Reflexive spaces of smooth functions : a logical account of linear partial differential equations

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Thèse de doctorat
de l'Université Sorbonne Paris Cité
Préparée à l'Université Paris Diderot
Ecole Doctorale 386 - Sciences Mathématiques de Paris Centre

Reflexive spaces of smooth functions: A logical account of linear partial differential equations

Thèse de doctorat d'Informatique
présentée et soutenue publiquement le 19 Octobre 2018 par

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Titre: Espaces réflexifs de fonctions lisses: un compte rendu logique des équations aux dérivées partielles linéaires.

Résumé

La théorie de la preuve se développe depuis la correspondance de Curry-Howard suivant deux sources d'inspirations: les langages de programmation, pour lesquels elle agit comme une théorie des types de données, et l'étude sémantique des preuves. Cette dernière consiste à donner des modèles mathématiques pour les comportements des preuves/programmes. En particulier, la sémantique dénotationnelle s'attache à interpréter ceux-ci comme des fonctions entre les types, et permet en retour d'affiner notre compréhension des preuves/programmes. La logique linéaire (LL) donne une interprétation logique des notions d'algèbre linéaire, quand la logique linéaire différentielle (DiLL) permet une compréhension logique de la notion de différentielle.

Cette thèse s'attache à renforcer la correspondance sémantique entre théorie de la preuve et analyse fonctionnelle, en insistant sur le caractère involutif de la négation dans DiLL. La première partie consiste en un rappel des notions de linéarité, polarisation et différentiation en théorie de la preuve, ainsi qu'un exposé rapide de théorie des espaces vectoriels topologiques. La deuxième partie donne deux modèles duaux de la logique linéaire différentielle, interprétant la négation d'une formule respectivement par le dual faible et le dual de Mackey. Quand la topologie faible ne permet qu'une interprétation discrète des preuves sous forme de série formelle, la topologie de Mackey nous permet de donner un modèle polarisé et lisse de DiLL. Enfin, la troisième partie de cette thèse s'attache à interpréter les preuves de DiLL par des distribuitons à support compact. Nous donnons un modèle polarisé de DiLL où les types négatifs sont interprétés par des espaces Fréchet Nucléaires. Nous montrons que enfin la résolution des équations aux dérivées partielles linéaires à coefficients constants obéit à une syntaxe qui généralise celle de DiLL, que nous détaillons.

Mots-clefs : théorie de la preuve, logique linéaire, sémantique dénotationnelle, espaces vectoriels topologies, théorie des distributions.

Title: Reflexive spaces of smooth functions: a logical account of linear partial differential equations.

Abstract

Around the Curry-Howard correspondence, proof-theory has grown along two distinct fields: the theory of programming languages, for which formulas acts as data types, and the semantic study of proofs. The latter consists in giving mathematical models of proofs and programs. In particular, denotational semantics distinguishes data types which serves as input or output of programs, and allows in return for a finer understanding of proofs and programs. Linear Logic (LL) gives a logical interpretation of the basic notions of linear algebra, while Differential Linear Logic allows for a logical understanding of differentiation.

This manuscript strengthens the link between proof-theory and functional analysis, and highlights the role of linear involutive negation in DiLL. The first part of this thesis consists in a quick overview of prerequisites on the notions of linearity, polarisation and differentiation in proof-theory, and gives the necessary background in the theory of locally convex topological vector spaces. The second part uses two classic topologies on the dual of a topological vector space and gives two models of DiLL: the weak topology allows only for a discrete interpretation of proofs through formal power series, while the Mackey topology on the dual allows for a smooth and polarised model of DiLL. Finally, the third part interprets proofs of DiLL by distributions. We detail a polarized model of DiLL in which negatives are Fréchet Nuclear spaces, and proofs are distributions with compact support. We also show that solving linear partial differential equations with constant coefficients can be typed by a syntax similar to the one of DiLL, which we detail.

key-words : proof-theory, linear logic, denotational semantics, locally convex topological vector spaces, distribution theory.

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Chapter 1

Introduction

Differentiation in mathematics historically deals with functions defined on continuous sets, in which infinitesimal variations of variables can be defined. Even though computation historically deals with discrete structures, the concept of differentiation has recently made its way into several applied domains of computer science, e.g. in numerical analysis [39], in incremental computing [65], or in machine learning [3].

Semantics of programs have however essentially focussed on programs operating on *discrete classes of resources*, relying on the guiding principle that programs compute using only a finite amount of resources. This concept has been prominent in the development of denotational semantics, even essential to obtain the first models of λ-calculus in domain theory [70].

The rise of probabilistic languages, infinite data structures and machine learning has more recently promoted a movement towards continuous semantics: theoretical computer science, which has traditionally been associated with discrete mathematics and algebra, is now steadily moving towards analysis.

Independently, differentiation also appeared in proof theory through Differential Linear Logic [23]. Linear Logic and its differential refinement are formal proof systems allowing for the characterization of the concept of *linearity of proofs*. Linear Logic [29] is a resource-aware refinement of intuitionnistic and classical logic, introduced after a study of denotational models of λ-calculus. Differential Linear Logic spawned from the careful study of models of Linear Logic based on vector spaces [18, 19] [74]. While the former enriches Logic with tools from Algebra, the latter transports those of Differential Calculus. When Linear Logic had applications in developments in Type systems [12, 80], Differential Linear Logic may now lead to developments in differential and probabilistic programming through differential λ-calculus [22].

This work draws formal links between Logic and the theory of topological vector spaces and distributions. It develops several *classical* and *smooth* models of Differential Linear Logic using the theory of topological vector spaces, and with these models embeds the theory of linear partial differential operators into a refinement of Differential Linear Logic.

The cornerstone of our approach is the intuition that semantics should carry over an essential aspect of the syntax of Differential Linear Logic, namely the involutive negation, and operate on continuous spaces. We will explain how this leads to the understanding of $!E$ *as a space of solutions to a Differential Equation.*

Models of Differential Linear Logic

Linear Types and Differentiation. Through the Curry-Howard correspondence (detailed in section 2.1) formulas A of a logical system can be understood as *types* of programs $p : A$. In particular, the logical implication $A \Rightarrow B$ is the type of programs $p : A \Rightarrow B$ producing data of type B from data of type A. Linear Logic is a logical system which distinguishes a linear implication $A \multimap B$, and a specific unary constructor $!A$, read as "of course A", or the exponential of A . The usual implication is then defined using this two connectives:

$$!A \multimap B \equiv A \Rightarrow B \tag{1.1}$$

where $\equiv$ stands for *logical equivalence*. Through the *dereliction* rule, Linear Logic expresses that a linear implication is in particular a non-linear one. That is, we have a proof d_A of the formula:

$$d_A : (A \multimap B) \multimap (A \Rightarrow B).$$

Differential Linear Logic contains a dual rule named *codereliction*, which expresses the fact that a linear proof can be extracted from a non-linear one. It is thought of as the best linear approximation of this proof at 0[1]. That is, we have a proof $\bar{d}_A$ of the formula:

$$\bar{d}_A : (A \Rightarrow B) \multimap (A \multimap B).$$

As a linear map is its own differentiation, a *cut-rule* - semantically a composition between d and $\bar{d}$ - must result in the identity:

$$d_A \circ \bar{d}_A = Id_A$$

In the last chapter (Chapter 8), we generalize this to interpret the theory of Linear Partial Differential Operators (LPDO) acting on distributions. We provide a dereliction rule $\bar{d}_{D,A}$ which applies an operator D to a proof $q : A \Rightarrow B$. The dereliction $d_{D,A}$ is then the rule which computes the solution $p : A \Rightarrow B$ such that

$$Dp = q.$$

This is justified as a model of Differential Linear Logic in which distributions with vectorial values [67] interpret proofs.

> *Differential Linear Logic is a Type System for several Differential Operators, and in particular for Linear Partial Differential Operators with Constant Coefficients.*

Semantics of Linear Logic The field of *Denotational semantics* studies programs by interpreting them by functions. In particular, a program $p : A \Rightarrow B$ is interpreted by a function $f : [\![A]\!] \longrightarrow [\![B]\!]$. Thus types A are interpreted by some spaces $[\![A]\!]$. These spaces must be stable under some operations. Typically, one must be able to *curry* and *uncurry*:

$$[\![A]\!] \times [\![B]\!] \longrightarrow [\![C]\!] \simeq [\![A]\!] \longrightarrow ([\![B]\!] \longrightarrow [\![C]\!]). \tag{1.2}$$

To interpret higher-order functional programming, one must consider the type of programs $A \Rightarrow B$ as a space itself, i.e we must have a category with *internal hom-set*. If $p : A \Rightarrow B$, then p is interpreted by a function $f : [\![A]\!] \longrightarrow [\![B]\!]$, which is itself an element of a space:

$$f \in \mathcal{C}([\![A]\!], [\![B]\!])$$

Through the Curry-Howard isomorphism, matching in particular types of programs to formulas, denotational semantics also applies to logical systems: we consider logical systems as type systems for certain programs. Models of *classical* logics must in particular interpret the logical equivalence between a formula A and its double negation $\neg\neg A$.

$$[\![A]\!] \equiv [\![\neg\neg A]\!].$$

Models of Linear Logic [59] distinguish between *linear* functions $\ell \in \mathcal{L}([\![A]\!], [\![B]\!])$and *non-linear functions* $f \in \mathcal{C}([\![A]\!], [\![B]\!])$. Moreover, Linear Logic is a classical logic with a *linear negation*. If $(\_)^{\perp}$ interprets this linear negation, then a model of Linear Logic should satisfy:

$$[\![A]\!] = [\![A]\!]^{\perp\perp}.$$

Equation 1.2 in the linear context translates into the usual monoidal closedness:

$$\mathcal{L}([\![A]\!] \otimes [\![A]\!], [\![A]\!]) \simeq \mathcal{L}([\![A]\!], \mathcal{L}([\![A]\!], [\![A]\!])).$$

Smooth semantics of Differential Linear Logic In denotational models of Differential Linear Logic, codereliction $\bar{d}$ is interpreted by the operator mapping a function $f \in \mathcal{C}([\![A]\!], [\![B]\!])$ to its differential at 0: $D_0(f) \in \mathcal{L}([\![A]\!], [\![B]\!])$. Thus functions must be *smooth*: everywhere infinitely differentiable. Differential Linear Logic also features sums of proofs, and thus one must interpret proofs in an additive category, where each hom-set $\mathcal{C}^{\infty}(A, B)$ is endowed with a commutative monoidal law $+_{A,B}$. This justifies searching for models of Differential Linear Logic within Algebra. In fact, Differential Linear Logic stemed from a study of vectorial models of Lineac Logic [18, 19] inspired from domain theory and coherent spaces. In these models, formulas were interpreted as vector

[1] and is interpreted in topological vector spaces with the differential at 0 of a function.

spaces of sequences, and proofs as power series between this spaces. The differentiation of a power series can then be computed immediately:

$$\bar{d}_A : x \mapsto (f = \sum_n f_n \in \mathcal{C}(A, B)) \mapsto f_1(x)),$$

where $x \in A$, and f_n is a n-monomial resulting from a n-linear map $\tilde{f} \in \mathcal{L}(A^{\otimes^n}, B)$. In this thesis, we emphasize the fact that formulas of Differential Linear Logic should be be interpreted by *continuous objects* , so as to rejoin the mathematical intuitions about differentiation. A natural question to ask is then: is there of a model of Differential Linear Logic in which formulas and proofs are interpreted by continuous objects and differentiation on smooth functions between, let's say, Banach spaces ?

Trying to interpret Linear Logic with traditional objects of analysis was tackled by Girard by constructing Coherent Banach spaces [32]. This attempt fails, as imposing a norm on spaces of smooth functions[2] is too strong a requirement. Thus one must relax the condition on normed spaces and consider more generally topological vector spaces. Ehrhard [18] considers specific spaces of sequences, called Köthe spaces, which are in particular complete, and thus construct a model of DiLL with an involutive linear negation. This models however relies on a discrete setting, as operations are defined on bases of the considered vector spaces. Blute, Ehrhard, and Tasson used the convenient analysis setting introduced by Frölicher, Kriegl and Michor [26, 53]. They construct a model of Intuitionistic Differential Linear Logic interpreting formulas by *bornological Mackey-complete* locally convex and Hausdorff topological vector spaces. Linear proofs are modeled by linear *bounded* maps and general proofs by a certain class of smooth functions which verifies equation 1.2.

Classical and smooth semantics of Differential Linear Logic Differential Linear Logic is also (linear) classical. In the setting of vector spaces, the linear negation of a formula is interpreted by the space of linear forms on the interpretation of a formula:

$$A^{\perp} \simeq \mathcal{L}([\![A]\!], \mathbb{K}) := A'.$$

In the setting of topological vector spaces, a model of classical Linear Logic must interpret formulas by *reflexive* spaces, which by definition are a the topological vector spaces such that:

$$E \simeq E''$$

However, the only smooth model of Differential Linear Logic [6] did not interpret classicality. In fact, reflexivity and smoothness work as opposite forces in the theory of topological vectors spaces. While interpreting smoothness requires some notion of *completeness*[3] — that is spaces with a topology fine enough to make Cauchy filters converge — reflexivity requires a topology coarse enough so as to keep the dual E' small enough. At the beginning of this thesis was thus the question of finding a smooth model for DiLL which would also feature a linear involutive negation.

Reflexivity and (co-)dereliction We argue moreovoer that computations in Differential Linear Logic strongly call for reflexivity. In that perspective, works by Mellies [62], inspired by semantics and which focuses on the geometric nature of linear negation in logic, is clarifying. On one hand, smoothness and a linear involutive negation gives us an exponential interpreted as a space of *distributions with compact support*. Consider indeed reflexive spaces E and F. Then the interpretation of equation 1.1 gives us:

$$!E \simeq \mathcal{C}^{\infty}(E, \mathbb{R})'$$

and this interpretation of the exponential as a central space of functional analysis leads the way to an exciting transfer of techniques between Proof Theory and Analysis. One the other hand, in a smooth model of DiLL, reflexivity leads to an elegant and general interpretation of dereliction and co-dereliction:

$$d_E : \begin{cases} \mathcal{C}^{\infty}(E, \mathbb{R})' \to E \\ \qquad \phi \mapsto \phi_{|\mathcal{L}(E,\mathbb{R})} \in \mathscr{E}'' \simeq E \end{cases} \tag{1.3}$$

$$\bar{d}_E : \begin{cases} \qquad E \simeq E'' \to \mathcal{C}^{\infty}(E, \mathbb{R})' \\ \phi \in \mathcal{L}(E, \mathbb{R})' \mapsto \phi \circ D_0 = (f \in \mathcal{C}^{\infty}(E, \mathbb{R}) \mapsto \phi(D_0(f))) \end{cases} \tag{1.4}$$

[2] In fact Girard uses analytical functions.

[3] see section 3.1.5

The equation on d_E is made possible by the fact that $\mathcal{L}(E,\mathbb{R}) \subset \mathcal{C}^\infty(E,\mathbb{R})$, and the second by the fact that $D_0(f)$ is by definition a linear function for every $f \in \mathcal{C}^\infty(E,\mathbb{R})$. This understanding of dereliction and co-dereliction as operators on *spaces of functions* allows a generalization from D_0 to Differential Operators in Chapter 8.

Functional analysis and distribution theory This thesis heavily uses tools from the theory of topological vector spaces [66] [44] [51] and distribution theory [69] to construct denotational models of (classical) Differential Linear Logic. The algebraic constructions interpreting the linear implication, the multiplicative conjunction $\otimes$ and the duality are straightforward: they consist respectively in taking spaces of linear continuous maps $\mathcal{L}(E,F)$, tensor product $E \otimes F$, and the spaces of scalar linear continuous maps E'. The difficulty lies in choosing the good topology to put on these spaces. We interpret the exponential ! by spaces of distributions with compact support $\mathcal{C}^\infty(E,\mathbb{R})'$. On these spaces one can use the theory of Linear Partial Differential Operators [43].

While functional analysis was developed with an emphasis on cartesian structures and smooth functions, the search for models of Linear Logic stresses the role of tensor products and reflexivity. Under this light, several results of the theory of locally convex vector spaces find a nice interpretation: Schwartz's Kernel theorem 7.3.5 interprets Seely's isomorphism, and the Mackey-Arens theorem 3.5.3 gives an adjunction with CHU which generates classical models of DiLL (see Part II). Likewise, distribution theory gives an interpretation for the *deduction rules* of Differential Linear Logic, and in particular those which are added to Linear Logic: the contraction rule $\bar{c}$ (see figure 2.6) is interpreted by the convolution 7.3.17, and co-weakening $\bar{w}$ in the case of Linear Partial Differential Equations (see figure 2.6) corresponds to input of a fundamental solution (see definition 8.1.11).

> *Differential Linear Logic provides a polarized syntax for the theories of Topological vector spaces, Distributions, and Linear Partial Differential Equations.*

Content of the thesis

The first part consists in extensive preliminaries:

- Chapter 2 details the syntax and categorical semantics of LL, $DiLL$ and their polarized versions, thus revisiting works and surveys by Girard [30], Mellies [59, 60], Ehrhard [20, 24], Laurent [54, 55]. We detail a denotational model for LL (Köthe spaces [18]) and a intuitionistic denotational model for DiLL (Convenient spaces [6]).

- Chapter 3 exposes the results from the theory of vector spaces used in this thesis. We mainly borrow material from the textbook by Jarchow [44].

The second part develops two classical models of DiLL through polar topologies on the dual:

- Chapter 4 is a quick perspective on quantitative semantics and duality in vector space. In particular, we detail work by Barr [2] in which he understands the Weak and Mackey topologies as right and left adjoint to the embedding of vector spaces in dual pairs.

- Chapter 5 is adapted from the published article [48]. It provides a classical and quantitative model for DiLL using *weak topologies* on the topological vector spaces interpreting formulas of DiLL.

- Chapter 6 refines the model of convenient vector spaces [6] into a *polarized*, *classical* ans *smooth* model of DiLL. This work is inspired by, but distinct from, a submitted work in collaboration with Y. Dabrowski (appendix B) focusing on unpolarized model of DiLL where the dual of $\otimes$ is interpreted by Schwartz' ε product.

The third part applies the theory of distributions and Linear Partial Differential Equations to Differential Linear Logic.

- Chapter 7 exposes a model of DiLL where formulas are interpreted as nuclear spaces and exponentials as spaces of distributions with compact support. The chapter begins with an exposition of the theory of nuclear spaces and the theory of distributions. Then it develops a model without higher-order for DiLL_0[4]. This is an adaptation of work recently published [47]. We then expose the work of a recent collaboration with J.-S. Lemay, generalizing this model to Higher-order.

- Chapter 8 builds two sequent calculi for Linear Partial Differential Equations, a non-deterministic and a deterministic one. The two are based on the same principle: the introduction of a new exponential $!_D$, which corresponds to the distributions ϕ solutions to the equation:

$$D\phi = \psi$$

  where ψ a distribution with compact support. The symbol D denotes either a Linear Partial Differential Operator with constant coefficients, or the differentiation at 0[5]. When $D = D_0$, we have $!_D E \simeq E'' \simeq E$ in a classical model, and the introduced calculi specialises to the standard syntax of DiLL.

[4] Which is DiLL without promotion, the first historical version of DiLL [24].
[5] which is denoted D_0 throughout this thesis.

Models of DiLL: a panorama

This thesis brings a contribution to the semantic study of DiLL, by constructing several models of it. The main body of the manuscripts focuses on polarized models of DiLL_0. Polarization in Linear Logic distinguishes two classes of formulas: the *positive* ones, preserved by the positive connectives, and the *negatives* ones, accordingly preserved by the negative connectives. Thus *polarized models* of DiLL distinguish two kind of spaces, and relaxes topological conditions as not all spaces are required to bear all stability properties.

We give a survey of the characteristics of the existing models and of the ones constructed in this thesis in the following figures:

Models of DiLL

	Polarized	Continuous spaces $E \not\subset \mathbb{K}^{\mathbb{N}}$	Quantitative semantics $f = \sum f_n$
KOTHE [18], recalled in Section 2.2.3			✓
Finiteness spaces [19]			✓
Weak spaces [48]		✓	✓
Convenient spaces [6], recalled in Section 2.4.3		✓	
Mackey-complete spaces and Power series [49],		✓	✓
k-reflexive spaces [17], recalled in section 6.1		✓	
Mackey-complete Schwartz spaces [17], recalled in section 6.1		✓	
Polarized convenient model, Chapter 6	✓	✓	
Nuclear Fréchet spaces, Chapter 7	✓	✓	

Models of DiLL

	Smooth Functions $!E = \mathcal{C}^\infty(E,\mathbb{R})'$	Involutive Linear Negation $E \simeq E''$	Higher-order $!!E$
KOTHE [18], recalled in Section 2.2.3		✓	✓
Finiteness spaces [19]		✓	✓
Wear spaces [48]		✓	✓
Convenient spaces [6],	✓		✓
Mackey-complete spaces [49]			✓
k-reflexive spaces [17],	✓	✓	✓
Mackey-complete Schwartz spaces [17],	✓	✓	✓
Polarized convenient model,	✓	✓	✓
Nuclear Fréchet spaces,	✓	✓	✓

Logic and Functional Analysis

The author believes that this semantic study allows one to draw solid ties between polarized Differential Linear Logic and the theory of topological vector spaces. We sum up these correspondences, while referring to a concrete model in which they are verified.

Operations of DiLL

Logical operation	Interpretation	Example
A	E a lcs	That's the case for every model considered above
Linear negation $A^{\perp}$	Linear topological dual E'	In all models except convenient spaces [6], where $A^{\perp}$ is interpreted by the bounded dual $E^{\times}$.
$\parr$	ε	In [17] and its summary in section 6.1, and also chapter 6
$?A$	$\mathcal{C}^{\infty}(E', \mathbb{R}^n)$	In [6], [17], Chapters 6 and 7
$!A$	$\mathscr{E}'(E) := \mathcal{C}^{\infty}(E, \mathbb{R})'$	When $E = \mathbb{R}^n$, this is the case in models of [6], [17] and Chapter 7
$\uparrow A$	E_w	The lcs E endowed with its weak topology in Chapter 5
$\uparrow A$	$\tilde{E}$ or $\hat{E}^M$, a *completion of* E	Any model with smooth functions
$\downarrow A$	$\bar{E}^{born}$, the bornologification of E	In Chapter 6

Exponentials for Differentials Equations

The last chapter details how exponentials should be understood as spaces of solutions for Differential Equations. We sum up the different interpretation of the rules of DiLL in the following table, indexing the different interpretations of the rules of DiLL or $\mathrm{D}-\mathrm{DiLL}$ according to the differential operator considered:

Rules for Differentials Equations

Differential Operator	D_0	D a LPDOcc	Id
Spaces of functions	Linear functions E'	functions $f = Dg$ $D(\mathcal{C}^{\infty}(A))$	all smooth functions $\mathcal{C}^{\infty}(A)$
Exponential $!_D E = D^{-1}(!E)$	$!_{D_0} E := E'' \simeq E$	$!_D A = D(\mathcal{C}^{\infty}(A))'$	$!_{Id} A = !A = \mathcal{C}^{\infty}(A)'$
Interpretation of $d_D : !E \longrightarrow !_D E$	$\phi \mapsto \phi_{\vert (A)'}$	$\phi \mapsto \phi_{\vert D(\mathcal{C}^{\infty}(A))}$	
Interpretation of $\bar{d} : !_D \longrightarrow !_E$	$x \mapsto (f \mapsto d(f)(0)(x))$	$\phi \mapsto (f \mapsto \phi(D(f)))$	
Interpretation of $w : !E \longrightarrow 1$	$\phi \mapsto \int \phi$		
Interpretation of $\bar{w} : 1 \longrightarrow !_E$	$1 \mapsto \delta_0$		
Interpretation of $w_D : !_D E \longrightarrow 1$		$\phi \mapsto \int D\phi$	
Interpretation of $\bar{w}_D : 1 \longrightarrow !_D E$		$1 \mapsto E_D$	
Interpretation of $c : ?E\hat{\otimes}?E \longrightarrow ?E$	$f \otimes g \mapsto f \cdot g$		
Interpretation of $\bar{c} : !E\hat{\otimes}!E \longrightarrow !E$	$\psi \otimes \phi \mapsto \psi * \phi$		
Interpretation of $c_D : ?_D E\hat{\otimes}?E \longrightarrow ?E$		$f \otimes g \mapsto f \cdot g$	
Interpretation of $\bar{c}_D : !_D E\hat{\otimes}!E \longrightarrow !E$		$\psi \otimes \phi \mapsto \psi * \phi$	

Part I

Preliminaries

Chapter 2

Linear Logic, Differential Linear Logic and their models

Tout ce qui est vrai dans le livre est vrai, dans le livre. Tout ce qui n'est pas dans le livre n'est pas dans le livre. Bien sûr tout ce qui est faux dans le livre est faux dans le livre. Tout ce qui n'est pas dans le livre, n'est pas dans le livre, vrai ou faux.
Claude Ponti, La course en livre, 2017.

Linear Logic and its differential extension come from a rich entanglement of syntax and semantics. In particular, Linear Logic comes from the understanding and decomposition of a model of propositional logic. We give in Section 2.2 give an overview of the syntax of Linear Logic, its cut-elimination procedure, and its categorical semantics. We will detail a model of LL constructed by Ehrhard, based on spaces of sequences. Section 2.3.1 details polarized linear logics as introduced by Laurent [54]. We give a categorical semantics following definitions by Melliès [61]. We introduce Differential Linear Logic in Section 2.4 as a sequent calculus, define its categorical semantics following definitions by Fiore [25] and give an example of a smooth Intuitionistic model [6]. In the last Section 2.5, we detail a polarized version of Differential Linear Logic, and detail a categorical semantics for it. It is an adaptation in a sequent calculus of the polarized differential nets by Vaux [79]. The categorical semantics extends the definitions of Melliès [61].

As preliminaries, we want to introduce the Curry-Howard-Lambek correspondence. This correspondence between theoretical programming languages, logic and categories justifies all the research conducted in this thesis: we look in semantics for intuitions about programming theory. Linear Logic and its Differential extension are typical examples of this movement.

Notation 2.0.1. *In this Chapter, we use a few times the basic vocabulary of Chapter 3, in order to give intuitions on the syntax. Section 3.1 covers these definitions. In a first approach, the reader can skip the few reference to the theory of topological vector spaces. Let us just say that the term lcs denotes a locally convex and Hausdorff topological vector space.*

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2.1 The Curry-Howard correspondence

In this section, we will give two examples of the Curry-Howard correspondence: first we show the correspondence between minimal logic, the simply-typed λ-calculus and cartesian closed categories. Then we detail the sequent calculus LK, thus omiting the Curry-Howard correspondance with the $\lambda\mu\tilde{\mu}$-calculus [16] and control categories [72].

2.1.1 Minimal logic and the lambda-calculus

This section consists in a very basic introduction to the Curry-Howard-Lambek correspondence. Programs are understood as terms of the λ-calculus. They are typed by formulas of minimal logic, and these types are interpreted in a cartesian closed category.

Definition 2.1.1. Consider $\mathfrak{A}$ a set of atoms. Formulas of minimal implicative logic are terms constructed via the following syntax: $A, B := a \in \mathfrak{U} \mid A \longrightarrow B$

Definition 2.1.2. Consider X a set of variables. Terms of the λ-calculus are constructed via the following syntax: $t, u := x \mid (t)u \mid \lambda x.t$

Typing judgements are denoted by $\Gamma \vdash t : A$, where t is a term and A a formula, and Γ a list of typing assignments $t' : A'$. Then the λ-calculus is typed according to the following rules: to each variable x is assigned a type A, and then:

$$\frac{}{\Gamma, x : a \vdash x : a}\; x \in X, a \in \mathfrak{A} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \longrightarrow B}\; \longrightarrow \qquad \frac{\Gamma \vdash t : A \longrightarrow B, u : A}{\Gamma \vdash (t)u : B}\; \text{app}$$

The β-reduction rule of λ- calculus is the following:

$$(\lambda x.t)u \longrightarrow t[u/x].$$

It preserves typing according to the cut rule above.

Categories The categorical semantics of a proof system interprets formulas by objects of a given category $\mathcal{C}$, and sequents $A \vdash B$ as morphisms from an object A to an object B. We will see that the morphisms of a cartesian closed category interpret the typing judgements of the simply typed λ-calculus.

Definition 2.1.3. [56] A category $\mathcal{C}$ consists in:

- A collection $\mathcal{O}(\mathcal{C})$ of objects,
- A collection $\mathcal{A}(\mathcal{C})$ of arrows such that each arrow f has a domain $A \in \mathcal{O}(\mathcal{C})$ and $B \in \mathcal{O}(\mathcal{C})$: we write $f : A \longrightarrow B$ and the collection of all arrows with domain A and codomain B is denoted by $\mathcal{C}(A, B)$.
- For each objects A, B and C a binary associative law: $\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$.
- For each object A an arrow $1_A \in \mathcal{C}(A, A)$ such that for any object B, any arrows $f : A \longrightarrow B$ and $g : B \longrightarrow A$ we have:

$$f \circ 1_A \;=\; f \text{ and } 1_A \circ g \;=\; g.$$

A category is said to be *small* if the collection of its objects is a set. A category is said to be *locally small* if every hom-set $\mathcal{C}(A, B)$ is a set. If $\mathcal{C}$ is a category, one construct its opposite category $\mathcal{C}^{op}$ with the same objects but with reverse arrows $\mathcal{C}^{op}(B, A) \;=\; \mathcal{C}(A, B)$.

Definition 2.1.4. A (covariant) *functor* $\mathcal{F}$ from categories $\mathcal{C}$ to $\mathcal{D}$ is a map between the collections of objects and the collection of arrows fo $\mathcal{C}$ and $\mathcal{D}$ respectively such that:

$$\mathcal{F}(f : A \longrightarrow B) : \mathcal{F}(A) \longrightarrow \mathcal{F}(B) \text{ and } \mathcal{F}(f \circ g) \;=\; \mathcal{F}(f) \circ \mathcal{F}(g).$$

A *contravariant* functor $\mathcal{F}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a covariant functor from $\mathcal{C}$ to $\mathcal{D}^{op}$. Without any further specifications, a functor is always supposed to be covariant.

Example 2.1.5. Consider $\mathcal{C}$ a locally small category. Then for every object $C \in \mathcal{C}$ we have a covariant functor

$$\mathcal{C}(C, \_) : \mathcal{C} \longrightarrow Set, B \mapsto \mathcal{C}(C, B), f \in \mathcal{C}(A, B) \mapsto f^* : g \in \mathcal{C}(C, A) \mapsto f \circ g \in \mathcal{C}(C, B)$$

and a contravariant functor

$$\mathcal{C}(\_, C) : \mathcal{C} \longrightarrow Set, A \mapsto \mathcal{C}(A, C), f \in \mathcal{C}(A, B) \mapsto {}^*f : g \in \mathcal{C}(A, C) \mapsto g \circ f \in \mathcal{C}(B, C).$$

Definition 2.1.6. We write $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$. A *natural transformation* η between two functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ is a collection of morphisms indexed by objects of $\mathcal{C}$ such that:

$$\eta_A : \mathcal{F}(A) \longrightarrow \mathcal{G}(A) \in \mathcal{D}(\mathcal{F}(A), \mathcal{G}(A))$$

and such that for every object $A, B \in \mathcal{C}$, arrows $f : A \longrightarrow B$ we have $\eta_B \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_A$. This diagram more precisely states the naturality in A of the collection of morphisms η.

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_A"] & \mathcal{F}(B) \arrow[d, "\eta_B"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(B)
\end{tikzcd}$$

Definition 2.1.7. Two functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ are respectively left adjoint and right adjoint if for each object $C \in \mathcal{C}$ and $D \in \mathcal{D}$ we have a bijection:

$$\mathcal{C}(C, \mathcal{G}(D)) \simeq \mathcal{D}(\mathcal{F}(C), D)$$

which is natural in C and D. We write $\mathcal{F} \dashv \mathcal{G}$ and we denote this as:

$$\begin{tikzcd}
C \arrow[r, bend left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}$$

Through this adjunction, there is a natural transformation $d : \mathcal{F} \circ \mathcal{G} \longrightarrow Id_{\mathcal{D}}$ which is called the *co-unit*, such that for any $D \in \mathcal{D}$ $d_D : \mathcal{F}(\mathcal{G}(D)) \longrightarrow D$ is the morphism of $\mathcal{D}$ corresponding to $1_{\mathcal{G}(D)} \in \mathcal{C}(\mathcal{G}(D), \mathcal{G}(D))$ through the adjunction. Similarly, there is a natural transformation $d : Id_{\mathcal{C}} \longrightarrow \mathcal{G} \circ \mathcal{F} \longrightarrow Id_{\mathcal{C}}$ which is called the *unit*, such that for any $C \in \mathcal{C}$ $d_C : C \longrightarrow \mathcal{G}(\mathcal{F}(C))$ is the morphism of $\mathcal{C}$ corresponding to $1_{\mathcal{F}(V)} \in \mathcal{D}(\mathcal{F}(C), \mathcal{F}(C))$ through the adjunction.

Definition 2.1.8. The categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent* through an adjunction if the unit and the co-unit are isomorphisms.

The definition of product categories and bifunctors is straightforward [56]. A terminal object in a category $\mathcal{C}$ is an object $\perp$ of C such that for every object A of $\mathcal{C}$, there is a unique arrow $t_A : A \longrightarrow \perp$.

Definition 2.1.9. A cartesian category is a category $\mathcal{C}$ endowed with a terminal object $\perp$ and binary law $\times : C \times C \longrightarrow C$ such that for every objects $A, B \in \mathcal{C}$, there are arrows (*projections*)

$$\pi_A : A \times B \longrightarrow A,$$

$$\pi_B : A \times B \longrightarrow B$$

such that for every object C of $\mathcal{C}$, for every arrows $f \in \mathcal{C}(C, A)$, $g \in \mathcal{C}(C, B)$, there is a unique arrow $\langle f, g\rangle \in \mathcal{C}(C, A \times B)$ satisfying $f = \pi_A \circ \langle f, g\rangle$ and $g = \pi_B \circ \langle f, g\rangle$.

Then one can show that this binary rule $\times$ is commutative and associative. The definition of a n-ary product is thus immediate. This category is *cartesian closed* if for every object $B \in \mathcal{C}$ we have a functor $[B, \_] : \mathcal{C} \longrightarrow \mathcal{C}$ and an adjunction:

$$\mathcal{C}(A \times B, C) \simeq \mathcal{C}(A, [B, C])$$

Thus to the morphism $id_{[A,B]} \in \mathcal{C}([A, B], [A, B])$ corresponds a unique morphism:

$$ev_{A,B} : [A, B] \times A \longrightarrow B.$$

Interpretation of the simply typed λ-calculus in categories Consider $\mathcal{C}$ a cartesian closed category. With every typing judgement one assigns an object in $\mathcal{C}$:

- If x is a variable, $x : A$ is matched to an arbitrary object $A \in \mathcal{C}$,
- $t : A \longrightarrow B$ is matched to $[A, B]$.

To a typing context $x_1 : A_1, ..., x_n : A_n$ one assigns the product of the interpretation of the types. To every typing derivation $\Gamma \vdash t : B$ one assigns a morphism in $\mathcal{C}(\Gamma, B)$ according to the following rules:

- To the axiom $x_1 : A_1, ..., x_n : A_n \vdash x_i : A_i$ corresponds the i-th projection $\pi_i : A_1 \times ... \times A_n \longrightarrow A_i$ on the last item of a product.
- From an arrow $f \in \mathcal{C}(A_1 \times ... \times A_n \times A, B)$ corresponding to $x_1 : A_1, ..., x_n : A_n, t : A \vdash u : B$, one constructs via the closedness property an arrow in $\mathcal{C}(A_1 \times ... \times A_n, [A, B])$.
- From an arrow $f \in \mathcal{C}(A_1 \times ... \times A_n, [A, B])$ corresponding to $x_1 : A_1, ..., x_n : A_n \vdash u : A \longrightarrow B$ and an arrow $g \in \mathcal{C}(B_1 \times ... \times B_n, A)$ corresponding to $y_1 : B_1, ..., y_n : B_m \vdash t : A$, one constructs via $ev_{A,B}$ an arrow in $\mathcal{C}(A_1 \times ... \times A_n \times B_1 \times ... \times B_m, B)$.

2.1.2 LK and classical Logic

Classical logic is the logic satisfying the excluded middle: the formula $A \vee \neg A$ is provable, even if one cannot construct a proof of A or a proof of $\neg A$. This is equivalent as being allowed to eliminate the double-negation: if we can prove the negation of the negation of A, then one can prove A. In other words, if we have a proof that the negation of A implies the absurd, then this is a proof for A. Classical logic have been criticized for being to non-constructive: a witness for the fact that the negation of A is impossible is not really an inhabitant of A. However, it was highlighted by Griffin [35] that classical logics has a computational content, as it allows to type exceptions handlers.

This will be illustrated here by detailing the propositional classical sequent calculus LK introduced by Gentzen [28]. A *sequent calculus* consists in formulas, sequents, and rules allowing to deduce sequents from axioms. A *formula* is a tree constructed from atoms, logical connectives and constants. Let $\mathfrak{A}$ be a set of atoms $a \in \mathfrak{A}$.

A *sequent* is an expression $\Gamma \vdash \Delta$, where Γ and Δ are both finite sequences of formulas. The (non-linear, usual) intuition is that writing $\Gamma \vdash \Delta$ means that from the conjunction of the formulas of Γ (*i.e.* from all the formulas of Γ), one can deduce the disjunction of the formulas of Δ.

Definition 2.1.10. The formulas of LK are defined according to the following grammar:

$$A, B := \top \mid \bot \mid a \mid A \vee B \mid A \wedge B.$$

In the above definition, $\vee$ denotes the disjunction (A or B) while $\wedge$ denotes the conjunction (A and B). One also defines an involutive negation $\neg(\cdot)$ on formulas:

$$\neg\top = \bot \qquad \neg\bot = \top \qquad \neg(A \vee B) = \neg A \wedge \neg B \qquad \neg(A \wedge B) = \neg A \vee \neg B.$$

As we are in a classical logic, A should be provable under the hypotheses of Δ if and only if no contradiction arise from the set of hypotheses $\Delta, A^{\perp}$, or likewhise if one can prove $\Delta^{\perp}, A$. This justifies a monolateral presentation $\vdash \Gamma$ of sequents, were $\vdash \Gamma^{\perp}, \Delta$ stands for $\Gamma \vdash \Delta$. The deduction rules for LK are detailed in figure 2.1.2. What happens between a bilateral and a monolateral presentation is the following: while in a bilateral presentation, one would need to introduce a rule (for example a logical rule introducing a connective) to the left and to the right, it is now enough to introduce it to the right. For example, a bilateral version of LK features the rules:

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge^R add \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge^{L1} add \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge^{L2} add$$

and

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee^L add \qquad \frac{\Gamma \vdash A\Delta}{\Gamma \vdash A \vee B\Delta} \vee^{R1} add \qquad \frac{\Gamma, \vdash B\Delta}{\Gamma \vdash A \vee B\Delta} \vee^{R2} add$$

Notice the symmetry between the introduction of a connective to the left and the introduction of its dual on the right. In the monolateral presentation of LK in figure 2.1.2, rules on the left are replaced by their dual rules on the right. What makes this logic classical is precisely the fact that $\neg A$ is involutive: a formula is logically isomorphic to its double negation.

The identity group

$$\frac{}{\vdash A, \neg A}\ ax \qquad\qquad \frac{\vdash A, \Delta \qquad \vdash \neg A, \Gamma}{\Delta, \Gamma}\ \text{cut}$$

The logical group

$$\frac{}{\vdash 1}\ \top^{mull} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot^{mul} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee^{mull} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B}\ (\wedge^{mull})$$

$$\frac{}{\vdash \Gamma, \top}\ \top^{add} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge^{add} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\ \vee_L^{add} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \vee B}\ \vee_R^{add}$$

The structural group

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ \text{contraction} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ \text{weakening} \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)\ \text{ for } \sigma \in \mathfrak{S}(|\ \Gamma\ |).}\ \text{commutation}$$

Figure 2.1: The inference rules for LK.

Structural rules Structural rules of LK are the ones describing the operations on Γ and Δ that do not change the logical content of the sequent $\Gamma \vdash \Delta$. Namely, contraction allows the duplication of a formula on the left (an hypothesis can be used twice), or on the right (if one can prove a formula once, then one can do it twice).

Additive and multiplicative rules When such structural rules are allowed, then each rule has two versions, an additive one and a multiplicative one. Additive ones are the one that do not make a distinction between the context of two given sequents of a rule, while multiplicative ones are those which concatenate the context of the premisses. These additive and multiplicative presentations are equivalent because of the structural rules. We will see later in Section 2.2 how linear logic, which restrains the use of structural rules on specific formulas, imposes a distinction between an additive and a multiplicative disjunction or conjunction.

Reversible rules Another distinction is to be made between *reversible* and *irreversible* rules. Reversible rules are the one such that if the provability of the conclusion is equivalent to provability of the premisses. Irreversible rules are those for which this is not true. In LK, the additive rule for $\wedge$ and the multiplicative rule for $\vee$ are reversible, while the other logical rules are irreversible. Again, polarized linear logic will refine this by distinguishing negatives and positives formulas.

Proofs A *proof* consists in a proof tree, with axioms as leaves, inference rules at each branching and a sequent at the root. Let us introduce a few more definitions. An inference rule between two sequents is *derivable* if the conclusion sequent can be derived from the premisse. A sequent is *provable* in a proof system if it can be derived from axioms using the inference rules of the proof system.

Example 2.1.11. Here are a few examples of proof-trees for derivable sequents in LK.

$$\frac{\dfrac{}{\vdash A, \neg A}\ ax}{\vdash A \vee \neg A}\ \vee^{add} \qquad\qquad \frac{\dfrac{\vdash \Delta, A \qquad \vdash \Delta, B}{\vdash \Delta, \Delta, A \wedge B}\ (\wedge^{mul})}{\vdash \Delta, A \wedge B}\ \text{contraction}$$

$$\frac{\dfrac{\dfrac{}{\vdash \neg B, B} \qquad \vdash \Delta, A}{\vdash \Delta, B, A \wedge \neg B}\ (\wedge^{mul}) \qquad \vdash \Gamma, \neg A \vee B}{\vdash \Gamma, B}\ \text{cut}$$

A fundamental result of Gentzen is the *cut-elimination* theorem. It states that any sequent which is provable in LK_{add} is provable without the cut rule, and most importantly gives an deterministic algorithm to eliminate the cut-rules from a proof. That is, any provable sequent can be proved using only rules which will introduce its connectives. This theorem also implies that LK is *coherent*, meaning that $\vdash \bot$ is not provable. Indeed, a cut-free proof of $\bot$ would imply an introduction rule for $\bot$.

2.2 Linear Logic

In this thesis LL (resp. ILL) always refers to *classical* (resp. Intuitionistic) Linear Logic. Likewise, DiLL (resp. $IDiLL$) always refers to *classical* (resp. Intuitionistic) Differential Linear Logic. We refer to a written course by O. Laurent [55] (in french) for a rigourous and detailed exposition of the notions which will be introduced in this first section.

2.2.1 Syntax and cut -elimination

Linear Logic [29] implements the above remarks about the different computational behaviour of the rule by restricting the structural rules on formulas marked by a unary operator $?$. Then additive rules and multiplicative rules are no longer equivalent, since formulas in a sequent cannot be duplicated or inserted at will. Linear Logic introduces two structural rules allowing for the special role of the connective $?$ and its dual $!$. The dereliction d says that A imply $?A$: a linear proof is in particular a non-linear one. Dually, it means that as an object of the context, the hypothesis $!A$ is stronger A. The promotion rule says that if the hypothesis are all reusable, then the conclusion can be made reusable.

Definition 2.2.1. The formulas of Linear Logic are

$$A, B := a \mid a^{\perp} \mid 0 \mid 1 \mid \top \mid \perp \mid A \otimes B \mid A \parr B \mid A \oplus B \mid A \times B \mid !A \mid ?A$$

where $\otimes$ (resp $\&$) denotes the multiplicative (resp. additive) conjunction and $\parr$ (resp $\oplus$) denotes the multiplicative (resp. additive) disjunction.

The *linear negation* of a formula A is denoted $A^{\perp}$ and defined inductively as follows:

$$\begin{array}{rclrcl}
(A \& B)^{\perp} & = & A^{\perp} \oplus B^{\perp} & (A \oplus B)^{\perp} & = & A^{\perp} \& B^{\perp} \\
(A \parr B)^{\perp} & = & A^{\perp} \otimes B^{\perp} & (A \otimes B)^{\perp} & = & A^{\perp} \parr B^{\perp} \\
!A^{\perp} & = & ?A^{\perp} & ?A^{\perp} & = & !A^{\perp} \\
1^{\perp} = \perp & & \perp^{\perp} = 1 & 0^{\perp} = \top & & \top^{\perp} = 0
\end{array}$$

We extend the unary connective of LL to list of formulas: if Γ denotes the list $A_1, ...A_n$, then $!\Gamma$ is $!A_1, ..., !A_n$. We give in figure 2.2 the inference rules for the sequents of Linear Logic. Linear Logic enjoys a terminating cut-elimination procedure [29]: we detail in figure 2.3 the cut-elimination rules for the exponential rules. The logical cut-elimination rules for the other rules follow the pattern of the duality defined above.

2.2.2 Categorical semantics

We refer to a survey by Melliès [59] for an exhaustive study of the categorical semantics of Linear Logic. We choose here to detail the axiomatization by Seely, known as Seely categories, as it will fit well our study of Differential Linear Logic.

Monoidal and $*$-autonomous categories We first describe the structure of monoidal closed categories, which are the good axiomatization for models of MLL. Although this is one the simplest categorical structures, finding examples of those in functional analysis is not at all straightforward, as it requires to solve *"Grothendieck's problème des topologies"* (see Section 3.6).

Definition 2.2.2. A symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ is a category $\mathcal{C}$ endowed with a bifunctor $\otimes$ and a unit 1 with the following isomorphisms:

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$$

$$\rho_A : A \otimes 1 \simeq A, \lambda_A : I \otimes A \simeq A$$

$$sym_{A,B} : A \otimes B \simeq B \otimes A$$

with the following triangle and pentagon commutative diagrams assuring the coherence of the associativity and the symmetry.

The Identity rule

$$\dfrac{}{\vdash A, A^\perp}\ \text{axiom} \qquad\qquad \dfrac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ \text{cut}$$

The multiplicative rules

$$\dfrac{}{\vdash 1}\ (1) \qquad\qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot$$

$$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad\qquad \dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes$$

The additive rules

$$\dfrac{}{\vdash \Gamma, \top}\ \top \qquad\qquad \dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \&$$

$$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_L \qquad\qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_R$$

The Exponential Rules

$$\dfrac{\vdash \Gamma}{\vdash \Gamma, ?A}\ w \qquad\qquad \dfrac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ c$$

$$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ d \qquad\qquad \dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ !$$

Figure 2.2: The inferences rules for Linear Logic

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \dfrac{\vdash \Delta, A^\perp}{\vdash \Delta, ?A^\perp}\ d}{\vdash \Delta, ?\Gamma}\ \text{cut} \quad\rightsquigarrow\quad \dfrac{\vdash ?\Gamma, A \qquad \vdash \Delta, A^\perp}{\vdash \Delta, ?\Gamma}\ \text{cut}$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \dfrac{\vdash \Gamma, ?A^\perp, ?A^\perp}{\vdash \Gamma, ?A^\perp}\ c}{\vdash \Delta, ?\Gamma}\ \text{cut} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \vdash \Gamma, ?A^\perp, ?A^\perp}{\vdash \Delta, \Gamma, ?A^\perp}\ \text{cut} \qquad \dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ !}{\vdash \Delta, ?\Gamma}\ \text{cut}$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \dfrac{\vdash \Delta}{\vdash \Delta, ?A^\perp}\ w}{\vdash \Delta, ?\Gamma}\ \text{cut} \quad\rightsquigarrow\quad \dfrac{\vdash \Delta}{\vdash \Delta, ?\Gamma}\ w$$

$$\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \dfrac{\vdash ?\Delta, ?A^\perp, B}{\vdash ?\Delta, ?A^\perp, !B}\ !}{\vdash ?\Delta, ?\Gamma, !}\ \text{cut} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \vdash ?\Delta, ?A^\perp, B}{\vdash ?\Delta, ?\Gamma, B}\ \text{cut}}{\vdash ?\Delta, ?\Gamma, !B}\ !$$

Figure 2.3: The logical cut-elimination rules for the exponential rules of LL

$$
\begin{tikzcd}
 & A \otimes B & \\
(A \otimes 1) \otimes B \arrow[ur, "\rho_A \otimes 1_B"] \arrow[rr, "\alpha_{A,1,B}"] & & A \otimes (1 \otimes B) \arrow[ul, "1_A \otimes \lambda_B"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C \otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A \otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C \otimes D}"] & & (A \otimes (B \otimes (C \otimes D))) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B \otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u]
\end{tikzcd}
$$

Definition 2.2.3. A monoidal closed category is a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ such that for each object A we have a adjunction

$$\_ \otimes A \dashv [A, \_]$$

which is natural in A. The object $[A, B]$ is called an internal-hom in $\mathcal{C}$.

Definition 2.2.4. A $*$-*autonomous category* [1] is a symmetric monoidal closed category

$$(\mathcal{C}, {}_{\mathcal{C}}, 1_{\mathcal{C}}, (\cdot \multimap \cdot)_{\mathcal{C}})$$

with an object $\perp$ such that the transpose $A \longrightarrow (A \multimap \perp) \multimap \perp$ to the natural transformation $ev_{A,\perp} : A \otimes (A \multimap \perp) \longrightarrow \perp$ is an isomorphism for every A.

Notation 2.2.5. *We write* $\delta_A \ = \ ev_{A,\perp}$.

Example 2.2.6. The category of *finite-dimensional* real vector spaces and linear functions between them, endowed with the algebraic tensor product is $*$-autonomous.

The previous example allows us to introduce informally notations which will be recalled and formally defined in Chapter 3 and widely used throughout this thesis: when E is a real locally convex and Haussdorf topological vector space (for example, any finite-dimensional or any normed space will do), then we denote by E' the vector space consisting of all the linear continuous scalar maps $\ell : E \longrightarrow \mathbb{R}$. Several topologies can be put on this vector space, although one has a canonical one which corresponds to the topology of uniform convergence on bounded subsets of E. This topology is called the strong topology. When E is normed, this corresponds to endowing E' with the following norm :

$$\|\cdot\|_1 : f \mapsto sup_{\|x\| \leqslant 1} |f(x)|$$

Thus a space is said to be *reflexive* when it is linearly homeomorphic to its double dual.

Example 2.2.7. The category of Hilbert spaces is *not* $*$-autonomous: if indeed any Hilbert space is reflexive, that is ismorphic to its double dual, Hilbert spaces and linear continuous functions between them is not a monoidal closed category. Indeed, the space of linear continuous functions from a Hilbert space to itself is not a Hilbert, nor it is reflexive in general. More generaly, the category of reflexive spaces and linear continuous maps between them is not $*$-autonomous, as reflexive spaces are not stable by topological tensor products nor by linear hom-sets.

Remark 2.2.8. We have $(\perp \multimap \perp) \simeq 1_{\mathcal{C}}$.

A particular degenerate example of $*$-autonomous categories are those where the duality is a strong monoidal endofunctor (see Definition 2.2.11) on $(\mathcal{C}, {}_{\mathcal{C}}, 1_{\mathcal{C}})$:

Definition 2.2.9. A compact-closed category is a $*$-autonomous category where for each objects A and B there is an isomorphism natural in A and B:

$$A^* \otimes_{\mathcal{C}} B^* \simeq (A \otimes_{\mathcal{C}} B)^*.$$

Example 2.2.10. The category of *finite-dimensional* real vector spaces is compact closed, as any finite dimensional vector space is linearly isomorphic to its dual.

Interpreting MLL Consider $A_1, ..., A_n$ formulas of LL and $\vdash A_1, ..., A_n$ a sequent of formulas of LL. In a monoidal closed category $\mathcal{C}$, we interpret A_i as an object $[\![A_i]\!]$:

- We interpret a formula $A \otimes B$ by $[\![A]\!]_{\mathcal{C}}[\![B]\!]$ ans thus the formula 1 by $1_{\mathcal{C}}$.
- We interpret $\perp$ by 1^*.
- We interpret $[\![A \parr B]\!]$- as $[\![A^{\perp}]\!] \multimap [\![B]\!]$.

Thus, in a $*$-autonomous category we have $([\![A^{\perp}]\!] \otimes [\![B]\!]^*)^* = [\![A]\!] \multimap [\![B]\!]$, and $[\![A^{\perp}]\!] = [\![A]\!]^*$.

Therefore, there is a natural isomorphism between the set of morphisms

$$f : 1_{\mathcal{C}} \longrightarrow [\![A_1]\!]^* \multimap (.... \multimap [\![A_n]\!])_{\mathcal{C}}$$

of $\mathcal{C}$ interpreting $\vdash A_1, ..., A_n$ and the set of morphisms $f : A_1 \otimes ...A_{n-1} \longrightarrow A_n$. This being said, one constructs by induction on their proof tree the interpretation $[\![\pi]\!]]$ of a proof π ofMLL. We interpret the sequent $\vdash A_1, ..., A_n$ as a morphism $1 \longrightarrow [\![A_1 \parr \cdots \parr A_n]\!]$.

- The interpretation of the axiom rule is $\vdash A^{\perp}, A$ is the morphism $1 \longrightarrow (1_{[\![A]\!]} \in]\!]A[\![\multimap [\![A]\!]$.
- If $f : [\![\Gamma^{\perp}]\!] \longrightarrow [\![A]\!]$ interprets proof π of $\vdash \Gamma, A$ and $g : [\![A]\!] \longrightarrow [\![\Delta]\!]$ interprets a proof π' of $\vdash A^{\perp}, \Delta$, then the proof of the sequent Γ, Δ resulting from the cut between the π and π' is $g \circ f : [\![\Gamma^{\perp}]\!] \longrightarrow [\![\Delta]\!]$.
- The interpretation of the introductions of 1 and $\perp$ correspond respectively to the identity map $1_{1_{\mathcal{C}}} : 1_{\mathcal{C}} \longrightarrow 1_{\mathcal{C}}$ an to the post-composition by the isomorphism $[\Gamma] \simeq [\![\Gamma]\!]^* \multimap \perp$.
- The interpretation of $\vdash \Gamma, A, B$ and of $\vdash \Gamma, A \parr B$ are the same.
- From maps $f : [\![\Gamma^{\perp}]\!] \longrightarrow [\![A]\!]$ and $g : [\![\Delta^{\perp}]\!] \longrightarrow [\![B]\!]$, one constructs the image of f and g by the bifunctor $\otimes$: $f \otimes g : [\![\Gamma^{\perp}]\!] \otimes [\![\Delta^{\perp}]\!] \longrightarrow [\![A]\!] \otimes [\![B]\!]$.

Interpreting MALL The interpretation of the additive connectives and rules are done via a cartesian structure $(\times, \top, t)$ on $\mathcal{C}$, as detailed in Section 2.1.1. Recall that $t_A : A \longrightarrow \perp$ describe the fact that $\perp$ is a terminal object.

In a $*$-autonomous category with a cartesian product, the dual $(A^* \times B^*)^*$ of a product is a co-product, interpreting the connective $\oplus$.

- If $f : [\![\Gamma^{\perp}]\!] \longrightarrow [\![A]\!]$ interprets a proof of $\vdash \Gamma, A$ then if we denote by $\iota_1 : A \longrightarrow A \otimes B$ the canonical injection of A in $A \oplus B$, the maps $\iota \circ f : [\![\Gamma^{\perp}]\!] \longrightarrow [\![A \oplus B]\!]$ interprets the proof of $\vdash \Gamma, A \oplus B$. The rule $\oplus_R$ is treated likewise.
- If $f : [\![\Gamma^{\perp}]\!] \longrightarrow [\![A]\!]$ interprets a proof of $\vdash \Gamma, A$ and $g : [\![\Gamma^{\perp}]\!] \longrightarrow [\![B]\!]$ interprets a proof of $\vdash \Gamma, B]\!]$, the product $\langle f, g \rangle : [\![\Gamma^{\perp}]\!] \longrightarrow [\![A]\!] \times [\![B]\!]$ interprets the proof of $\vdash \Gamma, A \,\&\, B$.

Strong monoidal co-monads Once the *linear* part of LL is interpreted, one needs to interpret non-linear proofs. This is done through linear/non-linear adjunctions [4], or equivalently through strong-monoidal co-monads [71]. The second point of view is the one developed here.

Definition 2.2.11. A strong monoidal functor between two monoidal categories $(\mathcal{C}, _{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, _{\mathcal{D}}, 1_{\mathcal{D}})$ is a functor equipped with natural isomorphisms:

$$m_{A,B} : \mathcal{F}(A \otimes_{\mathcal{C}} B) \simeq \mathcal{F}(A) \otimes_{\mathcal{D}} \mathcal{F}(B). \text{ and } m_0 : \mathcal{F}(1_{\mathcal{C}}) \simeq 1_{\mathcal{D}}$$

such that the following diagrams commute:

$$\begin{tikzcd}
(\mathcal{F}(x) \otimes_{\mathcal{D}} \mathcal{F}(y)) \otimes_{\mathcal{D}} \mathcal{F}(z) \arrow[r, "\alpha^{\mathcal{D}}_{\mathcal{F}(x),\mathcal{F}(y),\mathcal{F}(z)}"] \arrow[d] & \mathcal{F}(x) \otimes_{\mathcal{D}} (\mathcal{F}(y) \otimes_{\mathcal{D}} \mathcal{F}(z)) \arrow[d] \\
\mathcal{F}(x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{D}} \mathcal{F}(z) \arrow[r, "\mathcal{F}(\alpha^{\mathcal{C}}_{x,y,z})"] \arrow[d] & \mathcal{F}(x) \otimes_{\mathcal{D}} \mathcal{F}(y \otimes_{\mathcal{C}} z) \arrow[d] \\
\mathcal{F}((x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{C}} z) \arrow[r, "\mathcal{F}(\alpha^{\mathcal{C}}_{x,y,z})"] & \mathcal{F}(x \otimes_{\mathcal{C}} (y \otimes_{\mathcal{C}} z))
\end{tikzcd}$$

$$\begin{tikzcd}
1_{\mathcal{D}} \otimes_{\mathcal{D}} \mathcal{F}(x) \arrow[r, "m_0 \otimes id"] \arrow[d] & \mathcal{F}(1_{\mathcal{C}}) \otimes_{\mathcal{D}} \mathcal{F}(x) \arrow[d] \\
\mathcal{F}(x) \arrow[r, "\mathcal{F}(\lambda_x^{\mathcal{C}})"] & \mathcal{F}(1 \otimes_{\mathcal{C}} x)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(x) \otimes_{\mathcal{D}} 1_{\mathcal{D}} \arrow[r, "id \otimes m_0"] \arrow[d] & \mathcal{F}(x) \otimes_{\mathcal{D}} \mathcal{F}(1_{\mathcal{C}}) \arrow[d] \\
\mathcal{F}(x) \arrow[r, "\mathcal{F}(\rho_x^{\mathcal{C}})"] & \mathcal{F}(x \otimes_{\mathcal{C}} 1)
\end{tikzcd}$$

Definition 2.2.12. A comonad on a category $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \longrightarrow \mathcal{C}$ with natural transformations $\mu : T \longrightarrow T \circ T$ and $d : T \longrightarrow Id$ satisfying the following commutative diagrams for each object A of $\mathcal{C}$.

$$\begin{tikzcd}
T^3(A) & T^2(A) \arrow[l, "T(\mu_A))"'] \\
T^2(A) \arrow[u, "\mu_{T(A)}"] & T(A) \arrow[l, "\mu_A"] \arrow[u, "\mu_A"']
\end{tikzcd}$$

$$\begin{tikzcd}
T(A) & T^2(A) \arrow[l, "d_{T(A)}"'] \\
T^2(A) \arrow[u, "T(d_A)"] & T(A) \arrow[l, "\mu_A"] \arrow[u, "\mu_A"'] \arrow[ul, equal, "Id"]
\end{tikzcd}$$

From every adjunction $\mathcal{F} \dashv \mathcal{G}$, one gets a comonad on $\mathcal{D}$ by composing F and G: $\mathcal{F} \circ \mathcal{G} : \mathcal{D} \to \mathcal{D}$. With the above notations, the co-unit is $d_D : \mathcal{F} \circ \mathcal{G}(D) \longrightarrow D$ is the image of $1_{\mathcal{G}(D)}$ via the isomorphism $\mathcal{D}(\mathcal{F} \circ \mathcal{G}(D), D) \simeq \mathcal{C}(\mathcal{G}(D), \mathcal{G}(D))$, and the comultiplication is $\mu : \mathcal{F} \circ \mathcal{G} \to (\mathcal{F} \circ \mathcal{G})^2$.

Definition 2.2.13. The coKleisli category of a co-monad T is the category $\mathcal{C}_T$ whose objects are objects of $\mathcal{C}$, and such that $\mathcal{C}_T(A, B) = \mathcal{C}(TA, B)$.
Then the identity in $\mathcal{C}_T$ of an object A corresponds in $\mathcal{C}$ to $d_A : TA \longrightarrow A$, and the composition of two arrows $f : TA \longrightarrow B$ and $g : TB \longrightarrow C$ corresponds in $\mathcal{C}_T$ to the arrow:

$$g \circ^T f \ = \ g \circ Tf \circ \mu_A.$$

Then from every co-monad one constructs an adjunction between a category and its co-Kleisli:

$$\begin{tikzcd}
\mathcal{C}_T \arrow[r, bend left, "\mathbf{T}"] \arrow[r, phantom, "\perp"] & \mathcal{C}_T \arrow[l, bend left, "U"]
\end{tikzcd}$$

in which the functors $\mathbf{T}$ and U are deduced from T:

$$\mathbf{T} : \begin{cases} \mathcal{C}_T \to \mathcal{C} \\ A \mapsto T(A) \\ f \in \mathcal{C}_T(A, B) \mapsto Tf \circ \mu_A \in \mathcal{C}(T(A), T(B)) \end{cases}$$

$$\mathcal{U} : \begin{cases} \mathcal{C} \to \mathcal{C}_T \\ A \mapsto A \\ f : A \longrightarrow B \mapsto f \circ d_A : T(A) \longrightarrow B \end{cases}$$

Interpreting the structural rules of LL Consider $(\mathcal{C}, \otimes, 1, (.)^*)$ a *Seely Category*, that is a $*$-autonomous category with a cartesian product $(\times, \top)$ and endowed with a strong monoidal comonad $! : (\mathcal{C}, \times, \top) \longrightarrow (\mathcal{C}, \otimes, 1)$. Then we interpret the formulas of LL as previously and $[\![!A]\!] = ![\![A]\!]$ and $[\![?A]\!] = (!([\![A]\!]^*))^*$.

Remark 2.2.14. This strong monoidal functor is said to satisfy *Seely's isomorphism*:

$$!(A \times B) \simeq !A \otimes !B \tag{2.1}$$

The strong monoidal functor ! provides natural isomorphisms:

$$m_{A,B} : !(A \times B) \simeq !A \otimes !B, \tag{2.2}$$
$$m_0 : !\top \simeq 1. \tag{2.3}$$

Then one defines the natural transformations:

$$c_A : !A \xrightarrow{!\nabla_A} !(A \times A) \overset{m_{A,A}}{\simeq} !A \otimes !A \tag{2.4}$$
$$w_A : !A \xrightarrow{!n_A} !\top \overset{m_0}{\simeq} 1 \tag{2.5}$$

The natural transformation c models the contraction rule by pre-composition. Likewise, w gives us the interpretation for the weakening rule. The co-unit d gives us the interpretation of the dereliction rule by pre-composition.

Proposition 2.2.15. *[59] The morphisms w_A and d_A define natural transformations w and d.*

If moreover ! is a co-monad, then one gets the interpretation of the promotion rule: from the interpretation $f : ![\![\Gamma^\perp]\!] \longrightarrow A$ of a sequent $\vdash ?\Gamma, A$, one constructs:

$$prom(f) : ![\![\Gamma^\perp]\!] \xrightarrow{\mu_{[\![\Gamma^\perp]\!]}} !![\![\Gamma^\perp]\!] \xrightarrow{!f} !A.$$

Remark 2.2.16. Notice that the categorical interpretation of the promotion rule is the only one using the co-multiplication μ.

Remark 2.2.17. [59, 5.17.14] A co-monad which is a strong monoidal endofunctor on a category $\mathcal{L}$ leads to a monoidal adjunction between $\mathcal{L}$ and its co-Kleisli category $\mathcal{L}_!$:

$$\begin{tikzcd} \mathcal{L}_! \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & \mathcal{L} \arrow[l, bend left, "U"] \end{tikzcd}$$

A smooth classical semantics, exponentials as distributions. In this paragraph we introduce informally the intuition which will guide our understanding of Differential Linear Logic. Indeed, the interpretation of a differentiation operator imposes a smoothness condition of the maps of the co-Kleisli category of the exponential.

Let us observe from a functional analysis point of view what has been defined previously. We interpret formulas A, B by $\mathbb{R}$-vector space E and F with some topology allowing us to speak about limits, continuity and differentiability. The category $\mathcal{L}$ is then the category of these vector spaces and linear continuous maps between them.

Let us denote by $\mathcal{C}$ the co-Kleisli category $\mathcal{L}_!$. The maps $f : E \longrightarrow F$ of this category are linear maps $f : !E \multimap F$, and can be described as power series between E and F in classic vectorial models of LL [18, 19]. The multiplicative conjunction $A \otimes \_$ is then interpreted by a (topological) tensor product, whose right adjoint is the hom-set $\mathcal{L}(A, \_)$.

The interpretation for 1, neutral for $\otimes$, is thus the field $\mathbb{R}$. Following remark 2.2.8, the interpretation for $\perp$ is such that $\mathcal{L}(\perp, \perp) = \mathbb{R}$, thus $\perp$ is one-dimensional and:

$$\perp \simeq 1 \simeq \mathbb{R} \tag{2.6}$$

The duality $(\_)'$ corresponds to some topological dual $E^* = \mathcal{L}(E, \mathbb{R})$, and thus via the definition of $\mathcal{C}$ and the properties of a $*$-autonomous category we get:

$$!E \simeq (!E)'' \simeq \mathcal{L}(!E, \mathbb{R})' \simeq \mathcal{C}(E, \mathbb{R})' \tag{2.7}$$

Thus the exponential $!E$ must be understood as the dual of the space of non-linear scalar functions defined on E, as a topological vector space of linear continuous scalar functions (also called *linear forms*) acting on non-linear continuous scalar functions. Likewise, we get an intuition of the dual of the exponential as a space of non-linear functions.

$$?(E^*) \simeq \mathcal{C}(E, \mathbb{R}) \tag{2.8}$$

If $\mathcal{C}$ is the category of vector spaces and smooth functions, then the exponential is interpreted as a space of distributions [67], see Section 7.3.2 and Chapter 7.

2.2.3 An example: Köthe spaces

In this section we give an overview of the structure ofKOTHE spaces [18] which were used to construct the first vector-space model of linear logic and from which differential linear logic is inspired. We will make use in this section of some basic notions from the theory of topological vector space. These are vector spaces endowed with a locally convex and Hausdorff topology making addition and scalar multiplication continuous. The first two sections of Chapter 3 cover all the notions used here.

If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, $\mathbb{K}^{\mathbb{N}}$ denotes the vector space of all sequences on $\mathbb{K}$. Following Ehrhard, we write for $E \subset \mathbb{K}^{\mathbb{N}}$:

$$E^{\perp} := \{(\alpha_n)_n \in \mathbb{K}^{\mathbb{N}} \mid \forall \lambda \in E, (\lambda_n \mid \alpha_n \mid)_n \in \ell_1\}.$$

Let us remark that this space is always a vector space. It is endowed with the initial topology induced[1] by the semi-norm q_λ for $\lambda \in E$:

$$q_\lambda : \alpha \in E^{\perp} \mapsto \sum_n |\lambda_n \alpha_n|.$$

Proposition 2.2.18. *[44, 1.7.E] A space of* positive *sequence P is said to be a Köthe set*[2] *if it satisfies the following conditions:*

$$\forall \alpha \in P, \exists n \in \mathbb{N} : \alpha_n > 0 \textit{ and } \forall \alpha, \beta \in P, \exists \gamma \in P, \forall n : \max \alpha_n, \beta_n \leqslant \gamma_n.$$

If P is a Köthe set then $P^{\perp}$ is a lcs (that is, a locally convex and Hausdorff topological vector space, see Section 3.1).

This duality operation satisfies the classical axioms for an orthogonality, as detailed at the beginning of Chapter 3:

$$\begin{aligned} E &\subset E^{\perp\perp} \\ E^{\perp} &= E^{\perp\perp\perp} \\ E \subset F &\Rightarrow F^{\perp} \subset E^{\perp} \end{aligned}$$

We now give the definition of Köthe spaces as used by Ehrhard, which coincide with the definition of *perfect sequence space* by Schaefer and Köthe [51, 66]. We therefore call them perfect sequences spaces, as Köthe spaces will be studied more generally in Chapter 3, Section 3.2.

Definition 2.2.19. A perfect sequence space is the data (X, E_X) of a subset $X \subset \mathbb{N}$ and $E_X \subset \mathbb{K}^X$ such that $E_X^{\perp\perp} = E_X$. It is endowed with its *normal topology*, that is with the projective topology induced by the semi-norms:

$$q_\alpha : (\lambda_n)_n \mapsto \|(\lambda_n \alpha_n)_n\|_1 = \sum | \lambda_n \alpha_n |$$

for all $\alpha \in {E_X}^{\perp}$. As the index will be clear form the context, we abusively note E_X to denote the perfect sequence space (X, E_X).

We recall now the setting that makes the category of a model of DiLL.

[1] that is, the coarsest topology such that the q_λ are continuous
[2] this notion is introduced independently in Section 3.2.3

Monoidal and cartesian structure. Let us denote by KOTHE the category of perfect sequence spaces and linear continuous function between them. We consider E_X and F_Y two perfect sequence space, and denote by α_i the sequence null at every index but i and equals 1 at i.

Linear maps ℓ between E_X and F_Y can be seen as some matrix $M \in \mathbb{K}^{X \times Y}$, where $M_{i,j} = l(e_i)_j$.

Proposition 2.2.20. *[18, 2.10,2.11] The space $E \multimap F$ of linear continuous maps from E_X to F_Y correspond to the subset $\mathbb{K}^{X \times Y}$ of all M such that the sum:*

$$\sum_{i,j} M_{i,j} x_i y'_j$$

is absolutely converging for all $x \in E$ and $y' \in F^{\perp}$.

In particular we have $E^{\perp} = \mathcal{L}(E, \mathbb{K})$. The *tensor product* of two pfs E_X and F_Y is the pfs $(E \multimap F^{\perp})^{\perp}$. In particular, if for $x \in E$ and $y \in F$ we denote by $x \otimes y$ the sequence $(x_a y_b)_{a,b} \in \mathbb{K}^{X \times Y}$ we have:

$$E \otimes F = \{x \otimes y \mid x \in E, y \in F\}^{\perp\perp}.$$

This makes KOTHE a monoidal closed category.

The product and co-product constructions are defined as the product and co-product of topological vector spaces and preserve perfect sequence spaces. In particular, the normal topology of the product of two perfect sequence space corresponds to the product of the two normal topologies. However, as it is usual in locally convex spaces, the product and co-product constructions differ only on infinite sets.

Exponentials. The interpretation of exponentials formulas in Kothe spaces embodies with the intuition that non-linear proofs should be represented as analytic functions, and thus Kothe spaces are *quantitative models*. Consider a set X and $\mathcal{M}(X)$ the set of all finite multi-sets of X. If μ is a finite multiset of X and $x \in E$, we write:

$$x^{\mu} = \prod_n x_n^{\mu(n)}.$$

We define the set of scalar entire maps $E \Rightarrow \mathbb{K}$ as the vector space of matrices $M \in \mathbb{K}^{\mathcal{M}(X)}$ such that for all $x \in E$, the following sum converges absolutely:

$$f(x) = \sum_{\mu \in \mathcal{M}(X)} M_{\mu} x^{\mu}.$$

Then we define $!E$ as the perfect sequence space

$$!E = (E \Rightarrow \mathbb{K})^{\perp}.$$

The fact that ! defines a strong monoidal functor follows from combinatorial considerations, which are at stake in every quantitative model of LL ([19], [49], [32]). Let us remark that when X is a singleton, then this definition for entire maps between $\mathbb{K}^X = \mathbb{K}$ and $\mathbb{K}$ is the usual definition for absolutely converging power series with infinite radius of convergence.

Köthe spaces are in fact also a model of Differential Linear Logic (see Section 2.4). The mapping interpreting differentiations on maps of the co-Kleisli category, namely the codereliction, is then interpreted by the linear continuous morphism $\bar{d}_E : E \longrightarrow !E$ such that

$$\bar{d}_E(x) : M \in E \Rightarrow \mathbb{K} \mapsto \sum_{a \in X} M_{\{a\}} x_a.$$

Let us note that from a more analytic point of view, $M_{\{a\}}$ corresponds to $\frac{\partial f}{\partial x_a}(0)$ and M_{μ} to $\frac{\partial^{|\mu|} f}{\partial x_1^{\mu(1)} \ldots x_n^{\mu(n)} \ldots}(0)$.

2.3 Polarized Linear Logic

In this thesis we will consider denotational models of LL in which formulas and their dual cannot be interpreted by the same type of spaces (*e.g.* Fréchet and DF spaces in Chapter 7), or where two different linear negations coexist, the composition of which being the identity (as in chapter 6). This difference witnesses for polarities in the syntax. We recall the definition of polarized Linear Logic in Section 2.3.1. The semantic of polarized Linear Logic is intricate: because of the difference of polarities one must decompose the notions of $\star$-autonomous categories and monoidal closedness along adjunctions. This is done in terms of chiralities in Section 2.5.2.

The Identity rule

$$\frac{}{\vdash A, A^{\perp}}\ (\text{axiom}) \qquad\qquad \frac{\vdash \mathcal{N}, A \qquad \vdash A^{\perp}, \mathcal{M}}{\vdash \mathcal{N}, \mathcal{M}}\ (\text{cut})$$

The multiplicative rules

$$\frac{}{\vdash 1}\ (1) \qquad \frac{\vdash \mathcal{N}}{\vdash \mathcal{N}, \perp}\ (\perp) \qquad \frac{\vdash \mathcal{N}, N, M}{\vdash \mathcal{N}, N ⅋ M}\ (⅋) \qquad \frac{\vdash \mathcal{N}, P \qquad \vdash \mathcal{M}, Q}{\vdash \mathcal{N}, \mathcal{M}, P \otimes Q}\ (\otimes)$$

The additive rules

$$\frac{}{\vdash \mathcal{N}, \top}\ \top \qquad \frac{\vdash \mathcal{N}, N \qquad \vdash \mathcal{N}, M}{\vdash \mathcal{N}, N \,\&\, M}\ \& \qquad \frac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, P \oplus Q}\ \oplus_L \qquad \frac{\vdash \mathcal{N}, Q}{\vdash \mathcal{N}, P \oplus Q}\ \oplus_R$$

The Exponential Rules

$$\frac{\vdash \mathcal{N}}{\vdash \mathcal{N}, ?P}\ w \qquad \frac{\vdash \mathcal{N}, ?P, ?P}{\vdash \mathcal{N}, ?P}\ c \qquad \frac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, ?P}\ d \qquad \frac{\vdash ?\mathcal{N}, N}{\vdash ?\mathcal{N}, !N}\ !$$

Figure 2.4: The inferences rules for $\mathrm{LL}_{\mathrm{pol}}$

Remark 2.3.1. As a disclaimer, let us point out that the smooth polarized model we develop in Chapter 6 (adjunction between CONV and MACKEYCOMPL) and Chapter 7 (adjunction between NF and NDF) are not chiralities, as we have no good interpretation for the shifts. This is not an issue as all categories considered embed in TOPVEC and proofs are interpreted as arrows in TOPVEC (that is, as plain linear continuous maps). Thus a proof of $\vdash \mathcal{N}$ is interpreted as a function $f \in \mathcal{L}(\mathbb{K}, \llbracket \mathcal{N} \rrbracket)$ (and not $f \in \mathcal{L}(\llbracket \uparrow 1 \rrbracket, \llbracket \mathcal{N} \rrbracket)$ as axiomatized in section 2.2.2) and a proof of $\vdash P, \mathcal{N}$ as an arrow $f \in \mathcal{L}((\llbracket P \rrbracket'_{\mu}, \llbracket \mathcal{N} \rrbracket)$ as usually.

2.3.1 $\mathrm{LL}_{\mathrm{pol}}$ and LLP.

In this section we introduce the polarized syntax for LL. Linear Logic, as we have seen, distinguishes an additive and a multiplicative version of conjunction and disjunction. The polarized fragment of Linear Logic $\mathrm{LL}_{\mathrm{pol}}$ [54] refines Linear Logic by distinguishing formulas whose introduction rule is reversible and those whose introduction rule is not. The first are called negative formulas, and the second are called positive formulas.

Definition 2.3.2. Formulas of $\mathrm{LL}_{\mathrm{pol}}$ are constructed according to the following grammar, with a set of negative atoms $\mathfrak{N}$ denoted by a or b.

$$\text{Negative Formulas: } N, M := a \mid ?P \mid N ⅋ M \mid \perp \mid N \,\&\, M \mid \top \mid$$
$$\text{Positive Formulas: } P, Q := a^{\perp} \mid !N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1$$

Negation is defined as before on formulas. However, so as to agree with semantical consideration to come, we will denote $(\quad)^{\perp_L}$ the negation operating on the positives, and $(\quad)^{\perp_R}$ the negation operating on the negatives. Remark then that negation transforms a negative formula into a positive formula, and conversely.

Sequents of $\mathrm{LL}_{\mathrm{pol}}$ are traditionally presented as *focused* sequents $\vdash N_1, ..., N_n \mid P$, where a positive formula is isolated. This correspond to a *proof-search* presentation, where one tries to mechanically construct the proof-tree of a sequent. We will not emphasize this important aspect of proof-theory here, therefore we will use the same sequents as for LL. Note however that by construction, proofs contains at most one positive formula.

Definition 2.3.3. [55, 10.1] We say that a formula A is ?-fix if $?A \vdash A$ is derivable in $\mathrm{LL}_{\mathrm{pol}}$. Then for any A, $?A, \top, \perp$ are ?-fix, and ?-fixeness is preserved by ⅋ and &.

Observe now that, when all atoms in $\mathfrak{N}$ are negative, for all negative formulas N the sequent $?A \vdash A$ is prouvable.

Proposition 2.3.4. *If A is a formula which is ?-fixe, and if $\mathcal{N}$ consists only in negative formulas, then the following rules are admissible:*

$$\frac{\vdash \mathcal{N}}{\vdash \mathcal{N}, A}\ w_N \qquad \frac{\vdash \mathcal{N}, A, A}{\vdash \mathcal{N}, A}\ c_N \qquad \frac{\vdash \mathcal{N}, B}{\vdash \mathcal{N}, !B}\ \bar{d}_{\mathrm{pol}}$$

The Identity rule

$$\frac{}{\vdash A, A^\perp}\ (\text{axiom}) \qquad\qquad \frac{\vdash \mathcal{N}, A \qquad \vdash A^\perp, \mathcal{M}}{\vdash \mathcal{N}, \mathcal{M}}\ (\text{cut})$$

The multiplicative rules

$$\frac{}{\vdash 1}\ (1) \qquad \frac{\vdash \mathcal{N}}{\vdash \mathcal{N}, \bot}\ (\bot) \qquad \frac{\vdash \mathcal{N}, N, M}{\vdash \mathcal{N}, N \parr M}\ (\parr) \qquad \frac{\vdash \mathcal{N}, P \qquad \vdash \mathcal{M}, Q}{\vdash \mathcal{N}, \mathcal{M}, P \otimes Q}\ (\otimes)$$

The additive rules

$$\frac{}{\vdash \mathcal{N}, \top}\ \top \qquad \frac{\vdash \mathcal{N}, N \qquad \vdash \mathcal{N}, M}{\vdash \mathcal{N}, N \,\&\, M}\ \& \qquad \frac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, P \oplus Q}\ \oplus_L \qquad \frac{\vdash \mathcal{N}, Q}{\vdash \mathcal{N}, P \oplus Q}\ \oplus_R$$

The Exponential Rules

$$\frac{\vdash \mathcal{N}}{\vdash \mathcal{N}, N}\ w \qquad \frac{\vdash \mathcal{N}, N, N}{\vdash \mathcal{N}, N}\ c \qquad \frac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, ?P}\ d \qquad \frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\ !_P$$

Figure 2.5: The inferences rules for LLP

Proof. The admissibility is proved using cut-elimination between the structural rules of LL $-$ *pol* and the proof of $?A^\perp \vdash A^\perp$. $\square$

Thus, one can consistently add to $\mathrm{LL}_{\mathrm{pol}}$ the previous rules, which consists respectively in weakening for all the negative, contraction for all the negative, and a general promotion. This leads to the syntax of Polarized Linear Logic LLP detailed in figure 2.5.

Remark 2.3.5. **Comparing generalized promotion anc co-dereliction** For the readers already familiar with Differential Linear Logic (see Section 2.4), let us notice that the generalized promotion figuring in LLP constructs the same proof trees than a polarized co-dereliction $\bar{d}$, although the two behaves differently under cut-elimination:

- A cut between a generalized promotion and dereliction results in the *same* sequent than the cut-elimination between a co-dereliction rule and a dereliction rule (semantically, the composition between the two is the identity). This basically says that generalized promotion could act as a differentiation operator, as it is an operation on smooth maps which is the identity on linear maps:

$$\frac{\dfrac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\ !_P \qquad \dfrac{\vdash \mathcal{M}, N^\perp}{\vdash \mathcal{M}, ?N^\perp}\ d}{\vdash \mathcal{M}, \mathcal{N}}\ \text{cut} \qquad \rightsquigarrow \qquad \frac{\vdash \mathcal{N}, N \qquad \vdash \mathcal{M}, N^\perp}{\vdash \mathcal{M}, \mathcal{N}}\ \text{cut}$$

  Chapter 8 brings more intuitions on the interpretation of this cut-elimination.

- Cut-elimination between a generalized promotion rule and a weakening rule results in several weakening rules followed by a cut with the admissible sequent $?\mathcal{N} \vdash \mathcal{N}$. On the contrary, cut-elimination between $\bar{d}$ and a weakening results in the null proof tree $\mathfrak{o}$.

- The difference between generalized promotion and codereliction rules is better understood when looking at the cut-elimination between one of them and the contraction rule. Consider the cut-elimination between generalized promotion and contraction:

$$\frac{\dfrac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\ !_P \qquad \dfrac{\vdash \mathcal{M}, ?N^\perp, ?N^\perp}{\vdash \mathcal{M}, ?N^\perp}\ c}{\vdash \mathcal{N}, \mathcal{M}}\ cut$$

  After cut-elimination, it results in a proof-tree where promotion is applied twice to N, and cut-elimination applied successively to every copy of $N^\perp$. On the contrary, cut-elimination between a co-dereliction rule and a contraction will applied twice the co-dereliction rule, but it will sum the possibilities for cut-elimination between N and $N^\perp$, and then apply weakening:

$$
\dfrac{\dfrac{\vdash \mathcal{N}, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R}}{\vdash \Gamma, K_{f,g} : A \Rightarrow \mathbb{R}}\,c \qquad \dfrac{\vdash \mathcal{M}, v : A}{\vdash \mathcal{M}, D_0(\_)(v) : !A}\,\bar{d}}{\vdash \Gamma, \mathcal{M}, D_0(K_{f,g})(v) \;=\; D_0(f)(v).g(0) + f(0).D_0(g)(v) : \mathbb{R}}\,\text{cut} \quad \rightsquigarrow \tag{2.9}
$$

$$
\dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R} \qquad \dfrac{\vdash \mathcal{M}, v : A}{\vdash \mathcal{M}, D_0(\_)(v) : !A}\,\bar{d}}{\vdash \Gamma, \mathcal{M}, D_0(f)(v) : \mathbb{R}, g : A \Rightarrow \mathbb{R}}\,\text{cut} \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\,\bar{w}}{\vdash \mathcal{M}, \Gamma, D_0(f)(v) : \mathbb{R}, g(0) : \mathbb{R}}\,\text{cut}
$$

$$
+ \;\dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R} \qquad \dfrac{\vdash \mathcal{M}, v : A}{\vdash \mathcal{M}, D_0(\_)(v) : !A}\,\bar{d}}{\vdash \Gamma, \mathcal{M}, g : A \Rightarrow \mathbb{R}, D_0(g)(v) : \mathbb{R}}\,\text{cut} \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\,\bar{w}}{\vdash \mathcal{M}, \Gamma, f(0) : \mathbb{R}, D_0(g)(v) : \mathbb{R}}\,\text{cut}
$$

This invites to consider co-dereliction as a different way to deal with promotion and resources, and may legitimize DiLL without promotion.

Shifts. Negation changes the polarity of a formula but also its role: from hypothesis to conclusion and vice-versa in logic, or from context to term and vice-versa in the calculus. It appears in game semantics [54] that one can add a shift operation $\uparrow$, which does change the polarity but not the role of the formula. It corresponds to adding a dummy move at the beginning of the play. In our topological model, it will correspond adding points to our space, in order to make it complete (that is, in order to make any Cauchy sequence converge). Thus $\uparrow$ is interpreted by completion functor (see Proposition 3.1.24) from positive to negative, and the forgetful functor from negative to positive. We detail below the syntax of $\mathrm{LL}_{pol,\uparrow\downarrow}$, corresponding to $\mathrm{LL}_{\mathrm{pol}}$ with shifts.

Definition 2.3.6. The formulas of $\mathrm{LL}_{pol,\uparrow\downarrow}$ are defined from a set $\mathfrak{A}$ of negative atoms, and the following grammar:
Negative Formulas $N, M := a \in \mathfrak{A} \mid \uparrow P \mid ?P \mid\mid N \parr M \mid \bot \mid N \times M \mid \top \mid$
Positive Formulas: $P, Q := a^{\perp} \mid \downarrow N \mid !N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1$

Definition 2.3.7. The inference rules for $\mathrm{LL}_{pol,\uparrow\downarrow}$ are the ones of $\mathrm{LL}_{\mathrm{pol}}$ to which is added the following. $\mathcal{N}$ is a list of negative formulas.

$$
\dfrac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, \downarrow N}\,\downarrow \qquad\qquad \dfrac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, \uparrow P}\,\uparrow
$$

Then one defines:

$$
\uparrow P^{\perp_L} \;=\; \uparrow(P^{\perp_R}) \; \downarrow N^{\perp_R} \;=\; \uparrow(N^{\perp_L})
$$

Cut-elimination then represents the fact that $\uparrow\downarrow N$ is equivalent to N:

$$
\dfrac{\dfrac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, \downarrow N}\,\downarrow \qquad \dfrac{\vdash \mathcal{N}', N^{\perp_R}}{\vdash \mathcal{N}', \uparrow N^{\perp_R}}\,\uparrow}{\vdash \mathcal{N}, \mathcal{N}'}\,\text{cut} \qquad \text{reduces to} \qquad \dfrac{\vdash \mathcal{N}, N \qquad \vdash \mathcal{N}', N^{\perp_R}}{\vdash \mathcal{N}, \mathcal{N}'}\,\text{cut}
$$

2.3.2 Categorical semantics

In this section we give a definition for categorical models of $\mathrm{LL}_{\mathrm{pol}}$. Several categorical axiomatizations of $\mathrm{LL}_{\mathrm{pol}}$ exist. Let us mention the one using control or co-control categories [72] by Laurent [54], and the semantic study of polarized languages with effect by Curien, Fiore and Munch [15], which extends to effects.

Traditionally, the importance of Intuitionistic Linear Logic is justified semantically: the categorical definition of ILL is first done for Intuitionist Logic via a Seely Category, and the dualizing object requirement is added as a topcoat. The same phenomena appears for the different axiomatizations of models of DiLL, see Section 2.4. However, the new results of this thesis all stem from the tentative to place involutive linear negation at the center of the syntax and the semantics of DiLL. As polarization appears naturally while looking for classical models od DiLL in functional analysis (see Chapters 6 and 7), we want to focus here on giving categorical models of $\mathrm{LL}_{\mathrm{pol}}$

which emphasizes on this involutive negation. Chiralites, develloped by Melliès [60], are the good answer to this point of view.

We will show in Chapter 7 how the $\parr$ can be seen as a specific topological tensor product. Therefore, the following axiomatization treats $\otimes$ and $\parr$ symmetrically, instead of defining $\parr$ from the internal hom $\multimap$ as it was done in Section 2.2. This has been done by Melliès when he introduced the notion of *chiralities*. They interpret in particular the polarized version of the multiplicative connectives of Intuitionistic Linear Logic. This will help us distinguish between a negative interpretation of $\mathrm{LL}_{\mathrm{pol}}$ as in Chapter 5, and a polarized symmetrical one as in Chapter 7.

All the definitions given in this section follow closely the ones of Curien, Fiore and Munch [15], but differ as we interpret classical $\mathrm{LL}_{\mathrm{pol}}$, and thus requires the interpretations of the negation to be involutive on the category interpreting the negatives.

Remark 2.3.8. In Chapters 6 and 7 we give polarized models of Differential Linear Logic. As these models are in particular full subcategories of the category of topological vector spaces and linear continuous function, the definition below is not strictly necessary to describe them. However, it clarifies the role played by the different negations (interpreted as topological duals, which may or may not be completed), and the open the way to a possible categorical axiomatization of $\mathrm{D}-\mathrm{DiLL}$ (see Chapter 8).

2.3.2.1 Mixed chiralities

Definition 2.3.9. A linear distributive category [14] is a category $\mathcal{C}$ with two monoidal structures $(\mathcal{C}, \otimes, 1)$ and $(\mathcal{C}, \parr, \perp)$ such that for every object A, B and C we have a map, natural in A, B, and C:

$$A \otimes (B \parr C) \longrightarrow (A \otimes B) \parr C.$$

In particular, if $(\mathcal{C}, , 1, \multimap, \perp)$ is a $*$-autonomous category, and the bifunctor $\parr$ is defined on objects as $A \parr B := (A \multimap \perp) \multimap B$, then the category $\mathcal{C}$ is linearly distributive. The previous definition can then be seen as a generalization of a $*$-autonomous category.

Definition 2.3.10. A (right) duality on a linear distributive category is a functor $(-)^* : \mathcal{C} \longrightarrow \mathcal{C}^{op}$ such that we have adjunctions $A \otimes - \dashv A^* \parr -$ and $- \parr B \dashv - \otimes B^*$.

Proposition 2.3.11. *[14] A symmetric linear distributive category with a duality is a $*$-autonomous category.*

Melliès [61] extends the definition of linear distributive categories to the one of dialogue chirality, which defines a polarized version of linear distributive categories. Beware that the definition used in [61] is more restrictive than the following. Indeed, Melliès uses a definition where the interpretation for the negation $(-)^*$ and ${}^*(-)$ form a monoidal *equivalence*. We only ask for a monoidal adjunction, as we will study models of LL in which objects are not necessarily self-dual, or in which *only the interpretation of the negative connectives are self-dual*. This would correspond to a particular case of the mixed chirality introduced by Melliès [60]. We recall that a *strong monoidal adjunction* corresponds to an adjunction between two strong monoidal functors[3].

Definition 2.3.12. A mixed chirality consists in a pair of monoidal categories $(\mathscr{A}, \oslash, \mathbf{true})$ and $(\mathscr{B}, \obslash, \mathbf{false})$ equipped with:

- a strong monoidal adjunction $(-)^* : \mathscr{A} \longrightarrow \mathscr{B}^{op} \dashv {}^*(-) : \mathscr{B}^{op} \longrightarrow A$,

$$\begin{tikzcd} (\mathscr{A}, \oslash) \arrow[rr, bend left, "(-)^*"] \arrow[rr, phantom, "\perp"] & & (\mathscr{B}^{op}, \obslash) \arrow[ll, bend left, "{}^*(-)"] \end{tikzcd}$$

- an adjunction $L : \mathscr{A} \longrightarrow \mathscr{B} \dashv R : \mathscr{B} \longrightarrow \mathscr{A}$,

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

[3] It is in fact enough to suppose the strong monoidality of one functor. This is also equivalent to the strong monoidality of the unit and co-unit transformations derived from the adjunction [59, 5.17.14]

- a family of bijections:
$$\chi : \mathscr{B}(La, b \ovee b') \simeq \mathscr{B}(L(a \owedge {}^*(b)), b')$$
natural in m, a, b such that χ respects the various associativity morphisms by making the following diagrams commute:

$$\begin{tikzcd}
\mathscr{B}(L((m \owedge n) \owedge a), b) \arrow[rr, "\chi"] \arrow[d, "\text{assoc}"'] & & \mathscr{B}(La, (m \owedge m)^* \ovee b) \\
\mathscr{B}(L(m \owedge (n \owedge a)), b) \arrow[r, "\chi"] & \mathscr{B}(L(n \owedge a), m^* \ovee b) \arrow[r, "\chi"] & \mathscr{B}(L(a), n^* \ovee (m^* \ovee b)) \arrow[u, "\text{assoc., monoidality of negation}"']
\end{tikzcd} \tag{2.10}$$

A *dialogue chirality* is a mixed chirality where the monoidal adjunction leads to an equivalence of categories. However this is too strong a requirement for us, and in the forthcoming definitions we will ask for the composition of functors to be the identity only on $\mathscr{B}$ (negative interpretation) or $\mathscr{A}$ (positive interpretation).

In the development of this thesis, mixed chiralities will interpret polarized MALL with categories $(\mathscr{P}, \otimes, 1)$ and $(\mathscr{N}, ⅋, \perp)$ for positive and negative connectives respectively. The functors L and R are then shifts $\uparrow$ and $\downarrow$.

Remark 2.3.13. We will show below that polarized MALL is indeed interpreted by a negative chirality (with some coherence diagrams), consisting in two adjunctions, one of them being monoidal. When interpreting Differential Linear Logic, we get a strong monoidal adjunction between $(\mathcal{L}, \otimes, 1)$ and $(\mathcal{C}, \times, \top)$ by ! and a forgetful functor. Dereliction and co-dereliction can then be interpreted an adjunction between $(\mathcal{L}, \otimes, 1)$ and $(\mathcal{C}, \times, \top)$, which is the identity on $\mathcal{L}$ (see Section 2.5.2.3).

2.3.2.2 Negative chiralities.

We are in a case where, whether it is between shifts or (co)-derelictions, the adjunction between L and R always result in a reflection on $\mathscr{B}$, that is $L \circ R \ = \ Id_{\mathscr{B}}$. This corresponds to a (left) polarized version of (left) closure adjunction as detailed in Definition 3.0.2:

$$\begin{tikzcd}
\mathscr{B} \arrow[r, bend left, "\bar{(\cdot)}"] \arrow[r, phantom, "\perp"] & \mathscr{A} \arrow[l, bend left, "U"]
\end{tikzcd}$$

The typical topological example for this is the Cauchy completion in the category TopVec of locally convex separated vector spaces and continuous linear maps: if F is complete, to any linear map $f : E \longrightarrow F$ extends uniquely to a continuous linear map $\tilde{F} : \tilde{E} \longrightarrow F$.

Notation 2.3.14. *We now use $\mathscr{N}$ to denote the category on the right-hand side of the adjunctions in a chirality (previously denoted $\mathscr{B}$, and $\mathscr{P}$ to denote the category on the left-hand side of the adjunctions in a chirality (previously denoted $\mathscr{A}$).*

Definition 2.3.15. A left polarized closure is an adjunction $L : \mathscr{P} \longrightarrow \mathscr{N} \dashv R : \mathscr{N} \longrightarrow \mathscr{P}$ such that $L \circ R \ = \ Id_{\mathscr{N}}$. The use of *left* here emphasizes the fact that *the left adjoint is the one performing the closure*, while the right one is thought of as a forgetful functor.

We thus modify the definition by Melliès of dialogue chiralitiesby breaking the symmetry between $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.3.16. A *negative chirality* consists in a pair of monoidal categories $(\mathscr{P}, \owedge, \mathbf{true})$ and $(\mathcal{N}, \ovee, \mathbf{false})$ equipped with:

- a strong monoidal left polarized closure, $(-)^* : \mathscr{P} \longrightarrow \mathscr{N}^{op} \dashv {}^*(-) : \mathscr{N}^{op} \longrightarrow P$,
- a left polarized closure, $L : \mathscr{P} \longrightarrow \mathscr{N} \dashv R : \mathscr{N} \longrightarrow \mathscr{P}$,
- a family of bijections
$$\chi : \mathscr{N}(Lp, n \ovee m') \simeq \mathscr{N}(\mathcal{L}(p \owedge {}^*n), m') \tag{2.11}$$
natural in p, n, m such that χ respects the various associativity morphisms by making the diagram 2.10 commutes.

Remark 2.3.17. We leave to the reader a definition for *positive chiralities*, where the composition of adjoint functors are the identity on $\mathscr{P}$, and where equation 2.11 takes place in $\mathscr{P}$ with $\downarrow$ and p^*.

2.3.2.3 Interpreting $\mathrm{MALL}_{\mathrm{pol}}$

A negative chirality is the basic structure to interpret $\mathrm{MLL}_{\mathrm{pol}}$. We need an additional component to interpret formulas of $\mathrm{LL}_{\mathrm{pol}}$, which basically says that there is only one closure operation between the interpretation of negative and positive formulas in our models of $\mathrm{LL}_{\mathrm{pol}}$. What is done here corresponds to a version with closures (for the classical interpretation), and without effects of [15]. Thus here we interpret proofs as maps in the category of negatives connectives: through covariant adjunctions between $\mathscr{N}$ and $\mathscr{P}$ we could also have interpreted them as maps in the category of positive connectives.

Definition 2.3.18. A classical negative model of $\mathrm{MLL}_{\mathrm{pol}}$ consists in

- A negative chirality $(\mathscr{P}, \otimes, 1)$ and $(\mathscr{N}, \parr, \perp)$ with a strong monoidal closure

$$(-)^{\perp_L} : \mathscr{P} \longrightarrow \mathscr{N}^{op} \dashv (-)^{\perp_R} : \mathscr{N}^{op} \longrightarrow \mathscr{P},$$

$$\begin{tikzcd} (\mathscr{P}, \otimes) \arrow[r, bend left, "(-)^{\perp_L}"] \arrow[r, phantom, "\perp"] & (\mathscr{N}^{op}, \parr) \arrow[l, bend left, "(-)^{\perp_R}"] \end{tikzcd}$$

  and a polarized closure

$$\uparrow : \mathscr{P} \longrightarrow \mathscr{N} \dashv \downarrow : \mathscr{N} \longrightarrow \mathscr{P}$$

$$\begin{tikzcd} \mathscr{P} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \mathscr{P} \arrow[l, bend left, "\downarrow"] \end{tikzcd}$$

  with a family of bijections:

$$\chi : \mathscr{N}(Lp, n \parr m) \simeq \mathscr{N}(\mathcal{L}(p \otimes {}^*n), m).$$

- A family of isomorphisms in $\mathscr{P}$:

$$clos_P : \downarrow P^{\perp_L} \simeq (\uparrow P)^{\perp_R}$$

  natural in P.

Then as $\uparrow\downarrow \simeq Id_{\mathscr{N}} \simeq (-^{\perp_L})^{\perp_R}$ one has the isomorphisms $\downarrow\uparrow P \simeq P^{\perp_L \perp_R}$.

Remark 2.3.19. From the rest of the section it follows that a negative chirality is in particular a *negative interpretation* [54] of LL_{pol}.

Remark 2.3.20. The shifts L and R are used here to handle morphisms between object $\mathscr{N}$ and objects of $\mathscr{P}$: applying a shift to $P \in \mathscr{P}$ allows to consider $f \in \mathscr{N}(\uparrow P, N)$ for $N \in \mathscr{N}$. A much simpler situation is the one where $\mathscr{N}$ and $\mathscr{P}$ are both sub-monoidal categories of a larger linear distributive category (Definition 2.3.9). This will be the case latter in Chapter 7, where we will interpret positives and negatives in subcategories of the category of complete lcs and linear continuous maps.

The first difficulty in the theory of topological vector spaces will be to find chiralities for which the $\otimes$ and $\parr$ are associative. The semantics will rely on the fact negatives are those for which the parr is associative, positive are those for which the tensor product is associative. Once this difficulty is overcome, we work towards an equivalence of categories between the chiralities.

Definition 2.3.21. Consider $\mathcal{M} = M_1, ..., M_n$ a list of negative formulas and P a positive formula. Then one interprets a proof

$$\dfrac{\begin{array}{c}[\pi] \\ \vdots\end{array}}{\vdash \mathcal{M}, P}$$

of $\mathrm{LL_{pol}}$ as a morphism in $\mathscr{N}$:

$$[\![\pi]\!] \in \mathcal{N}(P^{\perp_L}, M_1 \parr ... \parr M_n).$$

Through the strong monoidal closure between the negations, this is equivalent to interpreting the proof of the sequent in $\mathscr{P}({M_1}^{\perp_R} \otimes ... \otimes {M_N}^{\perp_R}, P)$.
The proof π' of the sequent $\vdash \mathcal{M}$ is interpreted as:

$$[\![\vdash \pi']\!] \in \mathscr{N}(\uparrow 1, M_1 \parr ... \parr M_n)$$

Remark 2.3.22. Here we make use in our categorical interpretation and in our interpretation of sequents of a shift $\uparrow$. As we required that $\downarrow\uparrow P \simeq P^{\perp_L \perp_R}$, we could have replaced the use of $\uparrow$ by the functor $(.)^{\perp_L \perp_R \perp_L}$. We keep shifts for conciseness in the formulas, but keep in mind that this is not necessary when the composition of the negation is the identity on the negatives.

The interpretation of the formulas follows immediately from the notation we used. The interpretation of derivable sequent is defined by induction on the last derivation rule used exactly as for LL. We detail now the interpretation of the proofs:

Axioms, shifts and cuts

- The interpretation of an axiom $\vdash N, N^{\perp_R}$ corresponds to the identity morphism in $\mathscr{N}$, as $(N^{\perp_R})^{\perp_L} \simeq N$.
- If $f \in \mathscr{N}(P^{\perp_L}, M_1 \parr ... \parr M_n)$ interprets $\vdash M_1, ..., M_n, P$, let us construct the interpretation of the proof π of $\vdash M_1, ..., M_n, \uparrow P$ in $\mathscr{N}(\uparrow 1, M_1 \parr ... \parr M_N \parr \uparrow P)$. As in $\mathscr{N}$, we have via $clos_P$ the isomorphisms
$$P^{\perp_L} \simeq \uparrow\downarrow P^{\perp_L} \simeq \uparrow((\uparrow P)^{\perp_R}) \simeq \uparrow((\uparrow P)^{\perp_R} \otimes 1),$$
and in to f corresponds a morphism $\tilde{f} \in \mathscr{N}(\uparrow((\uparrow P)^{\perp_R} \otimes 1), M_1 \parr ... \parr M_n)$, and thus via associativity of $\parr$ and $\chi_{1,\uparrow P, M_1 \parr ... \parr M_n} : \mathscr{N}(L1, \uparrow P \parr M_1 \parr ... \parr M_n) \simeq \mathscr{N}(\mathcal{L}(1 \otimes {}^*\uparrow P), M_1 \parr ... \parr M_n)$ we construct a morphism:
$$[\![\pi]\!] \in \mathscr{N}(\uparrow 1, M_1 \parr ... \parr M_N \parr \uparrow P).$$
- If $f \in \mathscr{N}(\uparrow 1, M_1 \parr ... \parr M_n)$ interprets a proof π of $\vdash M_1, ..., M_n$, let us construct the interpretation of the proof π' of $\vdash M_1, ..., \downarrow M_n$ in $\mathscr{N}(\downarrow {M_n}^{\perp_L}, M_1 \parr ... \parr M_{n_1})$. Via the associativity of $\parr$ and $\chi_{1, M_n, M_1 \parr ... \parr M_{n-1}}$ we construct a morphism $\tilde{f} \in \mathscr{N}(\uparrow(1 \otimes {M_N}^{\perp_R}), M_1 \parr ... \parr M_{n_1})$. As $1 \otimes {M_N}^{\perp_R} \simeq {M_N}^{\perp_R}$, via $clos_{\downarrow M_n}$ we construct a morphism:
$$[\![\pi']\!] \in \mathscr{N}(\downarrow {M_n}^{\perp_L}, M_1 \parr ... \parr M_{n_1}).$$
- As detailed before, the cut-rule corresponds to the composition in $\mathscr{N}$. Consider $f \in \mathscr{N}([\![N]\!], [\![\mathcal{M}]\!])$ interpreting a proof of $\vdash \mathcal{M}, N^{\perp_R}$ as $(N^{\perp_R})^{\perp_L} \simeq N$, and $g \in \mathscr{N}(\uparrow 1, [\![\mathcal{M}']\!] \parr N)$ interpreting a proof of $\vdash \mathcal{M}', N$. Then the proof π of the sequent resulting from the cut of the two previous ones is interpreted by functoriality of $\parr$:
$$[\![\pi]\!] : 1 \xrightarrow{g} [\![\mathcal{M}']\!] \parr [\![N]\!] \xrightarrow{id_{[\![\mathcal{M}']\!]} \parr g} [\![\mathcal{M}']\!] \parr [\![\mathcal{M}]\!].$$

Multiplicatives.

- If $f \in \mathscr{N}(\uparrow 1, M_1 \parr ... \parr M_n)$ interprets $\vdash M_1, ..., M_n$, then the interpretation of the proof of $\vdash M_1, ..., M_{n-1} \parr M_n$ is still f.
- If $f \in \mathscr{N}(P^{\perp_L}, [\![\mathscr{N}]\!])$ interprets a proof of $\vdash N, P$ and $g \in \mathscr{N}(Q^{\perp_L}, [\![\mathcal{M}]\!])$ interprets a proofs of $\vdash N, Q$, then $f \parr g \in \mathscr{N}(P^{\perp_L} \parr Q^{\perp_L}, [\![\mathscr{N}]\!] \parr [\![M]\!])$ interprets (up to the isomorphism $({}^{\perp_L}P \parr Q^{\perp_L}) \simeq ({}^{\perp_L}P \otimes Q)$on the domain) the proof of $\vdash N, \mathcal{M}, P \otimes Q$.

Additives. In order to deal with additives, one needs the distributivity of the multiplicative connectives over the additive connectives:

$$N \parr (M_1 \& ... \& M_n) \simeq (N \parr M_1) \& ... \& (N \parr M_n).$$

Definition 2.3.23. A classical model of $\mathrm{MALL}_{\mathrm{pol}}$ consists in

- A negative chirality $(\mathscr{P}, \otimes, 1)$ and $(\mathscr{N}, \parr, \perp)$ with a strong monoidal left closure $(-)^{\perp_L} : \mathscr{P} \longrightarrow \mathscr{N}^{op} \dashv (-)^{\perp_R} : \mathscr{N}^{op} \longrightarrow \mathscr{P}$, with a polarized closure $\downarrow : \mathscr{N} \longrightarrow \mathscr{P} \dashv \uparrow : \mathscr{P} \longrightarrow \mathscr{N}$ such $\uparrow \circ \downarrow = Id_{\mathscr{N}}$, and with a family of bijections:
$$\chi : \mathscr{N}(Lp, n \parr m') \simeq \mathscr{N}(\mathcal{L}(p \otimes {}^* n), m').$$
- A cartesian product $(\times, \top)$ on $\mathscr{N}$ $(\mathscr{N}, \times, \top)$ such that $\parr$ is distributive over $\times$, through a family of natural bijections:
$$distr_{N,\mathcal{M}} : N \parr (M_1 \& ... \& M_n) \simeq (N \parr M_1) \& ... \& (N \parr M_n).$$
- A family of isomorphisms in $\mathscr{P}$:
$$clos_P : \downarrow P^{\perp_L} \simeq (\uparrow P)^{\perp_R}$$
natural in P.

Then as $\uparrow\downarrow \simeq Id_{\mathscr{N}} \simeq (-^{\perp_L})^{\perp_R}$ one has the isomorphisms $\downarrow\uparrow P \simeq P^{\perp_L \perp_R}$.

Then we interpret $\oplus$ as the dual of $\times$: $[\![P \oplus Q]\!] := ([\![P]\!]^{\perp_L} \times [\![G]\!]^{\perp_L})^{\perp_R}$. Because of the monoidal adjunction between ${}^{\perp_L}$ and ${}^{\perp_R}$, it is a co-product such that $\otimes$ distributes over it.

Let us detail how the additives rules are interpreted in this context, following the pattern detailed in Definition 2.3.21.

- Consider $f \in \mathscr{N}(\uparrow 1, [\![\Gamma]\!] \parr [\![N]\!])$ interpreting the proof of $\vdash \Gamma, N$ and $g \in \mathscr{N}(\uparrow 1, [\![\Gamma]\!] \parr [\![M]\!])$ interpreting the proof of $\vdash \Gamma, M$. By composition of f with $\chi_{1,\Gamma,N}$ we get a morphism $f' = f \circ \chi_{1,\Gamma,N} \in \mathscr{N}(\uparrow [\![\Gamma]\!], [\![N]\!])$ and likewise we get a morphism $g' = g \circ \chi_{1,\Gamma,M} \in \mathscr{N}(\uparrow [\![\Gamma]\!], [\![M]\!])$. By definition of a product we have $\langle f', g' \rangle \in \mathscr{N}(\uparrow [\![\Gamma]\!], ([\![M]\!] \times [\![N]\!]))$, which composed by $\chi^{-1}_{1,\Gamma,N \times M}$ results in the desired morphism.
- If $f \in \mathscr{N}([\![^{\perp_L} P]\!], [\![\Gamma]\!])$ interprets a proof of $\vdash \Gamma, P$, then we interpret a proof of $\vdash \Gamma, P \oplus Q$ by precomposing with the map $\pi_1 : [\![P]\!]^{\perp_L} \times [\![Q]\!]^{\perp_L} \longrightarrow [\![P]\!]^{\perp_L}$. As $[\![P]\!]^{\perp_L} \times [\![Q]\!]^{\perp_L} \simeq (([\![P]\!]^{\perp_L} \times [\![Q]\!]^{\perp_L})^{\perp_R})^{\perp_L} = [\![P]\!] \oplus [\![Q]\!]^{\perp_L}$ by definition, we have indeed that $f \circ \pi_1$ interprets $\vdash \Gamma, P \oplus Q$.
- The case for the right rule of the introduction of $\oplus$ is treated likewise.

2.3.2.4 Interpreting $\mathrm{LL}_{\mathrm{pol}}$

In Section 2.2.2 we interpreted ! as a co-monad on $\mathcal{L}$. Here, we take the point of view of a strong monoidal adjunction between ! and the forgetful functor U.

Usually one requires Seely's isomorphism :

$$!(N \times M) \simeq !N \otimes !M. \tag{2.12}$$

We are going to require the strong monoidality of ? instead of !: this is justified in a polarized setting by the fact that the above isomorphisms take place in the category interpreting the negative formulas, while the strong monoidiality of ! would be interpreted in the category interpreting the positives formulas. Indeed, as it will appear in Chapters 6 and 7, the negatives formulas are the one interpreted by some complete spaces (completeness in understood in this particular exeample in a wide sense: spaces may be Mackey-complete, or quasi-complete). Complete spaces play the role of co-domains F of smooth function $f \in \mathcal{C}^\infty(E, F)$. Positive one the contrary may not be complete, but may verify other properties preserved by inductive limits : in particular, they may be interpreted by barreled spaces 3.4.22 or bornological spaces 6.2.13. Thus in a setting rich enough to interpret differentiation, and thus with some notion of completeness, negative formulas are interpret as complete spaces, and positive ones as the formulas which need not be complete.

However, in the theory of topological vector spaces, the Seely isomorphism is not true with a non-completed tensor product $\otimes$. More specifically, when interpreting $!E$ as a space of distribution, the Kernel theorem 7.3.5 states this isomorphism for a completed tensor product. It does state the density of[4] $\mathcal{C}^\infty(E,\mathbb{R}) \otimes \mathcal{C}^\infty(F,\mathbb{R})$ in $\mathcal{C}^\infty(E \times F,\mathbb{R})$ and the fact that the topology induced by $\mathcal{C}^\infty(E \times F,\mathbb{R})$ on $\mathcal{C}^\infty(E,\mathbb{R}) \otimes \mathcal{C}^\infty(F,\mathbb{R})$ is the injective topology. It is by completing the tensor product that we obtain thus an isomorphism, which is dualized to be stated in the above form 2.12.

