar{\omega}\right) \\ \Psi_2 &=& \bar{K}\wedge\left(e^{i\nu}\cos\varphi\,\bar{\omega} - \sin\varphi\,e^{i\,j}\right) \end{array} \tag{1.61}$$

This leads to the following normalization of the generalized spinors:

$$\langle\Psi_1,\bar{\Psi}_1\rangle = \langle\Psi_2,\bar{\Psi}_2\rangle = 8i\,v_6 \tag{1.62}$$

From there, several possibilities can arise. When $\sin\varphi = 0$, both spinors are collinear so that they define the same $SU(3)$-structure (up to the phase ν) on the tangent space. If this condition is true on all of M we get a strict $SU(3)$-structure and the generalized spinors simplify to:

$$\begin{array}{rcl} \Psi_1 &=& e^{-i\nu}\,e^{i\,J} \\ \Psi_2 &=& e^{i\nu}\,\bar{\Omega} \end{array} \tag{1.63}$$

On the contrary when $\sin\varphi \neq 0$, K is well defined and so is the $SU(2)$-structure (1.59). If this condition is valid everywhere, then this $SU(2)$-structure is globally defined and further reduces the structure group of M. The extreme case where $\cos\varphi = 0$ is called static $SU(2)$.

2 Supergravity and string theory

As the previous chapter was entirely devoted to mathematics, let us come back to some physics. The main subject of this thesis is the study of certain solutions of $\mathcal{N} = 2$ supergravity in 10 dimensions. Even though supergravity is interesting in its own right, we should remember that our motivation comes here from the study of vacua of string theory. Thus I will first give a very quick and incomplete introduction to string theory, as several very good books have been written on the subject: see [13, 14, 15] or more recently [16]. Most results presented here are extracted from there, if not references will be added. The discussion will go straight to the construction of the string spectrum. This will enable to better understand how supergravity arises in string theory, and why supergravity solutions will play such an important role.

I will then give a few details about supergravity in 10 dimensions. An emphasis is put on the structure of fluxes, and their interplay with branes. Eventually I will talk about how to get a 4 dimensional effective theory, and supersymmetric compactification in general. This will make the connection with the mathematical tools introduced in chapter 1 as supersymmetric compactification if better understood in the context of G-structures.

2.1 A few words about superstrings

2.1.1 The closed oriented string

There are different types of strings, each leading to a different string theory. I will speak exclusively of type II superstrings, which are closed supersymmetric strings. The starting point is a (super) conformal field theory on the two dimensional worldsheet of the string. For the type II string the worldsheet is a cylinder $\mathcal{W} = \mathbb{R} \times S^1$. The field content will describe an embedding of the worldsheet in a D dimensional space-time $\mathbb{R}^{1,D-1}$. Indices $\alpha, \beta, \ldots$ will refer to indices on the worldsheet while $M, N, \ldots$ are indices on the target space.

X^M will be the coordinates of the string in space-time. From the worldsheet point of view they are just D independent scalars. Their fermionic counterparts are ψ^M, a set of D independent Majorana spinors. On top of these fields, the worldsheet is also equipped with a metric h and a Majorana-Weyl gravitino χ_α. The full classical action presents a lot of symmetries:

- Diffeomorphism invariance on the worldsheet
- Weyl and super Weyl transformations (or superconformal symmetry)
- Worldsheet supersymmetry

- Poincaré transformations on space-time

Conformal invariance in particular is especially rich in two dimensions and plays a fundamental role in string theory. At the quantum level, anomalies are governed by the number (and type) of fields, which is here directly linked to the dimension of space-time. An important consequence of conformal invariance, is that the cancellation of conformal anomalies imposes $D = 10$, which we now take for granted. All these symmetries give (more than) enough freedom to gauge away the worldsheet metric and gravitino. In this so-called conformal gauge, the action can be written:

$$S = \frac{1}{4\pi\alpha'} \int_{\mathcal{W}} \mathrm{d}^2\sigma \left(\partial_\alpha X_M \partial^\alpha X^M + \alpha' \bar{\psi}_M \mathcal{D} \psi^M \right) \tag{2.1}$$

where $\mathcal{D} = \gamma^\alpha \partial_\alpha$ is the Dirac operator in $1+1$ dimensions.

2.1.2 NS and R sectors

Before going further we need to make a comment on the spinor fields ψ. In order to define spinors on a manifold (here $\mathcal{W}$), the manifold shall have a spin structure (see the quick discussion in section 1.2. For the cylinder $\mathcal{W}$ there are actually two distinct spin structures. These structures come from the non trivial topology of the S^1 part, and are related to periodicity conditions when turning around the circle. One structure gives rise to a trivial spinor bundle, whose sections are periodic, while the other gives anti-periodic spinors. The trivial, periodic structure will be called the Ramond (R) sector, while the anti-periodic one the Neveu-Schwarz (NS) sector.

A priori there seems to be an ambiguity in the choice of spin structure. However it happens that String theory will need both sectors to be represented in order to get a coherent picture.

2.1.3 Type II string spectrum

To get to the string spectrum the action 2.1 is quantized: the solution to the equations of motion for X and ψ are decomposed in modes that become creation and annihilation operators in the quantum theory. For X the equations imply that it can be decomposed in the form $X(\sigma) = X_R(\sigma_0 - \sigma_1) + X_L(\sigma_0 + \sigma_1)$. $X_{L/R}$ are respectively called Left/Right movers. Define $a_n, \bar{a}_n$ their Fourier modes [1] for $n \in \mathbb{Z}$. Note that a_0 corresponds to the momentum of the string. Similarly ψ can be decomposed in positive and negative chirality Weyl spinors: $\psi = \psi_- + \psi_+$. The equations of motion then imply that ψ_- is actually a left mover while ψ_+ is a right mover; call $b_n, \bar{b}_n$ their modes. In the R sector n is an integer, while $n \in \frac{1}{2} + \mathbb{Z}$ in the NS sector. The $n < 0$ operators will be taken to be the creation operators, acting on a ground state. The indexing by n has been defined so that the action of a_{-n}, b_{-n} adds $2n$ units of (mass)^2.

R-sector The action of b_0 does not change the mass so the vacuum is degenerate. In fact it can be seen that the b_0^M act as space-time gamma matrices so that the ground state is a space-time spinor $|\eta\rangle$. Note also that the R ground state is massless.

[1] In this section the bar will refer to the right movers, not to be confused with the complex conjugation

NS-sector There is no b_0 here so that the ground state is unique and will thus be denoted $|0\rangle$. However the NS-ground state has negative mass squared, such that the first excitations $b_{1/2}|0\rangle$ are massless.

Note that the sectors for the left and right movers are independent, which leads to four different sectors for the closed type II string. In each sector the states are computed by acting with both left and right creation operators on the corresponding ground state. In fact, the NS sector leads to bosonic states while the R states are fermionic. Thus the bosonic spectrum comes for the (NS,NS) and (R,R) sectors while the fermionic one from the (NS,R) and (R,NS) sectors. Besides, left movers and right movers are not entirely independent so that the states should obey the level matching condition, namely the same amount of mass should be created in both sides. Nevertheless it can be shown that the full spectrum cannot be consistent and must be truncated. This is done using the Gliozzi-Scherk-Olive (GSO) projection. There are two nonequivalent projections, leading to two different type II string theories. This projection enables to remove the tachionic state (namely the (NS,NS) ground state) and to get a supersymmetric spectrum from the space-time point of view. We will only need the massless spectrum in the following, but the massive states can be easily computed, though the number of states rises quickly with the mass level. In the massless case, the GSO projection amounts to choosing a specific chirality for the R sector and the spectrum is listed below. To better understand their properties from the space-time point of view, they are further decomposed in representations of $\mathfrak{so}(1,9)$.

Type IIA strings: positive chirality for the left movers and negative chirality for the right movers

(NS,NS) States are space-time bivectors $b^M_{-1/2}|0\rangle \otimes \bar{b}^M_{-1/2}|0\rangle$. They can be decomposed into its symmetric traceless part, antisymmetric part and trace.

(R,R) States are bispinors $|\eta_+\rangle \otimes |\lambda_-\rangle$. Using a Fierz decomposition the matrix $\eta_+\tilde{\lambda}_-$ is an even self-dual polyform, which can thus be decomposed into a 0-form, a 2-form and a 4-form.

(R,NS) Here states are a product of a vector and a spinor: $|\eta_+\rangle \otimes \bar{b}^M_{-1/2}|0\rangle$. This can be decomposed into a positive chirality traceless vector spinor, and a negative chirality spinor.

(NS,R) This is the same case with opposite chirality: $b^M_{-1/2}|0\rangle \otimes |\lambda_-\rangle$ is decomposed into a negative chirality traceless vector spinor and a positive chirality spinor.

Type IIB strings positive chirality for both left and right movers

(NS,NS) This sector is the same as for the IIA case.

(R,R) States are bispinors $|\eta_+\rangle \otimes |\lambda_+\rangle$. Using a Fierz decomposition the matrix $\eta_+\tilde{\lambda}_+$ is an odd self-dual polyform, which can thus be decomposed into a 1-form, a 3-form and a self-dual 5-form.

(R,NS) This sector is the same as for the IIA case.

(NS,R) Unlike the IIA case, the chirality is identical to the previous sector: $b^M_{-1/2}|0\rangle \otimes |\lambda_+\rangle$ is decomposed into a positive chirality traceless vector spinor and a negative chirality spinor.

This spectrum still contains some unphysical states, since the Minkowski metric of the target space leads to negative norm states. To get the correct degrees of freedom, one needs to eliminate these states. Once this is done, the spectrum can be decomposed into representations of $\mathfrak{so}(8)$, as it should be for massless states in $\mathbb{R}^{1,9}$. For the (R,R) sector, this implies that the p-forms should rather be interpreted as the field strength of $(p-1)$-form potentials (in particular that the 0-form of the IIA string is not dynamical).

It can be seen that the symmetric traceless tensor of the (NS,NS) sector should be identified with the graviton. Doing so would imply that the string length is of the order of the Planck mass $\frac{1}{\alpha'} \sim M_P^2$. This means that the tower of massive states would be inaccessible in a phenomenological model of string theory. This is the main reason we will consider the massless spectrum only.[2]

2.2 Type II Supergravity

In 10 dimensions, maximal supersymmetry corresponds to $\mathcal{N} = 2$. There are actually two ways to achieve this supersymmetry, namely type IIA or IIB supergravity, which are related to the corresponding string theories (hence the names). In this section I will describe briefly both theories (with more emphasis on the IIA side as this thesis is mainly about type IIA supergravity). It is highly remarkable that the field content of type II supergravity matches exactly the massless spectrum. This is of course not a coincidence, as supergravity arises as a low energy limit of string theory (in a sense that will be shortly explained).

2.2.1 Type IIA supergravity

As a theory a gravitation, type IIA supergravity is defined on a 10 dimensional lorentzian manifold M. In physical words, the lorentzian metric g is dynamical and is called the graviton. The $\mathcal{N} = 2$ supersymmetry then imposes the existence of two gravitini. They are Majorana-Weyl vector-spinors and in IIA they are of opposite chirality, hence it is possible to regroup them into a single Majorana vector-spinor ψ_M. These are the basic ingredients of supergravity, but in order to get a supersymmetric theory several supplementary fields need to be introduced.

The bosons are a dilaton ϕ and a 2-form, the B-field; together with fluxes, namely a 1-form A and a 3-form C. All the differential forms here should be considered as potential forms, the physical observables are therefore their field strengths. The spectrum is completed by a pair of Majorana-Weyl spinors, the dilatini whose chirality should be opposite to the gravitini. Hence we also regroup them into a single Majorana spinor λ. Since both chiralities are represented for the gravitini, type IIA supergravity is non chiral and is also denoted $\mathcal{N} = (1,1)$ supergravity. Let us now write the action, parametrized by a constant parameter m called the Romans mass:

$$\begin{aligned} S_{IIA} = \frac{1}{2\kappa_{10}^2} \int_M \Big[& e^{-2\phi} \left(R v_{10} + 4\mathrm{d}\phi \wedge \star \mathrm{d}\phi - \frac{1}{2} H \wedge \star H \right) - \frac{1}{2} m^2 v_{10} - \frac{1}{2} F \wedge \star F - \frac{1}{2} G \wedge \star G \\ & - \frac{1}{2} B \wedge \mathrm{d}C^2 - \frac{m}{3} B^2 \wedge \mathrm{d}C - \frac{m^2}{10} B^5 \Big] + S_2 + S_4 \end{aligned} \tag{2.2}$$

[2]This argument is a bit quick here, since the Planck mass is usually defined in 4d. A better argument should take into account the compactification

where only the bosonic part of the action is explicit here. $\star$ is the Hodge star operator with respect to g, v_{10} is the volume form of M and could be written $v_{10} = \star(1) = \sqrt{-g}\mathrm{d}x^0 \wedge \cdots \mathrm{d}x^9$, and R is the scalar curvature. H, F, G are respectively the field strength of B, A, C, however the B-field will have a particular role. Indeed the presence of the B twists the structure of the fluxes, so that the field strength are written:

$$H = \mathrm{d}B \quad , \quad F = \mathrm{d}A + mB \quad , \quad G = \mathrm{d}C + A \wedge H + \frac{1}{2}mB^2 \tag{2.3}$$

This leads to the following Bianchi identities for the field strengths:

$$\mathrm{d}H = 0 \quad , \quad \mathrm{d}F = mH \quad , \quad \mathrm{d}G = H \wedge F \tag{2.4}$$

Note also that the Romans mass could considered as a 0-form field strength, with Bianchi identity $\mathrm{d}m = 0$. In fact the bosonic action contains the standard kinetic terms for a graviton, the scalar dilaton and the field strengths, apart from the non trivial coupling of the dilaton. The first three terms in the second line of (2.2) are called the Chern Simons terms S_{CS}, they are combinations of the fluxes that form a top form and thus do not need the metric to be integrated. Since they do not depend on the metric, they will not appear in Einstein's equations and are said to be topological.

I will not describe in detail the fermionic part as it is not particularly illuminating. The necessary terms will be precisely defined when necessary (namely in chapter 3). S_2 contains the quadratic fermion terms: they are basically kinetic terms of the generic form $\bar{\Psi}\mathcal{D}\Psi$, where $\mathcal{D}$ is a Dirac operator in a wide sense. $\mathcal{D}$ actually depends on the fluxes and thus gives the interaction terms between the fermions and the fluxes. S_4 is a quite involved contribution of quartic fermion terms, but we will need only a few of them (cf chapter 3). Note that these are self interaction terms that do not see the fluxes. A comprehensive expression can be found in [17, 18, 19, 20] however these papers seem to be in disagreement [21]. Recently the dilatino quartic terms were computed in [22] using the superspace formalism and found agreement with [17]. We will thus trust the latter in the following.

Supersymmetry transformations The only information missing are the supersymmetry transformations. Since we are considering $\mathcal{N} = 2$ supergravity the supersymmetry parameters should be two Majorana-Weyl spinors. In type IIA they should be of opposite chirality and are thus grouped in a single Majorana spinor ϵ. The supersymmetric variations of the bosons involve spinor bilinears between ϵ and the fermions. The variation of the fermion, which will be the one relevant for us, are given schematically by:

$$\begin{aligned} \delta_\epsilon \psi_M &= \hat{\nabla}_M \epsilon \\ \delta_\epsilon \lambda &= \not{\mathcal{F}} \epsilon \end{aligned} \tag{2.5}$$

where $\hat{\nabla}$ is the spin connection, modified by the fluxes, and $\not{\mathcal{F}}$ is a section of the Clifford bundle, constructed by contracting all the fluxes to gamma matrices (including the dilaton).

Einstein frame It can be quite confusing to see that the Einstein-Hilbert term of the action is coupled to the dilaton with a factor $e^{-2\phi}$. However it could be reabsorbed in the metric, at the price of a field redefinition. This is just a rescaling of the metric by a dilaton-dependent factor. Such a redefinition of the metric is called a change of frame. The action (2.2) is currently in the string frame, and the frame in which the scalar curvature is not coupled to the dilaton is called Einstein's frame. The transition between both frames can be done through a rescaling:

$$g^{st} = e^{\phi/2} g^E \tag{2.6}$$

Since we will also need the Einstein's frame action, let us spell out the changes:

$$\begin{aligned} S^E_{IIA} = \frac{1}{2\kappa_{10}^2}\int_M & (Rv_{10} - \frac{1}{2}\mathrm{d}\phi\wedge\star\mathrm{d}\phi - \frac{1}{2}e^{-\phi}H\wedge\star H - \frac{1}{2}e^{5\phi/2}m^2 v_{10} - \frac{1}{2}e^{3\phi/2}F\wedge\star F - \\ & \frac{1}{2}e^{\phi/2}G\wedge\star G) + S_{CS} + S_2 + S_4 \end{aligned} \tag{2.7}$$

2.2.2 Type IIB supergravity

The other possibility is type IIB supergravity. This thesis is not so much about type IIB, but since several solutions will be considered throughout the thesis, it would be welcome to at least have an action written somewhere. This case is very similar to the IIA case (at least for our purposes) and I will show here some of the main differences.

First of all this theory is chiral, as the gravitini have same chirality (say positive). Then both dilatini, whose chirality should be opposite to the gravitini, have negative chirality. This is also seen at the level of the supersymmetry parameters, which should both be of positive chirality.

Regarding the bosons, the graviton, B-field and dilaton are identical to the IIA case. However the fluxes are now a 0-form (a scalar) C_0, a 2-form C_2 and a 4-form C_4. Hence their field strengths F_1, F_3, F_5 are odd forms. Without much surprise, the IIB bosonic action is then (in the string frame):

$$\begin{aligned} S_{IIB} = \frac{1}{2\kappa_{10}^2}\int_M \Big[& e^{-2\phi}\left(Rv_{10} + 4\mathrm{d}\phi\wedge\star\mathrm{d}\phi - \frac{1}{2}H\wedge\star H\right) - \frac{1}{2}F_1\wedge\star F_1 - \frac{1}{2}F_3\wedge\star F_3 \\ & - \frac{1}{4}F_5\wedge\star F_5 - \frac{1}{2}C_4\wedge H\wedge F_3\Big] \end{aligned} \tag{2.8}$$

The only subtlety at first sight is coming from the Chern-Simons term, but there is also an issue concerning F_5. Indeed in type IIB supergravity, F_5 should be self-dual:

$$\star F_5 = F_5 \tag{2.9}$$

and this condition does not follow from this action. For this reason, (2.8) is called a pseudo-action: not all the equations of motion follow from the action which needs to be supplemented with the self duality condition.

Moreover, as was the case for type IIA supergravity, the field strength are twisted in presence of B-field:

$$H = \mathrm{d}B \quad , \quad F_1 = \mathrm{d}C_0 \quad , \quad F_3 = \mathrm{d}C_2 - C_0 H \quad , \quad H_5 = \mathrm{d}C_4 - H\wedge C_2 \quad , \tag{2.10}$$

which lead to the Bianchi identities:

$$\mathrm{d}H = 0 \quad , \quad \mathrm{d}F_1 = 0 \quad , \quad \mathrm{d}F_3 = H\wedge F_1 \quad , \quad \mathrm{d}F_5 = H\wedge F_3 \tag{2.11}$$

Note also that plugging (2.9) in the Bianchi identity for F_5 leads exactly to the equation of motion for C_4 coming from the action[3]. This means that the selfdual condition is more than just an algebraic constraint on the field strength: it is actually the equation of motion for C_4.

2.2.3 Low energy effective action of string theory

Compare now both type II supergravities to the spectrum derived in 2.1.3. As previously advertised, the field content is in exact correspondence with the massless spectrum of the corresponding string theory. The (NS,NS) sector, common to types IIA and IIB, gives the graviton, the B-field and the dilaton. The (R,R) sector will give the fluxes, with even field strength in IIA and odd in IIB, including the self-dual F_5. For this reason H will be called the (NS,NS) flux while the other forms are (R,R) fluxes. Now looking at fermions, the (NS,R) and (R,NS) sectors give each a gravitino and a dilatino with the right chiralities.

This matching between the massless spectrum of string theory and supergravity hints toward a resemblance between both theories, at least at low energies compared the string mass $E^2 \ll \frac{1}{\alpha'}$ so that the massive states can be ignored. One way to check this is to compute scattering amplitudes for string states and derive an effective action. The latter should recover at tree level the string amplitudes. Note that in general an interacting quantum theory should generates n-points functions for arbitrary n, so that an effective theory will have *a priori* an infinite number of terms. Thus we expect an expansion in powers of α'.

The computation of string amplitudes is done using tools from conformal field theory. Indeed the superconformal invariance of string theory gives a one to one correspondence between states and certain operators, called vertex operators. A scattering amplitude is then given by the correlators of the corresponding vertex operators. These correlators are computed in the worldsheet two dimensional theory by a path integral over the possible embeddings into a given background. This path integral, called the Polyakov integral, also sums over the topology of the worldsheet. For the closed string, this is a sum over all compact two dimensional oriented manifolds without boundaries. They are classified by an integer g called the genus: $g = 0$ is the sphere, $g = 1$ is a torus, and so on.

This path integral should be compared to the perturbative expansion in Feynman diagrams of quantum field theory. Higher genus surfaces correspond then to higher order loop corrections. A fundamental aspect of this is that these higher genus terms are governed by powers of the dilaton. Thus the expectation value of the dilaton is often called the string coupling constant:

$$g_s = e^{\langle \phi \rangle} \tag{2.12}$$

The effective action can thus be expanded in powers of α' and g_s, to which should be added non perturbative effects that cannot be captured by the Polyakov path integral (for example fermionic condensates that I will discuss in chapter 3). The computation of higher order terms is of course extremely involved. The most important result is that, at leading order in α', the effective Lagrangian boils down to the type II (A or B) Lagrangian:

$$\mathcal{L}_{eff} = \frac{1}{g_s^2}\mathcal{L}_{IIA/B} + \mathcal{O}(\alpha'^3) \tag{2.13}$$

[3]The fermions are set to zero here. (2.9) is modified in presence of fermions and the equation of motion for C_4 is sourced, so that the remark still holds in presence of fermions

Where the gravitational constant κ_{10} can be related to the string length:

$$2\kappa_{10}^2 = (2\pi)^7 \alpha'^4 \tag{2.14}$$

This means that type IIA and IIB supergravity are the low energy limits of the correspond string theory. Note also that the next order correction in g_s appears only at the order α'^3 at least, so that this approximation is perturbatively exact in the string coupling.

2.2.4 Democratic formulation

There exists another formulation of type II supergravity, developed in [23] and called the democratic formulation. It does not change the (NS,NS) part but unifies the (R,R) fluxes in a single polyform F. The bosonic part of the democratic action be written:

$$S = \frac{1}{2\kappa_{10}^2} \int_M \left[e^{-2\phi} \left(Rv_{10} + 4\mathrm{d}\phi \wedge \star \mathrm{d}\phi - \frac{1}{2} H \wedge \star H \right) - \frac{1}{4} F \wedge \star F \right] \tag{2.15}$$

where F is an even polyform in IIA and odd in IIB. In order to get the same number of degrees of freedom, a self-duality constraint must be imposed on F:

$$\star \, \lambda \, F = F \tag{2.16}$$

where λ has been defined in 1.5.1. To make contact with the previous description of (R,R) fluxes, write:

$$F = (1 + \star\lambda)(F_0 + F_2 + F_4) \quad \text{(IIA)} \quad , \quad F = (1 + \star\lambda)(F_1 + F_3) + F_5 \quad \text{(IIB)} \tag{2.17}$$

F can also be written in term of a polyform potential A:

$$F = \mathrm{d}_H A + F_0 e^{-B} \tag{2.18}$$

where $\mathrm{d}_H = \mathrm{d} + H\wedge$ is the twisted derivative[4] and $F_0 = m$ is the Romans mass, present only in IIA. The Bianchi identities can now be written in a compact form:

$$\mathrm{d}_H F = 0 \tag{2.19}$$

For the component F_p of degree $p \leq 5$, this gives the usual Bianchi identities. But for $p \geq 5$, F_p are the Hodge duals of the (R,R) fluxes, so that these equations contain also the equations of motion for all (R,R) fluxes. In fact the democratic formulation also generalizes the issue we had with F_5 in type IIB: the equations of motion result from the Bianchi identities and the self duality condition, which cannot be derived from the Lagrangian. Thus (2.15) is only a pseudo-Lagrangian.

Still the democratic formulation remains extremely useful, as it is necessary to make contact with generalized complex geometry, since polyforms and the twisted derivative will play an important role. This will be used extensively in section 4 to derive the transformation rules for the (R,R) fluxes under non abelian T-duality, and also in 4.3.1 to solve the supersymmetry equations.

[4]Here we take a slightly different convention for B, that amounts to changing its sign, in order to make contact with section 4.3.1

2.3 Fluxes and branes

Fluxes will be at the heart of several discussions in this thesis, and are intimately linked with the concept of branes. They can be understood as a generalization of Maxwell theory, to higher order forms (Maxwell potential is a 1-form). Then in all generality a flux is a field defined by p-form $F = \mathrm{d}A$, the field strength of a $(p-1)$-form potential A. The kinetic term of F in an action will take the form

$$S_{kin} = \int \frac{1}{2} F \wedge \star F \tag{2.20}$$

so that the Bianchi identity and equation of motion read respectively:

$$\mathrm{d}F = 0 \quad , \quad \mathrm{d} \star F = 0 \tag{2.21}$$

A source could then be inserted into these equations: a source for the equation of motion is called electric, while a source for the Bianchi identity is called magnetic. Note that by accepting the existence of a magnetic source one must give up the notion of potential[5]. These sources are extended and their dimensions depend on the degree of the flux form. Such extended objects are called branes in supergravity and string theory. Implicitly branes will always be extended in the time direction and will thus be denoted by their spacial dimension. Hence a 0-brane is the worldline of a particle, and a 1-brane is the worldsheet of a string for example.

In the equations branes will be modeled by currents $\mathcal{J}_{e/m}$, generalization of distributions for differential forms, which contain the volume form of the brane (the directions in which it is extended) and the charge distribution (usually a delta function stating the position of the brane). The equations become:

$$\mathrm{d}F = \star \mathcal{J}_m \quad , \quad \mathrm{d} \star F = \star \mathcal{J}_e \tag{2.22}$$

Which shows directly that a p-form is sourced electrically by a $(p-2)$-brane and magnetically by a $(n-p-2)$-brane. Another consequence of this is the charge conservation equation:

$$\mathrm{d} \star \mathcal{J} = 0 \tag{2.23}$$

The most straightforward example is when branes are inserted in a flat space-time, say $\mathbb{R}^{1,9}$. Then decompose space-time into the world volume of the p-brane and the space transverse to the brane (in this picture the brane is flat too and can be identified with $\mathbb{R}^{1,p}$). In the transverse space the brane is point like, it is then possible to compute the total charge by integrating the current on a ball containing the brane:

$$\begin{aligned} Q_e &= \int_{\mathcal{B}} \star \mathcal{J}_e = \int_S \star F \\ Q_m &= \int_{\mathcal{B}} \star \mathcal{J}_m = \int_S F \end{aligned} \tag{2.24}$$

[5]The existence of a potential, even locally, implies by definition that the flux is closed. However most of the time the source will be localized, it is then possible to remove the locus of the source from space-time. By an abuse of definition, a potential can then be defined on this excised space, but never on the neighborhood of the source.

Where $S = \partial\mathcal{B}$. Thus the charge can also be computed by integrating the flux on a cycle surrounding the charge. This is actually the definition given to charges in the following: charges are integrals of the fluxes along specific cycles. Note that in a non topologically trivial space-time, a flux can carry charge even without the actual presence of brane. Indeed the integral over a non exact cycle is not necessarily 0 even if the flux is closed.

Until now the discussion has remained general, let us now see how the concept of branes fits in supergravity and string theory.

2.3.1 Branes in supergravity

Strictly speaking, branes are not ingredients of supergravity. They should be considered as ad hoc objects introduced as a source to the fluxes in order to understand how they modify the equations and their solutions. NS1 and NS5 branes are respectively the electric and magnetic sources for the (NS,NS) 3-form H while Dp branes are sources for the corresponding (R,R) fluxes. Two important subtleties will arise when manipulating branes in the context of supergravity.

Backreaction on the metric

The first issue is that we are now looking at a theory of gravity, where the geometry of space-time itself is dynamical. Thus branes will also source Einstein's equations, as any object in supergravity. Inserting a brane in a background will backreact on the metric and possibly change the topology or introduce singularities. In this new background, all the equations of motion can be solved without the need for a source term. The interpretation of this brane can then be confusing: is the brane an actual physical object, or was it just a mathematical tool to get a new genuine supergravity solution ? A practical answer would be to look at the charges which are more robust: the brane itself is not that important, what matters is the charge carried by the fluxes. However there are cases where the brane can actually be tracked down. For example the backreaction of branes on a flat background can be computed and leave quite a peculiar signature:

NS5 branes

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}s^2_{//} + h(r)\mathrm{d}s^2_{\perp} \\ e^{\phi(r)} &= h(r)^{1/2} \\ H &= \star_{\perp}\mathrm{d}h \\ h(r) &= a + \frac{b}{r^2} \end{aligned} \tag{2.25}$$

Dp branes

$$\begin{aligned} \mathrm{d}s^2 &= h(r)^{-1/2}\mathrm{d}s^2_{//} + h(r)^{1/2}\mathrm{d}s^2_{\perp} \\ e^{\phi(r)} &= h(r)^{\frac{3-p}{4}} \\ F_{p+2} &= d(h^{-1}) \wedge v_{//} \\ h(r) &= a + \frac{b}{r^{7-p}} \end{aligned} \tag{2.26}$$

where the metrics are given here in the string frame. $\mathrm{d}s^2_{//}$ and $\mathrm{d}s^2_{\perp}$ are respectively the flat metrics of the worldvolume of the brane and of the transverse space. $\star_{\perp}$ is the Hodge star operator in the transverse space only and $v_{//}$ is the volume form on the brane. Each time $h(r)$ is a harmonic function on the transverse space[6], whose radius coordinate is r. The pattern here is that the brane backreacts on the metric through a warp factor that is singular at $r = 0$. Such a singularity in the metric does not necessarily mean that space-time is singular, but at least it is a hint that the global topology has changed.

The B-field and Page charges

The other issue concerns the presence of the B-field that twists the Bianchi identities and equations of motion. In the democratic formulation both can be regrouped into (2.19). Adding a source term to this equation would not be consistent: since $\mathrm{d}_H F$ is not closed this would violate the charge conservation equation (2.23). The way out is by defining the Page flux $\tilde{F} = e^B F$. Then the Bianchi identity becomes:

$$\mathrm{d}\tilde{F} = e^B \mathrm{d}_H F = 0 \tag{2.27}$$

and this can be consistently sourced. Thus the correct charges are rather the integrals of the Page fluxes, called Page charges. This construction comes at a price though, since now the charges depend explicitly on B and can change under a gauge transformation $\delta B \mathrm{d}\Lambda$.

2.3.2 Stringy origin

Contrary to supergravity, string theory is expected to naturally contain branes. For example Dp branes can be understood as the locus on which open strings can end. The NS1 branes are actually the fundamental strings themselves. The interpretation of the NS5 is on the the other hand less clear but they are also expected to appear, for example as dual to D5 branes in type IIB. Now when looking at the low energy effective action, branes can arise as non perturbative corrections to the supergravity action. It is important to keep that in mind when solving the supergravity equations to get string vacua. One restriction of string theory on the supergravity solutions is the quantization of the fluxes which we will state here as a fact.

Flux quantization The charges, or Page charges when appropriate, should be integers when normalized in the following way:

$$Q_p = \frac{1}{2\kappa_{10}^2 T_p} \int F \tag{2.28}$$

where T_p is the tension of a p-brane, given by:

$$\frac{1}{T_p} = (2\pi)^p \sqrt{\alpha'}^{p+1} \tag{2.29}$$

Note that this result is valid for Dp-branes as well as for NS5 branes when specializing to $p = 5$.

[6] For $p = 7$, $h(r)$ is logarithmic

2.4 Supersymmetric flux vacua

In order to connect string theory to the real world, much more work has to be done. For string theory to describe quantum gravity the string scale $\frac{1}{\alpha'}$ should be of the order of M_P^2. This means that the standard model, as well as any physical process that can be probed with particle accelerators, lie far below the string scale. Thus a realistic phenomenological model of string theory will most likely rely on a low energy approximation: for us it is type II supergravity. If string theory were indeed a good description of reality, physics at our energy scales should be described by small perturbations around a vacuum, that is a classical bosonic solution of supergravity[7]. The properties of the theory will then depend on the choice of vacuum. The study of string vacua is thus of paramount importance to understand the potential physical applications of string theory.

In this thesis we will be mainly concerned by vacua with the following properties:

- We expect the effective theory to describe our 4 dimensional world. So the 10 dimensional space-time is written as a direct product $M_4 \times M_6$, where M_4 is a 4 dimensional homogeneous space and M_6 is the compact internal manifold of characteristic length l_6. Supergravity will then be compactified on M_6 : this gives a refinement of the relation between α' and the Planck mass. Indeed the Planck mass is the coefficient of the Einstein-Hilbert term. Integrating will over M_6 gives a coefficient $\frac{v_6}{2\kappa_{10}^2}$ so that $M_P^2 \sim \frac{l_6^6}{\alpha'^4}$. Moreover for stringy effects to be neglected in the supergravity approximation, l_6 should be larger than the string length $l_6^2 \gg \alpha'$. On the other side no sign of extra dimensions has been detected to this day so that l_6 should be small compared to the characteristic length probed by actual experiments.

- The vacua are supersymmetric. This means that there exists spinors ϵ such that the supersymmetric variation of fields with parameter ϵ, evaluated at the configuration, vanish. This will imply supersymmetry of the effective theory and the number of independent spinors satisfying this condition is the number of preserved supersymmetries. Once again no sign supersymmetry has been detected, so that supersymmetry should then be broken at some point between the compactification and standard model scales. At first glance this seems like an additional constraint to an already involved problem but integrability theorems [24] show that, together with mild assumptions, supersymmetry of a configuration implies the equations of motion.

- Fluxes will *a priori* not vanish. Vacua without fluxes where originally considered, compactification then leads to many massless scalar fields in the effective theory. Turning on the fluxes can give a mass to these so-called moduli and thus helps deriving better phenomenological predictions. Anyway fluxes are genuine fields of the theory and it is thus necessary to consider them at some point in order to fully understand string vacua.

Let us now explore the constraints supersymmetry will impose on the vacua. For more details on supersymmetric flux vacua see the review [25]. Consider an arbitrary configuration for which all fermionic fields are put to zero. Since the supersymmetric variation of the bosons

[7] For such vacua to be trusted, it is important to keep the approximations under control.

is constructed from spinor bilinears between the fermions and the parameter ϵ, they vanish trivially. The constraints thus come from the fermionic variations, in the form of an equation on the spinor ϵ.

Define the following ansatz for type IIA supergravity in the democratic formalism, in accordance with our previous description. The metric is given by a warped product:

$$g = e^{2A} g_4 + g_6 \tag{2.30}$$

where g_4 is the maximally symmetric metric on M_4 and g_6 defines the geometry of the internal space M_6. To respect the symmetries of the 4 dimensional space the dilaton ϕ and the warp factor A depend only on the internal coordinates. Moreover the fluxes either have no leg on M_4 or are proportional to the volume-form v_4 of g_4. Thus H is a 3-form on M_6 and the flux polyform can be decomposed into:

$$F = F_i + e^{4A} v_4 \wedge \wedge F_e \tag{2.31}$$

where F_i is an even polyform on M_6. In order to respect the self-duality condition (2.16), F_e is related to F_i by $F_e = \star_6 \lambda(F_i)$. Finally the supersymmetry parameter will be factorized into spinors of M_4 and M_6. In 10 dimensions ϵ is a Majorana spinor, whose two chiralities are independent. Then take the ansatz:

$$\begin{aligned} \epsilon_+ &= \zeta_1 \otimes \eta_1 + \zeta_1^c \otimes \eta_1^c \\ \epsilon_- &= \zeta_2 \otimes \eta_2^c + \zeta_2^c \otimes \eta_2 \end{aligned} \tag{2.32}$$

where ζ_i, η_i are positive chirality spinors in respectively 4 and 6 dimensions. Then plug the ansatz in the supersymmetry variations with parameter (2.32). The external part tells us that ζ_i are killing spinors for M_4. The internal part will involve covariant derivatives of the η_i and will be the core of the discussion. The first consequence is that they cannot vanish. As we have seen in section 1.5.3 two non vanishing spinors on M_6 define an $SU(3) \times SU(3)$ structure. It can then be shown that the supersymmetry equation can be expressed in terms of the $SU(3) \times SU(3)$ structure data for which the (R,R) fluxes act as a source. This seems to lead to conditions on a supplementary structure, but an $SU(3) \times SU(3)$ structure actually determines the metric. In practice, the supersymmetry constraints are first order differential conditions on the bosonic fields. These equations will be much easier to solve than the second order supergravity equations, and thanks to the integrability theorems their solutions will be genuine vacua.

If both spinors η_1, η_2 are collinear the $SU(3) \times SU(3)$ structure boils down to a single $SU(3)$ structure. The supersymmetry conditions then determine the torsion classes, which are sourced by the fluxes. If the fluxes are set to zero, the $SU(3)$ structure becomes integrable and M_6 is Calabi-Yau. This also implies that the warp factor and dilaton vanish and the four dimensional space-time is Minkowski. Also ζ_1, ζ_2 can be chosen independently and this gives $\mathcal{N} = 2$ supersymmetry. On the contrary non vanishing fluxes break the Calabi-Yau condition and supersymmetry is possible only if $\zeta_1 = \zeta_2$, *ie* $\mathcal{N} = 1$ supersymmetry. Moreover it can be shown that fluxes give in general negative contributions to the curvature of the four dimensional space-time. This leads for example to the no-go theorem from [26] that excludes a Minkowski or de Sitter external space for vacua with non vanishing flux and no other ingredients. We are thus left with AdS: the conditions on the $SU(3)$ structure for $\mathcal{N} = 1$ supersymmetric AdS_4 have been computed in [24].

In the general case however η_1 and η_2 need not be related. Then the $SU(3)\times SU(3)$ structure is not a strict $SU(3)$. Of course each spinor defines an $SU(3)$ structure but rewriting the supersymmetry conditions in a unified way requires the framework of generalized complex geometry. Once again fluxes break supersymmetry to $\mathcal{N}=1$. The translation of the supersymmetry equations for $\mathcal{N}=1$ supersymmetric flux vacua in the language of $SU(3)\times SU(3)$ structure was done in [7, 9].

Note that all the results cited here fit in the present ansatz. Of course is is possible to look for flux vacua with a different ansatz. Then the supersymmetry equations need to be modified accordingly. This is done for example in section 4.3.1 in the case of a domain wall ansatz.

3 Consistent truncation in the presence of fermionic condensates

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[2] R. Terrisse and D. Tsimpis, “Consistent truncation with dilatino condensation on nearly Kähler and Calabi-Yau manifolds,” JHEP **1902**, 088 (2019),
[3] R. Terrisse and D. Tsimpis, “Consistent truncation and de Sitter space from gravitational instantons,” JHEP **1907**, 034 (2019)

This chapter will present consistent truncations of type IIA supergravity to 4 dimensions in presence of fermionic condensates and fluxes. The motivation stems from [22] that shows that quartic dilatino condensates could lead to de Sitter vacua, which are otherwise extremely difficult to get in string theory [27, 28]. In fact no-go theorems [29, 26] show that supergravity do not admit de Sitter vacua, at least when considering only its classical ingredients. The only way out is thus to include quantum corrections, here in the form of fermionic condensates.

Before diving straight into the discussion, let us say a few words about the two framework that will be used in this chapter:

Consistent truncation A truncation of a theory reduces its degrees of freedom by removing parts of its spectrum. If the truncated modes are not sourced by the remaining modes the truncation is called consistent. Then all solutions of the truncated theory can be lifted to solutions of the original theory. In practice a truncation is an ansatz on the fields that will be plugged in the equations of motion. If the resulting equations can be regrouped into a Lagrangian on the remaining fields, this Lagrangian gives a consistent truncation.

In string compactification consistent truncations can be used to build lower dimensional theories (here 4D) by truncating the modes living in the internal space. Finding solutions for the truncated theory will be much easier than for the full 10 dimensional supergravity, and consistency of the truncation ensures that these solutions are genuine vacua. The downside is that the stability of the solution can be checked only along the directions of the remaining fields, while truncated modes could still source instability.

Fermionic condensates In a maximally-invariant vacuum of the theory, all fermion vacuum expectation values (VEV) are assumed to vanish, but quadratic and quartic fermion terms may still develop nonvanishing VEV’s. Schematically,

$$\langle\lambda\rangle = 0 \; ; \quad \langle\bar{\lambda}\lambda\rangle := \int [\mathcal{D}\Phi](\bar{\lambda}\lambda)e^{-S[\Phi]} \neq 0 \; , \tag{3.1}$$

where λ collectively denotes the fermions and Φ stands for all fields in the action $S[\Phi]$. The vacuum $\langle\Phi\rangle$ is obtained by minimizing the effective action S_{eff} with respect to the fields,

$$\left.\frac{\delta S_{\text{eff}}}{\delta\Phi}\right|_{\langle\Phi\rangle} = 0 \; , \tag{3.2}$$

where, at tree level in the coupling, the effective action coincides with S.

Moreover, in the case of the critical IIA superstring, the two-derivative effective action S_{eff} coincides with the action of ten-dimensional IIA supergravity to all orders in string perturbation.[1] However, nonperturbatively, S_{eff} may develop non-vanishing VEV's for the quadratic and quartic fermion terms.

The idea of fermionic condensate is not new. The phenomenon has already been observed in supersymmetric Yang-Mils theories, for example in [30]. In the context of string theory studies have mainly focused on gaugino condensation in the heterotic case [31, 32, 33, 34, 35, 36, 37, 38]. Note that gaugino condensation did not seem to help getting a positive cosmological constant [39]. The situation seems to be different in the IIA case as [22] and this chapter point out.

At first this chapter builds on the work of [22] by constructing a consistent truncation in the presence of dilatini condensates. The truncation admits a limit in which the internal space is Calabi-Yau. The ansatz is extended in a second time to get a consistent truncation to the universal sector of Calabi-Yau compactification. Eventually gravitini condensates are added to the truncation which then admits de Sitter solutions. It is suggested that such condensates can be generated by gravitational instantons.

3.1 Dilatini condensation

We will not examine the mechanism for the generation of fermionic condensates here: we will simply assume their presence and examine the implications. In the following we will look in particular for *dilatonic solutions*, i.e. for solutions of the *dilatino-condensate action* of [22]. This is obtained from the IIA supergravity action by setting the Einstein-frame gravitino to zero. Moreover, the quadratic and quartic dilatino terms in the action should be thought of as replaced by their condensate VEV's, and thus become (constant) parameters of the action. The dilatino-condensate action should therefore be regarded as a book-keeping device whose variation with respect to the bosonic fields gives the correct bosonic equations of motion in the presence of dilatino condensates; the fermion equations of motion are trivially satisfied in the maximally-invariant vacuum, and need not be considered.

In [22] the fermionic terms of IIA supergravity were determined in the ten-dimensional superspace formalism previously developed in [40], resolving an ambiguity in the original literature [17, 18, 19] concerning the quartic fermions, and finding agreement with [17]. In the conventions of [22], the dilatino-condensate action of (massive) IIA reads,[2]

$$\begin{aligned} S = -S_b + \int \mathrm{d}^{10}x\sqrt{g}\Big\{ &(\bar\Lambda\Gamma^M\nabla_M\Lambda) - \frac{21}{16}e^{5\phi/4}m(\bar\Lambda\Lambda) + \frac{3}{512}(\bar\Lambda\Lambda)^2 \\ &- \frac{5}{32}e^{3\phi/4}F_{MN}(\bar\Lambda\Gamma^{MN}\Gamma_{11}\Lambda) + \frac{1}{128}e^{\phi/4}G_{MNPQ}(\bar\Lambda\Gamma^{MNPQ}\Lambda)\Big\}\,, \end{aligned} \tag{3.3}$$

[1] Indeed loop corrections in the string coupling are expected to modify the terms in S_{eff} with eight or more derivatives. In general, this will no longer be the case in the compactified theory.

[2] We have rescaled the Romans mass: $m \to 5m/4$ with respect to [22]. Moreover $\hat g_{mn}$ of that reference is denoted g_{mn} here. We have also changed conventions for the Riemann tensor so that $\hat R$ of [22] is $-R$ here.

where Λ is the dilatino; S_b is the bosonic sector of Romans supergravity [41],

$$S_b = \int \mathrm{d}^{10}x\sqrt{g}\Big(- R + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2\cdot 2!}e^{3\phi/2}F^2 + \frac{1}{2\cdot 3!}e^{-\phi}H^2 + \frac{1}{2\cdot 4!}e^{\phi/2}G^2 + \frac{1}{2}m^2e^{5\phi/2}\Big) + \mathrm{CS} \ , \tag{3.4}$$

where CS denotes the Chern-Simons term. We emphasize that, as mentioned previously, the dilatino terms in (3.3) are not dynamical but should be thought of as parameters of the action. In particular $(\bar\Lambda\Lambda)^2$ should be thought of as the VEV $\langle(\bar\Lambda\Lambda)^2\rangle$ and is therefore a priori independent of $(\bar\Lambda\Lambda)$, which should be thought of as the VEV $\langle\bar\Lambda\Lambda\rangle$.

The dilaton and Einstein equations following from action (3.3) read,

$$\begin{aligned} 0 = &-\nabla^2\phi + \frac{3}{8}e^{3\phi/2}F^2 - \frac{1}{12}e^{-\phi}H^2 + \frac{1}{96}e^{\phi/2}G^2 + \frac{5}{4}m^2e^{5\phi/2} \\ &- \frac{105}{64}e^{5\phi/4}m(\bar\Lambda\Lambda) - \frac{15}{128}e^{3\phi/4}F_{MN}(\bar\Lambda\Gamma^{MN}\Gamma_{11}\Lambda) + \frac{1}{512}e^{\phi/4}G_{MNPQ}(\bar\Lambda\Gamma^{MNPQ}\Lambda) \ , \end{aligned} \tag{3.5}$$

and,

$$\begin{aligned} R_{MN} = &\frac{1}{2}\partial_M\phi\partial_N\phi + \frac{1}{16}m^2e^{5\phi/2}g_{MN} + \frac{1}{4}e^{3\phi/2}\Big(2F^2_{MN} - \frac{1}{8}g_{MN}F^2\Big) \\ &+ \frac{1}{12}e^{-\phi}\Big(3H^2_{MN} - \frac{1}{4}g_{MN}H^2\Big) + \frac{1}{48}e^{\phi/2}\Big(4G^2_{MN} - \frac{3}{8}g_{MN}G^2\Big) \\ &+ \frac{1}{2}(\bar\Lambda\Gamma_{(M}\nabla_{N)}\Lambda) + \frac{1}{16}g_{MN}(\bar\Lambda\Gamma^P\nabla_P\Lambda) - \frac{1}{8}g_{MN}\Big[\frac{21}{16}e^{5\phi/4}m(\bar\Lambda\Lambda) - \frac{3}{512}(\bar\Lambda\Lambda)^2\Big] \\ &- \frac{5}{32}e^{3\phi/4}F_{(M}{}^P(\bar\Lambda\Gamma_{N)P}\Gamma_{11}\Lambda) + \frac{1}{128}e^{\phi/4}\Big[2G_{(M}{}^{PQR}(\bar\Lambda\Gamma_{N)PQR}\Lambda) - \frac{1}{8}g_{MN}G_{(4)}(\bar\Lambda\Gamma^{(4)}\Lambda)\Big] \ . \end{aligned} \tag{3.6}$$

The form equations read,

$$\begin{aligned} 0 &= \mathrm{d}\star[e^{3\phi/2}F - \frac{5}{16}e^{3\phi/4}(\bar\Lambda\Gamma_{(2)}\Gamma_{11}\Lambda)] + e^{\phi/2}H\wedge\star[e^{\phi/2}G + \frac{3}{16}e^{\phi/4}(\bar\Lambda\Gamma_{(4)}\Lambda)] \\ 0 &= \mathrm{d}\star e^{-\phi}H + e^{\phi/2}F\wedge\star[e^{\phi/2}G + \frac{3}{16}e^{\phi/4}(\bar\Lambda\Gamma_{(4)}\Lambda)] - \frac{1}{2}G\wedge G \\ &\quad + m\star[e^{3\phi/2}F - \frac{5}{16}e^{3\phi/4}(\bar\Lambda\Gamma_{(2)}\Gamma_{11}\Lambda)] \\ 0 &= \mathrm{d}\star[e^{\phi/2}G + \frac{3}{16}e^{\phi/4}(\bar\Lambda\Gamma_{(4)}\Lambda)] - H\wedge G \ , \end{aligned} \tag{3.7}$$

where: $(\bar\Lambda\Gamma_{(p)}\Lambda) := \frac{1}{p!}(\bar\Lambda\Gamma_{M_1\dots M_p}\Lambda)\mathrm{d}x^{M_p}\wedge\dots\wedge\mathrm{d}x^{M_1}$.

3.1.1 Dilatonic solutions

Bosonic AdS$_4$ solutions

The equations of motion (3.5)-(3.7) together with (2.4) admit bosonic solutions of the form AdS$_4\times M_6$, where M_6 is nearly Kähler, cf. section 11.4 of [42]. Let us now review these solutions, before switching on the dilatino condensates in section 3.1.1.

