Contributions to the Theory of Time-Delay Systems: Stability and Stabilisation

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Résumé

Le but de cette thèse est de présenter de nouveaux résultats sur l'analyse et la synthèse de systèmes à retard. Dans la première partie, nous étendons l'utilisation du système de dimension finie invariant dans le temps, appelé *système de comparaison*, à la conception d'un contrôleur qui dépend non seulement de la sortie du système à l'instant t ainsi que du retard maximal, mais également d'un nombre arbitraire de valeurs entre celles-ci. Cette approche nous permet d'augmenter le retard maximal stable sans exiger d'informations supplémentaires. Les méthodes présentées ici concernent la conception de systèmes de contrôle avec des retards en utilisant des routines numériques classiques basées sur la théorie $\mathcal{H}_\infty$. La deuxième partie de ce travail traite d'une nouvelle approche pour développer une enveloppe englobant tous les pôles d'un système à retard. Grâce aux LMIs (*Linear Matrix Inequalities*), nous sommes en mesure de déterminer les enveloppes pour les systèmes à retard du type retardé et du type neutre. Les enveloppes proposées sont non seulement plus étroites que celles de la littérature, mais, avec notre procédure, elles peuvent également être appliquées pour vérifier la stabilité du système et pour déterminer des contrôleurs par retour d'état qui sont robustes face aux incertitudes paramétriques. Les systèmes fractionnaires sont également discutés dans les deux chapitres mentionnés ci-dessus. La troisième et dernière partie étudie les systèmes stochastiques avec des retards. Nous discutons d'abord des systèmes à temps continu soumis à des sauts de Markov. Nous définissons la stabilité et obtenons des LMIs pour le contrôle par retour d'état de telle sorte que la relation entre les taux de transition entre les modes soit affine, ce qui permet donc de traiter le cas dans lequel les taux sont incertains. Nous discutons ensuite des systèmes positifs avec retards, tant pour le cas continu que pour le cas discret. Un système linéaire qui modélise la dynamique du premier moment est obtenu et la stabilité dépendant du retard est traitée. De nombreux exemples sont illustrés tout au long de la thèse.

Mots clés : Systèmes à retard, Retour d'état, Retour de sortie, Norme $\mathcal{H}_\infty$, Système de comparaison, Systèmes linéaires avec du saut de Markov, Systèmes fractionnaires, Inégalités matricielles linéaires.

Abstract

The aim of this dissertation is to present new results on analysis and control design of time-delay systems. On the first part, we extend the use of a finite order LTI system, called *comparison system*, to design a controller which depends not only on the output at the present time and maximum delay, but also on an arbitrary number of values between those. This approach allows us to increase the maximum stable delay without requiring any additional information. The methods presented here consider time-delay systems control design with classical numeric routines based on $\mathcal{H}_\infty$ theory. The second part of this work deals with a new approach to develop an envelope that engulfs all poles of a time-delay system. By means of LMIs, we are able to determine envelopes for retarded and neutral time-delay systems. The envelopes proposed are not only tighter than the ones in the literature but, with our procedure, they can also be applied to verify the stability of the system and design state-feedback controllers which are robust in face of parametric uncertainties. Fractional systems are also discussed for both chapters mentioned above. The third and last part studies stochastic time-delay systems. First we discuss continuous-time systems that are subjected to Markov jumps. We define stability and obtain LMIs for the state-feedback control in such a way that the relation with the transition rates between the modes is affine, allowing, therefore, to treat the case in which the rates are uncertain. We then discuss positive systems with delays, both for the continuous case as for the discrete case. A linear system that models the first moment dynamics is obtained and delay dependent stability is addressed. A fair amount of examples are presented throughout the dissertation.

Keywords: Time-delay Systems, State Feedback, Output feedback, $\mathcal{H}_\infty$-norm, Comparison System, Markov Jump Linear Systems, Fractional Systems, Linear Matrix Inequalities.

Resumo

O objetivo desta tese é apresentar novos resultados na análise e na síntese de controladores para sistemas com atrasos. Na primeira parte, estendemos o uso de um sistema linear invariante no tempo de ordem finita, chamado *sistema de comparação*, para projetar um controlador que depende não apenas da saída no tempo presente e do atraso máximo, mas também de um número arbitrário de valores entre eles. Essa abordagem nos permite aumentar o atraso estável máximo sem exigir do sistema nenhuma informação adicional. Os métodos apresentados aqui consideram o projeto de controle de sistemas de atraso no tempo com rotinas numéricas clássicas baseadas na teoria $\mathcal{H}_\infty$. A segunda parte deste trabalho trata de uma nova abordagem para desenvolver um envelope que engloba todos os polos de um sistema com atrasos. Por meio de LMIs, podemos determinar envelopes para sistemas com atrasos do tipo retardo e para sistemas com atrasos do tipo neutro. Os envelopes propostos não são somente mais estreitos do que os presentes na literatura, mas, além disso, com nosso procedimento, eles também podem ser aplicados para verificar a estabilidade do sistema e empregados para se projetar controladores via realimentação de estado que são robustos perante a incertezas paramétricas. Sistemas fracionários também são discutidos em ambas as partes supracitadas. A terceira e última parte estuda sistemas estocásticos com atraso. Primeiro discutimos sistemas sujeitos a saltos markovianos a tempo contínuo. Definimos estabilidade e obtemos LMIs para o controle por realimentação de estado de tal forma que a relação com as taxas de transição entre modos é afim, permitindo, portanto, tratarmos o caso em que as taxas são incertas. Discutimos, em seguida, sistemas positivos com atrasos, tanto para o caso contínuo como para o caso discreto. Um sistema linear que modela a dinâmica de primeiro momento é obtido e a estabilidade dependente do atraso é abordada. Uma boa quantidade de exemplos, ao longo da tese, ilustram os resultados alcançados.

Palavras-chave: Sistemas com atraso, realimentação de estado, realimentação de saída, norma-$\mathcal{H}_\infty$, Sistema de Comparação, Sistemas Lineares sujeitos a Saltos Markovianos, Sistemas Fracionários, Desigualdades Matriciais Lineares.

"There is nothing impossible to him who will try."

Alexander the Great

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Contents

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List of Symbols

$\mathbb{N}$	-	Set of natural numbers with zero.
$\mathbb{N}^*$	-	Set of natural numbers without zero.
$\mathbb{N}_N$	-	Set of the first N+1 natural numbers $(0, \ldots, N)$.
$\mathbb{K}$	-	Set of the first N numbers $\in \mathbb{N}^*$ $(1, \ldots, N)$.
$\mathbb{K}_i$	-	$\mathbb{K}$ - {i}.
$\mathbb{Z}$	-	Set of integer numbers.
$\mathbb{R}$	-	Set of real numbers.
$\mathbb{R}^*$	-	Set of real numbers without zero.
$\mathbb{R}_+$	-	Set of nonnegative real numbers.
$\mathbb{R}_-$	-	Set of nonpositive real numbers.
$\mathbb{C}$	-	Set of complex numbers.
$\Re(.)$	-	The real part of a complex number or a complex matrix.
$\Im(.)$	-	The imaginary part of a complex number or a complex matrix.
$\lfloor x \rfloor$	-	The largest integer less than or equal to x, $x \in \mathbb{R}$.
$\lceil x \rceil$	-	The least integer greater than or equal to x, $x \in \mathbb{R}$.
I	-	The identity matrix of any dimension.
X'	-	The transpose of the matrix X.
X^*	-	The conjugated transpose of the matrix X.
X^{-1}	-	The inverse of a nonsingular square matrix X.
$X > 0$	-	The symmetric matrix X is positive definite.
$X \geq 0$	-	The symmetric matrix X is positive semi-definite.
$\Vert X \Vert_p$	-	The induced p-norm of a matrix $X \in \mathbb{C}^{n \times m}$.
$\det(X)$	-	Determinant of the square matrix X.
Tr(X)	-	Trace of the square matrix X.
$\ker(X)$	-	The null space of the matrix X, i.e., $\{\mathbf{v} \in V \vert X\mathbf{v} = \mathbf{0}\}$.
$\mathbf{diag}(X\ ,\ Y)$	-	Diagonal block matrix formed by the matrices X and Y.
$X \otimes Y$	-	Kronecker product.
$X \circ Y$	-	Hadamard product.
$\mathrm{vec}(X)$	-	Vectorization of a matrix X, i.e., $[x_{1,1}, \ldots, x_{m,1}, \ldots, x_{1,n}, \ldots, x_{m,n}]'$.
$\mathrm{Vec}(X_i)$	-	Stack matrices X_i, $i \in \mathbb{K}$, in a column block such as $[X_1', X_2', \ldots, X_N']'$.

$\leftarrow$	-	When in an algorithm indicates to update a value.
λ_i	-	The ith eigenvalue of a matrix.
$\lambda_{\min}$	-	Minimum eigenvalue of a symmetric matrix.
λ_{rme}	-	Rightmost eigenvalue of a matrix.
$\sigma(X)$	-	Set of singular values of the matrix X.
$\sigma_M(X)$	-	Maximum singular value of the matrix X.
x_L	-	Left eigenvector, with dimension $1 \times n$, of a matrix X, i.e., $x_L X = \lambda_L x_L$.
${}_0\mathbb{D}_t^{\alpha}$	-	Differintegral operator.
$\binom{N}{k}$	-	Binomial coefficient.
$n!$	-	Factorial of n, i.e., $n! = 1 \times 2 \times \cdots \times n$.
$co\{S\}$	-	The convex hull of a finite point set S.
$\mathcal{E}[\cdot]$	-	Mathematical expectance.
$\mathcal{L}$	-	Infinitesimal generator.
$\Vert z(t)\Vert_2^2$	-	Defined by $\mathcal{E}\left[\int_0^\infty z(t)'z(t)dt\right]$.
$\mathbb{L}_2$	-	Set of all stochastic signals $z(t) \in \mathbb{R}^n$ such that $\Vert z(t)\Vert_2^2 < \infty$.
A_{ki}	-	$A_k(\theta_t)$ whenever $\theta_t = i \in \mathbb{K}$.
$\Gamma(x)$	-	Gamma function, i.e., $\int_0^\infty e^{-y}y^{x-1}dy$.
$\mathbf{1}_A(\omega)$	-	The Dirac measure over a set A.
$\mathbb{1}_N$	-	Unit vector of order N, i.e., $[1\ 1 \ldots\ 1]'$.
α-stability	-	$\Re(\lambda_j) < -\alpha$, for all eigenvalues of a linear time-invariant system.
$o(\Delta)$	-	$f \in o(\Delta)$ iff $\lim_{\Delta\to 0} f(\Delta)/g(\Delta) = 0$.
$\bullet$	-	Each one of the Hermitian blocks related to the diagonal in a Hermitian matrix.
$\square$	-	End of proof.
$\triangleq$	-	Equal by definition.

Works Published by the Author

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[5] C. B. Cardeliquio, A. R. Fioravanti e A. P. C. Gonçalves, *$\mathcal{H}_2$ output-feedback control of continuous-time MJLS with uncertain transition rates*, IEEE 53rd Annual Conference on Decision and Control, 2014.

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Chapter 1

Introduction

1.1 Objectives

This dissertation has as its main objective the study of time-delay systems. Our goal is to use Linear Matrix Inequalities (LMIs) to obtain new methods for analysis and synthesis of controllers and, also, to improve existing methods. The time-delay systems that we focus on will vary from branches such as classical systems, fractional systems, stochastic systems and positive systems. We aim not only analysis, but state-feedback and output-feedback design, as well as delay-independent and delay-dependent stabilisation.

1.2 Preliminaries

1.2.1 Time-Delay Systems

Time-delay systems have instigated an increasingly interest from the control community [1, 2, 3, 4]. This can be due to several practical reasons, among which we highlight: the time necessary to acquire the information needed for the control, the time required to transport information, the processing time, the sampling period, amid many others. Moreover, due to environmental conditions, e.g., high temperatures inside a compartment, access difficulties, such as offshore underwater oil platforms [5, 6, 7], unhealthy areas, among others, one method that is being used to command dynamical systems is the approach of control via a network [8, 9, 10]. Controllers performing through a network have, intrinsically, delays embedded in its structure. Even though those delays, in all cases mentioned, are oftentimes neglected, they can be responsible for poor performance and, in worst scenarios, they may even lead the system to instability. For that reason, several studies considering the so called time-delay systems have being made through the last decades.

Models containing delays can likewise appear in a fairly amount of processes such as physical, biological [11, 12], economical [13, 14], mechanical [15] and so forth. A first extensive study about delays in differential equations, known as DDEs, is made in [16] while some examples for time-delay systems as much as their analysis can be seen in [17]. In time domain, a generic expression for linear systems with one delay only is described by the following differential equation:

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-\tau) + H\dot{x}(t-\tau) + Ew(t), \\ z(t) &= C_0x(t) + C_1x(t-\tau) + D_zw(t), \end{aligned} \tag{1.1}$$

in which, for all $t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$ is the state variable, $w(t) \in \mathbb{R}^m$ is the exogenous input, $z(t) \in \mathbb{R}^p$ is the output of interest, $\tau \in \mathbb{R}_+$ is the delay and A_0, A_1, H, E, C_0, C_1 and D_z are real matrices with appropriate dimensions. This system is called a neutral time-delay system due to the term containing the derivative of the state delayed. For the case where $H = 0$, we say that the system has a delay in a retarded form. In this case, the system is called a retarded time-delay system. Both retarded and neutral time-delay systems are discussed throughout this work.

In frequency domain, the transfer function of (1.1) is given by

$$T(s,\tau) = \left(C_0 + C_1e^{-s\tau}\right)\left(sI - A_0 - A_1e^{-s\tau} - sHe^{-s\tau}\right)^{-1}E + D_z. \tag{1.2}$$

The characteristic equation is then quasi-polynomial with, in general, infinite solutions. There are several possible frameworks to study stability and stabilisation for time-delay systems. Stability is discussed, among others, in [4], [18] and [19]. A simple necessary and sufficient LMI[1] condition for the strong delay-independent stability of LTI systems with single delay is the subject of [20]. The development of efficient control design techniques that cope with time delay has received much attention in the past decades; see the books [21] and [22] and the survey paper [23] for important theoretical results in the area. In this context, $\mathcal{H}_\infty$ control techniques play a key role in the design of controllers that attain a pre-specified worst case $\mathbb{L}_2$ gain for the closed-loop system whenever the time delay is given [24].

For the stabilisation through state feedback, delay-independent controllers can be devised using Riccati equations [25, 26], whereas the delay-dependent case is usually designed by means of Lyapunov-Krasoviskii functionals [27, 28, 29]. Similar results have been extended to the output-feedback framework; see [30, 31, 32, 28]. Lyapunov-Krasoviskii functionals are also utilised for robust control of state delay systems in [33]. Filtering and output feedback for time-delay systems can be seen in [34] and the design of observers in [35]. State and output feedback stability is dealt in [36]. A modified Riccati equation is used in [25] for the design of a memoryless $\mathcal{H}_\infty$ controller. The $\mathcal{H}_\infty$ control problem for multiple input-output delays is also discussed in [37]. A controller design approach through a finite LTI comparison system is developed in [38] and in [39]. Another type of approach based on a rational approximate to the infinite-dimensional system can be done using *Padé* techniques such as in [40]. Criteria for robust stability and stabilisation is dealt in [41]. Robust exponential stabilisation for systems with time-varying delays can be seen in [23]. Robust stability and stabilisation for singular systems with parametric uncertainties are discussed, among others, in [42] and [43]. Delay independent stability for uncertain systems can be seen in [44] and delay-dependent stability and stabilisation in [45], [46] and [47]. The discrete counterpart is studied in [48], for positive systems. Guaranteed LQR control is dealt in [49] and robust polytopic $\mathcal{H}_\infty$ static output feedback in [50]. For uncertain linear systems with multiple time-varying delays, robust filter is design in [51]. Additionally, α-stability is discussed in [52] for non commensurate delays and in [53] via LMIs. For non-linear time-delay systems see [54, 55, 56, 57].

For stochastic systems, one of the first works in the literature dealing with Markov jump

[1] Linear Matrix Inequality

linear systems, henceforth called MJLS, without delays, is [58] for discrete systems and [59] for continuous systems. For discrete-time systems, a large amount of theory and design procedures has been developed to extend the concepts of deterministic systems to this particular class. In particular, the concepts of stability and the conditions for testing them, which are discussed in [60], [61] and [62]. Considering MJLS in continuous-time there are also several results in the literature. In [63] the $\mathcal{H}_2$ control is treated through state feedback via convex analysis. Controllability and stability concepts are studied in [64] and the optimal quadratic control with solution via the Separation Theorem is presented in [65]. The MJLS control and filtering projects assume, for the most part, that transition rates between modes are known a priori. However, in practice, only estimated values of these rates are available and these uncertainties can generate instabilities or at least degrade the system performance in the same way that occurs when there are uncertainties in the matrix of the state space representation of the plant. For this case in which transition rates between the modes are not fully known, there are works in the literature that show stability conditions, as can be seen, for example, in [66], where the robust case is discussed. The state feedback can be seen in [67] and [68]. A major reference for MJLS with delays is the book [24]. Other important works are [69], [70] and [71].

1.3 Final Remarks

This brief chapter introduces the subject and some basic definitions. The three major chapters that follows are independent and can be read in any preferable order by the reader.

The structure of this dissertation is the following:

Chapter 2: In this chapter an extended Rekasius substitution [72] is applied to replace the delay operator by a rational transfer function; in [73], a useful technique for stability analysis of time-delay systems that combines the Rekasius substitution and the Routh-Hurwitz criterion is proposed. An important consequence of the Rekasius substitution, as we are going to present, is the definition of a finite order linear time invariant system, called *comparison system*, which provides a tight lower bound to the $\mathcal{H}_\infty$ norm of the time-delay system and allows the development of simple and efficient synthesis algorithms; see also [74, 75]. Applying this equivalency we may cope with state feedback and output feedback for time-delay systems. Filters can be designed as a particular case of the output-feedback problem. The technique is then adapted for fractional systems. The objective is divided into two categories: firstly, to increase the maximum delay allowed in time-delay linear systems for a given $\mathcal{H}_\infty$ level γ and secondly, when the delay is given, to minimise γ.

Chapter 3: This chapter deals with stability and stabilisation of time-delay systems through the design of an envelope that engulfs all poles of the system. The use of an envelope that ensures that all poles are contained inside it is discussed in [4]. Different types of envelopes are also discussed in [76] and [77]. In any case, no methods utilising envelopes were developed to test stability nor to design controllers. In fact, in general, the envelope extends to the right half-plane and therefore, it only provides a region where the poles are allowed to be without any guarantee about the stability of the system. In this work we provide a different analysis for the use of envelopes. Instead of using a singular value approach, such as in [4], our method is based on LMIs. We are able to provide a new procedure to test stability for both retarded and neutral time-delay systems. Furthermore, it allows to cope with some project requirements designing a state-feedback controller that guarantees robust α-stability. We develop new results through LMIs for both retarded and neutral cases. Furthermore, we extend the analysis result to fractional systems.

Chapter 4: On this chapter we leave the deterministic domain and we handle with stochastic time-delay systems. *Markov Jump Linear Systems*, or MJLS for short, with delays are the main target. Stability for stochastic systems is defined, see [78]. State-feedback control is then designed through LMIs with the novelty of achieving an affine relation with respect to the transition rate between modes, allowing polytopic uncertainty to be treated. We then obtain a linear system that models the dynamics of the first moment for positive-Markovian systems and propose a method to analyse delay-dependent stability for both the continuous and the discrete-time case.

Chapter 5: This final chapter ends with a summarisation of all that is dealt in the present dissertation. The conclusion of the work is presented as same as the perspectives for future works.

Appendix A: An introduction on Fractional Systems.

Appendix B: A summary of the dissertation in French.

Appendix C: A summary of the dissertation in Portuguese.

Chapter 2

Rational Comparison Systems

This chapter deals with the $\mathcal{H}_\infty$ control synthesis for time-delay linear systems using both state-feedback and output-feedback approaches. The filtering problem is also presented.

2.1 Introduction

Our goal is to increase the maximum delay allowed in time-delay linear systems for a given $\mathcal{H}_\infty$ level γ through state-feedback and through output-feedback control design. A second problem that is also addressed is to minimise γ whenever the time delay is given. In [79], which is the main work that this chapter relies on, the Rekasius substitution [80] for $k = 1$ was successfully applied to obtain a finite order LTI system, called *comparison system*, which was used to calculate a lower bound for the $\mathcal{H}_\infty$ norm of the time-delay system. Here, we extend this approach finding the linear dependence on the matrices of the system with its *comparison system* for a substitution of order N. We can then use this new system to design a controller which depends not only on the measured output at the present time and maximum delay, but also on an arbitrary number of intermediate values in between, for both minimising $\mathcal{H}_\infty$ norm or maximising the allowed delay. Hence, we are able to increase the maximum stable delay using information that is already in the buffer. Illustrative examples are presented to reinforce the theoretical results.

The main idea is based on the following observation. Should the control law to be devised be of the form

$$u(t) = K_0x(t) + K_1x(t - \tau), \tag{2.1}$$

for some $\tau > 0$, then a possible generalisation of such signal consists in also using the intermediate values $x(t - k\tau/N)$, $k \in \{1, \ldots, N - 1\}$, for feedback. Note that the reasoning for this approach is based on the fact that, as long as the delayed state $x(t - \tau)$ must be stored in a buffer, such buffer would also contain the intermediate values of interest and, thus, compared to the original approach, no additional information is required.

Compared to [75] and [79], the main novelties are:

- The commensurate delay problem demands a new parametrisation for the comparison system, which, to the best of the author's knowledge, has not been presented in the literature. Moreover, the design procedure is simple to be implemented and, when compared with the ones already cited, provides more accurate results.

- Most feedback design procedures demand the knowledge of the delayed state. This requires a memory buffer in order to store information from all sensors during this time. Nevertheless, for most part of procedures, including [75] and [79], the only useful information is the one that matches the time delay. In our procedure, we relaxed this constraint and showed that intermediary state information can be used for both minimising $\mathcal{H}_\infty$ norm or maximising the allowed delay.

The same idea is then used for output-feedback controllers and to solve the filtering problem.

2.2 Comparison system

Consider the time-delay linear system with M commensurate delays, whose realisation is given by

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{M} \bar{A}_k x\left(t - \bar{\tau}_k\right) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{M} \bar{C}_{zk} x\left(t - \bar{\tau}_k\right), \end{aligned} \tag{2.2}$$

in which, for all $t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ is the exogenous input, $z(t) \in \mathbb{R}^p$ is the output of interest and $\bar{\tau}_k = \tau(M-k+1)/M$, $k \in \{1 \cdots M\}$, for a given constant time delay $\tau \geq 0$.

We address the case of the commensurate delayed system (2.2) by applying the following substitution to the time delay operator concerning the largest delay:

$$e^{-\tau s} = \left(\frac{\lambda - s}{\lambda + s}\right)^N, \tag{2.3}$$

which is an exact relation for $s = \jmath\omega$, whenever τ, λ, $\omega \in \mathbb{R}_+$ and $N \in \mathbb{N}^*$ are such that

$$\omega\tau = 2N \arctan\left(\frac{\omega}{\lambda}\right). \tag{2.4}$$

When $N = 1$ this is known as Rekasius substitution [80]. We extend this result allowing $N = hM$, $h \in \mathbb{N}^*$. For the following developments, regarding the analysis of this system, it will be necessary that the number of delays be the same as the order of the approximation (2.3). Note however, that whenever $N = hM$ for some $h \in \{1, 2, \ldots\}$, system (2.2) can be equivalently restated as

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{N} A_k x\left(t - \tau_k\right) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{N} C_{zk} x\left(t - \tau_k\right), \end{aligned} \tag{2.5}$$

where $A_k \leftarrow \bar{A}_j$, $\tau_k \leftarrow \bar{\tau}_j$ whenever

$$\frac{N-k+1}{N} = \frac{M-j+1}{M}, \tag{2.6}$$

for all $k \in \{1 \cdots N\}$, $j \in \{1 \cdots M\}$ and $A_k \leftarrow 0$ otherwise. Thus, without loss of generality, hereafter we are going to work with the rearranged system (2.5) which satisfies $N = hM$ for some $h \in \{1, 2, \ldots\}$.

Remark 2.1. The analysis will be done using the number of delays as the order of the substitution. The change of variables (2.6) is used to circumvent this, restating the system (2.2) as (2.5), achieving a stronger result for the synthesis. We can now have M delays and $N = hM$ for the order of the substitution. Let us illustrate this with an example. Let us assume a system with two delays and let us use $N = 4$.

$$\dot{x}(t) = A_0 x(t) + \bar{A}_1 x\left(t - \tau\right) + \bar{A}_2 \left(t - \frac{\tau}{2}\right). \tag{2.7}$$

Applying (2.6) we can restate (2.7) as

$$\dot{x}(t) = A_0 x(t) + A_1 x\left(t - \tau\right) + A_2 \left(t - \frac{\tau}{4}\right) + A_3 x \left(t - \frac{2\tau}{4}\right) + A_4 \left(t - \frac{3\tau}{4}\right), \tag{2.8}$$

in which $A_1 = \bar{A}_1$, $A_2 = A_4 = 0$ and $A_3 = \bar{A}_2$. On the new variables the order of the system is the same as the order chosen for the Rekasius substitution allowing us to use an order higher than the amount of delays from the original system.

One of our goals is to determine the maximal time delay $\tau^\star > 0$ which ensures that the system is globally asymptotically stable for any $\tau \in [0, \tau^\star)$. To achieve this, one must analyse the non-rational transfer function of (2.5), which is given by

$$T(s, \tau) = \left(C_{z0} + \sum_{k=1}^{N} C_{zk} e^{-\tau_k s}\right) \left(sI - A_0 - \sum_{k=1}^{N} A_k e^{-\tau_k s}\right)^{-1} E_0. \tag{2.9}$$

Applying the substitution (2.3) to the transfer function $T(s, \tau)$ in (2.9), we can define the *comparison system* with transfer function $H(s, \lambda)$ such that $H(j\omega, \lambda) = T(j\omega, \tau)$, whenever (2.4) holds. In this case, the following lemma will help us define the comparison system and the comparison system's transfer function is going to be given by Lemma 2.2.

Lemma 2.1. For any finite $s \in \mathbb{C}$ and matrices $C_k \in \mathbb{R}^{p\times n}$, $A_k \in \mathbb{R}^{n\times n}$ and $E_0 \in \mathbb{R}^{n\times m}$

$$\left(\sum_{k=0}^{N} C_k s^k\right)\left(s^{N+1}I - \sum_{k=0}^{N} A_k s^k\right)^{-1} E_0$$
$$= \begin{bmatrix} C_0' \\ C_1' \\ \vdots \\ C_N' \end{bmatrix}' \left(sI - \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I \\ A_0 & A_1 & A_2 & \cdots & A_{N-1} & A_N \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_0 \end{bmatrix}. \tag{2.10}$$

Proof. First of all, we adopt the following partition of the $Nn \times Nn$ matrix appearing in the inverse of the second line of (2.10):

$$\left[\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right] = \left[\begin{array}{ccccc|c} sI & -I & 0 & \cdots & 0 & 0 \\ 0 & sI & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI & -I \\ \hline -A_0 & -A_1 & -A_2 & \cdots & -A_{N-1} & sI - A_N \end{array}\right]. \tag{2.11}$$

In order to calculate its inverse, we consider the following identity.

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}^{-1} = \begin{bmatrix} I & -X^{-1}Y \\ 0 & I \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} I & 0 \\ -ZX^{-1} & I \end{bmatrix}, \tag{2.12}$$

with $\Lambda = (W - ZX^{-1}Y)^{-1}$.

Once X in (2.11) is triangular superior and s is finite, X is non singular and we have

$$X^{-1} = \begin{bmatrix} s^{-1}I & s^{-2}I & s^{-3}I & \cdots & s^{-N}I \\ 0 & s^{-1}I & s^{-2}I & \cdots & s^{-(N-1)}I \\ 0 & 0 & s^{-1}I & \cdots & s^{-(N-2)}I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s^{-1}I \end{bmatrix}, \tag{2.13}$$

which leads to

$$X^{-1}Y = -\begin{bmatrix} s^{-N}I & s^{-(N-1)}I & s^{-(N-2)}I & \cdots & s^{-1}I \end{bmatrix}' \tag{2.14}$$

and then to

$$\begin{aligned}\Lambda^{-1} &= sI - A_N - \sum_{k=0}^{N-1} A_k s^{-(N-k)} \\ &= s^{-N}\left(s^{N+1}I - \sum_{k=0}^{N} A_k s^k\right). \end{aligned} \tag{2.15}$$

Finally, making all the multiplications involved in the second line of (2.10), we obtain

$$\begin{aligned} &\begin{bmatrix} C_0' \\ C_1' \\ \vdots \\ C_N' \end{bmatrix}' \begin{bmatrix} -X^{-1}Y\Lambda \\ \Lambda \end{bmatrix} E_0 = \\ &= \begin{bmatrix} C_0' \\ C_1' \\ \vdots \\ C_N' \end{bmatrix}' \begin{bmatrix} I & s^1 I & s^2 I & \cdots & s^N I \end{bmatrix}' \left(s^{N+1}I - \sum_{k=0}^{N} A_k s^k\right)^{-1} E_0 \\ &= \left(\sum_{k=0}^{N} C_k s^k\right)\left(s^{N+1}I - \sum_{k=0}^{N} A_k s^k\right)^{-1} E_0, \end{aligned} \tag{2.16}$$

which is the proposed equality. □

Lemma 2.2. For a given pair $(\tau, \lambda) \in \mathbb{R}_+$, using (2.3) and applying Lemma 2.1, one can put (2.9) in an equivalent form as

$$\begin{aligned} H(s,\lambda) &= \left[\begin{array}{c|c} A_\lambda & E \\ \hline C_z & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ \sum_{k=0}^{N} \alpha_k(0) A_k & \sum_{k=0}^{N} A_k \Gamma_k - \lambda \Gamma_\lambda & E_0 \\ \hline \sum_{k=0}^{N} \alpha_k(0) C_{zk} & \sum_{k=0}^{N} C_{zk}\Gamma_k & 0 \end{array}\right], \end{aligned} \tag{2.17}$$

in which $\Gamma_k, \Gamma_\lambda \in \mathbb{R}^{n \times Nn}$ are given by

$$\Gamma_k = \begin{bmatrix} \alpha_k(1) & \alpha_k(2) & \alpha_k(3) & \cdots & \alpha_k(N-1) & \alpha_k(N) \end{bmatrix} \otimes I, \tag{2.18}$$
$$\Gamma_\lambda = \begin{bmatrix} \alpha_0(0) & \alpha_0(1) & \alpha_0(2) & \cdots & \alpha_0(N-2) & \alpha_0(N-1) \end{bmatrix} \otimes I, \tag{2.19}$$

and $\alpha_0(i)$, $\alpha_k(i)$, for $k=0$ and $k \geq 1$, respectively, are given by

$$\alpha_0(i) = \binom{N}{i}, \tag{2.20}$$

$$\alpha_k(i) = \sum_{\ell=0}^{k-1} \binom{N-k+1}{i-\ell} \binom{k-1}{\ell} (-1)^{i-\ell}. \tag{2.21}$$

Proof. Substituting the Rekasius expression (2.3) in (2.9) we get

$$H(s,\lambda) = \left(C_{z0} + \sum_{k=1}^{N} C_{zk} \left(\frac{\lambda - s}{\lambda + s} \right)^{N-k+1} \right) \left(sI - A_0 - \sum_{k=1}^{N} A_k \left(\frac{\lambda - s}{\lambda + s} \right)^{N-k+1} \right)^{-1} E_0. \tag{2.22}$$

Then, we can multiply $H(s,\lambda)$ by $\frac{(\lambda+s)^N}{(\lambda+s)^N}$ to obtain

$$\begin{aligned} H(s,\lambda) = &\left(C_{z0} (\lambda + s)^N + \sum_{k=1}^{N} C_{zk} (\lambda - s)^{N-k+1} (\lambda + s)^{k-1} \right) \times \\ &\times \left((sI - A_0)(\lambda + s)^N - \sum_{k=1}^{N} A_k (\lambda - s)^{N-k+1} (\lambda + s)^{k-1} \right)^{-1} E_0. \end{aligned} \tag{2.23}$$

Expanding the binomials the previous expression becomes

$$H(s,\lambda) = C_z(s,\lambda)(A(s,\lambda))^{-1} E_0, \tag{2.24}$$

in which

$$\begin{aligned} C_z(s,\lambda) = &\, C_{z0} \sum_{i=0}^{N} \binom{N}{i} \lambda^{N-i} s^i + C_{z1} \sum_{i=0}^{N} \binom{N}{i} \lambda^{N-i} (-s)^i + \\ &+ C_{z2} \sum_{i=0}^{N-1} \binom{N-1}{i} \lambda^{N-1-i} (-s)^i \sum_{\ell=0}^{1} \binom{1}{\ell} \lambda^{1-\ell} s^\ell + \cdots \\ &+ C_{zN} \sum_{i=0}^{1} \binom{1}{i} \lambda^{1-i} (-s)^i \sum_{\ell=0}^{N-1} \binom{N-1}{\ell} \lambda^{N-1-\ell} s^\ell \\ = &\, C_{z0} \sum_{i=0}^{N} \binom{N}{i} \lambda^{N-i} s^i + \sum_{k=1}^{N} C_{zk} \sum_{i=0}^{N-k+1} \sum_{\ell=0}^{k-1} \binom{N-k+1}{i} \binom{k-1}{\ell} \lambda^{N-i-\ell} s^{i+\ell} (-1)^i \end{aligned} \tag{2.25}$$

and

$$
\begin{aligned}
A(s,\lambda) = {} & sI\sum_{i=0}^{N}\binom{N}{i}\lambda^{N-i}s^{i} - A_0\sum_{i=0}^{N}\binom{N}{i}\lambda^{N-i}s^{i} - A_1\sum_{i=0}^{N}\binom{N}{i}\lambda^{N-i}(-s)^{i} - \\
& - A_2\sum_{i=0}^{N-1}\binom{N-1}{i}\lambda^{N-1-i}(-s)^{i}\sum_{\ell=0}^{1}\binom{1}{\ell}\lambda^{1-\ell}s^{\ell} - \cdots \\
& - A_N\sum_{i=0}^{1}\binom{1}{i}\lambda^{1-i}(-s)^{i}\sum_{\ell=0}^{N-1}\binom{N-1}{\ell}\lambda^{N-1-\ell}s^{\ell},
\end{aligned}
\tag{2.26}
$$

which can be written in a more compact way:

$$
A(s,\lambda) = (sI - A_0)\sum_{i=0}^{N}\binom{N}{i}\lambda^{N-i}s^{i} - \sum_{k=1}^{N}A_k\sum_{i=0}^{N-k+1}\sum_{\ell=0}^{k-1}\binom{N-k+1}{i}\binom{k-1}{\ell}\lambda^{N-i-\ell}s^{i+\ell}(-1)^{i}.
\tag{2.27}
$$

One can immediately see that the powers of s are in the interval $[0 \quad N]$ and that the power of s and the power of λ always add to N. Hence, it is possible to group the terms that multiply the same power of s as

$$
H(s,\lambda) = \left(\sum_{i=0}^{N}\tilde{C}_{zi}\lambda^{N-i}s^{i}\right)\left(s^{N+1}I - \sum_{i=0}^{N}\tilde{A}_i\lambda^{N-i}s^{i}\right)^{-1}E_0,
\tag{2.28}
$$

in which

$$
\tilde{C}_{zi} = \sum_{k=0}^{N}C_{zk}\alpha_k(i),
\tag{2.29}
$$

$$
\tilde{A}_i = \sum_{k=0}^{N}A_k\alpha_k(i) - \lambda\alpha_0(i-1)I,
\tag{2.30}
$$

and $\alpha_k(i)$ is given by (2.20) when $k=0$ and by (2.21) when $k \geq 1$. Finally, for being able to apply Lemma 2.1 all we need to do is a similarity transformation on (2.10) using the following matrix M.

$$
M = \mathbf{diag}(\lambda^{-N}I, \lambda^{-N+1}I, \cdots, \lambda^{-1}I, I),
\tag{2.31}
$$

which results in

$$
\begin{bmatrix} C_0'\lambda^{-N} \\ C_1'\lambda^{-(N-1)} \\ \vdots \\ C_N' \end{bmatrix}'
\begin{bmatrix}
sI & -\lambda I & 0 & \cdots & 0 & 0 \\
0 & sI & -\lambda I & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & sI & -\lambda I \\
-A_0\lambda^{-N} & -A_1\lambda^{-(N-1)} & -A_2\lambda^{-(N-2)} & \cdots & -A_{N-1}\lambda^{-1} & sI - A_N
\end{bmatrix}^{-1}
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_0 \end{bmatrix}.
\tag{2.32}
$$

Applying Lemma 2.1 on equation (2.28), after this transformation, the terms in λ are cancelled and we obtain

$$H(s,\lambda) = \begin{bmatrix} \tilde{C}'_{z0} \\ \tilde{C}'_{z1} \\ \vdots \\ \tilde{C}'_{zN} \end{bmatrix}' \begin{bmatrix} sI & -\lambda I & 0 & \cdots & 0 & 0 \\ 0 & sI & -\lambda I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & sI & -\lambda I \\ -\tilde{A}_0 & -\tilde{A}_1 & -\tilde{A}_2 & \cdots & -\tilde{A}_{N-1} & sI - \tilde{A}_N \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_0 \end{bmatrix}. \tag{2.33}$$

Using (2.18)-(2.21), (2.29) and (2.30) we finally achieve (2.17) which concludes the proof. □

2.2.1 $\mathcal{H}_\infty$ Norm Calculation

We will now show how to approximate

$$\|T(s,\tau)\|_\infty = \sup_{\omega\in\mathbb{R}} \ \sigma_M(T(j\omega,\tau)) \tag{2.34}$$

for a given $\tau \in [0,\tau^\star)$. The purpose is to show that the rational transfer function $H(s,\lambda)$ can be successfully used for $\mathcal{H}_\infty$ norm calculation of the time-delay system.

In the light of the results presented in [75], we extract an important property relating the $\mathcal{H}_\infty$ norm for both the comparison system and the original time-delay one. To this end, we need to define the scalar $\lambda_o = \inf\{\lambda \mid A_\lambda \text{ is Hurwitz}\}$ and for each $\lambda \in (\lambda_o,\infty)$, we define an $\alpha \geq 0$ such that,

$$\alpha \in \arg\sup_{\omega\in\mathbb{R}} \sigma_M(H(j\omega,\lambda)). \tag{2.35}$$

Finally, determining the time delay $\tau(\lambda,\alpha)$ that satisfies

$$\alpha/\lambda = \tan(\alpha\tau/2N), \tag{2.36}$$

allows us to state the following theorem, extending Theorem 1 of [75].

Theorem 2.1. Consider the system (2.5) with no exogenous inputs. Assume that $\sum_{i=0}^{N} A_i$ is Hurwitz and let α be given by (2.35). If $\tau(\lambda,\alpha) \in [0,\tau^\star)$ such that λ satisfies (2.36) then,

$$\|H(s,\lambda)\|_\infty \leq \|T(s,\tau(\lambda,\alpha))\|_\infty. \tag{2.37}$$

Proof. The proof follow directly from the definition of the $\mathcal{H}_\infty$ norm. We have that

$$\|H(s,\lambda)\|_\infty = \sigma_M(H(j\alpha,\lambda)). \tag{2.38}$$

Since α is given by (2.35) and recalling that (2.4) makes (2.3) an exact relation, we have

$$H(j\alpha,\lambda) = T(j\alpha,\tau(\lambda,\alpha)) \tag{2.39}$$

and thus,

$$\begin{aligned}\|H(s,\lambda)\|_\infty &= \sigma_M(H(j\alpha,\lambda)) \\ &= \sigma_M(T(j\alpha,\tau(\lambda,\alpha))) \\ &\le \|T(s,\tau(\lambda,\alpha))\|_\infty . \end{aligned} \tag{2.40}$$

□

The following example illustrates the result presented in Theorem 2.1 and points out the behaviour of the bound (2.37) with respect to the Rekasius order $N \in \mathbb{N}^*$.

Example 2.1. Let us consider the time-delay system (2.5), whose realisation is defined by

$$\begin{bmatrix} A_0 \\ \hline A_1 \\ \hline C_{z0} \\ \hline C_{z1} \\ \hline E_0' \end{bmatrix} = \left[\begin{array}{rrr} -1.65 & 0.34 & 0.91 \\ 0.31 & -2.21 & 0.63 \\ 0.18 & 0.51 & -2.32 \\ \hline -2.03 & 0.43 & 0.49 \\ 0.05 & -1.42 & 0.89 \\ 0.50 & 0.81 & -2.28 \\ \hline 0.71 & 0.62 & 0.34 \\ \hline 0.94 & 0.12 & 0.73 \\ \hline 0.39 & 0.93 & 0.92 \end{array}\right].$$

We analyse the behaviour of both the $\mathcal{H}_\infty$ norm $\|T(s,\tau)\|_\infty$ as well as its lower bounds $\|H(s,\lambda)\|_\infty$ given by the comparison system, yielded by the Rekasius substitution of order $N = 1$ and $N = 3$. These results are shown in Figure 2.1; the solid line corresponds to the real norm of the time-delay system, whilst the dashed ones represent the result obtained from the comparison system. Note that the use of a higher order comparison system provides a tighter bound.

This example induces us to conjecture that, considering two integers $N_1 < N_2$, and letting $H_1(s_1,\lambda_1)$ and $H_2(s_2,\lambda_2)$ be the comparison systems associated with N_1 and N_2 respectively, then they satisfy $\|H_1(s_1,\lambda_1)\|_\infty \le \|H_2(s_2,\lambda_2)\|_\infty$ whenever $[0,\tau^\star) \ni \tau = 2N_1/\omega_1 \arctan(\omega_1/\lambda_1) = 2N_2/\omega_2 \arctan(\omega_2/\lambda_2)$.

2.3 State-Feedback Design

In this section, let us add some control to the rearranged time-delay system (2.5), which becomes

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + \sum_{k=1}^{N} A_k x(t-\tau_k) + B_0u(t) + E_0w(t), \\ z(t) &= C_{z0}x(t) + \sum_{k=1}^{N} C_{zk} x(t-\tau_k) + D_{zu}u(t). \end{aligned} \tag{2.41}$$

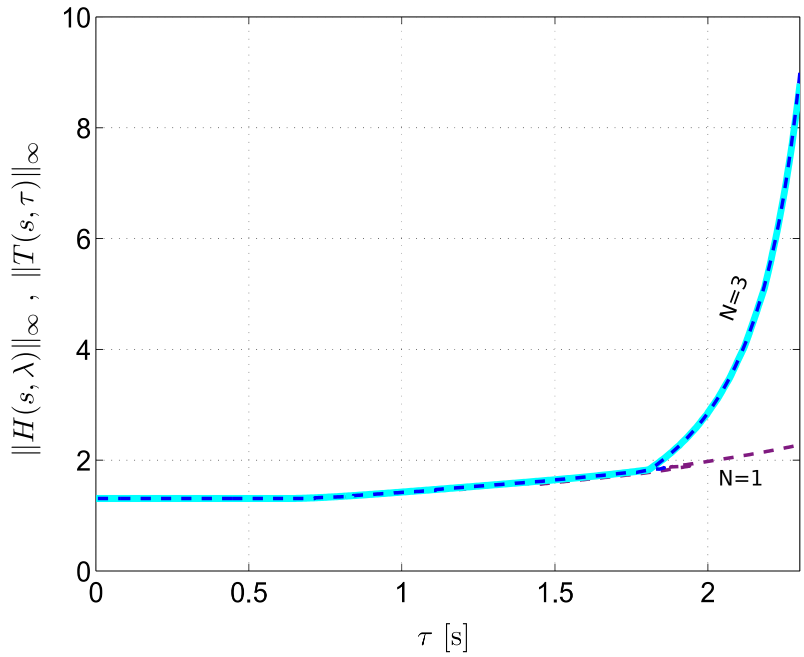

Figure 2.1: $\mathcal{H}_\infty$ norm and lower bounds as functions of τ.

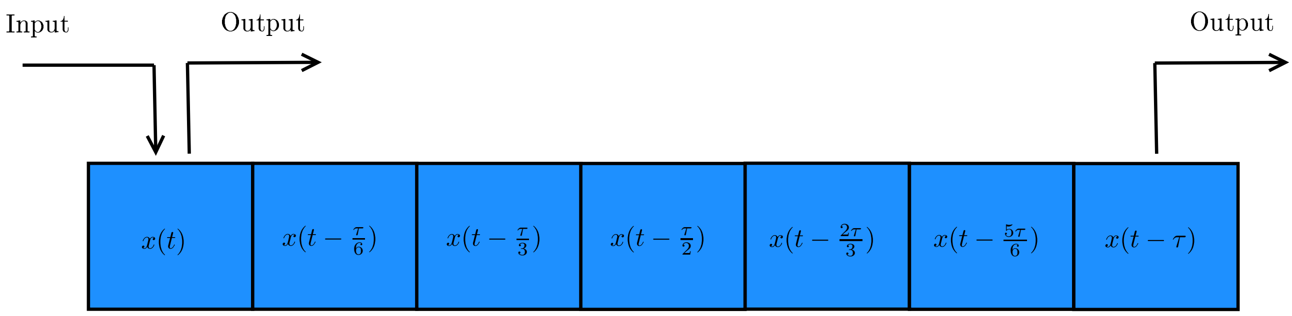

Figure 2.2: Buffer necessary to implement $u(t)$.

Our goal is to design a stabilising control rule of the form

$$u(t) = K_0 x(t) + \sum_{k=1}^{N} K_k x\left(t - \frac{N-k+1}{N}\tau\right), \tag{2.42}$$

in which the order N for the feedback law is chosen a priori and the corresponding gains K_k, for $1 \le k \le N$, must be properly designed. The reasoning for this approach is based on the fact that, as long as the state $x(t-\tau)$ can be held, if the choice of a sampling period of τ/N is feasible, it is possible to handle the states $x(t-\tau/N), x(t-2\tau/N), \cdots, x(t-\tau)$ in a buffer to be used to implement (2.42). Figure 2.2 illustrates this buffer for a designer choice of $N = 6$.

The unknown gains K_k together with the scalars $\alpha_k(i)$, for $(k,i) \in \{0, \cdots, N\}^2$, can be multiplied as

$$K = \begin{bmatrix} K_0' \\ K_1' \\ \vdots \\ K_N' \end{bmatrix}' \begin{bmatrix} \alpha_0(0) & \alpha_0(1) & \cdots & \alpha_0(N) \\ \alpha_1(0) & \alpha_1(1) & \cdots & \alpha_1(N) \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_N(0) & \alpha_N(1) & \cdots & \alpha_N(N) \end{bmatrix} \otimes I, \tag{2.43}$$

to obtain a gain matrix K which is exactly the gain that appears when we close the loop for the comparison system as can be seen on the following realisation

$$H(s,\lambda) = \left[\begin{array}{c|c} A_\lambda + BK & E \\ \hline C_z + D_{zu}K & 0 \end{array}\right], \tag{2.44}$$

in which the indicated matrices in the state-feedback framework are defined as

$$A_\lambda = \begin{bmatrix} 0 & \lambda I \\ \sum_{k=0}^{N} \alpha_k(0) A_k & \sum_{k=0}^{N} A_k \Gamma_k - \lambda \Gamma_\lambda \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix},$$
$$C_z = \begin{bmatrix} \sum_{k=0}^{N} \alpha_k(0) C_{zk} & \sum_{k=0}^{N} C_{zk}\Gamma_k \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ E_0 \end{bmatrix}. \tag{2.45}$$

The previous relations allow us to state the following lemma, which provides an important result that shall be exploited to yield design conditions for the state-feedback control law (2.42).