With these arguments in mind, we want to have an interpretation for ! and ? such that $!N = ?^{\perp_L}(N^{\perp_R})$, satisfying:

$$?(P \otimes Q) \simeq_{\mathscr{N}} ?P \parr ?Q \tag{2.13}$$

When the composition of the negations is not the identity on the positive (as it is the case in our negative chiralities), we have thus:

$$(!(N \times M))^{\perp_L \perp_R} \simeq_{\mathscr{P}} (!N^{\perp_L} \otimes !M^{\perp_R})^{\perp_R} \tag{2.14}$$

Thus, while Seely's isomorphism 2.12 is most of the time described as a linear/non-linear monoidal adjunction [59, Def. 21]:

$$\begin{tikzcd} (\mathscr{N}^\infty, \times) \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & (\mathscr{P}, \otimes) \arrow[l, bend left, "U"] \end{tikzcd}$$

we ask here for a *strong* monoidal adjunction:

$$\begin{tikzcd} (\mathscr{P}^{\infty,op}, \oplus) \arrow[r, bend left, "?"] \arrow[r, phantom, "\perp"] & (\mathscr{N}^{op}, \parr) \arrow[l, bend left, "U"] \end{tikzcd}$$

Remark 2.3.24. In an unpolarized setting, ! is a co-monad: $\mathcal{L} \longrightarrow \mathcal{L}$, and $\mathscr{N}^\infty$ is the co-Kleisli category $\mathcal{L}_!$. For any object $N \in \mathscr{N}^\infty$ we have an isomorphism between N and $U(!N) = !N$ in $\mathscr{N}^\infty$: the morphism $N \longrightarrow !N$ corresponds to $f = 1_{!N}$ in $\mathcal{L}$, while the morphism $!N \longrightarrow N$ corresponds to $g : !!N \xrightarrow{d_{!N}} !N \xrightarrow{d_N} N$. They are indeed inverse from one another in $\mathscr{N}^\infty$:

$$\begin{aligned} f \circ_! g &= f \circ !g \circ \mu_{!N} \text{by definition of } \circ_! && (2.15)\\ &= !d_N \circ !d_{!N} \circ \mu_{!N} && (2.16)\\ &= !d_N \text{by the second commutative diagram for comonads, see 2.2.12} && (2.17) \end{aligned}$$

However, in $\mathcal{N}^\infty$ the second commutative diagram for comonads say that $!d_N$ and $d_{!N}$ are the same arrow: they both act as a unit for the composition $\circ_!$, and thus are equal by the unicity of units. Moreover, we have:

$$g \circ_! f = d_N \circ d_{!N} g \circ \mu_N = d_N.$$

Thus g and f are inverse one another, and $N \simeq_{\mathscr{N}^\infty} U(!N)$ the adjunction between ! and U results in fact in a closure on $\mathscr{N}^\infty$. Likewise, the adjunction between ? and U (another functor denoted by U, as it is thought as a forgetful functor) is a closure on $\mathscr{P}^\infty$.

Definition 2.3.25. A classical model of $\mathrm{LL}_{\mathrm{pol}}$ consists in

- A negative chirality $(\mathscr{P}, \otimes, 1)$ and $(\mathscr{N}, \parr, \perp)$ with a strong monoidal left closure $(-)^{\perp_L} : \mathscr{P} \longrightarrow \mathscr{N}^{op} \dashv (-)^{\perp_R} : \mathscr{N}^{op} \longrightarrow \mathscr{P}$, with a polarized closure $\downarrow : \mathscr{N} \longrightarrow \mathscr{P} \dashv \uparrow : \mathscr{P} \longrightarrow \mathscr{N}$ such $\uparrow \circ \downarrow = Id_{\mathscr{N}}$,
- A cartesian structure on $\mathscr{N}$ $(\mathscr{N}, \times, \top)$ such that $\times$ is distributive over $\parr$
- A co-cartesian category $(\mathscr{P}^\infty, \oplus_\infty, 0_\infty)$ and a co-cartesian category $(\mathscr{N}^\infty, \times_\infty, \top_\infty)$ with a strong monoidal left closure
$$(-)^{\perp_{L,\infty}} : \mathscr{P}^\infty \longrightarrow \mathscr{N}^{\infty,op} \dashv (-)^{\perp_{R,\infty}} : \mathscr{N}^{\infty,op} \longrightarrow \mathscr{P}^\infty.$$

[4] In the context of topological vector spaces we have a biproduct and in particular $E \times F \simeq E \oplus F$

- A strong monoidal right closure adjunction

$$? : (\mathscr{P}^{\infty,op}, \oplus, 0) \longrightarrow (\mathscr{N}^{op}, \parr, 1) \dashv U : (\mathscr{N}^{op}, \parr, 1) \longrightarrow (\mathscr{P}^{\infty,op}, \oplus, \top).$$

- A family of isomorphisms in $\mathscr{P}$:

$$clos_P : \downarrow P^{\perp_L} \simeq (\uparrow P)^{\perp_R}$$

  natural in P.

Then as $\uparrow\downarrow \simeq Id_{\mathscr{N}} \simeq (-^{\perp_L})^{\perp_R}$ one has the isomorphisms $\downarrow\uparrow P \simeq P^{\perp_L \perp_R}$.

Interpreting the exponential connectives

Proposition 2.3.26. *We define a strong monoidal functor from* $(\mathscr{N}^{\infty}, \times)$ *to* $(\mathscr{P}, \otimes)$ *by*

$$!N = ? \circ U(N^{\perp_R})^{\perp_R}$$

$$!\ell = ? \circ U(\ell^{\perp_R})^{\perp_R}.$$

We interpret the formula $?P$ *for* $P \in \mathscr{P}$ *as*[5] $?U\uparrow P$.

Proof. It follows from the fact that,we have a closure between $\mathscr{N}^{\infty}$ and $\mathscr{P}^{\infty,op}$, that is $(-)^{\perp_{L,\infty}} \circ (-)^{\perp_{R,\infty}} = Id_{\mathscr{N}^{\infty}}$. □

Remark 2.3.27. The previously defined model is called *classical* as the double-negation is asked to be the identity on negatives. However, this definition would also suits an intuitionist setting by not asking the negation to define a closure. Then ! as defined above would not necessarily be a strong monoidal functor between $(\mathscr{N}^{\infty}, \times)$ and $(\mathscr{P}, \otimes)$.

The algebra structure of $?$ Let us detail how the exponential rules are interpreted in this context, following the pattern detailed in Definition 2.3.21. From the strong monoidality of !, one has natural isomorphisms in $\mathscr{N}$:

$$m^{\perp}_{P,Q} : ?P \parr ?Q \simeq ?(P \oplus Q)$$

$$m_0 : ?0 \simeq \perp$$

If we denote by $\nabla_P : P \oplus P \longrightarrow P$ the co-diagonal, then we obtain as previously, see 2.4:

$$c_P : ?P \parr ?P \longrightarrow ?P$$
$$w_P : \perp \longrightarrow ?P$$

These define natural transformations which interprets respectively the contraction and weakening rules, by precomposition.

Interpreting the dereliction rule By definition, the co-unit of the adjunction

$$? : (\mathscr{P}^{\infty,op}, \oplus, 0) \longrightarrow (\mathscr{N}^{op}, \parr, 1) \dashv U : (\mathscr{N}^{op}, \parr, 1) \longrightarrow (\mathscr{P}^{\infty,op}, \oplus, \top)$$

is a natural transformation $d^{\perp}_N \in \mathscr{N}^{op}(?U(N), N)$ which interprets the dereliction. We denote by

$$d_N : N \longrightarrow ?U(N)$$

the morphism corresponding to $d^{\perp}_{\mathbb{N}}$ in $\mathscr{N}$ via the isomorphism $\mathscr{N}^{op}(?U(N), N) \simeq \mathscr{N}(N, ?U(N))$.

Thus consider a morphism $f \in \mathscr{N}(P^{\perp_R}, M)$ interpreting the proof of $\vdash M, P$. One constructs the interpretation of the proof of $\vdash M, ?P$ as the morphism $\tilde{f} \in \mathscr{N}(\uparrow 1, M \parr ?U(\uparrow P)) \simeq \mathscr{N}(\uparrow(1 \otimes (?U(\uparrow P))^{\perp_L}$:

$$\uparrow(1 \otimes (?U(\uparrow P))^{\perp_L} \xrightarrow{\uparrow(d_{\uparrow P})^{\perp_L}} \uparrow((\uparrow P)^{\perp_L}) \simeq \uparrow\downarrow P^{\perp_L} \simeq P^{\perp_R} \xrightarrow{f} M$$

Promotion is interpreted as before by functoriality of ! and U.

[5] We need to apply a shift to P before constructing $!P$ because of our categorical definitions. In a interpretation where $?N^{\perp}$ represents the space of smooth scalar functions $\mathcal{C}^{\infty}(N, \mathbb{R})$, it amounts of saying that one needs a complete domain to define smoothness.

2.4 Differential Linear Logic

Differential Linear Logic[6] was introduced by Ehrhard and Regnier [23], based on the remark that, in Köthe spaces, the maps in the co-Kleisli category $\textsc{Kothe}(!X, Y)$ identify to power series:

$$f = \sum_{n \geqslant 0} f_n$$

where f_n is a n-monomial[7]. This intuition applies to any *quantitative* model of LL, and in particular to the histoic study by Girard of *normal functors* [30], which leads to Linear Logic. From a power series one can extract a linear-map f_1, which corresponds intuitively to its best linear approximation at 0. Indeed, if one considers the toy example of real power series converging absolutely:

$$f : x \mapsto \sum_{n \in \mathbb{N}} a_n x^n$$

the linear map $x \mapsto a_1 x$ is exactly $D_0(f) : x \mapsto f'(0)x$, the differential at 0 of f.

We saw that the dereliction rule d of LL in interpreted as a natural transformation $d_A : A \longrightarrow !A$, implying that a linear map $A \multimap B$ can be seen as a non-linear one $!A \multimap B$. In Köthe spaces and other quantitative models, we have a natural transformation $\bar{d}$ allowing to extract a linear map from a non-linear one, which corresponds intuitively to the differentiation at 0. This natural transformation is implemented syntactically and added to LL under the form:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{d}$$

Ehrhard and Regnier introduce two additional rules: the co-contraction $\bar{c}$ allowing to sum in the domain of non-linear maps (and thus to differentiate at other points than 0), and the co-weakening $\bar{w}$, representing the unit for the cocontraction, that is the Dirac distributions at 0 δ_0.

The first version of DiLL, introduced in [23] does not include the promotion rule, for the sake of a perfectly symmetrical calculus. The promotion rule is however essential to type the λ-calculus and its differential version [22], and was studied particularly by Michele Pagani [64]. In the survery by Ehrhard [20], the version of DiLL without promotion is called *finitary* DiLL. As usual in the literature, we will refer to DiLL *without* promotion as DiLL_0. Moreover, DiLL_0 will always denote the monolateral, thus linear classical, sequent calculus.

At the end of this section, we will get back to the denotational intuitions attached to DiLL (see Section 2.4.2): as explained above, the co-structural rules imply operations on function $f \in \mathcal{C}^\infty(E, F)$, and functions must be smooth (see Definition 3.2.4). This is the definition of distributions: linear continuous functions acting on smooth functions. In a model of DiLL where non linear proofs are interpreted by smooth functions $f \in \mathcal{C}^\infty(E, F)$ this will appear under a smooth version of equation 2.7:

$$!E = \mathcal{C}^\infty(E, \mathbb{R})'.$$

Concerning the categorical semantics, we refer mainly to the monograph by Ehrhard [20] and the abstract by Fiore [25]. The following papers are also fundamental: [7, 24]. Blute, Cockett, and Seely investigated the categorical axiomatization of differential calculus, thus generalizing models of DiLL, in a series of papers [8, 9].

2.4.1 Syntax and cut-elimination

2.4.1.1 The syntax

Definition 2.4.1. Formulas of DiLL_0 are the same as for LL, and negation is defined likewise.

As detailed in the introduction of this section, DiLL_0 does not feature a promotion rule but includes three new exponential rules, symmetrical to the ones of LL. While the weakening w, the contraction c and the dereliction d are called structural rules, accordingly to the intuitions described in Section 2.1.1, the newly introduced co-weakening $\bar{w}$, co-contraction $\bar{c}$ and co-dereliction $\bar{d}$ are called the *co-structural rules*.

[6] Beware that DiLL is introduced there under the form of interaction nets, while we discuss here a formulation with sequents. See [20] and Zimmerman's thesis [81] for a presentation of DiLL with sequents

[7] n-monomials from $\mathbb{R}$ to $\mathbb{R}$ are the functions $x \mapsto ax^n$ for $a \in \mathbb{R}$. In general, they are defined as $x \mapsto f(x, \ldots, x)$ where f is a continuous or bounded n-linear map

The Identity rule

$$\frac{}{\vdash A, B^{\perp}}\ \text{(axiom)} \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ \text{cut}$$

The multiplicative rules

$$\frac{}{\vdash 1}\ (1) \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ (\perp)$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ (\parr) \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ (\otimes)$$

The additive rules

$$\frac{}{\vdash \Gamma, \top}\ \top \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \&$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_L \qquad\qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_R$$

The Exponential Rules

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?E}\ w \qquad \frac{\vdash \Gamma, ?E, ?E}{\vdash \Gamma, ?E}\ c \qquad \frac{\vdash \Gamma, E}{\vdash \Gamma, ?E}\ d$$

$$\frac{}{\vdash !E}\ \bar{w} \qquad \frac{\vdash \Gamma, !E \qquad \vdash \Delta, !E}{\vdash \Gamma, \Delta, !E}\ \bar{c} \qquad \frac{\vdash \Gamma, E}{\vdash \Gamma, !E}\ \bar{d}$$

Figure 2.6: The deriving rules for sequents of DiLL

Some version of DiLL features a *Mix* rule which concatenates the contents of sequents. It is quite natural from a denotational point of view as it corresponds to a structure of commutative $\otimes$-monoid on $\bot$, *e.g.* in our models of vector space to the fact that we have a multiplication law on scalars [20, 2.3.1]. The version of DiLL used here does not feature the Mix rule.

Proofs of DiLL are then constructed from the rules in figure 2.6.

2.4.1.2 Typing: intuitions behind the exponential rules.

In this section, we give the intuitions from denotational semantics behind the rules of DiLL_0. The models figuring in sections 2.4.3 and sections 7.4.3 implement them concretely in terms of distributions. Following the intuitions on denotational models of LL, a proof of $\vdash\ ?A^\perp, B$ is denoted by a smooth function $f : A \longrightarrow B$. In particular, elements of $?A^\perp$ are scalar smooth functions $f : A \longrightarrow \mathbb{R}$ as the interpretation for $\bot$ is $\mathbb{R}$ (equation 2.6). Therefore, the denotational meaning of the dereliction rule d is that a linear map should in particular be a smooth map[8]:

$$\dfrac{\vdash f : A \multimap B}{\vdash f : !A \multimap B \simeq A \Rightarrow B}\ d$$

Weakening on a distribution $\phi \in !A$ consists in taking the value of ϕ at the function which is constant and equal to 1. Dually, the interpretation of a proof of $\vdash \Gamma, ?A$, where $?A$ was introduced through weakening, consists in the tensor product of the interpretation of the proof of Γ and of the smooth function constant at 1

$$\dfrac{\vdash \Gamma}{\vdash \Gamma, \mathrm{const}_1 : A \Rightarrow \bot}\ w$$

The contraction rules means that given two smooth scalar functions on the same domain $f : A \longrightarrow \bot$ and $g : A \longrightarrow \bot$ one can compute a unique smooth function on A: the function $f \cdot g : x \mapsto f(x)g(x)$. Remember from Section 2.2.2 that the interpretation of the contraction is a natural transformation $c_A : !A \longrightarrow !A \otimes !A$ which results from the diagonal $A \longrightarrow A \times A$ and Seely's isomorphism. It means that from the interpretation $f \in \mathcal{L}(!A \otimes !A, B)$ of a proof of $\vdash !A, !A, B$ one gets the interpretation of the proof of $\vdash !A, B$ by computing $f \circ c : x \in !A \mapsto f(x, x)$. We have thus a good interpretation of the contraction in terms of smooth functions and distributions theory. When $!A$ is a space of distributions on A (see Chapter 7) Seely's isomorphism is exactly Schwartz' Kernel theorem [67] (see 7.3.5), which says to any smooth function $h : A \times B \Rightarrow \mathbb{R}$, one can map an element of the *completed* tensor product $\lim_n f_n \otimes g_n \in (A \Rightarrow \mathbb{R}) \hat{\otimes} (B \Rightarrow \mathbb{R})$.

$$\dfrac{\vdash \Gamma, f : A \Rightarrow \bot, g : A \Rightarrow \bot}{\vdash \Gamma, f \cdot g : A \Rightarrow \bot}\ c$$

Let us now describe the denotational intuitions behind the new co-structural rules of DiLL_0. The most intuitive one is the co-dereliction: if $f \in \mathcal{L}(!A, B)$ interprets a non-linear proof of $\vdash\ ?A^\perp, B$, then the precomposition by the interpretation $\bar{d}_A : A \longrightarrow !A$ represents the best linear approximation of f.

$$\dfrac{\dfrac{\vdash A^\perp, A}{\vdash A^\perp, !A}\ \bar{d} \qquad \vdash f : ?A^\perp, B}{\vdash D_0 f : A^\perp, B}\ \text{cut}$$

Following the intuitions of equation 2.7, we understand elements of $!A$ as distributions ϕ, that is linear continuous forms, acting on smooth scalar functions in $\mathcal{C}^\infty(A, \mathbb{R})$. Co-contraction says that given two such distributions, one can combine their action on the domain of a function they apply to,. It corresponds to the convolution product $*$ between distributions[9]:

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \phi * \psi : !A}\ \bar{c} \qquad \vdash f : A \Rightarrow \mathbb{R}}{\vdash \Gamma, \Delta, \phi * \psi(f) \ = \ \phi(x : A \mapsto \psi(y : A \mapsto f(x +_A y))) : \mathbb{R}}\ \text{cut}$$

[8] In fact it means more: we should have an inclusion of the hom-sets, meaning that if these are topological vector spaces the topologies should agree

[9] we prove in Chapter 7, Proposition 7.4.12, that the concontraction is a model of DiLL with smooth functions interpreting non-linear proofs is indeed the convolution product

Lastly coweakening is the neutral for the co-contraction. It is interpreted denotationally by to applying a smooth function $f : A \Rightarrow B$ at a point $0_A : A$ in its domain, with 0_A being the neutral for the addition $+_A$ in A. It corresponds to the introduction of the Dirac distributions δ_0 as an element of $!A$.

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, \delta_0 : !A}\ \bar{w} \qquad \vdash f : A \Rightarrow \mathbb{R}}{\vdash \Gamma, f(0) : \mathbb{R}}\ \text{cut}$$

Remark 2.4.2. Through this cut-elimination rule, we can observe a fundamental feature of Differential Linear Logic. As $\perp$, the neutral for $⅋$ is interpreted by a vector space, and as $[\![⅋]\!]' = [|1|] = \mathbb{R}$ (the only possible neutral for $\otimes$), we have necessarily $[\![⅋]\!] = [\![⅋]\!]'' = \mathbb{R}$. Thus in the a sequent, one has ‘ìnvisible computations" in $\mathbb{R}$ happening, and which account for cut-elimination.

2.4.1.3 Cut-elimination and sums

A particular property of DiLL is that its cut-elimination is non-deterministic: this non-determinism is dealt with by considering *sums of proofs*. We will call *simple proofs* proofs of DiLL that are not a sum. Sums are generated by the cut-elimination procedure on simple proofs. We suppose that this sum is associative, commutative, and we suppose the existence of a neutral proof.

Definition 2.4.3. Simple proofs π of DiLL are proof-trees constructed from the proofs in figure 2.6. Proofs of DiLL are (associative and commutative) finite sums of simple proofs, with neutral o.

The fact that we have a zero proof o is very particular: o is in fact a proof for any sequent $\vdash \Gamma$. Thus, in this non-deterministic point of view, every sequent is derivable in DiLL. We will give in Chapter 8 a deterministic variant of DiLL.

The cut-elimination procedure between exponential rules is detailed below, with formulas typing functions and distributions in order to make intuitions clear. We detail only the principal cut-elimination rules between the *exponential rules*: that is, we detail the elimination of a cut rule happing over two formulas whose main connective was just introduced in the proof via a rule of DiLL_0. The cut-elimination between other logical rules follow the one of Linear Logic, see for example [59]. We refer to the thesis by Zimmerman [81] for a detailed exposition of the commutation rules at stakes for the cut-elimination procedure in DiLL_0. As we use mainly DiLL_0 and not DiLL we do not detail cut-elimination rules involving the promotion. The cut-elimination between the promotion and the co-structural rule $\bar{w}, \bar{d}, \bar{c}$ are quite intricate, and were studied in detail by Pagani [64].

Typing: the intuitions behind cut-elimination. Again, the cut-elimination rules need to be understood from a semantic perspective. As detailed in the previous paragraph, we see $?A^\perp$ as the type of smooth scalar functions $f \in \mathcal{C}^\infty(A, \mathbb{R})$, which we write $f : A \Rightarrow \mathbb{R}$. A formula $?A^\perp$ appearing in a sequent indicates that the proofs depends non linearly of a parameter A. We see $!A$ as an indication that the proofs outputs distributions ϕ in $\mathcal{C}^\infty(E, \mathbb{R})'$.

Cut-elimination between w and $\bar{w}$ take the value at 0 of the function constant at 1:

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, \text{const}_1 : A \Rightarrow \mathbb{R}}\ w \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\ \bar{w}}{\vdash \Gamma, 1 : \mathbb{R} \equiv \Gamma}\ \text{cut} \rightsquigarrow \vdash \Gamma \tag{2.18}$$

Cut-elimination between a contraction and a co-weakening consists in taking the value at 0 of the kernel of two functions: this is the tensor product of the values at 0 of each function. Equivalently, it consists in taking the value at 0 of a function $x \mapsto f(x, x)$, with $f \in \mathcal{C}^\infty(A \times A)$. This is the same as computing f at $(0, 0)$.

$$\dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R}}{\vdash \Gamma, f \cdot g : A \Rightarrow \mathbb{R}}\ c \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\ \bar{w}}{\vdash \Gamma \equiv \vdash \Gamma, f(0)g(0) : \mathbb{R}}\ \text{cut} \tag{2.19}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R} \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\,\bar{w}}{\vdash \Gamma, f(0) : \mathbb{R}, g : A \Rightarrow \mathbb{R}}\,\text{cut} \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\,\bar{w}}{\vdash \Gamma, f(0) : \mathbb{R}, g(0) : \mathbb{R} \equiv \vdash \Gamma, f(0).g(0) : \mathbb{R} \equiv \Gamma}\,\text{cut}$$

Surprisingly, the cut-elimination between the co-contraction and the weakening obeys to the same pattern[10]. Indeed, the cut-rule consists in applying two distributions on the domain of a constant function by summing points in the domain. For a constant function, these points do not matter, and it is the same as applying one distribution after the other.

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \psi : !A}\,\bar{c} \qquad \dfrac{\vdash \Delta}{\vdash \Delta, \text{const}_1 : A \Rightarrow \mathbb{R}}\,w}{\vdash \Delta, B}\,\text{cut} \tag{2.20}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \dfrac{\vdash \Delta}{\vdash \Delta, \text{const}_1 : A \Rightarrow \mathbb{R}}\,w}{\vdash \Gamma, \Delta, \phi(\text{const}_1) : \mathbb{R}}\,\text{cut}}{\vdash \Gamma, \Delta, \text{const}_{\phi(\text{const}_{1\in\mathbb{R}})} : A \Rightarrow \mathbb{R}}\,w \qquad \vdash \Gamma', \psi : !A}{\vdash \Delta, \Gamma, \Gamma', \psi(\text{const}_{\phi(\text{const}_{1\in\mathbb{R}})} ⅋ b) : \mathbb{R}}\,\text{cut}$$

The cut-elimination between a dereliction and a co-dereliction consist in taking the differential at 0 of a linear function: one gets back the linear function. This case is of fundamental importance for the development of a logic for differential equations.

$$\dfrac{\dfrac{\vdash \Delta, f : A \multimap B}{\vdash \Delta, f : A \Rightarrow B}\,d \qquad \dfrac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\,\bar{d}}{\vdash \Gamma, \Delta, D_0(f)(v) \;=\; f(v) : B}\,\text{cut} \;\rightsquigarrow\; \dfrac{\vdash \Delta, f : A \multimap B \qquad \vdash \Gamma, v : A}{\vdash \Gamma, \Delta, f(v) : B}\,\text{cut} \tag{2.21}$$

We now tackle the rules which make sums appear. Cut-elimination between w and $\bar{d}$ differentiates at 0 the function constant at 1, thus computing $0 \in \mathbb{R}$: the proof tree o needs to be thought as an empty proof of $\mathbb{R}$.

$$\dfrac{\dfrac{\vdash \Gamma}{\vdash \Gamma, \text{const}_1 : A \Rightarrow \mathbb{R}}\,w \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\,\bar{d}}{\vdash \Gamma, \Delta, 0 = D_0(\text{const}_1)(v) : \mathbb{R}}\,\text{cut} \;\rightsquigarrow\; \mathsf{o} \tag{2.22}$$

Likewise, a cut-rule between a $\bar{w}$ and d consists in taking the value at 0 of a map whose linearity we forgot: we are suppressing the input of a data of type $A^{\perp}$.

$$\dfrac{\dfrac{\vdash}{\vdash \delta_0 : !A}\,\bar{w} \qquad \dfrac{\vdash \Gamma, f : A \multimap \mathbb{R}}{\vdash \Gamma, f : A \Rightarrow \mathbb{R}}\,d}{\vdash \Gamma, f(0) = 0 : \mathbb{R}}\,\text{cut} \;\rightsquigarrow\; \mathsf{o} \tag{2.23}$$

The cut-elimination between a contraction and a co-dereliction consists in differentiating at 0 the kernel of two functions. As in the usual differential calculus, the derivative of a function of two variable is the sum of the derivative on each variable.

$$\dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R}}{\vdash \Gamma, f \cdot g : A \Rightarrow \mathbb{R}}\,c \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\,\bar{d}}{\vdash \Gamma, \Delta, D_0(f \cdot g)(v) \;=\; D_0(f)(v).g(0) + f(0).D_0(g)(v) : \mathbb{R}}\,\text{cut} \;\rightsquigarrow \tag{2.24}$$

[10]This is not a surprise from a categorical point of view, as the pair of rules c, w and $\bar{c}$, $\bar{w}$ all comes from a biproduct structure $\times \simeq \oplus$ and from the strong monoidality of !, see Section 2.4.2

$$\dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R} \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\,\bar{d}}{\vdash \Gamma, \Delta, D_0(f)(v) : \mathbb{R}, g : A \Rightarrow \mathbb{R}}\,\text{cut} \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\,\bar{w}}{\vdash \Delta, \Gamma, D_0(f)(v) : \mathbb{R}, g(0) : \mathbb{R}}\,\text{cut}$$

$$+\ \dfrac{\dfrac{\vdash \Gamma, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R} \qquad \dfrac{\vdash \Delta, v : A}{\vdash \Delta, D_0(\_)(v) : !A}\,\bar{d}}{\vdash \Gamma, \Delta, g : A \Rightarrow \mathbb{R}, D_0(g)(v) : \mathbb{R}}\,\text{cut} \qquad \dfrac{\vdash}{\vdash \delta_0 : !A}\,\bar{w}}{\vdash \Delta, \Gamma, f(0) : \mathbb{R}, D_0(g)(v) : \mathbb{R}}\,\text{cut}$$

The opposite case goes likewise: the cut-elimination between a co-contraction and a dereliction applies the convolution of two distributions on a linear function. As it is linear, this is exactly the sum of the values of each distribution applied to the linear function.

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \psi : !A}\,\bar{c} \qquad \dfrac{\vdash \Delta, f : A \multimap \mathbb{R}}{\vdash \Delta, f : A \Rightarrow \mathbb{R}}\,d}{\vdash \Gamma, \Gamma', \Delta, \phi * \psi(f) \;=\; \phi(x \mapsto \psi(y \mapsto f(x+y))) \;=\; \phi(x \mapsto f(x)) + \psi(y \mapsto f(y)) : \mathbb{R}}\,\text{cut} \rightsquigarrow \tag{2.25}$$

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma, \phi : !A \qquad \dfrac{\vdash \Delta, f : A \multimap B}{\vdash \Delta, f : A \Rightarrow B}\,d}{\vdash \Gamma, \Delta, \phi(f) : \mathbb{R} \parr B \equiv \vdash \phi(f) : \mathbb{R} \parr B, 1 : \mathbb{R}}\,\text{cut}}{\vdash \Gamma, \Delta, \phi(f) : \mathbb{R} \parr B, \text{const}_1 : A \Rightarrow \mathbb{R}}\,w \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma', \Gamma, \Delta, \phi(f) : \mathbb{R} \parr B, \psi(\text{const}_1) : \mathbb{R}}\,\text{cut}$$

$$+\ \dfrac{\dfrac{\dfrac{\vdash \Gamma', \psi : !A \qquad \dfrac{\vdash \Delta, f : A \multimap B}{\vdash \Delta, f : A \Rightarrow B}\,d}{\vdash \Gamma', \Delta, \psi(f) : \mathbb{R} \parr B \equiv \vdash \phi(f) : \mathbb{R} \parr B, 1 : \mathbb{R}}\,\text{cut}}{\vdash \Gamma', \Delta, \psi(f) : \mathbb{R} \parr B, \text{const}_1 : A \Rightarrow \mathbb{R}}\,w \qquad \vdash \Gamma, \phi : !A}{\vdash \Gamma, \Gamma', \Delta, \psi(f) : \mathbb{R} \parr B, \phi(\text{const}_1) : \mathbb{R}}\,\text{cut}$$

Finally, the cut-elimination between a contraction and a co-contraction results in applying a convolution on a kernel: by definition, each distribution applies to each sides of a kernel. We expose first an untyped version of the cut-elimination.

$$\dfrac{\dfrac{\vdash \Delta, ?A^\perp, ?A^\perp}{\vdash ?A^\perp}\,c \qquad \dfrac{\vdash \Gamma, \phi : !A \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', !A}\,\bar{c}}{\vdash \Gamma, \Gamma', \Delta}\,\text{cut} \rightsquigarrow \tag{2.26}$$

$$\begin{gathered}\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash ?A^\perp, !A} \qquad \dfrac{}{\vdash ?A^\perp, !A}}{\vdash ?A^\perp, ?A^\perp, !A}\,\bar{c} \qquad \vdash \Delta, ?A^\perp, ?A^\perp}{\vdash \Delta, ?A^\perp, ?A^\perp, ?A^\perp}\,cut \qquad \dfrac{\dfrac{}{\vdash ?A^\perp, !A} \qquad \dfrac{}{\vdash ?A^\perp, !A}}{\vdash ?A^\perp, ?A^\perp, !A}\,\bar{c}}{\vdash \Delta, ?A^\perp, ?A^\perp, ?A^\perp, ?A^\perp}\,\text{cut} \\ \vdots \\ \pi \\ \vdots\end{gathered}$$

where π stands for:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{gathered}\vdots\\ \vdash \Delta, ?A^\perp, ?A^\perp, ?A^\perp, ?A^\perp\end{gathered}}{\vdash \Delta, ?A^\perp, ?A^\perp, ?A^\perp}\,c \qquad \vdash \Gamma, !A}{\vdash \Delta, \Gamma, ?A^\perp, ?A^\perp}\,\text{cut}}{\vdash \Delta, \Gamma, ?A^\perp}\,c \qquad \vdash \Gamma', !A}{\vdash \Delta, \Gamma, \Gamma'}\,\text{cut}$$

The typed cut-elimination rule between the contraction and the co-contraction is :

$$\dfrac{\dfrac{\vdash \Delta, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R}}{\vdash f \cdot g : A \Rightarrow \mathbb{R}}\,c \qquad \dfrac{\vdash \Gamma, \phi : !A \qquad \vdash \Gamma', \psi : !A}{\vdash \Gamma, \Gamma', \phi * \phi : !A}\,\bar{c}}{\vdash \Gamma, \Gamma', \Delta, \phi * \psi(f \cdot g) \;=\; \phi(x \mapsto \psi(y \mapsto f(x-y)g(x-y))) : \mathbb{R}}\,\text{cut} \quad \rightsquigarrow \tag{2.27}$$

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash Id : !A \multimap !A} \qquad \dfrac{}{\vdash Id : !A \multimap !A}}{\vdash \bar{c} : \phi \otimes \psi \mapsto \phi * \psi \in !A \otimes !A \multimap !A}\,\bar{c} \qquad \vdash \Delta, f : A \Rightarrow \mathbb{R}, g : A \Rightarrow \mathbb{R}}{\vdash \Delta, g : A \Rightarrow \mathbb{R}, \bar{c}^f \;=\; \phi' \otimes \psi' \mapsto (\phi' * \psi')(f) : !A \otimes !A \multimap \mathbb{R}}\,cut \qquad \dfrac{\dfrac{}{\vdash Id : !A \multimap !A} \qquad \dfrac{}{\vdash Id : !A \multimap !A}}{\vdash \bar{c} : !A \otimes !A \multimap !A}\,\bar{c}}{\vdash \Delta, \bar{c}^f : !A \otimes !A \multimap \mathbb{R}, \bar{c}^g : !A \otimes !A \multimap \mathbb{R}}\,\text{cut} \\ \vdots \\ \pi \\ \vdots \end{array}$$

where π stands for:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ \vdash \Delta, \bar{c}^f : !A \otimes !\underline{A_2} \multimap \mathbb{R}, \bar{c}^g : !A \otimes !\underline{A_4} \multimap \mathbb{R}\end{array}}{\vdash \Delta, \bar{c}^f \cdot \bar{c}^g \circ c_{2,4} : !A \otimes !\underline{A} \otimes !A \multimap \mathbb{R}}\,c \qquad \vdash \Gamma, \phi : !A}{\vdash \Delta, \Gamma, \bar{c}^f \cdot \bar{c}^g(\_, c_{2,4}(\phi), \_) : \psi \otimes \psi' \mapsto (\psi * \phi * \psi'(f \cdot g) : !\underline{A_1} \otimes !\underline{A_3} \multimap \mathbb{R}}\,\text{cut}}{\vdash \Delta, \Gamma, \bar{c}^f \cdot \bar{c}^g(\_, c_{2,4}(\phi), \_) \circ c_{1,3} : !\underline{A} \multimap \mathbb{R}}\,c \qquad \vdash \Gamma', \psi : !A}{\vdash \Delta, \Gamma, \Gamma' : \bar{c}^f \cdot \bar{c}^g(c_{1,3}(\psi)c_{2,4}(\phi)) \;=\; \phi * \psi(f \cdot g) : \mathbb{R}}\,\text{cut}$$

Remark 2.4.4. This cut rule corresponds to the Hopf rule in Hopf algebras [5].

Remark 2.4.5. Typing monolateral axioms, that is saying that $Id_E \in \mathcal{L}(E, E)$ where E is a lcs (see 3) corresponds to an element of $E' \parr E \simeq \mathscr{E}'\hat{\otimes}E$ is by definition possible if and only if E has the approximation property (see [44, chapter 18]). When $E \multimap F$ is reflexive, this follows from the closed monoidal structure.

2.4.2 Categorical semantics

We describe in the first section a global method for constructing a model of DiLL (with promotion) from a Seely category. We describe in the second section a categorical axiomatization by Ehrhard of models of DiLL, making use of *exponential structure* and making a distinction between models of DiLL_0 and those also interpreting the promotion rule.

2.4.2.1 $*$-autonomous Seely categories with biproduct and co-dereliction

Following work by Fiore [25] we will describe how in a Seely category endowed with a biproduct, the interpretation for the co-contraction and the coweakening follow immediately. A model of DiLL will therefore be a model of LL where the cartesian product is a biproduct, provided with a natural transformation $\bar{d} : Id \longrightarrow !$ interpreting the codereliction rule. Remember from Section 2.2.2, that the interpretation for the weakening and the contraction are deduced from the strong monoidality of ! and the presence of a diagonal operator $\Delta : A \longrightarrow A \times A$.

Definition 2.4.6. A biproduct on a category $\mathcal{L}$ is a monoidal structure $(\diamond, I)$ together with natural transformations:

$$\begin{tikzcd} I \arrow[dr, "u_A"'] & & I \\ & A \arrow[ur, "n_A"'] \arrow[dr, "\Delta_A"] & \\ A \diamond A \arrow[ur, "\nabla_A"] & & A \diamond A \end{tikzcd}$$

such that (A, u, ∇) is a commutative monoid and (A, n, Δ) is a commutative comonoid.

Example 2.4.7. In the category of real vector spaces and linear maps, $\diamond$ is the binary product, which corresponds indeed the the binary co-product, and the unit is the null vector space : $I = \{0\}$.

Proposition 2.4.8. *[25, prop 2.1] If $\mathcal{C}$ is endowed with a biproduct, then the diagram: $A \simeq A \diamond 1 \xrightarrow{1_A \diamond u_B} A \diamond B \xleftarrow{n_A \diamond 1_B} 1 \diamond B \simeq B$ is a co-product with (I, u_A) as initial object, while the diagram $A \simeq A \diamond 1 \xleftarrow{1_A \diamond n_B} A \diamond B \xrightarrow{u_A \diamond 1_B} 1 \diamond B \simeq B$ is a product with (I, n_A) as terminal object.*

The following diagrams interpret in particular the cut-elimination rules 2.18 and 2.27:

Proposition 2.4.9. *[25, prop 2.2] In a category with biproduct, the following diagrams are commutative.*

$$\begin{tikzcd} & A \arrow[dr, "n_A"] & \\ I \arrow[ur, "u_A"] \arrow[rr, "1"] & & I \end{tikzcd}$$

$$\begin{tikzcd} A \diamond A \arrow[r, "\nabla_A"] \arrow[d, "\Delta_A \diamond \Delta_A"] & A \arrow[r, "\Delta_A"] & A \diamond A \\ A \diamond A \diamond A \diamond A \arrow[rr, "1 \diamond \gamma_{A,A} \diamond 1"] & & A \diamond A \diamond A \diamond A \arrow[u, "\nabla_A \diamond \nabla_A"'] \end{tikzcd}$$

where $\gamma_{A,B} : A \diamond B \simeq B \diamond A$ denotes the commutativity of $\diamond$.

Definition 2.4.10. [25, 2.3] A biproduct structure $(\diamond, I)$ on a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ is said to be compatible with the monoidal structure if the following diagrams commute:

$$\begin{tikzcd} A \otimes C \arrow[r, "1 \otimes n"] \arrow[d, "n"'] & A \otimes I \arrow[d, "1 \otimes u"] \arrow[dl, "n \otimes 1"'] \\ I \arrow[r, "u"'] & A \otimes C \end{tikzcd}$$

$$\begin{tikzcd} A \otimes C \arrow[r, "\Delta \otimes 1_C"] \arrow[d, "\Delta"] & (A \diamond A) \otimes C \arrow[d, "\nabla \otimes 1_C"] \\ (A \otimes C) \diamond (A \otimes C) \arrow[r, "\nabla"] & A \otimes C \end{tikzcd}$$

The previous definition interprets in particular the cut-elimination rules of co-contraction and dereliction, and contraction and co-dereliction (see Section 2.4.2.2).

A category $\mathcal{C}$ is *enriched over commutative monoids* if every hom-set $\mathcal{C}(a, b)$ is endowed with a structure of commutative monoid $(\mathcal{C}(a, b), +_{\mathcal{C}(a,b)}, 0_{\mathcal{C}(a,b)})$ compatible with composition. In the context of DiLL, this enrichment allows to interpret the sums of proofs. In fact, the enrichment over (commutative) monoids is equivalent to the presence on $\mathcal{C}$ of a biproduct:

Proposition 2.4.11. *[25, prop 2.3] A category $\mathcal{C}$ with finite product is enriched over commutative monoids if and only if it is endowed with a biproduct structure.*

Proof. Consider $\mathcal{C}$ a category endowed with a biproduct $(\diamond, I)$. One defines a monoid on $\mathcal{C}(A, B)$ by:

$$f + g : A \xrightarrow{\Delta} A \diamond A \xrightarrow{f \diamond g} B \diamond B \xrightarrow{\nabla} B.$$

The neutral $0_{\mathcal{C}(A,B)}$ corresponds to $u_B \circ n_A : A \longrightarrow I \longrightarrow B$.

Conversely, consider $\mathcal{C}$ a category such that for every objects A, B, we have a commutative monoids $(\mathcal{C}(A, B), +_{\mathcal{C}(A,B)}, 0_{\mathcal{C}(A,B)})$, and nite products $A \times B$. We show that this product is a biproduct. The map $n_A = 0_{\mathcal{C}(A,I)}$ makes I a terminal object. Then one constructs Δ_A as $A \times A \xrightarrow{1_A \times n_A + n_A \times 1} A \times 1 \simeq A$. □

The previous proposition has an important consequence for DiLL. In a monoidal closed and vectorial setting where hom-sets are in particular vector spaces, one cannot hope to distinguish the interpretation of the additive connective on finite indexes. This argues for a polarized semantics where, if we can't distinguish between the interpretation of the connective we can distinguish the interpretation of formulas they apply to.

Interpreting the (co-)contraction and the (co-)weakening Consider $(\mathcal{C}, \otimes, 1, (.)^*)$ a $*$-autonomous category with a biproduct $(\diamond, I, u, \Delta, n, \nabla)$ (following the notation of Definition 2.4.6) and endowed with a strong monoidal functor $! : (\mathcal{C}, \diamond, \top) \longrightarrow (\mathcal{C}, \otimes, 1)$. Then we interprets the formulas of LL as in Section 2.2.2 and $[\![!A]\!] = ![\![A]\!]$ and $[\![?A]\!] = (!([\![A]\!]^*))^*$. As in Section 2.2.2, we also write $⅋$ the interpretation in $\mathcal{C}$ of the multiplicative disjunction as the dual of $\otimes$.

Proposition 2.4.12. *In a $*$-autonomous category, the biproduct is self-dual: we have a family of isomorphisms natural in A and B such that:*

$$A^* \diamond B^* \simeq (A \diamond B)^*.$$

Let us recall the interpretation of the structural rules c and w. The strong monoidal functor ! provides natural isomorphisms:

$$m_{A,B} : !(A \times B) \simeq !A \otimes !B, \tag{2.28}$$

$$m_0 : !\top \simeq 1. \tag{2.29}$$

Then one defines the natural transformations, interpreting the structural morphisms:

$$c_A : !A \xrightarrow{!\Delta_A} !(A \times A) \overset{m_{A,A}}{\simeq} !A \otimes !A \tag{2.30}$$

$$w_A : !A \xrightarrow{!n_A} !\top \overset{m_0}{\simeq} 1 \tag{2.31}$$

Thanks to the biproduct structure, the co-structural rules are interpreted by similar morphisms. Indeed, one defines the natural transformations:

$$\bar{c}_A : !A \otimes !A \overset{m^{-1}_{A,A}}{\simeq} !(A \times A) \xrightarrow{!\nabla_A} !A \tag{2.32}$$

$$\bar{w}_A : 1 \overset{m_0^{-1}}{\simeq} !\top \xrightarrow{!u_A} !A \tag{2.33}$$

Proposition 2.4.13. *[25, 3.2] For every object A, we have a commutative bialgebra $(!A, w_A, c_A, \bar{w}_A, \bar{c}_A)$: $(!A, w_A, c_A)$ is a commutative comonoid, and $(!A, \bar{w}_A, \bar{c}_A)$ is a commutative monoid.*

Let us detail specifically the interpretation of proofs whose last derivable rule is amongst $c, w, \bar{c}, \bar{w}$:

- Consider a function $f : ![\![A]\!]^* \otimes ![\![A]\!]^* \longrightarrow [\![\Gamma]\!]$ interpreting the proof of $\vdash \Gamma, ?A, ?A$ equivalent to $\vdash \Gamma, ?A ⅋ ?A$. Then the proof of $\vdash \Gamma, ?A$, and deduced by the contraction rule, is interpreted by $f \circ c_{[\![A]\!]} : ![\![A]\!]^* \longrightarrow [\![\Gamma]\!]$.
- Consider a function $f : 1 \longrightarrow [\![\Gamma]\!]$ interpreting the proof of the sequent $\vdash \Gamma$. Then the sequent $\vdash \Gamma, ?A$ deduced by the weakening rule is interpreted by $f \circ w_{[\![A]\!]^*} : ![\![A]\!]^* \longrightarrow [\![\Gamma\,|$.
- Consider functions $f : 1 \longrightarrow ![\![A]\!] ⅋ [\![\Gamma]\!]$ and $g : 1 \longrightarrow ![\![A]\!] ⅋ [\![\Delta]\!]$ interpreteting respectively the proofs of the sequents $\vdash \Gamma, !A$ and $\vdash \Delta, !A$. Then the proof of the sequent $\vdash \Gamma, \Delta, !A$ deduced by the co-contraction rule, is interpreted by $\bar{c}_A(f ⅋ g) : 1 \simeq 1 ⅋ 1 \longrightarrow ![\![A]\!] ⅋ [\![\Gamma]\!] ⅋ [\![\Delta]\!]$, where we omitted to name explicitly the associativity morphism for $⅋$.
- The interpretation of the co-weakening rule $\vdash !A$ is simply $\bar{w}_{[\![A]\!]} : 1 \longrightarrow [\![!A]\!]$.

Interpreting dereliction and codereliction In a model of LL, the dereliction rule is interpreted by the co-unit $d_A : !A \longrightarrow A$ of the monad !, see Section 2.2.2. For a model of DiLL one needs however to add an interpretation for the codereliction. This is done by introducing a natural transformation

$$\bar{d} : Id \longrightarrow !$$

satisfying the following coherence diagrams [25, 4.3].

$$\begin{tikzcd} E \arrow[dr, "\bar{d}_E"] \arrow[dd, "1"'] & \\ & !E \arrow[dl, "d_E"] \\ E & \end{tikzcd} \tag{2.34}$$

$$\begin{tikzcd}
E \otimes !F \arrow[r, "\bar{d}_E \otimes !F"] \arrow[d, "1_E \otimes d_F"'] & !E \otimes !F \arrow[d, "\phi_{E,F}"] \\
E \otimes F \arrow[r, "\bar{d}_{E \otimes F}"'] & !(E \otimes F)
\end{tikzcd} \tag{2.35}$$

$$\begin{tikzcd}
E \arrow[r, "\bar{d}_E"] \arrow[d, "d_E"'] & !E \arrow[r, "\mu_E"] & !!E \\
1 \otimes E \arrow[r, "\bar{w}_E \otimes \bar{d}_E"] & !E \otimes !E \arrow[r, "\mu_E \otimes \bar{d}_{!E}"] & !!E \otimes !!E \arrow[u, "\bar{c}_{!E}"']
\end{tikzcd} \tag{2.36}$$

Diagram 2.35 features a natural lax monoidal morphism:

$$\phi_{E,F} : !E \otimes !F \longrightarrow !(E \otimes F)$$

which is deduced from the strong monoidal structure as follows:

$$\phi : !E \otimes !F \xrightarrow{m^{-1}_{E,F}} !(E \diamond F) \xrightarrow{\mu_{E \diamond F}} !!(E \diamond F) \xrightarrow{!c_{E \diamond F}} !(!(E \diamond F) \otimes !(E \diamond F)) \tag{2.37}$$

$$\xrightarrow{!(!(1_E \otimes n_F)) \otimes !(n_E \otimes 1_F))} !(!E \otimes !F)) \xrightarrow{!(d_E \otimes d_F)} !(E \otimes F) \tag{2.38}$$

Definition 2.4.14. An intuitionist model of DiLL is a Seely category equipped with a biproduct compatible with the monoidal structure and a co-dereliction satisfying the previous axioms.

From the previous definition follows a definition for a model of DiLL.

Definition 2.4.15. We define a Seely model of DILL to be a $\star$-autonomous category $(\mathcal{C}, \otimes, 1)$ which is a Seely model of LL, equipped with a biproduct compatible with the monoidal structure and a co-dereliction satisfying equations 2.35 2.36 2.34.

2.4.2.2 Invariance of the semantics over cut-elimination

Saying that the definition above is a model for DiLL means that we have an interpretation for the proofs of DiLL, and that these interpretations are coherent with cut-elimination, meaning that *the denotational interpretation is invariant under the cut-elimination procedure*. We detail the isomorphisms below. Again, we restrict our attention to the rules of DiLL_0. Remember first that the monoid structure on hom-sets is defined from the biproduct structure:

$$f + g : A \xrightarrow{\Delta} A \diamond A \xrightarrow{f \diamond g} B \diamond B \xrightarrow{\nabla} B$$
$$0_{\mathcal{C}(A,B)} \;=\; u_B \circ n_A : A \longrightarrow I \longrightarrow B$$

- To the cut-rule 2.18 between w and $\bar{w}$ corresponds the morphisms

  $$w_A \circ \bar{w}_A : 1 \overset{m_0^{-1}}{\simeq} !\top \xrightarrow{!u_A} !A \xrightarrow{!n_A} !\top \overset{m_0}{\simeq} 1,$$

  and $w_A \circ \bar{w}_A \;=\; 1_1$ as $!n_A \circ !u_A \;=\; !(n_A \circ u_A) \;=\; !1_A$ by functoriality of ! and by definition of a biproduct.

- To the cut-rule 2.22 between w and $\bar{d}$ corresponds the morphism

  $$w_A \circ \bar{d}_A : A \xrightarrow{\bar{d}_A} !A \xrightarrow{!n_A} !0 \overset{m_0}{\simeq} 1.$$

  As d is a natural transformation the following diagram is commutative, where I is neutral for $\diamond$:

$$
\begin{tikzcd}
A \arrow[r, "n_A"] \arrow[d, "\bar{d}_A"] & I \arrow[d, "\bar{d}_I"] \\
!A \arrow[r, "!n_A"] & !I
\end{tikzcd}
$$

As I is initial [25, 2.1], $d_I : I \longrightarrow !I \simeq 1$ is the only morphism from I to 1, thus $d_I = u_1$. Thus, through the previous commutative diagram we have $w_A \circ \bar{d}_A = u_I \circ n_A = 0_{\mathcal{C}(A,I)}$ by definition of the additive structure on hom-sets. This justifies the reduction to a cut between a weakening and a codereliction to a zero proof $\mathfrak{o}$, the logical interpretation of the 0 morphisms in an category enriched over commutative monoids.

- The same approach shows that the cut-rule 2.23 between $\bar{w}$ and d corresponds the morphism $\bar{w}_A \circ d_A = u_A \circ n_1 = 0_{\mathcal{C}_{1,A}}$, interpreting the zero proof $\mathfrak{o}$, proving A from no hypothesis.
- The cut-rule 2.21 between d and $\bar{d}$ corresponds to the morphism $d_A \circ \bar{d}_A$, which is the identity by hypothesis, see diagram 2.34.
- The cut-rule 2.19 between the contraction c and the co-weakening $\bar{w}$ in interpreted by the morphism:

$$c_A \circ \bar{w}_A : 1 \overset{m_0^{-1}}{\simeq} !\top \xrightarrow{!u_A} !A \xrightarrow{!\Delta_A} !(A \times A) \overset{m_{A,A}}{\simeq} !A \otimes !A.$$

As the product $\times$ comes from the biproduct $\diamond$, the terminal object $\top \simeq 0$ (which is the by definition the unit for the product) is the unit for I, the unit of the biproduct. Thus we have by [25, prop 2.2]

$$!\Delta \circ !u = !(\Delta \circ u) = u \diamond u,$$

where we omit the explicit description of the isomorphism between I and $I \diamond I$. Thus:

$$c_A \circ \bar{w}_A : 1 \overset{m_0^{-1}}{\simeq} !I \simeq !(I \diamond I) \xrightarrow{!(u_A \diamond u_a)} !(A \diamond A) \overset{m_{A,A}^{-1}}{\simeq} !A \otimes !A.$$

Through the strong monoidality of ! one can describe $!(u_A \diamond u_A)$ as $!u_A \otimes !u_A$, and the above morphisms correspond to the interpretation of the successive cut-rule between a co-weakening rule and one of the instances of $?A$ in the sequent $\vdash \Gamma, ?A, ?A$. This is exactly the interpretation of the cut-eliminated proof-tree in 2.19.

- The case concerning the cut-rule 2.20 between the co-contraction c and the weakening $\bar{w}$ behave exactly likewise, by symmetry of the definition of weakening (resp. contraction) with the one of the co-weakening (resp. co-contraction).
- The cut-rule 2.24 between the contraction and the co-dereliction is interpreted by:

$$c_A \circ \bar{d}_A : A \xrightarrow{\bar{d}_A} !A \xrightarrow{!\Delta_A} !(A \times A) \overset{m_{A,A}}{\simeq} !A \otimes !A.$$

By naturality of $\bar{d}$ we have that $!\Delta_A \circ \bar{d}_A = \bar{d}_{A \diamond A} \circ \Delta_A$, thus $c_A \circ \bar{d}_A = m_{A,A} \circ \bar{d}_{A \diamond D} \Delta_A$.
Moreover, the sum of proof trees resulting from the cut-rule is:

$$\bar{d}_A \otimes \bar{w}_A + \bar{w}_A \otimes \bar{d}_A : A \otimes 1 \simeq 1 \otimes A \simeq A \xrightarrow{\Delta} A \diamond A \xrightarrow{(\bar{d} \otimes \bar{w}) \diamond (\bar{w} \otimes \bar{d})} (!A \otimes !A) \diamond (!A \otimes !A) \xrightarrow{\nabla} !A \otimes !A.$$

To show equality between $c_A \circ \bar{d}_A$ and $\bar{d}_A \otimes w_A + w_A \otimes \bar{d}_A$ we must then prove:

$$\bar{d}_{A \diamond A} \circ m_{A,A} = \nabla_{!A \otimes !A} \circ (\bar{d} \otimes \bar{w}) \diamond (\bar{w} \otimes \bar{d})$$

which follows from the compatibility of the biproduct with $\otimes$ (Definition 2.4.10).

- The cut-rule 2.25 between the co-contraction and the dereliction is interpreted by:

$$d_A \circ \bar{c}_A : !A \otimes !A \overset{m_{A,A}^{-1}}{\simeq} !(A \diamond A) \xrightarrow{!\nabla_A} !A \xrightarrow{d_A} A.$$

By naturality of $\bar{d}$ we have that $d_A \circ !\nabla_A = \nabla_A \circ d_{A \diamond A}$, thus $d_A \circ \bar{c}_A = \nabla_A \circ d_{A \diamond A} \circ m_{A,A}^{-1}$.

Moreover, the sum of proof trees resulting from the cut-rule is:

$$\begin{aligned} d_A \otimes w_A + w_A \otimes d_A : {!A} \otimes {!A} &\xrightarrow{\Delta} ({!A} \otimes {!A}) \diamond ({!A} \otimes {!A}) \\ &\xrightarrow{(d\otimes w)\diamond(w\otimes d)} (A \otimes 1) \diamond (1 \otimes A) \simeq (A \otimes 1) \diamond (A \otimes 1) \simeq A \diamond A \xrightarrow{\nabla_A} A. \end{aligned}$$

To show the equality between $c_A \circ \bar{d}_A$ and $\bar{d}_A \otimes w_A + w_A \otimes \bar{d}_A$ we must then prove:

$$d_{A \diamond A} \circ m^{-1}_{A,A} = (d \otimes w) \diamond (w \otimes d) \circ .$$

which follows from the compatibility of the biproduct with $\otimes$ (Definition 2.4.10).

- The cut-rule 2.27 between the contraction and the co-contraction is interpreted by:

$$c_A \circ \bar{c}_A : {!A} \otimes {!A} \overset{m_{A,A}}{\simeq} {!(A \times A)} \xrightarrow{!\nabla_A} {!(A)} \xrightarrow{!\Delta_A} {!(A \times A)} \overset{m^{-1}_{A,A}}{\simeq} {!A} \otimes A.$$

However by properties of the biproduct (see [25, 2.2], recalled in Proposition 2.4.9)

$$!\nabla_A \circ !\Delta_A \;=\; (\Delta_A \diamond \Delta A) \circ (1_A \diamond \gamma_{A,A} \diamond_A) \circ (\nabla_A \diamond \nabla_A),$$

where $\gamma_{A,B} : A \diamond B \simeq B \diamond A$ is the commutation for $\diamond$. As $![(\Delta_A \diamond \Delta A) \circ (1_A \diamond \gamma_{A,A} \diamond_A) \circ (\nabla_A \diamond \nabla_A)]$ is exactly the interpretation of the reduced cut-rule 2.27, we have our result.

2.4.2.3 Exponential structures

Ehrhard defines [20] the notion of *exponential structures* on a (pre)-additive $*$-autonomous category a as tuple $(!, w, c, \bar{w}, \bar{c})$ such that ! is an endofunctor on $\mathcal{C}_{\text{iso}}$ (the subcategory of $\mathcal{C}$ with only isomorphisms as arrows), and such that for every object A, $(!A, w_A, c_A, \bar{w}_A, \bar{c}_A)$ is a commutative bialgebra. The isomorphisms are required to interpret the coherence rules. These morphisms moreover satisfy the following invariance diagrams, which axiomatize what has been proved before in Seely categories.

$$\begin{tikzcd} E \arrow[dr, "\bar{d}"] \arrow[dd, "Id"'] & \\ & {!E} \arrow[dl, "d"] \\ E & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[dr, "\bar{d}"] \arrow[dd, "0"'] & \\ & {!E} \arrow[dl, "w"] \\ 1 & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[dr, "c"] \arrow[dd, "\bar{d} \otimes \bar{w} + \bar{w} \otimes \bar{d}"'] & \\ & {!E} \arrow[dl, "\bar{d}"] \\ {!E} \otimes {!E} & \end{tikzcd}$$

$$\begin{tikzcd} {!E} \arrow[dr, "d"] \arrow[dd, "0"'] & \\ & E \arrow[dl, "\bar{w}"] \\ 1 & \end{tikzcd} \qquad \begin{tikzcd} {!E} \otimes {!E} \arrow[dr, "\bar{c}"] \arrow[dd, "d \otimes w + w \otimes d"'] & \\ & {!E} \arrow[dl, "d"] \\ E & \end{tikzcd}$$

Figure 2.7: Commutative diagrams in Exponential structures

Remark 2.4.16. In Chapter 7 and 8 we develop concrete models of DiLL_0, where ! is a functor but not an endofunctor. A proper categorical semantics for these models, intermediate between Seely categoriess and exponential structures, should be developed.

2.4.3 A Smooth intuitionistic model: Convenient spaces

We recall in this section how *Mackey-complete lcs*, *linear bounded functions* and *conveniently smooth functions* define a model of Intuitionistic DiLL. This is a simplification of the article of Blute, Ehrhard and Tasson [7], as it does not require spaces to be bornological.

The first models of DiLL were constructed by Ehrhard as a vectorial generalisation of coherent spaces in Köthe spaces [18] (see Section 2.2.3) and Finiteness spaces [19]. They are constructed from spaces of sequences. Once

DiLL was formalized a calculus of interaction nets [23] and its computational content axiomatized in the differential λ-calculus [22], a natural question was the geometric nature of the differential used. Indeed, differentiation in the models was done over power series: but there is more to the theory of differentiation than extracting the linear part of a power series. How far can the sequent calculus DILL model the general behaviour of differential geometry? This led Blute, Ehrhard and Tasson [7] to define a model of Intuitionistic DiLL, where functions are general smooth, i.e everywhere and iteratively differentiable, functions, with no further restriction. The model relies on the work of Frölicher, Kriegl and Michor [26, 53] who define completeness conditions and good notions of smoothness allowing for a cartesian closed category of smooth functions.

One of the specificites of this model is that linear maps are interpreted by linear *bounded* functions, which are not continuous in general. The necessity to relax continuity of functions into their boundedness comes from the specific definition of smoothness by Frölicher, Kriegl and Michor. The spaces satisfy a completeness condition known as *Mackey-completeness*, and are sometines asked to be moreover *bornological* ([7, 26]). The bornological conditions modifies the topology of spaces in order to ensure that every bounded linear map is still continuous. It is in fact unnecessary to the intuitionistic constructions of DiLL as we showed with Tasson [49]. We explain in Section 6, Chapter 6 how the bornological condition ensures a classical interpretation of DiLL.

We will *sketch* here the model of DiLL as detailed in [7], without the bornological condition. It corresponds thus to the summary made in sections 2, 3 and 4 of [49]. We will give more detail on the bornological condition later in section 6. We recall a few notion of Chapter 3. Contents of sections 3.4, 3.4.3 and 3.1 are enough to understand convenient vector spaces. We recall that an absolutely convex subset B of a vector space E is a subset such that: $\forall x, y \in B, \forall \lambda \in \mathbb{R}, \mu \in \mathbb{R}, |\lambda| + |\mu| < 1 : \lambda x + \mu y \in B$.

Definition 2.4.17. A *locally convex and separated topological vector space* (denoted as lcs) is a $\mathbb{R}$-vector space E, $\mathbb{R}$ being $\mathbb{R}$ or $\mathbb{C}$ endowed with a topology, admitting a pre-basis of absolutely convex neighbourhoods, making addition and scalar multiplication continuous.

Let E and F denote lcs from now on. A *bounded set* in a lcs E is a subset $B \subset E$ which is absorbed by any 0-neighbourhood. That is, for any open set U containing 0, there exists $\lambda \in \mathbb{R}$ such that $B \subset \lambda U$. A *bounded function* $f : E \longrightarrow F$ is then a function such that the image by f of any bounded set in E is bounded in F.

Definition 2.4.18. We denote by $\mathbf{L}(E, F)$ the space of all *bounded* linear functions. We endow it with the topology of uniform convergence on bounded subsets of E. This topology is generated by the open sets

$$\mathcal{W}_{B,U} = \{\ell \mid \ell(B) \subset U\}$$

where B is a bounded set in E and U an absolutely convex open set in F. In particular, a sequence of linear functions $(\ell_n)_n \in \mathbf{L}(E, F)$ converges towards $\ell \in \mathbf{L}(E, F)$ if and only if for every bounded set $B \subset E$, every 0-neighbourhood $U \subset F$, there is $N \in \mathbb{N}$ such that for $n \geqslant N$, we have $(\ell_n - \ell)(B) \subset U$.

The product and direct sum of two lcs are endowed with the usual product and co-product topologies (see Section 3.1.4). In particular, a bounded set in $E \times F$ is a product of bounded sets. Remember also that on finite indexes, products and direct sums coincide.

The use of bounded subsets simplifies the theory of topological vector spaces. If several notions of continuity can be defined on bilinear functions (for example continuity is not equivalent to separate continuity) there is only one notion of bounded bilinear function and thus of tensor product for them.

Definition 2.4.19. We endow the tensor product $E \otimes F$ of two lcs with the finest topology making the canonical bilinear map $h : E \times F \longrightarrow E \otimes F$ bounded. This makes $E \otimes F$ a lcs which is denoted $E \otimes_B F$.

Then one shows easily the following:

Proposition 2.4.20. *[49, 3.4] The category of* lcs *endowed with* $\otimes_B$ *is a monoidal closed category with internal hom* $E \multimap F \;=\; \mathbf{L}(E, F)$.

We write $E^\times := \mathbf{L}(E, \mathbb{R})$ the bornological dual of a lcs E. In general, it differs from the topological dual E'. It is the denotational interpretation of negation. Note that we won't have in this model $E \simeq E^{\times\times}$ in general, and thus we will not have a denotational model for classical DiLL.

In order to accommodate smooth functions, one needs a certain completeness condition on the lcs. Differentiation, and thus smoothness, is defined in term of limits of sequences in a given lcs F. If we want to compose smooth functions, or to prove cartesian closedness, most of the time we will be able to prove that the difference quotient is Cauchy, *i.e.* that the quotients are getting closer and closer, but we won't be able to prove directly that it converges. Knowing that Cauchy sequences converges is the definition of completeness. In their work Kriegl and Michor require a very weak completeness condition, which works well within the bounded setting.

Definition 2.4.21. A net $((x_\gamma)_\Gamma)$ is a family of elements of E indexed by a directed set Γ.

Definition 2.4.22. A *Mackey-Cauchy net* in E is a net $(x_\gamma)_{\gamma\in\Gamma}$ such that there is a net of scalars $\lambda_{\gamma,\gamma'}$ decreasing towards 0 and a bounded set B of E such that:

$$\forall\gamma,\gamma'\in\Gamma, x_\gamma - x_{\gamma'}\in\lambda_{\gamma,\gamma'}B.$$

A space where every Mackey-Cauchy net converges is called *Mackey-complete*.

There exists a procedure of *Mackey-completion*, that can be thought of as a the interpretation of a shift $\uparrow$ (see Section 2.3.1), and which allows to make a space Mackey-complete.

Proposition 2.4.23. *[53, I.4.29] For every lcs E there is a Mackey-complete lcs $\tilde{E}$ and a bounded embedding $\iota : E \to \tilde{E}$, unique up to bounded isomorphism, such that for every Mackey-complete lcs F, for every bounded linear map $f : E \to F$ there is a unique bounded linear map $\tilde{f} : \tilde{E} \to F$ extending f such that $f = \tilde{f}\circ\iota$.*

We denote by $\mathrm{M}\textsc{co}$ the category of Mackey-complete spaces and bounded linear maps.

Proposition 2.4.24. *[49, 3.5] The category $\mathrm{M}\textsc{co}$ endowed with the Mackey-completed tensor product $\hat{\otimes}$ is a monoidal closed category with internal hom $E \multimap F = \mathbf{L}(E,F)$.*

Products and co-products preserve Mackey-completeness, and thus $(\mathrm{M}\textsc{co},\otimes,\mathbb{R},\times,\{0\})$ is a model of MALL. We detail now the interpretation of the exponential ! as the dual of a space of smooth function. From now on we work with $\mathbb{R}$-vector spaces.

Definition 2.4.25. Let E be a Mackey-complete lcs. Consider a curve $c : \mathbb{R}\to E$, then c is said to be derivable if the following limit exist for every $t\in\mathbb{R}$:

$$c'(t) = \lim_{s\to 0}\frac{c(t+s)-c(t)}{s}.$$

The definition of smoothness is co-inductive: the curve c is *smooth* if it is derivable and its derivative in every point is smooth. We denote by $\mathcal{C}_E$ the set of smooth curves in E.

The space $\mathcal{C}_E$ is a Mackey-complete lcs when it is endowed with the topology of uniform convergence on bounded sets of each derivative separately [53, I.3.7]. A basis of 0-neighbourhoods for this topology is made of $\mathcal{W}_{b,i,U}$, where B is a bounded set in $\mathbb{R}$, $i\in\mathbb{N}$, U is a 0-neighbourhood in E, and

$$\mathcal{W}_{B,i,U} = \{c \mid \forall t\in b,\ c^{(i)}(t)\in U\}.$$

Definition 2.4.26. A function $f : E\to F$ is *smooth* if it preserves smooth curves: $\forall c\in\mathcal{C}_E,\ f\circ c\in\mathcal{C}_F$. We denote by $\mathcal{C}^\infty(E,F)$ the space of *smooth maps* from E to F. This definition of smoothness is a generalization of the usual one for finite dimension topological vector spaces (see Boman's theorem [10]).

This definition of smoothness *does not imply continuity*. In particular, the linear smooth maps are exactly the smooth bounded ones.

The space $\mathcal{C}^\infty(E,F)$ is endowed with the projective topology induced by the product $\prod_{c\in\mathcal{C}_E}\mathcal{C}_F$. In fact, the space $\mathcal{C}^\infty(E,F)$ is the closed subspace of $\prod_{c\in\mathcal{C}_E}\mathcal{C}_F$ whose elements $(f_c)_c$ are those such that for every $g\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, $f_{c\circ g} = f_c\circ g$. It is Mackey-complete when F is Mackey-complete [53, I.3.11]. Let us denote by $\mathbf{Smooth}$ the category of Mackey-complete lcs and smooth maps between them.

Theorem 2.4.27. *[53] The category $\mathbf{Smooth}$ is cartesian closed, meaning that for E, F, and G Mackey-complete lcs we have natural isomorphisms:*

$$\mathcal{C}^\infty(E\times F,G)\simeq\mathcal{C}^\infty(E,\mathcal{C}^\infty(F,G)).$$

As explained in remark 2.7, in a $*$-autonomous category modelizing classical LL, one would want to interpret $!E$ as $\mathcal{C}^\infty(E,\mathbb{R})'$. It is not the case here, but it is close : the exponential is defined as the completion of the linear span of all Dirac distributions δ_x in $\mathcal{C}^\infty(E,\mathbb{R})^\times$.

$$\delta : \begin{cases} E\to\mathcal{C}^\infty(E,\mathbb{C})^\times \\ x\mapsto\delta_x : f\mapsto f(x)\end{cases}$$

Definition 2.4.28. We define the Mackey-complete lcs $!E$ as the Mackey-completion $\widetilde{\langle\delta(E)\rangle}$ of the linear span of $\delta(E) = \{\delta_x \mid x \in E\}$. If $f \in \mathbf{Lin}(E,F)$ is a smooth linear map, its exponential $!f \in \mathbf{Lin}(!E,!F)$ is defined on the set $\delta(E)$ by $!f(\delta_x) = \delta_{f(x)}$. It is then extended to the linear span of $\delta(E)$ by linearity and to $!E$ by the universal property of the Mackey-completion.

Thus ! defines an endofunctor on MCO. It is moreover a co-monad, with co-unit and co-multiplication:

$$
\begin{aligned}
d_E &: \delta_x \in !E \mapsto x \in E \\
\mu_E &: \delta_x \in !E \mapsto \delta_{\delta x} \in !!E
\end{aligned}
$$

This interpretation for the exponential gives an interpretation of non-linear proofs of LL as smooth maps.

Theorem 2.4.29. *[7] The cokleisli category of the comonad ! over* MCO *is the category* $\mathbf{Smooth}$*. In particular, for any Mackey-complete spaces E and F,*

$$\mathcal{L}(!E,F) \simeq \mathcal{C}^\infty(E,F).$$

As $\mathbf{Smooth}$ is cartesian closed and MCO is monoidal closed, the previous theorem leads to the strong monoidality of the functor !, defined from $(\text{MCO}, \times, \{0\})$ to $(\mathbf{Smooth}, \otimes, \mathbb{R})$. If $!E$ were defined as $\mathcal{C}^\infty(E,\mathbb{R})^\times$ as in equation 2.7, and spaces were equal to their double bornological dual $E \simeq E^{\times\times}$ this would follow from the following computation:

$$
\begin{aligned}
\mathcal{C}^\infty(E \times F, \mathbb{R})^\times &\simeq \mathcal{C}^\infty(E, \mathcal{C}^\infty(F,\mathbb{R}))^\times && (2.39)\\
&\simeq \mathcal{L}(!E, (!F)^\times)^\times && (2.40)\\
&\simeq \mathcal{L}(!E \otimes !F, \mathbb{R})^\times && (2.41)\\
&\simeq !E \otimes !F && (2.42)
\end{aligned}
$$

In the intuitionistic setting of convenient vector spaces, the computations are slightly more subtle.

Theorem 2.4.30. *[7, 5.6] For any E, and F Mackey-complete spaces we have a natural isomorphism:*

$$m_{E,F} : !E \hat{\otimes} !F \simeq !(E \times F).$$

Proof. We sketch the proof detailed byBlute, Ehrhard and Tasson in [6]. The goal is to construct a bilinear bounded map $h : !E \times !F \longrightarrow !(E \times F)$, and to show that it is universal over bilinear maps on Mackey-complete spaces. The Mackey-completed tensor product $\hat{\otimes}$ enjoys indeed a universal property in MCO [6, 3.1] and is thus unique.

The bilinear map h is defined as $h(\delta_x, \delta_y) = \delta_{(x,y)}$. If $f : !E \times !F \longrightarrow G$ is another bilinear bounded map between Mackey-complete lcs, one first shows that f is smooth, and thus $f \circ (\delta, \delta) : E \times F \longrightarrow G$ is smooth and factors as a linear bounded map: $\hat{f} : !(E \times F) \longrightarrow G$. One then easily shows that $f = \hat{f} \circ h$. Thus $!(E \times F)$ with h is universal, and we have thus an isomorphism between $!(E \times F)$ and $!E \hat{\otimes} !F$. □

From the biproduct structure on MCO and the strong monoidality of !, we deduce as before a bialgebra structure on every object $!E$. The natural transformations making this bialgebraic structure are nicely defined on the Dirac distributions δ_x.

$$
\begin{aligned}
c &: \delta_x \in !E \mapsto \delta_x \otimes \delta_x \in !E \hat{\otimes} !E && (2.43)\\
w &: \delta_x \in !E \mapsto x \in E && (2.44)\\
\bar{c} &: \delta_x \otimes \delta y \in !E \hat{\otimes} !E \mapsto \delta_{x+y} \in !E && (2.45)\\
\bar{w} &: 1 \in \mathbb{R} \mapsto \delta_0 \in !E && (2.46)
\end{aligned}
$$

The co-dereliction is then the operator which extracts from a scalar map $f \in \mathcal{C}^\infty(E,\mathbb{R})$ its differential at 0:

$$\bar{d} : v \in E \mapsto \left(f \in \mathcal{C}^\infty(E,\mathbb{R}) \mapsto \lim_{t \to 0} \frac{f(tv) - f(0)}{t} \right) = \lim_{t \to 0} \frac{\delta_{tv} - \delta_0}{t} \in !E$$

The Identity rule

$$\frac{}{\vdash A, A^{\perp}}\ \text{(axiom)} \qquad \frac{\vdash \mathcal{N}, A \qquad \vdash A^{\perp}, \mathcal{M}}{\vdash \mathcal{N}, \mathcal{M}}\ \text{(cut)}$$

The additive rules

$$\frac{}{\vdash 1}\ (1) \qquad \frac{\vdash \mathcal{N}}{\vdash \mathcal{N}, \perp}\ (\perp) \qquad \frac{\vdash \mathcal{N}, N, M}{\vdash \mathcal{N}, N \parr M}\ (\parr) \qquad \frac{\vdash \mathcal{N}, P \qquad \vdash \mathcal{M}, Q}{\vdash \mathcal{N}, \mathcal{M}, P \otimes Q}\ (\otimes)$$

The multiplicative rules

$$\frac{}{\vdash \mathcal{N}, \top}\ \top \qquad \frac{\vdash \mathcal{N}, N \qquad \vdash \mathcal{N}, M}{\vdash \mathcal{N}, N \,\&\, M}\ \& \qquad \frac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, P \oplus Q}\ \oplus_L \qquad \frac{\vdash \mathcal{N}, Q}{\vdash \mathcal{N}, P \oplus Q}\ \oplus_R$$

The Exponential Rules

$$\frac{\vdash \mathcal{N}}{\vdash \mathcal{N}, ?P}\ w \qquad \frac{\vdash \mathcal{N}, ?P, ?P}{\vdash \mathcal{N}, ?P}\ c \qquad \frac{\vdash \mathcal{N}, P}{\vdash \mathcal{N}, ?P}\ d$$

$$\frac{}{\vdash !N}\ \bar{w} \qquad \frac{\vdash \mathcal{N}, !N \qquad \vdash \mathcal{M}, !N}{\vdash \mathcal{N}, \mathcal{M}, !N}\ \bar{c} \qquad \frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N}\ \bar{d}$$

Figure 2.8: The deduction rules for sequents of $\mathrm{DiLL}_{0,pol}$

2.5 Polarized Differential Linear Logic

2.5.1 A sequent calculus

We believe that the classical nature of DiLL is better understood in a polarized setting: we will support this claim in Chapter 7 by giving a polarized denotational model of DiLL, and in 8 by giving a computational interpretation in terms of differential equations. We describe here in terms of sequent calculus a polarized Differential Linear Logic, without promotion. This has been done in terms of interaction nets by Vaux [79].

Definition 2.5.1. The formulas of $\mathrm{DiLL}_{0,pol}$ are constructed over a set $\mathfrak{N}$ of negative atoms $\mathfrak{n}$, and makes a distinction between positive formulas P, Q and negative formulas N, M. They are constructed from the following grammar:

$$\begin{array}{r}\text{Negative Formulas: } N, M := \mathfrak{n} \mid ?P \mid N \parr M \mid \perp \mid N \times M \mid \top \mid \\ \text{Positive Formulas: } P, Q := \mathfrak{n}^{\perp} \mid !N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1\end{array}$$

Its inference rules are describes in figure 2.8 and restricts the rules of DiLL_0. The cut-elimination procedure for $\mathrm{DiLL}_{0,pol}$ is directly deduced from the one of DiLL and $\mathrm{LL}_{\mathrm{pol}}$.

$\mathrm{DiLL}_{\mathrm{pol}}$ with shift As $\mathrm{LL}_{\mathrm{pol}}$, $\mathrm{DiLL}_{\mathrm{pol}}$ can be enhanced with shift, changing the polarity of a formula.

Definition 2.5.2. The formulas of $\mathrm{DiLL}_{pol,\uparrow\downarrow}$ are defined from a set $\mathfrak{N}$ of negative atoms, and the following grammar:

$$\begin{array}{r}\text{Negative Formulas } N, M := \mathfrak{n} \in \mathfrak{N} \mid \uparrow P \mid ?P \mid N \parr M \mid \perp \mid N \times M \mid \top \mid \\ \text{Positive Formulas: } P, Q := \mathfrak{n}^{\perp} \mid \downarrow N \mid !N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1\end{array}$$

Definition 2.5.3. The inference rules for $\mathrm{DiLL}_{0,pol,\uparrow\downarrow}$ are the ones of $\mathrm{LL}_{\mathrm{pol}}$ extended by the following rules. $\mathcal{N}$ is a list of negative formulas.

$$\frac{\vdash N, N}{\vdash N, \downarrow N}\ \downarrow \qquad \frac{\vdash N, P}{\vdash N, \uparrow P}\ \uparrow$$

Then one defines:

$$\uparrow P^{\perp_L} \;=\; \downarrow(P^{\perp_R}) \;\downarrow N^{\perp_R} \;=\; \uparrow(N^{\perp_L})$$

2.5.2 Categorical semantics

In the categorical semantics of DiLL, co-dereliction is interpreted by a natural transformation $\bar{d}$ which is right inverse to the interpretation d of the dereliction on every object:

$$d_A \circ \bar{d}_A \;=\; id_A.$$

Dereliction is the rule which allows to see a linear map as a smooth one: it corresponds to the co-unit of the co-monad, or in a presentation using an adjunction:

$$\begin{tikzcd}
(\mathcal{L}^\infty, \times) \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & (\mathcal{L}, \otimes) \arrow[l, bend left, "\mathcal{U}"]
\end{tikzcd}$$

the dereliction of a linear map $f \in \mathcal{L}(A, B)$ is the smooth map $\mathcal{U}(f) \in \mathcal{L}(!A, B)$. Conversely, the co-dereliction should correspond to a forgetful functor $\bar{\mathcal{U}}$ mapping a smooth function $f \in \mathcal{L}(!A, B)$ to its differentiation at 0: $D_0(f) \in \mathcal{L}(A; B)$. However D_0 is notably not compositional, and thus it does not define a functor from $\mathcal{L}$ to $\mathcal{L}_!$. In Section 2.5.2.1 we detail the categories in which the equation

$$d \circ \bar{d} \;=\; Id$$

can be interpreted as a closure operation between functors. Then in Section 2.5.2.3 we give a categorical axiomatization for models of $\mathrm{DiLL}_{0,pol}$.

Outlook 1. This section on the categorical semantics of $\mathrm{DiLL}_{\mathrm{pol}}$ is a first step towards a more unified framework. Indeed, while $\mathrm{MLL}_{\mathrm{pol}}$ is interpreted categorically in chiralities, we would like to interpreted the exponential rules of $\mathrm{DiLL}_{\mathrm{pol}}$ in a chirality between the category of linear maps and the one of smooth maps, where the role of shifts are played by the dereliction and the co-dereliction.

2.5.2.1 Dereliction and co-dereliction as functors

In this section we explore to which extent the dereliction and co-dereliction could be seen as functors, acting as shifts in a chirality. To see Differentiation as a functor, we detail here the ideas of synthetic differential geometry [50]. We denote by $\circ$ the composition in $\mathcal{L}$ and by $\circ_!$ the composition in $\mathcal{L}_!$. We will first recall the properties of the forgetful functor which was defined after Definition 2.2.13.

Proposition 2.5.4. *Consider $\mathcal{L}$ a category and $(!,\, d,\, \mu)$ a co-monad on a small category $\mathcal{L}$. Then one defines the forgetful functor $\mathcal{U}_d : \mathcal{L} \longrightarrow \mathcal{L}_!$ such that $\mathcal{U}_d(A) \;=\; A$ and $\mathcal{U}_d(f : A \longrightarrow B) \;=\; f \circ d_A$.*

Proof. Let us define $\mathcal{U}_d(A) \;=\; A$ on objects of $\mathcal{L}$, and if $f \in \mathcal{L}(A, B), \mathcal{U}_d \;=\; f \circ d_A \in \mathcal{L}(!A, B) \simeq \mathcal{L}_!(A, B)$. Then $\mathcal{F}_d(1_A) \;=\; d_A$ which correspond by definition to the unit in the co-kleisli category. Likewise, we have for $g : B \longrightarrow C$:

$$\mathcal{U}_d(g \circ f) \;=\; g \circ f \circ d_A \in \mathcal{L}_!(A, C).$$

However, in the co-Kleisli category $\mathcal{L}_!$, $\mathcal{U}_d(g) \circ_! \mathcal{F}_d(f) \;=\; g \circ d_B \circ !(f \circ d_A) \circ \mu_A \;=\; g \circ d_B \circ !f \circ !d_A \circ \mu_A \;=\; g \circ d_B \circ !f \circ 1_{!A}$ by the second law of co-monads. From the last equality, and by naturality of d, we deduce $\mathcal{U}_d(g) \circ_! \mathcal{U}_d(f) \;=\; g \circ f \circ d_A$. Thus $\mathcal{U}_d$ is a functor.

□

While the previous proposition recalls that we can consider the dereliction as the forgetful functor between our category with linear maps and its co-Kleisli category of (thought-as) smooth maps, the converse for dereliction is more intricate. Indeed, consider the application D_0 which maps $f \in \mathcal{L}_!(A, B))$ to $D_0 f \in \mathcal{L}(A, B)$. Thus D_0 is not a functor as it not compositional:

$$D_0(f \circ g) \;=\; x \mapsto D_{f(0)}(g)(D(f)(0)(x)).$$

Thus to account for the arbitrary choice of 0, and view the codereliction as a functor, we introduce a new category:

Definition 2.5.5. Consider ! a co-monad on $\mathcal{L}$.

We denote $\mathcal{L}_!^\bullet$ the category whose object are those of $\mathcal{L}$, and whose morphisms are morphisms of $\mathcal{L}_!$ coupled with points of the domain:

$$\mathcal{L}_!^\bullet(A, B) \ = \ \mathcal{L}(!A, B) \times !A.$$

Composition is defined as follows: if $(f, \phi) \in \mathcal{L}_!^\bullet(A, B)$ and $(g, \psi) \in \mathcal{L}_!^\bullet(B, C)$, then:

$$(g, \phi) \circ_{!,\cdot} (f, \psi) \ = \ 0 \text{ if } \phi \neq !f\psi \ .$$

$$(g, (!f)\psi) \circ_{!,\cdot} (f, \phi) \ = \ (g \circ_! f, \psi)$$

This composition is associative by associativity of $\circ_!$. The unit is (d_A, δ_0).

Proposition 2.5.6. *Consider a model of $DiLL$ as described by Fiore ([25], Section 2.4.2). As usual, we denote by $\bar{d} : Id \longrightarrow !$ the creation operator interpreting the codereliction rule, that is the natural transformation such that $d \circ \bar{d} \ = \ Id$. We denote by $* : ! \times ! \longrightarrow !$ the binatural transformation interpreting the cocontraction. Then let us define:*

$$\bar{\mathcal{U}} : \begin{cases} \mathcal{L}_!^\bullet \to \mathcal{L} \\ A \mapsto A \\ (f : !A \longrightarrow B, \phi \in !A) \mapsto [v \in A \mapsto f \circ (\phi * \bar{d}_A(v))] \end{cases} \tag{2.47}$$

Then $\bar{\mathcal{U}}$ is a functor.

Example 2.5.7. One must think of $\bar{\mathcal{U}}$ as the functor mapping $f \in \mathcal{C}^\infty(A, B)$ and a (non-linear) point $x \in A$ to the linear map $v \mapsto D_x f(v)$. Then we have indeed:

$$D_x(f \circ g)(v) \ = \ D_{f(x)}(g) \circ D_x f(v).$$

Proof. We have $\bar{\mathcal{U}}(d_A, \delta_0) \ = \ [x \in A \mapsto d_A \circ (\delta_0 * \bar{d}_A(v)]$, thus as δ_0 is the neutral for $*$ we have:

$$\bar{\mathcal{U}}(d_A, \delta_0)(x) \ = \ d_A \circ \bar{d}_A(v) = v.$$

This shows that $\bar{\mathcal{U}}$ preserves the neutrals. Consider now $g \in \mathcal{L}(!B, C)$, $f \in \mathcal{L}(!A, B)$ and $\phi \in !A$. Then for $v \in A$ we have by the chain rule:

$$\begin{aligned} \bar{\mathcal{U}}(g \circ_! f, \phi))(v) \ &= \ g \circ !f \circ \mu_A \circ (((!f)\phi) * \bar{d}_A(v)) \\ \bar{\mathcal{U}}(g, !f\phi) \circ \bar{\mathcal{U}}(f, \phi)(v) \ &= \ g \circ (((!f)\phi) * \bar{d}_B(f \circ (\phi * \bar{d}_A(v)))) \end{aligned}$$

□

Outlook 2. Notice that syntactically Differential Linear Logic manages to avoid seing Differentiation as a functor, by using the co-contraction, which allows to sum in the domain of a function.

2.5.2.2 A polarized biproduct

In models of Differential Linear Logic, the addition which is necessary on hom-sets can be defined through a biproduct structure 2.4.6. In a polarized setting, this biproduct is defined through a cartesian equivalence of categories.

Definition 2.5.8. Consider a negative chirality $(\mathscr{P}, \otimes, 1)$ and $(\mathscr{N}, ⅋, \perp)$ with a strong monoidal closure $(-)^{\perp_L} : \mathscr{P} \longrightarrow \mathscr{N} \dashv (-)^{\perp_R} : \mathscr{N} \longrightarrow \mathscr{P}$, with a polarized closure $\downarrow : \mathscr{N} \longrightarrow \mathscr{P} \dashv \uparrow : \mathscr{P} \longrightarrow \mathscr{N}$ such $\uparrow \circ \downarrow \ = \ Id_{\mathscr{N}}$. A biproduct structure consists in two monoidal structures $(\diamond, I)$ on $\mathscr{N}$ and $(\blacklozenge, J)$ on $\mathscr{P}$ such that we have the strong monoidal equivalences:

$$\begin{tikzcd} (\mathscr{N}, \diamond) \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\perp"] & (\mathscr{P}^{op}, \blacklozenge) \arrow[l, bend left, "^*(-)"] \end{tikzcd}$$

$$\begin{tikzcd} (\mathscr{N}, \diamond) \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & (\mathscr{P}, \blacklozenge) \arrow[ll, bend left, "R"] \end{tikzcd}$$

with natural transformations $I \xrightarrow{u_N} N$, $N \xrightarrow{\nabla_N} N \diamond N$, $P \xrightarrow{n_P} J$ and $P \blacklozenge P \xrightarrow{\Delta_P} P$ such that $N, u, \nabla)$ is a commutative monoid and (P, n, Δ) is a commutative comonoid.

Then the additive structures on the sets $\mathscr{N}(\uparrow P, N)$ interpreting proofs of $\vdash N, P$ is defined as:

$$f + g : \uparrow P \longrightarrow \uparrow (P \blacklozenge P) \simeq (\uparrow P) \diamond (\uparrow P) \xrightarrow{f \diamond g} N \diamond N \longrightarrow N.$$

2.5.2.3 Categorical models of $\mathrm{DiLL}_{\mathrm{pol}}$

Proposition 2.5.9. *Consider a model of* $\mathrm{LL}_{\mathrm{pol}}$ *with a polarized biproduct:*

- *A negative chirality* $(\mathscr{P}, \otimes, 1)$ *and* $(\mathscr{N}, ⅋, \perp)$ *with a strong monoidal left closure* $(-)^{\perp_L} : \mathscr{P} \longrightarrow \mathscr{N}^{op} \dashv (-)^{\perp_R} : \mathscr{N}^{op} \longrightarrow \mathscr{P}$*, with a polarized closure* $\downarrow : \mathscr{N} \longrightarrow \mathscr{P} \dashv \uparrow : \mathscr{P} \longrightarrow \mathscr{N}$ *such* $\uparrow \circ \downarrow \; = \; Id_{\mathscr{N}}$,
- *A polarized biproduct* $(\diamond, I)$ *on* $\mathscr{N}$ *and* $(\blacklozenge, J)$ *on* $\mathscr{P}$*,*
- *A co-cartesian category* $(\mathscr{P}^{\infty}, \oplus_{\infty}, 0)$ *and a cartesian category* $(\mathscr{N}^{\infty, op}, \times_{\infty}, \top)$ *with a strong monoidal left closure*
$$(-)^{\perp_{L,\infty}} : \mathscr{P}^{\infty} \longrightarrow \mathscr{N}^{\infty, op} \dashv (-)^{\perp_{R,\infty}} : \mathscr{N}^{\infty, op} \longrightarrow \mathscr{P}^{\infty}.$$
- *A strong monoidal right closure*
$$? : (\mathscr{P}^{\infty, op}, \oplus, 0) \longrightarrow (\mathscr{N}^{op}, ⅋, 1) \dashv \mathcal{U} : (\mathscr{N}^{op}, ⅋, 1) \longrightarrow (\mathscr{P}^{\infty, \perp}, \oplus, \top).$$
- *A natural transformation:*
$$\bar{d} : ?U \longrightarrow Id_{\mathscr{N}}$$
which is thus defined between endofunctors of $\mathscr{N}$*.*

Then these four adjunctions, together with the monoidal and biproduct structures, define a denotational model of $\mathrm{DiLL}_{\mathrm{pol}}$.

Outlook 3. We would like to interpret dereliction and co-dereliction as shifts between the categories $\mathcal{P}^{\infty}$ and $\mathcal{N}$, and thus axiomatizing a categorical model of DiLL the data of two chiralities: one between positives in $\mathcal{P}$ and negatives in $\mathcal{N}$, and the other between $\mathcal{P}^{\infty}$ and $\mathcal{N}$. However, as the codereliction does not lead to a functor from $\mathcal{P}^{\infty}$ to $\mathcal{N}$ because of non-compositionality (see Section 2.5.2.1), this is still work in progress. It should incorporate the biproduct presentation, from which the interpretation of sums in hom-sets follows.

Chapter 3

Topological vector spaces

"Nej, jeg lever måske tusinder af dine dage, og min dag er hele årstider! Det er noget så langt, du slet ikke kan udregne det!"
"Nej, for jeg forstår dig ikke! Du har tusinder af mine dage, men jeg har tusinder af øjeblikke til at være glad og lykkelig i! Holder al denne verdens dejlighed op, når du dør?"
"Nej," sagde træet, "den bliver vist ved længere, uendeligt længere, end jeg kan tænke det!"
"Men så har vi jo lige meget, kun at vi regner forskelligt!"

Andersen, Det gamle egetræs sidste drøm, 1858.

Traditionally, one constructs a model of Linear Logic by defining the interpretation of formulas, then the one of linear proofs, and last one studies the possible interpretation for the exponential and non-linear proofs. We will inverse this order here: as we are looking for a nice and smooth interpretation of non-linear proofs, the necessary interpretation of the exponential connective will give us constraints on the topologies on our spaces.

This chapter tackles in its first section the basic definitions of the theory of topological vector spaces. In Section 3.2 we give examples of topological vector spaces made of functions (whether they are sequences or continuous measurable or smooth, functions). In Section 3.3 we introduce linear maps and the notions of dual pairs and weak topologies. Section 3.4.3 exposes notion of bounded subset, of bornologies, and explains how these bornologies characterize the different topologies on spaces of linear functions. Once that spaces of linear functions are well-understood, one can look for spaces on which the linear negation is involutive: the different possibilities for reflexivity are detailed in Section 3.5. At last, Section 3.6 explains the theory of topological tensor products. Nothing exposed in this chapter is new and every result is referred to the literature. The book by Jarchow [44] is the major reference.

As a foretaste for this chapter, we review the notion of orthogonalities and closures which are used throughout chapters 5, chapter 6 and 2.

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Closure and orthogonalities

Let us describe a the notions of orthogonality and closure operators [74]: they generalize for example the operations of completion, bornologification, of taking the Mackey or the Weak topology on a space.

Definition 3.0.1. Consider a set X. An orthogonality relation $\perp \subset X \times X$ is a symmetric relation. We define the dual of X: $X^{\perp} := \{y \in X \mid \forall x \in X, (x,y) \in \perp\}$. Then one has always $X \subset X^{\perp\perp}$, and $X^{\perp} = X^{\perp\perp\perp}$. A set is said to be $\perp$-reflexive when $X = X^{\perp\perp}$.

Examples of orthogonality relations include polars of subsets (Section 3.4.1) and duality in sequence spaces (Section 3.2). Orthogonalities are defined in general in the context of posets, but we give here a definition in the context of a category. A relevant example is the category of lcs and continuous linear injective maps between them.

Definition 3.0.2. Consider $\mathcal{C}$ a category. A closure operator $\bar{\cdot}$ is an idempotent endofunctor on $\mathcal{C}$, with a natural transformation $w : Id \longrightarrow \bar{\cdot}$ such that $\overline{w_A} = id_{\bar{A}} = w_{\bar{A}}$.

Note that according to this definition a closure operator is in particular an idempotent monad, where the multiplication is the identity.
Another way of looking to closure operators are thus adjunction between a forgetful functor and a completion:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\bar{\cdot}"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "U"] \end{tikzcd}$$

such that $\bar{\cdot} \circ U = Id_{\mathcal{C}}$.

Definition 3.0.3. A full subcategory $\mathcal{C}$ of a category $\mathcal{L}$ is said to be reflective when the inclusion functor $U : \mathcal{C} \longrightarrow \mathcal{L}$ has a left adjoint $\bar{\cdot}$. Then $U \circ \bar{\cdot}$ is a closure operator.

Closure operators and reflective subcategories are associated with many topological constructions: the topological closure $\bar{\cdot}$(making a subset closed), the absolutely convex closed closure $\bar{\cdot}^{abs}$, the completion $\tilde{\cdot}$ consists of particular of closure operators. In Chapter 2 we will in particular define a polarized version of this notion of closure.

Filters and topologies

The reader can be used to speak about topologies in terms of convergence of sequences: the set of open sets on a metric space F is entirely determined by the convergence of sequences $(x_n)_n$ in this space. Otherwise said, if for two norms on a vector space the convergent sequences are exactly the same, and have the same limits, then the norms are equivalent. This is true only when the topology is determined by a *metric* (Definition 3.1.2): then there exists a countable basis of open sets.
In general, one cannot describe a topology only by its convergent sequences. A more general, non-countable, notion is needed:

Definition 3.0.4. Consider E a set. Then a filter $\mathcal{F}$ on E is a family of subsets of E, such that:

- $\emptyset \notin \mathcal{F}$,
- for all $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$,
- If $A \in \mathcal{F}$ and $B \subset E$ is such that $A \subset B$, then $B \in \mathcal{F}$.

If $(x_n)_n \in E^{\mathbb{N}}$, then the family $\{S_n = \{x_k | k \geqslant \mathbb{N}\}\}$ is a filter on $(x_n)_n$. A *basis* of a filter $\mathcal{F}$ is a family $\mathcal{F}' \subset \mathcal{F}$ such that every element of $\mathcal{F}$ contains an element of $\mathcal{F}'$: a basis describes the behaviour of $\mathcal{F}$ in terms of convergence. The convergence is itself described in terms of filters.

Definition 3.0.5. A topological space is a set E, such that for every x we have a filter $\mathcal{F}(x)$ such that:

- every element of $\mathcal{F}(x)$ contains x,
- for every $U \in \mathcal{F}(x)$ there exists $V \in \mathcal{F}(x)$ such that for all $y \in V$, $U \in \mathcal{F}(y)$.

Then $\mathcal{F}(x)$ is called the filter of neighborhoods of x.

For example, a sub basis for the filter of neighborhood of a point $x \in F$ in a metric space F is the family of all open balls of center x and of radius $\frac{1}{n}$, for all $n \in \mathbb{N}^*$

Definition 3.0.6. Consider $\mathcal{F}$ a filter on a topological vector space E. $\mathcal{F}$ is said to be convergent towards $x \in E$ if $\mathcal{F}(x) \subset \mathcal{F}$.

In the example of a filter corresponding to a sequence $(x_k)_k$ in F, it means exactly that for every $n \geqslant 1$, there exists k_n such that for every $k \geqslant k_n$, $\|x_k - x\| < \frac{1}{n}$.
We have an equivalent definition of topological space, via a topology:

Definition 3.0.7. A topological space is a set E endowed with a collection $\mathcal{T}$ of subsets of E, called its *topology*, such that:

- $\varnothing \notin \mathcal{T}$ and $E \in \mathcal{T}$,
- $\mathcal{T}$ is stable by arbitrary union,
- $\mathcal{T}$ is stable by finite intersection.

When comparing different topologies on a same vector space E, we will say that the topology $\mathcal{T}$ is coarser than the topology $\mathcal{T}'$, and that $\mathcal{T}'$ is finer than $\mathcal{T}$, if any open set of $\mathcal{T}$ is in particular an open set of T'. We will denote this preorder on the topologies of E by $\leqslant$:

Definition 3.0.8. If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a space E, we say that $\mathcal{T} \leqslant \mathcal{T}'$ if and only if $\mathcal{T} \subset \mathcal{T}'$.

Convergence in topological vector spaces can also be characterized in terms of convergence of nets:

Definition 3.0.9. A net in a topological space E is a family $(x_a)_{a \in A}$ of elements of E indexed by an ordered set A. A net is said to be converging towards $x \in E$ is for every neighborhood U of x, there exists $b \in A$, such that for all $a \geqslant b$, $x_a \in U$. A sub-net of $(x_a)_{a \in A}$ is a family $(x_b)_{b \in B}$ with $B \subset A$.

3.1 First definitions

3.1.1 Topologies on vector spaces

Linear topologies

Definition 3.1.1. [44, 6.7] A topological vector space is a $\mathbb{K}$-vector space E over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, which is also a topological space, making addition and scalar multiplication continuous.

The topology of a lcs E, that is the collection of all its open sets, is denoted $\mathcal{T}_E$. A subset $U \subset E$ is a *neighborhood* of $x \in U$ if it contains an open set containing U.

Definition 3.1.2. A topological vector space is Hausdorff if its topology separates points: for any $x, y \in E$ such that $x \neq y$ there exists two open sets U, V of E such that $U \cap V = \varnothing$, $x \in U$ and $y \in V$.

Definition 3.1.3. A subset $U \in E$ is *absolutely convex* if for all $x, y \in Y$, all $\lambda, \mu \in \mathbb{K}$ such that $|\lambda| + |\mu| \leqslant 1$ we have $\lambda x + \mu v \in Y$. It is absorbent if for any $x \in E$ there exists $\lambda \in \mathbb{K}$ such that $x \in \lambda V$. A topological vector space is said to be *locally convex* if it admits a sub-basis of *absolutely convex absorbent* open sets.

Closed and absolutely convex subsets of a topological vector space E are also called *disks* in E. The role of absolutely convex neighborhoods is to allow for linear combinations: if Y is absolutely convex, then for any $\lambda, \mu \in \mathbb{K}$ we have the following equality between sets: $\lambda U + \mu U = (\lambda + \mu)Y$.

Notation 3.1.4. *If E is a locally convex topological vector space, we write $\mathcal{V}_E(x)$ the set of neighborhoods of x in E. If B is a subset of E, we write $acx(B)$ the absolutely convex hull of B in E, that is the smallest absolutely convex set containing B. We will write $\overline{B}$ for the closure of B is E, that is the smallest closed set containing B. When the topology on E is ambiguous, we will write $\overline{B}^{\mathcal{T}_E}$ for the closure of B in E endowed with $\mathcal{T}_E$.*

As $\mathcal{T}_E$ is stable under translations (addition is continuous), it is enough to specify a basis of 0-neighborhoods to understand the topology of E. It will be called a 0-*basis* for E.

Proposition 3.1.5. *If E admits a 0-basis of absorbing convex open sets, then the absolutely convex closures of these sets is also a 0-basis for E [44, 2.2.2]. Their closure also forms a 0-basis. As any 0-neighborhood is absorbent [44, 2.2.3], we will sometimes consider without any loss of a generality 0-basis consisting of closed convex sets.*

We sum up all the preceding definitions in the one of lcs, which will be our central object of study:

Terminology 3.1.6. *We will denote by lcs a topological vector space which is locally convex and Hausdorff.*

Proposition 3.1.7. *Any lcs is also a uniform space, whose entourage are the sets $\{x - y \mid x, y \in U\} \cup \{y - x \mid y, x \in U\}$ for every 0-neighborhood U.*

Isomorphisms

Notation 3.1.8. *We shall write $E \sim F$ to denote an algebraic isomorphism between the vector spaces E and F, and $E \simeq F$ to denote a bicontinuous isomorphism between the lcs E and F. We will sometimes refer to the first as linear isomorphism and to the second as linear homeomorphism.*

The following proposition is deduced from the stability of the topology under sums and scalar multiplication and captures the intuitions of lcs:

Proposition 3.1.9. *A linear function between two lcs is continuous if and only if it continuous at 0.*

3.1.2 Metrics and semi-norms

From now on and for the rest of the thesis, E, F and G will denote Hausdorff and locally convex topological vector spaces. Let us describe some specific classes of lcs that the reader may be more familiar with:

- Metrizable spaces are the lcs such that the topology is generated by a *metric*, that is a positive symmetric subadditive application $\mathtt{dist} : E \times E \longrightarrow \mathbb{R}_+$ which separates points. Open sets are then generated by the 0-neighborhoods $U_n = \{x \in E \mid \mathtt{dist}(0, x) < n\}$.
- Normed spaces are the lcs such that the topology is generated by a *norm*, that is a positive homogeneous[1] subadditive application $\|\cdot\|\, E \longrightarrow \mathbb{R}_+$ such that $\|x\| = 0$ if and only if $x = 0$. Open sets are then generated by the 0-neighborhoods $U_n = \{x \in E \mid \|x\| < n\}$. Normed spaces are in particular metrizable spaces.
- Euclidean spaces are finite-dimensional $\mathbb{R}$-vector spaces. They are normed vector spaces, with all norms equivalent to $\|\cdot\|_2 : x \in \mathbb{R}^n \mapsto \sqrt{\sum_i x_i^2}$.

Metrizable spaces are also the lcs which admit a countable basis, and that's what makes them useful:

Proposition 3.1.10. *[44, 2.8.1] A lcs is metrizable if and only if it admits a countable basis of 0-neighborhoods.*

Proposition 3.1.11. *[44, 2.9.2] Any lcs is linearly homeomorphic to a dense subset of a product of metrizable lcs.*

The notion of continuity on a lcs can be equivalently described in terms of topology or in terms of semi-norm. A *semi-norm* on a vector space is a subadditive homogeneous positive application $\mathtt{p} : E \longrightarrow \mathbb{R}_+$. It is notably not required that $\mathtt{p}(x) = 0$ implies that $x = 0$.

Local convergence. The following definition will be used to define precise notions of convergence and completeness. It allows to consider convergence locally, within normed spaces generated by some subsets of E. A typical example of a semi-norm is the *Minkowski gauge*, defined on a vector space E for an absorbing convex subset $A \subset E$:

$$q_A : x \mapsto \inf \{|\lambda| \,|\lambda \in \mathbb{R}_+^*, x \in \lambda A\}.$$

The gauge is not a norm as $q_A(x)$ may be null for $x \neq 0$. For x in the linear span of A we have however that q_A separates the points.

Definition 3.1.12. Consider U an absolutely convex subset of E. Then one defines the normed space E_U as the linear span of U, endowed with the norm q_U.

[1] that is, the norm must verify: $\forall \lambda > 0, \|\lambda x\| = \lambda \|x\|$

Let us explain the equivalence of the description of $\mathcal{T}_E$ in terms of topology or in terms of semi-norms (see for example the introduction of Chapter II.4 in [66] for a more detailed presentation). Consider a family $\mathtt{Q}$ of semi-norms on E. Then the topology induced by the semi-norms $q \in \mathtt{Q}$ is the projective topology generated by this family, that is the smallest topology containing the reverse image of any open set of $\mathbb{R}_+^*$ by $q \in \mathtt{Q}$. A 0-basis for this topology is the collection of all

$$B_{A_1,\dots,A_n,\epsilon} = \{x \in E \mid \forall i q_{A_i}(x) < \epsilon\}.$$

Consider conversely E a lcs and $\mathcal{U}$ a 0-basis consisting of absolutely convex subsets. Then the topology $\mathcal{T}_E$ of E coincides with the one generated by the family of semi-norms $(q_U)_{U \in \mathcal{U}}$.

3.1.3 Compact and precompact sets

Definition 3.1.13. A subset $K \subset E$ is:

1. *compact* if any net in K has a sub-net which converges in K.
2. *precompact* if for any 0-neighborhood U there exists a finite set $M \in E$ such that $K \subset U + M$.
3. *relatively compact* if its closure is compact in E.

Since a continuous function preserves open sets by inverse image, it follows that:

Proposition 3.1.14. *If $f : E \longrightarrow F$ is a continuous function, and if K is a compact in E, then $f(K)$ is a compact in F.*

The preceding proposition applies in particular to $Id : E_{\tau'} \longrightarrow E_\tau$, when τ' is finer than τ:

Corollary 3.1.15. Consider E a lcs endowed with a topology τ', and τ another topology on E which is coarser than τ'[2]. Then if K is compact in $E_{\tau'}$, τ and τ' coincide on K.

Because compact sets are preserved by direct images of continuous function, they are used to construct topologies on spaces of continuous function. This is one of the essential bricks of this chapter. In euclidean spaces, that is $\mathbb{R}$ and its finite products, compact sets are exactly the bounded and closed sets, but this is not the case in general.

3.1.4 Projective and inductive topolgies

We describe first products and coproducts of lcs.

Definition 3.1.16. Consider a I-indexed family of lcs $(E_i)_{i \in I}$. We define $\prod_{i \in I} E_i$ as the vector space product over I of the E_i, endowed with the coarsest topology on E making all p_i continuous.

If $\mathcal{U}_j$ is a basis of 0-neighborhoods in E_j, then the follwoing is a subbasis for the topology on $\prod_i E_i$:

$$U = \{U_{i_0} \times \textstyle\prod_{i \in I, i \neq i_0} E_i\} \text{ with } U_{i_0} \in \mathcal{U}_{i_0}.$$

Definition 3.1.17. We define $E := \bigoplus_{i \in I} E_i$ as the algebraic direct sum of the vector spaces E_i, endowed with the finest locally convex topology making every injection $I_j : E_j \to E$ is continuous. Remember that the algebraic direct sum E is the subspace of $\prod_i E_i$ consisting of elements (x_i) having finitely many non-zero x_i.

If $\mathcal{U}_i$ is a 0-basis in E_i, then a 0-basis for $\bigoplus_i E_i$ [44, 4.3] is described by all the sets:

$$U = \textstyle\bigcup_{n=1}^{\infty} \bigoplus_{k=1}^{n} \bigcup_j U_{j,k} \text{ with } U_{j,k} \in \mathcal{U}_j,\ j \in J,\ k \in \mathbb{N}.$$

Note that this topology is finer than the topology induced by $\prod E_i$ on $\bigoplus E_i$.

Proposition 3.1.18 ([44, 4.3.2])**.** *I is finite if and only if the canonical injection from $\bigoplus_{i \in I} E_i$ to $\prod_{i \in I} E_i$ is surjective.*

[2] That is we have an inclusion between the sets of open-sets: $\tau \subseteq \tau'$

Proof. By the definition of the algebraic co-product, as the sets of elements of the product with finitely many non-zero composites, we have $\bigoplus_{i \in I} E_i = \prod_{i \in I} E_i$ when I is finite. The description of the sub-basis of the inductive and projective topologies gives us the result. The converse proposition follows from the definition of the co-product as the set of finite tuples of the product: if product and co-product coincide, then the index I is finite. □

The previous constructions are in fact particular casse of projective and inductive topologies, defined on projective and inductive limits of topological vector spaces. On these notions, we refer respectively to [44, 2.6] and [44, 4.5, 4.6] and just recall the main definitions and results below.

Definition 3.1.19. Consider $(E_j)_{j \in J}$ a family of lcs indexed by an ordered set J and a system $(T_{j,k} : E_k \longrightarrow E_j)_{j \leqslant k}$ such that $T_{j,j} = Id_{E_j}$ and and for $i \leqslant j \leqslant k$ we have $T_{i,k} = T_{i,j} \circ T_{j,k}$. Then the projective limit $\varprojlim_J E_j$ is the sub-lcs of $\prod_j E_j$ such that $T_{j,k}(x_k) = x_j$ for all $j \leqslant k$. It is endowed with the topology induced by $\prod_{j \in J} E_j$.

Definition 3.1.20. Consider $(E_j)_{j \in J}$ a family of lcs indexed by a pre-ordered set J and a system of continuous linear maps $(S_{k,j} : E_j \longrightarrow E_k)_{j \leqslant k}$ such that $S_{j,j} = Id_{E_j}$ and such that for $i \leqslant j \leqslant k$ we have $S_{i,k} = S_{i,j} \circ S_{j,k}$.

- The inductive limit $\varinjlim_j E_j$ is defined as the quotient of $\bigoplus_J E_j$ by the relations $Id_{E_i} - Id_{E_j} \circ S_{j,i}$ for all$i \leqslant j$.
- It is endowed with the finest locally convex vector topology making all $Q \circ S_j : E_j \longrightarrow \oplus_i E_i \longrightarrow \varinjlim_i E_i$ continuous, where Q denotes the linear continuous projection $\oplus_i E_i \longrightarrow \varinjlim_i E_i$. This topology is not necessarily Hausdorff even when all the E_j are.
- The limit is said to be reduced when the maps S_i are injective, and regular when any bounded[3] set in $\varinjlim_i E_i$ is bounded in one of the E_i.
- A reduced and regular inductive limit $\varinjlim_j E_j$ is hausdorff if and only if all the E_j are.
- The limit is called strict when $J = \mathbb{N}$ and the maps $S_{i,j}$ are lcs inclusions $E_i \subset E_j$. If E_i is closed in E_j when $i \leqslant j$, then the limit is regular.

Remark 3.1.21. The fact that a condition on the bounded sets is necessary to guarantee a Hausdorff inductive limit is important. Bounded sets, and inductive limits, are well-behaving with positive connectives.

3.1.5 Completeness

The mere data of a topology is in general not enough to have a rich analytic theory. One needs another tool to deduce convergence of limits and then continuity of functions: we require some notion of *completeness* on spaces.

Definition 3.1.22. A net $(x_\alpha)_{\alpha \in \Lambda} \subset E$ is said to be a *Cauchy-net* if, for any 0-neighborhood U, there exists $\alpha \in \Lambda$ such that for all $\beta \geqslant \alpha$ we have: $x_\beta - x_\alpha \in U$. A lcs E is said to be *complete* if every Cauchy-net in E converges.

A normed space which is complete is called a *Banach* space. A metrizable space wich is complete is called a *Fréchet* space or (F)-space. A closed subset of a complete space is complete. Products and co-products of complete spaces are complete.

Theorem 3.1.23. *If E is any lcs, then there exists a complete lcs $\tilde{E}$ with a linear homeomorphism $I : E \longrightarrow \tilde{E}$ such that $I(E)$ is dense in $\tilde{E}$. Then $\tilde{E}$ is unique up to linear homeomorphism.*

Proof. We sketch the construction of [44, 3.3.3]. If E is metrizable, there exists an isometric embedding of E in $\ell_\infty(E)$, the Banach space of bounded sequences in E (see Section 3.2.3). We write $\tilde{E}$ the closure of $I(E)$ in $\ell_\infty(E)$. If E is not metrizable, we embed E in the product of the metrizable space generated by each 0-neighborhood of a 0-basis (by Proposition 3.1.11), and take the closure of E in product of the completions of these metrizable space. □

We recall the following classical result, which is proved in particular in [44, 3.9.1].

Proposition 3.1.24. *Consider $f : E \longrightarrow F$ a uniformly continuous map. Then there exists a unique uniformly continuous map $f : \tilde{E} \longrightarrow \tilde{F}$ which extends f. In particular, any continuous linear map in $E \longrightarrow F$ where F is complete admits a unique continuous linear extension in $\tilde{E} \longrightarrow F$.*

[3]That is, any set which is absorbed by every 0-neighborhood, see Section 3.4

The notion of completeness defined above is the most famous one and when it is necessary it will be specified as *Cauchy-completeness*. However, other notions of completeness can be defined. One can endow a lcs with other topologies inferred from its dual, such as the weak or the Mackey-topology, and from that define an alternative of completeness (see Definition 3.4.16). Other notions of completeness can also be defined as convergence of Cauchy-nets in some specific subsets of E: quasi-completeness is the convergence of Cauchy-nets in bounded sets (see Section 3.4).
Let us recall a fundamental properties relating completeness and compactness:

Proposition 3.1.25. *[44, 3.5.1] Consider E any lcs. A subset $K \subset E$ is precompact if and only if it is relatively compact in $\tilde{E}$, that is if and only if $\bar{K}$ is compact in $\tilde{E}$.*

Remark 3.1.26. We will later state Grothendieck' theorem **??**, which allows to see $\tilde{E}$ as a certain topology on a bidual of E. This is fundamental in our approach, and allows a model of a linear involutive negation, as it corresponds to the isomorphism:

$$\uparrow P \simeq (P^{\perp_R})^{\perp_L}$$

of a polarized linear logic allowing to interpret the involutive negation.

3.2 Examples: sequences and measures

We will detail here some other fundamental examples of lcs, which will serve as running examples in the rest of the Chapter.

3.2.1 Spaces of continuous functions

We denote by $\mathcal{C}(E, F)$ the vector space of all continuous functions from E to F. We now define two standard topologies on this space.

Definition 3.2.1. We denote by $\mathcal{C}_c(E, F)$ the vector space $\mathcal{C}(E, F)$ endowed with the topology of uniform convergence on compact subsets of E. It is also known as the compact-open topology, and a sub-basis for it is

$$\mathcal{W}_{K,U} = \{f : E \longrightarrow F | f(K) \subset U\}$$

where K is any compact set in E and U ranges over a basis $\mathcal{U}$ of $\mathcal{T}_F$. If a net $(f_\alpha)_{\alpha \in \Lambda}$ converges to f in $\mathcal{C}_c(E, F)$ we say it converges towards f *uniformly on compact subsets of* E. It is the case if and only if for every compact subset K of E, for every $U \in \mathcal{U}$, there exists $\alpha \in \Lambda$ such that for every $\beta \geqslant \alpha$ we have: $f_\beta(B) \in U$.

The neighborhoods $\mathcal{W}_{K,U}$ clearly are absolutely convex.