We take the ten-dimensional spacetime to be of direct product form AdS$_4\times M_6$,

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathrm{AdS}_4) + \mathrm{d}s^2(M_6) \ . \tag{3.8}$$

Let us parametrize,

$$R_{\mu\nu} = 3\lambda\, g_{\mu\nu} \ ; \quad R_{mn} = 20\omega^2 g_{mn} \ , \tag{3.9}$$

where $g_{\mu\nu}$, g_{mn} are the components of the metric in the external, internal space respectively; λ is negative for anti-de Sitter space; ω is related to the first torsion class of M_6 through (1.35).

Moreover we set the dilaton to zero, $\phi = 0$, and we parameterize the three-form and RR fluxes as follows,

$$H = f\mathrm{Re}\Omega \ ; \quad F = bJ \ ; \quad G = a\,\mathrm{vol}_4 + \frac{1}{2}cJ^2 \ ; \quad f, a, b, c \in \mathbb{R} \ , \tag{3.10}$$

where J is the Kähler form of M_6, and vol_4 is the volume element of AdS_4. It is then straightforward to see, using (1.35), that the Bianchi identities (2.4) are satisfied provided,

$$mf + 6b\omega = 0 \ . \tag{3.11}$$

The F-form equation in (3.7) is automatically satisfied, while the H-form equation reduces to,

$$2bc - ac + mb - 8f\omega = 0 \ . \tag{3.12}$$

The G-form equation in (3.7) reduces to,

$$af + 6c\omega = 0 \ . \tag{3.13}$$

Moreover the dilaton equation reduces to,

$$0 = 9b^2 + 3c^2 + 5m^2 - a^2 - 8f^2 \ . \tag{3.14}$$

The mixed (μ, m) components of the Einstein equations are automatically satisfied, while the internal (m, n) components of the Einstein equations reduce to,

$$20\omega^2 = 2b^2 + c^2 + m^2 - f^2 \ , \tag{3.15}$$

where we have taken (3.14) into account. Finally the (μ, ν) components of the Einstein equations reduce to,

$$\lambda = \frac{1}{6}f^2 - 10\omega^2 \ , \tag{3.16}$$

where we have used (3.14) and (3.15).

As noted in [42], these equations admit three general classes of solutions only one of which is supersymmetric and corresponds to the nearly-Kähler solutions first discovered in [43]; it reads,

$$a^2 = \frac{27}{5}m^2 \ ; \quad b = \frac{1}{9}a \ ; \quad c = \frac{3}{5}m \ ; \quad f = \pm\frac{2}{5}m \ ; \quad \omega = -\frac{1}{9}a \ ; \quad \lambda = -\frac{16}{25}m^2 \ . \tag{3.17}$$

In particular we see that the solution is parameterized by a single parameter (the Romans mass) and reduces to flat space without flux in the massless limit $m \to 0$.

Solutions with dilatino condensates

We will now allow for nonvanishing dilatino bilinear and quadratic condensates. Let $\Lambda_\pm$ be the positive-, negative-chirality components of the ten-dimensional dilatino. We decompose,

$$\Lambda_+ = \theta_+ \otimes \eta - \theta_- \otimes \eta^c \ ; \quad \Lambda_- = \theta'_+ \otimes \eta^c - \theta'_- \otimes \eta \ , \tag{3.18}$$

where θ_+, θ'_+ are arbitrary anticommuting four-dimensional Weyl spinors of positive chirality, see appendix A for our spinor conventions. Furthermore the reality of $\Lambda_\pm$ imposes the conditions,

$$\bar{\theta}_+ = \widetilde{\theta}_- \ ; \quad \bar{\theta}_- = -\widetilde{\theta}_+ \ , \tag{3.19}$$

which implies in particular,

$$(\widetilde{\theta}_+\theta'_+)^* = -(\widetilde{\theta}_-\widetilde{\theta}'_-) \ . \tag{3.20}$$

We define the following three complex numbers parameterizing the four-dimensional dilatonic condensate,

$$\mathcal{A} := (\widetilde{\theta}_+\theta'_+) \ ; \quad \mathcal{B} := (\widetilde{\theta}_+\theta_+) \ ; \quad \mathcal{C} := (\widetilde{\theta}'_+\theta'_+) \ . \tag{3.21}$$

In terms of these, the ten-dimensional dilaton bilinears decompose as follows,

$$\begin{aligned}
(\bar{\Lambda}_+\Lambda_-) &= 2\mathrm{Re}(\mathcal{A}) \\
(\bar{\Lambda}_+\Gamma_{mn}\Lambda_-) &= 2\mathrm{Im}(\mathcal{A})J_{mn} \\
(\bar{\Lambda}_+\Gamma_{mnrs}\Lambda_-) &= -6\mathrm{Re}(\mathcal{A})J_{[mn}J_{rs]} \\
(\bar{\Lambda}_+\Gamma_{mnp}\Lambda_+) &= 2\mathrm{Re}(\mathcal{B}\Omega_{mnp}) \\
(\bar{\Lambda}_-\Gamma_{mnp}\Lambda_-) &= -2\mathrm{Re}(\mathcal{C}\Omega^*_{mnp}) \\
(\bar{\Lambda}_+\Gamma_{\mu\nu\rho\sigma}\Lambda_-) &= 2\mathrm{Im}(\mathcal{A})\varepsilon_{\mu\nu\rho\sigma} \ ,
\end{aligned} \tag{3.22}$$

where we have used (1.28). For the "kinetic" bilinear terms we will assume that $(\bar{\Lambda}_\pm\Gamma_\mu\nabla_\nu\Lambda_\pm) = 0$. For a NK internal manifold such that (1.35) holds, we have,

$$\begin{aligned}
(\bar{\Lambda}_+\Gamma_{(m}\nabla_{n)}\Lambda_+) &= -2\mathrm{Im}(\mathcal{B}\omega)g_{mn} \\
(\bar{\Lambda}_-\Gamma_{(m}\nabla_{n)}\Lambda_-) &= -2\mathrm{Im}(\mathcal{C}\omega^*)g_{mn} \ .
\end{aligned} \tag{3.23}$$

Let us now substitute the above in the 10d equations of motion, while retaining the same form ansatz (3.10) as in section 3.1.1. The only difference is that we postulate a 10d metric of the form

$$\mathrm{d}s^2 = \mathrm{d}s^2(S^{1,3}) + \mathrm{d}s^2(M_6) \ , \tag{3.24}$$

where now $S^{1,3}$ can be any maximally symmetric four-dimensional space. I.e. (3.9) is still valid here, but we allow λ to also be positive or zero (corresponding to de Sitter or Minsowski, in addition to anti-de Sitter).

With this ansatz the 10d equations of motion are modified as follows: the Bianchi identities (2.4) are satisfied provided,

$$mf + 6b\omega = 0 \ , \tag{3.25}$$

as was the case for vanishing condensate. The F-form equation in (3.7) is automatically satisfied, while the H-form equation reduces to,

$$2bc - ac + mb - 8f\omega + \frac{5}{4}m\mathrm{Im}\mathcal{A} - \frac{3}{2}b\mathrm{Re}\mathcal{A} = 0 \ , \tag{3.26}$$

exactly as in the case of vanishing condensate. The G-form equation in (3.7) reduces to,

$$(a+\frac{3}{4}\mathrm{Im}\mathcal{A})f+6(c-\frac{3}{4}\mathrm{Re}\mathcal{A})\omega=0\ , \tag{3.27}$$

thus receiving a contribution from the condensate. The dilaton equation reduces to,

$$0=9b(b+\frac{5}{4}\mathrm{Im}\mathcal{A})+3c(c-\frac{3}{4}\mathrm{Re}\mathcal{A})+5m(m-\frac{21}{4}\mathrm{Re}\mathcal{A})-a(a+\frac{3}{4}\mathrm{Im}\mathcal{A})-8f^2\ . \tag{3.28}$$

The mixed (μ, m) components of the Einstein equations are automatically satisfied as before, while the internal (m, n) components of the Einstein equations reduce to,

$$\begin{aligned}20\omega^2=&\ \frac{1}{16}m^2+\frac{5}{16}b^2+\frac{1}{2}f^2+\frac{7}{16}c^2+\frac{3}{16}a^2\\&+\frac{5}{8}b\mathrm{Im}\mathcal{A}+\frac{3}{32}a\mathrm{Im}\mathcal{A}-\frac{15}{32}c\mathrm{Re}\mathcal{A}-\frac{7}{4}\mathrm{Im}(\mathcal{B}\omega+\mathcal{C}\omega^*)-\frac{21}{32}m\mathrm{Re}\mathcal{A}+\frac{3}{2^{12}}(\bar{\Lambda}\Lambda)^2\ ,\end{aligned} \tag{3.29}$$

where, as already mentioned, the last term above should be thought of as the VEV of a quartic fermion term, thus a priori different from the square of the bilinear VEV. Finally the (μ, ν) components of the Einstein equations reduce to,

$$\begin{aligned}\lambda=&\ \frac{1}{48}m^2-\frac{1}{16}b^2-\frac{1}{6}f^2-\frac{3}{16}c^2-\frac{5}{48}a^2\\&-\frac{3}{32}a\mathrm{Im}\mathcal{A}+\frac{3}{32}c\mathrm{Re}\mathcal{A}-\frac{1}{4}\mathrm{Im}(\mathcal{B}\omega+\mathcal{C}\omega^*)-\frac{7}{32}m\mathrm{Re}\mathcal{A}+\frac{1}{2^{12}}(\bar{\Lambda}\Lambda)^2\ .\end{aligned} \tag{3.30}$$

In the limit of vanishing condensates one recovers the bosonic $\mathrm{AdS}_4\times M_6$ solutions reviewed in section 3.1.1. Moreover one can obtain $\lambda>0$, and thus four-dimensional de Sitter space, provided $(\bar{\Lambda}\Lambda)^2$ is sufficiently large.

3.1.2 Consistent truncation

The solutions of section 3.1 can be recovered from the equations of motion of a four-dimensional consistent truncation of the ten-dimensional IIA dilatino-condensate action. In the following we will construct the consistent truncation in the case of vanishing condensates. The case of nonvanishing condensates will be considered in section 3.1.4.

Our ansatz for the ten-dimensional metric includes two scalars A, B with four-dimensional spacetime dependence,

$$\mathrm{d}s^2_{(10)}=e^{2A(x)}\left(e^{2B(x)}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu+g_{mn}\mathrm{d}y^m\mathrm{d}y^n\right)\ . \tag{3.31}$$

From the above we obtain the following formula for the ten-dimensional Laplacian of a scalar $S(x)$ with only four-dimensional spacetime dependence,

$$\nabla^2_{(10)}S(x)=e^{-2A-2B}\left(\nabla^2_{(4)}S(x)+8\partial^\rho A\partial_\rho S+2\partial^\rho B\partial_\rho S\right)\ , \tag{3.32}$$

where the contractions on the right-hand side are taken with respect to the unwarped four-dimensional metric. The Einstein tensor of (3.31) reads,

$$\begin{aligned}R^{(10)}_{mn}&=R^{(6)}_{mn}-e^{-2B}g_{mn}\left(\nabla^\rho\partial_\rho A+8\partial^\rho A\partial_\rho A+2\partial^\rho A\partial_\rho B\right)\\R^{(10)}_{\mu\nu}&=R^{(4)}_{\mu\nu}-g_{\mu\nu}\left(\nabla^\rho\partial_\rho A+\nabla^\rho\partial_\rho B+8\partial^\rho A\partial_\rho A+2\partial^\rho B\partial_\rho B+10\partial^\rho A\partial_\rho B\right)\\&\quad+8\partial_\mu A\partial_\nu A+2\partial_\mu B\partial_\nu B+16\partial_{(\mu}A\partial_{\nu)}B-8\nabla_\mu\partial_\nu A-2\nabla_\mu\partial_\nu B\ ,\end{aligned} \tag{3.33}$$

while the mixed components $R^{(10)}_{m\mu}$ vanish identically.

Our ansatz for the forms is such that the Bianchi identities (2.4) are automatically satisfied. It is given in terms of three scalars φ, χ, γ which are taken to only carry four-dimensional spacetime dependence. Explicitly,

$$F = m\chi J \ ; \quad H = \mathrm{d}\chi\wedge J - 6\omega\chi\mathrm{Re}\Omega \ ; \quad G = \varphi\mathrm{vol}_4 + \frac{1}{2}(m\chi^2+\gamma)J\wedge J - \frac{1}{8\omega}\mathrm{d}\gamma\wedge\mathrm{Im}\Omega \ . \tag{3.34}$$

In particular we obtain,

$$\begin{aligned}
F^2_{mn} &= m^2\chi^2 e^{-2A} g_{mn} \ ; \quad F^2 = 6m^2\chi^2 e^{-4A} \\
H^2_{mn} &= 2e^{-4A-2B}(\partial\chi)^2 g_{mn} + 144e^{-4A}\omega^2\chi^2 g_{mn} \ ; \quad H^2_{\mu\nu} = 6e^{-4A}\partial_\mu\chi\partial_\nu\chi \\
H^2 &= 18e^{-6A-2B}(\partial\chi)^2 + 864e^{-6A}\omega^2\chi^2 \\
G^2_{mn} &= 12e^{-6A}\Big(m\chi^2+\gamma\Big)^2 g_{mn} + \frac{3}{16\omega^2}e^{-6A-2B}(\partial\gamma)^2 g_{mn} \\
G^2_{\mu\nu} &= -6e^{-6A-6B}\varphi^2 g_{\mu\nu} + \frac{3}{8\omega^2}e^{-6A}\partial_\mu\gamma\partial_\nu\gamma \\
G^2 &= -24e^{-8A-8B}\varphi^2 + 72e^{-8A}\Big(m\chi^2+\gamma\Big)^2 + \frac{3}{2\omega^2}e^{-8A-2B}(\partial\gamma)^2 \ ,
\end{aligned} \tag{3.35}$$

where the contractions on the left-hand sides are taken with respect to the ten-dimensional metric while the contractions on the right-hand sides are taken with respect to the unwarped four- and six-dimensional metrics. The following expressions are also useful,

$$\begin{aligned}
\star_{10}F &= \frac{1}{2}m\chi e^{6A+4B}\mathrm{vol}_4\wedge J^2 \\
\star_{10}H &= \frac{1}{2}e^{4A+2B}\star_4\mathrm{d}\chi\wedge J^2 + 6\omega\chi e^{4A+4B}\mathrm{vol}_4\wedge\mathrm{Im}\Omega \\
\star_{10}G &= -\frac{1}{6}\varphi e^{2A-4B}J^3 + (m\chi^2+\gamma)e^{2A+4B}\mathrm{vol}_4\wedge J + \frac{1}{8\omega}e^{2A+2B}\star_4\mathrm{d}\gamma\wedge\mathrm{Re}\Omega \ ,
\end{aligned} \tag{3.36}$$

where the Hodge star is defined as in [24, 22]. Plugging the above ansatz into the equations of motion we obtain the following. The internal (m,n)-components of the Einstein equations (3.6) read,

$$\begin{aligned}
0 &= e^{-8A-2B}\nabla^\mu\Big(e^{8A+2B}\partial_\mu A\Big) + \frac{1}{16}m^2 e^{5\phi/2+2A+2B} + \frac{5}{16}e^{3\phi/2-2A+2B}m^2\chi^2 \\
&+ \frac{1}{8}e^{-\phi-4A}(\partial\chi)^2 + 18e^{-\phi-4A+2B}\omega^2\chi^2 + \frac{1}{16}e^{\phi/2}\Big(3e^{-6A-6B}\varphi^2 + 7e^{-6A+2B}(m\chi^2+\gamma)^2\Big) \\
&+ \frac{1}{256\omega^2}e^{\phi/2-6A}(\partial\gamma)^2 - 20e^{2B}\omega^2 \ ,
\end{aligned} \tag{3.37}$$

where we have taken (3.9) into account. The external (μ,ν)-components read,

$$\begin{aligned}
R^{(4)}_{\mu\nu} &= g_{\mu\nu}\Big(\nabla^2 A + \nabla^2 B + 8(\partial A)^2 + 2(\partial B)^2 + 10\partial A\cdot\partial B\Big) \\
&- 8\partial_\mu A\partial_\nu A - 2\partial_\mu B\partial_\nu B - 16\partial_{(\mu}A\partial_{\nu)}B + 8\nabla_\mu\partial_\nu A + 2\nabla_\mu\partial_\nu B \\
&+ \frac{3}{2}e^{-\phi-4A}\partial_\mu\chi\partial_\nu\chi + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{32\omega^2}e^{\phi/2-6A}\partial_\mu\gamma\partial_\nu\gamma \\
&+ \frac{1}{16}g_{\mu\nu}\Big(-\frac{3}{16\omega^2}e^{\phi/2-6A}(\partial\gamma)^2 - 6e^{-\phi-4A}(\partial\chi)^2 \\
&+ m^2 e^{5\phi/2+2A+2B} - 3m^2\chi^2 e^{3\phi/2-2A+2B} - 288e^{-\phi-4A+2B}\omega^2\chi^2 \\
&- 5e^{\phi/2-6A-6B}\varphi^2 - 9e^{\phi/2-6A+2B}(m\chi^2+\gamma)^2\Big) \ ,
\end{aligned} \tag{3.38}$$

while the mixed (μ, m)-components are automatically satisfied. The dilaton equation reads,

$$\begin{aligned}0 &= e^{-10A-4B}\nabla^\mu\left(e^{8A+2B}\partial_\mu\phi\right) - \frac{5}{4}m^2e^{5\phi/2} - \frac{9}{4}e^{3\phi/2-4A}m^2\chi^2 - \frac{1}{64\omega^2}e^{\phi/2-8A-2B}(\partial\gamma)^2 \\ &+ \frac{3}{2}e^{-\phi-6A-2B}(\partial\chi)^2 + 72e^{-\phi-6A}\omega^2\chi^2 + \frac{1}{4}e^{\phi/2}\left(e^{-8A-8B}\varphi^2 - 3e^{-8A}(m\chi^2+\gamma)^2\right) \ .\end{aligned} \tag{3.39}$$

The F-form equation of motion is automatically satisfied. The H-form equation reduces to,

$$\begin{aligned}0 &= -\nabla^\mu\left(e^{-\phi+4A+2B}\partial_\mu\chi\right) + 48\omega^2e^{-\phi+4A+4B}\chi + e^{3\phi/2+6A+4B}m^2\chi \\ &+ 2me^{\phi/2+2A+4B}(m\chi^2+\gamma)\chi - \varphi(m\chi^2+\gamma) \ .\end{aligned} \tag{3.40}$$

The G-form equation of motion reduces to,

$$0 = \nabla^\mu\left(e^{\phi/2+2A+2B}\partial_\mu\gamma\right) - 48\omega^2e^{\phi/2+2A+4B}(m\chi^2+\gamma) + 48\omega^2\chi\varphi \ , \tag{3.41}$$

together with the following constraint,

$$0 = \frac{1}{3}\mathrm{d}\left(e^{\phi/2+2A-4B}\varphi\right) + (m\chi^2+\gamma)\mathrm{d}\chi + \chi\mathrm{d}\gamma \ . \tag{3.42}$$

The latter can be immediately integrated to solve φ in terms of the remaining fields,

$$\varphi = \left(C - m\chi^3 - 3\gamma\chi\right)e^{-2A+4B-\phi/2} \ , \tag{3.43}$$

where C is an arbitrary constant.

The Lagrangian

As we can see from (3.31) the scalar $B(x)$ can be reabsorbed in the definition of the 4d metric. We have kept it arbitrary so far with the idea to use the associated freedom in order to obtain a 4d effective theory directly in the Einstein frame. This can be accomplished by choosing

$$B = -4A \ . \tag{3.44}$$

With this choice, and taking into account that φ is given in eq. (3.43), it is straightforward to check that the ten-dimensional equations given in (3.37)-(3.41) all follow from the 4d effective action,

$$S_4 = \int \mathrm{d}^4x\sqrt{g}\left(R - 24(\partial A)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{3}{2}e^{-4A-\phi}(\partial\chi)^2 - \frac{1}{32\omega^2}e^{-6A+\phi/2}(\partial\gamma)^2 - V\right) \ , \tag{3.45}$$

where the potential V is given by:

$$\begin{aligned}V = &-120\omega^2e^{-8A} + \frac{1}{2}m^2e^{-6A+5\phi/2} + \frac{3}{2}m^2\chi^2e^{-10A+3\phi/2} + 72\omega^2\chi^2e^{-12A-\phi} \\ &+ \frac{3}{2}(m\chi^2+\gamma)^2e^{-14A+\phi/2} + \frac{1}{2}\left(C - m\chi^3 - 3\gamma\chi\right)^2e^{-18A-\phi/2} \ .\end{aligned} \tag{3.46}$$

3.1.3 AdS$_4$ solutions revisited

The consistent truncation (3.45) captures all of the AdS$_4$ solutions of [42] reviewed in section 3.1.1. Indeed upon setting the warp factor and the dilaton to zero, $A = \phi = 0$, and the remaining fields γ, χ to constant values, imposing the equations of motion amounts to finding a minimum of the potential V of (3.46). We thus obtain the following three classes of solutions:

First class

$$H = \pm m\mathrm{Re}\Omega \ ; \quad F = \pm\frac{1}{\sqrt{3}}mJ \ ; \quad G = \mp\sqrt{3}m\,\mathrm{vol}_4 - \frac{1}{2}mJ^2 \ ; \quad \Omega = -\frac{2}{3}m^2 \ , \tag{3.47}$$

where it is understood that the sign of F is correlated with that of the external part of G, while the sign of H is arbitrary. This can be written equivalently:

$$\omega = \pm\frac{1}{2\sqrt{3}}m \ ; \quad \chi = \pm\frac{1}{\sqrt{3}} \ ; \quad \gamma = -\frac{4}{3}m \ ; \quad C = \mp\frac{20}{3\sqrt{3}}m \ . \tag{3.48}$$

Second class

$$H = 0 \ ; \quad F = 0 \ ; \quad G = \pm\sqrt{5}m\,\mathrm{vol}_4 \ ; \quad \Omega = -\frac{1}{2}m^2 \ . \tag{3.49}$$

Or, equivalently,

$$\omega = \pm\frac{1}{2\sqrt{5}}m \ ; \quad \chi = 0 \ ; \quad \gamma = 0 \ ; \quad C = \pm\sqrt{5}m \ . \tag{3.50}$$

Third class

$$H = \pm\frac{2}{5}m\mathrm{Re}\Omega \ ; \quad F = \pm\frac{1}{\sqrt{15}}mJ \ ; \quad G = \pm\sqrt{\frac{27}{5}}m\,\mathrm{vol}_4 + \frac{3}{10}mJ^2 \ ; \quad \Omega = -\frac{16}{25}m^2 \ , \tag{3.51}$$

where it is understood that the sign of F is correlated with that of the external part of G, while the sign of H is arbitrary. Equivalently:

$$\omega = \pm\frac{1}{\sqrt{15}}m \ ; \quad \chi = \pm\frac{1}{\sqrt{15}} \ ; \quad \gamma = \frac{8}{15}m \ ; \quad C = \pm\frac{32}{3\sqrt{15}}m \ . \tag{3.52}$$

In the above we have noted that Ω is given by the value of $V/6$ at the minimum, as follows from (3.45), (3.9). These coincide with the three classes of solutions presented in section 11.4 of [42], with the third class being the supersymmetric one, cf. (3.17).

3.1.4 Consistent truncation with condensates

In the presence of condensates, the internal (m,n)-components of the Einstein equations (3.37) get modified as follows,

$$\begin{aligned}
0 = {} & e^{-8A-2B}\nabla^\mu\left(e^{8A+2B}\partial_\mu A\right) + \frac{1}{16}m^2 e^{5\phi/2+2A+2B} + \frac{5}{16}e^{3\phi/2-2A+2B}m^2\chi^2 \\
& + \frac{1}{8}e^{-\phi-4A}(\partial\chi)^2 + 18e^{-\phi-4A+2B}\omega^2\chi^2 + \frac{1}{16}e^{\phi/2}\left(3e^{-6A-6B}\varphi^2 + 7e^{-6A+2B}(m\chi^2+\gamma)^2\right) \\
& + \frac{1}{256\omega^2}e^{\phi/2-6A}(\partial\gamma)^2 - 20e^{2B}\omega^2 - \frac{7}{4}e^{A+2B}\mathrm{Im}(\mathcal{B}\omega + \mathcal{C}\omega^*) \\
& - \frac{1}{32}e^{2A+2B}\left(21e^{5\phi/4}m\mathrm{Re}\mathcal{A} - \frac{3}{128}(\bar{\Lambda}\Lambda)^2\right) + \frac{5}{8}e^{3\phi/4+2B}m\chi\mathrm{Im}\mathcal{A} \\
& + \frac{3}{32}e^{\phi/4-2A-2B}\varphi\mathrm{Im}\mathcal{A} - \frac{15}{32}e^{\phi/4-2A+2B}(m\chi^2+\gamma)\mathrm{Re}\mathcal{A} \ ,
\end{aligned} \tag{3.53}$$

where we have taken (3.9) into account. The external (μ,ν)-components read,

$$\begin{aligned}
R^{(4)}_{\mu\nu} &= g_{\mu\nu}\Big(\nabla^2 A+\nabla^2 B+8(\partial A)^2+2(\partial B)^2+10\partial A\cdot\partial B\Big)\\
&-8\partial_\mu A\partial_\nu A-2\partial_\mu B\partial_\nu B-16\partial_{(\mu}A\partial_{\nu)}B+8\nabla_\mu\partial_\nu A+2\nabla_\mu\partial_\nu B\\
&+\frac{3}{2}e^{-\phi-4A}\partial_\mu\chi\partial_\nu\chi+\frac{1}{2}\partial_\mu\phi\partial_\nu\phi+\frac{1}{32\omega^2}e^{\phi/2-6A}\partial_\mu\gamma\partial_\nu\gamma\\
&+\frac{1}{16}g_{\mu\nu}\Big[-6e^{-\phi-4A}(\partial\chi)^2-\frac{3}{16\omega^2}e^{\phi/2-6A}(\partial\gamma)^2\\
&+m^2e^{5\phi/2+2A+2B}-3m^2\chi^2e^{3\phi/2-2A+2B}\\
&-288e^{-\phi-4A+2B}\omega^2\chi^2-5e^{\phi/2-6A-6B}\varphi^2-9e^{\phi/2-6A+2B}(m\chi^2+\gamma)^2\\
&-\frac{1}{2}e^{2A+2B}\Big(21e^{5\phi/4}m\mathrm{Re}\mathcal{A}-\frac{3}{128}(\bar\Lambda\Lambda)^2\Big)-12e^{A+2B}\mathrm{Im}(\mathcal{B}\omega+\mathcal{C}\omega^*)\\
&-\frac{9}{2}e^{\phi/4-2A-2B}\varphi\mathrm{Im}\mathcal{A}+\frac{9}{2}e^{\phi/4-2A+2B}(m\chi^2+\gamma)\mathrm{Re}\mathcal{A}\Big]\ ,
\end{aligned}\tag{3.54}$$

while the mixed (μ,m)-components are automatically satisfied. The dilaton equation reads,

$$\begin{aligned}
0 &= e^{-10A-4B}\nabla^\mu\Big(e^{8A+2B}\partial_\mu\phi\Big)+\frac{3}{2}e^{-\phi-6A-2B}(\partial\chi)^2-\frac{1}{64\omega^2}e^{\phi/2-8A-2B}(\partial\gamma)^2\\
&-\frac{5}{4}m^2e^{5\phi/2}-\frac{9}{4}e^{3\phi/2-4A}m^2\chi^2+72e^{-\phi-6A}\omega^2\chi^2\\
&+\frac{1}{4}e^{\phi/2}\Big(e^{-8A-8B}\varphi^2-3e^{-8A}(m\chi^2+\gamma)^2\Big)+\frac{105}{16}e^{5\phi/4}\mathrm{Re}\mathcal{A}\\
&-\frac{45}{16}e^{3\phi/4-2A}m\chi\mathrm{Im}\mathcal{A}+\frac{3}{16}e^{\phi/4-4A-4B}\varphi\mathrm{Im}\mathcal{A}+\frac{9}{16}e^{\phi/4-4A}(m\chi^2+\gamma)\mathrm{Re}\mathcal{A}\ .
\end{aligned}\tag{3.55}$$

The F-form equation of motion is automatically satisfied. The H-form equation (3.40) reads,

$$\begin{aligned}
0 &= -\nabla^\mu\Big(e^{-\phi+4A+2B}\partial_\mu\chi\Big)+48\omega^2e^{-\phi+4A+4B}\chi+e^{3\phi/2+6A+4B}m^2\chi+2me^{\phi/2+2A+4B}(m\chi^2+\gamma)\chi\\
&-\varphi(m\chi^2+\gamma)+\frac{5}{4}e^{3\phi/4-2B}m\mathrm{Im}\mathcal{A}-\frac{3}{2}e^{\phi/4-2A+2B}m\chi\mathrm{Re}\mathcal{A}\ .
\end{aligned}\tag{3.56}$$

The G-form equation of motion reads,

$$0=\nabla^\mu\Big(e^{\phi/2+2A+2B}\partial_\mu\gamma\Big)-48\omega^2e^{\phi/2+2A+4B}(m\chi^2+\gamma)+48\omega^2\chi\varphi+36\omega^2e^{\phi/4+6A+4B}\mathrm{Re}\mathcal{A}\ ,\tag{3.57}$$

together with the following constraint,

$$0=\mathrm{d}\left(\frac{1}{3}\varphi e^{\phi/2+2A-4B}+\frac{1}{4}e^{\phi/4+6A}\mathrm{Im}\mathcal{A}\right)+(m\chi^2+\gamma)\mathrm{d}\chi+\chi\mathrm{d}\gamma\ .\tag{3.58}$$

The latter can be immediately integrated to solve φ in terms of the remaining fields,

$$\varphi=\Big(C-m\chi^3-3\gamma\chi\Big)e^{-\phi/2-2A+4B}-\frac{3}{4}e^{-\phi/4+4A+4B}\mathrm{Im}\mathcal{A}\ ,\tag{3.59}$$

where C is an arbitrary constant.

Upon imposing (3.44) as before, a tedious but straightforward calculation then shows that all the above equations of motion can be obtained from the following four-dimensional action,

$$\boxed{S_4 = \int \mathrm{d}^4x\sqrt{g}\left(R - 24(\partial A)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{3}{2}e^{-4A-\phi}(\partial\chi)^2 - \frac{1}{32\omega^2}e^{-6A+\phi/2}(\partial\gamma)^2 - V\right)\ .} \tag{3.60}$$

The action has exactly the same kinetic terms as before, cf. (3.45), but the potential now reads,

$$\boxed{\begin{aligned} V = &-120\omega^2 e^{-8A} + \frac{1}{2}m^2 e^{-6A+5\phi/2} + \frac{3}{2}m^2\chi^2 e^{-10A+3\phi/2} + 72\omega^2\chi^2 e^{-12A-\phi} \\ &+ \frac{3}{2}(m\chi^2+\gamma)^2 e^{-14A+\phi/2} + \frac{1}{2}\varphi^2 e^{18A+\phi/2} - 12e^{-7A}\mathrm{Im}(\mathcal{B}\omega + \mathcal{C}\omega^*) \\ &+ \frac{15}{4}m\chi e^{3\phi/4-8A}\mathrm{Im}\mathcal{A} - \frac{21}{4}e^{5\phi/4-6A}m\mathrm{Re}\mathcal{A} - \frac{9}{4}e^{\phi/4-10A}(m\chi^2+\gamma)\mathrm{Re}\mathcal{A} \\ &+ \frac{3}{512}e^{-6A}(\bar{\Lambda}\Lambda)^2\ , \end{aligned}} \tag{3.61}$$

where φ is non-dynamical and is given by,

$$\boxed{\varphi = (C - m\chi^3 - 3\gamma\chi)\, e^{-\phi/2-18A} - \frac{3}{4}e^{-\phi/4-12A}\mathrm{Im}\mathcal{A}\ .} \tag{3.62}$$

It can also be seen that this consistent truncation contains the $S^{1,3}\times M_6$ solutions of section 3.1.1 as special cases.

3.1.5 The Calabi-Yau limit

It can be seen from the equations of motion that the limit $\omega \to 0$ can be taken consistently, provided that we first rewrite,

$$\gamma = \gamma_0 + 4\omega\xi\ , \tag{3.63}$$

where γ_0 is constant while ξ is dynamical. This corresponds to the Calabi-Yau (CY) limit, in the sense of the vanishing of all $SU(3)$ torsion classes.[3]

More explicitly, in this case our ansatz for the forms becomes,

$$F = m\chi J\ ;\quad H = \mathrm{d}\chi\wedge J\ ;\quad G = \varphi\mathrm{vol}_4 + \frac{1}{2}(m\chi^2+\gamma_0)J\wedge J - \frac{1}{2}\mathrm{d}\xi\wedge\mathrm{Im}\Omega\ , \tag{3.64}$$

and can be seen to automatically satisfy the BI's (2.4), taking into account that $\mathrm{d}J = \mathrm{d}\Omega = 0$. All remaining equations of motion can be obtained from (3.37)-(3.43) by first replacing γ using (3.63) and then taking the $\omega\to 0$ limit. Note that this rewriting allows to keep the dynamical field ξ in the limit.

Moreover it can be seen that all equations of motion can be integrated into the following Lagrangian,

$$S^{\mathrm{CY}} = \int \mathrm{d}^4x\sqrt{g}\left(R - 24(\partial A)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{3}{2}e^{-4A-\phi}(\partial\chi)^2 - \frac{1}{2}e^{-6A+\phi/2}(\partial\xi)^2 - V^{\mathrm{CY}}\right)\ . \tag{3.65}$$

[3]This is more general than the usual definition of a CY, as it allows for manifolds with nonvanishing fundamental group such as T^6.

The potential V^{CY} above is given by,

$$
\begin{aligned}
V^{\mathrm{CY}} = &+\frac{1}{2}m^2 e^{-6A+5\phi/2}+\frac{3}{2}m^2\chi^2 e^{-10A+3\phi/2}+\frac{3}{2}(m\chi^2+\gamma_0)^2 e^{-14A+\phi/2}\\
&+\frac{1}{2}\varphi^2 e^{18A+\phi/2}+\frac{15}{4}m\chi e^{3\phi/4-8A}\mathrm{Im}\mathcal{A}-\frac{21}{4}e^{5\phi/4-6A}m\mathrm{Re}\mathcal{A}\\
&-\frac{9}{4}e^{\phi/4-10A}(m\chi^2+\gamma_0)\mathrm{Re}\mathcal{A}+\frac{3}{512}e^{-6A}(\bar{\Lambda}\Lambda)^2 ,
\end{aligned}
\tag{3.66}
$$

where,

$$
\varphi = \Big(C - m\chi^3 - 3\gamma_0\chi\Big)\, e^{-\phi/2-18A} - \frac{3}{4}e^{-\phi/4-12A}\mathrm{Im}\mathcal{A} . \tag{3.67}
$$

As can be easily seen, in the absence of condensates, unless all flux is zero, the potential is non-negative and only has runaway minima. However this need no longer be the case in the presence of nonvanishing condensates.

To our knowledge, this is the first truncation on a CY manifold with massive four-dimensional fields, whose consistency has been rigorously proven.

Massless limit

A further truncation to two scalars, the warp factor A and the dilaton ϕ, can be obtained by taking the massless limit, $m = 0$, while at the same time setting χ, $\gamma = 0$. This amounts to the following flux ansatz:

$$
F = 0\ ; \quad H = 0\ ; \quad G = \varphi \mathrm{vol}_4\ , \tag{3.68}
$$

which is of Freund-Rubin type, and automatically satisfies the BI's (2.4). Moreover the remaining form equations reduce to a single constraint,

$$
\varphi = C\, e^{-\phi/2-18A} - \frac{3}{4}e^{-\phi/4-12A}\mathrm{Im}\mathcal{A}\ , \tag{3.69}
$$

where C is an arbitrary constant. It can then be seen that all equations of motion can be integrated to the following Lagrangian,

$$
S_0^{\mathrm{CY}} = \int \mathrm{d}^4x\sqrt{g}\left(R - 24(\partial A)^2 - \frac{1}{2}(\partial\phi)^2 - V^{\mathrm{CY}}\right)\ , \tag{3.70}
$$

where the potential V^{CY} is given by,

$$
V^{\mathrm{CY}} = \frac{1}{2}\varphi^2 e^{18A+\phi/2} + \frac{3}{512}e^{-6A}(\bar{\Lambda}\Lambda)^2\ . \tag{3.71}
$$

3.2 Calabi-Yau truncation

3.2.1 Review of IIA reduction on CY

To establish notation and conventions, let us briefly review the reduction of IIA on CY at the two-derivative level, in the absence of flux and condensates. As is well known, the KK reduction of (massless) IIA supergravity around the fluxless $\mathbb{R}^{1,3}\times Y$ vacuum results in a 4d $\mathcal{N}=2$ supergravity, whose bosonic sector consists of one gravity multiplet (containing the metric and one vector), $h^{1,1}$ vector multiplets (each of which consists of one vector and two real scalars) and $h^{2,1}+1$ hypermultiplets (each of which contains four real scalars), where $h^{p,q}$ are the Hodge numbers of the CY threefold Y. The $2h^{1,1}$ real scalars (v^A, χ^A) in the vector multiplets come from the NS-NS B field and deformations of the metric of the form,

$$B = \beta(x) + \sum_{A=1}^{h^{1,1}} \chi^A(x)\mathfrak{e}^A(y) \ ; \quad i\delta g_{a\bar{b}} = \sum_{A=1}^{h^{1,1}} v^A(x)\mathfrak{e}^A_{a\bar{b}}(y) \ , \tag{3.72}$$

where β is a two-form in $\mathbb{R}^{1,3}$; $\{\mathfrak{e}^A_{a\bar{b}}(y),\ A = 1, \ldots, h^{1,1}\}$ is a basis of harmonic (1,1)-forms on the CY, and x, y are coordinates of $\mathbb{R}^{1,3}$, Y respectively; we have introduced holomorphic, antiholomorphic internal indices from the beginning of the latin alphabet: $a = 1, \ldots, 3$, $\bar{b}, = 1, \ldots, 3$, respectively. Since every CY has a Kähler form (which can be expressed as a linear combination of the basis (1,1)-forms), there is a always at least one vector multiplet (which may be called "universal", in that that it exists for any CY compactification) whose scalars consist of the volume modulus v and one scalar χ.

The $2(h^{2,1}+1)$ complex scalars of the hypermultiplets, and the $h^{1,1}+1$ vectors of the gravity and the vectormultiplets arise as follows: from the one- and three-form RR potentials C_1, C_3 and the complex-structure deformations of the metric,[4]

$$\begin{aligned} \delta g_{\bar{a}\bar{b}} &= \sum_{\alpha=1}^{h^{2,1}} \zeta^\alpha(x)\Omega^{*cd}{}_{\bar{a}}\Phi^\alpha_{cd\bar{b}}(y) \ ; \quad C_1 = \alpha(x) \ ; \\ C_3 &= -\frac{1}{2}\Big(\xi(x)\mathrm{Im}\Omega + \xi'(x)\mathrm{Re}\Omega\Big) + \sum_{A=1}^{h^{1,1}} \gamma^A(x)\wedge\mathfrak{e}^A(y) + \Big(\sum_{\alpha=1}^{h^{2,1}} \xi^\alpha(x)\Phi^\alpha(y) + \text{c.c.}\Big) \ , \end{aligned} \tag{3.73}$$

where $\Omega(y)$ is the holomorphic threeform of the CY and $\{\Phi^\alpha_{ab\bar{c}}(y),\ \alpha = 1, \ldots, h^{2,1}\}$ is basis of harmonic (2,1) forms on the CY, we obtain the complex scalars $(\zeta^\alpha, \xi^\alpha)$ and the vectors (α, γ^A). Moreover the real scalars (ξ, ξ') together with the dilaton ϕ and the axion b combine into one universal hypermultiplet. Recall that if h is the 4d component of the NSNS three-form,

$$h = \mathrm{d}\beta \ , \tag{3.74}$$

the axion b is given schematically by $\mathrm{d}b \sim \star_4 h$ (the precise relation is eq. (3.93) below).

In summary, the universal bosonic sector of the 4d $\mathcal{N}=2$ supergravity arising from IIA compactification on Y contains the metric and the vector of the gravity multiplet $(g_{\mu\nu}, \alpha)$, the vector and the the scalars of one vectormultiplet (γ, v, χ), and the scalars of the universal hypermultiplet (ξ, ξ', ϕ, b).

In the last section we presented a universal consistent truncation on Nearly-Kähler and CY manifolds in the presence of dilatino condensates. As it turns out, this consistent truncation captures only part of the universal scalar sector of the $\mathcal{N}=2$ low-energy effective supergravity obtained from IIA theory compactified on CY threefolds. Therefore we must extend the ansatz to include the "missing" fields and also to take into account the gravitino condensates.

3.2.2 Action and equations of motion

In [22] the quartic dilatino terms of all (massive) IIA supergravities [17, 18, 19, 41, 44] were determined in the ten-dimensional superspace formalism of [40], and were found to agree

[4] The right-hand side of the first equation of (3.73) can be seen to be automatically symmetric in its two free indices.

with [17]. As follows from the result of [40], the quartic fermion terms are common to all IIA supergravities (massive or otherwise). In the following we will complete Romans supergravity (whose quartic fermion terms were not computed in [41]) by adding the quartic gravitino terms given in [17]. Furthermore we will set the dilatino to zero. Of course this would be inconsistent in general, since the dilatino couples linearly to gravitino terms. Here this does not lead to an inconsistency in the equations of motion, since we are ultimately interested in a maximally-symmettric vacuum, in which linear and cubic fermion VEV's vanish.

In the conventions of [22, 2], upon setting the dilatino to zero, the action of Romans supergravity reads,

$$\begin{aligned} S = S_b + \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x\sqrt{g}\Big\{ & 2(\tilde{\Psi}_M \Gamma^{MNP} \nabla_N \Psi_P) + \frac{1}{2} e^{5\phi/4} m (\tilde{\Psi}_M \Gamma^{MN} \Psi_N) \\ & - \frac{1}{2\cdot 2!} e^{3\phi/4} F_{M_1 M_2} (\tilde{\Psi}^M \Gamma_{[M} \Gamma^{M_1 M_2} \Gamma_{N]} \Gamma_{11} \Psi^N) \\ & - \frac{1}{2\cdot 3!} e^{-\phi/2} H_{M_1 \dots M_3} (\tilde{\Psi}^M \Gamma_{[M} \Gamma^{M_1 \dots M_3} \Gamma_{N]} \Gamma_{11} \Psi^N) \\ & + \frac{1}{2\cdot 4!} e^{\phi/4} G_{M_1 \dots M_4} (\tilde{\Psi}^M \Gamma_{[M} \Gamma^{M_1 \dots M_4} \Gamma_{N]} \Psi^N) + L_{\Psi^4} \Big\} , \end{aligned} \tag{3.75}$$

where Ψ_M is the gravitino; $S_b = -S^E_{\mathrm{IIA}}$ denotes the bosonic sector defined in (2.7). There are 24 quartic gravitino terms as given in [17], denoted L_{Ψ^4} in (3.75). Of these only four can have a nonvanishing VEV in an ALE space: they are discussed in more detail in section 3.3.4.

We emphasize that the action (3.75) should be regarded as a book-keeping device whose variation with respect to the bosonic fields gives the correct bosonic equations of motion in the presence of gravitino condensates. Furthermore, the fermionic equations of motion are trivially satisfied in the maximally-symmetric vacuum. The (bosonic) equations of motion (EOM) following from (3.75) are as follows:

Dilaton EOM,

$$\begin{aligned} 0 = & -\nabla^2\phi + \frac{3}{8} e^{3\phi/2} F^2 - \frac{1}{12} e^{-\phi} H^2 + \frac{1}{96} e^{\phi/2} G^2 + \frac{5}{4} m^2 e^{5\phi/2} \\ & + \frac{5}{8} e^{5\phi/4} m (\tilde{\Psi}_M \Gamma^{MN} \Psi_N) \\ & - \frac{3}{16} e^{3\phi/4} F_{M_1 M_2} (\tilde{\Psi}^M \Gamma_{[M} \Gamma^{M_1 M_2} \Gamma_{N]} \Gamma_{11} \Psi^N) \\ & + \frac{1}{24} e^{-\phi/2} H_{M_1 \dots M_3} (\tilde{\Psi}^M \Gamma_{[M} \Gamma^{M_1 \dots M_3} \Gamma_{N]} \Gamma_{11} \Psi^N) \\ & + \frac{1}{192} e^{\phi/4} G_{M_1 \dots M_4} (\tilde{\Psi}^M \Gamma_{[M} \Gamma^{M_1 \dots M_4} \Gamma_{N]} \Psi^N) \, . \end{aligned} \tag{3.76}$$

Einstein EOM,

$$
\begin{aligned}
R_{MN} &= \frac{1}{2}\partial_M\phi\partial_N\phi + \frac{1}{16}m^2 e^{5\phi/2}g_{MN} + \frac{1}{4}e^{3\phi/2}\Big(2F^2_{MN} - \frac{1}{8}g_{MN}F^2\Big)\\
&+ \frac{1}{12}e^{-\phi}\Big(3H^2_{MN} - \frac{1}{4}g_{MN}H^2\Big) + \frac{1}{48}e^{\phi/2}\Big(4G^2_{MN} - \frac{3}{8}g_{MN}G^2\Big)\\
&+ \frac{1}{24}e^{\phi/4}G_{(M|}{}^{M_1M_2M_3}(\tilde{\Psi}_P\Gamma^{[P}\Gamma_{|N)M_1M_2M_3}\Gamma^{Q]}\Psi_Q)\\
&- \frac{1}{96}e^{\phi/4}G_{M_1\dots M_4}\Big\{(\tilde{\Psi}_P\Gamma_{(M}\Gamma^{M_1\dots M_4}\Gamma^P\Psi_{N)}) - (\tilde{\Psi}_P\Gamma^P\Gamma^{M_1\dots M_4}\Gamma_{(M}\Psi_{N)}) + \frac{1}{2}g_{MN}(\tilde{\Psi}^P\Gamma_{[P}\Gamma^{M_1\dots M_4}\Gamma_{Q]}\Psi^Q)\\
&- \frac{1}{8}g_{MN}L_{\Psi^4} + \frac{\delta L_{\Psi^4}}{\delta g^{MN}}\ ,
\end{aligned}
\tag{3.77}
$$

where we have set: $\Phi^2_{MN} := \Phi_{MM_2\dots M_p}\Phi_N{}^{M_2\dots M_p}$, for any p-form Φ. We have refrained from spelling out explicitly the quartic gravitino terms in the Einstein equation above, as they are numerous and not particularly enlightening. We will calculate them explicitly later on in the case of the ALE space in section 3.3.4.

Form EOM's,[5]

$$
\begin{aligned}
0 &= \mathrm{d}\star\big[e^{3\phi/2}F - \frac{1}{2}e^{3\phi/4}(\tilde{\Psi}^M\Gamma_{[M}\Gamma^{(2)}\Gamma_{N]}\Gamma_{11}\Psi^N)\big] + H\wedge\star\big[e^{\phi/2}G + \frac{1}{2}e^{\phi/4}(\tilde{\Psi}^M\Gamma_{[M}\Gamma^{(4)}\Gamma_{N]}\Psi^N)\big]\\
0 &= \mathrm{d}\star\big[e^{-\phi}H - \frac{1}{2}e^{-\phi/2}(\tilde{\Psi}^M\Gamma_{[M}\Gamma^{(3)}\Gamma_{N]}\Gamma_{11}\Psi^N)\big] + e^{\phi/2}F\wedge\star\big[e^{\phi/2}G + \frac{1}{2}e^{\phi/4}(\tilde{\Psi}^M\Gamma_{[M}\Gamma^{(4)}\Gamma_{N]}\Psi^N)\big]\\
&- \frac{1}{2}G\wedge G + m\star\big[e^{3\phi/2}F - \frac{1}{2}e^{3\phi/4}(\tilde{\Psi}^M\Gamma_{[M}\Gamma^{(2)}\Gamma_{N]}\Gamma_{11}\Psi^N)\big]\\
0 &= \mathrm{d}\star\big[e^{\phi/2}G + \frac{1}{2}e^{\phi/4}(\tilde{\Psi}^M\Gamma_{[M}\Gamma^{(4)}\Gamma_{N]}\Psi^N)\big] - H\wedge G\ ,
\end{aligned}
\tag{3.78}
$$

where $\Gamma^{(p)} := \frac{1}{p!}\Gamma_{M_1\dots M_p}\mathrm{d}x^{M_p}\wedge\dots\wedge\mathrm{d}x^{M_1}$.

Note that we will use only the bosonic part during this section. Gravitini will be postponed to the next section.