Lemma 2.3. For any $N \in \mathbb{N}$ and the scalars $\alpha_k(i)$ defined in (2.20) and (2.21), the matrix $\tilde{\Gamma} \in \mathbb{N}^{(N+1)\times(N+1)}$, given by

$$\tilde{\Gamma} = \begin{bmatrix} \alpha_0(0) & \alpha_0(1) & \cdots & \alpha_0(N) \\ \alpha_1(0) & \alpha_1(1) & \cdots & \alpha_1(N) \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_N(0) & \alpha_N(1) & \cdots & \alpha_N(N) \end{bmatrix}, \tag{2.46}$$

is nonsingular.

Proof. Consider the polynomial vector

$$\Phi(s,\lambda) = \begin{bmatrix} (\lambda+s)^N \\ (\lambda-s)^N \\ (\lambda+s)(\lambda-s)^{N-1} \\ \vdots \\ (\lambda+s)^{N-1}(\lambda-s) \end{bmatrix} \in \mathbb{C}^{N+1}, \tag{2.47}$$

whose expansion as a sum of monomials $\lambda^{N-i}s^i$ can be expressed as

$$\Phi(s,\lambda) = \tilde{\Gamma}\Omega, \tag{2.48}$$

in which

$$\Omega = \begin{bmatrix} \lambda^N & \lambda^{N-1}s & \cdots & s^N \end{bmatrix}'. \tag{2.49}$$

Suppose M independent of (λ, s) satisfying

$$M\Phi(s,\lambda) = \Omega, \quad \forall\lambda \in \mathbb{R}_+, \forall s \in \mathbb{C}, \tag{2.50}$$

we can prove by *reductio ad absurdum* that M is invertible and, hence, is equal to $\tilde{\Gamma}^{-1}$.

Let us decompose M into its m_i rows as follows

$$M = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{N+1} \end{bmatrix}. \tag{2.51}$$

Supposing that M is not invertible, then for some j

$$m_j = \sum_{i\neq j} \alpha_i m_i \tag{2.52}$$

and

$$\begin{aligned} m_j\Phi(s,\lambda) &= \Omega_j, \\ \left(\sum_{i\neq j} \alpha_i m_i\right)\Phi(s,\lambda) &= \Omega_j, \end{aligned} \tag{2.53}$$

implying that

$$\begin{aligned} \sum_{i\neq j} \alpha_i \left(m_i\Phi(s,\lambda)\right) &= \Omega_j, \\ \sum_{i\neq j} \alpha_i\Omega_i &= \Omega_j, \end{aligned} \tag{2.54}$$

which, recalling that this relation must be valid for all (λ, s), is clearly not possible given the structure of (2.47). Therefore, M is invertible and we can write $\Phi(s,\lambda) = M^{-1}\Omega$. Which is identical to $\Phi(s,\lambda) = \tilde{\Gamma}\Omega$. □

Remark 2.2. The formation law of (2.47) is given by

$$\begin{aligned} \Phi_1 &= (\lambda + s)^N, \\ \Phi_i &= (\lambda + s)^{i-2}(\lambda - s)^{N-i+2}, \quad i \in \{2, \cdots, N+1\}. \end{aligned}$$

The most important consequence of this lemma is the non-singularity of the augmented matrix $\tilde{\Gamma} \otimes I$, implying that the state-feedback gains K_k, $k \in \{0, \cdots, N\}$, can be obtained from

$$\begin{bmatrix} K_0 & K_1 & \cdots & K_N \end{bmatrix} = K\left(\tilde{\Gamma} \otimes I\right)^{-1}. \tag{2.55}$$

This identity is of great importance for the design of the control rule (2.42). Indeed, we first observe that (2.44) represents a standard LTI system transfer function and, thus, the feedback gain K can be designed using classical techniques for systems of this class. In particular, the $\mathcal{H}_\infty$ *control feedback gain*

$$K = -(D'_{zu}D_{zu})^{-1}(PB + C'_z D_{zu})', \tag{2.56}$$

for which $P > 0$ is the stabilising solution to the Riccati equation

$$A'_\lambda P + PA_\lambda - (PB + C'_z D_{zu})(D'_{zu}D_{zu})^{-1}(PB + C'_z D_{zu})' + C'C + \gamma^{-2}PEE'P = 0, \tag{2.57}$$

ensures not only the stability of the transfer function, but also the bound $\|H(s,\lambda)\|_\infty \leq \gamma$; see [81] for details. Thus, identity (2.55), together with the comparison system and the $\mathcal{H}_\infty$ central feedback gain, provides the theoretical foundation for the methods described in Algorithms 1 and 2. They can be used to determine the gains K_k, $k \in \{0, \cdots, N\}$, for $N \geq 1$, each of them being associated with one of the particular important problem for time-delay systems:

- **Maximum delay problem**: For a pre-specified $\mathcal{H}_\infty$ level γ, find the state-feedback gains maximising the delay τ such that $T(s,\tau)$ is stable and $\|T(s,\tau)\|_\infty \leq \gamma$;
- **Minimum norm problem**: For a pre-specified delay τ, find the state-feedback gains minimising the $\mathcal{H}_\infty$ level γ such that $T(s,\tau)$ is stable and $\|T(s,\tau)\|_\infty \leq \gamma$.

It is of interest to point out that, since both algorithms are centred on Riccati-based methods, they are viable from the computational viewpoint. The following example illustrates the results presented hitherto.

Example 2.2. To illustrate the state-feedback design we consider a second order example borrowed from [28] where the matrices corresponding to the state space realisation (2.41) are as follows

$$\begin{bmatrix} A_0 \mid A_1 \mid E_0 \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -0.9 & 1 & 0 \end{array}\right],$$

$$\begin{bmatrix} B_0 \mid C_{z0} \mid C_{z1} \mid D_{zu} \end{bmatrix} = \left[\begin{array}{c|cc|cc|c} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0.1 \end{array}\right].$$

Our main purpose with this simple example is to point out the importance of the gain $K_i \neq 0$, $i \geq 1$, for performance improvement by comparing our results with those in [28] and [79], where state-feedback control laws of the form $u(t) = K_0 x(t)$ and $u(t) = K_0 x(t) + K_1 x(t-\tau)$ were designed. Setting $\gamma = 0.13$, for each $N \in \{1, \ldots, 9\}$ we have used Algorithm 1 to calculate $\tau_\gamma = \tau(\lambda_\gamma)$, as presented in Figure 2.3. In [28], for approximately the same value of γ and $\tau = 0.999$ the gain K_0 given has large modulus (of order 10^6), whereas in [79] a maximum delay of $\tau = 1.28$ was obtained respecting the desired norm level. For all $N \in \{1, \ldots, 9\}$ the proposed controllers guarantee stability for $\tau(\lambda) \in [0 \quad 1.5708)$. It is interesting to verify that, for $N = 8$, our method is able to guarantee the prescribed $\mathcal{H}_\infty$

Algorithm 1: Maximum delay problem

Data: System Matrices (2.41); maximum $\mathcal{H}_\infty$ level γ; $N \in \mathbb{N}^*$
Result: State-feedback Gains $(K_0, K_1, \dots K_N)$; maximum delay $\tau^\star$
Initialise: $\lambda \leftarrow \lambda_0$, λ_0 large;
  stop $\leftarrow 0$;
  $\tau_0,\ \tau_{-1} \leftarrow 0$;
  $\ell \leftarrow 0$;
while *stop* $= 0$ **do**
  Calculate $(A_\lambda, B, E, C_z, Dzu)$, from (2.45);
  Solve the LTI Riccati $\mathcal{H}_\infty$ problem, whose optimal solution K, given in (2.56), provides $(K_0, K_1, \dots K_N)$ via (2.55);
  Calculate $\tau_{(\ell)}$ from (2.4);
  Determine $\tau^\star_{(\ell)}$ and $\gamma_{(\ell)} = \|T(s, \tau_{(\ell)})\|_\infty$ for the closed-loop system;
  If $(\tau_{(\ell-1)} \leq \tau_{(\ell)} \in [0,\ \tau^\star_{(\ell)})$ and $\gamma_{(\ell)} \leq \gamma)$ **then**;
    $(K_0^{\text{opt}}, K_1^{\text{opt}}, \dots K_N^{\text{opt}}) \leftarrow (K_0, K_1, \dots K_N)$; $\tau^{\text{opt}} \leftarrow \tau_{(\ell)}$;
    $\ell \leftarrow \ell + 1$;
    $\lambda \leftarrow \lambda_\ell < \lambda_{\ell-1}$;
  Else stop $\leftarrow 1$;
end
Return $(K_0, K_1, \dots K_N) \leftarrow (K_0^{\text{opt}}, K_1^{\text{opt}}, \dots K_N^{\text{opt}})$ and $\tau^\star \leftarrow \tau^{\text{opt}}$;

Algorithm 2: Minimum norm problem

Data: System Matrices (2.41); delay τ; $N \in \mathbb{N}^*$
Result: State-feedback Gains $(K_0, K_1, \dots K_N)$; minimum $\mathcal{H}_\infty$ norm γ
Initialise: $\gamma \leftarrow \gamma_0$, γ_0 large enough;
  stop $\leftarrow 0$;
  $\ell \leftarrow 0$;
while *stop* $= 0$ **do**
  Solve the Maximum delay problem for the given state-space matrices, γ and $N \in \mathbb{N}^*$ obtaining $\tau_{(\ell)}$;
  If $\tau_{(\ell)} \geq \tau$ **then**;
    $\gamma^{\text{opt}} \leftarrow \gamma_{(\ell)}$;
    $\ell \leftarrow \ell + 1$;
    $\gamma \leftarrow \gamma_{(\ell)} < \gamma_{(\ell-1)}$;
  Else stop $\leftarrow 1$;
end
Return $\gamma \leftarrow \gamma^{\text{opt}}$ and $(K_0, K_1, \dots K_N) \leftarrow (K_0^{\text{opt}}, K_1^{\text{opt}}, \dots K_N^{\text{opt}})$ the solution related to τ of the maximum delay problem for the given state-space matrices, γ^{opt} and $N \in \mathbb{N}^*$;

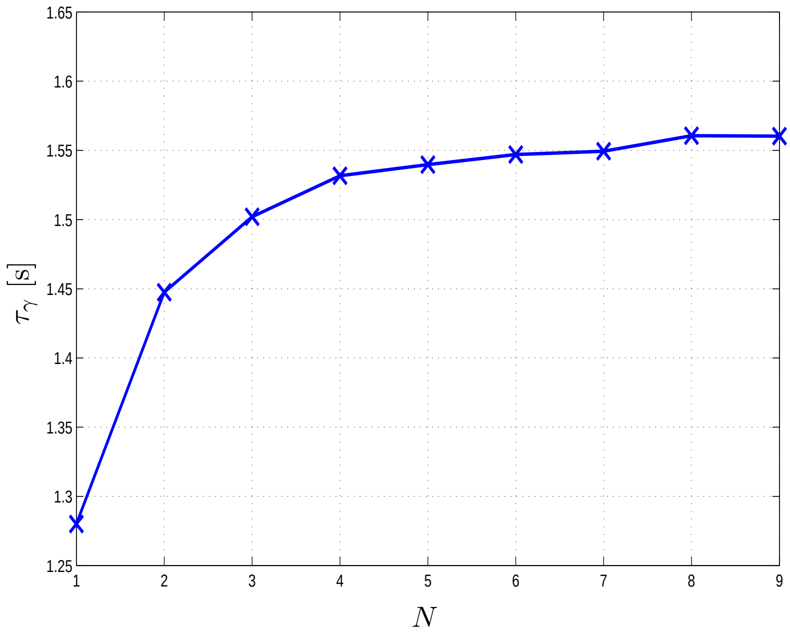

Figure 2.3: τ_γ as a function of N for $\gamma = 0.13$.

level for more than 99% of the complete stability delay interval.

We have also used Algorithm 2 to obtain the minimum value of $\gamma > 0$ for $\tau = 1$ and $N = 2$. The optimal solution, provided by the feedback gains

$$K_0 = \begin{bmatrix} 0.0000 & -31.0981 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0.0000 & 3.4589 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 0.0000 & 3.0459 \end{bmatrix},$$

guarantees that the bound $\|T(s,\tau)\|_\infty < 0.11$ holds for the values described previously.

2.4 Output-Feedback Design

In this section we address the output-feedback design. Lets consider the following time-delay system with realisation

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{M} \bar{A}_k x(t - \bar{\tau}_k) + B_0 u(t) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{M} \bar{C}_{zk} x(t - \bar{\tau}_k) + D_{zu} u(t), \\ y(t) &= C_{y0} x(t) + \sum_{k=1}^{M} \bar{C}_{yk} x(t - \bar{\tau}_k) + D_{yw} w(t), \end{aligned} \tag{2.58}$$

in which, in addition to the assumptions and the variables defined in previous sections, $y(t) \in \mathbb{R}^q$ is the measured signal. The aim at this point is to design a full order dynamic output-feedback controller with the following structure

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}_0\hat{x}(t) + \sum_{k=1}^{N} \hat{A}_k\hat{x}(t-\tau_k) + \hat{B}_0 y(t), \\ u(t) &= \hat{C}_0\hat{x}(t) + \sum_{k=1}^{N} \hat{C}_k\hat{x}(t-\tau_k), \end{aligned} \tag{2.59}$$

in which $\hat{x}(t) \in \mathbb{R}^n$ for all $t \in \mathbb{R}_+$ and $N = hM$ for some $h \in \{1, 2, \ldots\}$. We, once again, through a suitable change of indices and considering null the matrices where the respective delay is not present, rewrite (2.58) as

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{N} A_k x(t-\tau_k) + B_0 u(t) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{N} C_{zk} x(t-\tau_k) + D_{zu} u(t), \\ y(t) &= C_{y0} x(t) + \sum_{k=1}^{N} C_{yk} x(t-\tau_k) + D_{yw} w(t). \end{aligned} \tag{2.60}$$

After connecting (2.59) to (2.60), we obtain

$$\begin{aligned} \dot{\xi}(t) &= F_0\xi(t) + \sum_{k=1}^{N} F_k\xi(t-\tau_k) + G_0 w(t), \\ z(t) &= J_0\xi(t) + \sum_{k=1}^{N} J_k\xi(t-\tau_k), \end{aligned} \tag{2.61}$$

in which $\xi(t) = [x(t)' \quad \hat{x}(t)']' \in \mathbb{R}^{2n}$ is the state and the indicated matrices stand for

$$\begin{aligned} F_k &= \begin{bmatrix} A_k & B_0\hat{C}_k \\ \hat{B}_0 C_{yk} & \hat{A}_k \end{bmatrix}, \quad J_k = \begin{bmatrix} C_{zk} & D_{zu}\hat{C}_k \end{bmatrix}, \\ G_0 &= \begin{bmatrix} E_0' & D_{yw}'\hat{B}_0' \end{bmatrix}'. \end{aligned} \tag{2.62}$$

The transfer function $T_C(s,\tau)$ from the external input $w(t)$ to the controlled output $z(t)$ becomes exactly (2.9) if we consider $F_k \leftarrow A_k$, $J_k \leftarrow C_{zk}$ and $G_0 \leftarrow E_0$, in which the subindex C indicates its dependence on a given controller of the form (2.59). Hence, the goal is to design a controller such that $\|T_C(s,\tau)\|_\infty < \gamma$ for a given $\gamma > 0$, which is accomplished by the definition of the following rational comparison system

$$\begin{aligned} H_C(s,\lambda) &= \left[\begin{array}{c|c} F_\lambda & G \\ \hline J & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ \sum_{k=0}^{N} \alpha_k(0) F_k & \sum_{k=0}^{N} F_k\Gamma_k - \lambda\Gamma_\lambda & G_0 \\ \hline \sum_{k=0}^{N} \alpha_k(0) J_k & \sum_{k=0}^{N} J_k\Gamma_k & 0 \end{array}\right]. \end{aligned} \tag{2.63}$$

With this system, we can solve the corresponding $\mathcal{H}_\infty$ output-feedback design problem for each $\lambda > 0$ and extract the corresponding time delay $\tau(\lambda)$. Note that, even though the matrices of the state space realisation of $H_C(s,\lambda)$ depend on an intricate manner on the control state space realisation matrices, by applying the following similarity transformation

$$S = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \tag{2.64}$$

one can rewrite (2.63) in the equivalent form

$$\begin{aligned} H_C(s,\lambda) &= \left[\begin{array}{c|c} S^{-1}F_\lambda S & S^{-1}G \\ \hline JS & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_\lambda & B\hat{C} & E \\ \hat{B}C_y & \hat{A}_\lambda & \hat{B}D_{yw} \\ \hline C_z & D_{zu}\hat{C} & 0 \end{array}\right], \end{aligned} \tag{2.65}$$

in which the system matrices (A_λ, E, C_z) have been defined in (2.17),

$$B' = [0 \quad B_0'], \quad C_y = \left[\sum_{k=0}^{N} \alpha_k(0) C_{yk} \quad \sum_{k=0}^{N} C_{yk}\Gamma_k\right], \tag{2.66}$$

and the controller matrices are given by

$$\begin{aligned} \hat{A}_\lambda &= \begin{bmatrix} 0 & \lambda I \\ \sum_{k=0}^{N} \alpha_k(0)\hat{A}_k & \sum_{k=0}^{N} \hat{A}_k\Gamma_k - \lambda\Gamma_\lambda \end{bmatrix}, \; \hat{B} = \begin{bmatrix} 0 \\ \hat{B}_0 \end{bmatrix}, \\ \hat{C} &= \left[\sum_{k=0}^{N} \alpha_k(0)\hat{C}_k \quad \sum_{k=0}^{N} \hat{C}_k\Gamma_k\right], \end{aligned} \tag{2.67}$$

indicating that they are in the comparison form. Hence, the controller (2.59) whenever connected to the time-delay system (2.60) produces an LTI comparison system associated to the regulated output (2.61) whose transfer function can be alternatively determined from the connection of the LTI comparison system of the system (2.60) and the LTI comparison system of the controller (2.59).

Given the particular structure of (2.67), we propose a strategy similar to the one presented in [79] such that the controller matrices $(\hat{A}_\lambda, \hat{B}, \hat{C})$ will be replaced by general matrix variables (A_C, B_C, C_C). These two realisations are coupled by a nonsingular matrix $V \in \mathbb{R}^{(N+1)n\times(N+1)n}$ which defines the similarity transformation

$$\hat{A}_\lambda = VA_CV^{-1}, \tag{2.68}$$

$$\hat{B} = VB_C, \tag{2.69}$$

$$\hat{C} = C_CV^{-1}. \tag{2.70}$$

These equalities hold under the conditions stated on the following theorem.

Theorem 2.2. Let V be a nonsingular matrix such that

$$\text{vec}(V) \in \ker \begin{bmatrix} B_C' \otimes \begin{bmatrix} I & 0 \end{bmatrix} \\ A_C' \otimes \begin{bmatrix} I & 0 \end{bmatrix} - I \otimes \begin{bmatrix} 0 & \lambda I \end{bmatrix} \end{bmatrix} \tag{2.71}$$

and

$$n(1 + 1/N) > q. \tag{2.72}$$

Then, V satisfies

$$(\hat{A}_\lambda, \hat{B}, \hat{C}) = (VA_C V^{-1}, VB_C, C_C V^{-1}). \tag{2.73}$$

Proof. From (2.71) we have that

$$\begin{bmatrix} B_C' \otimes \begin{bmatrix} I & 0 \end{bmatrix} \\ A_C' \otimes \begin{bmatrix} I & 0 \end{bmatrix} - I \otimes \begin{bmatrix} 0 & \lambda I \end{bmatrix} \end{bmatrix} \text{vec}(V) = 0. \tag{2.74}$$

Using the fact that $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$, see [82], we can rewrite the matrix equality (2.74) into

$$\begin{bmatrix} I & 0 \end{bmatrix} VB_C = 0 \tag{2.75}$$

and

$$\begin{bmatrix} I & 0 \end{bmatrix} VA_C = \begin{bmatrix} 0 & \lambda I \end{bmatrix} V. \tag{2.76}$$

The first equation ensures that the first Nn rows of VB_C are zero. Hence, we have Nnq equations. The second equation says that the first Nn rows of $VA_C V^{-1}$ are in the form $\begin{bmatrix} 0 & \lambda I \end{bmatrix}$. Hence, we have more $Nn^2(N+1)$ equalities. For this system of equations to have a solution and guarantee the desired form of (2.67) we need more unknowns than equations, i.e.,

$$((N+1)n)^2 > Nnq + Nn^2(N+1), \tag{2.77}$$

which is satisfied by (2.72). Therefore, V can always be obtained. □

To obtain the matrices A_C, B_C e C_C we just solve the traditional LTI $\mathcal{H}_\infty$ problem for output feedback. This can be achieved through Riccati equations [83] under the usual assumptions $D_{zu}' C_z = 0$, $E_0 D_{yw}' = 0$, $D_{yw} D_{yw}' = I$ and $D_{zu}' D_{zu} = I$, or via LMIs such as in [84].

Now, to finally recover the controller matrices we define

$$\hat{A} = \begin{bmatrix} \sum_{k=0}^{N} \alpha_k(0) \hat{A}_k & \sum_{k=0}^{N} \hat{A}_k \Gamma_k \end{bmatrix}, \tag{2.78}$$

where $\hat{A}$ is obtained from the n last rows of $\hat{A}_\lambda$ added to $[0 \quad \Gamma_\lambda]$, and from

$$\tilde{\Gamma} = \begin{bmatrix} \alpha_0(0) & \alpha_0(1) & \cdots & \alpha_0(N) \\ \alpha_1(0) & \alpha_1(1) & \cdots & \alpha_1(N) \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_N(0) & \alpha_N(1) & \cdots & \alpha_N(N) \end{bmatrix}, \tag{2.79}$$

which is non singular, as proved in Lemma 2.3. Thus, we may recover the desired matrices as follows

$$\begin{bmatrix} \hat{A}_0 & \hat{A}_1 & \cdots & \hat{A}_N \end{bmatrix} = \hat{A}\left(\tilde{\Gamma} \otimes I\right)^{-1}, \tag{2.80}$$

$$\begin{bmatrix} \hat{C}_0 & \hat{C}_1 & \cdots & \hat{C}_N \end{bmatrix} = \hat{C}\left(\tilde{\Gamma} \otimes I\right)^{-1}, \tag{2.81}$$

and $\hat{B}_0$ is immediately obtained from $\hat{B}$.

Once we have the controller matrices at hand, it is a simple matter of computation to determine whether $\|T_C(s, \tau(\lambda))\|_\infty < \gamma$ holds, see [85] and [86].

Example 2.3. To illustrate the results for output-feedback design we consider a second order example borrowed from [28] and [79] where the matrices corresponding to the state space realisation (2.60) are as follows

$$\left[\begin{array}{c|c|c} A_0 & A_1 & E_0 \end{array}\right] = \left[\begin{array}{cc|cc|c|cc} 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -0.9 & 1 & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{c|c|c|c} B_0 & C_{z0} & C_{z1} & D_{zu} \end{array}\right] = \left[\begin{array}{c|cc|cc|c} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0.1 \end{array}\right],$$

$$\left[\begin{array}{c|c|c} C_{y0} & C_{y1} & D_{yw} \end{array}\right] = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0.1 \end{array}\right].$$

We can now solve two distinct problems, the maximum delay problem and the minimum norm problem. In the first one, for a fixed pre-specified $\mathcal{H}_\infty$ level γ, we find the output-feedback controller that maximise the delay. In the second one, for a fixed pre-specified maximum delay τ, we find the output-feedback controller matrices that minimise the $\mathcal{H}_\infty$ level γ. In both cases, $T(s,\tau)$ is stable and $\|T(s,\tau)\|_\infty \leq \gamma$.

To illustrate the first problem, lets set $\gamma = 1$. Using $N = 1$ in the expansion for the comparison system, we achieve for the maximum delay $\tau = 1.2324$. This result and the behaviour of both norms, $\|T_c(s,\tau(\lambda))\|_\infty$ and $\|H_c(s,\lambda)\|_\infty$, as a function of τ is exactly the same as in [79]. However, increasing the expansion by using $N = 2$ and $N = 3$ we get for the maximum delay $\tau = 1.4268$ and $\tau_\gamma = 1.5117$ respectively. Besides the fact that we increased the maximum delay allowed by a factor of 22%, we also have, that for every $0 < \tau < \tau_\gamma$, the norm of the comparison system is the same as the norm of the system with delays as depicted in Figure 2.4. These results can be better visualised in the table below.

N	$\tau_{\max}$
1	1.2324
2	1.4268
3	1.5117

For the maximum delay,

$$\|T_c(s,\tau(\lambda))\|_\infty = \|H_c(s,\lambda)\|_\infty = 0.2675. \tag{2.82}$$

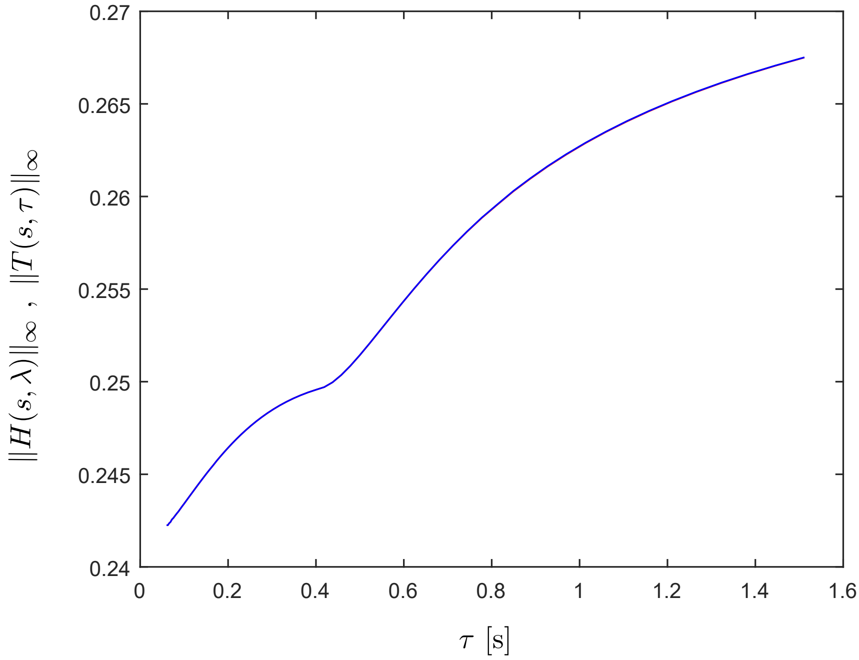

Figure 2.4: $\mathcal{H}_\infty$ performance versus time delay for $\gamma = 1$

Now, setting $\tau = 1$. We get $\|T_c(s, \tau(\lambda))\|_\infty = 0.2010$ which is 26% smaller than the $\mathcal{H}_\infty$ norm obtained by [79] and 76% smaller than the $\mathcal{H}_\infty$ norm obtained by [34]. We also have exactly the same norm for the comparison system, $\|H_c(s, \lambda)\|_\infty = 0.2010$. Finally, the controller matrices for this case are

$$\left[\ \hat{A}_0 \mid \hat{A}_1\ \right] = \left[\begin{array}{cc|cc} -15.1593 & 9.5743 & 1.2467 & -1.7115 \\ 18.5286 & -11.8869 & -0.6430 & 0.8165 \end{array}\right],$$
$$\left[\ \hat{A}_2 \mid \hat{A}_3\ \right] = \left[\begin{array}{cc|cc} 0.7977 & -2.8727 & -1.2294 & -1.1746 \\ -1.4215 & 2.7861 & 1.2287 & 1.6388 \end{array}\right],$$
$$\left[\ \hat{C}'_0 \mid \hat{C}'_1\ \right] = \left[\begin{array}{c|c} 39.1349 & -5.6235 \\ -24.8696 & 4.4252 \end{array}\right],$$
$$\left[\ \hat{C}'_2 \mid \hat{C}'_3\ \right] = \left[\begin{array}{c|c} -1.1655 & 3.0810 \\ 7.2716 & 2.7257 \end{array}\right],$$
$$\hat{B}_0 = \left[\ -2.0019 \quad 6.2982\ \right]'.$$

2.4.1 Filter Design

As a particular case of the output-feedback design, we can use the comparison system to design a filter that estimates a desired output using a measured output. Let us consider the following time-delay system with realisation

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{M} \bar{A}_k x\left(t-\bar{\tau}_k\right) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{M} \bar{C}_{zk} x\left(t-\bar{\tau}_k\right), \\ y(t) &= C_{y0} x(t) + \sum_{k=1}^{M} \bar{C}_{yk} x\left(t-\bar{\tau}_k\right) + D_{yw} w(t). \end{aligned} \tag{2.83}$$

The aim is to design a filter with the following structure

$$\begin{aligned} \dot{x}_f(t) &= A_{f0} x_f(t) + \sum_{k=1}^{N} A_{fk} x_f\left(t-\tau_k\right) + B_f y(t), \\ z_f(t) &= C_{f0} x_f(t) + \sum_{k=1}^{N} C_{fk} x_f\left(t-\tau_k\right), \end{aligned} \tag{2.84}$$

in which $x_f(t) \in \mathbb{R}^n$ for all $t \in \mathbb{R}_+$, $e(t) \triangleq z(t) - z_f(t)$ and $N = hM$ for some $h \in \{1, 2, \ldots\}$. We, once again, through a suitable change of indices and considering null the matrices where the respective delay is not present, rewrite (2.83) as

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{N} A_k x\left(t-\tau_k\right) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{N} C_{zk} x\left(t-\tau_k\right), \\ y(t) &= C_{y0} x(t) + \sum_{k=1}^{N} C_{yk} x\left(t-\tau_k\right) + D_{yw} w(t). \end{aligned} \tag{2.85}$$

After connecting (2.84) to (2.85), we obtain

$$\begin{aligned} \dot{\xi}_f(t) &= F_0 \xi_f(t) + \sum_{k=1}^{N} F_k \xi_f\left(t-\tau_k\right) + G_0 w(t), \\ z(t) &= J_0 \xi_f(t) + \sum_{k=1}^{N} J_k \xi_f\left(t-\tau_k\right), \end{aligned} \tag{2.86}$$

in which $\xi(t) = [x(t)' \quad e(t)']' \in \mathbb{R}^{n \times p}$ is the state and the indicated matrices stand for

$$\begin{aligned} F_k &= \begin{bmatrix} A_k & 0 \\ \hat{B}_0 C_{yk} & \hat{A}_k \end{bmatrix}, \quad J_k = \begin{bmatrix} C_{zk} & -\hat{C}_k \end{bmatrix}, \\ G_0 &= \begin{bmatrix} E_0' & D_{yw}' \hat{B}_0' \end{bmatrix}'. \end{aligned} \tag{2.87}$$

It is very easy to see that this is a special case of (2.62) when $B_u = 0$, $D_{zu} = -I$, $\hat{B}_0 = B_{fk}$, and $\hat{A}_k = A_{fk}$, $\hat{C} = C_{fk}$, for $k \in \{0 \cdots N\}$.

2.5 Fractional system

We can now extend our results to Fractional Order Systems, also known as *FOS*. Consider the time-delay fractional linear system with N commensurate delays, whose realisation is given by

$$\begin{aligned} {}_0\mathbb{D}_t^{\alpha} x(t) &= A_0 x(t) + \sum_{k=1}^{N} A_k x(t-\tau_k) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{N} C_{zk} x(t-\tau_k), \end{aligned} \tag{2.88}$$

in which, for all $t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ is the exogenous input, $z(t) \in \mathbb{R}^p$ is the output of interest and $\tau_k = \tau(N-k+1)/N, k = 1 \ldots N$, for a given constant time delay $\tau \geq 0$. Also, ${}_0\mathbb{D}_t^{\alpha}$ is the Differintegral operator, see Appendix A.

One of our goals, as extensively discussed in the integer case, is to determine the maximal time delay $\tau^\star > 0$ which ensures that the system is globally asymptotically stable for any $\tau \in [0, \tau^\star)$. To achieve this, one must analyse the non-rational transfer function of (2.88), which is given by

$$T(s,\tau) = \left(C_{z0} + \sum_{k=1}^{N} C_{zk} e^{-\tau_k s} \right) \left(s^{\alpha} I - A_0 - \sum_{k=1}^{N} A_k e^{-\tau_k s} \right)^{-1} E_0. \tag{2.89}$$

Applying the substitution (2.3) to the transfer function $T(s,\tau)$ in (2.89), we can define the *comparison system* with transfer function $H(s,\lambda)$ such that $H(j\omega,\lambda) = T(j\omega,\tau)$, whenever (2.4) holds. In this case, the fractional comparison system's transfer function is given by the following two lemmas.

Lemma 2.4. For any finite $s \in \mathbb{C}$, $\alpha = \dfrac{1}{M}$, $M \in \mathbb{N}^* - \{1\}$ and matrices $C_k \in \mathbb{R}^{p\times n}$, $A_k, L_k \in \mathbb{R}^{n\times n}$ and $E \in \mathbb{R}^{n\times m}$

$$\begin{aligned} &\left(\sum_{k=0}^{N} C_k s^k \right) \left(s^{N+\alpha} I - \sum_{k=0}^{N} A_k s^k - \sum_{k=0}^{N-1} L_k s^{(k+\alpha)} \right)^{-1} E_0 \\ &= \begin{bmatrix} C_0' \\ 0 \\ C_1' \\ \vdots \\ C_{N-1}' \\ 0 \\ C_N' \end{bmatrix}' \left(s^{\alpha} I - \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & I \\ A_0 & L_0 & 0 & \cdots & A_1 & \cdots & L_{N-1} & A_N \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_0 \end{bmatrix}, \end{aligned} \tag{2.90}$$

where the number of $0_{n\times p}$ between C_k' and C_{k+1}' is $M-1$ and the number of $0_{n\times n}$ between L_k and A_{k+1} is $M-2$.

Proof. First of all, adopting the following partition of the $\left(\frac{Nn}{\alpha} + n\right) \times \left(\frac{Nn}{\alpha} + n\right)$ matrix appearing in the inverse of the second line of (2.90):

$$\left[\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right] = \left[\begin{array}{ccccc|c} s^\alpha I & -I & 0 & \cdots & 0 & 0 \\ 0 & s^\alpha I & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s^\alpha I & -I \\ \hline -A_0 & -L_0 & 0 & \cdots & -L_{N-1} & s^\alpha I - A_N \end{array}\right]. \tag{2.91}$$

In order to calculate its inverse, we consider the following identity, based on the assumption of existing X^{-1}:

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}^{-1} = \begin{bmatrix} I & -X^{-1}Y \\ 0 & I \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} I & 0 \\ -ZX^{-1} & I \end{bmatrix}, \tag{2.92}$$

with $\Lambda = (W - ZX^{-1}Y)^{-1}$. Once X in (2.91) is triangular superior and s is finite, it is non singular and we have

$$X^{-1} = \begin{bmatrix} s^{-\alpha} I & s^{-2\alpha} I & s^{-3\alpha} I & \cdots & s^{-N} I \\ 0 & s^{-\alpha} I & s^{-2\alpha} I & \cdots & s^{-(N-\alpha)} I \\ 0 & 0 & s^{-\alpha} I & \cdots & s^{-(N-2\alpha)} I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s^{-\alpha} I \end{bmatrix}, \tag{2.93}$$

which leads to

$$X^{-1}Y = -\begin{bmatrix} s^{-N} I & s^{-(N-\alpha)} I & s^{-(N-2\alpha)} I & \cdots & s^{-\alpha} I \end{bmatrix}' \tag{2.94}$$

and then to

$$\begin{aligned} \Lambda^{-1} &= s^\alpha I - A_N - \sum_{k=0}^{N-1} A_k s^{-(N-k)} - \sum_{k=0}^{N-1} L_k s^{-(N-k)} s^\alpha \\ &= s^{-N} \left(s^{N+\alpha} I - \sum_{k=0}^{N} A_k s^k - \sum_{k=0}^{N-1} L_k s^{(k+\alpha)} \right). \end{aligned} \tag{2.95}$$

Finally, making all the multiplications involved in the second line of (2.90), we obtain

$$\begin{aligned}
&\begin{bmatrix} C_0' \\ 0 \\ C_1' \\ \vdots \\ C_{N-1}' \\ 0 \\ C_N' \end{bmatrix}' \begin{bmatrix} -X^{-1}Y\Lambda \\ \Lambda \end{bmatrix} E_0 = \\
&= \begin{bmatrix} C_0' \\ 0 \\ C_1' \\ \vdots \\ C_{N-1}' \\ 0 \\ C_N' \end{bmatrix}' \begin{bmatrix} I & s^{\alpha}I & s^{2\alpha}I & \cdots & s^{N-\alpha}I & s^{N}I \end{bmatrix}' \left(s^{N+\alpha}I - \sum_{k=0}^{N} A_k s^k - \sum_{k=0}^{N-1} L_i s^{(k+\alpha)} \right)^{-1} E_0 \\
&= \left(\sum_{i=0}^{N} C_k s^k \right) \left(s^{N+\alpha}I - \sum_{k=0}^{N} A_k s^k - \sum_{k=0}^{N-1} L_i s^{(k+\alpha)} \right)^{-1} E_0. \qquad (2.96)
\end{aligned}$$

which is the proposed equality. □

Lemma 2.5. For a given pair $(\tau, \lambda) \in \mathbb{R}_+$, using (2.3) and applying Lemma 2.4, one can put (2.89) in an equivalent form as

$$H(s,\lambda) = \left[\begin{array}{c|c} A_\lambda & E \\ \hline C_z & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & I & 0 \\ \sum_{k=0}^{N} \alpha_k(0) A_k & \sum_{k=0}^{N} A_k \Gamma_k - \Gamma_\lambda & E_0 \\ \hline \sum_{k=0}^{N} \alpha_k(0) C_{zk} & \sum_{k=0}^{N} C_{zk}\Gamma_k & 0 \end{array}\right], \qquad (2.97)$$

in which $\Gamma_k, \Gamma_\lambda \in \mathbb{R}^{n \times Nn}$ are given by

$$\Gamma_k = \begin{bmatrix} \Gamma_{k1} & \cdots & \Gamma_{kN/\alpha} \end{bmatrix} \otimes I, \qquad (2.98)$$

$$\Gamma_\lambda = \begin{bmatrix} \Gamma_{\lambda 1} & \cdots & \Gamma_{\lambda N/\alpha} \end{bmatrix} \otimes I, \qquad (2.99)$$

with

$$\Gamma_{ki} = \alpha_k(j)\lambda^{N-j}, 1 \le j \le N, \qquad (2.100)$$

on the N first positions where $(i\alpha) = \lfloor i\alpha \rfloor$, that is, $i\alpha \in \mathbb{Z}$ and $\Gamma_{ki} = 0$ otherwise.

$$\Gamma_{\lambda i} = \alpha_\lambda(j)\lambda^{N-j}, 0 \le j \le N-1, \qquad (2.101)$$

on the N first positions where $(i\alpha) - \lfloor i\alpha \rfloor = \alpha$ and $\Gamma_{\lambda i} = 0$ otherwise.

Also, $\alpha_0(i)$, $\alpha_k(i)$, for $k = 0$ and $k \geq 1$, respectively, are given by

$$\alpha_0(i) = \binom{N}{i}, \tag{2.102}$$

$$\alpha_k(i) = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \binom{N-k+1}{i-\ell} (-1)^{i-\ell}. \tag{2.103}$$

Proof. Substituting the Rekasius expression (2.3) in (2.89) we get

$$H(s,\lambda) = \left(C_{z0} + \sum_{k=1}^{N} C_{zk} \left(\frac{\lambda - s}{\lambda + s} \right)^{N-k+1} \right) \left(s^{\alpha} I - A_0 - \sum_{k=1}^{N} A_k \left(\frac{\lambda - s}{\lambda + s} \right)^{N-k+1} \right)^{-1} E_0. \tag{2.104}$$

Then, we can multiply $H(s,\lambda)$ by $\frac{(\lambda+s)^N}{(\lambda+s)^N}$ to obtain

$$\begin{aligned} H(s,\lambda) = & \left(C_{z0} (\lambda + s)^N + \sum_{k=1}^{N} C_{zk} (\lambda - s)^{N-k+1} (\lambda + s)^{k-1} \right) \times \\ & \times \left((s^{\alpha} I - A_0) (\lambda + s)^N - \sum_{k=1}^{N} A_k (\lambda - s)^{N-k+1} (\lambda + s)^{k-1} \right)^{-1} E_0. \end{aligned} \tag{2.105}$$

Expanding the binomials the previous expression becomes

$$H(s,\lambda) = C_z(s,\lambda)(A(s,\lambda))^{-1} E_0, \tag{2.106}$$

in which

$$\begin{aligned} C_z(s,\lambda) = & C_{z0} \sum_{i=0}^{N} \binom{N}{i} \lambda^{N-i} s^i + C_{z1} \sum_{i=0}^{N} \binom{N}{i} \lambda^{N-i} (-s)^i + \\ & + C_{z2} \sum_{i=0}^{N-1} \binom{N-1}{i} \lambda^{N-1-i} (-s)^i \sum_{\ell=0}^{1} \binom{1}{\ell} \lambda^{1-\ell} s^{\ell} + \cdots \\ & + C_{zN} \sum_{i=0}^{1} \binom{1}{i} \lambda^{1-i} (-s)^i \sum_{\ell=0}^{N-1} \binom{N-1}{\ell} \lambda^{N-1-\ell} s^{\ell} \\ = & C_{z0} \sum_{i=0}^{N} \binom{N}{i} \lambda^{N-i} s^i + \sum_{k=1}^{N} C_{zk} \sum_{i=0}^{N-k+1} \sum_{\ell=0}^{k-1} \binom{N+1-k}{i} \binom{k-1}{\ell} \lambda^{N-i-\ell} s^{i+\ell} (-1)^i \end{aligned} \tag{2.107}$$

and

$$\begin{aligned} A(s,\lambda) = s^{\alpha} I \sum_{i=0}^{N} \binom{N}{i} \lambda^{N-i} s^{i} - A_0 \sum_{i=0}^{N} \binom{N}{i} \lambda^{N-i} s^{i} - A_1 \sum_{i=0}^{N} \binom{N}{i} \lambda^{N-i} s^{i} (-1)^{i} - \\ - A_2 \sum_{i=0}^{N-1} \binom{N-1}{i} \lambda^{N-1-i} s^{i} (-1)^{i} \sum_{\ell=0}^{1} \binom{1}{\ell} \lambda^{1-\ell} s^{\ell} - \cdots \\ - A_N \sum_{i=0}^{1} \binom{1}{i} \lambda^{1-i} s^{i} (-1)^{i} \sum_{\ell=0}^{N-1} \binom{N-1}{\ell} \lambda^{N-1-\ell} s^{\ell}, \end{aligned} \tag{2.108}$$

which can be written in a more compact way:

$$A(s,\lambda) = (s^{\alpha} I - A_0) \sum_{i=0}^{N} \tbinom{N}{i} \lambda^{N-i} s^{i} - \sum_{k=1}^{N} A_k \sum_{i=0}^{N-k+1} \sum_{\ell=0}^{k-1} \tbinom{N+1-k}{i} \tbinom{k-1}{\ell} \lambda^{N-i-\ell} s^{i+\ell} (-1)^{i}. \tag{2.109}$$

One can immediately see that the powers of s are in the interval $[0 \quad N]$ and that the power of s and the power of λ always add to N. Hence, it is possible to group the terms that multiply the same power of s

$$\begin{aligned} H(s,\lambda) = \left(\sum_{i=0}^{N} \sum_{k=0}^{N} C_{zk} \alpha_k(i) \lambda^{N-i} s^{i} \right) \times \\ \times \left(s^{N+\alpha} I - \sum_{i=0}^{N} \sum_{k=0}^{N} A_k \alpha_k(i) \lambda^{N-i} s^{i} + \sum_{i=0}^{N-1} \binom{N}{i} \lambda^{N-i} s^{(i+a)} I \right)^{-1} E_0, \end{aligned} \tag{2.110}$$

which can be rewritten as

$$H(s,\lambda) = \left(\sum_{i=0}^{N} \tilde{C}_{zi} s^{i} \right) \left(s^{N+\alpha} I - \sum_{i=0}^{N} \tilde{A}_i s^{i} - \sum_{i=0}^{N-1} L_i s^{(i+\alpha)} \right)^{-1} E_0, \tag{2.111}$$

with

$$\tilde{C}_{zi} = \sum_{k=0}^{N} C_{zk} \alpha_k(i) \lambda^{N-i}, \tag{2.112}$$

$$\tilde{A}_i = \sum_{k=0}^{N} A_k \alpha_k(i) \lambda^{N-i}, \tag{2.113}$$

$$L_i = -\alpha_0(i) \lambda^{N-i} I, \tag{2.114}$$

in which $\alpha_k(i)$ is given by (2.102) when $k = 0$ and by (2.103) when $k \geq 1$. □

Remark 2.3. The $\mathcal{H}_\infty$ norm for the comparison system and the state-feedback algorithm can both be extended for this fractional case. However, these adaptations are not straightforward and they are going to be published in detail in the future.

2.6 Final Remarks

This chapter provides results that extend the ones presented in [38] and [39]. We use the procedure for time-delay control design based on a comparison system, obtained by Rekasius substitution, to implement state-feedback controllers, output-feedback controllers and filters. To the best of our knowledge, this is the first procedure able to better use the buffer necessary for implementing delayed state/output-feedback, and obtaining simultaneously more stability margin and lower $\mathcal{H}_\infty$ level.

Chapter 3

Stability and stabilisation using envelopes

On this third chapter we discuss stability and stabilisation for time-delay systems by means of an envelope that bounds the region where the poles of the characteristic equation of the system can be. Differently from the previous chapter, the commensurate delay constraint is not needed anymore, allowing the application of this method to more general systems. The goal is to use this envelope to check stability and to design state-feedback controllers. Using LMIs we are going to develop delay-independent stability and delay-dependent α-stability. Robustness will also be discussed for the case where we have parametric uncertainties. The method for the analysis is also valid for Fractional time-delay systems. All results will be presented through numerical examples.

3.1 Introduction

The use of an envelope that ensures that all poles are contained inside it is discussed in [4]. Different types of envelopes are also discussed in [76] and [77]. In those cases, the methods utilised to establish the envelopes are not used to test stability nor to design controllers. In fact, in general, the envelope extends to the right half plane and due to that, it only provides a region where the poles are allowed to be without any guarantee about the stability of the system. This work is based on LMIs instead of the singular value approach, see [4], and provides a different analysis regarding the use of envelopes. A procedure to test robust stability for retarded time-delay systems is established. In addition, a robust state-feedback controller coping with project requirements regarding α-stability, see Definition 3.1, can be designed.

Definition 3.1. Let $\alpha \in \Re_+$ and let λ_j, for all $j \in \{1, \ldots, n\}$ be all the eigenvalues of a linear time-invariant system of order n. Then, if

$$\Re(\lambda_j) < -\alpha, \tag{3.1}$$

for all $j \in \{1, \ldots, n\}$, the system is said to be α-stable.