Proposition 3.2.2. *The vector space $\mathcal{C}_c(E, F)$endowed with the compact-open topology is a lcs. It is metrizable when E admits a countable basis of compact subsets: this is the case for example when E is a euclidean space.*

Definition 3.2.3. We denote by $\mathcal{C}_s(E, F)$ the vector space $\mathcal{C}(E, F)$ endowed with the topology of simple convergence E. This can be described as the topology induced by the basis consisting of the following collection of subsets.

$$\mathcal{W}_{x_1,\dots x_n,U} = \{f : E \longrightarrow F | \forall i, f(x_i) \in U\}$$

where U ranges over a basis $\mathcal{U}$ of $\mathcal{T}_F$. In terms of nets, a net $(f_\alpha)_\alpha$ converges to f in $\mathcal{C}_s(E, F)$ if and only if for every $x \in E$, the net $(f_\alpha(x))_\alpha$ converges to $f(x)$ in F.

These are particular cases of convergences on bornologies: see Section 3.4.3 for more details.

3.2.2 Spaces of smooth functions

Consider $n \in \mathbb{N}$. We consider now the vector space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ of *smooth functions* from $\mathbb{R}^n$ to F.

Definition 3.2.4. A function $f : \mathbb{R}^n \longrightarrow F$ is smooth if it is differentiable, and if its differential $Df : \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ is smooth.

In the previous definition, the vector space $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ is endowed with the compact-open topology, as a subspace of $\mathcal{C}_c(\mathbb{R}^n, \mathbb{R})$. It is in this case equivalent to the bounded-open topology which will be considered in Section 3.4. We consider also the vector space of all *smooth functions with compact support*: $\mathcal{C}^\infty_{co}(\mathbb{R}^n, \mathbb{R})$. A function $f : E \longrightarrow F$ has compact support if and only if there exists $K \subset E$ compact such that f is null outside K.

Notation 3.2.5. *Consider $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and $\alpha \in \mathbb{N}^n$. We denote by $|\alpha|$ the quantity $\alpha_1 + ... + \alpha_n$. We write ∂^α the following linear operator from $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ to $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$*

$$\partial^\alpha : f \mapsto \left(x \mapsto \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} ... \partial x_1^{\alpha_1}}(x) \right).$$

By definition of smoothness, $\partial^\alpha f$ is again a smooth function.

By default, and if nothing else is mentioned, these spaces of smooth functions will be considered as a lcs endowed with the topology of uniform convergence of all derivatives on all compact subsets of $\mathbb{R}^n$:

Definition 3.2.6. Consider $\mathcal{U}$ a 0-basis of F consisting of absolutely convex subsets. The space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and its subspace $\mathcal{C}^\infty_{co}(\mathbb{R}^n, \mathbb{R})$ are endowed with the initial topology generated by the semi-norms[4].

$$q_{K,m} : f \mapsto \sup_{x \in K, |\alpha| < m} \{|\partial^\alpha f(x)|\}$$

where K ranges over all compact subsets of $\mathbb{R}^n$ and $m \in \mathbb{N}$.

Notation 3.2.7. *The lcs $\mathcal{C}^\infty_{co}(\mathbb{R}^n, \mathbb{R})$ is also called the space of test functions and denoted $\mathcal{D}(\mathbb{R}^n)$. The lcs $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is denoted $\mathcal{E}(\mathbb{R}^n)$. They are both the central objects of the theory of distributions (see Section 7.3.2) [69].*

3.2.3 Sequence spaces

Example 3.2.8. Let $p \in \mathbb{N}_+$ and ℓ_p the vector space of sequences $(x_n)_n \in \mathbb{R}^\mathbb{N}$ such that

$$(\sum_{i \in \mathbb{N}} |x_i|^p)^{\frac{1}{p}} < \infty.$$

Then ℓ_p is a normed vector space when endowed with the norm $\|\cdot\|_p$ defined as the finite-valued sum above. The vector space ℓ_∞ is the space of all bounded real sequences, endowed with the supremum norm $\|\cdot\|_\infty$. They all are Banach spaces.

The following definition will be fundamental for the theory of nuclear spaces described later in Section7.2.2.

Example 3.2.9. [44, 1.7.E] Consider the space of scalar rapidly decreasing sequences:

$$\mathfrak{s} = \{(\lambda_n)_n \in \mathbb{K}^\mathbb{N} | \forall k \in \mathbb{N}, (\lambda_n n^k)_n \in \ell_1\}.$$

It is a lcs when endowed with the projective topology induced by the seminorms:

$$q_k : (\lambda_n)_n \mapsto \left\| (\lambda_n n^k)_n \right\|_1.$$

It is metrizable as its topology is generated from a countable family of seminorms, and its completeness follows from the one of ℓ_1.

The preceding construction can be generalized to Köthe spaces, as defined in [44, 1.7.E].

Example 3.2.10. Consider $P \subset \mathbb{R}^\mathbb{N}_+$ a set of positive sequences. It is a poset when endowed with the pointwise ordering: $\alpha \leqslant \beta$ if and only if for all n, $\alpha_n \leqslant \beta_n$. Suppose that no sequence in P is null, and that P is directed:

$$\forall \alpha \in P, \exists n \in \mathbb{N} : \alpha_n > 0 \text{ and } \forall \alpha, \beta \in P, \exists \gamma \in P, \forall n : \alpha_n, \beta_n \leqslant \gamma_n. \tag{3.1}$$

Then one defines the Köthe sequence space $\Lambda(P)$ as:

$$\Lambda(P) := \{(\lambda_n)_n \in \mathbb{K}^\mathbb{N} | \forall \alpha \in P, (\lambda_n \alpha_n)_n \in \ell_1\}.$$

It is endowed with the projective topology induced by the seminorms:

$$q_\alpha : (\lambda_n)_n \mapsto \|(\lambda_n \alpha_n)_n\|_1 = \sum_n |\lambda_n| \alpha_n.$$

for all $\alpha \in P$. Thus if P is countable, $\Lambda(P)$ is metrizable. The space $\Lambda(P)$ is complete.

[4] That is, the coarsest topology on $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ making all the $q_{K,m}$ continuous

This definition is the basis for a first vectorial model of Linear Logic. In [18], Ehrhard interprets the formulas of Linear Logic as special Köthe spaces, with a more restrictive definition which is detailed in Section 2.2.3. The connectives of Linear Logic are then interpreted as constructions on the sequence set P underlying the Köthe space, and proofs as linear continuous maps. Ehrhard's Köthe spaces are in particular Köthe sets:

Remark 3.2.11. For all E, the space $|E^\perp| = \{(|\alpha_n|)_n \mid (\alpha_n)_n \in E^\perp\}$ satisfies the condition 3.1. It is indeed directed: if $\alpha, \beta \in |E^\perp|$, define the sequence γ as $\gamma_n = \max |\alpha_n|, |\beta_n|$. Then, for all $(\lambda_n)_n \in E$, as the series $\sum_n |\lambda_n| \alpha_n$ and $\sum_n |\lambda_n| \beta_n$ are absolutely converging, one has:

$$\sum_n \gamma_n \lambda_n = \frac{1}{2}(\sum_n |\alpha_n - \beta_n| \lambda_n + \sum_n |\alpha_n + \beta_n| \lambda_n) \tag{3.2}$$

$$\leqslant \sum_n |\alpha_n| \lambda_n + \sum_n |\beta_n| \lambda_n \tag{3.3}$$

thus $\gamma \in E^\perp$ and as it is positive we have $\gamma \in |E^\perp|$. The space $|E^\perp|$ also satisfies the first requirement: for $n \in \mathbb{N}$, the sequence α^n which is null at indexes $k \neq n$ and equals 1 at index n is of course in $E^\perp$. In particular, all finite sequences are in $E^\perp$.

Definition 3.2.12. A perfect sequence space is a pair (X, E_X), where X is a countable set, and E_X is a subset of $\mathbb{K}^X$ such that

$$E_X^{\perp\perp} = E_X.$$

It is endowed with its *normal topology*, that is with the projective topology induced by the semi-norms:

$$q_\alpha : (\lambda_n)_n \mapsto \|(\lambda_n \alpha_n)_n\|_1$$

for all $\alpha \in {E_X}^\perp$. Without loss of generality, we usually note E_X to denote the perfect sequence space (X, E_X).

Thus a perfect sequence space E_X is exactly the Köthe space $E_X^{\perp\perp} = \Lambda({E_X}^\perp)$ as defined previously in example 3.2.10. It is a lcs as ${E_X}^\perp$ fulfils the requirements for a Köthe set (Remark 3.2.11). Conversely, a Köthe space $\Lambda(P) = P^\perp$ is not necessarily a perfect sequence space, as the topology of $\Lambda(P)$ is induced by P, which in general differs from $P^{\perp\perp}$.

A Köthe spaces of periodical functions. We expose here a remark by Schwartz [66], detailed by Jarchow [44, 2.10.H].

Proposition 3.2.13. *There exists a linear homeomorphism $s \simeq \mathcal{C}^\infty_{per}(\mathbb{K})$ between the perfect sequence space of rapidly decreasing sequences and scalar smooth periodic functions.*

The homeomorphism is constructed by using the Fourier transform, which is recalled in Section 7.3.4.

3.3 Linear functions and their topologies

3.3.1 Linear continuous maps

Consider E and F two lcs. We write E^* the set of *all* linear forms $E \longrightarrow \mathbb{K}$. We denote E' the *dual* of E, that is the space of all continuous linear mappings from E to $\mathbb{K}$. We write $\mathcal{L}(E, F)$ the vector space of all continuous linear functions from E to F. We will see that several topologies make these vector spaces topological vector spaces.
Let us describe a few well-known examples:

Example 3.3.1. Consider $p \in \mathbb{R}^*_+$ and let us write $p^* = \frac{p}{p-1}$. Then for the spaces of sequences of measurable maps described in Section 3.2, one has $(\mathcal{L}_p(\mu))' = \mathcal{L}_{p*}(\mu)$, and in particular $(\ell_p)' = \ell_{p*}$.

Example 3.3.2. The dual of c_0, the space of sequences converging to 0 endowed with $\|.\|_\infty$, is ℓ_1. Note that by the preceding proposition the dual of ℓ_1 is ℓ_∞.

Proof. For $x \in \ell_1$, then $ev_x : y \in c_0 \mapsto \sum x_n y_n$ is well defined by boundedness of x, and thus continuous for the norm $\|\cdot\|_1$ on c_0. Conversely, consider $\phi \in c_0'$. As ϕ is linear continuous on c_0, there is $M > 0$ such that for every $x \in c_0$ $|\phi(x)| < M \|x\|_\infty$. Let us define $\phi_n = \phi(e^n)$, e^n being the sequence such that $e^n_k = 0$ if $n \neq k$ and $e^n_n = 1$. Define $x_n = \frac{\phi_n}{|\phi_n|}$ when ϕ_n is non-null, and $x_n = 0$ if ϕ_n is null. Let us denote $x^N = (x_1, ..., x_N, 0, \dots)$

the finite sequence whose N first terms are the x_n. As the sequence ϕ is converging towards 0, then $\phi(x^N)$ is well defined for every N, and $|\phi(x^N)| < 1$. We just proved that for all $N \in \mathbb{N}^*$:

$$\sum_{n=1}^{N} |\phi_n| < M.$$

We deduce that $(\sum_{n=1}^{N} |\phi_n|)_N$ is a monotonous bounded sequence, and it is then converging. Thus $\phi \in \ell_1$, and as $(\phi_n)'_n = \phi$ we have the linear homeomorphism between c_0' and ℓ_1. □

Example 3.3.3. Consider E a perfect sequence space as described in example 2.2.3. Then one has a linear isomorphism from $E^{\perp}$ to E', which maps a sequence $(\alpha)_n)_n$ to the linear continuous form:

$$\alpha : (x_n)_n \in E \mapsto \sum_n \alpha_n x_n.$$

In particular, one has ${\ell_p}^{\perp} = \ell_{p*}$ and ${\ell_1}^{\perp} = \ell_{\infty}$.

The central theorem of locally convex vector spaces is the Hahn-Banach theorem. We recall here two formulations of this theorem, the analytic one and the geometric one. We omit here some more general formulations that won't be needed in this thesis.

Theorem 3.3.4. *[44, 7.2.1] Consider E a locally convex topological $\mathbb{R}$-vector space (not necessarily Hausdorff), and F a subspace of E. Then every continuous linear form on F extends to a continuous linear form on E.*

Corollary 3.3.5. A locally convex topological vector space is Hausdorff iff for every $x \in E - \{0\}$ there exists $u \in E'$ such that $u(x) \neq 0$.

Theorem 3.3.6. *[44, 7.3.4] Consider E a (non necessarily Hausdorff) locally convex topological $\mathbb{R}$-vector space E, A a closed absolutely convex subset of E, and K a compact subset of E such that $A \cap K = \emptyset$. Then there exists $u \in E'$ such that for all $x \in A$, $y \in K$:*

$$u(x) < \alpha < u(y).$$

Corollary 3.3.7. Consider A a non-empty closed and absolutely convex subset of E. Then if $x \neq A$, there exists $u \in E'$ such that for every $y \in A$, $|u(y)| \leqslant 1$ and such that $|u(x)| > 1$.

In particular, if $x \in E$ is such that for every $\ell \in E'$, $\ell(x) = 0$, then $x = 0$.

3.3.2 Weak properties and dual pairs

Topological properties on E are usually considered with respect to the topology $\mathcal{T}_E$ of E, but they can also be defined *weakly*.

The weak topology

Definition 3.3.8. The weak topology $\sigma(E, E')$ on E is the topology of uniform convergence on finite subsets of E'. That is, a basis of 0-neighborhoods of $\sigma(E, E')$ is:

$$\mathcal{W}_{\ell_1,...,\ell_n,\epsilon} = \{x \in E \mid |l_1(x)| < \epsilon, ..., |l_n(x)| < \epsilon\}$$

for $\ell_1, ..., \ell_n \in E'$ and $\epsilon > 0$. We denote by E_w or E_σ the lcs E endowed with its weak topology. We will show later that this change of topology on E does not change the dual of E (proposition 3.3.13).

The weak topology on E is the inductive topology generated by all the $l \in E'$. This definition means that one can see the specifications of E_w as properties which are verified when elements are precomposed by every $l \in E'$:

Proposition 3.3.9. *A sequence $(x_n)_n \in E^{\mathbb{N}}$ is weakly convergent towards $x \in E$, that is $\sigma(E, E')$-convergent towards x, if and only if for every $l \in E'$ the sequence $l((x_n))_n$ converges towards $l(x)$ in $\mathbb{R}$.*

One checks easily that this topology makes addition and scalar multiplication continuous, and that the basis of 0-neighborhoods described above consists of convex sets. From the Hahn-Banach theorem 3.3.5, it follows that E is separated:

Proposition 3.3.10. *E endowed with $\sigma(E, E')$ is a lcs.*

Proof. As a consequence of Hahn-Banach separation theorem [44, 7.2.2.a], we have that E' separates the points of E: if $x, y \in E$ are distinct, then there exists $l \in E'$ such that $l(x) \neq l(y)$. This makes E endowed with its weak topology a Hausdorff topological vector space. It is locally convex as $\mathcal{W}_{\ell_1, \cdots, \ell_n, \epsilon}$ is convex as soon as U is convex. □

The weak* topology. We will heavily make use of another kind of weak topology. The weak* topology on E' is the weak topology generated on E' by E when the later is viewed as a dual of E'.

Definition 3.3.11. The weak* topology on E' is the topology $\sigma(E', E)$ of uniform convergence on finite subsets of E. A basis of 0-neighborhoods of $\sigma(E)$ is:

$$\mathcal{W}_{x_1,\ldots,x_n,\varepsilon} = \{\ell \in E' \mid |\ell(x_1)| < \epsilon, ..., |\ell(x_n)| < \varepsilon\}$$

for $x_1, ..., x_n \in E$ and $\varepsilon > 0$. Once E' is given or computed from a first topology on E, we construct lcs E endowed with its weak* topology and denote it E'_{w*} or $E'_{\sigma*}$.

As for the weak topology, we have the following:

Proposition 3.3.12. *E' endowed with $\sigma(E', E)$ is a lcs.*

When it can be deduced from the context without any ambiguity, we will denote E_σ as E endowed with its weak topology and $E'_{\sigma(E)}$ as E' endowed with its weak* topology. The weak topology induced by E' on E is the coarsest topology τ on E such that $(E_\tau)' = E$:

Proposition 3.3.13. *We have the linear isomorphisms $(E_\sigma)' \sim E'$ and $(E_{\sigma(E)})' \sim E$.*

The well-known demonstration of this proposition uses the following lemma:

Lemma 3.3.14. Consider E a vector space and $l, l_1, ...l_n$ linear forms on E. Then l lies in the vector space generated by the family $l_1, \ldots, l_n$ (denoted $\mathrm{Vect}(l_1, ...l_n)$) if and only if $\bigcap_{k=1}^n \mathrm{Ker}(l_k) \subset \mathrm{Ker}(l)$. □

Proof. If $l \in \mathrm{Vect}(l_1, ...l_n)$ then clearly $\bigcap_{k=1}^n \mathrm{Ker}(l_k) \subset \mathrm{Ker}(l)$. Conversely, suppose $\bigcap_{k=1}^n \mathrm{Ker}(l_k) \subset \mathrm{Ker}(l)$. Without loss of generality, we can suppose the family $\{l_k\}$ free. Let us show the result by induction on n. If $n = 1$, then $\mathrm{Ker}(l) = \mathrm{Ker}(l_1)$ as they have the same codimension, and one has $l = \frac{l(x_0)}{l_1(x_0)} l_1$ for any fixed $x_0 \notin \mathrm{Ker}(l)$. Consider now $l, l_1, ...l_n$ linear forms on E such that $\bigcap_{k=1}^n \mathrm{Ker}(l_k) \subset \mathrm{Ker}(l)$. Then by restricting l to $\mathrm{Ker}(l_n)$ we obtain scalars $\lambda_1, ..., \lambda_{n-1}$ such that

$$l_{|\mathrm{Ker}(l_n)} = \sum_{k=1}^{n-1} \lambda_k l_{k|\mathrm{Ker}(l)}.$$

Then $\mathrm{Ker}(l_n) \subset \mathrm{Ker}(l - \sum_{k=1}^{n-1} \lambda_k l_k)$, and we have our result by the case $n = 1$. □

Proof of Proposition 3.3.13. Let us show first that $(E_w)' \sim E'$. As the weak topology on E is coarser than the initial topology on E, we have $E' \subset (E_w)'$. Consider now a continuous linear form l on E_w. Then by continuity of l, and with the description of the weak topology given in Section 3.3.3, there exists $l_1, ..., l_n \in E'$ and $\epsilon > 0$ such that

$$l(\mathcal{W}_{l_1,...l_n,\epsilon}) \subset \{\lambda \in \mathbb{K} \mid |\lambda| < 1\}.$$

By homogeneity, we have $\bigcap_{k=1}^n \mathrm{Ker}(l_k) \subset \mathrm{Ker}(l)$ and the preceding lemma implies that $l \in E'$. Thus $(E_w)' \sim E'$. Both their topology being the weak* topology induced by points of E, we have $(E_w)' \simeq E'$. □

We can continue to write E' for the dual of a space E, regardless whether it may be endowed with its weak topology: $(E_w)' \sim E'$. Moreover we will write E'_w for $(E')_w$. The linear isomorphism $E \sim (E'_{w*})'$ can be lifted to a linear homeomorphism when E is endowed with the weak topology $\sigma(E')$ and E'' is endowed with the weak* topology $\sigma(E')$.

Proposition 3.3.15. *For any lcs E one has $E_\sigma \simeq (E'_{\sigma*})'_{\sigma*}$.*

From Proposition 3.3.13, one deduces the following criterion for continuity between weak spaces.

Proposition 3.3.16. *Consider E and F two open sets. A linear function $f : E \longrightarrow F_w$ is continuous if and only if for all $\ell \in F'$, we have $\ell \circ f \in E'$.*

Proof. If $f : E_w \longrightarrow F_w$ is continuous then for $\ell \in F'$ we have $\ell \circ f \in (E_w)' = E'$. Conversely, suppose that for all $\ell \in F'$, we have $\ell \circ f \in E'$. Then the reverse image of an open set $\mathcal{W}_{l_1,\dots l_n,\epsilon}$ by f contains $\mathcal{W}_{l_1 \circ f,\dots l_n \circ f,\epsilon}$, which is always an open set in E as the $l_i \circ f$ are continuous, thus f is continuous. □

Example 3.3.17. Beware that a perfect sequence space E as described in Section 2.2.3 is not endowed with its weak topology: the normal and the weak topologies induced by $E^\perp = E'$ (example 3.3.1) differ. Indeed, the first one is induced by all the semi-norms:

$$q_\alpha : x \mapsto \sum_n |\alpha_n x_n|$$

for all $\alpha \in \mathscr{E}'$ while the weak topology is induced by all the semi norms:

$$q_{\alpha^1,\dots,\alpha^k} : x \mapsto \sup_i \sum_n \alpha^i_n x_n$$

3.3.3 Dual pairs

Concerning topologies, a lot of notions do not depend strictly on the topology of a lcs E, but rather on its dual: the first example is the one of bounded sets (Proposition 3.4.10). Thus one can vary the topology of E, as long as it does not change the dual, the bounded sets stay the same. We have a precise knowledge of the topologies which do not change the dual (see Theorem 3.5.3). This is why the concept of dual pair is fundamental. Some notions however depends on the topology: this is the case of completeness, or of compactness.

The process described earlier between E and E' can be generalized to any pair of vector spaces forming a dual pair.

Definition 3.3.18. A dual pair consists of a pair of vector spaces E and F, and of a bilinear form $\langle \cdot, \cdot \rangle : E \times F \longrightarrow \mathbb{K}$ which is not degenerate.

In particular, if E is a lcs, then (E', E) endowed with the application $\langle \ell, x \rangle = \ell(x)$ is a dual pair. That this linear form is non-degenerate on the left follows by definition, and that it is non degenerate on the right follows from theorem 3.3.7: if x is such that for every ℓ we have $\ell(x) = 0$, then $x = 0$.

Then the previous definition for the weak* topology on E' can be generalized to any dual pair (E, F), and leads to a weak topology $\sigma(F, E)$ on F. This is done in chapter 8.1 of Jarchow's textbook [44]. Note that the role of the vector spaces in a dual pair are symmetric, and thus a dual pair also defines a weak topology $\sigma(E, F)$ on E.

In fact, from any separating sub-vector space F of E^* or even $\mathbb{K}^E$, one can define a dual pair (E, F): the application $(x \in E, \ell \in F) \mapsto \langle x, \ell \rangle = l(x)$ is then bilinear and non-degenerate.

Proposition 3.3.19. *The following function is a linear continuous injection of E into E'':*

$$\iota_E : \begin{cases} E \hookrightarrow E'^* \\ x \mapsto ev_x : (\ell \mapsto \ell(x)) \end{cases} \tag{3.4}$$

This injection and the idea that E (or $\tilde{E}$) can be considered as a subspace of E'' is fundamental in this thesis. In particular, note that (E, E') form a dual pair through this injection, with $\langle ev_x, \ell \rangle = ev_x(\ell) = \ell(x)$. Then the weak topology on E', inherited from the dual pair (E, E'), is exactly the same topology as the weak* topology, inherited from the dual pair (E', E).

More generally, for any dual pair (F, E), the non degenerate bilinear application allows for the consideration of F as a subspace of E^*.

Proposition 3.3.20. *[44, 8.1.4] Consider E a vector space and F a subspace of E^*. Then the bilinear form $\langle \cdot, \cdot \rangle : F \times E \longrightarrow \mathbb{K}$ is non-degenerate if and only if F is dense in E^* endowed with its weak* topology.*

Proof. Suppose that F is dense in E^*, and consider $x \in E$ such that for every $\ell \in F$, $\ell(x) = 0$. F being dense in E^* equipped with its weak topology we have in particular that for every $\ell \in E^*$, $\ell(x) = 0$. As E^* separates the points of E we have $x = 0$.

Conversely, if F is not dense in E^*, then by corollary 3.3.7 for $\ell \in E^* \backslash \bar{F}$ there exists $x \in E$ such that $\ell(x) \neq 0$ but $\ell'(x) = 0$ for all $\ell' \in F$. Since $x \neq 0$, it makes the application $F \times E \longrightarrow \mathbb{K}$ degenerate. □

Then we have:

Proposition 3.3.21. *[44, 8.1.5] Consider (E, F) a dual pair. F endowed with its weak* topology $\sigma(F, E)$ is complete if and only if $F = E^*$.*

Proof. $F = E^*$ is complete as $E^* \simeq \mathbb{K}^E$, which is complete as a product of complete spaces. Conversely, if F is complete it is in particular closed, and thus $F \simeq E^*$ by the previous proposition. □

This results means in particular that E'_{w*} is complete if and only if every linear scalar map $\ell \in E^*$ is continuous. This almost never happen: then when looking for reflexive space (interpreting classical Linear Logic) which are also complete (interpreting smoothness), we need therefore to look for other topologies on the dual.

3.4 Bornologies and uniform convergence

3.4.1 Polars and equicontinuous sets

Polars. Consider $A \subset E$. Its polar is defined as the set of all functions in E' which have values bounded by 1 on A:

$$A^\circ = \{l \in E' | \forall x \in A, |l(x)| \leqslant 1\}.$$

Notice that A° is absolutely convex and $\sigma(E)$-closed in E'. Symmetrically one can define the polar $B^\circ \subset E$ of a subset $B \subset E'$:

$$B^\circ = \{x \in E | \forall l \in B, |l(x)| \leqslant 1\}.$$

Thus B° is absolutely convex and $\sigma(E')$-closed in E. The following theorem is known as the bipolar theorem and makes the polarization a orthogonality relation:

Theorem 3.4.1. *Consider $A \subset E$. Then $A^{\circ\circ}$ is the $\sigma(E')$ closure of the absolutely convex hull of A:*

$$A^{\circ\circ} = \overline{acx(A)}^{\sigma(E')}.$$

Proof. As a polar is absolutely convex and weakly closed, one has $\overline{acx(A)}^{\sigma(E')} \subset A^{\circ\circ}$. The converse inclusion is shown by making use of corollary 3.3.7. □

Equicontinuity.

Notation 3.4.2. *Let T be any set of functions $f : E \longrightarrow F$. Then for $A \subset E$ we denote by $T(A)$ the set $\{f(x) \mid f \in T, x \in A\}$. For $B \subset F$ we denote by $T^{-1}(B)$ the set $\{x \in E \mid \exists f \in T, f(x) \in B\}$.*

Definition 3.4.3. A set of functions $K \subset \mathcal{L}(E, F)$ is equicontinuous if for all 0-neighborhood $W \subset F$, there exists a 0-neighborhood $V \subset E$ such that for all $l \in K$, $l(V) \subset W$.

Note that equicontinuity does not depend on the topology which may be defined on E', but only on the one of E.

Example 3.4.4. Consider U a 0-neighborhood in E. Then U° is equicontinuous in E'.

We recall below the fundamental properties of equicontinuous subsets:

Proposition 3.4.5. *[44, 8.5.1 and 8.5.2]*

1. *An equicontinuous subset $H \subset \mathcal{L}(E, F)$ is uniformly bounded, that is it sends a bounded set of E on a bounded set of F (this amounts to be bounded in the topology $\mathcal{L}_\beta(E, F)$ defined in Section 3.4.3).*
2. *The closure $\bar{H}$ of an equicontinuous subset $H \subset \mathcal{L}(E, F)$ in F^E is still contained in $\mathcal{L}(E, F)$ and is equicontinuous.*
3. *The Alaoglu-Bourbaki Theorem: the closure $\bar{H}$ of an equicontinuous subset $H \subset E'$ is weakly* compact.*

Proof. Consider B a bounded set in E and V a 0-neighborhood in F. Then $H^{-1}(V)$ is a 0-neighborhood in E, thus there exists λ such that $B \subset \lambda H^{-1}(V)$, and by linearity of functions of H we get $H(B) \subset \lambda H(H^{-1}(V)) \subset \lambda V$.

The second point follows immediately from the equicontinuity of H: one shows the equicontinuity of $\bar{H}$ by showing that the reverse image of any *closed* 0-neighborhood V in F is in the 0-neighborhood U in E such that $H(U) \subset V$.

The Alaoglu-Bourbaki theorem follows from the first two points. From 2 we have that it is enough to show that the closure of H in $\mathbb{K}^E$ is compact. But for that it is enough to show that it is a product of compact sets in $\mathbb{K}$, and since compact sets in $\mathbb{K}$ are closed bounded subsets the result follows from 1. □

3.4.2 Boundedness for sets and functions

Working with lcs which may not have a metric, there exists no possibility to define bounded sets as a collection of points which are uniformly at a finite distance from the origin. Instead, one defines a bornology, that is a collection of sets which behaves as a family of bounded sets.

Definition 3.4.6. Consider E a vector space. A *bornology* on a vector space E is a collection $\mathcal{B}$ of sets B called bounded, such that $\mathcal{B}$ is closed by downward inclusion, finite union and covers E.

Unlike continuity, boundedness works with direct images: a function is *bounded* if the direct image of a bounded set in its domain is bounded in its codomain.

Several bornologies can be defined from the topology $\mathcal{T}$ of a lcs. The most used one is the so-called Von-Neumann bornology:

Definition 3.4.7. A subset $B \subset E$ is said to be $\mathcal{T}_E$-bounded if for every 0-neighborhood $U \subset E$ there exists a scalar $\lambda \in \mathbb{K}$ such that

$$B \subset \lambda U.$$

This notion of boundedness is compatible with, but not equivalent to, continuity:

Proposition 3.4.8. *A linear continuous function $l : E \longrightarrow F$ is $\mathcal{T}_E$-bounded.*

Proof. Consider $B \subset E$ bounded, and U a 0-neighborhood in F. Then there exists $\lambda \in \mathbb{K}$ such that $B \subset \lambda l^{-1}(U)$. Thus $l(B) \subset l(\lambda l^{-1}(U)) \subset \lambda U$. □

Beware that the converse proposition is false. In a non-normed lcs, bounded linear functions may not be continuous.

Example 3.4.9. This example is given by Ehrhard in [18]. Consider the vector space ℓ_1. It is a sequence space and thus one can define the pfs ${\ell_1}^{\perp}$. Following example 3.3.1, there exists a linear isomorphism between ${\ell_1}^{\perp}$ and ℓ_∞. However as lcs, ${\ell_1}^{\perp}$ is not endowed with the norm ∞ but with the normal topology induced by $\ell_1^{\perp\perp} = {\ell_\infty}^{\perp}$.

A reasoning similar to the one used in example 3.3.1 shows that ${\ell_\infty}^{\perp} = c_0$, thus the dual of the pfs ℓ_∞ is the same as the one of ℓ_∞ normed by $\|\cdot\|_\infty$.

Through the Hahn-Banach theorem, one constructs a non-null linear function $f : \ell_\infty \longrightarrow \mathbb{K}$ mapping every sequence in c_0 to 0, which is continuous for $\|\cdot\|_\infty$. It is thus bounded for the Von-Neumann bornology induced by $\|\cdot\|_\infty$. As ℓ_∞ as a normed space and ℓ_∞ as the pfs ${\ell_1}^{\perp}$ have the same dual, they have the same bounded sets (see Proposition 3.4.10). Thus f is also a bounded linear form on ${\ell_1}^{\perp}$. It is however not continuous for the normal topology induced by c_0 on ${\ell_1}^{\perp}$: if it were continuous it would be null.

A fundamental property of $\mathcal{T}_E$-bounded sets is that one can test their boundedness scalarly:

Proposition 3.4.10. *[44, 8.3.4] A set B is $\mathcal{T}_E$-bounded if and only if it is weakly bounded, that is if and only if for every $l \in E'$, $l(B)$ is bounded in $\mathbb{K}$.*

Proof. One implication follows directly from Proposition 3.4.8. For the converse implication, we will only sketch here the proof of Jarchow done in his chapter 8.3 [44]. It relies on the fact that it is exactly the same property for B to be weakly bounded and for B° to be absorbent: it means exactly that for every $l \in E'$, there exists $\rho > 0$ such that for every $x \in B$, $|l(x)| < \rho$. But B° is also absolutely convex and weakly closed: when B is weakly bounded, it is thus a barrel (see definition 3.4.22). Then the Banach-Mackey theorem [44, 8.3.3] tells us that not only B° absorbs every point of E', but it also absorbs globally every Banach disk A of E. This is enough to be able to prove then that B will be absorbed by any closed absolutely convex neighborhood $U = U^{\circ\circ}$, as B° will absorb the Banach disk U°. □

This eases a lot the work on bounded sets of a lcs E as it allows, most of the times, to test a property in $\mathbb{K}$ and to infer it in E. For example, a function $f : E \longrightarrow F$ is bounded is and only if for every $l \in F'$, $l \circ f$ is bounded. Other bornologies can be defined, and each one will be used to define a new topology on spaces of linear functions in Section 3.4.3.

Definition 3.4.11. On any lcs E one defines the following bornologies:

- $\sigma(E)$, the bornology of all finite subsets of E.
- $\beta(E)$, the bornology of all $\mathcal{T}_E$ bounded sets of E, also called the Von-Neumann bornology of E. This is also the bornology of all weakly bounded sets of E (Proposition 3.4.10).
- $\mu(E)$ is the bornology of all absolutely convex compact sets in E_σ, that is of all the weakly compact absolutely convex sets. It is called the *Mackey bornology*, and plays an important role in Chapter 6.
- $\gamma(E)$ is the set of all absolutely convex compact subsets of E.
- $\mathtt{pc}(E)$ is the set of all absolutely convex precompact subsets of E.

All these bornologies relates to different notions of duality (Definition 3.4.16).

3.4.3 Topologies on spaces of linear functions

Let $\mathcal{B}$ a bornology on E. One defines on the space $\mathcal{L}(E, F)$ the topology of uniform convergence on sets of $\mathcal{B}$, whose basis of 0-neighborhood is:

$$\mathcal{W}_{B,U} = \{\ell \in \mathcal{L}(E, F) \mid l(B) \subset U\}$$

for $B \in \mathcal{B}$ and U a 0-neighborhood of E.

All the bornologies considered in section 3.4.2 are *saturated*, meaning they are stable by bipolar (and thus by absolutely convex closed hull by Theorem 3.4.1). If $\mathcal{B}$ is a bornology, then the topologies generated by $\mathcal{B}$ or by the set $\mathcal{B}^{sat} = \{B^{\circ\circ} | B \in \mathcal{B}\}$ are the same (see [44, 8.4.1]). Thus one can restrict one's attention to neighborhoods of the type $\mathcal{W}_{B^{\circ\circ},U}$.

We denote by $\mathcal{L}_{\mathcal{B}}(E, F)$ the vector space $\mathcal{L}(E, F)$ endowed with the topology $\mathcal{T}_{\mathcal{B}}$ described above. It is a locally convex topological vector space when F is. The fact that it is Hausdorff follows from the fact that we asked bornologies to cover E.

Proposition 3.4.12. *Let E and F be two lcs, and $\mathcal{B}$ a bornology on E. Then the space $\mathcal{L}_{\mathcal{B}}(E, F)$ is a lcs.*

Thus from a bornology on E one can define a topology on $\mathcal{L}(E, F)$, and in particular on $E' = \mathcal{L}(E, \mathbb{K})$. Symmetrically, from any bornology on E', one defines a topology on E:

Example 3.4.13. The weak* topology on E' is the topology of uniform convergence on finite subsets of E. The weak topology E_w on E is the topology of uniform convergence of $E = \mathcal{L}(E'_{w*}, \mathbb{K})$ on finite subsets of E'.

Note in particular that as $\tilde{E}' = E'$, (see 3.1.24) any bornology on $\tilde{E}$ also defines a topology on E'.

The strong, simple, compact-open, and Mackey topologies.

Definition 3.4.14. From the bornologies previously constructed in Section 3.4 one define the following locally convex and hausdorff vector topologies on the vector space $\mathcal{L}(E, F)$:

- the finite bornology on E defines the topology $\mathcal{L}_\sigma(E, F)$ of *simple* (sometimes called *pointwise*) convergence. A filter $\mathcal{F} \subset \mathcal{L}_\sigma(E, F)$ converges towards l if and only if, for every $x \in E$, $\mathcal{F}(x)$ converges towards $f(x)$ in F,
- the Von-Neumann bornology on E defines the *strong topology* $\mathcal{L}_\beta(E, F)$, also called the topology of uniform convergence on bounded subsets of E or the *bounded-open* topology,
- Likewise, one defines the lcs $\mathcal{L}_\gamma(E, F)$, $\mathcal{L}_{pc}(E, F)$ and $\mathcal{L}_\mu(E, F)$.

Proposition 3.4.15. *On the space $\mathcal{L}(E, F)$, $\mathcal{T}_s \leqslant \mathcal{T}_{pc} \leqslant \mathcal{T}_\mu \leqslant T_\beta$.*

Definition 3.4.16. Each one of these topologies leads in particular to a topology on the dual of E:

- the topology $E'_\beta = \mathcal{L}_\beta(E, \mathbb{K})$ is called the *strong topology* on E, is the most commonly used, and will be studied in section 3.5.2,
- the weak topology $E'_{\sigma(E)}$ was studied in Section 3.3.2,
- the Mackey-topology E'_μ will be studied in Section 3.5,
- the Arens-dual is E'_γ, and will be studied in Section 3.5.

Definition 3.4.17. Symmetrically:

- the Mackey topology $E_{\mu(E')}$ on E: it is the topology on E of uniform convergence on weakly* compact subsets of E'.
- the already known weak* topology $E_{\sigma(E')}$ on E,
- the strong topology $E_{\beta(E')}$ of uniform convergence on weakly* bounded subsets of E'.

While a space endowed with its weak topology is called a weak space, a space which is endowed with its Mackey topology is sometime called a *Mackey space* in the literature.

The equicontinuous bornology on E' defines also a topology on E, but it is exactly the original topology $\mathcal{T}_E$:

Proposition 3.4.18. *Consider E a lcs. Then $\mathcal{T}(E)$ corresponds to the topology of uniform convergence on equicontinuous subsets of E'.*

Proof. We write $\mathcal{T}(E_\mathcal{E})$ the topology on E of equicontinuous convergence on equicontinuous subsets of E. Then if $\mathcal{U}$ is a 0-basis in E, the sets $U^{\circ\circ}$ form a 0-basis of $\mathcal{T}(E_\mathcal{E})$ as polars are equicontinuous (example 3.4.4). Choosing a 0-basis of closed disks one has $U = U^{\circ\circ}$ by the bipolar theorem 3.4.1 and thus $\mathcal{T}(E) = \mathcal{T}(E_\mathcal{E})$. □

Note that finite subsets of E are in particular weakly-compact, which are in particular weakly bounded, thus bounded (Proposition 3.4.10). This leads to the following:

Proposition 3.4.19. *For E and F lcs, we have that the simple topology $\mathcal{T}_s$ on $\mathcal{L}(E, F)$ is coarser than the Mackey-topology $\mathcal{T}_\mu$, which is coarser than the strong topology $\mathcal{T}_\beta$.*

The preceding constructions for topologies differ in general, but some coincide on specific subsets of E:

Proposition 3.4.20. *[44, 8.5.1] On every equicontinuous subsets H of $\mathcal{L}(E, F)$, the topology $\mathcal{T}_S$ of simple convergence coincide with the topology $\mathcal{T}_{pc}$ of uniform convergence on precompact subsets .*

Proof. Any finite subset is precompact, therefore the topology of simple convergence is coarser than the precompact-open one $\mathcal{T}_s \leqslant \mathcal{T}_{pc}$. Let us prove that the converse is true. Consider $l_0 \in H$ and let us show that neighborhood of l_0 for the precompact-open topology are in particular neighborhoods for the simple topology. Consider thus a precompact set $S \subset E$ and a closed disk V in $\mathcal{U}_F(0)$ (thus $V = V^{\circ\circ}$ by the bipolar theorem 3.4.1). As H is equicontinuous, there exists $U \in \mathcal{U}_E(0)$ such that for all $l \in H$, $l(U) \subset \frac{1}{2}V$. Since S is precompact, there exists a finite subset A of H such that $S \subset A + \frac{1}{2}U$. Then if we set $M = 2A$, we can compute that $(l_0 + \mathcal{W}_{M,V}) \cap H \subset (l_0 + \mathcal{W}_{S,V}) \cap H$. □

All these different dual topologies characterize completeness according to a theorem proved by Grothendieck:

Theorem 3.4.21. *[51, 21,9.(2)] Consider $\langle E_2, E_1 \rangle$ a dual pair, and $\mathcal{B}$ a bornology on E_2. Then the completion of E_1 endowed with the topology of uniform convergence on $\mathcal{B}$ consists of all the linear functionals on E_2 whose restriction to sets $B \in \mathcal{B}$ are weakly continuous.*

This theorem relates completion (and thus the possiblity to work with smooth functions), with topologies on dual pairs. It will be used in particular to define chiralities of complete μ-reflexive spaces in Chapter 6, section 6.4.

3.4.4 Barrels

Barrels are yet another class of subsets of a lcs. The reader which is used to Banach spaces may have never heard of them: they coincide with the balls centred at 0 sets in normed lcs. They are important in this thesis as they characterize the topology of the double strong dual $(E'_\beta)'_\beta$, and thus the reflexivity of a lcs E (see Section 3.5.2).

Definition 3.4.22. A subset $U \subset E$ is a *barrel* if it is weakly closed, absorbent and absolutely convex. An lcs is said to be *barreled* if any barrel of E is a 0-neighborhood.

The following fact was already used in the proof of Proposition 3.4.10:

Proposition 3.4.23. *A subset B of E' is weakly* bounded if and only if B° is a barrel in E.*

Proof. Observe that by definition of boundedness, the polar B° of a weakly bounded subset B is absorbent: for all $x \in E$, there exists $\rho > 0$ such that for all $l \in B$, $|l(x)| < \rho$. Thus $x \in \rho B^\circ$. It is absolutely convex and weakly closed because it is a polar, and thus it is a barrel. Conversely if B° is a barrel, it absorbs any point of E, and thus B is weakly-bounded. $\square$

Thus a lcs is barreled if and only if it is endowed with the topology $\beta(E, E')$, that is the topology of uniform convergence on weakly-bounded sets of E'.

Proposition 3.4.24. *[44, 8.5.6] A metrisable complete lcs (that is, a Fréchet lcs) E is always barreled.*

In terms of models of polarized linear logic, barreledness is a positive property. It is preserved in general by inductive limits, and in specific cases by projective limits.

Proposition 3.4.25. *[44, 11.3] Barreledness of lcs is preserved by quotient, inductive limits, and cartesian products.*

3.5 Reflexivity

In this section, we will relate the theory of lcs with the fundamental equation of classical Linear Logic, that is the fact that a formula is equivalent to its double linear negation:

$$A \simeq A^{\perp\perp}.$$

This corresponds to reflexivity in denotational semantics based on lcs.:

$$E \simeq E''.$$

Note that the above linear homeomorphisms depends on the topology of E (which determines E'), *and* on the topology of E' (which determines E'').

Definition 3.5.1. Consider $\mathcal{B}$ a bornology on E (see Definition 3.4.6). A lcs is said to be *semi-$\mathcal{B}$-reflexive* if

$$\iota_E : \begin{cases} E \hookrightarrow E'^* \\ x \mapsto ev_x : (\ell \mapsto \ell(x)) \end{cases} \tag{3.5}$$

defines a linear isomorphisms $E \sim (E'_{\mathcal{B}})'$. It is said to be *$\mathcal{B}$-reflexive* if moreover the topologies of E and E'' corresponds, that is if $E \simeq (E'_{\mathcal{B}})'_{\mathcal{B}}$.

The restriction linear morphism ι_E is well defined: ev_x is continuous on $E'_{\mathcal{B}}$.
As the strong topology is the most common topology on the dual, we will by default call semi-reflexivity and reflexivity what is defined above as semi-β-reflexivity and β-reflexivity. In particular Chapter 7 deals with strong reflexivity.

3.5.1 Weak, Mackey and Arens reflexivities

In this section we expose some classes of topology on the dual for which semi-reflexivity always holds. We already showed that every lcs is semi-σ-reflexive (Proposition 3.3.13): $E \sim (E'_{w*})'$, and that any space endowed with its weak topology is σ-reflexive: $E_w \simeq (E'_{w*})'_{w*}$.
This is also true for the Mackey-topology, whose role is symmetric to that of the weak topology (see the fundamental Mackey-Arens Theorem 3.5.3).

Proposition 3.5.2. *Every lcs is semi-μ-reflexive:*

$$E \sim (E'_\mu)',$$

and any space endowed with its Mackey topology is μ-reflexive:

$$E_{\mu(E')} \simeq (E'_\mu)'_\mu.$$

Proof. Let us remark that the second linear homeomorphism follows directly form the first one: the Mackey topology on E is the topology of uniform convergence on absolutely convex compact sets in $E'_{\sigma(E)}$. If E is semi-μ-reflexive, the $\sigma(E', E)$ and $\sigma(E', (E'_\mu)')$ compact sets coincide, and thus the Mackey topology induced by E' on E coincides with Mackey topology induced by E' on $E \sim (E'_\mu)'$.

Let us prove the first equality. It follows from the fact that $(E'_\sigma) \sim E$ and from the fact that the Mackey topology is finer than the weak one (every finite subset is indeed weakly compact) that we have a continuous injection: $E'_\mu \hookrightarrow E'_\sigma$. Its transpose results in a linear inclusion:

$$E \sim (E'_\sigma)' \hookrightarrow (E'_\mu)'. \tag{3.6}$$

Let us prove that it is surjective. Consider $\phi \in (E'_\mu)'$. As ϕ is continuous, there exists a 0-neighborhood V in E'_μ such that for every $\ell \in V$, $|\phi(\ell)| < 1$. By definition of the Mackey topology, this means we have an absolutely weakly* compact subset K of E such that we can take $V = K^\circ$. This means that if we denote by $(\cdots)^\bullet$ the polar in the dual pair $((E'_\mu)', E',)$ we have $\phi \in K^{\bullet\bullet}$. K can indeed be considered as a subset of $(E'_\mu)'$ through the linear injection. However, thanks to the bipolar theorem 3.4.1, we have that $K^{\bullet\bullet}$ is the absolutely convex $\sigma((E'_\mu)', E')$-closed closure of K in $(E'_\mu)'$. As K is $\sigma(E, E')$-compact it is in particular $\sigma((E'_\mu)', E')$-compact and thus $\sigma((E'_\mu)', E')$-closed. As is it moreover absolutely convex, we have $K = K^{\bullet\bullet}$. As K is a subset of E, we obtain $\phi \in E$. ☐

This proposition implies in particular that for any topology $\mathcal{T}$ on E' which is finer than the weak topology and coarser than the Mackey-topology, we have $(E'_\mathcal{T})' \sim E$. This is proved by taking the double duals of the continuous injections: $E'_\sigma \hookrightarrow E'_\mathcal{T} \hookrightarrow E'_\mu$. But we have more:

Theorem 3.5.3. *The weak topology on E' is the coarsest locally convex Hausdorff vector topology $\mathcal{T}$ such that $(E'_\mathcal{T})' = E$, and the Mackey-topology is the finest.*

Proof. Let $\mathcal{T}$ be a locally convex Hausdorff vector topology on E' such that $(E'_\mathcal{T})' = E$. Then in particular any evalutation function $ev_x : E' \longrightarrow \mathbb{K}$ is continuous on $E'_\mathcal{T}$, thus $\mathcal{T}$ contains all the polars of the finite sets, and is finer than the weak topology. Moreover, we know that the topology $\mathcal{T}$ is also the topology of uniform convergence on equicontinuous sets of $(E'_\mathcal{T})' = E$ (Proposition 3.4.18). Thus any polar V° of a 0-neighborhood V in $\mathcal{T}$ is an equicontinuous set of E, thus is absolutely convex and weakly*-closed by the Alaoglu-Bourbaki theorem 3.4.5 . Therefore any closed absolutely convex 0-neighborhood $V = V^{\circ\circ}$ of $\mathcal{T}$ is a 0-neighborhood for the Mackey-topology. As $\mathcal{T}$ is also generated by its closed 0-neighborhood 3.1.5, we have that $\mathcal{T}$ is coarser than the Mackey-topology on E'. ☐

Remark 3.5.4. In particular, any Mackey-dual is endowed with its Mackey-topology. Indeed, as $(E'_{\mu(E)})' \sim E$ by proposition 3.5.2, we have that $E'_{\mu(E)}$, which is endowed by $\mu(E', E)$ by definition, is also endowed with its Mackey-topology $\mu(E', E'')$. With respect to what is done in the case of weak topologies, the topology $\mu(E', E)$ on E' should be called the Mackey* topology.

We now detail the stability properties of the weak and Mackey topologies:

Proposition 3.5.5. *[44, 8.8]*

- *The weak topology on a projective limit of lcs is the projective limit of the weak topologies.*

- *The Mackey topology of an inductive limit of lcs is the inductive limit of the Mackey topologies.*
- *The dual of an injective limit of lcs is* linearly isomorphic *to the dual of the projective limit.*
- *[44, 8.8.11] The Mackey dual of a Haussdorf injective limit of lcs* linearly homeomorphic *to the projective limite of the duals endowed with their respective Mackey topologies.*

Theorem 3.5.3 has an important corollary:

Corollary 3.5.6. If E'_τ is endowed with any topology comprised between the weak topology $\sigma(E', E)$ and the Mackey-topology $\mu(E', E)$, then the dual of E' is E.

Proof. It follows from the hypothesis on E that we have continuous linear injections $E'_w \hookrightarrow E'_\tau \hookrightarrow E'_\mu$. Taking the dual of these injections gives linear maps $E \longrightarrow (E'_\tau)' \longrightarrow E$. These maps are surjective by the Hahn Banach theorem, and thus leads to a linear isomorphism $(E'_\tau)' \sim E$. □

In particular, as the Arens topology (see Definition 3.4.16) satisfies this hypothesis:

Corollary 3.5.7. Any lcs E is semi-γ-reflexive, that is we have the linear isomorphism: $(E'_\gamma)' = E$.

Thus the Arens dual acts as the Mackey dual or the weak dual in terms of semi-reflexivity. We also have that an Arens dual is always γ-reflexive, meaning that we have a closure operator:

Proposition 3.5.8. *For any lcs E, we have*

$$E'_\gamma \simeq ((E'_\gamma)'_\gamma)'_\gamma.$$

Proof. This proof is done for example by Schwartz at the beginning of [67]. We already have the linear isomorphism by the previous corollary. Let us show that the two spaces in the isomorphism have the same topology. Consider any lcs F. Then the topology of $(F'_\gamma)'_\gamma = F$ induces on F the topology of uniform convergence on absolutely convex and compact subsets of F'_γ, while F is originally endowed with the topology of uniform convergence on equicontinuous subsets of F'. Without loss of generality, F is also endowed with the topology of uniform convergence on weakly closed equicontinuous subsets of F' . But by the Alaoglu-Bourbaki theorem 3.4.5, weakly closed equicontinuous sets are in particular weakly compact. As compact sets are weakly-compact, the topology of $(F'_\gamma)'_\gamma$ is always finer than the one of F.
However, when $F = E'_\gamma$, then equicontinuous subsets of F' are by definition generated by the bornology $\gamma(E, E')$, and therefore $E'_\gamma \simeq ((E'_\gamma)'_\gamma)'_\gamma$. □

3.5.2 Strong reflexivity

In the literature, reflexivity is usually defined with respect to the strong dual: the terms reflexive and semi-reflexive are used for β-reflexive and semi-β-reflexive. Let us describe how β-reflexivity can be understood through a description of the topology of E. The proof makes use of all the notion introduced above, relying on a fine understanding of the role played by the weak and Mackey-topology, and of when a space is endowed with these topologies.

Proposition 3.5.9. *[44, 11.4.1] The following propositions are equivalent:*

1. *E is semi-β-reflexive,*
2. *The strong topology and the Mackey topology on E' coincide: $E'_\beta \simeq E'_\mu$,*
3. *E_w is quasi-complete, that is every bounded Cauchy-filter converges weakly.*

Proof. (1) $\Leftrightarrow$ (2): As the strong topology is finer than the Mackey topology 3.4.15, and by the Mackey-Arens theorem 3.5.3, as soon as a space E is semi-β-reflexive its dual E' is endowed with its Mackey-topology. The converse follows immediately from the preceding Section 3.5.1.

(2) $\Leftrightarrow$ (3): If E'_β is endowed with its Mackey topology, it means that every bounded sets in E is weakly-compact, and in particular weakly complete. Conversely, if every bounded set B is weakly complete, from Proposition 3.1.25 it follows that its weak-closure is weakly-compact, and thus that the strong topology is coarser than the Mackey topology. As a weakly-compact set is always weakly-bounded, thus bounded by Proposition 3.4.10, the Mackey topology is always coarser than the strong topology and we have (2) $\Leftarrow$ (3). □

In terms of polarized linear logic, semi-reflexivity is a negative characterization: it is preserved by projective limits, and particular cases by inductive limits:

Proposition 3.5.10. *[44, 11.4.5] Semi-reflexivity of lcs is preserved by closed subspaces, projective limits and direct sums.*

As semi-β-reflexivity can be modelled through a completeness condition, there exists a closure operator which makes any E-semi-reflexive: the quasi-completion of E is semi-β-reflexive and enjoys a functorial property as described in Proposition 3.1.24. Obtaining reflexivity requires much more, and there exists no general closure operator for reflexivity in the context of lcs.
A semi-reflexive lcs E is reflexive if and only if E carries the same topology as $(E'_\beta)'_\beta$, that is the topology of uniform convergence on bounded subsets of E'_β. Those are exactly the uniformly bounded subsets of E', that is the sets of functions sending uniformly a bounded set on a bounded set. As E carries the topology of uniform convergence on equicontinuous subsets of E' (Proposition 3.4.18), we have:

Proposition 3.5.11. *A semi-reflexive space is reflexive if and only if the equicontinuous subsets of E' and the uniformly bounded ones coincide.*

In particular, let us recall from Section 3.4.4 that a lcs is barreled if and only if it is endowed with the topology $\beta(E, E')$, that is the topology of uniform convergence on weakly-bounded sets of E'. This topology is exactly the one induced by the strong bidual $(E'_\beta)'_\beta$ on E. Thus:

Proposition 3.5.12. *A semi-reflexive lcs E is reflexive if and only if it is barreled.*

Thus reflexivity combines a positive requirement (barreledness, stable by inductive limits) and a negative requirement (semi-reflexivity, stable by projective limits). This gives some intuition about where the difficulty lies for finding good models of LL made of reflexive spaces.

Proposition 3.5.13. *[44, 11.4.5] Reflexivity is preserved by cartesian products, direct sums, and strong duality. Consider $(E_j)_j$ an inductive system which is reduced (i.e. such that the maps $S_{k,j}$ are injective) and such that the E_j are all reflexive. Let $E = \varinjlim_j E_j$ be the inductive limit of the (E_j). Then if for every bounded subset $B \subset E$, there exists j such that B is a bounded subset of E_j, then E is reflexive.*

The following characterization is at the heart of Section 7.2.2. From the fact that a metrisable lcs is barreled 3.4.24, it follows that:

Proposition 3.5.14. *[44, 11.34.3] Consider E a metrisable lcs. Then if E is semi-β-reflexive, it is β-reflexive and complete.*

Outlook 4. In a model of MLL, we will be looking for conditions allowing for reflexivity which are preserved by tensor product. This requires finding a closure operator which makes a space reflexive (to use the Weak, Mackey or Arens dual), or to use a characterization restrictive enough so that it is preserved by some tensor product (as nuclearity, see Chapter 7). But even before that, one must choose the topology on the tensor product, and see under which condition this tensor product is associative. This is the content of the next Section 3.6 .

3.5.3 The duality of linear continuous functions

This short subsection is essential. It provides tools for proving adjunctions of the type $\mathcal{L}(E', F) \simeq \mathcal{L}(F', E)$ when E is reflexive for some notion of reflexivity, thus constructing chiralities and models of MLL. Consider a linear continuous function $f : E \longrightarrow F$. Then by precomposition we obtain a linear function $f' : F' \longrightarrow E'$. For which topologies is f' continuous ? The next proposition sums up results which are easy consequences of Section 3.5.1.

Proposition 3.5.15. *[44, 8.6.5]*
Consider $f : E \longrightarrow F$ linear. Then $f^* : F^* \longrightarrow E^*$ induces a linear map $f' : F' \longrightarrow E'$ if and only if f is weakly continuous (i.e continuous with respect to the weak topology on E and F, i.e. continuous with respect to weak topology on F and any topology on E compatible with the dual E'). In that case f' is weakly* continuous.*

Proof. The function f is weakly continuous if and only if f the reverse image of the polar of a finite subset of F' contains the polar of a finite subset of E', if and only if f' maps finite subsets of F' to finite subsets of E'. As f maps $(E'_w)'$ to $(F'_w)'$ we have then by the same reasoning that f' is weakly* continuous. □

In particular, any continuous linear map f is weakly continuous, and thus has a transpose $f' : F' \longrightarrow E'$. The converse however is not always true: a function can be weakly continuous but not continuous, as F might contains 0-neighborhoods which are not polars of finite subsets of F'. The previous proposition generalizes to:

Proposition 3.5.16. *Consider a linear map $f : E \longrightarrow F$ and $\mathcal{B}$ and $\mathcal{C}$ bornologies on E and F respectively. Then $f' : F'_{\mathcal{C}} \longrightarrow E'_{\mathcal{B}}$ is continuous if and only if for any $B \in \mathcal{B}$ $f(B)^{\circ\circ} \in \mathcal{C}$.*

Let us state very clearly the following easy proposition, which is fundamental throughout this thesis. It could also be deduced from Proposition 4.0.10, as reflexive spaces are barreled and thus endowed with their Mackey topology.

Proposition 3.5.17. *Consider E and F strongly reflexive spaces. Then we have a linear homeomorphism $\mathcal{L}_\beta(E'_\beta, F) \simeq \mathcal{L}_\beta(F'_\beta, E)$.*

Proof. Consider $f \in \mathcal{L}(E'_\beta, F)$. As f is continuous, it is weakly continuous, thus $f' : F'_\beta \longrightarrow (E'_\beta)'_\beta$ is continuous. As f is continuous and therefore bounded, it sends bounded sets to bounded sets, and thus f' is continuous. As E is reflexive, f' is continuous from $F'_\beta \longrightarrow E$. The mapping $f \mapsto f'$ is continuous: indeed, consider B_F an absolutely convex and weakly closed bounded set in F and B_E a bounded set in E. If $f'(B_F{}^\circ) \subseteq B_E^{\circ\circ}$, then $f(B_E{}^\circ) \subseteq B_F^{\circ\circ}$. The situation being completely symmetrical, we have that for $g \in \mathcal{L}(F'_\beta, E)$, we have $g' \in \mathcal{L}(E'_\beta, F)$, and $g \mapsto g'$ is continuous. $\square$

3.6 Topological tensor products and bilinear maps

This section is fundamental for this thesis, as it explores the different topologies which allow to interpret the connectives $\otimes$ and $\parr$ of linear logic. It also gives the bases for Schwartz' Kernel Theorem 7.3.9, which allows to see spaces of distributions as a strong monoidal functor interpreting the exponential.
In algebra, the tensor product is defined through its universal property:

Definition 3.6.1. Consider E and F two vector spaces. Then there exists a unique pair ($E \otimes F$, h) where $E \otimes F$ is a vector space and h a bilinear function from $E \times F$ to $E \otimes F$ such that, for any vector space G and any bilinear function $f : E \times F \longrightarrow G$, there exists a unique linear map $\tilde{f} : E \otimes F \longrightarrow G$ such that:

$$f = \tilde{f} \circ g.$$

This changes as soon as spaces are endowed with a topology: the notions of continuity exist for bilinear functions, and through them different topologies on a tensor product. We detail some important topological tensor products. Most of them ($\otimes_\pi$, $\otimes_i$, $\otimes_\gamma$) are directly linked with a notion of bilinearity, and will interpret the multiplicative conjunction $\otimes$ of DiLL in the next sections. The injective tensor product $\otimes_\varepsilon$ has the good behaviour of the dual $\parr^\perp$ of the multiplicative disjunction $\parr$ of DiLL.
In a classical setting, where $\otimes^{\perp\perp} = \parr^\perp = \otimes$, it makes then sense to require to have the interpretation of $\otimes$ equal $\otimes_\varepsilon$. This is at the heart of Chapter 7. We refer to the second book by Scwhartz on vectorial distributions for more details on various topological products and their associativity [68, II.1].

Notation 3.6.2. *In this section $\otimes$ denotes the algebraic tensor product between two vector spaces. It also denotes sometimes the multiplicative conjunction of* LL. *To design a lcs structure on the algebraic tensor product, we will use indices $\otimes_\pi$, $\otimes_i$, $\otimes_\varepsilon$.*

3.6.1 The projective, inductive and $\mathcal{B}$ tensor products

Definition 3.6.3. Let us denote $\mathbf{B}(E \times F, G)$ the space of all continuous, and not only separately continuous, bilinear functions from $E \times F$ (endowed with the product topology) to G. We denote by $\mathscr{B}(E, F)$ the space of all separately continuous bilinear functions from $E \times F$ (endowed with the product topology) to G.

Definition 3.6.4. Consider $\mathcal{B}(E)$ (resp. $\mathcal{B}(F)$) a bornology on E (resp. F). A bilinear function $u : E \times F \longrightarrow G$ is $\mathcal{B}$-hypocontinuous if for every bounded sets $B_E \in \mathcal{B}(E)$ (resp. $B_F \in \mathcal{B}(F)$), and for every open set V in G, there exists an open set W_F in F (resp. W_E open in E) such that $u(B_E \times W_F) \subset V$ (resp. $u(W_E \times B_F) \subset V$). This means that the linear functions $u(B_E, \_)$ and $u(\_, B_F)$ are continuous. We denote by $\mathscr{B}_{\mathcal{B}}(E \times F, G)$ the space of all $\mathcal{B}$-hypocontinuous functions from $E \times F$ to G.

Definition 3.6.5. Consider E and F two lcs. The projective tensor product $E \otimes_\pi F$ is the algebraic tensor product, endowed with the finest topology making the canonical bilinear map $E \times F \to E \otimes F$ continuous. The inductive tensor product $E \otimes_i F$ is the algebraic tensor product, endowed with the finest topology making the canonical bilinear map $E \times F \to E \otimes F$ separately continuous.

Definition 3.6.6. Consider $\mathcal{B}(E)$ (resp. $\mathcal{B}(F)$) a bornology on E (resp. F). Then the $\mathcal{B}$-tensor product is defined as the algebraic tensor product, endowed with the finest topology making the canonical bilinear map $E \times F \to E \otimes F$ a $\mathcal{B}$-hypocontinuous map. Remark that $E \otimes_\sigma F \simeq E \otimes_i F$.

Proposition 3.6.7. *([44, 15.1,16.1] and [68, II.1]) Consider E and F two lcs. The projective, injective , $\mathcal{B}$-topology and inductive topology make $E \otimes F$ a locally convex topological vector space. $E \otimes_\pi F$ and $E \otimes_\varepsilon F$ are always Hausdorff (when E and F are). The space $E \otimes_\mathcal{B} F$ is Hausdorff as $\mathcal{B}(E)$ and $\mathcal{B}(F)$ are total by hypothesis.*

Remark 3.6.8. A sub-basis of 0-neighborhoods on $E \otimes_\mathcal{B} F$ is given by a family of $B_E \times V_F \cup V_E \times B_F$ where B_E, B_F are in $\mathcal{B}_E$ and $\mathcal{B}_F$ respectively, and V_E and V_F are absolutely convex and weakly closed 0-neighborhoods in E an F respectively.

The projective topological tensor product can simply be characterized in terms of semi-norms:

Proposition 3.6.9. *A family of semi-norms for $E \otimes_\pi F$ is:*

$$\pi_{U,V} = q_{\overline{acx}(U \otimes V)}$$

when $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Then for each topological tensor product, we have a corresponding universal property:

Proposition 3.6.10. *Consider E and F two lcs. Then if we write $h \in \mathscr{B}_\mathcal{B}(E \times F, E \otimes_\mathcal{B} F)$ (resp. $\mathbf{B}(E \times F, E \otimes_\pi F)$ the canonical bilinear function from $E \times F$ to $E \otimes_\mathcal{B} F$ (resp. $E \otimes_\pi F$) we have that for any vector space G and any bilinear function $f \in \mathscr{B}_\mathcal{B}(E \times F, G)$ (resp.$f \in \mathbf{B}(E \times F, G)$), there exists a unique continuous linear map $\tilde{f} : E \otimes_\mathcal{B} F \longrightarrow G$ such that:*

$$f = \tilde{f} \circ h.$$

Proof. The existence of $\tilde{f}$ as a linear function follows from the universal property of the algebraic tensor product. It is defined on basic elements as $\tilde{f}(x \otimes y) = f(x, y)$. Its continuity is immediate, and it is unique as otherwise one would be able to define on $E \otimes F$ a finer topology with a canonical bilinear function into $\mathcal{B}$-hypocontinuous (resp. continuous). ☐

Then the computation of the dual follows easily.

Proposition 3.6.11.
- *$(E \otimes_\pi F)'$ is linearly isomorphic to $\mathbf{B}(E \times F, \mathbb{K})$.*
- *$(E \otimes_\mathcal{B} F)'$ is linearly isomorphic to $\mathscr{B}_\mathcal{B}(E \times F, \mathbb{K})$.*

Proposition 3.6.12. *[44, 16.2.8] The tensor products $\otimes_\pi$, $\otimes_\varepsilon$ and $\otimes_i$ are commutative and associative.*

Grothendieck's "problème des topologies". Having a bornology on spaces of linear functions such that $\mathcal{L}(E \hat{\otimes}_\mathcal{B} F, G) \simeq \mathcal{L}(E, \mathcal{L}_\mathcal{B}(F, G))$ is not at all immediate: even if the tensor product is associative, its dual may not be, or equivalently a bounded set in $E \hat{\otimes}_\mathcal{B} F$ may not be a tensor product of bounded sets. Let's take the example of the β-tensor product: a β-hypocontinuous bilinear map f defined on $E \times (F \otimes_\beta G)$ is continuous on $E \times B_1$ and $B_2 \times (F \otimes_\mathcal{B} G)$, where B_1 and B_2 are bounded sets of $F \otimes_\beta G$ and E respectively. Thus f is β-hypocontinuous on $(E \otimes_\beta F) \times G$ if and only if the bounded sets of the β-tensor product are exactly the finite sums of tensor products of bounded subsets. Indeed $B_E \otimes B_F$ is bounded in $E \otimes_\beta F$ (as it is absorbed by 0-neighborhoods $U \otimes B_F \cap B_E \otimes V$), and thus so are finite sums of tensor products of bounded subsets. The question of whether the converse was true in general corresponds to the generalisation of a question for bounded sets of the projective tensor product asked by Grothendieck [36], which was answered negatively [73].

In his second book about distributions with vectorial values [68, page 16], Schwartz describes the lcs for which this "problèmes des topologies" is answered positively, for different tensor products. We recall some particular cases of them below:

- The compact subsets of the completed γ-tensor $E \hat{\otimes}_\gamma F$ product of Fréchet lcs E and F (that is, metrisable complete lcs, see 7, Section 7.1.1) are tensor product of compact subsets of E and F.
- The completed β-tensor product is associative on duals of Fréchet spaces, and bounded subsets of it are tensor products of bounded subsets.

Comparing tensor products As explained before, in order to be able to work with smooth functions, some completeness notion will be needed in 6 and 7. However the tensor product of two complete space is not complete in general, it needs to be completed:

Definition 3.6.13. Consider $\theta \in \{\mathcal{B}, \pi, \varepsilon\}$. We denote by $E\hat{\otimes}_\theta F$ the completion of the lcs $E \otimes_\theta F$.

Proposition 3.6.14. *On $E \otimes F$, the inductive topology $\mathcal{T}_i$ is finer than the projective topology $\mathcal{T}_\pi$, which is finer than the injective one $\mathcal{T}_\varepsilon$.*

Example 3.6.15. We will show in Chapter 6 that the completion $\hat{\otimes}_\varepsilon$[5] provides a satisfactory interpretation of the $\invamp$ for complete topological vector spaces. The projective tensor product, one the other hand, is the interpretation for the multiplicative conjunction $\otimes$ as by definition it satisfies a universal property with respect to continuous bilinear functions.

Example 3.6.16. The difference between projective and injective tensor product is illustrated by the following classical example. Consider ℓ_1 the set of scalar sequences whose sum is absolutely bounded, E a lcs and $\mathcal{U}$ a basis of absolutely convex neighborhoods of 0 in E.

On the one hand, we define an absolutely Cauchy sequence in E as a sequence whose sum is absolutely bounded in each normed space E_U for all $U \in \mathcal{U}$. We write $l_1\{E\}$ the space of all such sequences. Then $l_1 \otimes_\pi E$ is a dense subspace of $l_1\{E\}$ [44, 15.7]. This is proved by describing $l_1 \otimes_\pi E$ as the vector space of all functions $\mathbb{N} \longrightarrow E$, endowed with the topology generated by the gauge functionals q_U. These functions which are described as finite sums are then dense in the functions $f : \mathbb{N} \longrightarrow E$ described as uniformly converging sums.

On the other hand, one defines unconditional Cauchy sequences in E as the sequences (x_n) such that for any permutation $\sigma \in \Sigma(\mathbb{N})$, the sequence $\left(\sum_{p \leqslant n} x_{\sigma(p)}\right)_n$ converges in the completion $\tilde{E}$ of E. We write $\ell_1\langle E\rangle$ the space of all such sequences, endowed with the topology generated by the semi-norms:

$$\epsilon_U((x_n)_n) := \sup_{a \in U^\circ} \sum_n | < a, x_n > |.$$

Notice that it is only by the hypothesis of absolute convergence on the sums of the (x_n) that we can define such semi-norms. Then it is easy to see that the linear injective embedding:

$$z = \sum_{i=1}^m (\lambda_n^i)_{n \in \mathbb{N}} \otimes x_i \in l_1 \otimes E \mapsto \left(\sum_{i=1}^m \lambda_n^i x_i\right)_n$$

is continuous from $\ell_1 \otimes_\epsilon E$ to $\ell_1\langle E\rangle$. As previously, we can identity as vector spaces $E^{(\mathbb{N})}$ and $l_1 \otimes E$, and we obtain the desired result. Thus $l_1 \otimes_\epsilon E$ identifies with a dense subspace of $\ell_1\langle E\rangle$ [44, 16.5].

3.6.2 The injective tensor product

There exists a canonical injection of the algebraic tensor product $E \otimes F$ into the space of continuous bilinear maps from $E' \times F'$ to $\mathbb{K}$:

$$\phi : \begin{cases} E \otimes F \longrightarrow \mathbf{B}(E'_w \times F'_w, \mathbb{K}) \\ x \otimes y \mapsto ((\ell, \ell') \mapsto \ell(x)\ell'(y)) \end{cases} \tag{3.7}$$

This definition makes sense: $\phi(x \otimes y)$ is indeed bilinear, but is also weakly continuous in ℓ and ℓ'. It is injective as E' (resp. F') separates the points of E (resp. F).

Lemma 3.6.17. If E and F are finite dimensional vector spaces, then $\mathbf{B}(E'_w \times F'_w) = E \otimes F$.

Proof. If $x \in E$ and $y \in F$, one maps the element $x \otimes y \in E \otimes F$ to the bilinear form:

$$\theta(x \otimes y) : (l_1, l_2) \in E' \times F' \mapsto l_1(x)l_2(y).$$

The linear function $\theta : E \otimes F \mapsto B(E' \times F')$ thus defined is injective. Indeed, consider $x \in E$ and $y \in F$ such that $x \otimes y \in \mathrm{Ker}(\theta)$. Then either $x = 0$ or $y = 0$, thus $x \otimes y = 0$: otherwise one can find $l_1 \in E'$ and $l_2 \in F'$ such that $l_1(x) \neq 0$ and $l_2(y) \neq 0$. As the dimension of $E \otimes F$ equals the dimension of $\mathbf{B}(E' \times F')$, θ defines an isomorphism between the two. □

[5] which identifies to the ε product of section 3.6.3 when spaces have the approximation property [44, Ch.18]

Proposition 3.6.18. *[76, 42.4] ϕ induces a linear isomorphism: $E \otimes F = \mathbf{B}(E'_w \times F'_w, \mathbb{K})$.*

Proof. Remember that E' and F' are endowed with the weak* topology. Consider again the linear mapping

$$\theta : \begin{cases} E \otimes F \to \mathbf{B}(E' \times F', \mathbb{K}) \\ x \otimes y \mapsto ev_{x\otimes y} : (l_1, l_2) \in E' \times F' \mapsto l_1(x) l_2(y). \end{cases}$$

It is injective, as E' (resp. F') separates E (resp. F). Let us show it is surjective. Consider $\phi \in \mathbf{B}(E', F')$. As ϕ is continuous, there exist $x_1, ..., x_n \in E$ and $y_1, ..., y_m \in F$ such that

$$\text{for all } i, j, |l_1(x_i)| < 1 \text{ and } |l_2(y_j)| < 1 \Rightarrow |\phi(l_1, l_2)| < 1.$$

Consider A the sub-vector space of E generated by the x_i, and B the sub-vector space of F generated by the u_j. Both A and B are finite dimensional. Thus, if A° (resp. B°) denotes the polar of A (resp. B), we have:

$$E' = A' \oplus A^\circ \text{ and } F' = B' \oplus B^\circ.$$

One sees by homogeneity that ϕ vanishes on $(E' \times B^\circ)$ and $(A^\circ \times F')$. Thus ϕ is uniquely determined by its restriction to $A' \times B'$. As A and B are finite dimensional, we have our result thanks to Lemma 3.6.17. □

From Proposition 3.6.18 it makes sense to define a topology on $E \otimes F$ from a topology on $\mathbf{B}(E'_w \times F'_w, \mathbb{K})$, or even on the space of separately continuous bilinear maps $\mathscr{B}(E'_w \times F'_w, \mathbb{K})$ which contains the previous one.

Definition 3.6.19. The injective tensor product $E \otimes_\varepsilon F$ is defined as the algebraic tensor product, endowed with the topology induced by $\mathscr{B}_e(E'_w \times F'_w, \mathbb{K})$, where $\mathscr{B}_e(E'_w \times F'_w, \mathbb{K})$ is the space $\mathscr{B}(E'_w \times F'_w, \mathbb{K})$ of separately continuous bilinear functions endowed with the topology of uniform convergence on the product of equicontinuous subsets of E'_w and F'_w. We write $E \hat{\otimes}_\epsilon F$ the completion of $E \otimes_\epsilon F$.

If $\mathcal{U}$ (resp. $\mathcal{V}$) is a basis of absolutely convex 0-neighborhoods in E (resp. F), then a family of semi-norms for $E \otimes_\epsilon F$ is:

$$\epsilon_{U,V}(\sum x_i \otimes y_i) = \sup\{|\sum_i x'(x_i) \cdot y'(y_i)| \mid x' \in U^\circ, y' \in V^\circ\}$$

where $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Proposition 3.6.20. *[44, 16.1] The injective tensor product of two lcs is a lcs, and preserves completeness, metrizability and normablity. The completed injective tensor product commutes with projective limits.*

Proposition 3.6.21. *[44, 16.2.1,16.2.7] The injective tensor product is functorial, commutative and associative in the category of lcs and continuous linear maps.*

While the projective and inductive tensor products make good interpretations for the multiplicative conjunction of Linear Logic, via their universal property 3.6.10, the injective tensor product provides a satisfactory interpretation of the $\parr$. The first clue is that it is well behaved with projective limits, see the above proposition. The second is that it is exactly the operator which glues well spaces of smooth functions: the following example can be found in [76, 44.1].

Example 3.6.22. For the topology described in Section 3.2 , we have a linear homeomorphism:

$$\mathcal{C}^\infty(\mathbb{R}^n, E) \simeq \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \hat{\otimes}_\varepsilon E.$$

This corresponds to the logical equivalence:

$$A \Rightarrow B \equiv (?A^\perp) \parr E^\perp.$$

The injective tensor product does not characterize some continuity on bilinear forms, but a subclass of the continuous bilinear forms known as the *integral* ones.

Definition 3.6.23. Consider $B \in \mathbf{B}(E \times F, G)$. It is an integral form if there exists U a 0-neighborhood in E and V a 0-neighborhood in F, μ a positive Radon measure on U° and V° such that for all $x, y \in E \times F$

$$B(x, y) = \int_{U^\circ \times V^\circ} y'(y).x'(x) d\mu(x', y').$$

We denote by $\mathcal{I}(E \times F, G)$ the vector space of all integral forms in $\mathbf{B}(E \times F, G)$.

Proposition 3.6.24. *[76, 49.1] The dual of the injective tensor product is the space of scalar integral forms:*

$$(E \otimes_\varepsilon F)' = \mathcal{I}(E \times F, \mathbb{K}).$$

3.6.3 The ε tensor product

With the injective tensor product comes another operator, which correspond to the completed injective tensor product (Definition 3.6.19) for a large class of lcs who have the approximation property (see [44, chapter 18]). Called the ε-tensor product, it gives a concrete representation of the elements of the completed injective tensor product whenever $E\varepsilon F \simeq E\hat{\otimes}_\varepsilon F$.

Definition 3.6.25. For E and F two lcs, we define $E\varepsilon F = (E'_\gamma \otimes_{\mathscr{E}} E'_\gamma)'$ as vector of $\mathscr{E}$-hypocontinuous bilinear forms on the duals E'_γ and F'_γ (the definition of hypocontinuity is done in 3.6.4, the one of the Arens dual E'_γ in 3.4.16). It is endowed with the topology of uniform convergence on products of equicontinuous sets in E', F'.

As on equicontinuous subsets of E' (resp. F'), the weak topology and the compact-open topology coincide (see Proposition 3.4.20). The vector space $(E'_\gamma \otimes_{\mathscr{E}} F'_\gamma)'$ coincides also to the $\mathscr{E}$-hypocontinuous functions on $E'_w \times F'_w$, which contains in particular the space of all continuous bilinear maps on $E'_w \times F'_w$.

Proposition 3.6.26. *The lcs $E\varepsilon F$ induces on the tensor product $E \otimes F$ the injective tensor product topology.*

The following proposition is proved by Jarchow in his textbook. It is interesting to notice that it is proved by using the semi-γ-reflexivity of any lcs:

$$E = (E'_\gamma)'_\gamma$$

and the fact that we have here a closure operator in the category of lcs: $E'_\gamma \simeq ((E'_\gamma)'_\gamma)'_\gamma$. This is concretely what is done every day in denotational models of LL, the equation $E \simeq E^{\perp\perp}$ is used to show that $L(E^\perp, F)$ is commutative and associative.

Proposition 3.6.27. *[44, 16.2.6,16.1.3] The ε product is commutative and associative on complete lcs, and $E\varepsilon F$ is complete if and only if E or F is complete.*

Beware that Jarchow in his proof describes the ε product by using on the dual E' the topology of uniform convergence on compact subsets of $\tilde{E}$. As we are dealing here with complete subsets our notations are coherent. Remember also that Jarchow uses the notation E'_γ to denote the topology of uniform convergence on absolutely convex compact subsets of $\tilde{E}$, which is coherent with our definition for E'_γ 3.4.16 only when spaces are complete.

Part II

Classical models of DiLL

Chapter 4

Mackey and Weak topologies as left and right adjoint to pairing

In this part, we describe two models of DiLL, each one using a specific topology on the dual allowing for an involutive linear negation. The first Chapter introduces the notion of quantitative versus smooth interpretation of proofs, and the Weak spaces and Mackey spaces through an adjunction with the category of Chu spaces. The second Chapter 5 is adapted from a published article by the author and details a quantitative model of DiLL with weak spaces [48]. We highlight the fact that this gives a polarized model of DiLL with a linear negation involutive on the negatives formulas. The third Chapter 6 takes inspiration from [17], so as to adapt the work of convenient space into a polarized model of DiLL with Mackey spaces, with a linear negation involutive on the positive formulas.

Quantitative semantics and Cartesian closed categories

Introduced by Girard [30], quantitative semantics refines the analogy between linear functions and linear programs (consuming exactly once their input). Indeed, programs consuming exactly n-times their resources are seen as monomials of degree n. General programs are seen as the disjunction of their executions consuming n-times their resources. Mathematically, one can apply this idea by interpreting non-linear proofs as sums of n-monomials. These sums may be converging [18, 32, 49], finite [19], or formal [48].

Power series are an efficient answer to the issue of finding cartesian closed categories of non-linear functions in a vectorial setting. The isomorphism between $\mathcal{C}^\infty(E \times F, G)$ and $\mathcal{C}^\infty(E, \mathcal{C}^\infty(F, G))$ consists of combinatorial manipulations on the monomials (see in particular Section 5.2). The convergence of the power series obtained as a result of these manipulations is proved in a second time, and uses the completeness of the spaces E, F and G.

However, one of the two guidelines for this thesis is the search for models of DiLL in which smooth functions are not necessarily power series, and most importantly spaces are general topological vector spaces. The motivation comes from the need to relate DiLL with functional analysis or differential geometry. One of the best settings for a cartesian closed category of smooth functions was developed by Frölicher, Kriegl and Michor [26, 53] , by defining smoothness as the preservation of smooth curves $c : \mathbb{R} \longrightarrow E$ (see Section 2.4.3). In Chapter 6, we adapt the results of [6] to a classical setting.

The Category of Chu spaces and its adjunctions to TopVec.

The category Chu of duals Pairs.

We describe the category Chu as described by Barr [2]. Chu spaces makes use of the monoidal closed structure of the category Vec of all vector space with linear maps between them, endowed with the usual tensor product, and transform it into a $*$-autonomous category. We recall first the notion of dual pairs as described previously 3.3.18.

Definition 4.0.1. A dual pair is a couple (E, F) of two $\mathbb{R}$-vector space together with a symetric bilinear and non degenerate form:

$$\langle \cdot, \cdot \rangle : \begin{cases} E \times F \longrightarrow \mathbb{R} \\ (x, y) \mapsto \langle x, y \rangle \end{cases} \tag{4.1}$$

As in the litterature [44], when E is a vector space, so in particular when E is a lcs, we denote by E^* the set of (not necessarily continuous) linear forms $\ell : E \longrightarrow \mathbb{K}$. Then for any dual pair $(E, F, \langle\cdot,\cdot\rangle)$ we have linear injections:

$$\begin{aligned} E &\hookrightarrow F^* \\ x &\mapsto (x^* : y \mapsto \langle x, y\rangle) \end{aligned}$$

$$\begin{aligned} F &\hookrightarrow {}^*E \\ Y &\mapsto ({}^*y : x \mapsto \langle x, y\rangle). \end{aligned}$$

The fact that these are injections comes from the non-degeneracy of the pairing. These definitions extend to functions, defining two endofunctors $(\cdot)^*$ and ${}^*(\cdot)$ on the category VEC of vector spaces and linear functions.

Definition 4.0.2. The category CHU has as objects dual pairs of vector spaces and as arrows pairs of linear maps:

$$(f : E_1 \longrightarrow E_2, f' : F_2 \longrightarrow F_1) : (E_1, F_1) \longrightarrow (E_2, F_2)$$

such that the following diagram commute:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow{\scriptstyle (\cdot)^*} & & \downarrow{\scriptstyle (\cdot)^*} \\ F_1^* & \xrightarrow{f'^*} & F_2^* \end{array}$$

We denote by $\text{CHU}((E_1, E_2), (F_1, F_2))$ the vector space of such pairs of linear functions. Thanks to the symmetry of $\langle ., .\rangle$, the commutation of this this diagram is equivalent to the one of the following:

$$\begin{array}{ccc} F_2 & \xrightarrow{f'} & F_1 \\ \downarrow{\scriptstyle {}^*(\cdot)} & & \downarrow{\scriptstyle {}^*(\cdot)} \\ {}^*E_2 & \xrightarrow{{}^*f} & {}^*E_1 \end{array}$$

Then one defines on CHU a tensor product and internal hom-set which makes it a duality which makes it a $*$-autonomous category.

Definition 4.0.3. Consider (E_1, F_1) and (E_2, F_2) two dual pairs with pairings denoted respectively by $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$. As in Chapter 3 we write $L(E, F)$ for the vector space of all linear functions between two vector spaces.

- $(E_1, F_1)^\perp = (F_1, E_1)$ and the pairing of $(E_1, F_1)^\perp$ is the symmetric to the pairing of (E_1, F_1).
- $(E_1, F_1) \otimes (E_2, F_2) = (E_1 \otimes E_2, \text{CHU}((E_1, F_2), (F_1, E_2)))$ with a pairing defined by: $\langle x_1 \otimes x_2, (\ell_1, \ell_2)\rangle = \langle \ell_1(x_1), x_2\rangle_2$ and then extended linearly on $E_1 \otimes E_2$. This definition is symmetric in (E_1, F_1) and (E_2, F_2) by the requirements on dual pairings.
- $(E_1, F_1) \multimap (E_2, F_2) = (\text{CHU}((E_1, E_2), (F_1, F_2)), E_1 \otimes F_2)$ with a pairing defined as $\langle(\ell_1, \ell_2), x_1 \otimes y_2\rangle = \langle \ell_1(x_1), y_2\rangle_2$.

Proposition 4.0.4. *The category* CHU *is monoidal closed with tensor product* $\otimes$ *and neutral* $(\mathbb{K}, \mathbb{K})$.

Proof. The monoidality of $\otimes$ in CHU follows from the one of the algebraic tensor product in the category of vector spaces and linear maps. By definition, maps in $\text{CHU}((E_1, F_1) \otimes (E_2, F_2), (E_3, F_3))$ are pairs $(f : E_1 \otimes E_2 \longrightarrow E_3, f' : F_3 \longrightarrow L(E_1, F_2))$ verifying the naturality condition in definition 4.0.2. By the universal property of the algebraic tensor product on VEC, we have thus natural isomorphisms between $\text{CHU}((E_1, F_1) \otimes (E_2, F_2), (E_3, F_3))$ and pairs $(f : E_1 \longrightarrow L(E_2, E_3), f' : F_3 \otimes E_1 \longrightarrow F_2)$, which is exactly the definition of $\text{CHU}((E_1, F_1), L(E_2, F_2), (E_3, F_3)))$. The commutative diagrams required for monoidality follow from the one of VEC. $\square$

Proposition 4.0.5. *The category* CHU *is cartesian with product* $(E_1, F_1) \times (E_2, F_2) = (E_1 \times E_2, F_1 \oplus F_2)$. *The neutral for the cartesian product is then* $(\{0\}, \{0\})$.

Proof. The projections and factorisation of maps is straightforward from the algebraic product and co-product of vector spaces. ☐

Proposition 4.0.6. *The category* CHU *is* $*$*-autonomous with duality the functor*

$$(\cdot)^\perp : (E,F) \longrightarrow (F,E), (f,f') \longrightarrow (f',f).$$

We have then $(E,F)^\perp = L((E,F),(\mathbb{R},\mathbb{R}))$.

Proof. The functor $(\cdot)^\perp$ defines indeed an equivalence of categories between CHU and CHU^{op}. As we have $(E,F)^\perp = L((E,F),(\mathbb{R},\mathbb{R}))$, this isomorphism is precisely induced by the transpose of the evaluation $((E,F) \times L((E,F),(\mathbb{R},\mathbb{R})) \longrightarrow (\mathbb{R},\mathbb{R})$. ☐

A topological adjunction

Barr in [2] states a categorical interpretation for the Mackey-Arens theorem 3.5.3. Recall that this theorem says that amongst all topologies on a certain lcs E preserving the dual E', the weak topology (Section 3.3.2) is the coarsest one and the Mackey*-topology is the finest one (the Mackey topology on E is the topology of uniform convergence on weak*-compact and absolutely convex subsets of E', see Section 3.4.3).

Definition 4.0.7. We denote by $\mathcal{P}$ the functor from TOPVEC to CHU sending a lcs E on the pair (E,E'), and a linear continuous function $f : E \longrightarrow F$ on the pair (f,f') :, where $f' : \ell \in F' \mapsto (\ell \circ f) \in E')$.

Recall that, as all polar topologies, the weak and the Mackey topologies are functorial: they define an endofunctor on the category TOPVEC which is the identity on linear continuous maps. Indeed, if $f \in \mathcal{L}(E,F)$ then f is continuous from $E_w \longrightarrow F_w$ as for $\ell \in F'$ we have $\ell \circ f \in E'$ (see Proposition 3.3.16). Likewise f is continuous from E_μ to F_μ as the image by F' of a weak* compact in F' is weak* compact in E'. We thus define two functors from CHU to TOPVEC. When E is a lcs, $E_{\sigma(E,F)}$ and $E_{\mu(E,F)}$ are indeed locally convex and separated topological vector spaces [44, 8.4].

Definition 4.0.8. The functor $\mathcal{W}$ maps a dual pair (E,F) to the lcs $E_{\sigma(E,F)}$ and acts as the identity on morphisms. The functor $\mathcal{M}$ maps a dual pair (E,F) to the lcs $E_{\mu(E,F)}$.

Theorem 4.0.9. *The weak functor* $\mathcal{W}$ *is right adjoint to* $\mathcal{P}$ *while the Mackey functor* $\mathcal{M}$ *is left adjoint to* $\mathcal{P}$.

$$\begin{tikzcd} \text{MACKEY} \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \text{CHU} \arrow[l, bend left, "\mathcal{M}"] \arrow[r, bend left, "\mathcal{W}"] \arrow[r, phantom, "\perp"] & \text{WEAK} \arrow[l, bend left, "\mathcal{P}"] \end{tikzcd}$$

Proof. Consider E and F two lcs. By Proposition 3.3.16, the linear continuous functions from E_w to F_w are exactly the linear functions such that, in CHU $(f,f') : (E,E') \longrightarrow (F,F')$. Thus if E is a space already endowed with its weak topology and (E_2,F_2) a dual pair, one has the linear isomorphisms $\mathcal{L}(E, F_{2,\sigma(F_2,E_2)}) = L((E,E'),(E_2,F_2))$.

Let us show that $\mathcal{M}$ is left adjoint to $\mathcal{P}$ by using directly the Mackey-Arens Theorem 3.5.3. We know that the Mackey-topology on a lcs E is the finest one preserving the dual E'. As it preserves the dual, we have that from any function f in $\mathcal{L}(E_\mu, F)$ one deduces a linear function $f' : F' \longrightarrow (E_\mu)'$. Thus for any dual pair (E_1,F_1) and lcs F one has an injection $\mathcal{L}(E_{1,\mu(E_1,F_1)}, F) \subset L((E_1,F_1),(F,F'))$, by mapping f to the pair $(f, \ell \in F' \mapsto \ell \circ f)$. Let us reason by contradiction and consider some arrow (f,f') between the dual pairs (E_1,F_1) and (F,F') which does not correspond to a linear continuous function $E_{1,\mu(E_1,F_1)} \longrightarrow F$. Then consider on E_1 the topology τ consisting of the open sets of $\mu(E_1,F_1)$ and those generated by the reverse image by f of the open sets of F. Then by definition we have $f' : F' \longrightarrow E'_\tau$, thus $E'_\tau \subset F_1$. But as the topology τ is by definition finer than $\mu(E_1,F_1)$ we have $F_1 = (E_{1,\mu(E_1,F_1)})' \subset E'_\tau$. Thus τ is a topology finer than the Mackey topology with dual F_1 which is absurd. ☐

Following this adjunction, we could have described two models of MALL. Each one would consist of lcs with the weak (resp. Mackey) topology inherited from their dual. The dual of a construction of LL would be defined alongside this construction. For example, if A and B are two formulas of LL, we would define in WEAK:

$$[\![A \otimes B]\!] = ([\![A]\!] \otimes [\![B]\!])_{\sigma([\![A]\!] \otimes [\![B]\!], \mathcal{L}([\![A]\!], [\![B]\!]'_\sigma)}$$
$$[\![A \parr B]\!] = (\mathcal{L}([\![A]\!]'_\sigma, [\![B]\!]))_{\sigma(\mathcal{L}([\![A]\!]'_\sigma, [\![B]\!]), [\![A]\!]' \otimes [\![B]\!]')}$$

and likewise in the category MACKEY using the Mackey topology:

$$[\![A \otimes B]\!] = ([\![A]\!] \otimes [\![B]\!])_{\mu([\![A]\!] \otimes [\![B]\!], \mathcal{L}([\![A]\!], [\![B]\!]'_\mu)}$$
$$[\![A \parr B]\!] = (\mathcal{L}([\![A]\!]'_\mu, [\![B]\!]))_{\mu(\mathcal{L}([\![A]\!]'_\mu, [\![B]\!]), [\![A]\!]' \otimes [\![B]\!]')}$$

As this is easily extended to the additive, we would construct this way two models of MALL. However, *what we do in Chapters 5 and 6 adds more features and refines the semantics*.

The adjunction between weak spaces and Mackey spaces allows to give an equivalent of Proposition 3.5.17 for Mackey spaces:

Proposition 4.0.10. *Consider E and F two lcs endowed with their Mackey-topology. Then we have a linear homeomorphism $\mathcal{L}_\mu(E'_\mu, F) \simeq \mathcal{L}_\mu(F'_\mu, E)$.*

Proof. A linear function $f : E'_{\mu(E)} \longrightarrow F$ is continuous, if and only if $f : E' \longrightarrow F_w$ is continuous, if and only if $f : F' \longrightarrow E''_w$ is continuous, if and only if $f : F'_\mu \longrightarrow E$ is continuous by Section 3.5.3. Thus we have a linear isomorphism $f \longrightarrow f'$ between $\mathcal{L}(E'_\mu, F) \simeq \mathcal{L}(F'_\mu, E)$. This isomorphism is continuous: let V be an absolutely convex 0-neighbourhood in E, and W an absolutely convex weakly compact set in F'_μ. As F is Mackey W° is a 0-neighborhood on F. Likewise, as E is Mackey we can take $V = K^\circ$, where K is absolutely convex and weakly compact in E'_μ. Then if $f'(W) \subset V$ we have $f(V^\circ) \subset W^\circ$, and thus $f(K) \subset W^\circ$ and conversely. Thus $(\mathcal{W}_{K,V^\circ})' \subset \mathcal{W}_{W,V}$ and $f \mapsto f'$ is continuous.

□

Remark 4.0.11. In particular, any Mackey-dual is endowed with its Mackey-topology. Indeed, as

$$(E'_{\mu(E)})' \sim E$$

by proposition 3.5.2, we have that $E'_{\mu(E)}$, which is endowed by $\mu(E', E)$ by definition, is also endowed with its Mackey-topology $\mu(E', E'')$. With respect to what it done for weak topologies, the topology $\mu(E', E)$ on E' should be called the Mackey* topology.

Chapter 5

Weak topologies and formal power series

We consider the category $\textsc{Weak}$ of lcs endowed with their weak* topology $\sigma^*(E, E')$ and linear continuous functions between them. We show in the two first sections that $\textsc{Weak}$ is a model of MALL, and in the third Section we construct a co-monad generating formal power series on it, interpreting DiLL. In the last Section we state the fact that we have here a negative model of $\mathrm{DiLL}_{0,pol}$, according to the categorical definitions in Chapter 2. This chapter is mainly built from the contents and the form of [48]. It is thus more detailed than most of the thesis, and may provide a nice introduction to the interpretation of DiLL in topological vector spaces.

Notation 5.0.1. *We use the notations introduced in Chapter 3, that is $E \sim F$ denotes a linear isomorphisms between the vector spaces E and F, while $E \simeq F$ denotes a linear homeomorphism between the lcs E and F (that is, they have the same vectorial and topological structure).*

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5.1 Multiplicative and additives connectives

Consider $\mathcal{F}(E,\mathbb{C})$ some vector space of functions from E to $\mathbb{C}$ and $ev : E \to \mathcal{F}(E,\mathbb{K})'$. When $\mathcal{F}(E,\mathbb{K})$ contains only linear functions, ev is linear. When $E' \subset \mathcal{F}(E,\mathbb{K})$, ev is injective, as E' separates the points of E.

Notation 5.1.1. *We write the following function as* $ev^{\mathcal{F}(E,\mathbb{K})}$*:*

$$ev : \begin{cases} E \to \mathcal{F}(E,\mathbb{K})' \\ x \mapsto (ev_x : f \mapsto f(x)) \end{cases}$$

If there is no ambiguity in the context, we will write ev for $ev^{E'} : E \mapsto E''$.

5.1.1 Spaces of linear maps

Among the topologies one can define on a space of linear maps (see Section 3.4.14), there is a very coarse one which leads to the weak topology on the dual.

Definition 5.1.2. Let us denote $\mathcal{L}_\sigma(E,F)$ the space of all continuous linear maps between E and F, endowed with the topology of simple convergence on points of E.

A basis for the topology of simple convergence on $\mathcal{L}_\sigma(E,F)$ is the collection of all

$$W_{x_1,\dots x_n,V} = \{l \in \mathcal{L}_\sigma(E,F) | l(x_1) \in V, \dots, l(x_n) \in V\}$$

where $n \in \mathbb{N}$, $x_i \in E$ and V is a neighbourhood of 0 in F.

Remark 5.1.3. For all lcs E, the weak* topology on E' is exactly the topology of simple convergence on points of E, thus, $E'_{w*} \simeq \mathcal{L}_\sigma(E,\mathbb{K})$.

We denote by $E \otimes F$ the algebraic tensor product between two lcs E and F. Later on, we will endow the tensor product with a suitable topology.

The next Proposition is fundamental and will allow us to show that the space of linear maps, endowed with the topology of simple convergence, is already endowed with its weak topology. This is the main fact that will allow to see in WEAK a negative interpretation of DiLL_{pol}.

Proposition 5.1.4 ([44, 15.3.5],[52, 39.7])**.** *$\mathcal{L}_\sigma(E_w,F_w)'$ is algebraically isomorphic to $E \otimes_i F'$.*

Proof. We will sketch here the proof of Köthe [52], as the proof by Jarchow uses the projective tensor product. Consider first the space $L(E,F)$ of all linear and not necessarily continuous maps between E and F, endowed with the topology of simple convergence on points of E. If we choose an algebraic basis X of E, we have $L(E,F) \sim \prod_{x\in X} F_x$ where F_x is a copy of F, and where the product $\prod_{x\in X} F_x$ is endowed with the product topology. Thus $L(E,F)' \sim (\prod_{x\in X} F_x)' \sim \bigoplus_X F'_x$ (the dual of a cartesian product is the direct sum of the duals, see Proposition 5.1.19). Linear forms on $\bigoplus_X F'_x$ are exactly finite sums of linear forms in F', each one with a different domain $F_x = \{f(x) \mid f \in L(E,F)\}$. When we consider linear forms on $\bigoplus_X F'_x$ as elements of $L(E,F)'$, we write them as finite sums $\sum_{1\leqslant i\leqslant n} l_i \circ ev_{x_i}$ with $x_i \in X$ and $l_i \in F'$. Thus the following linear application is well-defined and surjective:

$$\begin{cases} E \otimes_i F' \to L(E,F)' \\ \sum_{1\leqslant i\leqslant n} (x_i \otimes_i l_i) \mapsto \sum_{1\leqslant i\leqslant n} l_i \circ ev_{x_i} \end{cases}$$

Köthe shows in detail in his proof why this morphism is injective, proving that $L(E,F)'$ is algebraically isomorphic to $E \otimes_i F'$.

Now let us get back to $\mathcal{L}_\sigma(E,F)$. This space is dense in $L(E,F)$ when it is endowed with the topology of simple convergence on E, as for every pairwise distinct $x_1,\dots x_n \in E$ and for every open set V in F we can find a continuous linear map f such that $f(x_i) \in V$. Indeed, without loss of generality, we suppose the family $\{x_i\}$ free. Select $y \neq 0 \in V$, and for every $i \leqslant n$ $l_i \in E'$ such that for every $j \leqslant n$ $l_i(x_j) = \mu_{i,j}$. The function $f : x \mapsto \sum_i l_i(x)y$ is linear continuous, and sends x_i on y. Thus for any neighbourhood $\mathcal{W}_{x_1,\cdot,x_n,V}$ in $L(E,F)$, we have a function $f \in \mathbf{Lin}(E,F) \cap V$, and thus we proced the desired density.

Thus the dual of $\mathcal{L}_\sigma(E,F)$ is algebraically isomorphic to the dual of $L(E,F)$, that is to $E \otimes_i F'$.

□

This Proposition allows us to write every linear function $f \in \mathcal{L}_\sigma(E,F)'$ as a unique finite sum

$$f = \sum_{i=1}^{n} l_i \circ ev_{x_i}$$

where $l_i \in F'$ and $x_i \in E$. Let us now recall how linear functions behave with respect to weak topologies.

Lemma 5.1.5. Functions in $\mathcal{L}_\sigma(E,F_w)$ are exactly the linear maps from E to F which, when postcomposed with any map from F', results in a map into E'.

Proof. By definition of the weak topology on F, a function $f : E_w \to F_w$ is continuous if and only if for every $l \in F'$ $f \circ l : E \to \mathbb{K}$ is continuous. If f is linear, this means that $f \circ l \in E'$. □

The following proposition follows from Proposition 3.3.16:

Proposition 5.1.6. *For all E, F lcs, we have $\mathcal{L}(E,F_w) \sim \mathcal{L}(E_w,F_w)$, and thus $\mathcal{L}_\sigma(E,F_w) \simeq \mathcal{L}_\sigma(E_w,F_w)$*

Proof. A continuous linear map from E_w to F_w is continuous from E to F_w, as the weak topology is coarser than the initial topology on E. Consider now $f \in \mathcal{L}_\sigma(E,F_w)$. For every $l \in F'$ we have $f \circ l \in E'$, thus $f \circ l \in (E_w)'$. By the preceding lemma, we have $f \in \mathcal{L}_\sigma(E_w,F_w)$. □

5.1.2 Tensor and cotensor

Various ways exist to create a lcs from the tensor product of two lcs E and F. That is, several topologies exist on the vector space $E \otimes_i F$, the most prominent in the literature being the projective topology $\otimes_\pi$ [44, III.15] and the injective topology $\otimes_\varepsilon$ [44, III.16] which are recalled in Section 3.6 in Chapter 3. These topologies behave particularly well with respect to the completion of the tensor product, and were originally studied in Grothendieck's thesis [36].

However, we would like a topology on $E \otimes_i F$ that would endow WEAK with a structure of symmetric monoidal closed category. This is mainly why we use the inductive tensor product [36, I.3.1]. So as to define this topology, we need to mention the topological product of two lcs.

The tensor product

Definition 5.1.7. Consider E and F two lcs. $E \times F$ is the algebraic cartesian product of the two vector spaces, endowed with the product topology, that is the coarsest topology such that the projections $p_E : E \times F \to E$ and $p_F : E \times F \to F$ are continuous.

Neighbourhoods of 0 in $E \times F$ are generated by the sets $U \times V$, where U is a 0-neighbourhood in E and V is a 0-neighbourhood in F.

Definition 5.1.8. Let us recall that we denote by $\mathscr{B}(E \times F, G)$ the space of all bilinear and separately continuous functions from $E \times F$ to G. We endow it with the topology of simple convergence on $E \times F$. The vector space $\mathscr{B}(E \times F, G)$ is then a lcs.

Proposition 5.1.9. *Consider E, F and G three lcs, and f a bilinear map from $E \times F$ to G. Then $f \in \mathscr{B}(E \times F, G_w)$ if and only if for every $l \in G'$, $l \circ f \in \mathscr{B}(E \times F)$.*

Definition 5.1.10. Consider E and F two lcs. We endow $E \otimes F$ with the inductive topology , which is the finest topology making the canonical bilinear map $E \times F \to E \otimes_i F$ separately continuous. We denote this topological vector space $E \otimes_i F$.

Proposition 5.1.11. *[36, I.3.1.13] For every lcs G, we have $\mathcal{L}(E \otimes_i F, G) \sim \mathscr{B}(E \times F, G)$. Especially, $(E \otimes_i F)' \sim \mathscr{B}(E \times F)$.*

Proof. Let us write $B(E \times F, G)$ for the vector space of all bilinear maps from $E \times F$ to G. As $E \times F \to E \otimes_i F$ is separately continuous, the canonical isomorphism $L(E \otimes_i F, G) = B(E \times F, G)$ induces an injection from $\mathcal{L}_\sigma(E \otimes_i F, G)$ to $\mathscr{B}(E \times F, G)$. Let us show by contradiction that this injection is onto. Consider $f \in \mathscr{B}(E \times F, G)$ such that its linearisation $\tilde{f} \in L(E \otimes_i F, G)$ is not continuous. Let us denote $E \otimes_\tau F$ the vector space $E \otimes_i F$ endowed with the coarsest topology τ making $\tilde{f}$ continuous. Then, because f is separately continuous, $E \times F \to E \otimes_\tau F$ is separately continuous. Thus τ is coarser than the inductive topology. This would implies that $\tilde{f} : E \otimes_i F \to G$ would be continuous. We have a contradiction. □

Proposition 5.1.12 (Associativity of $\otimes$ in WEAK)**.** *Consider E, F, and G three lcs. Then*

$$(E_w \otimes_i (F_w \otimes_i G_w)_w)_w \simeq ((E_w \otimes_i F_w)_w \otimes_i G_w)_w.$$

Proof. As the algebraic tensor product is associative we have $(E_w \otimes_i (F_w \otimes_i G_w)_w)_w = ((E_w \otimes_i F_w)_w \otimes_i G_w)_w$. Let us show that the two spaces bear the same topology. The dual of the first space is $(E_w \otimes_i (F_w \otimes_i G_w)_w)' = \mathcal{B}(E_w \times (F_w \otimes_i G_w)_w)$ according to Proposition 5.1.11. One can show as for Proposition 5.1.11 that $\mathcal{B}(E_w \times (F_w \otimes_i G_w)_w)$ coincides to the space of all trilinear separately continuous functions on $E_w \times F_w \times G_w$. Likewise, the dual of the second space is $((E_w \otimes_i F_w)_w \otimes_i G_w) = \mathcal{B}((E_w \otimes_i F_w)_w \times G_w)$, which coincides also to the space of all trilinear separately continuous functions on $E_w \times F_w \times G_w$. Then $(E_w \otimes_i (F_w \otimes_i G_w)_w)_w$ and $((E_w \otimes_i F_w)_w \otimes_i G_w)_w$ are algebraically isomorphic and have the same dual, thus the same weak topology. □

If $f \in \mathcal{L}(E,F)$, $g \in \mathcal{L}(G,H)$, then one defines $f \otimes_i g \in \mathcal{L}(E \otimes_i G, F \otimes_i H)$ on basic elements as $(f \otimes_i g)(x \otimes_i y) = f(x) \otimes_i g(y)$, and then extends it by linearity. The associativity mapping obviously satisfies the coherence diagrams for a monoidal category [56, VII.1].

Monoidal closedness

Proposition 5.1.13. *Consider E, F and G three lcs. Then we have*

$$\mathcal{L}((E_w \otimes_i F_w)_w, G_w) \sim \mathscr{B}(E_w \times F_w, G_w).$$

Proof. A map f lies in $\mathcal{L}_\sigma((E_w \otimes_i F_w)_w, G_w)$ if and only if for every $l \in G'$, $l \circ f \in (E_w \otimes_i F_w)'$. But according to Proposition 5.1.11, we have $(E_w \otimes_i F_w)' = \mathscr{B}(E_w \times F_w)$. Thus $f \in \mathcal{L}_\sigma((E_w \otimes_i F_w)_w, G_w)$ if and only if the bilinear map corresponding to f is in $\mathscr{B}(E_w \times F_w, G_w)$. □

Proposition 5.1.14. *Consider E, F and G three lcs. Then we have*

$$\mathscr{B}(E_w \times F_w, G_w) \sim \mathcal{L}(E_w, \mathcal{L}_\sigma(F_w, G_w)_w).$$

Proof. Remember from Proposition 5.1.4 that $\mathcal{L}_\sigma(F_w, G_w)' = F \otimes_i G'$. Consider g a continuous linear function from E_w to $\mathcal{L}_\sigma(F_w, G_w)_w$. As the codomain of g is $\mathcal{L}_\sigma(F_w, G_w)$, we have that for $x \in E$ fixed, for all $l \in G'$, $y \mapsto l(g(x)(y))$ is continuous. To be continuous g must satisfy that for y and $l \in G'$ both fixed, $x \mapsto l(g(x)(y))$ is continuous. Consider $l \in G'$ fixed. We see that $l \circ g$ corresponds to a separately continuous map in $\mathscr{B}(E_w \times F_w)$. Thus g can be seen as a function $\tilde{g}$ in $\mathscr{B}(E_w \times F_w, G_w)$. The transformation of a map in $\mathscr{B}(E_w, F_w \times G_w)$ into a map of $\mathcal{L}_\sigma(E_w, \mathcal{L}_\sigma(F_w, G_w)_w)$ is done likewise. □

Thus we have an algebraic isomorphism between $\mathcal{L}_\sigma(E_w, \mathcal{L}_\sigma(F_w, G_w)_w)$ and $\mathcal{L}_\sigma((E_w \otimes_i F_w)_w, G_w)$. To show that they bear the same weak topology, we just have to show that they have the same dual. But according to Proposition 5.1.4, $\mathcal{L}_\sigma(E_w, \mathcal{L}_\sigma(F_w, G_w)_w)' = E_w \otimes_i \mathcal{L}_\sigma(F_w, G_w)' = E_w \otimes_i F_w \otimes_i G'_w = \mathcal{L}_\sigma((E_w \otimes_i F_w)_w, G_w)'$.

Theorem 5.1.15. *The category* WEAK *is symmetric monoidal closed, as we have for each lcs E_w, F_w, G_w:*

$$\mathcal{L}_\sigma(E_w, \mathcal{L}_\sigma(F_w, G_w)_w) \simeq \mathcal{L}_\sigma((E_w \otimes_i F_w)_w, G_w)_w$$

naturally in E and G.

The parr

Definition 5.1.16. The $\parr$ connective of linear logic is interpreted by the lcs $E \parr F := \mathcal{L}_\sigma(E', F)$.

Proposition 5.1.17. *The $\parr$ connective preserves the weak topology: indeed, for every lcs E and F, $(E \parr F)_w \simeq E_w \parr F_w$.*

Proof. As $E \parr F \simeq (E' \otimes_i F')'$, the result follows immediately from the fact that a dual endowed with its weak topology is also endowed with its weak* topology. □

5.1.3 A $*$-autonomous category

According to Theorem 5.1.15, WEAK is a symmetric monoidal closed category, with $ev_E : E \to E'' \simeq \mathcal{L}_\sigma(E, \mathcal{L}_\sigma(E, \mathbb{K}))$ being an isomorphism in this category for every object E by proposition 3.3.15. The use of weak topologies gives use a model of the classical part of Linear Logic, that is a $*$-autonomous category [1].

Theorem 5.1.18. WEAK *is a $*$-autonomous category, with dualizing object* $\mathbb{K}$.

Proof. Let us take $\mathbb{K} = \perp = 1$ the dualizing object. Then the evaluation map

$$(A \multimap \perp) \otimes_i A \to \perp$$

leads by symmetry of $\otimes$ and closure exactly to $ev : A \to ((A \multimap \perp) \multimap \perp$, that is $ev : A \to A''$. As shown in Proposition 3.3.15, $ev : A \to A''$ is an isomorphism in the category WEAK, and WEAK is $*$-autonomous. $\square$

5.1.4 Additive connectives

The additive connectives of linear logic are of course interpreted by the binary product and co-product between lcs. Finite product and co-product coincide (see Section 3.1.4). However, one infinite indexes, they behave differently with respect to weak topology: the product preserves the weak topology, while the direct sum doesn't. See Proposition 5.1.21 and Section 5.3 for an interpretation of this phenomenon in terms of polarities. This section completes the definitions and results exposed in Chapter 3, Section 3.1.4 on product and co-product, and discusses especially duality and weak topologies on (co)-products. Consider $(E_i)_i$ a family of lcs indexed by a set I. We denote by $E'_{i,w}$ the dual of E_i endowed with its weak topology.

Recall from Proposition 3.1.18 that an index I of any family of lcs $(E_i)_{i \in I}$ is finite if and only if the canonical injection from $\bigoplus_{i \in I} E_i$ to $\prod_{i \in I} E_i$ is surjective.

Proposition 5.1.19. *For any index I and all lcs E_i $(\bigoplus_{i \in I} E_i)' \sim \prod_{i \in I} E'_i$ and $(\prod_{i \in I} E_i)' \sim \bigoplus_{i \in I} E'_i$.*

Proof. Consider $l \in \prod_{i \in I} E'_i$. Then the function $x \in \bigoplus_{i \in I} E_i \mapsto \sum_i l_i(x_i)$ is well defined, linear and continuous. Reciprocally, to any $l \in (\bigoplus_{i \in I} E_i)'$ coincides the sequence $(l_i) \in \prod_{i \in I} E'_i$ with $l_i = l \circ I_i$.

Consider now $l \in (\prod_{i \in I} E_i)'$. Then by definition of the product topology, $l_i = l_{|E_i} \in E'_i$. As l is continuous, there is $H \subset I$ finite, and 0-neighbourhoods U_i for $i \in H$ such that

$$l(\prod_{i \in H} U_i \times \prod_{i \notin H} E_i) \subset \{\lambda \in \mathbb{K} | \, |\lambda| < 1\}.$$

By homogeneity, $l_i = 0$ for $i \notin H$, and l coincides to an element of $\bigoplus E'_i$. Conversely, an element of $\bigoplus E'_i$ acts on $\prod_i E_i$ as a continuous linear form. $\square$

Proposition 5.1.20 ([44, II.8.8 Theorem 5 and Theorem 10])**.** *We have always $(\prod_{i \in I} E_i)_w \simeq \prod_{i \in I} (E_i)_w$, but $(\bigoplus_{i \in I} E_i)_w \simeq \bigoplus_{i \in I} (E_i)_w$ holds only when I is finite.*

We can now characterize the dual of a product and of a direct sum in the category WEAK.