3.2.3 Consistent truncation

The truncation of section 3.1.2 contains the four real scalars (A, χ, ϕ, ξ), with A related to the volume modulus v of section 3.2.1: it does not capture all the scalars of the universal sector of $\mathcal{N}=2$ supergravity, since it does not include the vectors and it truncates the two scalars ξ', b of section 3.2.1. We must therefore expand the ansatz of 3.34 to include the "missing" fields, at the same time taking the limit to the massless IIA theory, $m\to 0$. Explicitly we set,

[5] We are using "superspace conventions" as in [24] so that,

$$
\Phi_{(p)} = \frac{1}{p!}\Phi_{m_1\dots m_p}\mathrm{d}x^{m_p}\wedge\dots\wedge\mathrm{d}x^{m_1}\ ;\quad \mathrm{d}\Big(\Phi_{(p)}\wedge\Psi_{(q)}\Big) = \Phi_{(p)}\wedge\mathrm{d}\Psi_{(q)} + (-1)^q\mathrm{d}\Phi_{(p)}\wedge\Psi_{(q)}\ .
$$

In D dimensions the Hodge star is defined as follows,

$$
\star(\mathrm{d}x^{a_1}\wedge\dots\wedge\mathrm{d}x^{a_p}) = \frac{1}{(D-p)!}\varepsilon^{a_1\dots a_p}{}_{b_1\dots b_{10-p}}\mathrm{d}x^{b_1}\wedge\dots\wedge\mathrm{d}x^{b_{10-p}}\ .
$$

$$F = \mathrm{d}\alpha\ ; \quad H = \mathrm{d}\chi\wedge J+\mathrm{d}\beta\ ; \quad G = \varphi\mathrm{vol}_4+\frac{1}{2}c_0 J\wedge J+J\wedge(\mathrm{d}\gamma-\alpha\wedge\mathrm{d}\chi)-\frac{1}{2}\mathrm{d}\xi\wedge\mathrm{Im}\Omega-\frac{1}{2}\mathrm{d}\xi'\wedge\mathrm{Re}\Omega\ , \tag{3.79}$$

where c_0 is a real constant and $\varphi(x)$ is a 4d scalar. We have chosen to express H in terms of the 4d potential β instead of the axion. Taking into account that for a CY we have $\mathrm{d}J = \mathrm{d}\Omega = 0$, this ansatz can be seen to automatically satisfy the Bianchi identities (2.4) in the massless limit. Our ansatz for the ten-dimensional metric is still (3.31). This gives,

$$\begin{aligned}
F^2_{\mu\nu} &= e^{-2A-2B}\mathrm{d}\alpha^2_{\mu\nu}\ ; \quad F^2 = e^{-4A-4B}\mathrm{d}\alpha^2\\
H^2_{mn} &= 2e^{-4A-2B}(\partial\chi)^2 g_{mn}\ ; \quad H^2_{\mu\nu} = 6e^{-4A}\partial_\mu\chi\partial_\nu\chi + e^{-4A-4B}h^2_{\mu\nu}\\
H^2 &= 18e^{-6A-2B}(\partial\chi)^2 + e^{-6A-6B}h^2\\
G^2_{mn} &= 3e^{-6A-2B}\Big[(\partial\xi)^2+(\partial\xi')^2\Big]g_{mn} + 12e^{-6A}c_0^2 g_{mn} + 3e^{-6A-4B}(\mathrm{d}\gamma-\alpha\wedge\mathrm{d}\chi)^2 g_{mn}\\
G^2_{\mu\nu} &= -6e^{-6A-6B}\varphi^2 g_{\mu\nu} + 6e^{-6A}(\partial_\mu\xi\partial_\nu\xi+\partial_\mu\xi'\partial_\nu\xi') + 18e^{-6A-2B}(\mathrm{d}\gamma-\alpha\wedge\mathrm{d}\chi)^2_{\mu\nu}\\
G^2 &= -24e^{-8A-8B}\varphi^2 + 24e^{-8A-2B}\Big[(\partial\xi)^2+(\partial\xi')^2\Big] + 72c_0^2 e^{-8A} + 36e^{-8A-4B}(\mathrm{d}\gamma-\alpha\wedge\mathrm{d}\chi)^2\ ,
\end{aligned} \tag{3.80}$$

where the contractions on the left-hand sides above are computed with respect to the ten-dimensional metric; the contractions on the right-hand sides are taken with respect to the unwarped metric. It is also useful to note the following expressions,

$$\begin{aligned}
\star_{10}F &= \frac{1}{6}e^{6A}\star_4\mathrm{d}\alpha\wedge J^3\\
\star_{10}H &= \tfrac{1}{2}e^{4A+2B}\star_4\mathrm{d}\chi\wedge J^2 + \tfrac{1}{6}e^{4A-2B}\star_4 h\wedge J^3\\
\star_{10}G &= -\tfrac{1}{6}\varphi e^{2A-4B}J^3 + c_0 e^{2A+4B}\mathrm{vol}_4\wedge J + \tfrac{1}{2}e^{2A}\star_4(\mathrm{d}\gamma-\alpha\wedge\mathrm{d}\chi)\wedge J^2\\
&\quad + \tfrac{1}{2}e^{2A+2B}\star_4\mathrm{d}\xi\wedge\mathrm{Re}\Omega - \tfrac{1}{2}e^{2A+2B}\star_4\mathrm{d}\xi'\wedge\mathrm{Im}\Omega\ ,
\end{aligned} \tag{3.81}$$

where the four-dimensional Hodge-star is taken with respect to the unwarped metric.

Plugging the above ansatz into the ten-dimensional EOM (3.76)-(3.78) we obtain the following: the internal (m,n)-components of the Einstein equations read,

$$\begin{aligned}
0 &= e^{-8A-2B}\nabla^\mu\left(e^{8A+2B}\partial_\mu A\right) - \frac{1}{32}e^{3\phi/2-2A-2B}\mathrm{d}\alpha^2 + \frac{1}{8}e^{-\phi-4A}(\partial\chi)^2 - \frac{1}{48}e^{-\phi-4A-4B}h^2\\
&- \frac{1}{32}e^{\phi/2-6A-2B}(\mathrm{d}\gamma-\alpha\wedge\mathrm{d}\chi)^2 + \frac{1}{16}e^{\phi/2-6A}\Big[(\partial\xi)^2+(\partial\xi')^2\Big]\\
&+ \frac{3}{16}e^{\phi/2-6A-6B}\varphi^2 + \frac{7}{16}e^{\phi/2-6A+2B}c_0^2\ .
\end{aligned} \tag{3.82}$$

The external (μ,ν)-components read,

$$
\begin{aligned}
R^{(4)}_{\mu\nu} = g_{\mu\nu}&\Big(\nabla^2 A + \nabla^2 B + 8(\partial A)^2 + 2(\partial B)^2 + 10\partial A\cdot\partial B\Big)\\
&- 8\partial_\mu A\partial_\nu A - 2\partial_\mu B\partial_\nu B - 16\partial_{(\mu}A\partial_{\nu)}B + 8\nabla_\mu\partial_\nu A + 2\nabla_\mu\partial_\nu B\\
&+ \frac{3}{2}e^{-\phi-4A}\partial_\mu\chi\partial_\nu\chi + \frac{1}{2}e^{3\phi/2-2A-2B}\mathrm{d}\alpha^2_{\mu\nu} + \frac{1}{4}e^{\phi-4A-4B}h^2_{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi\\
&+ \frac{1}{2}e^{\phi/2-6A}(\partial_\mu\xi\partial_\nu\xi + \partial_\mu\xi'\partial_\nu\xi') + \frac{3}{2}e^{\phi/2-6A-2B}(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi)^2_{\mu\nu}\\
&+ \frac{1}{16}g_{\mu\nu}\Big(- \frac{1}{2}e^{3\phi/2-2A-2B}\mathrm{d}\alpha^2 - \frac{1}{3}e^{\phi-4A-4B}h^2 - 3e^{\phi/2-6A}\Big[(\partial\xi)^2 + (\partial\xi')^2\Big]\\
&- 6e^{-\phi-4A}(\partial\chi)^2 - 5e^{\phi/2-6A-6B}\varphi^2 - 9c_0^2e^{\phi/2-6A+2B} - \frac{9}{2}e^{\phi/2-6A-2B}(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi)^2\Big) \ ,
\end{aligned}
\tag{3.83}
$$

while the mixed (μ,m)-components are automatically satisfied. The dilaton equation reads,

$$
\begin{aligned}
0 = e^{-10A-4B}\nabla^\mu\Big(e^{8A+2B}\partial_\mu\phi\Big) &- \frac{1}{4}e^{\phi/2-8A-2B}\Big[(\partial\xi)^2 + (\partial\xi')^2\Big] - \frac{3}{8}e^{3\phi/2-4A-4B}\mathrm{d}\alpha^2\\
+ \frac{3}{2}e^{-\phi-6A-2B}(\partial\chi)^2 &+ \frac{1}{12}e^{-\phi-6A-6B}h^2\\
+ \frac{1}{4}e^{\phi/2-8A-8B}\varphi^2 &- \frac{3}{4}c_0^2e^{\phi/2-8A} - \frac{3}{8}e^{\phi/2-8A-4B}(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi)^2 \ .
\end{aligned}
\tag{3.84}
$$

The F-form equation of motion reduces to the condition,

$$
\mathrm{d}(e^{3\phi/2+6A}\star_4\mathrm{d}\alpha) = \varphi e^{\phi/2+2A-4B}\mathrm{d}\beta - 3e^{\phi/2+2A}\mathrm{d}\chi\wedge\star_4(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi) \ . \tag{3.85}
$$

The H-form equation reduces to the following two equations,

$$
\mathrm{d}\Big(e^{-\phi+4A+2B}\star_4\mathrm{d}\chi\Big) = c_0\varphi\mathrm{vol}_4 + (\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi)\wedge(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi) - e^{\phi/2+2A}\mathrm{d}\alpha\wedge\star_4(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi) \ , \tag{3.86}
$$

and,

$$
\mathrm{d}\Big(e^{-\phi+4A-2B}\star_4\mathrm{d}\beta\Big) = 3c_0(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi) - \mathrm{d}\xi\wedge\mathrm{d}\xi' + e^{\phi/2+2A-4B}\varphi\mathrm{d}\alpha \ . \tag{3.87}
$$

The G-form equation of motion reduces to,

$$
\begin{aligned}
\mathrm{d}\Big(e^{\phi/2+2A+2B}\star_4\mathrm{d}\xi\Big) &= h\wedge\mathrm{d}\xi'\\
\mathrm{d}\Big(e^{\phi/2+2A+2B}\star_4\mathrm{d}\xi'\Big) &= -h\wedge\mathrm{d}\xi\\
\mathrm{d}\Big(e^{\phi/2+2A}\star_4(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi)\Big) &= 2\mathrm{d}\chi\wedge\mathrm{d}\gamma + c_0\mathrm{d}\beta \ ,
\end{aligned}
\tag{3.88}
$$

together with the constraint,

$$
0 = \mathrm{d}\Big(\varphi e^{\phi/2+2A-4B} + 3c_0\chi\Big) \ . \tag{3.89}
$$

This can be readily integrated to give,

$$
\varphi = e^{-\phi/2-18A}(c_1 - 3c_0\chi) \ . \tag{3.90}
$$

Since χ only appears in the equations of motion through its derivatives or through φ, we may absorb c_1 by redefining χ. This corresponds to a gauge transformation of the ten-dimensional B-field. We will thus set c_1 to zero in the following.

The Lagrangian

As we can see from (3.31) the scalar $B(x)$ can be redefined away by absorbing it in the 4d metric. This freedom can be exploited in order to obtain a 4d consistent truncation directly in the Einstein frame. The appropriate choice is,

$$B = -4A \ . \tag{3.91}$$

With this choice one can check that the ten-dimensional equations given in (3.82)-(3.89) all follow from the 4d action,

$$\begin{aligned} S_4 = \int \mathrm{d}^4x\sqrt{g}\Big(& R - 24(\partial A)^2 - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{3}{2}e^{-4A-\phi}(\partial\chi)^2 - \tfrac{1}{2}e^{-6A+\phi/2}\left[(\partial\xi)^2 + (\partial\xi')^2\right] \\ & - \tfrac{1}{4}e^{3\phi/2+6A}\mathrm{d}\alpha^2 - \tfrac{3}{4}e^{\phi/2+2A}(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi)^2 - \tfrac{1}{12}e^{-\phi+12A}\mathrm{d}\beta^2 - \tfrac{9}{2}e^{-\phi/2-18A}c_0^2\chi^2 - \tfrac{3}{2}e^{\phi/2-14A}c_0^2\Big) \\ & + \int 3c_0\mathrm{d}\gamma\wedge\beta + 3c_0\chi\ \alpha\wedge\mathrm{d}\beta + 3\chi\ \mathrm{d}\gamma\wedge\mathrm{d}\gamma - \beta\wedge\mathrm{d}\xi\wedge\mathrm{d}\xi' \ . \end{aligned} \tag{3.92}$$

Furthermore equation (3.87) can be solved in order to express $\mathrm{d}\beta$ in terms of a scalar b (the "axion"),

$$\mathrm{d}\beta = e^{\phi-12A}\star_4\left[\mathrm{d}b + \tfrac{1}{2}(\xi\mathrm{d}\xi' - \xi'\mathrm{d}\xi) + 3c_0(\gamma - \chi\alpha)\right] \ , \tag{3.93}$$

where we chose the gauge most symmetric in ξ, ξ'. The Lagrangian becomes, in terms of the axion,

$$\begin{aligned} S_4 = \int \mathrm{d}^4x\sqrt{g}\Big(& R - 24(\partial A)^2 - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{3}{2}e^{-4A-\phi}(\partial\chi)^2 - \tfrac{1}{2}e^{-6A+\phi/2}\left[(\partial\xi)^2 + (\partial\xi')^2\right] \\ & - \tfrac{1}{4}e^{3\phi/2+6A}\mathrm{d}\alpha^2 - \tfrac{3}{4}e^{\phi/2+2A}(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi)^2 - \tfrac{1}{2}e^{\phi-12A}\left(\mathrm{d}b + \omega\right)^2 \\ & - \tfrac{9}{2}e^{-\phi/2-18A}c_0^2\chi^2 - \tfrac{3}{2}e^{\phi/2-14A}c_0^2\Big) + \int 3\chi\ \mathrm{d}\gamma\wedge\mathrm{d}\gamma \ , \end{aligned} \tag{3.94}$$

where we have set,

$$\omega := \tfrac{1}{2}(\xi\mathrm{d}\xi' - \xi'\mathrm{d}\xi) + 3c_0(\gamma - \chi\alpha) \ . \tag{3.95}$$

Including background three-form flux

We can include background three-form flux by modifying the form ansatz (3.79) as follows,

$$\begin{aligned} & F = \mathrm{d}\alpha \ ; \quad H = \mathrm{d}\chi\wedge J + \mathrm{d}\beta + \frac{1}{2}\mathrm{Re}(b_0\Omega^*) \\ & G = \varphi\mathrm{vol}_4 + \frac{1}{2}c_0 J\wedge J + J\wedge(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi) - \frac{1}{2}D\xi\wedge\mathrm{Im}\Omega - \frac{1}{2}D\xi'\wedge\mathrm{Re}\Omega \ , \end{aligned} \tag{3.96}$$

where we have introduced a background charge $b_0 \in \mathbb{C}$. The covariant derivatives are given by,

$$D\xi := \mathrm{d}\xi + b_1\alpha \ ; \quad D\xi' := \mathrm{d}\xi' + b_2\alpha \ , \tag{3.97}$$

where we set $b_0 = ib_1 + b_2$. We see that the inclusion of a background charge for the three-form has the effect of gauging the isometries of the RR axions.

The modified form ansatz (3.96) is such that it automatically satisfies the Bianchi identities. At the level of the Lagrangian, the modification amounts to replacing: $\mathrm{d}\xi \to D\xi$, $\mathrm{d}\xi' \to D\xi'$, and adding a potential term,

$$S_4 \to S_4 - \int \mathrm{d}^4x\sqrt{g}\,\tfrac{1}{2}|b_0|^2 e^{-\phi-12A} \ . \tag{3.98}$$

3.3 Gravitini condensation

3.3.1 Derivative corrections

Four-derivative corrections to the 4d effective action resulting from compactification of the IIA superstring on CY threefolds have been known since [45]. Most recently they have been computed in [46] from compactification of certain known terms of the ten-dimensional IIA tree-level and one-loop superstring effective action at order α'^3. The authors of that reference take into account the graviton and B-field eight-derivative terms given in [47, 48], but neglect e.g. the dilaton derivative couplings and RR couplings of the form $R^2(\partial F)^2$ and $\partial^4 F^4$ calculated in [49]. Furthermore [46] neglects loop corrections from massive KK fields.[6]

In a low-energy expansion, the 4d effective action takes the schematic form [51],

$$2\kappa^2 S = \int \mathrm{d}x^4\sqrt{g}\left(R + \beta_1\alpha' R^2 + \beta_2\alpha'^2 R^3 + \beta_3\alpha'^3 R^4\right) \ , \tag{3.99}$$

where κ is the four-dimensional gravitational constant, and a Weyl transformation must be performed to bring the action to the 4d Einstein frame.[7] Moreover each coefficient in the series can be further expanded in the string coupling to separate the tree-level from the one-loop contributions. Although all the higher-derivative terms in (3.99) descend from the eight-derivative ten-dimensional α'^3-corrections, they correspond to different orders of the 4d low-energy expansion. Indeed if $l_s = 2\pi\sqrt{\alpha'}$, l_{4d} and l_Y are the string length, the four-dimensional low-energy wavelength and the characteristic length of Y respectively, we have,

$$l_s \ll l_Y \ll l_{4d} \ . \tag{3.100}$$

Moreover the term with coefficient β_n in (3.99) is of order,

$$\left(\frac{l_s}{l_{4d}}\right)^{2n}\left(\frac{l_s}{l_Y}\right)^{6-2n} \ ; \ n = 1, 2, 3 \ , \tag{3.101}$$

relative to the Einstein term, so that the $n = 1$ term dominates the $n = 2, 3$ terms in (3.99).

The ten-dimensional IIA supergravity (two-derivative) action admits solutions without flux of the form $\mathbb{R}^{1,3}\times Y$, where Y is of $SU(3)$ holonomy (which for our purposes we take to be a compact CY). A sigma model argument [52] shows that this background can be promoted to a solution to all orders in α', provided the metric of Y is appropriately corrected at each

[6] Presumably the KK loop corrections are subleading and vanish in the large-volume limit (see however [50] for an exception to this statement). At any rate these corrections are dependent on the specific CY and at the moment can only be computed on a case-by-case basis, e.g. around the orbifold limit where the CY reduces to T^6/Γ with Γ a discrete group. Winding modes are heavier than KK modes in a regime where (3.100) holds.

[7] As emphasized in [46], in computing the 4d effective action the compactification must be performed around the solution to the α'-corrected equations of motion. This procedure can thus generate α'-corrections also from the compactification of the ten-dimensional Einstein term.

order in such a way that it remains Kähler.[8] Indeed [46] confirms this to order α'^3 and derives the explicit corrections to the dilaton and the metric, which is deformed away from Ricci-flatness at this order. Their derivation remains valid for backgrounds of the form $M_4 \times Y$, where M_4 is any Ricci-flat four-dimensional space.

Within the framework of the effective 4d theory, nonperturbative gravitational instanton corrections arise from vacua of the form $M_4 \times Y$, where M_4 is an ALE space. These instanton contributions are weighted by a factor $\exp(-S_0)$, where S_0 is the 4d effective action evaluated on the solution $M_4 \times Y$. Subject to the limitations discussed above, and taking into account the Ricci-flatness of the metric of M_4, the IIA 4d effective action of [46] reduces to,

$$2\kappa^2 S_0 = \beta_1 \alpha' \int_{M_4} \mathrm{d}x^4 \sqrt{g} R_{\kappa\lambda\mu\nu} R^{\kappa\lambda\mu\nu} \ , \tag{3.102}$$

where in the conventions of [46],[9]

$$\kappa^2 = \pi\alpha' \ ; \quad M_{\mathrm{P}} = 2\sqrt{\pi}\, l_s^{-1} \ , \tag{3.103}$$

with $M_{\mathrm{P}} = \kappa^{-1}$ the (reduced) 4d Planck mass and β_1 given by,

$$l_s^6 \beta_1 = 2^9 \pi^4 \alpha'^2 \int_Y c_2 \wedge J \ , \tag{3.104}$$

where c_2 is the second Chern class of Y. For a generic Kähler manifold we have,

$$c_2 \wedge J = \frac{1}{32\pi^2} \left(R^2_{mnkl} - \mathfrak{R}^2_{mn} + \frac{1}{4}\mathfrak{R}^2 \right) \mathrm{vol}_6 \ , \tag{3.105}$$

where we have adopted real notation and defined $\mathfrak{R}_{mn} := R_{mnkl} J^{kl}$, $\mathfrak{R} := \mathfrak{R}_{mn} J^{mn}$. The contractions are taken with respect to the metric compatible with the Kähler form J and the connection of the Riemann tensor.

The information about Y enters the 4d effective action through the calculation of β_1. Since β_1 multiplies a term which is already a higher-order correction, it suffices to evaluate it in the CY limit (for which $\mathfrak{R}_{mn}$ vanishes). We thus obtain,

$$\beta_1 = \frac{1}{\pi^2 l_s^2} \int_Y \mathrm{d}^6 x \sqrt{g}\, R^2_{mnkl} > 0 \ . \tag{3.106}$$

Therefore the leading instanton contribution comes from the ALE space which minimizes the integral in (3.102). This is the EH space [54], cf. (3.113), so that,

$$S_0 = \frac{24}{\pi l_s^2} \int_Y \mathrm{d}^6 x \sqrt{g}\, R^2_{mnkl} > 0 \ . \tag{3.107}$$

[8] It should be possible to generalize the sigma-model argument of [52] to the case of backgrounds of the form $M_4 \times Y$, where M_4 is an ALE space, along the lines of [53].

[9] The ten-dimensional gravitational constant of [46] $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$, cf. (2.4) therein, is related to the four-dimensional one via $\kappa^2 = \kappa_{10}^2 / l_s^6$. Note in particular that eqs. (4.9) and (4.19) of that reference are given in units where $l_s = 2\pi\sqrt{\alpha'} = 1$: to reinstate engineering dimensions one must multiply with the appropriate powers of l_s. The 4d Einstein term in (3.99) has been canonically normalized via a Weyl transformation of the 4d metric. This affects the relative coefficient between two- and four-derivative terms in the action: note in particular that the right-hand side of (3.102) is invariant under Weyl transformations. We thank Kilian Mayer for clarifying to us the conventions of [46].

Note that S_0 does not depend on the dilaton: this is related to the fact that, starting from an action of the form $\int \mathrm{d}^4x\sqrt{g}(e^{-2\phi}R+\beta_1\alpha' R^2_{\mu\nu\rho\sigma})$, the dilaton exponential can be absorbed by a Weyl transformation of the form $g_{\mu\nu} \to e^{2\phi} g_{\mu\nu}$, cf. footnote 9. Therefore we have,

$$S_0 = c\left(\frac{l_Y}{l_s}\right)^2 , \tag{3.108}$$

with c a positive number of order one.

3.3.2 ALE instantons

Asymptotically locally Euclidean (ALE) spaces, see e.g. [55] for a review, are noncompact self-dual gravitational instantons, i.e. their Riemann tensor obeys,

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} R_{\kappa\lambda}{}^{\rho\sigma} = R_{\kappa\lambda\mu\nu} . \tag{3.109}$$

From the above and the identity $R_{[\kappa\lambda\mu]\nu} = 0$, it follows that the ALE spaces are Ricci-flat,

$$R_{\mu\nu} = 0 . \tag{3.110}$$

These spaces asymptote $S^3/\mathbb{Z}_{k+1}$ at spatial infinity, with $k \in \mathbb{N}$ (the case $k = 0$ corresponds to $\mathbb{R}^4$). The simplest nontrivial example in this class is the EH space [56], which corresponds to $k = 1$. Explicitly the metric reads,

$$\mathrm{d}s^2 = \mathrm{d}r^2\big(1-\tfrac{a^4}{r^4}\big)^{-1} + \tfrac{1}{4}r^2\Big(\sigma_1^2+\sigma_2^2+\big(1-\tfrac{a^4}{r^4}\big)\sigma_3^2\Big) , \tag{3.111}$$

where $a > 0$ is an arbitrary constant, and,

$$\sigma_1 = \sin\psi\mathrm{d}\theta - \sin\theta\cos\psi\mathrm{d}\phi ; \quad \sigma_2 = -\cos\psi\mathrm{d}\theta - \sin\theta\sin\psi\mathrm{d}\phi ; \quad \sigma_3 = \mathrm{d}\psi + \cos\theta\mathrm{d}\phi . \tag{3.112}$$

For the coordinate ranges $a \le r$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, $0 \le \psi \le 2\pi$, the manifold can be seen to be smooth with boundary given by $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$ at asymptotic infinity. (We would have an asymptotic S^3 if $0 \le \psi \le 4\pi$). The Hirzebruch signature τ of a self-dual space is given by,

$$\tau = \frac{1}{48\pi^2}\int \mathrm{d}x^4\sqrt{g}R_{\kappa\lambda\mu\nu}R^{\kappa\lambda\mu\nu} \in \mathbb{N} . \tag{3.113}$$

As can be verified using (3.111), the EH gravitational instanton is the ALE space with the smallest Hirzebruch signature, $\tau = 1$. More generally it can be shown that $\tau = k$, with k as given below eq. (3.110).

It is convenient to use a gauge in which not only the curvature but also the connection is self-dual [55],

$$\omega_{ab} = \frac{1}{2}\varepsilon_{abcd}\omega^{cd} . \tag{3.114}$$

In this gauge the covariant derivative reduces to a simple derivative on negative chirality spinors,

$$\nabla_\mu\theta^\alpha = \partial_\mu\theta^\alpha + \frac{1}{4}\omega_\mu^{ab}\left(\gamma_{ab}\right)^\alpha{}_\beta\theta^\beta = \partial_\mu\theta^\alpha , \tag{3.115}$$

where in the last equality we took (A.26), (3.114) into account. It follows in particular that covariantly-constant negative-chirality spinors are just constant. We may therefore choose their basis $\theta^\alpha_{(1)}$, $\theta^\alpha_{(2)}$ as follows, in the chiral gamma-matrix basis of appendix A,

$$\theta^\alpha_{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad \theta^\alpha_{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{3.116}$$

The Atiyah-Patodi-Singer theorem for ALE spaces predicts an equal number of positive- and negative-chirality spinor zeromodes for the Dirac operator $\nabla\!\!\!/$ [57]. On the other hand we have,

$$\begin{aligned}(\nabla\!\!\!/^2\theta)^\alpha &= (\nabla^2\theta + \gamma^{\mu\nu}\nabla_\mu\nabla_\nu\theta)^\alpha \\ &= (\nabla^2\theta + \frac{1}{8}R_{\mu\nu\rho\sigma}\gamma^{\mu\nu}\gamma^{\rho\sigma}\theta)^\alpha \\ &= (\nabla^2\theta)^\alpha \ ,\end{aligned} \tag{3.117}$$

where in the last equality we took (3.110) into account. It follows that negative-chirality zeromodes are (covariantly) constant, hence non-normalizable since the ALE space is non-compact. It thus follows from the index theorem that there are no (normalizable) spinor zeromodes of the Dirac operator.

For a spin-1 field[10] the index theorem predicts that the number of positive-chirality zeromodes of the Dirac operator minus the number of negative-chirality zeromodes is equal to the Hirzebruch signature of the ALE space. We now have,

$$(\nabla\!\!\!/^2\phi)_{\alpha\beta} = (\nabla^2\phi)_{\alpha\beta} + \frac{1}{8}R_{\mu\nu\rho\sigma}\,(\gamma^{\mu\nu})_\alpha{}^{\alpha'}\,(\gamma^{\rho\sigma})_\beta{}^{\beta'}\,\phi_{\alpha'\beta'} \ ; \quad (\nabla\!\!\!/^2\phi)^{\alpha\beta} = (\nabla^2\phi)^{\alpha\beta} \ , \tag{3.118}$$

where in the second equation we took (A.26) into account. By the same argument as before, it follows that $\phi^{\alpha\beta}$ is covariantly constant, hence non-renormalizable. Therefore there are no spin-1 fields of negative chirality. By the index theorem it follows that there are τ spin-1 zeromodes of positive chirality (i.e. one zeromode for the EH space).

A massless gravitino ψ_μ is also a zeromode of the Dirac operator $\nabla\!\!\!/$, in the gauge $\gamma^\mu\psi_\mu = 0$. By a similar argument as before, there are 2τ spin-3/2 zeromodes of positive chirality. These can be constructed as follows,

$$\psi_{(i)\mu\alpha} = \phi_{\alpha\beta}\theta^\gamma_{(i)}\left(C^{-1}\gamma_\mu\right)_\gamma{}^\beta \ ; \ i = 1, 2 \ , \tag{3.119}$$

where $\theta_{(i)}$ are the covariantly-constant spinors of (3.116), and $\phi_{\alpha\beta}$ are the positive-chirality spin-1 zeromodes of (3.118). Indeed we verify that the $\psi_{(i)\mu\alpha}$ are traceless,

$$(\gamma^\mu)^{\alpha\beta}\,\psi_{(i)\mu\alpha} = 0 \ ; \ i = 1, 2 \ , \tag{3.120}$$

as follows from (3.119) and the identity $(C^{-1}\gamma_\mu)_{(\gamma}{}^\beta\,(C^{-1}\gamma^\mu)_{\delta)}{}^\alpha = 0$. Moreover they obey the zeromode equation,

$$(\nabla\!\!\!/^2\psi_\mu)_\alpha = (\nabla^2\psi_\mu + \frac{1}{2}\gamma^{\rho\sigma}R_{\mu\rho\sigma}{}^\nu\psi_\nu)_\alpha = 0 \ , \tag{3.121}$$

where we used (3.119) and the Hodge duality relations (A.24).

3.3.3 Gravitino condensates in 4d $\mathcal{N} = 1$ supergravity

Within the context of 4d $\mathcal{N} = 1$ supergravity, the condensate $\langle\psi^\mu\psi'_\mu\rangle$ was shown in [57] to be proportional to the zeromode bilinear. From (3.119) we get,

$$\tilde{\psi}_{(1)\mu}\psi^\mu_{(2)} = f \ ; \quad f := 2C^{\alpha\beta}\phi_{\beta\gamma}C^{\gamma\delta}\phi_{\delta\alpha} \ , \tag{3.122}$$

[10] By a "spin-1 field" we understand a field transforming in the three-dimensional irreducible representation of the $su(2)$ algebra. It can be thought of as a field with two symmetric spinor indices of the same chirality, $\phi_{\alpha\beta} = \phi_{\beta\alpha}$ (positive chirality) or $\phi^{\alpha\beta} = \phi^{\beta\alpha}$ (negative chirality).

where f is a positive function on the ALE space, and we have normalized $\tilde{\theta}_{(1)}\theta_{(2)} = 1$. In deriving the above we have noted that $\phi_{\alpha\gamma}C^{\gamma\delta}\phi_{\delta\beta}$ is antisymmetric in its free indices, therefore it is necessarily proportional to $C_{\alpha\beta}$, since there is a unique scalar in the decomposition of the antisymmetric product of two spinors of positive chirality. For the EH space, cf. (3.111), f can be given explicitly as in [58],

$$f = 16\Big(\frac{a}{r}\Big)^8 \ . \tag{3.123}$$

The zeromode normalization can thus be inferred from,

$$\int \mathrm{d}^4x\sqrt{g}\,\tilde{\psi}_{(1)\mu}\psi^{\mu}_{(2)} = \frac{1}{2}\mathrm{Vol}(S^3)\int_a^\infty \mathrm{d}r\, r^3 f = 4\pi^2 a^4 \ , \tag{3.124}$$

where the "spherical" coordinates in (3.111) are related to the cartesian coordinates x^μ in the usual way, except that antipodal points on S^3 are identified, see below (3.112).

To calculate the gravitino bilinear we follow [58] who adopt the prescription of [59] for the functional integration over metrics. As shown explicitly in [58] in the case of 4d $\mathcal{N} = 1$ supergravity, expanding the action around the EH instanton saddle point and performing the Gaussian integrations, the one-loop determinants from all massive modes cancel out thanks to supersymmetry. One is then left with the integration over zeromodes. The latter reduces to an integration over the instanton size,

$$\begin{aligned}\langle\tilde{\psi}_\mu\psi^\mu\rangle &= \text{const.}\, M_{\mathrm{P}}\, e^{-S_0}\int \mathrm{d}a\, a^5\,\tilde{\psi}_{(1)\mu}\psi^{\mu}_{(2)} \\ &= \text{const.}\, M_{\mathrm{P}}\, e^{-S_0}\int \mathrm{d}a\, a^5\Big(\frac{a}{r}\Big)^8 a^{-4} \ ,\end{aligned} \tag{3.125}$$

where we have used (3.122), (3.123) and normalized $\psi_\mu \to \psi_\mu/(2\pi a^2)$, cf. (3.124); the remaining power of a comes from the Jacobian of the transformation from the integration over metric zeromodes to the integration over the instanton moduli.

The integration in (3.125) would seem to depend on the spacetime position, since a is bounded above by the radial distance r. In order to overcome this problem, [58] performs a coordinate transformation,

$$\tilde{x}^\mu = \frac{u}{r}x^\mu \ ; \quad u := r\sqrt{1-\left(\tfrac{a}{r}\right)^4} \ , \tag{3.126}$$

which has the effect of changing the radial coordinate from $r \geq a$ to $u \geq 0$. We can then rewrite (3.125) as follows,

$$\langle\tilde{\psi}_\mu\psi^\mu\rangle = \text{const.}\, M_{\mathrm{P}}\, e^{-S_0}\int_0^\infty \mathrm{d}a\, a^9\big(u^2+\sqrt{4a^4+u^4}\big)^{-4} \ . \tag{3.127}$$

This integral diverges for $a \to \infty$ at fixed u. In contrast, the same calculation for the gravitino fieldstrength bilinear $\langle(\nabla_{[\mu}\psi_{\nu]})^2\rangle$ yields a finite result [58]. This is due to the fact that the two derivatives bring about an extra $(u/r^2)^2$ factor compared to the integrand in (3.127), which contributes an extra a^{-4} factor in the $a \to \infty$ limit. However even this finite result seems to rely on the coordinate system (3.126). This does not seem satisfactory: for diffeomorphism invariance to be respected, the result should be independent of the coordinate system used for its calculation.

One may argue that the divergence/ambiguity encountered is not surprising since the 4d theory is nonrenormalizable and should anyway be thought of as an effective low-energy limit of string theory. On general grounds, at one loop in the gravitational coupling, one expects a gravitational instanton contribution to the condensate of the form,

$$\langle\tilde{\psi}_\mu\psi^\mu\rangle \propto M_{\rm P}\, e^{-S_0} \propto l_s^{-1} e^{-c\,(l_Y/l_s)^2} \ , \tag{3.128}$$

up to proportionality constants of order one, where in the second proportionality we have taken (3.103), (3.108) into account. Similarly, the quartic gravitino condensate receives contributions from the ALE with $\tau = 2$ and scales as the square of the bilinear condensate above.

3.3.4 Consistent truncation with condensates

In Euclidean signature the supersymmetric IIA action is constructed via the procedure of holomorphic complexification, see e.g. [60]. This amounts to first expressing the Lorentzian action in terms of $\tilde{\Psi}_M$ instead of $\bar{\Psi}_M$ (which makes no difference in Lorentzian signature) and then Wick-rotating, see appendix A for our spinor and gamma-matrix conventions. In this way one obtains a (complexified) Euclidean action which is formally identical to the Lorentzian one, with the difference that now the two chiralities $\Psi^\pm_M$, should be thought of as independent complex spinors (there are no Majorana Weyl spinors in ten Euclidean dimensions). Although the gravitino Ψ_M is complex in Euclidean signature, its complex conjugate does not appear in the action, hence the term "holomorphic complexification".

Since we are interested in the case where only the 4d gravitino condenses, we expand the 10d gravitino as follows,

$$\Psi_m = 0\ ; \quad \Psi_{\mu+} = \psi_{\mu+}\otimes\eta - \psi_{\mu-}\otimes\eta^c\ ; \quad \Psi_{\mu-} = \psi'_{\mu+}\otimes\eta^c - \psi'_{\mu-}\otimes\eta\ , \tag{3.129}$$

so that,

$$\tilde{\Psi}_{\mu+} = \tilde{\psi}_{\mu+}\otimes\tilde{\eta} + \tilde{\psi}_{\mu-}\otimes\tilde{\eta^c}\ ; \quad \tilde{\Psi}_{\mu-} = \tilde{\psi}'_{\mu+}\otimes\tilde{\eta^c} + \tilde{\psi}'_{\mu-}\otimes\tilde{\eta}\ . \tag{3.130}$$

In Lorentzian signature the positive- and negative-chirality 4d vector-spinors above are related though complex conjugation: $\bar{\theta}^\mu_+ = \tilde{\theta}^\mu_-$, $\bar{\theta}^\mu_- = -\tilde{\theta}^\mu_+$, so that Ψ_M is Majorana in 10d: $\bar{\Psi}_M = \tilde{\Psi}_M$. Upon Wick-rotating to Euclidean signature this is no longer true, and the two chiralities transform in independent representations. As already mentioned, in the present paper we focus on the contribution of ALE gravitational instantons to the fermion condensate. In this case there are no negative-chirality zeromodes and we can set,

$$\psi^\mu_- = \psi'^\mu_- = 0\ . \tag{3.131}$$

For any two 4d positive-chirality vector-spinors, θ^μ_+, χ^μ_+, the only nonvanishing bilinears read,

$$(\theta_+^{[\mu_1}\gamma^{\mu_2\mu_3}\chi_+^{\mu_4]}) = \frac{i^s}{12}\varepsilon^{\mu_1\mu_2\mu_3\mu_4}(\theta^\lambda_+\gamma_{\lambda\rho}\chi^\rho_+)\ ; \quad (\theta^\lambda_+\chi_{\lambda+})\ , \tag{3.132}$$

where we used the Fierz identity (A.23) and the Hodge duality relations (A.24); $s = 1, 2$ for Lorentzian, Euclidean signature respectively. Ultimately we will be interested in gamma-traceless vector-spinors,

$$\gamma_\mu\theta^\mu_+ = \gamma_\mu\chi^\mu_+ = 0\ , \tag{3.133}$$

since all ALE zeromodes can be put in this gauge [57]. In this case we obtain the additional relation,

$$(\theta^\lambda_+\gamma_{\lambda\rho}\chi^\rho_+) = -(\theta^\lambda_+\chi_{\lambda+})\ . \tag{3.134}$$

Assuming, as is the case for ALE spaces, that only positive-chirality zeromodes exist in four dimensions, cf. (3.131), the only nonvanishing bilinear condensates that appear in the equations of motion are proportional to,

$$\mathcal{A} := \left(\tilde{\psi}_{\mu+}\gamma^{\mu\nu}\psi'_{\nu+}\right) = -\left(\tilde{\psi}^{\mu}_{+}\psi'_{\mu+}\right) \ , \tag{3.135}$$

where in the second equality we have assumed that ψ^{μ}_{+}, ψ'^{μ}_{+} are gamma-traceless, cf. (3.133).

Furthermore we note the following useful results,

$$\begin{aligned}
\left(\tilde{\Psi}_{\rho}\Gamma_{(\mu}\Gamma^{M_1\dots M_4}\Gamma^{\rho}\Psi_{\nu)}\right) G_{M_1\dots M_4} &= 24(3c_0e^{-4A} + \varphi e^{-4A-4B})\mathcal{A}g_{\mu\nu} \\
\left(\tilde{\Psi}_{\rho}\Gamma_{\sigma}\Gamma_{(\mu}{}^{M_2M_3M_4}\Gamma^{\rho}\Psi^{\sigma}\right) G_{\nu)M_2M_3M_4} &= 24\varphi e^{-4A-4B}\mathcal{A}g_{\mu\nu} \\
\left(\tilde{\Psi}_{\rho}\Gamma_{\sigma}\Gamma_{(m}{}^{M_2M_3M_4}\Gamma^{\rho}\Psi^{\sigma}\right) G_{n)M_2M_3M_4} &= 48c_0\mathcal{A}e^{-4A-2B}g_{mn} \ ,
\end{aligned} \tag{3.136}$$

where on the left-hand sides above we used the warped metric for the contractions, while on the right-hand sides we used the unwarped metric. In the 4d theory, these bilinears receive contributions from the EH instanton at one loop in the gravitational coupling.

In the presence of gravitino condensates the equations of motion (3.82)-(3.89) are modified as follows: the internal (m, n)-components of the Einstein equations read,

$$0 = e^{-8A-2B}\nabla^{\mu}\left(e^{8A+2B}\partial_{\mu}A\right) + \dots + \frac{1}{4}\left(\varphi e^{\phi/4-4A-4B} - c_0e^{\phi/4-4A}\right)\mathcal{A} - \frac{1}{8}e^{2A+2B}L_{\Psi^4} \ , \tag{3.137}$$

where the ellipses stand for terms that are identical to the case without fermion condensates. The external (μ, ν)-components read,

$$R^{(4)}_{\mu\nu} = \dots - \frac{1}{2}g_{\mu\nu}e^{\phi/4-4A-4B}\varphi\mathcal{A} \ , \tag{3.138}$$

while the mixed (μ, m)-components are automatically satisfied. The dilaton equation reads,

$$0 = e^{-10A-4B}\nabla^{\mu}\left(e^{8A+2B}\partial_{\mu}\phi\right) + \dots + \tfrac{1}{4}(3c_0e^{\phi/4+2A} + \varphi e^{\phi/4+2A-4B})\mathcal{A} \ . \tag{3.139}$$

The F-form and H-form equations are modified as follows,

$$\mathrm{d}(e^{3\phi/2+6A} \star_4 \mathrm{d}\alpha) = \dots + e^{\phi/4+4A-2B}\mathcal{A}\ \mathrm{d}\beta \ , \tag{3.140}$$

and,

$$\mathrm{d}\left(e^{-\phi+4A-2B} \star_4 \mathrm{d}\beta\right) = \dots + e^{\phi/4+4A-2B}\mathcal{A}\ \mathrm{d}\alpha \ , \tag{3.141}$$

respecively. The G-form equation of motion remains unchanged except for the constraint,

$$0 = \mathrm{d}\left(\varphi e^{\phi/2+2A-4B} + 3c_0\chi + e^{\phi/4+4A-2B}\mathcal{A}\right) \ . \tag{3.142}$$

In deriving the above we have taken into account that,

$$(\tilde{\Psi}^M\Gamma_{[M}\Gamma^{(4)}\Gamma_{N]}\Psi^N) = 2\mathcal{A}e^{2A+2B}\left(\mathrm{vol}_4 - \frac{1}{2}e^{-4B}J\wedge J\right) \ . \tag{3.143}$$

At this stage it is important to notice that the new $\mathcal{A}$ terms in the flux equations (3.140) and (3.141) exactly compensate the modification of φ in (3.142), so that the form equations are ultimately unchanged in the presence of fermion condensates.

Of the 24 quartic gravitino terms that appear in the action of [17] only the following are nonvanishing,

$$
\begin{aligned}
\left(\tilde{\Psi}_\mu\Gamma_{11}\Psi_\nu\right)\left(\tilde{\Psi}^\mu\Gamma_{11}\Psi^\nu\right) &= 4\left(\tilde{\psi}_+^{[\mu}\psi_+^{\prime\nu]}\right)^2 e^{-4A-4B}\\
\left(\tilde{\Psi}^{\mu_1}\Gamma_{11}\Gamma_{\mu_1\dots\mu_4}\Psi^{\mu_2}\right)\left(\tilde{\Psi}^{\mu_3}\Gamma_{11}\Psi^{\mu_4}\right) &= -\frac{1}{6}\left(\tilde{\Psi}^{\mu_1}\Gamma_{\mu_1\dots\mu_4 mn}\Psi^{\mu_2}\right)\left(\tilde{\Psi}^{\mu_3}\Gamma^{mn}\Psi^{\mu_4}\right)\\
&= -\left(8\tilde{\psi}_{[\nu+}\psi^{\prime}_{\rho]+} + 4\tilde{\psi}^\mu_+\gamma_{\rho\nu}\psi^{\prime}_{\mu+}\right)\left(\tilde{\psi}^\rho_+\psi^{\prime\nu}_+\right)e^{-4A-4B}\\
\left(\tilde{\Psi}^{[M_1}\Gamma^{M_2M_3}\Psi^{M_4]}\right)^2 &= 4\left(\tilde{\psi}_+^{[\mu_1}\gamma^{\mu_2\mu_3}\psi_+^{\prime\mu_4]}\right)^2 e^{-4A-4B} - \frac{2}{3}\left(\tilde{\psi}_+^{[\mu}\psi_+^{\prime\nu]}\right)^2 e^{-4A-4B}\ ,
\end{aligned}
\tag{3.144}
$$

where for the contractions on the left-, right-hand sides above we have used the warped, unwarped metric respectively. We thus obtain, cf. (3.75),

$$
\begin{aligned}
L_{\Psi^4} &= \frac{1}{4}(\tilde{\Psi}_M\Gamma_{11}\Psi_N)^2 + \frac{1}{8}\tilde{\Psi}^{M_1}\Gamma_{11}\Gamma_{M_1\cdots M_4}\Psi^{M_2}\ \tilde{\Psi}^{M_3}\Gamma_{11}\Psi^{M_4}\\
&+ \frac{1}{16}\tilde{\Psi}^{M_1}\Gamma_{M_1\cdots M_6}\Psi^{M_2}\ \tilde{\Psi}^{M_3}\Gamma^{M_4M_5}\Psi^{M_6} + \frac{3}{4}(\tilde{\Psi}_{[M_1}\Gamma_{M_2M_3}\Psi_{M_4]})^2\\
&= e^{-4A-4B}\mathcal{B}\ ,
\end{aligned}
\tag{3.145}
$$

where we have defined,

$$
\mathcal{B} := -\frac{3}{2}(\tilde{\psi}_{[\mu}\psi^{\prime}_{\nu]})^2 + (\tilde{\psi}^\mu\gamma_{\rho\nu}\psi^{\prime}_\mu)(\tilde{\psi}^\rho\psi^{\prime\nu}) + 3(\tilde{\psi}_{[\mu_1}\gamma_{\mu_2\mu_3}\psi^{\prime}_{\mu_4]})^2\ , \tag{3.146}
$$

which does not depend on the warp factor. In the 4d theory, at one-loop order in the gravitational coupling, the quartic gravitino term receives contributions from the ALE instanton with $\tau = 2$ (four spin-3/2 zeromodes).

The Lagrangian

Imposing (3.91) as before, and solving once again for φ,

$$
\varphi = e^{-\phi/2-18A}\left(-3c_0\chi - e^{\phi/4+12A}\mathcal{A}\right)\ , \tag{3.147}
$$

it can now be seen that the ten-dimensional equations in the presence of gravitino condensates all follow from the 4d action,

$$
\begin{aligned}
S_4 = \int \mathrm{d}^4x\sqrt{g}\Big(&R - 24(\partial A)^2 - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{3}{2}e^{-4A-\phi}(\partial\chi)^2 - \tfrac{1}{2}e^{-6A+\phi/2}\left[(D\xi)^2 + (D\xi^{\prime})^2\right]\\
&- \tfrac{1}{4}e^{3\phi/2+6A}\mathrm{d}\alpha^2 - \tfrac{3}{4}e^{\phi/2+2A}(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi)^2 - \tfrac{1}{12}e^{-\phi+12A}\mathrm{d}\beta^2 - V\Big)\\
&+ \int 3c_0\mathrm{d}\gamma\wedge\beta + 3c_0\chi\ \alpha\wedge\mathrm{d}\beta + 3\chi\ \mathrm{d}\gamma\wedge\mathrm{d}\gamma - \beta\wedge D\xi\wedge D\xi^{\prime}\ ,
\end{aligned}
\tag{3.148}
$$

where the potential of the theory is given by,

$$
\boxed{
\begin{aligned}
V(\chi,\phi,A) = &\tfrac{9}{2}c_0^2\chi^2 e^{-\phi/2-18A} + \tfrac{3}{2}c_0^2 e^{\phi/2-14A} + \tfrac{1}{2}|b_0|^2 e^{-\phi-12A}\\
&+ 3c_0\chi\mathcal{A}e^{-\phi/4-6A} - 3c_0\mathcal{A}e^{\phi/4-4A} + e^{6A}(\mathcal{B} + \tfrac{1}{2}\mathcal{A}^2)\ .
\end{aligned}
}
\tag{3.149}
$$

Note that in integrating the 4d Einstein equation (3.138), care must be taken to first substitute in the right-hand side the value of φ from (3.147), and take into account the variation of the condensates $\mathcal{A}$, $\mathcal{B}$ with respect to the metric.

The relation (3.93) between β and the axion is unchanged. In terms of the axion, the action reads,

$$\boxed{\begin{aligned} S_4 = \int \mathrm{d}^4x\sqrt{g}\Big(&R - 24(\partial A)^2 - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{3}{2}e^{-4A-\phi}(\partial\chi)^2 - \tfrac{1}{2}e^{-6A+\phi/2}\left[(D\xi)^2 + (D\xi')^2\right] \\ &- \tfrac{1}{4}e^{3\phi/2+6A}\mathrm{d}\alpha^2 - \tfrac{3}{4}e^{\phi/2+2A}(\mathrm{d}\gamma - \alpha\wedge\mathrm{d}\chi)^2 - \tfrac{1}{2}e^{\phi-12A}(\mathrm{d}b + \tilde{\omega})^2 - V\Big) + \int 3\chi\ \mathrm{d}\gamma\wedge\mathrm{d}\gamma\ ,\end{aligned}} \tag{3.150}$$

where we have set,

$$\tilde{\omega} := \tfrac{1}{2}(\xi D\xi' - \xi' D\xi) + 3c_0(\gamma - \chi\alpha)\ . \tag{3.151}$$

Note that the three axionic scalars ξ, ξ', b, remain flat directions even in the presence of the flux and the condensate.