3.2 Retarded Systems

Consider the retarded linear time-delay system with N delays, whose minimal realisation is given by

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x\left(t - \tau_i\right), \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $0 = \tau_0 < \tau_1 < \cdots < \tau_N$ are the delays and $A_i \in \mathbb{R}^{n \times n}$ for all $i \in \{0, \ldots, N\}$. This system is exponentially stable if and only if all roots of its characteristic equation

$$\det\left(sI - \sum_{i=0}^{N} A_i e^{-s\tau_i}\right) = 0 \tag{3.3}$$

are in the open left half-plane [16].

The following proposition introduces an envelope that engulfs all of its poles.

Proposition 3.1. Let λ be any real number. If there exist matrices $T = T' > 0$, $Q_i = Q_i' > 0$, for all $i \in \{0, \ldots, N\}$ and a scalar μ that satisfy

$$\mu T \geq \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} \tag{3.4}$$

and

$$\begin{bmatrix} T & T & \ldots & T \\ \bullet & Q_0 & & 0 \\ \bullet & \bullet & \ddots & 0 \\ \bullet & \bullet & \bullet & Q_N \end{bmatrix} \geq 0, \tag{3.5}$$

then any characteristic root s_0 of equation (3.3) such that $s_0 = \lambda + j\omega$ verifies

$$|s_0| \leq \sqrt{\mu}. \tag{3.6}$$

Proof. The following inequality is always true, which is easily verifiable applying Schur's complement

$$\begin{bmatrix} A_i Q_i A_i' e^{-2\lambda\tau_i} & \bullet \\ A_i' e^{-(\lambda - j\omega)\tau_i} & Q_i^{-1} \end{bmatrix} \geq 0. \tag{3.7}$$

Adding them for all $i \in \{0, \ldots, N\}$ leads to

$$\begin{bmatrix} \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} & \bullet \\ \sum_{i=0}^{N} A_i' e^{-(\lambda - j\omega)\tau_i} & \sum_{i=0}^{N} Q_i^{-1} \end{bmatrix} \geq 0, \tag{3.8}$$

for which we can apply Schur's complement and utilise (3.4) to get

$$\mu T \geq \Sigma \left(\sum_{i=0}^{N} Q_i^{-1} \right)^{-1} \Sigma^*, \tag{3.9}$$

where $\Sigma \triangleq \sum_{i=0}^{N} A_i e^{-(\lambda + j\omega)\tau_i}$.

Notice that from (3.5)

$$T \geq \sum_{i=0}^{N} T Q_i^{-1} T. \tag{3.10}$$

Now, multiplying (3.10), through the left and through the right, by T^{-1} and taking the inverse on both sides of the inequality, we get

$$T \leq \left(\sum_{i=0}^{N} Q_i^{-1} \right)^{-1}. \tag{3.11}$$

Then, using this result in (3.9), it implies that

$$\mu T \geq \Sigma T \Sigma^*. \tag{3.12}$$

Finally, let $s_0 = \lambda + j\omega$ be an eigenvalue of Σ associated with a right-eigenvector v. It is well known, [87] and [88], that left and right eigenvalues are equal. Hence, s_0 is also an eigenvalue of Σ associated with a left-eigenvector x_L, with dimension $1 \times n$. In this case, we can multiply inequality (3.12) to the left by x_L and to the right by its conjugated transpose, x_L^*, obtaining

$$\mu x_L T x_L^* \geq x_L \Sigma T \Sigma^* x_L^* \tag{3.13}$$

and since $x_L \neq 0$ and $T > 0$,

$$\mu \geq (\lambda + j\omega)(\lambda - j\omega), \tag{3.14}$$

leading to

$$|s_0| \leq \sqrt{\mu}, \tag{3.15}$$

which concludes the proof. □

This result produces a better envelope than previous works such as [4]. This will be proved in the next lemma and evidenced in Example 3.1.

Let us analyse the envelope obtained from the eigenvalue approach to hereafter introduce a lemma that compares it with the LMI approach. The envelope in [4] is given by

$$|s_0| \leq \sum_{i=0}^{N} \|A_i\|_2 \, e^{-\lambda \tau_i}, \tag{3.16}$$

which is equivalent, see [89], to $|s_0| \leq \nu$, where ν is the optimal solution of the following optimisation problem

$$\begin{aligned}
\min_{\nu,\nu_i} \quad & \nu, \\
\text{subject to} \quad & \nu \geq \sum_{i=0}^{N} \nu_i, \\
& \nu_i^2 I \geq A_i' A_i e^{-2\lambda\tau_i},
\end{aligned} \tag{3.17}$$

for all $i \in \{0, \ldots, N\}$ and $\nu_i \geq 0$, for all $i \in \{0, \ldots, N\}$.

The next lemma shows that the envelope in [4] is a particular case of the class of envelopes defined by Proposition 3.1.

Lemma 3.1. Let $s_0 = \lambda + j\omega$ be a characteristic root of equation (3.3) and let ν and ν_i be the optimal solution of the optimisation problem (3.17). Then (3.4) and (3.5) are both satisfied with the particular choice of $T = \nu^{-1}I$, $Q_i = \nu_i^{-1}I$ and $\mu = \nu^2$.

Proof. From Schur Complement, (3.5) is equivalent to

$$T \geq \sum_{i=0}^{N} T Q_i^{-1} T. \tag{3.18}$$

It is easy to see that (3.18) is satisfied whenever $T = \nu^{-1}I$ and $Q_i = \nu_i^{-1}I$.

Applying the same substitutions on (3.4), we get

$$\mu\nu^{-1} I \geq \sum_{i=0}^{N} A_i \nu_i^{-1} A_i' e^{-2\lambda\tau_i}, \tag{3.19}$$

and remembering that $\mu = \nu^2$, we have

$$\sum_{i=0}^{N} \nu_i I \geq \sum_{i=0}^{N} A_i \nu_i^{-1} A_i' e^{-2\lambda\tau_i}, \tag{3.20}$$

which satisfies the conditions in (3.17). □

Therefore, the envelope in [4] is a particular case of Proposition 3.1 for specific choices of T, Q_i and μ. Having flexibility on those three variables the new envelope proposed is always tighter (or at least equal) than the aforementioned envelope.

3.2.1 Implementation

First of all, let us introduce the definition of closedness of an envelope. Let μ and λ be defined by Proposition 3.1 and let $\lambda \in [\lambda_{\min},\ \lambda_{\max}]$. If there is a point λ^* in this interval such that $\mu = (\lambda^*)^2$, we define $\lambda^* + \varepsilon$, with $\varepsilon > 0$ arbitrarily small, as the closure point of the envelope. This means that the envelope lies completely on the left side of the vertical line of the form $\Re(s) = \lambda^* + \varepsilon$, for $\varepsilon > 0$. Furthermore, we say that the envelope is closed whenever $\mu < \lambda^2$.

The choice of $\lambda_{\min}$ is completely free. In [76], a simple bound for the rightmost root of (3.3) was given, which can easily be generalised to N delays:

$$\Re(s) \leq \bar{\mu}(A_0) + \sum_{i=1}^{N} \|A_i\| = \ell, \tag{3.21}$$

where $\bar{\mu}(.)$ is a matrix measure, see [76] and [90]. We suggest to take $\lambda_{\max} = 2|\ell|$.

The following propositions illustrate, respectively, how to depict the envelope and how one can use the envelope to analyse the stability of a time-delay system. Also, it shows the behaviour of the envelope as a function of λ.

Proposition 3.2. Let $\lambda \in [\lambda_{\min},\ \lambda_{\max}]$ and let μ be given by Proposition 3.1. If $\mu \geq \lambda^2$ then the envelope on the complex plane is defined by the set of points (λ, ω) where $\omega = \pm\sqrt{\mu - \lambda^2}$. If for a particular λ^*, $\mu^* < (\lambda^*)^2$ then the envelope is closed for every $\lambda > \lambda^*$.

Proof. From equation (3.15) we have that $\lambda^2 + \omega^2 \leq \mu$ which directly implies that $\omega = \pm\sqrt{\mu - \lambda^2}$, for $\mu \geq \lambda^2$. Obviously, (λ, ω) belongs to the envelope. Now, suppose that for a certain λ^*, we have $\mu^* < (\lambda^*)^2$. As $A_i Q_i A_i' \geq 0$ and $e^{-2\lambda\tau_i}$ is non-increasing, we have that $\mu < \mu^*$ for every $\lambda > \lambda^*$ which means, by definition, that the envelope is closed. □

Proposition 3.3. Let $\lambda_0 \in \mathbb{R}$ and $\mu = \lambda_0^2 - \varepsilon$, for some $\varepsilon > 0$. If there exist $T, Q_i > 0$, for all $i \in \{0, \ldots, N\}$ such that (3.4) and (3.5) are both satisfied, then the envelope lies entirely on the left side of the vertical axis crossing λ_0.

Proof. From (3.15) we have that if $\lambda + j\omega$ is a root of the system, then

$$|\lambda + j\omega| \leq \sqrt{\lambda_0^2 - \varepsilon}, \tag{3.22}$$

which can be rewritten as

$$\lambda^2 + \omega^2 \leq \lambda_0^2 - \varepsilon. \tag{3.23}$$

Notice that this expression is never going to be satisfied with $\lambda \geq \lambda_0$, which implies that there cannot exist parts of the envelope to the right side of the vertical axis passing through λ_0. □

The computational procedure to obtain the envelope is summarised in the Algorithm 3. The minimisation of μ, for the retarded case, is achieved through the traditional generalised eigenvalue minimisation under LMI constraints, [91]. For the neutral case, which will be discussed further ahead, the minimisation of μ is done with a linear search, i.e., we choose a μ_0 using the generalised eigenvalue problem (*gevp*) and proceed through a linear search on μ checking on each step the feasibility of the LMIs. Since LMIs are convex and *gevp* is quasi-convex, there is no need for initial values for convergence to the optimal solution.

In spite of the fact that this envelope is tighter than [4], for $\lambda = 0$, it follows from (3.4) that $\mu \geq 0$, and therefore, the envelope is never closed on the left half-plane, which implies that stability cannot be assessed with the envelope in this present form. To circumvent this, we

Algorithm 3: Envelope Procedure

Data: System Matrices A_i, Delays τ_i
Initialise: Define a real interval $[\lambda_{min}, \lambda_{max}]$ for λ and a step p, $0 < p \leq (\lambda_{max} - \lambda_{min})$
Let $\kappa \in \mathbb{N}$
For each $\kappa \in \left\{0, \cdots, \left\lfloor \frac{\lambda_{max} - \lambda_{min}}{p} \right\rfloor \right\}$
Define $\lambda_\kappa = \lambda_{min} + \kappa p$
  Minimise μ_κ subject to (3.4) and (3.5)
  If $\mu_\kappa \geq \lambda_\kappa^2$ **then**
    $\omega_\kappa \leftarrow \sqrt{\mu_\kappa - \lambda_\kappa^2}$
  Else
    End Procedure
  End If
  Return $Q_{i\kappa}, T_\kappa, \mu_\kappa$ and ω_κ
End For

propose a change of coordinates through the new variable $s = z - d$, with $d > 0$ and hereafter calculate the envelope for z. With this change of variables, (3.3) becomes

$$\det\left(zI - (A_0 + dI) - \sum_{i=1}^{N} A_i e^{-z\tau_i} e^{d\tau_i}\right) = 0, \tag{3.24}$$

allowing us to work with an equivalent problem on the new parameters

$$\begin{aligned} \tilde{A}_0 &= A_0 + dI, \\ \tilde{A}_i &= A_i e^{d\tau_i}, \text{ for all } i \in \{1, \ldots, N\}. \end{aligned} \tag{3.25}$$

On the z-plan the envelope will remain open for $\lambda = 0$, however, if it is closed before $z = d$, it will be closed before the origin on the s-plan, guaranteeing stability for the original system.

Example 3.1. Consider the following system matrices

$$\left[\begin{array}{c|c} A_0 & A_1 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0.5413 \\ -2 & -3 & -1.0827 & -1.6240 \end{array}\right].$$

Applying Algorithm 3, for $\tau_1 = 1$, to this system, we calculate the envelope and compare the result with reference [4]. Figure 3.1 shows this comparison and it also illustrates the behaviour of the envelope for different values of d. An interesting remark is that for $d = 3$ we achieved a tighter envelope closer to the poles and we can also see that the point where the envelope ends is on the left side of the plane. This allows us to use the envelope as a stability criteria as will be seen in the stabilisation section. All system poles here and throughout this work are calculated via QPmR [a].

[a] QPmR is a Matlab function for computation and analysis of the spectrum of characteristic quasi-polynomials for both retarded and neutral time-delay systems. It utilises algorithms described in [92], [93]

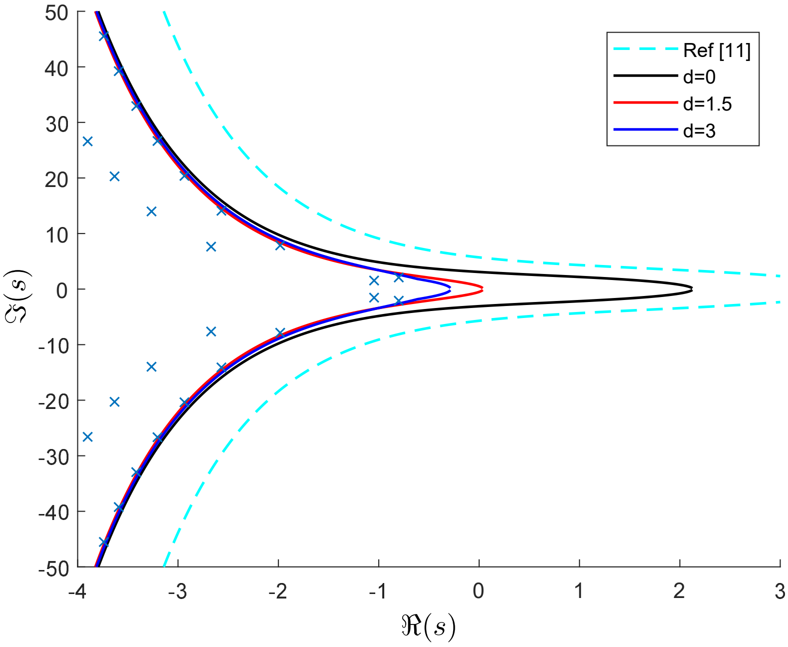

Figure 3.1: Envelopes for different values of d and from previous work in the literature

and [94].

3.2.2 Stability

There are two types of stability that can be analysed and obtained using the envelope.

Delay-independent stability

Proposition 3.3 shows that the existence of a solution for (3.4) and (3.5), for the modified system (3.25), with $\mu = d^2 - \varepsilon$ and $\lambda = d$, for some $d > 0$ and $\varepsilon > 0$ implies that the original system (3.2) is stable. Note also that, for $\lambda = d$, after the change of variables (3.25), all terms of inequality (3.4) that have delays cancel each other. This implies that the criteria is delay independent.

Delay-dependent α-stability

Is it possible to go one step ahead and design a controller that guarantees α-stability. Making the change of variables $z = s + d$, with $d = d^* + \alpha$, $d^* > 0$, $\alpha > 0$, it implies that if an envelope lies completely before d^* on the z-plane, then it will lie completely on the left side of the vertical line $\Re(s) = -\alpha$ on the s-plan.

For this case, with $\mu = (d^*)^2 - \varepsilon$ and $\lambda = d^*$, (3.4) becomes

$$\mu T \geq \tilde{A}_0 Q_0 \tilde{A}_0' + \sum_{i=1}^{N} A_i Q_i A_i' e^{2\alpha\tau_i}. \tag{3.26}$$

Now the criteria is delay dependent. Furthermore, if (3.26) is satisfied for $\alpha = \alpha^*$ and $\tau_i = \tau_i^*$ for all $i \in \{1 \cdots N\}$ then it will remain α-stable for all $\tau_i \leq \tau_i^*$.

3.2.3 State feedback for Retarded systems

We now address the stabilisation problem. Consider the system

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x\left(t - \tau_i\right) + Bu(t), \tag{3.27}$$

which we want to be controlled by means of a state-feedback control law

$$u(t) = \sum_{i=0}^{N} K_i x\left(t - \tau_i\right) \in \mathbb{R}^m, \tag{3.28}$$

to be designed through LMIs. This controller copes with project requirements, i.e., α-stability, and adds a certain degree of robustness to the closed-loop system. As will be shown the controller can be memoryless, i.e., $K_i \leftarrow 0$, $\forall i \in \{1, \ldots, N\}$ or can even use only some of the delayed states.

Theorem 3.1. Consider the time-delay system (3.27). If there exist matrices $T = T' > 0$, $Q_i = Q_i' > 0$, Y_i, $\forall i \in \{0, \ldots, N\}$ and positive scalars d, ε, with $\mu = d^2 - \varepsilon$, $\lambda = d$, such that

$$\begin{bmatrix} \mu T & \left(\tilde{A}_0 Q_0 + B_0 Y_0\right) e^{-\lambda\tau_0} & \ldots & \left(\tilde{A}_N Q_N + B_N Y_N\right) e^{-\lambda\tau_N} \\ \bullet & Q_0 & & 0 \\ \bullet & \bullet & \ddots & 0 \\ \bullet & \bullet & \bullet & Q_N \end{bmatrix} \geq 0 \tag{3.29}$$

and (3.5) are all satisfied, where $\tilde{A}_i$ is given by (3.25) and $B_i = Be^{d\tau_i}$ for all $i \in \{0, \ldots, N\}$, then the state-feedback control law (3.28), where the controller matrices are given by $K_i = Y_i Q_i^{-1}$, stabilises the system.

Proof. Applying Schur's complement in (3.29) we get exactly (3.4) with $A_i \leftarrow \tilde{A}_i + B_i K_i$, which completes the proof. □

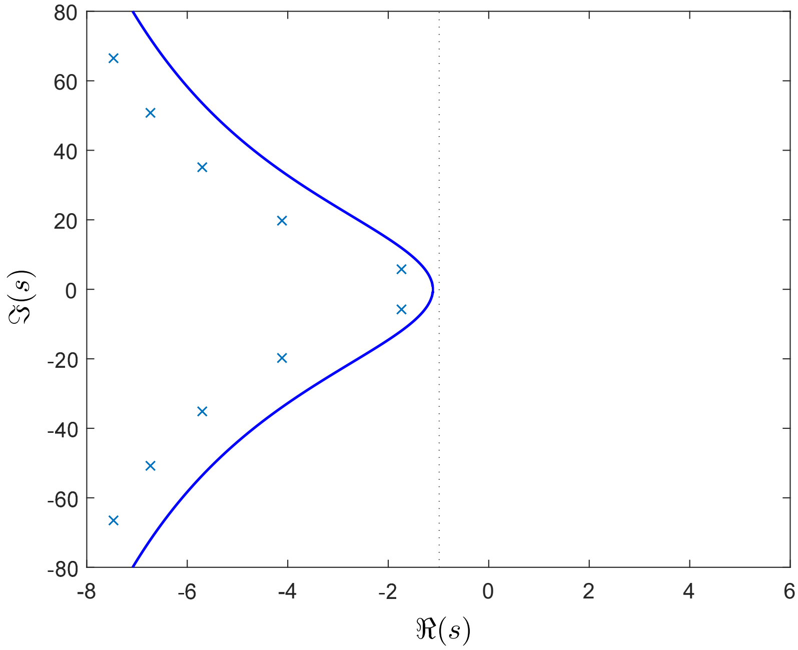

Figure 3.2: α-stability, $\alpha = 1$, $d = 31$

Example 3.2. Taking the matrices $-A_0$ and $-A_1$, from Example 3.1, with $\tau = 0.4$ and $B = [0 \quad 1]'$, the uncontrolled system is unstable with poles at 1.3194 and 2.4125. Choosing $\alpha = 1$, $d = 31$, $\lambda = d - \alpha$ and applying Theorem 3.1 we achieve 1-stability as can be seen in Figure 3.2. The gains for the controller are

$$\left[\begin{array}{c|c} K_0 & K_1 \end{array}\right] = \left[\begin{array}{cc|cc} 230.1100 & -41.5189 & -1.0849 & -4.8371 \end{array}\right].$$

We can also impose, for example, $K_1 = 0$ and still achieve 1-stability. In that case $K_0 = [298.9831 \; -43.7406]$.

Using (3.21) for the change of variables, i.e., $d = \bar{\mu}(A_0) + \|A_1\|$, we get

$$\left[\begin{array}{c|c} K_0 & K_1 \end{array}\right] = \left[\begin{array}{cc|cc} 35.2114 & -15.2267 & -1.0869 & -4.5434 \end{array}\right].$$

which has not only a smaller gain norm but also a tighter envelope as can be seen in the Figure 3.3.

Remark 3.1. Results presented here are in some sense complementary to those obtained from Lyapunov-Krasoviskii functionals. In general, Lyapunov-Krasoviskii methods are able to cope with a larger class of systems, such as time-varying delays [95] and [96], as well as providing guaranteed performance metrics such as $\mathcal{H}_\infty$ [97]. On the other hand, results built on frequency methods are more restricted with respect to the class of systems they

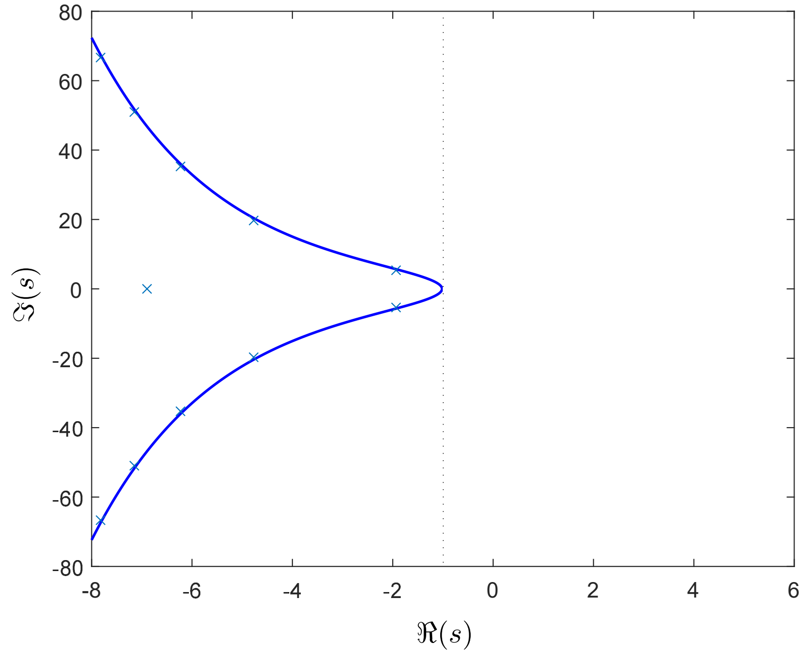

Figure 3.3: α-stability, $\alpha = 1$, d suggested by [76]

may be applied, but are able to provide information on the position of poles, and therefore on performance metrics which are more directly related to them.

3.2.4 Robust case

We are now going to show how to adapt the presented methodology for the robust case. Consider the uncertain retarded linear time-delay system with N delays, whose minimal realisation is given by

$$\dot{x}(t) = \sum_{i=0}^{N} \bar{A}_i x\left(t - \tau_i\right), \tag{3.30}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $0 = \tau_0 < \tau_1 < \cdots < \tau_N$ are the delays, $\bar{A}_i \in \mathbb{R}^{n\times n}$ for all $i \in \{0, \ldots, N\}$ and the system matrices belong to a convex polytope

$$\mathcal{P} \triangleq co\left\{\left[\bar{A}_0^\ell, \cdots, \bar{A}_N^\ell\right], \ell \in 1, \ldots, N_v\right\}, \tag{3.31}$$

defined by the convex combination of N_v vertices. Each matrix can be individually defined as [98]

$$\bar{A}_i \triangleq \left\{\sum_{\ell=1}^{N_i} \xi_i^\ell \bar{A}_i^\ell, \quad \sum_{\ell=1}^{N_i} \xi_i^\ell = 1, \quad \xi_i^\ell \geq 0\right\}. \tag{3.32}$$

Hence, $N_v = \Pi_{i=0}^{N} N_i$.

Example 3.3. As an example consider the matrix

$$\bar{A}_1 = \begin{bmatrix} -2 & [0 \;\; 1] \\ [-1 \;\; 3] & 2 \end{bmatrix}, \tag{3.33}$$

in which [0 1] and [-1 3] represents the parametric uncertainty on A_1. This matrix can be rewritten as

$$\bar{A}_1 = \xi_1^1 \bar{A}_1^1 + \xi_1^2 \bar{A}_1^2 + \xi_1^3 \bar{A}_1^3 + \xi_1^4 \bar{A}_1^4, \tag{3.34}$$

with $\xi_1^1 + \xi_1^2 + \xi_1^3 + \xi_1^4 = 1,\ \xi_1^1,\ \xi_1^2,\ \xi_1^3,\ \xi_1^4 \geq 0$.

It is easy to see that the convex combination that creates A_1 for this particular transition matrix is

$$\begin{aligned} \bar{A}_1 = \xi_1^1 &\begin{bmatrix} -2 & 0 \\ -1 & 2 \end{bmatrix} + \xi_1^2 \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} \\ + \xi_1^3 &\begin{bmatrix} -2 & 0 \\ 3 & 2 \end{bmatrix} + \xi_1^4 \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix}. \end{aligned} \tag{3.35}$$

System (3.30) is exponentially stable if and only if all zeros of its characteristic equation

$$\det\left(sI - \sum_{i=0}^{N} \bar{A}_i e^{-s\tau_i} \right) = 0 \tag{3.36}$$

are in the open left half-plane [16]. This would imply on checking an infinity amount of constraints, which is, obviously, computationally unfeasible. This issue can be circumvented by the use of LMIs on the vertices of the uncertainty polytope guaranteeing exponentially stability to all uncertainties within it.

The following proposition introduces an envelope that completely surround all of its poles.

Proposition 3.4. Let λ be any real number. If there exist a characteristic root s_0 of equation (3.36), such that $s_0 = \lambda + j\omega$ and if there exist matrices $T = T' > 0$, $Q_i = Q_i' > 0$, for all $i \in \{0, \ldots, N\}$ and a scalar μ that satisfy

$$\begin{bmatrix} \mu T & \bar{A}_0^\ell Q_0 e^{-\lambda\tau_0} & \ldots & \bar{A}_N^\ell Q_N e^{-\lambda\tau_N} \\ \bullet & Q_0 & 0 & 0 \\ \bullet & 0 & \ddots & 0 \\ \bullet & 0 & 0 & Q_N \end{bmatrix} \geq 0 \tag{3.37}$$

and

$$\begin{bmatrix} T & T & \dots & T \\ & Q_0 & & \\ & & \ddots & \\ & & & Q_N \end{bmatrix} \geq 0, \tag{3.38}$$

then

$$|s_0| \leq \sqrt{\mu}. \tag{3.39}$$

Proof. Employing the same idea of the proof of Proposition 3.4, it follows immediately from the linear dependence of the LMIs that the solution of the problem in question is obtained calculating it in each one of the vertices, $\ell \in \{0, \dots, N_v\}$, of the polytope of uncertainties given by (3.32), which concludes the proof. ☐

Stability and State feedback

For the stability, we can apply the same change of variables proposed earlier

$$\det\left(zI - (\bar{A}_0 + dI) - \sum_{i=1}^{N} \bar{A}_i e^{-z\tau_i} e^{d\tau_i}\right) = 0, \tag{3.40}$$

allowing us to work with an equivalent problem on the new parameters

$$\begin{aligned} \tilde{A}_0 &= \bar{A}_0 + dI, \\ \tilde{A}_i &= \bar{A}_i e^{d\tau_i}, \text{ for all } i \in \{1, \dots, N\}. \end{aligned} \tag{3.41}$$

Then, one must only apply the LMIs of Proposition 3.4 on each one of the vertices of the uncertain polytope.

For the stabilisation problem, consider the system

$$\dot{x}(t) = \sum_{i=0}^{N} \bar{A}_i x\left(t - \tau_i\right) + \bar{B}u(t), \tag{3.42}$$

with $\bar{A}_i$ given by (3.32), $\bar{B}$ defined as

$$\bar{B} \triangleq \left\{\sum_{\ell=1}^{M} \eta^\ell \bar{B}^\ell, \ \sum_{\ell=1}^{M} \eta^\ell = 1, \ \eta^\ell \geq 0\right\}, \tag{3.43}$$

and the polytope redefined as

$$\mathcal{P} \triangleq co\left\{\left[\bar{A}_0^\ell, \cdots, \bar{A}_N^\ell, \bar{B}^\ell\right], \ell \in 1, \dots, N_v\right\}, \tag{3.44}$$

with $N_v = M\Pi_{i=0}^{N} N_i$.

We want to control this system by means of a state-feedback control law, including the delayed states, to be designed through LMIs. This controller copes with project requirements and adds a certain degree of robustness to the closed-loop system. As will be shown the controller can be memoryless, i.e., $K_i \leftarrow 0$, for $i \geq 1$ or can even use some of the delayed states.

Theorem 3.2. There is a state-feedback control of the form $u(t) = \sum_{i=0}^{N} K_i x(t - \tau_i) \in \mathbb{R}^m$, $\forall t$, that stabilises the system (3.30) if there exist matrices $T = T' > 0$, $Q_i = Q_i' > 0$, Y_i $\forall i \in \{0, \ldots, N\}$ and positive scalars d, ε, with $\mu = d^2 - \varepsilon$, $\lambda = d$, such that

$$\begin{bmatrix} \mu T & \left(\tilde{A}_0^\ell Q_0 + \tilde{B}^\ell Y_0\right) e^{-\lambda \tau_0} & \ldots & \left(\tilde{A}_N^\ell Q_N + \tilde{B}^\ell Y_N\right) e^{-\lambda \tau_N} \\ \bullet & Q_0 & 0 & 0 \\ \bullet & 0 & \ddots & 0 \\ \bullet & 0 & 0 & Q_N \end{bmatrix} \geq 0 \qquad (3.45)$$

and (3.38) are all satisfied, where $\tilde{A}_i$ is given by (3.41) and $\tilde{B}_i = \bar{B} e^{d\tau_i}$ for all $i \in \{0, \ldots, N\}$. In this case, the controller matrices are given by $K_i = Y_i Q_i^{-1}$.

Proof. Applying Schur's complement in (3.45) we get exactly (3.37) with $\bar{A}_i \leftarrow \tilde{A}_i + \tilde{B}_i K_i$, which completes the proof. □

One remark that is interesting to highlight, it is that in computational terms, for systems without uncertainties is possible to minimise μ through a standard generalised eigenvalue problem approach. However, for the uncertain case this is no longer possible. The problem is then solved by a linear search on μ. Let us illustrate the results when we take into account parametric uncertainties on the system matrices.

For the next examples, we are going to describe the uncertainty as $\bar{A} = A^1 + \Delta A^2$, with $0 \leq \Delta \leq \Delta_{\max}$. Hence, the vertices of the polytope are A^1 and $A^1 + \Delta_{\max} A^2$. Now, for comparison purposes, we will design two controllers. One for the nominal plant neglecting the uncertainty and, therefore, fixing $\Delta = 0$, and the second one taking the uncertainty into account. For both controllers designed we execute the following procedure: Starting with $\Delta = 0$ we calculate all roots of the closed-loop system characteristic equation using QPmR [92]. Incrementing $\Delta = \Delta + p$, where p is a small step, e.g., $p = 0.05$, we recalculate the poles and we repeat this procedure until the system reach instability. With this procedure we can verify that the robustness is not only valid for $0 \leq \Delta \leq \Delta_{\max}$ but for a higher interval. Two numerical examples are shown below.

Example 3.4. Consider the following matrices for the system (3.42)

$$\bar{A}_0 = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} + \Delta \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & -0.5413 \\ 1.0827 & 1.6240 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 \end{bmatrix}',$$

in which $0 \leq \Delta \leq 2.5$ represents a parametric uncertainty and $\tau = 0.4$. Imposing $\Delta = 0$ and applying Theorem 3.1 to the system obtained, we have $K_1 = [7.1766 \quad -15.0622]$ and $K_2 = [-1.0764 \quad -1.9799]$. For the designed K we get that the system is stable for

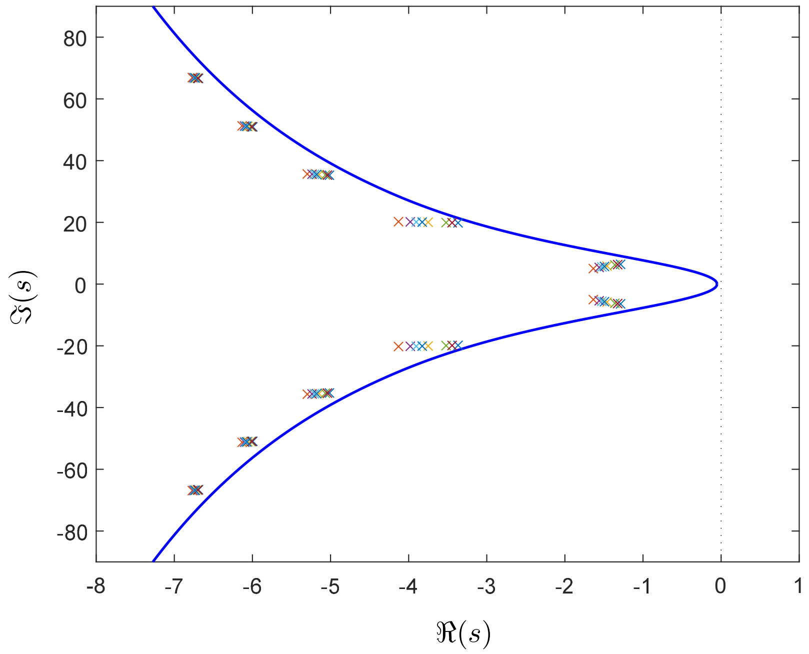

Figure 3.4: Envelope for an Uncertain Retarded Time-Delay System

$0 \leq \Delta \leq 1.1$.

Now, lets apply Theorem 3.2 considering all vertices.

$$\left[\begin{array}{c|c} A_0^1 & A_0^1 + \Delta_{\max} A_0^2 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & -1 & 2.5 & -1 \\ 2 & 3 & 7 & -4.5 \end{array}\right].$$

The new controller gains obtained are $K_1 = [103.7664 \quad -19.2698]$ and $K_2 = [-1.0823 \quad -6.1536]$. Applying once again the procedure described above, we have that the system is stable for $0 \leq \Delta \leq 4.9$. Furthermore, for $0 \leq \Delta \leq 2.5$, not only the poles are on the left half-plane, but our procedure ensures that they are all inside the envelope. A plot with a variety of linear combinations of (3.32), i.e., different values of Δ, is ploted altogether with the envelope on Figure 3.4.

Example 3.5. Let us consider $\bar{A}_0$, A_1 and B from the previous example, with $\tau = 0.2$, $0 \leq \Delta \leq 1.5$ and $\alpha = 1$. Designing a controller for the nominal system we get $K_1 = [43.8852 \quad -18.9928]$, $K_2 = [-1.0825 \quad -3.6587]$. Calculating the poles for each increment of Δ we verify stability for $0 \leq \Delta \leq 2.9$. Designing the robust controller through Theorem 3.2, with $\Delta = 1.5$, we get $K_1 = [126.8156 \quad -23.0812]$, $K_2 = [-1.0826 \quad -6.9346]$ and stability for $0 \leq \Delta \leq 5.2$. The envelope and the poles of the characteristic equation of (3.30) for different values of Δ, can be seen in Figure 3.5.

It is then clear the superiority of the robust controller in comparison to the controller designed for the nominal system when there are parametric uncertainties present.

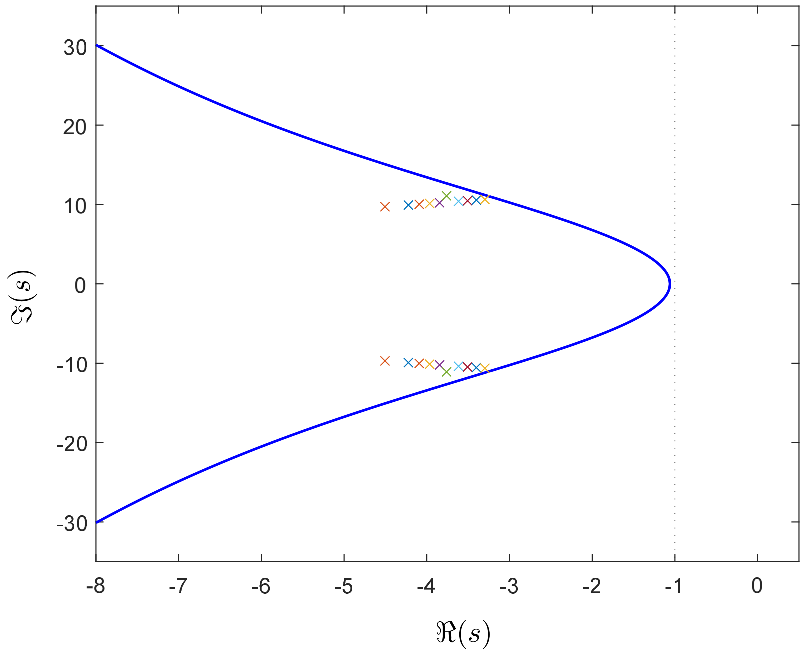

Figure 3.5: α-stability Envelope for an Uncertain Retarded Time-Delay System

Example 3.6. For one final example regarding robustness, let us consider the matrices [99], [100] and [45]

$$A_0 = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} + \Delta \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -1 & 0 \\ -0.8 & -0.1 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & 1 \end{bmatrix}', \ \|\Delta\| \leq 1.$$

In [99] the largest upper bound for the delay is $\tau_{\max} = 0.5557$. In [100] they showed that, in fact, the system is stabilisable for all delays. Applying Theorem 3.2, choosing properly the vertices that bounds Δ, we confirm that the system is stable for all delays and we get $K_1 = [-18.1576 \quad -1.8716]$, $K_2 = [1.0399 \quad 0.1020]$. We went one step further and achieved independent stabilisation for $\|\Delta\| \leq 12$.

3.3 Neutral Systems

Our goal here is to develop the envelopes for neutral-type systems. Consider the neutral time-delay linear system with $N+1$ delays, whose minimal realisation is given by

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x\left(t-\tau_i\right) + H\dot{x}\left(t-\tau_h\right), \tag{3.46}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $0 = \tau_0 < \tau_1 < \cdots < \tau_N$ and τ_h are the delays, $A_i \in \mathbb{R}^{n\times n}$, for all $i \in \{0,\ldots,N\}$ and H are real matrices. One sufficient condition for the exponential stability of this system is that all roots of the characteristic equation

$$\det\left(sI - \sum_{i=0}^{N} A_i e^{-s\tau_i} - sHe^{-s\tau_h}\right) = 0, \tag{3.47}$$

are on the left side of a vertical line $\Re(s) = -\alpha$, with $\alpha > 0$ [101].

For the neutral case, let us first do a digression and generalise the retarded envelope from the literature, i.e., the one obtained by the singular value approach, through the following proposition.

Proposition 3.5. Let s_0 be a characteristic root of the system (3.46) such that

$$\Re(s_0) > \log(\|H\|)/\tau_h \tag{3.48}$$

then, it satisfies

$$|s_0| \leq \frac{\sum_{i=0}^{N} \|A_i\|\, e^{-\Re(s_0)\tau_i}}{1-\|H\|e^{-\Re(s_0)\tau_h}}. \tag{3.49}$$

Proof. Equation (3.47) is equivalent to

$$s_0 \in \sigma\left(\sum_{i=0}^{N} A_i e^{-s_0\tau_i} + s_0 He^{-s_0\tau_h}\right), \tag{3.50}$$

and since the spectral radius of a matrix is the infimum of all its induced norms [102], it follows that

$$\begin{aligned} |s_0| &\leq \left\|\sum_{i=0}^{N} A_i e^{-s_0\tau_i} + s_0 He^{-s_0\tau_h}\right\| \\ &\leq \sum_{i=0}^{N} \|A_i\| e^{-\Re(s_0)\tau_i} + |s_0|\|H\|e^{-\Re(s_0)\tau_h}. \end{aligned} \tag{3.51}$$

Therefore

$$|s_0| \leq \frac{\sum_{i=0}^{N} \|A_i\| e^{-\Re(s_0)\tau_i}}{1-\|H\|e^{-\Re(s_0)\tau_h}}, \tag{3.52}$$

in which (3.48) follows from the positive denominator. □

It is interesting to note that if $\|H\| < 1$, we always have $s = j\omega$ in the domain of the function $\forall\omega \in \mathbb{R}$. Thus, we can deal with the imaginary axis which is a major concern for stability.

Now, let us apply our LMI approach. The following proposition generalises the envelope for neutral systems.

Proposition 3.6. Let λ be any real number. If there exist matrices $T = T' > 0$, $Q_i = Q_i' > 0, \forall i \in \{0, \ldots, N\}$, $Q_h = Q_h' > 0$ and a scalar μ such that

$$\mu T \geq \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} + \mu H Q_h H' e^{-2\lambda\tau_h} \tag{3.53}$$

and

$$\begin{bmatrix} T & T & \ldots & T & T \\ \bullet & Q_0 & & 0 & 0 \\ \bullet & \bullet & \ddots & 0 & 0 \\ \bullet & \bullet & \bullet & Q_N & 0 \\ \bullet & \bullet & \bullet & \bullet & \dfrac{\mu}{|s_0|^2} Q_h \end{bmatrix} > 0, \tag{3.54}$$

then any characteristic root s_0 of equation (3.47) such that $s_0 = \lambda + j\omega$ verifies

$$|s_0| \leq \sqrt{\mu}. \tag{3.55}$$

Proof. The following inequality is always true, which is easily verifiable applying Schur's complement

$$\begin{bmatrix} H Q_h H' e^{-2\lambda\tau_h} & \bullet \\ s_0^* H' e^{-(\lambda - j\omega)\tau_h} & |s_0|^2 Q_h^{-1} \end{bmatrix} \geq 0. \tag{3.56}$$

Multiplying both sides by $\mathbf{diag}(\sqrt{\mu}, \dfrac{1}{\sqrt{\mu}})$ and then adding the result to (3.8) we get

$$\begin{bmatrix} \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} + \mu H Q_h H' e^{-2\lambda\tau_h} & \bullet \\ \sum_{i=0}^{N} A_i' e^{-(\lambda - j\omega)\tau_i} + s_0^* H' e^{-(\lambda - j\omega)\tau_h} & \sum_{i=0}^{N} Q_i^{-1} + \dfrac{|s_0|^2}{\mu} Q_h^{-1} \end{bmatrix} \geq 0, \tag{3.57}$$

for which we can apply Schur's complement and utilise (3.53) to get

$$\mu T \geq \Sigma \left(\sum_{i=0}^{N} Q_i^{-1} + \frac{|s_0|^2}{\mu} Q_h^{-1} \right)^{-1} \Sigma^*, \tag{3.58}$$

in which $\Sigma \triangleq \sum_{i=0}^{N} A_i e^{-(\lambda + j\omega)\tau_i} + s_0 H e^{-(\lambda + j\omega)\tau_h}$. Furthermore, from (3.54) we have that

$$T > \sum_{i=0}^{N} T Q_i^{-1} T + \frac{|s_0|^2}{\mu} T Q_h^{-1} T, \tag{3.59}$$

which implies

$$T < \left(\sum_{i=0}^{N} Q_i^{-1} + \frac{|s_0|^2}{\mu} Q_h^{-1} \right)^{-1}. \tag{3.60}$$

Therefore, using this result with (3.58), it implies that

$$\mu T \geq \Sigma T \Sigma^*. \tag{3.61}$$

Proceeding in the same manner as the retarded case, multiplying the inequality (3.61) to the left by x_L and to the right by x_L^*, where x_L is the left-eigenvector of Σ associated with the eigenvalue s_0, leads to

$$|s_0| \leq \sqrt{\mu}. \tag{3.62}$$

□

One difficulty on applying this method lies on the necessity of knowing $|s_0|$ to implement the LMI (3.54). Nevertheless, as a consequence of the result $\mu \geq |s_0|^2$, we have that $\frac{\mu}{|s_0|^2} \geq 1$, implying that whenever

$$\begin{bmatrix} T & T & \dots & T & T \\ \bullet & Q_0 & & 0 & 0 \\ \bullet & \bullet & \ddots & 0 & 0 \\ \bullet & \bullet & \bullet & Q_N & 0 \\ \bullet & \bullet & \bullet & \bullet & Q_h \end{bmatrix} > 0 \tag{3.63}$$

is satisfied, then (3.54) is also true. Hence, one can use (3.63) in place of (3.54) to obtain the results derived from the Propostion 3.6.

3.3.1 Implementation

Now, we can make one more time the change of variables $s = z - d$, with $d > 0$ and calculate the envelope for z. After this, (3.47) becomes

$$\det \left(zI - \tilde{A}_0 - \sum_{k=1}^{N+1} \tilde{A}_k e^{-z\tau_k} - z\tilde{H} e^{-z\tau_h} \right) = 0, \tag{3.64}$$

in which

$$\begin{aligned} \tilde{A}_0 &= A_0 + dI, \\ \tilde{A}_i &= A_i e^{d\tau_i}, \forall i \in \{0, \dots, N\}, \\ \tilde{A}_{N+1} &= -dHe^{d\tau_h}, \\ \tilde{H} &= He^{d\tau_h}, \\ \tau_{N+1} &= \tau_h. \end{aligned} \tag{3.65}$$

This allows us to perform the same technique used for retarded systems. It remains valid, for the neutral case, the conclusions of subsection 3.2.2 regarding delay-independent stability and delay-dependent α-stability.

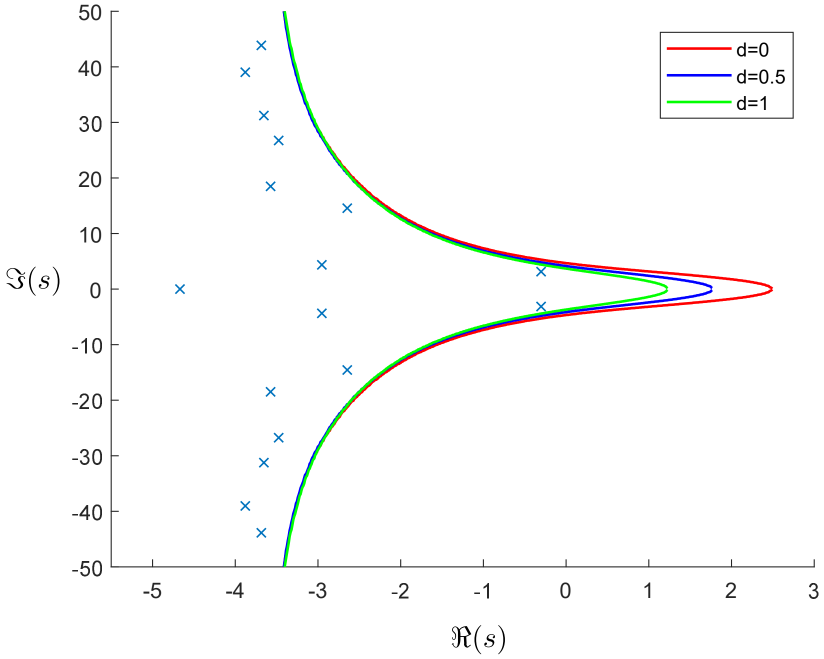

Figure 3.6: Envelopes for different values of d - Neutral-type

Example 3.7. Consider the matrices

$$\left[\begin{array}{c|c} A_0 & A_1 \end{array}\right] = \left[\begin{array}{cc|cc} -1.7073 & 0.6856 & -2.5026 & -1.0540 \\ 0.2279 & -0.6368 & -0.1856 & -1.5715 \end{array}\right] \tag{3.66}$$

and

$$H = \begin{bmatrix} 0.0558 & 0.0360 \\ 0.2747 & -0.1084 \end{bmatrix}. \tag{3.67}$$

Figure 3.6 illustrates the envelopes for $\tau_1 = \tau_h = 0.5$.