Proposition 5.1.21. *We have always $(\bigoplus_{i \in I} E_i)'_w \simeq \prod_{i \in I} E'_{i,w}$ but $(\prod_{i \in I} E_i)'_w \simeq \bigoplus_{i \in I} E'_{i,w}$ holds only when I is finite.*

Proof. According to Proposition 5.1.19 we have $(\bigoplus_{i \in I} E_i)' = \prod_{i \in I} E'_i$. The first bears the weak topology induced by $\bigoplus_{i \in I} E_i$, that is $(\bigoplus_{i \in I} E_i)' \simeq (\prod_{i \in I} E'_i)_w$ and the second bears the product topology induced by all the E_i, that is $\prod_{i \in I} E'_i \simeq \prod_{i \in I} (E'_i)_w$. The previous Proposition gives us a linear homeomorphism between the two. Likewise, we have $(\prod_{i \in I} E_i)' = \bigoplus_{i \in I} E'_i$, $(\prod_{i \in I} E_i)' \simeq (\bigoplus_{i \in I} E'_i)_w$ and $\bigoplus_{i \in I} E'_i \simeq \bigoplus_{i \in I} (E'_i)_w$. Proposition 5.1.20 tells us that $(\prod_{i \in I} E_i)'_w \simeq \bigoplus_{i \in I} E'_{i,w}$ if and only if I is finite. $\square$

5.2 A quantitative model of Linear Logic

The structure presented here is very algebraic, Weak spaces providing us with practically no tools except the Hahn-Banach theorem. In particular, as they satisfy no completeness condition, one cannot work easily with the notion of converging power series. Power series are converging sums of monomials, and convergence in topological vector spaces is mainly possible thanks to completeness[1]. This is why we simply chose to represent non-linear maps as finite sequences over $\mathbb{N}$ of n-monomials. We also explore another possible exponential, inspired by what happens in the theory of formal power series, in Section 5.2.5.

Outlook 5. Can we prove that compositionality or cartesian closedness on a category of lcs as objects and power series as arrows implies the Mackey-completeness[2] of the lcs?

The exponential we define here has a lot of similarities with the free symmetric algebra studied by Mellies, Tabareau and Tasson [63]. The difference here is that we consider sequences of monomials in the co-Kleisli category and not n-linear symmetric maps. Therefore our exponential is the direct sum over $n \in \mathbb{N}$ of the dual spaces of the spaces of n-monomials, and not a direct sum of symmetric n-tensor product of A.

5.2.1 The exponential

Monomials

Definition 5.2.1. $\mathcal{L}^n(E,F)$ is the space of symmetric n-linear separately continuous functions from E^n to F, We write $L_n(E,F)$ for the space of all symmetric n-linear maps from E to F.

An n-monomial from E to F is a function $f : E \to F$ such that there is $\hat{f} \in \mathcal{L}^n(E,F)$ verifying that for all $x \in E$ $f(x) = \hat{f}(x,...,x)$. It is symmetric when for every permutation $\sigma \in \mathbf{S}_n$, for every $x_1,..x_n \in E$ we have $f(x_{\sigma(1)},...,x_{\sigma(n)}) = f(x_1,...,x_n)$.

Proposition 5.2.2 (The Polarization formula [53, 7.13])**.** *Consider f a n-monomial from E to F. Then we have $f(x) = \hat{f}(x,...,x)$ where $\hat{f}$ is a symmetric n-linear function from E to F defined by:*

For every $x_1,...x_n \in E$, $\hat{f}(x_1,...,x_n) = \frac{1}{n!}\sum_{d_1,...,d_n=0}^{1}(-1)^{n-\sum_k d_k} f(\sum_k d_k x_k)$.

Thus the sum in the polarization formula is indexed by the subsets of $[1,n]$. Another way to write it would be the following:

For every $x_1,...x_n \in E$, $\hat{f}(x_1,...,x_n) = \frac{1}{n!}\sum_{I\subset[\![1,n]\!]}^{1}(-1)^{n-\mathrm{card} I} f(\sum_{k\in I} x_k)$.

Proof. Let us write for the multinomial coefficient:

$$\binom{n}{k_1,k_2,\ldots,k_m} = \frac{n!}{k_1!\,k_2!\cdots k_m!} = \binom{k_1}{k_1}\binom{k_1+k_2}{k_2}\cdots\binom{k_1+k_2+\cdots+k_m}{k_m}.$$

for every $x_1,...,x_n$, we have

$$f(\sum_{j=1}^{n} x_j) = \sum_{j_1+...+j_n=n}\binom{n}{k_1,k_2,\ldots,k_n}\hat{f}(\underbrace{x_1,...,x_1}_{k_1 \text{ times}},...,\underbrace{x_n,...,x_n}_{k_n \text{ times}})$$

Thus

$$f(\sum_j d_j x_j) = \sum_{j_1+...+j_n=n} d_1^{k_1}...d_n^{k_n}\binom{n}{k_1,k_2,\ldots,k_n}\hat{f}(\underbrace{x_1,...,x_1}_{k_1 \text{ times}},...,\underbrace{x_n,...,x_n}_{k_n \text{ times}})$$

and

$$\begin{aligned}
&\frac{1}{n!}\sum_{d_1,...,d_n=0}^{1}(-1)^{(n-\sum_j d_j)} f(\sum_j d_j x_j)\\
&= \sum_{d_1,...,d_n=0}^{1}\sum_{j_1+...+j_n=n}(-1)^{(n-\sum_j d_j)} d_1^{k_1}...d_n^{k_n}\frac{1}{j_1!\cdots j_n!}\hat{f}(\underbrace{x_1,...,x_1}_{k_1 \text{ times}},...,\underbrace{x_n,...,x_n}_{k_n \text{ times}})\\
&= \sum_{j_1+...+j_n=n}\frac{1}{j_1!\cdots j_n!}\hat{f}(\underbrace{x_1,...,x_1}_{k_1 \text{ times}},...,\underbrace{x_n,...,x_n}_{k_n \text{ times}})\sum_{d_1,...,d_n=0}^{1}(-1)^{(n-\sum_j d_j)} d_1^{k_1}...d_n^{k_n}
\end{aligned}$$

[1] It appears that the weakest completeness condition necessary to model quantitative linear logic should be Mackey completeness [49].

[2] A very weak completeness condition, studied in sections 2.4.3 and 6, which was shown to be enough for power series in $\mathbb{C}$-vector spaces in [49].

Let us show that $\sum_{d_1,...,d_n=0}^{1}(-1)^{n-\sum_j d_j} d_1^{k_1}...d_n^{k_n}$ is non-zero if and only if $k_1 = \cdots = k_n = 1$. Indeed, if there is an i such that $k_i > 1$, then there is j such that $k_j = 0$, as $k_1 + ... + k_n = n$. Let us suppose $k_1 = 0$. Then

$$\begin{aligned}\sum_{d_1,...,d_n=0}^{1}(-1)^{(n-\sum_j d_j)} d_1^{k_1}...d_n^{k_n} &= \sum_{d_2,...,d_n=0}^{1}(-1)^{(n-1-d_2-...-d_n)} d_2^{k_2}...d_n^{k_n} \\ &+ \sum_{d_2,...,d_n=0}^{1}(-1)^{n-d_2-...-d_n} d_2^{k_2}...d_n^{k_n} \\ &= 0\end{aligned}$$

□

Thus $\frac{1}{n!}\sum_{d_1,...,d_n=0}^{1}(-1)^{(n-\sum_j d_j)} f(\sum_j d_j x_j) = \hat{f}(x_1,...,x_n)$.

Definition 5.2.3. Let us write $\mathcal{H}^n(E,F)$ for the space of n-monomials over E endowed with the topology of simple convergence on points of E. For every lcs E and F, $\mathcal{H}^n(E,F)$ is a lcs.

As a consequence of the polarization formula 5.2.2, we know that there is is a unique symmetric n-linear map $\hat{f}$ associated to a n-monomial f.

Corollary 5.2.4. There is a bijection between $\mathcal{H}^n(E,F)$ and $\mathcal{L}^n(E,F)$.

As we will endow $\mathcal{H}^n(E,F)$ with its weak topology, we need to get a better understanding of its dual. To do so, we retrieve information from the dual of $\mathcal{L}^n_\sigma(E,F)$.

Proposition 5.2.5. *For every lcs E and F, for every $n \in \mathbb{N}$, we have*

$$\mathcal{H}^n(E,F) \simeq \mathcal{L}^n_\sigma(E,F).$$

Proof. The algebraic isomorphism between the two vector spaces follows from the previous corollary, as the function mapping a n-linear symmetric mapping to the corresponding n-monomial is clearly linear. As they are both endowed with the topology of simple convergence of points of E (resp $E \times ... \times E$), this mapping is bicontinuous. □

Notation 5.2.6. *We write $E_s^{\otimes^n}$ for the symmetrized n^{th}-tensor product of E with itself[3]. We denote by $\mathcal{L}^n_s(E,F)$ the vector space of all n-linear symmetric functions from E to F. Thus*

$$\mathcal{L}^n_s(E,F)_w \simeq \mathcal{L}_\sigma(E_s^{\otimes^n},F).$$

As also we have $\mathcal{H}^n(E_w,F_w) \simeq \mathcal{H}^n(E,F_w) \simeq \mathcal{L}^n_\sigma(E,F_w)$ by Proposition 5.2.5, the dual of $\mathcal{H}^n(E,F)'$ is the dual of $\mathcal{L}_\sigma(E_s^{\otimes^n},F)$. Proposition 5.1.4 thus gives us a way to compute it:

Proposition 5.2.7. *For every lcs E and F, $\mathcal{H}^n(E,\mathbb{K})' = E_s^{\otimes^n} \otimes_i F'$. That is, every continuous linear form θ on $\mathcal{H}^n(E,F)$ can be written as a finite sum of functions of the type $l \circ ev_{x_1 \otimes ... \otimes_i x_n}$ with $l \in F'$ and $x_1,...x_n \in E$.*

From this, we deduce that $\mathcal{H}^n(E_w,F_w)$ is a weak space: it is already endowed with its weak topology.

Corollary 5.2.8. For every lcs E and F, we have that $\mathcal{H}^n(E_w,F_w)_w \simeq \mathcal{H}^n(E,F_w) \simeq \mathcal{H}^n(E_w,F_w)$.

Proof. The topology on $\mathcal{H}^n(E_w,F_w)$ is the topology of simple convergence on $E_{sym}^{\otimes^n}$, with weak convergence on F. This is exactly the topology induced by its dual $E_{sym}^{\otimes^n} \otimes_i F'$. □

[3] That is, the vector space $E \otimes_i \cdots \otimes_i E$, quotiented by the equivalence relation $x_1 \otimes_i ... \otimes_i x_n \equiv x_{\sigma(1)} \otimes_i \cdots \otimes_i x_{\sigma(n)}$ for all $\sigma \in \mathbf{S}_n$.

The exponential The exponential $! : \textsc{Weak} \to \textsc{Weak}$ is defined as a functor on the category of linear maps. In Equation 2.7, we detailed how the exponential in a model with smooth functions should be interpreted by a space of distributions. The same reasonning applies here. Indeed, suppose we want non-linear proofs $E \Rightarrow F$ to be interpreted in some space of functions $\mathcal{F}(E, F)$. As the category of weak spaces and these functions is the co-Kleisli category $\textsc{Weak}_!$, we have:

$$\begin{aligned}(!E)_w &\simeq ((!E)_w)'' \\ &\simeq \mathcal{L}_\sigma(!E, \mathbb{K})' \\ &\simeq \mathcal{F}(E, \mathbb{K})'\end{aligned}$$

As we want our non-linear proofs to be interpreted by sequences of monomials, the definition of $!E$ is straightforward.

Definition 5.2.9. Let us define $!E$ as the lcs $\bigoplus_{n\in\mathbb{N}} \mathcal{H}^n(E, \mathbb{K})'$.

As usual, we need to endow $!E$ with its weak topology.

Proposition 5.2.10. *We have $(!E)' = \prod_n \mathcal{H}^n(E, \mathbb{K})$, and thus $(!E)_w \simeq (\prod_n \mathcal{H}^n(E, \mathbb{K}))'$.*

Proof. According to Proposition 5.1.19, we have that

$$(!E)' = \prod_n \mathcal{H}^n(E, \mathbb{K})'' = \prod_n \mathcal{H}^n(E, \mathbb{K}).$$

Thus, $(!E)' \simeq (\prod_n \mathcal{H}^n(E, \mathbb{K}))_w$, as both spaces in this equality are endowed by the topology of simple convergence on $!E$. Then $(!E)' \simeq \prod_n \mathcal{H}^n(E, \mathbb{K})_w \simeq \prod_n \mathcal{H}^n(E, \mathbb{K})$. Taking the dual of these spaces, we get $!E_w \simeq (\prod_n \mathcal{H}^n(E, \mathbb{K}))'$. ∎

As in spaces of linear functions, see Proposition 5.1.6, we have always that $\mathcal{H}^n(E, F_w) \simeq \mathcal{H}^n(E_w, F_w)$. Thus $!(E_w) \simeq \bigoplus_{n\in\mathbb{N}} \mathcal{H}^n(E_w, \mathbb{K})' \simeq \bigoplus_{n\in\mathbb{N}} \mathcal{H}^n(E, \mathbb{K})' \simeq !E$.

Notation 5.2.11. *We will write without any ambiguity $!E$ for $!(E_w)$ and $!E_w$ for $(!E)_w$.*

Definition 5.2.12. For $f \in \mathcal{L}_\sigma(E_w, F_w)$ we define

$$!f : \begin{cases} !E_w \to !F_w \\ \quad \phi \mapsto ((g_n) \in \prod_n \mathcal{H}^n(F, \mathbb{K}) \mapsto \phi((g_n \circ f)_n) \end{cases}$$

Proposition 5.2.13. *This makes $!$ a covariant functor on* $\textsc{Weak}$.

Proof. One has immediatly that for any lcs E, $!Id_E = Id_{ocE}$. Now consider three lcs E, F and G, and two linear continuous maps $f \in \mathcal{L}_\sigma(E_w, F_w)$ and $gf \in \mathcal{L}_\sigma(F_w, G_w)$. Then by definition, for $\phi \in !E$ one has :

$$!(g \circ f)(\phi) = (h_n) \in \prod_n \mathcal{H}^n(G, \mathbb{K}) \mapsto \phi((h_n \circ g \circ f)_n).$$

On the other hand, one has immediatly:

$$\begin{aligned}!g \circ !f(\phi) = (h_n) \in \prod_n \mathcal{H}^n(G, \mathbb{K}) &\mapsto (!f(\phi))((h_n \circ g)_n) \\ (h_n) &\mapsto \phi((h_n \circ g \circ f)_n.\end{aligned}$$

∎

Arithmetic of the composition $\circ_!$ We now would like to endow ! with its co-monadic structure whose structure is based on a good notion of composition in the co-Kleisli category $\textsc{Weak}_!$. For $f \in \prod_m \mathcal{H}^m(E,F)$ and $g \in \prod_n \mathcal{H}^n(F,G)$, we would like to define $f \circ_! g \in \prod_p \mathcal{H}^p(E,G)$ as

$$(g \circ_! f)_p = \sum_{k|p} g_k \circ f_{\frac{p}{k}}.$$

Proposition 5.2.14. *The operation $\circ_! : \prod_m \mathcal{H}^m(E,F) \times \prod_n \mathcal{H}^n(E,G) \longrightarrow \prod_p \mathcal{H}^p(E,G)$ is indeed a commutative and associative operation.*

Proof. Commutativity is immediate. For formal power series f, g, and h one has :

$$\begin{aligned}
((f \circ_! g) \circ_! h)_p &= \sum_{k|p} (f \circ_! g)_k \circ h_{\frac{p}{k}} \\
&= \sum_{k|p} \sum_{j|k} f_j \circ g_{\frac{k}{j}} \circ h_{\frac{p}{k}} \\
&= \sum_{j|p} f_j \circ \left(\sum_{j|k, k|p} g_{\frac{k}{j}} \circ h_{\frac{p}{k}} \right) \\
&= \sum_{j|p} f_j \circ \left(\sum_{j|k, k|p} g_{i=\frac{k}{j}} \circ h_{\frac{p}{ij}} \right) \text{ renaming } i = \frac{k}{j} \\
&= \sum_{j|p} f_j \circ \left(\sum_{i|\frac{p}{j}} g_i \circ h_{\frac{p}{ij}} \right) \\
&= \sum_{j|p} f_j \circ (g \circ_! h)_{\frac{p}{j}}.
\end{aligned}$$

□

Remark 5.2.15. At this point we must pay attention to the arithmetic employed here. So as to avoid infinite sums and a diverging term for $(g \circ f)_0$[4], we allow for only 0 to divide 0. Thus $(g \circ f)_0 = g_0 \circ f_0$.

Let us detail what happens when $E = F = G = \mathbb{R}$. Then we are in presence of *formal* power series $f = (x \mapsto a_n x_n)_n$ and $g = (y \mapsto b_n y^n)$. Then

$$(g \circ_! f)_p : z \mapsto (\sum_{k|p} b_k a_{\frac{p}{k}}) z^p.$$

Beware that even in the case of finite sums, this composition does not behave as the traditional composition between functions from $\mathbb{R}$ to $\mathbb{R}$. If we consider $f : x \mapsto x + x^2$ and $g : y \mapsto y^2$, we have $g \circ_! f : z \mapsto x^2 + x^4$, while as functions of $\mathbb{R}$ one has $g \circ f = x^2 + 2x^3 + x^4$.

Remark 5.2.16. Another composition of Formal power series, which coincide with the composition of real functions for converging power series, is given by the Faa di Bruno formula and detailed in Section 5.2.5.

The co-monadic structure

Theorem 5.2.17. *The functor $! : Lin \to Lin$ is a co-monad. Its co-unit $d : ! \to 1$ is defined by*

$$d_E \begin{cases} !E_w \to E_w \\ \quad \phi \mapsto \phi_1 \in E'' \simeq E \end{cases}$$

The co-unit is the operator extracting from $\phi \in !E$ its part operating on linear maps. The co-multiplication $\mu : ! \to !!$ is defined as:

[4] The problem of the possible divergence of the nonzero term can be found also in the theory of formal power series [38, IV.4], where composition is only allowed for series with no constant component.

$$\mu_E \begin{cases} !E_w \simeq (\prod_p \mathcal{H}^p(E,\mathbb{K}))' \to !!E_w \simeq \left(\prod_n \mathcal{H}^n([\prod_m \mathcal{H}^m(E,\mathbb{K})]',\mathbb{K})\right)' \\ \phi \in (\prod_p \mathcal{H}^p(E,\mathbb{K}))' \mapsto \left[(g_n)_n \mapsto \phi(\left(x \in E \mapsto \sum_{k|p} g_k[(f_m)_m \mapsto f_{\frac{p}{k}}(x)] \right)_p) \right] \end{cases}$$

We have indeed $(f_m)_m \mapsto f_{\frac{p}{k}}(x) \in [\prod_m \mathcal{H}^m(E,\mathbb{K})]'$ for all $x \in E$ and k dividng p. As explained before, we want to have on our co-Kleisli category a composition such that $(g \circ f)_p = \sum_{k|p} g_k \circ f_{\frac{p}{k}}$. The co-multiplication $\mu : ! \to !!$ can be seen as a continuation-passing style transformation of this operation. Indeed, consider $\phi \in !E$. We construct $\mu(\phi)$ as a function in $(\prod_n \mathcal{H}^n(!E,\mathbb{K}))'$ mapping a sequence $(g_n)_n$ to ϕ applied to the sequences of p-monomials on E defined as

$$x \in E \mapsto \sum_{k|p} g_k[(f_m)_m \mapsto f_{\frac{p}{k}}(x)].$$

This co-multiplication corresponds indeed to the composition $\circ_!$ between power series: if $f \in \prod_n \mathcal{H}^n(E,F)$ and $g \in \prod_n \mathcal{H}^n(F,G)$, then:

$$g \circ_! f = g \circ !f \circ \mu \tag{5.1}$$

So as to show that ! is in fact a co-monad, we need to understand better the elements of $!E$. The space $!E$ is defined as $\oplus_n \mathcal{H}^n(E,\mathbb{K})'$, so $\phi \in !E$ can be described as a finite sum $\phi = \sum_{n=1}^N \phi_n$ with $\phi_n \in \mathcal{H}^n(E,\mathbb{K})'$. The proofs presented below are based more on the idea of non-linear continuations than on a combinatoric point of view. The space $!E_w = (\prod_p \mathcal{H}^p(E,\mathbb{K}))'$ can be thought of as a space of quantitative-linear continuations, $\mathbb{K}$ being the space of the result of a computation.

Proof. We have to check the two equations of a co-monad, that is:

1. $d_! \mu = (!d)\mu = Id_!$
2. $\mu_! \mu = (!\mu)\mu$

• Let us detail the computations of the first equation. Remember that we write $ev^{\mathcal{H}^n(F,\mathbb{K})}$ for $ev : F \mapsto \mathcal{H}^n(F,\mathbb{K})'$. For every $\phi = \sum \phi_p \in !E$, we have:

$$\begin{aligned} d_{!E}\mu_E(\phi) &= d_{!E}\left((g_n)_n \in \prod \mathcal{H}^n(!E,\mathbb{K}) \mapsto \phi\left([x \in E \mapsto \sum_{k|p} g_k((f_m)_m \in !E \mapsto f_{\frac{p}{k}}(x))]_p \right)\right) \\ &= d_{!E}\left((g_n)_n \mapsto \phi\left([x \in E \mapsto \sum_{k|p} g_k(ev_x^{\mathcal{H}^{p/k}(E,\mathbb{K})})]_p \right)\right) \\ &= d_{!E}\left((g_n)_n \mapsto \sum_p \phi_p(x \mapsto \sum_{k|p} g_k(ev_x^{\mathcal{H}^{p/k}(E,\mathbb{K})}))\right) \end{aligned}$$

As $d_{!E}$ maps a function in $!E_w \simeq (\prod_n \mathcal{H}^n(E,\mathbb{K}))'$ to its restriction to $\mathcal{L}_\sigma(E,\mathbb{K})$, and then to the corresponding element in $!E$, we have without using the isomorphism $!E'' \simeq !E$:

$$\begin{aligned} d_{!E}\mu_E(\phi) &= g_1 \in !E' \mapsto \sum_p \phi_p(x \mapsto \sum_{k|p} \underbrace{g_k(ev_x^{\mathcal{H}^{p/k}(E,\mathbb{K})})}_{\neq 0 \text{ if and only if } k=1}) \\ &= g_1 \in !E' \mapsto \sum_p \phi_p(x \mapsto g_1(ev_x^{\mathcal{H}^p(E,\mathbb{K})})) \end{aligned}$$

As g_1 lives in $!E' \simeq \prod_m \mathcal{H}^m(E,\mathbb{K})$, we can write g_1 as a sequence $(g_{1,m})_m$ of m-monomials:

$$
\begin{aligned}
d_{!E}\mu_E(\phi) &= g_1 \in !E' \mapsto \sum_p \phi_p(x \mapsto ev_x^{\mathcal{H}^{p/k}(E,\mathbb{K})}(g_{1,p})) \\
&= g_1 \in !E' \mapsto \sum_p \phi_p(x \mapsto g_{1,p}(x)) \\
&= g_1 \in !E' \mapsto \sum_p \phi_p(g_{1,p}) \\
&= g_1 \in !E' \mapsto \phi(g_1)
\end{aligned}
$$

With the isomorphism $!E'' \simeq !E$ we obtain $d_!\mu = Id_!$.
The equation $!d\mu = Id$ is proved likewise: consider $\phi = \sum \phi_p \in !E$. Then

$$
\begin{aligned}
!d\mu(\phi) &= !d\left((g_n)_n \in \prod \mathcal{H}^n(!E,\mathbb{K}) \mapsto \sum_p \phi_p(x \mapsto \sum_{k|p} g_k(ev_x^{\mathcal{H}^{p/k}(E,\mathbb{K})}))\right) \\
&= (h_m)_m \in \prod \mathcal{H}^m(E,\mathbb{K}) \mapsto \sum_p \phi_p(x \mapsto \sum_{k|p} h_k \circ \underbrace{d(ev_x^{\mathcal{H}^{p/k}(E,\mathbb{K})})}_{\neq 0 \text{ if an only if } \frac{p}{k}=1}) \\
&= (h_m)_m \in \prod \mathcal{H}^m(E,\mathbb{K}) \mapsto \sum_p \phi_p(x \mapsto h_p(x)) \\
&= (h_m)_m \mapsto \phi((h_m)_m) \\
&= \phi
\end{aligned}
$$

So $!d\mu = Id$.

• The computations of the second equations follow immediatly from the functoriality of ! (proposition 5.2.13) and the associativity of the composition $\circ_!$ (proposition 5.2.14). Indeed consider E_w a weak lcs, and Id_E as an element of $\mathcal{L}(E,E)$, and thus as an element of $\prod_n \mathcal{H}^n(E,E)$. One has by associativity:

$$
\begin{aligned}
(Id_{!!!E} \circ_! Id_{!!E}) \circ_! Id_{!E} &= Id_{!!!E} \circ_! (Id_{!!E} \circ_! Id_{!E}) \\
Id_{!!!E} \circ !Id_{!!E} \circ \mu_{!E} \circ !Id_{!E} \circ \mu_E &= Id_{!!!E} \circ_! !(Id_{!!E} \circ_! Id_{!E}) \circ \mu_E \\
Id_{!!!E} \circ !Id_{!!E} \circ \mu_{!E} \circ !Id_{!E} \circ_! \mu_E &= Id_{!!!E} \circ !Id_{!!E} \circ !Id_{!E} \circ !\mu_E \circ \mu_E
\end{aligned}
$$

As $!Id_E = Id_{!E}$ we have thus $\mu_{!E}\, circ \mu_E = !\mu_E \circ \mu_E$. □

This co-monad is in fact strong monoidal by proposition 5.2.24.

Definition 5.2.18. The ? connective of linear logic is interpreted as the dual of !, that is

$$
?E \simeq (!E)' \simeq \prod_n \mathcal{H}^n(E',\mathbb{K}).
$$

We will write WEAK$_!$ for the co-Kleisli category of WEAK with !. We first show that morphisms of this category are easy to understand, as they are just sequences of n-monomials.

5.2.2 The co-Kleisli category

The exponential above was chosen because of its co-Kleisli category. Indeed, we want to decompose non-linear proofs as formal sums of n-linear proofs, and the simplest way to do that is to interpret non-linear maps from E to F, that is linear maps from $!E$ to F, as sequences of n-monomials from E to F.

Theorem 5.2.19. *For all lcs E and F, $\mathcal{L}_\sigma(!E_w, F_w) \sim \prod_{n\in\mathbb{N}} \mathcal{H}^n(E_w, F_w)$.*

Proof. Consider $f \in \mathcal{L}_\sigma(!E_w, F_w)$. Define, for each $n \in \mathbb{N}$, $f_n : x \in E_w \mapsto f(ev_x^{\mathcal{H}^n(E_w,\mathbb{K})})$. Then f_n is clearly the diagonalisation of an n-linear continuous function. Let us show that it is continuous from E_w to F_w. Consider $l \in F'$. Then $x \mapsto ev_x^{\mathcal{H}^n(E_w,\mathbb{K})}$ is continuous from E_w to $\mathcal{H}^n(E_w,\mathbb{K})'$, as the latter space is endowed with the topology of simple convergence on $\mathcal{H}^n(E_w,\mathbb{K})$. The injection $\mathcal{H}^n(E_w,\mathbb{K})' \hookrightarrow !E_w$ is continuous, as $!E_w \simeq (\bigoplus_k \mathcal{H}^k(E_w,\mathbb{K})')_w$ according to Proposition 5.1.21, and as $\mathcal{H}^n(E_w,\mathbb{K})' \hookrightarrow \bigoplus_k \mathcal{H}^k(E_w,\mathbb{K})'$ and $\bigoplus_k \mathcal{H}^k(E_w,\mathbb{K})'_w \hookrightarrow (\bigoplus_k \mathcal{H}^k(E_w,\mathbb{K})')_w$ are continuous. The following n-monomial is continuous:

$$f_n : E_w \xrightarrow{ev} \mathcal{H}^n(E_w,\mathbb{K})' \hookrightarrow \bigoplus_k \mathcal{H}^k(E_w,\mathbb{K})' \hookrightarrow (\bigoplus_k \mathcal{H}^k(E_w,\mathbb{K})')_w \xrightarrow{f} F_w$$

Thus $f_n \in \mathcal{H}^n(E_w,F_w)$. To every $f \in \mathcal{L}_\sigma(!E_w,F_w)$ we associate this way $(f_n) \in \prod_n \mathcal{H}^n(E_w,F_w)$.

Consider now $(f_n) \in \prod_n \mathcal{H}^n(E_w,F_w)$ and define $f : \phi \in !E_w \mapsto (l \in F' \mapsto \phi((l \circ f_n)_n))$. The function f is well-defined as $l \circ f_n \in \mathcal{H}^n(E_w,\mathbb{K})$ for every $n \in \mathbb{N}$ and every $l \in F'$. When ϕ is fixed, let us denote ϕ_f the function $l \in F' \mapsto \phi((l \circ f_n)_n))$. Then:

- the function $l \in F' \mapsto l \circ f_n \in \mathcal{H}^n(E,\mathbb{K})$ is continuous as F' (resp. $\mathcal{H}^n(E,\mathbb{K})$) is endowed with the topology of simple convergence on points of F (resp. on points of E);
- the function $l \mapsto (l \circ f_n)_n \in \prod_n \mathcal{H}^n(E_w,\mathbb{K})$ is then continuous by definition of the product topology;
- ϕ_f is then continuous.

Thus $\phi_f \in F'' \simeq F$. For each ϕ, there is $y \in F$ such that $\phi_f = ev_y$. We can now consider $f : \phi \in !E \mapsto y \in F$. f is clearly linear in ϕ. It is continuous as $!E_w$ is endowed with the topology of simple convergence on $\prod_{n\in\mathbb{N}} \mathcal{H}^n(E_w,\mathbb{K})$.

Finally, one can check that the mappings $\theta : f \in \mathcal{L}_\sigma(!E_w,F_w) \mapsto (f_n) \in \prod_n \mathcal{H}^n(E_w,F_w)$ and $\Delta : (f_n) \in \prod_n \mathcal{H}^n(E_w,F_w) \mapsto f \in \mathcal{L}_\sigma(!E_w,F_w)$ just described are mutually inverse. □

Let us show now that the isomorphism described above is a homeomorphism.

Theorem 5.2.20. *For all lcs E and F,*

$$\mathcal{L}_\sigma(!E_w,F_w) \simeq \prod_{n\in\mathbb{N}} \mathcal{H}^n(E_w,F_w),$$

and therefore

$$\mathcal{L}_\sigma(!E_w,F_w)_w \simeq (\prod_{n\in\mathbb{N}} \mathcal{H}^n(E_w,F_w))_w \simeq \prod_{n\in\mathbb{N}} \mathcal{H}^n(E_w,F_w)_w.$$

Proof. Let us show first that the function $\theta : f \in \mathcal{L}_\sigma(!E_w,F_w) \mapsto (f_n) \in \prod_n \mathcal{H}^n(E_w,F_w)$ is continuous. It is enough to show that $f \mapsto f_n$ is continuous. Consider $(f_\gamma)_{\gamma\in\Gamma}$ a net converging towards f in $\mathcal{L}_\sigma(!E_w,F_w)$. Thus for every $\phi \in !E_w$ we have that $f_\gamma(\phi)$ converges towards $f(\phi)$ in F_w. For every $x \in E$ the net $(f_\gamma(ev_x \in \mathcal{H}^n(E_w,\mathbb{K})'))_\gamma$ converges towards $f(ev_x)$ in F_w. Thus the net $(f_{\gamma,n})$ converges towards f_n and θ in continuous. The proof that $\Delta : (f_n) \in \prod_n \mathcal{H}^n(E_w,F_w) \mapsto f \in \mathcal{L}_\sigma(!E_w,F_w)$ is continuous is done likewise. □

Composition in WEAK$_!$ is thus given by the definition of a co-Kleisli category. If $f \in \mathcal{L}_\sigma(!E,F)$ and $g \in \mathcal{L}_\sigma(!F,G)$ we define:

$$g \circ f : !E \xrightarrow{\mu_E} !!E \xrightarrow{!f} !F \xrightarrow{g} G.$$

Notation 5.2.21. *For $f \in \mathcal{L}_\sigma(!E,F)$, we will write $(\widetilde{f_m})_m$ the corresponding sequences of monomials in $\prod_m \mathcal{H}^m(E,F)$.*

Proposition 5.2.22. *For every $f \in \mathcal{L}_\sigma(!E,F)$ and $g \in \mathcal{L}_\sigma(!F,G)$, we have*

$$\widetilde{(g \circ f)}_p = \sum_{k|p} \widetilde{g}_k \circ \widetilde{f}_{\frac{p}{k}}.$$

Proof. By definition, for $\phi \in !E$,

$$\begin{aligned} g \circ f(\phi) &= g(!f(\delta(\phi)) \\ &= g((g_n) \in \prod_n \mathcal{H}^n(F,\mathbb{K}) \mapsto \delta(\phi)((g_n \circ f)_n) \end{aligned}$$

For every $p \in \mathbb{N}^*$, and $x \in E$, we have:

$$
\begin{aligned}
(\widetilde{g \circ f})_p(x) &= g \circ f(ev_x^{\mathcal{H}^p(E,\mathbb{K})}) \\
&= g((g_n) \in \prod_n \mathcal{H}^n(F,\mathbb{K}) \mapsto \delta(ev_x^{\mathcal{H}^p(E,\mathbb{K})})((g_n \circ f)_n))
\end{aligned}
$$

Now $\delta(ev_x^{\mathcal{H}^p(E,\mathbb{K})}) = (h_j)_j \in \mathcal{H}^j(!E,\mathbb{K}) \mapsto \sum_{k|p} h_k(ev_x^{\mathcal{H}^{p/k}(E,\mathbb{K})})$. Thus

$$
\begin{aligned}
(\widetilde{g \circ f})_p(x) &= g((g_n) \in \prod_n \mathcal{H}^n(F,\mathbb{K}) \mapsto \delta(ev_x^{\mathcal{H}^p(E,\mathbb{K})})((g_n \circ f)_n) \\
&= \sum_{k|p} \tilde{g}_k(f(ev_x^{\mathcal{H}^{p/k}(E,\mathbb{K})})) \\
&= \sum_{k|p} \tilde{g}_k \circ \tilde{f}_{p/k}
\end{aligned}
$$

□

5.2.3 Cartesian closedness

Let us show that $\mathrm{W{\small EAK}}_!$, endowed with the cartesian product described in Section 5.1.4, is cartesian closed.

Theorem 5.2.23. *For every lcs E, F and G, we have:*

$$\prod_{p\in\mathbb{N}} \mathcal{H}^p(E_w \times F_w, G_w) \simeq \prod_{n\in\mathbb{N}} \mathcal{H}^n(E_w, \prod_{m\in\mathbb{N}} \mathcal{H}^m(F_w, G_w)).$$

The equality above means also that

$$(\prod_p \mathcal{H}^p(E_w \times F_w, G_w))_w \simeq [\prod_n \mathcal{H}^n(E_w, [\prod_m \mathcal{H}^m(F_w, G_w)]_w)]_w,$$

as $(\prod_m \mathcal{H}^m(F_w, G_w))_w) \simeq \prod_m \mathcal{H}^m(F_w, G_w)_w$ by Proposition 5.1.20, and since $\mathcal{H}^p(E_w, F_w)$ is already endowed with its weak topology by Proposition 5.2.8.

Proof. For every $n \in \mathbb{N}$, for every lcs E and F we have $\mathcal{H}^n(E,F) \simeq \mathcal{L}_s^n(E,F)$. We are therefore going to prove that for every lcs E, F and G:

$$\prod_p \mathcal{L}_s^p(E_w \times F_w, G_w) \simeq \prod_n \mathcal{L}_s^n(E_w, \prod_m \mathcal{L}_s^m(F_w, G_w)_w).$$

In the following, we will write $\vec{x}$ for some tuple $(x_1, ...x_n)$ in $E \times ... \times E$. Let us fix E, F and G, and define:

$$\phi : \begin{cases} \prod_p \mathcal{L}_s^p(E_w \times F_w, G_w) \to \prod_n \mathcal{L}_s^n(E_w, \prod_m \mathcal{L}_s^m(F_w, G_w)_w)) \\ \qquad (f_p) \mapsto [\vec{x} \mapsto (\vec{y} \mapsto f_{n+m}((x_1,0),...,(x_n,0),(0,y_1),...,(0,y_m)))_m]_n \end{cases}$$

Let us show that ϕ is well-defined.

- Consider $(f_p)_p \in \prod_p \mathcal{L}_s^p(E_w \times F_w, G_w)$, $n \in \mathbb{N}$, $x \in E$, and $m \in \mathbb{N}$. Then

  $$y \in F \mapsto f_{n+m}((x_1,0),...,(x_n,0),(0,y_1),...,(0,y_m))$$

  is m-linear and symmetric, and continuous from F_w to G_w as $f_{n+m} : E_w \times F_w \to G_w$ is continuous.
- Consider $(f_p)_p \in \prod_p \mathcal{L}_s^p(E_w \times F_w, G_w)$ and $n \in \mathbb{N}$. Then

  $$x_1,...,x_n \in E_w \mapsto (y_1,...,y_m \in F \mapsto f_{n+m}((x_1,0),...,(x_n,0),(0,y_1),...,(0,y_m)))$$

  is clearly n-linear and symmetric. It is continuous from E_w to $\mathcal{L}_s^m(F_w, G_w)$ as the latter bears the topology of simple convergence, and as f_{n+m} is continuous from $E_w \times f_w$ to G_w. Since the weak topology on $\mathcal{L}_s^m(F_w, G_w)$ is coarser than the strong topology, the function considered is also continuous from E_w to $\mathcal{L}_s^m(F_w, G_w)_w$.

We want to define the inverse function ψ of ϕ. Thus ψ is a function from $\prod_n \mathcal{L}_s^n(E_w, \prod_m \mathcal{L}_s^m(F_w, G_w))$ to $\prod_p \mathcal{L}_s^p(E_w \times F_w, G_w)$. Consider

$$f_n \in \mathcal{L}_s^n(E_w, \prod_m \mathcal{L}_s^m(F_w, G_w))$$

and let us write $f_{n,\vec{x},m}$ for $(f_n(\vec{x}))_m \in \mathcal{L}^m(F_w, G_w)$. If $p \geqslant \max n, m$, then the following function is $n+m$-linear:

$$((x_1, y_1), ..., (x_p, y_p)) \mapsto f_{n,(x_1,...,x_n),m}(y_1, ..., y_m).$$

When p is fixed, $\psi(f)_p$ collects all possible ways to produce a p-linear function as above, with $p = n + m$. As all possible permutations are considered, $\psi(f)_p$ is symmetric.

$$\psi : \begin{cases} \prod_n \mathcal{L}_s^n(E_w, \prod_m \mathcal{L}_s^m(F_w, G_w)) \to \prod_p \mathcal{L}_s^p(E_w \times F_w, G_w) \\ \qquad [f_n : \vec{x} \mapsto (f_{n,\vec{x},m})_m]_n \mapsto [\overrightarrow{(x,y)} \mapsto \sum_{\substack{I,J \subset [\![1,p]\!] \\ card(I)=n \\ card(J)=m \\ n+m=p}} \frac{1}{\binom{p}{n}} f_{n,\{x_i\}_{i\in I},m}(\{y_j\}_{j\in J})]_p \end{cases}$$

where in the index of the sum I and J are *disjoints* subsets of $[\![1, n]\!]$. Thus I and J forms a partition of $[\![1, n]\!]$. If $I = \{i_1, \ldots, i_n\}$ and $J = \{j_1, ..., j_m\}$, $f_{n,\{x_i\}_{i\in I},m}(\{y_j\}_{j\in J})$ is a notation for $f_{n,(x_{i_1},...,x_{i_n}),m}(y_{j_1}, \ldots, y_{j_m})$.

Let us show that ψ is well defined. Consider

$$[f_n : x_1, ..x_n \mapsto (f_{n,\{x_i\},m})_m]_n \in \prod_n \mathcal{L}_s^n(E_w, \prod_m \mathcal{L}_s^m(F_w, G_w)_w).$$

The function mapping $((x_1, y_1), ..., (x_p, y_p)) \in (E_w \times F_w)^p$ to $f_{n,\{x_i\}_{i\in I},m}(\{y_j\}_{j\in J})$ is $n + m$-linear and symmetric. For example, if $n = m = 1$, then the possible bilinear functions are:

$$((x, y), (x'y') \mapsto f_{1,x,1}(y')$$

and

$$((x, y), (x'y') \mapsto f_{1,x',1}(y).$$

Consider $\lambda \in \mathbb{K}$, then applied to $(\lambda \cdot (x_1, y_1), (x_2, y_2), ..., (x_p, y_p)) = ((\lambda x_1, \lambda y_1), (x_2, y_2), ..., (x_p, y_p))$ this function results in $\lambda f_{n,\{x_i\}_{i\in I},m}(\{y_j\}_{j\in J})$, as I and J are disjoints.

It is continuous, as the restrictions to fixed terms in E_w or F_w are continuous. So ψ is well defined. Note that both ϕ and ψ are continuous as the spaces $\mathcal{L}_s^n(E, F)_w$ are endowed with the topology induced by their dual $E_{sym}^{\otimes^n} \otimes_i F'$. Finally, one checks that ϕ and ψ are each other's inverse. Consider $f \in \prod_p \mathcal{L}_s^p(E_w \times F_w, G_w)$. Then $\psi(\phi(f))$ coincides to the function mapping p to the function in $\mathcal{L}_s^p(E_w \times F_w, G_w)$ mapping $((x_1, y_1), \ldots, (x_p, y_p))$ to:

$$\sum_{\substack{I,J \subset [\![1,p]\!] \\ card(I)=n \\ card(J)=m \\ n+m=p}} \frac{1}{\binom{p}{n}} f((x_{i_1}, 0), \ldots, (x_{i_n}, 0), (0, y_{j_1}), \ldots, (0, y_{j_m})).$$

By n-linearity of f_p this sum equals

$$f_p((x_1, y_1), \ldots, (x_p, y_p)).$$

Thus $\psi \circ \phi = \mathrm{Id}$. Consider now

$$g = [g_n : x_1, ..x_n \mapsto (g_{n,\{x_1,...,x_n\},m})_m]_n \in \prod_n \mathcal{L}_s^n(E_w, \prod_m \mathcal{L}_s^m(F_w, G_w)).$$

Let us show that $\phi(\psi(g)) = g$. The function $\psi(g)$ maps p, $((z_1, w_1), \ldots, (z_p, w_p))$ to

$$\sum_{\substack{I,J \subset [\![1,p]\!] \\ card(I)=a \\ card(J)=b \\ a+b=p}} \frac{1}{\binom{p}{a}} g_{n,\{z_i\}_{i\in I},m}(\{w_j\}_{j\in J}).$$

The function $\phi(\psi(g))$ maps $n \in \mathbb{N}$, $x_1, ..., x_n \in E$, $m \in \mathbb{N}$ and $y_1, ..., y_m \in F$ to this function applied to $n+m$ and $((x_1, 0), ..., (x_n, 0), (0, y_1), ..., (0, y_m))$. But notice that $g_{n,\{z_i\}_{i\in I},m}(\{w_j\}_{j\in J})$ is null as soon as one of the z_i or one of the w_i is null. So $\phi(\psi(g))$ applied to $n \in \mathbb{N}$, $x_1, ..., x_n \in E$, $m \in \mathbb{N}$ and $y_1, ..., y_m \in F$ results in

$$\frac{1}{\binom{n+m}{n}} \sum_{\substack{I,J\subset[\![1,n+m]\!] \\ I=1,n \\ J=\{1,m\}}} g_{n,\{z_i\}_{i\in I},m}(\{w_j\}_{j\in J})$$

which is exactly $g_{n,\{x_1,...,x_n\},m}(y_1, \ldots, y_m)$.

□

The Seely isomorphism

Proposition 5.2.24. *For all lcs E and F we have:*

$$!(E_w \times F_w) \simeq !E_w \otimes_i !F_w$$

Proof. This follows from the cartesian closedness of $\mathrm{W{\small EAK}}_!$, the monoidal closedness of $\mathrm{W{\small EAK}}$, and the description of $\mathrm{W{\small EAK}}_!$ obtained in Theorem 5.2.19. Indeed

$$\begin{aligned} !(E_w \times F_w) &\simeq \prod_p \mathcal{H}^p(E_w \times F_w, \mathbb{K})' \\ &\simeq \prod_n \mathcal{H}^n(E_w, \prod_m \mathcal{H}^m(F_w, \mathbb{K})_w))' \\ &\simeq \mathcal{L}_\sigma(!E_w, \prod_m \mathcal{H}^m(F_w, \mathbb{K})_w))' \\ &\simeq \mathcal{L}_\sigma(!E_w, \mathcal{L}_\sigma(!F_w, \mathbb{K})_w) \\ &\simeq (!E_w \otimes_i !F_w)'' \\ &\simeq !E_w \otimes_i !F_w. \end{aligned}$$

□

5.2.4 Derivation and integration

As a quantitative model of linear logic, this model interprets differential linear logic [20]. However, the interpretation of derivation remains combinatorial, and not as close to the usual differentiation operation as one would wish.

Definition 5.2.25. The co-dereliction rule of Differential linear logic is interpreted by:

$$\bar{d}_E : \begin{cases} E_w \to !E_w \\ \quad x \mapsto ((f_n) \in \prod_n \mathcal{H}^n(E, \mathbb{K}) \mapsto f_1(x)) \end{cases}$$

Proposition 5.2.26. *For every space E, $\bar{d}_E$ is a linear continuous function from E_w to $!E_w$.*

Proof. Let us fix $\phi = (\phi_n)_n \in (!E_w)' \simeq \prod_n \mathcal{H}^n(E, \mathbb{K})$. Then $\phi \circ \bar{d}_E$ maps $x \in E$ to $\phi_1(x)$. As $\phi_1 \in E'$, $\bar{d}_E$ is continuous from E_w to $!E_w$. □

For every lcs E, $!E$ bears a structure of bialgebra. The following morphisms are derived from the biproduct structure of $\mathrm{W{\small EAK}}$:

- The $\Delta : !E \to !E \otimes_i !E$ interprets the contraction rules of Linear Logic. It is coincides as $!E \to !(E \times E) \simeq !E \otimes_i !E$, where the first computation is the functor ! composed with the diagonalisation morphism, and the second coincides to the Seely isomorphism.
- $e : !E \to \mathbb{C}$ is $e(\phi) = \phi(1) \in S(E, \mathbb{C})$. It interprets the weakening rule of Linear Logic.

- $\nabla : !E \otimes_i !E \to !E$ is defined by $\nabla(h \mapsto \phi(x \mapsto \psi(y \mapsto h(x+y))))$. It interprets the co-contraction rule of Differential Linear Logic.
- $\nu : \mathbb{C} \to !E$ is $\nu(1) = ev_0$. It interprets the co-weakening rule of Differential Linear Logic.

Recall that ! is also a symmetric lax monoidal functor, see propositions 5.2.17. All these morphisms are necessary to build a differential structure.

Proposition 5.2.27. WEAK *endowed with coder is a differential category [8].*

Proof. As shown by Fiore [25], it is enough to prove that the following diagrams hold:

- Strenght:

$$\begin{tikzcd}
E \otimes_i !F \arrow[rr, "\bar{d}_E \otimes_i Id"] \arrow[dr, "Id \otimes_i d_F"'] & & !E \otimes_i !F \arrow[rr, "\mu_{E,F}"] & & !(E \otimes_i F) \\
& E \otimes_i F \arrow[urrr, "\bar{d}_{E \otimes_i F}"'] & & &
\end{tikzcd}$$

- Comonad:

$$\begin{tikzcd}
E \arrow[r, "\bar{d}_E"] \arrow[d, "Id"'] & !E \arrow[dl, "d_E"] \\
E &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\bar{d}_E"] \arrow[d] & !E \arrow[r, "\mu_E"] & !!E \\
E \otimes_i 1 \arrow[r, "\bar{d}_E \otimes_i \nu"'] & !E \otimes_i !E \arrow[r, "\bar{d}_{!E} \otimes_i \mu_E"'] & !!E \otimes_i !!E \arrow[u, "\nabla"']
\end{tikzcd}$$

In our category, both branches of the strength diagrams computes the following function:

$$(x \otimes_i \phi) \mapsto ((f_n)_n \mapsto \phi(y \mapsto f_1(x \otimes_i y)))\,.$$

The first comonad diagram is immediate by the definition of d. The second diagram computes the function

$$x \in E \mapsto (g_p)_p \in \prod_p \mathcal{H}^p(!E, \mathbb{K}) \mapsto g_1((f_n)_n \mapsto f_1(x)).$$

$\square$

We do not have an interpretation of a syntactic integration in this category. Indeed, the existence of Ehrhard's anti-derivative operator [20, 2.3] would imply some sort of integration. We do not have a way to integrate in our spaces, as no completeness condition is verified. It is noticeable that if our spaces were β-reflexive, that is isomorphic to their bidual when the dual is endowed with the topology of uniform convergence over bounded sets, a weak integration would be available.

5.2.5 An exponential with non-unit sequences

Inspired by the substitution problem in the theory of formal power series [38, Chapter 1], we could have used another composition between sequences of monomials. Indeed, such sequences can be considered as generalized formal power series. That is, to a sequence $(f_n)_n$ coincides a formal sum $\sum f_n$, where no notion of convergence is employed. A formal power series is a denumerable sum $A(X) = \sum_m a_m X^n$ where the a_i are coefficients in some commutative ring R. If $B = \sum_p B_n X^p$ is another formal power series, one has:

$$B(A(x)) = \sum_p b_p (A(x))^p$$

For every $n \in \mathbb{N}$, $A(x)^p$ can be computed as if the sum in A were convergent. That is, $B(A) = \sum_k c_k^p X^k$ with

$$c_k^p = \sum_{n \geqslant 0} \sum_{k_1 + \ldots + k_n = k} a_{k_1} \times a_{k_2} \times \ldots \times a_{k_n} \times a_0.$$

This sum is infinite, which causes a problem since no notion of convergence is employed here. However, if A is non-unit, that is if $a_0 = 0$ this sum becomes finite:

$$c_k^p = \sum_{p \geqslant n \geqslant 1} \sum_{\substack{m \geqslant k_i \geqslant 1 \\ k_1 + \ldots + k_n = p}} a_{k_1} \times a_{k_2} \times \ldots \times a_{k_n}.$$

With the same ideas, one can construct a comonad with a non-unit version of functor $!$: WEAK $\rightarrow$ WEAK such that, in the co-Kleisli category WEAK$_!$ coincides with substitution if its morphisms are seen as formal power series $f = \sum_{n \geqslant 1} f_n$.

$$(g \circ f)_p : x \mapsto \sum_{p \geqslant n \geqslant 1} \sum_{\substack{m \geqslant k_i \geqslant 1 \\ k_1 + \ldots + k_n = p}} g_n(f_{k_1}(x), \ldots, f_{k_n}(x))$$

Remark 5.2.28. This formula is called the *Faa di Bruno* fomula. A categorical account of the construction below was given by Cockett and Seely [13].

Let us state briefly these definitions.

Definition 5.2.29. Let us define $!_1 E$ as the lcs $\bigoplus_{n \geqslant 1} \mathcal{H}^n(E, \mathbb{K})'$.

Definition 5.2.30. For $f \in \mathcal{L}_\sigma(E_w, F_w)$ we define

$$!_1 f : \begin{cases} !_1 E_w \rightarrow !_1 F_w \\ \qquad \phi \mapsto ((g_n) \in \prod_{n \geqslant 1} \mathcal{H}^n(F, \mathbb{K}) \mapsto \phi((g_n \circ f)_n) \end{cases}$$

This makes $!_1$ a functor on WEAK. Such an exponential leads to a co-Kleisli category of non-unit sequences of monomials.

$$\mathcal{L}_\sigma(!_1 E_w, F_w) \simeq \prod_{n \geqslant 1} \mathcal{H}^n(E_w, F_w)_w.$$

It is endowed with the usual co-unit, and co-multiplication allowing for a composition which coincides intuitively to a substitution.

Proposition 5.2.31. *The functor* $!_1$: *Lin* $\rightarrow$ *Lin is a co-monad. Its co-unit* $d : !_1 \rightarrow 1$ *is defined by*

$$d_E \begin{cases} !_1 E_w \rightarrow E_w \\ \qquad \phi \mapsto \phi_1 \in E'' \simeq E \end{cases}$$

The co-unit is the operator extracting from $\phi \in !_1 E$ *its part operating on linear maps. The co-multiplication* $\mu : !_1 \rightarrow !_1 !_1$ *is defined by*

$$\mu_E \begin{cases} !_1 E_w \simeq (\prod_{p \geqslant 1} \mathcal{H}^p(E, \mathbb{K}))' \rightarrow !_1 !_1 E_w \simeq \left(\prod_{n \geqslant 1} \mathcal{H}^n([\prod_{m \geqslant 1} \mathcal{H}^m(E, \mathbb{K})]', \mathbb{K}) \right)' \\ \qquad \phi \in (\prod_p \mathcal{H}^p(E, \mathbb{K}))' \mapsto (g_n)_n \mapsto \\ \qquad\qquad \phi\left(\left(x \in E \mapsto \sum_{p \geqslant n \geqslant 1} \sum_{\substack{m \geqslant k_i \geqslant 1 \\ k_1 + \ldots + k_n = p}} g_n[(f_m)_m \mapsto g_n(f_{k_1}(x), \ldots, f_{k_n}(x))] \right)_p \right) \end{cases}$$

The co-Kleisli category remains cartesian closed, and thus we obtain likewise a Seely isomorphism

$$!_1(E_w \times F_w) \simeq !_1 E_w \otimes_i !_1 F_w.$$

As WEAK is $*$-autonomous, we obtain this way another model of linear logic.

5.3 A negative interpretation of $\mathrm{DiLL}_{\mathrm{pol}}$

In this Section we revisit the previous results in terms of chiralities and negative interpretations. We show that WEAK is a negative interpretation of LL_{pol}, through its adjunction with CHU. According to Section 4, we have an adjunction between the categories WEAK and CHU:

$$\begin{tikzcd} \mathrm{WEAK} \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \mathrm{CHU} \arrow[l, bend left, "\mathcal{W}"] \end{tikzcd}$$

This adjunction is a positive closure: if a lcs E is already endowed with its weak topology, then $\mathcal{W}(\mathcal{P}(E)) \simeq E$. One also has a contravariant adjunction:

$$\begin{tikzcd} \mathrm{WEAK} \arrow[r, bend left, "\neg"] \arrow[r, phantom, "\perp"] & \mathrm{CHU} \arrow[l, bend left, "(-)'_w"] \end{tikzcd} \tag{5.2}$$

where, for $E_w \in$ WEAK $\neg E_w = (E', E) \in$ CHU and for $(E, F) \in$ CHU we define $(E, F)'_w = F_{\sigma(F,E)}$. Then this is indeed a contravariant adjunction which results in a negative closure (it is in fact a contravariant equivalence).

Proposition 5.3.1. *Consider E and F weak spaces. By defining:*

$$E \parr F := \mathcal{L}_\sigma(E'_w, F)$$

we get an associative and commutative operator on WEAK

Proof. By Proposition 5.1.6, we have that for any weak space F, the lcs $\mathcal{L}_\sigma(E, F_w)$ is also a weak space.

The operation is commutative: by duality, for any $f \in \mathcal{L}_\sigma(E'_w, F)$ we construct a linear continuous functions $f' : F'_w \longrightarrow (E'_w)'_w \simeq E$, and this operation is involutive and preserves the topologies. Thus $E \parr F \simeq F \parr E$.

Let us show associativity. From commutativity we only need to show that $E \parr (F \parr G)$ is linearly homeomorphic to $G \parr (E \parr F)$. Consider $T \in E \parr (F \parr G)$ and define $\hat{T} : G' \longrightarrow \mathcal{L}(E'_w, F)$. We define its values in F through their image under linear form $\ell_F \in F'$ by Hahn-Banach theorem: $\ell_F(\hat{T}(\ell_G, \ell_E)) = \ell_G(T(\ell_E, \ell_F)$. Let us show that the linear map $\hat{T}$ takes indeed its values in $\mathcal{L}(E'_w, F)$: as F is endowed with its weak topology, $\hat{T}$ is continuous if and only if any $\ell_F \in F'$, $\ell \circ \hat{T}$ is continuous, and this is the case by hypothesis. One sees easy that the linear isomorphism we defined is then an homeomorphism. $\square$

Thus we want to see WEAK as the interpretation for the negatives. We consider thus the adjunctions:

$$\begin{tikzcd} \mathrm{TOPVEC} \arrow[r, bend left, "(\_)_w"] \arrow[r, phantom, "\perp"] & \mathrm{WEAK} \arrow[l, bend left, "U"] \end{tikzcd}$$

$$\begin{tikzcd} (\mathrm{TOPVEC}, \otimes_i) \arrow[r, bend left, "(\_)'_w"] \arrow[r, phantom, "\perp"] & (\mathrm{WEAK}^{op}, \parr) \arrow[l, bend left, "(\_)'_w"] \end{tikzcd} \tag{5.3}$$

These are indeed adjunctions: if F is a vector space already endowed with its weak topology, then the spaces $\mathcal{L}(E, F)$ and $\mathcal{L}(E_w, F)$ are exactly the same as they only depend of the dual $E' = (E_w)'$.

Proposition 5.3.2. TOPVEC *and* WEAK *form a negative chirality, which is extended without difficulties to products and co-products.*

As it is defined, the co-monad $!E$ is naturally endowed with its weak topology. One has thus an adjunction between TOPVEC and $\mathrm{WEAK}^!$,

$$\begin{tikzcd} \mathrm{WEAK}^! \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & \mathrm{TOPVEC} \arrow[l, bend left, "U"] \end{tikzcd}$$

were ! and U are strong monoidal functors, ! mapping spaces E to $(!E, ?E')$ while U is the identity on objects and maps an arrow f to $f \circ d$. It is strong monoidal by the Seely isomorphism.

Chapter 6

Mackey topologies on convenient spaces

In this Chapter, we refine DiLL intuitionnistic model of convenient spaces [6] into a *classical polarized* model of DiLL where spaces are endowed with their Mackey topology and non-linear proofs are interpreted by smooth functions. In further work with Tasson [49], we argued that the bornological condition on the topologies was not necessary to the constructions of intuitionnistic $DiLL$. However, as pointed out to the author by Dabrowski [17], bornological spaces are in particular endowed with their Mackey topologies. We thus detail in this Chapter how convenient spaces (that is bornological and Mackey-complete lcs) are the interpretation of positive connectives in a polarized, smooth and classical model of DiLL.

In Section 6.1 we give an introduction to our work with Dabrowski [17] on unpolarized classical and smooth models of DiLL where $⅋$ is interpreted Schwartz ε product. In Section 6.2 we recall and prove some classical results about bornologies and Mackey-complete lcs. In Section 6.3 we give a positive model of MALL with a chirality between bornological lcs and (some specific class of) Chu pairs. Finally, in Section 6.4 where we bring completeness in the picture, and the ε product appears as the interpretation of $⅋$. We describe a positive model of MALL with a chirality between convenient lcs and Complete and Mackey lcs. This model is extended to a model of DiLL through the methods of [6] in Section 6.4.2

Notation 6.0.1. *In this chapter and in the following ones, we only consider vector spaces on $\mathbb{K} = \mathbb{R}$.*

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Introduction Looking at the adjunctions between CHU and TOPVEC (see Section 4),

$$\begin{tikzcd} \mathrm{MACKEY} \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \mathrm{CHU} \arrow[l, bend left, "\mathcal{M}"] \arrow[r, bend left, "\mathcal{W}"] \arrow[r, phantom, "\perp"] & \mathrm{WEAK} \arrow[l, bend left, "\mathcal{P}"] \end{tikzcd}$$

and the categorical models of polarized linear logic (see Section 2.3.2),

$$\begin{tikzcd} \mathscr{P} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \mathscr{P} \arrow[l, bend left, "\downarrow"] \end{tikzcd}$$

we see that while weak topologies allow for a negative interpretation of DiLL_{pol}, Mackey topologies should offer a positive interpretation of DiLL_{pol}.

We will introduce in this chapter the concept of spaces endowed with bornologies, which is to be put in adjunction with the concept of topological spaces which are spaces endowed with topologies. We already explained in Chapter 3, section 3.4 how bounded sets can be described from a given topology. The converse is also possible, resulting in the following adjunction (Proposition 6.2.5):

$$\begin{tikzcd} \mathrm{TOP} \arrow[r, bend left, "\mathsf{Born}"] \arrow[r, phantom, "\perp"] & \mathrm{BORN} \arrow[l, bend left, "\mathsf{Top}"] \end{tikzcd}$$

In order to obtain a nice interpretation of the positive[1], one must work with spaces on which there is no hiatus between the topology and the Von-Neumann bornology. Such spaces are called *bornological*.

More precisely, we revisit in this Chapter the model of Blute, Ehrhard and Tasson [6], which was exposed under a simplified form in Section 2.4.3, into a classical positive interpretation of DiLL. We start from the fact that every convenient space is endowed with its Mackey-topology, to construct a positive chirality were negations are interpreted by the Mackey dual and the Mackey-completion of the dual. Let us stress that the name *convenient space* denotes spaces which are not only Mackey-complete spaces as in [49] or [53], but the spaces which are Mackey-complete and bornological as in [6] and [26]. We showed in [49] that Mackey-completion was enough to have a smooth intuitionist model, and we show here that in fact the bornological condition allows a (polarized) positive interpretation.

This should be put in perspective with the work of the author with Y. Dabrowski [17], where the same decomposition could be possible for Schwartz and Mackey-complete or Nuclear and Mackey-complete spaces. Thus the chirality between convenient spaces and complete and Mackey spaces is a classical refinement of [6] through chiralities and Mackey-duals, and a polarized refinement of [17] which allows for smooth maps which are always differentiable.

Notation 6.0.2. *As in Chapter 3, we denote by $\mathcal{L}(E,F)$ the vector space of all linear continuous functions between two lcs. We will also use the notation $\mathbf{L}(E,F)$ of Section 2.4.3 to denote the space of all linear* bounded *functions between two lcs, endowed with the bornology of all equibounded sets. In particular, in E is a vector space endowed with a bornology, we denote by $E^\times$ the* bounded dual $\mathbf{L}(E,\mathbb{K})$ *of E, whose bounded sets are thus the polars B° of bounded sets of E.*

6.1 Models of Linear Logic Based on Schwartz' ε product

In this Section we recall the results of [17], presented in Appendix B, where the authors obtained smooth and classical models of Differential Linear Logic.

k-reflexive spaces. This work features three models of LL with smooth maps, where smooth maps are then restricted those with continuous differentials in order to have a good interpretation for the codereliction. The first one, to be considered as the one with less restrictions, is obtained by requiring the spaces to be k-complete: the absolute convex closed hull of a compact set must be compact. This is noticeably a very lax condition allowing for

[1] Actually at time of writing this thesis, one only has a nice interpretation of the $\otimes$

the associativity of the ε product [17, section 4]. This condition is preserved by the Arens dual (that is the dual E' equipped with the topology of uniform convergence on absolutely convex compact subsets of E):

$$E_k^\star := \hat{E}_c'^{\,k}$$

allowing for a duality

$$((E_k^\star)_k^\star)_k^\star \simeq E_k^\star$$

for any k-complete space. Spaces which are invariant under double duality are then called *k-reflexive*. Defining on k-complete spaces a notion of smooth maps following Meise [58], with iterated Gateau-differentials continuous on compacts, one obtains a cartesian closed category of smooth functions. Although this setting is not ideal for differentiation, an important property is that by defining smoothness relatively to some subcategories of the category of k-reflexive spaces, one defines new smooth models of LL.

Topologies induced by spaces of smooth functions [17, Section 6.1]. Based on this preceding notions of smoothness and completeness, one defines (at least) two smooth and classical models of LL. For two lcs E and F, we denote by $\mathcal{C}^\infty_{co}(E,F)$ the space of infinitely many times Gâteaux-differentiable functions with derivatives continuous on compacts with value in the space $L^{n+1}_{co}(E,F) = L_{co}(E, L^n_{co}(E,F))$ with at each stage the topology of uniform convergence on compact sets. We endow this space with the topology of uniform convergence on compact sets of all derivatives in the space $L^n_{co}(E,F)$.

Consider $\mathscr{C} \subset k-\mathbf{Ref}$ a small category, and define for E and F Mackey-complete spaces:

$$\mathcal{C}^\infty_{\mathscr{C}}(E,F) := \{f : E \longrightarrow F | \forall X \in \mathscr{C}, \forall c \in \mathcal{C}^\infty_{co}(X,E), f \circ c \in \mathcal{C}^\infty_{co}(X,F)\}$$

This space is endowed with the inductive topology induced by the family of $\mathcal{C}^\infty_{co}(X,F)$, for all $X \in \mathscr{C}$ and all $c \in \mathcal{C}^\infty_{co}(X,E)$. We show that linear functions are in particular of that type of smooth functions, and thus the inclusion $E' \subset \mathcal{C}^\infty_{\mathscr{C}}(E,\mathscr{R})$ induces a topology on E', which we denote by $E'_{\mathscr{C}}$. This new definition for smooth functions defines also a new topology on E which we describe now.

We first consider $\mathscr{C} \subset k-\mathbf{Ref}$ a full Cartesian subcategory.

Let $\mathscr{C}^\infty$ be the smallest class of locally convex spaces containing $C^\infty_{co}(X,\mathcal{K})$ for $X \in \mathscr{C}$ ($X = \{0\}$ included) and stable by products and subspaces. Consider $\mathscr{S}_{\mathscr{C}}$ the functor on $\mathbf{LCS}$ of associated topology in this class described by [45, 2.6.4]. This functor maps a lcs E to the vector space E endowed with the finest topology coarser than the original one on E, such that $\mathscr{S}_{\mathscr{S}}(E) \in \mathscr{C}^\infty$.

With these definitions, it is possible to show if E is Mackey complete we have $E'_{\mathscr{C}} \simeq \mathscr{S}_{\mathscr{C}}(E'_c)$. In the article, we construct moreover an inductive Mackey-completion procedure $\hat{\_}^M$ (see proposition 6.2.22) which is functorial with respect to continuous linear maps. This is fundamental, as it allows to work well with the Mackey-Completion within TopVec. Let us point out that in [49], the authors only knew about a Mackey-completion with a universal property with respect to bounded maps, and that this stopped the progression towards a classical bornological model.

Then one defines on a Mackey-complete lcs E:

$$E^{\perp_{\mathscr{C}}} := \widehat{\mathscr{S}_{\mathscr{C}}(E'_c)}^M$$

We say that a space is $\mathscr{C}$-reflexive if $E^{\perp_{\mathscr{C}}\perp_{\mathscr{C}}} \simeq E$. In particular, $E^{\perp_{\mathscr{C}}}$ is always $\mathscr{C}$-reflexive.

Theorem 6.1.1. *Consider $\mathscr{C}$ a small category of Banach spaces containing finite dimensional vector spaces. The category of $\mathscr{C}$-reflexive spaces is $*$-autonomous, with tensor product $E \hat{\otimes}_{\mathscr{C}} F := (E^{\perp_{\mathscr{C}}} \varepsilon E^{\perp_{\mathscr{C}}})^{\perp_{\mathscr{C}}}$. It is a model of LL, with the exponential defined by:*

$$!_{\mathscr{C}} E := \mathcal{C}^\infty_{\mathscr{C}}(E,\mathscr{R})^{\perp_{\mathscr{C}}}.$$

Two concrete examples of models smooth models of LL are generated in this way:

Example 6.1.2. If $\mathscr{C}$ consists of the category of all finite dimensional spaces, then $C^\infty_{co}(\mathscr{R}^n,\mathbb{K}) \simeq \mathfrak{s}^{\mathcal{N}}$ [78, (7) p 383], where $\mathfrak{s}$ denotes the Köthe space of rapidly decreasing sequences. This space is a universal generator for nuclear lcs, meaning that every nuclear lcs is a subset of a product of copies of $\mathfrak{s}$. Thus the associated topology functo [45] is $\mathscr{N}(E) = \mathscr{S}_{Fin}(E)$, and the induced model consists of Nuclear Mackey-complete spaces which equals their double $\mathscr{C}$-dual.

Example 6.1.3. If $\mathscr{C}$ is the category of Banach spaces, then $\mathscr{C}^\infty$ is the category of all Schwartz spaces, and the induced model consists of all Mackey-complete Schwartz spaces which equal their double $\mathscr{C}$-dual.

Perspective : Mackey topologies for reflexivity and completeness for the negatives We give here another perspective of these models. A first condition allows for reflexivity: the Arens-dual is the one used in k-reflexive spaces, which could thus for a model of Mall where the spaces are the one for which $(E'_c)'_c \simeq E$. These spaces are the γ-reflexive ones (see Section 3.5). However, associativity of ε or of a dual tensor product requires more, and requires in particular a second condition which is the completeness one. By proving that $(E^*_k)^*_k$ is k-complete when E is, we are able to define a "new" duality as the k-completion of the Arens dual, which still leads to the property that every dual is reflexive.

Schwartz Mackey-complete spaces and Nuclear Mackey-complete spaces behave likewise. If a space E is Schwartz, then absolutely convex weakly-compact subsets are compact [17, 3.4], and thus $E'_\mu \simeq E'_c$. Writing $E^*_{\mathcal{S}} = \mathscr{S}(E'_c)$, this leads to:

$$((E^*_{\mathcal{S}})^*_{\mathcal{S}})^*_{\mathcal{S}} \simeq E^*_{\mathcal{S}},$$

allowing for a reflexivity condition. Mackey-completeness is then well-behaved with respect to this duality: if we denote $E^*_\rho = \mathcal{S}(\hat{(E'_c)}^M)$ where E is Schwartz Mackey-complete, we have that $\mathscr{S}((E'_c)'_c)$ is Mackey-complete when E is, thus allowing for:

$$((E^*_\rho)^*_\rho)^*_\rho \simeq E^*_\rho$$

and for a good definition of smooth maps.

A drawback: differentiation The smooth maps $f : E \longrightarrow F$ have differentials which are continuous on compact subsets,thus bounded but necessarily continuous. As the spaces E on which these smooth maps are defined are not bornological in general, the smooth maps are not continuous and no interpretation of the co-dereliction is available: the differential at 0 of a conveniently smooth maps is a bounded linear map, but again in general not a continuous one. However in the setting of DiLL one requires an interpretation

$$\bar{d} : E \longrightarrow \mathcal{C}^\infty(E, \mathbb{R})',$$

i.e. the differential at one point must be a linear continuous morphism.This is solved in [17] by restricting the use of conveniently smooth functions whose iterated differential is continuous at every point.

Replacing Schwartz and Mackey-complete by *bornological* and Mackey-complete What is done in the remaining sections of this Chapter can be understood as a way to replace the Schwartz condition by a bornological condition, allowing for an easier definition of differentiation on the smooth maps, which are then in particular continuous. We also consider a categorical perspective via chirality which we believe clarifies the situation.

Outlook 6. Can the bornologification of a space be understood as the embedding of a space E into a space of smooth functions $\mathcal{C}^\infty_{\mathcal{C}}(E', \mathbb{R})$, for $\mathcal{C}$ a small cartesian category of k-reflexive lcs? This would amount to find a universal generator for bornological spaces, and characterize it as some space of smooth functions $\mathcal{C}^\infty_{co,\mathcal{C}}(\mathbb{R}, \mathbb{R})$.

Remark 6.1.4. We only included the work of [17] in the appendix. We made this choice with to keep the main body of the thesis as concise as possible. We will not use the results of this paper, except for the construction of a Mackey-completion functorial with respect with continuous linear functions, which is recalled and proved (see Theorem 6.2.21).

Outlook 7. Let us emphasize that bornological spaces behave remarkably well with respect to Schwartzification and Nuclearification [44, 13.2.4], and future work should try and interpret the structures of [17] in a chirality where positive formulas are interpreted as bornological spaces.

6.2 Preliminaries: bornological notions

6.2.1 Bornological spaces

In Chapter 3, we defined bounded sets in a lcs as the sets which were absorbed by open sets (Definition 3.4.7). They define a bornology (Definition 3.4.6), which is thus defined from the topology of E. The converse is also possible: from a bornology one can define a topology.

Definition 6.2.1. Consider $\mathcal{B}_E$ a bornology on a vector space E. We define the topology $\mathsf{T}(\mathcal{B})$ as the collection of sets which absorbs all the elements of $\mathcal{B}$ (the so-called bounded sets). That is, a subset $U \subset E$ is an open set of E if and only if for every $B \in \mathcal{B}_E$ we have a scalar $\lambda \in \mathbb{K}$ such that:

$$B \subset \lambda U.$$

Note that this topology on E might enrich the bornology on E. The Von-Neumann bornology for $\mathsf{T}(\mathcal{B}_E)$ includes $\mathcal{B}_E$ but might be larger.

The theory of vector spaces endowed primarily with bounded sets (and not open sets) was intensively studied by Hogbe-Nlend [40]. They are an analogue to the theory of topological vector spaces. The following definition of BORN can be found in [53, 52.1] [26, 1.2] or [6].

Notation 6.2.2. *We write* BORN *the category of vector spaces with bornologies and linear bounded maps, and* TOP *the category of vector spaces with (non-necessarily separated locally convex vector) topologies and linear continuous maps.*

From an object of BORN one can construct a lcs whose open sets are exactly the one absorbing the bounded sets. The subsets absorbing all the bounded sets are called *bornivorous*.

Definition 6.2.3. Consider $E \in$ BORN with bornology $\mathcal{B}_E$. Then a subset $U \subset E$ is said to be bornivorous if for every $B \in \mathcal{B}_E$ there is a scalar $\lambda \in \mathbb{K}$ such that $B \subset \lambda U$.

We define Born as the functor from TOP to BORN matching a vector space E with a topology to he same vector space endowed with its (Von-Neuman) bornology, and which maps a linear continuous to itself. It is well defined as linear continuous functions are in particular bounded. Symmetrically, we consider also the functor Top from BORN to TOP wich maps E to the lcs E with the topology of bornivorous subsets, and which is the identity on linear bounded functions. We check that this functor is well-defined in the following Proposition.

Proposition 6.2.4. *A linear bounded map between two vector spaces E and F endowed with respective bornologies $\mathcal{B}_E$ and $\mathcal{B}_F$ defines a linear continuous maps between E endowed with $\mathsf{T}(\mathcal{B}_E)$ and F endowed with $\mathsf{T}(\mathcal{B}_F)$.*

Proof. Consider a linear bounded map from E to F and V a subset of V wich absorbs every element of $\mathcal{B}_F$. Then one sees immediately that ℓ^{-1} absorbs every element of $\mathcal{B}_F$ and thus that ℓ is continuous between E endowed with $\mathsf{T}(\mathcal{B}_E)$ and F endowed with $\mathsf{T}(\mathcal{B}_F)$. ☐

Proposition 6.2.5. *We have an adjunction:*

$$\begin{tikzcd} \text{BORN} \arrow[r, bend left, "\mathsf{Top}"] \arrow[r, phantom, "\perp"] & \text{TOP} \arrow[l, bend left, "\mathsf{Born}"] \end{tikzcd}$$

Proof. Consider $E \in$ BORN and $F \in$ TOP. Then a linear function $f : E \longrightarrow \mathsf{Born}(F)$ is bounded if and only if it sends every bounded set of $\mathcal{B}_E$ in a set $B' \subset F$ which is absorbed by every open set of the topology of F. Thus consider $U' \subset F$ an open set of F. Then for any bounded set B of E, we have a scalar λ such that $f(B) \subset \lambda U'$, thus $B \subset \lambda f^{-1}(U)$, thus f is continuous from $\mathsf{Top}(E) \longrightarrow F$. One show likewise that to continuous linear functions $\mathsf{Top}(E) \longrightarrow F$ coincides bounded linear ones from $E \longrightarrow \mathsf{Born}(F)$. ☐

Bornological lcs On some lcs, not-only do the open sets absorb the bounded subsets (by definition of boundedness in a lcs), but all the subsets that absorb the bounded sets are open sets. This means that the lcs E is invariant under the composition of functor $\mathsf{Top} \circ \mathsf{Born}$. These spaces are called bornological [44, 13.1], and they have a nice characterization in terms of bounded linear maps.

Proposition 6.2.6. *[44, 13.1.1] A lcs E is said to be* bornological *if one of these following equivalent propositions is true:*

1. *For any other lcs F, any bounded linear map $f : E \longrightarrow F$ is continuous, that is $\mathcal{L}(E, F) = \mathbf{L}(E, F)$,*

2. *E is endowed with the topology $\mathsf{Top} \circ \mathsf{Born}(E)$,*

3. *E is the topological inductive limits of the spaces E_B (see Definition 3.1.12), for B bounded, closed and absolutely convex.*

4. *E has the Mackey* topology of uniform convergence on the weak compact and absolutely convex subspaces of E', and any bounded linear form $f : E \longrightarrow \mathbb{K}$ is continuous.*

This definition coincides in fact to the one of a β-bornological, but we won't make use of bornologicality for other bornologies.

Proof. The equivalence between the first two Propositions follows from the adjunction 6.2.5.

If we have (2), then we have indeed that a 0-neighbourhood in $indE_B$ absorbs every bounded set of E, thus is a 0-neighbourhood in E. As a 0-neighbourhood in E is always one in the E_B, we have $E = indE_B$.

If we have (3), then as the Mackey topology is preserved by inductive limits [44, 11.3.1] and normed spaces are endowed with their Mackey-topology, we have that E is endowed with its Mackey-topology. Now a linear bounded form on E is in particular linear bounded on each E_B, thus linear continuous on E_B. By definition of the inductive topology, f is linear continuous on E.

If we have (4), consider any bounded linear function $f : E \longrightarrow F$. Then from any $\ell_i n F'$, $\ell \circ f$ is bounded, thus continuous. Thus f is continuous from E to F_{w*}. Thus f is continuous from E_μ to F by the adjunction 4.0.9. □

Notation 6.2.7. *We denote by* BTOPVEC *the category of bornological lcs and continuous (equivalently bounded) linear maps between them.*

Vector bornologies

Definition 6.2.8. A bornology is said to be a vector bornology if it is stable under addition and scalar multiplication. It is said to be convex if it is stable under convex closure. It is said to be separated if the only bounded sub-vector space in $\mathcal{B}$ is $\{0\}$.

Definition 6.2.9. We consider the category BORNVEC of vector spaces endowed with a convex separated vector bornology, with linear bounded maps as arrows.

One of the reasons why working with bornologies instead of topologies is not that popular is because the image by Top of $E \in$ BORNVEC may not have a separated topology. Counter examples are given in [26, 2.2], where a separation procedure is used afterwards. However we have:

Proposition 6.2.10. *If E is in* BORNVEC *and* $\mathsf{Top}(E)$ *is bornological, then it is a lcs.*

Proof. The fact that the topology $\mathsf{T}(\mathcal{B}_E)$ is convex and makes addition and scalar multiplication continuous is immediate. Let us show that $\mathsf{T}(\mathcal{B}_E)$ is separated: consider x and y two different points in E. Suppose that all open sets containing x contain y. Then the vector space generated by x and y is absorbed by any open set. Indeed, consider an open set U in $\mathsf{T}(\mathcal{B}_E)$. As the point $\{x\}$ is bounded there is a scalar λ such that $x \in \lambda U$. Thus, as $y \in U$ by hypothesis, every element $\mu x + \mu' y$ is in $\mu'' U$ for some scalar μ''. Thus the vector space generated by x and y is absorbed by every absorbing open set, it is thus bounded by Proposition 6.2.6. □

Bornologication

Proposition 6.2.11. *Consider E a lcs. Then* $\mathsf{Top} \circ \mathsf{Born}(E)$ *has the same bounded sets as E (meaning* $\mathsf{Born} \circ \mathsf{Top} \circ \mathsf{Born}(E) = \mathsf{Born}(E)$ *) and is thus bornological.*

What makes this Proposition work is the fact that, when the bornology is already defined as a Von-Neumann Bornology of subsets absorbed by open sets, then considering the topology of *all* bornivorous subsets won't change the bornology.

Proof. The bounded sets of $\mathsf{Top} \circ \mathsf{Born}(E)$ are those which are absorbed by every bornivorous subsets of E. Thus in particular they are absorbed by the open sets of E, and thus they are bounded in E. Now *by definition* of $\mathsf{Top} \circ \mathsf{Born}(E)$, any open set of this lcs absorbs the bounded sets of E. Thus the bounded sets of E are also bounded in $\mathsf{Top} \circ \mathsf{Born}(E)$.

The fact that it is bornological follows directly: we have $\mathsf{Top} \circ \mathsf{Born} \circ \mathsf{Top} \circ \mathsf{Born}(E) = \mathsf{Top} \circ \mathsf{Born}(E)$, thus $\mathsf{Top} \circ \mathsf{Born}(E)$ satisfies the second criterion of Proposition 6.2.6. □

Proposition 6.2.12. *If E is a lcs,* $\mathsf{Top} \circ \mathsf{Born}(E)$ *is the coarsest bornological lcs topology on E which preserves the bounded sets of E.*

Proof. Any bornological topology on E absorbs the bounded sets of E and thus contains the open sets of $\mathsf{Top} \circ \mathsf{Born}(E)$. □

Definition 6.2.13. We denote by $\bar{E}^{born}$ the lcs $\mathsf{Top} \circ \mathsf{Born}(E)$, which is called the *bornologification* of E.

From the previous Proposition it follows:

Proposition 6.2.14. *We have the left polarized closure:*

$$\begin{tikzcd} \mathrm{BTOPVEC} \arrow[r, bend left, "U"] \arrow[r, phantom, "\perp"] & \mathrm{BTOPVEC} \arrow[l, bend left, "\mathsf{Top}\circ\mathsf{Born}"] \end{tikzcd}$$

in which U denotes the forgetful functor, which leaves objects and maps unchanged.

One could likewise characterize the vector spaces E of BORNVEC in which every set absorbed by the open sets of $\mathsf{Top}(E)$ are exactly the bounded ones. These are the sets for which the bounded linear maps onto are exactly those continuous onto $\mathsf{Top}(E)$. See [26, 2.1] or the introduction of [6] for more detail.

Proposition 6.2.15. *[44, 13.5] The direct sum of any family of bornological lcs is bornological, as is the quotient of a bornological lcs by a closed subspace, as is the finite product of bornological lcs.*

Because bounded sets in a metrizable lcs are generated by closed balls, one shows easily:

Proposition 6.2.16. *Metrisable lcs are in particular bornological lcs, and thus they are endowed with their Mackey-topology.*

6.2.2 Mackey-completeness

Mackey-complete spaces were defined in Section 2.4.3 as those lcs in which Mackey-Cauchy nets were convergent. This definition is equivalent to the following one:

Definition 6.2.17. A lcs E is said to be *Mackey-complete* if for every bounded and absolutely convex subset B of E, the normed space E_B is complete. E_B coincides with the linear span of B with the norm $p_B : x \mapsto \inf\{\lambda \in \mathbb{K}, \frac{x}{\lambda} \in B\}$, see 3.1.12.

Thus we have in particular that Mackey-completeness is inherited by closed bounded inclusions, and thus by closed continuous inclusions.

Let us note that this definition allows to use Mackey-completeness on vector spaces endowed with bornologies and not topologies. Indeed Mackey-completeness depends only of the bounded subsets of a lcs E (and thus depends only of its dual E' by Proposition 3.4.10), and does not depends directly of the topology of E.

Proposition 6.2.18. *[53, I.2.15] Mackey-completeness is preserved by limits, direct sums, strict inductive limits of sequences of closed embeddings. It is not preserved in general by quotient nor general inductive limits.*

Proposition 6.2.19. *[53, I.2.15] Let E and F be lcs. If F is Mackey-complete, then so is $\mathbf{L}_\beta(E, F)$.*

Proof. Consider $(f_\gamma)_{(\gamma\in\Gamma)}$ a Mackey-Cauchy net in $\mathcal{B}(E, F)$. Each one of the nets $(f_\gamma(x))_{\gamma\in\Gamma}$ converges towards $f(x) \in F$ due to the Mackey-completeness of F. The function f thus defined is bounded and $(f_\gamma)_{(\gamma\in\Gamma)}$ converges towards f. Consider $(f_\gamma)_{(\gamma\in\Gamma)}$ a Mackey-Cauchy net in $\mathcal{B}(E, F)$: we are given a net $(\lambda_{\gamma,\gamma'}) \subset \mathbb{R}$ decreasing towards 0 and an equibounded B in $\mathcal{B}(E, F)$ such that

$$f_\gamma - f_{\gamma'} \in \lambda_{\gamma,\gamma'} B.$$

Consider also $x \in E$. As $B(\{x\})$ is bounded in F, $(f_\gamma(x))_{\gamma\in\Gamma}$ is also a Mackey-Cauchy net. Besides, F is Mackey-complete, so each of these Mackey-Cauchy nets converges towards $f(x) \in F$. Let us show that f is bounded. Indeed, consider b a closed bounded set in E, and U a 0-neighbourhood in F. As B is equibounded, there is $\lambda \in \mathbb{C}$ such that $B(b) \subset \lambda U$. Consider $\gamma_0 \in \Gamma$ such that, if $\gamma, \gamma' \geqslant \gamma_0$ then $|\lambda_{\gamma,\gamma'}| < \lambda$. Consider $\mu \in \mathbb{C}$ such that $f_{\gamma_0}(b) \subset \mu U$. Then for all $\gamma \geqslant \gamma_0$, $f_\gamma(b) \subseteq \mu U + \lambda U$. Thus $f(b)$ is in $(\lambda + \mu)\bar{U}$, thus in $3(\lambda + \mu)\bar{U}$. We proved that $f(b)$ is a bounded set, and so f is bounded. □

Definition 6.2.20. We denote by MCO the category of Mackey-complete vector spaces with and continuous linear functions between them.

Kriegl and Michor only make use of a Mackey-completion which is functorial on bounded linear maps. This is very restrictive and made impossible to extend the results of [49] to modelize classical Linear Logic. The developments in this Chapter are made possible by the following key proposition:

Theorem 6.2.21. *[17, 3.11] The full subcategory* $\textsc{Mco} \subset \textsc{TopVec}$ *of Mackey-complete spaces is a reflective subcategory with the Mackey completion* $\widehat{\_}^M$ *as left adjoint to inclusion .*

Lemma 6.2.22. The intersection $\widehat{E}^M$ of all Mackey-complete spaces containing E and contained in the completion $\tilde{E}$ of E, is Mackey-complete and called the Mackey-completion of E.

We define $E_{M;0} = E$, and for any ordinal λ, the subspace

$$E_{M;\lambda+1} = \cup_{(x_n)_{n\geqslant 0}\in M(E_{M;\lambda})}\overline{\Gamma(\{x_n, n \geqslant 0\})} \subset \tilde{E}$$

where the union runs over all Mackey-Cauchy sequences $M(E_{M;\lambda})$ of $E_{M;\lambda}$, and the closure is taken in the completion. We also let for any limit ordinal $E_{M;\lambda} = \cup_{\mu<\lambda}E_{M;\mu}$. Then for any ordinal λ, $E_{M;\lambda} \subset \widehat{E}^M$ and eventually for $\lambda \geqslant \omega_1$ the first uncountable ordinal, we have equality.

Proof. The first statement comes from stability of Mackey-completeness by intersection. It is easy to see that $E_{M;\lambda}$ is a subspace. At stage $E_{M;\omega_1+1}$, by uncountable cofinality of ω_1 any Mackey-Cauchy sequence has to be in $E_{M;\lambda}$ for some $\lambda < \omega_1$ and thus each term of the union is in some $E_{M;\lambda+1}$, therefore $E_{M;\omega_1+1} = E_{M;\omega_1}$.

Moreover if at some λ, $E_{M;\lambda+1} = E_{M;\lambda}$, then by definition, $E_{M,\lambda}$ is Mackey-complete (since we add with every sequence its limit that exists in the completion which is Mackey-complete) and then the ordinal sequence is eventually constant. Then, we have $E_{M,\lambda} \supset \widehat{E}^M$. One shows for any λ the converse by transfinite induction. For, let $(x_n)_{n\geqslant 0}$ is a Mackey-Cauchy sequence in $E_{M;\lambda} \subset F := \widehat{E}^M$. Consider A a closed bounded absolutely convex set in F with $x_n \longrightarrow x$ in F_A. Then by [44, Prop 10.2.1], F_A is a Banach space, thus $\overline{\Gamma(\{x_n, n \geqslant 0\})}$ computed in this space is complete and thus compact (since $\{x\} \cup \{x_n, n \geqslant 0\}$ is compact in the Banach space), thus its image in $\tilde{E}$ is compact and thus agrees with the closure computed there. Thus every element of $\overline{\Gamma(\{x_n, n \geqslant 0\})}$ is a limit in E_A of a sequence in $\Gamma(\{x_n, n \geqslant 0\}) \subset E_{M;\lambda}$ thus by Mackey-completeness, $\overline{\Gamma(\{x_n, n \geqslant 0\})} \subset F$. We thus conclude to the successor step $E_{M;\lambda+1} \subset \widehat{E}^M$, the limit step is obvious.

□

Thus we have a left polarized closure

$$\begin{tikzcd} \textsc{TopVec} \arrow[r, bend left, "\widehat{\_}^M"] \arrow[r, phantom, "\perp"] & \textsc{Mco} \arrow[l, bend left, "U"] \end{tikzcd}$$

We will also make use of Lemma 3.7 of [17], which relates Mackey-completions and Mackey-duals.

Proposition 6.2.23. *Consider* E *a space endowed with its Mackey-topology. Then* $\hat{E}^M$ *is still endowed with the Mackey-topology* $\mu(\hat{E}^M, E')$.

Proof. Remember that our Mackey-completion preserves the dual, thus $\mu(\hat{E}^M, E')$ is indeed the Mackey topology on $\hat{E}^M$. Moroever $\hat{E}^M$ is constructed as the intersection $\widehat{E}^M$ of all Mackey-complete spaces containing E and contained in the completion $\tilde{E}$ of E . Therefore an absolutely convex weakly compact set in F' coincide for the weak topologies induced by F and $\widetilde{F}$ and therefore also $\widehat{F}^M$, which is in between them. Thus the continuous inclusions $((F'_\mu)'_\mu) \longrightarrow (\hat{F}^M)'_\mu)'_\mu \longrightarrow ((\widetilde{F})'_\mu)'_\mu$ have always the induced topology. In the transfinite description of the Mackey completion, the Cauchy sequences and the closures are the same in $((\widetilde{F})'_\mu)'_\mu$ and $\widetilde{F}$ (since they have same dual hence same bounded sets), therefore one finds the stated topological isomorphism. □

6.2.3 Convenient spaces

In this section, we develop the theory of convenient spaces as defined by Frolicher and Kriegl [26], and studied in [6].

Definition 6.2.24. A lcs is said to be convenient if it is Mackey-complete and bornological.

Again, beware that the spaces called convenenient in [53] or [49] are just Mackey-complete, and not bornological. We defined Mackey-complete spaces as those spaces for which E_B is always a Banach, when B is an absolutely convex and closed bounded subset of E (definition 6.2.17).

By definition 2.4.22, a Mackey-Cauchy net is exactly a net which is Cauchy for one of the norms p_B. Thus by Proposition 6.2.6 a Mackey-complete bornological space is an inductive limit of Banach spaces.

Definition 6.2.25. A lcs E is said to be *ultra-bornological* if it can be represented as an inductive limit of Banach spaces (see [44, 13.1]).

Ultrabornological are in particular barrelled (see Section 3.4.4). Barrelledness is a very strong property, which allows for a Banach-Steinhauss theorem [44, 11.1.3] saying that simple convergence and strong convergence are the same for linear functions.

Definition 6.2.26. A set of functions $\mathsf{B} : \{f : E \longrightarrow F\}$ between two lcs is said to be simply bounded if for every $x \in E$, the set $\{f(x) | f \in \mathsf{B}\}$ is bounded in F. It is said to be strongly bounded, or equibounded, if for any bounded set $B \subset E$, the set $\bigcup f(B)$ is bounded in F. Simply bounded sets of linear continuous functions are exactly the bounded subsets of $\mathcal{L}_\sigma(E, F)$, while strongly bounded sets of linear continuous functions are exactly bounded subset of $\mathcal{L}_\beta(E, F)$.

It is clear that strongly bounded sets of functions are in particular simply bounded. We show that under an assumption of Mackey-completeness, the converse is true for linear bounded functions

Proposition 6.2.27. *Consider E a Mackey-complete space. Then a subset $\mathsf{B} \subset \mathbf{L}(E, F)$ is simply bounded if and only if it is strongly bounded.*

Proof. Let b be a bounded set of E. Taking the absolutely convex closed closure of b, and because E is Mackey-complete, we can assume without loss of generality that E_B is a Banach space. Since any bounded linear maps on a Banach space is also continuous, we can apply the classical Banach Steinhaus theorem on the restriction of B to linear bounded functions on E_B: it is equicontinuous. In particular, it sends bounded sets of E_B to bounded sets of F and $B(b)$ is bounded. $\square$

Corollary 6.2.28. If E is Mackey-complete and bornological, then a subset $\mathsf{B} \subset \mathcal{L}(E, F)$ is simply bounded if and only if it is strongly bounded.

A fundamental property of convenient spaces is that Mackey-completeness is preserved by bornologification:

Proposition 6.2.29. *If E is a Mackey-complete lcs, then $\bar{E}^{born}$ is Mackey-complete.*

Proof. By Proposition 6.2.11 the lcs $\bar{E}^{born} := \mathsf{Top} \circ \mathsf{Born}(E)$ has the same bounded sets as E. As Mackey-completeness only depends on the bounded sets, if E is Mackey-complete so is $\bar{E}^{born}$. $\square$

Thus if we Mackey-complete and then bornologify a lcs E we obtain a Mackey-complete and bornological vector space.

Notation 6.2.30. *We denote by* $\textsc{Conv}$ *the category of bornological Mackey-complete lcs, also called* convenient spaces*, and continuous linear maps between them. These are the convenient spaces in [6] and [26]. We denote by*

$$\bar{E}^{Conv} = \bar{E}^{M,born}$$

the closure operation which makes a lcs convenient, that is the bornologification of the Mackey-completion. Thanks to Theorem 6.2.21, it enjoys a universal property with respect to linear bounded (hence continuous) maps from E to a Mackey-complete lcs F.

In diagrams, this amounts to say that we have a polarised closure:

$$\begin{tikzcd} \textsc{Conv} \arrow[r, bend left, "U"] \arrow[r, phantom, "\top"] & \textsc{Mco} \arrow[l, bend left, "\bar{\cdot}^{born}"] \end{tikzcd}$$

6.3 A positive model of MALL with bornological tensor

6.3.1 Co-products of bornological spaces

By Proposition 6.2.15, we have that Mackey tensor and the co-product of lcs preserves the bornologicality of spaces. That is, bornological spaces are well behaved with respect to the *positive connectives* of Linear Logic. It is not the case however for the negative connectives of linear logic:

Proposition 6.3.1. *[44, 13.5.4] The product $\prod_{i\in I} E_i$ of any family $(E_i)_I$ of bornological spaces is bornological if and only if $\mathbb{K}^I$ is bornological, if and only if the cardinal of I does not admit a Ulam measure [77].*

Remark 6.3.2. The question of whether there are sets which carry a Ulam measure is an open question. In ZFC however, if the cardinal of a set is accessible, then it does not admits a Ulam measure. There are models of ZFC in which every cardinal is accessible.

6.3.2 A tensor product preserving the Mackey-topology

Spaces of linear functions. The space of linear maps endowed with the topology of uniform convergence on bounded subsets is not necessarily bornological: we need to bornologize it.

Remark 6.3.3. The topological dual does not necessarily preserve the bornological condition. As recalled by Erhhard [18], one can construct on the vector space ℓ_∞ a non-null bounded linear continuous function which sends c_0 (*i.e.* all sequences converging to 0) to the scalar 0. This function cannot be continuous on $\ell_\infty = {\ell_1}^\perp$, endowed with its normal topology, as c_0 is dense in ${\ell_1}^\perp$. Thus ℓ_∞ is not bornological.

Remember from Section 2.2.3 that the normal topology of $\ell_\infty = {\ell_1}^\perp$ is the topology of uniform convergence on equicontinuous subsets of ℓ_1.

More generally, one cannot prove without further hypothesis on a lcs E that if F is bornological, then so is $\mathcal{L}_\beta(E, F)$.

Definition 6.3.4. We denote by $\mathcal{L}_{born}(E, F)$ the lcs of all linear continuous functions between two lcs E and F, endowed with the bornologification of the topology $\mathcal{L}_\beta(E, F)$.

The bounded tensor product. Let us recall more precisely the monoidal structure of BORNVEC as explained by Kriegl and Michor [53, 5.7].

Proposition 6.3.5. *The bounded tensor product on* BORNVEC *is symmetric and associative, and $E \otimes_\beta \_$ is left adjoint to the hom-set functor which maps F to $\mathbf{L}(F, \_)$.*

From this monoidal structure on BORNVEC, we deduce the monoidal structure of BTOPVEC:

Proposition 6.3.6. *The β tensor product on* BTOPVEC *is symmetric ans associative, and $E \otimes_\beta \_$ is left adjoint to the hom-set functor which maps F to $\mathcal{L}_{born}(F, \_)$.*

Definition 6.3.7. Consider E and F two lcs. According to Section 3.6, the β-tensor $E \otimes_\beta F$ is the vector space $E \otimes F$ endowed with the finest lcs topology such that the canonical bilinear map $h_\beta : E \times F \longrightarrow E \otimes_\beta F$ is β-hypocontinuous.

Neighbourhoods of 0 in $E \otimes_\beta F$ are then generated by the prebasis consisting of product of bounded sets in E (resp. F) and 0-Neighbourhoods in F (resp. E). This tensor product enjoys a universal property with respect to β-hypocontinuous functions: these are the bilinear function which, restricted to a bounded set of E (resp. F) are continuous on F (resp. E). Then we recall from Section 3.6:

Proposition 6.3.8. *If $f : E \times F \longrightarrow G$ is β-hypocontinuous, then there is a unique linear continuous map $f_\beta : E \otimes_\beta F \longrightarrow G$ such that $f = f_\beta \circ h_\beta$.*

If E and F are bornological, then β-hypocontinuous bilinear functions $f : E \times F \longrightarrow G$ are exactly those which send a product of bounded sets $B_E \times B_F \in E \times F$ to a bounded sets of G. As the bounded sets of a product are exactly product of bounded sets (this is straightforward from the definition of the product topology, see Section 3.1.4), we have then the following fact:

Lemma 6.3.9. If E and F are bornological, then the β-hypocontinuous bilinear functions from $E \times F$ to any lcs are exactly the bilinear bounded ones.

Thus on bornological lcs the β-tensor product is the bounded tensor product described by Kriegl and Michor [53, I.5.7]: $E \otimes_\beta F$ is the algebraic tensor product with the finest locally convex topology such that $E \times F \to E \otimes F$ is bounded.

Proposition 6.3.10. *Consider E and F two bornological lcs. Then $E \otimes_\beta F$ is bornological.*

Proof. Consider G a lcs. By definition of the β-tensor product, we have the algebraic equality:

$$\mathcal{L}(E \otimes_\beta F, G) \sim \mathbf{B}_\beta(E \otimes_\beta F, G).$$

By the previous lemma, we have that $\mathbf{B}_{beta}(E \times F, G) \sim \mathbf{B}(E \times F, G)$, the space of bilinear bounded maps between $E \times F$ and G.

Consider now a bounded linear map $f \in \mathbf{L}(E \otimes_\beta F, G)$. This function coincide with a bounded bilinear map $\tilde{f}$ on $E \times F$: products of bounded sets in B_E and B_F respectively are bounded in $E \otimes_\beta F$, and thus $\tilde{f}$ sends a $B_E \times B_F$ on a bounded set in G. Thus $\tilde{f} \in \mathbf{B}_{beta}(E \otimes_\beta F, G)$, and thus f is continuous on $E \otimes_\beta F$. Therefore $E \otimes_\beta F$ is bornological.

□

We have thus a monoidal structure on BTOPVEC, where the *tensor product need not be submitted to any closure operation*.

6.3.3 A bornological $\parr$ for CHU

A model in MACKEY We proved in Section 6.3.2 that BTOPVEC endowed with $\otimes_\beta$ is a symmetric monoidal category. Following our quest for classical models of DiLL, the natural idea then would be to define an interpretation of $\parr$ as the dual of $\otimes_\beta$ in MACKEY: $E \parr_{\text{MACKEY}} F := ((\overline{(E)'_\mu}^{born} \otimes_\beta \overline{(F)'_\mu}^{born})'_\mu$. This would result in a model of MALL in the category MACKEY, in which *positive connectives are bornological lcs*. We try now to formalize this idea that positives are interpreted by bornological lcs in a model made of Mackey lcs, through a chirality between bornological lcs and Chu pairs. The following developments must then be read in analogy with Section 5.3 on Weak spaces.

Preconvenient dual pairs To obtain dual pairs wich result on bornological lcs, and not just Mackey lcs, a little more material is needed. Indeed, considering a dual pair (E_1, E_2), the dual of $\bar{E}^{born}_{1,\mu(E_1,E_2)}$ contains but is not restricted to E_2. To have an adjunction, one would need to consider bornological dual pairs , which are the one called preconvenient by Frolicher and Kriegl in [26, 2.4.1].

Remember from Section 4 that we have the following right polarized closures, where $\mathcal{P}$ is the functor which maps a lcs E to the pair (E, E') and $\mathcal{M}$ maps a pair (E_1, E_2) to $E_{1,\mu(E_1,E_2)}$:

$$\begin{tikzcd} \text{TOPVEC} \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\top"] & \text{CHU} \arrow[l, bend left, "\mathcal{M}(\_)"] \end{tikzcd}$$

$$\begin{tikzcd} \text{TOPVEC} \arrow[r, bend left, "\mathcal{P}^\perp"] \arrow[r, phantom, "\top"] & \text{CHU}^{op} \arrow[l, bend left, "(\mathcal{M}(\_))'_\mu"] \end{tikzcd}$$

Definition 6.3.11. We denote by PRECONV the category of dual pairs (E_1, E_2) which are invariant the composition of the functors $\mathcal{P} \circ \text{Born} \circ \text{Top} \circ \mathcal{M}$. That is, the dual of the bornologification of $E_{1,\mu(E_1,E_2)}$ is still E_2. According to [26, 2.4.1] these are exactly the dual pairs such that E_2 contains all the bounded linear forms on E_1 endowed with the bornology of $\sigma(E_1, E_2)$ weakly bounded subsets. We denote by DPRECONV the category of dual pairs (E_2, E_1) such that $(E_1, E_2) \in$ PRECONV.

Remark 6.3.12. Following Frölicher and Michor, we remark that a dual pair (E_1, E_2) is bornological if and only if the bornologification of the Mackey-topology on E_1 is exactly the Mackey topology on E_1.

Proposition 6.3.13. *The following diagrams define a right polarized chirality:*

$$\begin{tikzcd} \text{BTOPVEC} \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \text{PRECONV} \arrow[l, bend left, "\bar{\mathcal{M}(\_)}^{born}"] \end{tikzcd}$$

$$
\begin{tikzcd}
(\text{BTOPVEC}, \otimes_\beta, \mathbb{R}) \arrow[r, bend left, "\mathcal{P}^\perp"] \arrow[r, phantom, "\perp"] & (\text{PRECONV}^{op}, \parr, (\mathbb{R}, \mathbb{R})) \arrow[l, bend left, "(\overline{\mathcal{M}(\_)})'_\mu{}^{born}"]
\end{tikzcd}
$$

Proof. Remember that morphisms in CHU between (E_1, E_2) and (F_1, F_2) are pairs (f, f') in $(L(E_1, E_2), L(F_2, F_1))$ such that such that the following diagram commute:

$$
\begin{tikzcd}
E_1 \arrow[r, "f"] \arrow[d, "(\cdot)^*"] & E_2 \arrow[d, "(\cdot)^*"] \\
F_1^* \arrow[r, "f'^*"] & F_2^*
\end{tikzcd}
.
$$

Let us show then that we have natural isomorphisms, for F a bornological lcs and (E_1, E_2) a bornological pair:

$$\mathcal{L}(F, \bar{E}^{born}_{1,\mu(E_1,E_2)}) \simeq \text{CHU}((E_1, E_2), (F, F')).$$

By definition, as (E_1, E_2) is a preconvenient dual pair, we have $\bar{E}^{born}_{1,\mu(E_1,E_2)} \simeq E_{1,\mu(E_1,E_2)}$, and the adjunction follows from Section 4. The adjunction for the second diagram goes likewise, because we reversed the order of the vector spaces in the dual pair.

The second diagrams features strong monoidal adjunctions: by the universal property of $\otimes_\beta$ in BORNVEC and the fact that when E is bornological we have $\mathbf{L}(E, F) = \mathcal{L}(E, F)$. Moreover, when E and F are bornological then $(E \otimes_\beta F)' = \mathbf{L}(E \otimes_\beta F, \mathbb{R}) = \mathbf{L}(E, F^\times) = \mathcal{L}(E, F')$, and thus

$$\mathcal{P}^\perp(E \otimes_\beta F) = (E \otimes F, \mathcal{L}(E, F'))^\perp = (\mathcal{L}(E, F'), E \otimes F) = \mathcal{P}^\perp(E) \parr \mathcal{P}^\perp(F).$$

Likewise, we have that when E and F are bornological $E \otimes_\beta F$ is bornological, and thus already endowed with its Mackey-topology. Thus $\mathcal{M}\overline{(E', E)}^{born} \otimes_\beta \mathcal{M}\overline{(F', F)}^{born} = \mathcal{M}(E \otimes F, \mathcal{L}(E, F'))$ $\square$

Notice that because the category PRECONV is not symmetric when it concerns its dual pairs, we are not here in the setting of the negative chiralities defined in Section 2.3.2. Indeed, the second adjunction is between BTOPVEC and DPRECONV when it should be between BTOPVEC and PRECONV^{op}

Theorem 6.3.14. *These adjunctions between BTOPVEC and DPRECONV define a positive interpretation of* MALL*.*

Proof. We showed the adjunctions in Proposition 6.3.13. We need now to show that the two closures define the same action on the negatives. With the categorical notations of Section 2.5.2.3, this would amount to:

$$clos_N : \uparrow N^{\perp_R} \simeq (\downarrow N)^{\perp_L}.$$

Thus we must prove that for any pair $(N_1, N_2) \in D$PRECONV we have:

$$\mathcal{P}((\mathcal{M}(\overline{(P_1, P_2)}))'_\mu{}^{born} = P^\perp(\mathcal{M}((\bar{N_1}, N_2))^{born}$$

This equation is straighforward, as $(\mathcal{M}((P_1, P_2)))'_\mu = (P_2)_{\mu(P_1)}$, which is by definition bornological thus $\mathcal{P}((\mathcal{M}(\overline{(P_1, P_2)}))'_\mu{}^{born} = (P_2, P_1)$, and likewise $P^\perp(\mathcal{M}((\bar{N_1}, N_2))^{born} = (P_2, P_1)$ $\square$

Remark 6.3.15. In this proof, we showed that the closure operation is the identity even on the interpretation of the negatives. Thus one cannot argue that any computation is done here.

6.4 CONV and COMPL, a positive interpretation of DiLL

What has been done above for bornological spaces needs to be lifted to convenient spaces, that is bornological and Mackey-complete spaces, in order to interpret smooth functions.

As explained in Section 6.1, we conducted in [17] a study on classical smooth models of Linear Logic in which the $\parr$ is interpreted as Swchartz' ε product. What is done here is very similar, but takes the view point of chiralities and employs bornological spaces instead of Schwartz spaces. The construction with Schwartz spaces leads to a $*$-autonomous category as the Schwartzification preserves the dual. This is not possible on bornological spaces: the bornologification does not preserve the dual. Thus we have indeed for a bornological space E:

$$(E^{\perp_L})^{\perp_R} = (\overline{(E'_\mu)'_\mu})^{born} \simeq E$$

but writing $E^\perp = \overline{(E'_\mu)}^{born}$ we have

$$E^{\perp\perp} \not\simeq E$$

as $\overline{(E'_\mu)}^{born}$ may have as dual a vector space strictly included in E.

Thus we make use of a polarized setting and of a dual characterization of bornological spaces (see proposition 6.4.9): a lcs is bornological if and only it is Mackey and its strong dual is complete.

6.4.1 Multiplicative connectives

6.4.1.1 An internal hom-set on convenient sets

This Section reviews results which can be found in [44, 13.1] or [26].

Proposition 6.4.1. *If E is convenient (i.e. bornological and Mackey-complete), then E is endowed with the topology $\beta(E, E')$ of uniform convergence on weakly bounded subsets of E' (i.e on subsets B such that for every $x \in E$ $B(x)$ is bounded in $\mathbb{K}$). That is, a convenient lcs is barrelled , see Section 3.4.4.*

Proof. If E is bornological it is linearly homeomorphic to the inductive limit of the E_B, where B is an absolutely convex and weakly closed bounded subset of E (see Proposition 6.2.6). When E is Mackey-complete these E_B are Banach spaces by definition. However a Banach space E_B is always barrelled: it is as all lcs endowed with the topology of uniform convergence on equicontinuous subsets of E'_B and these are exactly the simply bounded sets, due to Banach Steinhaus theorem. Moreover, E_B has its Mackey-topology, and the Mackey-topology $\mu(E, E')$ is preserved by inductive limits [44, 8.9.11]. Thus E as a bornological space is endowed with the topology $\mu(E, E') = ind_B\mu(E_B, E'_B) = ind_B\beta(E_B, E'_B)$. As a weakly bounded set in E' is in particular a product of weakly bounded subsets of E'_B, and weak topologies are preserved by projective limits [44, 8.8.6], we have our result. □

Thus, as a bornological space is always endowed in particular with its Mackey-topology $\mu(E, E')$ (see Proposition 6.2.6), we have that through the bipolar theorem:

Proposition 6.4.2. *When E is a convenient space, the bornology $\mu(E')$ of absolutely convex and weakly compact subsets of E' and the one of absolutely convex and weakly closed bounded sets (i.e. of the bipolars of weakly bounded sets in E') coincide.*

From the fact that the space of linear bounded maps to a Mackey-complete lcs from another lcs is Mackey-complete 6.2.19, we have immediately:

Proposition 6.4.3. *If E is bornological and F is Mackey-complete, then $\mathcal{L}_\beta(E, F)$ is Mackey-complete for any lcs E.*

From Proposition 6.4.2, we have thus:

Proposition 6.4.4. *If E is convenient and F is Mackey-complete, then $\mathcal{L}_\mu(E, F)$ is Mackey-complete for any lcs E.*

In particular, when we bornologise the hom-set we obtain a convenient space:

Corollary 6.4.5. If E and F are convenient vector spaces, the lcs $\mathcal{L}_{born}(E, F)$ is also convenient.

This results hints for a good chirality between convenient spaces (for the positives) and Mackey-complete spaces (for the negatives). In fact, in order to ensure a good behaviour of the Mackey duals (for the negation), we will need to consider complete and Mackey lcs as interpretation for the negatives, see Theorem 6.4.16.

6.4.1.2 A duality theory for convenient spaces

We showed in Proposition 6.4.4 that the space of linear maps between a convenient space and a Mackey-complete space is Mackey-complete. When we consider linear scalar forms, we have a stronger result which appears for example in [44, 13.2.6]:

Proposition 6.4.6. *If E is bornological, then E'_β is complete. Thus if E is bornological and Mackey-complete, E'_μ is complete.*

Proof. The second assertion follows from the first by using Proposition 6.4.2. The first assertion is straightforward. Consider $\ell \in \widetilde{E'_\beta}$. It is linear as the pointwise limit of linear map. It is bounded: consider B bounded in E, then there is $\ell_n \in E'$ such that $(\ell - \ell_n) \in B^\circ$, thus $\ell(B) \subset \ell_(B) + B_{0,1}$ where $B_{0,1}$ is the unit ball in $\mathbb{R}$. Thus $\ell(B)$ is bounded in $\mathbb{R}$ for any B bounded, thus ℓ is bounded and thus continuous as E is bornological. thus $\tilde{E'}_\beta = E'_\beta$. □

Recall that the ε product (see Section 3.6.3), defined as $E\varepsilon F := \mathcal{L}_\varepsilon(E'_\gamma, F)$ is associative and commutative on complete spaces.

Proposition 6.4.7. *When E and F are convenient spaces, then we have a linear homeomorphism $(E \otimes_\beta F)'_\mu \simeq E'_\mu \varepsilon F'_\mu$.*

Proof. We have the following computations:

$$\begin{aligned}
(E \otimes_\beta F)'_\beta &\simeq \mathbf{B}(E \times F, \mathbb{R}) \text{as E and F are bornological} \\
&\simeq \mathbf{L}(E, F^\times) \\
&\simeq \mathcal{L}_\beta(E, F'_\beta) \text{as } E \text{ and } F \text{ are bornological}
\end{aligned}$$

As E and F are convenient, we have $\mathcal{L}_\beta(E, F'_\beta) \simeq \mathcal{L}_\mu(E, F'_\mu)$ (Proposition 6.4.2). Moroever any weakly compact set in E'_μ is compact as E'_μ is endowed with its Mackey topology (see [68]), and thus $(E'_\mu)'_\gamma \simeq (E'_\mu)'_\mu$. Finally, the equicontinuous subset of E seen as the dual of E'_μ are exactly the subsets of $\mu(E, E')$: equicontinuous subsets of E are polars U° of open sets in E'_μ, and open U sets in E'_μ are by definition polars W° of weakly compact and absolutely convex subsets of E. By the bipolar theorem 3.4.1, equicontinuous subset of E are exactly the subsets of $\mu(E, E')$, and thus $\mathcal{L}_\mu(E, F'_\mu) \simeq \mathcal{L}_\varepsilon(E, F'_\mu) \simeq (E'_\mu)\varepsilon(F'_\mu$ by the preceding reasoning. □

We will also make use of Lemma 3.7 of [17], which relates Mackey-completions and Mackey-duals. We recall it below:

Proposition 6.4.8. *Consider E a space endowed with its Mackey-topology. Then $\hat{E}^M$ is still endowed with the Mackey-topology $\mu(\hat{E}^M, E')$.*

Proof. Remember that our Mackey-completion preserves the dual, thus $\mu(\hat{E}^M, E')$ is indeed the Mackey topology on $\hat{E}^M$. Moroever $\hat{E}^M$ is constructed as the intersection $\hat{E}^M$ of all Mackey-complete spaces containing E and contained in the completion $\tilde{E}$ of E . Therefore an absolutely convex weakly compact set in F' coincide for the weak topologies induced by F and $\widetilde{F}$ and therefore also $\hat{F}^M$, which is in between them. Thus the continuous inclusions $((F'_\mu)'_\mu) \longrightarrow (\hat{F}^M)'_\mu)'_\mu \longrightarrow ((\widetilde{F})'_\mu)'_\mu$ have always the induced topology. In the transfinite description of the Mackey completion, the Cauchy sequences and the closures are the same in $((\widetilde{F})'_\mu)'_\mu$ and $\widetilde{F}$ (since they have same dual hence same bounded sets), therefore one finds the stated topological isomorphism. □

We adapt now a proof which can be found in [51, 28.5.4] or [44, 13.2.4]. The same generalisation can be found in the unpublished Master's report [27] by Gach, directed by Kriegl.