3.3.5 Vacua

Maximally-symmetric solutions of the effective 4d theory (3.150) can be obtained by setting the vectors to zero,

$$\alpha = \gamma = 0\ , \tag{3.152}$$

and minimizing the potential of the theory,

$$\partial_\chi V(\chi_0,\phi_0,A_0) = \partial_\phi V(\chi_0,\phi_0,A_0) = \partial_A V(\chi_0,\phi_0,A_0) = 0\ , \tag{3.153}$$

where $(\chi,\phi,A) = (\chi_0,\phi_0,A_0)$ is the location of the minimum in field space. Then the Einstein equations determine the scalar curvature of the 4d spacetime to be,[11]

$$R = 9c_0^2\chi_0^2 e^{-\phi_0/2-18A_0} + 3c_0^2 e^{\phi_0/2-14A_0} + |b_0|^2 e^{-\phi_0-12A_0} + 3c_0\chi_0\mathcal{A}e^{-\phi_0/4-6A_0} - 3c_0\mathcal{A}e^{\phi_0/4-4A_0}\ , \tag{3.154}$$

and we assume that a Wick rotation has been performed back to Minkowski signature.

Condition (3.153) admits two classes of solutions.

Case 1: $c_0 = 0$

In this case, imposing the vanishing of $\partial_\phi V$ sets $b_0 = 0$, and the potential only depends on the warp factor A. A minimum is obtained at finite value of A provided,

$$\mathcal{B} = -\tfrac{1}{2}\mathcal{A}^2\ , \tag{3.155}$$

and requires the quartic condensate to be negative. From (3.154) it then follows that $R = 0$, and we obtain a Minkowski 4d vacuum. In fact the potential vanishes identically.

Case 2: $c_0 \neq 0$

In this case (3.153) can be solved for finite values of ϕ and A. The value of χ at the minimum is given by,

$$\chi_0 = -\frac{\mathcal{A}}{3c_0}\, g_s^{1/4}e^{12A_0}\ , \tag{3.156}$$

where we have set $g_s := e^{\phi_0}$. The values of ϕ_0 and A_0 at the minimum can be adjusted arbitrarily, and determine $|b_0|$ and c_0 in terms of the condensates,

$$\begin{aligned} |b_0|^2 &= \frac{3}{400}\, g_s e^{18A_0}\left(40\mathcal{B} - 21\mathcal{A}^2 \mp 3\mathcal{A}\sqrt{49\mathcal{A}^2 + 80\mathcal{B}}\right) \\ c_0 &= \frac{1}{20}\, g_s^{-1/4}e^{10A_0}\left(7\mathcal{A} \pm \sqrt{49\mathcal{A}^2 + 80\mathcal{B}}\right)\ , \end{aligned} \tag{3.157}$$

[11]Note that (3.154) is different from the standard relation $R = 2V_0$. This is because the condensates $\mathcal{A}$, $\mathcal{B}$ have non-trivial variations with respect to the metric.

where the signs in b_0 and c_0 are correlated. Henceforth we will set $e^{A_0} = 1$, since the warp factor at the minimum can be absorbed in l_Y.

Consistency of (3.157) requires the quartic condensate to obey the constraint,

$$\mathcal{B} > 0 \ , \tag{3.158}$$

and correlates the sign of $\mathcal{A}$ with the two branches of the solution: the upper/lower sign in (3.157) corresponds to $\mathcal{A}$ negative/positive, respectively.[12]

From (3.154) it then follows that,

$$R_{\rm dS} = 3g_s^{-1}|b_0|^2 \propto l_s^{-2} e^{-2c\,(l_Y/l_s)^2} \ , \tag{3.159}$$

up to a proportionality constant of order one. We thus obtain a de Sitter 4d vacuum, provided (3.158) holds. In the equation above we have taken into account that the quadratic and quartic condensates are expected to be of the general form, cf. the discussion around (3.128),

$$\mathcal{A} \propto l_s^{-1} e^{-c\,(l_Y/l_s)^2} \ ; \quad \mathcal{B} \propto l_s^{-2} e^{-2c\,(l_Y/l_s)^2} \ , \tag{3.160}$$

up to proportionality constants of order one.

We have verified numerically, as a function of $\mathcal{A}^2/\mathcal{B}$, that all three eigenvalues of the Hessian of the potential are positive at the solution. I.e. the solution is a local minimum of the potential (3.149).

Flux quantization

The four-form flux is constrained to obey,[13]

$$\frac{1}{l_s^3}\int_{\mathcal{C}_A} G \in \mathbb{Z} \ , \tag{3.161}$$

where $\{\mathcal{C}_A \ ; \ A = 1, \dots, h^{2,2}\}$ is a basis of integral four-cycles of the CY, $\mathcal{C}_A \in H_4(Y, \mathbb{Z})$. From (3.96), (3.157), (3.160) we then obtain,

$$n_A \propto g_s^{-1/4} \Big(\frac{l_Y}{l_s}\Big)^4 e^{-c\,(l_Y/l_s)^2} \mathrm{vol}(\mathcal{C}_A) \ , \tag{3.162}$$

up to a proportionality constant of order one; $\mathrm{vol}(\mathcal{C}_A)$ is the volume of the four cycle $\mathcal{C}_A$ in units of l_Y, and $n_A \in \mathbb{Z}$. Since the string coupling can be tuned to obey $g_s \ll 1$ independently of the l_Y/l_s ratio, (3.162) can be solved for $\mathrm{vol}(\mathcal{C}_A)$ of order one, provided we take n_A sufficiently close to each other. Given a set of flux quanta n_A, this equation fixes the Kähler moduli in units of l_Y; the overall CY volume is set by l_Y, which remains unconstrained.

Note that even if we allow for large flux quanta in order to solve the flux quantization constraint, it can be seen that higher-order flux corrections are subdominant in the $g_s \ll 1$ limit. Indeed the parameter that controls the size of these corrections is $|g_s G|$, which scales as $g_s^{3/4}$.

Similarly, the three-form flux is constrained to obey,

$$\frac{1}{l_s^2}\int_{\mathcal{C}_A} H \in \mathbb{Z} \ , \tag{3.163}$$

[12] If $\mathcal{B} > 3\mathcal{A}^2/2$, we may also take the upper/lower sign in (3.157) for $\mathcal{A}$ positive/negative, respectively. Equation (3.158) is the weakest condition on the quartic condensate that is sufficient for consistency of the solution.

[13] The Page form corresponding to G is given by $\hat{G} := G - H \wedge \alpha$, which is closed. The difference between G and $\hat{G}$ vanishes when integrated over four-cycles of Y.

where $\{\mathcal{C}_A\ ;\ A=1,\ldots,h^{2,1}\}$ is a basis of integral three-cycles of the CY, $\mathcal{C}_A \in H_3(Y,\mathbb{Z})$. From (3.96) we can see that this equation constrains the periods of Ω, and hence the complex-structure moduli of Y.

3.3.6 Discussion

The validity of the de Sitter solutions presented here requires the higher-order string-loop corrections in the 4d action to be subdominant with respect to the ALE instanton contributions to the gravitino condensates. Since the latter do not depend on the string coupling, cf. (3.160), there is no obstruction to tuning g_s to be sufficiently small, $g_s \ll 1$, in order for the string-loop corrections to be negligible with respect to the instanton contributions.

The l_Y/l_s ratio can be tuned so that the condensates are of the order of the Einstein term in the 4d action, thus dominating 4d higher-order derivative corrections. This requires,

$$l_{4d}^{-2} \sim R_{\rm dS} \propto l_s^{-2} e^{-2c\,(l_Y/l_s)^2}\ , \tag{3.164}$$

where we have taken (3.159) into account. Current cosmological data give,

$$\frac{R_{\rm dS}}{M_{\rm P}^2} \sim \Big(\frac{l_s}{l_{4d}}\Big)^2 \sim 10^{-122}\ . \tag{3.165}$$

From (3.164) we then obtain $l_Y/l_s \sim 10$ for c of order one, cf. (3.108).

In addition to the higher-order derivative corrections, the 4d effective action receives corrections at the two-derivative level, of the form $(l_s/l_Y)^{2n}$ with $n \geq 1$. These come from a certain subset of the 10d tree-level α' corrections (string loops are subleading), which include the $R^2(\partial F)^2$ corrections of [49]. Given the l_Y/l_s ratio derived above, these corrections will be of the order of one percent or less.

As is well known, the vacua computed within the framework of consistent truncations, such as the one constructed in the present section, are susceptible to destabilization by modes that are truncated out of the spectrum. This is an issue that needs to be addressed before one can be confident of the validity of the vacua presented here. The stability issue is particularly important given the fact that, in the presence of a non-vanishing gravitino condensate, supersymmetry will generally be broken.

Ultimately, the scope of the path integral over metrics approach to quantum gravity is limited, since the 4d gravity theory is non-renormalizable. Rather it should be thought of as an effective low-energy limit of string theory. A natural approach to gravitino condensation from the string/M-theory standpoint, would be to try to construct brane-instanton analogues of the four-dimensional gravitational instantons. The fermion condensates might then be computed along the lines of [61, 62, 63].

Another interesting direction would be to try to embed the consistent truncation of the present paper within the framework of $\mathcal{N}=2$ 4d (gauged) supergravity. On general grounds [64], we expect the existence of a consistent truncation of a higher-dimensional supersymmetric theory to the bosonic sector of a supersymmetric lower-dimensional theory, to guarantee the existence of a consistent truncation to the full lower-dimensional theory. The condensate would then presumably be associated with certain gaugings of the 4d theory. It would be very interesting to investigate if such gaugings can be accomodated within the framework of [65].

4 D-branes and Non Abelian T Duality

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Non Abelian T-Duality (NATD) is a transformation relating different solutions of type II supergravity. The duality relies on the existence of an isometry group on the initial solution. When the rank of the isometry group is odd, NATD sends a IIA solution to a IIB solution and conversely, while a group of even rank preserves the type. Thus NATD can be used as a powerful solution generating tool in supergravity. In general solutions constructed with NATD are highly non trivial, and it is unlikely that such solutions could have been determined through other techniques. When the group is abelian (typically $U(1)$) one rather speak of T-duality which, unlike NATD, can be lifted to a duality between type IIA and type IIB string theory.

Non abelian dualities where introduced in [66]. The most direct way to compute the dual solution is to use the Büscher rules, which are given explicitly in appendix C. These are a direct generalization of the Büsher rules of [67] for T-duality. Originally these rules only determine the transformation for the (NS,NS) sector. The transformation rules of the (R,R) fluxes were derived more recently in [68], after which the interest for NATD got renewed, see for example [69, 70, 71, 72, 73, 74, 75, 76, 77].

The point of this chapter is to study the effect of NATD on brane configurations. The isometry group for the duality will be $SU(2)$: see [70] for an explicit expression of the Büscher rules for this case. The idea is that looking at full brane solutions will enable to follow the interpolation between the near horizon and spatial infinity limits and give more freedom to handle the dual configurations. The branes will then be tracked down by computing the flux charges.

Throughout the chapter NATD will be applied to several brane configurations that are known to be solutions of type II supergravity. For all these configurations the dual of the near horizon limit has already been studied in the literature. Following those endeavors we compute and study here the dual of the full interpolating solutions. The basic $D3$ brane solution of type IIB supergravity will be considered first as a warm-up. The insights thus acquired will be used to study several more involved configurations, namely type IIA $D2$ brane solutions coming from the reduction of $M2$ branes in 11 dimensions. Special attention will be devoted to the supersymmetric $D2$. Its supersymmetry will be explicitly checked using the generalized spinor formalism. The same framework will also be used to investigate on the existence of a massive deformation.

4.1 D3 brane

The metric describing a stack of parallel D3 branes is given by,

$$\mathrm{d}s^2 = H(r)^{-1/2}\mathrm{d}s^2(\mathbb{R}^{1,3}) + H(r)^{1/2}[\mathrm{d}r^2 + r^2\mathrm{d}s^2(S^5)] \;, \tag{4.1}$$

where $H(r) = (1 + \frac{L^4}{r^4})$. The S^5 is parameterized as follows,

$$ds^2(S^5) = \mathrm{d}\alpha^2 + \sin^2\!\alpha\,\mathrm{d}\theta^2 + \tfrac{1}{4}\cos^2\!\alpha\,(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \;, \tag{4.2}$$

where $\alpha \in [0, \frac{\pi}{2}]$, $\theta \in [0, 2\pi]$, and σ_i are left-invariant $SU(2)$ Maurer Cartan one-forms given by,

$$\begin{aligned}\sigma_1 &= -\sin\psi_1\mathrm{d}\theta_1 + \cos\psi_1\sin\theta_1\mathrm{d}\phi_1\\ \sigma_2 &= \cos\psi_1\mathrm{d}\theta_1 + \sin\psi_1\sin\theta_1\mathrm{d}\phi_1\\ \sigma_3 &= \cos\theta_1\mathrm{d}\phi_1 + \mathrm{d}\psi_1 \;,\end{aligned} \tag{4.3}$$

with ranges $\psi_1 \in [0, 4\pi]$, $\theta_1 \in [0, \pi]$, $\phi_1 \in [0, 2\pi]$. This background is supported by a constant dilaton and an F_5 flux given by,

$$F_5 = (1+\star)\mathrm{d}x_0\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\mathrm{d}H(r)^{-1} = \mathrm{d}x_0\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\mathrm{d}H(r)^{-1} - 4L^4\mathrm{d}\Omega_5 \;. \tag{4.4}$$

Upon quantization of the five-form flux, one obtains the well-known relation between the constant L in the harmonic function and the number N_{D3} of D3 branes: $L^4 = 4\pi\alpha'^2 N_{D3}$.

The D3 branes lie along the $\mathbb{R}^{1,3}$ directions. This can be seen in a probe approach. Consider the same expression as (4.4) for a flux living now in $\mathbb{R}^{1,9}$. The coordinates now refer to the metric (4.1), but with $H = 1$. Since in spherical coordinates the S^5 collapses at $r = 0$, its volume form $\mathrm{d}\Omega_5$ is ill-defined. However F_5 is a well-defined current (i.e. a distribution-valued form) and we can compute:

$$\mathrm{d}F_5 = \mathrm{d}\star F_5 = 4L^4\delta(r)\mathrm{d}r\wedge\mathrm{d}\Omega_5 \tag{4.5}$$

This means that a brane is inserted in $r = 0$. In this coordinate system this is a codimension 6 space, and thus a D3 lying along $\mathbb{R}^{1,3}$. In the transverse space $\mathbb{R}^6$ the brane looks like a point.

The D3 now acts as a source for the flux F_5, which backreacts on the metric through Einstein's equations to give (4.1). The global geometry has changed, and the S^5 no longer collapses. The supergravity equations are solved without sources and the brane cannot be seen anymore. Nevertheless the information about the brane is still present in the charge carried by the flux.

4.1.1 Near-Horizon and spatial infinity

Taking (4.1) as an ansatz for the metric, the supergravity equations reduce to an equation on H. In the probe interpretation this amounts to saying that H is harmonic in the transverse space. If we further constrain H to depend on r only, the general solution is of the form,

$$H(r) = a + \frac{b}{r^4} \;, \tag{4.6}$$

where a and b are two integration constants. b can readily be interpreted as the brane charge. Then two limiting cases arise:

Spatial infinity: If $b = 0$, H is a constant and the space is flat without flux: no brane is inserted. Since $H \to a$ when $r \to \infty$, this case is called the spatial infinity limit.

Near Horizon: If $a = 0$, the solution becomes,

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}s^2(AdS_5) + L^2\mathrm{d}s^2(S^5) = \frac{r^2}{L^2}\mathrm{d}s^2(\mathbb{R}^{1,3}) + \frac{L^2}{r^2}\mathrm{d}r^2 + L^2\mathrm{d}s^2(S^5) \\ F_5 &= (1+\star)\frac{4r^3}{L^4}\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}r \ , \end{aligned} \tag{4.7}$$

which is the well-known $AdS_5 \times S^5$ background. For $r \to 0$, $H \sim \frac{b}{r^4}$ so that this case is called the near horizon limit.

It is remarkable at first sight that both limits correspond to genuine backgrounds. The reason behind this is that they ultimately correspond to different choices of integration constants. These considerations might seem trivial for now, but they will be relevant in the following, when the brane configurations become more involved.

4.1.2 The NATD

After performing NATD along the SU(2) isometry in the σ_i, cf. (4.3), the background (4.1), (4.4) becomes,

$$\begin{aligned} ds^2 &= H(r)^{-1/2}ds^2(\mathbb{R}^{1,3}) + H(r)^{1/2}[dr^2 + r^2(d\alpha^2 + \sin^2\alpha d\theta^2)] \\ &\quad + \frac{1}{4}\Big[\frac{\alpha'^2}{\Xi}d\rho^2 + \frac{\Xi^2}{64\alpha'\Delta}\rho^2(d\chi^2 + \sin^2\chi d\xi^2)\Big] \\ B_2 &= -\frac{\rho^3\Xi}{256\Delta}\sin\chi d\chi \wedge d\xi \\ e^{-2\phi} &= \Delta, \quad \Delta = \frac{\Xi}{64\alpha'^3}(\alpha'^2\rho^2 + \Xi^2), \quad \Xi = r^2\cos^2\alpha\sqrt{H(r)} \ , \end{aligned} \tag{4.8}$$

and nonzero RR fluxes given by,

$$\begin{aligned} F_2 &= -\frac{\Xi}{8\alpha'^{3/2}}\frac{H'(r)}{\sqrt{H(r)}}r^3\cos\alpha\sin\alpha d\alpha \wedge d\theta \\ F_4 &= \frac{\Xi^2}{2048\alpha'^{3/2}\Delta}\frac{H'(r)}{\sqrt{H(r)}}r^3\rho^3\cos\alpha\sin\alpha\sin\chi d\alpha \wedge d\theta \wedge d\chi \wedge d\xi \ . \end{aligned} \tag{4.9}$$

In particular we see that the NATD has resulted in a metric which is singular at $\alpha = \frac{\pi}{2}$. Moreover the duality has generated a nonvanishing Kalb-Ramond field B_2 and a varying dilaton ϕ.

Note that the background (4.8) contains a family of solutions, inheriting its degrees of freedom from the $D3$ solutions before duality: for each choice of harmonic function H, NATD generates a different solution. We will keep the same denomination for the different limits, namely the near-horizon for $H = \frac{L^4}{r^4}$ and spatial infinity for $H = 1$ (i.e. $L = 0$). However their interpretation as different limits of the interpolating dual background is less meaningful. We will study them separately to get a better view on the brane configurations.

For later use let us rewrite the metric in (4.8) in terms of the coordinates defined by,

$$x = r\sin\alpha\cos\theta \ ; \quad y = r\sin\alpha\sin\theta \ ; \quad u = r\cos\alpha \ . \tag{4.10}$$

Recalling the ranges of the α, θ coordinates, cf. (4.2), we see that $u \geq 0$, while $x, y \in \mathbb{R}$. Simplifying with $r^2 = x^2 + y^2 + u^2$ then gives,

$$ds^2 = H^{-1/2}\Big[ds^2(\mathbb{R}^{1,3}) + \frac{\alpha'^2}{4u^2}d\rho^2\Big] + H^{1/2}\Big[dx^2 + dy^2 + du^2 + \frac{\alpha'^2\rho^2u^2}{4(\alpha'^2\rho^2 + Hu^4)}(d\chi^2 + \sin^2\chi d\xi^2)\Big] . \tag{4.11}$$

In these coordinates, the metric is singular at $u = 0$.

Brane configuration and charges

The non-vanishing fluxes might indicate the presence of branes. Here we could expect NS5, D4 and D6 branes as magnetic sources for H, F_4 and F_2. The first clue is given by the corresponding charges.

Let us start with the NS flux. An appropriate cycle would be the following: start at constant $\alpha = \alpha_0$ and integrate along ρ, χ, ξ where ρ goes from 0 to ρ_0. At $\rho = 0$ the cycle closes but we need to close it at ρ_0. To do so, keep ρ constant and vary α from α_0 to $\pi/2$. The resulting charge will be independent of α_0 so we can take the limit $\alpha_0 \to \pi/2$.

Along the cycle $(\Sigma_3 = [\rho, \chi, \xi], \alpha = \frac{\pi}{2})$, H_3 simplifies to

$$H_3 = \frac{1}{4}\alpha' \sin\chi d\xi \wedge d\chi \wedge d\rho \ . \tag{4.12}$$

Integrating H_3 yields,

$$Q_{NS5} = \frac{1}{2\kappa_{10}^2 T_{NS5}}\frac{\alpha'}{4}\int_0^{\rho_0} d\rho \int_0^{\pi} \sin\chi d\chi \int_0^{2\pi} d\xi = \frac{\rho_0}{4\pi} = N_{NS5} \ , \tag{4.13}$$

In fact the charge will depend only on the value of ρ when the cycle reaches $\alpha = \pi/2$. As we will see more explicitly in the simpler case of the spatial infinity limit in section 4.1.2, this suggests a continuous distribution of NS5 branes at $\alpha = \pi/2$ along the ρ direction, with constant charge density. For the flux to be quantized we need to close the cycle at quantized values of ρ, namely $\rho_0 = 4n\pi$. The NS5 branes are thus located at the singularity: this can be seen from the form of the metric and NS-NS fields in the limit $\alpha \to \frac{\pi}{2}$, which is consistent with the general form expected from the harmonic superposition rule [78]. After defining $\nu = (\pi/2 - \alpha)^2$ we find, in the $\alpha \to \frac{\pi}{2}$ limit,

$$\begin{aligned} ds^2 &= H^{-1/2}ds^2(\mathbb{R}^{1,3}) + H^{1/2}\Big(dr^2 + r^2d\theta^2 + \frac{r^2}{4\nu}\Big[d\nu^2 + \frac{\alpha'^2}{Hr^4}d\rho^2 + \nu^2(d\chi^2 + \sin^2\chi d\xi^2)\Big]\Big) \\ e^{2\phi} &= \frac{64\alpha'}{r^2\sqrt{H}\rho^2\nu} \ ; \quad H_3 = \frac{\alpha'}{4}\sin\chi d\rho \wedge d\chi \wedge d\xi \ . \end{aligned} \tag{4.14}$$

The harmonic function in the space transverse to the NS5 is proportional to ν^{-1}, indicating the presence of NS5 branes at $\nu = 0$. However this is not a point in the transverse space. Since ρ is still unconstrained, this is consistent with a distribution of charge along ρ.

In order to determine the configuration of the remaining branes we follow the same strategy. Recall that in solutions with nonzero B_2, the quantized charges are the Page charges, defined as integrals of the Page forms,

$$\tilde{F}_p = F_p e^{-B_2}. \tag{4.15}$$

As can be seen from this definition, the Page charges depend on the cohomology class of B_2, i.e. they are not invariant under large gauge transformations of B_2.

Integrating the Page forms in the D3 brane solution gives,

$$\begin{aligned} Q_{D6} &= \frac{1}{2\kappa_{10}^2 T_{D6}} \frac{L^4}{2\alpha'^{3/2}} \int_0^{\frac{\pi}{2}} \cos^3\alpha \sin\alpha d\alpha \int_0^{2\pi} d\theta = N_{D6} \\ Q_{D4} &= 0 \ , \end{aligned} \tag{4.16}$$

which leads to $L^4 = 8\alpha'^2 N_{D6}$. If we denote by ΔQ_{D4} the change of D4 brane charge under a large gauge transformation of B_2,

$$\Delta B_2 = -n\pi\alpha' \sin\chi d\xi \wedge d\chi \ , \tag{4.17}$$

we find,

$$\begin{aligned} \Delta Q_{D4} &= \frac{1}{2\kappa_{10}^2 T_{D4}} \int -\Delta B_2 \wedge F_2 \\ &= \frac{1}{2\kappa_{10}^2 T_{D4}} \frac{n\pi L^4}{8\sqrt{\alpha'}} \int_0^{\frac{\pi}{2}} \cos^3\alpha \sin\alpha d\alpha \int_0^{\pi} d\theta \int_0^{\pi} \sin\chi d\chi \int_0^{2\pi} d\xi \\ &= \Delta N_{D4} \ , \end{aligned} \tag{4.18}$$

which leads to $L^4 = \frac{1}{n} 8\alpha'^2 \Delta N_{D4}$. From this we readily see that

$$\Delta Q_{D4} = n N_{D6}. \tag{4.19}$$

This is nothing other than the creation of D4 branes via a Hanany-Witten effect [79], as will be reviewed in the following in section 4.1.2. In order to get a probe interpretation of these brane charges we would need to know in which background the branes are inserted, but the situation is not entirely clear here. The expression for the fluxes suggests that the $D6$ is transverse to r, α, θ and that the $D4$ is transverse to $r, \alpha, \theta, \chi, \xi$. This would lead to the following brane configuration:

	0	1	2	3	r	α	θ	ρ	χ	ξ
$NS5$	×	×	×	×	×		×			
$D6$	×	×	×	×				×	×	×
$D4$	×	×	×	×				×		

Spatial infinity limit

The spatial infinity limit gives the following background:

$$\begin{aligned} ds^2 &= ds^2(\mathbb{R}^{1,3}) + dr^2 + r^2(d\alpha^2 + \sin^2\alpha d\theta^2) + \frac{1}{4}\Big[\frac{\alpha'^2}{\Xi} d\rho^2 + \frac{\Xi^2}{64\alpha'\Delta}\rho^2(d\chi^2 + \sin^2\chi d\xi^2)\Big] \\ B_2 &= -\frac{\rho^3 \Xi}{256\Delta} \sin\chi d\chi \wedge d\xi \\ e^{-2\phi} &= \Delta, \quad \Delta = \frac{\Xi}{64\alpha'^3}(\alpha'^2\rho^2 + \Xi^2), \quad \Xi = r^2\cos^2\alpha \ . \end{aligned} \tag{4.20}$$

Here there are no RR fluxes anymore so all the D-brane charges vanish. The configuration is thus much simpler. In fact it will now be possible to understand the exact brane configuration, as is done for the D3. Moreover this background is the NATD of the spatial infinity limit of the D3 brane solution: the NATD (4.20) is simply the NATD of flat space along an $S^3 \subset \mathbb{R}^4$ factor. This decomposition is thus better suited for the spatial infinity limit than the $S^5 \subset \mathbb{R}^6$ decomposition of the D3 brane solution. Accordingly the seed metric before NATD reads,

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathbb{R}^{1,5}) + \mathrm{d}u^2 + u^2 \mathrm{d}s^2(S^3) \ , \tag{4.21}$$

which is simply the spatial infinity limit of the metric (4.1) written in the coordinates of (4.10). In these coordinates the NATD metric (4.20) is given by,

$$ds^2 = ds^2(\mathbb{R}^{1,5}) + du^2 + \frac{\alpha'^2}{4u^2} d\rho^2 + \frac{\alpha'^2 \rho^2 u^2}{4(\alpha'^2\rho^2 + u^4)} (d\chi^2 + \sin^2\chi d\xi^2) \ . \tag{4.22}$$

Let us now make a further change of variable,

$$\begin{aligned} u &= R^{1/4}\sqrt{\sin\frac{\theta}{2}} \\ \alpha'\rho &= R^{1/2}\cos\frac{\theta}{2} \ , \end{aligned} \tag{4.23}$$

upon which the metric becomes,

$$\begin{aligned} ds^2 &= ds^2(\mathbb{R}^{1,5}) + \frac{1}{16R^{3/2}\sin\frac{\theta}{2}} \left[dR^2 + R^2 d\theta^2 + R^2\sin^2\theta(d\chi^2 + \sin^2\chi d\xi^2)\right] \\ &= ds^2(\mathbb{R}^{1,5}) + f(R,\theta) ds^2(\mathbb{R}^4) \ , \end{aligned} \tag{4.24}$$

where in order to obtain a complete metric on $\mathbb{R}^4$ we must have $\theta \in [0, \pi]$. In the second line above we have introduced the function,

$$f(R,\theta) = \frac{1}{16R^{3/2}\sin\frac{\theta}{2}} \ , \tag{4.25}$$

which is harmonic in $\mathbb{R}^4$ except for $\theta = 0$. The NS-NS two-form and dilaton are given by,

$$\begin{aligned} B_2 &= -\frac{R^{1/2}}{4}\cos^3\frac{\theta}{2}\sin\chi d\chi \wedge d\xi \\ H_3 &= -\frac{\cos^3\frac{\theta}{2}}{8R^{1/2}} \mathrm{d}R \wedge \sin\chi d\chi \wedge d\xi + \frac{3}{8}R^{1/2}\cos^2\frac{\theta}{2}\sin\frac{\theta}{2} \mathrm{d}\theta \wedge \sin\chi d\chi \wedge d\xi \\ e^{2\phi} &= 1024\alpha'^3 f \ . \end{aligned} \tag{4.26}$$

This clearly shows the presence of NS5 branes along the $\mathbb{R}^{1,5}$ directions, located at $\theta = 0$ (or alternatively at $u = 0$), in accordance with the harmonic superposition rule [78]. However this is not enough to determine the exact position of the branes since they could be anywhere on this half line. Integrating H_3 on a spherical shell of radius R gives,

$$\int H_3 = \int_{\theta=0}^{\pi} \int_{\chi=0}^{\pi} \int_{\xi=0}^{2\pi} dB_2 = \pi\sqrt{R} \ . \tag{4.27}$$

The branes are thus smeared along the $\theta = 0$ direction, leading to a linear distribution of charge in the transverse space, whose charge density is proportional to $\frac{1}{\sqrt{R}}$ or constant in ρ (recall that at $\theta = 0$, $\sqrt{R} = \rho$).

More explicitly the NS5 distribution can be read off of the harmonic function f in (4.25) as follows. First it will be convenient to parameterize the $\mathbb{R}^4$ transverse to the NS5 by introducing the cylindrical coordinates $\vec{r} \in \mathbb{R}^3$, $w := R\cos\theta$, so that $R^2 = \vec{r}^{\,2} + w^2$ and,

$$\mathrm{d}s^2(\mathbb{R}^4) = \mathrm{d}\vec{r}\cdot\mathrm{d}\vec{r} + \mathrm{d}w^2 \ . \tag{4.28}$$

The new coordinates (R, w) are related to (u, ρ) by,

$$u^4 = \frac{1}{2}(R - w) \ ; \quad \alpha'^2\rho^2 = \frac{1}{2}(R + w) \ . \tag{4.29}$$

Moreover it can easily be verified that the function f can be represented as an integral over the Green's function for the Laplacian on $\mathbb{R}^4$,

$$f = \frac{\sqrt{2}}{16R\sqrt{R-w}} = \frac{1}{8\pi}\int_0^\infty \mathrm{d}w' \frac{\sigma(w')}{\vec{r}^{\,2} + (w - w')^2} \ , \tag{4.30}$$

with linear charge density $\sigma(w) = w^{-\frac{1}{2}}$ along the half line $w \geq 0$.

An alternative way to find the charge distribution is to compute the source for the H_3 Bianchi identity,

$$\mathrm{d}H_3 = j \ . \tag{4.31}$$

However $H_3 = \mathrm{d}B_2$ is closed as a form, and we thus need to consider this equation on currents. Indeed H_3 is not defined for $\theta = 0$, which is precisely the locus where we expect to find the brane. As a current, $\mathrm{d}H_3$ acts as a linear form (distribution) on six-forms. Consider a test six-form Ω,

$$\Omega = \omega v_6 \ , \tag{4.32}$$

where v_6 is the volume form of $\mathbb{R}^{(1,5)}$. After integration against H_3, the only components of $\mathrm{d}\Omega$ we need consider are,

$$\mathrm{d}\Omega = \partial_R\omega \mathrm{d}R \wedge v_6 + \partial_\theta\omega \mathrm{d}\theta \wedge v_6 + \cdots \ , \tag{4.33}$$

so that,

$$\begin{aligned}
\mathrm{d}H_3(\Omega) &= H_3(\mathrm{d}\Omega) \\
&= \int H_3 \wedge \mathrm{d}\Omega \\
&= -\frac{1}{8}\int \frac{\cos^3\frac{\theta}{2}}{R^{1/2}}\partial_\theta\omega \mathrm{d}R \wedge \sin\chi \mathrm{d}\chi \wedge \mathrm{d}\xi \wedge \mathrm{d}\theta \wedge v_6 \\
&+ \frac{3}{8}\int R^{1/2}\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}\partial_R\omega \mathrm{d}\theta \wedge \sin\chi \mathrm{d}\chi \wedge \mathrm{d}\xi \wedge \mathrm{d}R \wedge v_6 \ .
\end{aligned} \tag{4.34}$$

Integrating each term by parts (respectively in θ and R), the derivatives cancel out since H_3 is closed as a form. The charge can then be seen in the boundary terms. $R^{1/2}$ vanishes at $R = 0$, ω vanishes at $R \to \infty$ because it is a test function, and $\cos^3\frac{\theta}{2}$ vanishes at $\theta = \pi$. Note also that at $\theta = 0$, ω cannot depend on χ, ξ. We thus obtain,

$$\begin{aligned}
\mathrm{d}H_3(\Omega) &= \frac{1}{8}\int \frac{\omega(\theta = 0)}{R^{1/2}}\mathrm{d}R \wedge \sin\chi \mathrm{d}\chi \wedge \mathrm{d}\xi \wedge v_6 \\
&= \frac{\pi}{2}\int \frac{\mathrm{d}R}{R^{1/2}} \wedge \Omega(\theta = 0) \ .
\end{aligned} \tag{4.35}$$

From this we can read off the current,

$$j = \frac{1}{16}\frac{\delta(\theta)}{R^{1/2}}\mathrm{d}R \wedge \mathrm{d}\theta \wedge \sin\chi \mathrm{d}\chi \wedge \mathrm{d}\xi \ , \tag{4.36}$$

which gives the exact distribution of NS5 charge. This distribution is remarkable since it is entirely created by NATD from an empty flat space. It will be characteristic of the behavior of NATD near a fixed point of the $SU(2)$ isometry. For instance we found the same kind of distribution when looking close to the $\alpha = \pi/2$ singularity in the full dual solution (4.14).

The near-horizon limit

The NATD of the near horizon solution is [68],

$$\begin{aligned} ds^2 &= ds^2(AdS_5) + L^2(d\alpha^2 + \sin^2\alpha d\theta^2) \\ &\quad + \frac{1}{4}\Big(\frac{\alpha'^2}{\Xi}d\rho^2 + \frac{\Xi^2}{64\alpha'\Delta}\rho^2(d\chi^2 + \sin^2\chi d\xi^2)\Big) \\ B_2 &= -\frac{\rho^3\Xi}{256\Delta}\sin\chi d\chi\wedge d\xi \\ e^{-2\tilde{\Phi}} &= \Delta, \quad \Delta = \frac{\Xi}{64\alpha'^3}(\alpha'^2\rho^2 + \Xi^2), \quad \Xi = L^2\cos^2\alpha \ , \end{aligned} \tag{4.37}$$

and the nonzero RR fluxes are given by,

$$\begin{aligned} F_2 &= \frac{\Xi}{2\alpha'^{3/2}}L^2\cos\alpha\sin\alpha d\alpha\wedge d\theta \\ F_4 &= -\frac{\Xi^2}{512\alpha'^{3/2}\Delta}L^2\rho^3\cos\alpha\sin\alpha\sin\chi d\alpha\wedge d\theta\wedge d\chi\wedge d\xi \ . \end{aligned} \tag{4.38}$$

Field Theory interpretation of near horizon NATD

In [80] a holographic interpretation of the background (4.37)-(4.38) was proposed. It was pointed out that the background belongs to a class of Gaiotto-Maldacena geometries [81] dual to $\mathcal{N} = 2$ superconformal linear quivers with gauge groups of increasing rank. Their argument crucially involved constraining the range of the dual coordinate ρ in quantizing the NS5 brane charge. Let us briefly summarize the main points of the arguments originally presented in [80] and extended to further examples in [82, 83]. Related examples with flavor branes include [71, 84] and [72, 85].

In the NATD a new set of dual coordinates arise, which we have labeled (ρ, χ, ξ). The coordinates (χ, ξ) are naturally interpreted as compact angles on an S^2, i.e. $\chi \in [0, \pi]$, $\xi \in [0, 2\pi]$. The question remains how to interpret the ρ coordinate, as NATD currently lacks the global information needed to constrain the dual coordinates. Using insight from string theory the authors of [80] were led to impose the boundedness of the following quantity,

$$b_0 = \frac{1}{4\pi^2\alpha'}\oint_{\Sigma_2} B_2 \in [0,1] \ , \tag{4.39}$$

where in the case of (4.37) b_0 is maximal along $\Sigma_2 = [\chi, \xi], \alpha = \frac{\pi}{2}$. This leads to the coordinate ρ varying in $n\pi$ intervals, i.e. $\rho \in [n\pi, (n+1)\pi]$. To keep the relation (4.39) satisfied, a large gauge transformation must be performed on B_2 at each $n\pi$ interval, i.e.

$$B_2 \to B_2 - n\pi\alpha'\sin\chi d\chi\wedge d\xi \ . \tag{4.40}$$

As reviewed in section 4.1.2, this has the effect of changing the Page charges: quantizing Q_{D6} and Q_{D4} by integrating the RR fluxes in (4.38) above leads to $Q_{D6} = N_{D6}$ and $Q_{D4} = 0$. However under a large gauge transformation of B_2, we find $\Delta Q_{D6} = 0$ and $\Delta Q_{D4} = nN_{D6}$, where $Q_{NS5} = N_{NS5} = n$. Putting all this together suggests that there are parallel NS5 branes, each located at a π interval in ρ. Between each π interval n horizontal D4 branes are suspended between them. That is, as we move towards larger ρ, an increasing number of D4 branes appear. In the field theory interpretation this corresponds to an infinite linear quiver with increasing gauge group ranks. Interestingly, the field theory analysis of [80] suggested that there should be a cutoff to the ρ coordinate in order to terminate the quiver with a flavor brane. The intuitive way to see this is to start with parallel NS5 branes and a D6 flavor brane on one of the ends of the array. When one moves this flavor brane across the NS branes, D4 branes are created across the NS branes via the Hanany-Witten effect [79]. This completion of the quiver corresponds to giving ρ a finite range and it was shown that this is necessary to make sense of the dual field theory as a 4d CFT.[1]

Thus the "stringy" picture is consistent with the spatial infinity limit of section 4.1.2, provided we replace the supergravity approximation of a continuous linear distribution of NS5 branes along a half line, by a grid of localized NS5's so that there is one unit of NS5 charge per $\rho \in [n\pi, (n+1)\pi]$ interval.

4.2 M2 branes

The M2 brane solutions of eleven-dimensional supergravity can be reduced in various ways in order to obtain ten-dimensional IIA D2 brane solutions. Let us start from the M2-brane solution in flat space,

$$
\begin{aligned}
ds^2 &= H^{-2/3} ds^2(\mathbb{R}^{1,2}) + H^{1/3}(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_7^2) \\
G &= -\mathrm{d}H^{-1} \wedge \mathrm{vol}_3 \\
H &= 1 + \frac{\hat{Q}}{r^6} \ ,
\end{aligned}
\tag{4.41}
$$

where $\hat{Q}$ is a constant related to the number of parallel M2-branes, $\mathrm{d}\Omega_7^2$ is the metric of the round seven-sphere, and vol_3 is the volume element of $\mathbb{R}^{1,2}$. We will adopt the parameterization of the metric on S^7 given by,

$$
\begin{aligned}
\mathrm{d}\Omega_7^2 &= \frac{1}{4}\Big(d\mu^2 + \frac{1}{4}\big(\sin^2\mu\omega_i^2 + \lambda^2(\nu_i + \cos\mu\omega_i)^2\big)\Big), \\
\nu_i &= \sigma_i + \Sigma_i, \qquad \omega_i = \sigma_i - \Sigma_i \ ,
\end{aligned}
\tag{4.42}
$$

where $\mu \in [0, \pi]$, σ_i are the left-invariant $SU(2)$ Maurer Cartan one-forms given in (4.3), while the Σ_i have exactly the same form but with coordinates $(\theta_2, \phi_2, \psi_2)$. We will only treat the round S^7 case, i.e. $\lambda = 1$. In the near-horizon limit, we have $H = \frac{\hat{Q}}{r^6}$ and the space becomes $\mathrm{AdS}_4 \times S^7$.

This solution preserves 16 real supercharges (enhanced to 32 in the near-horizon limit), *i.e.* $\mathcal{N} = 4$ in four dimensions. In (4.41) we have written the flat metric on the space $\mathbb{R}^8$ transverse to the M2 as an eight-dimensional cone over the seven-sphere. We may replace

[1] It was suggested in [80] and further considered in [86] that the dual field theory could actually be higher dimensional through deconstruction.

the base of the cone by any Sasaki-Einstein seven-manifold[2], and still obtain a solution of eleven-dimensional supergravity. The amount of preserved supersymmetry depends on the number of Killing spinors of the Sasaki-Einstein.

Replacing the round sphere metric $\mathrm{d}\Omega_7^2$ by the $Y^{p,q}(B_4)$ metric of [87, 88], reduces supersymmetry to $\mathcal{N}=1$ in four dimensions, enhanced to $\mathcal{N}=2$ in the near-horizon limit. After a change of coordinates to bring us to the conventions of [77], the metric reads,

$$\mathrm{d}s^2(Y_7) = \frac{1}{4}\mathrm{d}s^2(M_6) + w(\theta)\left[\mathrm{d}\alpha + f(\theta)(\mathrm{d}\psi + A)\right]^2 \ , \tag{4.43}$$

for some functions w, f of θ that will be specified below, where $\mathrm{d}s^2(M_6)$ is the metric of the $S^2(B_4)$ bundle,

$$\mathrm{d}s^2(M_6) = \mathrm{d}s^2(B_4) + \frac{1}{(1+\cos^2\theta)^2}\mathrm{d}\theta^2 + \sin^2\theta(\mathrm{d}\psi + A)^2 \ , \tag{4.44}$$

with $\theta \in [0,\pi]$, $\psi \in [0,\pi]$ the coordinates of the S^2 fiber; the connection A is a one-form on the base B_4 obeying,

$$\mathrm{d}A = J \ , \tag{4.45}$$

with J the Kähler form of B_4 and α parametrizes a circle fibered over the basis M_6. Later we will consider the special case $B_4 = \mathbb{CP}^2$ for concreteness and in order to perform an SU(2) NATD.

The corresponding eleven-dimensional solution reads,

$$\begin{aligned} ds^2 &= H^{-2/3}ds^2(\mathbb{R}^{1,2}) + H^{1/3}\left(\mathrm{d}r^2 + \frac{1}{4}r^2\mathrm{d}s^2(M_6)\right) + r^2H^{1/3}w(\theta)(\mathrm{d}\alpha + A')^2 \\ G &= -\mathrm{d}H^{-1}\wedge \mathrm{vol}_3 \\ H &= 1 + \frac{\hat{Q}}{r^6} \ , \end{aligned} \tag{4.46}$$

where we have set $A' := f(\theta)(\mathrm{d}\psi + A)$.

4.2.1 Brane configuration and charges

We expect the M2 branes to lie along the $\mathbb{R}^{1,2}$ directions. The transverse space would then be $\mathbb{R}^8$ or the cone over $Y^{p,q}$ depending on the choice of 7-dimensionnal space. In both cases we find:

$$\star G = -6\hat{Q}v_7 \tag{4.47}$$

However since the 7-dimensional cycle collapses in the transverse space when $r=0$, $\star G$ is not closed and:

$$\mathrm{d}\star G = -6\hat{Q}\delta(r)\mathrm{d}r\wedge v_7 \tag{4.48}$$

We can also compute the M2 brane charge, which is defined by:

$$Q_{M2} = \frac{1}{2\kappa_{11}^2 T_{M2}}\int \star G = N_{M2} \ , \tag{4.49}$$

[2]The metric of the Sasaki-Einstein manifold must be normalized so that the cone over it is Ricci-flat.

with the M2 brane tension given by $T_{M2} = \frac{2\pi}{(2\pi l_p)^3}$ and $2\kappa_{11}^2 = (2\pi)^8 l_p^9$, where the Planck length is given by $l_p = g_s^{1/3}\sqrt{\alpha'}$. For example, in the $Y^{p,q}$ case, we compute:

$$\begin{aligned} Q_{M2} &= -\frac{1}{2\kappa_{11}^2 T_{M2}} \frac{27\hat{Q}}{256} \int_0^{2\pi} d\alpha \int_0^{\pi} d\theta \frac{\sin\theta}{a(\theta)^{3/2}} \int_{S^3} d\Omega_3 \int_0^{\pi} d\psi \int_0^{\frac{\pi}{2}} d\mu \sin^3\mu , \\ &= -\frac{27\hat{Q}}{4096\pi^2 l_p^6} = N_{M2} \ . \end{aligned} \tag{4.50}$$

This relates the constant in the harmonic function to the number of M2 branes,

$$\hat{Q} = \frac{4096}{27}\pi^2 l_p^6 N_{M2} \ . \tag{4.51}$$

We will now proceed to track the M2, first through dimensional reduction, then through NATD.

4.3 Supersymmetric D2 from reduction on $Y^{p,q}$

Here and in the following section we will need to make the choice $B_4 = \mathbb{CP}^2$ so that a non-abelian SU(2) isometry is manifest in the metric acting on the σ_i,

$$ds^2(\mathbb{CP}^2) = 3\big[d\mu^2 + \frac{1}{4}\sin^2\mu(\sigma_1^2 + \sigma_2^2 + \cos^2\mu\,\sigma_3^2)\big] \ , \tag{4.52}$$

where $\mu \in [0, \frac{\pi}{2}]$, and the σ_i are given in (4.3).

Reducing the M2 brane solution (4.46) to IIA on the circle parameterized by α preserves supersymmetry, as will be explicitly verified in section 4.3.2. Let us set,

$$\begin{aligned} e^{-2\phi/3}\mathrm{d}s_{\mathrm{A}}^2 &= H^{-2/3} ds^2(\mathbb{R}^{1,2}) + H^{1/3}\left(\mathrm{d}r^2 + \frac{1}{4}r^2 \mathrm{d}s^2(M_6)\right) \\ e^{4\phi/3} &= \frac{r^2}{l_p^2} H^{1/3} w(\theta) \ , \end{aligned} \tag{4.53}$$

so that upon reduction to ten dimensions $\mathrm{d}s_{\mathrm{A}}^2$ and the function $\phi(r,\theta)$ are identified with the IIA string-frame metric and dilaton respectively. Moreover the nonvanishing fluxes of the solution are given by,

$$F_2 = l_p \mathrm{d}A' \ ; \quad F_4 = -\mathrm{d}H^{-1} \wedge \mathrm{vol}_3 \ , \tag{4.54}$$

where A' was given below (4.46). F_2 carries a magnetic charge, but we will not interpret it as coming from a brane. Since this charge is not related to the M2 charge, but rather to the dimensional reduction, we will say that the flux is only geometric, and it will not be of interest here. The flux F_4 on the other hand carries an electric charge and is sourced by a stack of parallel D2 branes filling $\mathbb{R}^{1,2}$ and placed at $r = 0$ in the transverse space. Note that the H function is inherited from the M2 solution, so that it does not need to be harmonic in the new transverse space. We also inherit the usual parameters for a brane solution, which allow us to define the near-horizon and spatial infinity limit.

We can then obtain the explicit form of the functions $w(\theta)$, $f(\theta)$ by taking the near-horizon limit ($H = \frac{\hat{Q}}{r^6}$) of (4.53), (4.54) and comparing with [77]:

$$\begin{aligned} \mathrm{d}s_{\mathrm{A}}^2 &= \frac{1}{4}\hat{Q}^{1/2}\sqrt{w(\theta)}\left(ds^2(\mathrm{AdS}_4) + \mathrm{d}s^2(M_6)\right) \\ e^{4\phi/3} &= \hat{Q}^{1/3} w(\theta) \ ; \quad F_2 = l_p \mathrm{d}\left[f(\theta)(\mathrm{d}\psi + A)\right] \ , \end{aligned} \tag{4.55}$$

where $ds^2(\mathrm{AdS}_4)$ is the metric of an AdS_4 space of unit radius, so that its scalar curvature is normalized to $R = -12$. Comparing with (2.22), (2.23), (2.24) of [77] we read off,

$$w(\theta) = \frac{g_s^2\, e^{4A_0}}{8(1+\cos^2\theta)} \ ; \quad f(\theta) = \frac{\cos\theta}{2\sqrt{w(\theta)}} \ ; \quad \hat{Q} = \frac{64}{g_s^2} \ . \tag{4.56}$$

To summarize, the ten-dimensional D2-brane solution is given by (4.53), (4.54), where $\mathrm{d}s^2(M_6)$ is given in (4.44), H is given in (4.46) and f, w are given in (4.56). In the near-horizon limit the metric becomes a warped $\mathrm{AdS}_4 \times M_6$ product, cf. (4.55).