3.3.2 State feedback for Neutral systems

We can now adapt the previous result in devise a procedure able to design a state-feedback control law (3.28) for the linear neutral time-delay system

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x\left(t - \tau_i\right) + H\dot{x}\left(t - \tau_h\right) + Bu(t). \tag{3.68}$$

Theorem 3.3. Consider the time-delay system (3.68). If there exist matrices $T = T' > 0$, $Q_i = Q_i' > 0$, $\forall i \in \{0, \dots, N+1\}$, Y_i, $\forall i \in \{0, \dots, N\}$, $Q_h = Q_h' > 0$ and positive scalars d, ε, with $\mu = d^2 - \varepsilon$, $\lambda = d$, such that

$$\begin{bmatrix} \mu T & \left(\tilde{A}_0 Q_0 + B_0 Y_0\right) e^{-\lambda\tau_0} & \dots & \tilde{A}_{N+1} Q_{N+1} e^{-\lambda\tau_h} & \tilde{H} Q_h e^{-\lambda\tau_h} \\ \bullet & Q_0 & & 0 & 0 \\ \bullet & \bullet & \ddots & 0 & 0 \\ \bullet & \bullet & \bullet & Q_{N+1} & 0 \\ \bullet & \bullet & \bullet & \bullet & \frac{1}{\mu} Q_h \end{bmatrix} \geq 0 \tag{3.69}$$

and (3.54) are all satisfied, where $\tilde{H}$ and $\tilde{A}_i$ for all $i \in \{0, \dots, N+1\}$ are given by (3.65) and $B_i = Be^{d\tau_i}$ for all $i \in \{0, \dots, N\}$, then the state-feedback controller (3.28) obtained with the gain matrices $K_i = Y_i Q_i^{-1}$, $\forall i \in \{0, \dots, N\}$, stabilises the system.

Proof. Applying Schur's complement in (3.69) with $H \leftarrow \tilde{H}$ and $A_i \leftarrow \tilde{A}_i + B_i K_i$ $\forall i \in \{0, \dots, N\}$, we get

$$\begin{aligned} \mu T &\geq \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} + \tilde{A}_{N+1} Q_{N+1} \tilde{A}_{N+1}' e^{-2\lambda\tau_h} + \mu H Q_h H' e^{-2\lambda\tau_h} \\ &\geq \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} + \mu H Q_h H' e^{-2\lambda\tau_h}, \end{aligned} \tag{3.70}$$

which is exactly (3.53) completing the proof. □

Remark 3.2. The stabilisation of neutral delay systems is much more involved than the one of retarded systems due to the possible presence of an infinite number of poles in the right half-plane. We already know from [103] (in the particular case of commensurate delays) that no solution will be provided by Theorem 3.3 if there is a chain of poles clustering the imaginary axis in the right half-plane.

Example 3.8. For the matrices (3.66) and (3.67), see [99], [104], [105] and [106], the upper bound for the delay was given as 0.8418. Applying Theorem 3.3, we designed the following controller $K_0 = [-37.7924 \quad -20.7712]$, $K_1 = [5.3363 \quad 3.7375]$, which guarantees stability for all delays. We illustrate the envelope for $\tau_1 = \tau_h = 2$ in Figure 3.7.

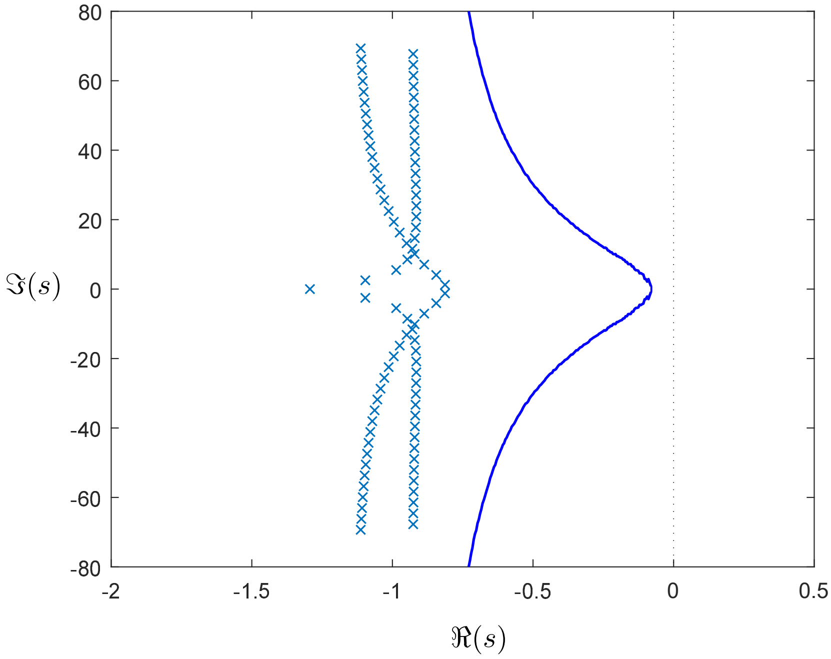

Figure 3.7: State-feedback - $\tau_1 = \tau_h = 2$

3.4 Fractional Case

For the analysis, it is possible to extend our results to the fractional case. Consider the following fractional state space representation.

$$ {}_0\mathbb{D}_t^{\alpha} x(t) = \sum_{i=0}^{N} A_i x\left(t - \tau_i\right), \tag{3.71} $$

in which $x(t) \in \mathbb{R}^n$ is the state variable, $0 = \tau_0 < \tau_1 < \cdots < \tau_N$ are the delays and $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \{0, \ldots, N\}$. Its characteristic polynomial is given by

$$ \det\left(s^{\alpha} I - \sum_{i=0}^{N} A_i e^{-s\tau_i}\right) = 0. \tag{3.72} $$

Let us apply our LMI method for designing envelopes through the following corollaries. First the retarded case.

Corollary 3.1. Let λ be any real number. If there exist matrices $T = T' > 0$, $Q_i = Q_i' > 0$, for all $i \in \{0, \ldots, N\}$ and a scalar μ that satisfy

$$ \mu T \geq \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} \tag{3.73} $$

and

$$\begin{bmatrix} T & T & \dots & T \\ & Q_0 & & \\ & & \ddots & \\ & & & Q_N \end{bmatrix} \geq 0, \tag{3.74}$$

then any characteristic root s_0 of equation (3.72), such that $s_0 = \lambda + j\omega$ verifies

$$|s_0| \leq \sqrt{\mu^{\frac{1}{\alpha}}}. \tag{3.75}$$

Proof. See Corollary 3.2. □

Example 3.9. Let us consider the matrices

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{3.76}$$

and

$$\alpha = 1/3, \quad \tau = 1. \tag{3.77}$$

We can see in Figure 3.8 that the poles of the system respects the envelope. We can also see the tightness of the envelope and the fact that the poles are really inside it by zooming in as showed by Figure 3.9.

Fractional linear time-delay neutral systems can also be analysed by the envelope method with a simple adaptation. Consider the system

$${}_0\mathbb{D}_t^\alpha x(t) = \sum_{i=0}^{N} A_i x\left(t - \tau_i\right) + H\dot{x}\left(t - \tau_h\right), \tag{3.78}$$

in which $x(t) \in \mathbb{R}^n$ is the state variable, $0 = \tau_0 < \tau_1 < \dots < \tau_N$ are the delays and $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \{0, \dots, N\}$. This system is exponentially stable if and only if all roots of its characteristic equation

$$\det\left(s^\alpha I - \sum_{i=0}^{N} A_i e^{-s\tau_i} - sHe^{-s\tau_h}\right) = 0 \tag{3.79}$$

are in the open left α-plane. The adaptation result is described by the Corollary 3.2 below.

Corollary 3.2. Let λ be any real number. If there exist matrices $T = T' > 0$, $Q_i = Q_i' > 0, \forall i \in \{0, \dots, N\}$, $Q_h = Q_h' > 0$ and a scalar μ such that

$$\mu T \geq \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} + \mu H Q_h H' e^{-2\lambda\tau_h} \tag{3.80}$$

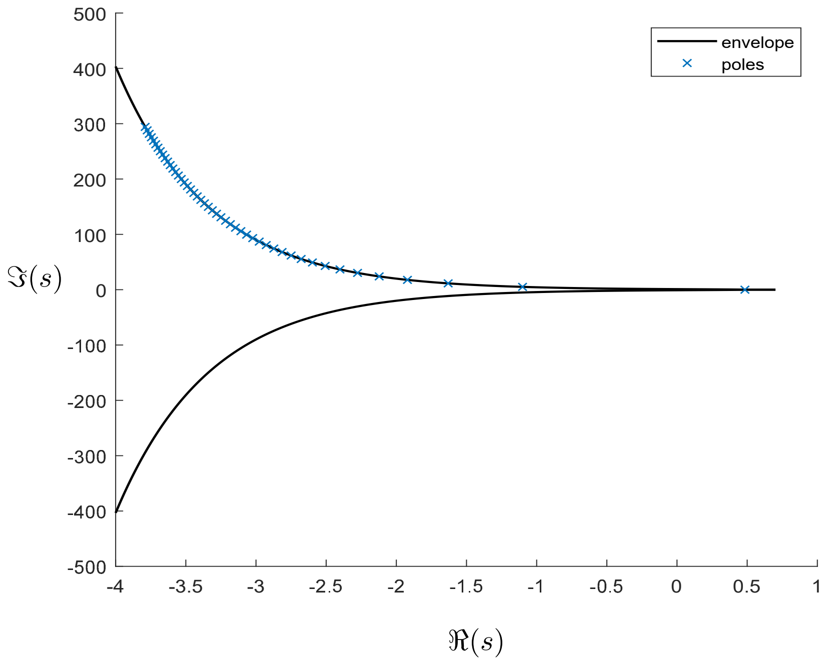

Figure 3.8: Stability Envelope for Fractional Retarded Time-Delay System

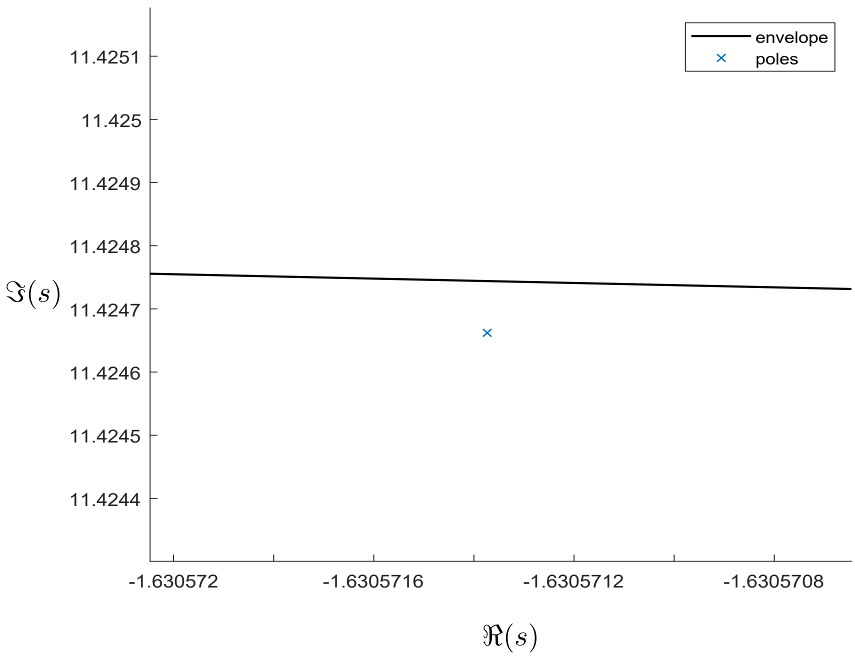

Figure 3.9: Zoom in on pole inside the envelope

and

$$\begin{bmatrix} T & T & \dots & T & T \\ \bullet & Q_0 & & 0 & 0 \\ \bullet & \bullet & \ddots & 0 & 0 \\ \bullet & \bullet & \bullet & Q_N & 0 \\ \bullet & \bullet & \bullet & \bullet & \frac{\mu}{|s_0|^2} Q_h \end{bmatrix} > 0, \tag{3.81}$$

then any characteristic root s_0 of equation (3.79) such that $s_0 = \lambda + j\omega$ verifies

$$|s_0| \leq \sqrt{\mu^{\frac{1}{\alpha}}}. \tag{3.82}$$

Proof. Both for the retarded as for the neutral case, defining $\bar{s} = s^\alpha$ and letting $\bar{s}_0 = \lambda + j\omega$ be an eigenvalue of Σ associated with a right-eigenvector v, we can proceed exactly as the integer case.

$$\mu T \geq \Sigma T \Sigma^*, \tag{3.83}$$

$$\mu x_L T x_L^* \geq x_L \Sigma T \Sigma^* x_L^* \tag{3.84}$$

and

$$\mu \geq (\lambda + j\omega)^\alpha (\lambda - j\omega)^\alpha, \tag{3.85}$$

leading to

$$|s_0| \leq \sqrt{\mu^{1/\alpha}}, \tag{3.86}$$

which concludes the proof. □

Once again, one difficulty that arises on applying this method for the neutral case, is the necessity of knowing $|s_0|$ to implement the LMI (3.81). This time, we cannot replace

$$\frac{\mu}{|s_0|^2} \tag{3.87}$$

by 1. However, from (3.82) we have that

$$\frac{\mu^{1/\alpha}}{|s_0|^2} \geq 1. \tag{3.88}$$

Multiplying both sides of it by $\mu^{1-1/\alpha}$, we get that $\frac{\mu}{|s_0|^2} \geq \mu^{1-1/\alpha}$, implying that whenever

$$\begin{bmatrix} T & T & \dots & T & T \\ \bullet & Q_0 & & 0 & 0 \\ \bullet & \bullet & \ddots & 0 & 0 \\ \bullet & \bullet & \bullet & Q_N & 0 \\ \bullet & \bullet & \bullet & \bullet & \mu^{1-1/\alpha} Q_h \end{bmatrix} > 0 \tag{3.89}$$

is satisfied, then (3.81) is also true. Hence, one can use (3.89) in place of (3.81) to obtain the results derived from the Corollary 3.2. This time we do not have a *generalised eigenvalue problem* anymore. Nonetheless, this problem can be easily treated by linear search and LMI feasibility on the other variables.

In spite of the fact that the stability analysis through the use on an envelope that surrounds all poles of a system was adapted to the fractional case without further difficulties, the same does not occurs to the synthesis analysis. The change of variables $s = z - d$ is not suitable anymore, due to the α exponent, i.e., the first equation of (3.65) cannot be isolated anymore because of the term $(z - d)^{\alpha}$. Nevertheless, we stress the fact that even if the technique can be applied for analysis only, this approach is entirely new, especially for fractional systems.

3.5 Final Remarks

The use of envelopes to study stability and even further to design feedback controllers is, to the best of our knowledge, entirely new. The approach to calculate those envelopes through LMIs differs from previous techniques such as the eigenvalue approach and is able to cope for the first time with neutral delay systems. Through an LMI approach, it is possible to use envelopes not only to study stability but to design robust feedback controllers for retarded and neutral time-delay systems. The controller designed can guarantee delay-independent stability, delay-dependent $\alpha - \textit{stability}$ for every $\tau < \tau^*$ and robustness to parametric uncertainties. Those results are presented in [107] and [108].

Chapter 4

Stochastic Time-Delay Systems

In this chapter we are going to leave the deterministic domain and we are going to study the so called stochastic time-delay systems. Stability conditions for continuous-time stochastic systems when they are subjected to delays will be discussed. We design a state-feedback controller via LMIs that stabilises the system even for the case where the transition rates between Markovian modes are uncertain. Furthermore, we are going to develop a linear system that models the first moment dynamics for stochastic positive systems with delays both for continuous and for discrete time. Those systems can then be used for mean-stability analysis in place of the original stochastic systems. Numerical simulation is performed to validate all the results.

4.1 Introduction

Dynamical systems that exhibit sudden changes in their structure due to environmental factors, sensor and actuator failures, changes in the operating point for the nonlinear case, among others, may not be well represented by the classic time invariant linear models. Among the various forms of modeling such systems, one that is of great interest is known as *linear systems subject to Markovian jumps*, usually denoted by Markov Jump Linear Systems or simply MJLS, which constitutes an important class of stochastic systems. In this model, these abrupt changes are represented as different subsystems. Each subsystem, also called as a mode of operation, is described by a set of linear equations and the randomness to be taken into account is modelled as a change, or jump, between the different modes of operation.

In order to better understand MJLS that we deal with throughout this section, consider a system that can present more than one mode of operation, each one of them governed by a set of linear and time invariant differential equations. The system changes its mode of operation according to a continuous Markov chain, i.e., the jump rate from one mode of operation to another depends only on its current state. It is important to stress the importance of being cautious with those type of systems because even when all modes are stable independently, the system all together can be unstable. That is, the stability from each mode does not imply any conclusions about the stability of the system as a whole.

One of the first works in the literature to consider continuous-time Markovian systems, without delays involved, is presented in [59]. In [63] the $\mathcal{H}_2$ control is treated through state feedback via convex analysis. Controllability and stability concepts are studied in [64] and the optimal quadratic control with solution via the Separation Theorem is presented in [65]. For the case in which delays are included in the dynamics, the book [24] is our major reference. Stability and stabilisation, for a singular delay, is viewed in [109], with partially unknown transition rates in [70] while robust $\mathcal{H}_\infty$ filtering is discussed in [71]. Delay-dependent stability and output feedback can be seen in [69].

The MJLS control and filtering projects assume, for the most part, that the transition rates between modes are known a priori. However, in practice, only estimated values of these rates are available and these uncertainties can generate instabilities or at least degrade system performance in the same way that occurs when there are uncertainties in the state space representation matrix of the plant. For this case, in which the transition rates between the modes are not fully known, there are works in the literature that present stability conditions, as can be seen, for example, in [66], where the robust case for the delay free case is discussed. The state feedback, also for the delay free case, can be seen in [67] and [68]. The case for continuous time-delay MJLS with uncertain transition rates is dealt in [110] for state feedback considering $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms and in [111] for output feedback considering the $\mathcal{H}_2$ norm. We are going to extend, on this dissertation, the state-feedback result for the case where some delay is present on the system.

Consider the following system

$$\begin{array}{rcl} \dot{x}(t) & = & A_0(\theta_t)x(t), \\ z(t) & = & C_0(\theta_t)x(t), \end{array} \tag{4.1}$$

in which $x(t) \in \mathbb{R}^n$ is the state variable, $\theta(t) = \theta_t \in \mathbb{K}$ is a random variable generated by a continuous Markovian process and $z(t) \in \mathbb{R}^p$ is the controlled output.

The process $\{\theta_t, t \in [0, +\infty)\}$ is a Markovian stochastic process with transition rates given by

$$\begin{aligned} p_{ij}(\Delta) &= \text{Prob}(\theta_{t+\Delta} = j | \theta_t = i) \\ &= \left\{ \begin{array}{rl} \lambda_{ij}\Delta + o(\Delta) & i \neq j, \\ 1 + \lambda_{ii}\Delta + o(\Delta) & i = j, \end{array} \right. \end{aligned} \tag{4.2}$$

with $\lambda_{ij} \geq 0$ for $i \neq j$, $\lambda_{ii} \leq 0$ and

$$\sum_{j \in \mathbb{K}} \lambda_{ij} = 0. \tag{4.3}$$

We can define the transition matrix as

$$\Lambda = [\lambda_{ij}] \in \mathbb{R}^{N \times N}, \tag{4.4}$$

which is formed by the transition rates between the states of the Markov chain that governs the evolution of $\theta_t \in \mathbb{K}$. Because the process is Markovian, we have that the model presented in (4.2) totally establishes the behaviour of the chain. In addition, it is essential to note that no additional information is needed, since the transition depends only on the current state and transition rate between states.

Proceeding, we also define $p(t)$ as

$$p(t) \triangleq \begin{bmatrix} p_1(t) & p_2(t) & \dots & p_N(t) \end{bmatrix}', \tag{4.5}$$

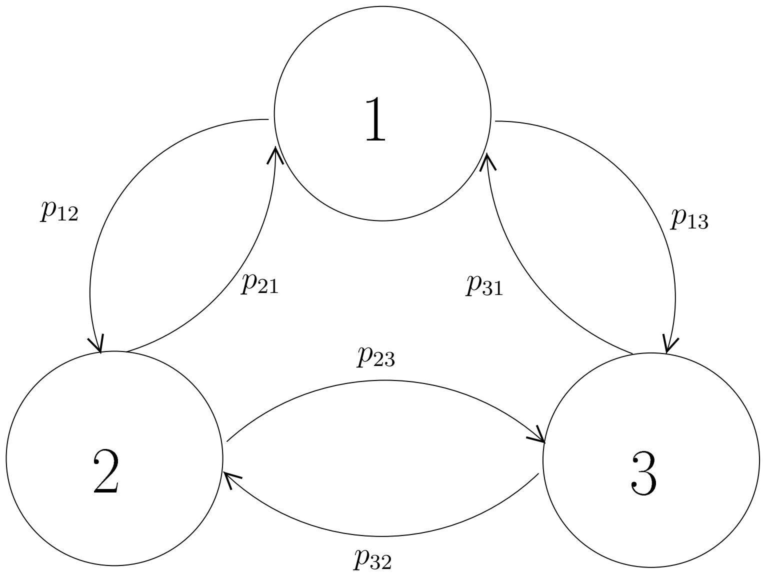

Figure 4.1: Markov chain with three states

in which

$$p_i(t) \triangleq \text{Prob}(\theta(t) = i). \tag{4.6}$$

It can be shown that [78]

$$p(t) = \Pi(t)'p(0), \tag{4.7}$$

where $p(0)$ is the initial probability distribution of θ, $\Pi(t)$ is a transition matrix that satisfies the semigroup property

$$\Pi(t+s) = \Pi(t)\Pi(s), \tag{4.8}$$

and is the unique solution of the backward Kolmogorov differential equation

$$\frac{d\Pi(t)}{dt} = \Lambda\Pi(t), \quad t \in \mathbb{R}_+, \quad \Pi(0) = I, \tag{4.9}$$

which can be explicitly calculated as

$$\Pi(t) \triangleq [\Pi_{ij}(t)] = e^{\Lambda t}, \quad t \in \mathbb{R}_+. \tag{4.10}$$

It is important to interpret the meaning of the equation (4.2). The sojourn time in each mode, for all continuous-time Markov processes, is given by an exponential random variable, see [112]. This random variable has a parameter given by $|\lambda_{ii}|$ and consequently, for $\lambda_{ii} < 0$, the mean time to stay in i mode is given by $1/|\lambda_{ii}|$. In the case of $\lambda_{ii} = 0$ we have an absorbing mode, implying that, once reached, the system will remain indefinitely in that mode. When the sojourn time finishes, the next mode j is selected by a discrete Markov chain, as depicted in Figure 4.1, where the transition probabilities are given by $p_{ij} = \lambda_{ij}/|\lambda_{ii}|$, $\forall j \neq i$. This interpretation is relevant and necessary so that the time simulation of Markovian processes can be implemented.

4.1.1 Uncertain rates

After this brief introduction, let us also consider that the transition rates are not fully known. We assume that $\Lambda = \{\lambda_{ij}\}$ is not known but that each row Λ_i, $i \in \mathbb{K}$, is contained in a convex set of known N_i vertices, i.e.,

$$\Lambda_i = \sum_{l=1}^{N_i} \alpha_l \Lambda_i^{(l)}, \tag{4.11}$$

with $\sum_{l=1}^{N_i} \alpha_l = 1$, $\alpha_l \geq 0$, $\forall l \in \{1, \ldots, N_i\}$, always respecting the normalisation

$$\sum_{j=1}^{N} \lambda_{ij}^{(l)} = 0, \forall i \in \mathbb{K},\ \forall l \in \{1, \ldots, N_i\}. \tag{4.12}$$

For example, for the following transition matrix

$$\Lambda = \begin{bmatrix} -2 & \Lambda_{12} & \Lambda_{13} \\ 2 & -4 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \tag{4.13}$$

in which $0 \leq \Lambda_{12} \leq 1$ and $1 \leq \Lambda_{13} \leq 2$, representing that the parameter is unknown but belongs to the given interval. The first row can be rewritten as

$$\Lambda_1 = \alpha_1 \Lambda_1^{(1)} + \alpha_2 \Lambda_1^{(2)}, \tag{4.14}$$

where $\alpha_1 + \alpha_2 = 1$, $\alpha_1, \alpha_2 > 0$ and

$$\Lambda_1^{(l)} = \begin{bmatrix} \lambda_{11}^{(l)} & \lambda_{12}^{(l)} & \lambda_{13}^{(l)} \end{bmatrix}. \tag{4.15}$$

It is easy to see that the convex combination that creates the first row for this particular transition matrix is

$$\Lambda_1 = \alpha_1 \begin{bmatrix} -2 & 0 & 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 & 1 & 1 \end{bmatrix}. \tag{4.16}$$

Another usual way of representing uncertainties in the transition matrix is by considering some of its elements as unknown, see [67], [68]. If these elements are not on the main diagonal of the transition matrix, it is immediate to represent such uncertainties as convex polytopes as in (4.11). However, when the unknown element is on the main diagonal of the transition matrix, this input can become unlimited. For this particular case, $\lambda_{ii} \to \infty$, which causes the expected value of the sojourn time in mode i to zero. In this case, the jump frequency can be arbitrarily large (a phenomenon known as chattering), which can create sliding surfaces in the system space [113]. One way to work around this problem, used by [68], is to set a lower bound for the unknown element λ_{ii}.

4.2 Stability

Before further developments, we need to establish some definitions, including the ones for stability. There are several notions of exponential stability for a stochastic system, such as the ones presented in [114], [24] and [78]. The next definitions, adapted from [115], addresses the stability of a continuous-time MJLS. It is important to stress that those notions are, in general, not equivalent and therefore can lead to different test conditions.

Definition 4.1. For any real scalar $\delta > 0$, system (4.1) is said to be exponentially δ-moment stable if there exist positive real scalars α and β such that

$$\mathcal{E}\{\|x(t)\|^{\delta}\} \leq \alpha e^{-\beta t}\|x(0)\|^{\delta} \tag{4.17}$$

for any initial condition $x(t) > 0,\ t \in [-\tau, 0]$, and any initial probability distribution $p(0)$. In particular, $\delta = 2$ defines the so-called Exponential Mean-Square Stability.

Definition 4.2. System (4.1) is said to be exponentially almost-sure stable if there exists a positive real scalar ρ such that

$$\text{Prob}\left\{\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t)\| \leq \rho\right\} = 1 \tag{4.18}$$

for any initial condition $x(t) > 0,\ t \in [-\tau, 0]$ and any initial probability distribution $p(0)$.

Definition 4.3. We say that system (4.1), is exponentially mean stable (EMS) if there exist positive real scalars α and β such that

$$\mathcal{E}\{x(t)\} \leq \alpha e^{-\beta t}\|x(0)\|\mathbb{1}_n \tag{4.19}$$

for any initial condition $x(t) > 0,\ t \in [-\tau, 0]$, and any initial probability distribution $p(0)$.

4.2.1 Stability of Markovian time-delay systems

On the previous section we introduced stochastic systems with the Markov property. Let us now increase the complexity of this system, including a delay. Restating the system (4.1) with a delay τ, we have the following system

$$\begin{aligned} \dot{x}(t) &= A_0(\theta_t)x(t) + A_1(\theta_t)x(t-\tau) + J(\theta_t)w(t), \\ z(t) &= C_0(\theta_t)x(t), \end{aligned} \tag{4.20}$$

in which the initial condition of the system is specified as $(\theta_0, \phi(.))$, with $x(s) = \phi(s) \in \mathbb{L}_2[-\tau, 0]$, $x_0 = x(0) = \phi(0)$. Furthermore, to ease the readability, the following notation will be used interchangeably, $A_k(\theta_t) \triangleq A_{ki}$, $J(\theta_t) \triangleq J_i$, $C_0(\theta_t) \triangleq C_{0i}$, and so on, whenever $\theta_t = i \in \mathbb{K}$.

The definition of stochastic stability, with the inclusion of the delay, is given by the following definition.

Definition 4.4. The system (4.20), with $w(t) \equiv 0$, is said to be stochastically stable if there exists a constant $M(\theta_0, \phi(.)) > 0$ such that

$$\mathcal{E}\left[\int_0^\infty x(t)'x(t)dt \middle| \theta_0\right] \leq M(\theta_0, \phi(.)). \tag{4.21}$$

Utilising those definitions we can state the following theorem.

Theorem 4.1. If there exist symmetric, positive definite matrices $P_i = P_i' > 0$, $Q = Q' > 0$, $\forall i \in \mathbb{K}$, such that the following LMIs are simultaneously verified

$$\Xi_i \triangleq A_{0i}'P_i + P_iA_{0i} + \sum_{j\in\mathbb{K}} \lambda_{ij}^{(l)} P_j + P_i + Q < 0, \tag{4.22}$$

$$A_{1i}'P_iA_{1i} \leq Q, \tag{4.23}$$

$\forall i \in \mathbb{K}$, $\forall l \in N_i$. Then the system (4.20), with $w(t) \equiv 0$, is stochastically stable.

The proof can be easily achieved by choosing an appropriate Lyapunov-Krasoviskii functional as can be seen in [24].

4.3 Stabilisation

4.3.1 $\mathcal{H}_\infty$ Norm

For the study of stabilisation let us add a control input u to the previous system.

$$\begin{aligned} \dot{x}(t) &= A_0(\theta_t)x(t) + A_1(\theta_t)x(t-\tau) + B(\theta_t)u(t) + J(\theta_t)w(t), \\ z(t) &= C_0(\theta_t)x(t) + C_1(\theta_t)x(t-\tau) + D(\theta_t)u(t) + E_z(\theta_t)w(t), \\ u(t) &= K(\theta_t)x(t). \end{aligned} \tag{4.24}$$

An essential property in any control project is stability. However, in many applications, other design goals must be considered, such as robustness and performance. In this section, we present the design of a controller that not only guarantees stochastic stability but also optimises the $\mathcal{H}_\infty$ norm of the closed-loop system. We present conditions that depend in an affine way on the transition rates between modes and this is, in our view, the main distinction of this theorem in relation to the current literature. Thus, with this new proposal, it was possible to treat adequately the case where the transition rates between modes is uncertain. Similar work is done for stochastic systems without delays in [110].

Before stating the next theorem, let us define the infinitesimal generator $\mathcal{L}$.

Definition 4.5. Let X_t, $t \in [0, +\infty)$, be a stochastic process and $f(X_t)$ a function. Then, the infinitesimal generator of X_t applied to f is given by

$$\mathcal{L}f = \lim_{\Delta \to 0^+} \mathcal{E}\left[\frac{f(X_{t+\Delta}) - f(X_t)}{\Delta}\right]. \tag{4.25}$$

Theorem 4.2. Let $\gamma > 0$ be a constant scalar. If there exist matrices $P_i = P_i' > 0$ and $Q = Q' > 0$, such that the following LMIs hold for every $i \in \mathbb{K}$ and for every $l \in N_i$,

$$\begin{bmatrix} \Pi_i & P_i A_{1i} + C_{0i}' C_{1i} & P_i J_i + C_{0i}' E_{zi} \\ A_{1i}' P_i + C_{1i}' C_{0i} & -Q + C_{1i}' C_{1i} & C_{1i}' E_{zi} \\ J' P_i + E_{zi}' C_{0i} & E_{zi}' C_{1i} & -\gamma^2 I + E_{zi}' E_{zi} \end{bmatrix} < 0, \tag{4.26}$$

in which $\Pi_i \triangleq A_{0i}' P_i + P_i A_{0i} + \sum_{j \in \mathbb{K}} \lambda_{ij}^{(l)} P_j + Q + C_{0i}' C_{0i}$, then the system (4.24) with $u(t) \equiv 0$ is stochastically stable and satisfies

$$\|z\|_2^2 \leq \gamma^2 \|w\|_2^2 + x_0' P(\theta_0) x_0 + \int_{-\tau}^{0} \phi'(s) Q \phi(s) ds, \tag{4.27}$$

in which $x_0 = \phi(0)$, meaning that the system is stochastically stable with γ-disturbance attenuation.

Proof. To cast our model into the framework of Markov systems let us define

$$\mathbf{x}_t(s) = x(t+s), \quad t - \tau \leq s \leq t, \tag{4.28}$$

allowing us to choose as Lyapunov-Krasovskii functional

$$V(\mathbf{x}_t, \theta_t) = x(t)' P(\theta_t) x(t) + \int_{-\tau}^{0} x'(t+s) Q x(t+s) ds. \tag{4.29}$$

Applying the infinitesimal generator

$$\begin{aligned} \mathcal{L}V(x, \theta_t = i) \;=\; & (A_{0i} x(t) + A_{1i} x(t-\tau) + J_i w(t))' P_i x(t) + \\ & + x(t)' P_i (A_{0i} x(t) + A_{1i} x(t-\tau) + J_i w(t)) + \\ & + x(t)' \left(\textstyle\sum_{j \in \mathbb{K}} \lambda_{ij}^{(l)} P_j + Q\right) x(t) - x'(t-\tau) Q x(t-\tau). \end{aligned} \tag{4.30}$$

Adding to (4.30) the terms

$$- z(t)' z(t) + \gamma^2 w(t)' w(t) + z(t)' z(t) - \gamma^2 w(t)' w(t), \tag{4.31}$$

the result remains the same and can be rearranged as

$$\begin{aligned}
\mathcal{L}V(x,\theta_t=i) &= (A_{0i}x(t)+A_{1i}x(t-\tau)+J_iw(t))'P_ix(t)+ \\
&+ x(t)'P_i(A_{0i}x(t)+A_{1i}x(t-\tau)+J_iw(t))+x(t)'\left(\textstyle\sum_{j\in\mathbb{K}}\lambda_{ij}^{(l)}P_j+Q\right)x(t)+ \\
&+ (C_{0i}x(t)+C_{1i}x(t-\tau)+E_{zi}w(t))'(C_{0i}x(t)+C_{1i}x(t-\tau)+E_{zi}w(t))- \\
&- \gamma^2w(t)'w(t)-z(t)'z(t)+\gamma^2w(t)'w(t)-x'(t-\tau)Qx(t-\tau).
\end{aligned}$$

Rewriting this result on matrix notation

$$\begin{bmatrix} x(t)' & x(t-\tau) & w(t)' \end{bmatrix}\begin{bmatrix} \Pi_i & P_iA_{1i}+C_{0i}'C_{1i} & P_iJ_i+C_{0i}'E_{zi} \\ A_{1i}'P_i+C_{1i}'C_{0i} & -Q+C_{1i}'C_{1i} & C_{1i}'E_{zi} \\ J'P_i+E_{zi}'C_{0i} & E_{zi}'C_{1i} & -\gamma^2+E_{zi}'E_{zi} \end{bmatrix}\begin{bmatrix} x(t) \\ x(t-\tau) \\ w(t) \end{bmatrix}-$$

$$- \quad z(t)'z(t)+\gamma^2w(t)'w(t),$$

we can see that

$$\mathcal{L}V(x,\theta_t=i) < -z(t)'z(t)+\gamma^2w(t)'w(t). \tag{4.32}$$

Imposing $w(t)=0$ we can see that the system is stochastically stable. Furthermore, integrating and applying mathematical expectation on both sides

$$-x_0'P(\theta_0)x_0-\int_{-\tau}^{0}\phi'(s)Q\phi(s)ds < -\mathcal{E}\left\{\int_0^\infty(z(t)'z(t)dt)\right\}+\gamma^2\mathcal{E}\left\{\int_0^\infty(w(t)'w(t)dt)\right\}, \tag{4.33}$$

that is,

$$\|z\|_2^2 < \gamma^2\|w\|_2^2+x_0'P(\theta_0)x_0+\int_{-\tau}^{0}\phi'(s)Q\phi(s)ds. \tag{4.34}$$

For null initial conditions, $\phi(s)=0,\forall s\in(-\tau,0)$, we get the standard form

$$\|z\|_2^2 < \gamma^2\|w\|_2^2. \tag{4.35}$$

□

Remark 4.1. This result is standard and can be seen, for instance, in [24]. However, we presented it here as an auxiliary result for the next theorem.

4.3.2 State Feedback

On this section we deal with state-feedback control for Markovian systems with delay. We achieve an affine relation with respect to the transition rates between modes allowing to tackle the problem where the rates are not fully known but belong to a convex polytope.

The following theorem states how to obtain a controller $u(t) = K_i x(t)$ for each mode of the system.

Theorem 4.3. There is a feedback controller that satisfies

$$\|z\|_2^2 < \gamma^2\|w\|_2^2 + x_0' P(\theta_0) x_0 + \int_{-\tau}^{0} \phi'(s) Q \phi(s) ds, \tag{4.36}$$

if there exist symmetric matrices $X_i > 0$, $U > 0$, $Z_{ij} > 0$ and matrices Y_i and H_i with compatibles dimensions satisfying

$$\begin{bmatrix} A_{0i}X_i + X_i A_{0i}' + B_i Y_i + Y_i' B_i' + \lambda_{ii}^{(l)} X_i & \bullet & \bullet & \bullet & \bullet & \bullet \\ U A_{1i}' & -U & \bullet & \bullet & \bullet & \bullet \\ J_i' & 0 & -\gamma^2 I & \bullet & \bullet & \bullet \\ X_i & 0 & 0 & -H_i - H_i' + \sum_{j\in\mathbb{K}_i} \lambda_{ij}^{(l)} Z_{ij} & \bullet & \bullet \\ X_i & 0 & 0 & 0 & -U & \bullet \\ C_{0i}X_i + D_i Y_i & C_{1i}U & E_{zi} & 0 & 0 & -I \end{bmatrix} < 0, \tag{4.37}$$

$$\begin{bmatrix} Z_{ij} & \bullet \\ H_i & X_j \end{bmatrix} > 0, \quad i \neq j, \tag{4.38}$$

for all $(i,j) \in \mathbb{K}\times\mathbb{K}$. In that case, the feedback gains are $K_i = Y_i X_i^{-1}$, $i \in \mathbb{K}$.

Proof. Suppose that (4.37)–(4.38) are valid. Applying Schur complement at (4.38), we get

$$Z_{ij} > H_i' X_j^{-1} H_i. \tag{4.39}$$

Multiplying (4.39) by $\lambda_{ij}^{(l)}$ and adding for all $j \in \mathbb{K}$, $j \neq i$, it follows that

$$\sum_{j\in\mathbb{K}_i} \lambda_{ij}^{(l)} Z_{ij} \geq H_i' \left(\sum_{j\in\mathbb{K}_i} \lambda_{ij}^{(l)} X_j^{-1} \right) H_i. \tag{4.40}$$

Denoting, for positive definite matrices, X_{qi} as the inverse of the linear combination of inverses X_j^{-1} weighted by $\lambda_{ij}^{(l)}$, $\forall j \in \mathbb{K}_i$

$$X_{qi} = \left(\sum_{j\in\mathbb{K}_i} \lambda_{ij} X_j^{-1} \right)^{-1}, \tag{4.41}$$

we have from (4.39) that

$$\begin{aligned} H_i + H_i' - \sum_{j\in\mathbb{K}_i} \lambda_{ij}^{(l)} Z_{ij} &\leq H_i + H_i' - H_i' X_{qi}^{-1} H_i \\ &\leq X_{qi} - (H_i - X_{qi})' X_{qi}^{-1} (H_i - X_{qi}) \\ &\leq X_{qi}. \end{aligned} \tag{4.42}$$

Therefore, by (4.42) we can replace $-H_i - H_i' + \sum_{j\in\mathbb{K}_i} \lambda_{ij}^{(l)} Z_{ij}$ by $-X_{qi}$ in (4.37) and the expression will remain valid. Considering $\tilde{A}_{0i} = A_{0i} + B_i K_i$ and $\tilde{C}_{0i} = C_{0i} + D_i K_i$ we can rewrite the inequality obtained as

$$\begin{bmatrix} \tilde{A}_{0i}X_i + X_i\tilde{A}_{0i}' + \lambda_{ii}^{(l)}X_i & \bullet & \bullet & \bullet & \bullet & \bullet \\ UA_{1i}' & -U & \bullet & \bullet & \bullet & \bullet \\ J_i' & 0 & -\gamma^2 I & \bullet & \bullet & \bullet \\ X_i & 0 & 0 & -X_{qi} & \bullet & \bullet \\ X_i & 0 & 0 & 0 & -U & \bullet \\ \tilde{C}_{0i}X_i & C_{1i}U & E_{zi} & 0 & 0 & -I \end{bmatrix} < 0. \tag{4.43}$$

Multiplying the left side by $\mathbf{diag}(P_i, Q, I, I, I, I)$ and the right side by its transpose, we have

$$\begin{bmatrix} P_i\tilde{A}_{0i} + \tilde{A}_{0i}'P_i + \lambda_{ii}^{(l)}P_i & \bullet & \bullet & \bullet & \bullet & \bullet \\ A_{1i}'P_i & -Q & \bullet & \bullet & \bullet & \bullet \\ J_i'P_i & 0 & -\gamma^2 I & \bullet & \bullet & \bullet \\ I & 0 & 0 & -X_{qi} & \bullet & \bullet \\ I & 0 & 0 & 0 & -U & \bullet \\ \tilde{C}_{0i} & C_{1i} & E_{zi} & 0 & 0 & -I \end{bmatrix} < 0, \tag{4.44}$$

in which $P_i = X_i^{-1}$ and $U = Q^{-1}$. Applying Schur's complement we get (4.26)

$$\begin{bmatrix} P_i\tilde{A}_{0i} + \tilde{A}_{0i}'P_i + \sum_{j\in\mathbb{K}} \lambda_{ij}^{(l)}P_j + \tilde{C}_{0i}'\tilde{C}_{0i} + Q & \bullet & \bullet \\ A_{1i}'P_i + C_{1i}'\tilde{C}_{0i} & -Q + C_{1i}'C_{1i} & \bullet \\ J_i'P_i + E_{zi}'\tilde{C}_{0i} & E_{zi}'C_{1i} & -\gamma^2 I + E_{zi}'E_{zi} \end{bmatrix} < 0, \tag{4.45}$$

for the closed loop system. □

Remark 4.2. For $l = 1$, we can also prove necessity. Assuming that (4.22) and (4.23) are valid for the values of A_{0i} and C_{0i} concerning the closed loop, that is, for $A_{0i} + B_i K_i$ and $C_{0i} + D_i K_i$ respectively, we need to show that (4.37) and (4.38) are both satisfied. Let us choose $H_i = X_{qi}$ and $Z_{ij} = X_{qi}X_j^{-1}X_{qi} + \varepsilon I$, with $\varepsilon > 0$. Substituting them on (4.38) we have

$$X_{qi}X_j^{-1}X_{qi} + \varepsilon I > X_{qi}X_j^{-1}X_{qi}. \tag{4.46}$$

Furthermore, with this choice for H_i and Z_{ij}, we get

$$\begin{aligned} H_i + H_i' - \sum_{j\in\mathbb{K}_i} \lambda_{ij} Z_{ij} &= X_{qi} + X_{qi} - X_{qi} \sum_{j\in\mathbb{K}_i} \lambda_{ij} X_j^{-1} X_{qi} - \sum_{j\in\mathbb{K}_i} \lambda_{ij} \varepsilon I \\ &= X_{qi} + X_{qi} - X_{qi} X_{qi}^{-1} X_{qi} + \lambda_{ii} \varepsilon I \\ &= X_{qi} + \lambda_{ii} \varepsilon I. \end{aligned} \tag{4.47}$$

Thus, choosing $\varepsilon > 0$ sufficiently small, the equation (4.47) implies that (4.37) is valid and the initial hypothesis is true concluding the proof of necessity for the certain case.

Let us illustrate those results with some numerical examples.

Example 4.1. Consider the system (4.24) described by the system matrices

$$\left[\begin{array}{c|c|c|c} A(1) & A(2) & A_1(1) & A_1(2) \end{array}\right] = \left[\begin{array}{cc|cc|cc|cc} 2 & 0.1 & -1 & 0.2 & -0.8 & 0 & -0.4 & 0 \\ 0 & -4 & -1 & 3 & 0.3 & 0.2 & 0.3 & 0.2 \end{array}\right],$$

$$\left[\begin{array}{c|c|c|c} B(1) & B(2) & J(1) & J(2) \end{array}\right] = \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & -1 & 0.1 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{c|c|c|c} C(1) & C(2) & C_1(1) & C_1(2) \end{array}\right] = \left[\begin{array}{cc|cc|cc|cc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{array}\right],$$

$$\left[\begin{array}{c|c|c|c} D(1) & D(2) & E_z(1) & E_z(2) \end{array}\right] = \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0.2 & 1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right].$$

Consider also the following transition matrix:

$$\Lambda = \begin{bmatrix} -5 & 5 \\ 3 & -3 \end{bmatrix}. \tag{4.48}$$

Applying Theorem 4.3 we achieved the disturbance attenuation level $\gamma = 2.0311$. However, the controller obtained has large gains. To minimise this we apply the same LMIs, only this time for a given $\gamma = 2.0717$ two percent higher. Then we maximise the trace of X_i aiming on reducing the gain of the controller. With this procedure, we get

$$K(1) = \begin{bmatrix} -11.9717 & -6.9016 \\ 7.0399 & 6.8936 \end{bmatrix},$$

$$K(2) = \begin{bmatrix} -52653 & -72901 \\ -9 & -11 \end{bmatrix}.$$

This example was extracted from [24] where it is guaranteed $\gamma = 80.0408$. Our theorem guarantees $\gamma = 2.0717$ and closing the loop the $\mathcal{H}_\infty$ norm is 2.0428.

The advantage of this method relies on the fact that it can be applied on the case where the transition rate is not completely known. This is due to the affine relation with respect to the transition rate λ instead of a square root relation as is commonly used in the literature. Let us apply this method for this scenario.

Example 4.2. Let us consider the same matrices from the previous example except this time we have the following uncertainty on the transition rates

$$\Lambda = \begin{bmatrix} [-7 - 3] & [\ 3 \quad 7] \\ 3 & -3 \end{bmatrix}. \tag{4.49}$$

Figure 4.2 shows the guarantee cost and the closed loop norm for all values in the uncertain range.