Proposition 6.4.9. *Let E be a Mackey lcs such that E'_μ is complete. Then E is bornological.*

This uses Grothendieck' characterization of completeness (Theorem 3.4.21).

Proof. Let E be a Mackey space whose strong dual is complete. By proposition 6.2.6 we just have to show that $E' = E^\times$, *i.e.* that any bounded linear function on E is continuous. Consider $f \in E^\times$, let us show that f is continuous. We make use of Grothendieck's Theorem 3.4.21, which says that $E'_\mu \simeq \tilde{E}'_\mu \sim \{f : E \longrightarrow \mathbb{K} \mid \forall K \subset E, K$ abs. convex weakly compact$f_{|K} : K_{\sigma(E,E')} \longrightarrow \mathbb{K}\}$ is continuous. Thus consider K a weakly compact absolutely convex subset of E. We consider the normed space E_K, normed with p_K (see Definition 3.1.12). As K is in particular bounded, we have that f is bounded and thus continuous on the normed space E_K endowed with p_K.

However on $K \subset E$ the weak topology and p_K coincide. Indeed the weak topology is coarser than the one induced by p_K on E_K (that is the intersection between E_K and a weak-open set is an open set for p_K), K is compact in E_K endowed with p_K, and thus on $K \subset E$ the weak topology and p_K coincide [51, 28.5.2].

Thus on K p_K and the weak topology $\sigma(E, E')$ correspond, and a bounded linear function f is in particular linear continuous on $K_{\sigma(E,E')}$. Thus $E^\times \sim E'$. □

Corollary 6.4.10. Let F be a Mackey lcs which is complete. Then F'_μ is bornological.

Proof. Let us denote $E := F'_\mu$ the Mackey-dual of E. Then $E'_\mu \simeq F$ as F is Mackey. Moreover, E'_β is endowed with a topology which is finer (any weakly compact absolutely convex set is in particular weakly bounded and thus bounded by Proposition 3.4.10). Thus as E'_μ is complete, so is E'_β. Thus by proposition 6.4.9, as E is Mackey as it is defined as a Mackey-dual, we have that E is bornological. □

Corollary 6.4.11. When E and F are Mackey and complete, their respective dual E'_μ and F'_μ are bornological. Since our Mackey-completion 6.2.22 preserve the dual, we have that:

$$E \simeq (\widehat{E'_\mu}^M)'_\mu,$$

$$F \simeq (\widehat{F'_\mu}^M)'_\mu,$$

that is E and F are the Mackey dual of convenient lcs.

Corollary 6.4.12. In particular, if E and F are Mackey and Complete

$$E \varepsilon F$$

is Mackey, as $E \varepsilon F \simeq (E'_\mu \otimes_\beta F'_\mu)'_\mu$.

6.4.1.3 A chirality between convenient and Complete and Mackey spaces

In the previous section, we showed a dual characterization between bornological spaces, whose strong dual is complete, and complete and Mackey spaces, whose Mackey dual is bornological. We know prove a few more results in order to show that we are indeed in the context of a strong monoidal adjunction of duals.

Proposition 6.4.13. *If E is a convenient space and F is complete and Mackey, then we have natural bijections:*

$$\mathcal{L}^{op}(E, \widehat{F'_\mu}^M) \simeq \mathcal{L}(E'_\mu, F),$$

which thus leads to and adjunction between

$$(\_)'_\mu : \textsc{Conv} \longrightarrow \textsc{ComplMackey}$$

and

$$\hat{(\_)}^M_\mu : \textsc{ComplMackey} \longrightarrow \textsc{Conv}.$$

Proof. The functors are is well defined: if E is convenient the E'_μ is complete by proposition 6.4.6. Conversely if F is Mackey and complete, $\widehat{F'_\mu}^M$ is convenient by corollary 6.4.11. Then the adjunction follows from the universal property of the Mackey-completion for continuous linear maps (see Proposition 6.2.22):

$$\mathcal{L}(\widehat{F'_\mu}^M, E) \sim \mathcal{L}(F'_\mu, E)$$

as E is Mackey-complete, and from proposition 4.0.10 as E and F'_μ are Mackey:

$$\mathcal{L}(F'_\mu, E) = \mathcal{L}(E'_\mu, F).$$

□

Remark 6.4.14. Beware that the terminology is particularly uneasy to handle: positives are interpreted by convenient lcs, which are the bornological and Mackey-complete ones, while negatives are interpreted by Mackey and Complete lcs. These Mackey *and* Complete lcs are Mackey-complete as they are complete, but they are more than that.

Proposition 6.4.15. *When E and F are complete Mackey spaces, then we have a bounded linear isomorphism (that is an isomorphism in* $\textsc{Conv}$*):*

$$\widehat{(E\varepsilon F)'_\mu}^M \simeq \widehat{(E'_\mu)}^M \hat{\otimes}_\beta^M \widehat{(F'_\mu)}^M .$$

Proof. If E and F are complete and Mackey, we have by definition

$$(E\varepsilon F)'_\mu = \mathcal{L}_\varepsilon(E'_\gamma, F) = \mathcal{L}_\mu(E'_\mu, F)$$

as equicontinuous in E'_γ are the absolutely convex compact subsets of E, which are exactly the weakly compact as E is Mackey. As $\widehat{(E'_\mu)}^M$ and $\widehat{(F'_\mu)}^M$ are convenient, we have on the other hand that: $\widehat{(E'_\mu)}^M \hat{\otimes}_\beta^M \widehat{(F'_\mu)}^M$ is convenient, thus in particular Mackey and linearly homeomorphic to its double Mackey dual. Thus we get

$$\widehat{(E'_\mu)}^M \hat{\otimes}_\beta^M \widehat{(F'_\mu)}^M \simeq [\mathcal{L}_\mu(\widehat{E'_\mu}, F)'_\mu]^M \simeq (E\hat{\varepsilon}F)'^{M} z_\mu$$

via the functoriality of Mackey-Completion 6.2.22 and Lemma 6.4.8. □

Theorem 6.4.16. *To sum up the two previous propositions, we have a strong monoidal adjunction:*

$$\begin{tikzcd}(\textsc{Conv}, \hat{\otimes}_\beta^M) \arrow[rr, bend left, "(\_)'_\mu"] \arrow[rr, phantom, "\perp"] & & (\textsc{ComplMackey}^{op}, \varepsilon) \arrow[ll, bend left, "\widehat{(\_)'_\mu}^M"]\end{tikzcd}$$

where the composition of the adjoint functors is the identity on $\textsc{Conv}$*. They satisfy for E and F objects of* $\textsc{Conv}$ *and $G \in$* $\textsc{ComplMackey}$*:*

$$\mathbf{L}(E\hat{\otimes}_\beta^M F, \hat{G}^M) \simeq \mathbf{L}(E, F'_\mu \parr G). \tag{6.1}$$

Proof. We are left with proving equation 6.1. Consider E and F convenient spaces and G a complete space. Then:

$$\begin{aligned}
\mathbf{L}((E\hat{\otimes}_\beta F), G) &\simeq \mathbf{L}(E \otimes_\beta F, G) \text{ as } E \otimes_\beta F \text{ is bornological (prop. 6.3.10}\\
&\simeq \mathbf{L}(E, \mathbf{L}(F, G'_\mu)) \text{ by the universal property of } \otimes_\beta \text{ in } \textsc{BornVec}\\
&\simeq \mathcal{L}(E, \mathcal{L}_\beta(F, G'_\mu)) \text{ as } E \text{ and } F \text{ are bornological}\\
&\simeq \mathcal{L}(E, \mathcal{L}_\mu(F, G'_\mu)) \text{ by Proposition 6.4.2}\\
&\simeq \mathcal{L}(E, F'_\mu \parr G) \text{ as } F \text{ and } G \text{ are endowed with their Mackey-topology}
\end{aligned}$$

By functoriality of the Mackey-completion, and as F is already Mackey-complete, this is extended to an isomorphism

$$\mathcal{L}(E\hat{\otimes}_\beta^M F, \bar{G}^M) \simeq \mathcal{L}(E, (F'_\mu \parr G)^M).$$

□

Remark 6.4.17. We *not* are here in presence of a *positive chirality*, as we *do not have* an adjunction:

$$\begin{tikzcd}\textsc{Conv} \arrow[rr, bend left, "\tilde{(\_)}"] \arrow[rr, phantom, "\perp"] & & \textsc{ComplMackey} \arrow[ll, bend left, "\bar{(\_)}^{born}"]\end{tikzcd}$$

The functors are well defined : the completion $\tilde{E}$ of a complete space is till endowed by its Mackey-topology by [44, 8.5.8]. The bornologification $\bar{F}^{born}$ of a complete space (thus in particular Mackey-complete) is still Mackey complete as $\bar{F}^{born}$ and F have the same bounded subsets.

Moreover, by the universal property of the completion one has $\mathcal{L}(\tilde{E}, F) = \mathcal{L}(E, F)$ when F is complete. As E is bornological we have $\mathcal{L}(E, F) = \mathbf{L}(E, F)$ which embeds boundedly in $\mathbf{L}(E, \bar{F}^{born})$ as F and $\bar{F}^{born}$ have the same bounded subsets. However this embedding is not a bijection. Thus the categorical setting developed in Section 2.2.2 does not apply strictly speaking. It is however not an issue: both CONV and COMPLMACKEY embed fully and faithfully in the category TOPVEC and we interpret proofs as arrows in TOPVEC (that is, as plain linear continuous maps). Thus a proof of $\vdash \mathcal{N}$ is interpreted as a function $f \in \mathcal{L}(\mathbb{R}, [\![\mathcal{N}]\!])$ (and not $f \in \mathcal{L}(]\!]\uparrow 1[\![, [\![\mathcal{N}]\!])$ as axiomatized in Section 2.2.2) and a proof of $\vdash P, \mathcal{N}$ as an arrow $f \in \mathcal{L}(([\![P]\!]'_\mu, [\![\mathcal{N}]\!])$ as usually.

Example 6.4.18. A typical complete and Mackey lcs is any Banach or Fréchet space. In particular, spaces of smooth functions $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ are complete and Mackey as they are metrisable and thus bornological. A typical convenient space include any of the previous example, as well as any space of distribution which could be Mackey-complete without being complete.

Outlook 8. It may be enough to consider ultrabornological spaces instead of convenient ones as an interpretation for positive connectives.

6.4.2 Additive connectives and exponentials

In the previous section we constructed a polarized model of MALL. The interpretation of the additive connectives is straightforward via the biproduct on vector spaces. Concerning the exponential, we choose an easy yet disapointing solution by using the free exponential of Frolicher and Kriegl [**?**].

We recall a few results from the literature:

Proposition 6.4.19. *[44, 8.8] The Mackey topology is preserved via products, co-products and inductive limits.*

Proposition 6.4.20. *[53, 2.15] Mackey-completeness is preserved by products, co-products and projective limits.*

Proposition 6.4.21. *Bornological lcs are preserved by inductive limits of lcs ([44, 13.1.5]) and Mackey-completed tensor product (Proposition 6.3.10)*

Thus CONVMACKEY is endowed with a co-product and MACKEYMCO by product, which interprets the additive connectives of linear logic. As in all lcs, they correspond and finite indexes, and gives us a bi-product.

In this Section, we first explain how the exponential detailed in [6] would give us an ad-hoc exponential in our model, and then develop a new one more fitting to the polarized setting.

A space of functions in COMPLMACKEY. The preceding Section gives us a positive model of MALL, which should be extended into a model of DiLL or DiLL_0. One could build a model of $DiLL$ where positive would be interpreted as bornological spaces, and negatives as pre-bornological Chu spaces as in Section 5.2. In this setting without any completeness requirements on the negative, no smooth interpretation for the proofs should be possible.

Let us sketch how we could construct a good interpretation of ? in COMPLMACKEY. One of the point of Frolicher Kriegl and Michor is that "Smoothness of curves is a bornological concept" ([53, 1.8]). Consider E a Mackey-complete and bornological lcs. , thus the definition of Kriegl and Michor apply for spaces of smooth functions applies to E'_μ:

Definition 6.4.22. [53, 3.11] Consider E a Mackey-complete and bornological lcs. Remember that then $E'_\beta \simeq E'_\mu$ is complete (see 6.4.6), and in particular Mackey-complete. The lcs $\mathcal{C}^\infty(E'_\mu, \mathbb{R})$ is the vector space of all smooth functions $f : E' \longrightarrow \mathbb{R}$ such that for all *smooth curve* $c : \mathbb{R} \longrightarrow E'_\mu$, $f \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. It is endowed with the coarsest locally convex and separated topology making:

$$c^* : \mathcal{C}^\infty(E'_\mu, \mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$$

continuous, when $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is endowed with its usual topology of uniform convergence of all derivatives of finite order on every compact of $\mathbb{R}$.

Thus $\mathcal{C}^\infty(E', \mathbb{R})$ is the projective limit of the spaces $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, indexed by curves $c : \mathbb{R} \longrightarrow \mathbb{R}$ (Lemma [53, 3.15]). This definition is to be put in relation with the higher order exponential constructed in Nuclear Spaces (see Section 7.5), where spaces of scalar function $\mathscr{E}(E)$ are also constructed as projective limits[2]. As a projective limit of complete spaces, $\mathcal{C}^\infty(E', \mathbb{R})$ is complete, however we have no proof that it is Mackey, and thus no good construction of the interpretation of ? on the category COMPLMACKEY.

Outlook 9. If $\mathcal{C}^\infty(E', \mathbb{R})$ was to be Mackey, then its Mackey dual would be bornological (Proposition 6.4.9) and thus we could define this way a functor ! : COMPLMACKEY $\longrightarrow$ CONV, which would have to be proved strong monoidal.

The Free exponential of Frolicher, Kriegl and Michor. The definitions of [6], recalled in Section 2.4.3 are however adapted easily to an adjunction between CONV and COMPLMACKEY. The exponential is the bornologification of the Mackey-completion of the vector space having as basis the set of all $\delta_x \in \mathcal{C}^\infty(E, \mathbb{R})'$, $x \in E$.

$$!E := < \overline{\delta(E)} >^{conv} \subset \mathcal{C}^\infty(E, \mathbb{R})_\mu,$$

This is a little disappointing, as the exponential is *as hoc*: it needs to be bornologified and completed.

$$!E := < \overline{\delta(E)} >^{conv} \subset \mathcal{C}^\infty(E, \mathbb{R})_\mu,$$

Then we deduce from [26, 5.1.1], that we have a linear isomorphism preserving bounded subsets:

$$(!E)'_\mu \sim (!E)'_\beta \sim \mathcal{C}^\infty(E, \mathbb{R}).$$

As the spaces considered are not in general bornological (they are *duals* of bornological spaces), this is however not an isomorphism in TOPVEC. We need to bornologize these topologies to make this isomorphisms a topological one.

We begin by pointing out that, in the construction of [26, 5.1.1] and [6], bornologicality is not necessary to define $!E$, and that only Mackey-completeness is necessary. This was already discussed in Section 2.4.3, and the definition on a theory of smooth functions on Mackey-complete spaces in [53]. Thus the functor ! described as an endofunctor of CONV in [6] is well defined as a functor ! : MCO $\longrightarrow$ CONV.

Remark 6.4.23. The functor ! also defines a functor ! : MCO $\longrightarrow$ CONV. and thus in particular an endofunctor of MCO

$$\begin{cases} \text{MCO} \longrightarrow \text{CONV} \\ \quad E \mapsto < \overline{\delta(E)} >^{conv} \\ \quad f \mapsto (\delta_x \mapsto \delta_{\ell(x)}) \end{cases} \tag{6.2}$$

Following [26, 5.1.1], when $E, F \in$ MCO we have a natural *bounded* linear isomorphism:

$$\mathcal{C}^\infty(E, F) \sim \mathbf{L}_\beta(!E, F)$$

resulting in an adjunction with bounded unit $\eta_E \in \mathcal{C}^\infty(E, E)$ and co-unit $d_F : !F \longrightarrow F$. Thus for $E \in$ MCO, η_E is a smooth, and for $F \in$ CONV, d_F is a bounded, equivalently continuous as $!F$ is bornological, linear map. More specifically, as $!E$ is bornological, we have $\mathbf{L}_\beta(!E, F) \simeq \mathcal{L}(!E, F)$. The co-multiplication of this adjunction, which is defined as $\mu_E : !E \longrightarrow !!E; \delta_x \mapsto \delta_{\delta_x}$ is thus also continuous.

Proposition 6.4.24. *For every $E, F \in$ MCO we have a bijection natural in E and F:*

$$\mathcal{L}_\beta(!E, F) \sim \mathcal{C}^\infty(E, F)$$

Definition 6.4.25. Let us denote by COMPLMACKEY$^\infty$ the category of Complete Mackey spaces and *smooth functions* between them, as defined in Definition 6.4.22.

Definition 6.4.26. We define the following functors between COMPLMACKEY$^\infty$ and CONV:

$$! : \begin{cases} \text{COMPLMACKEY}^\infty \longrightarrow \text{CONV} \\ \quad E \mapsto < \overline{\delta(E)} >^{conv} \\ \quad f \mapsto !(f \in \mathcal{L}(!E, F)) \circ \mu_E \end{cases} \tag{6.3}$$

$$V : \begin{cases} \text{CONV} \longrightarrow \text{COMPLMACKEY}^\infty \\ \quad F \mapsto \tilde{F}_{\mu(\tilde{F}, F')} \\ \quad f \mapsto f \circ \iota_F \circ d_{V(F)} \text{ as defined in proof of prop. 6.4.24, and } \iota_F : F \hookrightarrow \tilde{F} \end{cases} \tag{6.4}$$

[2] They differs on the indexes of these projective limits

One easily check that these functors are covariant. The lcs $V(F)$ is well defined as the Mackey-topology on a complete lcs is still complete [44, 8.5.8]. From Proposition 6.4.24 and the strong monoidality of ! (see Proposition [6, 5.6] or Section 2.4.3) follows:

Proposition 6.4.27. *We have a strong monoidal adjunction:*

$$\begin{tikzcd}
(\textsc{ComplMackey}^\infty, \times) \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & (\textsc{Conv}, \otimes_\beta) \arrow[l, bend left, "?_\beta"]
\end{tikzcd}$$

The dual of this adjunction is:

$$\begin{tikzcd}
(\textsc{ComplMackey}, \varepsilon) \arrow[r, bend left, "W"] \arrow[r, phantom, "\perp"] & (\textsc{Conv}^\infty, \oplus) \arrow[l, bend left, "?_\beta"]
\end{tikzcd}$$

with $\textsc{Conv}^\infty$ the category of convenient spaces and smooth functions, and:

$$W : \begin{cases} \textsc{ComplMackey} \longrightarrow \textsc{Conv}^\infty \\ \qquad E \mapsto \bar{E}^{conv} \\ \qquad f \mapsto (d_{E'})' \circ f \end{cases} \tag{6.5}$$

$$? : \begin{cases} \textsc{Conv}^\infty \longrightarrow \textsc{ComplMackey} \\ \qquad E \mapsto \widetilde{\mathcal{C}^\infty(E,\mathbb{R})}_{\mu(!E)} \\ \qquad f \mapsto (\mu_{E'})' \circ \mathcal{C}^\infty(f, \_) \end{cases} \tag{6.6}$$

Co-dereliction is interpreted as in [6], thus resulting in a polarized and classical model of Differential Linear Logic.

Outlook: Exponentials as spaces of compact support distributions The preceding constructions are however deceptive, as neither $!F$ nor $?E$ are proved to be intrinsically convenient (resp. Mackey). In Section 7.5, we construct an exponential $!_{distr} E$ (denote $!E$ in that Section) as an inductive limit of spaces $\mathscr{E}'(\mathbb{R}^n)$ of distributions with compact support.

We hint in Section 7.5.5 how this exponential could lead to a model of $DiLL$, agreeing with the adjunction:

$$\begin{tikzcd}
(\textsc{ComplMackey}^\infty, \times) \arrow[r, bend left, "!_{distr}"] \arrow[r, phantom, "\perp"] & (\textsc{Conv}, \otimes_\beta) \arrow[l, bend left, "?_\beta"]
\end{tikzcd}$$

Part III

Distributions and Linear Partial Differentials Equations in DiLL

Chapter 7

Distributions, a model of Smooth DiLL

As we pointed out in throughout this thesis, in a denotational model with reflexive spaces the exponential should be thought of as a space of distributions with compact support, that is:

$$!E \simeq \mathcal{C}^\infty(E, \mathbb{R})'.$$

Spaces of smooth functions are defined on euclidean spaces $E = \mathbb{R}^n$, and endowed with the topology of uniform convergence of all derivatives on every compact subset of $\mathbb{R}^n$ (Definition 3.2.4). Their main property is to be *Fréchet* and *Nuclear*, and as such they are *reflexive* (see Proposition 7.2.25). Thus we begin this Chapter by exposing the theory of Fréchet spaces and their dual in Section 7.1.1, then recall the definition and main properties of Schwartz and Nuclear spaces in Section 7.2.2, thus allowing for a polarized model of MALL. To interpret the exponentials, we detail in Section 7.3 the theory of distributions and their kernel theorems. This gives a model of DiLL without higher-order, which is detailed in Section 7.4. We then extends this model to a model of DiLL_0 in Section 7.5, by defining the exponential as a projective limit of spaces of distributions.

Notation 7.0.1. *Unless it is mentioned otherwise, the notation E' here always refer to the strong dual E'_β of the lcs E. However, when E' is a reflexive space, it is Mackey and barrelled, and thus $E'_\beta \simeq E'_\mu$.*

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7.1 (F)-spaces and (DF)-spaces

We begin this Chapter by briefly exposing the duality theory for metrisable spaces. Indeed, the strong dual E'_β of a metrisable space E is not necessarily metrisable. In fact, if E and E' are metrisable, then E is normable [44, 12.4.4]. Thus we will characterize the dual of metrisable spaces, which are necessarily complete, and built a model of $IMALL$ in which $\parr$ is in particular interpreted by the ε-product.

7.1.1 The duality of (F)-spaces

Definition 7.1.1. Recall from Chapter 3, section 3.1, that a lcs is metrisable if and only if it admits a countable basis of 0-neighbourhoods. A (F)-space, or *Fréchet* lcs, is a complete and metrisable lcs.

Proposition 7.1.2. *A metrizable lcs, and in particular a (F)-space, is endowed with its Mackey topology.*

Proof. A metrizable set is in particular bornological, and thus is Mackey by proposition 6.2.6. Indeed, in a lcs metrizable with a metric d, bounded sets are generated by the closed ball B of the points at distance $\mathrm{d}(0, x) \leqslant 1$ of the origin. Thus a set absorbing all bounded sets contains in particular λB for $\lambda > 0$, and thus contains the open set $\frac{\lambda}{2}\dot{B}$, and as such is a 0-neighbourhood. □

Definition 7.1.3. A (DF)-space (*dual of Fréchet*) is a lcs E

- admitting a countable basis of bounded sets $\mathcal{A} = (A_n)_n$, that is a collection of bounded set such that every bounded set in included into an object of $\mathcal{A}$,
- and such that if $(U_n)_n$ is a sequence of closed and absolutely convex neighbourhoods of 0 whose intersection U is bornivorous, then U is a neighbourhood of 0.

Remark 7.1.4. It is costless to ask that for every n, the set A_n is absolutely convex and $A_n + A_n \subset A_{n+1}$. We will therefore always suppose that this is the case.

Although definition 7.1.3 may seem obscure, it is the right one for interpreting the dual and pre-dual of (F)-spaces.

Proposition 7.1.5 ([37] IV.3.1)**.**
- *If F is metrisable, then its strong dual E' is a complete (DF)-space.*
- *If E is a (DF)-space and F an (F)-space, then $\mathcal{L}_\beta(E, F)$ is an (F)-space. In particular, E' is an (F)-space.*

Proposition 7.1.6 ([44] 12.4.2 and 15.6.2)**.** *The class of (DF)-spaces is preserved by countable inductive limits, countable direct sums, quotient, completions and projective tensor product. The class of (F)-spaces is preserved by products and completed projective tensor products $\hat{\otimes}_\pi$.*

Remark 7.1.7. A (F)-space which is also a (DF)-space is necessarily a normable lcs. Thus a nuclear (Definition 7.2.18) (F)-space and (DF)-space is a finite dimensional vector space (see proposition 7.2.21).

$$\begin{tikzcd} \textsc{TopVec} \arrow[r, bend left, "\tilde{}"] \arrow[r, phantom, "\top"] & \textsc{Compl} \arrow[l, bend left, "U"] \end{tikzcd}$$

7.1.2 (F)-spaces and the ε product

Remember that (F)-spaces are the complete metrisable topological vector spaces. From the literature [44, 16.1.3, 16.1.5, 16.2.6] we have:

Proposition 7.1.8. *The ε product of two (F)-spaces is again a (F)-space, and it is associative and symmetric on the category* $\textsc{Fre}$ *of (F)-spaces and linear continuous maps.*

An important result is that Grothendieck's problem (see section 3.6) is resolved for the γ-tensor product on Fréchet spaces.

Lemma 7.1.9. When E and F are (F)-spaces, the compact subsets of $E \otimes_\gamma F$ are included in tensor products of compact subsets of E and F [44, 15.6.3]

We now detail a remark appearing in the introduction of the first book of Schwartz on distributions with vectorial values [67, Ch I, 1.].

Proposition 7.1.10. *Complete Mackey-spaces, and in particular (F)-spaces are γ-reflexive.*

Proof. Remember from section 3.5.1, corollary 3.5.7 that any lcs is semi-γ-reflexive. A lcs E is γ-reflexive, that is linearly homeomorphic to $(E'_\gamma)'_\gamma$ if and only if it is endowed by the topology of uniform convergence on absolutely convex compact subsets of E'_γ.

As a lcs E is always endowed by the topology of uniform convergence on equicontinuous subsets of E', and equicontinuous sets are in particular compact in E'_γ (as they are weakly compact, and on equicontinuous sets the topology of relatively compacts and the weak topology coincides, see 3.4.5 and 3.1.25), the topology induced by $(E'_\gamma)'_\gamma \sim E$ on E is finer than the topology of E.

As compact sets are in particular weakly compact, the topology induced by $(E'_\gamma)'_\gamma \sim E$ on E is coarser than the Mackey topology on E. Thus when E is Mackey and complete, the topology induced by $(E'_\gamma)'_\gamma \sim E$ on E is exactly the topology of E:

$$(E'_\gamma)'_\gamma \simeq E.$$

□

We endow FRE with the completion $\tilde{\otimes}_\pi$ of the projective tensor product, which preserves the class of (F)-spaces [44, 15.1.1]. Then according to the generalization of a result of Buchwalter [44, 16.1.7], ε represents indeed the dual of the tensor product:

Proposition 7.1.11. *(Buchwalter equalities[44, 16.1.7]) When E and F are (F)-spaces, we have:*

$$(E\tilde{\otimes}_\pi F)'_\gamma \simeq (E'_\gamma \varepsilon F'_\gamma)$$

and

$$(E\varepsilon F)'_\gamma \simeq E'_\gamma \hat{\otimes}_\pi F'_\gamma.$$

Again, as already noticed in section 3.4.3, beware that the notation of Jarchow is not the same as ours, but coincides on complete spaces. We don't detail the proof of the previous proposition, which is detailed in the litterature: what makes the Buchwalter equalities work is the fact that (F)-spaces are γ-reflexive, but also that, due to their metrisability, separately continuous and continuous bilinear maps defined on them coincide.

Proposition 7.1.12. *Consider E, F (F)-spaces and G a complete space. Then we have: $\mathcal{L}_\gamma(E\tilde{\otimes}_\pi F, G) \simeq \mathcal{L}_\gamma(E, \mathcal{L}_\gamma(F, G))$.*

Proof. This is due to a particular case where Grothendieck's "problème des topologies" is solved: any compact set in $E\tilde{\otimes}_\pi F$ is the convex closure of the tensor product of a compact set in E and a compact set in F [68][Chapter II. 1. proposition 1]. Thus if we have the algebraic equality between $\mathcal{L}_c(E\tilde{\otimes}_\pi F, G)$ and $\mathcal{L}_c(E, \mathcal{L}(F, G))$, their topology correspond. Now a linear continuous function between $E\tilde{\otimes}_\pi F$ to G, coincide to a bilinear continuous function from $E \times F$ to G. This function is in particular γ-hypocontinuous, and thus coincides to an element of $\mathcal{L}_c(E, \mathcal{L}(F, G))$. Conversely, as E and F are (F)-spaces, a bilinear separately continuous function from $E \otimes F$ to G in continuous, thus we have the injection $\mathcal{L}_c(E, \mathcal{L}(F, G)) \longrightarrow \mathcal{L}(E\tilde{\otimes}_\pi F, G)$. □

Thus we have a strong monoidal functor: $(\_)'_\gamma : (\text{FRE}, \hat{\otimes}_\pi, \mathbb{R}) \longrightarrow (\text{COMPL}^{op}, \varepsilon, \mathbb{R})$ and a strong-monoidal functor $(\_)'_\gamma : (\text{FRE}, \varepsilon, \mathbb{R}) \longrightarrow (\text{COMPL}^{op}, \hat{\otimes}_\pi, \mathbb{R})$. We would like to describe one of this functors as the left adjoint in a strong monoidal adjunction, thus describing a polarized model of MALL. However the γ-dual (*i.e.* the Arens dual) of a complete space is not necessarily metrizable, but the β-dual of a (DF)-space is. However we saw that the class of DF subspaces preserved by $\otimes_\pi$ and its completion. Thus we want to state the above monoidal functors using the strong dual β, for which we have a good duality theory between (F)-spaces and (DF)-spaces.

7.2 Nuclear and Schwartz' spaces

7.2.1 Schwartz spaces

Other convergence can be defined on the dual of E', and plays an important role in this thesis. For example, Grothendieck's theorem allows to understand the completion $\tilde{E}$ of a lcs E as some double dual of E. We begin by defining a notion of continuity by restricting it on equicontinuous subsets of E'.

Definition 7.2.1. A map $f : E' \longrightarrow F$ is said to be γ-continuous if, for every equicontinuous subset $H \subset E'$, the restriction of f to H is continuous for the relativised weak* topology $\sigma(E', E)$ on H.

To this notion of continuity coincides a notion of convergence for filters in E':

Definition 7.2.2. A filter $\mathcal{F} \subset E'$ (see Definition 3.0.4) is said to be γ-convergent, or to be *continuously converging* towards ℓ, if there is an equicontinuous subset $H \subset E'$, $H \in \mathcal{F}$ such that $\ell \in H$, and the restriction of $\mathcal{F}$ to H converges weakly in H.

Remark 7.2.3. [44, 9.1.3] This notion of convergence never coincides with a topology on E' if E is infinite dimensional.

A stronger notion of convergence can be defined, relating the convergence of a filter locally in an equicontinuous subset, see definition 3.1.12. Remember that the polar of a 0-neighborhood U° is absolutely convex and weakly closed in E'_w, and thus one can define local convergence in the normed space E'_{U° (see definition 3.1.12). In particular, as U° is weakly-complete, the normed space E'_{U° is a Banach.

Definition 7.2.4. Consider a filter $\mathcal{F} \subset E'$. Then we say that $\mathcal{F}$ converges equicontinuously to ℓ if there is a 0-neighbourhood U such that $\mathcal{F} - \ell$ converges to 0 in the Banach space E'_{U°.

Then of course equicontinuous convergence implies continuous convergence to the same limit. Schwartz spaces characterize the lcs where the converse is true.

Definition 7.2.5. A lcs E is a Schwartz space if every continuous filter in E' converges equicontinuously.

Equivalently, it means that for every 0-neighbourhood U there is $V \subset U$ such that U° is compact in E'_{V°. Indeed, the filter whose trace is weakly-convergent in U° is then going to converge in E'_{V°.

Example 7.2.6. Let us consider some example of Schwartz spaces:

- A space E_{w*} endowed with its weak* topology is always a Schwartz space: indeed, equicontinuous subsets of (E_{w*}) are bipolars of finite subsets of E', thus absolutely convex weak closures of finite subsets of E'. If a filter $(\ell_\gamma)_\gamma$ converges weakly in some $\{\ell_1, ..., \ell_n\}^{\circ\circ}$ it will converge for the norm on $E_{\{\ell_1,...,\ell_n\}^{\circ\circ}}$, which is:
$$\ell \mapsto \sup\{\lambda \mid \lambda\ell \in \{\ell_1, ..., \ell_n\}^{\circ\circ}\}.$$
- The spaces of smooth functions $\mathcal{C}^\infty(\mathbb{R}^n)$, $\mathcal{C}^\infty(\mathbb{R}^n)$ and the euclidean spaces are Schwartz spaces, but no infinite dimensional Banach space is.

Proposition 7.2.7. *[44, 21.2.3] The class of Schwartz spaces is preserved by completions, cartesian products, countable direct sums, projective tensor products, subspaces and quotients.*

Proposition 7.2.8. *[44, 10.4.1] A lcs E is Schwartz if and only if it is endowed with the topology of uniform convergence on the sequences in E' which converges equicontinuously to* 0*.*

The previous proposition, which we won't prove, leads to a closure operation making any lcs a Schwartz space:

Definition 7.2.9. Consider E any lcs. We denote by E_0 the vector space E endowed with the topology $\mathcal{T}_{c_0}$ of uniform convergence on the the sequences in E' with converges equicontinuously to 0.

Then E_0 is a Schwartz space. Its topology $\mathcal{T}_{c_0}$ is coarser than the original one $\mathcal{T}_E$ as $\mathcal{T}_E$ is the topology of uniform convergence on *every* equicontinuous subset (proposition 3.4.18). But $\mathcal{T}_{c_0}$ is also finer than the weak* topology on E, which can be seen as the topology of uniform convergence on all finite sequences of E' (as finite sequences can be considered in particular as equicontinuously converging to 0). Thus we have continuous linear embeddings:
$$E \hookrightarrow E_0 \hookrightarrow E_{w*}.$$
Taking the dual of these embeddings leads by Hahn-Banach theorem 3.3.4 to linear continuous injections $E' \longrightarrow (E_0)' \longrightarrow E'$, thus:

Proposition 7.2.10. *The dual of E_0 is still $(E_0)' = E'$.*

Hence every weakly continuous linear map $f : E \longrightarrow F$ also defines a weakly continuous linear map $f : E_0 \longrightarrow F$, and $E \mapsto E_0$ is a closure operation (definition 3.0.2) in the category of weak spaces and continuous linear maps.

Proposition 7.2.11. *[44, 10.4.4] $\mathcal{T}_{c_0}$ is the finest Schwartz topology on E which is coarser than $\mathcal{T}$.*

We will now recall two fundamental properties of Schwartz spaces:

Proposition 7.2.12. *[44, 10.4.3] A Schwartz space is normed if and only if it is finite dimensional.*

Theorem 7.2.13. *[44, 10.5.1] Every Schwartz space E is linearly homeomorphic to a subspace of the cartesian product $(c_0)_0^I$ for some set I.*

Theorem 7.2.14. *[44, 21.1.7] Schwartz spaces are preserved by arbitrary cartesian product, countable co-product and arbitrary subspaces.*

Proposition 7.2.15. *[44, 10.4.3] Bounded subsets of a Schwartz space are precompact. Thus a complete Schwartz space is γ-reflexive, and a (F)-space which is also a Schwartz space is β-reflexive.*

Proof. When E is complete and Schwartz we have thus that the bounded sets of E are compact, and thus E' is endowed with the γ-topology of uniform convergence on absolutely convex compact subsets of E. $\square$

Proposition 7.2.16. *[44, 12.5.8] The Arens dual E'_γ of a metrizable complete space is a Schwartz DF space.*

Consider FRESCHW (resp. DFSCHW) the subcategory of TOPVEC made of (F)-space Schwartz (resp. (DF)-space) lcs. Because (F)-spaces are γ-reflexive, as metrizable spaces are in particular Mackey (see proposition 7.1.2), we have thus a left polarized closure:

$$\begin{tikzcd} (\textsc{FreSchw}, \tilde{\otimes}_\pi, \mathbb{R}) \arrow[r, bend left, "(\cdot)'_\gamma"] \arrow[r, phantom, "\top"] & (\textsc{DFSchw}^{op}, \varepsilon, \mathbb{R}) \arrow[l, bend left, "(\cdot)'_\gamma"] \end{tikzcd}$$

Proof. The adjunction follows from the fact that a (F)-space is endowed with its γ-topology, or said otherwise that a (F)-space is γ-reflexive. Thus the dual f' of a linear map $f : E'_\gamma \longrightarrow F$ is a linear map $f' : F'_\gamma \longrightarrow (E'_\gamma)'_\gamma$, as E is endowed with its Mackey topology (see [44, 8.6.5]). See section 3.5.3 $\square$

We would like to extend this closure to a model of DiLL. This asks for a strong monoidal functor playing the role of an exponential. However, to prove this strong monoidality the setting of nuclear spaces is better-suited. The Kernel theorem 7.3.9, proving the strong monoidality of an exponential interpreted as a space of distributions, is also valid in our setting as spaces of distributions are Schwartz, but it uses the nuclearity of the spaces of distributions.

7.2.2 Nuclear spaces

The theory of nuclear spaces will allow us to interpreted the involutive negation of DiLL, and as the same time the theory of exponentials as distributions. Nuclear lcs can be understood through two approaches: one uses the theory of nuclear operators [44, 17.3] [76, 47] while the other uses the theory of topological vector spaces. We refer to the book by Treves [76] for a nice introduction to these notions, although [44] contains all the necessary material also.

Definition 7.2.17. A linear map f from a lcs E to a Banach X is said to be nuclear if there is an equicontinuous sequence (a_n) in E', a bounded sequence (y_n) in X, and a sequence $(\lambda_n) \in l_1$ such that for all $x \in E$:

$$f(x) = \sum_n \lambda_n a_n(x) y_n.$$

Definition 7.2.18. Consider E a lcs. We say that E is nuclear if every continuous linear map of E into any Banach space is nuclear.

Proposition 7.2.19. *[76, Thm 50.1] The following propositions are equivalent:*

- *E is nuclear,*

- *For every for every Hausdorff space E, there is a linear homeomorphism*

$$E \otimes_\pi F \simeq E \otimes_\epsilon F,$$

- *For every semi-norm p on E, there is a semi-norm q finer than p such that the continuous linear injection $(\tilde{E_p})' \hookrightarrow (\tilde{E_q})'$ is nuclear.*

Remember from section 3.1.2 that E_p denotes the normed space $E/Ker(p)$ normed by p, that $\tilde{E_p}$ is thus a Banach space, and $(\tilde{E_p})'$ endowed with the dual Banach topology. $(\tilde{E_p})'$ is also the linear span in E' of the polar ${U_p}^\circ$ of the closed unit ball U_p of p.

One of the interests of nuclearity lies in its remarkable stability property:

Proposition 7.2.20 ([44] 21.2.3)**.** *The class of nuclear spaces is preserves by completions, projective limits, countable inductive limits, projective tensor products, subspaces and quotients.*

Proposition 7.2.21. *The only nuclear normed spaces are the finite-dimensional ones.*

Proof. This uses the fact that bounded sets of nuclear lcs are precompact [76, prop. 50.2]: nuclear maps are in particular compact (sending bounded onto relatively compact subsets). If E is nuclear, then for every absolutely convex neighbourhood U of E the linear mapping $E \longrightarrow \tilde{E_U}$ is compact, where E_U denotes the normed associated with U, that is the quotient of E by the kernel of the semi-norm associated with U. In particular, the image of a bounded set B is compact in any of the $\tilde{E_U}$, thus precompact in E.

If moreover E is normed, then every bounded set is relatively compact, that is E has the Heine-Borel property. But only finite-dimensional normed space have the Heine-Borel property. □

This proposition is important for the development of Smooth DiLL in section 7.4. It means that we have a gap between finite data ($\mathbb{R}^n$) and the smooth computations we make from these data $\mathcal{C}^\infty(\mathbb{R}^n)$.

Example 7.2.22. The following lcs, introduced in 3.2, are nuclear (see [76, ch.51] for the proofs):

- Any finite dimensional vector space $\mathbb{R}^n$,
- The spaces of smooth functions $\mathcal{D}(\mathbb{R}^n) = \mathcal{C}^\infty_c(\mathbb{R}^n, \mathbb{R})$, $\mathscr{E}(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and their duals $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ (see section 3.2).
- The Köthe space $\mathfrak{s}$ of rapidly decreasing sequences:

$$\mathfrak{s} = \{(\lambda_n)_n \in \mathbb{K}^\mathbb{N} \mid \forall k \in \mathbb{N}, (\lambda_n n^k)_n \in \ell_1\}.$$

  This definition generalizes to spaces of rapidly decreasing sequences of tuples in $(\mathbb{R}^m)^\mathbb{Z}$.

- The space of tame functions

$$\mathcal{S}(\mathbb{R}^n) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n), \forall P, Q \in \mathbb{R}[X_1, ..., X_n], \sup_x |P(x)Q(\partial/\partial x)f(x)| < \infty\}^1$$

  and its dual $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. These spaces will be studied in section 7.3.4.

Interestingly, the nuclearity of the considered spaces of functions relies on the nuclearity of the Köthe space $\mathfrak{s}$. Indeed, the space of tempered functions is isomorphic to a subspace of $\mathfrak{s}$, the space of compact supported smooth functions is a subspace of the space of tempered functions; and the space of smooth functions is the projective limit of the space of smooth functions with compact support.

Outlook 10. This fact hints for a general correspondence between smooth models of LL and Köthe spaces, via Fourier transform (see [76, Theorem 51.3]).

Lemma 7.2.23. [66, III.7.2.2] Every bounded subset of a nuclear space is precompact.

[1] The notation $Q(\partial/\partial x)$ will be used in Chapter 8, and represents a linear partial differential operator with finite coefficients. If $Q(X_1, ...X_n) := \sum_{\alpha\in\mathbb{N}^n} a_\alpha X_1^{\alpha_1}...X_n^{\alpha_n}$, then $Q(\partial/\partial x)$ represents the operator

$$f \in \mathcal{C}^\infty(\mathbb{R}^n) \mapsto \left(x \mapsto \sum_\alpha a_\alpha \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}(x)\right).$$

Proof. Consider E a nuclear space and B bounded in E. Consider $\mathcal{U}$ a basis of absolutely convex closed neighbourhoods of E. As a lcs, E embeds in the product of Banach $\prod_{U\in\mathcal{U}}\tilde{E_U}$, and thus B embeds into $\prod_{U\in\mathcal{U}}\tilde{E_U}$ as a product of bounded subsets via maps $q_U : E \longrightarrow \widetilde{E_U}$. These maps are in particular nuclear. Now we use the fact that nuclear maps are compact [76, 47.3], that is send a 0-neighbourhood U (the polar of the closure of the (a_n) in definition 7.2.17) into a precompact set (the closed hull of the y_k) by definition **??**. Thus B, absorbed by U, is precompact as its image is precompact, and we can recover B by reverse image by q_U of open sets (thus open sets). □

Proposition 7.2.24. *A complete nuclear lcs E is semi-β-reflexive.*

Proof. By lemma 7.2.23, the bounded sets of E are compact, and thus F' is endowed with the γ-topology of uniform convergence on absolutely convex compact subsets of F. By the corollary to the Mackey-Arens theorem 3.5.7, this makes F semi-reflexive. □

It follows then immediately that:

Theorem 7.2.25. *An (F)-space F which is also nuclear is reflexive.*

Proof. A semi-β-reflexive metrisable space is β-reflexive: when F is metrisable E-equicontinuous sets and E-weakly bounded sets coincides in E' [44, 8.5.1]. □

Corollary 7.2.26. A fundamental consequence of the previous lemma is that spaces of smooth functions and spaces distributions with compact support, respectively denoted by $\mathcal{E}(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$, defined in section are reflexive.

Another property of nuclear spaces is that they allow for a polarized model of MALL with nuclear (F)-spaces and nuclear (DF)-spaces, as the class of nuclear (DF)-spaces is stable by *tensor product*.

Proposition 7.2.27.
- *Consider E a lcs which is either an (F)-space or a (DF)-space. Then E is nuclear if and only if E' is nuclear [36, Chap II, 2.1, Thm 7].*
- *IF E is a complete (DF)-space and if F is nuclear, then $\mathcal{L}_b(E,F)$ is nuclear. If moreover F is an (F)-space or a (DF)-space, then $\mathcal{L}_b(E,F)'$ is nuclear [36, Chapter II, 2.2, Thm 9, Cor. 3]. As a corollary, the dual of a nuclear (DF)-space is a nuclear (F)-space.*

Proposition 7.2.28 ([36] Chapter II, 2.2, Thm 9)**.** *If E and F are both nuclear (DF)-spaces, then so is $E\otimes_\pi F$.*

A central result of the theory of nuclear spaces is the following proposition. It is proved by applying the hypothesis that E is reflexive and thus E' is complete and barrelled, and thus applying the hypothesis of [76, 50.4].

Proposition 7.2.29 ([76] prop. 50.4)**.** *Consider E a nuclear (F)-space, and F a complete space. Then $E\hat{\otimes}_\pi F \simeq \mathcal{L}_\beta(E',F)$.*

Proposition 7.2.29 follows from the fact that $E\otimes F$ is dense in $\mathcal{L}_\epsilon(E',F)$, due to the fact that nuclear spaces satisfy a good *approximation property* (see [44, Chapter 18]). Thus when F is complete we have an isomorphism between the completed tensor product and the space of linear functions. As E is an (F)-space and thus barrelled, $\mathcal{L}_\varepsilon(E',F) \simeq \mathcal{L}_\beta(E',F)$. The following result specifies Buchwalter equalities 7.1.11:

Proposition 7.2.30. *[76, 50.7] When E and F are nuclear (F)-spaces, then writing $\hat{\otimes}$ equivalently for the completed projective or completed injective tensor product, we have:*

$$(E\hat{\otimes}F)' \simeq \mathbf{B}(E,F) \simeq E'\hat{\otimes}F'.$$

Proof. Remember that $\mathbf{B}(E,F)$ represents the bilinear continuous scalar functions on $E\times F$. As E and F are in particular metrizable, we have that $\mathbf{B}(E,F) = \mathscr{B}(E,F)$, the space of separately continuous bilinear maps. Thus $\mathbf{B}(E,F) = \mathcal{L}(E,F')$, and via Proposition 7.2.29 we have $\mathcal{L}(E,F') \simeq E'\hat{\otimes}_\pi F'$ as E is reflexive. Moreover, $\otimes_\pi$ is universal for the continuous bilinear maps, and via the universal property for the completion we have moreover: $(E\hat{\otimes}F)' \simeq \mathbf{B}(E,F)$ when the last space is endowed with the topology of uniform convergence on bounded subsets of $E\hat{\otimes}F$. Let us show that the linear isomorphism $\mathbf{B}(E,F) = \mathcal{L}(E,F')$ is also a linear homeomorphism when $\mathbf{B}(E,F)$ is endowed with this topology convergence on bounded subsets of $E\hat{\otimes}F$. As open subsets of $\mathcal{L}(E,F')$ are generated by the $\mathcal{W}_{B_E,B_F{}^\circ}$ where B_E and B_F are bounded respectively in E and F, we must then show that: bounded subsets of $E\otimes F$ are contained in the absolutely convex closure of tensor products $B_E\otimes B_F$ of bounded subsets of E and F respectively.

However, we know from proposition 7.2.20 that as E and F are nuclear, so is $E \otimes F$. Thus bounded subsets of $E \otimes F$, E and F are precompact thanks to lemma 7.2.23, and thus relatively compact as these spaces are (F)-space and in particular complete. However, (F)-spaces satisfy Grothendieck problèmes des topologies for compact subsets, as already seen in the proof of 7.1.12: a compact subset of $E\hat{\otimes}_\pi F$ is the convex hull of a tensor of compact sets. This can be proved simply by considering that the elements of a projective tensor product of (F)-space are absolutely convergent series

$$\sum_n \lambda_n x_n \otimes y_n$$

where $\sum |\lambda_n| < 1$ and (x_n) and (y_n) are sequences converging to 0 in E and F respectively [76][45.1, corrolary 2]. Thus bounded subsets of $E \otimes F$ are contained in the convex balanced hull of tensor products $B_E \otimes B_F$ of compacts, thus bounded, subsets of E and F respectively, and we have $\mathbf{B}(E, F) \simeq \mathcal{L}(E, F')$. $\square$

7.2.3 A polarized model of MALL

Notation 7.2.31. *We write* $\text{N}\textsc{f}$ *the category of Nuclear (F)-spaces and continuous linear maps,* $\text{N}\textsc{df}$ *the category of complete Nuclear (DF)-spaces and continuous linear maps, and* $\text{E}\textsc{ucl}$ *the subcategory of both formed of euclidean spaces.*

Let us begin with an important negative statement:

Remark 7.2.32. The categories of Nuclear (F)-spaces and the one of nuclear (DF)-spaces *do not* form a dialogue chirality (definition 2.3.12). Indeed, the strong dual $E \in \text{N}\textsc{df} \mapsto E'_\beta \in \text{N}\textsc{df}$ is in adjunction with $E \in \text{N}\textsc{df} \mapsto E'_\beta \in \text{N}\textsc{f}$ (see next proposition). However, there is no shift $\downarrow : \text{N}\textsc{f} \longrightarrow \text{N}\textsc{df}$ allowing to see Nuclear (F)-spaces as a particular case of Nuclear DF spaces. In fact, a nuclear (F)-space which is also a (DF)-space is necessarily a euclidean space.

Proof. This proof follows [44, 12.4.4]: a lcs which is an (F)-space and a (DF)-space is normable. Following [44, 9.31], if there is a linear topology on E' such that the bilinear form $(x, \ell) \in E \times E' \mapsto \ell(x)$ is continuous (and not only separately continuous), then E is normable. In our case, the strong topology makes E' a Fréchet space, and thus the separately continuous map on $E \times E'$ is continuous. As normable nuclear space, E is thus finite dimensional. $\square$

Thus one must interpret proofs as maps in a category including $\text{N}\textsc{f}$ and its duals, for example the category of complete lcs.

Notation 7.2.33. *As in Chapter 6, we denote by* $\text{C}\textsc{ompl}$ *the category of complete spaces and continuous linear maps between them. We denote by* $\tilde{\_} : \text{T}\textsc{op}\text{V}\textsc{ec} \longrightarrow \text{C}\textsc{ompl}$ *the completion functor. As we are considering nuclear spaces, we denote by* $\hat{\otimes}$ *the completed projective or equivalently injective tensor product. Following the usual convention in 6,* $\text{N}\textsc{df}\text{C}\textsc{ompl}$ *denotes the category of complete and nuclear (DF)-spaces.*

Proposition 7.2.34. *We have a strong monoidal adjunction:*

$$\begin{tikzcd} (\text{N}\textsc{df}\text{C}\textsc{ompl}, \otimes, \mathbb{R}) \arrow[r, bend left, "(\cdot)'_\beta"] \arrow[r, phantom, "\top"] & (\text{N}\textsc{f}^{op}, \hat{\otimes}, \mathbb{R}) \arrow[l, bend left, "(\cdot)'_\beta"] \end{tikzcd}$$

such that we have a natural isomorphism for $E \in \text{N}\textsc{df}\text{C}\textsc{ompl}$, $F, G \in \text{N}\textsc{f}$*:*

$$\mathcal{L}_\beta(E, F\hat{\otimes}G) \simeq \mathcal{L}_\beta(E\hat{\otimes}F', G)$$

Proof. The fact that the left and right negations $(\_)'_\beta$ send indeed an (F)-space on a (DF)-space follows from proposition 7.1.6 and section 7.1.1. They are strong monoidal functors thanks to proposition 7.2.30. They are adjoint as the spaces in $\text{N}\textsc{f}$ and $\text{N}\textsc{df}$ are reflexive: $\mathcal{L}(E, F') \simeq \mathcal{L}(F, E')$ by proposition 3.5.17.

Consider $E \in$ NDFCOMPL, and $F, G \in$ NF. The equation $\mathcal{L}(E, F\hat{\otimes}G) \simeq \mathcal{L}(E\hat{\otimes}E'_\beta, G)$ follows from proposition 7.2.29 and the associativity of the projective tensor product:

$$\begin{aligned}
\mathcal{L}_\beta(E, F\hat{\otimes}G) &\simeq \mathcal{L}_\beta((F\hat{\otimes}G)', E') \text{ by 3.5.17}\\
&\simeq (F\hat{\otimes}G)\hat{\otimes}E' \text{ by proposition 7.2.29, as } F\hat{\otimes}G \text{ is a nuclear (F)-space}\\
&\simeq G\hat{\otimes}(E'\hat{\otimes}F'') \text{ by reflexivity of } F \text{ and associativity of } \hat{\otimes}\\
&\simeq G\hat{\otimes}(F'\hat{\otimes}E)' \text{ by proposition 7.2.30}\\
&\simeq \mathcal{L}(E\hat{\otimes}F', G) \text{ by proposition 7.2.29.}
\end{aligned}$$

□

Interpreting MALL Thus we have a polarized model of MALL, where formulas are interpreted as complete spaces. Negative formulas are interpreted by Nuclear (F)-spaces. Positive spaces are interpreted as duals of nuclear (F)-spaces, thus complete nuclear (DF)-spaces. We interpret the formulas of MALL as in negative chiralities (definition 2.3.21). Consider $\mathcal{M} = M_1, ..., M_n$ a list of negative formulas and P a positive formula. Then one interprets a proof

$$\dfrac{\begin{matrix}[\pi]\\ \vdots\end{matrix}}{\vdash \mathcal{M}, P}$$

of LL_{pol} as a morphism in COMPL:

$$[\![\pi]\!] \in \mathcal{L}(P', M_1\hat{\otimes}...\hat{\otimes}M_n).$$

Through the strong monoidal closure between the negations, this is equivalent to interpreting a proof of the sequent in $\text{COMPL}(P' \otimes ...\hat{\otimes}M'_n, P)$. A proof of the sequent $\vdash \mathcal{M}$ is interpreted as:

$$[\![\pi]\!] \in \mathcal{L}(\mathbb{R}, M_1\hat{\otimes}...\hat{\otimes}M_n)$$

The interpretation of connectives follows the one of section 2.3.2.3. The only difference is that the cut-rule coincides to the composition of linear continuous functions in COMPL. The $\uparrow$ is interpreted as the completion making a space in NDF complete.

7.3 Kernel theorems and Distributions

Traditionally, one tackles the search for a denotational model of LL by looking for a cartesian closed category, which will allow for an interpretation of the Seely isomorphism. Here, we take another point of view by looking for strong monoidal functors. Let us denote informally spaces of distributions are the duals of spaces of smooth functions: $\mathscr{E}'(\mathbb{R}^n) := (\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}))'$. In particular, if f is a function with compact support, then for any smooth function h the integral

$$\int f(x)h(x)dx$$

is well defined, and defines a distribution $T_f : h \mapsto \int f(x)h(x)dx$. Now if you consider natural numbers n and m, then any distribution k in $\mathcal{C}^\infty(\mathbb{R}^{n+m}, \mathbb{R})'$ defines a linear operator, called *Kernel*, $K : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})'$:

$$K(h)(h') = k((x_1, ..., x_n, y_1, ..., y_m) \mapsto h(x)h'(y)).$$

Schwartz's theorem says that the converse operation always possible: from a kernel K one can constructs a distribution k. In linear logic, if we denote $!\mathbb{R}^n := (\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}))'$ this amounts to say that

$$!(\mathbb{R}^n \times \mathbb{R}^m) \sim !\mathbb{R}^n \parr !\mathbb{R}^m.$$

In fact, because we will work with nuclear spaces, this $\parr$ is a completed tensor product, which is associative and commutative on (F)-spaces.

In this Chapter we will try to look for denotational models of LL and DiLL in which the exponential is interpreted by spaces of distributions. This will lead us to some intermediate syntax (Smooth DiLL), of DiLL without higher order, and we will then exploit a categorical extension of distributions to interpret higher order terms.

Notation 7.3.1. *If X is an open subset of $\mathbb{R}^n$, we denote by $\mathcal{C}^\infty_C(X)$ the vector space of all smooth functions $f : X \longrightarrow \mathbb{R}$ with compact support (i.e. there is K compact in X such that if $x \in X \backslash K$, $f(x) = 0$). It is a lcs when endowed with the topology of uniform convergence of all derivatives of finite order on every compact of X. This topology has been described in Definition 3.2.6. This space is sometimes called the space of* test functions*, and denoted by $\mathcal{D}(\mathbb{R}^n)$ or $\mathcal{D}(X)$.*

Notation 7.3.2. *If X is an open subset of $\mathbb{R}^n$, we denote by $\mathcal{C}^\infty(X)$ the vector space of all smooth functions $f : X \longrightarrow \mathbb{R}$ It is a lcs when endowed with the topology of uniform convergence of all derivatives of finite order on every compact of X. This topology has been described Definition 3.2.6. Following the literature it is also denoted by $\mathscr{E}(X)$ or $\mathscr{E}(\mathbb{R}^n)$.*

7.3.1 Kernel theorems for functions

Kernel theorems come from the fact that one can approximate functions with compact support by polynomials (their Taylor sums in fact, see Chapter 15 of [76]).

Proposition 7.3.3. *[76, 39.2] Consider X (resp. Y) an open subset of $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). Then the algebraic tensor product $\mathcal{C}^\infty_c(X) \otimes \mathcal{C}^\infty_c(Y)$ is sequentially dense in $\mathcal{C}^\infty_c(X \times Y)$.*

The proof of the previous theorem consists in approximating a function in $\mathcal{C}^\infty_c(X \times Y)$ by polynomials $(P_k)_k$, and then using partition of the unity g and h onthe projection on X or Y of the support of f, to define a sequence of polynomials $(g \otimes h)P_k$ which converge to f in $\mathcal{C}^\infty_c(X \times Y)$. This theorem is also true under a refined statement: $\mathcal{C}^k_c(\infty, X) \otimes \mathcal{C}^l_c(Y)$ is sequentially dense in $\mathcal{C}^{k+l}_c(X \times Y)$[2].

From this theorem we deduce:

Proposition 7.3.4. *Consider X (resp. Y) an open subset of $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). Then the algebraic tensor product $\mathcal{C}^\infty(X) \otimes \mathcal{C}^\infty(Y)$ is sequentially dense in $\mathcal{C}^\infty(X \times Y)$.*

The following kernel theorem for functions is a direct consequence of the fact that the considered spaces of functions are nuclear.

Theorem 7.3.5. *[76, Theorems 39.2 and 51.6] Consider X (resp. Y) an open subset of $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). Then we have the linear homeomorphism:*

$$\mathcal{C}^\infty(X) \hat{\otimes} \mathcal{C}^\infty(Y) \simeq \mathcal{C}^\infty(X \times Y).$$

Proof. In the previous theorem, the connector $\hat{\otimes}$ equivalently denotes the completed projective tensor product or the completed injective tensor product, as it involves nuclear spaces. Let us show that the lcs $\mathcal{C}^\infty(X \times Y)$ induces on the vector space $\mathcal{C}^\infty(X) \otimes \mathcal{C}^\infty(Y)$ the topology $\otimes_\pi = \otimes_\epsilon$. As the second is dense in the first, and the first is complete, the completion of the second will be linearly homeomorphic to the first.

The topology induced by $\mathcal{C}^\infty(X \times Y)$ is coarser than the projective topology: indeed the bilinear mapping $\mathcal{C}^\infty(X) \times \mathcal{C}^\infty(Y) \longrightarrow \mathcal{C}^\infty(X \times Y)$ is separately continuous and thus continuous as we are dealing with (F)-spaces.

The topology induced by $\mathcal{C}^\infty(X \times Y)$ is finer than the injective topology: tensor product $A \otimes B$ of equicontinuous sets in $\mathcal{C}^\infty(X)'$ and $\mathcal{C}^\infty(Y)'$ respectively are equicontinuous in $\mathcal{C}^\infty(X \times Y)'$.

Thus the topology induced by $\mathcal{C}^\infty(X \times Y)$ on $\mathcal{C}^\infty(X) \otimes \mathcal{C}^\infty(Y)$ is exactly the projective and injective topology. □

Kernel theorems for other spaces of functions Various forms of kernel theorems exists for formal power series, holomorphic functions (spaces of holomorphic functions are also nuclear), spaces $\mathcal{C}^\infty_c(\mathbb{R}^n)$ of smooth functions with compact support, and measurable functions.

Let us emphasize what happens for measurable functions. From the same techniques one deduces that, for $1 \leqslant p < \infty$, $L^p(X) \otimes L^p(X)$ is dense in $L^p(X \times Y)$ where X and Y are measurable spaces (showing that measurable maps forms a good basis to model probabilistic programs). However this density result on measurable maps could not lead to an isomorphism as introduced in the introduction of section 7.3, by detailing the particular case of $L^2(X)$. The dual of $L^2(X)$ is also $L^2(X)$, and any linear map $k \in L^2(X \times Y)'$ induces indeed a "kernel" $K : L^2(X) \longrightarrow L^2(Y)' \simeq L^2(X)$ through the computation exposed in the introduction of the section:

$$K(f \in L^2(X)) : g \in L^2(Y) \mapsto k((x, y) \mapsto f(x)g(y)).$$

[2] that is, every point of $\mathcal{C}^{k+l}_c(X \times Y)$ is the limit of a sequence, and not only a filter, in $\mathcal{C}^k_c(X) \otimes \mathcal{C}^l_c(Y)$

This embedding of $L^2(X \times Y)'$ into $\mathcal{L}(L^2(X), L^2(Y)')$ is however not surjective. Indeed when $X = Y$ the identity $K = Id_{L^2(X)}$ is an element of $\mathcal{L}(L^2(X), L^2(X)') = \mathcal{L}(L^2(X), L^2(X))$, while it would coincide to

$$k : h \in L^2(X, Y) \mapsto \delta_{x-y} (\neq 0 \text{ iff } x = y)$$

which is not a measurable *function* on $X \times Y$.

This coincides to the necessity to consider spaces of distributions in generality, spaces which includes in particular Dirac maps.

7.3.2 Distributions and distributions with compact support

Interpreting the exponential by spaces of distributions with compact support is recurrent in models of Linear Logic. Following a remark by Frölicher and Kriegl, the authors of [6] point out that their exponential coincides with the space of distributions with compact support when the exponential is defined on a euclidean space $\mathbb{R}^n$ [26, 5.1.8]. In Köthe spaces, Ehrhard notes that $!1$ contains the distributions with compact support on $\mathbb{R}$.

Notation 7.3.6. *The strong dual of $\mathcal{D}(X)$ is called the space of distributions and denoted $\mathcal{D}'(X)$. The strong dual of $\mathscr{E}(X)$ is called the space of distributions with compact support and denoted $\mathscr{E}'(X)$.*

The idea is that distributions are generalized smooth functions and distributions with compact support are generalized functions with compact support.

Indeed, any smooth scalar function f defines a distribution

$$T_f : g \in \mathcal{D}(\mathbb{R}^n) \mapsto \int fg$$

while smooth scalar function with compact support defines a distribution with compact support:

$$T_f : g \in \mathscr{E}(\mathbb{R}^n) \mapsto \int fg.$$

The integral is well-defined as in each case, f or g has compact support.

Proposition 7.3.7. *For any $n \in \mathbb{N}$, $\mathcal{E}(\mathbb{R}^n)$ is an $(F) - space$ and $\mathcal{E}'(\mathbb{R}^n)$ is a complete $(DF) - space$.*

Example 7.3.8. A distribution must be considered as a generalized function, and acts as such. The key idea is that, if $f \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ then on defines a compact distribution by

$$g \in \mathcal{C}^\infty(\mathbb{R}^n) \mapsto \int f(x)g(x)dx.$$

Typical examples of distributions which do not follow this pattern are the dirac distribution or its iterated derivatives: for $x \in \mathbb{R}^n$ one defines the dirac at x as: $\delta_x : f \in \mathcal{E}(\mathbb{R}^n) \mapsto f(x)$. Then $\delta_0^{(k)} : f \mapsto (-1)^k f^{(k)}(0)$.

Taking the strong dual of the kernel theorem for functions 7.3.5 gives:

$$\mathscr{E}'(\mathbb{R}^n \times \mathbb{R}^m) \simeq ((\mathscr{E}(\mathbb{R}^n)\hat{\otimes}\mathscr{E}(\mathbb{R}^m))'.$$

Let us recall that as $\mathscr{E}(\mathbb{R}^n)$ is a nuclear space, the operator $\hat{\otimes}$ equivalently denotes the projective tensor product $\hat{\otimes}_\pi$ or the injective tensor product $\hat{\otimes}_\varepsilon$. However as $\mathscr{E}(\mathbb{R}^n)$ is a (F)-space we have through proposition 7.2.30 that $(\mathscr{E}(\mathbb{R}^n)\hat{\otimes}\mathscr{E}(\mathbb{R}^m))' \simeq \mathscr{E}'(\mathbb{R}^n)\hat{\otimes}\mathscr{E}'(\mathbb{R}^m)$ and thus:

Theorem 7.3.9 ([76] 51.6)**.** *For any $n, m \in \mathbb{N}$ we have:*

$$\mathcal{E}'(\mathbb{R}^{n+m}) \simeq \mathcal{E}'(\mathbb{R}^n)\hat{\otimes}_\pi\mathcal{E}'(\mathbb{R}^m) \simeq \mathcal{L}(\mathcal{E}'(\mathbb{R}^m), \mathcal{E}(\mathbb{R}^n))$$

Definition 7.3.10. The support of a distribution $\phi \in \mathcal{D}'(\mathbb{R}^n)$ is the set of points $x \in \mathbb{R}^n$ such that there is no neighbourhood U of x, with a non-null function $f \in \mathcal{C}^\infty_c(\mathbb{R}^n)$whose support is included in U such that $\phi(f) = 0$.

Then the terminology is coherent: the distributions in $\mathcal{D}'(\mathbb{R}^n)$ which have compact support are exactly the distributions which extends to $\mathscr{E}(\mathbb{R}^n)$. This is proved for example by Hormander in [41, 1.5.2].

7.3.3 Distributions and convolution product

Definition 7.3.11. Consider $n \in \mathbb{N}$ and Ω an open set in $\mathbb{R}^n$. We consider the space of smooth functions with compact support $\mathcal{D}(\Omega) := \mathcal{C}_c^\infty(\Omega, \mathbb{R})$ endowed with the topology described in 3.2.2, and its strong dual $\mathcal{D}'(\Omega) := \mathcal{C}_c^\infty(\Omega, \mathbb{R})'_\beta$.

Then we have in particular $\mathcal{D}(\Omega) \subset \mathscr{E}(\Omega)$, the first being dense in the second, and thus $\mathscr{E}'(\Omega) \subset \mathcal{D}'(\Omega)$.

Definition 7.3.12. Consider f and g two continuous functions on $\mathbb{R}^n$, one of which has compact support. Then we have a continuous function $f * g$ with compact support:

$$f * g : x \mapsto \int f(x-y)g(y)dy.$$

This operation called the convolution of functions, and is commutative and associative on functions when at leat two of the functions considered have compact suupport[41, Thm 1.6.2].

Definition 7.3.13. We define the convolution between a distribution $\phi \in \mathcal{D}'(\mathbb{R}^n)$ and a function $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ as the function $\phi * f : x \mapsto \phi(y \mapsto f(x-y))$. Then $\phi * f \in \mathcal{C}^\infty(\mathbb{R}^n)$. In particular, beware that the function resulting from the convolution does not necessarily have a compact support.

Remark 7.3.14. For any $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ we have $\delta_0 * f = f$.

Convolution is now extended to a convolution product between distributions by the following unicity results.

Proposition 7.3.15. *[41, Thm 1.6.3] $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is sequentially dense in $\mathcal{C}^\infty(\mathbb{R}^n)$, and thus $\mathcal{C}^\infty(\mathbb{R}^n)$ is weakly sequentially dense in $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. The first assertion follows by definition of the topology on $\mathcal{C}^\infty(\mathbb{R}^n)$, and by multiplying a function f by partition of the unity. Thus by Hahn-Banach $\mathcal{C}^\infty(\mathbb{R}^n)$ is sequentially dense in $\mathcal{D}'(\mathbb{R}^n)$ endowed with its weak topology. ☐

Proposition 7.3.16. *[41, Thm 1.6.4] Consider a continuous linear mapping $U : \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R})$ which commutes with translations. Then there is a unique $\phi \in \mathcal{D}'(\mathbb{R}^n)$ such that $U(f) = \phi * f$ for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.*

Proof. The function $\phi : f \mapsto U(f)(0)$ is linear and continuous and thus $\phi \in \mathcal{D}'(\mathbb{R}^n)$. We have indeed for $x \in \mathbb{R}^n$: $\phi * u(x) = \phi(y \mapsto f(x-y)) = U(y \mapsto f(x-y))(0)$, and this equals $U(f)(x)$ as U is invariant by translation. The unicity of ϕ follows from the previous theorem. ☐

Definition 7.3.17. Consider $\psi \in \mathcal{E}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}'(\mathbb{R}^n)$. Then $\phi * \psi$ is the unique distribution in $\mathcal{D}'(\mathbb{R}^n)$ such that:

$$\forall f \in \mathcal{D}(\mathbb{R}^n), (\phi * \psi) * f = \phi * (\psi * f).$$

This definition is made possible by the fact that $f \mapsto \phi * (\psi * f)$ is invariant by translation. Thus $\phi * \psi$ is defined as $\phi * \psi : f \mapsto \phi * (\psi * f)(0) = \phi(y \mapsto \psi * f(0-y)) = \phi(y \mapsto \psi(z \mapsto f(y-z))$.

Proposition 7.3.18. *[41, Thm 1.6.5] The convolution product defines a commutative operation on $\mathcal{E}'(\mathbb{R}^n)$, and the support of $\phi * \psi$ is included in the sum of the support of ϕ and ψ respectively. If $\phi, \phi' \in \mathcal{E}'(\mathbb{R}^n)$ and $\psi \in \mathcal{D}'(\mathbb{R}^n)$, then we have associativity:*

$$\phi * (\psi * \phi') = (\phi * \psi) * \phi'$$

Remark 7.3.19. By definition and proposition 7.3.16, the convolution product is a bilinear continuous operation $* : \mathcal{D}'(\mathbb{R}^n) \times \mathscr{E}'(\mathbb{R}^n) \longrightarrow \mathscr{E}'(\mathbb{R}^n)$. The previous theorem implies that it is a commutative and associative operation, and moreover that it preserves $\mathscr{E}'(\mathbb{R}^n)$: if ϕ and ψ have compact support, then so has $\phi * \psi$.

As a corollary, we have the following domains of definition for the convolution product. This classification will be important for the model considered below in section 7.4, and later in Chapter 8.

Proposition 7.3.20. *The convolution product$*$ defines a bilinear continuous function:*

- *from $\mathscr{E}'(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$,*
- *from $\mathscr{E}'(\mathbb{R}^n) \times \mathscr{E}'(\mathbb{R}^n)$ to $\mathscr{E}'(\mathbb{R}^n)$.*

7.3.4 Other spaces of distributions and Fourier transforms

We refer to [76] for this section, and in particular to Chapter 25.

Definition 7.3.21. The space of tempered functions is defined as

$$\mathcal{S}(\mathbb{R}^n) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n), \forall P, Q \in \mathbb{R}[X_1, ..., X_n], \sup_{x \in \mathbb{R}^n} |P(x)Q(\partial/\partial x)f(x)| < \infty\}$$

and is endowed with the topology generated by the semi-norms $\|\_\|_P : f \mapsto \sup_x |P(x)Q(\partial/\partial x)f(x)|$.

We define on tempered functions a *Fourier transform* $\mathcal{F}$ and an *inverse Fourier transform* $\bar{F}$:

$$\mathcal{F} : f \mapsto (\hat{f} : \zeta \in \mathbb{R}^n \mapsto \int exp(-2i\pi\langle x, \zeta\rangle)f(x)dx),$$

$$\bar{\mathcal{F}} : f \mapsto (\check{f} : x \in \mathbb{R}^n \mapsto \int exp(2i\pi\langle x, \zeta\rangle)f(\zeta)d\zeta).$$

Theorem 7.3.22. *$\mathcal{F}$ and $\bar{\mathcal{F}}$ defines linear homeomorphisms of $\mathcal{S}(\mathbb{R}^n)$ into itself.*

Moreover, we have continuous linear injections: $\mathcal{D} \hookrightarrow \mathcal{S} \hookrightarrow \mathscr{E}$, and thus if we define $\mathcal{S}'(\mathbb{R}^n)$ as the strong dual of $\mathcal{S}(\mathbb{R}^n)$, we have the continuous linear injections

$$\mathscr{E}' \hookrightarrow \mathcal{S}' \hookrightarrow \mathcal{D}'.$$

The lcs $\mathcal{S}'(\mathbb{R}^n)$ is called the space of *tempered distributions* on $\mathbb{R}^n$. The Fourier transform of a tempered distribution ϕ is then $\hat{\phi} : f \mapsto \phi(\hat{f})$. Likewise, Fourier transform is a linear homeomorphism of $\mathcal{S}'(\mathbb{R}^n)$. Moreover, this operation behaves remarkably with respect to convolution and partial differential operators:

Proposition 7.3.23. *[76, Chapter 30]*

- *Consider $\phi, \psi \in \mathscr{E}'(\mathbb{R}^n)$. Then the Fourier transform of the convolution is the scalar multiplication of the Fourier transforms $\widehat{\phi * \psi} = \hat{\phi}\hat{\psi}$[3].*
- *Consider $f \in \mathcal{E}(\mathbb{R}^n)$ and $j \in \{1, ..., n\}$. Then $\mathcal{F}(\frac{\partial f}{\partial x_j})(\zeta) = i\zeta_j \mathcal{F}(f)(\zeta)$.*

Generalized to polynomials, the second points thus gives for any polynomial $P \in \mathbb{R}[x_1, ..., X_n]$: $\mathcal{F}(P(\frac{\partial}{\partial x})(f)) = P(i)\mathcal{F}(f)$.

7.4 Smooth Differential Linear Logic and its models

In this section we introduce a toy syntax called Smooth Differential Linear Logic for which Nuclear spaces and distributions form a classical and smooth model. This is a calculus with no-higher order. We distinguish two classes of formulas, the finitary ones on which an exponential can be applied, and the smooth ones, which represents those on which an exponential has already been applied.

Outlook 11. This calculus should be seen as a toy calculus, allowing to play with the mathematical tools of distributions and (F)-spaces. This is useful, as it allows to highlight the traditional mathematical objects attached to Differential Linear Logic. In section 7.5, we provide a higher order negative model of DiLL where the exponential is interpreted as a space of higher order distributions.

7.4.1 The categorical structure of Nuclear and (F)-spaces.

Nuclear (F)-spaces gather all the stability properties to be a (polarized) model of LL, except that we do not have an interpretation for higher-order smooth functions. If $!\mathbb{R}^n$ is interpreted as $\mathcal{E}'(\mathbb{R}^n)$, we do not have an immediate way to define $!!\mathbb{R}^n$.

[3] Beware that the multiplication between distribution is in general not possible [69], and that it is possible here only because the Fourier transform of a convolution with compact support is a generalized function.

Definition 7.4.1. One defines the exponential as a space of Distributions:

$$! \begin{cases} \mathbb{R}^n \mapsto !\mathbb{R}^n = \mathscr{E}'(\mathbb{R}^n, \mathbb{R}) \\ \ell : \mathbb{R}^n \longrightarrow \mathbb{R}^m \mapsto (?(\ell) : \phi \in \mathcal{E}'(\mathbb{R}^n) \mapsto \phi(\_ \circ \ell) \in \mathcal{E}'(\mathbb{R}^m) \end{cases} \tag{7.1}$$

From the kernel theorem 7.3.9 and example 7.2.22 it follows:

Theorem 7.4.2. *The exponential* $! : \textsc{Eucl} \longrightarrow \textsc{Ndf}$ *is a strong monoidal functor.*

Definition 7.4.3. Dually, one defines the following functor:

$$? \begin{cases} \mathbb{R}^n \mapsto ?\mathbb{R}^n = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \\ \ell : \mathbb{R}^n \longrightarrow \mathbb{R}^m \mapsto (?(\ell) : f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto f \circ \ell' \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})) \text{ where } \ell' \text{ is the dual of } \ell \end{cases} \tag{7.2}$$

The the Kernel theorem on functions 7.3.5 states that $?$ is strong monoidal from $(\textsc{Eucl}, \oplus)$ to $(\textsc{Nucl}, \hat{\otimes})$.

Remark 7.4.4. The Kernel Theorem is all about proving the *strong* monoidality of $?$, thus allowing for an interpretation of the co-structural rules $\bar{c}$ and $\bar{w}$. The fact that there is a natural transformation from $?\mathbb{R}^n \otimes ?\mathbb{R}^m$ is the easy part, which allows for the interpretation of the structural rules c and w.

Thus the structure of Nuclear (F)-space or Nuclear (F)-spaces gives us a strong monoidal closure $(\_)'_\beta$: $\textsc{Nf} \longrightarrow \textsc{Ndf} \vdash (\_)'_\beta : \textsc{Ndf} \longrightarrow \textsc{Nf}$:

$$\begin{tikzcd} (\textsc{Ndf}, \hat{\otimes}_\pi, \mathbb{R}) \arrow[r, phantom, "\top"] \arrow[r, bend left, "(\cdot)'_\beta"] & (\textsc{Nf}^{op}, \hat{\otimes}_\epsilon, \mathbb{R}) \arrow[l, bend left, "(\cdot)'_\beta"] \end{tikzcd}$$

and a strong monoidal functor:

$$\begin{tikzcd} (\textsc{Eucl}, \times, \mathbb{R}) \arrow[r, phantom, "\top"] \arrow[r, bend left, "(\cdot)'_\beta"] & (\textsc{Ndf}^{op}, \hat{\otimes}_\pi, \mathbb{R}) \arrow[l, bend left, "(\cdot)'_\beta"] \end{tikzcd}$$

but no interpretation for the shift. We must interpret proofs in the category of complete spaces.

7.4.2 Smooth Differential Linear Logic

In this section, we construct a version of DILL for which Nuclear spaces and distributions are a model, by distinguishing several classes of formulas. We introduce now SDiLL: its grammar, defined in figure 7.1, separates formulas into finitary ones and polarized smooth ones.

Definition 7.4.5. We call Smooth DiLL, denoted SDiLL, the sequent calculus whose formulas are constructed according to the grammar of figure 7.1, and whose rules are the ones of DiLL_0 (without promotion, as usual).

Its rules are those of DiLL. Thus, the cut-elimination procedure is the same as the one defined originally [23]. What makes it different is the grammar of its formulas: the construction of the formula $!!A$ is not possible. We see this Smooth Differential Linear Logic as a first step towards Chapter 8. In fact, we construct in the next section a model of DiLL with higher-order, base on Nuclear and (F)-spaces.

If we forget about the polarisation of SDiLL, a model of it would be a model of DiLL where the object $!!A$ does not need to be defined. It is thus a model of DiLL where $!$ does not need to be an endofunctor, but just a strong monoidal functor $! : \textsc{Fin} \longrightarrow \textsc{Smooth}$ between two categories. The categories $\textsc{Fin}$ and $\textsc{Smooth}$ need to be both a model of MALL.

This distinction is necessary here to account for spaces of distributions and spaces of smooth functions, which cannot be understood as part of the same $*$-autonomous category. Indeed, if both type of space are reflexive,

$$
\begin{array}{c}
E, F := A \mid N \mid P \\
\text{Finitary formulas: } A, B := \mathfrak{a} \in \mathfrak{A} \mid 0 \mid 1 \mid \top \mid \bot \mid A^{\perp} \mid A \otimes B \mid A \parr B \mid A \oplus B \mid A \times B. \\
\text{Negative Smooth Formulas: } N, M := A | ?A \mid |N \parr M \mid N \times M \mid P^{\perp} \\
\text{Positive Smooth Formulas: } P, Q := A \mid !A \mid P \otimes Q \mid P \oplus Q \mid N^{\perp}
\end{array}
$$

Figure 7.1: The syntax of Smooth DiLL

$$
\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d
$$

$$
\frac{\vdash}{\vdash !A}\, \bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}
$$

Figure 7.2: Exponential Rules of SDiLL

reflexive lcs are *not preserved* by topological tensor products nor spaces of linear maps. One must then refine this category into two monoida subcategories of REFL, and construct an adjunction between the negation functors.

We also give a categorical semantics for an unpolarized version of SDiLL. Remember that considering the development by Fiore [25], if the product on FIN is a biproduct, the interpretation of w, c, $\bar{w}$, $\bar{c}$ follow from the strong monoidality of !. One then needs to make precise the interpretation of d and $\bar{d}$.

7.4.3 A model of Smooth DiLL with distributions

A categorical model of DILL is clearly a model of SDILL. On the other hand, to interpret SDILL one does not need the co-monadic structure on !. But then, in order to interpret the dereliction, one must ask for a natural transformation accounting for the lost co-unit of the co-monad.

Definition 7.4.6. A unpolarized categorical model of SDiLL consists of a model of MALL EUCL, and a model of MALL with biproduct NUCL, equipped with wa strong monoidal functor

$$
! : (\text{EUCL}, \times) \longrightarrow (\text{NUCL}, \otimes),
$$

a forgetful functor U : EUCL $\longrightarrow$ NUCL strong monoidal in $\otimes, \parr, \&, \oplus$ and two natural transformation $d : ! \longrightarrow U$ and $\bar{d} : ! \longrightarrow U$ such that $d \circ \bar{d} = Id_{\text{EUCL}}$.

The model of smooth formulas is the one with biproduct, as the biproduct accounts for the necessity to sum the interpretation of non-linear proofs, interpreted as maps in NUCL. Thus a polarized model of SDiLL would consist into two polarized model of MALL, with a forgetful functor respecting polarities and an exponential inversing the polarities. This would amount to describe 4 chiralities, and their respective coherence laws. We leave this to future work, and describe a concrete polarized model of Smooth DiLL with Nuclear (F)-space or Nuclear DF spaces.

A concrete model of SDiLL. Positive formulas of SDiLL are interpreted as complete nuclear DF spaces. Negative formulas are interpreted as Nuclear (F)-spaces. A sequent $\vdash \Gamma, A$ of Smooth DiLL is intepreted as a continuous linear map in $\mathcal{L}(\llbracket \Gamma \rrbracket'_{\beta}, A)$.

Remark 7.4.7. Notice that we are not here in the categorical context of chiralities. Nuclear (F)-space and Nuclear DF lcs are not a chirality, as we don't have of a (covariant) adjunction between the two interpreting the shifts. However, as the two are subcategories of TOPVEC, we don't need these shifts to interpret the sequents, as they can be interpreted as arrows of TOPVEC, that is continuous linear arrows.

Theorem 7.4.8. *The categories of Euclidean, and (F)-space Nuclear, and DF Nuclear spaces, defines a model of SmoothDiLL.*

Proof. We interpret finitary formulas A as euclidean spaces. Without any ambiguity, we denote also by A the interpretation of a finitary formula into euclidean spaces. The exponential is interpreted as $!A = \mathcal{E}'(A)$, extended by precomposition to functions. We explain the interpretation for the rules, which follow the intuition of [23]. We define:

$$
d : \begin{cases} !A & \longrightarrow A'' \\ \phi & \mapsto \phi_{|A'} \end{cases} \qquad \bar{d} : \begin{cases} A'' \longrightarrow !A \\ ev_x \mapsto (f \mapsto ev_x(D_0(f))) \end{cases}
$$

Then we have indeed: $d \circ \bar{d} = Id_{A''}$. The interpretation of w, c, $\bar{w}$, $\bar{c}$ follows from the biproduct structure on EUCL and from the monoidality of !, as explained in 2.4.2. □

We now detail the description of the bialgebraic structure on $!\mathbb{R}^n$.

Proposition 7.4.9. *The weakening and contraction have the following interpretation in* NUCL*:*

$$w : \begin{cases} !\mathbb{R}^n \longrightarrow \mathbb{R} \\ \phi \mapsto \int \phi \end{cases} \tag{7.3}$$

$$c : \begin{cases} !\mathbb{R}^n \longrightarrow !\mathbb{R}^n \hat{\otimes} !\mathbb{R}^n \simeq !(\mathbb{R}^n \times \mathbb{R}^n) \\ \phi \mapsto (f \in \mathscr{E}(\mathbb{R}^n \times \mathbb{R}^n) \mapsto \phi(x \mapsto f(x,x))) \end{cases} \tag{7.4}$$

Proof. The contraction rules is interpreted from the cartesian structure on EUCL and the Kernel Theorem: if we write $\triangle : A \longrightarrow A \times A$ the diagonal morphisms, we have $!\triangle : !A \longrightarrow !(A \times A) \simeq !A \hat{\otimes} !A$. From the introduction of section 7.3, remember that the isomorphism $!(A \times A) \simeq !A \hat{\otimes} !A \simeq (?A^{\perp} \hat{\otimes} ?A^{\perp})'$ coincides to $k \in !(A \times A) \mapsto (h \otimes h' \in ?A^{\perp} \hat{\otimes} ?A^{\perp}) \mapsto k((x,x') \mapsto h(x)h'(x'))$.

Weakening is then interpreted by the exponentiation of the terminal morphism $\bar{w} = !(n_{\mathbb{R}^n})$, thus $\bar{w}(\phi \in !\mathbb{R}^n) = const_a \in \mathscr{E}(\{0\}) \mapsto \phi(const_a \circ n_{\mathbb{R}^n})$. Thus $\bar{w}$, as a constant scalar function, identifies to a scalar in $\mathbb{R}$. This scalar is its constant value. If ϕ coincides with the generalization of a function $g \in \mathcal{C}^{\infty}_c(\mathbb{R}^n)$ then we would have:

$$\bar{w}(\phi) = a \mapsto a \int_{\mathbb{R}^n} g(x)dx,$$

from which follows the notation used in the proposition. □

Proposition 7.4.10. *Thus the dual of the contraction is :*

$$c' : \begin{cases} ?\mathbb{R}^n \hat{\otimes} ?\mathbb{R}^n \longrightarrow ?\mathbb{R}^n \simeq !(\mathbb{R}^n \times \mathbb{R}^n) \\ f \otimes g \mapsto (x \mapsto f(x)g(x)) \textit{ and then extended via the universal property of completion} \end{cases} \tag{7.5}$$

Proof. The dual c' of c corresponds to the composition of the diagonalisation morphism and of the isomorphism resulting from the Kernel theorem on functions 7.3.5, which is exactly proved by density of $?\mathbb{R}^n \longrightarrow ?\mathbb{R}^n$ in $?(\mathbb{R}^n \times \mathbb{R}^n)$. □

Remark 7.4.11. During the proofs of this Chapter and of Chapter 8, we will equally use the interpretation c or c' when interpreting the contraction.

We show that $\bar{w}$, $\bar{c}$ have a direct interpretation which follows the intuitions of [6].

Proposition 7.4.12. *The cocontraction and codereliction defined through the kernel theorem coincide in fact with the convolution product of distributions and the introduction of* δ_0*.*

$$\bar{c} : \begin{cases} !A \otimes !A & \longrightarrow !A \\ \phi \otimes \psi & \mapsto \phi * \psi \end{cases} \qquad \bar{w} : \begin{cases} \mathbb{R} \longrightarrow !A \\ 1 \mapsto \delta_0 : (f \in \mathcal{E}(A) \mapsto f(0)) \end{cases}$$

Proof. $\bar{w}_A = !(u : \{0\} \longrightarrow A)$ coincides with $\bar{w}_A(1) = (f \in \mathcal{E}(A) \mapsto f \circ u = f(0))$, thus $\bar{w} = \delta_0$.

During the rest of the proof we use Fourier transforms and tamed distributions, as exposed in section 7.3.4. The co-contraction is defined categorically as $\bar{c} = !\nabla \circ m^{-1}_{A,A}$. In the categorical setting, addition in hom sets is defined through the biproduct. But here the reasoning is done backward. We know that $\oplus = \times$ is a biproduct thanks to $\nabla : A \times A \longrightarrow A; (x,y) \mapsto x+y$, and thus $!\nabla : \phi \in !(A \times A) \mapsto (f \in \mathcal{E}(A) \mapsto \phi((x,y) \mapsto f(x+y))$. Moreover if $f \in \mathcal{E}(A \times A)$ is the sequential limit of $(f_n \otimes g_n)_n \in (\mathcal{E}(A) \otimes \mathcal{E}(A))^{\mathbb{N}}$ (see theorem 7.3.9)

$$m^{-1}_{A,A}(\phi \otimes \psi)(f) = \lim_n (\phi(f_n)\psi(g_n)).$$

If we write by $\mathcal{F}\phi$ the Fourier transform of a distribution, we have that of $\mathcal{F}\phi * \psi = \mathcal{F}\phi\mathcal{F}\psi$. From the details above we deduce moreover:

$\mathcal{F}\bar{c}(\phi,\psi)(f) = m^{-1}_{A,A}(\phi \otimes \psi)(\widehat{(x,y) \mapsto f(x+y)}) = m^{-1}_{A,A}(\phi \otimes \psi)\hat{f}_x\hat{f}_y = \hat{\phi}(f)\hat{\psi}(f)$.

As distributions with compact support are temperate, we can apply the inverse Fourier transform to $\mathcal{F}\phi * \psi$ and and $\mathcal{F}\bar{c}(\phi,\psi)$, and thus $\bar{c}$ coincides to the convolution. □

Outlook 12. Let us notice that we could use Sobolev spaces [76] as a model for exponentials in Smooth DiLL. This would lead to the possibility of greater applications in the theory of Linear partial differential operators. Indeed, Sobolev spaces are reflexive when one considers derivatives in L^2. However, we must refine the Kernel theorem to do so. This is work in progress.

7.5 Higher-Order models with Distributions

In Section 7.4 we constructed a strong monoidal functor $! : \textsc{Eucl} \longrightarrow \textsc{Ndf}$. This functor is also defined in an unpolarized way as $! : \textsc{Eucl} \longrightarrow \textsc{Refl}$, the category of reflexive spaces. In this section, we construct a strong monoidal co-monad on reflexive lcs with isomorphisms between them $! : \textsc{Refl}_{iso} \longrightarrow \textsc{Refl}_{iso}$ which generalizes the exponential $! : \textsc{Eucl} \longrightarrow \textsc{Refl}$.

This construction is based on the one for differential algebras developed by Kriegl, Michor [46], and bears similarity with the free exponential of Mellies, Tabareau and Tasson [63]. It differs however by the fact that the exponential is constructed as an injective limit, in order to have a good interpretation of the $⅋$ on negative spaces (see Section 7.5.4). The originality here consists in proving reflexivity of $!E$, and using it to construct the interpretation of the rules of Differential Linear Logic, and on the fact that we index the limit defining $\mathcal{C}^\infty(E,\mathbb{R})$ by injections of euclidean spaces *in E'*, allowing thus to prove covariance of a functor ?.

This section emphasizes a behaviour of structural morphisms on $!E$ which are lost in the case $E = \mathbb{R}^n$ as $\mathbb{R}^n$ is its own dual. Let us also note that what we do later should be further explored in terms of Kan extensions.

Notation 7.5.1. *In this section, we will especially distinguish over linear continuous* injective *indexes $f : \mathbb{R}^n \multimap E$ and smooth scalar functions indexed by f : $\boldsymbol{f}_f \in \mathscr{E}(\mathbb{R}^n)$. We thus borrow the Linear Logic notation $\multimap$ to oppose linear functions to non-linear ones.*

This work is inspired by the construction of differential algebras in [46]. Note that while the authors take an inductive definition of spaces of functions, we define here inductively the space of distributions. Moreover, while in their article the inductive limit of functions space is indexed by finite bases of a lcs, here we index the limit by linear continuous *injective* functions $f : \mathbb{R}^n \multimap E$.

Categories with isomorphisms Let us highlight an important drawback in our work. Injectivity of the indices is needed in order to have an order on these indices. Therefore, the space of functions we construct cannot be functorial with respect to every linear continuous morphism in $\textsc{TopVec}$, and is only defined on linear homeomorphisms between lcs. Thus, we do not have an exponential described as an endofunctor of $\textsc{Refl}$, but a functor of $\textsc{Refl}_{iso}$, the category of Reflexive spaces and linear continuous isomorphisms between them towards. Indeed, one needs to compose injective indices f with maps ℓ of the category (resp. their dual ℓ'), and these composition $\ell \circ f$ (resp $\ell' \circ g$) must stay injective. As shown by Treves [76, 23.2], ℓ' is injective if and only if ℓ has dense image, and therefore we have no choice but to ask for isomorphisms.

In his survey on Differential Linear Logic [23], Ehrhard encountered the same issue and proved that this models in particular the finitary part of Differential Linear Logic. The basic idea is that *functoriality on the isomorphisms is necessary to guaranty an involutive linear negation, but is not neededd to interpreted* $w, c, d, \bar{w}, \bar{c}, \bar{d}$. That is, we do not have a model of LL nor a model of DiLL but we have a model of DiLL_0.

Outlook 13. In this section, we will construct an exponential $!E$ of a reflexive space E as an inductive limit of spaces $\mathscr{E}'(\mathbb{R}^n)$, indexed by *linear continuous* functions $f : \mathbb{R}^n \multimap E$.

$$\mathscr{E}'(E) := \varinjlim_{f:\mathbb{R}^n \multimap E} \mathscr{E}'_f(\mathbb{R}^n).$$

We also consider the space $\mathscr{E}(E)$, thought of as the space of smooth scalar functions on E $\mathcal{C}^\infty(E,\mathbb{R})$, as a projective limit of spaces $\mathscr{E}(\mathbb{R}^)$, indexed by inclusions $f : \mathbb{R}^n \multimap E$. The interpretation of the "why not" connective is thus :

$$?E := \varprojlim_{f:\mathbb{R}^n \multimap E'} \mathscr{E}_f(\mathbb{R}^n).$$

7.5.1 Higher-order distributions and Kernel

In all this section the term reflexive refers to β-reflexive spaces. Without further indication, the notation E' will refer to the lcs E'_β, the strong dual of the lcs E.

Definition 7.5.2. Consider E a lcs and two continuous linear injective functions $f : \mathbb{R}^n \multimap E$ and $g : \mathbb{R}^m \multimap E$ such that $n \geqslant m$. We say that $f \geqslant g$ when

$$f = g_{|\mathbb{R}^n}$$

or said otherwise, when $f = g \circ \iota_{n,m}$ where $\iota_{n,m} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the canonical injection. This defines indeed an order on the set of all linear continuous injective functions from an euclidean space to E. Then we consider the family of spaces $(\mathscr{E}'(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n)', f : \mathbb{R}^n \multimap E)_f$, directed by the order above mentioned. If $f \leqslant g$ then we have an inclusion seen as a linear continuous injective map $S_{f,g} : \mathscr{E}'_g(\mathbb{R}^n) \longrightarrow \mathscr{E}'_f(\mathbb{R}^m), phi \mapsto (h \mapsto \phi(h \circ \iota_{n,m}))$.

Definition 7.5.3. Consider E any lcs. We define then the space of distributions on E as the inductive limit[4] in the category TOPVEC:

$$\mathscr{E}'(E) := \varinjlim_{f:\mathbb{R}^n \multimap E} \mathscr{E}'_f(\mathbb{R}^n)$$

where $f : \mathbb{R}^n \multimap E$ is *linear continuous injective* , and $\mathscr{E}'_f(\mathbb{R}^n)$ denotes the copy of $\mathscr{E}'(\mathbb{R}^n)$ corresponding to $f : \mathbb{R}^n \multimap E$.

Remark 7.5.4. With this definition we are basically saying that distributions with compact support on E are the distributions with a finite dimensional compact support $K \subset \mathbb{R}^n$.

Proposition 7.5.5. *For any lcs E, $\mathscr{E}'(E)$ is reflexive lcs.*

Proof. As the inductive limit of reflexive and thus barrelled spaces, $\mathscr{E}'(E)$ is barrelled (see [44, 11.3.1]). Thus we just need to prove that $\mathscr{E}'(E)$ is semi-reflexive, that is that every bounded closed subset of $\mathscr{E}'(E)$ is weakly compact (then we will have that the strong dual E'_β coincides to the Mackey dual, and thus that we have the linear isomorphism $(E'_\beta)' \sim E$, see section 3.5.2).

Let use a well known method and show that this inductive limit is regular, that is any bounded set B is contained in one of the $\mathscr{E}'_f(\mathbb{R}^n)$. Suppose that it is not. Then we have some bounded set B such that for every n there is f_n such that we have $b_n \in B \backslash \mathscr{E}'_f(\mathbb{R}^n)$. The lcs $F = \varinjlim_n \mathscr{E}'_{f_n}(\mathbb{R}^n)$ is a strict (*i.e.* indexed by $\mathbb{N}$) inductive limit of spaces such that $\mathscr{E}'_{f_n}(\mathbb{R}^{n+1})$ is closed in $\mathscr{E}'_{f_n}(\mathbb{R}^{n+1})$, and as such it is regular ([44, 4.6.2]). But the topology on F is the one induced its inclusion in $\mathscr{E}(E)$, and thus the image B_F of B in F is bounded. So as F is a regular limit B_F should be included in one of the $\mathscr{E}'_{f_n}(\mathbb{R}^n)$, and we have a contradiction.

Thus any bounded set B of $\mathscr{E}'(E)$ is bounded in one $\mathscr{E}'_f(\mathbb{R}^n)$. As these spaces are reflexive (as duals of nuclear (F)-spaces, see section 7.2.2), B is in particular $\mathscr{E}'_f(\mathbb{R}^n)$-weakly compact. But the dual of an inductive limit is contained in the product of the duals ([44, 8.8]) and as such B is also weakly compact in $\mathscr{E}(E)$. Thus $\mathscr{E}'(E)$ is semi-reflexive, and as it is barrelled it is reflexive. □

In fact, $\mathscr{E}'(E)$ is can be characterized as an inductive limit of nuclear DF spaces. What we showed in the previous proof is that any inductive limit of nuclear DF space is reflexive. By analogy with the *strict* inductive limit of Fréchet spaces which are denoted *LF-spaces* in the literature, we introduce in section 7.5.4 the characterization of LNDF spaces. Now we show that our notation is indeed coherent with the one used in the previous sections.

Proposition 7.5.6. *If $E \simeq \mathbb{R}^m$ for some m, then $\mathscr{E}'(E) \simeq \mathscr{E}'(\mathbb{R}^m)$, where $\mathscr{E}'(\mathbb{R}^m)$ denotes here the usual space of distributions on $\mathbb{R}^m$ with compact support.*

Proof. If $E \simeq \mathbb{R}^m$, the linear continuous injective map $id : \mathbb{R}^m \multimap \mathbb{R}^m$, results, by definition of the inductive limit, in a linear continuous injective map $\mathscr{E}'_i d(\mathbb{R}^m) \simeq \mathscr{E}'(\mathbb{R}^m) \longrightarrow \mathscr{E}'(E)$. Now consider $\phi \in \mathscr{E}'(E)$. Suppose that we have an index $f : \mathbb{R}^n \longrightarrow E$ on which ϕ is non-null. Then as f is an injection one has $n \leqslant m$ and thus a linear injection $\mathscr{E}'(\mathbb{R}^n) \hookrightarrow \mathscr{E}'(\mathbb{R}^m)$. This injection is continuous: indeed any compact in $\mathbb{R}^n$ is compact in $\mathbb{R}^m$, and thus the topology induced by $\mathscr{E}'(\mathbb{R}^m)$ on $\mathscr{E}'(\mathbb{R}^n)$ is exactly the strong topology on $\mathscr{E}'(\mathbb{R}^n)$. As we have in particular a linear map $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$, we have the desired equality. □

Definition 7.5.7. For any reflexive space E we denote by $\mathscr{E}(E)$ the strong dual $(\mathscr{E}'(E))'_\beta$ of $\mathscr{E}'(E)$. As the strong dual of a reflexive space it is reflexive.

We recall the duality between projective and inductive limits: the dual of a projective limit is *linearly isomorphic* to the inductive limit of the duals, however the topology may not coincide by [44, 8.8.12]. When E is endowed with its Mackey topology (which is the case in particular when E is reflexive), then the topologies coincides.

Notation 7.5.8. *We consider the projective system $(\mathscr{E}(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n), f : \mathbb{R}^n \multimap E)_f$, directed by the order above mentioned. We denote by $\mathscr{E}_f(\mathbb{R}^n)$ the lcs copy of $\mathscr{E}(\mathbb{R}^n)$.*

Proposition 7.5.9. *When E is reflexive, the lcs $\mathscr{E}(E)$ is linearly homeomorphic to the projective limit*

$$\varprojlim_{f:\mathbb{R}^n \multimap E} \mathscr{E}_f(\mathbb{R}^n),$$

[4] see section 3.1.4

indexed by

$$T_{g,f} = S'_{f,g} : \mathscr{E}_g(\mathbb{R}^m) \longrightarrow \mathscr{E}_f(\mathbb{R}^n), \boldsymbol{g} \mapsto \boldsymbol{g} \circ \iota_{n,m}$$

for $f \leqslant g$*, that is for* $f = g \circ \iota_{n,m}$.

Proof. According to [44, 8.8.7], as $F = \varprojlim_{f:\mathbb{R}^n \multimap E} \mathcal{C}^\infty_f(\mathbb{R}^n)$ is reduced, then its dual is linearly isomorphic to

$$F' = \varinjlim_{f:\mathbb{R}^n \multimap E} \mathcal{C}^\infty_f(\mathbb{R}^n)$$

by reflexivity of the spaces $\mathcal{C}^\infty_f(\mathbb{R}^n)$. Let us prove that this is a linear homeomorphism. As a reflexive space, $\mathscr{E}_f(\mathbb{R}_n)_f$ is endowed with its Mackey topology. As the Mackey-topology is preserved by inductive limits, we have that the topology on the lcs $F' \simeq \mathscr{E}'(E) \simeq \varprojlim_{f:\mathbb{R}^n \multimap E} \mathcal{C}^\infty_f(\mathbb{R}^n)'$ is also the Mackey-topology $\mu(F, F'')$. But as we know that $\mathscr{E}(E)$ is reflexive, this topology is exactly the strong dual topology.

□

Remark 7.5.10. Thus elements $\mathbf{f} \in \mathscr{E}(E)$ are families $(\mathbf{f}_f)_{f:\mathbb{R}^n \multimap E}$ such that if $f : \mathbb{R}^n \multimap E$, $g : \mathbb{R}^m \multimap F$ and $f = g \circ \iota_{n,m}$, we have :

$$\mathbf{f}_f = \mathbf{f}_g \circ \iota_{n,m}.$$

Remark 7.5.11. Thus we are saying that distributions a lcs E are in fact distributions with compact support in an euclidean space, or equivalently that smooth functions a lcs E are those with are smooth when restricted to one $\mathbb{R}^n$. This makes it possible to define multinomials on lcs E :

$$P(x \in \mathbb{R}^k) = \sum_{I \subset [|1,n|]} a_\alpha x_1^{\alpha^1} \dots x_n^{\alpha_n^I}$$

where we embed $\mathbb{R}^k$ in $\mathbb{R}^n$ for $k < n$, and project it for $k > n$. This definition, which as we said is inspired by work on differential algebras [46]. One could thought of another setting restricted specifically to higher order spaces of distributions and not to every reflexive space. Indeed, we would like to describe smooth scalar functions on a lcs $E := \mathscr{E}(\mathbb{R}^n)$ as :

$$h \in \mathscr{E}(\mathbb{R}^n) \mapsto f(0)^2,$$

thus taking into account that we have as input non-linear functions. This seems to indicate that we would like to construct smooth functions indexed by dirac functions, that is by functions $\delta : \mathbb{R}^n \multimap E' = \mathscr{E}'(\mathbb{R}^n)$.

The Kernel Theorem As what was argued for negative interpretations of LL_{pol} in 2.3.2, here we need to prove the Kernel theorem on negative spaces, that is on spaces of functions as $\mathscr{E}(E)$. Indeed, the spaces of functions are the one which can be described as projective limits, and projective limits are the one which commutes with a completed projective tensor product. Let us highlight the fact that this theorem strongly depend of the fact that the considered spaces of functions are nuclear.

Theorem 7.5.12. *For every lcs* E *and* F *we have a linear homeomorphism:*

$$\mathscr{E}(E) \hat{\otimes}_\pi \mathscr{E}(F) \simeq \mathscr{E}(E \oplus F).$$

Proof. By section 3.6 or [44, 15.4.2] we have that the completed projective tensor product of two projective limits indexed respectively by I and J is the projective limit, indexed by $I \times J$ with the pointwise order, of the completed tensor product. Thus:

$$\begin{aligned}
\mathscr{E}'(E) \hat{\otimes}_\pi \mathscr{E}'(F) &\simeq (\varprojlim_{f:\mathbb{R}^n \multimap E} \mathscr{E}'_f(\mathbb{R}^n)) \hat{\otimes}_\pi (\varprojlim_{g:\mathbb{R}^m \multimap F} \mathscr{E}'_g(\mathbb{R}^m)) \\
&\simeq \varprojlim_{(f,g)} \mathscr{E}'_f(\mathbb{R}^n)(\hat{\otimes}_\pi \mathscr{E}'_g(\mathbb{R}^m)) \\
&\simeq \varprojlim_{(f,g)} \mathscr{E}'(\mathbb{R}^n \oplus \mathbb{R}^m) \text{ through the Kernel theorem 7.3.9}
\end{aligned}$$

However, the direct sum $\mathbb{R}^n \times \mathbb{R}^m$ is isomorphic to $\mathbb{R}^{n+m}$ as we deal here with finite indices. Thus any linear continuous injective function $h : \mathbb{R}^k \longrightarrow E \oplus F$ is by definition of the biproduct topology a sum $f + g : \mathbb{R}^n \oplus \mathbb{R}^m \multimap$

$E \oplus F$[5]. Indeed, as h is linear we have that $h^{-1}(E, 0)$ is a sub-linear space of $\mathbb{R}^k$, that is an euclidean space $\mathbb{R}^n$ for $n \leqslant k$. Thus the indexing by pairs $(f, g) : \mathbb{R}^{n+m} \longrightarrow E \times F$ coincide with the indexation by linear smooth functions $h : \mathbb{R}^k \longrightarrow E \times F'$, with corresponding pre-order, and thus:

$$\begin{aligned} \mathscr{E}'(E) \hat{\otimes}_\pi \mathscr{E}'(F) &\simeq \varprojlim_{f:\mathbb{R}^k \longrightarrow E \times F} \mathscr{E}'(\mathbb{R}^k) \\ &\simeq \mathscr{E}'(E \times F) \end{aligned}$$

□

Now we proceed to the definition of a functor $?$, which agrees to the previous characterization $?E \simeq \mathcal{C}^\infty(E', \mathbb{R})$ when $E \simeq \mathbb{R}^n$.

Notation 7.5.13. *We denote by* REFL_{iso} *and linear homeomorphism between them.*

Proposition 7.5.14. *We extend the definition of* $\mathscr{E}(E)$ *to a functor* $? : \mathrm{REFL}_{iso} \longrightarrow \mathrm{REFL}_{iso}$*, by:*

$$? : \begin{cases} \mathrm{REFL}_{iso} \longrightarrow \mathrm{REFL}_{iso} \\ E \mapsto \mathscr{E}(E') \\ \ell : E \longrightarrow F \mapsto ?\ell : \mathscr{E}(E') \longrightarrow \mathscr{E}(F') \end{cases} \tag{7.6}$$

As $\mathscr{E}(F')$ *is a projective limit,* $?\ell$ *is defined by postcomposition with the projections* $\pi_g : \mathscr{E}(E') \longrightarrow \mathscr{E}_g(\mathbb{R}^m)$*, for* $g : \mathbb{R}^m \multimap F'$ *as*

$$\left(?\ell((\boldsymbol{f}_f)_f \in \mathscr{E}_f(\mathbb{R}^n))\right)_g = \boldsymbol{f}_{\ell' \circ g}.$$

Here $\ell' : F' \multimap E'$ *denotes the transpose of* ℓ.

This is well defined as $\mathcal{C}^\infty_{\ell \circ f : \mathbb{R}^n \multimap F}(\mathbb{R}^n, \mathbb{R}) \hookrightarrow \mathscr{E}(F')$*. The linear map* $?\ell$ *is continuous as the composition of* $?\ell$ *with the projections* π_f *is continuous, and injective as* ℓ' *is injective.*

Proof. We check immediately that indeed $?Id = Id$. Let us see that $?$ is covariant: consider $\ell_1 : E \longrightarrow \mathcal{F}$ and $\ell_2 : F \longrightarrow G$ linear continuous surjective functions between the lcs E, F and G. Consider $h : \mathbb{R}^k \longrightarrow G'$, and $\mathbf{f} = (\mathbf{f}_f)_f \in ?E$. Let us write $f = (\ell_2 \circ \ell_1)' \circ h : \mathbb{R}^k \longrightarrow E'$. Then by definition:

$$((?(\ell_2 \circ \ell_1))(\mathbf{f}))_h = \mathbf{f}_f$$

Consider $g : \mathbb{R}^m \multimap F'$. Then

$$((?\ell_1)(\mathbf{f}))_g = \mathbf{f}_{\ell_1' \circ g}$$

Moreover one has for any $\mathbf{g} \in \mathscr{E}(F)$:

$$(?(\ell_2)(\mathbf{g}))_h = \mathbf{g}_{\ell_2' \circ h}.$$

Thus

$$((?\ell_2 \circ ?\ell_1)\mathbf{f})_h = (?\ell_2[(?\ell_1)(\mathbf{f})])_h = (?\ell_1(\mathbf{f}))_{\ell_2' \circ h} = \mathbf{f}_{\ell_1' \circ \ell_2' \circ h} = \mathbf{f}_f.$$

Thus $?$ is covariant. □

Notation 7.5.15. *We denote by* REFL_{iso} *the category of reflexive lcs and linear continuous isomorphisms between them.*

Definition 7.5.16. We define the functor $! : \mathrm{REFL}_{iso} \longrightarrow \mathrm{REFL}_{iso}$ defined as:

$$! : \begin{cases} \mathrm{REFL}_{iso} \longrightarrow \mathrm{REFL}_{iso} \\ E \mapsto \mathscr{E}'(E) \\ \ell : E \longrightarrow F \mapsto !\ell \in \mathscr{E}(F') \end{cases} \tag{7.7}$$

where $!\ell$ is defined by precomposition with the injections $\iota_{f:\mathbb{R} \longrightarrow E} : \mathscr{E}_f'(\mathbb{R}^n) \hookrightarrow \mathscr{E}'(E)$ by

$$!\psi \circ \iota_f(\mathbf{f}_f \in \mathcal{C}^\infty_f(\mathbb{R}^n, \mathbb{R})) = \mathbf{f}_{\ell \circ f : \mathbb{R}^n \multimap F}.$$

The functoriality of $!$ follows from the one of $?$, as we have indeed $!E := (?E')'$ and $!\ell = (?\ell')'$.

[5]Notice here the link between direct sum, biproduct and sums of smooth functions $f : \mathbb{R}^K \longrightarrow E$ and $g : \mathbb{R}^k \longrightarrow F$, as exposed in section 2.4.2.

Remark 7.5.17. We showed previously that when $E \simeq \mathbb{R}^n$, then $?E \simeq \mathscr{E}(\mathbb{R}^n)$. The notation $!E := \mathscr{E}'(E)$ is still coherent with the euclidean case, as $(\mathbb{R}^n)' \simeq \mathbb{R}^n$ and thus $\mathscr{E}(E')' \simeq \mathscr{E}'(\mathbb{R}^n)$ when $E \simeq \mathbb{R}^n$.

Remark 7.5.18. Let us point out that constructing $\mathscr{E}(E)$ and $\mathscr{E}'(E)$ as projective and inductive limits over linear functions $f : \mathbb{R}^n \multimap \mathscr{E}'$ (thus embedding into the dual of the codomain E and not into E itself) would have made the definition of the codereliction and the co-multiplication much easier. Indeed, for example, the codereliction would have been defined as :

$$x \in E \mapsto \mathbf{f} \in \mathscr{E}(E) \mapsto D_0(\mathbf{f})(f'(x)),$$

where $f' : E'' \simeq E \longrightarrow \mathbb{R}^n$ denotes the transpose of f. However, we must index $\mathscr{E}'(E)$ by $f : \mathbb{R}^n \multimap \mathscr{E}$ in order to make $\mathscr{E}$ a covariant functor (proposition 7.5.14).

Outlook 14. In order to avoid the use of isomorphisms in REFL, and to have the functoriality of ! with respect to any linear continuous function $E \multimap F$, we could try to define the following. Consider E and F two lcs and $\ell : E \multimap F$ a linear continuous map. Let $\phi \in \mathscr{E}'(E)$ and we denote by $f : \mathbb{R}^n \multimap E$ the linear continuous injections indexing $\mathscr{E}'(E)$.

Then one defines a linear continuous injection as :

$$\widehat{\ell \circ F} : \frac{\mathbb{R}^n}{Ker(\ell \circ f)} \longrightarrow E.$$

The quotient vector space $\frac{\mathbb{R}^n}{Ker(\ell \circ f)}$ is finite dimensional and thus isomorphic to some $\mathbb{R}^p$, for $p \leqslant n$. We denote by $\pi_{n,p}$ the projection of $\mathbb{R}^n$ onto $\mathbb{R}^p$. Then one could define:

$$\mathscr{E}(\ell)(\phi)(\mathbf{g}) = (\mathbf{g}_{\widehat{\ell \circ f}} \circ \pi_{n,p})_f.$$

The linear continuous map $\widehat{\ell \circ f}$ is indeed injective, while $\mathbf{g}_{\widehat{\ell \circ f}} \circ \pi_{n,p} \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. However, one still needs to check that the dereliction, co-multiplication and co-dereliction stated below are still natural with that definition of ! on maps.