At spatial infinity ($H = 1$) the metric becomes a warped product $\mathbb{R}^{1,2} \times C(M_6)$,

$$ds_{\mathrm{A}}^2 = \frac{r}{l_p}\sqrt{w(\theta)}\left(\mathrm{d}s^2(\mathbb{R}^{1,2}) + \mathrm{d}r^2 + \frac{1}{4}r^2\mathrm{d}s^2(M_6)\right) \ , \tag{4.57}$$

where $C(M_6)$ is the metric cone over M_6, while the remaining fields are given by,

$$\begin{aligned} e^{4\phi/3} &= \frac{r^2}{l_p^2}w(\theta) \\ F_2 &= l_p\mathrm{d}\left[f(\theta)(\mathrm{d}\psi + A)\right] \\ F_4 &= 0 \ . \end{aligned} \tag{4.58}$$

It can be verified that this is an exact supergravity solution in its own right. Contrary to the case of the D3 brane, here spacetime is neither flat nor empty at spatial infinity.

The solution (4.53), (4.54) describes D2 branes with worldvolume along the $\mathbb{R}^{1,2}$, as inherited from the M2 solution. Looking at F_4, we find:

$$\star\, F_4 = -\frac{3\hat{Q}}{32 l_p}\sqrt{w(\theta)}v_6 \tag{4.59}$$

However, contrary to the standard brane configurations (such as the D3 and M2 presented previously), the probe interpretation is not straightforward. In order to understand this configuration we take the transverse space to be the cone over M_6. There the 6-cycle collapses at $r = 0$, so that:

$$\mathrm{d}\star F_4 = -\frac{3\hat{Q}}{32 l_p}\sqrt{w(\theta)}\delta(r)\mathrm{d}r \wedge v_6 \tag{4.60}$$

Here again this equation must be considered on the transverse space. The D2 background is a genuine solution of IIA supergravity, in which the 6-cycle does not collapse anymore. Then $\star F_4$ is closed, as required by the equations of motion, and the brane is not visible. We can however compute the brane charge, which requires the D-brane tension $T_{Dp}^{-1} = ((2\pi)^p \alpha'^{\frac{(p+1)}{2}})$ and $2\kappa_{10}^2 = (2\pi)^7\alpha'^4$. We obtain,

$$Q_{D2} = \frac{27\tilde{Q}}{8192\pi^5 l_p \alpha'^{5/2}}\int_0^\pi d\psi \int_0^\pi d\theta \frac{\sin\theta}{a(\theta)^{3/2}}\int_{S^3} d\Omega_3 \int_0^{\frac{\pi}{2}} d\mu \sin^3\mu\cos\mu, \tag{4.61}$$

The flux quantization condition $Q_{D2} = N_{D2}$ then leads to

$$\hat{Q} = \frac{4096}{27}\pi^2 l_p \alpha'^{5/2} N_{D2} \ . \tag{4.62}$$

Note that there are no D6 branes associated with the F_2 flux. Indeed in the present case spacetime is smooth[3] and the metric singularity expected in the vicinity of a D6 is absent.

[3] The geometry and topology of the M-theory reduction along the α-cycle is discussed in detail in [89].

As we will see in section 4.5, this is in contrast to the case of the D2 brane coming from the reduction of M-theory on S^7. Similarly one sees that there are no $D4$ branes sourced by the F_4 flux.

4.3.1 Domain wall supersymmetry equations

As already mentioned, (4.57) is a supersymmetric domain wall (DW) solution in four-dimensional space, where the latter is viewed as a foliation, parameterized by r, with $\mathbb{R}^{1,2}$ leaves. The supersymmetry conditions for $\mathcal{N}=1$ domain walls were written in [90] in generalized $G_2 \times G_2$ form in eqs. (2.5), (2.6) therein. For our purposes it would be more useful to recast these equations in terms of generalized pure spinors on M_6. Such a rewriting is indeed given in [90], *cf.* (2.7) therein. We will now review their results adapting them to our case.

The ansatz for the splitting of the metric and the flux is given by:

$$\begin{aligned} \mathrm{d}s^2 &= e^{2A}\mathrm{d}s^2(\mathbb{R}^{1,2}) + \mathrm{d}s^2(M_7) \\ F_t &= F + v_3 \wedge \star\lambda F \ , \end{aligned} \tag{4.63}$$

where the warp factor A and the dilaton ϕ are not constrained at this point; λ is an involution reversing the order of wedge products. The NS-NS form H is assumed to be internal, i.e. to only have legs along M_7, and likewise for the internal RR flux F. The total flux F_t is then chosen to be self-dual: for F internal we get $\star\lambda F = v_3 \wedge \star_7\lambda F$ and in ten Lorentzian dimensions $(\star\lambda)^2 = 1$. Unbroken supersymmetry of the solution implies on M_7 the existence of two Majorana spinors χ_1, χ_2 normalized so that $\chi_a^\dagger \chi_a = 1$. This leads us to define a bispinor Ψ, which can also be viewed as a polyform via the Clifford map:

$$\Psi = 8\chi_1 \otimes \chi_2^\dagger = \Psi_+ + i\,\Psi_- \ , \tag{4.64}$$

where Ψ_+ and Ψ_- are respectively the real-even and imaginary-odd parts of Ψ. We should be careful however about how the identification is imposed: odd dimensional Fierzing does not provide an isomorphism between bispinors and polyforms because the Clifford representation is not faithful, as can be confirmed by a simple count of dimensions. We thus need to choose the range of our identification. Here we take Ψ to be self-dual as a polyform: $-i\star_7\lambda\Psi = \Psi$. This also means that the decomposition (4.64) is only valid in the polyform space and that Ψ_+, Ψ_- are not independent:

$$\Psi_+ = \star_7\lambda\Psi_- \ . \tag{4.65}$$

These choices lead to the normalization:

$$\langle \Psi_+, \Psi_- \rangle = \frac{i}{2}\langle \Psi, \bar{\Psi} \rangle = 8v_7 \ . \tag{4.66}$$

We now have all the necessary ingredients to write the supersymmetry for IIA in terms of generalized spinors:

$$\begin{aligned} \mathrm{d}_H(e^{3A-\phi}\Psi_+) &= -e^{3A}\star_7\lambda F \\ \mathrm{d}_H(e^{2A-\phi}\Psi_-) &= 0 \\ \langle \Psi_-, F \rangle &= 0 \ . \end{aligned} \tag{4.67}$$

In order to match (4.53), (4.54) we need to further split M_7 to M_6 plus a transverse direction parameterized by the coordinate r. The metric and fluxes thus decompose as follows:

$$\begin{aligned} \mathrm{d}s^2 &= e^{2Z}(e^{2a}\mathrm{d}s^2(\mathbb{R}^{1,2}) + \mathrm{d}r^2) + \mathrm{d}s^2(M_6) \\ F &= F_i + \mathrm{d}r \wedge F_r \\ H &= H_i + \mathrm{d}r \wedge H_r \ , \end{aligned} \tag{4.68}$$

where a depends only on r, and F_i, F_r, H_i, H_r only have legs on M_6. Note also that the expression $\mathrm{d}s^2(M_6)$ can depend on r, since it can include a warp factor for instance. The same split must then be performed for the spinors, by expressing 7D spinors in terms of 6D chiral spinors. Since we are splitting along r, γ_r (in flat basis) becomes the chirality matrix for spinors of M_6. Thus we take:

$$\begin{array}{rclcrcl} \eta_1 & := & \sqrt{2}\,P_+\chi_1 & , & \eta_2 & := & \sqrt{2}\,P_-\chi_2 \\ \chi_1 & = & \frac{1}{\sqrt{2}}(\eta_1+\eta_1^c) & , & \chi_2 & = & \frac{1}{\sqrt{2}}(\eta_2+\eta_2^c)\ , \end{array} \tag{4.69}$$

where $P_\pm := \frac{1}{2}(1\pm\gamma_r)$. Introducing the following bispinors on M_6 (which can be viewed equivalently, via 6D Fierzing and the Clifford map, as polyforms or generalized spinors):

$$\Phi_1 := 8e^{3Z-\phi}\eta_1\otimes\eta_2^\dagger \quad , \quad \Phi_2 := 8e^{3Z-\phi}\eta_1\otimes\tilde{\eta}_2\ , \tag{4.70}$$

we get:

$$\Psi_+ = e^{-3Z+\phi}(\Re\Phi_2 + e^Z\mathrm{d}r\wedge\Re\Phi_1) \quad , \quad \Psi_- = e^{-3Z+\phi}(\Im\Phi_1 + e^Z\mathrm{d}r\wedge\Im\Phi_2)\ . \tag{4.71}$$

The factor $e^{3Z-\phi}$ is introduced here for future convenience; it is simply another choice of normalization:

$$i\langle\Phi_1,\bar{\Phi}_1\rangle = i\langle\Phi_2,\bar{\Phi}_2\rangle = 8e^{6Z-2\phi}v_6\ . \tag{4.72}$$

We then substitute (4.68) and (4.71) into (4.67), and decompose along $\mathrm{d}r$. We look for an expression solely in terms of polyforms on the internal space M_6, where r is now considered as an external parameter:

$$\begin{array}{rcl} \mathrm{d}^H e^Z\Re\Phi_1 & = & e^{4Z}\star\lambda F_i + e^{-3a}\partial_r^H e^{3a}\Re\Phi_2 \\ \mathrm{d}^H\Re\Phi_2 & = & -e^{2Z}\star\lambda F_r \\ \mathrm{d}^H e^{-Z}\Im\Phi_1 & = & 0 \\ \mathrm{d}^H\Im\Phi_2 & = & e^{-2a}\partial_r^H e^{2a-Z}\Im\Phi_1 \\ \langle\Im\Phi_1,F_r\rangle + e^Z\langle\Im\Phi_2,F_i\rangle & = & 0\ , \end{array} \tag{4.73}$$

where now $\mathrm{d}^H = \mathrm{d} + H_i\wedge$, $\partial_r^H = \partial_r + H_r\wedge$, and d acts only on the coordinates of M_6. Note also that $\langle,\rangle$ now refers to the 6D Mukai pairing (1.44).

4.3.2 Supersymmetric D2

We now want to check explicitly that the solution in (4.53), (4.54) is compatible with the equations (4.73). This amounts to defining two polyforms Φ_1, Φ_2, whose $SU(3)\times SU(3)$-structure carries the 6D part of the metric (4.53), and which is solution of (4.73). First we need to identify the various fields. Comparing (4.53) and (4.68) we find,

$$e^a = \frac{1}{\sqrt{H}} \quad \text{and} \quad e^Z = e^{\phi/3}H^{1/6}\ . \tag{4.74}$$

Since the fluxes are not given in the same formalism, we need to retrieve F_6 and F_8 from F_2 and F_4 by Hodge duality, in order to build the total flux polyform F_t in the democratic formalism. If we write $F_{nd} = F_2 + F_4$, the total flux of the solution (4.54) we find,

$$F_t = F_{nd} + \star_{10}\lambda F_{nd}\ .$$

This leads to,

$$\begin{aligned} F_i &= F_2 + \star_{10}\lambda F_4 = \mathrm{d}f(\mathrm{d}\psi + A) + \tfrac{H'}{\sqrt{H}}e^{-4Z}v_6 \\ F_r &= 0 \\ H_i &= 0 \\ H_r &= 0 \ , \end{aligned} \tag{4.75}$$

where v_6 is the volume form of M_6 taking into account the warp factor. We can now use the results from section 1.5 to define our polyforms Φ_1 and Φ_2. Our ansatz will introduce several functions of θ as supplementary degrees of freedom that should enable us to find a solution of the DW equations.

We begin with the local $SU(2)$-structure, given by the Kähler structure of B_4. We denote by $\hat{j}$ the Kähler form and $\hat{\omega}$ a holomorphic 2-form normalized so that,

$$\begin{aligned} \hat{j} \wedge \hat{\omega} = \hat{\omega} \wedge \hat{\omega} = 0 \\ \hat{\omega} \wedge \hat{\omega}^* = 2\hat{j} \wedge \hat{j} \ . \end{aligned} \tag{4.76}$$

Note that $\hat{j}$ is global but $\hat{\omega}$ can only be defined locally. Furthermore we define,

$$\begin{aligned} \tilde{\omega} &= e^{2i(\psi+\zeta)}\hat{\omega} \\ j &= \frac{1}{4}r^2 e^{2Z}\left(\cos\theta\hat{j} + \sin\theta\Re\tilde{\omega}\right) \\ \omega &= \frac{1}{4}r^2 e^{2Z+2i\alpha}\left(\cos\theta\Re\tilde{\omega} - \sin\theta\hat{j} + i\Im\tilde{\omega}\right) \\ K &= \frac{1}{2}re^{Z+i\beta}\left(\frac{1}{1+\cos^2\theta}\mathrm{d}\theta + i\sin\theta(\mathrm{d}\psi + A)\right) \ . \end{aligned} \tag{4.77}$$

Finally the polyforms (or, equivalently, the generalized spinors) defining the $SU(3)\times SU(3)$-structure are given by,

$$\begin{aligned} \Phi_1 &= \sqrt{H}\,\bar{K} \wedge \left(e^{i\nu}\cos\varphi\,\bar{\omega} - \sin\varphi\, e^{i\,j}\right) \\ \Phi_2 &= \sqrt{H}\,e^{-\frac{1}{2}K\wedge\bar{K}}\,\left(e^{-i\nu}\cos\varphi\, e^{i\,j} + \sin\varphi\,\bar{\omega}\right) \ , \end{aligned} \tag{4.78}$$

where the factor $\sqrt{H}$ has been added to match the normalization (4.72),

$$i\langle\Phi_1, \bar{\Phi}_1\rangle = i\langle\Phi_2, \bar{\Phi}_2\rangle = 8e^{6Z-2\phi}v_6 = 8H\,v_6 \ . \tag{4.79}$$

We are thus left with five undetermined functions of θ ($\alpha, \beta, \zeta, \nu$ and φ) that should provide enough freedom for a solution of the DW supersymmetry equations: α and ζ act as rotation of the local $SU(2)$ and, since the $SU(2)$-structures span a two-sphere, they can be respectively seen as the intrinsic rotation and precession; β is merely a modification of the phase of the vielbein one-form K; the meaning of ν and φ is explained in section 1.5, recall in particular that φ must vanish at $\theta = 0, \pi$. Note also that a global phase of Φ_2 can be absorbed in ν and α whereas a global phase of Φ_1 can be absorbed in β.

Solution

Note first that in the near horizon limit, the $SU(3)\times SU(3)$ is in fact pure $SU(3)$, i.e. $\varphi = 0$. Thus if our ansatz is correct (φ is function of θ only), φ should remain constant to match the near horizon limit. Looking at the first equation of (4.73), the scalar term gives straightforwardly,

$$\cos\varphi\cos\nu = 1 \ .$$

This is consistent with the ansatz, and also gives information about ν. We get

$$\varphi = 0 \quad , \quad \nu = 0 \; .$$

The structure is then pure $SU(3)$ all along the r coordinate. Moreover α and β now play the same role: a global phase shift of the holomorphic 3-form. β can thus be absorbed by a redefinition of α, and be set to 0. At this point, the second and fifth equation of (4.73) are satisfied. Moreover the three-form part of the fourth equation of (4.73) leads to $2\alpha = -\frac{\pi}{2}$. All the remaining terms are then proportional to ζ' so that ζ has to be constant. Taking $\zeta = 0$ then solves (4.73).

4.3.3 Mass Deformation

The background (4.54) is a solution to the type IIA supergravity equation with vanishing Romans mass $m = F_0 = 0$. Its near-horizon limit (4.55) has already been studied as an example of AdS compactification and it was shown in [91, 92] that it admits a massive deformation. That is there exists a family of type IIA backgrounds, parametrized by the Romans mass, whose limit $m \to 0$ is precisely the background (4.55). Let us now investigate if this massive deformation can be extended to the full interpolating brane solution. To do so we deform the $SU(3) \times SU(3)$-structure ansatz presented in the previous section and try to solve (4.73) with the deformed ansatz, in presence of non vanishing F_0.

The only difference between the full interpolating brane solution and its AdS_4 near horizon limit is a modification of the function $H(r)$. The background (4.53), (4.54) is a genuine solution under the sole condition that $H(r)$ is harmonic in the transverse space $\mathbb{R}^7$. The interpolating solution corresponds to the most general choice of $H(r)$, whereas the near horizon limit and the spatial infinity limit correspond respectively to the choices $H(r) = Q/r^6$ and $H(r) = 1$. We would then expect that finding a massive deformation of the interpolating solution would amount to adding the correct r-dependence in the different functions of the massive deformation.

Consequences of the massive deformation: For the massive deformation of the near horizon limit, the $SU(3) \times SU(3)$ structure is no longer pure $SU(3)$ [91, 92], and this will obviously be the case for the interpolating solution. It will now be necessary to switch on the function ν, φ and ζ, leading to our first source of complication. It is also important to notice that the base and fiber of M_6 get a different warp factor, and our ansatz should take that into account. We also expect the massive deformation to switch on all fluxes. Switching on the three-form H also impacts equation (4.73) by twisting the derivative.

Taking all the above into account, let us define the following ansatz for the $SU(3) \times SU(3)$ structure and the fluxes. We first define the local $SU(2)$ and one-form similarly to (4.77),

$$\begin{aligned} \tilde{\omega} &= e^{2i(\psi+\zeta)}\hat{\omega} \\ j &= e^{2B}\left(\cos\gamma\hat{j} + \sin\gamma\Re\tilde{\omega}\right) \\ \omega &= e^{2B+2i\alpha}\left(\cos\gamma\Re\tilde{\omega} - \sin\gamma\hat{j} + i\Im\tilde{\omega}\right) \\ K &= e^{C+i\beta}\left(f(\theta)\mathrm{d}\theta + i\sin\theta(\mathrm{d}\psi + A)\right) \; . \end{aligned} \tag{4.80}$$

The $SU(3) \times SU(3)$ structure is now given by,

$$\begin{aligned} \Phi_1 &= e^{3Z-\phi}\,\bar{K} \wedge \left(e^{i\nu}\cos\varphi\,\bar{\omega} - \sin\varphi\, e^{i\,j}\right) \\ \Phi_2 &= e^{3Z-\phi}\, e^{-\frac{1}{2}K\wedge\bar{K}}\,\left(e^{-i\nu}\cos\varphi\, e^{i\,j} + \sin\varphi\,\bar{\omega}\right) \; . \end{aligned} \tag{4.81}$$

This corresponds to the ansatz of (4.68). The internal metric of M_6 follows from the $SU(3)\times SU(3)$-structure,

$$\mathrm{d}s^2(M_6) = e^{2B}\mathrm{d}s^2(B_4) + e^{2C}\left(f(\theta)^2\mathrm{d}\theta^2 + \sin^2\theta(\mathrm{d}\psi + A)^2\right) \ . \tag{4.82}$$

For The Bianchi Identities to be automatically satisfied we will rather define the fluxes from their potentials: the two-form B and the odd polyform C,

$$\begin{aligned} B &= h\hat{j} \\ C_1 &= f_2 A \\ C_3 &= f_4 A\wedge\hat{j} \\ C_5 &= f_6 A\wedge\hat{j}^2 \ . \end{aligned} \tag{4.83}$$

The fluxes are then, in the formulation of (4.68),

$$\begin{aligned} H_i &= \mathrm{d}B \\ H_r &= \partial_r B \\ F_i &= \mathrm{d}^H C + me^{-B} \\ F_r &= \partial_r^H B \ . \end{aligned} \tag{4.84}$$

All the warp factors Z, B, C, the dilaton ϕ, the phases $\alpha,\beta,\gamma,\zeta,\varphi,\nu$ and the fluxes h, f_2, f_4, f_6 are allowed to depend on r,θ.

Despite its generality, we have checked that this ansatz does not solve the supersymmetry equations (except for the solutions we already know, which are special cases thereof) and thus does not qualify for a massive deformation. Unfortunately, without further input, relaxing the ansatz to allow for a dependence on more variables quickly becomes intractable.

4.3.4 The NATD

The NATD of the supersymmetric D2 brane (4.53) is obtained by an SU(2) action on the σ_i, cf. (4.52). The NS-NS sector reads,

$$\begin{aligned} \hat{ds}^2 &= \frac{r}{l_p}\sqrt{w(\theta)}H^{-1/2}ds^2(\mathbb{R}^{1,2}) + \Lambda^2\Big(\frac{4}{r^2}dr^2 + 3d\mu^2 + \frac{1}{(1+\cos^2\theta)^2}d\theta^2 \\ &\quad + \frac{4}{Q}\sin^2\theta\cos^2\mu d\psi^2\Big) + \frac{3\alpha'^2\Xi}{4M}[d(\rho\sin\chi)]^2 \\ &\quad + \frac{81}{4096\alpha'\Delta}\left[\frac{\Xi^2\rho^2\sin^2\chi}{Q}(d\xi\psi)^2 + \frac{1}{M}\left(\alpha'^2\rho^2\cos\chi d\rho + 4\Xi^2 d(\rho\cos\chi)\right)^2\right] \\ \hat{B}_2 &= \frac{81\rho^2\Xi\sin\chi}{8192Q\Delta}d\xi\psi\wedge d\rho\chi + \frac{3\alpha'\sin^2\theta}{2Q}d(\rho\cos\chi)\wedge d\psi \\ e^{-2\hat{\phi}} &= e^{-2\phi}\Delta, \quad \Delta = \frac{27\Xi}{1024\alpha'^3}(4\Xi^2 Q + \alpha'^2\rho^2 K), \quad \Xi = \sin^2\mu\Lambda^2, \quad \Lambda = \frac{1}{2}e^{\phi/3}rH^{1/6} \ , \end{aligned} \tag{4.85}$$

where we have defined the following one-forms,

$$\begin{aligned} d\xi\psi &= (Qd\xi - 4\sin^2\theta d\psi) \\ d\rho\chi &= \left(\rho K d\chi + \cos\chi\sin\chi(Q-4)d\rho\right) \\ d\theta\mu &= (f'(\theta)\sin\mu d\theta + 2\cos\mu f(\theta)d\mu) \ , \end{aligned} \tag{4.86}$$

and included the following definitions,

$$\begin{aligned} Q &= 4\cos^2\mu + 3\sin^2\mu\sin^2\theta \\ K &= Q\cos^2\chi + 4\sin^2\chi \\ M &= \alpha'^2\rho^2\cos^2\chi + 4\Xi^2 \ . \end{aligned} \tag{4.87}$$

The RR sector is given by

$$\begin{aligned} \hat{F}_1 &= \frac{9l_p}{32\sqrt{\alpha'}}\sin\mu\Big[f(\theta)\sin\mu d(\rho\cos\chi) - \rho\cos\chi d\theta\mu\Big] \\ \hat{F}_3 &= -\Big(\frac{9l_p\sqrt{\alpha'}\rho\cos^2\mu f'(\theta)}{16Q}d\rho + \frac{9l_p^2\Lambda^6 H'\cos\mu\sin^3\mu\sin\theta}{4r^2\alpha'^{3/2}H^{3/2}w(\theta)a(\theta)}d\mu\Big)\wedge d\theta\wedge d\psi \\ &+ \frac{729l_p\rho^3\sin^3\mu\Lambda^2}{262144\sqrt{\alpha'}Q\Delta}\Big[-\cos\chi\sin\chi\sin\mu Q d\rho\wedge d\chi\wedge d\xi\psi \\ &+ 2\cos\mu f(\theta)Q(\sin^2\chi d\xi - \cos^2\chi\sin^2\mu d\psi)\wedge d\mu \\ &- \sin\mu\sin^2\chi f'(\theta)d\theta\wedge d\xi\psi\Big]\wedge d\rho \\ &+ \frac{729l_p\rho\sin^7\mu\Lambda^6}{65536\alpha'^{5/2}\Delta}[\sin\chi d\theta\mu\wedge d(\rho\sin\chi)\wedge d\xi\psi \\ &- 8\cos\mu\sin^2\theta f(\theta)d\mu\wedge d\rho\wedge d\psi] \\ \hat{F}_5 &= \frac{9l_p^2\sqrt{\alpha'}H'\rho}{64r^2H^{3/2}w(\theta)}v_4\wedge d\rho \\ &+ \frac{9l_p\Lambda^2\sin^3\mu}{16\alpha'^{3/2}\sin\theta a(\theta)}v_4\wedge\Big(2f(\theta)\sin^2\theta\sin\mu d\theta + a(\theta)^2\cos\mu f'(\theta)d\mu\Big) \\ &+ \frac{729l_p\Lambda^6\rho^2\cos\mu\sin^5\mu\sin\chi}{65536\alpha'^{3/2}r^2H^{3/2}w(\theta)a(\theta)\Delta}\Big[2l_p\Lambda^2H'\sin\theta d\theta\wedge d\mu\wedge d\rho\chi \\ &- r^2H^{3/2}w(\theta)a(\theta)\sin\mu(3f(\theta)\sin^2\theta\sin\mu d\mu \\ &- 2\cos\mu f'(\theta)d\theta)\wedge d\rho\wedge d\chi\Big]\wedge d\xi\wedge d\psi \ , \end{aligned} \tag{4.88}$$

where $a(\theta) = 2(1+\cos^2\theta)$ and $v_4 = -\frac{r^2w(\theta)}{l_p^2\sqrt{H}}dr\wedge dx_0\wedge dx_1\wedge dx_2$.

Brane configuration and charges

The D-brane background before the NATD was a D2 brane solution, therefore we expect to see the presence of D3, D5, and NS5 branes from the general lore Dp→D(p+1)-D(p+3)-NS5. We will follow the same strategy as in section 4.1.2 to better understand the brane configuration.

We first compute the NS5 charge. In the same spirit as in the D3 brane example, cf. (4.12), we integrate H_3 along the cycle $(\Sigma_3[\rho,\chi,\xi],\mu=0)$, on which H_3 simplifies as,

$$H_3 = \frac{3}{8}\alpha'\sin\chi d\xi\wedge d\chi\wedge d\rho \ . \tag{4.89}$$

We get,

$$Q_{NS5} = \frac{1}{2\kappa_{10}^2 T_{NS5}}\int H_3 = \frac{3}{8\pi}\rho_0 \ , \tag{4.90}$$

where we cut off the integration at $\rho = \rho_0$. For the charge to be correctly quantized, we need $\rho_0 = \frac{8n\pi}{3}$. This is compatible with the condition of boundedness of b_0 given in (4.39). Modulo a large gauge transformation on B_2, this condition is satisfied if the range of ρ is taken to be $[\frac{8(n-1)\pi}{3}, \frac{8n\pi}{3}]$. Once again we can see that there is, at least from the supergravity point of view, a continuous distribution of charge at the singularity created by NATD (here $\mu = 0$). This distribution is smeared along the ρ direction and is constant in ρ. As was the case in section 4.1.2, this can be seen directly in the metric by zooming in at the singularity. Close to $\mu = 0$ and after making the substitution $\nu = \mu^2$, the metric becomes,

$$\begin{aligned} ds^2_{\mu\to 0} = &\frac{r}{2l_p\sqrt{H(r)}\sqrt{a(\theta)}}\Big[ds^2(\mathbb{R}^{1,2}) + H(r)(dr^2 + r^2(\frac{1}{a(\theta)^2}d\theta^2 + \frac{1}{4}\sin^2\theta d\psi^2))\Big] \\ &+ \frac{1}{\nu}\Big[\frac{3}{32l_p r^3\sqrt{H(r)}\sqrt{a(\theta)}}\Big(16l_p^2\alpha'^2 a(\theta)^2 d\rho^2 + r^6 H(r)\Big[d\nu^2 + \nu^2(d\chi^2 \\ &+ \sin^2\chi d\xi(d\xi - 2\sin^2\theta d\psi))\Big]\Big)\Big] \ , \end{aligned} \tag{4.91}$$

where ν^{-1} is the harmonic function in the transverse space for NS5 branes along the $(\mathbb{R}^{1,2}, r, \theta, \psi)$ directions.

For the D-branes we need to consider the Page forms, given by,

$$\begin{aligned} \tilde{F}_3 = &\frac{9l_p\sqrt{\alpha'}\rho}{256}[4f'(\theta)d\theta\wedge d\rho\wedge d\psi - 3\sin\chi\sin\mu d\theta\mu\wedge d(\rho\sin\chi)\wedge d\xi] \\ &- \frac{27\hat{Q}\sqrt{w(\theta)}}{128l_p\alpha'^{3/2}a(\theta)}\cos\mu\sin^3\mu\sin\theta d\theta\wedge d\mu\wedge d\psi \\ \tilde{F}_5 = &-\frac{27\hat{Q}l_p^2\sqrt{\alpha'}}{32r^9H^{3/2}w(\theta)}\rho d\rho\wedge v_4 + \frac{9r^3\sqrt{H}\sqrt{w(\theta)}}{64\alpha'^{3/2}a(\theta)\sin\theta}\sin^3\mu(2f(\theta)\sin^2\theta\sin\mu d\theta \\ &+ a(\theta)^2\cos\mu f'(\theta)d\mu)\wedge v_4 \\ &+ \frac{27l_p\alpha'^{3/2}}{512}\rho^2\sin\chi f'(\theta)d\theta\wedge d\rho\wedge d\chi\wedge d\xi\wedge d\psi \ , \end{aligned} \tag{4.92}$$

where $\tilde{F}_1 = \hat{F}_1$ given in (4.88) is unchanged.

We can readily see that the Page forms have two contributions: one coming from the geometric flux F_2 and the other one from F_4.[4] Since we want to trace the fate of the M2 branes we will consider the first part as geometric fluxes and focus on the second. The relevant components are thus those proportional to $\hat{Q}$ (i.e. those that vanish when there is no M2).

Ignoring the geometric fluxes, the only non-vanishing Page charge is Q_{D5}, which can be found by integrating $\tilde{F}_3$. Namely, we keep only the (θ, μ, ψ) term,

$$Q_{D5} = \frac{1}{2\kappa_{10}^2 T_{D5}}\frac{27\hat{Q}}{128l_p\alpha'^{3/2}}\int_0^{4\pi}d\psi\int_0^{\pi}\frac{\sin\theta\sqrt{w(\theta)}}{a(\theta)}d\theta\int_0^{\frac{\pi}{2}}\cos\mu\sin^3\mu d\mu = N_{D5} \ . \tag{4.93}$$

The quantization condition of Q_{D5} then leads to a relation between the constant in the harmonic function and the number of D5 branes,

$$\hat{Q} = \frac{2048}{27}\pi l_p\alpha'^{5/2}N_{D5} \ . \tag{4.94}$$

[4]Recall that NATD acts linearly on the RR fluxes so we can isolate each contribution.

However as it was pointed out in the D3 example, the page charges depend on the choice of B_2, and may change under a large gauge transformation. Here under a large gauge transformation given by,

$$\Delta B_2 = -n\pi\alpha' \sin\chi \mathrm{d}\chi \wedge \mathrm{d}\xi \ , \tag{4.95}$$

Q_{D3} receives a new contribution,

$$\begin{aligned}\Delta Q_{D3} &= \int -\Delta B_2 \wedge F_3 \\ &= \frac{1}{2T_{D3}\kappa^2}\frac{27n\pi\hat{Q}}{128 l_p\sqrt{\alpha'}}\int_0^{\frac{\pi}{2}} \cos\mu\sin^3\mu \int_0^{\pi} \frac{\sin\theta\sqrt{w(\theta)}}{a(\theta)}\int_0^{4\pi} d\psi d\theta \int_0^{\pi}\sin\chi d\chi \int_0^{2\pi} d\xi \ .\end{aligned} \tag{4.96}$$

Evaluating this and comparing to (4.94), we then find,

$$\Delta Q_{D3} = nN_{D5}. \tag{4.97}$$

This is analogous to the relation found in (4.19) above.

Assuming that the r coordinate still describes the radius of the cycles wrapped by the RR-fluxes in the transverse space of the D-branes, the brane configuration is given by the table below.

	0	1	2	r	μ	θ	ψ	ρ	χ	ξ
$NS5$	×	×	×	×		×	×			
$D5$	×	×	×					×	×	×
$D3$	×	×	×					×		

As we will find in additional examples throughout this paper, this relationship seems to be universal for D-brane backgrounds generated by SU(2) non-Abelian T-duality. When a Dp-brane background is transformed, the D(p+3) brane charges are easily found from integrating the appropriate term in the Page form. The D(p+1) brane charges then are found from restricting B_2 to a cycle containing (χ, ξ) cycle, performing a large gauge transformation, computing the change in the Page form under this transformation, and finally integrating to obtain $\Delta Q_{D(p+1)}$. This results in the general relation $\Delta Q_{D(p+1)} = nQ_{D(p+3)}$. The new D2 brane examples presented in this paper are not only distinct from the original D3 brane example where this relation was proposed, but also highly nontrivial.

The spatial infinity limit

The spatial infinity limit of the supersymmetric D2 brane NATD solution (4.85)-(4.88) is given by,

$$\begin{aligned}\tilde{ds}^2 &= \frac{r}{l_p}\sqrt{w(\theta)}\Big[(\mathbb{R}^{1,2}) + dr^2 + \frac{r^2}{4}(3d\mu^2 + \frac{1}{(1+\cos^2\theta)^2}d\theta^2 + \frac{4}{Q}\sin^2\theta\cos^2\mu d\psi^2) \\ &+ \frac{3l_p\alpha'^2\sqrt{w(\theta)}\Xi}{4M}[d(\rho\sin\chi)]^2\Big] + \frac{81}{16384 l_p^2\alpha'\Delta}\Big[\frac{w(\theta)\Xi^2\rho^2\sin^2\chi}{4Q}(d\xi\psi)^2 \\ &+ \frac{1}{M}\Big(4l_p^2\alpha'^2\rho^2\cos\chi d\rho + w(\theta)\Xi^2 d(\rho\cos\chi)\Big)^2\Big],\end{aligned} \tag{4.98}$$

$$\hat{B}_2 = \frac{81\sqrt{w(\theta)}\Xi\rho^2\sin\chi}{32768 l_p Q\Delta}d\xi\psi\wedge d\rho\chi + \frac{3\alpha'\sin^2\theta}{2Q}d(\rho\cos\chi)\wedge d\psi,$$

$$e^{-2\hat{\phi}} = e^{-2\phi}\Delta, \quad \Delta = \frac{27\sqrt{w(\theta)}\Xi}{16384 l_p^3\alpha'^3}(4l_p^2\alpha'^2\rho^2 K + w(\theta)Q\Xi^2), \quad \Xi = r^3\sin^2\mu,$$

where we have defined the following one-forms,

$$
\begin{aligned}
d\xi\psi &= (Qd\xi - 4\sin^2\theta d\psi), \\
d\rho\chi &= \Big(\rho K d\chi + \cos\chi\sin\chi(Q-4)d\rho\Big), \\
d\theta\mu &= (f'(\theta)\sin\mu d\theta - 2\cos\mu f(\theta)d\mu),
\end{aligned}
\tag{4.99}
$$

with Q, K defined in (4.87) and $M = 4l_p^2\alpha'^2\rho^2\cos^2\chi + w(\theta)\Xi^2$. The RR sector is given by

$$
\begin{aligned}
\hat{F}_1 &= \frac{9l_p}{32r^3\sqrt{\alpha'}}\Big[f(\theta)\Xi d(\rho\cos\chi) - \rho\cos\chi d\theta\mu\Big], \\
\hat{F}_3 &= \frac{9l_p\sqrt{\alpha'}}{16Q}\rho\cos^2\mu f'(\theta)d\theta\wedge d\rho\wedge d\psi \\
&+ \frac{729r^3\sqrt{w(\theta)}\rho^3\sin^3\mu}{1048576\sqrt{\alpha'}Q\Delta}\Big[-\cos\chi\sin\chi\sin\mu Q d\rho\wedge d\chi\wedge d\xi\psi \\
&+ 2\cos\mu f(\theta)Q(\sin^2\chi d\xi - \cos^2\chi\sin^2\mu d\psi)\wedge d\mu \\
&- \sin\mu\sin^2\chi f'(\theta)d\theta\wedge d\xi\psi\Big]\wedge d\rho \\
&+ \frac{729r^9 w(\theta)^{3/2}\rho\sin^7\mu}{4194304 l_p^2\alpha'^{5/2}\Delta}[\sin\chi d\theta\mu\wedge d(\rho\sin\chi)\wedge d\xi\psi \\
&- 8\cos\mu\sin^2\theta f(\theta)d\mu\wedge d\rho\wedge d\psi] \\
\hat{F}_5 &= \frac{9r^3\sqrt{w(\theta)}\sin^3\mu}{64\alpha'^{3/2}\sin\theta a(\theta)}v_4\wedge\Big(2f(\theta)\sin^2\theta\sin\mu d\theta + a(\theta)^2\cos\mu f'(\theta)d\mu\Big) \\
&\frac{729r^9 w(\theta)^{3/2}\rho^2\cos\mu\sin^6\mu\sin\chi}{4194304 l_p^2\alpha'^{3/2}\Delta}(3f(\theta)\sin^2\theta\sin\mu d\mu \\
&+ 2\cos\mu f'(\theta)d\theta)\wedge d\rho\wedge d\chi\wedge d\xi\wedge d\psi\ .
\end{aligned}
\tag{4.100}
$$

As was already the case for the D2 solution, the spatial infinity is neither flat nor empty. In a probe interpretation, this configuration could be interpreted as the space in which the branes are inserted. As can be seen from (4.98), it is a foliation over the "radial" coordinate r with leaves of the form of a warped product $\mathbb{R}^{1,2}\times\tilde{M}_6$. At fixed r, the space $\tilde{M}_6$ can be thought of as a fibration of the space $\tilde{N}_3$ parameterized by (ρ,χ,ξ) fibered over the base $\tilde{M}_3$ parameterized by (μ,θ,ψ). The topology of $\tilde{M}_3$ can be deduced from the line element,

$$
\mathrm{d}s^2(\tilde{M}_3) := 3\mathrm{d}\mu^2 + \frac{1}{(1+\cos^2\theta)^2}\mathrm{d}\theta^2 + \frac{4\cos^2\mu}{Q}\sin^2\theta\mathrm{d}\psi^2\ , \tag{4.101}
$$

and it is that of an S^2 parameterized by (θ,ψ), fibered over the interval parameterized by μ. Indeed at fixed μ, $\mathrm{d}s^2(\tilde{M}_3)$ is of the form $g(\theta)\mathrm{d}\theta^2 + h(\theta)\mathrm{d}\psi^2$, for some positive functions g, h of θ. This has the topology of a circle parameterized by ψ, fibered over the interval parameterized by θ. Moreover at the endpoints of the θ-interval, Q is equal to $4\cos^2\mu$ and the metric becomes $\frac{1}{4}\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\psi^2$ in the vicinity of $\theta = 0,\pi$. This is smooth given that the period of ψ is equal to π. In other words the ψ-circle degenerates to a point at the endpoints of the θ-interval so that the total space remains smooth. We thus obtain the topology of an S^2, as advertised.

The range of the coordinate ρ was constrained by flux quantization to be the interval specified in section 4.3.4. Moreover over a fixed base point $(\mu,\theta,\psi)\in\tilde{M}_3$, the coordinates

(χ, ξ) parameterize a smooth S^2 provided we take $\xi \in [0, 2\pi]$, $\chi \in [0, \pi]$. This can already be seen from the geometry near the location of the NS5 branes, cf. (4.91). More generally the geometry of the $\tilde{N}_3$ fiber over a fixed point in $\tilde{M}_3$ is rather complicated, as can be seen from (4.98). Topologically it is an S^2 parameterized by (χ, ξ) fibered over the interval parameterized by ρ. Indeed at constant ρ the line element of $\tilde{N}_3$ is proportional to,

$$\frac{3l_p\alpha'^2\sqrt{w(\theta)}\Xi}{4M}\left(\cos^2\chi + \frac{27\sqrt{w(\theta)}\Xi}{2^{12}l_p^3\alpha'^3\Delta}\sin^2\chi\right)\mathrm{d}\chi^2 + \frac{81w(\theta)\Xi^2\sin^2\chi}{2^{16}l_p^2\alpha'\Delta Q}d\xi^2 \ , \tag{4.102}$$

which is a circle parameterized by ξ fibered over the interval parameterized by χ. Moreover it can be seen that near the endpoints of the interval $\chi = 0, \pi$ the line element above reduces to,

$$\frac{3l_p\alpha'^2\sqrt{w(\theta)}\Xi}{4(\alpha'^2\rho^2\cos^2\chi + 4\Xi^2)}(\mathrm{d}\chi^2 + \sin^2\chi\mathrm{d}\xi^2) \ , \tag{4.103}$$

so that the S^2 parameterized by (ξ, χ) is smooth for the ranges given above.

We have thus been able to specify the ranges of all coordinates parameterizing the NATD space. Once this result has been established for the leaf of the r-foliation at spatial infinity, it remains valid for finite r and applies also to the full interpolating solution (4.85). In particular the smoothness of the S^2 parameterized by (θ, ψ) is shown by the same argument following (4.101). The smoothness of the S^2 parameterized by (ξ, χ) also follows as above, upon modifying (4.102), (4.103) to account for the interpolating metric (4.85).

The near-horizon limit is obtained by substituting $H \to \frac{\hat{Q}}{r^6}$ in (4.85). As is clear from the previous analysis, the general structure of the leaves of the r-foliation described above remains unchanged. Moreover the $\mathbb{R}^{1,2}$ space combines with the radial coordinate to form an AdS$_4$ factor exactly as before the NATD.

4.4 Non-supersymmetric D2 from reduction on $Y^{p,q}$

We will now reduce along the "obvious" Sasaki-Einstein S^1 cycle, thereby completely breaking supersymmetry. Let us rewrite the $Y^{p,q}(B_4)$ metric (4.43) as follows,

$$\mathrm{d}s^2(Y_7) = \frac{1}{4}\left(\mathrm{d}s^2(\tilde{M}_6) + (\mathrm{d}\psi + \tilde{A})^2\right) \ , \tag{4.104}$$

where the base $\tilde{M}_6$ is topologically an $\mathbb{CP}^2 \times S^2$ with metric given by,

$$\mathrm{d}s^2(\tilde{M}_6) = \mathrm{d}s^2(B_4) + \frac{1}{(1+\cos^2\theta)^2}\mathrm{d}\theta^2 + 4w(\theta)\sin^2\theta\,\mathrm{d}\alpha^2 \ , \tag{4.105}$$

and we have defined,

$$\tilde{A} := A + 2\sqrt{w(\theta)}\cos\theta\mathrm{d}\alpha \ . \tag{4.106}$$

Note that, as follows from (4.56), for the S^2 parameterized by (θ, α) to be smooth α must have period $2\pi/(g_s\, e^{2A_0})$. Alternatively we may redefine $\alpha \to g_s\, e^{2A_0}\alpha$, so that $\alpha \in [0, 2\pi]$. In terms of the redefined coordinate,

$$\mathrm{d}s^2(\tilde{M}_6) = \mathrm{d}s^2(B_4) + \frac{1}{(1+\cos^2\theta)^2}\mathrm{d}\theta^2 + \frac{\sin^2\theta}{2(1+\cos^2\theta)}\mathrm{d}\alpha^2 \ . \tag{4.107}$$

The corresponding eleven-dimensional solution reads,

$$
\begin{aligned}
ds^2 &= H^{-2/3}ds^2(\mathbb{R}^{1,2}) + H^{1/3}\left(\mathrm{d}r^2 + \frac{1}{4}r^2\mathrm{d}s^2(\tilde{M}_6)\right) + \frac{1}{4}r^2H^{1/3}(\mathrm{d}\psi + \tilde{A})^2 \\
G &= -\mathrm{d}H^{-1}\wedge \mathrm{vol}_3 \\
H &= 1 + \frac{\hat{Q}}{r^6} \ .
\end{aligned} \tag{4.108}
$$

Reducing along the S^1 cycle parameterized by ψ results in a non-supersymmetric ten-dimensional D2-brane solution given by,

$$
\begin{aligned}
ds^2_{\mathrm{A}} &= e^{2\phi/3}\left(H^{-2/3}\mathrm{d}s^2(\mathbb{R}^{1,2}) + H^{1/3}\mathrm{d}r^2 + \frac{1}{4}H^{1/3}r^2\mathrm{d}s^2(\tilde{M}_6)\right) \\
e^{4\phi/3} &= \frac{r^2}{4l_p^2}H^{1/3} \\
F_2 &= l_p\mathrm{d}\tilde{A} \\
F_4 &= -\mathrm{d}H^{-1}\wedge \mathrm{vol}_3 \ .
\end{aligned} \tag{4.109}
$$

In the spatial infinity limit, $H = 1$, the metric reduces to,

$$
ds^2_{\mathrm{A}} = \frac{r}{2l_p}\left(\mathrm{d}s^2(\mathbb{R}^{1,2}) + \mathrm{d}r^2 + \frac{1}{4}r^2\mathrm{d}s^2(\tilde{M}_6)\right) \ , \tag{4.110}
$$

while the remaining fields reduce to,

$$
\begin{aligned}
e^{4\phi/3} &= \frac{r^2}{4l_p^2} \\
F_2 &= l_p\mathrm{d}\tilde{A} \\
F_4 &= 0 \ .
\end{aligned} \tag{4.111}
$$

Once again we see that, contrary to the D3 case, the spacetime is neither flat nor empty in the spatial infinity limit: rather it is conformal to $\mathbb{R}^{1,2}\times C(\tilde{M}_6)$, where the latter factor is the metric cone over $\tilde{M}_6$.

Upon dimensional reduction on ψ the M2 branes become D2 along $\mathbb{R}^{1,2}$, whose transverse space would be the cone over $\tilde{M}_6$. As in section 4.3 we find,

$$
\star F_4 = -\frac{3\hat{Q}}{32l_p}\sqrt{w(\theta)}v_6 \ . \tag{4.112}
$$

Since the cycle $\tilde{M}_6$ collapses in the transverse space at $r = 0$,

$$
\mathrm{d}\star F_4 = -\frac{3\hat{Q}}{32l_p}\sqrt{w(\theta)}\delta(r)\mathrm{d}r\wedge v_6 \ . \tag{4.113}
$$

We compute the quantized D2 charge and obtain a similar result to the supersymmetric case, up to a factor of 2 difference arising from the different ranges of α and ψ,

$$
Q_{D2} = \frac{27\hat{Q}}{8192\pi^5 l_p\alpha'^{5/2}}\int_0^{2\pi}d\alpha\int_0^{\pi}d\theta\frac{\sin\theta}{a(\theta)^{3/2}}\int_{S^3}d\Omega_3\int_0^{\frac{\pi}{2}}d\mu\sin^3\mu\cos\mu, \tag{4.114}
$$

leading to a relation between the constant in the harmonic function and the number of D2 branes,

$$\hat{Q} = \frac{2048}{27}\pi^2 l_p \alpha'^{5/2} N_{D2} \ . \tag{4.115}$$

Since F_2 is not related to the D2 brane charge, we will only consider it as a geometric flux. For the same reasons as in the supersymmetric case, cf. section 4.3, there are no D4/D6 branes.

4.4.1 The NATD

We can now take the NAT dual of the background (4.109). The NS-NS sector of the resulting background is given by,

$$\begin{aligned}
\hat{ds}^2 &= \frac{r}{2l_p} H^{-1/2} ds^2(\mathbb{R}^{1,2}) + \Lambda^2 \Big(\frac{4}{r^2} dr^2 + 3d\mu^2 + \frac{1}{(1+\cos^2\theta)^2} d\theta^2 \\
&\quad + 4w(\theta)\sin^2\theta \mathrm{d}\alpha^2\Big) + \frac{3\alpha'^2\Xi}{4M}[d(\rho\sin\chi)]^2 \\
&\quad + \frac{81}{256\alpha'\Delta}\Big[\rho^2\Xi^2\cos^2\mu\sin^2\chi(d\xi)^2 + \frac{1}{M}\Big(\alpha'^2\rho^2\cos\chi d\rho + \Xi^2 d(\rho\cos\chi)\Big)^2\Big] \\
\hat{B}_2 &= \frac{81\rho^2\sin\chi\Xi}{256\Delta} d\xi\wedge d\rho\chi \\
e^{-2\hat{\phi}} &= e^{-2\phi}\Delta, \quad \Delta = \frac{27\Xi}{64\alpha'^3}\big[\cos^2\mu\Xi^2 + \alpha'^2\rho^2 K\big], \quad \Xi = \sin^2\mu\Lambda^2 \ , \quad \Lambda = \frac{1}{2}e^{\phi/3} r H^{1/6} \ ,
\end{aligned} \tag{4.116}$$

where we have defined the following one-form,

$$d\rho\chi = \Big(\rho K d\chi - \cos\chi\sin\chi\sin^2\mu d\rho\Big) \ , \tag{4.117}$$

and included the following definitions,

$$\begin{aligned}
K &= \cos^2\mu\cos^2\chi + \sin^2\chi, \\
M &= \alpha'^2\rho^2\cos^2\chi + \Xi^2 \ .
\end{aligned} \tag{4.118}$$

The RR sector is given by

$$\begin{aligned}
\hat{F}_1 &= -\frac{9l_p\sqrt{\Xi}}{16\sqrt{\alpha'}\Lambda^2}\Big[\sqrt{\Xi}d(\rho\cos\chi) + 2\rho\cos\mu\cos\chi\Lambda d\mu\Big], \\
\hat{F}_3 &= \Big(\frac{9l_p\sqrt{\alpha'}\rho\sin\theta}{16a(\theta)^2\sqrt{w(\theta)}}d\rho + \frac{18l_p^2\sqrt{w(\theta)}\Lambda^3\Xi^{3/2}H'}{\alpha'^{3/2}r^2a(\theta)H^{3/2}}\cos\mu\sin\theta d\mu\Big)\wedge d\theta\wedge d\alpha \\
&\quad + \frac{729 l_p\rho\Xi^{3/2}\cos\mu\sin\chi}{4096\alpha'^{5/2}\Lambda^2\Delta}\Big(-2\cos^2\mu\Lambda\Xi^2 d\mu\wedge d(\rho\sin\chi) \\
&\quad + \alpha'^2\rho^2\Big[2\Lambda\sin\chi d\mu + \cos\mu\cos\chi\sqrt{\Xi}d\chi\Big]\wedge d\rho\Big)\wedge d\xi, \\
\hat{F}_5 &= \frac{9}{32\alpha'^{3/2}}v_4\wedge\Big(8\frac{l_p^2\alpha'^2\rho H'}{r^2H^{3/2}}d\rho + \frac{l_p\cos\mu\sin\theta\Xi^{3/2}}{a(\theta)w(\theta)\Lambda\sin\theta}d\mu\Big) \\
&\quad - \frac{729 l_p\rho^2\cos\mu\sin\chi\Xi^{5/2}}{4096r^2\alpha'^{3/2}a(\theta)^2\sqrt{w(\theta)}H^{3/2}\Delta}d\alpha\wedge d\theta\wedge d\xi\wedge \\
&\quad \Big(r^2\cos\mu\sin\theta\sqrt{\Xi}H^{3/2}d\rho\wedge d\chi + 32l_p a(\theta)w(\theta)\sin\theta H'\Lambda^3 d\mu\wedge d\rho\chi\Big) \ ,
\end{aligned} \tag{4.119}$$

where $a(\theta) = 2(1+\cos^2\theta)$ and $v_4 = -\frac{r^2}{4l_p^2\sqrt{H}} dr \wedge dx_0 \wedge dx_1 \wedge dx_2$.