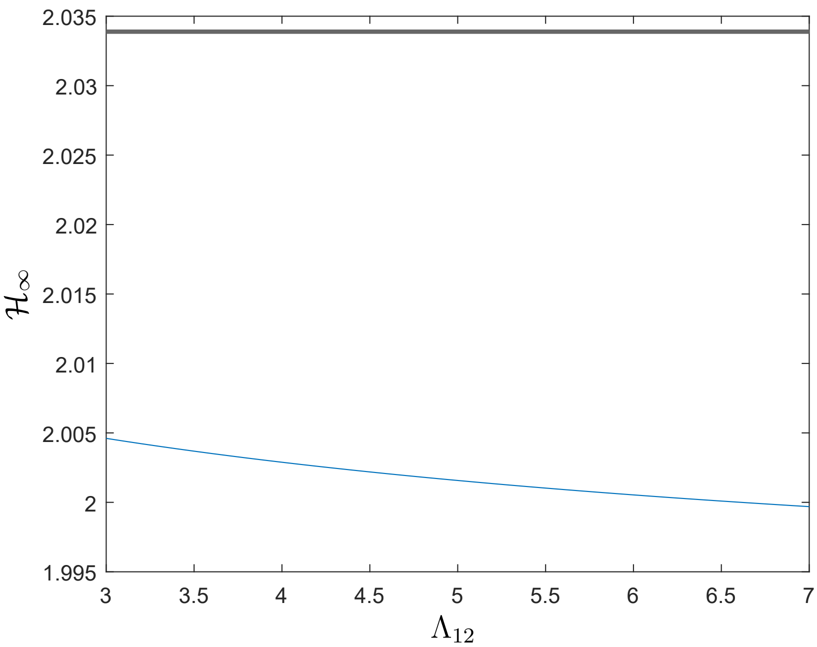

Figure 4.2: System norm for one uncertain parameter

Example 4.3. For the next example, consider the same matrices from Example 4.1 and let us add a third mode given by

$$\left[\begin{array}{c|c|c|c} A(3) & B(3) & A_1(3) & J(3) \end{array}\right] = \left[\begin{array}{cc|cc|cc|cc} -1 & 0.2 & -1 & 0.1 & -0.4 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 & 0.3 & 0.2 & 1 & 0 \end{array}\right],$$

$$\left[\begin{array}{c|c|c|c} C(3) & C_1(3) & D(3) & E_z(3) \end{array}\right] = \left[\begin{array}{cc|cc|cc|cc} 0 & 1 & 1 & 1 & 0 & 0.2 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right].$$

Consider also the following transition matrix:

$$\Lambda = \begin{bmatrix} -13 & [1 \;\; 12] & [1 \;\; 12] \\ [1 \;\; 8] & -10 & [2 \;\; 9] \\ 1 & 1 & -2 \end{bmatrix}. \tag{4.50}$$

Applying Theorem 4.3, the bound obtained by the project is $\gamma = 2.0726$. Figure 4.3 shows the norm of the system for changes on the uncertain parameters, all respecting the upper bound γ.

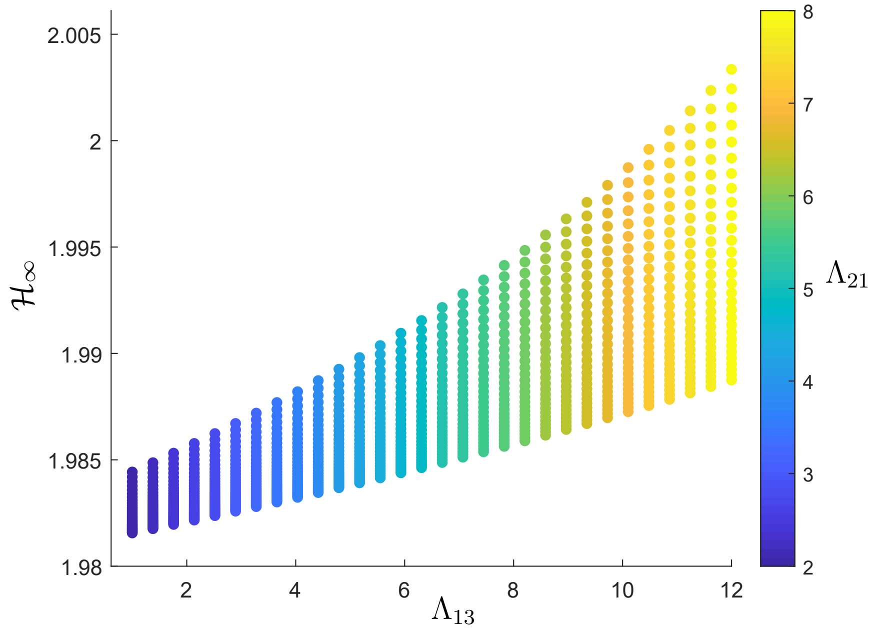

Figure 4.3: System norm for two uncertain parameters

4.4 Positive systems

Internally positive systems [116] are defined by the property of having its state and output both nonnegative, given a nonnegative initial state and a nonnegative input. This interesting property can be availed to synthesise controllers as for example in [117]. One important matrix, as we are going to show, for those systems, is the so called Metzler Matrix defined as follows.

Definition 4.6. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be Metzler if its off-diagonal entries are all nonnegative, i.e., $A_{ij} \geq 0 (i \neq j)$.

4.4.1 Continuous-time Case

Consider now the internally positive time-delay system

$$\begin{aligned} \dot{x}(t) &= A_0(\theta_t)x(t) + A_1(\theta_t)x(t-\tau),\ t \in \mathbb{R}_+, \\ x(t_0) &= x_0, \end{aligned} \tag{4.51}$$

in which A_{0i} and A_{1i} are, respectively, Metzler and nonnegative $\forall i \in \mathbb{K}$.

Let us define the Dirac measure [78], that will be necessary on Lemma 4.1, as follows:

Definition 4.7. In a probability space $(\Omega, \mathcal{F}, P)$, the Dirac measure over a set $A \in \mathcal{F}$ is defined by $\mathbf{1}_A(\cdot)$, meaning that

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise.} \end{cases} \tag{4.52}$$

From Definition 4.7, it follows directly that

$$x(t) = \sum_{i\in\mathbb{K}} x(t)\mathbf{1}_{\{\theta(t)=i\}}. \tag{4.53}$$

In order to obtain convergence results to the first moment of the state variable $x(t)$, let us define, for each $t \in \mathbb{R}_+$,

$$q(t) \triangleq \mathcal{E}(x(t)) \in \mathbb{R}^n. \tag{4.54}$$

Derived from this definition, we also consider that

$$q_i(t) \triangleq \mathcal{E}(x(t)\mathbf{1}_{\{\theta(t)=i\}}) \in \mathbb{R}^n, \tag{4.55}$$

$$\hat{q}(t) \triangleq \begin{bmatrix} q_1(t) & \dots & q_N(t) \end{bmatrix}' \in \mathbb{R}^{Nn}, \tag{4.56}$$

and therefore

$$q(t) = \sum_{i\in\mathbb{K}} q_i(t). \tag{4.57}$$

All those previous definitions can, and are going to, be used interchangeably for discrete-time just replacing (t) by $[k]$.

Lemma 4.1. Consider a stochastic process $\{f(t)\}$ such that $f(t)$ is $\mathcal{F}_t$-measurable and

$\mathcal{E}\{f(t)\mathbf{1}_{\{\theta(t)=i\}}\} \triangleq f_i(t)$ exists. Then

$$\mathcal{E}\{f(t)(\mathbf{1}_{\{\theta(t+h)=i\}} - \mathbf{1}_{\{\theta(t)=i\}})\} = \sum_{j\in\mathbb{K}} \lambda_{ji} f_j(t)h + o(h). \tag{4.58}$$

The proof can be viewed in [78] but it is rewritten here to ease the readout.

Proof. We have from (4.2) that

$$\begin{aligned}
&\mathcal{E}\{f(t)(\mathbf{1}_{\{\theta(t+h)=i\}} - \mathbf{1}_{\{\theta(t)=i\}})\} = \\
&= \sum_{j\in\mathbb{K}} \mathcal{E}\{\mathcal{E}\{f(t)\mathbf{1}_{\{\theta(t+h)=i\}}\mathbf{1}_{\{\theta(t)=j\}}|\mathcal{F}_t\}\} - \mathcal{E}\{f(t)\mathbf{1}_{\{\theta(t)=i\}}\} \\
&= \sum_{j\in\mathbb{K}} P(\theta(t+h)=i|\theta(t)=j)f_j(t) - f_i(t) \\
&= \sum_{j\in\mathbb{K}} \lambda_{ji} f_j(t)h + o(h),
\end{aligned}$$

achieving the desired result. □

For positive systems, the concept of mean stability is equivalent to the 1^{st}-moment stability, which in turn implies almost-sure stability. On the other hand, it is a weaker condition when compared to the classic mean-square stability. See [115] for more details and discussions on this matter.

The next theorem provides a delay differential equation able to compute the first moment of the state variable $x(t)$ of system (4.51).

Theorem 4.4. For all t, $\tau \in \mathbb{R}_+$, the 1^{st}-moment of $x(t)$ from (4.51) is modeled by the following delay differential equation:

$$\dot{\hat{q}}(t) = F\hat{q}(t) + G(\tau)\hat{q}(t-\tau), \tag{4.59}$$

in which, $\hat{q}$ is given by (4.56) and F, $G(\tau)$ are given by

$$\begin{aligned}
F &= \Lambda' \otimes I_n + \mathrm{diag}(A_{0i}), \\
G(\tau) &= \mathrm{diag}(A_{1i})(\Pi(\tau)' \otimes I_n),
\end{aligned} \tag{4.60}$$

where Λ is the transition matrix and $\Pi(\tau)$ was defined in (4.10).

Proof. We proceed following the steps from [78]. Applying Itô's rule to the first equation in

(4.55), we have that

$$\begin{aligned} dq_j(t) &= \mathcal{E}\{dx(t)\mathbf{1}_{\{\theta(t)=j\}} + x(t)d\mathbf{1}_{\{\theta(t)=i\}}\} \\ &= A_{0j}\mathcal{E}\{x(t)\mathbf{1}_{\{\theta(t)=j\}}\}dt + A_{1j}\mathcal{E}\{x(t-\tau)\mathbf{1}_{\{\theta(t)=j\}}\}dt + \sum_{i\in\mathbb{K}}\lambda_{ij}q_i(t)dt \\ &= (A_{0j}q_j(t) + \sum_{i\in\mathbb{K}}\lambda_{ij}q_i(t))dt + A_{1j}\mathcal{E}\{\sum_{i\in\mathbb{K}}x(t-\tau)\mathbf{1}_{\{\theta(t)=j\}}\mathbf{1}_{\{\theta(t-\tau)=i\}}\}dt \\ &= (A_{0j}q_j(t) + \sum_{i\in\mathbb{K}}\lambda_{ij}q_i(t) + A_{1j}\sum_{i\in\mathbb{K}}\Pi_{ij}(\tau)q_i(t-\tau))dt. \end{aligned} \tag{4.61}$$

and thus (4.59) follows. □

Remark 4.3. It is essential to note that, if all matrices A_{0i}, for $i \in \mathbb{K}$, are Metzler, and all matrices A_{i1}, for $i \in \mathbb{K}$, are nonnegative, then F is Metzler and $G(\tau)$ is nonnegative. This implies that the delay system (4.59) is positive. The converse, on the other hand, is not always true.

The property presented in Remark 4.3 will be of great importance on the design of a numerical method able to provide the mean-stability windows for the positive Markov system with delay.

The next result relates mean-stability of the positive stochastic system with delay (4.51) with the stability of the deterministic delay system (4.59).

Lemma 4.2. Consider the positive continuous-time MJLS with delay given in (4.51). This system is exponentially mean-stable if, and only if, the delay system (4.59) is exponentially stable.

Proof. It follows directly from the fact that $\mathcal{E}\{x(t)\} = \sum_{i\in\mathbb{K}} q_i(t)$ and the definition of $\hat{q}(t)$. □

For any given constant delay $\tau \geq 0$, stability of system (4.59) is given by the position of its poles. There are a number of numerical procedures able to determine stability [73] and stability windows [118] for such systems. Nonetheless, due to the fact that matrices F and $G(\tau)$, together with the initial condition, define a positive delay system, a simpler test can be used:

Lemma 4.3. A positive delay system

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau), \tag{4.62}$$

in which A_0 is Metzler and A_1 is a nonnegative matrix, is delay independent stable if, and only if, $A_0 + A_1$ is Hurwitz.

Proof. The proof is based on the existence of a positive vector p and the Lyapunov-Krasovskii functional

$$V(\Psi) = p'\Psi(0) + \int_{-\tau}^{0} p' A_1 \Psi(\theta) d\theta. \tag{4.63}$$

See [119] for all the details. □

Remark 4.4. Note that this interesting result is only valid for positive systems as illustrated in the next example, borrowed from [120].

Example 4.4. Consider the matrices

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4.64}$$

in which A_0 is not Metzler and A_1 is not non-negative, hence the system is not positive.

Note that $A_0 + A_1$ is Hurwitz, however, it is shown in [120], that this system in unstable when $\tau = (2n+1)\pi$, for $(n = 0, 1, 2, ...)$, and stable otherwise.

The previous lemma implies a strong result for deterministic positive linear system with delay. It says that, if the delay-free system (i.e., $\tau = 0$) is stable, then the system is stable for all positive values of the delay, which is known as delay-independent stability. It also implies that if the delay-free system is unstable, the same is true for each positive value of τ. Therefore, for such class of system, we cannot find stability windows, a common effect for general linear systems with delay.

Remark 4.5. All these results can be easily extended to the multi-delay case. In fact, if a system is described by the stochastic equations

$$\dot{x}(t) = A_0(\theta_t)x(t) + \sum_{k=1}^{N} A_k(\theta_t)x(t - \tau_k), \tag{4.65}$$

in which $\tau_k > 0$, $A_0(\theta_t)$ is Metzler and $A_k(\theta_t)$ is nonnegative, for each $\theta_t \in \mathbb{K}$ and $k \in \{1, \ldots, N\}$, then the same procedures can be performed to show that this system is mean-stable if, and only if, the matrix $F + \sum_{k=1}^{N} G(\tau_k)$, where F and G are defined in (4.60), is Hurwitz.

Example 4.5. As our first example, let us consider a positive MJLS with state-space matrices given by

$$A_0(1) = \begin{bmatrix} -1.0 & 0.4 \\ 0 & -2.8 \end{bmatrix}, \quad A_0(2) = \begin{bmatrix} -0.9 & 0 \\ 0 & -0.1 \end{bmatrix},$$
$$A_1(1) = \begin{bmatrix} 0.5 & 0.2 \\ 1.3 & 0.7 \end{bmatrix}, \quad A_1(2) = \begin{bmatrix} 0.2 & 0.1 \\ 1.7 & 0.1 \end{bmatrix}, \tag{4.66}$$

and

$$\Lambda = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{4.67}$$

Applying Theorem 4.4, as it is show in Figure 4.4, this system is mean-stable for $\tau \in [0, 0.12)$ and unstable outside this interval. This examples illustrates a case where the delay has a destabilizing effect on the mean-stability of the system.

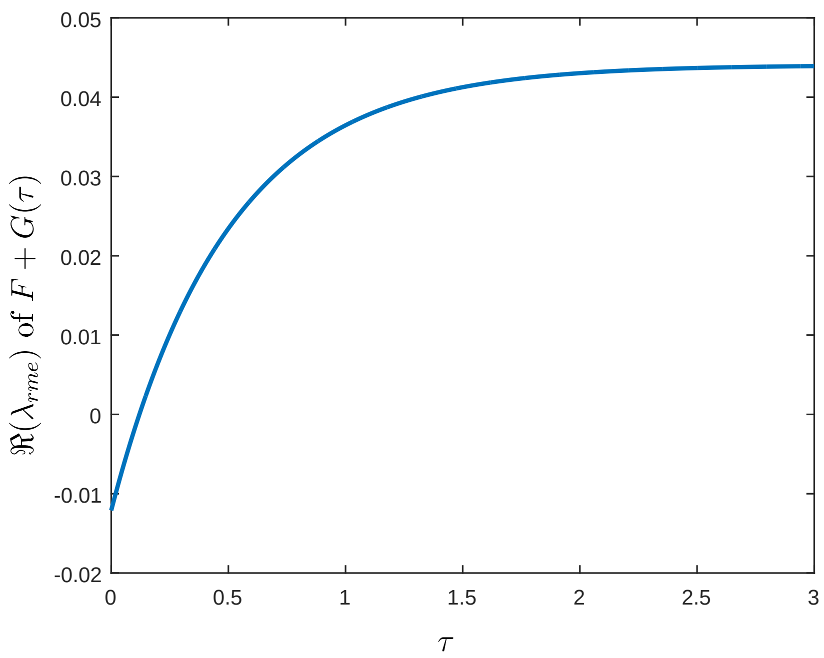

Figure 4.4: First Example: Real Part of Rightmost Eigenvalue of $F + G(\tau)$

Example 4.6. For the second example, consider the positive MJLS with state-space matrices given by

$$A_0(1) = \begin{bmatrix} -3.9 & 0.4 \\ 0.2 & -1.9 \end{bmatrix}, \quad A_0(2) = \begin{bmatrix} -1.5 & 0.3 \\ 0.4 & -3.2 \end{bmatrix},$$
$$A_1(1) = \begin{bmatrix} 1.3 & 1.4 \\ 0.1 & 1.1 \end{bmatrix}, \quad A_1(2) = \begin{bmatrix} 1.9 & 0.4 \\ 0.8 & 1.0 \end{bmatrix}, \tag{4.68}$$

and the same transition matrix Λ as in the previous example. For this case, Figure 4.5 shows that the system is unstable for $\tau = 0$, and mean-stability is only achieved for $\tau > 0.2957$.

This example illustrates a situation where the delay works as a stabilizing effect for the system.

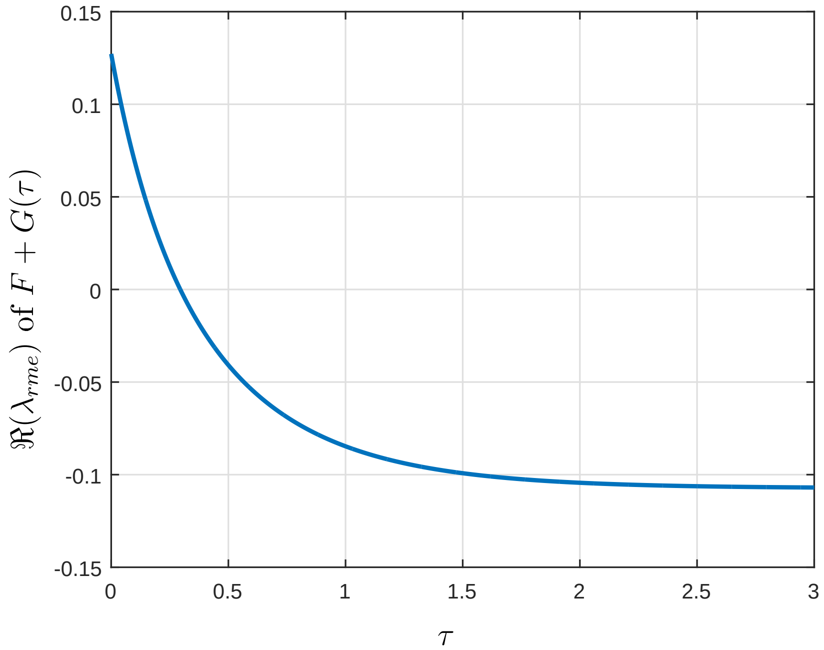

Figure 4.5: Second Example: Real Part of Rightmost Eigenvalue of $F + G(\tau)$

For two different values of the delay ($\tau = 2.0$, inside the stability region, and $\tau = 0.1$, outside the stability region), we performed a Monte-Carlo simulation with 5000 realisations, and the mean for each state variable is presented in Figures 4.6 and 4.7, respectively. These results support the mean-stability analysis performed.

Let us see one more example.

Example 4.7. Consider the positive MJLS with state-space matrices given by

$$A_0(1) = \begin{bmatrix} -4.88 & 0.61 & 0.87 \\ 0.48 & -5.28 & 0.24 \\ 1.13 & 0.88 & -3.16 \end{bmatrix}, \quad A_0(2) = \begin{bmatrix} -3.62 & 0.60 & 0.90 \\ 0.46 & -3.74 & 1.09 \\ 0.70 & 0.20 & -5.80 \end{bmatrix},$$
$$A_1(1) = \begin{bmatrix} 1.8 & 0.3 & 1.7 \\ 0.5 & 0.2 & 0.8 \\ 1.0 & 1.4 & 1.0 \end{bmatrix}, \quad A_1(2) = \begin{bmatrix} 0 & 0.5 & 2.5 \\ 0.5 & 0.8 & 1.7 \\ 0.6 & 0.6 & 0.3 \end{bmatrix}, \tag{4.69}$$

and

$$\Lambda = \begin{bmatrix} -0.229 & 0.229 \\ 0.314 & -0.314 \end{bmatrix}. \tag{4.70}$$

For this case, Figure 4.8 shows that the system is mean-stable only for the interval

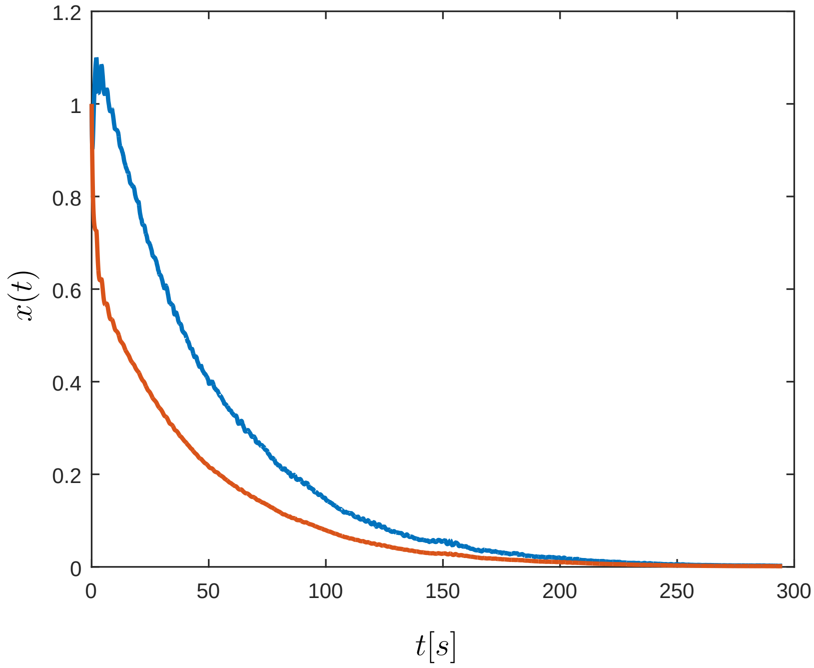

Figure 4.6: Expected value for the state variables for $\tau = 2$

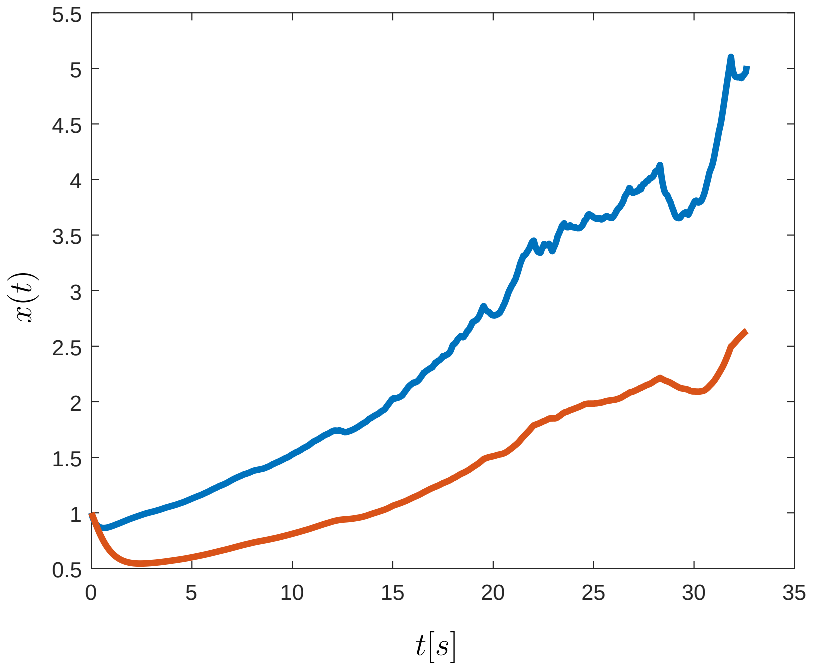

Figure 4.7: Expected value for the state variables for $\tau = 0.1$

$\tau \in [1.932,\ 5.385]$. This example illustrates a situation where a stability window is present even when the delay-free system is unstable.

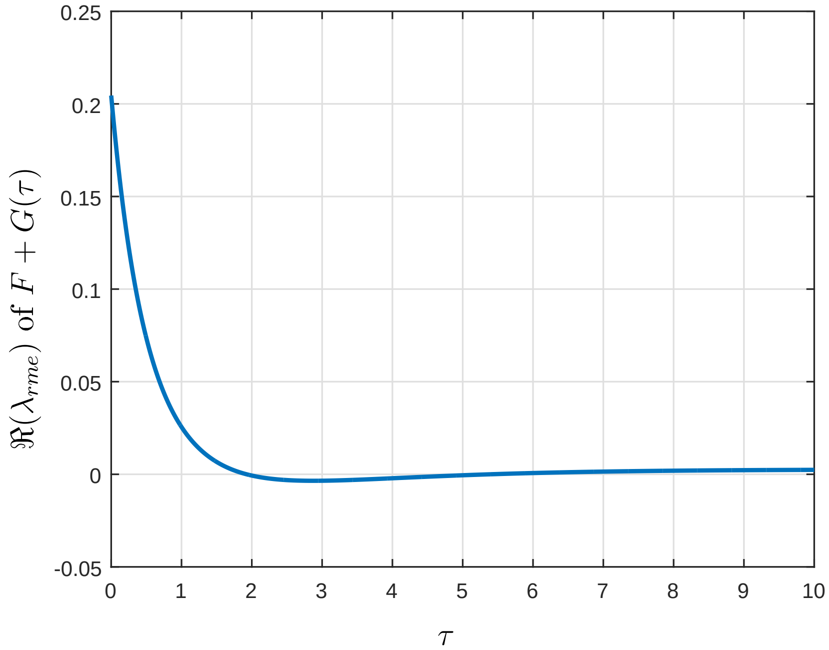

Figure 4.8: Third Example: Real Part of Rightmost Eigenvalue of $F+G(\tau)$

4.4.2 Discrete-time Case

A discrete-time MJLS with delay is described by the following stochastic state-space model

$$x[k+1] = A_0(\theta_k)x[k] + A_1(\theta_k)x[k-\tau]. \tag{4.71}$$

Lemma 4.4. For all k, $\tau \in \mathbb{N}^*$, the 1^{st}-moment of $x[k]$ of (4.71) is modeled by the following delay difference equation:

$$\hat{q}[k+1] = F\hat{q}[k] + G\hat{q}[k-\tau], \tag{4.72}$$

$$G = G(\tau), k \geq \tau, \tag{4.73}$$
$$G = G(k,\mu), k < \tau, \tag{4.74}$$

in which, $\hat{q}$ is the analogous discrete of (4.56) and F, G are given respectively by

$$\begin{aligned} F &= (\Pi' \otimes I_n)\mathrm{diag}(A_{0i}), \\ G(\tau) &= (\mathbb{1}'_N \otimes (\Pi' \otimes I_n))(\mathrm{diag}(\mathrm{vec}(\Pi'(\tau))) \otimes I_n)(I_N \otimes \mathrm{Vec}(A_{1i})), \end{aligned} \tag{4.75}$$

for $k \geq \tau$ and

$$G(k,\mu) = (\mathbb{1}'_N \otimes (\Pi' \otimes I_n))((\mathrm{diag}(\mathrm{vec}(\Pi'(k)) \circ (\mu' \otimes \mathbb{1}_N)) \otimes I_n)(I_N \otimes \mathrm{Vec}(A_{1i})), \tag{4.76}$$

for $k < \tau$. In addition, $\Pi(k) \triangleq \Pi^k$ and $\mu_i \triangleq \mathrm{Prob}(\theta_0 = \mathrm{i})$.

Remark 4.6. Remembering that the Hadamard product is defined by

$$X \circ Y \triangleq \begin{bmatrix} x_{11}y_{11} & \cdots & x_{1n}y_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1}y_{m1} & \cdots & x_{mn}y_{mn} \end{bmatrix}. \tag{4.77}$$

Proof. We start the proof using the discrete counterpart of the definition (4.55).

$$\begin{aligned} q_j[k+1] &= \mathcal{E}\{x[k+1]\mathbf{1}_{\{\theta(k+1)=j\}}\} \\ &= \sum_{i\in\mathbb{K}} \mathcal{E}\{A_{0i}x[k]\mathbf{1}_{\{\theta(k)=i\}}\mathbf{1}_{\{\theta(k+1)=j\}}\} + \\ &+ \sum_{i\in\mathbb{K}} \mathcal{E}\{A_{1i}x[k-\tau]\mathbf{1}_{\{\theta(k)=i\}}\mathbf{1}_{\{\theta(k+1)=j\}}\}. \end{aligned} \tag{4.78}$$

The first term of the right side of this equation, the one who does not depend on τ, is very easily obtained as

$$\begin{aligned} \sum_{i\in\mathbb{K}} \mathcal{E}\{A_{0i}x[k]\mathbf{1}_{\{\theta(k)=i\}}\mathbf{1}_{\{\theta(k+1)=j\}}\} &= \sum_{i\in\mathbb{K}} \Pi_{ij}A_{0i}\mathcal{E}\{x[k]\mathbf{1}_{\{\theta(k)=i\}}\} \\ &= \sum_{i\in\mathbb{K}} \Pi_{ij}A_{0i}q_i[k], \end{aligned} \tag{4.79}$$

which written in matricial form is exactly $Fq_i[k]$. For the term containing τ, we shall consider two distinct cases, i.e., $k \geq \tau$ and $k < \tau$. For the first scenario we can introduce a Dirac's measure $\mathbf{1}_{\{\theta(k-\tau)=\ell\}}$ and sum up to all ℓ.

$$\begin{aligned} &\sum_{i\in\mathbb{K}} \mathcal{E}\{A_{1i}x[k-\tau]\mathbf{1}_{\{\theta(k)=i\}}\mathbf{1}_{\{\theta(k+1)=j\}}\} = \\ &= \sum_{\ell\in\mathbb{K}}\sum_{i\in\mathbb{K}} \mathcal{E}\{A_{1i}x[k-\tau]\mathbf{1}_{\{\theta(k-\tau)=\ell\}}\mathbf{1}_{\{\theta(k)=i\}}\mathbf{1}_{\{\theta(k+1)=j\}}\} \\ &= \sum_{\ell\in\mathbb{K}}\sum_{i\in\mathbb{K}} \Pi_{ij}A_{1i}\mathcal{E}\{x[k-\tau]\mathbf{1}_{\{\theta(k-\tau)=\ell\}}\mathbf{1}_{\{\theta(k)=i\}}\}, \end{aligned} \tag{4.80}$$

in which we used the total expectance law, $\mathcal{E}(X) = \mathcal{E}(\mathcal{E}(X|Y))$. Doing this a second time

$$\begin{aligned} &\sum_{i\in\mathbb{K}} \mathcal{E}\{A_{1i}x[k-\tau]\mathbf{1}_{\{\theta(k)=i\}}\mathbf{1}_{\{\theta(k+1)=j\}}\} = \\ &= \sum_{\ell\in\mathbb{K}}\sum_{i\in\mathbb{K}} \Pi_{\ell i}(\tau)\Pi_{ij}A_{1i}\mathcal{E}\{x[k-\tau]\mathbf{1}_{\{\theta(k-\tau)=\ell\}}\} \\ &= \sum_{\ell\in\mathbb{K}}\sum_{i\in\mathbb{K}} \Pi_{\ell i}(\tau)\Pi_{ij}A_{1i}q_\ell[k-\tau]. \end{aligned} \tag{4.81}$$

For $k < \tau$ we cannot use the same strategy because the Dirac's measure would be undefined for $k < 0$, i.e., the system is deterministic in such situation. The development then becomes

$$\begin{aligned}
&\sum_{i\in\mathbb{K}} \mathcal{E}\{A_{1i}x[k-\tau]\mathbf{1}_{\{\theta(k)=i\}}\mathbf{1}_{\{\theta(k+1)=j\}}\} = \\
= &\sum_{\ell\in\mathbb{K}}\sum_{i\in\mathbb{K}} \Pi_{ij}A_{1i}x[k-\tau]\mathrm{Prob}(\theta(\mathrm{k})=\mathrm{i}|\theta(0)=\ell)\mathrm{Prob}(\theta(0)=\ell) \\
= &\sum_{\ell\in\mathbb{K}}\sum_{i\in\mathbb{K}} \Pi_{\ell i}(k)\Pi_{ij}A_{1i}x[k-\tau]\mu(\ell).
\end{aligned} \tag{4.82}$$

Equations (4.81) and (4.82) can be written in a more compact matricial form. Notice that

$$(I_N \otimes \mathrm{Vec}(A_{1i})) = \begin{bmatrix} \begin{matrix} A_{11} \\ \vdots \\ A_{1N} \end{matrix} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \begin{matrix} A_{11} \\ \vdots \\ A_{1N} \end{matrix} \end{bmatrix} \tag{4.83}$$

and

$$(\mathrm{diag}(\mathrm{vec}(\Pi'(\tau))) \otimes I_n) = \begin{bmatrix} \Pi_{11}(\tau) & & & & & & \\ & \Pi_{11}(\tau) & & & & \mathbf{0} & \\ & & \ddots & & & & \\ & & & \Pi_{12}(\tau) & & & \\ & & & & \ddots & & \\ & \mathbf{0} & & & & \Pi_{NN}(\tau) & \\ & & & & & & \Pi_{NN}(\tau) \end{bmatrix}. \tag{4.84}$$

Left-Multiplying (4.83) by (4.84) and right-multiplying the result by $\hat{q}[k-\tau]$ gives

$$(\mathrm{diag}(\mathrm{vec}(\Pi'(\tau))) \otimes I_n)(I_N \otimes \mathrm{Vec}(A_{1i}))\hat{q}[k-\tau] = \begin{bmatrix} \Pi_{11}(\tau)A_{11}q_1 \\ \vdots \\ \Pi_{1N}(\tau)A_{1N}q_1 \\ \vdots \\ \Pi_{N1}(\tau)A_{11}q_N \\ \vdots \\ \Pi_{NN}(\tau)A_{1N}q_N \end{bmatrix}. \tag{4.85}$$

Noticing also that

$$(\Pi' \otimes I_n) = \begin{bmatrix} \Pi_{11} & & \Pi_{21} & & & \Pi_{N1} & \\ & \ddots & & \ddots & \cdots & & \ddots \\ & & \Pi_{11} & & \Pi_{21} & & & \Pi_{N1} \\ \Pi_{12} & & \Pi_{22} & & & \Pi_{N2} & \\ & \ddots & & \ddots & \cdots & & \ddots \\ & & \Pi_{12} & & \Pi_{22} & & & \Pi_{N2} \\ & \vdots & & \vdots & & \vdots & \\ \Pi_{1N} & & \Pi_{2N} & & & \Pi_{NN} & \\ & \ddots & & \ddots & \cdots & & \ddots \\ & & \Pi_{1N} & & \Pi_{2N} & & & \Pi_{NN} \end{bmatrix}, \tag{4.86}$$

we can finally left-multiply (4.85) by $(\mathbb{1}'_N \otimes (\Pi' \otimes I_n))$ giving for the j-th row of the result exactly the summation notation on the last line of (4.81). The last line of (4.82) can be written in matrix form using a similar approach completing then the proof. □

Remark 4.7. Parallel to the development of this section, similar results were developed in [121] and [122]. Note however that they did not explicitly state the necessity of (4.82) missing the fact that the Dirac measure is not defined for this case. This has theoretical and practical consequences. A precise numerical simulation, for instance, is only possible when this is being taken into consideration.

Example 4.8. For the system (4.71), consider the following matrices

$$\begin{aligned} A_0(1) &= \begin{bmatrix} 0.10 & 0.10 \\ 0.15 & 0.05 \end{bmatrix}, \quad A_0(2) = \begin{bmatrix} 0.50 & 0.05 \\ 0.10 & 0.05 \end{bmatrix}, \\ A_1(1) &= \begin{bmatrix} 0.20 & 0.00 \\ 0.05 & 0.05 \end{bmatrix}, \quad A_1(2) = \begin{bmatrix} 0.05 & 0.50 \\ 0.10 & 0.05 \end{bmatrix}, \end{aligned} \tag{4.87}$$

and

$$\Pi = \begin{bmatrix} 0.5 & 0.5 \\ 0.7 & 0.3 \end{bmatrix}. \tag{4.88}$$

Using Lemma 4.4, it was tested for $\tau \in \{1, 2 \ldots, 10\}$ and it was verified that this system is mean-stable for all those delays. For $\tau = 1$, Figure 4.9 shows that the states in fact converge to zero. This example also shows that the mean value of the state $x[k]$ converges to $q[k]$.

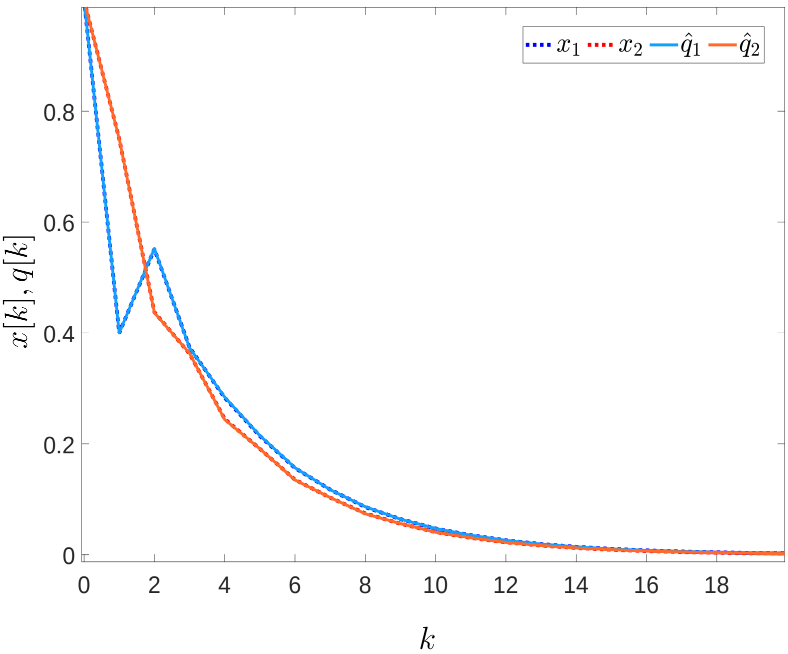

Figure 4.9: $x[k]$ and $\hat{q}[k]$, for $\tau = 1$

4.5 Final Remarks

A new set of LMIs for state feedback of continuous Markovian systems with delays is provided. The main difference to the literature is that, usually, the term for the transition rates is non-linear. Augmented matrices with the square root of those transitions are used for state feedback. For the cases in which these rates are well known this is not an issue. Our method has an affine relation with respect to the transition parameter λ. Due to this linearity, uncertain transition rates can be dealt, applying the method on the vertices of the uncertainty polytope. The chapter proceeds with the introduction of internally positive systems and methods to model the 1^{st}-moment of the state vector of the time-delay system in order to analyse the delay-dependent stability for the original system. Both continuous and discrete-time systems are discussed. Parallel to this work, similar results were published on this matter. Note, however, that as stated on Remark 4.7, some definitions are not the same and they are important for the precise simulation and validation of the results.

Chapter 5

Conclusions

This dissertation contributes to the theory of time-delay systems providing new methods for stability analysis and also for stabilisation of those systems. The work is divided into three parts. On the first segment we extend the procedure for time-delay control design based on a LTI system, called *comparison system*, that provides a lower bound for the $\mathcal{H}_\infty$ norm of the time-delay system. Increasing the order of the Rekasius substitution, this is the first procedure able to make a better use of the buffer necessary for implementing delayed feedback. The method we propose can obtain simultaneously more stability margin and a lower $\mathcal{H}_\infty$ level. Classical routines such as the Riccati equation can be used for design the controller for the time-delay system. Stabilisation by state feedback and by output feedback are discussed, as the implementation of two algorithms, one for minimising the $\mathcal{H}_\infty$ when the delay is given and the other for maximising the maximum delay when a lower bond γ for the $\mathcal{H}_\infty$ is given. Filters can also be design with the methodology presented. The results for state feedback were published in [38] and the ones for output feedback in [39]. Finally, for this first part, we extend the analysis of our procedure for Fractional time-delay systems. *Comparison systems* for fractional systems are developed; we stress that it is possible to approximate the $\mathcal{H}_\infty$ norm of the LTI system for the time-delay system likewise for this case. The state feedback for fractional time-delay systems is under developing and is going to be submitted for publication in the near future.

On the second segment, we developed a new strategy to design an envelope that engulfs all poles of a time-delay system. Using LMIs, we can obtain envelopes that are less conservative than the ones developed by the eigenvalue approach presented in the literature. Furthermore, the novelty here, is that we can use this envelope to study the stability and to design feedback controllers for linear time-delay systems. For retarded systems, we discuss state-feedback design and adapt the method to deal with parametric uncertainties on the system matrices. Design requirements can also be tackled such as pole allocation to the left of the vertical line $s = -\alpha$ with $\alpha > 0$ on the s-plane. Those results were published in [107]. Furthermore, for the first time, an envelope is proposed for neutral time-delay systems. Stability, stabilisation and robustness is discussed for this type of system and the results were published in the journal [108]. The analysis part is extended to fractional time-delay systems. However, the synthesis part cannot be applied directly and it is going to be topic of study for the future. Output feedback using envelopes is also an open problem that is under our attention.

For the third and last part, we deal with stochastic systems. Firstly, for continuous Markovian systems with delays, we propose LMIs, for the $\mathcal{H}_\infty$ state-feedback design, that are affine with respect to the transition rates between different Markov modes. This affine relation allows top incorporate polytopic uncertainty on those parameters. The idea is based in our previous work [110]. The output feedback for the system without delays is discussed in [111] while for the case with delays is a work in progress. A system that models the 1st moment of an internally positive Markov time-delay system is then presented. This new system is used to analyse stability both for continuous and discrete time. Parallel to this work, similar results were published in the literature by [121], [122] and [123]. Nonetheless, we decided to keep the results here due to the following reasons. The results obtained are slightly different and are still very recently. Equation (4.76) seems to be missing on the results published. It is important to stress the necessity of this equation for the case in which $k < \tau$, because the dirac's measure is not defined in this scenario. For time simulations that are both correct and precise this needs to be taken into consideration. Therefore, this is a minor contribution on our part.

Stochastic time-delay systems go beyond the field of the Engineering reaching areas such as finances [13]. Due to intrinsic lag information stochastic equations such as the famous Black-scholes [124] equation can be adapted to take into account the delays such as in [14]. Ito's theory generalised for the scenario with delays is the starting point. Future works are intended to carry on this interesting and challenging line of work. Besides the continuity of all work developed here, as mentioned on the paragraphs above, analysis and synthesis of controllers for stochastic time-delay systems and its wide range of applications, in engineering, in biology and chiefly in finances will be at sight.

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Appendix A

Fractional Systems

A.0.1 Fractional calculus and control

This appendix brings a brief introduction on Fractional systems. The LMIs methods that are presented on Chapters 2 and 3 were extended for those type of systems. Thus, we state some basic concepts about this topic with some generalisations of the results obtained for the classical case. Fractional systems are defined in terms of fractional derivatives. The idea is to extend

$$\frac{d^\alpha}{dt^\alpha} f(t), \tag{A.1}$$

to every α real. The subject started in a conversation by letter between Gottfried Leibniz and Guillaume de l'Hôpital, around 1695. Leibniz wrote in response to the possibility of this generalisation [125]: "*An apparent paradox, from which one day useful consequences will be drawn*". Indeed he was right. Fractional control gives us a new angle to numerous problems. Allowing for instance $\alpha \neq 1$ on a PID gives us a new tunable parameter increasing the control flexibility and permitting to attain better performance levels.

Among the several definitions related to fractional calculus, on this dissertation, we shall choose the Caputo definition [126] as presented below.

$$ {}_c\mathbb{D}_t^\alpha f(t) = \begin{cases} \displaystyle\int_c^t \frac{(t-\tau)^{-\alpha-1}}{\Gamma(-\alpha)} f(\tau) d\tau, & \text{if } \alpha \in \mathbb{R}_-^* \\ f(t), & \text{if } \alpha = 0 \\ {}_c\mathbb{D}_t^{\alpha-\lceil\alpha\rceil} \dfrac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} f(t), & \text{if } \alpha \in \mathbb{R}_+^* \end{cases} \tag{A.2}$$

in which

$$\Gamma(x) = \int_0^{+\infty} e^{-y} y^{x-1} dy \tag{A.3}$$

is the Gamma funtion.

The relevance of Fractional control relies on achieving a performance that would be very hard, or even impossible, to achieve with regular integer order control. The degrees of freedom for design are increased when we drop out the restriction that derivatives must have an integer order. A starter on the subject of Fractional Calculus and Fractional control can be viewed in [127]. Analysis of fractional delay systems both for retarded as for neutral type can be viewed in [128]. Stability and stabilisation of retarded fractional delay differential systems is dealt in [129]. Still for fractional delay systems of neutral type, stabilisation and $\mathcal{H}_\infty$ stability are discussed respectively in [130] and [129]. A PID controller design for fractional-order systems with time delays is studied in [131] and a method for tuning a fractional-PID in [132].

In frequency domain, one can take the unilateral Laplace Transform of (A.2) to get

$$\mathcal{L}\left({}_c\mathbb{D}_t^\alpha f(t)\right) = \begin{cases} s^\alpha F(s), & \text{if } \alpha \in \mathbb{R}_-^* \\ F(s), & \text{if } \alpha = 0 \\ s^\alpha F(s) - \sum\limits_{k=0}^{\lceil\alpha\rceil-1} s^{\alpha-k-1}\mathbb{D}^k f(0), & \text{if } \alpha \in \mathbb{R}_+^*. \end{cases} \tag{A.4}$$

This definition is very interesting, because all initial conditions come from derivatives of integer order, at 0, which, in general, have direct physical meaning and, therefore, are simpler to be obtained.

Moreover, a fractional transfer function can be given by

$$G(s) = \frac{\sum\limits_{k=0}^{m} b_k s^{\beta_k}}{\sum\limits_{k=0}^{n} a_k s^{\alpha_k}}, \tag{A.5}$$

where $\alpha_k, \beta_k \geq 0$, $\alpha_n \neq 0$ and there are no zero-pole cancelations, i.e,. the transfer function is in its minimal realisation. Finally, we also assume that (A.5) is in commensurate form, i.e., α_k and β_k can be expressed as multiples of $1/\nu$, where $\nu \in \mathbb{N}^*$.

A.0.2 Stability

Stability for fractional systems is similar to the well known integer case. The following theorem defines necessary and sufficient conditions for stability of those systems for the case where the transfer function $G(s)$ is commensurate, also known as Matignon's Theorem. This result was first introduced in [133] and further discussed in books such as [127].

Theorem A.1. Let the transfer function $G(s)$ given by (A.5) be commensurate and let σ_k, $k = \{1 \dots n\}$, be the roots of the polynomial $A(\sigma) := \sum\limits_{i=0}^{n} a_i \sigma^i$ built with the coefficients of the denominator of (A.5). Then $G(s)$ is stable if and only if

$$|\angle\sigma_k| > \alpha\pi/2,\ \forall k, \tag{A.6}$$

$|\angle \sigma_k| \in \ ]-\pi, +\pi]$.