7.5.2 Dereliction and co-dereliction

Remark 7.5.19. A guideline in defining the structural morphisms on $!E$ are the structural morphisms on the convenient exponential of [6], as detailed in section 2.4.3. Indeed, in this model of DiLL, the exponential is the Mackey-completion (and bornologification) of the vector spaces generated by all the dirac maps $\delta_x \in \mathcal{C}^\infty(E)'$. What is important is that every structural operation is defined on the dirac map. For example, the codereliction d_{conv} maps δ_x to x, while the co-multiplication maps δ_x to δ_{δ_x}.

In our context, the mapping δ_x must be understood as the linear continuous function which maps $x \in E$ to $((\mathbf{f}_f)_f \in \mathscr{E}(E') \mapsto \mathbf{f}(f^{-1}(x))$ in $\mathscr{E}'(E)$, which is well defined as we show below.

Definition 7.5.20. We define the dereliction as the natural transformation $d : ! \longrightarrow Id_{\text{REFL}}$ such that:

$$d_E : \begin{cases} !(E) \longrightarrow E'' \simeq E \in \text{REFL} \\ \quad \phi \mapsto (\ell \in E' \mapsto \phi((\ell \circ f)_{f:\mathbb{R}^n \multimap E} \in \mathscr{E}(E)) \end{cases} \tag{7.8}$$

This is well defined as $\ell \circ f$ is a linear continuous injectivefunction from $\mathbb{R}^n$ to $\mathbb{R}$, and thus is smooth and belongs in particular to $\mathscr{E}(\mathbb{R}^n)$.

Remark 7.5.21. As we are working with reflexive spaces, d_E could have been described equivalently as a natural transformation

$$d : Id_{\text{REFL}} \longrightarrow ?$$

by the following:

$$d_E : \begin{cases} E \longrightarrow \mathscr{E}(E') \\ x \mapsto (ev_x \circ f \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}))_{f:\mathbb{R}^n \multimap E'} \end{cases} \tag{7.9}$$

Proposition 7.5.22. *The map $d_E :_o cE \longrightarrow E$ is linear continuous, and natural with respect to linear homeomorphisms.*

Proof. d_E is clearly linear, and it is continuous as it is continuous on every $\mathscr{E}'_f(\mathbb{R}^n)$. Let us show that it is a natural transformation. Consider $\ell : E \longrightarrow F$ linear continuous. We must then show:

$$d_F \circ !\ell = \ell \circ d_E$$

Consider $\phi \in !E$. Then $!\ell(\phi) : \mathbf{g} \in \mathscr{E}(F) \mapsto \phi(\mathbf{g}_{\ell \circ f})_{f:\mathbb{R}^n \multimap E}$, ans thus

$$d_F \circ !\ell(\phi) = l_F \in F' \mapsto \phi(l_F \circ \ell \circ f)_{f:\mathbb{R}^n \multimap E}.$$

Conversely, $\ell \circ d_E(\phi) = \ell(l_E \in E' \mapsto \phi((l \circ f)_f)$, thus :

$$\ell \circ d_E(\phi) = \ell(l_E \in E' \mapsto \phi((l_E \circ f)_f).$$

Seeing $\ell \circ d_E(\phi) \in F$ as an element of the dual of F', we have thus

$$\ell \circ d_E(\phi) =_F \in F' \mapsto \phi(l_F \circ \ell \circ f)_f.$$

□

Let us study the interpretation of the codereliction. We denote by D_0 the operator which maps a function to its differential at 0.

$$D_0 : \begin{cases} \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow (\mathbb{R}^n)' \\ \qquad \mathbf{f} \mapsto \left(v \in \mathbb{R}^n \mapsto \lim_{t \longrightarrow 0} \dfrac{f(tx) - f(0)}{t} = \sum_{i=1}^n \dfrac{\partial f}{\partial x_i}(0) v_i \right) \end{cases}$$

The operator D_0 is then linear in f. It is continuous as the reciprocal image of the polar $B_{0,1}$ of the unit ball contains the set of all functions in $\mathcal{C}^\infty(\mathbb{R}^n)$ whose partial derivatives of order one have maximal value 1 on the compact $\{0\}$.

Definition 7.5.23. We define the codereliction with respect to indices f which contains x in their image. We prove that this is well-defined.

$$\bar{d}_E : \begin{cases} E \longrightarrow !E \simeq (\mathscr{E}(E))' \\ x \mapsto (\mathbf{f}_f \in \mathcal{C}^\infty_f(\mathbb{R}^n, \mathbb{R}))_{f:\mathbb{R}^n \multimap E'} \mapsto D_0 \mathbf{f}_f(f^{-1}(x)) \text{ where } f \text{ is injective such that } x \in Im(f) \; . \end{cases} \tag{7.10}$$

Before proving that the codereliction is indeed well-defined, let us point out that the arbitrary in the choice of $f^{-1}(x)$ does not matter morally : $f^{-1}(x)$ is the *linear* argument of the differentiation.

Proof. This is well defined thanks to the same reasoning as before: suppose that $f \leqslant g$. Then we have $f = g \circ \iota_{n,m}$. Thus by definition of the projective limit we have $\mathbf{f}_f = \mathbf{f}_g \circ \iota_{n,m}$ and:

$$\begin{aligned} D_0 \mathbf{f}_f(f^{-1}(x)) &= D_0(\mathbf{f}_g \circ \iota_{n,m})((g \circ \iota_{n,m})^{-1}(x)) \\ &= D_0 \mathbf{f}_g(D_0 \iota_{n,m}(\iota_{n,m}^{-1}(g^{-1}(x)))) \\ &= D_0 \mathbf{f}_g(\iota_{n,m}(\iota_{n,m}^{-1}(g^{-1}(x))) \text{ as } \iota_{n,m} \text{ is linear} \\ &= D_0 \mathbf{f}_g(g^{-1}(x))) \end{aligned}$$

As any couple of linear functions $f : \mathbb{R}^n \multimap E$ and $g : \mathbb{R}^m \longrightarrow E$ is bounded by $f + g : \mathbb{R}^{n+m} \longrightarrow E$, we have the unicity required. □

Proposition 7.5.24. *The map $\bar{d}_E : E \longrightarrow !E$ is linear continuous and natural with respect to linear homeomorphisms..*

Proof. The linearity is deduced by the linearity of the indices $f : \mathbb{R}^n \multimap E$. The continuity follows from the one of D_0. To prove its naturality, we must prove that for any linear continuous function $\ell : E \longrightarrow F$ we have: $!\ell \circ d_E = d_F \circ \ell$. Consider a family $(\mathbf{g}_g)_g \in \mathscr{E}(F) \simeq \varinjlim_g \mathcal{C}^\infty_g(\mathbb{R}^m)$ and $x \in E$. On one hand $d_F \circ \ell(x)$ maps $(\mathbf{g}_g)_g$ to $D_0 \mathbf{g}_g(g^{-1}(\ell(x)))$ when $\ell(x)$ is in the image of g. On the other hand, $!\ell \circ d_E(x)$ maps $(\mathbf{g}_g)_g$ to $d_E(x)((\mathbf{g}_{\ell \circ f})_{f:\mathbb{R}^n \multimap E}) = D_0 \mathbf{g}_{\ell \circ f}(f^{-1}(x))$ when x is in the image of f. Thus when we consider $g = \ell \circ f$, which has indeed x in his image, we have indeed $(!\ell \circ d_E(x))((\mathbf{g}_g)_g) = (d_F \circ \ell(x))((\mathbf{g}_g)_g)$. □

Remark 7.5.25. As we are working with reflexive spaces, $\bar{d}_E$ could have been described equivalently by its transpose, a natural transformation $d : ? \longrightarrow Id_{\text{REFL}}$ by:

$$\bar{d}^t_E : \begin{cases} \mathscr{E}(E') \longrightarrow E'' \simeq E \\ (\mathbf{f}_f)_{f:\mathbb{R}^n \multimap E'} \mapsto (\ell \in E' \mapsto D_0\mathbf{f}_f(f^{-1}(\ell)) \end{cases} \tag{7.11}$$

Proposition 7.5.26. *For any lcs E we have:*

$$d_E \circ \bar{d}_E = Id_E.$$

Proof. Consider $x \in E$. Then $d_E \circ \bar{d}_E(x) = d_E\left((\mathbf{f}_f)_f \mapsto D_0\mathbf{f}(f^{-1}(x)\right) = \ell \mapsto D_0(\ell \circ f)(f^{-1}(x)$ for any f with x in its image, thus as$\ell \circ f$ is linear we have $\ell \mapsto D_0(\ell \circ f)(f^{-1}(x)) = \ell \mapsto \ell \circ f(f^{-1}(x)) = \ell(x)$. □

7.5.3 Co-multiplication, and the bialgebraic natural transformations

We define co-multiplication as:

$$\mu_E : \begin{cases} !E \longrightarrow !!E \\ \phi \mapsto \left((\mathbf{g}_g)_g \in \mathscr{E}(!E) \simeq \varinjlim_g \mathcal{C}^\infty_g(\mathbb{R}^m) \right) \mapsto \mathbf{g}_g(g^{-1}(\phi)) \text{ when } \phi \in Im(g) \text{ and } g \text{ is injective} \end{cases} \tag{7.12}$$

This is well defined, as we can show as for the dereliction 7.8 and the codereliction 7.10 that the term $\mathbf{g}_g(g^{-1}(\phi))$ is unique when $g : \mathbb{R}^m \longrightarrow !E$ linear and $\mathbf{g}_g \in \mathcal{C}^\infty_g(\mathbb{R}^m)$ vary. There is necessarily at least one linear function $g : \mathbb{R}^m \longrightarrow !E$ which has ϕ in its image.

Proposition 7.5.27. *The co-multiplication $\mu_E : !E \longrightarrow !!E$ is linear continuous and natural with respect to linear homeomorphisms.*

Proof. Consider a linear continuous function $\ell : E \longrightarrow F$ between two lcs. We need to show that we have an equality between the functions $\mu_F \circ !\ell$ and $!!\ell \circ \mu_E$. On one hand, we have for $\phi \in !E$ and $(\mathbf{g}_g)_g \in \mathscr{E}(!F)$,

$$(\mu_F \circ !\ell)(\phi)((\mathbf{g}_g) = \mathbf{g}_g(g^{-1}([(\mathbf{h}_h)_h \in \mathscr{E}(F) \mapsto \phi(\mathbf{h}_{\ell \circ f})_h])$$

On the other hand,

$$\begin{aligned} (!!\ell \circ \mu_E)(\phi) &= !!\ell(((\mathbf{h}_{h:\mathbb{R}^m \longrightarrow !E} \mapsto \mathbf{h}_h(h^{-1}((\phi_f)_f))) \\ &= (\mathbf{g}_g)_g \mapsto \left[(((\mathbf{h}_{h:\mathbb{R}^m \longrightarrow !E} \mapsto \mathbf{h}_h(h^{-1}((\phi_f)_f))\right](\mathbf{g}_{!\ell \circ h})_h \\ &= (\mathbf{g}_g)_g \mapsto \mathbf{g}_{!\ell \circ h}((h)^{-1}(\phi_f)_f))) \end{aligned}$$

By denoting $g = !\ell \circ h$, we have $h = (!\ell)^{-1} \circ g$ on the image of $!\ell$, and thus

$$\begin{aligned} (!!\ell \circ \mu_E)(\phi)(\mathbf{g}_g)_g &= \mathbf{g}_{!\ell \circ h}((h)^{-1}(\phi_f)_f))) \\ &= \mathbf{g}_g(g^{-1} \circ !\ell((\phi_f)_f)) \end{aligned}$$

and the last line results exactly in $\mathbf{g}_g(g^{-1}([(\mathbf{h}_h)_h \in \mathscr{E}(F) \mapsto \phi(\mathbf{h}_{\ell \circ f})_h])$ by definition of $!\ell$. □

Theorem 7.5.28. *The structure $(!, d, \mu)$ defines a co-monad on* REFL.

Proof. Let us check that the morphisms of 2.2.12 are satisfied.

1. Consider $\phi \in !E$. On one hand, we have $d_{!E} \circ \mu_E(\phi) : \mathbf{h} \in \mathscr{E}'(E)' \mapsto (\mu^E(\phi))(\mathbf{h} \circ g)_{g:\mathbb{R}^n \multimap !E}$, thus $d_{!E} \circ \mu_E(\phi)(\mathbf{h}) = (\mathbf{h} \circ g(g^{-1}(\phi))$ when $\phi \in Im(g)$, thus $d_{!E} \circ \mu_E(\phi)(\mathbf{h}) = \mathbf{h}(\phi)$. Through the isomorphism $!E \simeq (!E')'$, we have thus

$$d_{!E} \circ \mu_E = Id_{!E}.$$

On the other hand, we have $!d_E : \psi \in !!E \mapsto ((\mathbf{f}_f)_f \in \mathscr{E}(E) \mapsto \psi((\mathbf{f}_{d_E \circ g})_{g:\mathbb{R}^m \longrightarrow !E}))$. Thus

$$!d_E \circ \mu_E(\phi) = (\mathbf{f}_f)_f \in \mathscr{E}(E) \mapsto \mathbf{f}_{d_E \circ g}(g^{-1}(\phi))$$

when $\phi \in Im(g)$. Consider $f : \mathbb{R}^m \multimap E$ linear continuous injective such that $d_E(\phi) \in Im(f)$. Then writing $g = \bar{d}_E \circ f$ in the above equation we have, as $g^{-1} = f^{-1} \circ d_E$

$$!d_E \circ \mu_E(\phi)(\mathbf{f}_f)_f = \mathbf{f}_f(f^{-1}(d_E(\phi))) \tag{7.13}$$

Remember that $d_E(\phi)$ coincides to the vector $x \in E$ such that for all $\ell \in E'$ we have $\ell(x) = \phi((\ell \circ f)_{f:\mathbb{R}^n \multimap E})$. However in equation 7.13 $\mathbf{f}_f \circ f^{-1}$ is a linear continuous function from $Im(f)$ to $\mathbb{R}$. It can be extended through Hahn-Banach to a linear continuous injectivefunction also denoted $\mathbf{f}_f \circ f^{-1}$ from E to $\mathbb{R}$, that is $\mathbf{f}_f \circ f^{-1} \in E'$. Thus we have $!d_E \circ \mu_E(\phi)(\mathbf{f}_f)_f = (\phi((\mathbf{f}_f \circ f^{-1} \circ f)_f) = \phi(\mathbf{f}_f)$ and we have the desired identity.

2. Let us show that $!(\mu_E) \circ \mu_E = \mu_{!E} \circ \mu_E$ for all E. One the one hand, $!((\mu_E) : !!E \longrightarrow !!!E$ sends ϕ and a projective limit of functions $(\mathbf{g}_g)_g \in \mathscr{E}(ocE!E)$ to $\phi((\mathbf{g}_{\mu_E \circ h})_h \in \mathscr{E}(!E)$. On the other hand $\mu_{!E}$ sends $\phi \in !!E$ and $(\mathbf{g}_g)_g \in \mathscr{E}(!!E)$ to $\mathbf{g}_g(g^{-1}(\phi))$.

Consider $\psi \in !E$ and $\phi = \mu_E(\psi)$, we know that $\phi((\mathbf{h}_h) \in \mathscr{E}(!E)) = h_h(h^-1(\psi))$ for any $h : \mathbb{R}^n \longrightarrow !E$ containing ψ in its image. Consider a projective limit of functions $(\mathbf{g}_g)_g \in \mathscr{E}(!!E)$. Thus on the one hand we have:

$$\begin{aligned} !(\mu_E)(\phi)((\mathbf{g}_g)_g) &= \phi((\mathbf{g}_{\mu_E \circ h})_h) \in \mathscr{E}(!E) \\ &= \mathbf{g}_{\mu_E \circ h}(h^{-1}(\psi)) \text{ for any } h \text{ containing } \psi \text{ in its image.} \end{aligned}$$

On the other hand we have:

$$\mu_{!E}(\phi)((\mathbf{g}_g)_g) = \mathbf{g}_g(g^{-1}(\phi)) \text{ for } g \text{ such that } \phi \in Im(g)$$

Thus we have the desired equality. □

Theorem 7.5.29. *The endo-functor* $! : \textsc{Refl} \longrightarrow \textsc{Refl}$ *forms a co-monad with co-dereliction* d *and co-multiplication* μ*. We also have a natural transformation* $\bar{d} : Id \longrightarrow !$ *such that for any* $E \in \textsc{Refl}$*,* $d_E \circ \bar{d}_E = Id_E$*.*

7.5.4 A model of MALL for our higher-order exponential

The product and biproduct of topological vector spaces 3.1.4 are linearly homeomorphic on finite indexed and forms a biproduct, which leads to the usual sum on hom-sets as described in proposition 2.4.8.

Let us explore the monoidal structure. It should be clear now that the difficulty in constructing a model of MLL in topological vector spaces is to choose the topology which will make the tensor product associative and commutative on the already chosen category of lcs. The kernel theorem implies that one should interpret $⅋$ as the completed tensor $\hat{\otimes}_\varepsilon$, and thus the tensor product as its dual (which may not equal the completed projective product $\hat{\otimes}_\pi$ in general). Thus we consider the category $C\textsc{Refl}$ of complete reflexive lcs and continuous linear maps, which contains in particular nuclear (F)-spaces and their duals, and thus all spaces of smooth functions and their dual, the spaces of compact-supported distributions. Following [44, 16.2.7], $\hat{\otimes}_\varepsilon$ is associative and commutative on complete spaces.

If we study more closely the definition of spaces of Higher-order smooth functions, we see that they reflexivity follows the one of a more restrictive of spaces.

Definition 7.5.30. A lcs is said to be a LNF-space if it is a regular projective limit of nuclear Fréchet spaces. The category of LNF-spaces and linear continuous injective maps is denoted LNF.

Example 7.5.31. A typical example of LNF-space are the spaces $\mathscr{E}(E)$, when E is any lcs.

Remark 7.5.32. Observe that this definition is again the result of a good interplay between topologies and bornologies. This should be characterized in terms of the categories TopVec and BornVec.

Proposition 7.5.33. *A* LNF*-space* E *is always* β*-reflexive.*

Proof. As nuclear (F)-space are reflexive (proposition 7.2.27), and as reflexive spaces are stables by projective limit (proposition 3.5.13), a LNDF space is reflexive. □

By using the same proofs as those computing the dual of $\mathscr{E}(E)$, one can characterize the duals of $\textsc{Lnf}$-space.

Definition 7.5.34. A lcs E is said to be a $\textsc{Lndf}$-space if it is an inductive limit of nuclear complete (DF)-spaces.

Corollary 7.5.35. As an inductive limit of reflexive and thus barrelled lcs, a $\textsc{Lndf}$ space is barrelled. Following the methods of the proof 7.5.5, we show that any $\textsc{Lndf}$ space is semi-reflexive, and thus any $\textsc{Lndf}$ space is reflexive.

Proposition 7.5.36. *The class of* $\textsc{Lndf}$ *spaces is stable by completed projective tensor product (equivalently, completed tensor product).*

Proof. This follows from the fact that completed tensor product preserves projective limits [44], and by the fact that nuclear and (DF)-spaces are stable by projective tensor product (propositions 7.1.6, 7.2.20). $\square$

Proposition 7.5.37. *The dual of a* $\textsc{Lnf}$*-space is a* $\textsc{Lndf}$*-space.*

Proof. Again, by [44, 8.8.12], the dual E' of a $\textsc{Lnf}$-space $E = \varinjlim_i E_i$ identifies as a linear space to a projective limit of complete nuclear (DF)-spaces. As the limit $\varinjlim_i E_i$ is regular, we have that any bounded set in E is bounded in some of the E_i. Thus the strong topology on E' coincide with the projective topology. $\square$

Notation 7.5.38. *We denote by* $\textsc{Lndf}$ *(resp.* $\textsc{Lnf}$*) the category of* $\textsc{Lndf}$*-spaces (resp.* $\textsc{Lnf}$*-spaces) and* isomorphisms *between them.*

Because we defined spaces of functions $\mathscr{E}(E)$ as projective limit, we have still a good knowledge of the interpretation of the $⅋$ between $\textsc{Lnf}$ spaces (which are thus the interpretation of negatives formulas). Indeed, the completed injective tensor product $\hat{\otimes}_\varepsilon$ of a projective limit of lcs is the projective limit of the completed injective tensor products [44, 16.3.2]. Taking the duals of theorem 7.5.12 applied to E' and F' gives us immediately:

Proposition 7.5.39. *For any reflexive spaces E and F we have a linear homeomorphism:*

$$?E \hat{\otimes}_\varepsilon ?F \simeq ?(E \otimes F).$$

Thus we have a model of polarized DiLL_0 with distributions, from which proofs are interpreted as arrows in $\textsc{TopVec}$ as in our previous section (see also [23]).

7.5.5 An exponential for convenient spaces

In this section we sketch how the exponential constructed here also could also fit the polarized refinement of convenient spaces developed in Chapter 6. We recall in the diagram below the model of MALL at stakes:

$$\begin{tikzcd}
(\textsc{Conv}, \hat{\otimes}_\beta^M) \arrow[r, bend left, "(_)'_\mu"] \arrow[r, phantom, "\perp"] & (\textsc{ComplMackey}^{op}, \varepsilon) \arrow[l, bend left, "\overline{(_)'_\mu}^{conv}"]
\end{tikzcd}$$

Proposition 7.5.40. *The lcs* $!_E = \mathscr{E}'(E) =:= \varinjlim_{f:\mathbb{R}^n \multimap E} \mathscr{E}'_f(\mathbb{R}^n)$ *(see Definition 7.5.3) is a Mackey-complete and bornological lcs.*

Proof. Bornological spaces are stable by inductive limits (Section 3.4.2 and [44, 13.1]), thus $!E$ is bornological. We proved also that $!_1 E$ is reflexive (Proposition 7.5.5), and as such it is weakly quasi-complete (every bounded Cauchy sequence converges weakly), and thus Mackey-complete (if B is bounded in E, a Cauchy sequence in $E - B$ is in particular bounded, and if it converges weakly it converges in E_B). $\square$

Proposition 7.5.41. *The lcs $\mathscr{E}(E)$ is in particular Nuclear, and thus has the Approximation property (that is, if E is Nuclear, for every lcs F we have $E' \otimes F$ dense in $\mathcal{L}(E, F)$).*

Proof. This follows directly from the fact that the class of Nuclear lcs is stable by projective limits [44, 21.2.3], and from the fact that nuclear spaces have the approximation property ([44, 21.2.2]). $\square$

As the injective tensor product is exactly the topology induced on the tensor product $F_1 \otimes F_2$ by $F_1 \varepsilon F_2$, we have for any lcs E_1 and E_2 :

$$\mathscr{E}(E_1)\hat{\otimes}_\varepsilon \mathscr{E}(E_2) \simeq \mathscr{E}(E_1)\varepsilon\mathscr{E}(E_2).$$

Thus the higher-order kernel theorem 7.5.12 gives us a strong monoidal functor :

$$? : (\textsc{Refl}, \times) \longrightarrow (\textsc{MackeyCompl}, \varepsilon).$$

Chapter 8

LPDE and $\mathrm{D}-\mathrm{DiLL}$

We provide in this chapter a sequent calculus whose syntax and semantics describe the resolution of linear partial differential equations with constants coefficients. In section 8.1, we review the theory of linear partial differential operators on distributions, and introduce a specific space of distributions $!_D E$. We then discuss in Section 8.2 the possible extensions of DiLL to Linear Partial Differential Equations. In Section 8.3, we give a non-deterministic sequent calculus for which D_0 and Linear Partial Differential Equations with constant coefficients give a model. In 8.4, we give a deterministic sequent calculus for which Linear Partial Differential Equations with constant coefficients give a model. For the two sequent calculi detailed above, the cut-elimination rule between dereliction and co-dereliction is interpreted by the resolution of the Differential Equation at stake.

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8.1 Linear Partial Equations

In this section we give very quick and partial background in the theory of linear partial differential equations with constant coefficients. We are not considering in this Section border conditions, regularity of the solutions to equations with non-constant coefficients, nor modern research subjects in this theory such as non-linear equations.

Outlook 15. As in the previous chapter, we are almost always considering distributions and equations defined on a whole euclidean space $\mathbb{R}^n$. The Kernel theorem also restrict to open subsets of euclidean subsets, and the study of partial differential equations which are not necessarily linear with constant coefficients requires to work on specific subsets of euclidean spaces. Translating this in logic and type systems, via subtyping, is work which is still to be done.

We refer mainly to the books by Hormander for this section [41], [42]. We also refer to the book by Treves [75] for a more categorical point of view.

8.1.1 Linear Partial Differential operators

Functions with compact support Let us recall the properties of $\mathcal{D}(\mathbb{R}^n)$, which differ from the ones of $\mathcal{C}^\infty(\mathbb{R}^n)$.

Proposition 8.1.1. *[76, Ch.13, Ex.II] The lcs $\mathcal{D}(\mathbb{R}^n)$ is linearly homeomorphic to the strict inductive limit $\varinjlim_{K_n} \mathcal{C}^\infty(K_n, \mathbb{R})$ indexed by the closed balls of radius n $K_n \subset\in \mathbb{R}^n$, ordered by inclusion.*

This limit is in particular regular (see Definition 3.1.20), as $\mathcal{C}^\infty(K_n)$ is closed in $\mathcal{C}^\infty(K_{n+1})$. The spaces which are strict countable inductive limits of (F)-spaces are called LF-spaces in the literature. They are in particular complete (see [44, 4.6]) and barreled (see Proposition 3.4.25). A countable inductive limit of nuclear spaces is also nuclear (see Proposition 7.2.20). As such, a strict countable inductive limit of nuclear fréchet spaces is complete and nuclear, thus semi-reflexive (see proposition 7.2.25) and reflexive as it is barreled. This combination of stability properties proves the following proposition:

Proposition 8.1.2. *The lcs $\mathcal{D}(\mathbb{R}^n)$ is a complete nuclear and reflexive space.*

Linear Partial Differential operators For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ we write ∂^α the linear continuous function:

$$f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathscr{R}) \mapsto x \in \mathbb{R}^n \mapsto \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}(x)$$

where $|\alpha|$ denotes the sum $\alpha_1 + \cdots + \alpha_n$ of the coefficients of α.

Definition 8.1.3. Consider smooth functions $a_\alpha \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ indexed by multi-indices $\alpha \in \mathbb{N}^n$. Then a Linear Partial Differential Operator (LPDO) is defined as an operator $D : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{D}(\mathbb{R}^n)$ of the form:

$$D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha.$$

D is a LPDO with constant coefficients (denoted as *LPDOcc*) when the a_α are constants. Notice that D could also have been defined as an endomorphism of $\mathscr{E}(\mathbb{R}^n)$, as the support of $\partial^\alpha f$ is contained in the support of f. We will sometimes consider this point of of view, and in particular when defining $!_D E$ in Section 8.1.3.2.

Proposition 8.1.4. *[76] D is a linear continuous endomorphism of $\mathcal{D}(\mathbb{R}^n)$ (resp. $\mathscr{E}(\mathbb{R}^n)$).*

Proof. Remember that $\mathscr{E}(\mathbb{R}^n)$ is endowed with the topology generated by the semi-norms:

$$\|f\|_{N,K} := \sup_{x \in K, |\alpha| \leqslant N} |\partial^\alpha f(x)|$$

where $N \in \mathbb{N}$ and K is a compact subset of $\mathbb{R}^n$. If $\dot{B}_{N,K}$ denotes the open ball for $\|\_\|_{N,K}$, then $(\partial^\alpha)^{-1}(\dot{B}_{N,K})$ contains $\dot{B}_{N+|\alpha|,K}$. Thus any LPDO is continuous as the finite linear combination with continuous factors of continuous maps. □

Example 8.1.5. LPDO's with constant coefficients appear in several phenomena. Let us recall the definition of the Laplacian:

$$\nabla^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

or of the heat operator, for $\lambda > 0$:

$$f \in \mathcal{E}(\mathbb{R}^n + 1) \mapsto \frac{\partial f}{\partial t} - \lambda \nabla^2 f$$

where f takes as variables t and $x_1, ..., x_n$.

Remark 8.1.6. In the literature, LPDO's with constant coefficients are sometimes written

$$P(D) = \sum_{n=1}^{N} \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha$$

where $P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha X^\alpha \in \mathbb{R}[X_1, ...X_n]$ is a polynomial with n variables.

Definition 8.1.7. The definition of D is then extended to $\mathcal{D}'(\mathbb{R}^n)$ (resp. $\mathcal{E}'(\mathbb{R}^n)$) as follows: the derivative of a distribution $\psi \in \mathscr{E}'(E)$ or $\psi \in \mathcal{D}'(E)$ is defined as:

$$D(\psi) : f \mapsto \psi \left(\sum_\alpha (-1)^\alpha a_\alpha \frac{\partial^\alpha f}{\partial x^\alpha} \right).$$

We denote by $\check{D}$ the LPDOcc:

$$\check{D} := \sum_\alpha (-1)^\alpha a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$$

This definition agrees with the intuition that distributions are generalized functions. In that case indeed, the differentiation of the generalized function is the generalized differentiated function:

$$\begin{aligned} D\psi_g(f) &= \int_y g(y) \left(\sum_\alpha (-1)^\alpha a_\alpha \frac{\partial^\alpha f}{\partial x^\alpha}(y) \right) dy \\ &= \int_y \left(\sum_\alpha a_\alpha \frac{\partial^\alpha g}{\partial x^\alpha}(y) \right) f(y) dy \text{ by integration by part} \\ &= \psi_{Dg}(f) \end{aligned}$$

A fundamental example Consider the interpretation for the dereliction and co-dereliction in NUCL:

$$d_E : \phi \in \mathscr{E}'(E) \mapsto (\ell \in E' \mapsto \phi(\ell)) \in E'' \simeq E$$

$$\bar{d}_E : \psi \in E'' \mapsto (f \in \mathscr{E}(E) \mapsto \psi(D_0(f))$$

Remember that in the definition of the codereliction, we have $\psi = ev_x$ for some x. Moreover, one could define as previously D_0 on distributions $\psi \in \mathscr{E}'(E)$:

$$D_0(\psi) : f \mapsto (-1)\psi(D_0 f)$$

which is again a distribution $D_0(\psi) \in \mathscr{E}'(E)$.

Looking now at D_0 as a differential operator on distribution, we have in NUCL:

$$d_E : \phi \in \mathscr{E}'(E) \mapsto \phi_{D_0(\mathscr{E}(E))} \in E'' \simeq E$$

$$\bar{d}_E : \psi \in E'' \mapsto -D_0(\psi)$$

Here, we would like indeed to generalize by replacing D_0 by D and E'' by $!_D E$ (see Section 8.1.3.2).

$$D(\phi) = \phi \circ D = f \mapsto \sum_\alpha a_\alpha \partial^\alpha(f),$$

so that when $g \in \mathcal{C}^\infty_{co}(\mathbb{R}^n)$ we have $D(f \mapsto \int fg) = f \mapsto \int f D(g)$.

Definition 8.1.8. The *resolution* of the equation then consists, when a distribution $\phi \in \mathcal{D}'(\mathbb{R}^n)$ is fixed, in finding ψ such that $D\psi = \phi$.

The regularity of ψ regarding the one of ϕ is then a major field of study, through Sobolev spaces for example [11].

Definition 8.1.9. Consider D a LPDO. By $D(\mathscr{E}(\mathbb{R}^n))$ we denote the image of D in $\mathscr{E}(\mathbb{R}^n)$. It is a vector space by linearity of D. We endow it with the lcs topology induced by $\mathscr{E}(\mathbb{R}^n)$. Beware that D is of course not injective in general.

8.1.2 Differentiation and convolution

Convolution and solutions to LPDE's with constant coefficients behave particularly well:

Proposition 8.1.10 ([42] 4.2.5)**.** *Consider $f \in \mathcal{E}(\mathbb{R}^n)$ and $\phi \in \mathcal{E}'(\mathbb{R}^n)$. Then $\partial^\alpha(\phi * f) = \phi * (\partial^\alpha f) = (\partial^\alpha \phi) * f$.*

Proof. We prove the result for $|\alpha| = 1$ and the general result will follow by induction. Consider $x \in \mathbb{R}^n$. By definition

$$\begin{aligned}
\frac{\partial(\phi * f)}{\partial v}(x) &= \lim_{t \longrightarrow 0} \frac{(\phi * f)(x + tv)}{t} \\
&= \lim_{t \longrightarrow 0} \frac{\phi(y \mapsto f(x + tv - y))}{t} \\
&= \phi\left(y \mapsto \lim_{t \longrightarrow 0} \frac{f(x + tv - y)}{t}\right) \text{ as } \phi \text{ is linear and continuous.}
\end{aligned}$$

Thus the first equality is proved. The second equality follows from Definition (8.1.7):

$$\begin{aligned}
(\frac{\partial \phi}{\partial x} * f)(x) &= \frac{\partial \phi}{\partial v}(y \mapsto f(x - y)) \\
&= -\phi\left(\frac{\partial(y \mapsto f(x - y)}{\partial v}\right) \\
&= \phi\left(y \mapsto \frac{\partial f}{\partial v}(x - y)\right) \\
&= \left(\phi * \frac{\partial f}{\partial v}\right)(x)
\end{aligned}$$

∎

This result extends to convolutions of distributions via Definition 7.3.17: for $\phi \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{E}'(\mathbb{R}^n)$ the distribution $\partial^\alpha(\phi * \psi)$ maps $f \in \mathcal{D}(\mathbb{R}^n)$ to $(-1)^{|\alpha|}(\phi * \psi)(\partial^\alpha f)$. Thus:

$$\begin{aligned}
\partial^\alpha(\phi * \psi)(f) &= ((\partial^\alpha \phi) * \psi)(f) && (8.1)\\
&= \phi * ((\partial^\alpha \psi)(f)) \text{ by commutativity of the convolution} && (8.2)
\end{aligned}$$

8.1.3 Solving linear partial differential equations with constant coefficients

8.1.3.1 Fundamental solutions

Among the solutions to LPDE's, some are particularly studied: these are the fundamental solutions, that is solutions E to the equation:

$$DE = \delta_0$$

where the parameter is a Dirac distribution. Because partial differential linear operators with constant coefficients behave particularly well with respect to convolution, the answer to this particular input is enough to compute the answers to any input f.

Definition 8.1.11. A fundamental solution for the LPDO D consists in a distribution $\psi \in \mathcal{D}'(\mathbb{R}^n)$[1] to the equation:

$$D\psi = \delta_0.$$

Remark 8.1.12. Notice that such a fundamental solution is in general not unique: if ϕ is such that $D\phi = 0$, then $E_D + \phi$ is also a fundamental solution to D.

Example 8.1.13. Because of linear partial differential operators, we are working with distributions whose support is not necessarily compact. Indeed, the existence of a fundamental solution is not ensured when distributions must apply to any smooth function. The typical example is

$$D : f \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}) \mapsto f',$$

where f' is the derivative of the real, one-variable function f. If f has compact support, one can define:

$$E_D : f \mapsto \int_{\infty}^{0} f$$

and one has indeed $DE_D(f) = f(0)$. This however is not possible in full generality when $f \in \mathcal{C}^{\infty}(\mathbb{R}^n)$. Observe that it would be enough for the partial derivative of f to be bounded by an integrable function for x big enough. Then the derivative of f would be integrable, and E_D could be defined. Extended to any partial differential operator, this says that if any partial derivative $\partial^{\alpha} f$, $\alpha \in \mathbb{N}^n$ of f are integrable on $\mathbb{R}^n$, then E_D can be defined. This idea is implemented in the construction of a tamed fundamental solution $E_D \in \mathscr{S}'(\mathbb{R}^R)$ [43].

Notation 8.1.14. *Consider $D : \mathscr{E}(\mathbb{R}^n) \longrightarrow \mathscr{E}(\mathbb{R}^n)$ a LPDO. When a fundamental solution for D exists and it is fixed, it is denoted by $E_D \in \mathcal{D}'(\mathbb{R}^n)$.*

Remark 8.1.15. We will recall later in theorem 8.1.18 that when D is a LPDO with constant coefficients there is always of at least one fundamental solution for D.

The resolution of LPDOs with constant coefficients is always possible, and particularly elegant, due to the behaviour of convolution with respect to partial differentiation.

Proposition 8.1.16. *Consider $D = \sum_{\alpha} a_{\alpha}\partial^{\alpha}$ a LPDO with constant coefficients. Suppose that D admits a fundamental solution E_D. Then for any $\phi \in \mathcal{E}'(\mathbb{R}^n)$ we have:*

$$D(E_D * \phi) = \phi. \tag{8.3}$$

Proof. Thanks to the bilinearity of the convolution product

$$\begin{aligned} D(E_D * \phi) &= \sum_{\alpha} a_{\alpha}[\partial^{\alpha}(E_D * \phi)] \\ &= (\sum_{\alpha} a_{\alpha}\partial^{\alpha} E_D) * \phi \text{ thanks to equation 8.1 and the bilinearity of } *, \\ &= \delta_0 * \phi \text{ by definition of } E_D, \\ &= \phi \end{aligned}$$

□

Remark 8.1.17. Beware that equation 8.3 is valid in $\mathcal{D}'(\mathbb{R}^n)$, but not in $\mathscr{E}'(\mathbb{R}^n)$. Indeed, the convolution between $\phi \in \mathscr{E}'(\mathbb{R}^n)$ and $\psi \in \mathcal{D}'(\mathbb{R}^n)$ only yields a distribution $\psi * \phi \in \mathcal{D}'(\mathbb{R}^n)$. This is by construction of the convolution following proposition 7.3.16. It is also a particular case of the theorem of support [42, Thm 4.2.4] which says that the support of convolution $\phi * \psi$ of two distributions $\phi \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in \mathscr{E}'(\mathbb{R}^n)$ is contained is the sum of the respective supports of ψ and ϕ. We do not emphasize on support and singularities of distributions here.

Theorem 8.1.18. *Every LPDEcc admits a fundamental solution $E_D \in \mathcal{D}'(\mathbb{R}^n)$.*

[1] note that we do not ask for $\psi \in \mathscr{E}'(\mathbb{R}^n)$, as this is not possible in general, even for the LPDO with constant coefficients

Comments on the proof. Several constructions of a fundamental solution exist: we refer to the one of Malgrange, refined and exposed by Hormander [41, Thm 3.1.1] or to [42, 7.3.10] which gives a fundamental solution with optimal local growth. Others construct a fundamental solution which is temperate [43]. The proof starts with a technical Lemma [41, 3.1.1], which says that for any smooth function with *compact support* $f \in \mathcal{D}(\mathbb{R}^n)$, we have that

$$|f(0)| \leqslant C \, \|(cosh(\epsilon|x|)D(f)\|_{1,\frac{1}{\deg P}} \tag{8.4}$$

where C does not depend of f. Thus if $D(f) = D(g)$, one has $D(f-g) = 0$ and $f(0) = g(0)$. Then defines the function $\check{E}$ on $D(\mathcal{D}(\mathbb{R}^n))$ as $\check{E}(D(f)) = f(0)$. This function is well defined, linear and continuous. The above majoration allows then to extend this function to $E \in \mathcal{D}'(\mathbb{R}^n)$ through Hahn-Banach 3.3.6. The theorem is in fact much more precise as we have information about the local growth of E_D. We do not have in general that $E_D \in \mathcal{E}'(\mathbb{R}^n)$.

Remark 8.1.19. Notice that equation 8.4 ensures that E_D is well defined on $D(\mathcal{D}(\mathbb{R}^n))$, as for $f \in D(\mathcal{D}(\mathbb{R}^n))$ the function g such that $f = Dg$ is unique. This is not true for functions in $D(\mathscr{E}(\mathbb{R}^n))$

8.1.3.2 The space $!_D$

This section studies the question of which space is the good interpretation of $!_D\mathbb{R}^n$ for " the space of distributions which are solutions to a LPDEcc "

Let us sum up the situation: the computation $E_D(g)$ is well defined as soon as $g \in \mathcal{D}(\mathbb{R}^n)$, as Theorem 8.1.18 shows that $E_D \in D'(\mathbb{R}^n)$. If $f \in \mathcal{D}(\mathbb{R}^n)$, then obviously $\check{D}(f) \in \mathcal{D}(\mathbb{R}^n)$ and we have:

$$E_D(\check{D}(f)) = DE_D(f) = f(0).$$

If $f \in \mathscr{E}(\mathbb{R}^n)$ (that is, f does not necessarily have compact support) is such that $\check{(}D(f)) \in \mathcal{D}(\mathbb{R}^n)$, then $E_D(\check{D}(f))$ is well defined, and we have also $E_D(\check{D}(f)) = f(0)$. However, when $f \in \mathscr{E}(\mathbb{R}^n)$ and $\check{D}(f)$ does not have compact support, then $E_D(\check{D}(f))$ is *not defined*, and thus we can't compute $f(0)$ as E_D applied to $\check{D}(f)$. Let us export this this reasoning to distributions.

Definition 8.1.20. We denote by $!_D\mathbb{R}^n$ the sub-vector space of $\mathcal{D}'(\mathbb{R}^n)$ consisting of the distributions $\phi \in \mathcal{D}'(\mathbb{R}^n)$ such that $D\phi \in \mathscr{E}'(\mathbb{R}^n)$.

$$!_D\mathbb{R}^n := \{\phi \in \mathcal{D}'(\mathbb{R}^n) \mid D\phi \in \mathscr{E}'(\mathbb{R}^n)\}$$

Example 8.1.21. A typical example of distribution in $!_D\mathbb{R}^n$ is E_D , as $DE_D = \delta_0 \in \mathscr{E}'(\mathbb{R}^n)$.

Proposition 8.1.22. *Endowed with the topology inherited from $\mathcal{D}'(\mathbb{R}^n)$, the space $!_D\mathbb{R}^n$ is a lcs. Moreover, we have the topological embeddings*[2] $\mathscr{E}'(\mathbb{R}^n) \hookrightarrow !_D\mathbb{R}^n \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$.

Proof. If we consider D as linear continuous endomorphism:

$$D : \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$$

then $!_D\mathbb{R}^n$ is the inverse image of $\mathscr{E}'(\mathbb{R}^n)$ by D. As $\mathscr{E}(\mathbb{R}^n)$ is closed in $\mathcal{D}(\mathbb{R}^n)$, and as D is continuous, then $!_D$ is a closed sub-locally convex topological vector space of $\mathcal{D}'(\mathbb{R}^n)$. It is Hausdorff: consider $\phi \neq \phi'$ are both distributions, and if $D\phi$ and $D\phi'$ both have compact support. Consider V and V' disjoint neighbourhoods $\mathcal{D}'(\mathbb{R}^n)$ of ϕ and ϕ' respectively. Then as $\mathscr{E}'(\mathbb{R}^n)$ is dense in $\mathcal{D}'(\mathbb{R}^n)$, $V \cap \mathscr{E}'(\mathbb{R}^n)$ and $V \cap \mathscr{E}'(\mathbb{R}^n)$ are two non-empty open sets contained in $!_D\mathbb{R}^n$ whose intersection is empty and containing respectively ϕ and ϕ'. The topological embeddings follows from the fact that if ϕ has compact support, then $D\phi$ has compact support: thus $\mathscr{E}'(\mathbb{R}^n) \subset !_D\mathbb{R}^n$. This embedding is topological as $\mathcal{D}'(\mathbb{R}^n)$ and $\mathscr{E}'(\mathbb{R}^n)$ carry the same topology. The other embedding follows from the fact that by definition, $!_D\mathbb{R}^n$ is a sub-lcs of $\mathcal{D}'(\mathbb{R}^n)$. ☐

Notation 8.1.23. *We denote by $?_D\mathbb{R}^n$ the lcs $(!_D\mathbb{R}^n)'$ endowed with the strong topology on the dual.*

As D commutes with convolution (proposition 8.1.10) and as the convolution of two distributions with compact support has compact support, we have immediately:

Corollary 8.1.24. For any $\phi \in !_D\mathbb{R}^n$ and any $\psi \in \mathscr{E}(\mathbb{R}^n)$ we have $\phi * \psi \in !_D\mathbb{R}^n$.

Corollary 8.1.25. For $f \in ?_D\mathbb{R}^n$ and $g \in \mathcal{C}^\infty(\mathbb{R}^n)$, $D(f * g)$ is defined as $Df * g$. It is coherent with the notation when $f \in \mathcal{D}(\mathbb{R}^n)$, as partial differentiation commutes with convolution (Proposition 8.1.10).

[2] that is, the linear continuous injections

Analogy with DiLL Defining $!_{D_0}\mathbb{R}^n$ as the inverse image of $\mathscr{E}'(\mathbb{R}^n)$ by D_0 in $\mathcal{D}'(\mathbb{R}^n)$ does not make any sense. Indeed, $D_0 f$ is a linear function: it has compact support if and only if it is null. Thus with the definition used before, we have $?_{D_0}\mathbb{R}^n = \{0\}$ and $!_{D_0}\mathbb{R}^n = \mathbb{R}^n$.

Remark 8.1.26. Although D_0 is not a LPDOCC, δ_0 kind of behaves as a fundamental solution to D_0. Indeed, if a function $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is such that there is $g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that:

$$D_0 g = f,$$

then f is necessarily linear, and thus one can take $g = f = \delta_0 * f$.

The good definition for "the space of distributions solutions to $D_0\psi = \phi$ " is

$$?_{D_0}\mathbb{R}^n := D_0^{-1}(\mathscr{E}(\mathbb{R}^n)),$$

and thus

$$!_{D_0}\mathbb{R}^n \simeq \mathcal{L}(\mathbb{R}^n, \mathbb{R})' \simeq \mathbb{R}^n.$$

In that case dereliction and codereliction rules are interpreted in NUCL by:

$$d_{D_0} : !\mathbb{R}^n \longrightarrow !_{D_0}\mathbb{R}^n, \psi \mapsto (\ell \mapsto \psi(\ell))$$
$$\bar{d}_{D_0} : !_{D_0}\mathbb{R}^n \longrightarrow !\mathbb{R}^n, \phi = ev_x \mapsto (f \mapsto \phi(D_0(f)).$$

However if we look for a fundamental solution $E_{D_0} \in E''$, it must satisfy:

$$E_{D_0}(D_0 f) = f(0)$$

and thus in particular, for any $\ell \in E'$, we have $E_{D_0}(\ell) = E_{D_0}(D_0\ell) = \ell(0) = 0$. *Thus the analogy with fundamental solutions does not hold between D_0 and LPDOcc, outside of the interpretation of dereliction and co-dereliction.*

Outlook 16. One should explore the similarities between the work of Ehrhard on anti-derivatives [20], and the proof theory of fundamental solutions as described in this chapter.

8.2 Discussion: Linear Logic for differential equations

In the previous semantical study, we saw the appearance of a connective $!_D$ such that $E'(E) \subset !_D E \subset \mathcal{D}(E)'$ when $E \simeq \mathbb{R}^n$. This space is $D^{-1}(\mathscr{E}'(E))$, that is it represents the space of distributions ϕ which are solutions to the equation

$$D\phi = \psi,$$

when $\psi \in \mathscr{E}'(E)$. We noticed that defining $!_{D_0}E$ as $D_0^{-1}(\mathscr{E}'(E))$ was not meaningful and that the good analogy was $!_{D_0}E = (D_0(\mathscr{E}(\mathbb{R}^n)) = (\mathbb{R}^n)'' \simeq \mathbb{R}^n$.

We also saw that while $E \simeq E''$ represents the space of distributions with solutions to $D_0\phi = \psi$, the space $!_D E$ represents the space of distributions solutions to $D\phi = \psi$. In sections 8.3 and 8.4, we develop thus a syntax for LPDE, where the interactions between $!_D A$ and $!A$ mimic the ones between A and $!A$ in DiLL.

Remark 8.2.1. In the semantics developed in the previous section, (**??**) is only valid when E is euclidean, but it should extend to higher order in view of the results in section 7.5.

Remark 8.2.2. Beware that all the intuitions cannot be retrieved from the model of chapter 7. In particular while we have $E \simeq E'' \hookrightarrow !E$, we have here an embedding $!E \hookrightarrow !_D E$. The interpretation of $!E$ as $\mathcal{D}'(E)$ would have yield the inverse embedding between $!E$ and $!_D E$, thus alike the embedding $E \simeq E'' \hookrightarrow !E$. However, we really need to use distribution with compact support as the interpretation of the exponential, as they are needed in a convolution product (interpreting co-contraction).

Dereliction and co-dereliction Dereliction, which in a category of reflexive spaces was interpreted as

$$d : \psi \in \mathscr{E}'(E) \mapsto (\ell \in E' \mapsto \psi(\ell)) \in E'' \simeq E,$$

should now be interpreted as the restriction to $!_D$:

$$d_D : \psi \in \mathscr{E}'(E) \mapsto E_D * \psi \in !_D E. \tag{8.5}$$

This is well-defined thanks to 8.1.21 and 8.1.24. It coincides with the intuitions of DiLL. When $D = D_0$, then $E_D = \delta_0 \in E''$, thus $\psi * \delta_0 \in E''$ is just the restriction of f to linear forms in E'.
Conversely, the codereliction which was interpreted as

$$\bar{d} : \phi \in E'' \mapsto \phi \circ D_0 = D_0(\phi) \in !(E)$$

should now be interpreted as

$$\bar{d}_D : \phi \in !_D E \mapsto D\phi \in \mathscr{E}(E)'.$$

This is made possible by the very definition of $!_D$. Moreover, when in DiLL we had, for $x \in A$:

$$d \circ \bar{d} = Id_{E''}$$

we have now, by proposition 8.1.16, for $\phi \in !_D E$

$$d_D \circ \bar{d}_D(\phi) = E_D * D\phi = \phi.$$

We will see indeed in Section 8.4 that for any LPDOcc D, the interpretation for $!_D$ given in definition 8.1.20 results in a model (without higher-order) for the sequent calculus $\mathrm{D} - \mathrm{DiLL}$.

Remark 8.2.3. Distinguishing isomorphisms from identities is matter for future work. However, one should notice now that the isomorphism $E''' \simeq E$ is really used only for the generalization of structural rules (the interpretation of d,c and w) while the generalization of co-structural rules only uses the canonical injection $E \hookrightarrow E''$.

Remark 8.2.4. **Fixpoints and quantitative semantics** Thus elements of E' are exactly the functions $f \in \mathcal{C}^\infty(E, \mathbb{R})$ such that $D_0 f = f$ (which is equivalent, as pointed out earlier, to the fact that there is $g \in \mathcal{C}^\infty(E, \mathbb{R})$ such that $D_0 g = f$). That is, E' is exactly the space of fixpoints for $D_0 : ?E \longrightarrow ?E$. Thus:

$$E \simeq !_{D_0} E.$$

We have obviously

$$!E \simeq !_{Id} E.$$

We argue here that this exponential indexed by a Differential Operator has a meaning beyond D_0, Id and LPDOcc. Let us note that if T denotes a Taylor sum operator:

$$T : f \in \mathcal{C}^\infty(E, \mathbb{R}) \mapsto (x \longrightarrow \sum_n D_0^n f(x))$$

then *power series* are exactly the functions $f \in \mathcal{C}^\infty(E, \mathbb{R})$ such that $T(f)$ is well defined (that is $T(x)$ is a convergent sum for every x), and such that

$$T(f) = f.$$

Thus if we denote by $S(E)$ the set of scalar power series (taking for example the definition appearing in [49], over complex topological vector spaces) the *quantitative exponential* is exactly the space of fixpoints for T, and

$$S(E)' \simeq !_T E.$$

As T is idempotent, one can define dereliction and codereliction for T:

$$d_T^t : !E \longrightarrow !_T E; \phi \mapsto \phi_{S(E)}$$
$$\bar{d}_T : !_T E \longrightarrow !E; \psi \mapsto \psi \circ T$$

As T is idempotent we have indeed $d_T^t \circ \bar{d}_T = Id$.

The Identity rule

$$\dfrac{}{\vdash A, A^{\perp}}\text{(axiom)} \qquad \dfrac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\text{(cut)}$$

The multiplicative rules

$$\dfrac{}{\vdash 1}(1) \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, \perp}(\perp) \qquad \dfrac{\vdash \Gamma, N, M}{\vdash \Gamma, N \parr M}(\parr) \qquad \dfrac{\vdash \Gamma, P \quad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}(\otimes)$$

The additive rules

$$\dfrac{}{\vdash \Gamma, \top}\top \qquad \dfrac{\vdash \Gamma, N \quad \vdash \Gamma, M}{\vdash \Gamma, N \,\&\, M}\& \qquad \dfrac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q}\oplus_L \qquad \dfrac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q}\oplus_R$$

The Exponential Rules

$$\dfrac{\vdash \mathscr{N}}{\vdash \mathscr{N}, ?P}w \qquad \dfrac{\vdash \mathscr{N}, ?P, ?P}{\vdash \mathscr{N}, ?P}c \qquad \dfrac{\vdash \mathscr{N}, ?_D P}{\vdash \mathscr{N}, ?P}d_D$$

$$\dfrac{}{\vdash !N}\bar{w} \qquad \dfrac{\vdash \mathscr{N}, !N \quad \vdash \mathcal{M}, !N}{\vdash \mathscr{N}, \mathcal{M}, !N}\bar{c} \qquad \dfrac{\vdash \mathscr{N}, !_D N}{\vdash \mathscr{N}, !N}\bar{d}_D$$

Figure 8.1: The deriving rules for sequents of $\mathrm{D}_0 - \mathrm{DiLL}$

8.3 $\mathrm{D}_0 - \mathrm{DiLL}$

We introduce in this section a toy polarized sequent calculus, with sums, for which both D_0 and LPDOcc D are a model. This sequent calculus captures the idea that *dereliction and codereliction correspond to the resolution of a differential equation*. We justify this idea by giving two semantics for $\mathrm{D}_0 - \mathrm{DiLL}$ without higher order: one which corresponds to the semantics of Chapter 7, and another one corresponding to the resolution of LPDEcc.

8.3.1 Grammar and rules

Definition 8.3.1. The formulas of $\mathrm{D}_0 - \mathrm{DiLL}$ are constructed with the following polarized syntax:

$$N, M := ?A \mid ?_D A \mid N \parr M \mid N \times M$$

$$P, Q := !A \mid !_D A \mid P \otimes Q \mid P \oplus Q$$

Definition 8.3.2. Then the negation is the one of LL, with the addition definitions:

$$!_D A^{\perp} = ?_D A^{\perp} \qquad ?_D A^{\perp} = !_D A^{\perp}$$

Then the proofs of $\mathrm{D}_0 - \mathrm{DiLL}$ are constructed according to the rules in figure 8.3.1. Notice that these are the rules of polarized DiLL (see 2.8), with modified the dereliction and the codereliction accounting for the introduction of the newly introduced connectives $?_D$ and $!_D$.

Cut-elimination The cut elimination procedure in $\mathrm{D}_0 - \mathrm{DiLL}$ follows exactly the one of DiLL_{pol} (see section 2.4.1.3). In particular, the cut between the dereliction d_D and the codereliction $\bar{d}_D$ is:

$$\dfrac{\dfrac{\vdash \mathcal{N}, ?_D P}{\vdash \mathcal{N}, ?P}d_D \quad \dfrac{\vdash \mathcal{M}, !_D N}{\vdash \mathcal{M}, !N}\bar{d}_D}{\vdash \mathcal{N}, \mathcal{M}}\mathit{cut} \rightsquigarrow \dfrac{\vdash \mathcal{N}, ?_D P \quad \vdash \mathcal{M}, !_D N}{\vdash \mathcal{N}, \mathcal{M}}\mathit{cut}$$

and the one between contraction and codereliction involves the usual co-weakening:

$$\dfrac{\dfrac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}c \quad \dfrac{\vdash \Delta, !_D A}{\vdash \Delta, !A}\bar{d}_D}{\vdash \Gamma, \Delta}\text{cut} \rightsquigarrow$$

$$\cfrac{\cfrac{\vdash \Gamma, ?A, ?A \qquad \cfrac{\vdash \Delta, !_D A}{\vdash \Delta, !A}\,\bar{d}}{\vdash \Gamma, \Delta, ?A}\,\text{cut} \qquad \cfrac{\vdash}{\vdash !A}\,\bar{w}}{\vdash \Delta, \Gamma}\,\text{cut}$$

$$+ \quad \cfrac{\cfrac{\vdash \Gamma, ?A, ?A \qquad \cfrac{\vdash \Delta, A}{\vdash !A}\,\bar{d}}{\vdash \Gamma, \Delta}\,\text{cut} \qquad \cfrac{\vdash}{\vdash !A}\,\bar{w}}{\vdash \Delta, \Gamma}\,\text{cut}$$

Remark 8.3.3. With no introduction rule for $!_D$, this system is not finished. It should be enhanced with promotion and functorial promotion for $!_D$. However, as in this thesis we did not detail the commutative rules between promotion and the co-structural rules of DiLL, we do not do that here.

8.3.2 Models

8.3.2.1 A model with D_0.

We consider the semantics introduced in chapter 7, section 7.4. Thus formulas are interpreted as complete nuclear spaces, and negative formulas are in particular interpreted as nuclear (F)-space. Remember that the exponential was interpreted as:

$$! : \begin{cases} \text{EUCL} \longrightarrow \text{NUCL} \\ \mathbb{R}^n \longrightarrow \mathscr{E}'(\mathbb{R}^n) \\ \ell : \mathbb{R}^n \longrightarrow \mathbb{R}^m \mapsto (\phi \in \mathcal{E}'(\mathbb{R}^n) \mapsto \phi \circ \ell \in \mathcal{E}'(\mathbb{R}^m)) \end{cases} \tag{8.6}$$

We work here within the setting of a toy semantics without higher order, when $!_D$ and $!$ are only functor between EUCL and NF.

Outlook 17. The semantics developed in section 7.5 should give a model with higher order for $\mathrm{D}_0 - \mathrm{DiLL}$.

Definition 8.3.4. Following the discussions in section 8.2, we give the following interpretation:

$$!_{\mathcal{D}_0} : \begin{cases} \text{EUCL} \longrightarrow \text{NUCL} \\ \mathbb{R}^n \mapsto D_0^{-1}(\mathscr{E}'(\mathbb{R}^n)) \simeq \mathbb{R}^n \end{cases} \tag{8.7}$$

Then A and $!_{D_0} A$ have the same interpretation. Thus the semantics developed in chapter 7 gives a semantics for $\mathrm{D}_0 - \mathrm{DiLL}$, and we have in particular:

$$d_{D_0,\mathbb{R}^n} = d_{\mathbb{R}^n} : \phi \in \mathscr{E}'(\mathbb{R}^n) \mapsto (\ell \in (\mathbb{R}^n)' \mapsto \phi(\ell))$$

$$\bar{d}_{D_0,\mathbb{R}^n} = \bar{d}_{\mathbb{R}^n} : x \simeq ev_x \in (\mathbb{R}^n)'' \mapsto (f \in \mathscr{E}(\mathbb{R}^n) \mapsto ev_x(D_0(f))$$

8.3.2.2 A model with any LPDOcc D

Consider D a linear partial differential operator with constant coefficients, acting on functions in $\mathscr{E}(\mathbb{R})$.

$$D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha$$

We adopt the notations of chapter 7 which were used also for the model with D_0 in section 8.3.2.1: $! : \text{EUCL} \longrightarrow \text{NUCL}$ is the functor mapping $\mathbb{R}^n$ to $\mathscr{E}'(\mathbb{R}^n)$.

Conjecture 1. As for D_0, we give here a first order semantics which should extend to higher order following the method of 7.5.

We defined the lcs $!_D\mathbb{R}^n$ as the inverse image of $\mathscr{E}'(\mathbb{R}^n)$ under $D : \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$. The LPDOcc D applies to scalar functions of n real variables.. It can however be extended to functions defined on any euclidean space $\mathbb{R}^m$, following the basic idea that we should only consider the n first variables when applying D to a function $f \in \mathcal{D}(\mathbb{R}^m)$.

Definition 8.3.5. Consider $m \leqslant n$ and $f \in \mathscr{E}(\mathbb{R}^m)$, one defines an extension of f to $\mathbb{R}^n$ as $f : (x_1, ...x_n) \mapsto f(x_1, ..., x_m)$, and thus defines D on $\mathscr{E}(\mathbb{R}^m)$. It $m > n$ and $f \in \mathcal{D}(\mathbb{R}^m)$, one define $D(f)$ as D applied to the restriction of f to $\mathbb{R}^n$: $D(f) := D((x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n, 0, \dots, 0))$.

This definition is fundamental, at is accounts for the *deterministic* cut-elimination of the system $\mathrm{D}-\mathrm{DiLL}$, introduced in section 8.4.

Remark 8.3.6. We reused here a convention in mathematical physics (see for example the introduction of chapter 52 in [76]). One denotes by D_x the operator D applied to a function $f \in \mathcal{C}_c^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, to indicate that D applies only to the variable x of the function $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto f(x, y)$.

Definition 8.3.7. We define the functor: $!_D : \textsc{Eucl} \longrightarrow \textsc{TopVec}$.

$$!_D : \begin{cases} \textsc{Eucl} \longrightarrow \textsc{Ndf} \\ \mathbb{R}^n \mapsto !_D\mathbb{R}^n \\ \ell : \mathbb{R}^n \longrightarrow \mathbb{R}^m \mapsto (\phi \in !_D\mathbb{R}^n \mapsto \psi : (f \in ?_D\mathbb{R}^m) \mapsto \phi(f \circ \ell)) \end{cases} \tag{8.8}$$

This functor is well defined: consider $\ell : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $\phi \in !_D\mathbb{R}^n$, then for any $f \in ?_D\mathbb{R}^m : D((!_D\ell)(\phi))(f) = (!_D\ell)(\phi)(\check{D}(f)) = \phi(\check{D}(f \circ \ell) = \phi((\check{D}(f) \circ \ell)$. As $\check{D}$ has compact support and ℓ is linear continuous, so has $(\check{D}(f) \circ \ell$.

Definition 8.3.8. We define the following natural transformations $d_D : ! \longrightarrow !_D$ and $\bar{d}_D : !_D \mapsto !$.

$$d_{D,\mathbb{R}^m} : \begin{cases} \mathscr{E}'(\mathbb{R}^m) \longrightarrow !_D\mathbb{R}^m \\ \psi \mapsto E_D * \psi \end{cases}$$

$$\bar{d}_{D,\mathbb{R}^m} : \begin{cases} !_D\mathbb{R}^m \longrightarrow !\mathbb{R}^m \\ \psi \mapsto (f \in \mathscr{E}(\mathbb{R}^m) \mapsto D(\psi)(f)) \end{cases}$$

The codereliction is well defined thanks to Proposition **??**. The dereliction is well defined thanks to Corollary 8.1.24 and Example 8.1.21. Moreover, this denotational interpretation is unchanged under the cut-elimination procedure, as by Proposition 8.1.16:

$$\begin{aligned} \forall \phi \in !_D\mathbb{R}^n, d_{D,\mathbb{R}^n} \circ \bar{d}_{D,\mathbb{R}^n}(\phi) =& d_{D,\mathbb{R}^n}(D\phi) \in \mathscr{E}'(\mathbb{R}^n) \\ &= E_D * (D\phi) \\ &= \phi \in \mathcal{D}'(\mathbb{R}^n) \end{aligned}$$

Remark 8.3.9. Let us note that to extend this model to higher order, one cruelly misses a proof of the fact that $!_D\mathbb{R}^n$ is reflexive.

8.4 D-DiLL

We now depart from the intuitions conveyed by D_0 and DiLL, to study more specifically the case of Linear Partial Differential Operators with constant coefficients.

8.4.1 The sequent calculus $\mathrm{D}-\mathrm{DiLL}$

8.4.1.1 Grammar and rules

We introduce a generalisation of (higher-order, polarized, without promotion) DiLL, where the role of $A \simeq A^{\perp\perp}$ in the exponential rules $\bar{d}$ and d is played by a new formula $!_D A$. The other exponential rules are moreover replaced by equivalent rules involving $!_D$, in order to account for the specificities of the convolution and the fundamental equation. In particular, this modification implies that this sequent calculus does not involve sums of proofs (see Section 8.3). The formulas of $\mathrm{D}-\mathrm{DiLL}$ are constructed following the same grammar as those of $\mathrm{D}_0-\mathrm{DiLL}$:

Definition 8.4.1. The formulas of $\mathrm{D}-\mathrm{DiLL}$ are constructed with the following polarized syntax:

$$N, M := ?A \mid ?_D A \mid N \parr M \mid N \times M$$

$$P, Q := !A \mid !_D A \mid P \otimes Q \mid P \oplus Q$$

The Identity rule

$$\frac{}{\vdash A, A^{\perp}}\ (\text{axiom}) \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\text{cut})$$

The additive rules

$$\frac{}{\vdash 1}\ (1) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ (\perp) \qquad \frac{\vdash \Gamma, N, M}{\vdash \Gamma, N \parr M}\ (\parr) \qquad \frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\ (\otimes)$$

The multiplicative rules

$$\frac{}{\vdash \Gamma, \top}\ \top \qquad \frac{\vdash \Gamma, N \qquad \vdash \Gamma, M}{\vdash \Gamma, N \,\&\, M}\ \& \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q}\ \oplus_L \qquad \frac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q}\ \oplus_R$$

The exponential rules

$$\frac{\vdash \mathcal{N}}{\vdash \mathcal{N}, ?_D P}\ w_D \qquad \frac{\vdash \mathcal{N}, ?P, ?_D P}{\vdash \mathcal{N}, ?_D P}\ c \qquad \frac{\vdash \mathcal{N}, ?_D P}{\vdash \mathcal{N}, ?P}\ d$$

$$\frac{\vdash}{\vdash !_D N}\ \bar{w}_D \qquad \frac{\vdash \mathcal{N}, !N \qquad \vdash \mathcal{M}, !_D N}{\vdash \mathcal{M}, \mathcal{N}, !_D N}\ \bar{c}_D \qquad \frac{\vdash \mathcal{N}, !_D N}{\vdash \mathcal{N}, !N}\ \bar{d}_D$$

Figure 8.2: Rules for $\mathrm{D} - \mathrm{DiLL}$

Definition 8.4.2. The negation is the one of LL, with the additional definitions:

$$!_D A^{\perp} = ?_D A^{\perp} \qquad\qquad ?_D A^{\perp} = !_D A^{\perp}$$

Remark 8.4.3. Let us relate the specific form of the co-contraction with the "coabs" rule in the parcimonious types of Mazza [57]:

$$\frac{\Gamma; \Delta \vdash t : A \qquad \Gamma'; \Delta' \vdash u : !A}{\Gamma, \Gamma'; \Delta, \Delta' \vdash t :: u : !A}$$

which underlines that a modified version of the co-dereliction should type the constructions of lists. Now remember that semantically, $!_D A$ is interpreted as a space which contains $!A = \mathscr{E}'(A)$. Thus the D-co-contraction $\bar{c}_D$ could imply that one should see the type $!_D A$ as the type of lists of elements of type $!_A$. The operation extracting the first element of a list should then be typed by $\bar{d}_D$.

Remark 8.4.4. Thus the connectives $!_D$ and $?_D$ are only introduced via coweakening and weakening, which is quite weak. As for $\mathrm{D}_0 - \mathrm{DiLL}$, one should add to the present sequent calculus functorial promotions rules for $!_D$ (which are well interpreted in the semantics were $!_D : \textsc{Eucl} \longrightarrow \textsc{Nucl}$ is indeed a functor). This is not done here, as the meaning in terms of distributions of cut-elimination rules between a promotion for $!_D$ and structural rules is not clear.

8.4.1.2 Admissible rules

Before going to cut-elimination, let us show that the version of the structural rule c, w, $\bar{c}$, $\bar{w}$ involving only $!N$ and not $!_D N$ are admissible in $\mathrm{D} - \mathrm{DiLL}$.

Proposition 8.4.5. *The rules $\bar{w}$, w are admissible in* $\mathrm{D} - \mathrm{DiLL}$.

Proof. The following proof-trees can be understood through their interpretation in terms of distributions and partial differential operators. For example, $\bar{w}_D : \mathbb{R} \longrightarrow !_D E$ is the introduction of a fundamental solution $1 \mapsto E_D$. Thus the usual coweakening $\bar{w} : 1 \in \mathbb{R} \mapsto \delta_0 \in !E$ is exactly $\bar{w}(1) = \bar{d}_D \circ \bar{w}_D(1) = DE_D = \delta_0$.

$$\frac{}{\vdash !N}\ \bar{w} = \cfrac{\cfrac{\vdash}{\vdash !_D N}\ \bar{w}_D}{\vdash !N}\ \bar{d}_D \qquad\qquad \frac{\vdash \mathcal{N}}{\vdash \mathcal{N}, ?P}\ w = \cfrac{\cfrac{\vdash \mathcal{N}}{\vdash \mathcal{N}, ?_D P}\ w_D}{\vdash \mathcal{N}, ?P}\ d_D$$

□

Proposition 8.4.6. *The rules c and $\bar{c}$ are admissible in* $\mathrm{D-DiLL}$.

Proof. In terms of distributions and fundamental solutions, $\bar{c}_D$ and $\bar{c}$ are both interpreted as the convolution, but they are not defined on the same spaces. The usual cocontraction is the convolution between to distributions with compact support $\bar{c}(\phi, \psi) = \phi * \psi$ while $\bar{c}_D$ asks for one of them to be in $!_D E = D^{-1}(\mathscr{E}'(E)) \subset \mathcal{D}'(E)$. Thus $\bar{c}(\phi, \psi) = \phi * \psi = D(E_D * \phi) * \psi = \bar{d}_D \circ \bar{c}_D(d_D(\phi), \psi)))$.

$$\dfrac{\vdash \mathcal{N}, !N \qquad \vdash \mathcal{M}, !N}{\vdash \mathcal{N}, \mathcal{M}, !N}\,\bar{c} = \dfrac{\dfrac{\dfrac{\vdash \mathcal{N}, !N \qquad \dfrac{}{\vdash !_D N}\,\bar{w}_d}{\vdash \mathcal{N}, !_D N}\,\bar{c}_D \qquad \vdash \mathcal{M}, !N}{\vdash \mathcal{N}, \mathcal{M}, !_D N}\,\bar{c}_D}{\vdash \mathcal{N}, \mathcal{M}, !N}\,\bar{d}_D$$

$$\dfrac{\vdash \mathcal{N}, ?P, ?P}{\vdash \mathcal{N}, ?P}\,c = \dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathcal{N}, ?P, ?P}{\vdash \mathcal{N}, ?P, ?P, ?_D A}\,w_D}{\vdash \mathcal{N}, ?P, ?_D A}\,c_D}{\vdash, \mathcal{N}, ?_D P}\,c_D}{\vdash \mathcal{N}, ?P}\,d_D$$

□

Proposition 8.4.7. *The rules $\bar{w}_D$, $\bar{c}_D$, w_D and c_D are admissible in* DiLL, *when $!_D A$ is equivalent to A.*

8.4.2 The cut-elimination procedure in $\mathrm{D-DiLL}$

This cut-elimination is inspired by the one of DiLL and by the calculus of distributions, see section 8.1. We describe the cut-elimination rules and commutative rules in figure 8.3. They are best understood from the semantical point of view developed in section 8.4.3.

Notice that the differences between DiLL and $\mathrm{D-DiLL}$ makes the cut-elimination procedure much simpler: cuts between d_D and $\bar{w}_d$ or $\bar{d}_D$ and w_D are not possible. Moreover, the fact that the contraction and co-contraction rules are no longer symmetrical suppress the need for sums of proofs. Thus *proofs of* $\mathrm{D-DiLL}$ *are simple proofs.*

The commutative rules for the exponential rules of $\mathrm{D-DiLL}$ are the same as the ones of DiLL (see [81]), and are immediate as the exponential rules are multiplicative and do not interfere with the context.

8.4.3 A concrete semantics without higher order

Let us define formally the semantics of $\mathrm{D-DiLL}$ which was used informally throughout the last sections. This section is peculiar, as it uses the notations used in mathematical physics to interpret the rules of $\mathrm{D-DiLL}$, see theorem 8.4.11.

In this section we show that the categories EUCL, NDF and NF, together with distributions of compact support and a LPDO D with contant coefficients, form a first-order model of $\mathrm{D-DiLL}$. As for Smooth DiLL, we define $!_D$ as a strong monoidal functor between EUCL and NDF. Extending this to a higher-order model could be possible using the same techniques as in section 7.5.
Consider $D : \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$ a LPDOcc:

$$D(f, x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial^\alpha f(x).$$

In section 8.3.2.2 we saw that we could make D act on any euclidean space:

Definition 8.4.8. If $m < n$ and $f \in \mathcal{D}(\mathbb{R}^m)$, one defines an extension of f to $\mathbb{R}^m$ as $f : (x_1, ...x_n) \mapsto f(x_1, ..., x_m)$, and thus define D on $\mathcal{D}(\mathbb{R}^m)$. It $m > n$ and $f \in \mathcal{D}(\mathbb{R}^m)$, one define $D(f)$ as D applied to the restriction of f to $\mathbb{R}^n$: $D(f) := D((x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n, 0, \dots, 0))$.

Remark 8.4.9. In particular, the fundamental solution $E_D \in !_D\mathbb{R}^n$ can be restricted or extended to $E_D \in \mathbb{R}^m$, for any m.

$$\dfrac{\dfrac{\vdash \mathcal{N}}{\vdash \mathcal{N}, ?_D N^\perp}\,w_D \qquad \dfrac{\vdash}{\vdash\, !_D N}\,\bar{w}_D}{\vdash \mathcal{N}}\,cut \rightsquigarrow \vdash \mathcal{N}$$

$$\dfrac{\dfrac{\vdash \mathcal{N}, ?_D N^\perp, ?N^\perp}{\vdash \mathcal{N}, ?_D N^\perp}\,c_D \qquad \dfrac{\vdash}{\vdash\, !_D N}\,\bar{w}_D}{\vdash \mathcal{N}}\,cut \rightsquigarrow \dfrac{\dfrac{\vdash \mathcal{N}, ?_D N^\perp, ?N^\perp \qquad \dfrac{\vdash}{\vdash\, !N}\,\bar{w}}{\vdash \mathcal{N}, ?N_D^\perp}\,cut \qquad \dfrac{\vdash}{\vdash\, !_D N}\,\bar{w}_D}{\vdash \mathcal{N}}\,cut$$

$$\dfrac{\dfrac{\vdash \mathcal{N}, !_D N \qquad \vdash \mathcal{N}', !N}{\vdash \mathcal{N}, \mathcal{N}', !_D N}\,\bar{c}_D \qquad \dfrac{\vdash \mathcal{M}}{\vdash \mathcal{M}, ?_D N^\perp}\,w_D}{\vdash \mathcal{N}, \mathcal{N}', \mathcal{M}}\,cut$$

$$\rightsquigarrow \dfrac{\dfrac{\dfrac{\vdash \mathcal{N}, !_D N \qquad \dfrac{\vdash \mathcal{M}}{\vdash \mathcal{M}, ?_D N^\perp}\,w_D}{\vdash \mathcal{N}, \mathcal{M}}\,cut}{\vdash \mathcal{N}, \mathcal{M}, ?N^\perp}\,w \qquad \vdash \mathcal{N}', !N}{\vdash \mathcal{N}, \mathcal{N}', \mathcal{M}}\,cut$$

$$\dfrac{\dfrac{\vdash \mathcal{N}, ?_D N^\perp}{\vdash \mathcal{N}, ?N^\perp}\,d_D \qquad \dfrac{\vdash \mathcal{M}, !_D N}{\vdash \mathcal{M}, !N}\,\bar{d}_D}{\vdash \mathcal{N}, \mathcal{M}}\,cut \rightsquigarrow \dfrac{\vdash \mathcal{N}, ?_D N^\perp \qquad \vdash \mathcal{M}, !_D N}{\vdash \mathcal{N}, \mathcal{M}}\,cut$$

$$\dfrac{\dfrac{\vdash \mathcal{N}, ?_D N^\perp, ?N^\perp}{\vdash \mathcal{N}, ?_D N^\perp}\,c_D \qquad \dfrac{\vdash \mathcal{M}, !_D N \qquad \vdash \mathcal{M}', !N}{\vdash \mathcal{M}, \mathcal{M}', !_D N}\,\bar{c}_D}{\vdash \mathcal{N}, \mathcal{M}', \mathcal{M}}\,cut$$

$$\rightsquigarrow \dfrac{\dfrac{\vdash \mathcal{N}, ?_D N^\perp, ?N^\perp \qquad \vdash \mathcal{M}, !_D N}{\vdash \mathcal{N}, \mathcal{M}, ?N^\perp}\,cut \qquad \vdash \mathcal{M}', !N}{\vdash \mathcal{N}, \mathcal{M}, \mathcal{M}'}\,cut$$

Figure 8.3: Cut-elimination for the exponential rules of $\mathrm{D}-\mathrm{DiLL}$

We interpret finitary formulas A, B as euclidean spaces. One has indeed $1 \simeq \bot = \mathbb{R}$ and $\top \simeq 0 = \{0\}$. The connectives of LL are interpreted in EUCL, NF and NDF as in section 7.4.

Definition 8.4.10. We recall the definition of the functor $!_D$ which was made explicit in section 8.3.2.2:

$$!_D : \begin{cases} \text{EUCL} \longrightarrow \text{TOPVEC} \\ \mathbb{R}^m \mapsto !_D\mathbb{R}^m \\ \ell : \mathbb{R}^m \longrightarrow \mathbb{R}^p \phi \in !_D\mathbb{R}^m \mapsto (f \in D(\mathcal{D}(\mathbb{R}^p) \mapsto \phi(f \circ \ell)) \end{cases} \tag{8.9}$$

The introduction of $!_D$ breaks the symmetry of the Kernel theorem:

Theorem 8.4.11. *Consider $D : \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$ a LPDOcc. Then for any $m \geqslant 0$, we have natural isomorphisms*

$$m_{D,n} : !_D(\mathbb{R}^{n+n}) \simeq !_D\mathbb{R}^n \hat{\otimes}_\pi !\mathbb{R}^n.$$

Proof. This theorem encodes the definition 8.3.5, which allows to extends D defined on $\mathcal{E}(\mathbb{R}^n)$ to $\mathcal{E}(\mathbb{R}^{n+m})$. This theorem is then directly deduced from the Kernel theorem 7.3.5: as $\mathcal{E}(\mathbb{R}^{n+m}) \simeq \mathcal{E}(\mathbb{R}^n)\hat{\otimes}\mathcal{E}(\mathbb{R}^m)$ we have $D(\mathcal{E}(\mathbb{R}^{n+m})) \simeq D\mathcal{E}(\mathbb{R}^n)\hat{\otimes}\mathcal{E}(\mathbb{R}^m)$. Taking the dual gives us the desired result. $\square$

From this strong monoidal isomorphism and the biproduct structure on TOPVEC, one deduces the interpretation of w_D, $\bar{w}_D$, c_D and $\bar{c}_D$, following the categorical procedure introduced by Fiore (see section 2.4.2.1). These interpretations are to be compared to the ones for the model of DiLL with distributions of Chapter 7, which are detailed in Section 7.4.3.

Definition 8.4.12. We give the following interpretation for w_D, $\bar{w}_D$, c_D and $\bar{c}_D$:

- In Section 2.4.2), we interpreted the coweakening as the map $\bar{w}_A :! \longrightarrow !_A$ resulting of the application of the functor ! on the initial map $u_A : I \longrightarrow A$. Concretely, we have $I = \{0\}$ $u_{\mathbb{R}^m}(0 \in \{0\}) \mapsto 0 \in \mathbb{R}^m$, and $!_D(u_A)(1) = \delta_0$. Applying $!_D$ to u_A gives exaclty the same result : $!_D(u_A)(1) = \delta_0$, but *this is not what we want here*. Thus one defines:

$$\bar{w}_D : \begin{cases} \mathbb{R} \longrightarrow !_D\mathbb{R}^m \\ 1 \mapsto (E_D * !_D(u_A)(1)) = E_D \end{cases} \tag{8.10}$$

  Notice that E_D was given in $!_D\mathbb{R}^n$, but it is extended to $!_D\mathbb{R}^m$ following Definition 8.4.8.
- The cocontraction $\bar{c} : !_D \otimes ! \longrightarrow !_D$ is interpreted by the convolution product (see prop. 7.4.12)

$$\bar{c}_D : \begin{cases} !_D\mathbb{R}^m \hat{\otimes} !\mathbb{R}^m \longrightarrow !_D\mathbb{R}^m \\ \phi \otimes \psi \mapsto \phi * \psi \end{cases} \tag{8.11}$$

  and is well defined thanks to corollary 8.1.24.
- The interpretation of weakening $w_D : !_D \longrightarrow 1$ is defined as:

$$w_D : \phi \in !_D\mathbb{R}^m \mapsto \int D\phi.$$

  It is well defined as by definition $D\phi$ has compact support. As explained in Section 7.4.3, integration on distributions with compact support $\psi \in \mathscr{E}'(E)$ is defined as ψ applied to a function constant at 1:

$$\int \psi = \psi(const_1).$$

  When ψ is the generalization of a function f with compact support, then we have indeed

$$\psi(const_1) = \int 1 \cdot f = \int f.$$

- The contraction $c_D : !_D \longrightarrow !_D \otimes !$ is interpreted as in proposition 7.4.9 by:

$$c_D : \begin{cases} ?_D\mathbb{R}^m \hat{\otimes} ?\mathbb{R}^m \longrightarrow ?_D(\mathbb{R}^m) \\ f \otimes g \mapsto (x \mapsto f(x)g(x)) \end{cases} \tag{8.12}$$

Remark 8.4.13. Notice that the interpretation of c_D follows the intuitions of theorem 8.4.11:

$$f \in \check{D}(\mathscr{E}(\mathbb{R}^m \times \mathbb{R}^m))$$

is in fact in $\check{D}(\mathscr{E}(\mathbb{R}^m))\hat{\otimes}\mathscr{E}(\mathbb{R}^m)$, as differentiation occurs only on the first m variables. In particular, if $m > 0$, then the differentiation on $\mathbb{R}^m \times \mathbb{R}^n$ is defined as:

$$\check{D} \otimes Id_{\mathbb{R}^m} : \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m)$$

Thus the fundamental solution $E_D \in !_D(\mathbb{R}^{m+n})$ corresponds to $E_D \otimes \delta_0 \in !_D\mathbb{R}^n \otimes !\mathbb{R}^m$.

Dereliction and the codereliction are defined as previously for $\mathrm{D}_0 - \mathrm{DiLL}$ (see Section 8.3.2.1).

Definition 8.4.14. We interpret the dereliction $d_D : ! \longrightarrow !_D$ as

$$d_{D,E}(\phi \in \mathcal{E}'(\mathbb{R}^n)) \mapsto (E_D * \phi)$$

and codereliction $\bar{d}_D : ! \longrightarrow !_D$ as

$$\bar{d}_{D,E} : (\phi \in !_D\mathbb{R}^n) \mapsto (D\phi) \in \mathcal{E}'(\mathbb{R}^n).$$

Proposition 8.4.15. *This denotational semantics of the rules of* $\mathrm{D} - \mathrm{DiLL}$ *is preserved by cut-elimination (see figure* 8.3*).*

In figure 8.4.3, we provide the cut-elimination rules annotated with the interpretation of the proofs, to support the proof of proposition 8.4.15.

Proof.
- Cut elimination between d_D and $\bar{d}_D$. By equation 8.3 one has for every $\phi \in \mathcal{E}'(\mathbb{R}^n)$ and $f \in \mathcal{E}(\mathbb{R}^n)$:

$$\begin{aligned} d_{D,E} \circ \bar{d}_{D,E}(\phi)(f) &= E_D * (\phi(D(f)) \\ &= \phi(E_D * D(f)) \\ &= \phi(f) \end{aligned}$$

- Cut elimination between w_D and $\bar{w}_D$:

$$w_D \circ \bar{w}_D(1 \in \mathbb{R}) = \int DE_D = \int \delta_0 = 1$$

- Cut elimination between c_D and $\bar{w}_D$: by definition of the interpretation of c_D, one has:

$$c_D \circ \bar{w}_D(1) = (f_D, g) \in \check{D}(\mathscr{E}(\mathbb{R}^n)) \otimes \mathscr{E}(\mathbb{R}^n) \mapsto \bar{w}_D(x \mapsto f(x)g(x)).$$

Thus as $\bar{w}_D(1) = E_D$, and by making use of remark 8.4.13, we have:

$$c_D \circ \bar{w}_D(1) = E_D \hat{\otimes} \delta_0((x \mapsto f(x)g(x)))$$

which corresponds to the interpretation of the reduced cut-rule.

- Cut elimination between $\bar{c}_D$ and w_D: consider $\phi \in !_D\mathbb{R}^m$ and $\psi \in !\mathbb{R}^m$. We are used to see the cocontraction as the convolution between distribution, but remember that we can also understand it as the resulting distribution which sums in the codomain of its function (see Proposition 7.4.12). Then:

$$\begin{aligned} w_D \circ \bar{c}_D(\phi, \psi) &= \int D[f \in \mathscr{E}_D(\mathbb{R}^m) \mapsto \phi(x \mapsto \psi(x' \mapsto f(x + x')))] \\ &= \int [f \mapsto \phi(x \mapsto D\psi(x' \mapsto f(x + x')))] \\ &= \phi(x \mapsto D\psi(x' \mapsto const_1(x + x'))) \\ &= \phi(const_{D\psi(const_1)}) \end{aligned}$$

Again by definition 8.3.5, D applies only to ϕ. Thus the interpretation of the cut-rule and its reduced form corresponds.

- Cut elimination between $\bar{c}_D$ and c_D: consider $\phi \in !_D\mathbb{R}^m$ and $\psi \in !\mathbb{R}^m$. Then up to the kernel isomorphism we have:
$$c_D \circ \bar{c}_D = !_D(A \times A) \xrightarrow{!_D \triangle_A} !_D A \xrightarrow{!_D \nabla_A} !_D(A \times A)$$

  Thus

$$\begin{aligned} c_D \circ \bar{c}_D(\phi \in !_D(A \times A)) &= f \in \check{D}\mathscr{E}(A \times A) \mapsto (\bar{c}_D \phi)(x \mapsto f(x,x)) \\ &= f \mapsto \phi((x,y) \mapsto f(x,x)f(y,y)). \end{aligned}$$

  Through the Kernel isomorphism we can suppose by density that $\phi = \phi' \otimes \psi$, with $\phi' \in !_D A$ and $\psi \in !A$, and $f = f' \otimes g$ with $f' \in ?_D A^\perp$ and $g \in ?A^\perp$. Thus

$$\begin{aligned} (c_D \circ \bar{c}_D)(\phi \in !_D(A \times A)) &= f' \hat{\otimes} g \mapsto \phi' \otimes \psi((x,y) \mapsto f'(x)g(y)) \\ &= f' \hat{\otimes} g \mapsto \phi'(f')\phi(g), \end{aligned}$$

  and the last proposition corresponds to the interpretation of the reduced cut-rule between c_D and $\bar{c}_D$.

$\square$

$$\dfrac{\dfrac{\vdash \mathcal{N}}{\vdash \mathcal{N}, \phi \mapsto (D\phi)(const_1) : ?_D N^\perp}\,w_D \qquad \dfrac{\vdash}{\vdash E_D : !_D N}\,\bar{w}_D}{\vdash \mathcal{N}, \delta_0(const_1) = 1 : \mathbb{K}}\,cut \rightsquigarrow \vdash \mathcal{N}, 1 : \mathbb{K}$$

$$\dfrac{\dfrac{\vdash \mathcal{N}, \check{D}f : ?_D N^\perp, g :?N^\perp}{\vdash \mathcal{N}, \check{D}f\cdot g : ?_D N^\perp}\,c_D \qquad \dfrac{\vdash}{\vdash E_D : !_D N}\,\bar{w}_D}{\vdash \mathcal{N}, E_D(\check{D}f\cdot g) = \delta_0(f\cdot g) = f(0)g(0) : \mathbb{R}}\,cut$$

$$\rightsquigarrow \dfrac{\dfrac{\vdash \mathcal{N}, \check{D}f : ?_D N^\perp, g :?N^\perp \qquad \dfrac{\vdash}{\vdash \delta_0 : !N}\,\bar{w}}{\vdash \mathcal{N}, \check{D}f :?N_D^\perp, g(0) : \mathbb{R}}\,cut \qquad \dfrac{\vdash}{\vdash E_D !_D N}\,\bar{w}_D}{\vdash \mathcal{N}, E_D(\check{D}f) = f(0) : \mathbb{R}, g(0) : \mathbb{R}}\,cut$$

$$\dfrac{\dfrac{\vdash \mathcal{N}, \psi : !_D N \qquad \vdash \mathcal{N}', \phi : !N}{\vdash \mathcal{N}, \mathcal{N}', \psi * \phi : !_D N}\,\bar{c}_D \qquad \dfrac{\vdash \mathcal{M}, 1 : \mathbb{R}}{\vdash \mathcal{M}, \check{D}(const_1) :?N^\perp}\,w_D}{\vdash \mathcal{N}, \mathcal{N}', \mathcal{M}, (\psi * \phi)(\check{D}(const_1)) = (D\psi * \phi)(const_1) : \mathbb{R}}\,cut$$

$$\rightsquigarrow \dfrac{\dfrac{\dfrac{\vdash \mathcal{N}, \psi : !_D N \qquad \dfrac{\vdash \mathcal{M}}{\vdash \mathcal{M}, \check{D}(const_1) : ?_D N^\perp}\,w_D}{\vdash \mathcal{N}, \mathcal{M}, D(\psi)(const_1) : \mathbb{R}}\,cut}{\vdash \mathcal{N}, \mathcal{M}, const_{D(\psi)(const_1)} :?N^\perp}\,w \qquad \vdash \mathcal{N}', \phi : !N}{\vdash \mathcal{N}, \mathcal{N}', \mathcal{M}, \psi(const_{D(\psi)(const_1)}) : \mathbb{R}}\,cut$$

$$\dfrac{\dfrac{\vdash \mathcal{N}, \check{D}f : ?_D N^\perp}{\vdash \mathcal{N}, f : ?N^\perp}\,d_D \qquad \dfrac{\vdash \mathcal{M}, \psi : !_D N}{\vdash \mathcal{M}, D\psi : !N}\,\bar{d}_D}{\vdash \mathcal{N}, \mathcal{M}, D\psi(f) = \psi(\check{D}f) : \mathbb{R}}\,cut \rightsquigarrow \dfrac{\vdash \mathcal{N}, \check{D}f : ?_D N^\perp \qquad \vdash \mathcal{M}, \psi : !_D N}{\vdash \mathcal{N}, \mathcal{M}, \psi(\check{D}f) : \mathbb{R}}\,cut$$

$$\dfrac{\dfrac{\vdash \mathcal{N}, \check{D}f : ?_D N^\perp, g :?N^\perp}{\vdash \mathcal{N}, \check{D}f\cdot g : ?_D N^\perp}\,c_D \qquad \dfrac{\vdash \mathcal{N}, \psi : !_D N \qquad \vdash \mathcal{N}', \phi : !N}{\vdash \mathcal{N}, \mathcal{N}', \psi * \phi : !_D N}\,\bar{c}_D}{\vdash \mathcal{N}, \mathcal{M}', \mathcal{M}, (\psi * \phi)(\check{D}f\cdot g) = (\psi*\phi)\check{D}(f\cdot g) : \mathbb{R}}\,cut$$

$$\rightsquigarrow \dfrac{\dfrac{\vdash \mathcal{N}, \psi : ?_D N^\perp, \phi : ?N^\perp \qquad \vdash \mathcal{M}, !_D N}{\vdash \mathcal{N}, \mathcal{M}, ?N^\perp}\,cut \qquad \vdash \mathcal{M}', !N}{\vdash \mathcal{N}, \mathcal{M}, \mathcal{M}'}\,cut$$

Figure 8.4: Cut-elimination for the exponential rules of $\mathrm{D}-\mathrm{DiLL}$, annotated with the semantics

Chapter 9

Conclusion

In this thesis, we conducted a study of the semantics of Differential Linear Logic: our goal was to find a model of DiLL in which functions were smooth (making coincide the intuitions of analysis concerning differentiation and the requirements of logic), and spaces continuous and reflexive (as DiLL features an involutive linear negation).

We met this goal by constructing several model for it. During this study, it appeared that a *polarized* was better fit to interpret DiLL, as we may want to interpret differently the linear negation on the positives or and the negative (Chapter 6), or as a space and its dual may not belong to a same monoidal closed category (Chapter 7). Throughout this study, we highlighted the fact spaces distributions with compact support are the canonical interpretation for the exponential. In this spaces of distributions, one can resolve Linear Partial Differential Equations, and we provide a sequent calculus supporting the idea that the exponential is the space of solutions for a Linear Partial Differential Equation.

Directly following this thesis: Among other points, let us mention what could appear in this thesis and is not:

1. Categorical models of $\mathrm{DiLL}_{0,pol}$ presented in section 2.5.2.3 are not optimal. Indeed, they always provide for an interpretation of the promotion rule, even when this one does not figure in DiLL_0. Moreover, the symmetry between the dereliction and codereliction rules is broken in the categorical axiomatization: dereliction comes with the strong monoidal adjunction between ! and U, while codereliction is ad-hoc. In view of results in Chapter 8, we should have a symmetric axiomatization.
2. In Chapter 6, one would want an interpretation for the exponential which is intrinsically bornological, thus emphasizing on the fact that the adjunction between convenient spaces and complete Mackey spaces is the good linear classical refinement of convenient spaces.
3. In Chapter 7, we detailed a model of DiLL using the theory of Nuclear Spaces. One should explore the link between this model and the model of Köthe spaces, using in particular Fourier transformation. In particular, as any nuclear fréchet space is a subset of a denumerable product of the Kothe spaces of rapidly decreasing sequences [44, 21.7.1], introducing subtyping may lead to a complete semantics.
4. In Chapter 8, the proof that $!_D E$ is reflexive is cruelly missing. We also miss a good notion of categorical model for $\mathrm{D} - \mathrm{DiLL}$.
5. Chapter 8 gives a logical account for the notion of fundamental solution for Linear Partial Differential Operator. Let us note that the Cauchy Problem for Linear Partial Differential Equation behaves well with respect to solutions defined on cones, and that a link with the recent models of probabilistic programming by Ehrhard, Pagani and Tasson may exists [21].
6. This thesis emphasized the importance of reflexivity when constructing models of Differential Linear Logic. One should now investigate the differential λ-calculus in terms of linear continuations, and linear exceptions, in the spirit of [16] and [35].

What's more important: In Chapter 8, we build a deterministic sequent calculus $\mathrm{D} - \mathrm{DiLL}$ with a concrete first-order denotational model in which applying the dereliction rule corresponds to solving a linear partial differential equation, with the basic intuition that several exponential exists, and each one is associated with a Linear Differential Operator

1. The priority is to work towards completing the Curry-Howard-Lambek correspondence for Linear Partial Differential Equations. This means finding the good categorical axiomatization for models of $\mathrm{D} - \mathrm{DiLL}$, and most importantly refining the differential λ-calculus [22] into a deterministic calculus for Linear Partial Differential Equations. This should be done by understanding computationally the proof of the Existence of E_D.

2. In Chapter 7, we gave a sequent calculus D-DiLL, which corresponds to a *fixed* Linear Partial Differential Operator D. We would like to generalize this procedure, by constructed a graduated sequent calculus à la BLL [34], which would describe any Linear Partial Differential operator with constant coefficient.

3. Of course, the global objective is to work towards a computational understanding of non-linear Partial Differential Equations.

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Appendices

Appendix A

Index of symbols

We indicate symbols, and then give the name of the mathematical object they denote, and the page or section in which they are introduced.

Categories

- TopVec: the category of Hausdorff and locally convex topological vector spaces and linear continuous bounded maps between them.
- BornVec: the category of vector spaces endowed with a vector bornology, with linear bounded maps between them.

The following categories are full and faithful subcategories of TopVec.

- Kothe, model of DiLL : the category of kothe spaces, page 22.
- Mco, model of intuitionist DiLL: the category of Mackey-Complete spaces, page 45.
- Chu, model of MALL : the category of Chu spaces, Section 4.
- Weak, model of DiLL: the category oflcs endowed with their Weak topology, Chapter 5.
- bTopVec: the category of bornological lcs, page 107.
- Mackey : the category of lcs endowed with their Mackey topology, Section 4.
- Conv, model of iDiLL [6] and interpretation for the positives formulas of a model in Section 6.4 : the category of bornological and Mackey-Complete spaces.
- Compl : the category of Complete lcs, Section 3.1.5.
- MackeyCompl, interpretation for the negative formulas a model of DiLL Section **??** the category of lcs which are Mackey and Complete, see Section 6.4.
- Nucl: the category of Nuclear lcs, Section 7.2.2.
- Nf, Ndf interpreting negatives (resp. positives) in Chapter 7 : the category of Nuclear Fréchet (resp. Nuclear DF) lcs, Section 7.1.1.

Chapter 2

- $L(E,F)$, $\mathbf{Lin}(E,F)$, $\mathcal{L}(E,F)$: the vector space of all (resp. of all bounded, resp of all continuous) linear functions betwenn the lcs E and F.
- w, c, d : the weakening, contraction and dereliction rules of LL, page 17.
- (T,d,μ) : a co-monad with co-unit d and co-multiplication μ., page 20.

- $\mathcal{L}_!$ Co-Kleisli category of the co-monad !, page 20.
- $\uparrow, \downarrow$: the shifts of LL_{pol}, page 24.
- $\mathscr{N}$, $\mathscr{P}$: categories involved in a model of LL_{pol}, page 28.
- $\bar{w}$, $\bar{c}$, $\bar{d}$: the co-weakening, co-contraction, co-dereliction rules of DiLL, page 35.
- $\diamond$, ∇, u, $\triangle$, n : a biproduct structure on a category , page 40.

Chapter 3

- E_B : when B is a absolutely convex and weakly-closed subset of E, it is the vector space generated by B, normed by the distance to B, page 57
- $\tilde{E}$ the completion of a lcs E, page 59.
- $\varinjlim$, $\varprojlim$: the lcs injective (resp. projective) limit of lcs, page 58.
- $\mathcal{C}_c^\infty(\mathbb{R}^n) = \mathcal{D}(\mathbb{R}^n)$: the vector space of all smooth functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *with compact support,* endowed with the topology of uniform convergence of compact subsets of every derivatives of finite order, page 60
- $\mathcal{C}^\infty(\mathbb{R}^n) = \mathscr{E}(\mathbb{R}^n)$: the vector space of *all* smooth functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, endowed with the topology of uniform convergence of compact subsets of every derivatives of finite order, page 60.
- E_w, E'_{w*} : the weak and weak* topologies on E and E' respectively, page 63.
- $E_{\mu*}$, E'_μ : the Mackey* and Mackey topologies on E and E' respectively, page 68.
- E'_β : the strong dual, page 68
- $\otimes_\pi$, $\hat{\otimes}_\pi$: the projective tensor product, the completed projective tensor product, page 74.
- $\otimes_\varepsilon$, $\hat{\otimes}_\varepsilon$: the injective tensor product, the completed injective tensor product, page 74.

Chapter 5

- $\otimes_i$: the inductive tensor product, page 86.
- $\mathcal{H}^n(E, F)$: The space of n-linear symmetric separately continuous functions from E to F, page 89.
- ! : the exponential for which arrows in the co-kleisli category are formal power series.
- $!_1$: the exponential for which arrows in the co-kleisli category are formal power series without constant coefficients, whose composition correspond to the Faa di Bruno Formula.cde"

Chapter 6

- $\bar{(\_)}^{born}$: the bornologification of a lcs , page 108.
- $\hat{(\_)}^M$: the Mackey-completion of a lcs, page 110.
- $\bar{\_}^{conv}$: The bornologification of the Mackey-completion, making a space *convenient*, page 111.
- E_μ : The Mackey topology on the lcs E, considered as the dual (equivalently pre-dual) of E'_μ, page 69.
- ε: the ε product, page 78.

Chapter 7

- $\mathcal{D}'(\mathbb{R}^n) = \mathcal{C}_c^\infty(\mathbb{R}^n)'_\beta$, the space of distributions,, page 132.
- $\mathscr{E}'(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)'_\beta$, the space of distributions *with compact support*, page 132.

- $f, f', g, g', h, h'...$: smooth functions, sometimes with compact support. If it is not explicitly mentioned, f' will never designate the derivative of f, as f is not in general a function defined on $\mathbb{R}$.
- $\phi, \phi', \psi, \psi' \ldots$: Distributions, sometimes with compact support. Beware that Hormander [41, 42] uses reverse notations for functions and distributions.

Chapter 8

- D_0 : the operator mapping a smooth function to its differential at 0 (which is linear continuous).
- D : a Linear Partial Differential Operator with constant coefficients.

$$D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \frac{\partial^\alpha f}{\partial x^\alpha}$$

$$\check{D} := \sum_{\alpha \in \mathbb{N}^n} (-1)^\alpha a_\alpha \frac{\partial^\alpha f}{\partial x^\alpha}$$

- $D_0 - \mathrm{DiLL}$: a non-deterministic sequent calculus for which both D and D_0 results in a model.
- $D - \mathrm{DiLL}$: a deterministic sequent calculus for which D results in a model.

Appendix B

Models of Linear Logic based on the Schwartz ε-product.

This chapter consist in a submitted paper in collaboration with Y. Dabrowksi. In contrary to chapters 6, 7, we develop here an unpolarized, and therefore more difficult, approach to smooth and reflexive model of DiLL. From the interpretation of Linear Logic multiplicative disjunction as the ε-product defined by Laurent Schwartz, we construct several models of Differential Linear Logic based on usual mathematical notions of smooth maps. We isolate a completeness condition, called **k**-quasi-completeness, and an associated notion stable by duality called **k**-reflexivity, allowing for a $*$-autonomous category of **k**-reflexive spaces in which the dual of the tensor product is the reflexive version of the ε product. We adapt Meise's definition of Smooth maps into a first model of Differential Linear Logic, made of **k**-reflexive spaces. We also build two new models of Linear Logic with conveniently smooth maps, on categories made respectively of Mackey-complete Schwartz spaces and Mackey-complete Nuclear Spaces (with extra reflexivity conditions). Varying slightly the notion of smoothness, one also recovers models of DiLL on the same $*$-autonomous categories. Throughout the article, we work within the setting of Dialogue categories where the tensor product is exactly the ε-product (without reflexivization).

B.0.0.1 Introduction

Smooth models of classical Linear Logic. Since the discovery of linear logic by Girard [29], thirty years ago, many attempts have been made to obtain denotational models of linear logic in the context of some classes of vector spaces with linear proofs interpreted as linear maps [5, 6, 18, 19, 33]. Models of linear logic are often inspired by coherent spaces, or by the relational model of linear logic. Coherent Banach spaces [31], coherent probabilistic or coherent quantum spaces [33] are Girard's attempts to extend the first model, as finiteness spaces [19] or Köthe spaces [18] were designed by Ehrhard as a vectorial version of the relational model. Following the construction of Differential linear logic [24], one would want moreover to find natural models of it where non-linear proofs are interpreted by some classes of smooth maps. This requires the use of more general objects of functional analysis which were not directly constructed from coherent spaces. We see this as a strong point, as it paves the way towards new computational interpretations of functional analytic constructions, and a denotational interpretation of continuous or infinite data objects.

A consequent categorical analysis of the theory of differentiation was tackled by Blute, Cockett and Seely [8, 9]. They gave several structure in which a differentiation operator is well-behaved. Their definition then restricts to models of Intuitionistic Differential Linear Logic. Our paper takes another point of view as we look for models of classical DiLL, in which spaces equal some double dual. We want to emphasize on the classical computational nature of Differential Linear Logic.

Three difficulties appear in this semantical study of linear logic. The equivalence between a formula and its double negation in linear logic asks for the considered vector spaces to be isomorphic to their double duals. This is constraining in infinite dimension. This infinite dimensionality is strongly needed to interpret exponential connectives. Then one needs to find a good category with smooth functions as morphisms, which should give a Cartesian closed category. This is not at all trivial, and was solved by using a quantative setting, i.e. power series as the interpretation for non-linear proofs, in most of the previous works [18, 19, 31, 33]. Finally, imposing a reflexivity condition to respect the first requirement usually implies issues of stability by natural tensor products of this condition, needed to model multiplicative connectives. This corresponds to the hard task of finding $*$-autonomous categories [Ba79]. As pointed out in [20], the only model of differential Linear logic using smooth maps [6] misses annoyingly the $*$-autonomous property for classical linear logic.

Our paper solves all these issues simultaneously and produces several denotational models of classical linear logic with some classes of smooth maps as morphism in the Kleisli category of the monad. We will show that the constraint of finding a $*$-autonomous category in a compatible way with a Cartesian closed category of smooth maps is even relevant to find better mathematical notions of smooth maps in locally convex spaces. Let us explain this mathematical motivation first.

A framework for differential calculus. It seems that, historically, the development of differential calculus beyond normed spaces suffered from the lack of interplay between analytic considerations and categorical, synthetic or logic ones. Partially as a consequence, analysts often forgot looking for good stability properties by duality and focused on one side of the topological or bornological viewpoint.

Take one of the analytic summary of the early theory in the form of Keller's book [Kel]. It already gives a unified and simplified approach based on continuity conditions of derivatives in various senses. But it is well-known that in order to look for good categorical properties such as Cartesian closedness, the category of continuous maps is not a good starting point, the category of maps continuous on compact sets would be better. This appears strongly in all the developments made to recover continuity of evaluation on the topological product (instead of considering the product of a Cartesian closed category), which is unavoidable for full continuity of composition of derivatives in the chain rule. This leads to considering convergence notions beyond topological spaces on spaces of linear maps, but then, no abstract duality theory of those vector convergence spaces or abstract tensor product theory is developed. Either one remains with spaces of smooth maps that have tricky composition (of module type) between different notions of smoothness or composition within the classes involving convergence vector spaces whose general theory remained underdeveloped with respect to locally convex spaces. At the end, everything goes well only on restricted classes of spaces that lack almost any categorical stability properties, and nobody understands half of the notions introduced. The situation became slightly better when [58] considered k-space conditions and obtained what analysts call kernel representation theorems (Seely isomorphisms for linear logicians), but still the class of spaces considered and the k-space conditions on products limited having a good categorical framework for the hugest classes of spaces: the only classes stable by products were Fréchet spaces and (DFM)-spaces, which are by their very nature not stable by duality.

The general lesson here is that, if one wants to stay within better studied and commonly used locally convex spaces, one should better not stick to functions continuous on products, and the corresponding projective topological tensor product, but always take tensor products that come from a $*$-autonomous category, since one also needs duality, or at least a closed category, to control the spaces of linear maps in which the derivatives take values. $*$-autonomous categories are the better behaved categories having all those data. Ideally, following the development of polarization of Linear logic in [MT] inspired by game semantics, we are able to get more flexibility and allow larger dialogue categories containing such $*$-autonomous categories as their category of continuation. We will get slightly better categorical properties on those larger categories.

A better categorical framework was later found and summarized in [**?** KM] the so-called convenient smoothness. A posteriori, as seen [Ko], the notion is closely related to synthetic differential geometry as diffeological spaces are. It chooses a very liberal notion of smoothness, that does not imply continuity except on very special compact sets, images of finite dimensional compact sets by smooth maps. It gives a nice Cartesian closed category and this enabled [6] to obtain a model of intuitionistic differential linear logic. As we will see, this may give the wrong idea that this very liberal notion of smoothness is the only way of getting Cartesian closedness and it also takes the viewpoint of focusing on bornological properties. This is the main reason why, in our view, they don't obtain $*$-autonomous categories since bornological locally convex spaces have complete duals which gives an asymmetric requirement on duals since they only need a much weaker Mackey-completeness on their spaces to work with their notion of smooth maps. We will obtain in this paper several models of linear logic using conveniently smooth maps, and we will explain logically this Mackey-completeness condition in section 6.2. It is exactly a compatibility condition on F enabling to force our models to satisfy $!E \multimap F = (!E \multimap 1) \parr F$. Of course, as usual for vector spaces, our models will satisfy the mix rule making the unit for multiplicative connectives self-dual and this formula is interpreted mathematically as saying that smooth maps with value in some complete enough space are never a big deal and reduced by duality to the scalar case. But of course, this requires to identify the right completeness notion.

A smooth interpretation for the $\parr$. Another insight in our work is that the setting of models of Linear logic with smooth maps offers a decisive interpretation for the multiplicative disjunction. In the setting of smooth functions, the epsilon product introduced by Laurent Schwartz is well studied and behave exactly as wanted: under some completeness condition, one indeed has $\mathcal{C}^\infty(E,\mathscr{R})\varepsilon F \simeq \mathcal{C}^\infty(E,F)$. This required for instance in [58] some restrictive conditions. We reduce these conditions to the definition B.1.27 of k-complete spaces, which is also enough to get associativity and commutativity of ε. The interpretation of the tensor product follows as the negation of the ε product. We would like to point out that plenty of possibilities exists for defining a topological tensor product (see subsection 2.2 for reminders), and that choosing to build our models from the ε product offers a simplifying and intuitive guideline.

With this background in mind, we can describe in more detail our results and our strategy.

Organisation of the first part about MALL The first part of the paper will focus on building several $*$-autonomous categories. This work started with a negative lesson the first author learned from the second author's results in [Ker]. Combining lots of strong properties on concrete spaces as for instance in [BD, D] will never be enough, it makes stability of these properties by tensor product and duality too hard. The only way out is to get a duality functor that makes spaces reflexive for this duality in order to correct tensor products by double dualization. The lesson is that identifying a proper notion of duality is therefore crucial if one wants to get an interesting analytic tensor product. From an analytic viewpoint, the inductive tensor product is too weak to deal with extensions to completions and therefore the weak dual or the Mackey dual, shown to work well with this tensor product in [Ker], and which are the first duality functors implying easy reflexivity properties, are not enough for our purposes. The insight is given by a result of [S] that implies that another slightly different dual, the Arens dual always satisfies the algebraic equality $((E'_c)'_c)'_c = E'_c$ hence one gets a functor enabling to get reflexive spaces, in some weakened sense of reflexivity. Moreover, Laurent Schwartz also developed there a related tensor product, the so called ε-product which is intimately related. This tensor product is a dual tensor product, generalization of the (dual) injective tensor product of (dual) Banach spaces and logicians would say it is a negative connective (for instance, as seen from its commutation with categorical projective limits) suitable for interpreting $\parr$. Moreover, it is strongly related with Seely-like isomorphisms for various classes of non-linear maps, from continuous maps (see e.g. [76]) to smooth maps [58]. It is also strongly related with nuclearity and Grothendieck's approximation property. This is thus a well established analytic tool desirable as a connective for a natural model of linear logic. We actually realize that most of the general properties for the Arens dual and the ε-product in [S] are nicely deduced from a very general $*$-autonomous category we will explain at the end of the preliminary section 2. This first model of MALL that we will obtain takes seriously the lack of self-duality of the notion of locally convex space and notices that adjoining a bornology with weak compatibility conditions enables to get a framework where building a $*$-autonomous category is almost tautological. This may probably be related to some kind of Chu construction (cf. [Ba96] and appendix to [Ba79]), but we won't investigate this expectation here. This is opposite to the consideration of bornological locally convex vector spaces where bornology and topology are linked to determine one another, here they can be almost independently chosen and correspond to encapsulating on the same space the topology of the space and of its dual (given by the bornology).

Then, the work necessary to obtain a $*$-autonomous category of locally convex spaces is twofold, it requires to impose some completeness condition required to get associativity maps for the ε-product and then make the Arens dual compatible with some completion process to keep a reflexivity condition and get another duality functor with duals isomorphic to triple duals. We repeat this general plan twice in sections 4 and 5 to obtain two extreme cases where this plan can be carried out. The first version uses the notion of completeness used in [S], or rather a slight variant we will call k-quasi-completeness and builds a model of MALL without extra requirement than being k-quasi-complete and the Arens dual of a k-quasi-complete space. This notion is equivalent to a reflexivity property that we call k-reflexivity. This first $*$-autonomous category is important because its positive tensor product is a completed variant of an algebraic tensor product $_\gamma$ having universal properties for bilinear maps which have a so-called hypocontinuity condition implying continuity on product of compact sets (see section 2.2 for more preliminary background). This suggested us a relation to the well-known Cartesian closed category (equivalent to k-spaces) of topological spaces with maps all maps continuous on compact sets. Using strongly that we obtained a $*$-autonomous category, this enables us to provide the strongest notion of smoothness (on locally convex spaces) that we can imagine having a Cartesian closedness property. Contrary to convenient smoothness, it satisfies a much stronger continuity condition of all derivatives on compacts sets. Here, we thus combine the $*$-autonomous category with a Cartesian closed category in taking inspiration of the former to define the latter. This is developed in subsection 4.2.

Then in section 5, we can turn to the complementary goal of finding a $*$-autonomous framework that will be well-suited for the already known and more liberal notion of smoothness, namely convenient smoothness. Here, we need to combine Mackey-completeness with a Schwartz space property to reach our goals. This is partially based on preliminary work in section 3 that actually makes appear a strong relation with Mackey duals which can actually replace Arens duals in this context, contrary to the first author's original intuition alluded to before. Technically, it is convenient to decompose our search for a $*$-autonomous category in two steps. Once identified the right duality notion and the corresponding reflexivity, we produce first a Dialogue category that deduces its structure from a kind of intertwining with the $*$-autonomous category obtained in section 2. Then we use [MT] to recover a $*$-autonomous category in a standard way. This gives us the notion of ρ-dual and the $*$-autonomous category of ρ-Reflexive spaces. As before, those spaces can be described in saying that they are Mackey-complete with Mackey-complete Mackey dual (coinciding with Arens dual here) and they have the Schwartz topology associated to their Mackey topology. We gave the name ρ-dual since this was the first and more fruitful way (as seen its relation developed later with convenient smoothness) of obtaining a reflexive space by duality, hence the letter ρ for reflexive, while staying close to the letter σ that would have remembered the key Schwartz space property, but which was already taken by weak duals.

At the end of the first part of the paper, we have a kind of generic methodology enabling to produce $*$-autonomous categories of locally convex spaces from a kind of universal one from section 2. We also have obtained two examples that we want to extend to denotational models of full (differential) Linear logic in the second part.