Brane configuration and charges

As in the supersymmetric reduction, spacetime is singular at $\mu = 0$, which corresponds to the fixed locus of the $SU(2)$ isometry before duality. We thus first compute the NS5 charge by integrating H_3 on the cycle $(\Sigma_3[\rho,\chi,\xi], \mu = 0)$, on which H_3 simplifies to,

$$H_3 = \frac{3}{4}\alpha' \sin\chi d\xi \wedge d\chi \wedge d\rho \ , \tag{4.120}$$

so that,

$$Q_{NS5} = \frac{1}{2\kappa_{10}^2 T_{NS5}} \frac{3\alpha'}{4} \int_0^{\rho_0} d\rho \int_0^{\pi} \sin\chi d\chi \int_0^{2\pi} d\xi = \frac{3\rho_0}{4\pi} = N_{NS5} \ . \tag{4.121}$$

For the charge to be quantized we need $\rho_0 = \frac{4n\pi}{3}$. This is compatible with the condition (4.39) which leads to $\rho \in [\frac{4(n-1)\pi}{3}, \frac{4n\pi}{3}]$ and a large gauge transformation on B_2. We can now examine the metric close to $\mu = 0$, with $\nu = \mu^2$,

$$\begin{aligned} ds^2_{\mu\to 0} =& \frac{r}{2l_p\sqrt{H(r)}}\Big[ds^2(\mathbb{R}^{1,2}) + H(r)\big(dr^2 + r^2(\frac{1}{a(\theta)^2}d\theta^2 + \frac{1}{4a(\theta)}\sin^2\theta d\alpha^2)\big)\Big] \\ &+ \frac{1}{\nu}\Big[\frac{3}{32 l_p r^3\sqrt{H(r)}}\Big(16 l_p^2\alpha'^2 d\rho^2 + r^6 H(r)\Big[d\nu^2 + \nu^2\big(d\chi^2 + \sin^2\chi d\xi^2\big)\Big]\Big)\Big] \ , \end{aligned} \tag{4.122}$$

where ν^{-1} is the harmonic function for NS5 branes along the $(\mathbb{R}^{1,2}, r, \theta, \alpha)$ directions. As in the previous example, the NS5 branes are located at the singularity $\mu = 0$ and are smeared along the ρ direction.

The Page forms are given by

$$\begin{aligned} \tilde{F}_3 =& \frac{9 l_p\sqrt{\alpha'}\rho}{64 a(\theta)^2\sqrt{w(\theta)}}\big[-4\sin\theta d\alpha\wedge d\theta\wedge d\rho \\ &+ 6a(\theta)^2\sqrt{w(\theta)}\cos\mu\sin\mu\sin\chi d\mu\wedge d(\rho\sin\chi)\big] \\ &- \frac{27\hat{Q}}{256 l_p\alpha'^{3/2}a(\theta)}\cos\mu\sin^2\mu\sin\theta d\alpha\wedge d\theta\wedge d\mu \\ \tilde{F}_5 =& -\frac{27 l_p^2\sqrt{\alpha'}\hat{Q}}{2r^9 H^{3/2}}\rho d\rho\wedge v_4 + \frac{9r^3\sqrt{H}}{64\alpha'^{3/2}}\cos\mu\sin^3\mu d\mu\wedge v_4 \\ &- \frac{27 l_p\alpha'^{3/2}}{64 a(\theta)^2\sqrt{w(\theta)}}\rho^2\sin\theta\sin\chi d\alpha\wedge d\theta\wedge d\xi\wedge d\rho\wedge d\chi \ , \end{aligned} \tag{4.123}$$

with $\tilde{F}_1 = \hat{F}_1$ given in (4.119). We will focus on the components that are proportional to $\hat{Q}$, whereas the remaining term will only be considered as geometric flux. Integrating the (α,θ,μ) term in $\tilde{F}_3$ we obtain,

$$Q_{D5} = \frac{1}{2\kappa_{10}^2 T_{D5}}\frac{27\hat{Q}}{256 l_p\alpha'^{3/2}}\int_0^{2\pi} d\alpha \int_0^{\pi}\frac{\sin\theta}{a(\theta)^2}d\theta\int_0^{\frac{\pi}{2}}\cos\mu\sin^3\mu d\mu = N_{D5} \ , \tag{4.124}$$

leading to a relation between the constant in the harmonic function and the number of D5 branes,

$$\hat{Q} = \frac{4096}{27}\pi l_p\alpha'^{5/2}N_{D5} \ . \tag{4.125}$$

If we further consider the change in the Page forms under a large gauge transformation in B_2[5], it is the D3 charge which is created and we find $\Delta Q_{D3} = nN_{D5}$.

In the same spirit as before, we would then have the following brane configuration:

	0	1	2	r	μ	θ	α	ρ	χ	ξ
$NS5$	$\times$	$\times$	$\times$	$\times$		$\times$	$\times$			
$D5$	$\times$	$\times$	$\times$					$\times$	$\times$	$\times$
$D3$	$\times$	$\times$	$\times$					$\times$		

The spatial infinity limit

The NS-NS sector of the spatial infinity limit of the non-supersymmetric D2-brane NATD solution is obtained by setting $H(r) = 1$ in (4.116)-(4.119),

$$\begin{aligned}
\hat{ds}^2 &= \frac{r}{2l_p}\Big(ds^2(\mathbb{R}^{1,2}) + dr^2 + r^2(3d\mu^2 + \frac{1}{(1+\cos^2\theta)^2}d\theta^2 + 4w(\theta)\sin^2\theta d\alpha^2)\Big) \\
&+ \frac{6l_p\alpha'^2\Xi}{M}[d(\rho\sin\chi)]^2 + \frac{81}{16384 l_p^2\alpha'\Delta}\Big[\rho^2\Xi^2\cos^2\mu\sin^2\chi(d\xi)^2 \\
&+ \frac{1}{M}\Big(64l_p^2\alpha'^2\rho^2\cos\chi d\rho + \Xi^2 d(\rho\cos\chi)\Big)^2\Big] \qquad (4.126) \\
\hat{B}_2 &= \frac{81\rho^2\sin\chi\Xi}{2048 l_p\Delta}d\xi\wedge d\rho\chi \\
e^{-2\hat{\phi}} &= e^{-2\phi}\Delta, \quad \Delta = \frac{27\Xi}{32768 l_p^3\alpha'^3}\big[\cos^2\mu\Xi^2 + 64l_p^2\alpha'^2\rho^2 K\big], \quad \Xi = r^3\sin^2\mu\ ,
\end{aligned}$$

where we have defined the following one-form,

$$d\rho\chi = \Big(\rho K d\chi - \cos\chi\sin\chi\sin^2\mu d\rho\Big)\ , \tag{4.127}$$

and included the following definitions,

$$\begin{aligned}
K &= \cos^2\mu\cos^2\chi + \sin^2\chi, \\
M &= 64l_p^2\alpha'^2\rho^2\cos^2\chi + \Xi^2\ .
\end{aligned} \tag{4.128}$$

The RR sector is given by,

$$\begin{aligned}
\hat{F}_1 &= -\frac{9l_p\sqrt{\Xi}}{16r^3\sqrt{\alpha'}}\Big[\sqrt{\Xi}d(\rho\cos\chi) + 2r^{3/2}\rho\cos\mu\cos\chi d\mu\Big], \\
\hat{F}_3 &= \frac{9l_p\sqrt{\alpha'}\rho\sin\theta}{16a(\theta)^2\sqrt{w(\theta)}}d\theta\wedge d\alpha\wedge d\rho \\
&+ \frac{729\rho\Xi^{3/2}\cos\mu\sin\chi}{1048576 r^3 l_p^2\alpha'^{5/2}\Delta}\Big(r^{3/2}\cos^2\mu\Xi^2 d\mu\wedge d\xi\wedge d(\rho\sin\chi) \\
&+ 32l_p^2\alpha'^2\rho^2\Big[2r^{3/2}\sin\chi d\mu + \cos\mu\cos\chi\sqrt{\Xi}d\chi\Big]\wedge d\mu\wedge d\xi\Big), \qquad (4.129) \\
\hat{F}_5 &= \frac{9\cos\mu\sin\theta\Xi^{3/2}}{256r^{3/2}\alpha'^{3/2}a(\theta)w(\theta)\sin\theta}v_4\wedge d\mu \\
&+ \frac{729\rho^2\cos^2\mu\sin\chi\sin\theta\Xi^3}{2097152 l_p^2\alpha'^{3/2}a(\theta)^2\sqrt{w(\theta)}\Delta}d\theta\wedge d\alpha\wedge d\xi\wedge d\rho\wedge d\chi,
\end{aligned}$$

[5] The large gauge transformation has the same expression each time: see (4.17) or (4.95)

with $v_4 = -\frac{r^2}{4l_p^2} dr \wedge dx_0 \wedge dx_1 \wedge dx_2$.

The ten-dimensional spacetime is a foliation over the r-coordinate with leaves of the form of a warped product $\mathbb{R}^{1,2} \times \tilde{M}_6$. The general structure of the leaves is very similar to that of section 4.3.4, and can be analyzed in the same way: at fixed r, the space $\tilde{M}_6$ can be thought of as a fibration of the space $\tilde{N}_3$ parameterized by (ρ, χ, ξ) fibered over the base $\tilde{M}_3$ parameterized by (μ, θ, α). The topology of $\tilde{M}_3$ is that of an S^2 parameterized by (θ, α) times the interval parameterized by μ.

The range of the coordinate ρ was constrained by flux quantization to be the interval specified in section 4.4.1. Moreover, over a fixed base point $(\mu, \theta, \alpha) \in \tilde{M}_3$, the coordinates (χ, ξ) parameterize a smooth S^2 provided we take $\xi \in [0, 2\pi]$, $\chi \in [0, \pi]$. This can already be seen from the geometry near the location of the NS5 branes, cf. (4.122). More generally the geometry of the $\tilde{N}_3$ fiber over a fixed point in $\tilde{M}_3$ is a smooth S^2 parameterized by (χ, ξ) fibered over the interval parameterized by ρ.

As in the supersymmetric D2 case, we have thus been able to specify the ranges of all coordinates parameterizing the NATD space. Once this result has been established for the leaf of the r-foliation at spatial infinity, it remains valid for finite r and applies also to the full interpolating solution (4.116). The near-horizon limit is obtained by substituting $H \to \frac{\hat{Q}}{r^6}$ in (4.116), and results in an AdS$_4$ factor exactly as is the supersymmetric case.

4.5 D2 from reduction on S^7

Here we consider the reduction of the M2 brane background with S^7 internal space (4.41), to IIA along ψ_1,

$$\begin{aligned} ds_{10}^2 &= \frac{r}{2l_p}\cos\frac{\mu}{2}\Big[H(r)^{-1/2}ds^2(\mathbb{R}^{1,2}) + H(r)^{1/2}(\mathrm{d}r^2 + \frac{1}{4}r^2(\sin^2\frac{\mu}{2}\Sigma_i^2 + \cos^2\frac{\mu}{2}ds^2(\Omega_2) + d\mu^2))\Big] \\ B_2 &= 0, \quad e^{2\Phi} = \frac{r^3}{8l_p^3}\sqrt{H(r)}\cos^3\frac{\mu}{2} \\ F_2 &= -l_p d\Omega_2, \quad F_4 = -dH^{-1} \wedge \mathrm{dvol}_3 \ , \end{aligned} \tag{4.130}$$

with Ω_2 representing an S^2 with coordinates (θ_1, ϕ_1) leftover from the σ_i in (4.42). The near horizon limit of this solution and its NATD were given explicitly in [76]. It is not known however whether the reduction preserves supersymmetry.

We can see the presence of D2 branes from $\star F_4$,

$$\star F_4 = -\frac{3\hat{Q}}{64 l_p}\cos\frac{\mu}{2} v_6 \ , \tag{4.131}$$

where v_6 is the volume form of the 6-dimensional space M_6 (along $\mu, \mathrm{d}\Omega_2$ and $\mathrm{d}\Omega_3$). If we take the transverse space to be the cone over M_6, this cycle collapses at $r = 0$, where we can see the D2 brane,

$$\mathrm{d} \star F_4 = -\frac{3\hat{Q}}{64 l_p}\cos\frac{\mu}{2}\delta(r)\mathrm{d}r \wedge v_6 \ . \tag{4.132}$$

Upon quantizing the flux, we obtain

$$Q_{D2} = \frac{1}{2\kappa_{10}^2 T_{D2}}\int_{M_6} \star F_4 = -\frac{3\hat{Q}}{2048\pi^5 l_p \alpha'^{5/2}}\int_{S^2} d\Omega_2 \int_{S^3} d\Omega_3 \int_0^{\frac{\pi}{2}} d\mu \sin^3\frac{\mu}{2}\cos^3\frac{\mu}{2} \ , \tag{4.133}$$

leading to,

$$\hat{Q} = 128\pi^2 l_p \alpha'^{5/2} N_{D2} \ . \tag{4.134}$$

On the other hand, as is the case for the near-horizon limit, F_2 is sourced by a $D6$ brane along $\mathbb{R}^{(1,2)}, r, \Omega_3$ and located at $\mu = \pi$, where the 2-sphere Ω_2 collapses. As shown in [76], the metric in the vicinity of $\mu = \pi$ is singular and takes the precise form of the metric near a D6 brane source. The charge is given by:

$$Q_{D6} = \frac{1}{2\kappa_{10}^2 T_{D6}} \int F_2 = -\frac{2l_p}{\sqrt{\alpha'}} \ . \tag{4.135}$$

The brane configuration is thus the following:

	0	1	2	r	μ	θ_1	ϕ_1	θ_2	ϕ_2	ψ_2
$D2$	×	×	×							
$D6$	×	×	×	×				×	×	×

Note that the 3-sphere Ω_3, on which we will now dualize, is transverse to the D2 but parallel to the D6. We will now see how both will behave under NATD.

4.5.1 The NATD

The background resulting from the application of NATD on the Σ_i reads,

$$\begin{aligned}
\hat{ds}^2 &= \frac{r\cos\frac{\mu}{2}}{2l_p}\Big[H(r)^{-1/2}ds^2(\mathbb{R}^{1,2}) + H(r)^{1/2}\Big(dr^2 + r^2\big[d\mu^2 + \cos^2\frac{\mu}{2}ds^2(\Omega_2) \\
&+ \frac{9r^3\sqrt{H(r)}\rho^2\cos\frac{\mu}{2}\sin^4\frac{\mu}{2}}{4096 l_p^2\alpha'\Delta}ds^2(d\chi^2 + \sin^2\chi d\xi^2)\big]\Big)\Big] + \frac{9l_p\alpha'^2}{8r^3\sqrt{H(r)}\cos\frac{\mu}{2}\sin^2\frac{\mu}{2}}d\rho^2, \\
\hat{B}_2 &= \frac{27r^3\rho^3\cos\frac{\mu}{2}\sin^2\frac{\mu}{2}}{4096 l_p\Delta}\sin\chi d\xi\wedge d\chi, \quad e^{-2\hat{\Phi}} = \frac{8l_p^3\Delta}{r^3\sqrt{H(r)}\cos^3\frac{\mu}{2}}, \\
\Delta &= \frac{r^3\sqrt{H(r)}\cos\frac{\mu}{2}\sin^2\frac{\mu}{2}}{512 l_p^3\alpha'^3}(9l_p^2\alpha'^2\rho^2 + r^6 H(r)\cos^2\frac{\mu}{2}\sin^2\frac{\mu}{2}) \ ,
\end{aligned} \tag{4.136}$$

and,

$$\begin{aligned}
\hat{F}_3 &= -\frac{9}{64}l_p\sqrt{\alpha'}\rho d\Omega_2\wedge d\rho + \frac{3\hat{Q}}{64 l_p\alpha'^{3/2}}\cos^3\frac{\mu}{2}\sin^3\frac{\mu}{2}d\Omega_2\wedge d\mu \\
\hat{F}_5 &= -\frac{27\hat{Q}l_p^2\sqrt{\alpha'}}{8r^9 H(r)^{3/2}\cos^2\frac{\mu}{2}}\rho d\rho\wedge v_4 + \frac{r^3\sqrt{H}\sin^3\frac{\mu}{2}}{8\alpha'^{3/2}\cos\frac{\mu}{2}}d\mu\wedge v_4 \\
&+ \frac{27r^3\sqrt{H}\rho^2\cos^3\frac{\mu}{2}\sin^5\frac{\mu}{2}}{524288 l_p^2\alpha'^{3/2}\Delta}\Big(2r^6 H\sin\frac{\mu}{2}d\rho - 6\hat{Q}\rho\cos\frac{\mu}{2}d\mu\Big)\wedge d\Omega_2\wedge\sin\chi d\chi\wedge d\xi \ ,
\end{aligned} \tag{4.137}$$

with $v_4 = -\frac{r^2\cos^2\frac{\mu}{2}}{4l_p^2\sqrt{H}}dr\wedge dx_0\wedge dx_1\wedge dx_2$.

The Page five-form is given by,

$$\begin{aligned}\tilde{F}_5 = &-\frac{27\hat{Q}l_p^2\sqrt{\alpha'}}{8r^9\cos^2\frac{\mu}{2}H^{3/2}}\rho v_4\wedge d\rho + \frac{r^3\sqrt{H}\sin^3\frac{\mu}{2}}{8\alpha'^{3/2}}v_4\wedge d\mu \\ &-\frac{27}{512}l_p\alpha'^{3/2}\rho^2\sin\chi d\Omega_2\wedge d\rho\wedge d\chi\wedge d\xi\ .\end{aligned} \tag{4.138}$$

Brane configuration and charges

We first compute the NS5 charge by integrating H_3 on the cycle $([\rho,\chi,\xi],\mu=0)$,

$$\begin{aligned}H_3 &= -\frac{3}{8}\alpha'\sin\chi d\rho\wedge d\chi\wedge d\xi \\ Q_{NS5} &= \frac{1}{2\kappa_{10}^2T_{NS5}}\int H_3 = \frac{3\rho_0}{8\pi}\ .\end{aligned} \tag{4.139}$$

Q_{NS5} is quantized if $\rho_0 = L_n$, where we set $L_n := \frac{8}{3}\pi n$. With $\rho\in[L_n,L_{n+1}]$ and a suitable large gauge transformation on B_2, the relation (4.39) is satisfied.

The NS5 branes are also seen by zooming in on the singularity generated by the NATD at $\mu=0$,

$$\begin{aligned}ds^2_{\mu\to 0} = &\frac{r}{2l_p\sqrt{H(r)}}\Big[ds^2(\mathbb{R}^{1,2}) + H(r)\big(dr^2+\frac{r^2}{4}d\Omega_2^2\big)\Big] \\ &+\frac{1}{\nu}\Big[\frac{9l_p\alpha'^2}{2r^3\sqrt{H(r)}}d\rho^2 + \frac{r^3\sqrt{H(r)}}{32l_p}\Big(d\nu^2+\nu^2d\tilde{\Omega}\Big)\Big]\ .\end{aligned} \tag{4.140}$$

This is indeed consistent with the harmonic superposition rule, with harmonic function proportional to ν^{-1}. This gives the characteristic NS5 brane configuration: along the $(\mathbb{R}^{1,2},r,\Omega_2)$ directions, located at $\mu=0$ and smeared along ρ.

Next we compute the quantized Page charges. We start with the dual of F_4 to track the D2. This corresponds to the terms proportional to $\hat{Q}$. Here only the $\tilde{F}_3=\hat{F}_3$ gives a non zero charge and we integrate the (Ω_2,μ) term to find,

$$Q_{D5} = \frac{\hat{Q}}{256l_p\pi\alpha'^{5/2}} = \frac{1}{2}\pi N_{D2}\ , \tag{4.141}$$

where we took (4.134) into account. We see that, as already noted in the near-horizon limit [76], N_{D5} and N_{D2} differ by a factor of $\frac{\pi}{2}$ and thus cannot both be integers. Indeed it is known that NATD generically maps integer charges to non-integer ones [72]. In the dual theory we are thus led to impose a different quantization condition: $\frac{1}{2}\pi N_{D2}\in\mathbb{Z}$, so that (4.141) is satisfied with $Q_{D5}\in\mathbb{Z}$. Moreover, we may perform a large gauge transformation on B_2 and find the resulting change in the Page charge for F_5, ΔQ_{D3},

$$\Delta Q_{D3} = \frac{n\hat{Q}}{256\pi l_p\alpha'^{5/2}} = nQ_{D5}\ . \tag{4.142}$$

We can also track the D6 by looking at the dual of F_2, i.e. the remaining components of the Page forms. These are found by integrating the terms not proportional to $\hat{Q}$ in (4.137) and (4.138), which we label $\tilde{F}_{3'}$ and $\tilde{F}_{5'}$. We find,

$$Q_{D5'} = \frac{1}{2\kappa_{10}^2T_{D5}}\int\tilde{F}_{3'} = -\frac{9l_p(L_{n+1}^2-L_n^2)}{128\pi\sqrt{\alpha'}} = \frac{1}{4}\pi(2n+1)Q_{D6} \tag{4.143}$$

$$Q_{D3'} = \frac{1}{2\kappa_{10}^2T_{D3}}\int\tilde{F}_{5'} = -\frac{9l_p(L_{n+1}^3-L_n^3)}{512\pi^2\sqrt{\alpha'}} = \frac{1}{6}\pi(3n^2+3n+1)Q_{D6}\ , \tag{4.144}$$

where in the last equalities on the right hand sides above we have taken (4.135) into account and the quantization of ρ_0 given below (4.139). Similar to the case of Q_{D5} above, we see that $Q_{D5'}$, $Q_{D3'}$ cannot be integers if Q_{D6} is integer. In the dual theory we are thus led to impose a different quantization condition: $\frac{1}{12}\pi Q_{D6} \in \mathbb{Z}$. Moreover, under a large gauge transformation of B_2, $Q_{D3'}$ is modified in the same fashion as Q_{D3}, cf. (4.142),

$$\Delta Q_{D3'} = nQ_{D5'} \ . \tag{4.145}$$

The brane configuration is summarized in the following table.

	0	1	2	r	μ	θ_1	ϕ_1	ρ	χ	ξ
$NS5$	×	×	×	×		×	×			
$D3$	×	×	×					×		
$D5$	×	×	×					×	×	×
$D3'$	×	×	×	×						
$D5'$	×	×	×	×					×	×

The spatial infinity limit

The supergravity background corresponding to the NATD of the spatial infinity limit of (4.130) is presented here,

$$\begin{aligned}
\hat{ds}^2 &= \frac{r\cos\frac{\mu}{2}}{2l_p}\Big[ds^2(\mathbb{R}^{1,2}) + \Big(dr^2 + r^2\big[d\mu^2 + \cos^2\frac{\mu}{2}ds^2(\Omega_2) \\
&\quad + \frac{9r^3\rho^2\cos\frac{\mu}{2}\sin^4\frac{\mu}{2}}{4096 l_p^2\alpha'\Delta}ds^2(d\chi^2 + \sin^2\chi d\xi^2)\big]\Big)\Big] + \frac{9l_p\alpha'^2}{8r^3\cos\frac{\mu}{2}\sin^2\frac{\mu}{2}}d\rho^2, \\
\hat{B}_2 &= \frac{27r^3\rho^3\cos\frac{\mu}{2}\sin^2\frac{\mu}{2}}{4096 l_p\Delta}\sin\chi d\xi\wedge d\chi, \quad e^{-2\hat{\Phi}} = \frac{8l_p^3\Delta}{r^3\cos^3\frac{\mu}{2}}, \\
\Delta &= \frac{r^3\cos\frac{\mu}{2}\sin^2\frac{\mu}{2}}{512 l_p^3\alpha'^3}(9l_p^2\alpha'^2\rho^2 + r^6\cos^2\frac{\mu}{2}\sin^2\frac{\mu}{2}) \ ,
\end{aligned} \tag{4.146}$$

and,

$$\begin{aligned}
F_3 &= -\frac{9}{64}l_p\sqrt{\alpha'}\rho d\Omega_2\wedge d\rho \\
F_5 &= \frac{r^3\sin^3\frac{\mu}{2}}{8\alpha'^{3/2}\cos\frac{\mu}{2}}d\mu\wedge v_4 + \frac{27r^9\rho^2\cos^3\frac{\mu}{2}\sin^6\frac{\mu}{2}}{262144 l_p^2\alpha'^{3/2}\Delta}d\rho\wedge d\Omega_2\wedge\sin\chi d\chi\wedge d\xi \ ,
\end{aligned} \tag{4.147}$$

with $v_4 = -\frac{r^2\cos^2\frac{\mu}{2}}{4l_p^2}dr\wedge dx_0\wedge dx_1\wedge dx_2$. The surviving RR flux terms in the asymptotic limit ultimately arise from the charge created in the reduction of the parent M-theory background to Type IIA, and thus from the D6. The NS5 also survives since it comes from the singularity in the NATD.

4.6 Discussion

Having full-fledged brane solutions, interpolating between the near-horizon and spatial infinity limit, may give a better handle on the brane configurations and the global properties of the NATD. In particular certain general features emerge. The NATD of the spatial infinity limit of standard intersecting brane solutions is universal: it is given by a continuous linear distribution of NS5 branes along a half line with specific charge density. We have also provided additional examples where a general relation, observed previously in the NATD literature, between the Page charges generated by the NATD and their behavior under a large gauge transformation of the NS flux is obeyed. Since this behavior results from the non-trivial dependence of the Page charge on the B_2 field, NATD naturally furnishes several examples where the choice of B_2 plays an important role.

More generally in cases where the brane configuration before NATD is not flat at spatial infinity, the NATD contains highly nontrivial RR fluxes even at spatial infinity. If the charges before NATD are related to the presence of branes, the latter can be tracked throughout the NATD. On the other hand, the precise NS5-D(p+1)-D(p+3) brane intersections underlying these solutions cannot be systematically identified with this approach. Indeed, we have not been able to describe these brane fluxes as resulting from backreaction (as dictated by the harmonic superposition rule) on some initial spacetime without branes. The exception to this statement is the case of the geometry near the locus of the NS5 branes. Let us also note that the spatial infinity limits of the NATD backgrounds presented here are highly nontrivial exact supergravity solutions in their own right, and they can be considered independently from the full interpolating intersecting brane solution.

In the case of the NATD of the D2 branes, proceeding by analogy to the NATD of the D3 brane, cutting off the range of the ρ coordinate at a finite value, in order to impose NS5 charge quantization, provides a prescription for assigning well-defined ranges to all dual coordinates. On the other hand, from a purely geometrical point of view this procedure renders the space geometrically incomplete. Ultimately such a procedure should be justified through a physical interpretation. In the case of the NATD of the D3 brane, such an interpretation was provided by the field theory dual proposed in [80], as reviewed in section 4.1.2. It would be interesting to provide a similar interpretation for the NATD of the D2 branes of the present paper.

We have cast the supersymmetric D2 brane solution, arising from the reduction of M2 branes on seven-dimensional Sasaki-Einstein, in the language of generalized geometry pure spinor equations for domain walls. This framework allowed to look for massive supersymmetric deformations of the D2 brane solutions, and we have been able to rule out a certain class of ansätze. It would be interesting to try to construct these massive deformations explicitly, at least in a perturbative expansion in Romans mass as in [93]. If they exist, these would be full interpolating intersecting brane solutions whose near-horizon limit coincides with the class of massive IIA $\mathrm{AdS}_4 \times M_6$ solutions of [92]. Note also that it is not clear whether the D2 brane solution obtained through reduction on S^7 is supersymmetric: the formalism of generalized complex geometry could also help resolve this issue.

It would also be interesting to cast the NATD of the supersymmetric solutions in the generalized geometry formalism for domain walls, thus refining the general results of [94, 95]. Besides providing a check of supersymmetry, this might give insight into the global structure of the solutions. In certain cases the duals might fall within the class recently examined in [96].

5 $SU(3)$-structures on toric $\mathbb{CP}^1$ bundles

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As we have seen in 2.4, supersymmetric compactification require the existence of an $SU(3)$-structure on the internal manifold. Then in some cases the supersymmetry equations can be translated into conditions on the torsion classes of the $SU(3)$-structure. Depending on the flux content, the admissible torsion classes may vary. A classification of flux vacua would then go through a classification of manifolds with $SU(3)$ structure and the torsion classes they can carry.

Calabi-Yau manifolds correspond to the special case where all torsion classes vanish. In the context of compactification this is equivalent to vanishing fluxes. And this is only one of many reasons why Calabi-Yau manifolds are interesting. Thus Calabi-Yau manifolds have been the subject of intensive research, both in mathematics and theoretical physics. Other types of torsion classes have been less studied and relatively few things are known about their classification. This would require a significant pool of manifolds with $SU(3)$ structure.

For this reason it was proposed in [97] to use smooth compact toric varieties (SCTV) as a class of manifolds upon which a classification procedure could be started. Toric varieties stem from algebraic geometry, thus in the smooth case they lie at the intersection between differential and algebraic geometry and tools from both fields are available. SCTV come with a lot a structure and this should help constructing $SU(3)$-structures on general grounds. Indeed [97] gives a procedure to build $SU(3)$-structures that rely only on the existence of a specific one form K. This procedure was applied in many examples, first in [97] and then in [98], but the choice of K prevents the method to be fully systematic.

This chapter presents another procedure that can be systematically applied to a wide class of SCTV. The procedure defines a 3D SCTV as a $\mathbb{CP}^1$ bundle over any 2D SCTV, and uses the bundle structure to build an $SU(3)$ structure. The discussion starts with a review of the formalism that will be used throughout the chapter. Then the general construction is described, before being illustrated on the case where the base is $\mathbb{CP}^2$. Also the computation of the torsion classes is initiated.

5.1 Review of the formalism

In order to fix the notation and make the chapter self-contained, in this section we give a review of the SCTV formalism developed in [97]. Along the way we introduce the tools that will be useful in the rest of the chapter. The description of the toric $\mathbb{CP}^1$ bundles is given at the end of the section.

5.1.1 The symplectic quotient and coordinates

There are various equivalent ways to define a toric variety see *e.g.* [99], or [100] for an introduction for physicists. In the following we will use the symplectic quotient description, which turns out to be the best suited for the explicit construction of G-structures and the associated differential calculus. The starting point of the symplectic quotient description is a *parent space* $\mathbb{C}^k$, with coordinates $\{z_i,\ i=1,\ldots,k\}$, and a set of s linearly-independent integer k-vectors Q_i^a, $\{a=1,\cdots,s\}$ called the *charges*. Let $\tilde{M}$ be the real submanifold defined by the following set of *moment map equations*,

$$Q_i^a|z_i|^2 = \xi^a \ . \tag{5.1}$$

The real parameters ξ^a are the so-called, *Fayet-Iliopoulos* parameters: they correspond to Kähler moduli, parametrizing the sizes of cycles of the toric variety. On the other hand the topology of the variety is independent of the ξ^a as long as we stay inside the *Kähler cone*, defined by the conditions $\xi^a > 0$[1]. In the following we will always assume this to be the case. The associated toric variety M is given by the quotient $M = \tilde{M}/U(1)^s$ where the *phase vector* $\phi_a \in U(1)^s$ acts on the coordinates $z_i \in \tilde{M}$ through the following *gauge transformations*,

$$z_i \to \phi\cdot z_i := e^{iQ_i^a\phi_a} z_i \ . \tag{5.2}$$

Hence M is a manifold of complex dimension $d = k-s$: the equations (5.1) can be thought of as removing s real "radial" directions, whereas the action of (5.2) removes s real 'angular' directions. In total the equations (5.1), (5.2) remove s pairs consisting of one radial and one angular variable, which may be thought of as s complex variables.

Since the Q^a are independent as k-vectors, one may choose a set S of s indices such that Q_b^a, $b\in S$, is invertible. The open set $\{z_b \neq 0, b\in S\} \subset \mathbb{C}^k$ then descends to a well-defined open set in M, denoted by U_S. On this patch one can then use the z_b coordinates to compensate the $U(1)^s$ action on the z_α coordinates, where the index α takes values in the complement of S, $\alpha \in \complement S$. One may then define the following gauge-invariant quantities,

$$t_i := z_i \prod_{a\in S} z_a^{-\mathcal{Q}_b^a Q_i^b} \ , \tag{5.3}$$

where we have set,

$$\mathcal{Q}_b^a := (Q_b^a)^{-1} \ . \tag{5.4}$$

Thus, provided $|Q_S| := |\det Q_{b\in S}^a| = 1$, the map

$$\begin{array}{rcl} \varphi_S : U_S & \to & \mathbb{C}^d \\ {[z_i]} & \mapsto & (t_\alpha)_{\alpha\in\complement S} \end{array} \ , \tag{5.5}$$

is well defined and homeomorphic, while the transition functions $\varphi_S\circ\varphi_{S'}^{-1}$ are biholomorphic and rational. Now if $M = \bigcup_{|Q_S|=1} U_S$, the charts (U_S,φ_S) form a holomorphic atlas on M: the t_α, $\alpha\in\complement S$, define d gauge-invariant local holomorphic coordinates on U_S. Note that for $i = c\in S$, we thus find $t_c = 1$. This condition of covering of M is thought to be related to the condition of smoothness for general toric varieties and is satisfied for the cases we consider here.

To take a simple example, consider the case $s=1$ and $Q=(1,\cdots,1)$. The corresponding toric variety is the complex projective space $\mathbb{CP}^{k-1}$. Indeed (5.1) gives $\|z\|^2 = \xi$, *i.e.* $\tilde{M} =$

[1] This needs not always be the case

S^{2k-1}. Taking the $U(1)$ quotient, M can be written as $M = (\mathbb{C}^k \backslash \{0\})/\mathbb{C}^*$, the set of complex lines in $\mathbb{C}^k$. On the patch $U_j = \{z_j \neq 0\}$, the local coordinates take the form,

$$t_i = \frac{z_i}{z_j} \ , \tag{5.6}$$

which we recognize as the set of canonical coordinates of $\mathbb{CP}^{k-1}$. The z_i on the other hand correspond to homogeneous coordinates of $\mathbb{CP}^{k-1}$.

5.1.2 Differential forms

We have seen that toric varieties are equipped with systems of complex coordinates which can easily be made explicit. Moreover it is often advantageous to work directly in the parent space $\mathbb{C}^k$ using the homogeneous coordinates z_i. We will be interested in particular in globally-defined differential forms on the manifold M. One way to construct a differential form on M is to start from its local expression on a patch, and make sure a regular global extension exists by checking its compatibility with the transition functions of the cotangent bundle. Working directly in $\mathbb{C}^k$ drastically simplifies this problem: since the topology of the parent space is trivial, a single expression suffices to define differential forms globally. From this point of view the key question is to identify the differential forms of $\mathbb{C}^k$ which descend to well-defined forms on M.

In the following we review how the formalism of [97] can be used to treat this question. Let Φ be a differential form on $\mathbb{C}^k$. In order for Φ to descend to a well defined form on M, it should be well-defined on $\tilde{M}$. Hence it should be compatible with the moment map equations (5.1) which imply,

$$Q_i^a \bar{z}_i \mathrm{d}z_i + Q_i^a z_i \mathrm{d}\bar{z}_i = 0 \ . \tag{5.7}$$

Consequently Φ should not have any components along the $\Re\eta^a$, where we have defined,

$$\eta^a := Q_i^a \bar{z}_i \mathrm{d}z_i \ . \tag{5.8}$$

In other words, we require,

$$\iota_{\Re(V^a)}\Phi = 0, \tag{5.9}$$

where V^a is the dual of η^a (with respect to the canonical metric of $\mathbb{C}^k$),

$$V^a := Q_i^a z_i \partial_{z_i} \ . \tag{5.10}$$

Moreover, Φ should be compatible with the quotient (5.2). On the other hand the $U(1)^s$ action in (5.2) is generated by the vector fields $\Im(V^a)$. Hence the $U(1)^s$ invariance can be stated in terms of the following two conditions:

1. Φ must be constant along $U(1)^s$ orbits, *i.e.* $\mathcal{L}_{\Im(V^a)}\Phi = 0$.

2. Φ should not have any components along the orbits, *i.e.* $\iota_{\Im(V^a)}\Phi = 0$.

These conditions have a natural interpretation: first note that a form Φ has charge q^a if it is an eigenvector of the Lie derivative $\mathcal{L}_{\Im(V^a)}$,

$$\mathcal{L}_{\Im(V^a)}\Phi = q^a \Phi \ . \tag{5.11}$$

We then see that the first of the two conditions above is simply the gauge invariance of Φ, *i.e.* the condition that the total charge of Φ vanishes. Moreover the second condition combined with (5.9) gives,

$$\iota_{V^a}\Phi = \iota_{\bar{V}^a}\Phi = 0 \ , \tag{5.12}$$

which is equivalent to Φ being *vertical* with respect to V^a.

Thus in order to construct a well-defined form on M descending from a form Φ on $\mathbb{C}^k$, the gauge invariance of Φ must be imposed from the outset. On the other hand, the verticality condition is purely algebraic and can be imposed by projecting out the components along η^a.

Let us now come to the explicit construction of the vertical projector. We introduce the real symmetric matrix,

$$g^{ab} := \eta^a(V^b) = Q_i^a Q_i^b |z_i|^2 \ . \tag{5.13}$$

The projection P on a $(1,0)$-form Φ is then given by,

$$P(\Phi) = \Phi - \tilde{g}_{ab}\iota_{V^a}(\Phi)\eta^b \ , \tag{5.14}$$

where $\tilde{g} = g^{-1}$. This definition of P can be readily extended to all (k,l)-forms [97]. In the following it will be useful to define the vertical projections, $\mathcal{D}z_i$, of the one-forms $\mathrm{d}z_i$,

$$\mathcal{D}z_i := P(\mathrm{d}z_i) = \mathrm{d}z_i - \tilde{g}_{ab}Q_j^a Q_i^b \bar{z}_j z_i \mathrm{d}z_j = \mathrm{d}z_i - h_{ij} z_i \bar{z}_j \mathrm{d}z_j \ , \tag{5.15}$$

(no sum over i) where we have set,

$$h_{ij} := Q_i^a Q_j^b \tilde{g}_{ab} \ . \tag{5.16}$$

The $\mathcal{D}z_i$ are the building blocks that we will use to construct global forms on M. Note however that since they are not gauge invariant, one must compensate their charge by appropriate (charged) coefficients.

On the other hand the (singular) form $\mathcal{D}z_i/z_i$ is both gauge invariant and vertical and therefore admits an expression in terms of the local coordinates t_i. On the patch U_S we have,

$$\frac{\mathrm{d}t_i}{t_i} = \frac{\mathrm{d}z_i}{z_i} - \sum_{a\in S} \mathcal{Q}_b^a Q_i^b \frac{\mathrm{d}z_a}{z_a} \ , \tag{5.17}$$

where we took (5.3) into account. Setting $i = c \in S$ then gives $\mathrm{d}t_c = 0$, *cf.* (5.4). This leaves us with d linearly-independent one-forms $\mathrm{d}t_\alpha$, $\alpha \in {}^{\complement}S$. We can then compute,

$$\frac{\mathcal{D}z_i}{z_i} = \frac{\mathrm{d}t_i}{t_i} - h_{ij}|z_j|^2 \frac{\mathrm{d}t_j}{t_j} \ . \tag{5.18}$$

where we took into account that: $h_{ij}|z_j|^2 Q_j^b = \tilde{g}_{cd}\, Q_i^c Q_j^d\, |z_j|^2\, Q_j^b = \tilde{g}_{cd}\, Q_i^c\, g^{db} = Q_i^b$. As expected, given that the form on the left-hand side is vertical and gauge invariant, the result can be expressed in terms of the local coordinates alone. Note also that the gauge-invariant $|z_j|^2$ can be expressed as a function of t_i using (5.1).

Conversely, (5.17) can be used to express $\mathrm{d}t_i$ as a function of $\mathcal{D}z_i$, since $\mathrm{d}t_i$ is vertical by definition. We now have all the necessary tools to translate back and forth between the local coordinate system $\{t_i\}$ on M and the global coordinate system $\{z_i\}$ on $\mathbb{C}^k$.

5.1.3 Kähler structure

It is now possible to show that the toric variety will inherit a Kähler structure from the parent space. Indeed $\mathbb{C}^k$ is equipped with a canonical hermitian form:

$$h = \mathrm{d}z_i \otimes \mathrm{d}\bar{z}_i = g_{\mathbb{C}} - i\, J_{\mathbb{C}} \tag{5.19}$$

Where $g_{\mathbb{C}}$ is the euclidean metric and $J_{\mathbb{C}}$ the canonical Kähler form. The nice thing about this is that the product of z and $\bar{z}$ automatically compensates for the charges and makes $h_{\mathbb{C}}$ gauge invariant. We just need to project on the vertical component to get a tensor on M.

Thus a toric variety naturally comes with a hermitian structure:

$$\mathfrak{h}(\xi^a) = P(h_{\mathbb{C}}) = P\left(\mathrm{d}z_i \otimes \mathrm{d}\bar{z}_i\right) = \mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i \ . \tag{5.20}$$

As we have already noted, the $\mathcal{D}z_i$ above are not linearly independent and do not form a basis of the cotangent bundle of M. Using (5.18) it is not difficult to see that the hermitian metric takes the following form in local coordinates,

$$\mathfrak{h}(\xi^a) = \frac{|z_i|^2}{|t_i|^2}\mathrm{d}t_i \otimes \mathrm{d}\bar{t}_i - h_{jk}\frac{|z_j|^2}{|t_j|^2}\frac{|z_k|^2}{|t_k|^2}\bar{t}_j\mathrm{d}t_j \otimes t_k\mathrm{d}\bar{t}_k \ , \tag{5.21}$$

where we have made use of the identity $h_{ij}h_{ik}|z_i|^2 = h_{jk}$ which can be shown by taking into account the various definitions. As we have seen in section 1.3.3 the hermitian metrics is given by a metric and a symplectic form. Using the decomposition 1.18 define:

$$\mathfrak{h} = \mathfrak{g} - i\,\hat{J} \tag{5.22}$$

In this case $\hat{J}$ is in fact a Kähler form,

$$\hat{J} = \frac{i}{2}\mathcal{D}z_i \wedge \mathcal{D}\bar{z}_i \ . \tag{5.23}$$

Although $\mathcal{D}z_i$ are not closed, it can readily be verified that $\mathrm{d}\hat{J}$ vanishes as it should. Thus any toric variety is Kähler.

Let us illustrate the above with the example of $\mathbb{CP}^{k-1}$: on the patch U_k we have $g = \xi$ and $h_{ij} = \frac{1}{\xi}$. Moreover (5.1) gives $|z_k|^2 = \frac{|z_\alpha|^2}{|t_\alpha|^2} = \frac{\xi}{1+t^2}$, where $t^2 := \sum_\alpha |t_\alpha|^2$; $\alpha = 1, \ldots, k-1$. Hence,

$$\mathfrak{h}(\xi) = \xi\left(\frac{\mathrm{d}t_i \otimes \mathrm{d}\bar{t}_i}{1+t^2} - \eta \otimes \bar{\eta}\right) \ , \tag{5.24}$$

where $\eta := \frac{1}{1+t^2}\ \bar{t}_i\mathrm{d}t_i$. We thus recover the Fubini-Study metric and its associated Kähler form.

An hermitian metric also gives rise to a scalar product "$\cdot$" on differential forms on M. Since $P^2 = P$, the calculation of the scalar product on vertical forms can be done in the parent space $\mathbb{C}^k$ using the flat metric. Indeed the action of the metric on vertical forms η_1, η_2 can be written:

$$\mathfrak{g}(\eta_1, \eta_2) = g_{\mathbb{C}}(P\eta_1, P\eta_2) = g_{\mathbb{C}}(\eta_1, \eta_2) \tag{5.25}$$

Then, using (5.15) we can compute the useful relation:

$$\mathcal{D}\bar{z}_i \cdot \mathcal{D}z_j = 2(\delta_{ij} - h_{ij}\bar{z}_i z_j) \ , \tag{5.26}$$

which shows that the $\mathcal{D}z_i$ are not orthogonal.

5.1.4 SU(d) structures

To further reduce to $SU(d)$ we need an holomorphic d-form. Once again look at the parent space: $\mathbb{C}^k$ has a canonical $SU(k)$ structure given by,

$$J_{\mathbb{C}} = \frac{i}{2}\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i \tag{5.27}$$

$$\Omega_{\mathbb{C}} = \bigwedge_i \mathrm{d}z_i, \tag{5.28}$$

As we have just seen, projecting $J_{\mathbb{C}}$ does lead to a Kähler form on M. The question that remains is about $\Omega_{\mathbb{C}}$. However $P(\tilde{\Omega})$ vanishes trivially since there can be no $(k,0)$ forms on M. In order to obtain a $(d = k - s, 0)$-form on M we must contract $\tilde{\Omega}$ with each of the V^a vectors, so that,

$$\hat{\Omega} := \frac{1}{\sqrt{\det g}} \prod_a \iota_{V^a}\tilde{\Omega} \ . \tag{5.29}$$

This expression is naturally vertical, since the vertical components have been contracted. The V^a are not charged so that $\hat{\Omega}$ has the same charge as $\tilde{\Omega}$, *i.e.* $q^a = \sum_i Q_i^a$. Now if $\sum_i Q_i^a = 0$, $\hat{\Omega}$ is gauge invariant thus defining a holomorphic d-form on M, and then an $SU(d)$-structure. However M is also Kähler so that M is in fact Calabi-Yau and cannot be compact. This case will then be excluded for us.

Let us now suppose that $\sum_i Q_i^a \neq 0$. The pair $(\hat{J}, \hat{\Omega})$ does satisfy the compatibility equations (1.24), thus defining at least a *local* $SU(d)$ structure on M. Moreover $\hat{\Omega}$ admits a simple expression in terms of local coordinates[2] on U_S. After some straightforward manipulations we obtain,

$$\hat{\Omega} = (-1)^S Q_S \frac{\prod_i z_i}{\sqrt{\det g}} \bigwedge_\alpha \frac{\mathrm{d}t_\alpha}{t_\alpha} \ , \tag{5.30}$$

where $a \in S$, $\alpha \in \complement S$ and we have defined,

$$(-1)^S := (-1)^{\sum_{a\in S} a + \frac{(s+1)(s+2)}{2}} \ . \tag{5.31}$$

Remark Any manifold admits local $SU(d)$ structures since the issue is precisely the global definition. For example, for a manifold with $U(d)$-structure choose a holomorphic vielbein define the holomorphic d-form as the product of the holomorphic vielbein form. Thus the existence of $\hat{\Omega}$ is of course not surprising but it will still be very useful as I will show in the following. In order to define an $SU(d)$-structure it will be necessary to modify $\hat{\Omega}$ to make it gauge invariant. At the same time $\hat{J}$ should also change to keep the compatibility conditions intact. In [97] a prescription was given for the construction of *global* $SU(d)$ structures on M.[3] It relies on the existence of a one-form K on $\mathbb{C}^k$ with the following properties:

1. It is vertical and (1,0) with respect to the complex structure of $\mathbb{C}^k$.

2. It has half the charge of $\tilde{\Omega}$.

3. It is nowhere-vanishing.

[2]Thus defined, $\hat{\Omega}$ is compatible with the transition functions. But the z_i are not strictly functions on U_S since they are not gauge-invariant. Then, a rigorous local form could be achieved by substituting z_i with $|z_i|$, losing in the process the compatibility with transition functions.

[3]Originally presented for $d = 3$, the prescription of [97] is in fact directly generalizable to any dimension.

Given a one-form K on $\mathbb{C}^k$ satisfying the conditions above, [97] showed that a global $SU(3)$ structure on M can be constructed, and provided explicit examples of such a K for certain toric $\mathbb{CP}^1$ bundles. Many more examples of K were provided for other toric varieties in [98], which also provided explicit computations of the torsion classes of the associated $SU(3)$ structures. However there is no known construction for K that would be applicable in general, even for a subclass of SCTV, and the search for $SU(3)$ structures on SCTV had so far proceeded in a case by case fashion.