Proof. Suppose, first, that $A(\sigma)$ has no roots with multiplicity higher than one, $G(s)$ can be written as a partial fraction expansion:

$$G(s) = \sum_{k=1}^{n} \frac{\rho_k}{s^\alpha - \sigma_k}. \tag{A.7}$$

The impulsive response of (A.7) is given approximately by [127]

$$y(t) \simeq \sum_{k=1}^{n} \frac{\rho_k}{\alpha} \sigma_k^{\frac{1-\alpha}{\alpha}} e^{t\sigma_k^{1/\alpha}}. \tag{A.8}$$

This response goes to zero as time goes to infinity if

$$\Re(\sigma_k^{1/\alpha}) = \Re\left(|\sigma_k^{1/\alpha}|\left(\cos\frac{\angle\sigma_k}{\alpha} + \jmath \sin\frac{\angle\sigma_k}{\alpha}\right)\right) < 0. \tag{A.9}$$

That condition is satisfied when $\cos\frac{\angle\sigma_k}{\alpha} < 0$, i.e., $|\frac{\angle\sigma_k}{\alpha}| > \pi/2$.

If $A(\sigma)$ has roots with a multiplicity higher than one, (A.7) is replaced by

$$G(s) = \sum_{k=1}^{n_d} \sum_{q=1}^{m_k} \frac{\rho_{k,q}}{(s^\alpha - \sigma_k)^q}, \tag{A.10}$$

where n_d is the number of distinct roots and m_k the multiplicity of the root σ_k and the asymptotic impulsive response is now approximately given by [127]

$$y(t) \simeq \sum_{k=1}^{n_d} \sum_{q=1}^{m_k} \sum_{r=0}^{q-1} \frac{\rho_{k,q} a_{r,q-1}(\alpha)}{\alpha^q \Gamma(q)} \sigma_k^{\frac{1-q\alpha+r}{\alpha}} t^r e^{t\sigma_k^{1/\alpha}}. \tag{A.11}$$

Since the exponential tends to zero faster than the power function tends to infinity, all terms will again tend to zero if $\Re(\sigma_k^{1/\alpha}) < 0$, concluding the proof. □

Figure A.1 illustrates the region of stability for $0 < \alpha < 1$.

The methods for analysis and synthesis for those type of systems are discussed in Chapters 2 and 3.

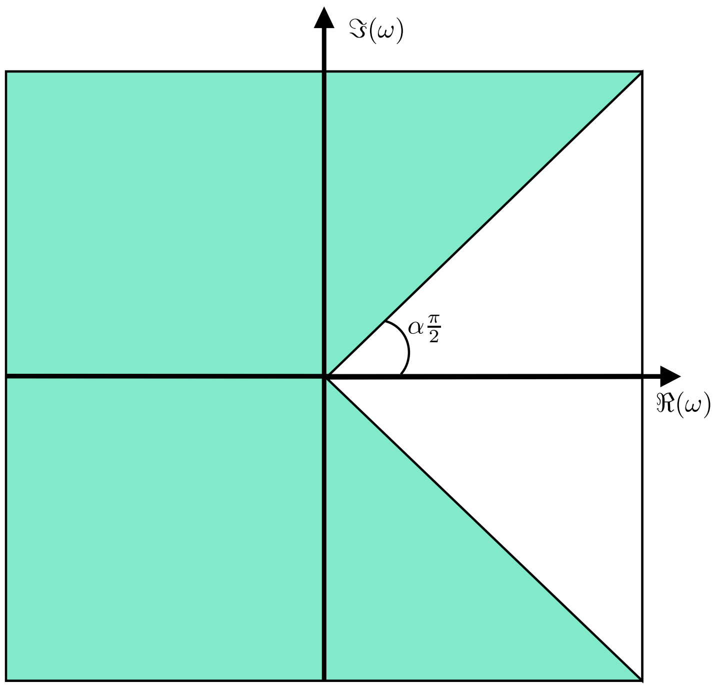

Figure A.1: The ω-stability region for fractional systems

Annexe B

Résumé en Français

B.1 Introduction

Cette thèse vise à étudier des méthodes d'analyse de la stabilité et la synthèse de contrôleurs pour des systèmes dynamiques à retards. Des retards sont intrinsèquement associés a la plupart des systèmes dynamiques. Cela peut être dû à plusieurs raisons, parmi lesquelles nous soulignons : le temps requis pour obtenir des informations nécessaires au contrôle, le temps nécessaire pour transporter des informations, le temps de traitement, la période d'échantillonnage, parmi beaucoup d'autres. De plus, en raison des conditions environnementales, par exemple des températures élevées à l'intérieur d'un compartiment, une méthode utilisée pour commander des systèmes dynamiques est l'approche de contrôle via un réseau [8, 9, 10]. Les contrôleurs mis en œuvre sur un réseau ont intrinsèquement des retards intégrés dans leur structure. Bien que ces retards, dans tous les cas mentionnés, soient souvent négligés, ils peuvent être responsables de mauvaises performances et, dans le pire des cas, conduire le système à l'instabilité. C'est pourquoi plusieurs études sur les systèmes dynamiques avec retards ont été réalisées au cours des dernières décennies.

Les modèles contenant des retards peuvent apparaître dans un nombre raisonnable de processus physiques, biologiques [11, 12], économiques [13, 14], mécaniques [15], et ainsi de suite. Une première étude approfondie sur des retards dans les équations différentielles, connue sous le nom de DDE, en anglais *delay differential equations*, a été réalisée dans [16], tandis que quelques exemples de systèmes avec retard et leur analyse est visible dans [17]. Dans le domaine temporel, une expression générique pour un seul retard est décrite par l'équation différentielle suivante :

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + H\dot{x}(t-\tau) + Ew(t), \\ z(t) &= C_0 x(t) + C_1 x(t-\tau) + D_z w(t), \end{aligned} \tag{B.1}$$

dans lequel, pour tout $t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$ est la variable d'état, $w(t) \in \mathbb{R}^m$ est un sortie exogène, $z(t) \in \mathbb{R}^p$ est la sortie d'intérêt, τ est le retard et A_0, A_1, H, E, C_0, C_1 et D_z sont des matrices réelles de taille appropriée.

Dans le domaine fréquentiel, la fonction de transfert de (B.1) est donnée par

$$T(s,\tau) = \left(C_0 + C_1 e^{-s\tau}\right)\left(sI - A_0 - A_1 e^{-s\tau} - sHe^{-s\tau}\right)^{-1} E + D_z. \tag{B.2}$$

L'équation caractéristique est alors un quasi-polynôme avec une quantité infinis de pôles. Il existe plusieurs manières possibles pour étudier la stabilité et le commande des systèmes avec des retards. La stabilité a été discutée, entre autres, dans [4], [18] et [19]. La mise au point de techniques de conception de contrôle efficaces permettant de traiter des retards a suscité beaucoup d'attention au cours des dernières décennies ; voir les livres [21] et [22] et l'article [23] pour d'importants résultats théoriques dans le domaine. Dans ce contexte, les techniques de commande $\mathcal{H}_\infty$ jouent un rôle clé dans la conception des contrôleurs qui réalisent un gain prédéterminé $\mathbb{L}_2$ pour le système en boucle fermée chaque fois que le retard est donné [24].

Pour la stabilisation par retour d'état, des contrôleurs indépendants du retard peuvent être conçus à l'aide des équations de Riccati [25, 26], tandis que le cas dépendant du retard à travers du fonctionnel de Lyapunov-Krasoviskii [27, 28, 29]. Les fonctionnelles de Lyapunov-Krasoviskii ont également été utilisées pour le commande robuste de systèmes avec retard dans [33]. Les critères de stabilité et de stabilisation robustes ont été discutés dans [41]. Une stabilisation exponentielle robuste pour des systèmes avec des retards variables dans le temps est visible dans [23]. Une stabilité et une stabilisation robustes pour des systèmes singuliers avec des incertitudes paramétriques ont été discutées, entre autres, dans [42] et [43]. La stabilité indépendante du retard pour des systèmes incertains peut être vue dans [44] et la stabilité dépendant du retard dans [45], [46] et [47]. La contrepartie discrète a été étudiée dans [48], pour des systèmes positifs. Le contrôle LQR à coût garanti a été traité dans [49] et le retour de sortie statique polytopique $\mathcal{H}_\infty$ dans [50].

Ce travail est divisé principalement en trois parties. La première propose un système LTI d'ordre fini, appelé *système de comparaison*, qui fournit une limite inférieure a la norme $\mathcal{H}_\infty$ du système à retards et peut être utilisé pour concevoir des commandes par retour d'état, retour de sortie et filtres pour le système d'origine. La deuxième partie de cette thèse introduit une nouvelle approche pour développer une enveloppe englobant tous les pôles d'un système dynamique avec des retards. Cette nouvelle technique utilise des LMIs au lieu de l'approche traditionnelle des valeurs propres. En outre, l'enveloppe proposée peut être utilisée pour analyser la stabilité et concevoir des contrôleurs robustes pour des systèmes à incertitudes paramétriques. La troisième partie traite des retards dans les systèmes markoviens, qui sont une branche particulière des systèmes stochastiques avec la propriété d'être sans mémoire, c'est-à-dire que la probabilité de sauts entre modes ne dépend que du mode actuel du système. Nous traitons le retour d'état pour le cas où les taux de transition ont des incertitudes paramétriques. Nous présentons également un système qui modélise le premier moment d'un système de Markov positif avec des retards et nous l'utilisons pour analyser la stabilité pour des systèmes à temps continu et pour des systèmes à temps discret.

B.2 Système de Comparaison

Considérons le système linéaire avec M retards commensurables, dont la réalisation est donnée par

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{M} \bar{A}_k x\left(t-\bar{\tau}_k\right) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{M} \bar{C}_{zk} x\left(t-\bar{\tau}_k\right), \end{aligned} \tag{B.3}$$

dans lequel, pour tout $t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$ est l'état, $w(t) \in \mathbb{R}^m$ est la sortie exogène, $z(t) \in \mathbb{R}^p$ est la sortie d'intérêt et $\bar{\tau}_k = \tau(M-k+1)/M, k = 1 \ldots M$ pour un retard constant $\tau \geq 0$.

Nous traitons le cas des retards commensurables pour le système (B.3) en appliquant le remplacement suivant :

$$e^{-\tau s} = \left(\frac{\lambda - s}{\lambda + s}\right)^N, \tag{B.4}$$

qui est exact si $s = j\omega$, avec τ, λ, $\omega \in \mathbb{R}_+$ et $N \in \mathbb{N}^*$, tel que

$$\omega\tau = 2N \arctan\left(\frac{\omega}{\lambda}\right). \tag{B.5}$$

Lorsque $N = 1$, cela est connu comme substitution de Rekasius [80]. Nous étendons ce résultat en permettant $N = hM$, $h \in \mathbb{N}^*$. Pour les développements suivants, en relation avec l'analyse de ce système, il faudra que le nombre de retards soit le même que l'ordre de l'approximation (B.4). Notez cependant que chaque fois que $N = hM$ pour un certain $h \in \{1, 2, \ldots\}$, le système (B.3) peut être réécrit de manière équivalente comme

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{N} A_k x\left(t - \tau_k\right) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{N} C_{zk} x\left(t - \tau_k\right), \end{aligned} \tag{B.6}$$

où $A_k \leftarrow \bar{A}_j$, $\tau_k \leftarrow \bar{\tau}_j$, chaque fois que

$$\frac{N-k+1}{N} = \frac{M-j+1}{M}, \tag{B.7}$$

pour tout $k \in \{1 \cdots N\}$, $j \in \{1 \cdots M\}$ et $A_k \leftarrow 0$ sinon. Ainsi, sans perte de généralité, nous continuerons à partir de là toujours avec le système réarrangé (B.6) qui respecte $N = hM$ pour certains $h \in \{1, 2, \ldots\}$.

L'un de nos objectifs est de déterminer le retard le plus grand $\tau^\star > 0$, ce qui garantit la stabilité asymptotique globale du système pour tout $\tau \in [0, \tau^\star)$. Pour cela, il faut analyser la fonction de transfert non rationnel de (B.3), donnée par

$$\begin{aligned} T(s,\tau) &= \left(C_{z0} + \sum_{k=1}^{N} C_{zk} e^{-\tau_k s}\right) \times \\ &\quad \times \left(sI - A_0 - \sum_{k=1}^{N} A_k e^{-\tau_k s}\right)^{-1} E_0. \end{aligned} \tag{B.8}$$

En appliquant la substitution (B.4) dans la fonction de transfert $T(s,\tau)$ dans (B.8), nous pouvons définir un *système de comparaison* avec la fonction de transfert $H(s,\lambda)$ tels que $H(j\omega,\lambda) = T(j\omega,\tau)$, chaque fois que (B.5) est valide. Dans ce cas, la fonction de transfert du système de comparaison est donnée par les lemmas suivants.

Lemme B.1. Pour tout $s \in \mathbb{C}$ fini et matrices $C_k \in \mathbb{R}^{p\times n}$, $A_k \in \mathbb{R}^{n\times n}$ et $E_0 \in \mathbb{R}^{n\times m}$

$$\left(\sum_{k=0}^{N} C_k s^k\right)\left(s^{N+1}I - \sum_{k=0}^{N} A_k s^k\right)^{-1} E_0$$
$$= \begin{bmatrix} C_0' \\ C_1' \\ \vdots \\ C_N' \end{bmatrix}' \left(sI - \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I \\ A_0 & A_1 & A_2 & \cdots & A_{N-1} & A_N \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_0 \end{bmatrix}. \tag{B.9}$$

Lemme B.2. Pour une paire $(\tau, \lambda) \in \mathbb{R}_+$, en utilisant (B.4) et en appliquant Lemma B.1, on peut mettre (B.8) sous la forme équivalente suivante

$$H(s,\lambda) = \left[\begin{array}{c|c} A_\lambda & E \\ \hline C_z & 0 \end{array}\right]$$
$$= \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ \sum_{k=0}^{N} \alpha_k(0) A_k & \sum_{k=0}^{N} A_k \Gamma_k - \lambda \Gamma_\lambda & E_0 \\ \hline \sum_{k=0}^{N} \alpha_k(0) C_{zk} & \sum_{k=0}^{N} C_{zk}\Gamma_k & 0 \end{array}\right], \tag{B.10}$$

dans lequel $\Gamma_k, \Gamma_\lambda \in \mathbb{R}^{n\times Nn}$ sont donnés par

$$\Gamma_k = \begin{bmatrix} \alpha_k(1) & \alpha_k(2) & \alpha_k(3) & \cdots & \alpha_k(N-1) & \alpha_k(N) \end{bmatrix} \otimes I, \tag{B.11}$$
$$\Gamma_\lambda = \begin{bmatrix} \alpha_0(0) & \alpha_0(1) & \alpha_0(2) & \cdots & \alpha_0(N-2) & \alpha_0(N-1) \end{bmatrix} \otimes I, \tag{B.12}$$

et $\alpha_0(i)$, $\alpha_k(i)$, pour $k = 0$ et $k \geq 1$, respectivement, donné par

$$\alpha_0(i) = \binom{N}{i}, \tag{B.13}$$
$$\alpha_k(i) = \sum_{\ell=0}^{k-1} \binom{N-k+1}{i-\ell}\binom{k-1}{\ell}(-1)^{i-\ell}. \tag{B.14}$$

Preuve: La preuve peut être faite en remplaçant l'expression du Rekasius, en développant les binômes et en regroupant les termes dans les pouvoirs de s. □

B.2.1 Calcul de la norme $\mathcal{H}_\infty$

Nous montrons maintenant comment approcher

$$\|T(s,\tau)\|_\infty = \sup_{\omega\in\mathbb{R}} \ \sigma_M(T(j\omega,\tau)) \tag{B.15}$$

pour un $\tau \in [0, \tau^\star)$. L'objectif est de montrer que la fonction de transfert rationnel $H(s, \lambda)$ peut être utilisée pour calculer la norme $\mathcal{H}_\infty$ du système avec des retards.

À la lumière des résultats présentés dans [75], nous extrayons une propriété importante concernant la norme $\mathcal{H}_\infty$ pour le système de comparaison et pour le système original avec des retards. Pour cela, nous devons définir le scalaire $\lambda_o = \inf\{\lambda \mid A_\lambda \text{ est Hurwitz}\}$, et pour chaque $\lambda \in (\lambda_o, \infty)$, nous définissons un $\alpha \geq 0$ tel que,

$$\alpha \in \arg\sup_{\omega\in\mathbb{R}} \sigma_M(H(j\omega, \lambda)). \tag{B.16}$$

Enfin, déterminez le délai $\tau(\lambda, \alpha)$ qui satisfait

$$\alpha/\lambda = \tan(\alpha\tau/2N), \tag{B.17}$$

ce qui nous permet d'affirmer le théorème suivant en étendant le Théorème 1 de [75].

Théorème B.1. Considérons le système (B.6) sans entrées exogènes. Supposons que $\sum_{i=0}^{N} A_i$ est Hurwitz et permet que α soit fourni par (B.16). Si $\tau(\lambda, \alpha) \in [0, \tau^\star)$ est tel que λ satisfait (B.17), conséquemment

$$\|H(s, \lambda)\|_\infty \leq \|T(s, \tau(\lambda, \alpha))\|_\infty. \tag{B.18}$$

B.3 Retour d'état

Dans cette section, nous allons ajouter le contrôle au système réarrangé (B.6), qui devient

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{N} A_k x\left(t - \tau_k\right) + B_0 u(t) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{N} C_{zk} x\left(t - \tau_k\right) + D_{zu} u(t). \end{aligned} \tag{B.19}$$

Notre objectif est de concevoir un contrôle stabilisant sous la forme

$$u(t) = K_0 x(t) + \sum_{k=1}^{N} K_k x\left(t - \frac{N-k+1}{N}\tau\right), \tag{B.20}$$

dans lequel K_k, pour $1 \leq k \leq N$, doit être correctement projeté. La justification de cette approche est basée sur le fait que tant que l'état $x(t-\tau)$ peut être maintenu, si le choix d'une période d'échantillonnage de τ/N est réalisable, il est possible d'utiliser $x(t - \tau/N), x(t - 2\tau/N), \cdots, x(t-\tau)$ pour la mise en oeuvre (B.20).

Les gains inconnus K_k et les scalaires $\alpha_k(i)$, pour $(k, i) \in \{0, \cdots, N\}^2$, peuvent être multiplié comme

$$K = \begin{bmatrix} K_0' \\ K_1' \\ \vdots \\ K_N' \end{bmatrix}' \begin{bmatrix} \alpha_0(0) & \alpha_0(1) & \cdots & \alpha_0(N) \\ \alpha_1(0) & \alpha_1(1) & \cdots & \alpha_1(N) \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_N(0) & \alpha_N(1) & \cdots & \alpha_N(N) \end{bmatrix} \otimes I. \tag{B.21}$$

et le gain de retour d'état $K \in \mathbb{R}^{m\times(N+1)n}$ est exactement ce qui apparaît lorsque l'on ferme la boucle avec le système de comparaison

$$H(s,\lambda) = \left[\begin{array}{c|c} A_\lambda + BK & E \\ \hline C_z + D_{zu}K & 0 \end{array}\right], \tag{B.22}$$

dans lequel les matrices indiquées dans la structure de retour d'état sont définies comme

$$\begin{aligned} A_\lambda &= \begin{bmatrix} 0 & \lambda I \\ \sum_{k=0}^{N} \alpha_k(0) A_k & \sum_{k=0}^{N} A_k \Gamma_k - \lambda \Gamma_\lambda \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \\ C_z &= \begin{bmatrix} \sum_{k=0}^{N} \alpha_k(0) C_{zk} & \sum_{k=0}^{N} C_{zk}\Gamma_k \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ E_0 \end{bmatrix}. \end{aligned} \tag{B.23}$$

Les relations précédentes nous permettent d'énoncer le lemme suivant, qui fournit un résultat important qui doit être exploité pour créer les conditions de conception de la loi de commande par retour d'état (B.20).

Lemme B.3. Pour tout $N \in \mathbb{N}$ et les scalaires $\alpha_k(i)$ définis dans (B.13) et (B.14), la matrice $\tilde{\Gamma} \in \mathbb{N}^{(N+1)\times(N+1)}$, donné par

$$\tilde{\Gamma} = \begin{bmatrix} \alpha_0(0) & \alpha_0(1) & \cdots & \alpha_0(N) \\ \alpha_1(0) & \alpha_1(1) & \cdots & \alpha_1(N) \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_N(0) & \alpha_N(1) & \cdots & \alpha_N(N) \end{bmatrix}, \tag{B.24}$$

est non singulier.

La conséquence la plus importante de ce lemme est la non-singularité de la matrice augmentée $\tilde{\Gamma} \otimes I$, ce qui implique que le gain du retour d'état K_k, $k \in \{0, \cdots, N\}$, peut être obtenu comme

$$\begin{bmatrix} K_0 & K_1 & \cdots & K_N \end{bmatrix} = K \left(\tilde{\Gamma} \otimes I\right)^{-1}. \tag{B.25}$$

Cette identité est d'une grande importance pour la conception de la règle de contrôle (B.20). En fait, notons d'abord que (B.22) représente une fonction de transfert de système LTI standard et que le gain de retour d'état K peut être projeté à l'aide des techniques classiques pour des systèmes de cette classe. En particulier, le gagne de retour d'état $\mathcal{H}_\infty$

$$K = -(D_{zu}' D_{zu})^{-1}(PB + C_z' D_{zu})', \tag{B.26}$$

pour qui $P > 0$ est la solution stabilisatrice de l'équation de Riccati

$$A_\lambda' P + PA_\lambda - (PB + C_z' D_{zu})(D_{zu}' D_{zu})^{-1}(PB + C_z' D_{zu})' + C'C + \gamma^{-2} PEE'P = 0, \tag{B.27}$$

garantit non seulement la stabilité de la fonction de transfert, mais également la limite $\|H(s,\lambda)\|_\infty \leq \gamma$; voir [81] pour plus de détails. Ainsi, l'identité (B.25), ainsi que le système de comparaison et le gain de retour d'état $\mathcal{H}_\infty$ central, peuvent être utilisés pour déterminer les gains K_k, $k \in \{0, \cdots, N\}$, pour $N \geq 1$, en permettant régler deux importants problèmes des systèmes avec des retards :

- **Problème du retard maximum** : Pour un niveau $\mathcal{H}_\infty$ prédéfini γ, cherchez les gains de retour d'état en maximisant le retard τ de sorte que $T(s,\tau)$ soit stable et $\|T(s,\tau)\|_\infty \leq \gamma$;
- **Problème de la norme minimum** : Pour un retard prédéfini τ, cherchez les gains de retour d'état en minimisant le niveau $\mathcal{H}_\infty$ γ tel que $T(s,\tau)$ est stable et $\|T(s,\tau)\|_\infty \leq \gamma$.

Il est intéressant de noter que, puisque les deux algorithmes sont centrés sur des méthodes basées sur Riccati, ils sont viables sur le plan du calcul.

Exemple B.1. Pour illustrer la conception du retour d'état, considérons un exemple de second ordre emprunté à [28] où les matrices correspondant à la réalisation de l'espace d'état (B.19) sont les suivantes :

$$\begin{bmatrix} A_0 \mid A_1 \mid E_0 \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -0.9 & 1 & 0 \end{array}\right],$$

$$\begin{bmatrix} B_0 \mid C_{z0} \mid C_{z1} \mid D_{zu} \end{bmatrix} = \left[\begin{array}{c|cc|cc|c} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0.1 \end{array}\right].$$

Notre objectif principal avec cet exemple simple est de souligner l'importance du gain $K_i \neq 0$, $i \geq 1$ pour l'amélioration des performances en comparant nos résultats avec ceux de [28] et de [79], où les lois de commande par retour d'état sous la forme $u(t) = K_0 x(t)$ et $u(t) = K_0 x(t) + K_1 x(t-\tau)$ ont été projetées. Pour chaque $N \in \{1, \ldots, 9\}$ nous calculons $\tau_\gamma = \tau(\lambda_\gamma)$, comme le montre la Figure B.1. Dans [28], pour approximativement la même valeur de γ et $\tau = 0.999$, le gain K_0 donné a un grand module (10^6), tandis que dans [79], le retard maximum de $\tau = 1,28$ a été obtenu en respectant le niveau de la norme souhaité. Pour tous les $N \in \{1, \ldots, 9\}$ les contrôleurs proposés garantissent la stabilité de $\tau(\lambda) \in [0 \;\; 1.5708)$. Il est intéressant de noter que, pour $N = 8$, notre méthode est en mesure de garantir le niveau prescrit de la norme $\mathcal{H}_\infty$ pour plus de 99% de l'intervalle complet de la stabilité du retard.

B.4 Enveloppes

Un système dynamique avec des retards a une quantité infinie des pôles. L'utilisation d'une enveloppe garantissant que tous les pôles sont contenus dedans cette enveloppe a été décrite dans [4]. Différents types d'enveloppes ont également été abordés dans [76] et [77]. Dans ces cas, les méthodes utilisées pour définir les enveloppes n'ont pas été utilisées pour tester la stabilité ou pour concevoir des contrôleurs. En fait, l'enveloppe s'étend généralement au demi-plan droit et ne constitue qu'une région où les pôles peuvent être sans garantie de la stabilité du système. Dans ce travail, l'enveloppe est conçue au moyen d'inégalités matricielles linéaires (LMIs) et fournit une procédure permettant de tester la stabilité robuste des systèmes du type retardé et du type neutre. De plus, un contrôleur de retour d'état robuste peut être conçu.

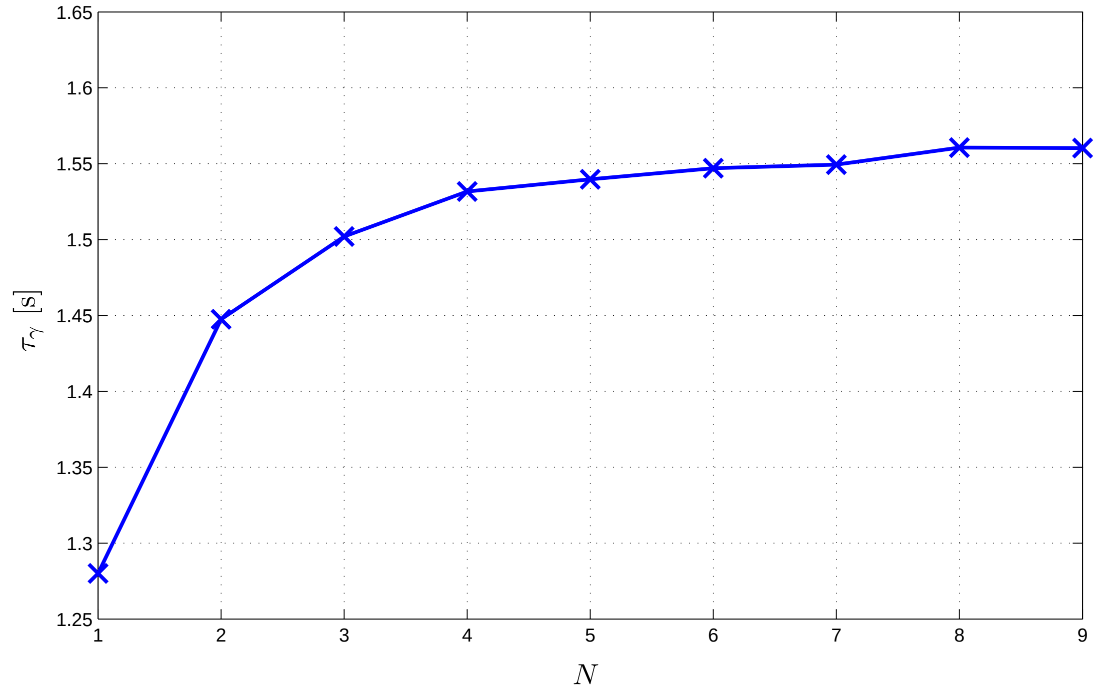

FIGURE B.1: τ_γ en tant que fonction de N pour $\gamma = 0.13$.

B.5 Systèmes retardés

Considérons le système linéaire du type retardé avec N retards, dont la réalisation minimale est donnée par

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x\left(t - \tau_i\right), \tag{B.28}$$

dans lequel $x(t) \in \mathbb{R}^n$ est la variable d'état, $0 = \tau_0 < \tau_1 < \cdots < \tau_N$ sont les retards et $A_i \in \mathbb{R}^{n \times n}$ pour tous $i \in \{0, \ldots, N\}$. Ce système est exponentiellement stable si et seulement si toutes les racines de son équation caractéristique

$$\det\left(sI - \sum_{i=0}^{N} A_i e^{-s\tau_i}\right) = 0 \tag{B.29}$$

sont dans le demi-plan gauche ouvert [16].

La proposition suivante introduit une enveloppe qui englobe tous ses pôles.

Proposition B.1. Soit λ n'importe quel nombre réel. S'il existe des matrices $T = T' > 0$,

$Q_i = Q_i' > 0$, pour tout $i \in \{0, \ldots, N\}$ et un scalaire μ satisfaisant

$$\mu T \geq \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} \tag{B.30}$$

et

$$\begin{bmatrix} T & T & \ldots & T \\ & Q_0 & & \\ & & \ddots & \\ & & & Q_N \end{bmatrix} \geq 0, \tag{B.31}$$

puis, toute racine s_0 de l'équation caractéristique (B.29) telle que $s_0 = \lambda + \jmath\omega$ satisfasse

$$|s_0| \leq \sqrt{\mu}. \tag{B.32}$$

Preuve: L'inégalité suivante est toujours vraie, ce qui est facilement vérifiable en appliquant le complément de Schur

$$\begin{bmatrix} A_i Q_i A_i' e^{-2\lambda\tau_i} & \bullet \\ A_i' e^{-(\lambda-\jmath\omega)\tau_i} & Q_i^{-1} \end{bmatrix} \geq 0. \tag{B.33}$$

En les ajoutant pour tout $i \in \{0, \ldots, N\}$ conduit à

$$\begin{bmatrix} \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} & \bullet \\ \sum_{i=0}^{N} A_i' e^{-(\lambda-\jmath\omega)\tau_i} & \sum_{i=0}^{N} Q_i^{-1} \end{bmatrix} \geq 0, \tag{B.34}$$

où nous pouvons appliquer le complément de Schur et utiliser (B.30) pour obtenir

$$\mu T \geq \Sigma \left(\sum_{i=0}^{N} Q_i^{-1} \right)^{-1} \Sigma^*, \tag{B.35}$$

où $\Sigma \triangleq \sum_{i=0}^{N} A_i e^{-(\lambda+\jmath\omega)\tau_i}$.

Notez que de (B.31)

$$T \geq \sum_{i=0}^{N} T Q_i^{-1} T. \tag{B.36}$$

Maintenant, en multipliant (B.31), par la gauche et par la droite, par T^{-1} et en prenant l'inverse des deux côtés de l'inégalité, on obtient

$$T \leq \left(\sum_{i=0}^{N} Q_i^{-1} \right)^{-1}. \tag{B.37}$$

Puis, en utilisant ce résultat dans (B.35), cela implique que

$$\mu T \geq \Sigma T \Sigma^*. \tag{B.38}$$

Enfin, supposons que $s_0 = \lambda + j\omega$ soit une valeur propre de Σ associée à un vecteur propre à droite v. Il est bien connu, [87] et [88], que les valeurs propres à gauche et à droite sont égales. Par conséquent, s_0 est également une valeur propre de Σ associée à un vecteur propre à gauche x_L, avec la dimension $1 \times n$. Dans ce cas, on peut multiplier l'inégalité (B.38) à gauche par x_L et à droite par sa transposée conjuguée, x_L^*, en obtenant

$$\mu x_L T x_L^* \geq x_L \Sigma T \Sigma^* x_L^* \tag{B.39}$$

et puisque $x_L \neq 0$ et $T > 0$,

$$\mu \geq (\lambda + \jmath\omega)(\lambda - \jmath\omega), \tag{B.40}$$

menant à

$$|s_0| \leq \sqrt{\mu}, \tag{B.41}$$

qui conclut la preuve. □

Ce résultat produit une meilleure enveloppe que les travaux précédents tels que [4].

B.5.1 Mise en œuvre

Tout d'abord, introduisons la définition de la fermeture d'une enveloppe. Soit μ et λ définis par la Proposition B.1 et $\lambda \in [\lambda_{\min},\ \lambda_{\max}]$. S'il existe un point λ^* dans cet intervalle tel que $\mu = (\lambda^*)^2$, nous définissons $\lambda^* + \varepsilon$, avec $\varepsilon > 0$ arbitrairement petit, comme le point de fermeture de l'enveloppe. Cela signifie que l'enveloppe se trouve complètement à gauche de la ligne verticale $\Re(s) = \lambda^* + \varepsilon$. De plus, on dit que l'enveloppe est fermée chaque fois que $\mu < \lambda^2$. Le choix de $\lambda_{\min}$ est totalement libre. Dans [76], une simple limitant a été donnée pour la racine la plus à droite de (B.29), qui peut facilement être généralisée à N retards :

$$\Re(s) \leq \bar{\mu}(A_0) + \sum_{i=1}^{N} \|A_i\| = \ell, \tag{B.42}$$

où $\bar{\mu}(.)$ est une mesure matricielle, voir [76] et [90]. Nous suggérons de prendre $\lambda_{\max} = 2|\ell|$.

La proposition suivante montre comment décrire l'enveloppe, ainsi que son comportement en fonction de λ.

Proposition B.2. Soit $\lambda \in [\lambda_{\min},\ \lambda_{\max}]$ et soit μ donné par Proposition B.1. Si $\mu \geq \lambda^2$, l'enveloppe sur le plan complexe est définie par l'ensemble des points (λ, ω) où $\omega = \pm\sqrt{\mu - \lambda^2}$. Si pour un λ^* particulier, $\mu^* < (\lambda^*)^2$, d'ailleurs l'enveloppe est fermée pour chaque $\lambda > \lambda^*$.

Preuve: De l'équation (B.41) nous avons $\lambda^2 + \omega^2 \leq \mu$ qui implique directement que $\omega = \pm\sqrt{\mu - \lambda^2}$, pour $\mu \geq \lambda^2$. Évidemment, (λ, ω) appartient à l'enveloppe. Supposons maintenant que pour un certain λ^*, nous avons $\mu^* < (\lambda^*)^2$. Comme $A_i Q_i A_i' \geq 0$ et $e^{-2\lambda\tau_i}$ ne croissent pas, nous avons $\mu < \mu^*$ pour chaque $\lambda > \lambda^*$ qui signifie , par définition, que l'enveloppe est fermée.
□

Bien que cette enveloppe soit plus étroite que [4], pour $\lambda = 0$, il résulte de (B.30) que $\mu \geq 0$, et par conséquent, l'enveloppe n'est jamais fermé sur le demi-plan gauche, ce qui implique que la stabilité ne peut pas être évaluée avec l'enveloppe sous cette forme. Pour contourner ce problème, nous proposons un changement de coordonnées avec la nouvelle variable $s = z - d$, avec $d > 0$, et calculons ensuite l'enveloppe pour z. Avec ce changement de variables, (B.29) devient

$$\det\left(zI - (A_0 + dI) - \sum_{i=1}^{N} A_i e^{-z\tau_i} e^{d\tau_i}\right) = 0, \tag{B.43}$$

nous permettant de travailler avec un problème équivalent sur les nouveaux paramètres

$$\begin{aligned} \tilde{A}_0 &= A_0 + dI, \\ \tilde{A}_i &= A_i e^{d\tau_i}, \text{ for all } i \in \{1, \dots, N\}. \end{aligned} \tag{B.44}$$

Sur le plan z, l'enveloppe restera ouverte pour $\lambda = 0$. Toutefois, si elle est fermée avant $z = d$, elle sera fermée avant l'origine sur le plan s, garantissant ainsi la stabilité pour le système d'origine.

Exemple B.2. Considérez les matrices système suivantes

$$\left[\begin{array}{c|c} A_0 & A_1 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0.5413 \\ -2 & -3 & -1.0827 & -1.6240 \end{array}\right].$$

La figure B.2 montre la comparaison de notre enveloppe avec [4], elle illustre également le comportement de l'enveloppe pour différentes valeurs de d. Une observation intéressante est que, pour $d = 3$, nous obtenons une enveloppe plus étroite plus près des pôles et nous pouvons également voir que le point auquel l'enveloppe se termine se trouve à gauche du plan. Cela nous permet d'utiliser l'enveloppe comme critère de stabilité, comme on le verra dans la section stabilisation. Tous les pôles du système ici et tout au long de ce travail ont été calculés via [92].

La stabilité

Voyons maintenant comment utiliser l'enveloppe pour analyser la stabilité d'un système à retards.

Proposition B.3. Soit $\lambda_0 \in \mathbb{R}$, $\mu = \lambda_0^2 - \varepsilon$, pour certains $\varepsilon > 0$. S'il existe $T, Q_i > 0$, pour tout $i \in \{0, \dots, N\}$ tels que (B.30) et (B.31) sont tous les deux satisfaits, alors l'enveloppe se trouve entièrement à gauche de l'axe vertical, traversant λ_0.

B.5.2 Retour d'état pour systèmes retardés

Nous abordons maintenant le problème de la stabilisation. Considérons le système

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - \tau_i) + Bu(t), \tag{B.45}$$

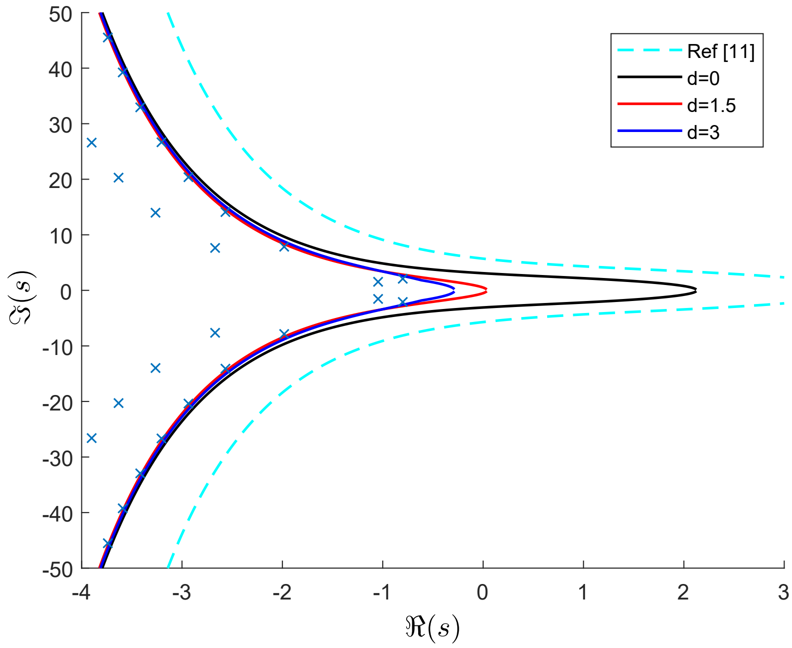

FIGURE B.2: Enveloppes pour différentes valeurs de d

que nous voulons que ce soit contrôlé au moyen d'une loi de contrôle par retour d'état

$$u(t) = \sum_{i=0}^{N} K_i x\left(t - \tau_i\right) \in \mathbb{R}^m, \tag{B.46}$$

à être conçu par des LMIs. Ce contrôleur répond aux exigences du projet, c'est-à-dire, α-stabilité, et ajoute un certain degré de robustesse au système en boucle fermée. Comme on le verra, le contrôleur peut être sans mémoire, c'est-à-dire que $K_i \leftarrow 0$, $\forall i \in \{1, \ldots, N\}$ ou ne peut même utiliser que certains des états retardés.

Théorème B.2. Considérons le système avec des retards (B.45). S'il y a des matrices $T = T' > 0$, $Q_i = Q_i' > 0$, Y_i $\forall i \in \{0, \ldots, N\}$ et des scalaires positifs d, ε, avec $\mu = d^2 - \varepsilon$, $\lambda = d$, de sorte que

$$\begin{bmatrix} \mu T & \left(\tilde{A}_0 Q_0 + B_0 Y_0\right) e^{-\lambda \tau_0} & \ldots & \left(\tilde{A}_N Q_N + B_N Y_N\right) e^{-\lambda \tau_N} \\ \bullet & Q_0 & 0 & 0 \\ \bullet & 0 & \ddots & 0 \\ \bullet & 0 & 0 & Q_N \end{bmatrix} \geq 0 \tag{B.47}$$

et (B.31) sont tous satisfaits, où $\tilde{A}_i$ est donné par (B.44) et $B_i = Be^{d\tau_i}$ pour chaque $i \in \{0, \ldots, N\}$, par conséquent, la loi de contrôle de retour d'état (B.46) où les matrices du contrôleur sont donnés par $K_i = Y_i Q_i^{-1}$, stabilise le système.

Preuve: En appliquant le complément de Schur dans (B.47), nous obtenons exactement (B.30) avec $A_i \leftarrow \tilde{A}_i + B_i K_i$, ce qui complète la preuve. □

B.6 Systèmes neutres

Notre objectif ici est de développer les enveloppes pour des systèmes du type neutre. Considérons le système linéaire à retard du type neutre avec $N+1$ retards, dont la réalisation minimale est donnée par

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x\left(t-\tau_i\right) + H\dot{x}\left(t-\tau_h\right), \tag{B.48}$$

dans lequel $x(t) \in \mathbb{R}^n$ est la variable d'état, $0 = \tau_0 < \tau_1 < \cdots < \tau_N$ et τ_h sont les retards, $A_i \in \mathbb{R}^{n\times n}$, pour tous $i \in \{0,\ldots,N\}$ et H sont matrices réelles. Une condition nécessaire à la stabilité exponentielle de ce système est que toutes les racines de l'équation caractéristique

$$\det\left(sI - \sum_{i=0}^{N} A_i e^{-s\tau_i} - sHe^{-s\tau_h}\right) = 0, \tag{B.49}$$

sont sur le côté gauche d'une ligne verticale $\Re(s) = -\alpha$, avec $\alpha > 0$ [101].

Proposition B.4. Soit λ n'importe quel nombre réel. S'il y a des matrices $T = T' > 0$, $Q_i = Q_i' > 0, \forall i \in \{0,\ldots,N\}$, $Q_h = Q_h' > 0$ et un scalaire μ tel que

$$\mu T \geq \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} + \mu H Q_h H' e^{-2\lambda\tau_h} \tag{B.50}$$

et

$$\begin{bmatrix} T & T & \cdots & & T \\ & Q_0 & & & \\ & & \ddots & & \\ & & & Q_N & \\ & & & & \frac{\mu}{|s_0|^2} Q_h \end{bmatrix} > 0, \tag{B.51}$$

par conséquent, toute racine s_0 de l'équation caractéristique (B.49) telle que $s_0 = \lambda + j\omega$ satisfasse

$$|s_0| \leq \sqrt{\mu}. \tag{B.52}$$

Preuve: L'inégalité suivante est toujours vraie, ce qui est facilement vérifiable en appliquant le complément de Schur

$$\begin{bmatrix} HQ_hH'e^{-2\lambda\tau_h} & \bullet \\ s_0^* H' e^{-(\lambda - \jmath\omega)\tau_h} & |s_0|^2 Q_h^{-1} \end{bmatrix} \geq 0. \tag{B.53}$$

En multipliant les deux côtés par $\mathbf{diag}(\sqrt{\mu}, \frac{1}{\sqrt{\mu}})$ puis en ajoutant le résultat à (B.34), nous obtenons

$$\begin{bmatrix} \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} + \mu H Q_h H' e^{-2\lambda\tau_h} & \bullet \\ \sum_{i=0}^{N} A_i' e^{-(\lambda-\jmath\omega)\tau_i} + s_0^* H' e^{-(\lambda-\jmath\omega)\tau_h} & \sum_{i=0}^{N} Q_i^{-1} + \frac{|s_0|^2}{\mu} Q_h^{-1} \end{bmatrix} \geq 0, \tag{B.54}$$

où nous pouvons appliquer le complément de Schur et utiliser (B.50) pour obtenir

$$\mu T \geq \Sigma \left(\sum_{i=0}^{N} Q_i^{-1} + \frac{|s_0|^2}{\mu} Q_h^{-1} \right)^{-1} \Sigma^*, \tag{B.55}$$

où $\Sigma \triangleq \sum_{i=0}^{N} A_i e^{-(\lambda+\jmath\omega)\tau_i} + s_0 H e^{-(\lambda+\jmath\omega)\tau_h}$.

De plus, de (B.51) nous avons que

$$T > \sum_{i=0}^{N} T Q_i^{-1} T + \frac{|s_0|^2}{\mu} T Q_h^{-1} T, \tag{B.56}$$

ce qui implique

$$T < \left(\sum_{i=0}^{N} Q_i^{-1} + \frac{|s_0|^2}{\mu} Q_h^{-1} \right)^{-1}. \tag{B.57}$$

Par conséquent, en utilisant ce résultat dans (B.55), cela implique que

$$\mu T \geq \Sigma T \Sigma^*. \tag{B.58}$$

En procédant de la même manière que le cas retardé, en multipliant l'inégalité (B.58) à gauche par x_L et à droite par x_L^*, où x_L est le vecteur propre de Σ associé à la valeur propre s_0, conduit à

$$|s_0| \leq \sqrt{\mu}. \tag{B.59}$$

□

La difficulté lors de l'application de cette méthode réside dans la nécessité de connaître $|s_0|$ pour mettre en œuvre le LMI (B.54). Néanmoins, en conséquence de ce résultat $\mu \geq |s_0|^2$, nous avons $\frac{\mu}{|s_0|^2} \geq 1$, ce qui implique que si

$$\begin{bmatrix} T & T & \dots & & T \\ & Q_0 & & & \\ & & \ddots & & \\ & & & Q_N & \\ & & & & Q_h \end{bmatrix} > 0 \tag{B.60}$$

est satisfait, l'inégalité (B.51) est également vrai. Par conséquent, (B.60) peut être utilisé à la place de (B.51).

B.6.1 Mise en œuvre

Maintenant, nous pouvons faire une fois de plus le changement des variables $s = z - d$, avec $d > 0$ et calculer l'enveloppe pour z. Cela forcera l'enveloppe à se déplacer vers la gauche du plan s. Avec ce changement de variable, (B.49) devient

$$\det\left(zI - (A_0 + dI) - \sum_{k=1}^{N} A_k e^{-z\tau_k} e^{d\tau_k} - zHe^{-z\tau_h} e^{d\tau_h} + dHe^{-z\tau_h} e^{d\tau_h}\right) = 0, \tag{B.61}$$

nous permettant de travailler sur les nouvelles variables

$$\det\left(zI - \tilde{A}_0 - \sum_{k=1}^{N+1} \tilde{A}_k e^{-z\tau_k} - z\tilde{H}e^{-z\tau_h}\right) = 0, \tag{B.62}$$

dans lequel

$$\begin{aligned} \tilde{A}_0 &= A_0 + dI, \\ \tilde{A}_i &= A_i e^{d\tau_i}, \forall i \in \{0, \dots, N\}, \\ \tilde{A}_{N+1} &= -dHe^{d\tau_h}, \\ \tilde{H} &= He^{d\tau_h}. \end{aligned} \tag{B.63}$$

Cela nous permet de réaliser la même technique que celle utilisée pour des systèmes retardés.