Organisation of the second part about LL and DiLL. In the second part of the paper, we develop a theory for variants of conveniently smooth maps, which we restrict to allow for continuous, and not only bounded, differentials. We start with the convenient smoothness setting in section 6. Actually we work with several topological variants of this setting (all having the same bornologification). To complement our identification of a logical meaning of Mackey-completeness, we also relate the extra Schwartz property condition with the logical interpretation of the transpose of the dereliction $\mathsf{d}^t E^* \multimap (!E)^*$. This asks for the topology on E^* to be finer than the one induced by $(!E)^*$. If moreover one wants to recover later a model of differential linear logic, we need a morphism: $\overline{\mathsf{d}} :!E \longrightarrow E$ such that $\mathsf{d} \circ \overline{\mathsf{d}} = Id_E$. This enforces the fact that the topology on E^* must equal the one induced by $(!E)^*$. In this way, various natural topologies on conveniently smooth maps suggest various topologies on duals. We investigate in more detail the two extreme cases again, corresponding to well-known functional analytic conditions, both invented by Grothendieck, namely Schwartz topologies and the subclass of nuclear topologies. We obtain in that way in section 6 two denotational models of LL on the same $*$-autonomous category (of ρ-reflexive spaces), with the same Cartesian closed category of conveniently smooth maps, but with two different comonads. We actually show this difference in remark B.2.24 using Banach spaces without the approximation property. This also gives an insight of the functional analytic significance of the two structures. Technically, we use dialogue categories again, but not trough the models of tensor logic from [MT], but rather with a variant we introduce to keep Cartesian closed the category equipped with non-linear maps as morphisms.

Finally, in section 7, we extend our models to models of (full) differential linear logic. In the k-reflexive space case, we have already identified the right notion of smooth maps for that in section 4, but in the ρ-reflexive case, which generalizes convenient vector spaces, we need to slightly change our notion of smoothness and introduce a corresponding notion of ρ-smoothness. Indeed, for the new ρ-reflexive spaces which are not bornological, the derivative of conveniently smooth maps are only bounded and need not be in spaces of continuous linear maps which are the maps of our $*$-autonomous categories. Taking inspiration of our use of dialogue categories and its interplay with Cartesian closed categories in section 6, we introduce in section 7.1 a notion merging dialogue categories with differential λ-categories of [BEM] and realize the correction of derivative we need in a general context in section 7.2. This enables us to get a class of models of DiLL with at least 3 new different models in that way, one on k-reflexive spaces (section 7.4) and two being on the same category of ρ-reflexive spaces with ρ-smooth maps (section 7.3). This is done concretely by considering only smooth maps whose derivatives are smooth in their non-linear variable with value in (iterated) spaces of continuous linear maps.

A first look at the interpretation of Linear Logic constructions For the reader familiar with other denotational models of Linear Logic, we would like to point out some of the constructions involved in the first model $k-\mathbf{Ref}$. Our two other main models make use of similar constructions, with a touch of Mackey-completeness.

First, we define a *k-quasi-complete* space as a space in which the closed absolutely convex cover of a compact subset is still compact. We detail a procedure of k-quasi-completion, which is done inductively.

We take as the interpretation $E^\perp$ of the negation the k-quasi completion of E'_c, the dual of E endowed with the compact-open topology, at least when E is k-quasi-complete. We define $!E$ as $\mathcal{C}^\infty_{co}(E,\mathcal{K})^\perp$, the k-quasicompletion of the dual of the space of scalar smooth functions. This definition is in fact enforced as soon as we have a $*$-autonomous category with a co-Kleisli category of smooth maps. Here we define the space of smooth functions as the space of infinitely many times Gâteaux-differentiable functions with derivatives continuous on compacts, with a good topology (see subsection B.1.3.2). This definition, adapted from the one of Meise, allows for Cartesian closedeness.

We then interpret the $⅋$ as the (double dual of) the ε product: $E\varepsilon F = \mathcal{L}_\epsilon(E'_c, F)$, the space of all linear continuous functions from E'_c to F endowed with the topology of uniform convergence on equicontinuous subsets of E'. The interpretation of $\otimes$ is the dual of ε, and can be seen as the k-quasi-completion of a certain topological tensor product $\otimes_\gamma$.

The additive connectives $\times$ and $\oplus$ are easily interpreted as the product and the co-product. In our vectorial setting, they coincide in finite arity.

In the differential setting, codereliction $\bar{\mathrm{d}}$ is interpreted as usual by the transpose of differentiation at 0 of scalar smooth maps.

B.1 Three Models of MALL

B.1.1 Preliminaries

We will be working with *locally convex separated* topological vector spaces. We will write in short lcs for such spaces, following [K] in that respect. We refer to the book by Jarchow [Ja] for basic definitions. We will recall the definitions from Schwartz [S] concerning the ε product. We write $E = F$ when two lcs are equal algebraically and $E \simeq F$ when the lcs equal topologically as well.

Remark B.1.1. We will call *embedding* a continuous linear map $E \longrightarrow F$ which is one-to-one and with the topology of E induced from this inclusion. In the functional analytic literature [K2, p 2] this is called topological monomorphism and abbreviated monomorphism, this is also the case in [S]. This disagrees with the categorical terminology, hence our choice of a more consensual term. A monomorphism in the category of separated locally convex vector spaces is an injective continuous linear map, and a regular monomorphism is a embedding with closed image (a *closed embedding*). A regular monomorphism in the category of non-separated locally convex spaces coincide with an embedding but we won't use this category.

Remark B.1.2. We will use projective kernels as in [K]. They are more general than categorical limits, which are more general than projective limits of [K], which coincide with those categorical limits indexed by directed sets.

B.1.1.1 Reminder on topological vector spaces

Definition B.1.3. Consider E a vector space. A bornology on E is a collection of sets (the bounded sets of E) such that the union of all those sets covers E, and such that the collection is stable under inclusion and finite unions.

When E is a topological vector space, one defines the Von-Neumann bornology β as those sets which are absorbed by any neighbourhood of 0. Without any other precision, the name bounded set will refer to a bounded set for the Von-Neumann bornology. Other examples of bornology are the collections γ of all absolutely convex compact subsets of E, and σ of all bipolars of finite sets. When E is a space of continuous linear maps, one can also consider on E the bornology ε of all equicontinuous parts of E. When E is a lcs, we only consider saturated bornologies, namely those which contain the subsets of the bipolars of each of its members.

Definition B.1.4. Consider E, F, G topological vector spaces and $h : E \times F \mapsto G$ a bilinear map.

- h is continuous if it is continuous from $E \times F$ endowed with the product topology to G.
- h is separately continuous if for any $x \in E$ and $y \in F$, $h(x, .)$ is continuous from F to G and $h(., y)$ is continuous from E to G.
- Consider $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) a bornology on E (resp. F). Then h is said to be $\mathcal{B}_1$,$\mathcal{B}_2$ hypocontinuous [S2] if for every 0-neighbourhood W in G, every bounded set A_E in E, and every bounded set A_F in F, there are 0-neighbourhoods $V_F \subset F$ and $V_E \subset E$ such that $h(A_E \times V_F) \subset W$ and $h(V_E \times A_F) \subset W$. When no precision is given, an hypocontinuous bilinear map is a map hypocontinuous for both Von-Neumann bornologies.

Consider A an absolutely convex and bounded subset of a lcs E. We write E_A for the linear span of A in E. It is a normed space when endowed with the Minkowski functional

$$||x||_A \equiv p_A(x) = \inf \left\{\lambda \in \mathscr{R}^+ \mid x \in \lambda A\right\}.$$

A lcs E is said to be *Mackey-complete* (or locally complete [Ja, 10.2]) when for every bounded closed and absolutely convex subset A, E_A is a Banach space. A sequence is *Mackey-convergent* if it is convergent in some E_B. This notion can be generalized for any bornology $\mathcal{B}$ on E: a sequence is said to be *$\mathcal{B}$-convergent* if it is convergent is some E_B for $B \in \mathcal{B}$.

Consider E a lcs and τ its topology. Recall that a filter in E' is said to be equicontinuously convergent if it is ε-convergent. E is a *Schwartz space* if it is endowed with a Schwartz topology, that is a space such that every continuously convergent filter in E' converges equicontinuously. We refer to [HNM, chapter 1] and [Ja, sections 10.4, 21.1] for an overview on Schwartz topologies. We recall some facts below.

The finest Schwartz locally convex topology coarser than τ is the topology τ_0 of uniform convergence on sequences of E' converging equicontinuously to 0. We write $\mathscr{S}(E) = \mathscr{S}(E, \tau) = (E, \tau_0)$. We have $\mathscr{S}(E)' = E'$, and $\mathscr{S}(E)$ is always separated. A lcs E is a Schwartz space if and only if $\mathscr{S}(E) = E$, if and only if the completion $\tilde{E}$ is a Schwartz space. We do know also that $\mathscr{S}(E)$ is Mackey-complete as soon as E is (as both space have the same dual, they have the same bounded sets by Mackey-Arens Theorem). Any subspace of a Schwartz space is a Schwartz space.

B.1.1.2 Reminder on tensor products and duals of locally convex spaces.

Several topologies can be associated with the tensor product of two topological vector space.

Definition B.1.5. Consider E and F two lcs.

- The projective tensor product $E \otimes_\pi F$ is the finest locally convex topology on $E \otimes F$ making $E \times F \to E \otimes_\pi F$ continuous.
- The inductive tensor product $E \otimes_i F$ is the finest locally convex topology on $E \otimes F$ making $E \times F \to E \otimes_i F$ separately continuous.
- The hypocontinous tensor product $E \otimes_\beta F$ is the finest locally convex topology on $E \otimes F$ making $E \times F \to E \otimes_\beta F$ hypocontinuous.

- The γ tensor product $E \otimes_\gamma F$ is the finest locally convex topology on $E \otimes F$ making $E \times F \to E \otimes_\gamma F$ γ-hypocontinuous.
- Suppose that E and F are duals. The ε-hypocontinous tensor product $E \otimes_{\beta e} F$ is the finest locally convex topology on $E \otimes F$ making $E \times F \to E \otimes_{\beta e} F$ ε-hypocontinuous.
- Consider $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) a bornology on E (resp. F).The $\mathcal{B}_1 - \mathcal{B}_2$-hypocontinous tensor product $E \otimes_{\mathcal{B}_1,\mathcal{B}_2} F$ is the finest locally convex topology on $E \otimes F$ making $E \times F \to E \otimes_{\mathcal{B}_1,\mathcal{B}_2} F$ $\mathcal{B}_1, \mathcal{B}_2$-hypocontinuous.

All the above tensor products, except the last one, are commutative and the $\otimes_\pi$ product is associative. With the last generic notation one gets ${}_i = {}_{\sigma,\sigma}, {}_\beta = {}_{\beta,\beta}, {}_\gamma = {}_{\gamma,\gamma}, \otimes_{\beta e} = {}_{\varepsilon,\varepsilon}$ and we will sometimes consider during proofs non-symmetric variants such as: ${}_{\varepsilon,\gamma}, {}_{\sigma,\gamma}$ etc. Note that the injective tensor product ${}_\varepsilon \neq {}_{\varepsilon,\varepsilon}$ is a dual version we will discuss later. It does not have the above kind of universal properties.

Definition B.1.6. One can define several topologies on the dual E' of a lcs E. We will make use of :

- The strong dual E'_β, endowed with the strong topology $\beta(E', E)$ of uniform convergence on bounded subsets of E.
- The *Arens dual* E'_c endowed with the topology $\gamma(E', E)$ of uniform convergence on absolutely convex compact subsets of E.
- The *Mackey dual* E'_μ, endowed with the Mackey topology of uniform convergence on absolutely convex weakly compact subsets of E.
- The weak dual E'_σ endowed with the weak topology $\sigma(E', E)$ of simple convergence on points of E.
- The ε-dual E'_ε of a dual $E = F'$ is the dual E' endowed with the topology of uniform convergence on equicontinuous sets in F'.

Remember that when it is considered as a set of linear forms acting on E', E is always endowed with the topology of uniform convergence on equicontinuous parts of E', equivalent to the original topology of E, hence $(E'_\mu)'_\epsilon \simeq (E'_c)'_\epsilon \simeq (E'_\sigma)'_\epsilon \simeq E$. A lcs is said to be reflexive when it is topologically equal to its strong double dual $(E'_\beta)'_\beta$.

The *Mackey-Arens theorem* [Ja, 8.5.5] states that whenever E' is endowed with a topology finer that the weak topology, and coarser than the Mackey topology, then $E = E''$ algebraically. Thus one has

$$E = (E'_c)'. \tag{B.1}$$

As explained by Laurent Schwartz [S, section 1], the equality $E \simeq (E'_c)'_c$ holds as soon as E is endowed with its γ topology, i.e. with the topology of uniform convergence on absolutely convex compact subsets of E'_c. He proves moreover that an Arens dual is always endowed with its γ-topology, that is: $E'_c \simeq ((E'_c)'_c)'_c$. This fact is the starting point of the construction of a $*$-autonomous category in section B.1.4.

The *ε-product* has been extensively used and studied by Laurent Schwartz [S, section 1]. By definition $E\varepsilon F = (E'_c \otimes_{\beta e} F'_c)'$ is the set of ε-hypocontinuous bilinear forms on the duals E'_c and F'_c. When E, F have their γ topologies this is the same as $E\varepsilon F = (E'_c \otimes_\gamma F'_c)'$.

The topology on $E\varepsilon F$ is the topology of uniform convergence on products of equicontinuous sets in E', F'. If E, F are quasi-complete spaces (resp. complete spaces , resp. complete spaces with the approximation property) so is $E\varepsilon F$ (see [S, Prop 3 p29, Corol 1 p 47]). The ε tensor product $E \otimes_\varepsilon F$ coincides with the topology on $E \otimes F$ induced by $E\varepsilon F$ (see [S, Prop 11 p46]), $\otimes_\varepsilon$ is associative, and $E\hat{\otimes}_\varepsilon F \simeq E\varepsilon F$ if E, F are complete and E has the approximation property.

The ε-product is also defined on any finite number of space as $\varepsilon_i E_i$, the space of ε-equicontinuous multilinear forms on $\prod_i (E_i)'_c$, endowed the the topology of uniform convergence on equicontinuous sets. Schwartz proves the associativity of the ε-product when the spaces are quasi-complete. We do so when the spaces are Mackey-complete and Schwartz, see lemma B.1.51.

B.1.1.3 Dialogue and $*$-autonomous categories

It is well known that models of (classical) linear logic requires building $*$-autonomous categories introduced in [Ba79]. If we add categorical completeness, they give models of MALL. We need some background about them, as well as a generalization introduced in [MT]: the notion of Dialogue category that will serve us as an intermediate in between a general $*$-autonomous category we will introduce in the next subsection and more specific ones requiring a kind of reflexivity of locally convex spaces that we will obtain by double dualization, hence in moving to the so-called continuation category of the Dialogue category.

Recall the definition (cf. [Ba79]):

Definition B.1.7. A *$*$-autonomous category* is a symmetric monoidal closed category $(\mathcal{C}, {}_{\mathcal{C}}, 1_{\mathcal{C}}, [\cdot,\cdot]_{\mathcal{C}})$ with an object $\perp$ giving an equivalence of categories $(\cdot)^* = [\cdot, \perp]_{\mathcal{C}} : \mathcal{C}^{op} \longrightarrow \mathcal{C}$ and with the canonical map $d_A : A \longrightarrow (A^*)^*$ being a natural isomorphism.

Since our primary data will be functional, based on space of linear maps (and tensorial structure will be deduced since it requires various completions), we will need a consequence of the discussion in [Ba79, (4.4) (4.5) p 14-15]. We outline the proof for the reader's convenience. We refer to [DeS, p 25] (see also [DL]) for the definition of symmetric closed category.

Lemma B.1.8. Let $(\mathcal{C}, 1_{\mathcal{C}}, [\cdot,\cdot]_{\mathcal{C}})$ a symmetric closed category, which especially implies there is a natural isomorphism $s_{X,Y,Z} : [X,[Y,Z]_{\mathcal{C}}]_{\mathcal{C}} \longrightarrow [Y,[X,Z]_{\mathcal{C}}]_{\mathcal{C}}$ and let $\perp = [1_{\mathcal{C}}, 1_{\mathcal{C}}]_{\mathcal{C}}$. Assume moreover that there is a natural isomorphism, $d_X : X \longrightarrow [[X,\perp]_{\mathcal{C}}, \perp]_{\mathcal{C}}$. Define $X^* = [X,\perp]_{\mathcal{C}}$ and $(X_{\mathcal{C}}Y) = ([X,Y^*]_{\mathcal{C}})^*$. Then $(\mathcal{C}, {}_{\mathcal{C}}, 1_{\mathcal{C}}, [\cdot,\cdot]_{\mathcal{C}}, (\cdot)^*)$ is a $*$-autonomous category.

Proof. Recall for instance that $i_X : X \longrightarrow [1_{\mathcal{C}}, X]_{\mathcal{C}}$ is an available natural isomorphism. Note first that there is a natural isomorphism defined by:

$$d_{X,Y} : [X,Y]_{\mathcal{C}} \xrightarrow{[X,d_Y]_{\mathcal{C}}} [X, Y^{**}]_{\mathcal{C}} \xrightarrow{s_{X,Y^*,\perp}} [Y^*, [X,\perp]_{\mathcal{C}}]_{\mathcal{C}}.$$

The assumptions give a natural isomorphism:

$$\begin{aligned} \mathcal{C}(X, [Y, Z^*]_{\mathcal{C}}) &\simeq \mathcal{C}(1, [X,[Y,Z^*]_{\mathcal{C}}]_{\mathcal{C}}) \simeq \mathcal{C}(1, [X,[Z,Y^*]_{\mathcal{C}}]_{\mathcal{C}}) \\ &\simeq \mathcal{C}(1, [Z,[X,Y^*]_{\mathcal{C}}]_{\mathcal{C}}) \simeq \mathcal{C}(Z, [Y,X^*]_{\mathcal{C}}). \end{aligned}$$

Moreover, we have a bijection $\mathcal{C}(X^*, Y^*) \simeq \mathcal{C}(1, [X^*, Y^*]_{\mathcal{C}}) \simeq \mathcal{C}(1, [Y,X]_{\mathcal{C}}) \simeq \mathcal{C}(Y,X)$ so that the assumptions in [Ba79, (4.4)] are satisfied. His discussion in (4.5) gives a natural isomorphism: $\pi_{XYZ} : \mathcal{C}(X_{\mathcal{C}}Y, Z) \longrightarrow \mathcal{C}(X, [Y,Z]_{\mathcal{C}}]_{\mathcal{C}})$.

We are thus in the third basic situation of [DeS, IV .4] which gives (from s) a natural transformation p_{XYZ} : $[X_{\mathcal{C}}Y, Z]_{\mathcal{C}} \longrightarrow [X, [Y,Z]_{\mathcal{C}}]_{\mathcal{C}}$. Then the proof of his Prop VI.4.2 proves his compatibility condition MSCC1 from SCC3, hence we have a monoidal symmetric closed category in the sense of [DeS, Def IV.3.1].

Then [DeS, Thm VI.6.2 p 136] gives us a usual symmetric monoidal closed category in the sense of [EK]. This concludes. ☐

We finally recall the more general definition in [MT]:

Definition B.1.9. A *Dialogue category* is a symmetric monoidal category $(\mathcal{C}, {}_{\mathcal{C}}, 1_{\mathcal{C}})$ with a functor, called tensorial negation: $\neg : \mathcal{C} \longrightarrow \mathcal{C}^{op}$ which is associated to a natural bijection $\varphi_{A,B,C} : \mathcal{C}(A_{\mathcal{C}}B, \neg C) \simeq \mathcal{C}(A, \neg(B_{\mathcal{C}}C))$ and satisfying the commutative diagram with associators $Ass^{\mathcal{C}}_{A,B,C} : A_{\mathcal{C}}(B_{\mathcal{C}}C) \longrightarrow (A_{\mathcal{C}}B)_{\mathcal{C}}C$:

$$\begin{tikzcd} \mathcal{C}((A_{\mathcal{C}}B)_{\mathcal{C}}C, \neg D) \arrow[r, "\varphi_{A_{\mathcal{C}}B,C,D}"] \arrow[d, "\mathcal{C}(Ass^{\mathcal{C}}_{A,B,C}, \neg D)"'] & \mathcal{C}(A_{\mathcal{C}}B, \neg(C_{\mathcal{C}}D)) \arrow[r, "\varphi_{A,B,C_{\mathcal{C}}D}"] & \mathcal{C}\Big(A, \neg\big[B_{\mathcal{C}}(C_{\mathcal{C}}D)\big]\Big) \\ \mathcal{C}(A_{\mathcal{C}}(B_{\mathcal{C}}C), \neg D) \arrow[rr, "\varphi_{A,BC,D}"] & & \mathcal{C}\Big(A, \neg\big[(B_{\mathcal{C}}C)_{\mathcal{C}}D\big]\Big) \arrow[u, "\mathcal{C}(A, \neg Ass^{\mathcal{C}}_{B,C,D})"'] \end{tikzcd} \tag{B.2}$$

B.1.1.4 A model of MALL making appear the Arens dual and the Schwartz ε-product

We introduce a first $*$-autonomous category that captures categorically the part of [S] that does not use quasi-completeness. Since bornological and topological concepts are dual to one another, it is natural to fix a saturated bornology on E in order to create a self-dual concept. Then, if one wants every object to be a dual object as in a $*$-autonomous category, one must consider only bornologies that can arise as the natural bornology on the dual, namely, the equicontinuous bornology. We could take a precompactness condition to ensure that, but to make appear the Arens dual and ε-product (and not the polar topology and Meise's variant of the ε-product), we use instead a compactness condition. A weak-compactness condition would work for the self-duality requirement by Mackey Theorem but not for dealing with tensor products.

We will thus use a (saturated, topological) variant of the notion of compactology used in [Ja, p 157]. We say that a saturated bornology B_E on a lcs E is a *compactology* if it consists of relatively compact sets. Hence, the bipolar of each bounded set for this bornology is an absolutely convex compact set in E, and it is bounded for this bornology. A separated locally convex space with a compactology will be called a *compactological locally convex space*.

Definition B.1.10. Let $\mathbf{LCS}$ be the category of separated locally convex spaces with continuous linear maps and $\mathbf{CLCS}$ the category of compactological locally convex spaces, with maps given by bounded continuous linear maps. For $E, F \in \mathbf{CLCS}$ the internal Hom $L_b(E,F)$ is the above set of maps given the topology of uniform convergence on the bornology of E and the bornology of equibounded equicontinuous sets. We call $E'_b = L_b(E,\mathcal{K})$ (its bornology is merely the equicontinuous bornology, see step 1 of next proof). The algebraic tensor product $E_H F$ is the algebraic tensor product with the topology having the universal property for B_E, B_F-hypocontinuous maps, and the bornology generated by bipolars of sets AC for $A \in B_E, C \in B_F$.

Note that we didn't claim that $E_H F$ is in $\mathbf{CLCS}$, it may not be. It gives a generic hypocontinuous tensor product. Note that composition of bounded continuous linear maps are of the same type, hence $\mathbf{CLCS}$ is indeed a category.

Recall also that $\mathbf{LCS}$ is complete and cocomplete since it has small products and coproducts, kernels and cokernels (given by the quotient by the closure $\overline{Im[f-g]}$) [K, 18.3.(1,2,5), 18.5.(1)].

In order to state simultaneously a variant adapted to Schwartz spaces, we introduce a variant:

Definition B.1.11. Let $\mathbf{Sch} \subset \mathbf{LCS}$ be the full subcategory of Schwartz spaces and $\mathbf{CSch} \subset \mathbf{CLCS}$ the full subcategory of Schwartz compactological lcs, namely those spaces which are Schwartz as locally convex spaces and for which E'_b is a Schwartz lcs too.

This second condition is well-known to be equivalent to the bornology being a Schwartz bornology [HNM], and to a more concrete one:

Lemma B.1.12. For $E \in \mathbf{CLCS}$, E'_b is a Schwartz lcs if and only if every bounded set in B_E is included in the closed absolutely convex cover of a B_E-null sequence.

Proof. E'_b is Schwartz if and only if $E'_b = \mathscr{S}(E'_b)$. But $\mathscr{S}(E'_b)$ is known to be the topology of uniform convergence on $(B_E)_{c_0}$ the saturated bornology generated by B_E-null sequences of $E = (E'_b)'$ [Ja, Prop 10.4.4]. Since both bornologies are saturated this means [K, 21 .1. (4)] that E'_b is a Schwartz space if and only if $B_E = (B_E)_{c_0}$. ☐

We call $\mathscr{S}L_b(E,F)$ the same lcs as $L_b(E,F)$ but given the bornology $(B_{L_b(E,F)})_{c_0}$ namely the associated Schwartz bornology. Note that $\mathscr{S}L_b(E,\mathcal{K}) = E'_b$ as compactological lcs for $E \in \mathbf{CSch}$.

Theorem B.1.13. ***CLCS*** *(resp.* $\mathbf{CSch}$*) is a complete and cocomplete $*$-autonomous category with dualizing object $\mathcal{K}$ and internal Hom $L_b(E,F)$ (resp. $\mathscr{S}L_b(E,F)$).*

1. *The functor $(.)'_c : \mathbf{LCS} \longrightarrow \mathbf{CLCS}^{op}$ giving the Arens dual the equicontinuous bornology, is right adjoint to $U((.)'_b)$, with U the underlying lcs and $U((.)'_b) \circ (.)'_c = Id_{\mathbf{LCS}}$. The functor $(.)'_\sigma : \mathbf{LCS} \longrightarrow \mathbf{CLCS}^{op}$ giving the weak dual the equicontinuous bornology, is left adjoint to $U((.)'_b)$ and $U((.)'_b) \circ (.)'_\sigma = Id_{\mathbf{LCS}}$.*

2. *The functor $U : \mathbf{CLCS} \longrightarrow \mathbf{LCS}$ is left adjoint and also left inverse to $(.)_c$, the functor $E \mapsto E_c$ the space with the same topology and the absolutely convex compact bornology. U is right adjoint to $(.)_\sigma$, the functor $E \mapsto E_\sigma$ the space with the same topology and the saturated bornology generated by finite sets. $U, (.)_c, (.)_\sigma$ are faithful.*

3. *The functor U : **CSch** $\longrightarrow$ **Sch** is left adjoint and also left inverse to $(.)_{sc}$, the functor $E \mapsto E_{sc}$ the space with the same topology and the Schwartz bornology associated to the absolutely convex compact bornology. U is again right adjoint to $(.)_\sigma$ (restriction of the previous one). $U, (.)_{sc}, (.)_\sigma$ are faithful.*

4. *The ε-product in* **LCS** *is given by $E\varepsilon F = U(E_c \mathbin{⅋}_b F_c)$ with $G \mathbin{⅋}_b H = L_b(G'_b, H)$ and of course the Arens dual by $U((E_c)'_b)$, and more generally $L_c(E, F) = U(L_b(E_c, F_c))$. The inductive tensor product $E_i F = U(E_{\sigma b} F_\sigma)$ with $G_b H = (G'_b \mathbin{⅋}_b H'_b)'_b$ and of course the weak dual is $U((E_\sigma)'_b)$.*

Proof. **Step 1:** Internal Hom functors $L_b, \mathscr{S}L_b$.

We first need to check that the equibounded equicontinuous bornology on $L_b(E, F)$ is made of relatively compact sets when $E, F \in$ **CLCS**. In the case $F = \mathcal{K}$, the bornology is the equicontinuous bornology since an equicontinuous set is equibounded for von Neumann bornologies [K2, 39.3.(1)]. Our claimed statement is then explained in [S, note 4 p 16] since it is proved there that every equicontinuous closed absolutely convex set is compact in $E'_c = (U(E))'_c$ and our assumption that the saturated bornology is made of relatively compact sets implies there is a continuous map $E'_c \longrightarrow E'_b$. This proves the case $F = \mathcal{K}$.

Note that by definition, $G = L_b(E, F)$ identifies with the dual $H = (E_H F'_b)'_b$. Indeed, the choice of bornologies implies the topology of H is the topology of uniform convergence on equicontinuous sets of F' and on bounded sets of E which is the topology of G. An equicontinuous set in H is known to be an equihypocontinuous set [S2, p 10], i.e. a set taking a bounded set in E and giving an equicontinuous set in $(F'_b)'$, namely a bounded set in F, hence the equibounded condition, and taking symmetrically a bounded set in F'_b i.e. an equicontinuous set and sending it to an equicontinuous set in E', hence the equicontinuity condition [K2, 39.3.(4)].

Let $\widetilde{E_H F'_b} \subset \widehat{E_H F'_b}$ the subset of the completion obtained by taking the union of bipolars of bounded sets. It is easy to see this is a vector subspace on which we put the induced topology. One deduces that $H = (\widetilde{E_H F'_b})'_b$ where the $\widetilde{EF'_b}$ is given the bornology generated by bipolars of bounded sets (which covers it by our choice of subspace). Indeed the completion does not change the dual and the equicontinuous sets herein [K, 21.4.(5)] and the extension to bipolars does not change the topology on the dual either. But in $\widetilde{E_H F'_b}$, bounded sets for the above bornology are included into bipolars of tensor product of bounded sets. Let us recall why tensor products AB of such bounded sets are precompact in $E_H F'_b$ (hence also in $\widetilde{E_H F'_b}$ by [K, 15.6.(7)]) if $E, F \in$ **CLCS**. Take U' (resp. U) a neighbourhood of 0 in it (resp. such that $U + U \subset U'$), by definition there is a neighbourhood V (resp. W) of 0 in E (resp. F'_b) such that $VB \subset U$ (resp. $AW \subset U$). Since A, B are relatively compact hence precompact, cover $A \subset \cup_i x_i + V$, $x_i \in A$ (resp. $B \subset \cup_j y_j + W$, $y_j \in B$) so that one gets the finite cover giving totally boundedness:

$$AB \subset \cup_i x_i B + VB \subset \cup_{i,j} x_i y_j + x_i W + VB \subset \cup_{i,j} x_i y_j + U + U \subset \cup_{i,j} x_i y_j + U'.$$

Note that we used strongly compactness here in order to exploit hypocontinuity, and weak compactness and the definition of Jarchow for compactologies wouldn't work with our argument.

Thus from hypocontinuity, we deduced the canonical map $E \times F'_b \longrightarrow \widetilde{E_H F'_b}$ send $A \times B$ to a precompact (using [K, 5.6.(2)]), hence its bipolar is complete (since we took the bipolar in the completion which is closed there) and precompact [K, 20.6.(2)] hence compact (by definition [K, 5.6]). Thus $\widetilde{E_H F'_b} \in$ **CLCS**, if $E, F \in$ **CLCS**. From the first case for the dual, one deduces $L_b(E, F) \in$ **CLCS** in this case. Moreover, once the next step obtained, we will know $\widetilde{E_H F'_b} \simeq E_b F'_b$.

Let us explain why **CSch** is stable by the above internal Hom functor. First for $E, F \in$ **CSch** we must see that $L_b(E, F)$ is a Schwartz lcs. By definition F, E'_b are Schwartz spaces, hence this is [Ja, Thm 16.4.1]. From the choice of bornology, $\mathscr{S}L_b(E, F) \in$ **CSch** since by definition $U((\mathscr{S}L_b(E, F))'_b) \simeq \mathscr{S}(U((L_b(E, F))'_b))$.

Step 2: **CLCS** and **CSch** as Closed categories.

It is well know that **Vect** the category of Vector spaces is a symmetric monoidal category and especially a closed category in the sense of [EK]. **CLCS** $\subset$ **Vect** is a (far from being full) subcategory, but we see that we can induce maps on our smaller internal Hom. Indeed, the linear map $L^E_{FG} : L_b(F, G) \longrightarrow L_b(L_b(E, F), L_b(E, G))$ is well defined since a bounded family in $L_b(F, G)$ is equibounded, hence it sends an equibounded set in $L_b(E, F)$ to an equibounded set in $L_b(E, G)$, and also equicontinuous, hence its transpose sends an equicontinuous set in

$(L_b(E,G))'$ (described as bipolars of bounded sets in E tensored with equicontinuous sets in G') to an equicontinuous set in $(L_b(E,F))'$. This reasoning implies L^E_{FG} is indeed valued in continuous equibounded maps and even bounded with our choice of bornologies. Moreover we claim L^E_{FG} is continuous. Indeed, an equicontinuous set in $(L_b(L_b(E,F),L_b(E,G)))'$ is generated by the bipolar of equicontinuous C set in G', a bounded set B in E and an equibounded set A in $(L_b(E,F))$ and the transpose consider $A(B) \subset F$ and C to generate a bipolar which is indeed equicontinuous in $(L_b(F,G))'$. Hence, L^E_{FG} is a map of our category. Similarly, the morphism giving identity maps $j_E : \mathcal{K} \longrightarrow L_b(E,E)$ is indeed valued in the smaller space and the canonical $i_E : E \longrightarrow L_b(\mathcal{K},E)$ indeed sends a bounded set to an equibounded equicontinuous set and is tautologically equicontinuous. Now all the relations for a closed category are induced from those in $\mathbf{Vect}$ by restriction. The naturality conditions are easy.

Let us deduce the case of $\mathbf{CSch}$. First, let us see that for $E \in \mathbf{CSch}$,

$$\mathscr{S}L_b(E,\mathscr{S}L_b(F,G)) = \mathscr{S}L_b(E,L_b(F,G)) \tag{B.3}$$

By definition of boundedness, a map $f \in L_b(E,L_b(F,G))$ sends a Mackey-null sequence in E to a Mackey-null sequence in $L_b(F,G)$ hence by continuity the bipolar of such a sequence is sent to a bounded set in $\mathscr{S}L_b(F,G)$, hence from lemma B.1.12, so is a bounded set in E. We deduce the algebraic equality in (B.3). The topology of $L_b(E,H)$ only depends on the topology of H, hence we have the topological equality since both target spaces have the same topology. It remains to compare the bornologies. But from the equal target topologies, again, the equicontinuity condition is the same on both spaces hence boundedness of the map $L_b(E,\mathscr{S}L_b(F,G)) \longrightarrow L_b(E,L_b(F,G))$ is obvious. Take a sequence f_n of maps Mackey-null in $L_b(E,L_b(F,G))$ hence in the Banach space generated by the Banach disk D of another Mackey-null sequence (g_n). Let us see that $\{g_n, n \in \mathcal{N}\}^{oo}$ is equibounded in $L_b(E,\mathscr{S}L_b(F,G))$. For take $B \subset L_b(E,L_b(F,G))$ the disk for (g_n) with $||g_n||_B \longrightarrow 0$ and take a typical generating bounded set $A = \{x_n, n \in \mathcal{N}\}^{oo} \subset E$ for x_n B_E-Mackey-null. Then $g_n(A) \subset \{g_m(x_n), m, n \in \mathcal{N}\}^{oo} =: C$ and $||g_m(x_n)||_{(B(A))^{oo}} \leqslant ||g_m||_B ||x_n||_A$ and since $B(A)$ is bounded by equiboundedness of B, $(g_m(x_n))$ is Mackey-null, hence C is bounded in $\mathscr{S}L_b(F,G)$ and hence $D = \{g_n, n \in \mathcal{N}\}^{oo}$ is equibounded as stated. But since D is also bounded in $L_b(E,L_b(F,G))$ it is also equicontinuous, hence finally, bounded in $L_b(E,\mathscr{S}L_b(F,G))$. This gives that f_n Mackey-null there which concludes to the bornological equality in (B.3).

As a consequence, for $E,F,G \in \mathbf{CSch}$, the previous map L^E_{FG} induces a map

$$\begin{aligned} L^E_{FG} : \mathscr{S}L_b(F,G) \longrightarrow \mathscr{S}L_b(L_b(E,F),L_b(E,G)) &\longrightarrow \mathscr{S}L_b(\mathscr{S}L_b(E,F),L_b(E,G)) \\ &= \mathscr{S}L_b(\mathscr{S}L_b(E,F),\mathscr{S}L_b(E,G)) \end{aligned}$$

coinciding with the previous one as map. Note that we used the canonical continuous equibounded map $L_b(L_b(E,F),G) \longrightarrow L_b(\mathscr{S}L_b(E$ obviously given by the definition of associated Schwartz bornologies which is a smaller bornology.

Step 3: $*$-autonomous property.

First note that $L_b(E,F) \simeq L_b(F'_b,E'_b)$ by transposition. Indeed, the space of maps and their bornologies are the same since equicontinuity (resp. equiboundedness) $E \longrightarrow F$ is equivalent to equiboundedness (resp. equicontinuity) of the transpose $F'_b \longrightarrow E'_b$ for equicontinuous bornologies (resp. for topologies of uniform convergence of corresponding bounded sets). Moreover the topology is the same since it is the topology of uniform convergence on bounded sets of E (identical to equicontinuous sets of $(E'_b)'$) and equicontinuous sets of F' (identical to bounded sets for F'_b). Similarly $\mathscr{S}L_b(E,F) \simeq \mathscr{S}L_b(F'_b,E'_b)$ since on both sides one considers the bornology generated by Mackey-null sequences for the same bornology.

It remains to check $L_b(E,L_b(F,G)) \simeq L_b(F,L_b(E,G))$. The map is of course the canonical map. Equiboundedness in the first space means sending a bounded set in E and a bounded set in F to a bounded set in G and also a bounded set in E and an equicontinuous set in G' to an equicontinuous set in F'. This second condition is exactly equicontinuity $F \longrightarrow L_b(E,G)$. Finally, analogously, equicontinuity $E \longrightarrow L_b(F,G)$ implies it sends a bounded set in F and an equicontinuous set in G' to an equicontinuous set in E' which was the missing part of equiboundedness in $L_b(F,L_b(E,G))$. The identification of spaces and bornologies follows. Finally, the topology on both spaces is the topology of uniform convergence on products of bounded sets of E,F.

Again, the naturality conditions of the above two isomorphisms are easy, and the last one induces from $\mathbf{Vect}$ again the structure of a symmetric closed category, hence lemma B.1.8 concludes to $\mathbf{CLCS}$ $*$-autonomous.

Let us prove the corresponding statement for $\mathbf{CSch}$. Note that (B.3) implies the compactological isomorphism

$$\mathscr{S}L_b(E,\mathscr{S}L_b(F,G)) \simeq \mathscr{S}L_b(E,L_b(F,G)) \simeq \mathscr{S}L_b(F,L_b(E,G)) \simeq \mathscr{S}L_b(F,\mathscr{S}L_b(E,G)).$$

Hence, application of lemma B.1.8 concludes in the same way.

Step 4: Completeness and cocompleteness.

Let us describe first coproducts and cokernels. This is easy in **CLCS** it is given by the colimit of separated locally convex spaces, given the corresponding final bornology. Explicitely, the coproduct is the direct sum of vector spaces with coproduct topology and the bornology is the one generated by finite sum of bounded sets, hence included in finite sums of compact sets which are compact [K, 15.6.(8)]. Hence the direct sum is in **CLCS** and clearly has the universal property from those of topolocial/bornological direct sums. For the cokernel of $f, g : E \longrightarrow F$, we take the coproduct in **LCS**, $Coker(f, g) = F/\overline{(f-g)(E)}$ with the final bornology, i.e. the bornology generated by images of bounded sets. Since the quotient map is continuous between Hausdorff spaces, the image of a compact containing a bounded set is compact, hence $Coker(f, g) \in$ **CLCS**. Again the universal property comes from the one in locally convex and bornological spaces. Completeness then follows from the $*$-autonomous property since one can see $\lim_i E_i = (\mathrm{colim}_i (E_i)'_b)'_b$ gives a limit.

Similarly in **CSch** the colimit of Schwartz bornologies is still Schwartz since the dual is a projective limit of Schwartz spaces hence a Schwartz space (cf lemma B.1.18). We therefore claim that the colimit is the Schwartz topological space associated to the colimit in **CLCS** with same bornology. Indeed this is allowed since there are more compact sets hence the compatibility condition in **CLCS** is still satisfied and functoriality of $\mathscr{S}$ in lemma B.1.18 implies the universal property.

Step 5: Adjunctions and consequences.

The fact that the stated maps are functors is easy. We start by the adjunction for U in (2): $\mathbf{LCS}(U(F), E) = L_b(F, E_c) = \mathbf{CLCS}(F, E_c)$ since the extra condition of boundedness beyond continuity is implied by the fact that a bounded set in F is contained in an absolutely convex compact set which is sent to the same kind of set by a continuous linear map. Similarly, $\mathbf{LCS}(E, U(F)) = L_b(E_\sigma, F) = \mathbf{CLCS}(E_\sigma, F)$ since the image of a finite set is always in any bornology (which must cover E and is stable by union), hence the equiboundedness is also automatic.

For (3), since $(E_\sigma)'_b = E'_\sigma$ is always Schwartz, the functor $(.)_\sigma$ restricts to the new context, hence the adjunction. Moreover $U((E_c)'_b) = \mathscr{S}(E'_c)$ by construction. The key identity $\mathbf{Sch}(U(F), E) = L_b(F, E_c) = \mathbf{CSch}(F, E_{sc})$ comes from the fact that a Mackey-null sequence in F is send by a continuous function to a Mackey-null sequence for the compact bornology hence to a bounded set in E_{sc}. All naturality conditions are easy.

Moreover, for the adjunction in (1), we have the equality as set (using involutivity and functoriality of $(.)'_b$ and the previous adjunction):

$$\mathbf{CLCS}^{op}(F, E'_c) = L_b((E_c)'_b, F) = L_b(F'_b, E_c) = \mathbf{LCS}(U(F'_b), E),$$

$$\mathbf{CLCS}^{op}(E'_\sigma, F) = L_b(F, (E_\sigma)'_b) = L_b(E_\sigma, F'_b) = \mathbf{LCS}(E, U(F'_b)).$$

The other claimed identities are obvious by definition. □

The second named author explored in [Ker] models of linear logic using the positive product $_i$ and $(.)'_\sigma$. We will use in this work the negative product ε and the Arens dual $(.)'_c$ appearing with a dual role in the previous result. Let us summarize the properties obtained in [S] that are consequences of our categorical framework.

Corollary B.1.14.

1. Let $E_i \in \mathbf{LCS}, i \in I$. The iterated ϵ-product is $\varepsilon_{i\in I} E_i = U(\parr_{b,i\in I}(E_i)_c)$, it is symmetric in its arguments and commute with limits.
2. There is a continuous injection $(E_1 \varepsilon E_2 \varepsilon E_3) \longrightarrow E_1 \varepsilon (E_2 \varepsilon E_3)$.
3. For any continuous linear map $f : F_1 \longrightarrow E_1$ (resp. continuous injection, closed embedding), so is $f \varepsilon Id : F_1 \varepsilon E_2 \longrightarrow E_1 \varepsilon E_2$.

Note that (3) is also valid for non-closed embeddings and (2) is also an embedding [S], but this is not a categorical consequence of our setting.

Proof. The equality in (1) is a reformulation of definitions, symmetry is an obvious consequence. Commutation with limits come from the fact that $U, ()_c$ are right adjoints and $\parr_b$ commutes with limits from universal properties.

Using associativity of $\parr_b$: $E_1\varepsilon(E_2\varepsilon E_3) = U((E_1)_c \parr_b [U((E_2)_c \parr_b (E_3)_c)]_c)$ hence functoriality and the natural transformation coming from adjunction $Id \longrightarrow (U(\cdot))_c$ concludes to the continuous map in (2). It is moreover a monomorphism since $E \longrightarrow (U(E))_c$ is one since $U(E) \longrightarrow U((U(E))_c)$ is identity and U reflects monomorphisms and one can use the argument for (3).

For (3) functorialities give definition of the map, and recall that closed embeddings in **LCS** are merely regular monomorphisms, hence a limit, explaining its commutation by (1). If f is a monomomorphism, in categorical sense, so is $U(f)$ using a right inverse for U and so is $(f)_c$ since $U((f)_c) = f$ and U reflects monomorphisms as any faithful functor. Hence it suffices to see $\parr_b$ preserves monomorphisms but $g_1, g_2 : X \longrightarrow E \parr_b F$ correspond by Cartesian closedness to maps $X_b F'_b \longrightarrow E$ that are equal when composed with $f : E \longrightarrow G$ if f monomorphism, hence so is $f \parr_b id_F$. □

In general, we have just seen that ε has features for a negative connective as $\parr$, but it lacks associativity. We will have to work to recover a monoidal category, and then models of LL. In that respect, we want to make our fix of associativity compatible with a class of smooth maps, this will be the second leitmotiv. We don't know if there is an extension of the model of MALL given by **CLCS** into a model of LL using a kind of smooth maps.

B.1.2 Mackey-complete spaces and a first interpretation for $\parr$

Towards our goal of obtaining a model of LL with conveniently smooth maps as non-linear morphisms, it is natural to follow [53] and consider Mackey-complete spaces as in [6, KT]. In order to fix associativity of ε in this context, we will see appear the supplementary Schwartz condition. This is not such surprising as seen the relation with Mackey-completeness appearing for instance in [Ja, chap 10] which treats them simultaneously. This Schwartz space condition will enable to replace Arens duals by Mackey duals (lemma B.1.19) and thus simplify lots of arguments in identifying duals as Mackey-completions of inductive tensor products (lemma B.1.23). This will strongly simplify the construction of the strength for our doubly negation monad later in section B.1.4. Technically, this is possible by various results of [K] which points out a nice alternative tensor product η which replaces ε-product exactly in switching Arens with Mackey duals. But we need to combine η with $\mathscr{S}$ in a clever way in yet another product ζ in order to get an associative product. Said in words, this is a product which enables to ensure at least two Schwartz spaces among three in an associativity relation. This technicality is thus a reflection of the fact that for Mackey-complete spaces, one needs to have at least 2 Schwartz spaces among three to get a 3 term associator for an ε-product. In course of getting our associativity, we get the crucial relation $\mathscr{S}(\mathscr{S}(E)\varepsilon F) = \mathscr{S}(E)\varepsilon\mathscr{S}(F)$ in corollary B.1.24. This is surprising because this seems really specific to Schwartz spaces and we are completely unable to prove an analogue for the associated nuclear topology functor $\mathscr{N}$, even if we expect it for the less useful associated strongly nuclear topology. We conclude in Theorem B.1.26 with our first interpretation of $\parr$ as ζ.

B.1.2.1 A Mackey-Completion with continuous canonical map

Note that for a γ-Mackey-Cauchy sequence, topological convergence is equivalent to Mackey convergence (since the class of bounded sets is generated by bounded closed sets).

Remark B.1.15. Note also that is $E \subset F$ is a continuous inclusion, then a Mackey-Cauchy/convergent sequence in E is also Mackey-Cauchy/convergent in F since a linear map is bounded.

We now recall two alternative constructions of the Mackey-completion, from above by intersection and from below by union. The first construction is already considered in [PC].

Lemma B.1.16. The intersection $\widehat{E}^M$ of all Mackey-complete spaces containing E and contained in the completion $\tilde{E}$ of E, is Mackey-complete and called the Mackey-completion of E.

We define $E_{M;0} = E$, and for any ordinal λ, the subspace $E_{M;\lambda+1} = \cup_{(x_n)_{n\geqslant 0}\in M(E_{M;\lambda})}\overline{\Gamma(\{x_n, n \geqslant 0\})} \subset \tilde{E}$ where the union runs over all Mackey-Cauchy sequences $M(E_{M;\lambda})$ of $E_{M;\lambda}$, and the closure is taken in the completion. We also let for any limit ordinal $E_{M;\lambda} = \cup_{\mu<\lambda}E_{M;\mu}$. Then for any ordinal λ, $E_{M;\lambda} \subset \widehat{E}^M$ and eventually for $\lambda \geqslant \omega_1$ the first uncountable ordinal, we have equality.

Proof. The first statement comes from stability of Mackey-completeness by intersection (using remark B.1.15). It is easy to see that $E_{M;\lambda}$ is a subspace. At stage $E_{M;\omega_1+1}$, by uncountable cofinality of ω_1 any Mackey-Cauchy sequence has to be in $E_{M;\lambda}$ for some $\lambda < \omega_1$ and thus each term of the union is in some $E_{M;\lambda+1}$, therefore $E_{M;\omega_1+1} = E_{M;\omega_1}$.

Moreover if at some λ, $E_{M;\lambda+1} = E_{M;\lambda}$, then by definition, $E_{M,\lambda}$ is Mackey-complete (since we add with every sequence its limit that exists in the completion which is Mackey-complete) and then the ordinal sequence is eventually constant. Then, we have $E_{M,\lambda} \supset \widehat{E}^M$. One shows for any λ the converse by transfinite induction. For, let $(x_n)_{n\geqslant 0}$ is a Mackey-Cauchy sequence in $E_{M;\lambda} \subset F := \widehat{E}^M$. Consider A a closed bounded absolutely convex set in F with $x_n \longrightarrow x$ in F_A. Then by [Ja, Prop 10.2.1], F_A is a Banach space, thus $\overline{\Gamma(\{x_n, n \geqslant 0\})}$ computed in this space is complete and thus compact (since $\{x\} \cup \{x_n, n \geqslant 0\}$ is compact in the Banach space), thus its image in $\tilde{E}$ is compact and thus agrees with the closure computed there. Thus every element of $\overline{\Gamma(\{x_n, n \geqslant 0\})}$ is a limit in E_A of a sequence in $\Gamma(\{x_n, n \geqslant 0\}) \subset E_{M;\lambda}$ thus by Mackey-completeness, $\overline{\Gamma(\{x_n, n \geqslant 0\})} \subset F$. We thus conclude to the successor step $E_{M;\lambda+1} \subset \widehat{E}^M$, the limit step is obvious. □

B.1.2.2 A ⅋ for Mackey-complete spaces

We first define a variant of the Schwartz ϵ-product:

Definition B.1.17. For two separated lcs E and F, we define $E\eta F = L(E'_\mu, F)$ the space of continuous linear maps on the Mackey dual with the topology of uniform convergence on equicontinuous sets of E'. We write $\eta(E, F) = (E'_\mu \beta e F'_\mu)'$ with the topology of uniform convergence on products of equicontinuous sets and $\zeta(E, F) \equiv E\zeta F$ for the same space with the weakest topology making continuous the canonical maps to $\eta(\mathscr{S}(E), F)$ and $\eta(E, \mathscr{S}(F))$.

This space $E\eta F$ has already been studied in [K2] and we can summarize its properties similar to the Schwartz ϵ product in the next proposition, after a couple of lemmas.

We first recall an important property of the associated Schwartz topology from [45]. These properties follow from the fact that the ideal of compact operators on Banach spaces in injective, closed and surjective. Especially, from [45, Corol 6.3.9] it is an idempotent ideal.

Lemma B.1.18. The associated Schwartz topology functor $\mathscr{S}$ commutes with arbitrary products, quotients and embeddings (and as a consequence with arbitrary projective kernels or categorical limits).

Proof. For products and (topological) quotients, this is [45, Prop 7.4.2]. For embeddings (that he calls topological injections), this is [45, Prop 7.4.8] based on the previous ex 7.4.7. The consequence comes from the fact that any projective kernel is a subspace of a product, as a categorical limit is a kernel of a map between products. □

We will also often use the following relation with duals

Lemma B.1.19. If E is a Schwartz lcs, $E'_c \simeq E'_\mu$ so that for any lcs F, $E\eta F \simeq E\varepsilon F$ topologically. Thus for any lcs E, $E'_\mu \simeq (\mathscr{S}(E))'_c$.

Proof. Take K an absolutely convex $\sigma(E', E)$-weakly compact in E, it is an absolutely convex closed set in E and precompact as any bounded set in a Schwartz space [Ho, 3, 15 Prop 4]. [Bo2, IV.5 Rmq 2] concludes to K complete since $E \longrightarrow (E'_\mu)'_\sigma$ continuous with same dual and K complete in $(E'_\mu)'_\sigma$, and since K precompact, it is therefore compact in E. As a consequence E'_c is the Mackey topology. Hence, $E\eta F = L(E'_\mu, F) = L(E'_c, F) = E\varepsilon F$ algebraically and the topologies are defined in the same way. The last statement comes from the first and $E'_\mu \simeq (\mathscr{S}(E))'_\mu$. □

Proposition B.1.20. *Let E, F, G, H, be separated lcs, then:*

1. *We have a topological canonical isomorphisms $\zeta(E, F) = \zeta(F, E)$,*

$$E\eta F = F\eta E \simeq \eta(E, F)$$

and we have a continuous linear map $E\varepsilon F \longrightarrow E\eta F \longrightarrow E\zeta F$ which is a topological isomorphism as soon as either E or F is a Schwartz space. In general, $E\varepsilon F$ is a closed subspace of $E\eta F$.

2. *$E\eta F$ is complete if and only if E and F are complete.*

3. *If $A : G \longrightarrow E$, $B : H \longrightarrow F$ are linear continuous (resp. linear continuous one-to-one, resp. embeddings) so are the tensor product map $(A\eta B), (A\zeta B)$ both defined by $(A\eta B)(f) = B \circ f \circ A^t$, $f \in L(E'_\mu, F)$.*

4. *If $F = \mathrm{K}_{i\in I}(A_i)^{-1}F_i$ is a projective kernel so are $E\eta F = \mathrm{K}_{i\in I}(1\eta A_i)^{-1}E\eta F_i$ and $E\zeta F = \mathrm{K}_{i\in I}(1\zeta A_i)^{-1}E\zeta F_i$. Moreover, both η, ζ, ε commute with categorical limits in* **LCS**.

5. *$\eta(E, F)$ is also the set of bilinear forms on $E'_\mu \times F'_\mu$ which are separately continuous. As a consequence, the Mackey topology $((E'_\mu{}_{\beta e}F'_\mu)'_\mu)'_\mu = E'_\mu{}_i F'_\mu$ is the inductive tensor product.*

6. *A set is bounded in $\eta(E, F)$ or $\zeta(E, F)$ if and only if it is ϵ-equihypocontinuous on $E'_\beta \times F'_\beta$.*

Proof. It is crucial to note that $\eta(E, F), \eta(\mathscr{S}(E), F), \eta(E, \mathscr{S}(F))$ are the same space algebraically since $(\mathscr{S}(E))'_\mu \simeq E'_\mu$.

(2) is [K, 40.4.(5)] and (1) is similar to the first statement there. With more detail functoriality of Mackey dual gives a map $L(E'_\mu, F) \longrightarrow L(F'_\mu, (E'_\mu)'_\mu)$ and since $(E'_\mu)'_\mu \longrightarrow E$ continuous we have also a map $L(F'_\mu, (E'_\mu)'_\mu) \longrightarrow L(F'_\mu, E)$. This explains the first map of the first isomorphism (also explained in [Ja, Corol 8.6.5]). The canonical linear map from a bilinear map in $\eta(E, F)$ is clearly in $E\eta F$, conversely, if $A \in E\eta F$, $\langle A(.), .\rangle_{F,F'}$ is right ϵ-hypocontinuous by definition and the other side of the hypocontinuity comes from the $A^t \in F\eta E$.

The closed subspace property is [K2, 43.3.(4)].

(3) for η is [K2, 44.4.(3,5,6)]. For ζ since A, B are continuous after taking the functor $\mathscr{S}$, one deduces $A\eta B$ is continuous (resp one-to-one, resp. an embedding using lemma B.1.18) on

$$\eta(\mathscr{S}(G), H) \longrightarrow \eta(\mathscr{S}(E), F), \ \ \eta(G, \mathscr{S}(H)) \longrightarrow \eta(E, \mathscr{S}(F))$$

and this conclude by universal properties of projective kernels (with two terms) for ζ. Since the spaces are the same algebraically, the fact that the maps are one-to-one also follows.

(4) The η case with kernels is a variant of [K2, 44.5.(4)] which is also a direct application of [K2, 39.8.(10)]. As a consequence $E\zeta F$ is a projective kernel of $\mathscr{S}(E)\eta F = \mathrm{K}_{i\in I}(1\eta A_i)^{-1}[\mathscr{S}(E)]\eta F_i$ and, using lemma B.1.18 again, of :

$$E\eta\mathscr{S}(F) = E\eta\Big(\mathrm{K}_{i\in I}A_i^{-1}\mathscr{S}(F_i)\Big) = \mathrm{K}_{i\in I}(1\eta A_i)^{-1}\big(E\eta\mathscr{S}(F_i)\big).$$

The transitivity of locally convex kernels (coming from their universal property) concludes.

For categorical limits, it suffices commutation with products and kernels. In any case the continuous map $I : (\lim E_i)\eta F \longrightarrow \lim(E_i\eta F)$ comes from universal properties , it remains to see it is an algebraic isomorphism, since then the topological isomorphism will follow from the kernel case. We build the inverse as follows, for $f \in F'_\mu$, the continuous evaluation map $E_i\eta F = L(F'_\mu, E_i) \longrightarrow E_i$ induces a continuous linear map $J_f : \lim(E_i\eta F) \longrightarrow (\lim E_i)$. It is clearly linear in f and gives a bilinear map $J : \lim(E_i\eta F) \times F'_\mu \longrightarrow (\lim E_i)$. We have to see it is separately continuous yielding a linear inverse map I^{-1} and then continuity of this map. We divide into the product and kernel case.

For products one needs for $g \in \prod_{i\in I}(E_i\eta F)$ $J(g, .)^t : (\prod_{i\in I} E_i)' \longrightarrow (F'_\mu)'$ send equicontinuous sets i.e. a finite sum of equicontinuous set in the sum $\sum_{i\in I} E'_i$ to an equicontinuous set in $(F'_\mu)'$. But absolutely convex weakly compact sets are stable by bipolars of sum, since they are stable by bipolars of finite unions [K, 20.6.(5)] (they don't even need closure to be compact, absolutely convex cover is enough), hence it suffices to see the case of images of equicontinuous sets $E'_i \longrightarrow F$ but they are equicontinuous by assumption. This gives the separate continuity in this case. Similarly, to see the continuity of I^{-1} in this case means that we take $A \subset F'$ equicontinuous and a sum of equicontinuous sets B_i in $(\prod_{i\in I} E_i)'$ and one notices that $(I^{-1})^t(A \times \sum B_i) \subset \sum(I^{-1})^t(A \times B_i)$ is a sum of equicontinuous sets in $(\prod_{i\in I}(E_i\eta F))'$ and it is by hypothesis equicontinuous.

For kernels, of $f, g : E \longrightarrow G$, $I : Ker(f - g)\eta F \longrightarrow Ker(f\eta id_F - g\eta id_F)$ is an embedding by (3) since source and target are embeddings in $E\eta F$, the separate continuity is obtained by restriction of the one of $E\eta F \times F'_\mu \longrightarrow E \supset Ker(f - g)$ and similarly continuity by restriction of $E\eta F \longrightarrow L(F'_\mu, E)$.

For ζ, this is then a consequence of this and lemma B.1.18 again. (5) is an easier variant of [S, Rmq 1 p 25]. Of course $\eta(E, F)$ is included in the space of separately continuous forms. Conversely, if $f : E'_\mu \times F'_\mu \longrightarrow \mathcal{K}$ is separately continuous, from [Ja, Corol 8.6.5], it is also separately continuous on $E'_\sigma \times F'_\sigma$ and the non-trivial implication follows from [K, 40.4.(5)]. For the second part, the fact that both algebraic tensor products have the same dual implies there is, by Arens-Mackey Theorem, a continuous identity map $((E'_\mu{}_{\beta e}F'_\mu)'_\mu)'_\mu \longrightarrow E'_\mu{}_i F'_\mu$. Conversely, one uses the universal property of the inductive tensor product which gives a separately continuous map $E'_\mu \times F'_\mu \longrightarrow E'_\mu{}_{\beta e}F'_\mu$. But applying functoriality of Mackey duals on each side gives for each $x \in E'_\mu$ a continuous map $F'_\mu \longrightarrow ((E'_\mu{}_{\beta e}F'_\mu)'_\mu)'_\mu$ and by symmetry, a separately continuous map $E'_\mu \times F'_\mu \longrightarrow ((E'_\mu{}_{\beta e}F'_\mu)'_\mu)'_\mu$. The universal property of the inductive tensor product again concludes.

(6) can be obtained for η with the same reasoning as in [K2, 44.3.(1)]. For ζ the first case gives by definition equivalence with ϵ-equihypocontinuity both on $(\mathscr{S}(E))'_\beta \times F'_\beta$ and on $E'_\beta \times (\mathscr{S}(F))'_\beta$. But the second implies that

for equicontinuous on E', one gets an equicontinuous family on $(\mathscr{S}(F))'_\beta \simeq F'_\beta$ and the first gives the converse, and the other conditions are weaker, hence the equivalence with the first formulation. $\square$

We then deduce a Mackey-completeness result:

Proposition B.1.21. *If L_1 and L_2 are separated Mackey-complete locally convex spaces, then so are $L_1\eta L_2$ and $L_1\zeta L_2$.*

Proof. Since both topologies on the same space have the same bounded sets (proposition B.1.20.(6)), it suffices to consider $L_1\eta L_2$. Consider a Mackey-Cauchy sequence $(x_n)_{n\geqslant 0}$, thus topologically Cauchy. By completeness of the scalar field, x_n converges pointwise to a multilinear form x on $\prod_{i=1}^2 (L_i)'_\mu$. Since the topology of the η-product is the topology of uniform convergence on products of equicontinuous parts (which can be assumed absolutely convex and weakly compact), $x_n \longrightarrow x$ uniformly on these products (since (x_n) Cauchy in the Banach space of continuous functions on these products). From proposition B.1.20.(5) we only have to check that the limit x is separately continuous. For each $y \in (L_2)'$, and B a bounded set in $L_1\eta L_2 = L((L_2)'_\mu, L_1)$, one deduces $B(y)$ is bounded in $((L_1)'_\mu)' = L_1$ with its original topology of convergence on equicontinuous sets of L_1'. Therefore, $(x_n(y))$ is Mackey-Cauchy in L_1, thus Mackey-converges, necessarily to $x(y)$. Therefore $x(y)$ defines an element of $((L_1)'_\mu)'$. With the similar symmetric argument, x is thus separately continuous, as expected. We have thus obtained the topological convergence of x_n to x in $L_1\eta L_2$. It is easy to see x_n Mackey converges to x in $L_1\eta L_2$ in taking the closure of the bounded set from its property of being Mackey-Cauchy. Indeed, the established topological limit $x_n \longrightarrow x$ transfers the Mackey-Cauchy property in Mackey convergence as soon as the bounded set used in Mackey convergence is closed. $\square$

We will need the relation of Mackey duals and Mackey completions:

Lemma B.1.22. For any separated lcs F, we have a topological isomorphism $\widehat{((F'_\mu)'_\mu)}^M \simeq ((\widehat{F}^M)'_\mu)'_\mu$.

Proof. Recall also from [K, 21.4.(5)] the completion of the Mackey topology has its Mackey topology $(\widetilde{(F'_\mu)'_\mu}) = ((\widetilde{F})'_\mu)'_\mu$ therefore an absolutely convex weakly compact set in F' coincide for the weak topologies induced by F and $\widetilde{F}$ and therefore also $\hat{F}^M$, which is in between them. Thus the continuous inclusions $((F'_\mu)'_\mu) \longrightarrow (\hat{F}^M)'_\mu)'_\mu \longrightarrow ((\widetilde{F})'_\mu)'_\mu$ have always the induced topology. In the transfinite description of the Mackey completion, the Cauchy sequences and the closures are the same in $((\widetilde{F})'_\mu)'_\mu$ and $\widetilde{F}$ (since they have same dual hence same bounded sets), therefore one finds the stated topological isomorphism. $\square$

Lemma B.1.23. If L, M are separated locally convex spaces we have embeddings:

$$L'_\mu{}_{\beta e} M'_\mu \longrightarrow (L\zeta M)'_\epsilon \longrightarrow (L\eta M)'_\epsilon \longrightarrow L'_\mu \widetilde{{}_{\beta e}} M'_\mu,$$

with the middle duals coming with their ϵ-topology as biduals of $L'_\mu{}_{\beta e} M'_\mu$. The same holds for:

$$L'_\mu{}_i M'_\mu \longrightarrow (L\eta M)'_\mu \longrightarrow L'_\mu \widetilde{{}_i} M'_\mu.$$

Finally, $(L\eta M)'_\epsilon \subset L'_\mu \widetilde{{}_{\beta e}}^M M'_\mu$ as soon as either L or M is a Schwartz space, and in any case we have $(L\zeta M)'_\epsilon \subset L'_\mu \widetilde{{}_{\beta e}}^M M'_\mu$.

Proof. **Step 1:** First line of embeddings.

From the identity continuous map $L\eta M \longrightarrow L\zeta M$, there is an injective linear map $(L\zeta M)' \longrightarrow (L\eta M)'$. Note that, on $L'_\mu M'_\mu$, one can consider the strongest topology weaker than $(\mathscr{S}(L))'_\mu{}_{\beta e} M'_\mu$ and $L'_\mu{}_{\beta e}(\mathscr{S}(M))'_\mu$. Let us call it $L'_\mu{}_\zeta M'_\mu$ and see it is topologically equal to $L'_\mu{}_{\beta e} M'_\mu$ by checking its universal property. We know by definition the map $L'_\mu{}_\zeta M'_\mu \longrightarrow L'_\mu{}_{\beta e} M'_\mu$. Conversely, there is an ε-equihypocontinuous map $(\mathscr{S}(L))'_\mu \times M'_\mu \longrightarrow L'_\mu{}_\zeta M'_\mu$ so that for every equicontinuous set in M', the corresponding family is equicontinuous $(\mathscr{S}(L))'_\mu = L'_\mu \longrightarrow L'_\mu{}_\zeta M'_\mu$ from the topological equality. Similarly, by symmetry, one gets for every equicontinuous set in L'_μ, an equicontinuous family of maps $M'_\mu \longrightarrow L'_\mu{}_\zeta M'_\mu$. As a consequence, the universal property gives the expected map $L'_\mu{}_{\beta e} M'_\mu \longrightarrow L'_\mu{}_\zeta M'_\mu$ concluding to equality. As a consequence, since by definition $(L'_\mu{}_\zeta M'_\mu)' = L\zeta M$ is the dual kernel for the hull defining the $_\zeta$ tensor product, one gets that an equicontinuous set in the kernel is exactly an equicontinuous set in $(L'_{\mu\zeta}M'_\mu)' = (L'_{\mu\beta e}M'_\mu)'$ namely an ϵ-equihypocontinuous family. This gives the continuity of our map $(L\zeta M)'_\epsilon \longrightarrow (L\eta M)'_\epsilon$ and even the embedding property (if we see the first as bidual of $L'_{\mu\zeta}M'_\mu$ but we only stated an obvious embedding in the statement).

We deduce that $L'_{\mu\beta e}M'_\mu \simeq L'_{\mu\zeta}M'_\mu \longrightarrow (L\zeta M)'_\epsilon$ is an embedding from [K, 21.3.(2)] which proves that the original topology on a space is the topology of uniform convergence on equicontinuous sets.

We then build a continuous linear injection $(L\eta M)'_\epsilon \longrightarrow L'^{\sim}_{\mu\beta e}M'_\mu$ to the full completion.Since both spaces have the same dual, it suffices to show that the topology on $L\eta M$ is stronger than Grothendieck's topology $\mathfrak{I}^{lf}(L'_{\mu\beta e}M'_\mu)$ following [K] in notation. Indeed, let C in $L\eta M$ equicontinuous. Assume a net in C converges pointwise $x_n \longrightarrow x \in C$ in the sense $x_n(a,b) \longrightarrow x(a,b), a \in L'_\mu, b \in M'_\mu$. For equicontinuous sets $A \subset L'_\mu, B \subset M'_\mu$ which we can assume absolutely convex weakly compact, it is easy to see C is equicontinuous on products $A \times B$. Thus it is an equicontinuous bounded family in $C^0(A \times B)$ thus relatively compact by Arzela-Ascoli Theorem [Ho, 3.9 p237]. Thus since any uniformly converging subnet converges to x, the original net must converge uniformly on $A \times B$ to x. As a consequence the weak topology on C coincides with the topology of $L\eta M$, and by definition we have a continuous identity map, $(L\eta M, \mathcal{I}^{lf}(L'_{c\beta e}M'_c)) \longrightarrow L\eta M$. By Grothendieck's construction of the completion, the dual of the first space is the completion and this gives the expected injection between duals. Since a space and its completion induce the same equicontinuous sets, one deduces the continuity and induced topology property with value in the full completion.

Step 2: Second line of embeddings.

It suffices to apply $((.)'_\mu)'_\mu$ to the first line. We identified the first space in proposition B.1.20.(5) and the last space as the completion of the first (hence of the second and this gives the induced topologies) in the proof of lemma B.1.22.

Step 3: Reduction of computation of Mackey completion to the Schwartz case.

It remains to see the $(L\zeta M)'_\epsilon$ is actually valued in the Mackey completion.

Note that as a space, dual of a projective kernel, $(L\zeta M)'_\epsilon$ is the inductive hull of the maps $A = (\mathscr{S}(L)\varepsilon M)'_\epsilon \longrightarrow (L\zeta M)'_\epsilon = C$ and $B = (L\varepsilon\mathscr{S}(M))'_\epsilon \longrightarrow (L\zeta M)'_\epsilon = C$. Therefore, it suffices to check that the algebraic tensor product is Mackey-dense in both these spaces A, B that span C since the image of a bounded set in A, B being bounded in C, there are less Mackey-converging sequences in A, B. This reduces the question to the case L or M a Schwartz space. By symmetry, we can assume L is.

Step 4: Description of the dual $(L\zeta M)' = (L\varepsilon M)'$ for L Schwartz and conclusion.

We take inspiration from the classical description of the dual of the injective tensor product as integral bilinear maps (see [K2, 45.4]). As in [S, Prop 6], we know any equicontinuous set (especially any point) in $(L\varepsilon M)'$ is included in the absolutely convex weakly closed hull Γ of AB with A equicontinuous in L', B in M'. Since the dual of $L'^{\sim M}_{\mu\beta e}M'_\mu$ is the same, this weakly closed hull can be computed in this space too. Moreover, since L is a Schwartz space, we can and do assume that $A = \{x_n, n \in \mathcal{N}\}$ is a ϵ-Mackey-null sequence in L'_μ, since they generate the equicontinuous bornology as a saturated bornology. We can also assume A, B are weakly compact and B absolutely convex.

Any element $f \in L\varepsilon M$ defines a continuous map on $A \times B$ (see e.g. [S, Prop 2] and following remark). We equip $A \times B$ with the above weakly compact topology to see $f|_{A\times B} \in C^0(A \times B)$. For μ a (complex) measure on $A \times B$ (i.e. $\mu \in (C^0(A \times B))'$, we use measures in the Bourbaki's sense, which define usual Radon measures [S4]) with norm $||\mu|| \leqslant 1$ so that $\int_{A\times B} f(z)d\mu(z) = \mu(f|_{A\times B}) =: w_\mu(f)$ make sense.

Note that $|w_\mu(f)| \leqslant ||f||_{C^0(A\times B)}$ which is a seminorm of the ϵ-product, so that μ defines a continuous linear map $w_\mu \in (L\varepsilon M)'$. Note also that if f is in the polar of AB, so that $|w_\mu(f)| \leqslant 1$ and thus by the bipolar theorem, $w_\mu \in \Gamma$. We want to check the converse that any element of $w \in \Gamma$ comes from such a measure. But if H is the subspace of $C^0(A \times B)$ made of restrictions of functions $f \in L\varepsilon M$, w induces a continuous linear map on H with $|w(f)| \leqslant ||f||_{C^0(A\times B)}$, Hahn-Banach theorem enables to extend it to a measure w_μ, $||\mu|| \leqslant 1$. This concludes to the converse.

Define the measure μ_n by $\int_{A\times B} f(z)d\mu_n(z) = \frac{1}{\mu(1_{\{x_n\}\times B})}\int_{A\times B} f(z)1_{\{x_n\}\times B}(z)d\mu(z)$ using its canonical extension to semicontinuous functions. Note that by Lebesgue theorem (dominated by constants)

$$w_\mu(f) = \sum_{n=0}^{\infty} \mu(1_{\{x_n\}\times B}))w_{\mu_n}(f)$$

As above one sees that the restriction of w_{μ_n} to $f \in L\varepsilon M$ belongs to the weakly closed absolute convex hull of $\{x_n\} \times B$. Thus since B absolutely convex closed $w_{\mu_n}(f) = f(x_n \otimes y_n)$ for some $y_n \in B$. We thus deduces that any $w_\mu \in \Gamma$ has the form:$w_\mu(f) = \sum_{n=0}^{\infty} \mu(1_{\{x_n\}\times B}))f(x_n \otimes y_n)$. Since the above convergence holds for any f, this means the convergence in the weak topology:

$$w_\mu = \sum_{n=0}^{\infty} \mu(1_{\{x_n\}\times B}))x_n \otimes y_n. \tag{B.4}$$

Let D the equicontinuous closed disk such that x_n tends to 0 in $(L')_D$. Consider the closed absolutely convex cover $\Lambda = \overline{\Gamma(DB)}$. The closed absolutely convex cover can be computed in $(L\varepsilon M)'_\epsilon$ or $(L\varepsilon M)'_\sigma$, both spaces having same dual [K, 20.7.(6) and 8.(5)], and DB being equicontinuous [S, Corol 4 p 27, Rmq p 28], so is Λ [K, 21.3.(2)] hence it is weakly compact by Mackey Theorem, so complete in $(L\varepsilon M)'_\epsilon$ [Bo2, IV.5 Rmq 2], so that Λ is therefore a Banach disk there. But $||x_n y_n||_{(L\varepsilon M)'_\Lambda} \leqslant 1$ so that since $\sum_{n=0}^{\infty} |\mu(1_{\{x_n\}\times B}))| \leqslant 1$ the above series is summable in $(L\varepsilon M)'_\Lambda$ and thus Mackey converges in $(L\varepsilon M)'_\epsilon$. As a conclusion, $\Gamma \subset L'^{\frown M}_{\mu\beta e}M'_\mu$ and this gives the final statement. □

The above proof has actually the following interesting consequence:

Corollary B.1.24. For any E, F separated locally convex spaces, we have the topological isomorphism:

$$\mathscr{S}\Big([\mathscr{S}(E)]\varepsilon F\Big) = [\mathscr{S}(E)]\varepsilon[\mathscr{S}(F)].$$

Proof. We have the canonical continuous map $[\mathscr{S}(E)]\varepsilon F \longrightarrow [\mathscr{S}(E)]\varepsilon[\mathscr{S}(F)]$, hence since the ε-product of Schwartz spaces is Schwartz (see below proposition B.1.50), one gets by functoriality the first continuous linear map:

$$\mathscr{S}\Big([\mathscr{S}(E)]\varepsilon F\Big) \longrightarrow [\mathscr{S}(E)]\varepsilon[\mathscr{S}(F)]. \tag{B.5}$$

Note that we have the algebraic equality $[\mathscr{S}(E)]\varepsilon F = L(E'_\mu, F) = L(E'_\mu, \mathscr{S}(F)) = [\mathscr{S}(E)]\varepsilon[\mathscr{S}(F)]$ where the crucial middle equality comes from the map (see [Ja, Corol 8.6.5])

$$L(E'_\mu, F) = L(E'_\mu, (F'_\mu)'_\mu) = L(E'_\mu, ([\mathscr{S}(F)]'_\mu)'_\mu) = L(E'_\mu, \mathscr{S}(F)).$$

To prove the topological equality, we have to check the duals are the same with the same equicontinuous sets. We can apply the proof of the previous lemma (and we reuse the notation there) with $L = \mathscr{S}(E)$, $M = F$ or $M = [\mathscr{S}(F)]$. First the space in which the Mackey duals are included $L'^{\sim}_{\mu i}M'_\mu$ is the same in both cases, and the duals are described as union of absolutely convex covers, it suffices to see those unions are the same to identify the duals. Of course, the transpose of (B.5) gives $\Big([\mathscr{S}(E)]\varepsilon[\mathscr{S}(F)]\Big)' \subset \Big([\mathscr{S}(E)]\varepsilon F\Big)'$ so that we have to show the converse. From (B.4) and rewriting $x_n y_n$ as $\frac{1}{\lambda_n}x_n\lambda_n y_n$ with $\lambda_n = \sqrt{||x_n||_{L'_C}}$, one gets that both sequences $x'_n = (\frac{1}{\lambda_n}x_n), y'_n(\mu) = (\lambda_n y_n)$ are null sequences for the equicontinuous bornology of E', F' and therefore included in equicontinuous sets for the duals of associated Schwartz spaces. This representation therefore gives the equality of duals. Finally, to identify equicontinuous sets, in the only direction not implied by (B.5), we must see that an ε-null sequence w_{ν_n} of linear forms in the dual is included in the closed absolutely convex cover of a tensor product of two such sequences in $[\mathscr{S}(E)]', [\mathscr{S}(F)]'$. From the null convergence, ν_n can be taken measures on the same $A \times B$, for each ν_n, we have a representation $w_{\nu_n} = \sum z_m(\nu_n)x'_m y'_m(\nu_n)$ where x'_m is a fixed sequence and $(y'_m(\nu_n))_m$ are null sequences in the same Banach space M'_B. Moreover $\sum |z_m(\nu_n)| \leqslant ||\nu_n|| \longrightarrow 0$ from the assumption that ν_n is a null sequence in the Banach space generated by Γ (we can assume $||\nu_n|| \neq 0$ otherwise $w_{\nu_n} = 0$). Therefore, we rewrite, the series as $w_{\nu_n} = \sum \frac{1}{||\nu_n||}z_m(\nu_n)x'_m y'_m(\nu_n)||\nu_n||$ and we gather all the sequence $(y'_m(\nu_n)||\nu_n||)_m$ into a huge sequence converging to 0 in M'_B which generates the equicontinuous set B' of $(\mathscr{S}(M))'$ we wanted. (x'_m) generates another such equicontinuous set A'. This concludes to $w_{\nu_n} \in \overline{\Gamma(A'B')}$ so that the equicontinuous set generated by our sequence (w_{ν_n}) must be in this equicontinuous set for $([\mathscr{S}(E)]\varepsilon[\mathscr{S}(F)])'$. □

We are ready to obtain the associativity of the ζ tensor product:

Proposition B.1.25. *Let L_1, L_2, L_3 be lcs with L_3 Mackey-complete, then there is a continuous linear map*

$$Ass : L_1\zeta(L_2\zeta L_3) \longrightarrow (L_1\zeta L_2)\zeta L_3.$$

If also L_1 is Mackey-complete, this is a topological isomorphism.

Proof. First note that we have the inclusion

$$L_1\zeta(L_2\zeta L_3) \subset L((L_1)'_\mu, L_\sigma((L_2)'_\mu, L_3)) = L((L_1)'_\mu i (L_2)'_\mu, L_3).$$

Since L_3 is Mackey-complete, such a map extends uniquely to the Mackey completion $L((L_1)'_\mu \widehat{i}^M (L_2)'_\mu, L_3)$ and since lemma B.1.23 gives $(L_1\eta L_2)'_\mu$ as a subspace, we can restrict the unique extension and get our expected linear map:

$$i : L_1\zeta(L_2\zeta L_3) \longrightarrow L((L_1\eta L_2)'_\mu, L_3) = (L_1\eta L_2)\eta L_3.$$

It remains to check continuity. Since the right hand side is defined as a topological kernel, we must check continuity after applying several maps. Composing with

$$J_1 : (L_1\eta L_2)\eta L_3 \longrightarrow (L_1\varepsilon\mathscr{S}(L_2))\varepsilon\mathscr{S}(L_3),$$

one gets a map $J_1 \circ i$ which is continuous since it coincides with the composition of the map obtained from corollary B.1.24:

$$I_1 : L_1\zeta(L_2\zeta L_3) \longrightarrow L_1\varepsilon\mathscr{S}(\mathscr{S}(L_2)\varepsilon L_3) = L_1\varepsilon(\mathscr{S}(L_2)\varepsilon\mathscr{S}(L_3))$$

with a variant $i' : L_1\varepsilon(\mathscr{S}(L_2)\varepsilon\mathscr{S}(L_3)) \longrightarrow (L_1\varepsilon\mathscr{S}(L_2))\varepsilon\mathscr{S}(L_3)$ of i in the Schwartz case, so that $i' \circ I_1 = i \circ J_1$. And i' is continuous since the equicontinuous set in their duals are generated by tensor products of equicontinuous sets for the base spaces (easy part in the corresponding associativity in [S]). The case of composition with $J_2 : (L_1\eta L_2)\eta L_3 \longrightarrow (\mathscr{S}(L_1)\varepsilon L_2)\varepsilon\mathscr{S}(L_3)$ is similar and easier.

The last two compositions are gathered in one using corollary B.1.24 again. We have to compose with the map

$$J_3 : (L_1\eta L_2)\eta L_3 \longrightarrow \Big(\mathscr{S}\big(\mathscr{S}(L_1)\varepsilon L_2\big)\Big)\varepsilon L_3 = \Big((\mathscr{S}(L_1))\varepsilon(\mathscr{S}(L_2))\Big)\varepsilon L_3 = \Big(\mathscr{S}\big(L_1\varepsilon(\mathscr{S}(L_2))\big)\Big)\varepsilon L_3.$$

Again we use the canonical continuous factorization via $L_1\zeta(L_2\zeta L_3) \longrightarrow (\mathscr{S}(L_1))\varepsilon\Big((\mathscr{S}(L_2))\varepsilon L_3\Big)$ and use the same argument as before between ε-products. $\square$

We can now summarize the categorical result obtained, which gives a negative connective, hence an interpretation of $⅋$.

Theorem B.1.26. *The full subcategory $\mathbf{Mc} \subset \mathbf{LCS}$ of Mackey-complete spaces is a reflective subcategory with reflector (i.e. left adjoint to inclusion) the Mackey completion $\widehat{\cdot}^M$. It is complete and cocomplete and symmetric monoidal with product ζ which commutes with limits.*

Proof. The left adjoint relation $\mathbf{Mc}(\widehat{E}^M, F) = \mathbf{LCS}(E, F)$ is obvious by restriction to $E \subset \widehat{E}^M$ and functoriality of $\widehat{\cdot}^M$ [PC, Prop 5.1.25]. As usual, naturality is easy. As a consequence, limits in $\mathbf{Mc}$ are those of $\mathbf{LCS}$ and colimits are the Mackey-completed colimits. The unit for ζ is of course $\mathcal{K}$. The associator has been built in Proposition B.1.25. With $E\zeta F = L(E'_\mu, F)$, we saw the braiding is the transpose map, left unit λ_F is identity and right unit is identification $\rho_E : (E'_\mu)'_\epsilon \simeq E$. Taking the Mackey-dual of expected maps in relations (pentagon, triangle and hexagon identities) one gets the transposed relations, which restrict to the known relations for $(\mathbf{LCS}, {}_i)$ as symmetric monoidal category. By Mackey-density obtained in proposition B.1.23, the relations extend to the expected relations for the transpose maps. Hence, transposing again (i.e. applying functor $(\cdot)'_\epsilon$ from dual spaces with linear maps preserving equicontinuous sets to $\mathbf{LCS}$) imply the expected relations. We already saw in lemma B.1.20 the commutation of limits with ζ. $\square$

B.1.3 Original setting for the Schwartz ε-product and smooth maps.

In his original paper [S], Schwartz used quasi-completeness as his basic assumption to ensure associativity, instead of restricting to Schwartz spaces and assuming only Mackey-completeness as we will do soon inspired by section 3. Actually, what is really needed is that the absolutely convex cover of a compact set is still compact. Indeed, as soon as one takes the image (even of an absolutely convex) compact set by a continuous bilinear map, one gets only what we know from continuity, namely compactness and the need to recover absolutely convex sets, for compatibility with the vector space structure, thus makes the above assumption natural. Since this notion is related to compactness and continuity, we call it k-quasi-completeness.

This small remark reveals this notion is also relevant for differentiability since it is necessarily based on some notion of continuity, at least at some level, even if this is only on $\mathscr{R}^n$ as in convenient smoothness. Avoiding the technical use of Schwartz spaces for now and benefiting from [S], we find a $*$-autonomous category and an adapted notion of smooth maps.

We will see this will give us a strong notion of differentiability with Cartesian closedness. We will come back to convenient smoothness in the next sections starting from what we will learn in this basic example with a stronger notion of smoothness.

B.1.3.1 $*$-autonomous category of k-reflexive spaces.

Definition B.1.27. A (separated) locally convex space E is said to be k-quasi-complete, if for any compact set $K \subset E$, its closed absolutely convex cover $\overline{\Gamma(K)}$ is complete (equivalently compact [K, 20.6.(3)]). We denote by $\mathbf{Kc}$ the category of k-quasi-complete spaces and linear continuous maps.

Remark B.1.28. There is a k-quasi-complete space which is not quasi-complete, hence our new notion of k-quasi-completeness does not reduce to the usual notion. Indeed in [78], is built a completely regular topological space W such that $C^0(W)$ with compact-open topology is bornological and such that it is an hyperplane in its completion, which is not bornological. If $C^0(W)$ were quasi-complete, it would be complete by [Ja, Corol 3.6.5] and this is not the case. $C^0(W)$ is k-quasi-complete since by Ascoli Theorem twice [Bo, X.17 Thm 2] a compact set for the compact open topology is pointwise bounded and equicontinuous, hence so is the absolutely closed convex cover of such a set, which is thus compact too.

The following result is similar to lemma B.1.16 and left to the reader.

Lemma B.1.29. The intersection $\widehat{E}^K$ of all k-quasi-complete spaces containing E and contained in the completion $\tilde{E}$ of E, is k-quasi-complete and called the k-quasi-completion of E.

We define $E_0 = E$, and for any ordinal λ, the subspace $E_{\lambda+1} = \cup_{K\in C(E_\lambda)}\overline{\Gamma(K)} \subset \tilde{E}$ where the union runs over all compact subsets $C(E_\lambda)$ of E_λ with the induced topology, and the closure is taken in the completion. We also let for any limit ordinal $E_\lambda = \cup_{\mu<\lambda}E_\mu$. Then for any ordinal λ, $E_\lambda \subset \widehat{E}^K$ and eventually for λ large enough, we have equality.

Definition B.1.30. For a (separated) locally convex space E, the topology $k(E', E)$ on E' is the topology of uniform convergence on absolutely convex compact sets of $\widehat{E}^K$. The dual $(E', k(E', E)) = (\widehat{E}^K)'_c$ is nothing but the Arens dual of the k-quasi-completion and is written E'_k. We let $E^*_k = \widehat{E'_k}^K$. A (separated) locally convex space E is said k-reflexive if E is k-quasi-complete and if $E = (E'_k)'_k$ topologically. Their category is written $k - \mathbf{Ref}$.

From Mackey theorem, we know that $(E'_k)' = (E^*_k)' = \widehat{E}^k$.

We first want to check that $k - \mathbf{Ref}$ is logically relevant in showing that $(E'_k)'_k$ and E^*_k are always in it. Hence we will get a k-reflexivization functor. This is the first extension of the relation $E'_c = ((E'_c)'_c)'_c$ that we need.

We start by proving a general lemma we will reuse several times. Of course to get a $*$-autonomous category, we will need some stability of our notions of completion by dual. The following lemma says that if a completion can be decomposed by an increasing ordinal decomposition as above and that for each step the duality we consider is sufficiently compatible in terms of its equicontinuous sets, then the process of completion in the dual does not alter any kind of completeness in the original space.

Lemma B.1.31. Let D a contravariant duality functor on $\mathbf{LCS}$, meaning that algebraically $D(E) = E'$. We assume it is compatible with duality ($(D(E))' = E$). Let $E_0 \subset E_\lambda \subset \widetilde{E_0}$ an increasing family of subspaces of the completion $\widetilde{E_0}$ indexed by ordinals $\lambda \leqslant \lambda_0$. We assume that for limit ordinals $E_\lambda = \cup_{\mu<\lambda}E_\mu$ and, at successor ordinals that every point $x \in E_{\lambda+1}$ lies in $\overline{\Gamma(L)}$, for a set $L \subset E_\lambda$, equicontinuous in $[D(E_{\lambda_0})]'$.

Then any complete set K in $D(E_0)$ is also complete for the stronger topology of $D(E_{\lambda_0})$.

Proof. Let $E = E_0$. Note that since $D(E) = D(\widetilde{E})$ we have $D(E) = D(E_\lambda)$ algebraically.

Take a net $x_n \in K$ which is a Cauchy net in $D(E_{\lambda_0})$. Thus $x_n \longrightarrow x \in K$ in $D(E_0)$. We show by transfinite induction on λ that $x_n \longrightarrow x$ in $D(E_\lambda)$.

First take λ limit ordinal. The continuous embeddings $E_\mu \longrightarrow E_\lambda$ gives by functoriality a continuous identity map $D(E_\lambda) \longrightarrow D(E_\mu)$ for any $\mu < \lambda$. Therefore since we know $x_n \longrightarrow x$ in any $D(E_\mu)$ the convergence takes place in the projective limit $D_\lambda = \mathrm{proj}\lim_{\mu<\lambda} D(E_\mu)$.

But we have a continuous identity map $D(E_\lambda) \longrightarrow D_\lambda$ and both spaces have the same dual $E_\lambda = \cup_{\mu<\lambda} E_\mu$. For any equicontinuous set L in $(D(E_\lambda))'$ x_n is Cauchy thus converges uniformly in $C^0(L)$ on the Banach space of weakly continuous maps. It moreover converges pointwise to x, thus we have uniform convergence to x on any equicontinuous set i.e. $x_n \longrightarrow x$ in $D(E_\lambda)$.

Let us prove convergence in $D(E_{\lambda+1})$ at successor step assuming it in $D(E_\lambda)$. Take an absolutely convex closed equicontinuous set L in $(D(E_{\lambda+1}))' = E_{\lambda+1}$, we have to show uniform convergence on any such equicontinuous set. Since L is weakly compact, one can look at the Banach space of weakly continuous functions $C^0(L)$. Let $\iota_L : D(E_{\lambda+1}) \longrightarrow C^0(L)$. $\iota_L(x_n)$ is Cauchy by assumption and therefore converges uniformly to some y_L. We want to show $y_L(z) = \iota_L(x)(z)$ for any $z \in L$. Since $z \in E_{\lambda+1}$ there is by assumption a set $M \subset E_\lambda$ equicontinuous in $[D(E_{\lambda_0})]'$ such that $z \in \overline{\Gamma(M)}$ computed in $E_{\lambda+1}$. Let $N = \overline{\Gamma(M)}$ computed in E_{λ_0}, so that $z \in N$. Since M is equicontinuous in $(D(E_{\lambda_0}))'$ we conclude that so is N and it is also weakly compact there. One can apply the previous reasoning to N instead of L (since x_n Cauchy in $D(E_{\lambda_0})$, not only in $D(E_{\lambda+1})$). $\iota_N(x_n) \longrightarrow y_N$ and since $z \in L \cap N$ and using pointwise convergence $y_L(z) = y_N(z)$. Note also $\iota_N(x)(z) = \iota_L(x)(z)$. Moreover, for $m \in M \subset E_\lambda$, $\iota_N(x_n)(m) \longrightarrow \iota_N(x)(m)$ since $\{m\}$ is always equicontinuous in $(D(E_\lambda))'$ so that $\iota_N(x)(m) = y_N(m)$. Since both sides are affine on the convex N and weakly continuous (for $\iota_N(x)$ since $x \in D(E_{\lambda_0}) = E'_{\lambda_0}$), we extend the relation to any $m \in N$ and thus $\iota_N(x)(z) = y_N(z)$. Altogether, this gives the expected $y_L(z) = \iota_L(x)(z)$. Thus K is complete as expected. $\square$

Lemma B.1.32. For any separated locally convex space, $E^*_k = ((E'_k)'_k)'_k$ is k-reflexive. A space is k-reflexive if and only if $E = (E'_c)'_c$ and both E and E'_c are k-quasi-complete. More generally, if E is k-quasi-complete, so are $(E'_k)'_c = (E'_c)'_c$ and $(E^*_k)'_c$ and $\gamma(E) = \gamma((E'_c)'_c) = \gamma((E^*_k)^*_k)$.

Remark B.1.33. The example $E = C^0(W)$ in Remark B.1.28, which is not quasi-complete, is even k-reflexive. Indeed it remains to see that E'_c is k-quasi-complete. But from [Ja, Thm 13.6.1], it is not only bornological but ultrabornological, hence by [Ja, Corol 13.2.6], E'_μ is complete (and so is $F = \mathscr{S}(E'_\mu)$). But for a compact set in E'_c, the closed absolutely convex cover is closed in E'_c, hence E'_μ, hence complete there. Thus, by Krein's Theorem [K, 24.5.(4)], it is compact in E'_c, making E'_c k-quasi-complete.

Proof. One can assume E is k-quasi-complete (all functors start by this completion) thus so is $(E'_c)'_c$ by [Bo2, IV.5 Rmq 2] since $(E'_c)'_c \longrightarrow E$ continuous with same dual (see [S]). There is a continuous map $(E^*_k)'_c \longrightarrow (E'_c)'_c$ we apply lemma B.1.31 to $E_0 = E'_c$, E_λ the λ-th step of the completion in lemma B.1.29. Any $\overline{\Gamma(K)}$ in the union defining $E_{\lambda+1}$ is equicontinuous in $((E_{\lambda+1})'_c)'$ so a fortiori in $((E_{\lambda_0})'_c)'$ for λ_0 large enough. We apply the lemma to another K closed absolutely convex cover of a compact set of $(E^*_k)'_c$ computed in $(E'_c)'_c$ therefore compact there by assumption. The lemma gives K is complete there contains the bipolar of the compact computed in $(E^*_k)'_c$ which must also be compact as a closed subset of a compact. In this case we deduced $(E^*_k)'_c = (E'_k)'_k$ is k-quasi-complete.

Clearly $((E'_k)'_k)'_k = ((E^*_k)'_k)'_k \longrightarrow E^*_k$ continuous. Dualizing the continuous $(E'_k)'_k \longrightarrow E$ one gets $E'_k \longrightarrow ((E'_k)'_k)'_k = ((E^*_k)'_k)'_k \longrightarrow E^*_k$ and since the space in the middle is already k-quasi-complete inside the last which is the k-quasi-completion, it must be the last space and thus E^*_k k-reflexive and we have the stated equality.

For the next-to-last statement, sufficiency is clear, the already noted $(E'_k)'_k = (E^*_k)'_c \longrightarrow (E'_c)'_c \longrightarrow E$ in the k-quasi-complete case which implies $(E'_c)'_c \simeq E$ if $(E'_k)'_k \simeq E$ and $E^*_k = ((E'_k)'_k)'_k = ((E'_k)'_k)'_c = E'_c$ implies this space is also k-quasi-complete. For the comparison of absolutely convex compact sets, note that $(E^*_k)^*_k \longrightarrow (E'_c)'_c$ ensures one implication and if $K \in \gamma((E'_c)'_c)$ we know it is equicontinuous in $(E'_c)'$ hence [K, 21.4.(5)] equicontinuous in $(\widehat{E'_c}^K)'$ and as a consequence included in an absolutely convex compact in $(\widehat{E'_c}^K)'_c = (E^*_k)^*_k$, i.e. $K \in \gamma((E^*_k)^*_k)$. $\gamma(E) = \gamma((E'_c)'_c)$ is a reformulation of $E'_c \simeq ((E'_c)'_c)'_c$. $\square$

We consider $\gamma - \mathbf{Kc}$ the full subcategory of $\mathbf{Kc}$ with their γ-topology, and γ-$\mathbf{Kb}$ the full subcategory of $\mathbf{LCS}$ made of spaces of the form E'_c with E k-quasi-complete.

We first summarize the results of [S]. We call $\gamma - \mathbf{LCS} \subset \mathbf{LCS}$ the full subcategory of spaces having their γ-topology, namely $E = (E'_c)'_c$. This is equivalent to saying that subsets of absolutely convex compact sets in E'_c

are (or equivalently are exactly the) equicontinuous subsets in E'. With the notation of Theorem B.1.13, this can be reformulated by an intertwining relation in **CLCS** which explains the usefulness of these spaces:

$$E \in \gamma - \mathbf{LCS} \Leftrightarrow (E'_c)_c = (E_c)'_b \Leftrightarrow (E_c)'_b = [U((E_c)'_b)]_c \tag{B.6}$$

Proposition B.1.34. *k-quasi-complete spaces are stable by ε-product, and $(\mathbf{Kc}, \varepsilon, \mathbb{K})$ form a symmetric monoidal category . Moreover, if E, F are k-quasi-complete, a set in $E\varepsilon F$ is relatively compact if and only if it is ε-equihypocontinuous. Therefore we have canonical embeddings:*

$$E'_{c\beta e}F'_c \longrightarrow (E\varepsilon F)'_c \longrightarrow E'^{\widehat{K}}_{c\beta e}F'_c.$$

Proof. The characterization of relatively compact sets is [S, Prop 2], where it is noted that the direction proving relative compactness does not use any quasi-completeness. It gives $(E\varepsilon F)'_c = (E\varepsilon F)'_\epsilon$ with the epsilon topology as a bidual of $E'_{c\beta e}F'_c$ and in general anyway a continuous linear map:

$$(E\varepsilon F)'_c \longrightarrow (E\varepsilon F)'_\epsilon \tag{B.7}$$

For a compact part in $E\varepsilon F$, hence equicontinuous in $(E'_{c\beta e}F'_c)'$, its bipolar is still ε-equihypocontinuous hence compact by the characterization, as we have just explained. This gives stability of k-quasi-completeness.

Associativity of ε is Schwartz' Prop 7 but we give a reformulation giving a more detailed proof that $(\mathbf{Kc}, \varepsilon)$ is symmetric monoidal. The restriction to $\mathbf{Kc}$ of the functor $(\cdot)_c$ of Theorem B.1.13 gives a functor we still call $(\cdot)_c : \mathbf{Kc} \longrightarrow \mathbf{CLCS}$. It has left adjoint $\widehat{\cdot}^K \circ U$. Note that for $E, F \in \mathbf{Kc}$, $E\varepsilon F = \widehat{\cdot}^K \circ U(E_c \mathbin{⅋}_b F_c)$ from our previous stability of $\mathbf{Kc}$. Moreover, note that

$$\forall E, F \in \mathbf{Kc}, \qquad (E\varepsilon F)_c = E_c \mathbin{⅋}_b F_c \tag{B.8}$$

thanks to the characterization of relatively compact sets, since the two spaces were already known to have same topology and the bornology on the right was defined as the equicontinuous bornology of $(E'_{c\beta e}F'_c)'$ and on the left the one generated by absolutely convex compact sets or equivalently the saturated bornology generated by compact sets (using $E\varepsilon F \in \mathbf{Kc}$). Lemma B.1.36 concludes to $(\mathbf{Kc}, \varepsilon, \mathbb{K})$ symmetric monoidal. They also make $(\cdot)_c$ a strong monoidal functor.

We could deduce from [S] the embeddings, but we prefer seeing them as coming from **CLCS**.

Let us apply the next lemma to the embedding of our statement. Note that by definition $E'_{c\beta e}F'_c = U((E_c)'_b{}_H(F_c)'_b)$, and $(E\varepsilon F)'_\varepsilon = U((E_c \mathbin{⅋}_b F_c)'_b) = U((E_c)'_b{}_b(F_c)'_b)$ so that we got the embeddings for $E, F \in \mathbf{LCS}$:

$$E'_{c\beta e}F'_c \longrightarrow (E\varepsilon F)'_\varepsilon \longrightarrow E'^{\widehat{K}}_{c\beta e}F'_c \tag{B.9}$$

which specializes to the statement in the k-quasi-complete case by the beginning of the proof to identify the middle terms. ☐

We have used and are going to reuse several times the following:

Lemma B.1.35. Let $E, F \in$ **CLCS** (resp. with E, F'_b having moreover Schwartz bornologies) we have the topological embedding (for U the map giving the underlying lcs):

$$U(E_H F'_b) \longrightarrow U(E_b F'_b) \longrightarrow [\widehat{\cdot}^K \circ U](E_H F'_b). \tag{B.10}$$

$$(resp. \qquad U(E_H F'_b) \longrightarrow U(E_b F'_b) \longrightarrow [\widehat{\cdot}^M \circ U](E_H F'_b). \qquad) \tag{B.11}$$

Proof. Recall that for $E, F \in$ **CLCS**, $E_H F$ has been defined before the proof of Theorem B.1.13 and is the algebraic tensor product. Let us explain that, even before introducing the notion of k-quasi-completion, we already checked the result of the statement. By construction we saw $(E_b F'_b)'_b = E'_b \mathbin{⅋}_b F = L_b(E, F) = (EF'_b)'_b$, hence by $*$-autonomy $E_b F'_b = ((E_H F'_b)'_b)'_b = (L_b(E, F))'_b$ and it has been described as a subspace $\widehat{E_H F'_b}$ inside the completion (in step 1 of this proof) with induced topology, obtained as union of bipolars of AB or $\overline{AB}$ (image of the product), for A bounded in E, B bounded in F'_b. Hence the embeddings follows from the fact we checked $\overline{AB}$ is precompact, and of course closed in the completion hence compact and the bipolar is one of those appearing in the first step of the inductive description of the k-quasicompletion.

For the case E, F'_b having Schwartz bornologies, bounded sets are of the form $A \subset \overline{\Gamma(x_n, n \in \mathcal{N})}, B \subset \overline{\Gamma(y_m, m \in \mathcal{N})}$ with $(x_n), (y_m)$ Mackey-null in their respective bornologies. Take C, D absolutely convex precompact sets bounded in the respective bornologies with $||x_n||_C \longrightarrow 0, ||y_m||_D \longrightarrow 0$, hence $||x_n y_m||_{(CD)^{oo}} \leqslant ||x_n||_C ||y_m||_D$ and since we checked in the proof of Theorem B.1.13 that $(CD)^{oo}$ is precompact hence bounded, $x_n y_m$ can be gathered in a Mackey-null sequence has the one whose bipolar appears in the first term of the Mackey-completion. $\square$

We have also used the elementary categorical lemma:

Lemma B.1.36. Let $(\mathcal{C}, {}_{\mathcal{C}}, I)$ a symmetric monoidal category and $\mathcal{D}$ a category. Consider a functor $R : \mathcal{D} \longrightarrow \mathcal{C}$ with left adjoint $L : \mathcal{C} \longrightarrow \mathcal{D}$ and define $J = L(I)$, and $E_{\mathcal{D}} F = L(R(E)_{\mathcal{C}} R(F))$. Assume that for any $E, F \in \mathcal{D}$, $L(R(E)) = E$, $R(J) = I$ and

$$R(E_{\mathcal{D}} F) = R(E)_{\mathcal{C}} R(F).$$

Then, $(\mathcal{D}, {}_{\mathcal{D}}, J)$ is a symmetric monoidal category.

Proof. The associator is obtained as $Ass^{\mathcal{D}}_{E,F,G} = L(Ass^{\mathcal{C}}_{R(E),R(F),R(G)})$ and the same intertwining defines the braiding and units and hence transports the relations which concludes. For instance in the pentagon we used the relation $L(Ass^{\mathcal{C}}_{R(E),R(F)_{\mathcal{C}}R(G),R(H)}) = Ass^{\mathcal{D}}_{E,F_{\mathcal{D}}G,H}$. $\square$

We deduce a description of internal hom-sets in these categories: we write $L_{co}(E, F)$, the space of all continuous linear maps from E to F endowed with the topology of uniform convergence on compact subsets of E. When E is a k-quasi-complete space, note this is the same lcs as $L_c(E, F)$, endowed with the topology of uniform convergence on absolutely convex compacts of E.

Corollary B.1.37. For $E \in \gamma - \mathbf{Kc}$ and $F \in \mathbf{Kc}$ (resp. $F \in \mathbf{Mc}$), one has $L_c(E'_c, F) \simeq E\varepsilon F$, which is k-quasi-complete (resp. Mackey-complete).

Proof. Algebraically, $E\varepsilon F = L(E'_c, F)$ and the first space is endowed with the topology of uniform convergence on equicontinuous sets in E'_c which coincides with subsets of absolutely convex compact sets since E has its γ-topology. $\square$

Using that for $E \in \gamma$-**Kb**, $E = F'_c$ for $F \in \mathbf{Kc}$, hence $E'_c = (F'_c)'_c \in \mathbf{Kc}$ by lemma B.1.32.

Corollary B.1.38. Consider $E \in \gamma$-**Kb** , $F \in \mathbf{Kc}$ (resp. $F \in \mathbf{Mc}$) then $L_c(E, F)$ is k-quasi-complete (resp. Mackey-complete).

Proposition B.1.39. $\gamma - \mathbf{Kc} \subset \mathbf{Kc}$ *is a coreflective subcategory with coreflector (right adjoint to inclusion)* $((\cdot)'_c)'_c$*, which commutes with* $\cdot\widehat{\ }^K$ *on* γ-***Kb***. *For* $F \in \gamma - \mathbf{Kc}$, $\cdot\widehat{\ }^K_\gamma F'_c : \mathbf{LCS} \longrightarrow \mathbf{Kc}$ *(resp.* $\mathbf{Kc} \longrightarrow \mathbf{Kc}, \gamma - \mathbf{Kc} \longrightarrow \gamma - \mathbf{Kc}$*) is left adjoint to* $F\varepsilon\, \cdot$ *(resp.* $F\varepsilon\, \cdot, ((F\varepsilon\, \cdot)'_c)'_c$*). More generally, for* $F \in \mathbf{Kc}, \cdot\widehat{\ }^K_{\gamma,\varepsilon} F'_c : \mathbf{LCS} \longrightarrow \mathbf{Kc}$ *is left adjoint to* $F\varepsilon\ \cdot$ *. Finally,* γ-***Kb*** *is stable by* $\widehat{\ }^K_\gamma$ *.*

Proof. (1) We start by proving the properties of the inclusion $\gamma - \mathbf{Kc} \subset \mathbf{Kc}$. Let $E \in \mathbf{Kc}$. We know the continuous map $(E'_c)'_c \longrightarrow E$ and both spaces have the same dual, therefore for K compact in $(E'_c)'_c$ its closed absolutely convex cover is the same computed in both by the bipolar Thm [K, 20.7.(6) and 8.(5)] and it is complete in E by assumption so that by [Bo2, IV.5 Rmq 2] again also in $(E'_c)'_c$ which is thus k-quasi-complete too. Hence, by functoriality of Arens dual, we got a functor: $((\cdot)'_c)'_c : \mathbf{Kc} \longrightarrow \gamma - \mathbf{Kc}$. Then we deduce from functoriality the continuous inverse maps $L(F, E) \longrightarrow L((F'_c)'_c, (E'_c)'_c) = L(F, (E'_c)'_c) \longrightarrow L(F, E)$ (for $F \in \gamma - \mathbf{Kc}, E \in \mathbf{Kc}$) which gives the first adjunction. The unit is $\eta = id$ and counit given by the continuous identity maps: $\varepsilon_E : ((E)'_c)'_c \longrightarrow E$.

(2) Let us turn to proving the commutation property with completion. For $H \in \gamma$-**Kb**, $H = G'_c = ((G'_c)'_c)'_c, G \in \mathbf{Kc}$ we thus have to note that the canonical map $((\widehat{H}^K)'_c)'_c \longrightarrow \widehat{H}^K$ is inverse of the map obtained from canonical map $H \longrightarrow \widehat{H}^K$ by applying functoriality: $H \longrightarrow (\widehat{H}^K)'_c)'_c$ and then k-quasi-completion (since we saw the target is in $\gamma - \mathbf{Kc}$:) $\widehat{H}^K \longrightarrow ((\widehat{H}^K)'_c)'_c$.

(3) For the adjunctions of tensor products, let us start with a heuristic computation. Fix $F \in \gamma - Kc, E \in \mathbf{LCS}, G \in \mathbf{Kc}$. From the discussion before (B.6), $L_\gamma(F'_c, G) \simeq F\varepsilon G$ thus, there is a canonical injection

$$\mathbf{Kc}(E\widehat{\ }^K_\gamma F'_c, G) = L(E_\gamma F'_c, G) \longrightarrow L(E, L_\gamma(F'_c, G)) = L(E, F\varepsilon G).$$

But an element in $L(E, F\varepsilon G)$ sends a compact set in E to a compact set in $F\varepsilon G$ therefore an ϵ-equihypocontinuous set by proposition B.1.34 which is a fortiori an equicontinuous set in $L(F'_c, G)$. This gives the missing hypocontinuity to check the injection is onto.

Let us now give a more abstract alternative proof of the first adjunction. Fix $F \in \gamma - \mathbf{Kc}$. Let us define $\cdot \hat{\ }^{K}_{\gamma} F'_c : \mathbf{LCS} \longrightarrow \mathbf{Kc}$ as the composition $\hat{\cdot}^{M} \circ U \circ (\cdot_b (F_c)'_b) \circ (.)_c$ so that we will be able to describe the unique adjunction by composing known adjunctions. (Similarly, for $F \in \mathbf{Kc}$ one can define $\cdot \hat{\ }^{K}_{\gamma,\varepsilon} F'_c : \mathbf{LCS} \longrightarrow \mathbf{Kc}$ as the same composition $\hat{\cdot}^{M} \circ U \circ (\cdot_b (F_c)'_b) \circ (.)_c$). We have to check this is possible by agreement on objects. This reads for $E \in \mathbf{LCS}$ as application of (B.10), (B.6) and reformulation of the definition $\cdot_\gamma \cdot = U((\cdot)_{cH}(\cdot)_c)$:

$$\hat{\cdot}^{K} \circ U(E_{cb}(F_c)'_b) = \hat{\cdot}^{K} \circ U(E_{cH}(F_c)'_b) = \hat{\cdot}^{K} \circ U(E_{cH}(F'_c)_c) = E \hat{\ }^{K}_{\gamma} F'_c.$$

The case $F \in \mathbf{Kc}$ is similar since by definition $\cdot_{\gamma,\varepsilon}(\cdot)'_c = U((\cdot)_{cH}((\cdot)_c)'_b)$.

Then, to compute the adjunction, one needs to know the adjoints of the composed functors, which are from Theorem B.1.13 and the proof of proposition B.1.34. This gives as adjoint $U \circ (\cdot \parr_b F_c) \circ (.)_c = \cdot\varepsilon F$.

(4) The second adjunction is a consequence and so is the last if we see $\cdot \hat{\ }^{K}_{\gamma} F'_c : \gamma - \mathbf{Kc} \longrightarrow \gamma - \mathbf{Kc}$ as composition of $i : \gamma - \mathbf{Kc} \longrightarrow \mathbf{Kc}$, $\cdot \hat{\ }^{K}_{\gamma} F'_c : \mathbf{Kc} \longrightarrow \mathbf{Kc}$ and the right adjoint of i (which we will see is not needed here). Indeed, by proposition B.1.34, for $E \in \gamma - \mathbf{Kc}$, $E \hat{\ }^{rK}_{c\gamma} F'_c = E \hat{\ }^{rK}_{c\beta e} F'_c$ is the k-quasi-completion of $(E\varepsilon F)'_c \in \gamma - \mathbf{Kb}$, and therefore from the commutation of γ-topology and k-quasi-completion in that case, that we have just established in (2), it is also in $\gamma - \mathbf{Kc}$. Hence, the adjunction follows by composition of previous adjunctions and we have also just proved that γ-**Kb** is stable by $\hat{\ }^{K}_{\gamma}$. □

We emphasize expected consequences from the $*$-autonomous category we will soon get since we will use them in slightly more general form.

Corollary B.1.40. For any $Y \in \mathbf{Kc}, X, Z_1, ..., Z_m, Y_1, ..., Y_n \in \gamma - \mathbf{Kc}, T \in k - \mathbf{Ref}$ the following canonical linear maps are continuous

$$ev_{X'_c} : (Y\varepsilon X) \hat{\ }^{K}_{\gamma} X'_c \longrightarrow Y, \quad comp^*_{T'_c} : (Y\varepsilon T) \hat{\ }^{K}_{\gamma} ((T'_c \varepsilon Z_1 \cdots \varepsilon Z_m)^*_k)^*_k \longrightarrow (Y\varepsilon Z_1 \cdots \varepsilon Z_m),$$

$$comp_{T'_c} : (Y_1 \varepsilon \cdots \varepsilon Y_n \varepsilon T)_\gamma (T'_c \varepsilon Z_1 \cdots \varepsilon Z_m) \longrightarrow (Y\varepsilon Y_1 \cdots \varepsilon Y_n \varepsilon Z_1 \cdots \varepsilon Z_m),$$
$$comp^\sigma_{T'_c} : (Y\varepsilon Y_1 \cdots \varepsilon Y_n \varepsilon T)_{\sigma,\gamma} (T'_c \varepsilon Z_1 \cdots \varepsilon Z_m) \longrightarrow (Y\varepsilon Y_1 \cdots \varepsilon Y_n \varepsilon Z_1 \cdots \varepsilon Z_m),$$

Moreover for any $F, G \in \mathbf{Kc}$, $V, W \in \gamma - \mathbf{Kb}$ and U, E any separated lcs, there are continuous associativity maps

$$Ass_\varepsilon : E\varepsilon(F\varepsilon G) \longrightarrow (E\varepsilon F)\varepsilon G, \quad Ass_\gamma : (U \hat{\ }^{K}_{\gamma} V) \hat{\ }^{K}_{\gamma} W \longrightarrow U \hat{\ }^{K}_{\gamma} (V \hat{\ }^{K}_{\gamma} W),$$

$$Ass_{\gamma,\varepsilon} : V \hat{\ }^{K}_{\gamma} (T\varepsilon X) \longrightarrow (V \hat{\ }^{K}_{\gamma} T)\varepsilon X.$$

Proof. (1) From the adjunction, the symmetry map in $L((Y\varepsilon X), (X\varepsilon Y)) = L((Y\varepsilon X) \hat{\ }^{K}_{\gamma} X'_c, Y)$ gives the first evaluation map.

(2) For the associativity Ass_ε, recall that using definitions and (B.8) (using $F, G \in \mathbf{Kc}$):

$$\begin{aligned} E\varepsilon(F\varepsilon G) = U(E_c \parr_b [F\varepsilon G]_c) = U(E_c \parr_b [F_c \parr_b G_c]) &\longrightarrow U([E_c \parr_b F_c] \parr_b G_c) \\ &\longrightarrow U([U(E_c \parr_b F_c)]_c \parr_b G_c) = (E\varepsilon F)\varepsilon G, \end{aligned}$$

where the first map is $U(Ass^{\parr_b}_{E_c, F_c, G_c})$ and the second obtained by functoriality from the unit $\eta_{E_c \parr_b F_c} : E_c \parr_b F_c \longrightarrow [U(E_c \parr_b F_c)]_c$.