In the following we will present a construction of $SU(3)$ structures valid for toric $\mathbb{CP}^1$ bundles over any 2d SCTV. As we will see, our method is not equivalent to the prescription of [97], although it also makes use of a certain (1,0)-form on $\mathbb{C}^k$.

5.2 $\mathbb{CP}^1$ over general SCTV

5.2.1 Toric $\mathbb{CP}^1$ bundles over SCTV

In [101], the classification of SCTV in three (complex) dimensions was shown to reduce to the classification of certain weighted triangulations of the two-dimensional sphere. In [97] it was shown how to systematically translate the results of [101] into the symplectic quotient language reviewed previously. In the following we will be interested in the subclass of the classification of [101] corresponding to $\mathbb{CP}^1$ bundles over a two-dimensional SCTV base. However the formalism applies generally to the case of $\mathbb{CP}^1$ bundles over SCTV, so in this subsection we will keep the dimension of the base arbitrary.

The $U(1)$ charges of these bundles are given by the following set of $(s+1)\times(k+2)$ matrices,

$$Q_I^A = \begin{pmatrix} q_i^a & -n^a & 0 \\ 0 & 1 & 1 \end{pmatrix} , \tag{5.32}$$

where $A = 1, \dots, s+1$, $I = 1, \dots, k+2$; $n_a \in \mathbb{N}$, $a = 1, \dots, s$, are integers specifying the twisting of the $\mathbb{CP}^1$ bundle over a SCTV M; q_i^a, $a = 1, \dots, s$, $i = 1, \cdots, k$, are the $U(1)$ charges of the symplectic quotient description of M, which is therefore of complex dimension $d = k - s$. (In subsequent subsections we will specialize to the case $d = 2$.)

The total space of the bundle is constructed by appending two coordinates and one new charge to those of M (given by the q_i^a), as in (5.32), thus obtaining a space of complex dimension $d+1$. We will hence use specific notations for the data related to the fiber, namely:

$$u := z_{k+1} \ ; \quad v := z_{k+2} \ ; \quad \xi := \xi^{s+1} \ . \tag{5.33}$$

The last charge Q_i^{s+1} defines a $\mathbb{CP}^1$ fiber over M, while the integers n^a determine the twisting of the bundle. Indeed the moment map equations for the total space read,

$$\sum_{i=1}^{k} q_i^a\, |z_i|^2 = \xi^a + n^a |u|^2 \ ; \quad |u|^2 + |v|^2 = \xi \ . \tag{5.34}$$

Thus the last two coordinates define a sphere of radius $\sqrt{\xi}$, while the first n coordinates define locally an M_ρ whose "radii" $(\rho^a)^2 := \xi^a + n^a|u|^2$ depend on the fiber. The twisting can be thought of as a consequence of the modified $U(1)^{s+1}$ action.

5.2.2 Decomposition of the metric

We would now like to construct a metric that exhibits the bundle structure, *i.e.* a metric of the form : $\mathfrak{h}_{d+1} = \mathfrak{h}_d + \mathfrak{h}_{\mathbb{CP}^1}$, where $\mathfrak{h}_d$ is a metric on M and $\mathfrak{h}_{\mathbb{CP}^1}$ is a metric on the fiber $\mathbb{CP}^1$, possibly modified by a connexion on the base.

We start by defining the vertical one-forms using the formalism introduced in section 5.1.2, for the total bundle. The relations are thus given by the charges Q_I^A. The same can be done by taking only the charges of the base q_i^a :

$$\begin{aligned}\hat{g}^{ab} &:= q_i^a q_i^b |z_i|^2 \\ \hat{h}_{ij} &:= \hat{g}_{ab} q_i^a q_j^b \\ \hat{\mathcal{D}} z_i &:= \hat{P}(\mathrm{d}z_i) = \mathrm{d}z_i - \hat{h}_{ij} z_i \bar{z}_j \mathrm{d}z_j \ ,\end{aligned}$$

hatted symbols are used to denote objects relatively to the base, in order to distinguish from the objects constructed in (5.1.2). Note that $\hat{g}, \hat{h}, \hat{\mathcal{D}}z_i$ live on the same space as their non-hatted counterparts, which are the relevant objects for the definition of forms in the symplectic quotient description. This means that they do not have, before further analysis, any trivial interpretation. For example, the $|z_i|^2$ do not verify the moment maps equations of the base but those of the total bundle, and thus $\hat{g}, \hat{h}$ and $\hat{\mathcal{D}}z_i$ depend on the radii. A quick calculation confirms now that the $\hat{\mathcal{D}}z_i$ do obey the expected algebraic relations :

$$\sum_{i=1}^{k} q_i^a \bar{z}_i \hat{\mathcal{D}} z_i = 0 \ .$$

Recall the form of the canonical metric on a SCTV (the generalization of the Fubiny-Study metric of $\mathbb{CP}^1$),

$$\mathfrak{h}_{d+1} = \sum_{I=1}^{k+2} \mathcal{D}z_I \otimes \mathcal{D}\bar{z}_I \ .$$

We will now decompose this metric into base and fiber components. Since the $\mathcal{D}z$ depends on the matrix g^{AB}, the key here will be to decompose it and its inverse along the different bundle directions.

The definition of Q_I^A leads to :

$$g^{AB} = \begin{pmatrix} \hat{g}^{ab} + n^a n^b |u|^2 & -n^a |u|^2 \\ -n^b |u|^2 & \xi \end{pmatrix} \ .$$

Moreover we need to express the inverse g_{AB} while keeping track of the inverse, $\hat{g}_{ab}$, of $\hat{g}^{ab}$. For this purpose we first need to compute the determinant $g = \det g^{AB}$,

$$g = \begin{vmatrix} \hat{g} + n\,n^T |u|^2 & n|u|^2 \\ n^T |u|^2 & \xi \end{vmatrix} = \begin{vmatrix} \hat{g} + n\,n^T |u|^2 (1 - \frac{|u|^2}{\xi}) & 0 \\ n^T |u|^2 & \xi \end{vmatrix} = \xi \det\left(\hat{g} + \frac{1}{\xi}|u|^2|v|^2 n\,n^T\right) \ .$$

We now use the property of multilinearity of the determinant to expand this expression. We then get all different terms of order $s - m$ in g and m in $n\,n^T$. But since $\operatorname{rank} n\,n^T = 1$, only the terms of order zero or one remain. The terms of order one are merely the determinant of $\hat{g}$ where the column a has been replaced by the vector $\frac{n^a}{\xi}|u|^2|v|^2\ n$. By expanding along this same column, we exhibit the cofactors of $\hat{g}$ which are independent of this exact column, and are related to the inverse matrix,

$$\det(\hat{g}, g^a \leftrightarrow n) = \sum_a \operatorname{cof}(\hat{g})_{ab} n^b = \hat{g}\, \hat{g}_{ab} n^b \ .$$

Thus we have :

$$g = \xi\left(\hat{g} + \frac{n^a}{\xi}|u|^2|v|^2\hat{g}\,\hat{g}_{ab}n^b\right) = \hat{g}(\xi + \hat{g}_{ab}n^a n^b|u|^2|v|^2)\ .$$

The same trick can be used to compute the inverse matrix :

$$g_{s+1\,s+1} = \frac{1}{g}\det(\hat{g} + n\,n^T|u|^2) = \frac{\hat{g}}{g}(1 + \hat{g}_{ab}n^a n^b|u|^2)\ .$$

Moreover,

$$g_{a\,s+1} = \frac{1}{g}\det(\hat{g}, g^a \leftrightarrow -|u|^2\,n) = \frac{\hat{g}}{g}|u|^2\,\hat{g}_{ab}n^b\ .$$

The last cofactors are somewhat more complicated, since they involve double cofactors. Eventually we get :

$$g_{ab} = \hat{g}_{ab} - \frac{\hat{g}}{g}|u|^2|v|^2\,\hat{g}_{ac}n^c\,\hat{g}_{bd}n^d\ .$$

It is now possible to compute the $h_{\mu\nu}$. Let us introduce the objects

$$V := \hat{g}_{ab}n^a n^b\ ,\quad V_i := \hat{g}_{ac}q_i^a n^c\ , \tag{5.35}$$

in terms of which we obtain,

$$\begin{aligned}
h_{ij} &= g_{ab}q_i^a q_j^b = \hat{h}_{ij} - \frac{\hat{g}}{g}|u|^2|v|^2\,V_i\,V_j \\
h_{ik+1} &= g_{a\,s+1}q_i^a - g_{ab}q_i^a n^b = -\frac{\hat{g}}{g}V_i|v|^2 \\
h_{ik+2} &= g_{a\,s+1}q_i^a = \frac{\hat{g}}{g}V_i|u|^2 \\
h_{k+1\,k+1} &= g_{s+1\,s+1} - 2g_{as+1}n^a + g_{ab}n^a n^b = \frac{\hat{g}}{g}(1 + V|v|^2) \\
h_{k+1\,k+2} &= g_{s+1\,s+1} - g_{as+1}n^a = \frac{\hat{g}}{g} \\
h_{k+2\,k+2} &= g_{s+1\,s+1} = \frac{\hat{g}}{g}(1 + V|u|^2)\ .
\end{aligned} \tag{5.36}$$

We can now compute the $\mathcal{D}z_I$,

$$\frac{\mathcal{D}z_i}{z_i} = \frac{\hat{\mathcal{D}}z_i}{z_i} + \frac{\hat{g}}{g}V_i|u|^2|v|^2\ \varepsilon\ , \tag{5.37}$$

where,

$$\varepsilon = \frac{\mathrm{d}u}{u} - \frac{\mathrm{d}v}{v} + V_j\bar{z}_j\mathrm{d}z_j\ . \tag{5.38}$$

The last two coordinates correspond to colinear one-forms,

$$\frac{\mathcal{D}u}{u} = \frac{\hat{g}}{g}|v|^2\ \varepsilon;\quad \frac{\mathcal{D}v}{v} = -\frac{\hat{g}}{g}|u|^2\ \varepsilon\ .$$

Finally the canonical metric reads,

$$\begin{aligned}\mathfrak{h}_{d+1} &= \mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i + \mathcal{D}u \otimes \mathcal{D}\bar{u} + \mathcal{D}v \otimes \mathcal{D}\bar{v} \\ &= \hat{\mathcal{D}}z_i \otimes \hat{\mathcal{D}}\bar{z}_i + \frac{\hat{g}}{g} V_i |u|^2 |v|^2 \; \hat{\mathcal{D}}z_i \otimes \bar{z}_i K^* + c.c \\ &\qquad + V_i V_i |z_i|^2 \frac{\hat{g^2}}{g^2} |u|^4 |v|^4 \; \varepsilon \otimes \varepsilon^* + \frac{\hat{g}^2}{g^2} |u|^2 |v|^2 \xi \; \varepsilon \otimes \varepsilon^* \; .\end{aligned} \tag{5.39}$$

On the other hand we have,

$$V_i \bar{z}_i \hat{\mathcal{D}} z_i = \hat{g}_{ab} n^a \, q_i^b \bar{z}_i \hat{\mathcal{D}} z_i = 0 \; ; \quad V_i^2 \, |z_i|^2 = V \; ,$$

so that the metric simplifies to,

$$\mathfrak{h}_{d+1}(\xi^A) \;\; = \;\; \mathfrak{h}_d((\rho^a)^2) + \frac{\hat{g^2}}{g^2} |u|^2 |v|^2 (\xi + V|u|^2|v|^2) \; \varepsilon \otimes \varepsilon^* \tag{5.40}$$

$$\;\; = \;\; \mathfrak{h}_d((\rho^a)^2) + \frac{\hat{g}}{g} |u|^2 |v|^2 \; \varepsilon \otimes \varepsilon^* \; . \tag{5.41}$$

Note that this decomposition remains valid in the complex local coordinates t_i, t_{k+1}, on the chart U_S defined by $S = \hat{S} \cup \{k+2\}$, in which ε can be written as,

$$\varepsilon = \frac{\mathrm{d}t_{k+1}}{t_{k+1}} + \sum_{i=1}^{k} V_i |z_i|^2 \frac{\mathrm{d}t_i}{t_i} \; .$$

The $\hat{\mathcal{D}}z_i$ happen to be the projections on the space generated by the $\mathrm{d}t_i$, in fact they are related to the $\mathrm{d}t_i$ by the relations (5.18) where we take $\hat{h}_{ij}$ instead of h_{IJ}. This justifies that in the decomposition (5.40), the metric on the base is exactly the canonical metric whose radii vary along the fiber.

5.2.3 $SU(3)$-structure

We will now show how to construct a globally-defined $SU(3)$ structure on a canonical (defined in eq. (5.42) below) $\mathbb{CP}^1$ bundle over a SCTV of complex dimension $d = 2$.

As we saw explicitly in section 5.2.2, the canonical metric of the SCTV, eq. (5.20), is smooth for any twisting of the bundle parameterized by $n^a \in \mathbb{N}$. On the other hand the existence of a globally-defined $SU(3)$ structure imposes a topological constraint and hence a constraint on the n^a, as we explain in the following. This constraint is automatically satisfied for the canonical $\mathbb{CP}^1$ bundle.[4]

We start with a $(d+1)$-dimensional toric $\mathbb{CP}^1$ bundle over a d-dimensional base M, whose charges were given in (5.32). The $\mathbb{CP}^1$ bundle will be called *canonical* if the charge of z_{k+1}, defining the twisting of the bundle, is taken to compensate exactly for the charges of the base, *i.e.*,

$$n^a = \sum_{i=1}^{k} q_i^a \; . \tag{5.42}$$

[4]We use the term *canonical metric* for the metric (5.20) of the SCTV, which is defined for all n^a, *i.e.* for all topologies. On the other hand we use the term *canonical* $\mathbb{CP}^1$ *bundle* for the topology defined in eq. (5.42). Hopefully this will not lead to confusion.

As emphasized in [98], the topological condition for the existence of an $SU(3)$ structure on the total space of the SCTV is that its first Chern class should be even. Condition (5.42) guarantees that there is no topological obstruction for the existence of an $SU(3)$ structure. This can be seen as follows: the first Chern class of the SCTV is given by,

$$c_1 = \sum_{I=1}^{k+2} D_I \ , \tag{5.43}$$

where we have denoted by D_I the divisors corresponding to $\{z_I = 0\}$. On the other hand on a toric variety there are as many linearly-independent divisors as there are $U(1)$ charges [100]. In our case the fact that the local coordinates defined by S in (5.3) are gauge-invariant is equivalent to the linear relations,

$$D_I - \sum_{A\in S}\sum_{B=1}^{s+1} \mathcal{Q}_B^A Q_I^B D_A = 0 \ . \tag{5.44}$$

Taking the charges (5.32) into account, and inserting into (5.43) then leads to,

$$c_1 = \sum_{A\in S}\left(\sum_{b=1}^{s} \mathcal{Q}_b^A (\sum_{i=1}^{k} q_i^b - n^b) + 2\mathcal{Q}_{s+1}^A\right) D_A \ , \tag{5.45}$$

which, as advertised, is even if the bundle is canonical. More generally, we see that a globally-defined $SU(3)$ structure exists provided $(\sum_{i=1}^k q_i^a - n^a)$ are even for all a [98].

We define the usual toric coordinates and a local $SU(d+1)$ structure $(\hat{J},\hat{\Omega})$ as explained in section 5.1.4. We recall that $\hat{\Omega}$ is not gauge-invariant: for the canonical $\mathbb{CP}^1$ bundle it has charge,

$$Q(\hat{\Omega}) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \ , \tag{5.46}$$

where we took (5.42) into account.

We first note that the $\mathbb{CP}^1$ fiber distinguishes a one-form K, which we normalize such that $K^* \cdot K = 2$,

$$K := \frac{1}{\sqrt{1 - h_{k+2\ k+2}|v|^2}}\mathcal{D}v = \sqrt{\frac{g}{\hat{g}}}\,\frac{\mathcal{D}v}{|u|} \ . \tag{5.47}$$

Note that K is not globally defined since it is not gauge-invariant. This can be seen explicitly by taking the $u \to 0$ limit, in which $\mathcal{D}v$ vanishes. Indeed in this limit we have,

$$K \sim \sqrt{\frac{g}{\hat{g}}}v\,\frac{\bar{u}}{|u|}\mathrm{d}u \sim e^{i(\varphi_v - \varphi_u)}\mathrm{d}u \ ,$$

where φ_u, φ_v denote the phases of u, v. However $K \wedge K^*$ does not suffer from any phase ambiguity, so that,

$$\hat{j} := \hat{J} - \frac{i}{2}K \wedge K^* \ , \tag{5.48}$$

is globally well-defined.

The next step is writing $\hat{\Omega}$ in terms of $\mathcal{D}z$. However this exercise is rather involved, since the $\mathcal{D}z$ are not independent and because of the ambiguity in the decomposition of wedge products. Our starting point is eq. (5.29),

$$\hat{\Omega} = \frac{1}{\sqrt{g}}\bigwedge_A Q_J^A z_J \partial_{z_J} \cdot \bigwedge_I \mathrm{d}z_I \ .$$

In this expression, we notice that the expansion of the contraction with the horizontal vectors amounts to choosing a set S of $s+1$ integers between 1 and $k+2$, corresponding to the indices of the contracted coordinates. We compute,

$$\hat{\Omega} = \frac{1}{\sqrt{g}} \sum_S (-1)^S Q_S \prod_{A\in S} z_A \bigwedge_{\alpha\in \complement S} \mathrm{d}z_\alpha \ ,$$

cf. (5.31), where Q_S is the determinant of the submatrix of Q_I^A whose columns are indexed by S. Notice that if S contains duplicates, or if it does not select independent columns, the determinant vanishes. Thus the sum selects only the sets S for which the matrix Q_A^B is invertible. The sign $(-1)^S$ is the signature of the permutation required to put the $s+1$ indices of S in the first position, namely :

$$(-1)^S = \sigma(S, \complement S) = (-1)^{\sum_{a\in S} + \frac{1}{2}(s+1)(s+2)} \ . \tag{5.49}$$

We would now like to decompose $\hat{\Omega}$ with respect to the bundle structure. We therefore distinguish four cases:

1. $S \subset [|1,k|]$
2. $S = \hat{S} \cup \{k+1\}$ where $\hat{S} \subset [|1,k|]$, $\sharp\hat{S} = s-1$
3. $S = \hat{S} \cup \{k+2\}$
4. $S = \check{S} \cup \{k+1, k+2\}$ where $\check{S} \subset [|1,k|]$, $\sharp\check{S} = s-2$

In the first case we get $Q_S = 0$, since $\operatorname{rank} q_i^a = d < d+1$. In cases 2 and 3 we can easily see that $Q_S = q_{\hat{S}} := \det(q_a^b)_{a\in\hat{S}}$, while $(-1)^S = (-1)^{\hat{S}}(-1)^d$ for case 2, and $(-1)^S = (-1)^{\hat{S}}(-1)^{d+1}$ for case 3. We can now write,

$$\hat{\Omega} = \frac{(-1)^{d+1}}{\sqrt{g}} \sum_{\hat{S}} (-1)^{\hat{S}} q_{\hat{S}} \prod_{a\in\hat{S}} z_a \bigwedge_{\alpha\in\complement\hat{S}} \mathrm{d}z_\alpha \wedge (v\mathrm{d}u - u\mathrm{d}v) + \frac{1}{\sqrt{g}}\Sigma_4 \ ,$$

with Σ_4 to be determined. In case 4 we get,

$$\begin{aligned} Q_S &= \det(q_a^b, -n^b)_{a\in\check{S}} \\ &= \det(q_a^b, -\sum_{i=1}^k q_i^b) \\ &= -\sum_{i=1}^k \det(q_a^b, q_i^b) \ . \end{aligned}$$

In the sum, if $i \in \check{S}$, the determinant cancels out, leaving only a sum over $\complement\check{S}$, so that,

$$\Sigma_4 = -\sum_{\check{S}} \sum_{\beta\in\complement\check{S}} (-1)^S \det(q_a, q_\beta)_{a\in\check{S}} \prod_{a\in\check{S}} z_a u\, v \bigwedge_{\alpha\in\complement\check{S}} \mathrm{d}z_\alpha \ .$$

We are now ready include this sum in the one over the $\hat{S}$, which appears in cases 2 and 3: we just need to make the change of variable $\hat{S} = \check{S} \cup \{\beta\}$. However $\mathrm{d}z_\beta$ appears in the product, thus we need to shift it to the last position. At the same time we need to move it to its right place inside $\det(q_a, q_\beta)$ so as to maintain the increasing order of $\hat{S}$. The number of shifts needed to do so is the number of shifts required to bring β from its place to the end in $\complement\check{S}$ plus the number of shifts to bring it from the end to its place in $\check{S}$; since $\complement\check{S} \cup \check{S} = [|1,k|]$, this is exactly the number of shifts required to bring β from its place to the end in $[|1,k|]$, *i.e.* $k-\beta$. The last sign we need to compute is,

$$\begin{aligned}(-1)^S &= (-1)^{\sum_{a\in\check{S}} a+(k+1)+(k+2)-\frac{1}{2}(s+1)(s+2)} \\ &= (-1)^{\sum_{a\in\hat{S}} a-\beta+(k+1)+(k+2)-\frac{1}{2}s(s+1)-(s+1)} \\ &= -(-1)^{\hat{S}}(-1)^{k-\beta+(d+1)} \ .\end{aligned}$$

Having expressed everything in terms of $\hat{S}$ and β, it is now possible to transform the sum $\sum_{\check{S}}\sum_{\beta\in\complement\check{S}}$ in $\sum_{\hat{S}}\sum_{b\in\hat{S}}$,

$$\Sigma_4 = (-1)^{d+1}\sum_{\hat{S}}(-1)^{\hat{S}} q_{\hat{S}} \prod_{a\in\hat{S}} z_a \bigwedge_{\alpha\in\complement\hat{S}} \mathrm{d}z_\alpha \ \wedge \left(u\,v\sum_{b\in\hat{S}}\frac{\mathrm{d}z_b}{z_b}\right) \ .$$

To get a more symmetrical expression we can simply complete the sum $\sum \mathrm{d}z_b/z_b$, since the missing terms can be trivially added thanks to the wedge product. The final expression is thus,

$$\hat{\Omega} = \frac{(-1)^{d+1}}{\sqrt{g}}\left(\sum_{\hat{S}}(-1)^{\hat{S}} q_{\hat{S}} \prod_{a\in\hat{S}} z_a \bigwedge_{\alpha\in\complement\hat{S}} \mathcal{D}z_\alpha\right) \wedge \left(v\mathcal{D}u - u\mathcal{D}v + u\,v\sum_{i=1}^{k}\frac{\mathcal{D}z_i}{z_i}\right) \ .$$

The $\mathrm{d}z$ were ultimately replaced by $\mathcal{D}z$ because $\hat{\Omega}$ is vertical. Now recall that the expression (5.37) decomposes $\mathcal{D}z_i$ into base and fiber parts. Since the metric decomposes correctly into (5.40), the $\hat{\mathcal{D}}z_i$ are orthogonal to K. Besides, the fiber part can be shown to cancel out in the first factor, so that the first factor is overall orthogonal to K. Thus we can take the second factor to be proportional to K, and the proportionality factor can be found by computing,

$$\begin{aligned}&K^*\cdot\Big(v\mathcal{D}u - u\mathcal{D}v + u\,v\sum_{i=1}^{k}\frac{\mathcal{D}z_i}{z_i}\Big) \\ &= \frac{2}{\sqrt{1-h_{k+2\ k+2}|v|^2}}\Big(v(0-h_{k+1,k+2}\bar{v}\,u) \\ &\qquad - u(1-h_{k+2,k+2}|v|^2) + u\,v\sum_{i=1}^{k}(0-h_{k+2\ i})\Big) \\ &= \frac{2u}{\sqrt{1-h_{k+2\ k+2}|v|^2}}\Big(-1+|v|^2(-h_{k+1,k+2}+h_{k+2,k+2}-\sum_{i=1}^{k}h_{k+2\ i})\Big) \ .\end{aligned} \tag{5.50}$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{k} h_{k+2\ i} &= g_{AB}Q^A_{k+2}\sum_{i=1}^{k}Q^B_i \\ &= g_{AB}Q^A_{k+2}(Q^B_{k+2}-Q^B_{k+1}) \\ &= h_{k+2,k+2}-h_{k+2,k+1} \ ,\end{aligned}$$

so that,

$$K^* \cdot \Big(v\mathcal{D}u - u\mathcal{D}v + u\, v \sum_{i=1}^{k} \frac{\mathcal{D}z_i}{z_i} \Big) = -2\sqrt{\frac{g}{\hat{g}}}\frac{u}{|u|} \ .$$

Hence $\hat{\Omega}$ simplifies to

$$\hat{\Omega} = \frac{(-1)^d}{\sqrt{\hat{g}}} \left(\sum_{\hat{S}} (-1)^{\hat{S}} q_{\hat{S}} \prod_{a\in\hat{S}} z_a \bigwedge_{\alpha\in\complement\hat{S}} \mathcal{D}z_\alpha \right) \wedge e^{i\varphi_u} K \ . \tag{5.51}$$

Its contraction with K is given by,

$$\frac{1}{2} K^* \cdot \hat{\Omega} = \frac{e^{i\varphi_u}}{\sqrt{\hat{g}}} \left(\sum_{\hat{S}} (-1)^{\hat{S}} q_{\hat{S}} \prod_{a\in\hat{S}} z_a \bigwedge_{\alpha\in\complement\hat{S}} \mathcal{D}z_\alpha \right) \ , \tag{5.52}$$

which is not gauge-invariant. A gauge-invariant local holomorphic form $\hat{\omega}$ on the base can be constructed as follows,

$$\hat{\omega} := \frac{1}{2} e^{-i\varphi_v} K^* \cdot \hat{\Omega} \ . \tag{5.53}$$

Let us now specialize to $d = 2$. We will now modify the local $SU(2)$ structure $(\hat{\jmath}, \hat{\omega})$ in order to construct a global $SU(3)$ structure. Since we have $|u|^2 + |v|^2 = \xi$, we can define a parameter $\theta \in [0, \pi]$ such that : $|u| = \sqrt{\xi_s}\, \sin\frac{\theta}{2}$ and $|v| = \sqrt{\xi_s}\, \cos\frac{\theta}{2}$. The $SU(3)$ structure (J, Ω) given by,

$$\begin{aligned} J &:= j + \frac{i}{2} K \wedge K^* \\ \Omega &:= \omega \wedge e^{-i\varphi_v} K \ , \end{aligned} \tag{5.54}$$

where,

$$\begin{aligned} j &:= \sin\theta\, \Re\hat{\omega} + \cos\theta\, \hat{\jmath} \\ \omega &:= \cos\theta\, \Re\hat{\omega} - \sin\theta\, \hat{\jmath} + i \Im\hat{\omega} \ , \end{aligned}$$

can be seen to be globally-defined. The argument will be detailed in section 5.3.3, where the procedure will be thoroughly applied on an illuminating example. Its associated metric is the canonical metric of the SCTV, given in (5.20), (5.40). The associated torsion classes will all be nonvanishing in general, *cf.* section 5.2.4 for more details.

This structure could be easily modified by multiplying (j, ω) and K by functions of the coordinates of the S^2 fiber. The associated metric will be modified accordingly to,

$$\mathfrak{h}_3 = |h|^2\, \mathfrak{h}_2 + |f|^2\, \frac{\hat{g}}{g} |u|^2 |v|^2 K \otimes K^* \ , \tag{5.55}$$

for some functions of the fiber coordinates, f, h. Indeed modifying the local $SU(2)$ structure via $\omega \to h^2\omega$, $j \to |h|^2\omega$, $K \to fK$ results in the metric (5.55). More generally, an orthogonal transformation can be applied on the triplet $(j, \Re\omega, \Im\omega)$, without changing the metric $\mathfrak{h}_2$ of the base.

Provided f, h are smooth, the topology of the total space is that of the SCTV $\mathbb{CP}^1$ over M. The metric (5.55) is smooth, since it is a smooth deformation of the canonical metric (5.40) of the SCTV. In some cases allowing f, h to have singularities or zeros can lead to a smooth metric on a total space of different topology. We will see an example of this phenomenon in section 5.4 where an apparently singular metric on S^2 over $\mathbb{CP}^2$ is in fact the local form of the round metric on S^6.

5.2.4 Torsion classes

For a generic SCTV base all torsion classes are nonvanishing. We will not write them down explicitly in this case, as they are rather cumbersome and not particularly illuminating. The computation boils down to determining the exterior differentials of $\hat{\omega}$ and K. In the following we give some details of the calculation.

In the notation of (5.33), K and $(\hat{j}, \hat{\omega})$ can be written as follows,

$$K = \sqrt{\frac{g}{\hat{g}}} \frac{\mathcal{D}v}{|u|} \ ; \quad \hat{j} = \hat{J} - \frac{i}{2} K \wedge K^* \ ; \quad \hat{\omega} = \frac{1}{2} e^{-i\varphi_v} K^* \cdot \hat{\Omega} \ . \tag{5.56}$$

In terms of the $\hat{\mathcal{D}}z_i$, we have,

$$\begin{aligned} \hat{j} &:= \tfrac{i}{2} \hat{\mathcal{D}}z_i \wedge \hat{\mathcal{D}}\bar{z}_i \\ \hat{\omega} &:= \frac{e^{i(\varphi_u - \varphi_v)}}{\sqrt{\hat{g}}} \textstyle\sum_{\hat{S}} (-1)^{\hat{S}} q_{\hat{S}} \prod_{a \in \hat{S}} z_a \ \bigwedge_{\alpha \in \complement \hat{S}} \hat{\mathcal{D}} z_\alpha \ . \end{aligned}$$

Up to a phase (required for gauge invariance) this coincides with the canonical local $SU(2)$-structure of the base. In particular this implies that $\hat{j}$ is Kähler at fixed fiber coordinates. The dependence of $\hat{j}$ on the fiber coordinates is such that $\hat{J}$ is Kähler.

We can also rewrite everything in local complex coordinates on the patch $S = \hat{S} \cup \{k+2\}$:

$$\begin{aligned} K &= -\sqrt{\frac{\hat{g}}{g}} |u| v \left(\frac{\mathrm{d}t_{k+1}}{t_{k+1}} + V_i |z_i|^2 \frac{\mathrm{d}t_i}{t_i} \right) \\ \hat{\omega} &= \frac{e^{i(\psi + \sum_\alpha \psi_\alpha)}}{\sqrt{\hat{g}}} (-1)^{\hat{S}} q_{\hat{S}} \prod_i |z^i| \bigwedge_\alpha \frac{\mathrm{d}t^\alpha}{t^\alpha} = f \bigwedge_\alpha \frac{\mathrm{d}t^\alpha}{t^\alpha} \ , \end{aligned} \tag{5.57}$$

where ψ, ψ_α are the phases of t_{k+1}, t_α. We can now introduce real coordinates θ, ψ on the fiber with $|u|^2 = \xi \sin^2 \frac{\theta}{2}$:

$$K = \frac{1}{2} \left(\gamma \mathrm{d}\theta + \frac{\xi}{\gamma} \sin\theta \ i(\mathrm{d}\psi + A) \right) \ ,$$

where $\gamma = \sqrt{\frac{g}{\hat{g}}} = \sqrt{\xi + \frac{1}{2}\xi^2 \, V \sin^2\theta}$ and $A = V_i \, |z^i|^2 \, \Im \frac{\mathrm{d}t^i}{t^i} = V_i \, |z^i|^2 \, \mathrm{d}\psi_i$. We also get,

$$\mathrm{d}A = \frac{i}{2} V_i \, \hat{\mathcal{D}}z_i \wedge \hat{\mathcal{D}}\bar{z}_i + \frac{i}{4} \sin\theta \mathrm{d}\theta \ \wedge V_i^2 \left(\bar{z}_i \hat{\mathcal{D}}z_i - z_i \hat{\mathcal{D}}\bar{z}_i \right) \ .$$

Differentiating $\hat{\omega}$ leads to another one-form,

$$\mathrm{d}\hat{\omega} = \frac{\mathrm{d}f}{f} \wedge \hat{\omega} = \left(-\frac{\xi}{2} V_i (1 - \hat{h}_{ii} |z_i|^2) \sin\theta \mathrm{d}\theta + i \mathrm{d}\psi + \sum_j i \mathrm{d}\psi_j + \frac{1}{2} (1 - \hat{h}_{jj} |z_j|^2) \left(\frac{\hat{\mathcal{D}}\bar{z}^j}{\bar{z}^j} - \frac{\hat{\mathcal{D}}z_j}{z_j} \right) \right) \wedge \hat{\omega} \ .$$

Alternatively, in terms of t^i,

$$\mathrm{d}\hat{\omega} = \left(-\frac{\xi}{2} V_i (1 - \hat{h}_{ii} |z_i|^2) \sin\theta \mathrm{d}\theta + i \left(\mathrm{d}\psi + A + \mathrm{d}\psi_i |z_i|^2 (h_{ii} - h_{ij} h_{jj} |z_j|^2) \right) \right) \wedge \hat{\omega} \ .$$

We can write,

$$A' := A + \mathrm{d}\psi_i |z_i|^2 (h_{ii} - h_{ij} h_{jj} |z_j|^2) = A + B \ ,$$

where B comes from the derivatives of $\hat{g}$ and is nonvanishing in general. For simple bases such as $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\hat{g}$ is constant and thus B vanishes. The $\mathrm{d}\theta$ term comes from the deformation of the base metric along the direction θ. Note also that at fixed θ, $\mathrm{d}A' \propto \mathcal{R}$ where $\mathcal{R}$ is the Ricci form of the base, *cf.* (B.4).

5.3 A thorough application: $\mathbb{CP}^1$ over $\mathbb{CP}^2$

Let us now examine in detail the construction of an $SU(3)$ structure on the $\mathbb{CP}^1$ bundle over $\mathbb{CP}^2$. This is the simplest example in the class of 3d SCTV of the form $\mathbb{CP}^1$ bundle over M, where M is a 2d SCTV, but it already captures the main idea of the construction.

5.3.1 Illustration of the symplectic quotient

The toric data in this case are: $k = 5$ (the complex dimension of the parent space), $s = 2$ (the number of charges), $d = 3$ (the complex dimension of the toric variety). Explicitly the charges are given by,

$$Q = \begin{pmatrix} 1 & 1 & 1 & -n & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} , \tag{5.58}$$

where $n \in \mathbb{N}$. The corresponding moment map equations read, using previous notations,

$$\begin{aligned} |z_1|^2 + |z_2|^2 + |z_3|^2 &= \xi_1 + n|u|^2 \\ |u|^2 + |v|^2 &= \xi \ . \end{aligned} \tag{5.59}$$

This is a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^2$, with twisting parameterized by n. We can make this more explicit in local coordinates: on the patch $U_{1,5} := \{z_1, v \neq 0\}$ we define,

$$t_2 := \frac{z_2}{z_1} \ ; \quad t_3 := \frac{z_3}{z_1} \ ; \quad t_4 := \frac{z_1^n u}{v} \ .$$

Hence t_2, t_3 are local coordinates parameterizing a $\mathbb{CP}^2$ whereas, for z_1 fixed, t_4 is a local coordinate on a $\mathbb{CP}^1$. For $n = 0$, the bundle becomes trivial and we obtain the direct product $\mathbb{CP}^2 \times \mathbb{CP}^1$. We can also see explicitly that the toric variety can be covered with patches of the form U_S, as in (5.3): in the present case S is given by the pair (i, j) where $i = 1, 2, 3$ and $j = 4, 5$, and the moment map equations (5.59) exclude the simultaneous vanishing of z_1, z_2, z_3 or that of u, v. To make contact with our previous discussion about local coordinates we can check here that $|Q_S| = 1$ for all these S. However for the patch $U_{4,5}$, we do not get in general compatible local coordinates since $Q_{S=\{4,5\}} = -n$, but this patch is not needed in the covering of the toric variety.

Let us now calculate explicitly the various objects introduced in section 5.1. Since the base is defined by only one charge $q_i = (1, 1, 1)$, the calculations are rather simple. We get at first :

$$\begin{aligned} \hat{g} &= \rho^2 \\ V_i &= \frac{n}{\rho^2} \\ V &= \frac{n^2}{\rho^2} \\ g &= \xi\rho^2 + n^2|u|^2|v|^2 \ . \end{aligned} \tag{5.60}$$

We find thus that :

$$\varepsilon = \frac{\mathrm{d}t_4}{t_4} + n\,\eta \ ,$$

where we have set,

$$\eta := \frac{1}{1+t^2}\left(\bar{t}_2 \mathrm{d}t_2 + \bar{t}_3 \mathrm{d}t_3\right) \ ; \quad t^2 := |t_2|^2 + |t_3|^2 \ . \tag{5.61}$$

If we now introduce

$$\Gamma := \frac{|u|^2|v|^2}{g} ,$$

the decomposition of the metric (5.40) can be written :

$$\mathfrak{h} = \rho^2 \mathfrak{h}_{\mathbb{CP}^2} + \Gamma\rho^2 \left| \frac{\mathrm{d}t_4}{t_4} + n\,\eta \right|^2 ; \tag{5.62}$$

$\mathfrak{h}_{\mathbb{CP}^2}$ is the hermitian Fubini-Study metric of $\mathbb{CP}^2$, *cf.* eq. (5.24) with radius 1.

We see the fibration structure appearing naturally in (5.62): the displacement along t_4 is modified by a connection, proportional to η, depending on the variables of the $\mathbb{CP}^2$ base, t_2, t_3. Moreover,

$$\mathrm{d}\eta = 2i\hat{\jmath} , \tag{5.63}$$

where $\hat{\jmath}$ is the Kähler form of $\mathbb{CP}^2$, *cf.* eq. (5.24). For vanishing n the connection piece drops out from the vertical displacement and the metric becomes that of a direct product as excpected.

5.3.2 Comparison with the literature

Endowed with the hermitian metric (5.24), the base $\mathbb{CP}^2$ of the $\mathbb{CP}^1$ fibration is a Kähler-Einstein manifold obeying,

$$\mathrm{d}\hat{\jmath} = 0 ; \quad R_{mn} = \lambda\, \mathfrak{g}_{mn} . \tag{5.64}$$

i.e. $\hat{\jmath}$ is closed and the Ricci tensor is proportional to the metric. With our conventions, setting $\xi = 1$ gives $\lambda = 6$. Identifying the $\mathbb{CP}^1$ fiber with S^2 (by forgetting the complex structure), M can be thought of as an S^2 fibration over a Kähler-Einstein base B_4, denoted by $S^2(B_4)$. These spaces appear naturally in the context of supersymmetric AdS_4 compactifications of M-theory on the so-called $Y^{p,q}(B_4)$ spaces [87, 89], which can be thought of as S^1 fibrations over $S^2(B_4)$. Compactifying M-theory on an appropriately chosen S^1 then leads to $\mathcal{N} = 2$ type IIA solutions of the form $AdS_4 \times S^2(B_4)$ [91]. The latter can be deformed to solutions of massive IIA for any Kähler-Einstein base B_4 [92], although regularity requires B_4 to have positive curvature.

In the conventions of [89] the $S^2(B_4)$ metric reads,

$$\mathfrak{g} = U^{-1}\mathrm{d}\tilde{\rho}^2 + \tilde{\rho}^2 \mathfrak{g}_{\mathbb{CP}^2} + q\,(\mathrm{d}\psi + A)^2 , \tag{5.65}$$

where $\tilde{\rho} \in [\tilde{\rho}_1, \tilde{\rho}_2]$ and $\psi \in [0, 2\pi/3]$ are the coordinates of the S^2 fiber (for general λ the period of ψ is $4\pi/\lambda$); U and q are positive functions of $\tilde{\rho}$, vanishing at $\tilde{\rho}_1$ et $\tilde{\rho}_2$. The circle parametrized by ψ is fibered over the $[\tilde{\rho}_1, \tilde{\rho}_2]$ interval. The connection A is a one-form on the base B_4 obeying,

$$\mathrm{d}A = 2\hat{\jmath} . \tag{5.66}$$

At the endpoints of the $\tilde{\rho}$ interval the ψ circle contracts to a point, thus resulting in a total space with the topology of S^2. The period of ψ is fixed by requiring the metric to be smooth at the endpoints, *i.e.* that,

$$U^{-1}\mathrm{d}\tilde{\rho}^2 + q\,\mathrm{d}\psi^2 \to \mathrm{d}u^2 + u^2\mathrm{d}\tilde{\psi}^2 , \quad \text{for } \tilde{\rho} \to \tilde{\rho}_1, \tilde{\rho}_2 , \tag{5.67}$$

where u is a function of $\tilde{\rho}$ that vanishes at the endpoints $\tilde{\rho}_1, \tilde{\rho}_2$, and we have defined an angular variable $\tilde{\psi} := \lambda\psi/2$ with period 2π.

Moreover the ψ coordinate parametrizes an S^1 fibration in the canonical bundle of B_4. To see this, note that the connection of the canonical bundle of a Kähler-Einstein space with curvature normalized as in (5.64) obeys,

$$\mathrm{d}\mathcal{P} = \lambda \hat{j} \ , \tag{5.68}$$

cf. appendix B. Comparing with (5.66) we see that $\mathcal{P} = \lambda A/2$, and so the vertical displacement along the S^1 fiber, *cf.* the last term in (5.65), is proportional to $(\mathrm{d}\tilde{\psi} + \mathcal{P})$, as required for the canonical bundle. The fact that λ is positive for $\mathbb{CP}^2$ guarantees that the total space of the S^1 fibration, written in local coordinates in (5.65), extends globally to a smooth five-dimensional (squashed) Sasaki-Einstein space.

Real coordinates

To make contact with the coordinates of (5.62), we must rewrite the $\mathbb{CP}^1$ fiber coordinate t_4 in terms of a pair of real coordinates. It is not necessary to do the same for t_2, t_3, since the coordinates of the $\mathbb{CP}^2$ base do not appear explicitly in (5.65). Using eq. (5.34), $|t_4|$ can be written in terms of ρ and the base coordinates,

$$|t_4|^2 = \frac{|z_1|^{2n}|z_4|^2}{|z_5|^2} = \frac{\rho^{2n}}{(1+t^2)^n}\frac{\rho^2-\rho_1^2}{\rho_2^2-\rho^2} \ . \tag{5.69}$$

Let $\varphi \in [0, 2\pi]$ be the phase of t_4, so that t_4 becomes a function of t_2, t_3, ρ, φ,

$$\begin{aligned} \frac{\mathrm{d}t_4}{t_4} &= \frac{\rho \mathrm{d}\rho}{\rho^2-\rho_1^2} + \frac{\rho \mathrm{d}\rho}{\rho^2-\rho_1^2} + n\,\frac{\mathrm{d}\rho}{\rho} - n\,\frac{\mathrm{d}(t^2)}{2(1+t^2)} + i\mathrm{d}\varphi \\ &= \frac{\mathrm{d}\rho}{n\,\rho\Gamma} - n\Re\eta + i\mathrm{d}\varphi \ . \end{aligned}$$

Moreover we set,

$$\varepsilon := \frac{\mathrm{d}t_4}{t_4} + n\,\eta = \frac{\mathrm{d}\rho}{n\,\rho\Gamma} + i\,(\mathrm{d}\varphi + n\,\Im\eta) \ . \tag{5.70}$$

The term $|\varepsilon|^2 := \varepsilon \otimes \bar{\varepsilon}$ appears naturally in (5.62) through the contribution,

$$\varepsilon \otimes \bar{\varepsilon} = \frac{1}{n^2\rho^2\Gamma^2}\mathrm{d}\rho^2 + (\mathrm{d}\varphi + n\,\Im\eta)^2 - i\frac{1}{n\,\rho\Gamma}\,\mathrm{d}\rho \wedge (\mathrm{d}\varphi + n\,\Im\eta) \ . \tag{5.71}$$

The last term on the right-hand side above contributes to the Kähler form, while the rest contributes to the metric. We can then rewrite the Riemannian metric $\mathfrak{g}$ and Kähler form J associated with (5.62) for $n \neq 0$. The result reads,

$$\mathfrak{g} = \frac{1}{n^2\Gamma}\mathrm{d}\rho^2 + \rho^2\mathfrak{g}_{\mathbb{CP}^2} + \Gamma\rho^2\,(\mathrm{d}\varphi + n\Im\eta)^2 \ , \tag{5.72}$$

and,

$$J = \rho^2\hat{j} + \frac{\rho}{n}\,\mathrm{d}\rho \wedge (\mathrm{d}\varphi + n\Im\eta) \ , \tag{5.73}$$

where we are using local coordinates on the patch $U_{1,5}$. The $\mathbb{CP}^1$ fiber is parametrized by the (ρ, φ) coordinates: φ parametrizes a circle, fibered over the interval $\rho \in [\rho_1, \rho_2] = [\sqrt{\xi_1}, \sqrt{\xi_1 + n\xi}]$, whose radius vanishes at the endpoints. Indeed Γ vanishes for $u = 0$ or $v = 0$ which corresponds respectively to $\rho = \rho_1$ and $\rho = \rho_2$, following from the moment maps equations (5.59). Moreover it can be checked that the metric is smooth there.

Deformation of the metric

Setting $\psi := \varphi/n$ and $A := \Im\eta$, we recover the terms appearing in (5.65), provided we set $n = 3$. Moreover the relative coefficient between the $\mathrm{d}\rho^2$ and the $(\mathrm{d}\psi + A)^2$ term is fixed in the expression of $|\varepsilon|^2$, and this determines the change of variables $\rho \to \tilde{\rho}(\rho)$ by comparing with (5.65). However, performing this change of variables in (5.62) does not directly bring us to the metric of (5.65): there remain two coefficients that still need to be adjusted. This can be achieved by introducing two warp factors F and G as we now show.

Let us go back to the expression of the metric in terms of $\mathcal{D}z_i$. In local coordinates we have,

$$
\begin{aligned}
\frac{\mathcal{D}z_1}{z_1} &= n\,\Gamma\varepsilon - \eta \\
\frac{\mathcal{D}z_2}{z_2} &= \frac{\mathrm{d}t_2}{t_2} + n\,\Gamma\varepsilon - \eta \\
\frac{\mathcal{D}z_3}{z_3} &= \frac{\mathrm{d}t_3}{t_3} + n\,\Gamma\varepsilon - \eta \\
\frac{\mathcal{D}z_4}{z_4} &= \rho^2\frac{\rho_2^2 - \rho^2}{n\,\det g}\,\varepsilon \\
\frac{\mathcal{D}z_5}{z_5} &= \rho^2\frac{\rho_1^2 - \rho^2}{n\,\det g}\,\varepsilon\ .
\end{aligned}
\tag{5.74}
$$

It follows that the term $\sum_{i=1}^{3}\mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i$ gives the hermitian metric of $\mathbb{CP}^2$ plus a $|\varepsilon|^2$ term, whereas $\mathcal{D}z_4, \mathcal{D}z_5$ only contribute to $|\varepsilon|^2$. Let us define,

$$
\begin{aligned}
\mathfrak{h} &= F(\rho)\sum_{i=1}^{3}\mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i + G(\rho)\sum_{i=4,5}\mathcal{D}z_i \otimes \mathcal{D}\bar{z}_i \\
&= F\rho^2\,\mathfrak{h}_{cp2} + \left(F + (\frac{1}{n^2\Gamma} - 1)G\right) n^2\rho^2\Gamma^2|\varepsilon|^2 \\
&= F\rho^2\,\mathfrak{h}_{cp2} + \left(F + (\frac{1}{n^2\Gamma} - 1)G\right)\left(\mathrm{d}\rho^2 + n^4\rho^2\Gamma^2\,(\mathrm{d}\psi + A)^2 - in^2\rho\Gamma\,\mathrm{d}\rho \wedge (\mathrm{d}\psi + A)\right)\ .
\end{aligned}
\tag{5.75}
$$

We can then adjust F,G, and ρ so that,

$$
\begin{aligned}
F\rho^2 &= \tilde{\rho}^2 \\
\left(F + (\frac{1}{n^2\Gamma} - 1)G\right)\,\mathrm{d}\rho^2 &= \frac{1}{U}\mathrm{d}\tilde{\rho}^2 \\
\left(F + (\frac{1}{n^2\Gamma} - 1)G\right) n^4\rho^2\Gamma^2 &= q\ .
\end{aligned}
\tag{5.76}
$$

These equations can easily be decoupled by first solving for ρ, then for F and finally for G. For this solution the real and imaginary parts of (5.75) reduce to the metric in (5.65) and the form J_+ of [89] respectively, provided we set $n = 3$.

The condition $n = 3$ is also important for the existence of a globally-defined $SU(3)$ structure. We turn to the construction of this structure in section 5.3.3. Note however that the canonical metric of the SCTV, eq. (5.20), is smooth by construction for all $n \in \mathbb{N}$. This can also be verified explicitly by examination of the local form of the metric in terms of the coordinates (5.3) in each patch U_S.

5.3.3 The SU(3) structure

In this section we will set $F = G = 1$ for simplicity of presentation: the two warp factors $F(\rho)$, $G(\rho)$ discussed in section 5.3.2 can be easily reinstated without changing any of the conclusions.

Specializing the formalism of section 5.1.4 to the present example we obtain a local $SU(3)$ structure $(\hat{J}, \hat{\Omega})$, where $\hat{J}$ is obtained from (5.73) by setting $n = 3$. On the other hand we have,

$$\hat{\Omega} = -\frac{z_5^2}{\sqrt{\det g}}\, \mathrm{d}t_2 \wedge \mathrm{d}t_3 \wedge \mathrm{d}t_4 \ , \tag{5.77}$$

which is not gauge invariant, so this $SU(3)$ structure is not globally defined. In fact neither of the two local $SU(3)$ structures $(J_\pm, \Omega_\pm)$ of [89] can be globally extended: in the following we will see how to make contact with their results.