B.6.2 Retour d'état pour systèmes neutres

Nous pouvons maintenant adapter le résultat des systèmes du type retardé aux systèmes du type neutre décrits par

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - \tau_i) + H\dot{x}(t - \tau_h) + Bu(t). \tag{B.64}$$

Théorème B.3. Considérons le système avec des retards (B.64). S'il y a des matrices $T = T' > 0$, $Q_i = Q_i' > 0$, $\forall i \in \{0, \dots, N+1\}$, Y_i, $\forall i \in \{0, \dots, N\}$, $Q_h = Q_h' > 0$ et des scalaires positifs d, ε, avec $\mu = d^2 - \varepsilon$, $\lambda = d$ tel que

$$\begin{bmatrix} \mu T & \left(\tilde{A}_0 Q_0 + B_0 Y_0\right) e^{-\lambda\tau_0} & \dots & \left(\tilde{A}_N Q_N + B_N Y_N\right) e^{-\lambda\tau_N} & \left(\tilde{H} Q_h + B_h Y_h\right) e^{-\lambda\tau_h} \\ \bullet & Q_0 & 0 & 0 & 0 \\ \bullet & 0 & \ddots & 0 & 0 \\ \bullet & 0 & 0 & Q_N & 0 \\ \bullet & 0 & 0 & 0 & \frac{1}{\mu} Q_h \end{bmatrix} \geq 0, \tag{B.65}$$

et (B.51) sont tous satisfaits, où $\tilde{H}$ et $\tilde{A}_i$ pour chaque $i \in \{0, \ldots, N+1\}$ sont donnés par (B.44) et $B_i = Be^{d\tau_i}$ pour chaque $i \in \{0, \ldots, N\}$ puis le contrôleur de retour d'état (B.46), obtenue avec les matrices de gain $K_i = Y_i Q_i^{-1}$ $\forall i \in \{0, \ldots, N\}$, stabilise le système.

Preuve: En appliquant le complément de Schur dans (B.65) nous obtenons exactement (B.50) avec $A_i \leftarrow \tilde{A}_i + B_i K_i$ et $H \leftarrow \tilde{H} + B_h K_h$, qui complète la preuve. □

Exemple B.3. Pour les matrices

$$\left[\begin{array}{c|c} A_0 & A_1 \end{array}\right] = \left[\begin{array}{cc|cc} -1.7073 & 0.6856 & -2.5026 & -1.0540 \\ 0.2279 & -0.6368 & -0.1856 & -1.5715 \end{array}\right] \tag{B.66}$$

et

$$H = \begin{bmatrix} 0.0558 & 0.0360 \\ 0.2747 & -0.1084 \end{bmatrix}, \tag{B.67}$$

voir [99], [104], [105] et [106], avec $\tau_1 = \tau_h = 2$ et en appliquant le Théorème B.3, nous avons conçu le contrôleur suivant

$$\begin{aligned} K_0 &= \begin{bmatrix} -37.7924 & -20.7712 \end{bmatrix}, \\ K_1 &= \begin{bmatrix} 5.3363 & 3.7375 \end{bmatrix}, \end{aligned}$$

ce qui assure la stabilité pour tous les retards. Nous illustrons l'enveloppe pour $\tau_1 = \tau_h = 2$ dans la Figure B.3.

B.7 Systèmes de Markov continus avec des retards

Un modèle MJLS à temps continu à retard est décrit par le modèle d'espace d'états stochastique suivant

$$\dot{x}(t) = A_0(\theta_t)x(t) + A_1(\theta_t)x(t-\tau), \tag{B.68}$$

dans lequel $x \in \mathbb{R}^n$ est la variable d'état, $\theta_t \in \mathbb{K}$ est une variable aléatoire et $\tau > 0$ est le retard. Les conditions initiales sont $x(t) = x_0(t)$ pour $t \in [-\tau, 0]$ et la distribution de probabilité initiale $p(0)$. Nous supposons que les matrices A_{0i}, pour $i \in \mathbb{K}$, sont Metzler et que les matrices A_{i1}, pour $i \in \mathbb{K}$, sont non-négatives. Cela garantit que le système (B.68) est un système positif et par conséquent, pour toute condition initiale positive, $x_0(t) > 0$, $t \in [-\tau, 0]$, le vecteur d'état $x(t)$ reste non-négatif pour tout $\theta(t)$ et tout $t \geq 0$.

Le processus $\{\theta_t, t \in [0, +\infty)\}$ est un processus markovien stochastique où

$$\begin{aligned} p_{ij}(\Delta) = &\ \text{Prob}(\theta_{t+\Delta} = j | \theta_t = i) \\ = &\ \begin{cases} \lambda_{ij}\Delta + o(\Delta) & i \neq j, \\ 1 + \lambda_{ii}\Delta + o(\Delta) & i = j, \end{cases} \end{aligned} \tag{B.69}$$

où $\lambda_{ij} \geq 0$ pour $i \neq j$, $\lambda_{ii} \leq 0$ et

$$\sum_{j \in \mathbb{K}} \lambda_{ij} = 0. \tag{B.70}$$

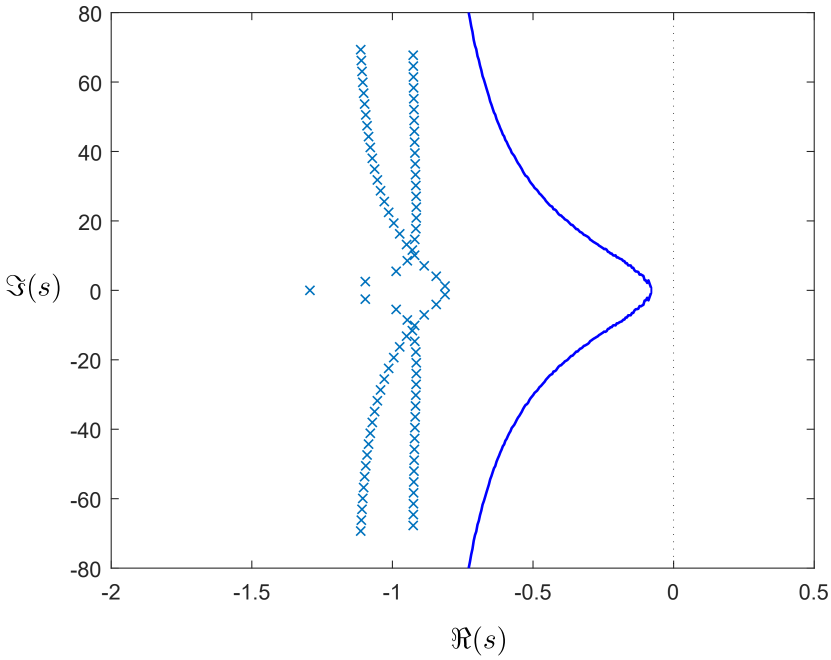

FIGURE B.3: Retour d'état - Type neutre - $\tau_1 = \tau_h = 2$

On peut ainsi définir la matrice de transition $\Lambda = [\lambda_{ij}]$ qui présente tous les taux de transition entre les états de la chaîne de Markov représentés par $\theta_t \in \mathbb{K}$.

Pour obtenir les résultats de convergence pour le premier moment de la variable d'état $x(t)$, définissons, pour chaque $t \in \mathbb{R}_+$,

$$q(t) \triangleq \mathcal{E}(x(t)) \in \mathbb{R}^n. \tag{B.71}$$

Définissons aussi

$$q_i(t) \triangleq \mathcal{E}(x(t)\mathbf{1}_{\{\theta(t)=i\}}) \in \mathbb{R}^n, \tag{B.72}$$

$$\hat{q}(t) \triangleq \left[q_1(t) \ \ldots \ q_N(t)\right]' \in \mathbb{R}^{Nn}. \tag{B.73}$$

Le lemme suivant fournit une équation différentielle à retard capable de calculer le premier moment de la variable d'état $x(t)$.

Lemme B.4. Pour tout t, $\tau \in \mathbb{R}_+$, le 1^{er}-moment de $x(t)$ de (B.68) est modélisé par l'équation différentielle à retard suivante :

$$\dot{\hat{q}}(t) = F\hat{q}(t) + G(\tau)\hat{q}(t-\tau), \tag{B.74}$$

dans laquelle $\hat{q}$ est donné par (B.73) et F, $G(\tau)$ sont donnés par

$$\begin{aligned} F &= \Lambda' \otimes I_n + \operatorname{diag}(A_{0i}), \\ G(\tau) &= \operatorname{diag}(A_{1i})(\Pi(\tau)' \otimes I_n). \end{aligned} \tag{B.75}$$

Preuve: Nous procédons en suivant les étapes de [78]. En appliquant la règle de Itô à la première équation de (B.72), nous avons

$$\begin{aligned} dq_j(t) &= \mathcal{E}\{dx(t)\mathbf{1}_{\{\theta(t)=j\}} + x(t)d\mathbf{1}_{\{\theta(t)=i\}}\} \\ &= A_{0j}\mathcal{E}\{x(t)\mathbf{1}_{\{\theta(t)=j\}}\}dt + A_{1j}\mathcal{E}\{x(t-\tau)\mathbf{1}_{\{\theta(t)=j\}}\}dt + \textstyle\sum_{i\in\mathbb{K}}\lambda_{ij}q_i(t)dt \\ &= (A_{0j}q_j(t) + \textstyle\sum_{i\in\mathbb{K}}\lambda_{ij}q_i(t))dt + A_{1j}\mathcal{E}\{\textstyle\sum_{i\in\mathbb{K}} x(t-\tau)\mathbf{1}_{\{\theta(t)=j\}}\mathbf{1}_{\{\theta(t-\tau)=i\}}\}dt \\ &= (A_{0j}q_j(t) + \textstyle\sum_{i\in\mathbb{K}}\lambda_{ij}q_i(t) + A_{1j}\textstyle\sum_{i\in\mathbb{K}}\Pi_{ij}(\tau)q_i(t-\tau))dt, \end{aligned} \tag{B.76}$$

et ainsi (B.74) suit. □

Remarque B.1. Il est essentiel de noter que, si toutes les matrices A_{0i}, pour $i \in \mathbb{K}$, sont Metzler et si toutes les matrices A_{i1}, pour $i \in \mathbb{K}$, sont non-négatives, ainsi F est Metzler et $G(\tau)$ est non-négatif. Cela implique que le système à retard (B.74) est positif. L'inverse, en revanche, n'est pas toujours vrai.

La propriété présentée dans Remarque B.1 sera d'une grande importance pour la conception d'une méthode numérique capable de fournir les fenêtres de stabilité moyenne du système de Markov positif avec retard.

Le résultat suivant associe la stabilité moyenne du système stochastique positif avec retard (B.68) à la stabilité du système à retard déterministe (B.74).

Lemme B.5. Considérons le MJLS positif avec retard à temps continu donné par (B.68). Ce système est exponentiellement stable en moyenne si, et seulement si, le système à retard (B.74) est exponentiellement stable.

Preuve: Il découle directement du fait que $\mathcal{E}\{x(t)\} = \sum_{i\in\mathbb{K}} q_i(t)$ et de la définition de $\hat{q}(t)$. □

Pour tout retard constant $\tau \geq 0$, la stabilité du système (B.74) est donnée par la position de ses pôles. Il existe un certain nombre de procédures numériques capables de déterminer la stabilité [73] et les fenêtres de stabilité [118] pour de tels systèmes. Néanmoins, étant donné que les matrices F et $G(\tau)$, associées à la condition initiale, définissent un système de retard positif, un test plus simple peut être utilisé :

Lemme B.6. Un système à retard positif

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau), \tag{B.77}$$

dans lequel A_0 est Metzler et A_1 est une matrice non-négative, est stable indépendamment des retards si et seulement si $A_0 + A_1$ est Hurwitz.

Preuve: La preuve est basée sur l'existence d'un vecteur positif p et du fonctionnel de Lyapunov-Krasovskii

$$V(\Psi) = p'\Psi(0) + \int_{-\tau}^{0} p'A_1\Psi(\theta)d\theta. \tag{B.78}$$

Voir [119] pour tous les détails. □

Le lemme précédent implique un résultat fort pour un système linéaire déterministe positif avec retard. Il indique que, si le système sans retard (c'est-à-dire, $\tau = 0$) est stable, il est stable pour toutes les valeurs positives du retard, ce qui est connu sous le nom de stabilité indépendante du retard. Cela implique également que si le système sans retard est instable, il en va de même pour chaque valeur positive de τ. Par conséquent, pour une telle classe de système, nous ne pouvons pas trouver de fenêtre de stabilité, un effet commun pour des systèmes linéaires à retard.

Remarque B.2. Tous ces résultats peuvent facilement être étendus au cas des retards multiples. En fait, si un système est décrit par les équations stochastiques

$$\dot{x}(t) = A_0(\theta_t)x(t) + \sum_{k=1}^{N} A_k(\theta_t)x(t - \tau_k), \tag{B.79}$$

dans lequel $\tau_k > 0$, $A_0(\theta_t)$ est Metzler et $A_k(\theta_t)$ est non-négatif, pour chaque $\theta_t \in \mathbb{K}$ et $k \in \{1, \ldots, N\}$, alors les mêmes procédures peuvent être effectuées pour montrer que ce système est stable en moyenne si, et seulement si, la matrice $F + \sum_{j=1}^{M} G(\tau_j)$, où F et G sont définis dans (B.75), est Hurwitz.

Exemple B.4. Comme premier exemple, considérons un MJLS positif avec les matrices dans l'espace d'états donné par

$$\begin{aligned} A_0(1) &= \begin{bmatrix} -1.0 & 0.4 \\ 0 & -2.8 \end{bmatrix}, \quad A_0(2) = \begin{bmatrix} -0.9 & 0 \\ 0 & -0.1 \end{bmatrix}, \\ A_1(1) &= \begin{bmatrix} 0.5 & 0.2 \\ 1.3 & 0.7 \end{bmatrix}, \quad A_1(2) = \begin{bmatrix} 0.2 & 0.1 \\ 1.7 & 0.1 \end{bmatrix}, \end{aligned} \tag{B.80}$$

et

$$\Lambda = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{B.81}$$

En appliquant Lemme B.4, comme le montre la Figure B.4, ce système est stable pour $\tau \in [0, 0.12)$ et instable en dehors de cet intervalle. Cet exemple illustre un cas où le retard a un effet déstabilisateur sur la stabilité moyenne du système.

Exemple B.5. Pour le deuxième exemple, considérons le MJLS positif avec les matrices dans l'espace d'états donné par

$$\begin{aligned} A_0(1) &= \begin{bmatrix} -3.9 & 0.4 \\ 0.2 & -1.9 \end{bmatrix}, \quad A_0(2) = \begin{bmatrix} -1.5 & 0.3 \\ 0.4 & -3.2 \end{bmatrix}, \\ A_1(1) &= \begin{bmatrix} 1.3 & 1.4 \\ 0.1 & 1.1 \end{bmatrix}, \quad A_1(2) = \begin{bmatrix} 1.9 & 0.4 \\ 0.8 & 1.0 \end{bmatrix}, \end{aligned} \tag{B.82}$$

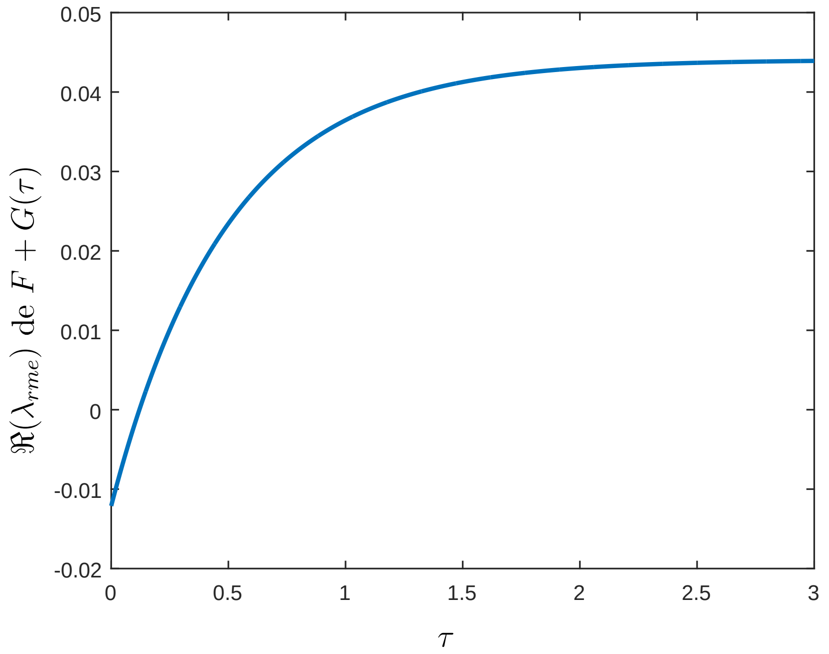

FIGURE B.4: 1er exemple : Partie réelle de la valeur propre la plus à droite de $F+G(\tau)$

et la même matrice Λ que dans l'exemple précédent. Dans ce cas, la figure B.5 montre que le système est instable pour $\tau = 0$ et que la stabilité en moyenne n'est atteinte que pour $\tau > 0.2957$. Cet exemple illustre une situation dans laquelle le retard sert d'effet stabilisateur pour le système.

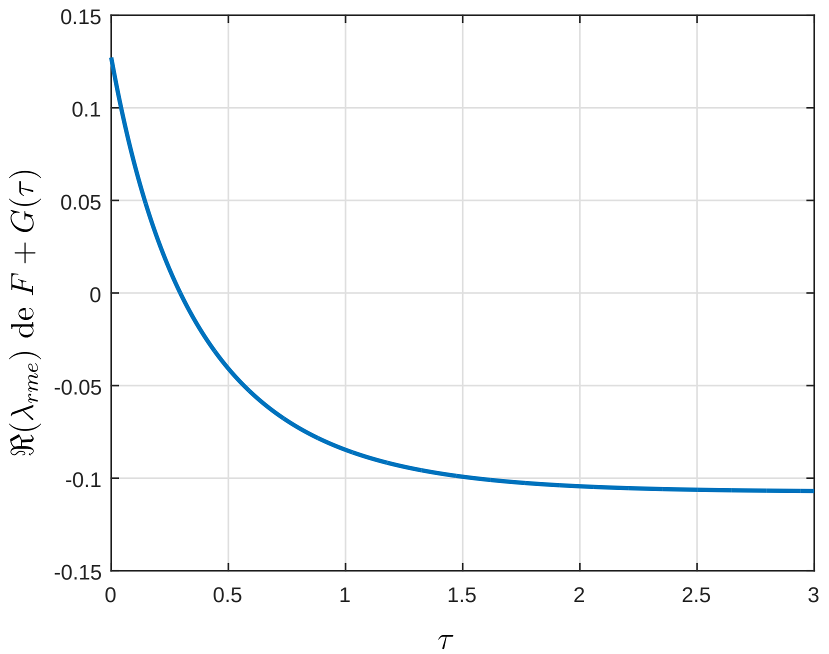

FIGURE B.5: 2ème exemple : Partie réelle de la valeur propre la plus à droite de $F+G(\tau)$

Pour deux valeurs différentes du retard ($\tau = 2.0$, à l'intérieur de la région de stabilité et $\tau = 0.1$, à l'extérieur de la région de stabilité), nous avons effectué une simulation de Monte-Carlo avec 5000 réalisations, la moyenne pour chaque variable d'état est présentée dans les figures B.6 et B.7, respectivement. Ces résultats confirment l'analyse de stabilité moyenne réalisée.

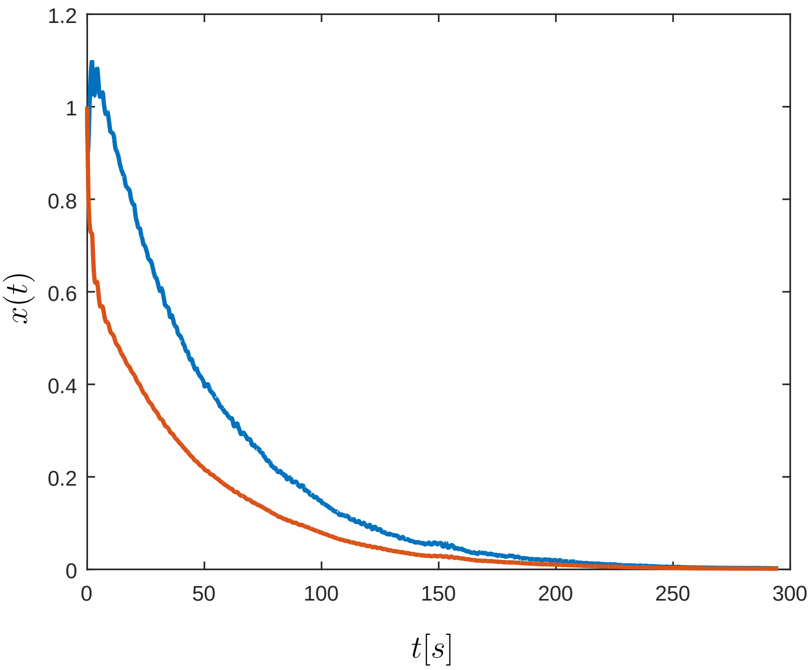

FIGURE B.6: Valeur attendue pour les variables d'état - $\tau = 2$

B.8 Conclusions

Cette thèse contribue à la théorie des systèmes à retards en proposant de nouvelles méthodes d'analyse de la stabilité et de stabilisation de ces systèmes. Le travail est divisé principalement en trois parties. Sur le premier segment, nous étendons la procédure de conception de contrôle basée sur un système LTI, appelé *système de comparaison*, qui fournit une limite inférieure pour le norme $\mathcal{H}_\infty$ du système avec retards. En augmentant l'ordre de la substitution de Rekasius, il s'agit de la première procédure permettant de mieux utiliser le mémoire tampon nécessaire à la mise en oeuvre du retour d'état. La méthode que nous proposons permet d'obtenir simultanément plus de marge de stabilité et un niveau inférieur de la norme $\mathcal{H}_\infty$. Des routines classiques telles que l'équation de Riccati peuvent être utilisées pour concevoir le contrôleur pour le système à retards. On discute la stabilisation par retour d'état et par retour de sortie, comme la mise en œuvre de deux algorithmes, l'un pour minimiser la norme $\mathcal{H}_\infty$ lorsque le retard est donné et l'autre pour maximiser le retard maximum quand une limite inférieur γ pour le norme $\mathcal{H}_\infty$ est donné. Les filtres peuvent également être conçus avec la méthodologie présentée. Les résultats concernant le retour d'état ont été publiés dans [38] et ceux concernant le retour de sortie dans [39]. Enfin, pour cette première partie, nous étendons l'analyse de notre procédure pour des systèmes fractionnaires avec des retards. *Systèmes de*

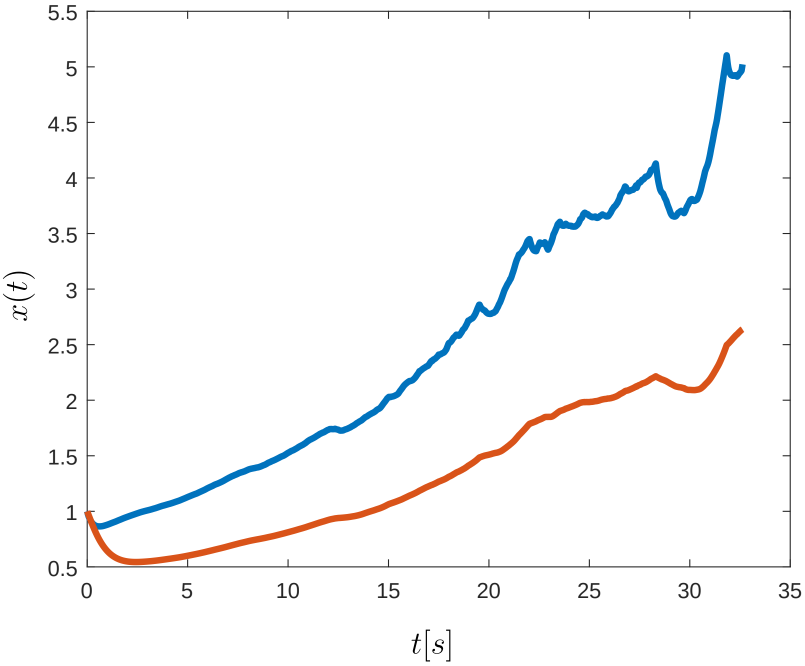

FIGURE B.7: Valeur attendue pour les variables d'état - $\tau = 0.1$

comparaison pour des systèmes fractionnaires sont développés ; nous soulignons qu'il est possible d'approcher la norme $\mathcal{H}_\infty$ du système LTI pour le système fractionnaires à retard dans ce cas aussi.

Sur le deuxième segment, nous développons une nouvelle stratégie pour concevoir une enveloppe englobant tous les pôles d'un système à retard. En utilisant des LMIs, nous pouvons obtenir des enveloppes moins conservatrices que celles développées par l'approche des valeurs propres présentée dans la littérature. De plus, la nouveauté réside dans le fait que nous pouvons utiliser cette enveloppe pour étudier la stabilité et concevoir des contrôleurs par retour d'état pour des systèmes linéaire avec retards. Pour des systèmes retardés, nous discutons de la conception à retour d'état et adaptons la méthode pour traiter les incertitudes paramétriques sur les matrices du système. Les exigences de conception peuvent également être abordées, telles que l'allocation des pôles à gauche de la ligne verticale $s = -\alpha$ avec $\alpha > 0$ sur le plan s. Ces résultats ont été publiés dans [107]. De plus, pour la première fois, une enveloppe est proposée pour des systèmes du type neutre. La stabilité, la stabilisation et la robustesse sont discutées pour ce type de système et les résultats ont été publiés dans la revue [108]. La partie de l'analyse est étendue aux systèmes à retard fractionnaires. Cependant, la partie de la synthèse ne peut pas être appliquée directement et elle fera l'objet d'une étude pour l'avenir.

Pour la troisième et dernière partie, nous traitons des systèmes stochastiques. Premièrement, pour des systèmes markoviens continus avec des retards, nous proposons des LMIs, pour la conception à retour d'état $\mathcal{H}_\infty$, qui sont affines en ce qui concerne les taux de transition entre différents modes de Markov. Cette relation affine permet d'incorporer une incertitude polytopique sur ces paramètres. L'idée est basée sur notre précédent travail [110]. Par la suite, nous développons un système qui modélise le premier moment d'un système de Markov positif avec retard. Ce nouveau système est utilisé pour analyser la stabilité du système markovien en temps continu et en temps discret.

Les travaux futurs sont destinés à poursuivre ce travail intéressant et stimulant. Outre la continuité de tous les travaux développés ici, comme mentionné dans les paragraphes ci-dessus, l'analyse et la synthèse des contrôleurs pour des systèmes stochastiques à retards et son large éventail d'applications, en ingénierie, en biologie et principalement en finances, seront à vue.

Apêndice C

Resumo em Português

C.1 Introdução

Esta tese tem como objetivo o estudo de métodos de análise de estabilidade e de síntese de controladores para sistemas dinâmicos com atrasos. Atrasos no tempo estão intrinsecamente associados a quase todos os sistemas dinâmicos. Isto pode ser devido a diversas razões, entre as quais destacamos: o tempo necessário para adquirir as informações necessárias para o controle, o tempo necessário para transportar informações, o tempo de processamento, o período de amostragem, entre muitos outros. Além disso, devido a condições ambientais, por exemplo, altas temperaturas dentro de um compartimento, um método que está sendo usado para controlar sistemas dinâmicos é a abordagem de controle através de uma rede de comunicação [8, 9, 10]. Controladores implementados através de uma rede têm, intrinsecamente, atrasos embutidos em sua estrutura. Embora esses atrasos, em todos os casos mencionados, sejam muitas vezes negligenciados, eles podem ser responsáveis por um desempenho insuficiente e, em piores cenários, podem até levar o sistema à instabilidade. Por essa razão, diversos estudos que consideram os chamados sistemas com atraso no tempo foram feitos nas últimas décadas.

Modelos contendo atrasos podem aparecer em uma quantidade razoável de processos físicos, biológicos [11, 12], econômicos [13, 14], mecânicos [15] e assim por diante. Um primeiro estudo extensivo sobre atrasos em equações diferenciais, conhecido como DDEs, do inglês *delay differential equations*, foi feito em [16] enquanto alguns exemplos de sistemas com atraso no tempo tanto quanto sua análise podem ser vistos em [17]. No domínio do tempo, uma expressão genérica para apenas um atraso é descrita pela seguinte equação diferencial:

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-\tau) + H\dot{x}(t-\tau) + Ew(t), \\ z(t) &= C_0x(t) + C_1x(t-\tau) + D_zw(t), \end{aligned} \tag{C.1}$$

na qual, para todo $t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$ é a variável de estado, $w(t) \in \mathbb{R}^m$ é uma saída exógena, $z(t) \in \mathbb{R}^p$ é a saída de interesse, τ é o atraso e A_0, A_1, H, E, C_0, C_1 e D_z são matrizes reais de dimensões apropriadas.

No domínio da freqüência, a função de transferência de (C.1) é dada por

$$T(s,\tau) = \left(C_0 + C_1e^{-s\tau}\right)\left(sI - A_0 - A_1e^{-s\tau} - sHe^{-s\tau}\right)^{-1}E + D_z. \tag{C.2}$$

A equação característica é então um quase-polinômio com infinitos polos. Existem várias estruturas possíveis para estudar a estabilidade e o controle de sistemas com atraso no tempo. A estabilidade foi discutida, entre outros, em [4], [18] e [19]. O desenvolvimento de técnicas eficientes de projeto de controle que lidam com o atraso têm recebido muita atenção nas últimas décadas; veja os livros [21] e [22] e o artigo [23] para importantes resultados teóricos na área. Neste contexto, as técnicas de controle $\mathcal{H}_\infty$ desempenham um papel fundamental no projeto de controladores que atingem um ganho $\mathbb{L}_2$ pré-especificado para o sistema de malha fechada sempre que o atraso é dado [24]. Para a estabilização através de realimentação de estado, controladores independentes do atraso podem ser projetados usando as equações de Riccati [25, 26], enquanto o caso dependente do atraso por meio dos funcionais de Lyapunov-Krasoviskii em [27, 28, 29].

Os funcionais de Lyapunov-Krasoviskii também foram utilizados para o controle robusto de sistemas com atraso em [33]. Critérios para estabilidade e estabilização robustas foram tratados em [41]. Estabilização exponencial robusta para sistemas com atrasos variáveis no tempo pode ser vista em [23]. Estabilidade robusta e estabilização para sistemas singulares com incertezas paramétricas foram discutidas, entre outras, em [42] e [43]. Estabilidade independente do atraso para sistemas incertos pode ser vista em [44] e estabilidade dependente do atraso em [45], [46] e [47]. A contraparte discreta foi estudada em [48], para sistemas positivos. O controle LQR de custo garantido foi tratado em [49] e a realimentação estática de saída politópica $\mathcal{H}_\infty$ em [50].

Este trabalho é dividido principalmente em três partes. A primeira propõe um sistema LTI de ordem finita, chamado de sistema de comparação, do inglês *comparison system*, que fornece um limitante inferior para a norma $\mathcal{H}_\infty$ do sistema com atraso e pode ser usado para projetar controladores por realimentação de estado, controladores de realimentação de saída e filtros para o sistema original. A segunda parte desta tese introduz uma nova abordagem para desenvolver um envelope que engloba todos os polos de um sistema dinâmico com atraso no tempo. Essa nova técnica utiliza LMIs em vez da tradicional abordagem por autovalor. Além disso, o envelope proposto pode ser usado para analisar a estabilidade e para projetar controladores robustos para sistemas com incertezas paramétricas. A terceira parte, trata de atrasos em sistemas markovianos, que são uma ramificação particular de sistemas estocásticos com a propriedade de serem sem memória, isto é, a probabilidade de saltos entre modos depende apenas do modo atual do sistema. Tratamos a realimentação de estado para o caso em que as taxas de transição possuem incertezas paramétricas. Apresentamos ainda, um sistema que modela o primeiro momento de um sistema positivo com atraso, tanto em tempo contínuo como em tempo discreto, podendo ser usado para análise de estabilidade no sentido médio.

C.2 Sistema de Comparação

Considere o sistema linear com M atrasos comensuráveis, cuja realização é dada por

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{M} \bar{A}_k x\left(t - \bar{\tau}_k\right) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{M} \bar{C}_{zk} x\left(t - \bar{\tau}_k\right), \end{aligned} \tag{C.3}$$

na qual, para todo $t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$ é o estado, $w(t) \in \mathbb{R}^m$ é a saída exógena, $z(t) \in \mathbb{R}^p$ é a saída de interesse e $\bar{\tau}_k = \tau(M-k+1)/M, k = 1 \dots M$, para algum atraso constante $\tau \geq 0$.

Nós lidamos com o caso de atrasos comensuráveis para o sistema (C.3) aplicando a seguinte substituição:

$$e^{-\tau s} = \left(\frac{\lambda - s}{\lambda + s}\right)^N, \tag{C.4}$$

que é uma relação exata para $s = j\omega$, com τ, λ, $\omega \in \mathbb{R}_+$ e $N \in \mathbb{N}^*$ tal que

$$\omega\tau = 2N \arctan\left(\frac{\omega}{\lambda}\right). \tag{C.5}$$

Quando $N = 1$ isto é conhecido como substituição de Rekasius [80]. Estendemos este resultado permitindo $N = hM$, $h \in \mathbb{N}^*$. Para os seguintes desenvolvimentos, em relação à análise deste sistema, será necessário que o número de atrasos seja o mesmo que a ordem da aproximação (C.4). Observe, no entanto, que sempre que $N = hM$ para algum $h \in \{1, 2, \dots\}$, o sistema (C.3) pode ser reescrito de forma equivalente

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{N} A_k x\left(t - \tau_k\right) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{N} C_{zk} x\left(t - \tau_k\right), \end{aligned} \tag{C.6}$$

onde $A_k \leftarrow \bar{A}_j$, $\tau_k \leftarrow \bar{\tau}_j$ sempre que

$$\frac{N-k+1}{N} = \frac{M-j+1}{M}, \tag{C.7}$$

para todo $k \in \{1 \cdots N\}$, $j \in \{1 \cdots M\}$ e $A_k \leftarrow 0$ caso contrário. Portanto, sem perda de generalidade, prosseguiremos a partir de então sempre com o sistema rearranjado (C.6) respeitando $N = hM$ para algum $h \in \{1, 2, \dots\}$.

Um de nossos objetivos é o de determinar o maior atraso $\tau^\star > 0$ que garante que o sistema seja globalmente assintoticamente estável para qualquer $\tau \in [0, \tau^\star)$. Para alcançar isso, deve-se analisar a função de transferência não racional de (C.3), dada por

$$\begin{aligned} T(s,\tau) &= \left(C_{z0} + \sum_{k=1}^{N} C_{zk} e^{-\tau_k s}\right) \times \\ &\quad \times \left(sI - A_0 - \sum_{k=1}^{N} A_k e^{-\tau_k s}\right)^{-1} E_0. \end{aligned} \tag{C.8}$$

Aplicando a substituição (C.4) na função de transferência $T(s,\tau)$ em (C.8), podemos definir um *sistema de comparação* com função de transferência $H(s,\lambda)$ tal que $H(j\omega,\lambda) = T(j\omega,\tau)$, sempre que (C.5) for válida. Neste caso, a função de transferência do sistema de comparação é dada pelos seguintes lemas.

Lema C.1. Para qualquer $s \in \mathbb{C}$ finito e matrizes $C_k \in \mathbb{R}^{p\times n}$, $A_k \in \mathbb{R}^{n\times n}$ e $E_0 \in \mathbb{R}^{n\times m}$

$$\left(\sum_{k=0}^{N} C_k s^k\right)\left(s^{N+1}I - \sum_{k=0}^{N} A_k s^k\right)^{-1} E_0$$
$$= \begin{bmatrix} C_0' \\ C_1' \\ \vdots \\ C_N' \end{bmatrix}' \left(sI - \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I \\ A_0 & A_1 & A_2 & \cdots & A_{N-1} & A_N \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_0 \end{bmatrix}. \tag{C.9}$$

Lema C.2. Para um determinado par $(\tau,\, \lambda) \in \mathbb{R}_+$, usando (C.4) e aplicando o Lemma C.1, pode-se colocar (C.8) na seguinte forma equivalente

$$H(s,\lambda) = \left[\begin{array}{c|c} A_\lambda & E \\ \hline C_z & 0 \end{array}\right]$$
$$= \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ \sum_{k=0}^{N} \alpha_k(0) A_k & \sum_{k=0}^{N} A_k \Gamma_k - \lambda \Gamma_\lambda & E_0 \\ \hline \sum_{k=0}^{N} \alpha_k(0) C_{zk} & \sum_{k=0}^{N} C_{zk}\Gamma_k & 0 \end{array}\right], \tag{C.10}$$

na qual $\Gamma_k, \Gamma_\lambda \in \mathbb{R}^{n\times Nn}$ são dados por

$$\Gamma_k = \begin{bmatrix} \alpha_k(1) & \alpha_k(2) & \alpha_k(3) & \cdots & \alpha_k(N-1) & \alpha_k(N) \end{bmatrix} \otimes I, \tag{C.11}$$
$$\Gamma_\lambda = \begin{bmatrix} \alpha_0(0) & \alpha_0(1) & \alpha_0(2) & \cdots & \alpha_0(N-2) & \alpha_0(N-1) \end{bmatrix} \otimes I, \tag{C.12}$$

e $\alpha_0(i)$, $\alpha_k(i)$, para $k = 0$ e $k \geq 1$, respectivamente, dados por

$$\alpha_0(i) = \binom{N}{i}, \tag{C.13}$$
$$\alpha_k(i) = \sum_{\ell=0}^{k-1} \binom{N-k+1}{i-\ell}\binom{k-1}{\ell}(-1)^{i-\ell}. \tag{C.14}$$

.

Prova: A prova pode ser feita substituindo a expressão de Rekasius, expandindo os binomiais e reagrupando os termos nas potências de s. □

C.2.1 Cálculo da norma $\mathcal{H}_\infty$

Mostraremos agora como aproximar

$$\|T(s,\tau)\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_M(T(j\omega,\tau)) \tag{C.15}$$

para um dado $\tau \in [0, \tau^\star)$. O objetivo é mostrar que a função de transferência racional $H(s, \lambda)$ pode ser usada para o cálculo da norma $\mathcal{H}_\infty$.

À luz dos resultados apresentados em [75], extraímos uma propriedade importante a respeito da norma $\mathcal{H}_\infty$ tanto para o sistema de comparação quanto para o sistema original com atrasos. Para este fim, precisamos definir o escalar $\lambda_o = \inf\{\lambda \mid A_\lambda \text{ Hurwitz}\}$ e para cada $\lambda \in (\lambda_o, \infty)$, definimos um $\alpha \geq 0$ tal que,

$$\alpha \in \arg \sup_{\omega \in \mathbb{R}} \sigma_M(H(j\omega, \lambda)). \tag{C.16}$$

Por fim, determinar o atraso $\tau(\lambda, \alpha)$ que satisfaz

$$\alpha/\lambda = \tan(\alpha\tau/2N), \tag{C.17}$$

nos permite declarar o seguinte teorema, estendendo o Teorema 1 de [75].

Teorema C.1. Considere o sistema (C.6) sem entradas exógenas. Suponha que $\sum_{i=0}^{N} A_i$ seja Hurwitz e permita que α seja fornecido por (2.35). Se $\tau(\lambda, \alpha) \in [0, \tau^\star)$ for tal que λ satisfaça (2.36), então

$$\|H(s, \lambda)\|_\infty \leq \|T(s, \tau(\lambda, \alpha))\|_\infty. \tag{C.18}$$

C.3 Realimentação de estado

Nesta seção, vamos adicionar o controle para o sistema rearranjado (C.6), que se torna

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{N} A_k x\left(t - \tau_k\right) + B_0 u(t) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{N} C_{zk} x\left(t - \tau_k\right) + D_{zu} u(t). \end{aligned} \tag{C.19}$$

Nosso objetivo é projetar um controle estabilizante sob a forma

$$u(t) = K_0 x(t) + \sum_{k=1}^{N} K_k x\left(t - \frac{N-k+1}{N}\tau\right), \tag{C.20}$$

em que os ganhos K_k, para $1 \leq k \leq N$, devem ser adequadamente projetados. O raciocínio para essa abordagem é baseado no fato de que, contanto que o estado $x(t-\tau)$ possa ser armazenado, se a escolha de um período de amostragem de τ/N for viável, é possível utilizar os estados $x(t - \tau/N), x(t - 2\tau/N), \cdots, x(t-\tau)$ presentes em um buffer de memória para implementar (C.20).

Os ganhos desconhecidos K_k e os escalares $\alpha_k(i)$, para $(k, i) \in \{0, \cdots, N\}^2$, podem ser multiplicados como

$$K = \begin{bmatrix} K_0' \\ K_1' \\ \vdots \\ K_N' \end{bmatrix}' \begin{bmatrix} \alpha_0(0) & \alpha_0(1) & \cdots & \alpha_0(N) \\ \alpha_1(0) & \alpha_1(1) & \cdots & \alpha_1(N) \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_N(0) & \alpha_N(1) & \cdots & \alpha_N(N) \end{bmatrix} \otimes I, \tag{C.21}$$

e ganho de realimentação de estado $K \in \mathbb{R}^{m\times(N+1)n}$ é exatamente o que aparece quando fechamos a malha com o sistema de comparação

$$H(s,\lambda) = \left[\begin{array}{c|c} A_\lambda + BK & E \\ \hline C_z + D_{zu}K & 0 \end{array}\right], \tag{C.22}$$

em que as matrizes indicadas na estrutura de realimentação do estado são definidas como

$$A_\lambda = \begin{bmatrix} 0 & \lambda I \\ \sum_{k=0}^{N} \alpha_k(0)A_k & \sum_{k=0}^{N} A_k\Gamma_k - \lambda\Gamma_\lambda \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix},$$
$$C_z = \begin{bmatrix} \sum_{k=0}^{N} \alpha_k(0)C_{zk} & \sum_{k=0}^{N} C_{zk}\Gamma_k \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ E_0 \end{bmatrix}. \tag{C.23}$$

As relações anteriores nos permitem afirmar o seguinte lema, que fornece um resultado importante que deve ser explorado para produzir condições de projeto para a lei de controle por realimentação de estado (C.20).

Lema C.3. Para qualquer $N \in \mathbb{N}$ e para os escalares $\alpha_k(i)$ definidos em (C.13) e (C.14), a matriz $\tilde{\Gamma} \in \mathbb{N}^{(N+1)\times(N+1)}$, dada por

$$\tilde{\Gamma} = \begin{bmatrix} \alpha_0(0) & \alpha_0(1) & \cdots & \alpha_0(N) \\ \alpha_1(0) & \alpha_1(1) & \cdots & \alpha_1(N) \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_N(0) & \alpha_N(1) & \cdots & \alpha_N(N) \end{bmatrix} \tag{C.24}$$

é não singular.

A consequência mais importante deste lema é a não-singularidade da matriz aumentada $\tilde{\Gamma} \otimes I$, o que implica que o ganho da realimentação de estado K_k, $k \in \{0, \cdots, N\}$, pode ser obtido em

$$\begin{bmatrix} K_0 & K_1 & \cdots & K_N \end{bmatrix} = K\left(\tilde{\Gamma} \otimes I\right)^{-1}. \tag{C.25}$$

Essa identidade é de grande importância para o projeto da regra de controle (C.20). De fato, primeiro observamos que (C.22) representa uma função de transferência do sistema linear invariante no tempo padrão e, assim, o ganho de realimentação K pode ser projetado usando técnicas clássicas para sistemas dessa classe. Em particular, o ganho $\mathcal{H}_\infty$ de realimentação de estado

$$K = -(D_{zu}'D_{zu})^{-1}(PB + C_z'D_{zu})', \tag{C.26}$$

no qual $P > 0$ é a solução estabilizante da equação de Riccati

$$A'_\lambda P + PA_\lambda - (PB + C'_z D_{zu})(D'_{zu} D_{zu})^{-1}(PB + C'_z D_{zu})' + C'C + \gamma^{-2} PEE'P = 0, \quad \text{(C.27)}$$

garante não apenas a estabilidade da função de transferência, mas também o limite $\|H(s,\lambda)\|_\infty \leq \gamma$; veja [81] para mais detalhes. Assim, a identidade (C.25), juntamente com o sistema de comparação e o ganho $\mathcal{H}_\infty$ de realimentação central, podem ser usados para se determinar os ganhos K_k, $k \in \{0, \cdots, N\}$, para $N \geq 1$, sendo possível tratar dois problemas importantes para sistemas com atraso:

- **Problema do atraso máximo**: Para um limitante $\mathcal{H}_\infty$ pré-determinado γ, encontrar o ganho de realimentação de estado que maximiza o atraso τ tal que $T(s,\tau)$ é estável e $\|T(s,\tau)\|_\infty \leq \gamma$;
- **Problema da norma mínima**: Para um atraso τ pré-determinado, encontrar o ganho de realimentação de estado que minimiza o nível $\mathcal{H}_\infty$, γ, tal que $T(s,\tau)$ é estável e $\|T(s,\tau)\|_\infty \leq \gamma$.

É interesse ressaltar que, uma vez que ambos os algoritmos são centrados em métodos baseados em equações de Riccati, eles são viáveis do ponto de vista computacional.

Exemplo C.1. Para ilustrar o projeto de realimentação de estado, consideramos um exemplo de segunda ordem de [28] onde as matrizes correspondentes à realização em espaço de estados (C.19) são as seguintes

$$\begin{bmatrix} A_0 \mid A_1 \mid E_0 \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -0.9 & 1 & 0 \end{array}\right],$$

$$\begin{bmatrix} B_0 \mid C_{z0} \mid C_{z1} \mid D_{zu} \end{bmatrix} = \left[\begin{array}{c|cc|cc|c} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0.1 \end{array}\right].$$

Nosso principal objetivo com este simples exemplo é apontar a importância do ganho $K_i \neq 0$, $i \geq 1$, para melhoria de desempenho comparando nossos resultados com aqueles em [28] e [79], onde as leis de controle por realimentação de estado da forma $u(t) = K_0 x(t)$ e $u(t) = K_0 x(t) + K_1 x(t-\tau)$ foram projetadas. Definindo $\gamma = 0.13$, para cada $N \in \{1, \ldots, 9\}$, calculamos $\tau_\gamma = \tau(\lambda_\gamma)$, como apresentado na Figura C.1. Em [28], para aproximadamente o mesmo valor de γ e $\tau = 0.999$ o ganho K_0 dado possui um módulo alto (de 10^6), enquanto em [79] o atraso máximo de $\tau = 1,28$ foi obtido respeitando o nível de norma desejado. Para todo $N \in \{1, \ldots, 9\}$ os controladores propostos garantem estabilidade para $\tau(\lambda) \in [0 \;\; 1.5708)$.

C.4 Envelopes

Um sistema dinâmico com atrasos no tempo possui infinitos polos. O uso de um envelope que garante que todos os polos estejam contidos nele foi discutido em [4]. Diferentes tipos de envelopes também foram discutidos em [76] e [77]. Nesses casos, os métodos utilizados para estabelecer os envelopes não foram usados para testar a estabilidade nem para projetar

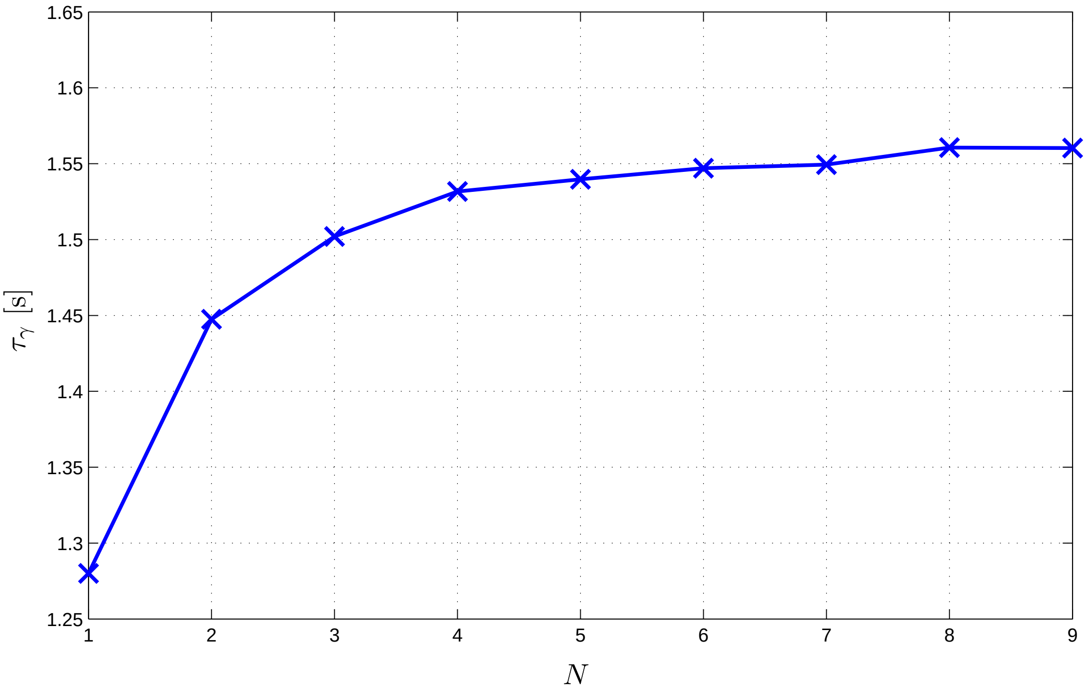

Figura C.1: τ_γ como função de N para $\gamma = 0.13$.

controladores. De fato, em geral, o envelope se estende até o semi-plano direito e, devido a isso, fornece apenas uma região onde os polos podem estar sem qualquer garantia sobre a estabilidade do sistema. Neste trabalho o envelope é projetado por intermédio de desigualdades matriciais lineares (LMIs) e fornece um procedimento para testar a estabilidade robusta para sistemas com atraso dos tipos retardo e neutro. Além disso, um controlador robusto pode ser projetado por realimentação de estado.

C.5 Sistemas com atrasos do tipo retardo

Considere o sistema linear com atraso do tipo retardo com N atrasos, cuja realização mínima é dada por

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x\left(t - \tau_i\right), \tag{C.28}$$

onde $x(t) \in \mathbb{R}^n$ é a variável de estado, $0 = \tau_0 < \tau_1 < \cdots < \tau_N$ são os atrasos e $A_i \in \mathbb{R}^{n\times n}$ para todo $i \in \{0, \ldots, N\}$. Este sistema é exponencialmente estável se e somente se todas as raízes de sua equação característica

$$\det\left(sI - \sum_{i=0}^{N} A_i e^{-s\tau_i}\right) = 0 \tag{C.29}$$

estiverem no semi plano esquerdo [16].

A seguinte proposição introduz um envelope que engloba todos os seus polos.

Proposição C.1. Seja λ um número real qualquer. Se existirem matrizes $T = T' > 0$, $Q_i = Q_i' > 0$, para todo $i \in \{0, \dots, N\}$ e um μ escalar que satisfaçam

$$\mu T \geq \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} \tag{C.30}$$

e

$$\begin{bmatrix} T & T & \dots & T \\ & Q_0 & & \\ & & \ddots & \\ & & & Q_N \end{bmatrix} \geq 0, \tag{C.31}$$

então, qualquer raiz s_0 da equação característica (C.29) tal que $s_0 = \lambda + \jmath\omega$, satisfaz

$$|s_0| \leq \sqrt{\mu}. \tag{C.32}$$

Prova: A seguinte inequação é sempre verdadeira, o que é facilmente verificável aplicando o complemento de Schur

$$\begin{bmatrix} A_i Q_i A_i' e^{-2\lambda\tau_i} & \bullet \\ A_i' e^{-(\lambda-\jmath\omega)\tau_i} & Q_i^{-1} \end{bmatrix} \geq 0. \tag{C.33}$$

Adicionando-os para todo $i \in \{0, \dots, N\}$ leva a

$$\begin{bmatrix} \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} & \bullet \\ \sum_{i=0}^{N} A_i' e^{-(\lambda-\jmath\omega)\tau_i} & \sum_{i=0}^{N} Q_i^{-1} \end{bmatrix} \geq 0, \tag{C.34}$$

onde podemos aplicar o complemento de Schur e utilizar (C.30) para obter

$$\mu T \geq \Sigma \left(\sum_{i=0}^{N} Q_i^{-1} \right)^{-1} \Sigma^*, \tag{C.35}$$

onde $\Sigma \triangleq \sum_{i=0}^{N} A_i e^{-(\lambda+\jmath\omega)\tau_i}$.

Note que de (C.31)

$$T \geq \sum_{i=0}^{N} T Q_i^{-1} T. \tag{C.36}$$

Agora, multiplicando (C.31), pela esquerda e pela direita, por T^{-1} e tomando o inverso em ambos os lados da desigualdade, obtemos

$$T \leq \left(\sum_{i=0}^{N} Q_i^{-1} \right)^{-1} . \tag{C.37}$$

Usando este resultado em (C.35), isso implica que

$$\mu T \geq \Sigma T \Sigma^*. \tag{C.38}$$

Finalmente, seja $s_0 = \lambda + j\omega$ um autovalor de Σ associado a um autovetor a direita v. É bem conhecido, [87] e [88], que os autovalores a esquerda e a direita são iguais. Portanto, s_0 também é um autovalor de Σ associado a um autovetor a esquerda x_L, com dimensão $1 \times n$. Neste caso, podemos multiplicar a desigualdade (C.38) pela esquerda por x_L e pela direita por seu transposto conjugado, x_L^*, obtendo

$$\mu x_L T x_L^* \geq x_L \Sigma T \Sigma^* x_L^* \tag{C.39}$$

e como $x_L \neq 0$ e $T > 0$,

$$\mu \geq (\lambda + j\omega)(\lambda - j\omega), \tag{C.40}$$

levando a

$$|s_0| \leq \sqrt{\mu}, \tag{C.41}$$

que conclui a prova. □

Este resultado produz um envelope mais estreito que trabalhos anteriores como [4].

C.5.1 Implementaçao

Primeiro de tudo, vamos introduzir a definição de fechamento de um envelope. Seja μ e λ definido pela Proposição C.1 e seja $\lambda \in [\lambda_{\min},\ \lambda_{\max}]$. Se houver um ponto λ^* nesse intervalo tal que $\mu = (\lambda^*)^2$, definimos $\lambda^* + \varepsilon$, com $\varepsilon > 0$ arbitrariamente pequeno, como o ponto de fechamento do envelope. Isso significa que o envelope está completamente no lado esquerdo da linha vertical de forma $\Re(s) = \lambda^* + \varepsilon$. Além disso, dizemos que o envelope está fechado sempre que $\mu < \lambda^2$. A escolha de $\lambda_{\min}$ é totalmente arbitrária. Em [76], um simples limitante para a raíz mais à direita de (C.29) foi dado, o que pode ser facilmente generalizado para N atrasos:

$$\Re(s) \leq \bar{\mu}(A_0) + \sum_{i=1}^{N} \|A_i\| = \ell, \tag{C.42}$$

onde $\bar{\mu}(.)$ é uma medida matricial, veja [76] e [90]. Sugerimos $\lambda_{\max} = 2|\ell|$.

A seguinte proposição ilustra como representar o envelope e também o comportamento do envelope como uma função de λ.

Proposição C.2. Seja $\lambda \in [\lambda_{\min},\ \lambda_{\max}]$ e deixe μ ser dado pela Proposição C.1. Se $\mu \geq \lambda^2$ então o envelope no plano complexo é definido pelo conjunto de pontos (λ, ω) onde $\omega = \pm\sqrt{\mu - \lambda^2}$. Se para um determinado λ^*, $\mu^* < (\lambda^*)^2$ então o envelope está fechado para cada $\lambda > \lambda^*$.

Prova: Da equação (C.41) temos que $\lambda^2 + \omega^2 \leq \mu$ que implica diretamente que $\omega = \pm\sqrt{\mu - \lambda^2}$, para $\mu \geq \lambda^2$. Obviamente, (λ, ω) pertence ao envelope. Agora, suponha que, para um determinado λ^*, tenhamos $\mu^* < (\lambda^*)^2$. Como $A_i Q_i A_i' \geq 0$ e $e^{-2\lambda\tau_i}$ é não crescente, nós temos

$\mu < \mu^*$ para cada $\lambda > \lambda^*$ que significa, por definição, que o envelope está fechado. □

Apesar do fato de que este envelope é mais estreito que em [4], para $\lambda = 0$, segue de (C.30) que $\mu \geq 0$ e, portanto, o envelope nunca está fechado no semi-plano esquerdo, o que implica que a estabilidade não pode ser avaliada com o envelope na presente forma. Para contornar isso, propomos uma mudança de coordenadas através da nova variável $s = z - d$, com $d > 0$ e em seguida calculamos o envelope para z. Com esta mudança de variáveis, (C.29) torna-se

$$\det\left(zI - (A_0 + dI) - \sum_{i=1}^{N} A_i e^{-z\tau_i} e^{d\tau_i}\right) = 0, \tag{C.43}$$

permitindo-nos trabalhar com um problema equivalente nos novos parâmetros

$$\begin{aligned} \tilde{A}_0 &= A_0 + dI, \\ \tilde{A}_i &= A_i e^{d\tau_i}, \text{ for all } i \in \{1, \ldots, N\}. \end{aligned} \tag{C.44}$$

No plano-z, o envelope permanecerá aberto para $\lambda = 0$, no entanto, se for fechado antes de $z = d$, ele estará fechado antes da origem no plano-s, garantindo estabilidade para o sistema original.

Exemplo C.2. Considere as seguintes matrizes de sistema

$$\left[\begin{array}{c|c} A_0 & A_1 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0.5413 \\ -2 & -3 & -1.0827 & -1.6240 \end{array}\right].$$

Para $\tau = 1$, a figura C.2 mostra a comparação do nosso envelope com [4], também ilustra o comportamento do envelope para diferentes valores de d. Uma observação interessante é que, para $d = 3$, conseguimos um envelope mais estreito, mais próximo dos polos, e também podemos ver que o ponto em que o envelope se fecha está no lado esquerdo do plano. Isso nos permite usar o envelope como um critério de estabilidade, como será visto na seção de estabilidade. Todos os polos do sistema aqui e durante todo este trabalho foram calculados via [92].

Estabilidade

Veremos agora como é possível usar o envelope para analisar a estabilidade de um sistema com atraso.

Proposição C.3. Seja $\lambda_0 \in \mathbb{R}$, $\mu = \lambda_0^2 - \varepsilon$ para algum $\varepsilon > 0$. Se existir $T, Q_i > 0$, para todo $i \in \{0, \ldots, N\}$, tais que (C.30) e (C.31) são ambas satisfeitas , então o envelope fica inteiramente à esquerda do eixo vertical que passa por λ_0.

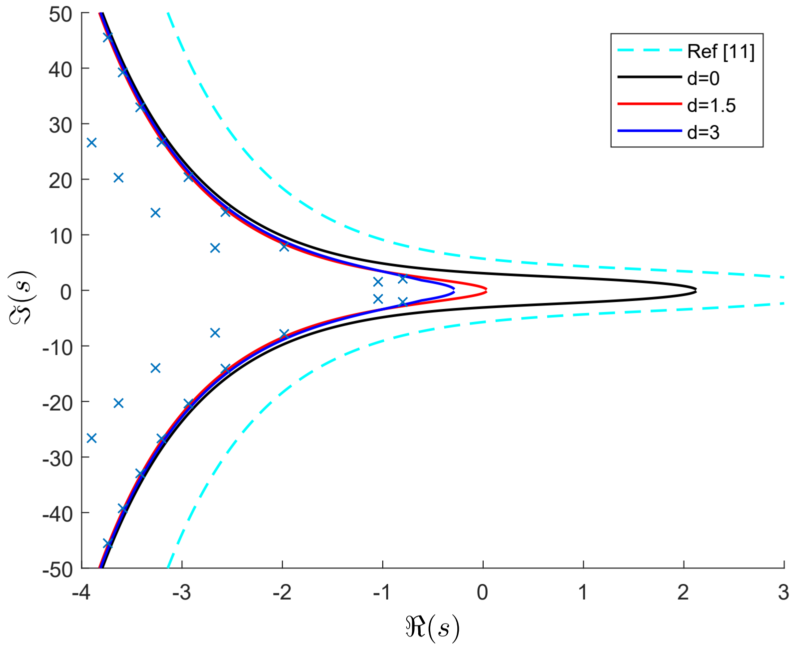

Figura C.2: Envelopes para diferentes valores de d

C.5.2 Realimentação de estado para sistemas do tipo retardo

Tratemos agora do problema de estabilização. Considere o sistema

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x\left(t-\tau_i\right) + Bu(t), \tag{C.45}$$

que queremos controlar por meio de uma lei de controle de realimentação de estado

$$u(t) = \sum_{i=0}^{N} K_i x\left(t-\tau_i\right) \in \mathbb{R}^m, \tag{C.46}$$

a ser projetada através de LMIs. Este controlador lida com requisitos de projeto, como por exemplo, estabilidade-α e adiciona um certo grau de robustez ao sistema de malha fechada. Como será mostrado, o controlador pode ser sem memória, isto é, $K_i \leftarrow 0$, $\forall i \in \{1, \ldots, N\}$ ou pode usar alguns, ou todos, estados atrasados.

Teorema C.2. Considere o sistema com atrasos (C.45). Se existirem matrizes $T = T' > 0$, $Q_i = Q_i' > 0$, Y_i, $\forall i \in \{0, \ldots, N\}$ e escalares positivos d, ε, com $\mu = d^2 - \varepsilon$, $\lambda = d$, de modo que

$$\begin{bmatrix} \mu T & \left(\tilde{A}_0 Q_0 + B_0 Y_0\right) e^{-\lambda\tau_0} & \dots & \left(\tilde{A}_N Q_N + B_N Y_N\right) e^{-\lambda\tau_N} \\ \bullet & Q_0 & & 0 \\ \bullet & \bullet & \ddots & 0 \\ \bullet & \bullet & \bullet & Q_N \end{bmatrix} \geq 0 \tag{C.47}$$

e (C.31) são todas satisfeitas, onde $\tilde{A}_i$ é dado por (C.44) e $B_i = Be^{d\tau_i}$ para todo $i \in \{0, \dots, N\}$, então a lei de controle de realimentação de estado (C.46), onde as matrizes do controlador são dadas por $K_i = Y_i Q_i^{-1}$, estabiliza o sistema.

Prova: Aplicando o complemento de Schur em (C.47) obtemos exatamente (C.30) com $A_i \leftarrow \tilde{A}_i + B_i K_i$, o que completa a prova. □

C.6 Sistemas com atrasos do tipo neutro

Nosso objetivo aqui é desenvolver os envelopes para sistemas do tipo neutro. Considere o sistema linear com atrasos do tipo neutro com $N + 1$ atrasos, cuja realização mínima é dada por

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x\left(t - \tau_i\right) + H\dot{x}\left(t - \tau_h\right), \tag{C.48}$$

onde $x(t) \in \mathbb{R}^n$ é a variável de estado, $0 = \tau_0 < \tau_1 < \cdots < \tau_N$ e τ_h são os atrasos, $A_i \in \mathbb{R}^{n\times n}$, para todo $i \in \{0, \dots, N\}$, e H são matrizes reais. Uma condição necessária para a estabilidade exponencial deste sistema é que todas as raízes da equação característica

$$\det\left(sI - \sum_{i=0}^{N} A_i e^{-s\tau_i} - sHe^{-s\tau_h}\right) = 0, \tag{C.49}$$

estejam no lado esquerdo de uma linha vertical $\Re(s) = -\alpha$, com $\alpha > 0$ [101].

Proposição C.4. Seja λ um número real qualquer. Se existirem matrizes $T = T' > 0$, $Q_i = Q_i' > 0, \forall i \in \{0, \dots, N\}$, $Q_h = Q_h' > 0$ e um escalar μ tal que

$$\mu T \geq \sum_{i=0}^{N} A_i Q_i A_i' e^{-2\lambda\tau_i} + \mu H Q_h H' e^{-2\lambda\tau_h} \tag{C.50}$$

e

$$\begin{bmatrix} T & T & \dots & & T \\ & Q_0 & & & \\ & & \ddots & & \\ & & & Q_N & \\ & & & & \frac{\mu}{|s_0|^2} Q_h \end{bmatrix} > 0, \tag{C.51}$$

então, qualquer raiz s_0 da equação característica (C.49) tal que $s_0 = \lambda + j\omega$, satisfaz

$$|s_0| \leq \sqrt{\mu}. \tag{C.52}$$

Prova: A seguinte inequação é sempre verdadeira, o que é facilmente verificável aplicando o complemento de Schur

$$\begin{bmatrix} HQ_hH'e^{-2\lambda\tau_h} & \bullet \\ s_0^*H'e^{-(\lambda-j\omega)\tau_h} & |s_0|^2Q_h^{-1} \end{bmatrix} \geq 0. \tag{C.53}$$

Multiplicando ambos os lados por $\mathbf{diag}(\sqrt{\mu}, \frac{1}{\sqrt{\mu}})$ e adicionando o resultado a (C.34) obtemos

$$\begin{bmatrix} \sum_{i=0}^{N} A_iQ_iA_i'e^{-2\lambda\tau_i} + \mu HQ_hH'e^{-2\lambda\tau_h} & \bullet \\ \sum_{i=0}^{N} A_i'e^{-(\lambda-j\omega)\tau_i} + s_0^*H'e^{-(\lambda-j\omega)\tau_h} & \sum_{i=0}^{N} Q_i^{-1} + \frac{|s_0|^2}{\mu}Q_h^{-1} \end{bmatrix} \geq 0, \tag{C.54}$$

onde podemos aplicar o complemento de Schur e utilizar (C.50) para obter

$$\mu T \geq \Sigma \left(\sum_{i=0}^{N} Q_i^{-1} + \frac{|s_0|^2}{\mu} Q_h^{-1} \right)^{-1} \Sigma^*, \tag{C.55}$$

onde $\Sigma \triangleq \sum_{i=0}^{N} A_ie^{-(\lambda+j\omega)\tau_i} + s_0He^{-(\lambda+j\omega)\tau_h}$. Além disso, de (C.51) temos que

$$T > \sum_{i=0}^{N} TQ_i^{-1}T + \frac{|s_0|^2}{\mu} TQ_h^{-1}T, \tag{C.56}$$

que implica em

$$T < \left(\sum_{i=0}^{N} Q_i^{-1} + \frac{|s_0|^2}{\mu} Q_h^{-1} \right)^{-1}. \tag{C.57}$$

Portanto, usando esse resultado em (C.55), isso implica que

$$\mu T \geq \Sigma T \Sigma^*. \tag{C.58}$$

Procedendo da mesma maneira que o caso do sistema com atraso do tipo retardo, multiplicando a desigualdade (C.58) para a esquerda por x_L e para a direita por x_L^*, em que x_L é o autovetor esquerdo de Σ associado ao autovalor s_0, leva a

$$|s_0| \leq \sqrt{\mu}. \tag{C.59}$$

□

A dificuldade em aplicar este método, reside na necessidade de conhecer $|s_0|$ para implementar a LMI (C.54). No entanto, como consequência deste resultado, $\mu \geq |s_0|^2$, temos que $\frac{\mu}{|s_0|^2} \geq 1$, implicando que se

$$\begin{bmatrix} T & T & \dots & & T \\ & Q_0 & & & \\ & & \ddots & & \\ & & & Q_N & \\ & & & & Q_h \end{bmatrix} > 0 \tag{C.60}$$

for satisfeita, então (C.51) também é verificada. Desta forma, (C.60) pode ser substituída por (C.51).

C.6.1 Implementação

Agora, podemos fazer mais uma vez a mudança de variáveis $s = z - d$, com $d > 0$ e calcular o envelope para z. Isso forçará o envelope a ser transladado para o lado esquerdo no plano s. Com esta mudança de variável, (C.49) torna-se

$$\det\left(zI - (A_0 + dI) - \sum_{k=1}^{N} A_k e^{-z\tau_k} e^{d\tau_k} - zHe^{-z\tau_h} e^{d\tau_h} + dHe^{-z\tau_h} e^{d\tau_h}\right) = 0, \tag{C.61}$$

permitindo-nos trabalhar nas novas variáveis

$$\det\left(zI - \tilde{A}_0 - \sum_{k=1}^{N+1} \tilde{A}_k e^{-z\tau_k} - z\tilde{H}e^{-z\tau_h}\right) = 0, \tag{C.62}$$

onde

$$\begin{aligned} \tilde{A}_0 &= A_0 + dI, \\ \tilde{A}_i &= A_i e^{d\tau_i}, \forall i \in \{0, \dots, N\}, \\ \tilde{A}_{N+1} &= -dHe^{d\tau_h}, \\ \tilde{H} &= He^{d\tau_h}. \end{aligned} \tag{C.63}$$

Isso nos permite executar a mesma técnica usada para sistemas com atraso do tipo retardo.

C.6.2 Realimentação de estado para sistemas do tipo neutro

Agora podemos adaptar o resultado de sistemas do tipo retardo para sistemas do tipo neutro descritos por

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - \tau_i) + H\dot{x}(t - \tau_h) + Bu(t). \tag{C.64}$$

Teorema C.3. Considere o sistema com atrasos (C.64). Se existirem matrizes $T = T' > 0$, $Q_i = Q_i' > 0$, $\forall i \in \{0, \ldots, N+1\}$, Y_i, $\forall i \in \{0, \ldots, N\}$, $Q_h = Q_h' > 0$ e escalares positivos d, ε, com $\mu = d^2 - \varepsilon$, $\lambda = d$, tais que

$$\begin{bmatrix} \mu T & \left(\tilde{A}_0 Q_0 + B_0 Y_0\right) e^{-\lambda\tau_0} & \ldots & \left(\tilde{A}_N Q_N + B_N Y_N\right) e^{-\lambda\tau_N} & \left(\tilde{H} Q_h + B_h Y_h\right) e^{-\lambda\tau_h} \\ \bullet & Q_0 & 0 & 0 & 0 \\ \bullet & 0 & \ddots & 0 & 0 \\ \bullet & 0 & 0 & Q_N & 0 \\ \bullet & 0 & 0 & 0 & \frac{1}{\mu} Q_h \end{bmatrix} \geq 0, \qquad \text{(C.65)}$$

e (C.51) forem todas satisfeitas, onde $\tilde{H}$ e $\tilde{A}_i$ para todo $i \in \{0, \ldots, N+1\}$ são dados por (C.44) e $B_i = Be^{d\tau_i}$ para todo $i \in \{0, \ldots, N\}$ então, o controlador de realimentação de estado (C.46) obtido com as matrizes de ganho $K_i = Y_i Q_i^{-1}$, $\forall i \in \{0, \ldots, N\}$ estabiliza o sistema.

Prova: Aplicando o complemento de Schur em (C.65) obtemos exatamente (C.50) com $\tilde{A}_i \leftarrow \tilde{A}_i + B_i K_i$ e $\tilde{H} \leftarrow \tilde{H} + B_h K_h$, o que completa a prova. □

Exemplo C.3. Para as matrizes

$$\left[\, A_0 \mid A_1 \,\right] = \left[\begin{array}{cc|cc} -1.7073 & 0.6856 & -2.5026 & -1.0540 \\ 0.2279 & -0.6368 & -0.1856 & -1.5715 \end{array}\right] \qquad \text{(C.66)}$$

e

$$H = \begin{bmatrix} 0.0558 & 0.0360 \\ 0.2747 & -0.1084 \end{bmatrix}, \qquad \text{(C.67)}$$

veja [99], [104], [105] e [106], com $\tau_1 = \tau_h = 2$ e aplicando o Teorema C.3, projetamos o seguinte controlador

$$K_0 = \begin{bmatrix} -37.7924 & -20.7712 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 5.3363 & 3.7375 \end{bmatrix},$$

que garante estabilidade para todos os atrasos. Ilustramos o envelope para $\tau_1 = \tau_h = 2$ na Figura C.3.

C.7 Sistemas de Markov contínuo com atrasos

Um sistema sujeito a saltos markovianos de tempo contínuo com atraso é descrito pelo seguinte modelo de espaço de estados estocástico

$$\dot{x}(t) = A_0(\theta_t)x(t) + A_1(\theta_t)x(t - \tau), \qquad \text{(C.68)}$$

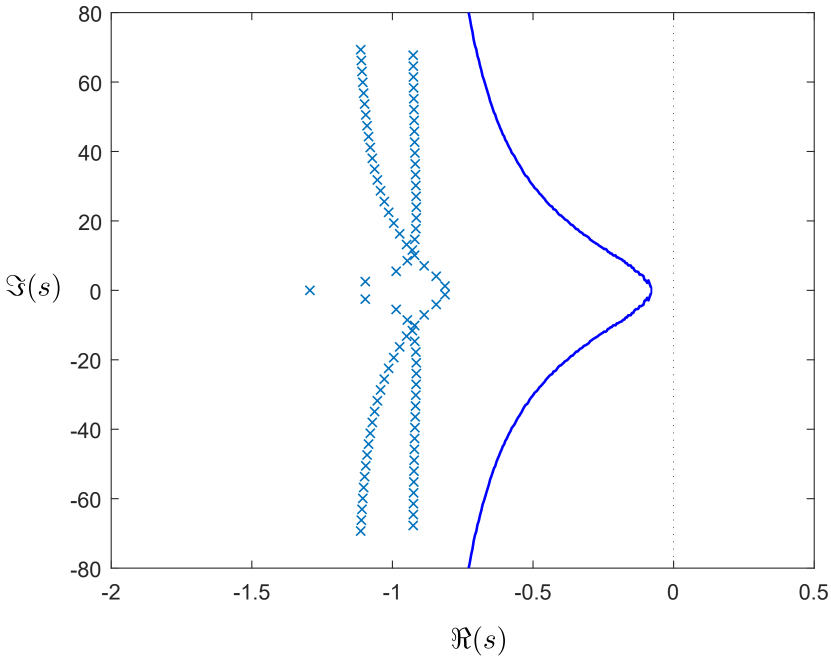

Figura C.3: Realimentação de estado - Tipo Neutro - $\tau_1 = \tau_h = 2$

onde $x \in \mathbb{R}^n$ é a variável de estado, $\theta_t \in \mathbb{K}$ é uma variável aleatória, e $\tau > 0$ é o atraso. As condições iniciais são $x(t) = x_0(t)$ para $t \in [-\tau, 0]$ e distribuição inicial de probabilidade $p(0)$. Assumimos que as matrizes A_{0i}, para $i \in \mathbb{K}$, são Metzler, e as matrizes A_{i1}, para $i \in \mathbb{K}$, são não-negativas. Isso garante que o sistema (C.68) seja um sistema positivo e, portanto, para quaisquer condições iniciais positivas $x_0(t) > 0$, $t \in [-\tau, 0]$, o vetor de estado $x(t)$ permanece não-negativo para qualquer $\theta(t)$ e todo $t \geq 0$.

O processo $\{\theta_t, t \in [0, +\infty)\}$ é um processo markoviano estocástico tal que

$$\begin{aligned} p_{ij}(\Delta) = & \ \mathrm{Prob}(\theta_{t+\Delta} = j | \theta_t = i) \\ = & \begin{cases} \lambda_{ij}\Delta + o(\Delta) & i \neq j, \\ 1 + \lambda_{ii}\Delta + o(\Delta) & i = j, \end{cases} \end{aligned} \tag{C.69}$$

onde $\lambda_{ij} \geq 0$ para $i \neq j$, $\lambda_{ii} \leq 0$ e

$$\sum_{j \in \mathbb{K}} \lambda_{ij} = 0. \tag{C.70}$$

Podemos, portanto, definir a matriz de transição $\Lambda = [\lambda_{ij}]$ que apresenta todas as taxas de transição entre os estados da cadeia de Markov representada por $\theta_t \in \mathbb{K}$.

Para obter resultados de convergência para o primeiro momento da variável de estado $x(t)$, vamos definir, para cada $t \in \mathbb{R}_+$,

$$q(t) \triangleq \mathcal{E}(x(t)) \in \mathbb{R}^n. \tag{C.71}$$

Derivados dessa definição, também consideramos que

$$q_i(t) \triangleq \mathcal{E}(x(t)\mathbf{1}_{\{\theta(t)=i\}}) \in \mathbb{R}^n, \tag{C.72}$$

$$\hat{q}(t) \triangleq \begin{bmatrix} q_1(t) & \dots & q_N(t) \end{bmatrix}' \in \mathbb{R}^{Nn}, \tag{C.73}$$

e, portanto,

$$q(t) = \sum_{i\in\mathbb{K}} q_i(t). \tag{C.74}$$

O próximo lema fornece uma equação diferencial com atraso capaz de calcular o primeiro momento da variável de estado $x(t)$.

Lema C.4. Para todo t, $\tau \in \mathbb{R}_+$, o 1°-momento de $x(t)$ do sistema (C.68) é modelado pela seguinte equação diferencial com atraso:

$$\dot{\hat{q}}(t) = F\hat{q}(t) + G(\tau)\hat{q}(t-\tau), \tag{C.75}$$

na qual $\hat{q}$ é dado por (C.73) e F, $G(\tau)$ são dados por

$$\begin{array}{rcl} F & = & \Lambda' \otimes I_n + \text{diag}(A_{0i}), \\ G(\tau) & = & \text{diag}(A_{1i}).(\Pi(\tau)' \otimes I_n). \end{array} \tag{C.76}$$

Prova: Seguindo os passos de [78] e aplicando a regra de Itô à primeira equação em (C.72), temos que

$$\begin{array}{rcl} dq_j(t) & = & \mathcal{E}\{dx(t)\mathbf{1}_{\{\theta(t)=j\}} + x(t)d\mathbf{1}_{\{\theta(t)=i\}}\} \\ & = & A_{0j}\mathcal{E}\{x(t)\mathbf{1}_{\{\theta(t)=j\}}\}dt + A_{1j}\mathcal{E}\{x(t-\tau)\mathbf{1}_{\{\theta(t)=j\}}\}dt + \sum_{i\in\mathbb{K}}\lambda_{ij}q_i(t)dt \\ & = & (A_{0j}q_j(t) + \sum_{i\in\mathbb{K}}\lambda_{ij}q_i(t))dt + A_{1j}\mathcal{E}\{\sum_{i\in\mathbb{K}}x(t-\tau)\mathbf{1}_{\{\theta(t)=j\}}\mathbf{1}_{\{\theta(t-\tau)=i\}}\}dt \\ & = & (A_{0j}q_j(t) + \sum_{i\in\mathbb{K}}\lambda_{ij}q_i(t) + A_{1j}\sum_{i\in\mathbb{K}}\Pi_{ij}(\tau)q_i(t-\tau))dt, \end{array} \tag{C.77}$$

logo, (C.75) se segue. □

Observação C.1. É essencial observar que, se todas as matrizes A_{0i}, para $i \in \mathbb{K}$, forem Metzler, e todas as matrizes A_{i1}, para $i \in \mathbb{K}$, forem não-negativas , então F é Metzler e $G(\tau)$ é não-negativo. Isto implica que o sistema com atraso (C.75) é positivo. O contrário, por outro lado, não é sempre verdadeiro.

A propriedade apresentada na Observação C.1 será de grande importância no projeto de um método numérico capaz de fornecer as janelas de estabilidade média para o sistema de Markov positivo com atraso.

O próximo resultado relaciona a estabilidade média do sistema estocástico positivo com atraso (C.68) com a estabilidade do sistema determinístico com atraso (C.75).

Lema C.5. Considere o MJLS de tempo contínuo positivo com atraso dado em (C.68). Este sistema é exponencialmente estável na média se, e somente se, o sistema com atraso (C.75) for exponencialmente estável.

Prova: Segue diretamente do fato de que $\mathcal{E}\{x(t)\} = \sum_{i\in\mathbb{K}} q_i(t)$ e da definição de $\hat{q}(t)$. □

Para qualquer atraso constante dado $\tau \geq 0$, a estabilidade do sistema (C.75) é dada pela posição de seus polos. Existem vários procedimentos numéricos capazes de determinar a estabilidade [73] e as janelas de estabilidade [118] para tais sistemas. No entanto, devido ao fato de que as matrizes F e $G(\tau)$, juntamente com a condição inicial, definirem um sistema positivo com atraso, um teste mais simples pode ser usado:

Lema C.6. Um sistema positivo com atraso

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \tag{C.78}$$

onde A_0 é Metzler e A_1 é uma matriz não-negativa, é estável independente do atraso se, e somente se, $A_0 + A_1$ for Hurwitz.

Prova: A prova é baseada na existência de um vetor positivo p e no funcional de Lyapunov-Krasovskii

$$V(\Psi) = p'\Psi(0) + \int_{-\tau}^{0} p' A_1 \Psi(\theta) d\theta. \tag{C.79}$$

Veja [119] para todos os detalhes. □

O lema anterior implica um resultado mais forte para o sistema linear positivo determinístico com atraso. Ele diz que, se o sistema sem atraso (isto é, $\tau = 0$) é estável, então o sistema é estável para todos os valores positivos de atraso, o que é conhecido como estabilidade independente do atraso. Isso também implica que, se o sistema sem atrasos for instável, o mesmo vale para cada valor positivo de τ. Portanto, para tal classe de sistema, não podemos encontrar janelas de estabilidade, um efeito comum para sistemas lineares com atraso.

Observação C.2. Todos esses resultados podem ser facilmente estendidos para o caso de vários atrasos. De fato, se um sistema é descrito pelas equações estocásticas

$$\dot{x}(t) = A_0(\theta_t) x(t) + \sum_{k=1}^{N} A_k(\theta_t) x(t-\tau_k), \tag{C.80}$$

onde $\tau_k > 0$, $A_0(\theta_t)$ é Metzler e $A_k(\theta_t)$ é não-negativa, para cada $\theta_t \in \mathbb{K}$ e $k \in \{1, \dots, N\}$, então os mesmos procedimentos podem ser realizados para mostrar que este sistema é estável na média se, e somente se, a matriz $F + \sum_{k=1}^{N} G(\tau_k)$, onde F e G são definidas em (C.76), for Hurwitz.

Exemplo C.4. Como nosso primeiro exemplo, vamos considerar um sistema MJLS positivo com matrizes de espaço de estados dadas por

$$A_0(1) = \begin{bmatrix} -1.0 & 0.4 \\ 0 & -2.8 \end{bmatrix}, \quad A_0(2) = \begin{bmatrix} -0.9 & 0 \\ 0 & -0.1 \end{bmatrix}, \\ A_1(1) = \begin{bmatrix} 0.5 & 0.2 \\ 1.3 & 0.7 \end{bmatrix}, \qquad A_1(2) = \begin{bmatrix} 0.2 & 0.1 \\ 1.7 & 0.1 \end{bmatrix}, \tag{C.81}$$

e

$$\Lambda = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{C.82}$$

Aplicando o Lema C.4, como é mostrado na Figura C.4, este sistema é estável na média para $\tau \in [0, 0.12]$ e instável fora desse intervalo. Este exemplo ilustra um caso em que o atraso tem um efeito desestabilizador na estabilidade média do sistema.

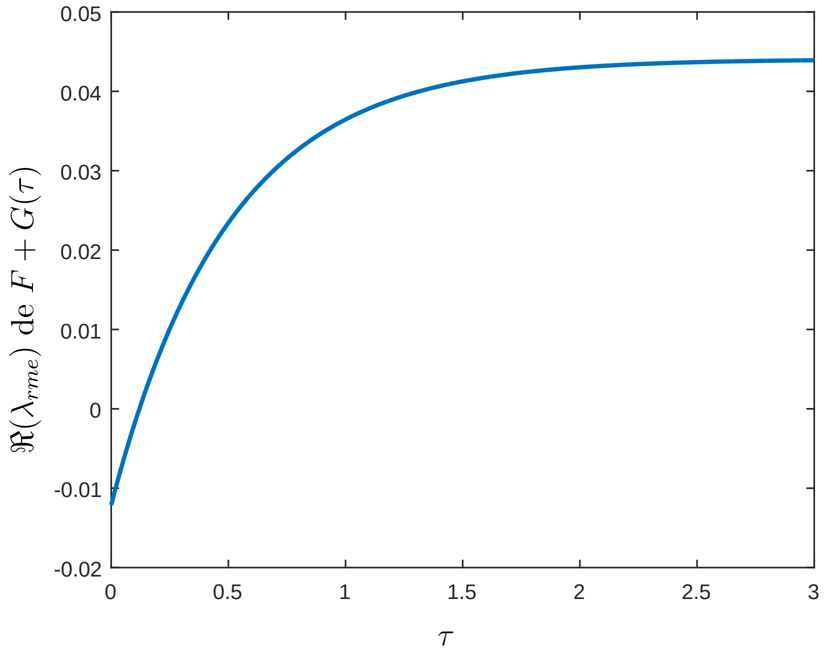

Figura C.4: Primeiro Exemplo: Parte real do autovalor mais a direita de $F + G(\tau)$

Exemplo C.5. Para o segundo exemplo, considere o sistema MJLS positivo com matrizes de espaço de estados dadas por

$$A_0(1) = \begin{bmatrix} -3.9 & 0.4 \\ 0.2 & -1.9 \end{bmatrix}, \quad A_0(2) = \begin{bmatrix} -1.5 & 0.3 \\ 0.4 & -3.2 \end{bmatrix}, \\ A_1(1) = \begin{bmatrix} 1.3 & 1.4 \\ 0.1 & 1.1 \end{bmatrix}, \qquad A_1(2) = \begin{bmatrix} 1.9 & 0.4 \\ 0.8 & 1.0 \end{bmatrix}, \tag{C.83}$$

e a mesma matriz Λ como no exemplo anterior. Para este caso, a Figura C.5 mostra que o sistema é instável para $\tau = 0$, e a estabilidade média é alcançada apenas para $\tau > 0.2957$. Este exemplo ilustra uma situação em que o atraso funciona como um efeito estabilizador para o sistema.

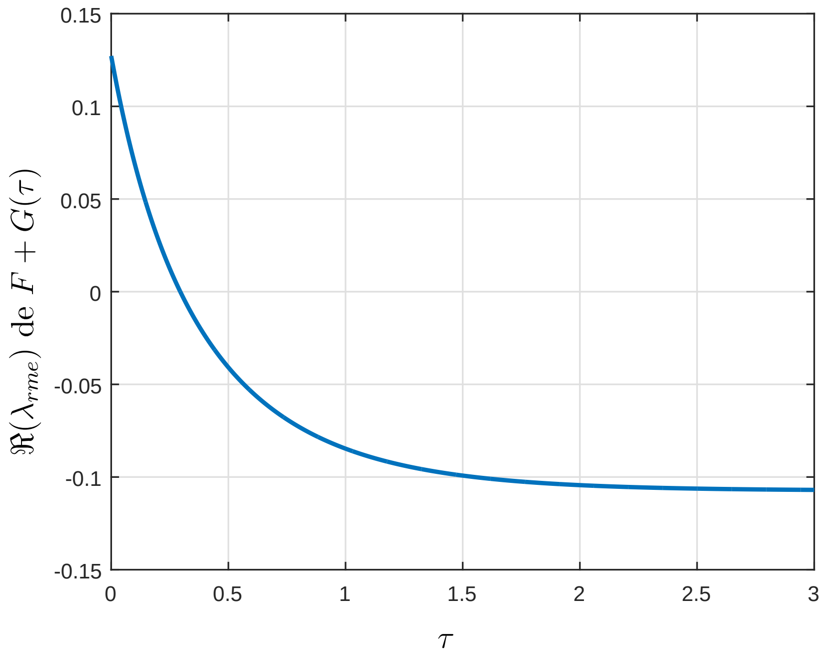

Figura C.5: Segundo Exemplo: Parte real do autovalor mais a direita de $F + G(\tau)$

Para dois valores diferentes do atraso ($\tau = 2.0$, dentro da região de estabilidade e $\tau = 0.1$, fora da região de estabilidade), realizamos uma simulação de Monte-Carlo com 5.000 realizações; a média para cada variável de estado é apresentada nas Figuras C.6 e C.7, respectivamente. Estes resultados suportam a análise de estabilidade média realizada.

C.8 Conclusões

Esta dissertação traz contribuições para a teoria de sistemas com atrasos, fornecendo novos métodos para análise de estabilidade e também para estabilização desses sistemas. O trabalho está dividido principalmente em três partes. No primeiro segmento, estendemos o procedimento para o projeto de controle de sistemas com atrasos, baseados em um sistema LTI, chamado *sistema de comparação*, que fornece um limite inferior para a norma $\mathcal{H}_\infty$ do sistema com atraso. Aumentando a ordem da substituição de Rekasius, este é o primeiro procedimento capaz de fazer um melhor uso do buffer necessário para implementar a realimentação de estado. O método que propomos pode obter simultaneamente mais margem de estabilidade e um menor nível $\mathcal{H}_\infty$. Rotinas clássicas como a equação de Riccati podem ser usadas para projetar o controlador para o sistema com atrasos. A estabilização por realimentação de estado e por realimentação de saída são discutidas, assim como a implementação de dois algoritmos, um para minimizar a norma $\mathcal{H}_\infty$ quando o atraso é dado e o outro para maximizar

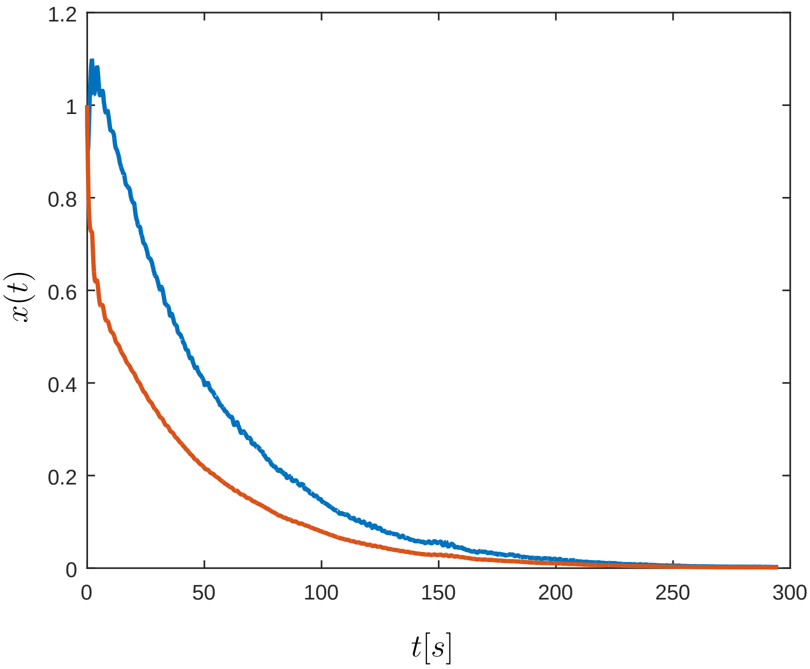

Figura C.6: Valor esperado para as variáveis de estado - $\tau = 2$

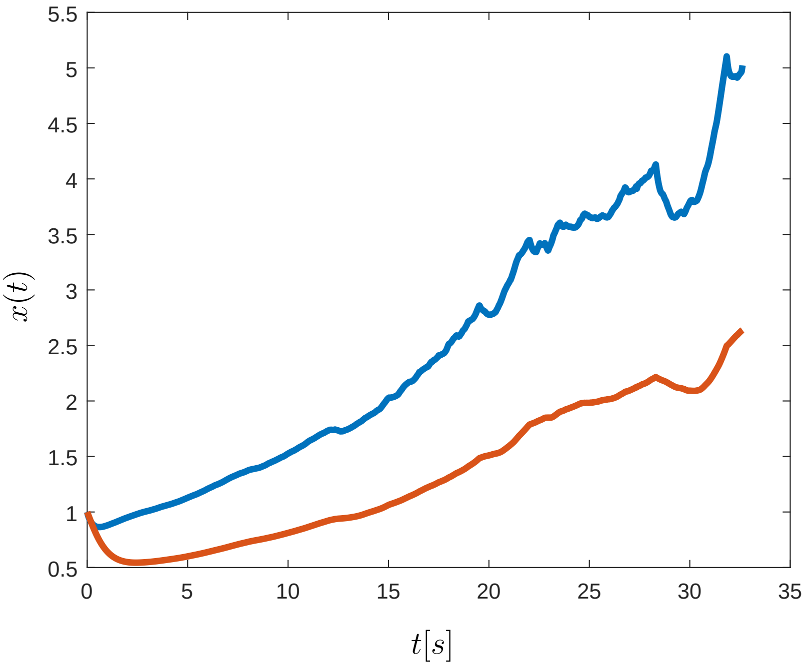

Figura C.7: Valor esperado para as variáveis de estado - $\tau = 0.1$

o atraso máximo quando for dado um limitante inferior γ para a norma $\mathcal{H}_\infty$. Filtros também podem ser projetados com a metodologia apresentada. Os resultados para realimentação de estado foram publicados em [38] e para realimentação de saída em [39]. Finalmente, nesta primeira parte, estendemos a análise de nosso procedimento para sistemas fracionários com atrasos. *Sistemas de comparação* para sistemas fracionários são desenvolvidos; enfatizamos que é possível aproximar a norma $\mathcal{H}_\infty$ do sistema LTI para o sistema com atrasos da mesma forma para este caso.

No segundo segmento, desenvolvemos uma nova estratégia para projetar um envelope que envolve todos os pólos de um sistema com atrasos. Usando LMIs, podemos obter envelopes menos conservadores do que os desenvolvidos pela abordagem de autovalores apresentada na literatura. Além disso, a novidade aqui é que podemos usar esse envelope para estudar a estabilidade e projetar controladores de realimentação para sistemas lineares com atrasos. Para sistemas do tipo retardo, discutimos a realimentação de estado e adaptamos o método para lidar com incertezas paramétricas nas matrizes do sistema. Os requisitos de projeto também podem ser abordados, como a alocação de pólos à esquerda de uma linha vertical $s = -\alpha$ com $\alpha > 0$ no plano s. Esses resultados foram publicados em [107]. Além disso, pela primeira vez, é proposto um envelope para sistemas com atrasos do tipo neutro. Estabilidade, estabilização e robustez são discutidas para esse tipo de sistema e os resultados foram publicados na revista [108]. A parte de análise é estendida aos sistemas fracionários com atrasos. No entanto, a parte da síntese não pode ser aplicada diretamente e será um tópico de estudo para o futuro.

Para a terceira e última parte, lidamos com sistemas estocásticos. Primeiramente, para sistemas markovianos a tempo contínuo com atrasos, propomos LMIs, para o projeto de realimentação de estado $\mathcal{H}_\infty$, que são afins em relação às taxas de transição entre os diferentes modos de Markov. Essa relação afim permite incorporar incertezas politópicas nesses parâmetros. A ideia é baseada em nosso trabalho anterior [110]. Posteriormente, desenvolvemos um sistema que modela o primeiro momento de um sistema de Markov positivo com atraso. Este novo sistema é usado para analisar a estabilidade na média tanto em tempo contínuo como em discreto.

Trabalhos futuros têm como objetivo continuar esta linha de trabalho interessante e desafiadora. Além da continuidade de todo o trabalho desenvolvido aqui, como mencionado nos parágrafos acima, estará à vista a análise e síntese de controladores para sistemas estocásticos com atrasos e sua ampla gama de aplicações, em engenharia, em biologia e principalmente em finanças.