We can now apply the construction of 5.2.3. Let us first define a local $SU(2)$ structure $(\hat{j}, \hat{\omega})$ on $\mathbb{CP}^2$ where[5],

$$\hat{\omega} = \frac{1}{(1+t^2)^{3/2}}\, \mathrm{d}t_2 \wedge \mathrm{d}t_3 \ , \tag{5.78}$$

and $\hat{j}$ is the Kähler form of $\mathbb{CP}^2$, *cf.* eq. (5.24), so that,

$$\begin{aligned} \hat{\omega} \wedge \hat{j} &= 0 \\ \hat{\omega} \wedge \hat{\omega}^* &= 2\hat{j} \wedge \hat{j} \ . \end{aligned} \tag{5.79}$$

This $SU(2)$ structure is only locally defined since $\hat{\omega}$ has a singularity at $z_1 = 0$, as can be seen by using the transition functions to rewrite $\hat{\omega}$ in a patch where z_1 is allowed to vanish. The $SU(3)$ structures of [89] are then obtained by appending the contribution of the fiber coordinate,

$$J_\pm := \rho^2 \hat{j} \pm \frac{i}{2} K \wedge K^* \ ; \quad \Omega_+ := \rho^2\, \hat{\omega} \wedge K \ ; \quad \Omega_- := \rho^2\, \hat{\omega} \wedge K^* \ , \tag{5.80}$$

where,

$$K := \rho\sqrt{\Gamma}\, \varepsilon \ . \tag{5.81}$$

We see that exchanging $K \leftrightarrow K^*$ is equivalent to $(J_+, \Omega_+) \leftrightarrow (J_-, \Omega_-)$.

To better understand the global properties of the $\Omega_\pm$, let us start from their local expression on the patch $U_{1,5}$,

$$\begin{aligned} \Omega_+ &= e^{-i\varphi} \frac{|z_5|^2}{\sqrt{\det g}}\, \mathrm{d}t_2 \wedge \mathrm{d}t_3 \wedge \mathrm{d}t_4 \\ \Omega_- &= e^{i\varphi} \frac{|z_5|^2}{\sqrt{\det g}}\, \mathrm{d}t_2 \wedge \mathrm{d}t_3 \wedge \left(\mathrm{d}\bar{t_4} + 3\bar{t_4}\bar{\eta}\right) \ . \end{aligned}$$

We can see that the singularity in $\hat{\omega}$ has been compensated by wedging with K, K^*. On the other hand, we can rewrite $\Omega_\pm$ in the patch $U_{1,4}$ by using the transition function $t_5 = 1/t_4$,

$$\begin{aligned} \Omega_+ &= e^{i\varphi} \frac{|z_1|^6 |z_4|^2}{\sqrt{\det g}}\, \mathrm{d}t_2 \wedge \mathrm{d}t_3 \wedge (-\mathrm{d}t_5) \\ \Omega_- &= e^{-i\varphi} \frac{|z_1|^6 |z_4|^2}{\sqrt{\det g}}\, \mathrm{d}t_2 \wedge \mathrm{d}t_3 \wedge \left(-\mathrm{d}\bar{t_5} + 3\bar{t_5}\bar{\eta}\right) \ . \end{aligned}$$

[5]Note that these definitions match (5.48) and (5.53)

We see that $\Omega_\pm$ has singularities of the form $e^{i\varphi} = t_4/|t_4| = |t_5|/t_5$ at $t_4 = 0$ and $t_5 = 0$: indeed the phase of a complex number z is ambiguous at $z = 0$. It is always possible to soak up one of the two singularities by multiplying or dividing by $e^{i\varphi}$, but never both at the same time. Hence $e^{\pm i\varphi}\Omega_\pm$ are well-defined at $t_4 = 0$ but singular at $t_5 = 0$, whereas $e^{\mp i\varphi}\Omega_\pm$ are well defined at $t_5 = 0$ but singular at $t_4 = 0$. This problem does not arise for $J_\pm$, since $K \wedge K^*$ does not suffer from any phase ambiguities.

The way out is then to construct an Ω which combines both $e^{\pm i\varphi}\Omega_\pm$ and $e^{\mp i\varphi}\Omega_\pm$. We can take a hint from the supersymmetric $SU(3)$ structure of [92] which we know is globally well-defined. We use a new coordinate θ instead of ρ, defined by $|u|^2 = \xi \sin^2 \frac{\theta}{2}$. Thus we see that $|v|^2 = \xi \cos^2 \frac{\theta}{2}$ and $\rho^2 = \xi^1 + n\,\xi \sin^2 \frac{\theta}{2}$, which means that $\theta = 0$ or π for $\rho = \rho_1$ (corresponding to $t_4 = 0$) or $\rho = \rho_2$ (corresponding to $t_5 = 0$), respectively. The idea is then to modify $\hat{\omega} \to \omega$ by including the problematic phase $e^{i\varphi}$, then define another form $\tilde{\omega}$ with the property that $\tilde{\omega}$ varies from ω to ω^* as θ varies from 0 to π. More specifically we define,

$$
\begin{aligned}
\omega &:= e^{i\varphi}\hat{\omega} \\
\tilde{j} &:= \sin\theta\, \Re\hat{\omega} + \cos\theta\, \hat{j} \\
\tilde{\omega} &:= \cos\theta\, \Re\hat{\omega} - \sin\theta\, \hat{j} + i\Im\hat{\omega} \ ,
\end{aligned}
\tag{5.82}
$$

so that the $SU(3)$ is given by,

$$
\begin{aligned}
J &:= \rho^2 \tilde{j} + \frac{i}{2} K \wedge K^* \\
\Omega &:= \rho^2\ \tilde{\omega} \wedge K \ .
\end{aligned}
\tag{5.83}
$$

The relations (5.79) ensure that (1.24) is satisfied. Moreover at $\theta = 0$ we have $\Omega = e^{i\varphi}\Omega_+$, whereas at $\theta = \pi$ we have $\Omega = -\left(e^{i\varphi}\Omega_-\right)^*$. The two singularities have thus been regularized and Ω is globally defined. Thus the pair (J, Ω) is a globally-defined structure $SU(3)$ on the manifold.

Let us make one final comment: the prescription of [97] for constructing global $SU(3)$ structures, reviewed at the end of section 5.1.4, gives a form Ω which is of type (2,1) with respect to the integrable complex structure of the toric variety. We see that the prescription used here can never coincide with that of [97]: the form Ω defined in eq. (5.83) is of mixed type, varying from (3,0) at $\theta = 0$ to (1,2) at $\theta = \pi$, with respect to the integrable complex structure.

5.4 LT structures on $S^2(B_4)$

We will now show that the sphere bundles of the form $S^2(B_4)$, where B_4 is any four-dimensional Kähler-Einstein space of positive curvature, admit regular globally-defined $SU(3)$ structures of LT type, *i.e.* such that all torsion classes vanish except for W_1 and W_2. This is the generic type of $SU(3)$ structure that appears in supersymmetric AdS$_4$ compactifications of massive IIA supergravity [24].

Let $\hat{j}$ be the Kähler form of B_4, normalized as in (B.7), (B.8) with $\lambda = 6$, and let $(\hat{j}, \hat{\omega})$ be a local $SU(2)$ structure on B_4 so that,

$$
\begin{aligned}
&\hat{\omega} \wedge \hat{\omega}^* = 2\hat{j} \wedge \hat{j} \ ; \quad \hat{j} \wedge \hat{\omega} = 0 \ ; \\
&\mathrm{d}\mathcal{P} = 6\ \hat{j} \ ; \quad \mathrm{d}\hat{j} = 0 \ ; \quad \mathrm{d}\hat{\omega} = i\mathcal{P} \wedge \hat{\omega} \ ,
\end{aligned}
\tag{5.84}
$$

where $\mathcal{P}$ is the canonical bundle of B_4, *cf.* appendix B. We define the following $SU(3)$ structure,

$$\begin{aligned} J &= |h|^2 j + \frac{i}{2} K \wedge K^* \\ \Omega &= h^2 \omega \wedge K \ , \end{aligned} \tag{5.85}$$

where h is a complex function of θ and,

$$\begin{aligned} j &:= \cos\theta\ \hat{j} + \sin\theta\ \Re(e^{i\psi}\hat{\omega}) \\ \omega &:= -\sin\theta\ \hat{j} + \cos\theta\ \Re(e^{i\psi}\hat{\omega}) + i\ \Im(e^{i\psi}\hat{\omega}) \\ K &:= f\mathrm{d}\theta + ig(\mathrm{d}\psi + \mathcal{P}) \ , \end{aligned} \tag{5.86}$$

with $\psi \in [0, 2\pi)$ and f, g real functions of θ. The associated metric reads,

$$\mathfrak{g} = |h|^2 \mathfrak{g}_4 + f^2 \mathrm{d}\theta^2 + g^2 (\mathrm{d}\psi + \mathcal{P})^2 \ , \tag{5.87}$$

with $\mathfrak{g}_4$ the Kähler-Einstein metric of B_4.

Using eq. (5.84), one can then compute the torsion classes of the $SU(3)$ structure (5.85),

$$\begin{aligned} W_1 &= -\frac{i}{3}\frac{h}{h^*}\left(\frac{1}{f} + \frac{\sin\theta}{g} + 6\frac{g\sin\theta}{|h|^2}\right) \\ W_2 &= \frac{i}{3}\frac{h}{h^*}\left(\frac{1}{f} + \frac{\sin\theta}{g} - 12\frac{g\sin\theta}{|h|^2}\right) J^\perp \\ W_3 &= \frac{1}{2}\left(\frac{1}{f} - \frac{\sin\theta}{g} + 6\frac{g\sin\theta}{|h|^2}\right) \Re\Omega^\perp \\ W_4 &= \left(\frac{|h^2|'}{f|h^2|} - 6\cos\theta\frac{g}{|h^2|}\right)\mathrm{d}\theta \\ W_5 &= \left(\frac{h'}{fh} + \frac{g'}{2fg} - \frac{\cos\theta}{2g}\right) K \ , \end{aligned} \tag{5.88}$$

where we have introduced the primitive forms,

$$\begin{aligned} J^\perp &= |h|^2\ j - iK \wedge K^* \\ \Omega^\perp &= |h|^2\ \omega \wedge K^* \ . \end{aligned} \tag{5.89}$$

Moreover, as we show in appendix D, one can impose $W_3 = W_4 = W_5 = 0$ provided,

$$f = \alpha\left(1 - 6\alpha^2\frac{\sin^2\theta}{H}\right)^{-1} \ ; \quad g = \alpha\sin\theta\ ; \quad h = \sqrt{H(\theta)}\ e^{i\beta} \ , \tag{5.90}$$

with,

$$\begin{aligned} H(\theta) &:= \frac{1}{3}\left(\tilde{x} + \frac{\tilde{x}^2}{B} + B\right) \\ B &:= \left(\frac{27H_0^3}{2} + \tilde{x}^3 + 3\sqrt{3}\sqrt{\frac{27H_0^6}{2} + \tilde{x}^3 H_0^3}\right)^{1/3} \\ \tilde{x} &:= 9\alpha^2\sin^2\theta \ , \end{aligned} \tag{5.91}$$

where the real constants α, β and $H_0 \geq 0$ are the parameters of the solution.

For $H_0 > 0$ the functions f, h are nowhere vanishing. Moreover the $\theta \to 0, \pi$ limit gives a regular metric, provided the period of ψ is 2π. Then by the same argument as in [87, 92], the $SU(3)$ structure (5.85) is globally-defined and the associated metric (5.87) is regular and complete: the (ψ, x^μ) space, where x^μ are the coordinates of B_4, parametrizes a circle fibration in the canonical bundle $\mathcal{L}$ over B_4; it extends to a complete, regular five-dimensional Sasaki-Einstein manifold provided B_4 is Kähler-Einstein of positive curvature [102]. The (ψ, θ) space parameterizes a smooth S^2, so that the total space has the same topology as $\mathcal{L} \times_{U(1)} \mathbb{CP}^1$, in the notation of [87]. The nonvanishing torsion classes read,

$$
\begin{aligned}
W_1 &= -\frac{2i}{3}\frac{e^{2i\beta}}{\alpha} \\
W_2 &= \frac{2i}{3}\frac{e^{2i\beta}}{\alpha}\left(1 - \frac{9\alpha^2 \sin^2\theta}{H}\right) J^\perp \ .
\end{aligned}
\tag{5.92}
$$

Therefore the $S^2(B_4)$ bundles admit $SU(3)$ structures of LT type, rendering them suitable as compactification spaces for supersymmetric AdS$_4$ solutions of massive IIA [24]. Note that unlike the LT $SU(3)$ structures on $S^2(\mathbb{CP}^2)$ discussed in [103] from the point of view of twistor spaces (*cf.* appendix E) or in [104] from the point of view of cosets, the structure (5.92) does not obey $\mathrm{d}W_2 \in (3,0) \oplus (0,3)$.[6] Indeed a direct calculation gives,

$$
\mathrm{d}W_2 = e^{2i\beta}\left(1 - \frac{9\alpha^2\sin^2\theta}{H}\right)\left(\frac{2i}{3\alpha^2}\left(1 - \frac{9\alpha^2\sin^2\theta}{H}\right)\Re(e^{-2i\beta}\Omega) - \frac{6i\sin^2\theta}{H}\Re\Omega^\perp\right) \ . \tag{5.93}
$$

As a consequence, if these manifolds are to be used as compactification spaces for massive IIA, the Bianchi identity for the RR two-form will require the introduction of (smeared) six-brane sources. Another difference from the LT structures of [103, 104] is that the discussion of this section applies to any $S^2(B_4)$ bundle with Kähler-Einstein base, not only to $B_4 = \mathbb{CP}^2$.

In the case $H_0 = 0$, on the other hand, one obtains the solution,

$$
f = 3\alpha \ ; \quad g = \alpha\sin\theta \ ; \quad h = 3\alpha\sin\theta\ e^{i\beta} \ . \tag{5.94}
$$

This corresponds to the *nearly Kähler* limit, in which also W_2 vanishes. Moreover the $\theta \to 0, \pi$ limit results in a conical metric of the form,

$$
\mathfrak{g} \sim \mathrm{d}\theta^2 + \theta^2 \mathrm{d}s_5^2 \ , \tag{5.95}
$$

where,

$$
\mathrm{d}s_5^2 := \mathfrak{g}_4 + \frac{1}{9}(\mathrm{d}\psi + \mathcal{P})^2 \ , \tag{5.96}
$$

is the canonically normalized metric of a five-dimensional Sasaki-Einstein base written as a circle fibration on the canonical bundle over B_4; the normalization is such that the cone metric (5.95) is Ricci-flat. Hence for $H_0 = 0$ the metric presents conical singularities in general, unless B_4 is $\mathbb{CP}^2$, in which case the associated Sasaki-Einstein metric (5.96) is that of the round sphere, and the associated cone (5.95) is not only Ricci-flat but also flat. Going back to the metric (5.87) we obtain,

$$
\mathfrak{g} = 9\alpha^2\left(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}s_5^2\right) \ . \tag{5.97}
$$

[6] It should be possible to make contact with the results of [103, 104] by suitably acting on the vielbein by an orthogonal transformation. There does not seem to exist a simple ansatz for this transformation, which may be rather involved as it could *a priori* depend on all coordinates.

We thus see that in the smooth case, $B_4 = \mathbb{CP}^2$, we obtain a round S^6 of radius 3α. We thus recover the well-know result that the round S^6 admits an associated nearly-Kähler structure.

Let us finally note that we may relax the condition on B_4, so that B_4 is any four-dimensional Kähler manifold (not necessarily toric, or Einstein). In this case the torsion classes can also be explicitly calculated, *cf.* appendix F, however we do not expect the structure to admit a global extension to a complete space with a regular metric.

Conclusion

In this thesis flux vacua have been studied through several approaches. The $SU(3)$-structure was often at the core of the discussion.

Consistent truncations of type IIA supergravity to four dimensions were presented. The truncation ansätze are guided by the $SU(3)$-structure of the internal manifold. Quadratic and quartic fermionic terms where added in the Lagrangian to model the effects of fermionic condensation. At first the internal manifold is Kähler Einstein and the condensates are dilatini. Subsequently a consistent truncation to the universal sector of Calabi-Yau compactification is constructed then supplemented with gravitini condensates. In both cases the truncations admitted, at least formally, solutions with positive cosmological constant and non vanishing fluxes supported by the quartic condensates. By consistency of the ansatz these solutions can be lifted to de Sitter vacua. At this time it is not clear whether such fermionic condensates are realistic from a string theory point of view. In the gravitini case it is suggested that the condensates originate from gravitational instantons. If this intuition is correct the condensates are under control and can lead to realistic values for the cosmological constant. Still the de Sitter solution relies on the quartic term being positive. The spin 3/2 zero modes of the Dirac operator of the $\tau = 2$ ALE instanton should be determined to elucidate on this point. Moreover the stability of the vacua cannot be checked in the truncated theory as the truncated modes may lead to instabilities. This issue needs to be further addressed.

The non abelian T-dual of several brane configurations were computed. The dual of the flat background is a NS5 brane configuration given by a continuous distribution of charge along a half-line. This distribution is bound to the singularity generated by the fixed point of the $SU(2)$-isometry. In fact such a smeared NS5 brane is expected to appear whenever the orbits of the $SU(2)$-isometry collapse in a similar way, which is then confirmed in the non trivial cases considered later. Moreover D branes were studied by computing the Page charges in the dual configuration. This leads to general relations between the charges and their behavior under a large gauge transformation of the B field. The supersymmetry equations for the D2 solution were solved in the formalism of $SU(3) \times SU(3)$-structure. It would be interesting to do the same thing for its dual, thus furnishing a check for supersymmetry and hopefully providing new insights into its global structure. The same tools were also used to investigate on the existence of a massive deformation for the supersymmetric D2 solution, ruling out a class of ansätze. Further work is needed to draw definitive conclusions.

The construction of SU(3)-structures on SCTV had up to now proceeded on a case-by-case basis. The last chapter presented a systematic construction valid on the canonical $\mathbb{CP}^1$-bundle of any two-dimensional SCTV. It was also shown how to modify this construction by introducing certain functions, thus opening up more possibilities for the torsion classes. Such general constructions are necessary steps for a systematic scan of flux vacua. The admissibility of a manifold with $SU(3)$-structure as internal space for compactification relies on its torsion. In the most general case the $SU(3)$-structure presented here has very involved torsion classes, so that is it difficult to get conclusive results. Note also that the construction has been extended to cases where the base manifold is not necessarily toric but Kähler-Eintein. These $S^2(\mathcal{B}_4)$ bundles add to the pool of potential internal manifolds for flux compactification.

A Conventions for spinors

Before specializing to specific signatures and dimensions, let us say a few general words on spinors. We will focus on the even dimensional case $D = 2n$. Consider a metric g with signature $(t, 2n-t)$ where t is the number ot timelike directions (here 0 or 1), corresponding to the negative eigenvalues of g. The gamma matrices γ_μ satisfy the Clifford relation:

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu,\nu} \tag{A.1}$$

The chirality matrix

$$\gamma_{2n+1} = i^{n-t}\gamma_1 \cdots \gamma_{2n} \tag{A.2}$$

satisfies $\gamma_{2n+1}^2 = 1$ and decomposes spinors into positive and negative chiralities: $S = S^+ \oplus S^-$.

Fierzing

Moreover the Clifford algebra is isomorphic to $\mathcal{L}(S)$ and a unitary basis is given by the set of γ_I where I is a multi-index whose cardinal will be note $|I|$. Since they are unitary, $\operatorname{tr}\gamma_I^\dagger\gamma^J = 2^n\delta_I^J$ where δ is the generalized Kronecker delta. For two spinors η, ψ this gives a decomposition:

$$\eta\psi^\dagger = \frac{1}{2^n}\sum_I \frac{1}{|I|!}\,\psi^\dagger\gamma_I^\dagger\eta\;\gamma^I \tag{A.3}$$

From there several relations can be derived, depending on the chirality and reality properties of the spinors, which will all together be referred to as Fierz decompositions. They enable to relate bispinors to gamma matrices, and also to differential forms by sending $\gamma^{\mu_1\cdots\mu_k}$ to $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}$.

A.1 Spinors in $D = 6$, Euclidean

The charge conjugation operator is such that

$$ {}^tC = C = -C^{-1} \quad , \quad {}^t\gamma_m = -C^{-1}\gamma_m C \tag{A.4}$$

The conjugates of a spinor η are then defined by

$$\eta^c = C\eta^* \quad , \quad \tilde{\eta} = {}^t\eta C^{-1} = (\eta^c)^\dagger \tag{A.5}$$

Moreover

$$C\gamma_7 = -\gamma_7 C \ , \tag{A.6}$$

so that conjugation changes the chirality of the spinor. Thus spinors can be Majorana or Weyl but not both.

We note the following useful properties of spinor bilinears in six dimensions,

$$\begin{aligned} (\widetilde{\psi}_\pm \gamma_{(2p)} \chi_\pm) = (\widetilde{\psi}_\pm \gamma_{(2p+1)} \chi_\mp) = 0 \\ (\widetilde{\psi} \gamma_{(p)} \chi) = (-1)^{\frac{1}{2}p(p+1)} (\widetilde{\chi} \gamma_{(p)} \psi) \ , \end{aligned} \tag{A.7}$$

where ψ_+, χ_+ are arbitrary commuting Weyl spinors of positive chirality; ψ_-, χ_- are arbitrary commuting Weyl spinors of negative chirality; ψ, χ are arbitrary commuting Dirac spinors.

A.2 Spinors in $D = 4$, Minkowski

The charge conjugation operator is such that

$$^tC = -C = -C^{-1} \quad , \quad ^t\gamma_m = C^{-1} \gamma_m C \tag{A.8}$$

The conjugates of a spinor θ are then defined by

$$\theta^c = C\gamma_0 \theta^* \quad , \quad \tilde{\theta} = {}^t\theta C^{-1} \tag{A.9}$$

The definition is such that

$$\widetilde{\theta^c} = \bar{\theta} \ , \tag{A.10}$$

for $\bar{\theta} = \theta^\dagger \gamma_0$. If θ_+ is a spinor a positive chirality, define

$$\theta_- = \theta_+^c \tag{A.11}$$

of negative chirality.

Spinor bilinears in four dimensions obey,

$$\begin{aligned} (\widetilde{\psi}_\pm \gamma_{(2p+1)} \chi_\pm) = (\widetilde{\psi}_\pm \gamma_{(2p)} \chi_\mp) = 0 \\ (\widetilde{\psi} \gamma_{(p)} \chi) = (-1)^{\frac{1}{2}p(p-1)} (\widetilde{\chi} \gamma_{(p)} \psi) \ , \end{aligned} \tag{A.12}$$

where ψ_+, χ_+ are now arbitrary anti-commuting Weyl spinors of positive chirality; ψ_-, χ_- are arbitrary anti-commuting Weyl spinors of negative chirality; ψ, χ are arbitrary anti-commuting Dirac spinors.

A.3 Spinors in $D = 10 \to 4 + 6$, Minkowski

Decompose $S_{10} = S_4 \otimes S_6$. Gamma matrices Γ_M in ten dimension are then:

$$\begin{aligned} \Gamma_\mu = \gamma_\mu \otimes 1 \\ \Gamma_m = \gamma_5 \otimes \gamma_m \end{aligned} \tag{A.13}$$

The charge conjugation and chirality operator are:

$$C = C_4\gamma_5 \otimes C_6 \quad , \quad \Gamma_{11} = \gamma_5 \otimes \gamma_7 \tag{A.14}$$

Thus C satisfies:

$${}^tC = -C = C^{-1} \quad , \quad {}^t\Gamma_M = -C^{-1}\Gamma_M C \tag{A.15}$$

The conjugates of a spinor ε are defined by

$$\varepsilon^c = C\Gamma_0\varepsilon^* \quad , \quad \tilde{\varepsilon} = {}^t\varepsilon C^{-1} \quad , \quad \bar{\varepsilon} = \varepsilon^\dagger\Gamma_0 \ , \tag{A.16}$$

such that $\bar{\varepsilon} = \widetilde{\varepsilon^c}$. A Majorana-Weyl spinor $\varepsilon_\pm$ satisfies:

$$\varepsilon^c_\pm = \varepsilon_\pm = \pm\Gamma_{11}\varepsilon_\pm \tag{A.17}$$

A.4 Spinors in $D = 4$, Euclidean

The charge conjugation operator C is the same as in the Minkowski case. However the definition of θ^c changes:

$$\theta^c = C\theta^* \tag{A.18}$$

which does not change the chirality. Thus spinors of opposite chirality are not related anymore. Moreover

$$(\theta^c)^c = -\theta \tag{A.19}$$

so that there are no Majorana spinors.

For this case we will also need to make the spinor indices explicit. Our conventions are as follows. A positive-, negative-chirality 4d Weyl spinor is indicated with a lower, upper spinor index respectively: θ_α, χ^α. We never raise or lower the spinor indices on spinors, so that the position unambiguously indicates the chirality. The 4d gamma matrices, the charge conjugation and chirality matrices are decomposed into chiral blocks,

$$\gamma_\mu = \begin{pmatrix} 0 & (\gamma_\mu)_{\alpha\beta} \\ (\gamma_\mu)^{\alpha\beta} & 0 \end{pmatrix} \ ; \quad C^{-1} = \begin{pmatrix} C^{\alpha\beta} & 0 \\ 0 & C_{\alpha\beta} \end{pmatrix} \ ; \quad \gamma_5 = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & -\delta^\alpha{}_\beta \end{pmatrix} \ . \tag{A.20}$$

It is the "Pauli matrices" ($C^{-1}\gamma_{\mu_1\ldots\mu_n}$) which act as Clebsch-Gordan coefficients between spinor bilinears and n-forms. For example, the structure of indices of the charge conjugation matrix reflects the fact that scalars can only be formed as spinor bilinears of Weyl spinors of the same chirality,

$$v = \theta_\alpha C^{\alpha\beta}\chi_\beta \ ; \quad u = \theta^\alpha C_{\alpha\beta}\chi^\beta \ . \tag{A.21}$$

As another example, the structure of indices of $C^{-1}\gamma_\mu$ reflects the fact that vectors can only be formed as spinor bilinears of Weyl spinors of opposite chirality,

$$v_\mu = \theta^\alpha \left(C^{-1}\gamma_\mu\right)_\alpha{}^\beta \chi_\beta \ ; \quad u_\mu = \theta_\alpha \left(C^{-1}\gamma_\mu\right)^\alpha{}_\beta \chi^\beta \ . \tag{A.22}$$

We also make use of the Fierz relation for two positive-chirality 4d spinors,

$$\theta_\alpha \chi_\beta = -\frac{1}{2}(\tilde{\theta}\chi)C_{\alpha\beta} - \frac{1}{8}(\tilde{\theta}\gamma_{\mu\nu}\chi)\,(\gamma^{\mu\nu}C)_{\alpha\beta} \ , \tag{A.23}$$

where $\tilde{\theta} \equiv \theta^{\mathrm{Tr}} C^{-1}$, and similarly for negative chirality.

The Hodge duality relations read,

$$\tfrac{1}{(4-l)!}\,\varepsilon_{\mu_1\ldots\mu_l}{}^{\nu_1\ldots\nu_{4-l}}\gamma_{\nu_1\ldots\nu_{4-l}} = -(-1)^{\frac{1}{2}l(l-1)}\gamma_{\mu_1\ldots\mu_l}\gamma_5 \ , \tag{A.24}$$

With explicit spinor indices in Euclidean signature we have,

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\,(\gamma^{\rho\sigma})_\alpha{}^\beta = (\gamma_{\mu\nu})_\alpha{}^\beta \ ; \quad \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\,(\gamma^{\rho\sigma})^\alpha{}_\beta = -\,(\gamma_{\mu\nu})^\alpha{}_\beta \ . \tag{A.25}$$

In particular if $T_{\mu\nu}$ is a self-dual tensor, $\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}T^{\rho\sigma} = T_{\mu\nu}$, it follows that $T \cdot \gamma$ vanishes when acting on negative-chirality spinors,

$$T^{\mu\nu}\,(\gamma_{\mu\nu})^\alpha{}_\beta = 0 \ . \tag{A.26}$$

B Kähler-Einstein manifolds

A Kähler manifold of real dimension $2d$ corresponds to the case of a *local* $SU(d)$-structure where W_5 is the only nonvanishing torsion class,

$$\mathrm{d}J = 0\ ; \quad \mathrm{d}\Omega = i\mathcal{P}\wedge\Omega\ , \tag{B.1}$$

cf. (1.33). The local structure (J, Ω) can also be expressed in terms of bilinears of a locally-defined spinor η on M. In terms of this spinor eq. (B.1) can be written equivalently,

$$\nabla_m\eta = \frac{i}{2}\mathcal{P}_m\eta\ , \tag{B.2}$$

where $\mathcal{P} := 2\Im W_5$ is a real one-form. (Note that the existence of the complex structure allows us to reconstruct the torsion W_5 from its imaginary part alone.) Moreover (B.2) can be inverted to obtain $\mathcal{P}$ from the covariant spinor derivative,

$$\mathcal{P}_m = -2i\eta^\dagger\nabla_m\eta\ . \tag{B.3}$$

From (4.41),(B.2), using $\nabla_{[m}\nabla_{n]}\eta = \frac{1}{8}R_{mnpq}\gamma^{pq}\eta$ we obtain,

$$\mathrm{d}\mathcal{P} = \mathcal{R}\ , \tag{B.4}$$

where $\mathcal{R}$ is the Ricci form. Hence $\mathcal{P}$ can be identified with the connection of the canonical bundle of M. On the other hand, the Ricci tensor is obtained from the Riemann tensor via,

$$\mathcal{R}_{mn} = \frac{1}{2}R_{mnpq}J^{pq} = R_{mpnq}J^{pq}\ . \tag{B.5}$$

On a Kähler manifold the Ricci form, the Ricci tensor and the Ricci scalar obey,

$$\mathcal{R}_{mn} = J_m{}^pR_{pn}\ ; \quad \mathcal{R}_{mn}J^{mn} = R\ . \tag{B.6}$$

Furthermore for a Kähler-Einstein manifold such that,

$$R_{mn} = \lambda g_{mn}\ , \tag{B.7}$$

eqs. (B.7),(B.6) imply,

$$\mathcal{R} = \lambda J\ , \tag{B.8}$$

but in general the Ricci form need *not* be proportional to the Kähler form.

The above relations are valid for arbitrary dimension. Specializing to four real dimensions we adopt the notation $(J, \Omega) \to (\hat{\jmath}, \hat{\omega})$, in accordance with the main text. We may decompose any two-form Φ on the basis of a local $SU(2)$-structure $(\hat{\jmath}, \hat{\omega})$ as follows:

$$\Phi = \varphi \hat{\jmath} + \widetilde{\Phi} + \chi \hat{\omega} + \psi \hat{\omega}^* \ , \tag{B.9}$$

where $\varphi := \frac{1}{4}\hat{\jmath}^{mn}\Phi_{mn}$ is the trace of Φ, and $\widetilde{\Phi}$ is (1,1)-traceless: $\hat{\jmath}^{mn}\widetilde{\Phi}_{mn} = 0$. Equivalently,

$$\hat{\jmath} \wedge \widetilde{\Phi} = 0 \ . \tag{B.10}$$

It is also straightforward to show that $(\hat{\jmath}, \hat{\omega})$ are selfdual forms while (1,1)-traceless forms are anti-selfdual,

$$\star\, (\hat{\jmath}, \hat{\omega}) = (\hat{\jmath}, \hat{\omega}) \ ; \quad \star\, \widetilde{\Phi} = -\widetilde{\Phi} \ . \tag{B.11}$$

In particular for the Ricci form the expansion reads,

$$\mathcal{R} = \frac{1}{4} R \hat{\jmath} + \tilde{R} \ . \tag{B.12}$$

Moreover the above properties can be used to calculate,

$$\mathcal{R} \wedge \mathcal{R} = \left(\frac{1}{4} R^2 - \frac{1}{2} R_{mn} R^{mn} \right) \mathrm{vol}_4 \ , \tag{B.13}$$

where the volume is given by,

$$\mathrm{vol}_4 = \frac{1}{2} \hat{\jmath} \wedge \hat{\jmath} \ . \tag{B.14}$$

C Büscher rules for NATD

Let us recall here the rules for Non Abelian T Duality from which the backgrounds in (4.1.2),(4.3.4),(4.4.1),(4.5.1) are computed. NATD can be applied along a generic isometry group but all the cases considered here have $SU(2)$ isometry.

C.1 NS-NS sector

The metric g and B-field can be combined in a single tensor $Q = g + B$. As Q is invariant under $SU(2)$, it can be decomposed into:

$$Q = Q_{\mu\nu}(x)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + Q_{i\nu}(x)\sigma^i \otimes \mathrm{d}x^\nu + Q_{\mu j}(x)\mathrm{d}x^\mu \otimes \sigma^j + Q_{ij}(x)\sigma^i \otimes \sigma^j$$

Where x^μ are the spectator coordinates (unaffected by the $SU(2)$ action) and σ^i are the Maurer Cartan forms given in (4.3). In this basis the isometry amounts to saying that the components of Q depend only on x^μ. From there NATD will dualize along the σ^i directions. Introduce new coordinates v_i (note that v actually lives in the Lie algebra of $SU(2)$) and define the matrix

$$M_{ij} = Q_{ij} + \alpha' f_{ij}{}^k v_k \quad ,$$

where $f_{ij}{}^k$ are the structure constants of $SU(2)$. Then the dual $\hat{Q}$ to Q is given by:

$$\begin{aligned}
\hat{Q}_{\mu\nu} &= Q_{\mu\nu} - Q_{\mu i} M^{-1\,ij} Q_{j\nu} \\
\hat{Q}_\mu{}^j &= Q_{\mu i} M^{-1\,ij} \\
\hat{Q}^i{}_\nu &= -M^{-1\,ij} Q_{j\nu} \\
\hat{Q}^{ij} &= M^{-1\,ij}
\end{aligned}$$

The dual metric $\hat{g}$ (resp. $\hat{B}$-field) are recovered by taking the symmetric (resp. anti-symmetric) part of $\hat{Q}$. Note that $\hat{Q}$ is given in the basis $(\mathrm{d}x^\mu, \mathrm{d}v^i)$.

The dual dilaton $\hat{\phi}$ is given by the formula:

$$\log \hat{\phi} = \phi - \frac{1}{2} \log \frac{\det M}{\alpha'^3}$$

C.2 R-R sector

Identify the flux polyform F with a bispinor through Fierz isomorphism. For this a choice for a vielbein is needed, take:

$$e^A = e^A_\mu \mathrm{d}x^\mu \quad , \quad e^a = \kappa^a_i \sigma^i + \lambda^a_\mu \mathrm{d}x^\mu$$

Then define

$$P = e^\phi F \quad .$$

The bispinor P will be transformed under NATD by the action of a specific Clifford matrix. Define

$$\Omega = \frac{\Gamma^{123} + \zeta_a \Gamma^a}{\sqrt{\alpha'^3}\sqrt{1+\zeta^2}} \Gamma_{11} \quad .$$

The gamma matrices involved here have flat indices and

$$\zeta^a = \kappa^a_i (b^i + v^i) \quad .$$

Where b^i is the three-dimensional dual of B: $b^i = \epsilon^{ijk} B_{jk}$ [1]. The dual $\hat{P}$ to P is then:

$$\hat{P} = P\Omega^{-1}$$

[1] B_{ij} is the antisymmetric part of Q_{ij}, *ie* the components of B along $\sigma^i \wedge \sigma^j$

D LT structures

In this section we fill out the details leading up to eq. (5.91). Plugging the following general ansatz in the decomposition of $\mathrm{d}J, \mathrm{d}\Omega$,

$$\begin{aligned}
\mathrm{d}J &= \frac{3\alpha_1}{2}\Im\Omega - \frac{3\alpha_2}{2}\Re\Omega + \alpha_3\Re K + \alpha_4\Im K + \alpha_5\Re\Omega^\perp + \alpha_6\Im\Omega^\perp \\
\mathrm{d}\Omega &= a_1 J\wedge J + a_2 K^*\wedge\Omega + a_3 J^\perp\wedge J \ ,
\end{aligned} \tag{D.1}$$

for some real and complex parameters $\alpha_1,\ldots,\alpha_6$ and $a_1,\ldots,a_3$ respectively, and using eqs. (5.84), (5.85), (5.86), we arrive at the torsion classes given in (5.88). Imposing $W_3 = W_4 = W_5 = 0$ leads to,

$$\begin{array}{lclcl}
W_3 & : & \frac{1}{f} - \frac{\sin\theta}{g} + 6\frac{g\sin\theta}{|h|^2} & = & 0 \\
W_4 & : & \frac{|h^2|'}{f|h^2|} - 6\cos\theta\frac{g}{|h^2|} & = & 0 \\
W_5 & : & \frac{h'}{fh} + \frac{g'}{2fg} - \frac{\cos\theta}{2g} & = & 0 \ .
\end{array} \tag{D.2}$$

From $W_5 - \bar{W}_5$ we see that the phase of h must be constant but is otherwise unconstrained by the equations, *i.e.*,

$$h = |h|e^{i\beta} \ , \tag{D.3}$$

for some real constant $\beta \in [0, 2\pi)$. Moreover we set $H := |h|^2$, for some nonnegative function H. Since $\mathrm{d}\psi$ is not defined at $\theta = 0, \pi$, regularity requires that the coefficient of $\mathrm{d}\psi + A$ should vanish at the poles. It is therefore convenient to set $g := G\sin\theta$ for some function G. The equations now read,

$$\begin{aligned}
\frac{1}{f} - \frac{1}{G} + 6\frac{G\sin^2\theta}{H} &= 0 \\
H' - 3\sin 2\theta\ Gf &= 0 \\
\frac{H'}{H} + \frac{G'}{G} + \cot\theta - \frac{f\cot\theta}{G} &= 0 \ ,
\end{aligned} \tag{D.4}$$

where we have assumed that f, h are nonvanishing. Plugging the first two into the third then implies,

$$G = \alpha \ , \tag{D.5}$$

for some real constant α. The system is then solved as in eq. (5.91), where H satisfies,

$$H'\left(1 - 6\alpha^2\frac{\sin^2\theta}{H}\right) = 3\alpha^2\sin 2\theta \ . \tag{D.6}$$

We immediately see that $H(\theta) = 9\alpha^2 \sin^2\theta$ is a special solution. Moreover the differential equation imposes $H(\pi - \theta) = H(\theta)$. It is thus convenient to introduce a new function $\varphi(x)$, where $x := \sin^2\theta$ and $H := 9\alpha^2 x\varphi(x)$, in terms of which the equation becomes,

$$\frac{\varphi - \frac{2}{3}}{\varphi - \varphi^2}\ \varphi' = \frac{1}{x}\ . \tag{D.7}$$

Integrating over x between X_0 and X we obtain,

$$\int_{X_0}^{X} \frac{\varphi - \frac{2}{3}}{\varphi - \varphi^2}\ \varphi' \mathrm{d}x = \log\frac{X}{X_0}\ , \tag{D.8}$$

where $\varphi_0 := \varphi(X_0)$. On the other hand,

$$\frac{\varphi - \frac{2}{3}}{\varphi - \varphi^2} = -\frac{2}{3}\frac{1}{\varphi} - \frac{1}{3}\frac{1}{\varphi - 1}\ . \tag{D.9}$$

Since $\varphi \geq 0$ and $\varphi - 1$, $\varphi_0 - 1$ have the same sign, we find,

$$-\frac{2}{3}\log\frac{\varphi}{\varphi_0} - \frac{1}{3}\log\frac{\varphi - 1}{\varphi_0 - 1} = \log\frac{X}{X_0}\ , \tag{D.10}$$

which leads to,

$$\varphi^2(\varphi - 1) = \frac{X_0^3}{X^3}\varphi_0^2(\varphi_0 - 1)\ . \tag{D.11}$$

Rewriting the above in terms of H which, contrary to φ, is necessarily everywhere well-defined, we obtain,

$$H^2(H - 9\alpha^2 X) = H_0^2(H_0 - 9\alpha^2 X_0) = \text{constant}\ . \tag{D.12}$$

We can henceforth assume $X_0 = 0$ without loss of generality, which leads to,

$$H^2(H - 9\alpha^2 X) - H_0^3 = 0\ . \tag{D.13}$$

It is easy to see that the above polynomial in H is increasing for negative H, until it atteins the value $-H_0^3 \leq 0$ at $H = 0$. It then decreases until $H = 6\alpha^2 X$, from which point on it becomes increasing. Therefore if we impose $H_0 > 0$ the polynomial only vanishes once, for $H > 6\alpha^2 X \geq 0$. For $H_0 = 0$, there are two solutions: $H = 0$ (which must be discarded) and the special solution $H = 9\alpha^2 X$. We conclude that for any $H_0 \geq 0$, there is a unique solution to the differential equation with the boundary conditions $H(0) = H_0 = H(\pi)$; it is given in eq. (5.91) of the main text.

E Twistor spaces

There is an alternative description of the total space of the $\mathbb{CP}^1$ fibration over $\mathbb{CP}^2$ in terms of twistor spaces. More generally, for the purposes of the present section we may replace the $\mathbb{CP}^2$ base by any four-dimensional Kähler space B_4.

Consider B_4 equipped with its canonical complex structure $\hat{I}$ and a hermitian metric $\mathfrak{g}$. Let us introduce a complex zweibein z_1, z_2, so that $\hat{I}z_k = i\,z_k$, for $k = 1, 2$. These forms are of course only locally defined, since B_4 is not parallelizable in general. We can thus express the metric and the local $SU(2)$ structure on B_4 in terms of the complex zweibein,

$$\begin{aligned} \mathfrak{g} &= z_1\bar{z}_1 + z_2\bar{z}_2 \\ \hat{j} &= \tfrac{i}{2}\,(z_1 \wedge \bar{z}_1 + z_2 \wedge \bar{z}_2) \\ \hat{\omega} &= z_1 \wedge z_2 \ . \end{aligned}$$

At any one point $x \in B_4$, $\hat{j}_x, \hat{\omega}_x$ form an $SU(2)$ structure on the tangent space T_xB_4. The latter is equipped with a complex structure and a scalar product given by $\hat{I}_x$ and $\mathfrak{g}_x$ respectively. Moreover the relation,

$$\hat{I}_m^k = \mathfrak{g}^{kn}\hat{j}_{mn} \ ,$$

allows us to identify the complex structure with a real selfdual form. The latter are parameterized as follows, see appendix B,

$$j_x = \alpha\hat{j}_x + \frac{\beta}{2}\hat{\omega}_x + \frac{\beta^*}{2}\hat{\omega}_x^* \ ,$$

where α is real and $\alpha^2 + |\beta|^2 = 1$. Hence the space of complex structures I_x compatible with the metric $\mathfrak{g}_x$ forms a sphere whose coordinates $\theta \in [0, \pi]$, $\psi \in [0, 2\pi)$ are defined by $\alpha = \cos\theta$, $\beta = \sin\theta e^{i\psi}$, so that I_x is associated with the two-form,

$$j_x = \cos\theta\ \hat{j}_x + \sin\theta\ \Re(e^{i\psi}\hat{\omega}_x) \ .$$

Extending this procedure to each point on B_4 then defines an almost complex structure I over the whole manifold (unlike $\hat{I}$, I will not be integrable in general). Over each point on B_4 an almost complex structure compatible with the metric of B_4 can be thought of as a point on the sphere S^2 parameterized by (θ, ψ). Hence the space of almost complex structures on B_4 is a fiber bundle S^2 over B_4 denoted by $\mathrm{Tw}(B_4)$, the *twistor space* of B_4.

The zweibein z_1, z_2 is no longer compatible with the almost complex structure I associated with the real two-form j given above. Rather we define,

$$\begin{aligned} f_1 &:= \cos\frac{\theta}{2}e^{i\frac{\psi}{2}}\,z_1 + i\sin\frac{\theta}{2}e^{-i\frac{\psi}{2}}\,\bar{z}_2 \\ f_2 &:= \cos\frac{\theta}{2}e^{i\frac{\psi}{2}}\,z_2 - i\sin\frac{\theta}{2}e^{-i\frac{\psi}{2}}\,\bar{z}_1 \ , \end{aligned} \tag{E.1}$$

so that $If_k = if_k$. In terms of the new zweibein the local $SU(2)$ structure and the metric read,

$$\begin{aligned} \mathfrak{g} &= f_1\bar{f}_1 + f_2\bar{f}_2 \\ j &= \tfrac{i}{2}\left(f_1\wedge\bar{f}_1 + f_2\wedge\bar{f}_2\right) \\ \omega &= f_1\wedge f_2 = \cos\theta\ \Re(e^{i\psi}\hat{\omega}) - \sin\theta\ \hat{j} + i\ \Im(e^{i\psi}\hat{\omega})\ , \end{aligned}$$

which is precisely of the form of (5.82). Let us also note that the choice of zweibein compatible with I is only determined up to a phase. The latter leaves j and the metric invariant but acts nontrivially on ω, thus changing the $SU(2)$ structure.

We have seen that $I_x(\theta,\psi)$ defines an almost complex structure on the base. Together with the natural complex structure of the sphere (thought of as a $\mathbb{CP}^1$) we can construct an almost complex structure on the the total space,

$$I_\pm = \begin{pmatrix} I_x(\theta,\psi) & 0_{4\times 2} \\ 0_{2\times 4} & \begin{matrix} 0 & \pm\frac{1}{\sin\theta} \\ \mp\sin\theta & 0 \end{matrix} \end{pmatrix}\ ,$$

so that f_1, f_2 and $K = \mathrm{d}\theta + i\sin\theta(\mathrm{d}\psi + A)$ are eigenforms of $I_\pm$ with eigenvalue $\pm i$. We can thus take (f_1, f_2, K) as the vielbein on $\mathrm{Tw}(B_4)$. More generally we could modify (f_1, f_2, K) by introducing "warp factors" as in (5.86) below.

F Torsion classes for Kälher base

As mentioned in section 5.4 we may relax the condition on the base of $S^2(B_4)$, so that B_4 is a generic four-dimensional Kähler manifold. The torsion classes can also be straightforwardly calculated in this case. Note however that this is only a local calculation: without additional constraints, we do not expect there to exist a global extension to a complete space.

Let us postulate a globally-defined $SU(3)$ structure as in (5.86) on a $\mathbb{CP}^1$ bundle with metric,

$$g_6 = |h|^2 g_4 + KK^* \; ; \quad K = f \mathrm{d}\theta + ig(\mathrm{d}\psi + A) \; , \tag{F.1}$$

where f, g, h are *a priori* complex functions; θ and ψ parameterize the S^2 fiber; the one-form A satisfies (B.1), (B.4) for $(J, \Omega) \to (\hat{j}, \hat{\omega})$. We will impose further restrictions on f, g, h; these functions must be regular and non-vanishing, except for g which must vanish at $\theta = 0$ and $\theta = \pi$. The most general situation we will consider here is that $\mathrm{d}f$, $\mathrm{d}g$, $\mathrm{d}h$ live on the space spanned by K, K^* (this restricts the dependance on the coordinates). Explicitly we expand,

$$\mathrm{d}f = f_1 K + f_2 K^* \; , \tag{F.2}$$

and similarly for g, h. It is also possible restrict the dependance on θ alone.

The calculation of the torsion classes proceeds in the same fashion as in appendix D, with the following result,

$$\begin{array}{rcl} W_1 &=& -\frac{2i}{3}\frac{h}{h^*}\left(\frac{g+f\sin\theta}{fg^*+f^*g} + \frac{R}{2}g\frac{\sin\theta}{|h|^2}\right) \\ W_2 &=& \frac{2i}{3}\frac{h}{h^*}\left(\frac{g+f\sin\theta}{fg^*+f^*g} - Rg\frac{\sin\theta}{|h|^2}\right) J^\perp \\ W_3 &=& -\frac{1}{2}(fg^* + f^*g)\mathrm{d}\theta \wedge \tilde{R} + \Re\left(\frac{g-f\sin\theta}{fg^*+f^*g} + \frac{R}{2}g\frac{\sin\theta}{|h|^2}\right)\Omega^\perp \\ W_4 &=& \mathrm{d}(\log|h^2|) - \frac{R}{2|h^2|}(fg^* + f^*g)\cos\theta\mathrm{d}\theta \\ W_5^* &=& \frac{1}{fg^*+f^*g}\left(f\cos\theta + f_1 g - fg_1 - (f^*g_2 + f_2 g^*) - 2(fg^* + f^*g)\frac{h_2}{h}\right) K^* \; . \end{array} \tag{F.3}$$

Our degrees of freedom in the above are a somewhat redundant: a phase change of K can be absorbed in h so that f or g can be taken real. Let us also note that in general a cross term $(fg^* - f^*g)\mathrm{d}\theta(\mathrm{d}\psi + A)$ appears in the metric. If we want this to vanish, we must impose f and g to be colinear, so that they can both be taken real.

Furthermore if we want to impose $W_4 = 0$, we must restrict h to depend only on θ, in which case we get,

$$h_1 = \frac{g^* h'}{fg^* + f^*g} \; ; \quad h_2 = \frac{gh'}{fg^* + f^*g} \; . \tag{F.4}$$

Therefore f and g must also be restricted so that $R\,(f^*g + fg^*)$ is a function of θ alone.

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