(3) For the associativity Ass_γ, we know from the adjunction again, since $V'_c, W'_c \in \gamma - \mathbf{Kc}, V = (V'_c)'_c, W = (W'_c)'_c$:

$$L((U \hat{\ }^{K}_{\gamma} V) \hat{\ }^{K}_{\gamma} W, U \hat{\ }^{K}_{\gamma} (V \hat{\ }^{K}_{\gamma} W)) = L((U \hat{\ }^{K}_{\gamma} V), W'_c \varepsilon \Big(U \hat{\ }^{K}_{\gamma} (V \hat{\ }^{K}_{\gamma} W)\Big)) = L(U, V'_c \varepsilon \Big(W'_c \varepsilon \Big(U \hat{\ }^{K}_{\gamma} (V \hat{\ }^{K}_{\gamma} W)\Big)\Big)).$$

Then composing with Ass_ε (note the γ tensor product term is the term requiring nothing but k-quasi-completeness for the adjunction to apply) gives a map:

$$L(U \hat{\ }^{K}_{\gamma,\varepsilon} \Big(V'_c \varepsilon W'_c\Big)'_c, \Big(U \hat{\ }^{K}_{\gamma} (V \hat{\ }^{K}_{\gamma} W)\Big)) \simeq L(U, \Big(V'_c \varepsilon W'_c\Big) \varepsilon \Big(U \hat{\ }^{K}_{\gamma} (V \hat{\ }^{K}_{\gamma} W)\Big)) \longrightarrow L(U, V'_c \varepsilon \Big(W'_c \varepsilon \Big(U \hat{\ }^{K}_{\gamma} (V \hat{\ }^{K}_{\gamma} W)\Big)\Big))$$

Since an equicontinuous set in $\left(V_c'\varepsilon W_c'\right)_c'$ is contained in an absolutely convex compact set, one gets by universal properties a continuous linear map: $U_{\gamma,\varepsilon}^{K}\left(V_c'\varepsilon W_c'\right)_c' \longrightarrow U_{\gamma}^{K}\left(V_c'\varepsilon W_c'\right)_c'$.

Finally by functoriality and the embedding of proposition B.1.34 there is a canonical continuous linear map: $U_{\gamma}^{K}\left(V_c'\varepsilon W_c'\right)_c' \longrightarrow U_{\gamma}^{K}(V_{\gamma}^{K}W)$. Dualizing, we also have a map which we can evaluate at the identity map composed with all our previous maps to get Ass_γ:

$$L(U_{\gamma}^{K}(V_{\gamma}^{K}W), \left(U_{\gamma}^{K}(V_{\gamma}^{K}W)\right)) \longrightarrow L(U_{\gamma,\varepsilon}^{K}\left(V_c'\varepsilon W_c'\right)_c', \left(U_{\gamma}^{K}(V_{\gamma}^{K}W)\right))$$

(4) We treat similarly the map $comp_{T_c'}^{*}$ in the case $m = 2$, for notational convenience. It is associated to $ev_{T_c'} \circ (idev_{(Z_1)_c'}) \circ (idev_{(Z_2)_c'}id)$ via the following identifications. One obtains first a map between Hom-sets using the previous adjunction:

$$L\Big(\Big[((Y\varepsilon T)\rtimes_{\gamma}^{K}(((T_c'\varepsilon Z_1)\varepsilon Z_2)_k^{*})_k^{*})\rtimes_{\gamma}^{K}(Z_2)_c'\Big]\rtimes_{\gamma}^{K}(Z_1)_c', Y\Big) = L((Y\varepsilon T)\rtimes_{\gamma}^{K}(((T_c'\varepsilon Z_1)\varepsilon Z_2)_k^{*})_k^{*}, (Y\varepsilon Z_1)\varepsilon Z_2).$$

We compose this twice with Ass_γ and the canonical map $(E_k^*)_k^* \longrightarrow E$ for E k-quasi-complete:

$$L\Big((Y\varepsilon T)\rtimes_{\gamma}^{K}\Big[\Big(((T_c'\varepsilon Z_1)\varepsilon Z_2)\rtimes_{\gamma}^{K}(Z_2)_c'\Big)\rtimes_{\gamma}^{K}(Z_1)_c'\Big], Y\Big) \longrightarrow L\Big((Y\varepsilon T)\rtimes_{\gamma}^{K}\Big[\Big((((T_c'\varepsilon Z_1)\varepsilon Z_2)_k^{*})_k^{*}\rtimes_{\gamma}^{K}(Z_2)_c'\Big)\rtimes_{\gamma}^{K}(Z_1)_c'\Big], Y\Big) \longrightarrow$$
$$L\Big(\Big[(Y\varepsilon T)\rtimes_{\gamma}^{K}\Big((((T_c'\varepsilon Z_1)\varepsilon Z_2)_k^{*})_k^{*}\rtimes_{\gamma}^{K}(Z_2)_c'\Big)\Big]\rtimes_{\gamma}^{K}(Z_1)_c', Y\Big) \longrightarrow L\Big(\Big[\Big((Y\varepsilon T)\rtimes_{\gamma}^{K}(((T_c'\varepsilon Z_1)\varepsilon Z_2)_k^{*})_k^{*}\Big)\rtimes_{\gamma}^{K}(Z_2)_c'\Big]\rtimes_{\gamma}^{K}(Z_1)_c', Y\Big).$$

Note that the first associativity uses the added $((\cdot)_k^*)_k^*$ making the Arens dual of the space k-quasi-complete as it should to use Ass_γ and the second since $(((T_c'\varepsilon Z_1)\varepsilon Z_2)_k^{*})_k^{*}\rtimes_{\gamma}^{K}(Z_2)_c' \in \gamma - \mathbf{Kb}$ from Proposition B.1.39.

Note that $T_c' \in \mathbf{Kc}$ is required for definition of $ev_{(Z_i)_c'}$ hence the supplementary assumption $T \in k - \mathbf{Ref}$ and not only $T \in \gamma - \mathbf{Kc}$.

(5)By the last statement in lemma B.1.32, we already know that $((T_c'\varepsilon Z_1 \cdots \varepsilon Z_m)_k^*)_k^*$ and $T_c'\varepsilon Z_1 \cdots \varepsilon Z_m$ have the same absolutely convex compact sets. Hence for any absolutely compact set in this set $comp_{T_c'}^{*}$ induces an equicontinuous family in $L(Y_1 \cdots \varepsilon Y_n \varepsilon T, Y \varepsilon Y_1 \cdots \varepsilon Y_n \varepsilon Z_1 \cdots \varepsilon Z_m)$. But now by symmetry on ε product and of the assumption on Y_i, Z_j one gets the second hypocontinuity to define $comp_{T_c'}$ by a symmetric argument.

(6)One uses $comp_{T_c'}$ on $((Y_c')_c'\varepsilon Y_1 \cdots \varepsilon Y_n \varepsilon T) = (Y \varepsilon Y_1 \cdots \varepsilon Y_n \varepsilon T)$ algebraically, since $(Y_c')_c' \in \gamma - \mathbf{Kc}$. This gives the separate continuity needed to define $comp_{T_c'}^{\sigma}$, the one sided γ-hypocontinuity follows from $comp_{T_c'}^{*}$ as in (5).

(7) We finish by $Ass_{\gamma,\varepsilon}$. We know from the adjunction again composed with Ass_ε and symmetry of ε that we have a map:

$$L(T\varepsilon X, (V\rtimes_{\gamma}^{K}T)\varepsilon\left(V_c'\varepsilon X\right)) \longrightarrow L(T\varepsilon X, V_c'\varepsilon\left((V\rtimes_{\gamma}^{K}T)\varepsilon X\right)) = L(V\rtimes_{\gamma}^{K}(T\varepsilon X), (V\rtimes_{\gamma}^{K}T)\varepsilon X)$$

Similarly, we have canonical maps:

$$L(\left(T\varepsilon X\right)\rtimes_{\gamma}^{K}(V\rtimes_{\gamma}^{K}T)_c', \left(V_c'\varepsilon X\right)) \simeq L(T\varepsilon X, ((V\rtimes_{\gamma}^{K}T)_c')_c'\varepsilon\left(V_c'\varepsilon X\right)) \longrightarrow L(T\varepsilon X, (V\rtimes_{\gamma}^{K}T)\varepsilon\left(V_c'\varepsilon X\right))$$

$$L(\left(X\varepsilon T\right)\rtimes_{\gamma}^{K}(T_c'\varepsilon V_c'), \left(X\varepsilon V_c'\right)) \longrightarrow L(\left(T\varepsilon X\right)\rtimes_{\gamma}^{K}\left((V_c'\varepsilon T_c')_c'\right)_c', \left(V_c'\varepsilon X\right)) \longrightarrow L(\left(T\varepsilon X\right)\rtimes_{\gamma}^{K}(V\rtimes_{\gamma}^{K}T)_c', \left(V_c'\varepsilon X\right)).$$

The image of $comp_{T_c'} \in L(\left(X\varepsilon T\right)\rtimes_{\gamma}^{K}(T_c'\varepsilon V_c'), \left(X\varepsilon V_c'\right))$ gives $Ass_{\gamma,\varepsilon}$ since $X, V_c' \in \gamma - \mathbf{Kc}$. $\square$

We refer to [MT, T] for the study of dialogue categories from their definition, already recalled in subsection 2.3. Note that $*$-autonomous categories are a special case.

We state first a transport lemma for dialogue categories along monoidal functors, which we will use several times.

Lemma B.1.41. Consider $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$, and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ two symmetric monoidal categories, $R : \mathcal{C} \longrightarrow \mathcal{D}$ a functor, and $L : \mathcal{D} \longrightarrow \mathcal{C}$ the left adjoint to R which is assumed strictly monoidal. If $\neg$ is a tensorial negation on $\mathcal{C}$, then $E \mapsto R(\neg L(E))$ is a tensorial negation on $\mathcal{D}$.

Proof. Let $\varphi^{\mathcal{C}}$ the natural isomorphism making $\neg$ a tensorial negation. Let us call the natural bijections given by the adjunction
$$\psi_{A,B} : \mathcal{D}(A, R(B)) \simeq \mathcal{C}(L(A), B).$$

Define
$$\varphi^{\mathcal{D}}_{A,B,C} = \psi^{-1}_{A,\neg(L(B\otimes_{\mathcal{D}}C))} \circ \varphi^{\mathcal{C}}_{L(A),L(B),L(C)} \circ \psi_{A\otimes_{\mathcal{D}}B,\neg(L(C))} : \mathcal{D}(A \otimes_{\mathcal{D}} B, F(\neg(L(C)))) \longrightarrow \mathcal{D}(A, F(\neg(L(B \otimes_{\mathcal{D}} C))))$$

It gives the expected natural bijection:
$$\begin{aligned}\mathcal{D}(A \otimes_{\mathcal{D}} B, F(\neg(L(C)))) &\simeq \mathcal{C}(L(A \otimes_{\mathcal{D}} B), \neg(L(C))) = \mathcal{C}(L(A) \otimes_{\mathcal{C}} L(B)), \neg(L(C)))\\ &\simeq \mathcal{C}(L(A), \neg(L(B \otimes_{\mathcal{D}} C)))) \simeq \mathcal{D}(A, F(\neg(L(B \otimes_{\mathcal{D}} C))))\end{aligned}$$

where we have used strict monoidality of L: $L(B \otimes_{\mathcal{D}} C) = L(B) \otimes_{\mathcal{C}} L(C)$, and the structure of dialogue category on C.

It remains to check the compatibility relation (B.2). For it suffices to note that by naturality of the adjunction, one has for instance:
$$\mathcal{D}(Ass^{\mathcal{D}}_{A,B,C}, F(\neg(L(D)))) = \psi^{-1}_{A\otimes_{\mathcal{D}}(B\otimes_{\mathcal{D}}C),\neg(L(D))} \circ \mathcal{C}(L(Ass^{\mathcal{D}}_{A,B,C}), \neg(L(D))) \circ \psi_{(A\otimes_{\mathcal{D}}B)\otimes_{\mathcal{D}}C,\neg(L(D))}.$$

and since $L(Ass^{\mathcal{D}}_{A,B,C}) = Ass^{\mathcal{C}}_{L(A),L(B),L(C)}$ from compatibility of a strong monoidal functor, the new commutative diagram in $\mathcal{D}$ reduces to the one in $\mathcal{C}$ by intertwining. ☐

Remark B.1.42. Note that we have seen or will see several examples of such monoidal adjunctions:

- between $(\mathbf{Kc}^{op}, \varepsilon)$ and $(\mathbf{CLCS}^{op}, \parr_b)$ through the functors $L = (\cdot)_c$ and $R = (\hat{\cdot}^K \circ U)$ (proof of proposition B.1.34 and (B.8)),
- between $(\mathbf{CSch}^{op}, \parr_b)$ and $(\mathbf{McSch}^{op}, \varepsilon)$ through the functors $L = (\cdot)_{sc}$ and $R = (\hat{\cdot}^M \circ U)$(proposition B.1.65).

Theorem B.1.43. $\mathbf{Kc}^{op}$ *is a dialogue category with tensor product* ε *and tensorial negation* $(\cdot)^*_k$ *which has a commutative and idempotent continuation monad* $((\cdot)^*_k)^*_k$*.*

Its continuation category is equivalent to the $*$*-autonomous category* $k - \mathbf{Ref}$ *with tensor product* $E_k F = (E^*_k \varepsilon F^*_k)^*_k$*, dual* $(.)^*_k$ *and dualizing object* $\mathcal{K}$*. It is stable by arbitrary products and direct sums.*

Proof. The structure of a dialogue category follows from the first case of the previous remark since $(\mathbf{CLCS}^{op}, \parr_b, (\cdot)'_b)$ is a $*$-autonomous category, hence a Dialogue category by Theorem B.1.13 and then the new tensorial negation is $R(\neg L(\cdot)) = \hat{\cdot}^K \circ (\cdot)'_c$ which coincides with $(\cdot)^*_k$ on $\mathbf{Kc}$. The idempotency of the continuation monad comes from lemma B.1.32.

In order to check that the monad is commutative, one uses that from [T, Prop 2.4], the dialogue category already implies existence of right and left tensor strengths say $t_{X,Y}, \tau_{X,Y}$. Note that in order to see they commute, it suffices to see the corresponding result after applying $(\cdot)^*_k$. Then from proposition B.1.34, the two maps obtained on $\widehat{X'_c}^K \hat{\otimes}^K_{\beta e} \widehat{Y'_c}^K$ must be extensions by continuity of an ϵ-hypocontinuous multilinear map on $X'_c \otimes_{\beta e} Y'_c$, which is unique by [K2, 40.3.(1)] which even works in the separately continuous case but strongly requires known the separate continuity of the extension. Hence we have the stated commutativity.

The $*$-autonomous property follows from the following general lemma. ☐

Lemma B.1.44. Let $(\mathcal{C}^{op}, \parr_{\mathcal{C}}, I, \neg)$ a dialogue category with a commutative and idempotent continuation monad and $\mathcal{D} \subset \mathcal{C}$ the full subcategory of objects of the form $\neg C, C \in \mathcal{C}$. Then $\mathcal{D}$ is equivalent to the Kleisli category of the comonad $T = \neg\neg$ in $\mathcal{C}$. If we define $\cdot_{\mathcal{D}}\cdot = \neg(\neg(\cdot) \parr_{\mathcal{C}} \neg(\cdot))$, then $(\mathcal{D}, _{\mathcal{D}}, I, \neg)$ is a $*$-autonomous category and $\neg : \mathcal{C}^{op} \longrightarrow \mathcal{D}$ is strongly monoidal.

Proof. From the already quoted [T, Prop 2.9] of Hagasawa, the cited Kleisli category (or Continuation category $\mathcal{C}^{\neg}$) is a $*$-autonomous category since we start from a Dialogue category with commutative and idempotent continuation monad. Consider $\neg : \mathcal{C}^{\neg} \longrightarrow \mathcal{D}$. $\mathcal{D}(\neg A, \neg B) = \mathcal{C}^{op}(\neg B, \neg A) = \mathcal{C}^{\neg}(A, B)$ which gives that $\neg$ is fully faithful on the continuation category. The map $\neg : \mathcal{D} \longrightarrow \mathcal{C}^{\neg}$ is the strong inverse of the equivalence since $\neg \circ \neg \simeq Id_{\mathcal{D}}$ by choice of $\mathcal{D}$, and idempotency of the continuation and the canonical map

$J_{\neg A} \in \mathcal{C}^{\neg}(\neg\neg(A), id_{\mathcal{C}^{\neg}}(A)) = \mathcal{C}^{op}(\neg A, \neg(T(a))$ is indeed natural in A and it is an isomorphism in $\mathcal{C}^{\neg}$. Therefore we have a strong equivalence. Recall that the commutative strength $t_{A,B} : A \parr_{\mathcal{C}} T(B) \longrightarrow T(A \parr_{\mathcal{C}} B), t'_{A,B} : T(A) \parr_{\mathcal{C}} B \longrightarrow T(A_{\mathcal{C}} B)$ in $\mathcal{C}^{op}$, implies that we have isomorphisms

$$I_{A,B} = \neg\Big(T(t'_{A,B}) \circ t_{T(A),B}\Big) \circ J^{2op}_{\neg(A \parr_{\mathcal{C}} B)} : \neg(A \parr_{\mathcal{C}} B) \simeq \neg(\neg\neg A \parr_{\mathcal{C}} B) \simeq \neg(\neg\neg A \parr_{\mathcal{C}} \neg\neg B)$$

with commutation relations $I_{A,B} = \neg\Big(T(t_{A,B}) \circ t'_{A,T(B)}\Big) \circ J^{2op}_{\neg(A \parr_{\mathcal{C}} B)}$. This gives in $\mathcal{D}$ the compatibility map for the strong monad: $\mu_{A,B} = I^{op-1}_{A,B} : \neg(A \parr_{\mathcal{C}} B) \simeq \neg A_{\mathcal{D}} \neg B$. Checking the associativity and unitarity for this map is a tedious computation left to the reader using axioms of strengths, commutativity, functoriality. This concludes. $\square$

B.1.3.2 A strong notion of smooth maps

During this subsection, $\mathcal{K} = \mathscr{R}$ so that we deal with smooth maps and not holomorphic ones while we explore the consequence of our $*$-autonomy results for the definition of a nice notion of smoothness.

We recall the definition of (conveniently) smooth maps as used by Frolicher, Kriegl and Michor: a map $f : E \longrightarrow F$ is smooth if and only if for every smooth curve $c : \mathbb{R} \longrightarrow E$, $f \circ c$ is a smooth curve. See [53]. They define on a space of smooth curves the usual topology of uniform convergence on compact subsets of each derivative. Then they define on the space of smooth functions between Mackey-complete spaces E and F the projective topology with respect to all smooth curves in E (see also section 6.1 below).

As this definition fits well in the setting of bounded linear maps and bounded duals, but not in our setting using continuous linear maps, we make use of a slightly different approach by Meise [58]. Meise works with k-spaces, that is spaces E in which continuity on E is equivalent to continuity on compacts subsets of E. We change his definition and rather use a continuity condition on compact sets in the definition of smooth functions.

Definition B.1.45. For X, F separated lcs we call $C^{\infty}_{co}(X, F)$ the space of infinitely many times Gâteaux-differentiable functions with derivatives continuous on compacts with value in the space $L^{n+1}_{co}(E, F) = L_{co}(E, L^{n}_{co}(E, F))$ with at each stage the topology of uniform convergence on compact sets. We put on it the topology of uniform convergence on compact sets of all derivatives in the space $L^{n}_{co}(E, F)$.

We denote by $C^{\infty}_{co}(X)$ the space $\mathcal{C}^{\infty}_{c}(X, \mathbb{K})$.

One could treat similarly the case of an open set $\Omega \subset X$. We always assume X k-quasi-complete.

Our definition is almost the same as in [58], except for the continuity condition restricted to compact sets. Meise works with k-spaces, that is spaces E in which continuity on E is equivalent to continuity on compacts subsets of E. Thus for X a k-space, one recovers exactly Meise's definition. Since a (DFM) space X is a k-space ([KM, Th 4.11 p 39]) his corollary 7 gives us that for such an X, $C^{\infty}_{co}(X, F)$ is a Fréchet space as soon as F is. Similarly for any (F)-space or any (DFS)-space X then his corollary 13 gives $C^{\infty}_{co}(X, \mathscr{R})$ is a Schwartz space.

As in his lemma 3 p 271, if X k-quasi-complete, the Gâteaux differentiability condition is automatically uniform on compact sets (continuity on absolutely convex closure of compacts of the derivative is enough for that), and as a consequence, this smoothness implies convenient smoothness. We will therefore use the differential calculus from [53].

One are now ready to obtain a category.

Proposition B.1.46. *$k-\mathbf{Ref}$ is a category with $C^{\infty}_{co}(X, F)$ as spaces of maps, that we denote $k-\mathbf{Ref}_{\infty}$. Moreover, for any $g \in C^{\infty}_{co}(X, Y), Y, X \in k-\mathbf{Ref}$, any F Mackey-complete, $\cdot \circ g : C^{\infty}_{co}(Y, F) \longrightarrow C^{\infty}_{co}(X, F)$ is linear continuous.*

Proof. For stability by composition, we show more, consider $g \in C^{\infty}_{co}(X, Y), f \in C^{\infty}_{co}(Y, F)$ with $X, Y \in k-\mathbf{Ref}$ and $F \in \mathbf{Mc}$ we aim at showing $f \circ g \in C^{\infty}_{co}(X, F)$. We use stability of composition of conveniently smooth maps, we can use the chain rule [KM, Thm 3.18]. This enables to make the derivatives valued in F if F is Mackey-complete so that, up to going to the completion, we can assume $F \in \mathbf{Kc}$ since the continuity conditions are the same when the topology of the target is induced. This means that we must show continuity on compact sets of expressions of the form

$$(x, h) \mapsto df^{l}(g(x))(d^{k_1}g(x), ..., d^{k_l}g(x))(h_1, ..., h_m), m = \sum_{i=1}^{l} k_i, h \in Q^m.$$

First note that $L_{co}(X,F) \simeq X'_c\varepsilon F$, $L^n_{co}(X,F) \simeq (X'_c)^{\varepsilon n}\varepsilon F$ fully associative for the spaces above.

Of course for K compact in X, $g(K) \subset Y$ is compact, so $df^l \circ g$ is continuous on compacts with value in $(Y'_c)^{\varepsilon l}\varepsilon F$ so that continuity comes from continuity of the map obtained by composing various $Comp^*_Y, Ass_\varepsilon$ from Corollary B.1.40 (note Ass_γ is not needed with chosen parentheses):

$$\left(\left(...\left(((Y'_c)^{\varepsilon l}\varepsilon F) \otimes_\gamma \left(\left(((X'_c)^{\varepsilon k_1}\varepsilon Y\right)^*_k\right)^*_k\right) \otimes_\gamma ...\right) \otimes_\gamma \left(\left(((X'_c)^{\varepsilon k_l}\varepsilon Y\right)^*_k\right)^*_k\right) \longrightarrow ((X'_c)^{\varepsilon m}\varepsilon F)$$

and this implies continuity on products of absolutely convex compact sets of the corresponding multilinear map even without $((\cdot)^*_k)^*_k$ since from lemma B.1.32 absolutely convex compact sets are the same in both spaces (of course with same induced compact topology). We can compose it with the continuous function on compacts with value in a compact set (on compacts in x):$x \mapsto (df^l(g(x)), d^{k_1}g(x), ..., d^{k_l}g(x))$. The continuity in f is similar and uses hypocontinuity of the above composition (and not only its continuity on products of compacts). ☐

We now prove the *Cartesian closedeness* of the category $k-\mathbf{Ref}$, the proofs being slight adaptation of the work by Meise [58]

Proposition B.1.47. *For any $X \in k-\mathbf{Ref}$, $C^\infty_{co}(X,F)$ is k-quasi-complete (resp. Mackey-complete) as soon as F is.*

Proof. This follows from the projective kernel topology on $\mathcal{C}^\infty_c(X,F)$, Corollary B.1.38 and the corresponding statement for $C^0(K,F)$ for K compact. In the Mackey-complete case we use the remark at the beginning of step 2 of the proof of Theorem B.1.56 that a space is Mackey-complete if and only if the bipolar of any Mackey-Cauchy sequence is complete. We treat the two cases in parallel, if F is k-quasi-complete (resp Mackey-complete), take L a compact set (resp. a Mackey-Cauchy sequence) in $C^0(K,F)$, M its bipolar, its image by evaluations L_x are compact (resp. a Mackey-null sequence) in F and the image of M is in the bipolar of L_x which is complete in F hence a Cauchy net in M converges pointwise in F. But the Cauchy property of the net then implies as usual uniform convergence of the net to the pointwise limit. This limit is therefore continuous, hence the result. ☐

The following two propositions are an adaptation of the result by Meise [58, Thm 1 p 280]. Remember though that his ε product and E'_c are different from ours, they correspond to replacing absolutely convex compact sets by precompact sets. This different setting forces him to assume quasi-completeness to obtain a symmetric ε-product in his sense.

Proposition B.1.48. *For any k-reflexive space X, any compact K, and any separated k-quasi-complete space F one has $\mathcal{C}^\infty_c(X,F) \simeq \mathcal{C}^\infty_c(X)\varepsilon F, C^0(K,F) \simeq C^0(K)\varepsilon F$. Moreover, if F is any lcs, we still have a canonical embedding $J_X : \mathcal{C}^\infty_c(X)\varepsilon F \longrightarrow \mathcal{C}^\infty_c(X,F)$.*

Proof. We build a map $ev_X \in C^\infty_{co}(X, (C^\infty_{co}(X))'_c)$ defined by $ev_X(x)(f) = f(x)$ and show that $\cdot \circ ev_X : C^\infty_{co}(X)\varepsilon F = L_\epsilon((C^\infty_{co}(X))'_c, F) \longrightarrow C^\infty_{co}(X,F)$ is a topological isomorphism and an embedding if F only Mackey-complete. The case with a compact K is embedded in our proof and left to the reader.

(a) We first show that the expected j-th differential $ev^j_X(x)(h)(f) = d^j f(x).h$ indeed gives a map:

$$ev^j_X \in C^0_{co}(X, L^j_{co}(X, (C^\infty_{co}(X))'_c)).$$

First note that for each $x \in X$, $ev^j_X(x)$ is in the expected space. Indeed, by definition of the topology $f \mapsto d^j f(x)$ is linear continuous in $L(C^\infty_{co}(X), L^j_{co}(X, \mathscr{R})) \subset L(((C^\infty_{co}(X))'_c)'_c, L^j_{co}(X,\mathscr{R})) = (C^\infty_{co}(X))'_c\varepsilon(X'_c)^{\varepsilon j}$. Using successively Ass^ϵ from Corollary B.1.40 (note no completeness assumption on $(C^\infty_{co}(X))'_c$ is needed for that) hence $ev^j_X(x) \in (\cdots((C^\infty_{co}(X))'_c\varepsilon X'_c)\cdots\varepsilon X'_c) = L^j_{co}(X, (C^\infty_{co}(X))'_c)$. Then, once the map well-defined, we must check its continuity on compacts sets in variable $x \in K \subset X$, uniformly on compacts sets for $h \in Q$, one must check convergence in $(C^\infty_{co}(X))'_c$. But everything takes place in a product of compact sets and from the definition of the topology on $C^\infty_{co}(X)$, $ev^j_X(K)(Q)$ is equicontinuous in $(C^\infty_{co}(X))'$. But from [K, 21.6.(2)] the topology $(C^\infty_{co}(X))'_c$ coincides with $(C^\infty_{co}(X))'_\sigma$ on these sets. Hence we only need to prove for any f continuity of $d^j f$ and this follows by assumption on f. This concludes the proof of (a).

(b) Let us note that for $f \in L_\varepsilon((C^\infty_{co}(X))'_c, F)$, $f \circ ev_X \in C^\infty_{co}(X,F)$. We first note that $f \circ ev^j_X(x) = d^j(f \circ ev_X)(x)$ as in step (c) in the proof of [58, Thm 1]. This shows for $F = (C^\infty_{co}(X))'_c$ that the Gâteaux derivative is $d^j ev_X = ev^j_X$ and therefore the claimed $ev_X \in C^\infty_{co}(X, (C^\infty_{co}(X))'_c)$.

(c)$f \mapsto f \circ ev_X$ is the stated isomorphism. The monomorphism property is the same as (d) in Meise's proof. Finding a right inverse j proving surjectivity is the same as his (e) . Let us detail this since we only assume k-quasi-completeness on F. We want $j : C^\infty_{co}(X,F) \longrightarrow C^\infty_{co}(X)\varepsilon F = L(F'_c, C^\infty_{co}(X))$ for $y' \in F'$, $f \in C^\infty_{co}(X,F)$

we define $j(f)(y') = y' \circ f$. Note that from convenient smoothness we know that the derivatives are $y' \circ d^j f(x)$ and $d^j f(x) \in (X'_c)^{\varepsilon n} \varepsilon F = (X'_c)^{\varepsilon n} \varepsilon (F'_c)'_c$ algebraically so that, since $(F'_c)'_c$ k-quasi-complete, one can use $ev_{F'_c}$ from Corollary B.1.40 to see $y' \circ d^j f(x) \in (X'_c)^{\varepsilon n}$ and one even deduces (using only separate continuity of $ev_{F'_c}$) its continuity in y'. Hence $j(f)(y')$ is valued in $C^\infty_{co}(X)$ and from the projective kernel topology, $j(f)$ is indeed continuous. The simple identity showing that j is indeed the expected right inverse proving surjectivity is the same as in Meise's proof. □

Proposition B.1.49. *For any space $X_1, X_2 \in k - \mathbf{Ref}$ and any Mackey-complete lcs F we have:*

$$C^\infty_{co}(X_1 \times X_2, F) \simeq C^\infty_{co}(X_1, C^\infty_{co}(X_2, F)).$$

Proof. Construction of the curry map Λ is analogous to [58, Prop 3 p 296]. Since all spaces are Mackey-complete, we already know from [KM, Th 3.12] that there is a Curry map valued in $C^\infty(X_1, C^\infty(X_2, F))$, it suffices to see that the derivatives $d^j\Lambda(f)(x_1)$ are continuous on compacts with value $C^\infty_{co}(X_2, F)$. But this derivative coincides with a partial derivative of f, hence it is valued pointwise in $C^\infty_{co}(X_2, F) \subset C^\infty_{co}(X_2, \widehat{F}^K)$. Since we already know all the derivatives are pointwise valued in F, we can assume F k-quasi-complete. But the topology for which we must prove continuity is a projective kernel, hence we only need to see that $d^k(d^j\Lambda(f)(x_1))(x_2)$ continuous on compacts in x_1 with value in $L^j_c(X_1, C^0(K_2, L^j_c(X_2, F)))$. But we are in the case where the ε product is associative, hence the above space is merely $C^0(K_2)\varepsilon L^j_c(X_1, L^j_c(X_2, F)) = C^0(K_2, L^j_c(X_1, L^j_c(X_2, F)))$. We already know the stated continuity in this space from the choice of f. The reasoning for the inverse map is similar using again the convenient smoothness setting (and Cartesian closedness $C^0(K_1, C^0(K_2)) = C^0(K_1 \times K_2)$). □

B.1.4 Schwartz locally convex spaces, Mackey-completeness and ρ-dual.

In order to obtain a $*$-autonomous category adapted to convenient smoothness, we want to replace k-quasi-completeness by the weaker Mackey-completeness and adapt our previous section.

In order to ensure associativity of the dual of the ε-product, Mackey-completeness is not enough as we saw in section B.1.2. We have to restrict simultaneously to Schwartz topologies. After some preliminaries in subsection 5.1, we thus define our appropriate weakened reflexivity (ρ-reflexivity) in subsection 5.2, and investigate categorical completeness in 5.3.

We want to put a $*$-autonomous category structure on the category $\rho - \mathbf{Ref}$ of ρ-reflexive (which implies Schwartz Mackey-complete) locally convex spaces with continuous linear maps as morphisms.

It turns out that one can carry on as in section B.1.3 and put a Dialogue category structure on Mackey-complete Schwartz spaces. Technically, the structure is derived via an intertwining from the one in $\mathbf{CSch}$ in Theorem B.1.13. This category can even be seen as chosen in order to fit our current Mackey-complete Schwartz setting. We actually proved all the results first without using it and made appear the underlying categorical structure afterwards.

Then the continuation monad will give the $*$-autonomous category structure on $\rho - \mathbf{Ref}$ where the internal hom is described as

$$E \multimap_\rho F = (([E^*_\rho]\varepsilon F)^*_\rho)^*_\rho$$

and based on a twisted Schwartz ε-product. The space is of course the same (as forced by the maps of the category) but the topology is strengthened. Our preliminary work on double dualization in section 6.2 make this construction natural to recover an element of $\rho - \mathbf{Ref}$ anyway.

B.1.4.1 Preliminaries in the Schwartz Mackey-complete setting

We define $\mathbf{Mc}$ (resp. $\mathbf{Sch}$) the category of Mackey-complete spaces (resp Schwartz space) and linear continuous maps. The category $\mathbf{McSch}$ is the category of Mackey-complete Schwartz spaces.

We first recall [Ja, Corol 16.4.3]. Of course it is proven their for the completed variant, but by functoriality, the original definition of this product is a subspace and thus again a Schwartz space.

Proposition B.1.50. *If E and F are separated Schwartz locally convex spaces, then so is $E\varepsilon F$.*

We can benefit from our section 3 to obtain associativity:

Proposition B.1.51. $(\mathbf{McSch}, \varepsilon)$ *is a symmetric monoidal complete and cocomplete category.* $\mathbf{McSch} \subset \mathbf{Mc}$ *is a reflective subcategory with reflector (left adjoint to inclusion) $\mathscr{S}$ and the inclusion is strongly monoidal. Moreover on* $\mathbf{LCS}$, *$\mathscr{S}$ and $\widehat{\cdot}^M$ commute and their composition is the reflector of* $\mathbf{McSch} \subset \mathbf{LCS}$.

Proof. From Theorem B.1.26, we know $(\mathbf{Mc}, \zeta)$ is of the same type and for $E, F \in \mathbf{McSch}$, lemma B.1.20 with the previous lemma gives $E\varepsilon F = E\zeta F \in \mathbf{McSch}$. Hence, we deduce $\mathbf{McSch}$ is also symmetric monoidal and the inclusion strongly monoidal. The unit of the adjunction is the canonical identity map $\eta_E : E \longrightarrow \mathscr{S}(E)$ and counit is identity satisfying the right relations hence the adjunction. From the adjunction the limits in $\mathbf{McSch}$ are the limits in $\mathbf{Mc}$ and colimits are obtained by applying $\mathscr{S}$ to colimits of $\mathbf{Mc}$. It is easy to see that $\mathbf{McSch} \subset \mathbf{Sch} \subset \mathbf{LCS}$ are also reflective, hence the two ways of writing the global composition gives the commutation of the left adjoints. □

B.1.4.2 ρ-reflexive spaces and their Arens-Mackey duals

We define a new notion for the dual of E, which consists of taking the Arens-dual of the Mackey-completion of the Schwartz space $\mathscr{S}(E)$, which is once again transformed into a Mackey-complete Schwartz space E^*_ρ.

Definition B.1.52. For a lcs E, the topology $\mathscr{S}\rho(E', E)$ on E' is the topology of uniform convergence on absolutely convex compact sets of $\widehat{\mathscr{S}(E)}^M$. We write $E'_{\mathscr{S}\rho} = (E', \mathscr{S}\rho(E', E)) = (\widehat{\mathscr{S}(E)}^M)'_c$. We write $E^*_\rho = \widehat{\mathscr{S}(E'_{\mathscr{S}\rho})}^M$ and $E'_{\mathbb{R}} = \mathscr{S}(E'_{\mathscr{S}\rho})$.

Remark B.1.53. Note that $E'_{\mathscr{S}\rho}$ is in general not Mackey-complete: there is an Arens dual of a Mackey-complete space (even of a nuclear complete space with its Mackey topology) which is not Mackey-complete using [BD, thm 34, step 6]. Indeed take Γ a closed cone in the cotangent bundle (with 0 section removed) $\dot{T}^*R^n$. Consider Hörmander's space $E = \mathcal{D}'_\Gamma(\mathscr{R}^n)$ of distributions with wave front set included in Γ with its normal topology in the terminology of [BD, Prop 12,29]. It is shown there that E is nuclear complete. Therefore the strong dual is $E'_\beta = E'_c$. Moreover, [BD, Lemma 10] shows that this strong dual if $\mathcal{E}'_\Lambda$, the space of compactly supported distributions with a wave front set in the open cone $\Lambda = -\Gamma^c$ with a standard inductive limit topology. This dual is shown to be nuclear in [BD, Prop 28]. Therefore we have $E'_c = E'_{\mathscr{S}\rho}$. Finally, as explained in the step 6 of the proof of [BD, Thm 34] where it is stated it is not complete, as soon as Λ is not closed (namely by connectedness when $\Gamma \notin \{\varnothing, \dot{T}^*R^n\}$), then E'_c is not even Mackey-complete. This gives our claimed counter-example. The fact that E above has its Mackey topology is explained in [D].

First note the functoriality lemma :

Lemma B.1.54. $(\cdot)^*_\rho$ and $(\cdot)'_{\mathbb{R}}$ are contravariant endofunctors on $\mathbf{LCS}$.

Proof. They are obtained by composing $\mathscr{S}$, $(\cdot)'_c$ and $\widehat{\cdot}^M$ (recalled in Theorem B.1.26). □

From Mackey theorem and the fact that completion does not change the dual, we can deduce immediately that we have the following algebraic identities $\mathscr{S}((E'_{\mathscr{S}\rho})'_{\mathscr{S}\rho}) = (E'_{\mathscr{S}\rho})'_{\mathscr{S}\rho} = \widehat{\mathscr{S}(E)}^M$.

From these we deduce the fundamental algebraic equality:

$$(E^*_\rho)^*_\rho = \widehat{\mathscr{S}(E)}^M \tag{B.12}$$

Definition B.1.55. A lcs E is said *ρ-reflexive* if the canonical map $E \longrightarrow \widehat{\mathscr{S}(E)}^M = (E^*_\rho)^*_\rho$ gives a topological isomorphism $E \simeq (E^*_\rho)^*_\rho$.

We are looking for a condition necessary to make the above equality a topologically one. The following theorem demonstrates an analogous to $E'_c = ((E'_c)'_c)'_c$ for our new dual. For, we now make use of lemma B.1.31.

Theorem B.1.56. *Let E be a separated locally convex space, then E^*_ρ is ρ-reflexive. As a consequence, if E is ρ-reflexive, so is E^*_ρ and $\mathscr{S}((E'_c)'_c) \simeq E \simeq \mathscr{S}((E'_\mu)'_\mu)$ topologically. Moreover, when E is Mackey-complete $(E^*_\rho)^*_\rho = (E^*_\rho)'_{\mathbb{R}}$ and E have the same bounded sets.*

Proof. Note that the next-to-last statement is obvious since if E ρ-reflexive, we have $(E^*_\rho) = ((E^*_\rho)^*_\rho)^*_\rho$ and the last space is always ρ-reflexive. Moreover, from the two first operations applied in the duality, one can and do assume $\mathscr{S}(E)$ is Mackey-complete.

Let us write also $\mathscr{C}_M(.) = \widehat{\cdot}^M$ for the Mackey-completion functor and for an ordinal λ,$\mathscr{C}^\lambda_M(E) = E_{M,\lambda}$ from lemma B.1.16.

Note also that since the bounded sets in E and $\mathscr{S}(E)$ coincide by Mackey Theorem [Ho, Th 3 p 209], one is Mackey-complete if and only if the other is.

Step 1: $\mathscr{S}((E'_\mu)'_\mu)$ is Mackey-complete if $\mathscr{S}(E)$ is Mackey-complete.

This follows from the continuity $\mathscr{S}((E'_\mu)'_\mu) \longrightarrow \mathscr{S}(E)$ and the common dual, they have same bounded sets, hence same Mackey-Cauchy/converging sequences.

Step 2: $\mathscr{S}((E'_{\mathscr{S}\rho})'_{\mathscr{S}\rho})$ is Mackey-complete if $\mathscr{S}(E)$ is Mackey-complete.

First note that a space is Mackey-complete if and only if any K, closed absolutely convex cover of a Mackey-Cauchy sequence, is complete. Indeed, if this is the case, since a Mackey-Cauchy sequence is Mackey-Cauchy for the saturated bornology generated by Mackey-null sequences [Ja, Thm 10.1.2], it is Mackey in a normed space having a ball (the bipolar of the null sequence) complete in the lcs, hence a Banach space in which the Cauchy sequence must converge. Conversely, if a space is Mackey-complete, the sequence converges in some Banach space, hence its bipolar in this space is compact, and thus also in the lcs and must coincide with the bipolar computed there which is therefore compact hence complete.

We thus apply lemma B.1.31 to K the closed absolutely convex cover of a Mackey-Cauchy sequence in $\mathscr{S}((E'_{\mathscr{S}\rho})'_{\mathscr{S}\rho})$, $E_0 = \mathscr{S}(E'_{\mathscr{S}\rho})$, $D = \mathscr{S}((\cdot)'_c)$, $E_\lambda = \mathscr{C}_M^\lambda(E_0)$ eventually yielding to the Mackey completion so that $D(E_{\lambda_0}) = \mathscr{S}((E'_{\mathscr{S}\rho})'_{\mathscr{S}\rho})$ for λ_0 large enough and with $D(E_0) = \mathscr{S}((E'_\mu)'_\mu)$ using lemma B.1.19. The result will conclude since the above bipolar K computed in $D(E_{\lambda_0})$ must be complete by Mackey-completeness of this space hence complete in $\mathscr{S}((E'_{\mathscr{S}\rho})'_{\mathscr{S}\rho})$ by the conclusion of lemma B.1.31 and hence the bipolar computed in there which is a closed subset will be complete too. We thus need to check the assumptions of lemma B.1.31. The assumption at successor ordinal comes from the definition of $\mathscr{C}_M^{\lambda+1}$ since any point there z satisfy $z \in N = \overline{\Gamma(L)}$ with $L = \{t_n, n \in \mathcal{N}\}$ a Mackey-Cauchy sequence in E_λ. Thus there is an absolutely convex bounded $B \subset E_\lambda$ with (t_n) Cauchy in the normed space $(E_\lambda)_B \subset (E_{\lambda_0})_{\overline{B}}$. We know $t_n \longrightarrow t$ in the completion so $t \in N$.

But since E_{λ_0} is Mackey-complete, this last space is a Banach space, $t_n \longrightarrow t$ and it is contained in $C_1 = \{s_0 = 2t, s_n = 2(t_n - t), n \in \mathcal{N}\}^{oo}$. $||s_n||_{\overline{B}} \longrightarrow 0$ we can define $r_n = s_n/\sqrt{||s_n||_{\overline{B}}}$ which converges to 0 in $(E_{\lambda_0})_{\overline{B}}$. Hence $\{r_n, n \in \mathcal{N}\}$ is precompact as any converging sequence and so is its bipolar say C computed in the Banach space $(E_{\lambda_0})_{\overline{B}}$, which is also complete thus compact. C is thus compact in E_{λ_0} too. Since $||s_n||_C \leqslant \sqrt{||s_n||_{\overline{B}}} \longrightarrow 0$ it is Mackey-null for the bornology of absolutely convex compact sets of E_{λ_0}. Thus C_1 is equicontinuous in $(D(E_{\lambda_0}))'$ and so is t_n as expected.

Step 3: Conclusion.

Note we will use freely lemma B.1.19. If $\mathscr{S}(E)$ is Mackey-complete, and $Z = (E'_{\mathscr{S}\rho})'_{\mathscr{S}\rho}$ then from step 2, $\mathscr{S}(Z), \mathscr{S}(Z'_{\mathscr{S}\rho})$ are Mackey-complete and as a consequence $Z'_{\mathscr{S}\rho} = Z'_\mu$ and then $(Z'_{\mathscr{S}\rho})'_{\mathscr{S}\rho} = (Z'_{\mathscr{S}\rho})'_\mu = (Z'_\mu)'_\mu$ topologically. In particular we confirm our claimed topological identity:

$$(E^*_\rho)^*_\rho \equiv \mathscr{C}_M(\mathscr{S}(Z)) = \mathscr{S}(Z) \equiv \mathscr{S}((E^*_\rho)'_{\mathscr{S}\rho}).$$

From the continuous linear identity map: $(Z'_\mu)'_\mu \longrightarrow Z$ one gets a similar map $\mathscr{S}((Z'_{\mathscr{S}\rho})'_{\mathscr{S}\rho}) \longrightarrow \mathscr{S}(Z)$.

Similarly, since there is a continuous identity map $Z \longrightarrow \mathscr{S}(Z)$, one gets a continuous linear map $Z'_\mu \longrightarrow Z'_c = (H'_c)'_c \longrightarrow \widehat{\mathscr{S}(E'_{\mathscr{S}\rho})}^M \equiv H$. Since the last space is a Schwartz topology on the same space, one deduces a continuous map $\mathscr{S}(Z'_\mu) \longrightarrow \widehat{\mathscr{S}(E'_{\mathscr{S}\rho})}^M$. Finally, an application of Arens duality again leads to a continuous identity map: $Z \longrightarrow (Z'_\mu)'_\mu = ((Z)'_{\mathscr{S}\rho})'_{\mathscr{S}\rho}$. This concludes to the equality $\mathscr{S}((Z'_{\mathscr{S}\rho})'_{\mathscr{S}\rho}) = \mathscr{S}((Z'_\mu)'_\mu) = \mathscr{S}((Z'_\mu)'_c) = \mathscr{S}(Z)$. As a consequence if E is ρ-reflexive, it is of the form $E = \mathscr{S}(Z)$ and one deduces $\mathscr{S}((E'_c)'_c) = E = \mathscr{S}((E'_\mu)'_\mu)$.

Consider E a Mackey-complete and Schwartz space. Then $(E^*_\rho)^*_\rho = \mathscr{S}\left[\left(\widehat{\mathscr{S}(E'_c)}^M\right)'_c\right]$ and we have continuous linear maps $E'_c \longrightarrow \mathscr{S}(E'_c) \longrightarrow \widehat{\mathscr{S}(E'_c)}^M \longrightarrow \widetilde{\mathscr{S}(E'_c)}$ which by duality and functoriality give continuous linear maps:

$$\left(\widetilde{\mathscr{S}(E'_c)}\right)'_c \longrightarrow (E^*_\rho)^*_\rho \longrightarrow \mathscr{S}((E'_c)'_c) \longrightarrow E. \tag{B.13}$$

Let us show that a ρ-dual $Y = E^*_\rho$ is always ρ-reflexive (for which we can and do assume E is Mackey-complete and Schwartz). According to equation (B.12), as Y is Mackey-complete and Schwartz we already have the algebraic equality $(Y_\rho^*)_\rho^* = Y$. The above equation gives a continuous identity map $(Y_\rho^*)_\rho^* \to Y$. Now according to step 2 of this proof $Y_c' \equiv (E'_{\mathscr{S}\rho})'_{\mathscr{S}\rho} = Z$ and $(Y_\mu')_\mu'$ are Mackey-complete. Thus $(Y_\rho^*)_\rho^* = \mathscr{S}([\mathscr{S}(Y_c')]_c') = \mathscr{S}([Y_\mu']_\mu')$. However the equation (B.13) gives a continuous identity map $(E_\rho^*)_\rho^* \to \widehat{\mathscr{S}(E)}^M$, which by duality and functoriality of $\mathscr{S}$ leads to a continuous identity map $Y \to (Y_\rho^*)_\rho^*$. Every ρ-dual is thus ρ-reflexive.

Let us show the last statement, since a space and its associated Schwartz space have the same bounded sets, we can assume E Mackey-complete and Schwartz. As a consequence of the equation (B.13) and of the next lemma, the bounded sets in the middle term $(E_\rho^*)_\rho^*$ have to coincide too, and the last statement of the proposition is shown. □

Lemma B.1.57. If E is Mackey-complete lcs, then $[\widetilde{\mathscr{S}(E_c')}]_c'$ has the same bounded sets as E.

Proof. Since E is Mackey-complete bounded sets are included in absolutely convex closed bounded sets which are Banach disks. On E' the topology $\mathcal{T}_{\mathcal{B}_b}$ of uniform convergence on Banach disk (bornology $\mathcal{B}_b$) coincides with the topology of the strong dual E_β'.

Moreover, by [Ja, Th 10.1.2] $\mathcal{B}_b$-Mackey convergent sequences coincide with $(\mathcal{B}_b)_0$-Mackey convergent sequences but the closed absolutely convex cover of a null sequence of a Banach space is compact in this Banach space, therefore compact in E, thus they coincide with $\varepsilon((E_c')')$-null sequences (i.e. null sequences for Mackey convergence for the bornology of absolutely convex compact sets). Therefore $\mathscr{S}(E_c') = \mathscr{S}(E', \mathcal{T}_{\mathcal{B}_b}) = \mathscr{S}(E_\beta')$. As a consequence, combining this with [Ja, Th 13.3.2], the completion of $\mathscr{S}(E_c')$ is linearly isomorphic to the dual of both the bornologification and the ultrabornologification of E. Therefore, the bounded sets in $(\widetilde{\mathscr{S}(E_c')})_c'$ are by Mackey theorem the bounded sets for $\sigma(E, \widetilde{\mathscr{S}(E_c')}) = \sigma(E, (E_{bor})')$ namely the bounded of E_{bor} or E. □

The ρ-dual can be understood in a finer way. Indeed, the Mackey-completion on $E_\mathbb{R}' = \mathscr{S}(E'_{\mathscr{S}\rho})$ is unnecessary, as we would get a Mackey-complete space back after three dualization.

Proposition B.1.58. *For any lcs E,*

$$((E_\mathbb{R}')_\mathbb{R}')_\mathbb{R}' \simeq \widehat{E_\mathbb{R}'}^M \equiv E_\rho^*$$

and if E Mackey-complete, $(E_\mathbb{R}')_\mathbb{R}' = (E_\rho^*)_\rho^*$.

Proof. We saw in step 3 of our theorem B.1.56 that, for any Mackey-complete Schwartz space E, first $(E_\mathbb{R}')_\mathbb{R}'$ is Mackey-complete hence $(E_\mathbb{R}')_\mathbb{R}' = (E_\rho^*)_\rho^*$ and then (B.13) gives a continuous identity map $((E_\mathbb{R}')_\mathbb{R}') \longrightarrow E$. By functoriality one gets a continuous linear map:$E_\mathbb{R}' \longrightarrow ((E_\mathbb{R}')_\mathbb{R}')_\mathbb{R}'$. Moreover $((E_\mathbb{R}')_\mathbb{R}')_\mathbb{R}' = ((\widehat{E_\mathbb{R}'}^M)_\mathbb{R}')_\mathbb{R}'$ is Mackey-complete by step 2 of our previous theorem, thus the above map extends to $\widehat{E_\mathbb{R}'}^M \longrightarrow ((E_\mathbb{R}')_\mathbb{R}')_\mathbb{R}'$. This is of course the inverse of the similar continuous (identity) map given by (B.13): $((\widehat{E_\mathbb{R}'}^M)_\mathbb{R}')_\mathbb{R}' \longrightarrow \widehat{E_\mathbb{R}'}^M$ which gives the topological identity. □

We finally relate our definition with other previously known notions:

Theorem B.1.59. *A lcs is ρ-reflexive, if and only if it is Mackey-complete, has its Schwartz topology associated to the Mackey topology of its dual $\mu_{(s)}(E,E')$ and its dual is also Mackey-complete with its Mackey topology. As a consequence, Arens=Mackey duals of ρ-reflexive spaces are exactly Mackey-complete locally convex spaces with their Mackey topology such that their Mackey dual is Mackey-complete.*

Remark B.1.60. A k-quasi-complete space is Mackey-complete hence for a k-reflexive space E, $\mathscr{S}((E_\mu')_\mu')$ is ρ-reflexive (since E_c' k-quasi-complete implies that so is E_μ' which is a stronger topology). Our new setting is a priori more general than the one of section 4. We will pay the price of a weaker notion of smooth maps. Note that a Mackey-complete space need not be k-quasi-complete (see lemma B.1.61 below).

Proof. If E is ρ-reflexive we saw in Theorem B.1.56 that $E \simeq \mathscr{S}((E_\mu')_\mu')$ and both $E, E_\mathbb{R}' = \mathscr{S}(E_\mu')$ (or E_μ') are Mackey-complete with their Mackey topology.

Conversely, if E with $\mu_{(s)}(E,E')$ is Mackey-complete as well as its dual, $E'_{\mathscr{S}\rho} = E_c'$ and thus $E_\mathbb{R}' = \mathscr{S}(E_c')$ which has the same bornology as the Mackey topology and is therefore Mackey-complete too, hence $E_\mathbb{R}' = E_\rho^*$. Therefore we have a map $(E', \mu_{(s)}(E', E)) \longrightarrow E_\mathbb{R}' = \mathscr{S}(E_c')$. Conversely, note that $E_c' = (E', \mu(E', E))$ from lemma B.1.19 so that one gets a continuous isomorphism.

From the completeness and Schwartz property and dualisation, and then lemma B.1.19 again, there is a continuous identity map $(E_\rho^*)_\mathscr{R}' = \mathscr{S}([\mathscr{S}(E_c')]_c') = \mathscr{S}((E_\mu')_\mu') = E$, which is Mackey-complete. Therefore $(E_\rho^*)_\rho^* = (E_\mathbb{R}')_\mathscr{R}' = E$, i.e. E is ρ-reflexive.

For the last statement, we already saw the condition is necessary, it is sufficient since for F Mackey-complete with its Mackey topology with Mackey-complete Mackey-dual, $\mathscr{S}(F)$ is ρ-reflexive by what we just saw and so that $(\mathscr{S}(F))'_c$ is the Mackey topology on F', by symmetry $[\mathscr{S}([\mathscr{S}(F)]'_c)]'_c = F$ and therefore F is both Mackey and Arens dual of the ρ-reflexive space $\mathscr{S}([\mathscr{S}(F)]'_c)$. $\square$

Several relevant categories appeared. $\mathcal{M} \subset \mathbf{LCS}$ the full subcategory of spaces having their Mackey topology. $\mu\mathbf{Sch} \subset \mathbf{LCS}$ the full subcategory of spaces having the Schwartz topology associated to its Mackey topology. $\mathbf{Mb} \subset \mathbf{LCS}$ the full subcategory of spaces with a Mackey-complete Mackey dual. And then by intersection always considered as full subcategories, one obtains:

$$\mathbf{Mc}\mu\mathbf{Sch} = \mathbf{Mc} \cap \mu\mathbf{Sch},\ \ \mathbf{Mb}\mu\mathbf{Sch} = \mathbf{Mb} \cap \mu\mathbf{Sch},,\ \ \mathbf{McMb} = \mathbf{Mb} \cap \mathbf{Mc},$$

$$\mu\mathbf{McMb} = \mathbf{McMb} \cap \mathcal{M},\ \ \rho - \mathbf{Ref} = \mathbf{McMb} \cap \mu\mathbf{Sch}.$$

We can summarize the situation as follows: There are two functors $(.)'_c$ and μ the associated Mackey topology (contravariant and covariant respectively) from the category $\rho - \mathbf{Ref}$ to $\mu\mathbf{McMb}$ the category of Mackey duals of ρ-Reflexive spaces (according to the previous proposition). There are two other functors $(.)^*_\rho, \mathscr{S}$ and they are the (weak) inverses of the two previous ones.

Finally, the following lemma explains that our new setting is more general than the k-quasi-complete setting of section 4:

Lemma B.1.61. There is a space $E \in \mathbf{Mc}\mu\mathbf{Sch}$ which is not k-quasi-complete.

Proof. We take $\mathcal{K} = \mathscr{R}$ (the complex case is similar). Let $F = C^0([0,1])$ the Banach space with the topology of uniform convergence. We take $G = \mathscr{S}(F'_\mu) = F'_c$ which is complete since F ultrabornological [Ja, Corol 13.2.6]. Consider $H = Span\{\delta_x, x \in [0,1]\}$ the vector space generated by Dirac measures and $E = \widehat{H}^M$ the Mackey completion with induced topology (since we will see E identifies as a subspace of G). Let K be the unit ball of F', the space of measures on $[0,1]$. It is absolutely convex, closed for any topology compatible with duality, for instance in G, and since G is a Schwartz space, it is precompact, and complete by completeness of G, hence compact. By Krein-Millman's theorem [K, 25.1.4] it is the closed convex cover of its extreme points. Those are known to be $\delta_x, -\delta_x, x \in [0,1]$ [K, 25.2.(2)]. Especially, E is dense in G, which is therefore its completion. By the proof of lemma B.1.22, the Mackey-topology of E is induced by G and thus by lemma B.1.18, $\mathscr{S}(E_\mu)$ is also the induced topology from G. Hence $E \in \mathbf{Mc}\mu\mathbf{Sch}$. But by Maharam decomposition of measures, it is known that F' has the following decomposition (see e.g. [Ha, p 22]) as an ℓ^1-direct sum:

$$F' = L^1(\{0,1\}^\omega)^{\oplus_1 2^\omega} \oplus_1 \ell^1([0,1])$$

and the Dirac masses generate part of the second component, so that $H \subset \ell^1([0,1])$ in the previous decomposition. But the bounded sets in G are the same as in F'_β (by principle of uniform boundedness), hence Mackey-convergence in G implies norm convergence in F'_β, so that by completeness of $\ell^1([0,1])$, $E \subset \ell^1([0,1])$. Hence Lebesgue measure (which gives one of the summands $L^1(\{0,1\}^\omega)$) gives $\lambda \notin E$. Finally, consider $\delta : [0,1] \longrightarrow K \subset G$ the dirac mass map. It is continuous since a compact set in F is equicontinuous by Ascoli theorem, which gives exactly uniform continuity of δ on compact sets in F. Hence $\delta([0,1])$ is compact in E while its absolutely convex cover in G contains λ so that the intersection with E cannot be complete, hence E is not k-quasi-complete. $\square$

B.1.4.3 Relation to projective limits and direct sums

We now deduce the following stability properties from Theorem B.1.59.

Corollary B.1.62. The class of ρ-reflexive spaces is stable by countable locally convex direct sums and arbitrary products.

Proof. Let $(E_i)_{i\in I}$ a countable family of ρ-reflexive spaces, and $E = \oplus_{i\in I} E_i$. Using Theorem B.1.59, we aim at proving that E is Mackey-complete, has its Schwartz topology associated to the Mackey topology of its dual $\mu_{(s)}(E, E')$ and its dual is also Mackey-complete with its Mackey topology.

From the Theorem B.1.59, E_i itself has the Schwartz topology associated to its Mackey topology. From [K, 22. 5.(4)], the Mackey topology on E is the direct sum of Mackey topologies. Moreover the maps $E_i \longrightarrow \mathscr{S}(E_i)$ give a direct sum map $E \longrightarrow \oplus_{i\in I} \mathscr{S}(E_i)$ and thus a continuous map $\mathscr{S}(E) \longrightarrow \oplus_{i\in I} \mathscr{S}(E_i)$ since a countable direct sum of Schwartz spaces is a Schwartz space. Conversely the maps $E_i \longrightarrow E$ give maps $\mathscr{S}(E_i) \longrightarrow \mathscr{S}(E)$

and by the universal property this gives $\mathscr{S}(E) \simeq \oplus_{i\in I}\mathscr{S}(E_i)$. Therefore, if all spaces E_i are ρ-reflexive, E carries the Schwartz topology associated to its Mackey topology. From [KM, Th 2.14, 2.15], Mackey-complete spaces are stable by arbitrary projective limits and direct sums, thus the Mackey-completeness condition on the space and its dual (using the computation of dual Mackey topology from [K, 22. 5.(3)]) are also satisfied.

For an arbitrary product, [K, 22. 5.(3)] again gives the Mackey topology, universal properties and stability of Schwartz spaces by arbitrary products give the commutation of $\mathscr{S}$ with arbitrary products and the stability of Mackey-completeness can be safely used (even for the dual, uncountable direct sum).

$\square$

Lemma B.1.63. For $(E_i, i \in I)$ a (projective) directed system of Mackey-complete Schwartz locally convex space if $E = \text{proj } \lim_{i\in I} E_i$, then:

$$((E)^*_\rho)^*_\rho \simeq \Big[\Big[\text{proj } \lim_{i\in I}((E_i)^*_\rho)^*_\rho)\Big]^*_\rho\Big]^*_\rho.$$

The same holds for general locally convex kernels and categorical limits.

Proof. The bidualization functor and universal property of projective limits give maps $(E)^*_\rho)^*_\rho \longrightarrow ((E_i)^*_\rho)^*_\rho$ and then $((E^*_\rho)^*_\rho \longrightarrow \text{proj } \lim_{i\in I}((E_i)^*_\rho)^*_\rho$, (see [K, 19.6.(6)] for l.c. kernels) and bidualization and ρ-reflexivity concludes to the first map. Conversely, the canonical continuous linear map in the Mackey-complete Schwartz case $((E_i)^*_\rho)^*_\rho) = ((E_i)'_c)^*_\rho) \longrightarrow E_i$ gives the reverse map after passing to the projective limit and double ρ-dual. The localy convex kernel case and the categorical limit case are identical. $\square$

Proposition B.1.64. *The category $\rho - \mathbf{Ref}$ is complete and cocomplete, with products and countable direct sums agreeing with those in* $\mathbf{LCS}$ *and limits given in lemma* B.1.63

Proof. Bidualazing after application of $\mathbf{LCS}$-(co)limits clearly gives (co)limits. Corollary B.1.62 gives the product and sum case. $\square$

B.1.4.4 The Dialogue category McSch.

We first deduce from Theorem B.1.13 and a variant of [S, Prop 2] a useful:

Lemma B.1.65. Let $\invamp_{sb}$ be the $\invamp$ of the complete $*$-autonomous category $\mathbf{CSch}$ given by $A\invamp_{sb}B = \mathscr{S}(L_b((A)'_b, B)$. Then we have the equality in $\mathbf{CSch}$:

$$\forall E, F \in \mathbf{McSch}, \quad E_{sc} \invamp_{sb} F_{sc} = (E\varepsilon F)_{sc}. \tag{B.14}$$

As a consequence, $(\mathbf{McSch}, \varepsilon, \mathcal{K})$ is a symmetric monoidal category.

Proof. We already know that $(\mathbf{McSch}, \varepsilon, \mathcal{K})$ is symmetric monoidal but we give an alternative proof using lemma B.1.36.

All spaces E, F are now in $\mathbf{McSch}$. Note that $(E_{sc})'_b$ is $\mathscr{S}(E'_c)$ with equicontinuous bornology, which is a Schwartz bornology, hence a continuous linear map from it to any F sends a bounded set into a bipolar of a Mackey-null sequence for the absolutely convex compact bornology. Hence

$$U(E_{sc} \invamp_b F_{sc}) = U(L_b((E_{sc})'_b, F_{sc})) = L_\epsilon(\mathscr{S}(E'_c), F) = L_\epsilon(E'_c, F) = E\varepsilon F$$

since the boundedness condition is satisfied hence equality as spaces, and the topology is the topology of convergence on equicontinuous sets, and the next-to-last equality since F Schwartz. Moreover an equicontinuous set in $L_\epsilon(\mathscr{S}(E'_c), F)$ coincide with those in $L_\epsilon(E'_c, F)$ and an equibounded set in $L_b((E_{sc})'_b, F_{sc})$ only depends on the topology on E, hence in $\mathbf{CLCS}$:

$$L_b((E_{sc})'_b, F_{sc}) = L_b((E_c)'_b, F_{sc})$$

Now in $\mathbf{CSch}$ we have $E_{sc} \invamp_{sb} F_{sc} = \mathscr{S}L_b((E_{sc})'_b, F_{sc}) = \mathscr{S}L_b((E_c)'_b, F_{sc})$ and $F_{sc} = \mathscr{S}L_b((F_c)'_b, \mathcal{K})$ hence (B.3) gives:

$$E_{sc} \invamp_{sb} F_{sc} = \mathscr{S}L_b((E_c)'_b, \mathscr{S}L_b((F_c)'_b, \mathcal{K})) = \mathscr{S}L_b((E_c)'_b, L_b((F_c)'_b, \mathcal{K})) = \mathscr{S}L_b((E_c)'_b, F_c)$$

so that the bornology is the Schwartz bornology associated to the ϵ-equicontinuous bornology of $E\varepsilon F$ (the one of $E_c \invamp_b F_c$). It remains to identify this bornology with the one of $[E\varepsilon F]_{sc}$. Of course from this description the identity map $E_{sc} \invamp_{sb} F_{sc} \longrightarrow [E\varepsilon F]_{sc}$ is bounded, one must check the converse.

This is a variant of [S, Prop 2]. Thus take an absolutely convex compact $K \subset E\varepsilon F = L(E'_c, F)$ and a sequence $\{x_n, n \in \mathcal{N}\} \subset (E\varepsilon F)_K$, with $||x_n||_K \longrightarrow 0$. We must check it is Mackey-null in $E_c \parr_b F_c$. For take as usual $\{y_n, n \in \mathcal{N}\}$ another sequence with $||y_n||_K \longrightarrow 0$ and $C = \{y_n, n \in \mathcal{N}\}^{oo}$ such that $||x_n||_C \longrightarrow 0$. It suffices to check that C is ε-equicontinous in $E\varepsilon F$, the bornology of $E_c \parr_b F_c$.

For instance, one must show that for A equicontinuous in E', $D = (C(A))^{oo}$ is absolutely convex compact in F (and the similar symmetric condition). But since E is Schwartz, it suffices to take $A = \{z_n, n \in \mathcal{N}\}^{oo}$ with z_n ϵ-null in E' and especially, Mackey-null. But $D \subset \{y_n(z_m), n, m \in \mathcal{N}\}^{oo}$ so that it suffices to see that $(y_n(z_m))_{n,m\in\mathcal{N}^2}$ is Mackey-null (since F is Mackey-complete, this will imply Mackey-null for the bornology of Banach disk, hence with compact bipolar). But from [S, Prob 2bis p 28] since C is bounded in $E\varepsilon F$ it is ε-equihypocontinuous on $E'_\beta \times F'_\beta$ and hence it sends an equicontinuous set as A to a bounded set in F, so that D is bounded in F. Finally, $||(y_n(x_m))||_D \leqslant ||x_n||_A||y_m||_C$ hence the claimed Mackey-null property.

Let us prove again that $(\mathbf{McSch}, \varepsilon, \mathcal{K})$ is symmetric monoidal using lemma B.1.36 starting from $(\mathbf{CSch}, \parr_{sb}, \mathcal{K})$. We apply it to the adjunction $(\cdot)_{sc} : \mathbf{McSch} \longrightarrow \mathbf{CSch}$ with left adjoint $\widehat{\cdot}^M \circ U$ using U from Theorem B.1.13.(3). The lemma concludes since the assumptions are easily satisfied, especially $E\varepsilon F = U([E\varepsilon F]_{sc}) = \widehat{\cdot}^M \circ U([E\varepsilon F]_{sc})$ from stability of Mackey-completeness and using the key (B.14) $\square$

We will now use lemma B.1.41 to obtain a Dialogue category.

Proposition B.1.66. *The negation $(.)^*_\rho$ gives $\mathbf{McSch}^{op}$ the structure of a Dialogue category with tensor product ε.*

Proof. Proposition B.1.51 or lemma B.1.65 gives $\mathbf{McSch}^{op}$ the structure of a symmetric monoidal category. We have to check that $(.)^*_\rho : \mathbf{McSch}^{op} \longrightarrow \mathbf{McSch}$ is a tensorial negation on $\mathbf{McSch}^{op}$.

For, we write it as a composition of functors involving $\mathbf{CSch}$. Note that $(\cdot)_{sc} : \mathbf{McSch} \longrightarrow \mathbf{CSch}$ the composition of inclusion and the functor of the same name in Theorem B.1.13 is right adjoint to $L := \widehat{\cdot}^M \circ U$ in combining this result with the proof of Proposition B.1.51 giving the left adjoint to $\mathbf{McSch} \subset \mathbf{Sch}$. Then on $\mathbf{McSch}$,

$$(.)^*_\rho = \widehat{\cdot}^M \circ \mathscr{S} \circ (\cdot)'_c = L \circ \mathscr{S} \circ (\cdot)'_b \circ (\cdot)_c = L \circ (\cdot)'_b \circ (\cdot)_{sc}.$$

Lemma B.1.41 and the following remark concludes. $\square$

B.1.4.5 Commutation of the double negation monad on McSch

Tabareau shows in his theses [T, Prop 2.9] that if the continuation monad $\neg\neg$ of a Dialogue category is commutative and idempotent then, the continuation category is $*$-autonomous. Actually, according to a result attributed to Hasegawa [MT], for which we don't have a published reference, it seems that idempotency and commutativity are equivalent in the above situation. This would simplify our developments since we chose our duality functor to ensure idempotency, but we don't use this second result in the sequel.

Thus we check $((\cdot)^*_\rho)^*_\rho$ is a commutative monad. We deduce that from the study of a dual tensor product. Let us motivate its definition first.

As recalled in the preliminary section the ε-product is defined as $E\varepsilon F = (E'_c \otimes_{\beta e} F'_c)'$. Moreover, we saw in Theorem B.1.56 that when E is ρ-reflexive (or $E \in \mu\mathbf{Sch}$) then $E = \mathscr{S}((E'_c)'_c)$. Recall also from [Ja, 10.4] that a Schwartz space is endowed with the topology of uniform convergence on the ϵ-null sequences of E'.

Thus when $E \in \mu\mathbf{Sch}$, its equicontinuous subsets $\varepsilon(E')$ are exactly the collection $\mathbb{R}(E'_c)$ of all sets included in the closed absolutely convex cover of a $\varepsilon(((E'_c)'_c)')$-Mackey-null sequence.[1]

Remember also from section B.1.1.1 that every Arens-dual E'_c is endowed with its γ-topology of uniform convergence on absolutely convex compact subsets of E. Thus if $\gamma(E'_c)$ is the bornology generated by absolutely convex compact sets, the equicontinuous sets of $(((E'_c)'_c)')$ equals $\gamma(E'_c)$, as E is always endowed with the topology of uniform convergence on equicontinuous subsets of E'. Thus $\mathbb{R}(E'_c) = (\gamma(E'_c))_0$ is the bornology generated by bipolars of null sequences of the bornology $\gamma(E'_c)$ (with the notation of [Ja, subsection 10.1]). We write in general $\mathbb{R}(E) = (\gamma(E))_0$.

We call $\mathbb{R}(E)$ the saturated bornology generated by γ-null sequences. Note that they are the same as null sequences for the bornology of Banach disks hence [Ja, Th 8.4.4 b] also for the bornology of absolutely convex weakly compact sets.

[1] Remember that a Mackey-null sequence is a sequence which Mackey-converges to 0. By [Ja, Prop 10.4.4] any such Mackey-null sequence is an equicontinuous set: indeed the associated Schwartz topology is the topology of uniform convergence on those sequences and conversely using also the standard [K, 21.3.(2)].

Definition B.1.67. The $\mathbb{R}$-tensor product $E \otimes_{\mathbb{R}} F$ is the algebraic tensor product endowed with the finest locally convex topology making $E \times F \to E \otimes F$ a $(\mathbb{R}(E) - \mathbb{R}(F))$-hypocontinuous bilinear map. We define $L_{\mathbb{R}}(E, F)$ the space of continuous linear maps with the topology of convergence on $\mathscr{R}(E)$.

Note that with the notation of Theorem B.1.13, for any $E, F \in \mathbf{LCS}$, this means

$$E \otimes_{\mathbb{R}} F = U(E_{sc\,H} F_{sc}), \quad L_{\mathscr{R}}(E, F) = U(L_b(E_{sc}, F_{sc})).$$

Pay attention $E'_{\mathbb{R}} = L_{\mathscr{R}}(\mathscr{S}(\widehat{E}^M), \mathcal{K}) \neq L_{\mathscr{R}}(E, \mathcal{K})$ in general, which may not be the most obvious convention when $E \notin \mathbf{McSch}$.

For the reader's convenience, we spell out an adjunction motivating those definitions even if we won't really use it.

Lemma B.1.68. Let E, F, G separated lcs. If F is a Schwartz space, so is $L_{\mathbb{R}}(E, F)$. Moreover, if we also assume $E \in \mathbf{Mc}\mu\mathbf{Sch}$, then:

$$L_{\mathbb{R}}(E, F) \simeq E'_{\mathbb{R}} \varepsilon F.$$

Finally if E, F, G are Schwartz spaces and $F \in \mathbf{Mb}\mu\mathbf{Sch}$, then we have an algebraic isomorphism:

$$L(E \otimes_{\mathbb{R}} F, G) \longrightarrow L(E, L_{\mathbb{R}}(F, G)).$$

Proof. For the Schwartz property, one uses [Ja, Th 16.4.1], it suffices to note that $L_{\mathbb{R}}(E, \mathcal{K})$ is a Schwartz space and this comes from [Ja, Prop 13.2.5]. If E is Mackey-complete Schwartz space with $E = \mathscr{S}((E'_\mu)'_\mu)$ then $(E'_{\mathbb{R}})'_c = (E'_\mu)'_\mu$ and therefore $E'_{\mathbb{R}} \varepsilon F = L((E'_{\mathbb{R}})'_c, F) = L(E, F)$ and the topology is the one of convergence on equicontinuous sets, namely on $\mathbb{R}(E) = \mathbb{R}((E'_{\mathbb{R}})'_c)$ since Mackey-null sequences coincides with $\gamma(E)$-null ones since E Mackey-complete and thus does not depend on the topology with same dual.

Obviously, there is an injective linear map

$$L(E \otimes_{\mathbb{R}} F, G) \longrightarrow L(E, L_{\mathscr{R}}(F, G))$$

Let us see it is surjective. For $f \in L(E, L_{\mathbb{R}}(F, G))$ defines a separately continuous bilinear map and if $K \in \mathbb{R}(F)$ the image $f(.)(K)$ is equicontinuous by definition. What is less obvious is the other equicontinuity. For $(x_n)_{n \geqslant 0}$ a $\gamma(E)$-null sequence, i.e null in E_K for K absolutely convex compact set, we want to show $\{f(x_n), n \geqslant 0\}$ equicontinuous, thus take U° in G' an equicontinuous set, since G is a Schwartz space, it is contained in the closed absolutely convex cover of a $\varepsilon(G')$-null sequence, say $\{y_n, n \geqslant 0\}$ with $||y_n||_{V^\circ} \longrightarrow 0$. $f(K)$ is compact thus bounded, thus $f(K)^t(V^\circ)$ is bounded in $L_{\mathscr{R}}(F, \mathcal{K}) = \mathscr{S}(F'_c)$ or in F'_c. Thus $(f(x_n)^t(y_m))_{n,m}$ is Mackey-null in F'_c. Since $F = \mathscr{S}((F'_c)'_c)$, $F'_c = F'_\mu$ and as recalled earlier $\mathbb{R}(F'_c) = \varepsilon(F')$. If moreover, F'_c is Mackey-complete, $(f(x_n)^t(y_m))_{n,m}$ is Mackey for the bornology of Banach disks hence in $\mathbb{R}(F'_c)$, thus it is equicontinuous in F'. ☐

We continue with two general lemmas deduced from lemma B.1.35.

Lemma B.1.69. Let $X, Y \in \mathbf{Sch}$ and define $G = (X \varepsilon Y)'_\varepsilon$ the dual with the topology of convergence on equicontinuous sets from the duality with $H = X'_c \otimes_{\beta e} Y'_c$. Then we have embeddings

$$H \subset G \subset \widehat{H}^M, \; ((H)'_\mu)'_\mu \subset ((G)'_\mu)'_\mu \subset ((\widehat{H}^M)'_\mu)'_\mu, \; \mathscr{S}(((H)'_\mu)'_\mu) \subset \mathscr{S}(((G)'_\mu)'_\mu) \subset \mathscr{S}(((\widehat{H}^M)'_\mu)'_\mu).$$

Proof. We apply lemma B.1.35 to $(X_c)'_b, (Y_c)'_b$ which have a Schwartz bornology since $X, Y \in \mathbf{Sch}$. Note that $H = U((X_c)'_{bH}(Y_c)'_b)$ and that

$$U((X_c)'_{bb}(Y_c)'_b) = U\left(\left[((X_c)'_b)'_b \parr_b ((X_c)'_b)'_b\right]'_b\right) = U\left(\left[X_c \parr_b Y_c\right]'_b\right) = G.$$

Lemma B.1.35 concludes exactly to the first embedding. The second follows using lemma B.1.22 and the third from lemma B.1.18. ☐

Lemma B.1.70. Let $X, Y \in \mu\mathbf{Sch}$ and define $G = (X \varepsilon Y)'_\varepsilon$ the dual with the topology of convergence on equicontinuous sets from the duality with $H = X'_\mu \otimes_{\mathbb{R}} Y'_\mu$. Then we have embeddings

$$H \subset G \subset \widehat{H}^M, \; ((H)'_\mu)'_\mu) \subset ((G)'_\mu)'_\mu \subset ((\widehat{H}^M)'_\mu)'_\mu, \; \mathscr{S}(((H)'_\mu)'_\mu) \subset \mathscr{S}(((G)'_\mu)'_\mu) \subset \mathscr{S}(((\widehat{H}^M)'_\mu)'_\mu).$$

As a consequence for $X, Y \in \mathbf{Mc}\mu\mathbf{Sch}$, we have topological identities $(X \varepsilon Y)^*_\rho \simeq \widehat{\mathscr{S}(H)}^M$ and

$$((X \varepsilon Y)^*_\rho)^*_\rho \simeq (X'_c \otimes_{\mathbb{R}} Y'_c)^*_\rho.$$

Proof. This is a special case of the previous result. Indeed since $X \in \mu\mathbf{Sch}$ so that $X'_c = X'_\mu, X = \mathscr{S}((X'_c)'_\mu) = \mathscr{S}((X'_c)'_c)$, equicontinuous sets in its dual are those in $\mathbb{R}(X'_c) = \mathbb{R}(X'_\mu)$, hence:

$$(X\varepsilon Y) = (X'_{c\beta e}Y'_c)' = (X'_{c\mathbb{R}}Y'_c)', \qquad X'_{c\beta e}Y'_c \simeq X'_{c\mathbb{R}}Y'_c.$$

The Mackey-complete case is a reformulation using only the definition of $(\cdot)^*_\rho$ (and the commutation in Proposition B.1.51). $\square$

Let us state a consequence on $X \otimes_\kappa Y := ((X \otimes_\mathbb{R} Y)'_\mu)'_\mu \in \mu - \mathbf{LCS}$. We benefit from the work in lemma B.1.23 that made appear the inductive tensor product.

Proposition B.1.71. *For any $X \in \mathbf{Mb} \cap \mu - \mathbf{LCS}, Y \in \mu - \mathbf{LCS}$, then the canonical map is a topological isomorphism:*

$$I : X\widehat{\otimes}^M_\kappa Y \simeq X\widehat{\otimes}^M_\kappa(\widehat{Y}^M). \tag{B.15}$$

Proof. Let us write for short $\mathcal{F} = \mathscr{S}((\cdot)'_\mu), \mathcal{G} = \widehat{\cdot}^M \circ (\cdot)'_\mu, (\cdot)_\mu = ((\cdot)'_\mu)'_\mu$. Note that from the canonical continuous linear map $\mathcal{F}(X) \longrightarrow \mathcal{G}(X)$ one deduces a continuous identity map $\mathcal{F}(\mathcal{G}(X)) \longrightarrow X = \mathcal{F}(\mathcal{F}(X))$.

Similarly, using lemma B.1.70 for the equality, one gets by functoriality the continuous linear map:

$$I : X\widehat{\otimes}_\kappa Y = \mathcal{G}\Big(\mathcal{F}(X)\varepsilon\mathcal{F}(Y)\Big) \longrightarrow X\widehat{\otimes}_\kappa(\widehat{Y}^M).$$

For the converse, we apply lemma B.1.23 to $L = \mathcal{F}(X)$ and $M = \mathcal{F}(Y)$, we know that $[\mathcal{F}(X)\varepsilon\mathcal{F}(Y)]'_\epsilon = [\mathcal{F}(X)\eta\mathcal{F}(Y)]'_\epsilon$ induces on $L'_\mu M'_\mu$ the ϵ-hypocontinuous tensor product. Using the reasoning of the previous lemma to identify the tensor product, this gives a continuous map

$$L'_{\mu\beta e}M'_\mu = X_{\mu\mathbb{R}}Y_\mu \longrightarrow [\mathcal{F}(X)\varepsilon\mathcal{F}(Y)]'_\epsilon \longrightarrow X_{\mu\mathbb{R}}^{\widehat{\ }M}Y_\mu.$$

This gives by definition of hypocontinuity a continuous linear map in $L(Y_\mu, L_\mathbb{R}(X_\mu, X_{\mu\mathbb{R}}^{\widehat{\ }M}Y_\mu)$. Note that from the computation of equicontinuous sets and lemma B.1.19, we have the topological identity:

$$L_\mathbb{R}(X_\mu, X_{\mu\mathbb{R}}^{\widehat{\ }M}Y_\mu)) = L_\epsilon((\mathscr{S}(X'_\mu))'_c, X_{\mu\mathbb{R}}^{\widehat{\ }M}Y_\mu)) \simeq \mathscr{S}(X'_\mu)\varepsilon(X_{\mu\mathbb{R}}^{\widehat{\ }M}Y_\mu).$$

From this identity, one gets $L_\mathbb{R}(X_\mu, X\widehat{\otimes}_\kappa Y)) = \mathscr{S}(X'_\mu)\varepsilon(X_{\mu\mathbb{R}}^{\widehat{\ }M}Y_\mu)$ is Mackey-complete since $\mathscr{S}(X'_\mu) = \mathcal{F}(X)$ is supposed so and $X_{\mu\mathbb{R}}^{\widehat{\ }M}Y_\mu$ is too by construction.

As a consequence by functoriality of Mackey-completion, the map we started from has an extension to $L(\widehat{Y_\mu}^M, \mathscr{S}(X'_\mu)\varepsilon(X_{\mu\mathbb{R}}^{\widehat{\ }M}Y_\mu)) = L(\widehat{Y_\mu}^M, L_\mathbb{R}(X_\mu, (X_{\mu\mathbb{R}}^{\widehat{\ }M}Y_\mu)))$. A fortiori, this gives a separately continuous bilinear map and thus a continuous linear map extending the map we started from:

$$J : X_{\mu i}\widehat{Y_\mu}^M \longrightarrow (X_{\mu\mathbb{R}}^{\widehat{\ }M}Y_\mu)$$

We apply lemma B.1.23 again to $L = \mathcal{F}(X)$ and $M = \mathcal{F}(\widehat{Y}^M)$, we know that $[\mathcal{F}(X)\varepsilon\mathcal{F}(\widehat{Y}^M)]'_\mu = [\mathcal{F}(X)\eta\mathcal{F}(\widehat{Y}^M))]'_\mu$ induces on $L'_\mu M'_\mu$ the inductive tensor product. Therefore, using also [Ja, Corol 8.6.5], one gets a continuous linear map

$$J : X_\mu \otimes_i [\widehat{Y_\mu}^M]_\mu \longrightarrow \Big[X_{\mu\mathbb{R}}^{\widehat{\ }M}Y_\mu\Big]_\mu = X\widehat{\otimes}_\kappa Y.$$

In turn this maps extends to the Mackey completion $X_\mu\widehat{\otimes}^M_i\widehat{Y_\mu}^M = [X\widehat{\otimes}^M_\kappa(\widehat{Y}^M)]_\mu$ and our map $J : X_\mu\widehat{\otimes}_\kappa(\widehat{Y}^M) \longrightarrow X\widehat{\otimes}^M_\kappa Y$ which is the expected inverse of I.

$\square$

Corollary B.1.72. $T = ((\cdot)^*_\rho)^*_\rho$ is a commutative monad on $(\mathbf{McSch}^{op}, \varepsilon, \mathcal{K})$.

Proof. Fix $X, Y \in \mathbf{McSch}$. Hence there is a continuous identity map $J_Y : (Y^*_\rho)^*_\rho \longrightarrow Y$. In order to build the strength, we use lemma B.1.23 and $(Y^*_\rho)^*_\rho = (Y^*_\rho)'_\mathbb{R}$ to get the the identity

$$(X\varepsilon((Y^*_\rho)^*_\rho)^*_\rho = \mathcal{C}_M\Big(\mathscr{S}\Big(\big[X\varepsilon((Y^*_\rho)'_\mathbb{R})\big]'_\mu\Big)\Big) = \mathscr{S}\Big(\big[X'^{\widehat{\ }M}_{\mu i}[(Y^*_\rho)'_c]'_\mu\Big) = \mathscr{S}\Big(X'^{\widehat{\ }M}_{\mu i}(\widehat{Y'_\mu}^M)\Big) = \mathscr{S}\Big(X'_\mu\widehat{\otimes}^M_\kappa(\widehat{Y'_\mu}^M)\Big)$$

and similarly $(X\varepsilon Y)^*_\rho = \mathscr{S}\Big(X'_\mu \widehat{\otimes}^M_\kappa Y'_\mu\Big)$.

Hence applying proposition B.1.71 to X'_μ, Y'_μ one gets that the canonical map is an isomorphism: $(X\varepsilon Y)^*_\rho \longrightarrow (X\varepsilon((Y^*_\rho)^*_\rho)^*_\rho$ hence by duality the topological isomorphism: $I_{X,Y} : ((X\varepsilon((Y^*_\rho)^*_\rho)^*_\rho)^*_\rho \simeq ((X\varepsilon Y)^*_\rho)^*_\rho$ and we claim the expected strength is

$$t_{X,Y} = J_{X\varepsilon T(Y)} \circ I^{-1}_{X,Y} \in \mathbf{McSch}^{op}(X\varepsilon T(Y), T(X\varepsilon Y)).$$

Instead of checking the axioms directly, one uses that from [T, Prop 2.4], the dialogue category already implies existence of a strength say $\tau_{X,Y}$ so that it suffices to see $\tau_{X,Y} = t_{X,Y}$ to get the relations for a strength for t. Of course we keep working in the opposite category. From the axioms of a strength, see e.g.[T, Def 1.19,(1.10),(1.12)], and of a monad, we know that $\tau_{X,Y} = J_{X\epsilon T(Y)} \circ T(\tau_{X,Y}) \circ J^{-1}_{T(X\epsilon Y)}$. Hence it suffices to see $I^{-1}_{X,Y} = T(\tau_{X,Y}) \circ J^{-1}_{T(X\epsilon Y)}$ or equivalenty $((I_{X,Y})^*_\rho)^{-1} = (J^{-1}_{T(X\epsilon Y)})^*_\rho \circ (\tau_{X,Y})^*_\rho$. But recall that the left hand side is defined uniquely by continuous extension, hence it suffices to see the restriction agrees on $X'_\mu Y'_\mu$ and the common value is determined for both sides by axiom [T, (1.12)].

Finally with our definition, the relation for a commutative monad ends with the map $J_{T(X)\varepsilon T(Y)}$ and the map obtained after removing this map and taking dual of both sides is determined as a unique extension of the same map, hence the commutativity must be satisfied. $\square$

B.1.4.6 The $*$-autonomous category $\rho - \mathbf{Ref}$.

Definition B.1.73. We thus consider $\rho - \mathbf{Ref}$, the category of ρ-reflexive spaces, with tensor product $E \otimes_\rho F = ((E_\mu \otimes_{\mathbb{R}} F_\mu)^*_\rho)^*_\rho$ and internal hom $E \multimap_\rho F = (((E^*_\rho)\varepsilon F)^*_\rho)^*_\rho$.

Recall $E_\mu = (E'_\mu)'_\mu$. For $E \in \rho - \mathbf{Ref}$ we deduce from lemma B.1.68 that $E \multimap_\rho F \simeq ((L_{\mathbb{R}}(E,F))^*_\rho)^*_\rho$.
The tensor product $E \otimes_\rho F$ is indeed a ρ-reflexive space by Theorem B.1.56.
We are ready to get that $\rho - \mathbf{Ref}$ is $*$-autonomous.

Theorem B.1.74. *The category $\rho - \mathbf{Ref}$ endowed with the tensor product $\otimes_\rho$, and internal Hom $\multimap_\rho$ is a complete and cocomplete $*$-autonomous category with dualizing object $\mathcal{K}$. It is equivalent to the Kleisli category of the comonad $T = ((\cdot)^*_\rho)^*_\rho$ in* $\mathbf{McSch}$.

Proof. Corollary B.1.64 has already dealt with categorical (co)completeness. $(\mathbf{McSch}, \epsilon, K)$ is a dialogue category by proposition B.1.66 with a commutative and idempotent continuation monad by Corollary B.1.72 and Theorem B.1.56.

The lemma B.1.44 gives $*$-autonomy. As a consequence, the induced $⅋_\rho$ is $E ⅋_\rho F = ((E\varepsilon F)^*_\rho)^*_\rho$ and the dual is still $(\cdot)^*_\rho$. The identification of $\multimap_\rho$ is obvious while $_\rho$ comes from lemma B.1.70. $\square$

B.2 Models of LL and DiLL

From now on, to really deal with smooth maps, we assume $\mathcal{K} = \mathscr{R}$.

B.2.1 Smooth maps and induced topologies. New models of LL

Any denotational model of linear logic has a morphism interpreting dereliction on any space E: $d_E : E \longrightarrow ?E$. In our context of smooth functions and reflexive spaces, it means that the topology on E must be finer than the one induced by $\mathcal{C}^\infty(E^*, \mathcal{K})$. From the model of k-reflexive spaces, we introduce a variety of new classes of smooth functions, each one inducing a different topology and a new smooth model of classical Linear Logic. We show in particular that each time the $⅋$ is interpreted as the ε-product.

We want to start from the famous Cartesian closedness [KM, Th 3.12] and its corollary, but we want an exponential law in the topological setting, and not in the bornological setting. We thus change slightly the topology on (conveniently)-smooth maps $C^\infty(E,F)$ between two locally convex spaces. We follow the simple idea to consider spaces of smooth curves on a family of base spaces stable by product, thus at least on any $\mathscr{R}^n$. Since we choose at this stage a topology, it seems reasonable to look at the induced topology on linear maps, and singling out smooth varieties indexed by $\mathscr{R}^n$ does not seem to fit well with our Schwartz space setting for ρ-reflexive spaces, but rather with a stronger nuclear setting. This suggests that the topology on smooth maps could be a guide to the choice of a topology even on the dual space. In our previous developments, the key property for us was stability by

ε product of the topology we chose, namely the Schwartz topology. This property is shared by nuclearity but there are not many functorial and commonly studied topologies having this property. We think the Seely isomorphism is crucial to select such a topology in transforming stability by tensor product into stability by product.

B.2.1.1 $\mathscr{C}$-Smooth maps and $\mathscr{C}$-completeness

We first fix a small Cartesian category $\mathscr{C}$ that will replace the category of finite dimensional spaces $\mathscr{R}^n$ as parameter space of curves.

We will soon restrict to the full category $\mathbf{F} \times \mathbf{DFS} \subset \mathbf{LCS}$ consisting of (finite) products of Fréchet spaces and strong duals of Fréchet-Schwartz spaces, but we first explain the most general context in which we know our formalism works. We assume $\mathscr{C}$ is a full Cartesian small subcategory of $k - \mathbf{Ref}$ containing $\mathscr{R}$, with smooth maps as morphisms.

Proposition B.1.49 and the convenient smoothness case suggests the following space and topology. For any $X \in \mathscr{C}$, for any $c \in C^\infty_{co}(X, E)$ a ($k - \mathbf{Ref}$ space parametrized) curve we define $C^\infty_{\mathscr{C}}(E, F)$ as the set of maps f such that $f \circ c \in C^\infty_{co}(X, F)$ for any such curve c. We call them *$\mathscr{C}$-smooth maps*. Note that $. \circ c$ is in general not surjective, but valued in the closed subspace:

$$[C^\infty_{co}(X, F)]_c = \{g \in C^\infty_{co}(X, F) : \forall x \neq y : c(x) = c(y) \Rightarrow g(x) = g(y)\}.$$

One gets a linear map $. \circ c : C^\infty_{\mathscr{C}}(E, F) \longrightarrow C^\infty_{co}(X, F)$. We equip the target space of the topology of uniform convergence of all differentials on compact subsets as before. We equip $C^\infty_{\mathscr{C}}(E, F)$ with the projective kernel topology of those maps for all $X \in \mathscr{C}$ and c smooth maps as above, with connecting maps all smooth maps $C^\infty_{co}(X, Y)$ inducing reparametrizations. Note that this projective kernel can be identified with a projective limit (indexed by a directed set). Indeed, we put an order on the set of curves $C^\infty_{co}(\mathscr{C}, E) := \sqcup_{X \in \mathscr{C}} C^\infty_{co}(X, E)/\sim$ (where two curves are identified with the equivalence relation making the preorder we define into an order). This is an ordered set with $c_1 \leqslant c_2$ if $c_1 \in C^\infty_{co}(X, E), c_2 \in C^\infty_{co}(Y, E)$ and there is $f \in C^\infty_{co}(X, Y)$ such that $c_2 \circ f = c_1$. This is moreover a directed set. Indeed given $c_i \in C^\infty_{co}(X_i, E)$, one considers $c_i' \in C^\infty_{co}(X_i \times \mathscr{R}, E)$, $c_i'(x, t) = tc_i(x)$ so that $c_i' \circ (., 1) = c_i$ giving $c_i \leqslant c_i'$. Then one can define $c \in C^\infty_{co}(X_1 \times \mathscr{R} \times X_2 \times \mathscr{R}, E)$ given by $c(x, y) = c_1'(x) + c_2'(y)$. This satisfies $c \circ (., 0) = c_1', c \circ (0, .) = c_2'$, hence $c_i \leqslant c_i' \leqslant c$. We claim that $C^\infty_{\mathscr{C}}(E, F)$ identifies with the projective limit along this directed set (we fix one c in each equivalence class) of $[C^\infty_{co}(X, F)]_c$ on the curves $c \in C^\infty_{co}(X, E)$ with connecting maps for $c_1 \leqslant c_2$, $. \circ f$ for one fixed f such that $c_2 \circ f = c_1$. This is well-defined since if g is another curve with $c_2 \circ g = c_1$, then for $u \in [C^\infty_{co}(X_2, F)]_{c_2}$ for any $x \in X_1$, $u \circ g(x) = u \circ f(x)$ since $c_2(g(x)) = c_1(x) = c_2(f(x))$ hence $\cdot \circ g = \cdot \circ f : [C^\infty_{co}(X_2, F)]_{c_2} \longrightarrow [C^\infty_{co}(X_1, F)]_{c_1}$ does not depend on the choice of f.

For a compatible sequence of such maps in $[C^\infty_{co}(X, F)]_c$, one associates the map $u : E \longrightarrow F$ such that $u(x)$ is the value at the constant curve c_x equal to x in $C^\infty_{co}(\{0\}, E) = E$. For, the curve $c \in C^\infty_{co}(X, E)$ satisfies for $x \in X$, $c \circ c_x = c_{c(x)}$, hence $u \circ c$ is the element of the sequence associated to c, hence $u \circ c \in [C^\infty_{co}(X, F)]_c$. Since this is for any curve c, this implies $u \in C^\infty_{\mathscr{C}}(E, F)$ and the canonical map from this space to the projective limit is therefore surjective. The topological identity is easy.

We summarize this with the formula:

$$C^\infty_{\mathscr{C}}(E, F) = \mathrm{proj} \lim_{c \in C^\infty_{co}(X, E)} [C^\infty_{co}(X, F)]_c \tag{B.16}$$

For $\mathscr{C} = Fin$ the category of finite dimensional spaces, $C^\infty_{Fin}(E, F) = C^\infty(E, F)$ is the space of conveniently smooth maps considered by Kriegl and Michor. We call them merely smooth maps. Note that our topology on this space is slightly stronger than theirs (before they bornologify) and that any $\mathscr{C}$-smooth map is smooth, since all our $\mathscr{C} \supset Fin$. Another important case for us is $\mathscr{C} = Ban$ the category of Banach spaces (say, to make it into a small category, of density character smaller than some fixed inaccessible cardinal, most of our considerations would be barely affected by taking the category of separable Banach spaces instead).

Lemma B.2.1. We fix $\mathscr{C}$ any Cartesian small and full subcategory of $\mathbf{k} - \mathbf{Ref}$ containing $\mathscr{R}$ and the above projective limit topology on $C^\infty_{\mathscr{C}}$. For any E, F, G lcs, with G k-quasi-complete, there is a topological isomorphism:

$$C^\infty_{\mathscr{C}}(E, C^\infty_{\mathscr{C}}(F, G)) \simeq C^\infty_{\mathscr{C}}(E \times F, G) \simeq C^\infty_{\mathscr{C}}(E \times F) \varepsilon G.$$

Moreover, the first isomorphism also holds for G Mackey-complete, and $C^\infty_{\mathscr{C}}(F, G)$ is Mackey-complete (resp. k-quasi-complete) as soon as G is. If $X \in \mathscr{C}$ then $C^\infty_{\mathscr{C}}(X, G) \simeq C^\infty_{co}(X, G)$ and if only $X \in k - \mathbf{Ref}$ there is a continuous inclusion: $C^\infty_{co}(X, G) \longrightarrow C^\infty_{\mathscr{C}}(X, G)$.

Proof. The first algebraic isomorphism comes from [KM, Th 3.12] in the case $\mathscr{C} = Fin$ (since maps smooth on smooth curves are automatically smooth when composed by "smooth varieties" by their Corollary 3.13). More generally, for any $\mathscr{C}$, the algebraic isomorphism works with the same proof in using Proposition B.1.49 instead of their Proposition 3.10. We also use their notation $f^\vee, f^\wedge$ for the maps given by the algebraic Cartesian closedness isomorphism.

Concerning the topological identification we take the viewpoint of projective kernels, for any curve $c = (c_1, c_2) : X \longrightarrow E \times F$, one can associate a curve $(c_1 \times c_2) : (X \times X) \longrightarrow E \times F$, $(c_1 \times c_2)(x, y) = (c_1(x), c_2(y))$ and for $f \in C^\infty(E, C^\infty(F, G))$, one gets $(\cdot \circ c_2)(f \circ c_1) = f^\wedge \circ (c_1 \times c_2)$ composed with the diagonal embedding gives $f^\wedge \circ (c_1, c_2)$ and thus uniform convergence of the latter is controlled by uniform convergence of the former. This gives by taking projective kernels, continuity of the direct map.

Conversely, for $f \in C^\infty(E \times F, G)$, $(\cdot \circ c_2)(f^\vee \circ c_1) = (f \circ (c_1 \times c_2))^\vee$ with c_1 on X_1, c_2 on X_2 is controlled by a map $f \circ (c_1 \times c_2)$ with $(c_1 \times c_2) : X_1 \times X_2 \longrightarrow E \times F$ and this gives the converse continuous linear map (using proposition B.1.49).

The topological isomorphism with the ε product comes from its commutation with projective limits as soon as we note that $[C^\infty_{co}(X, G)]_c = [C^\infty_{co}(X, \mathscr{R})]_c \varepsilon G$ but these are also projective limits as intersections and kernels of evaluation maps. Therefore this comes from lemma B.1.20 and from proposition B.1.48.

Finally, $C^\infty_{\mathscr{C}}(F, G)$ is a closed subspace of a product of $C^\infty_{co}(X, G)$ which are Mackey-complete or k-quasi-complete if so is G by proposition B.1.47.

For the last statement, since $id : X \longrightarrow X$ is smooth, we have a continuous map $I : C^\infty_{\mathscr{C}}(X, G) \longrightarrow C^\infty_{co}(X, G)$ in case $X \in \mathscr{C}$. Conversely, it suffices to note that for any $Y \in \mathscr{C}$, $c \in C^\infty_{co}(Y, X)$, $f \in C^\infty_{co}(X, G)$, then $f \circ c \in C^\infty_{co}(Y, G)$ by the chain rule from proposition B.1.46 and that this map is continuous linear in f for c fixed. This shows I is the identity map and gives continuity of its inverse by the universal property of the projective limit. □

We now want to extend this result beyond the case G k-quasi-complete in finding the appropriate notion of completeness depending on $\mathscr{C}$.

Lemma B.2.2. Consider the statements:

1. F is Mackey-complete.
2. For any $X \in \mathscr{C}$, $J_X : C^\infty_{co}(X)\varepsilon F \longrightarrow C^\infty_{co}(X, F)$ is a topological isomorphism
3. For any lcs E, $J^{\mathscr{C}}_E : C^\infty_{\mathscr{C}}(E)\varepsilon F \longrightarrow C^\infty_{\mathscr{C}}(E, F)$ is a topological isomorphism.
4. For any $X \in \mathscr{C}$, $f \in (C^\infty_{co}(X))'_c$, any $c \in C^\infty_{co}(X, F) \subset C^\infty_{co}(X, \tilde{F}) = C^\infty_{co}(X)\varepsilon \tilde{F}$, we have $(f \varepsilon Id)(c) \in F$ instead of its completion (equivalently with its k-quasi-completion).

We have equivalence of (2),(3) and (4) for any $\mathscr{C}$ Cartesian small and full subcategory of $k - \mathbf{Ref}$ containing $\mathscr{R}$. They always imply (1) and when $\mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$, (1) is also equivalent to them.

This suggests the following condition weaker than k-quasi-completeness:

Definition B.2.3. A locally convex space E is said $\mathscr{C}$-complete (for a $\mathscr{C}$ as above) if one of the equivalent conditions (2),(3),(4) are satisfied.

This can be the basis to define a $\mathscr{C}$-completion similar to Mackey completion with a projective definition (as intersection in the completion) based on (2) and an inductive construction (as union of a chain in the completion) based on (4).

Proof. (2) implies (3) by the commutation of ε product with projective limits as in lemma B.2.1 and (3) implies (2) using $C^\infty_{\mathscr{C}}(X, F) = C^\infty_{co}(X, F)$, for $X \in \mathscr{C}$. (2) implies (4) is obvious since the map $(f \varepsilon Id)$ gives the same value when applied in $C^\infty_{co}(X)\varepsilon F$. Conversely, looking at $u \in C^\infty_{co}(X, F) \subset C^\infty_{co}(X, \tilde{F}) = L\big((C^\infty_{co}(X))'_c, \tilde{F}\big)$, (4) says that the image of the linear map u is valued in F instead of $\tilde{F}$, so that since continuity is induced, one gets $u \in L\big((C^\infty_{co}(X))'_c, F\big)$ which gives the missing surjectivity hence (2) (using some compatibility of J_X for a space and its completion).

Let us assume (4) and prove (1). We use a characterization of Mackey-completeness in [KM, Thm 2.14 (2)], we check that any smooth curve has an anti-derivative. As in their proof of (1) implies (2) we only need to check any smooth curve has a weak integral in E (instead of the completion, in which it always exists uniquely by their lemma 2.5). But take $Leb_{[0,x]} \in (C^\infty_{co}(\mathscr{R}))'$, for a curve $c \in C^\infty(\mathscr{R}, \tilde{F})$ it is easy to see that $(Leb_{[0,x]} \epsilon Id)(c) = \int_0^x c(s)ds$

is this integral (by commutation of both operations with application of elements of F'). Hence (4) gives exactly that this integral is in F instead of its completion, as we wanted.

Let us show that (1) implies (2) first in the case $\mathscr{C} = Fin$ and take $X = \mathscr{R}^n$. One uses [**?** , Thm 5.1.7] which shows that $S = \text{Span}(ev_{\mathscr{R}^n}(\mathscr{R}^n))$ is Mackey-dense in $C^\infty(\mathscr{R}^n)'_c$. But for any map $c \in C^\infty(\mathscr{R}^n, F)$, there is a unique possible value of $f \in L(C^\infty(\mathscr{R}^n)'_c, F)$ such that $J_X(f) = c$ once restricted to $\text{Span}(ev_{\mathscr{R}^n}(\mathscr{R}^n))$. Moreover $f \in L(C^\infty(\mathscr{R}^n)'_c, \tilde{F})$ exists and Mackey-continuity implies that the value on the Mackey-closure of S lies in the Mackey closure of F in the completion, which is F. This gives surjectivity of J_X.

In the case $X \in \mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$, it suffices to show that $S = \text{Span}(\cup_{k \in \mathcal{N}} ev_X^{(k)}(X^{k+1}))$ is Mackey dense in $C^\infty(X)'_c$. Indeed, one can then reason similarly since for $c \in C^\infty_{co}(X, F)$ and $f \in L(C^\infty_{co}(X)'_c, \tilde{F})$ with $J_X(f) = c$ satisfies $f \circ ev_X^{(k)} = c^{(k)}$ which takes value in F by convenient smoothness and Mackey-completeness, hence also Mackey limits so that f will be valued in F. Let us prove the claimed density. First recall that $C^\infty_{co}(X)$ is a projective kernel of spaces $C^0(K, (X'_c)^{\epsilon k})$ via maps induced by differentials and this space is itself a projective kernel of $C^0(K \times L^k)$ for absolutely convex compact sets $K, L \subset X$. Hence by [K, 22.6.(3)], $(C^\infty_{co}(X))'$ is a locally convex hull (at least a quotient of a sum) of the space of signed measures $(C^0(K \times L^k))'$. As recalled in the proof of [58, Corol 13 p 279], every compact set K in $X \in \mathbf{F} \times \mathbf{DFS}$ is a compact subset of a Banach space, hence metrizable. Hence the space of measure signed measures $(C^0(K \times L^k))'$ is metrizable too for the weak-* topology (see e.g. [DM]), and by Krein-Millman's Theorem [K, 25.1.(3)] every point in the (compact) unit ball is a weak-* limit of an absolutely convex combination of extreme points, namely Dirac masses [K, 25.2.(2)], and by metrizability one can take a sequence of such combinations, which is bounded in $(C^0(K \times L^k))'$. Hence its image in $E = (C^\infty_{co}(X))'$ is bounded in some Banach subspace, with equicontinuous ball B (by image of an equicontinuous sets, a ball in a Banach space by the transpose of a continuous map) and converges weakly. But from [58, Prop 11 p 276], $C^\infty_{co}(X)$ is a Schwartz space, hence there is an other equicontinuous set C such that B is compact in E_C hence the weakly convergent sequence admitting only at most one limit point must converge normwise in E_C. Finally, we have obtained Mackey convergence of this sequence in $E = (C^\infty_{co}(X))'$ and looking at its form, this gives exactly Mackey-density of S. $\square$

B.2.1.2 Induced topologies on linear maps

In the setting of the previous subsection, $E' \subset C^\infty_{\mathscr{C}}(E, \mathscr{R})$. From Mackey-completeness, this extends to an inclusion of the Mackey completion, on which one obtains an induced topology which coincides with the topology of uniform convergence on images by smooth curves with source $X \in \mathscr{C}$ of compacts in this space. Indeed, the differentials of the smooth curve is also smooth on a product and the condition on derivatives therefore reduces to this one. This can be described functorially in the spirit of $\mathscr{S}$.

We first consider $\mathscr{C} \subset k - \mathbf{Ref}$ a full Cartesian subcategory.

Let $\mathscr{C}^\infty$ be the smallest class of locally convex spaces containing $C^\infty_{co}(X, \mathcal{K})$ for $X \in \mathscr{C}$ ($X = \{0\}$ included) and stable by products and subspaces. Let $\mathscr{S}_{\mathscr{C}}$ the functor on **LCS** of associated topology in this class described by [45, 2.6.4]. This functor commutes with products.

Example B.2.4. If $\mathscr{C} = \{0\}$ then $\mathscr{C}^\infty = \mathbf{Weak}$ the category of spaces with their weak topology, since $\mathcal{K}$ is a universal generator for spaces with their weak topology. Thus the weak topology functor is $\mathscr{S}_{\{0\}}(E)$.

Example B.2.5. If $\mathscr{C}^\infty \subset \mathscr{D}^\infty$ (e.g. if $\mathscr{C} \subset \mathscr{D}$) then, from the very definition, there is a natural transformation $id \longrightarrow \mathscr{S}_{\mathscr{D}} \longrightarrow \mathscr{S}_{\mathscr{C}}$ with each map $E \longrightarrow \mathscr{S}_{\mathscr{D}}(E) \longrightarrow \mathscr{S}_{\mathscr{C}}(E)$ is a continuous identity map.

Lemma B.2.6. For any lcs E, $(\mathscr{S}_{\mathscr{C}}(E))' = E'$ algebraically.

Proof. Since $\{0\} \subset \mathscr{C}$, there is a continuous identity map $E \longrightarrow \mathscr{S}_{\mathscr{C}}(E) \longrightarrow \mathscr{S}_{\{0\}}(E) = (E'_\sigma)'_\sigma$. The Mackey-Arens theorem concludes. $\square$

As a consequence, E and $\mathscr{S}_{\mathscr{C}}(E)$ have the same bounded sets and therefore are simultaneously Mackey-complete. Hence $\mathscr{S}_{\mathscr{C}}$ commutes with Mackey-completion. Moreover, the class $\mathscr{C}^\infty$ is also stable by ε-product, since this product commutes with projective kernels and $C^\infty_{co}(X, \mathcal{K})\varepsilon C^\infty_{co}(Y, \mathcal{K}) = C^\infty_{co}(X \times Y, \mathcal{K})$ and we assumed $X \times Y \in \mathscr{C}$.

We now consider the setting of the previous subsection, namely we also assume $\mathscr{R} \in \mathscr{C}$, $\mathscr{C}$ small and identify the induced topology $E'_{\mathscr{C}} \subset C^\infty_{\mathscr{C}}(E, \mathscr{R})$.

Lemma B.2.7. For any lcs E, there is a continuous identity map: $E'_{\mathscr{C}} \longrightarrow \mathscr{S}_{\mathscr{C}}(E'_c)$.

If moreover E is $\mathscr{C}$-complete, this is a topological isomorphism.

Proof. For the direct map we use the universal property of projective kernels. Consider a continuous linear map $f \in L(E'_c, C^\infty_{co}(X,\mathcal{K})) = C^\infty_{co}(X,\mathcal{K})\varepsilon E$ and the corresponding $J_X(f) \in C^\infty_{co}(X,E)$, then by definition of the topology $\cdot \circ J_X(f) : C^\infty_{\mathscr{C}}(X,E) \longrightarrow C^\infty_{co}(X,\mathscr{R})$ is continuous and by definition, its restriction to E' agrees with f, hence $f : E'_{\mathscr{C}} \longrightarrow C^\infty_{co}(X,\mathcal{K})$ is also continuous. Taking a projective kernel over all those maps gives the expected continuity.

Conversely, if E is $\mathscr{C}$-complete, note that $E'_c \longrightarrow E'_{\mathscr{C}}$ is continuous using again the universal property of a kernel, it suffices to see that for any $X \in \mathscr{C}, c \in C^\infty_{\mathscr{C}}(X,E)$ then $\cdot \circ c : E'_c \longrightarrow C^\infty_{\mathscr{C}}(X,K)$ is continuous, and this is the content of the surjectivity of J_X in lemma B.2.2 (2) since $\cdot \circ c = J_X^{-1}(c)$. Hence since $E'_{\mathscr{C}} \in \mathscr{C}^\infty$ by definition as projective limit, one gets by functoriality the continuity of $\mathscr{S}_{\mathscr{C}}(E'_c) \longrightarrow E'_{\mathscr{C}}$. ☐

We are going to give more examples in a more restricted context. We now fix $Fin \subset \mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$. But the reader may assume $\mathscr{C} \subset Ban$ if he or she wants, our case is not such more general. Note that then $C^\infty_{\mathscr{C}}(E,F) = C^\infty(E,F)$ algebraically. For it suffices to see $C^\infty_{co}(X,F) = C^\infty(X,F)$ for any $X \in \mathbf{F} \times \mathbf{DFS}$ (since then the extra smoothness condition will be implied by convenient smoothness). Note that any such X is ultrabornological (using [Ja, Corol 13.2.4], [Ja, Corol 13.4.4,5] since a DFS space is reflexive hence its strong dual is barrelled [Ja, Prop 11.4.1] and for a dual of a Fréchet space, the quoted result implies it is also ultrabornological, for products this is [Ja, Thm 13.5.3]). By Cartesian closedness of both sides this reduces to two cases. For any Fréchet space X, Fréchet smooth maps are included in $C^\infty_{co}(X,F)$ which is included in $C^\infty(X,F)$ which coincides with the first space of Fréchet smooth maps by [KM, Th 4.11.(1)] (which ensures the continuity of Gateaux derivatives with value in bounded linear maps with strong topology for derivatives, those maps being the same as continuous linear maps as seen the bornological property). The case of strong duals of Fréchet-Schwartz spaces is similar using [KM, Th 4.11.(2)]. The index $\mathscr{C}$ in $C^\infty_{\mathscr{C}}(E,F)$ remains to point out the different topologies.

Example B.2.8. If $\mathscr{C} = \mathbf{F} \times \mathbf{DFS}$ (say with objects of density character smaller than some inaccessible cardinal) then $\mathscr{C}^\infty \subset \mathbf{Sch}$, from [58, Corol 13 p 279]. Let us see equality. Indeed, $(\ell^1(\mathcal{N}))'_c \subset C^\infty_{co}(\ell^1(\mathcal{N}),\mathcal{K})$ and $(\ell^1(\mathcal{N}))'_c = (\ell^1(\mathcal{N}))'_\mu$ (since on $\ell^1(\mathcal{N})$ compact and weakly compact sets coincide [HNM, p 37]), and $(\ell^1(\mathcal{N}))'_\mu$ is a universal generator of Schwartz spaces [HNM, Corol p 36], therefore $C^\infty_{co}(\ell^1(\mathcal{N}),\mathcal{K})$ is also such a universal generator. Hence we even have $\mathscr{C}^\infty = Ban^\infty = \mathbf{Sch}$. Let us deduce even more of such type of equalities.

Note also that $Sym(E'_c\varepsilon E'_c) \subset C^\infty_{co}(E,\mathcal{K})$ is a complemented subspace given by quadratic forms. In case $E = H$ is an infinite dimensional Hilbert space, by Buchwalter's theorem $H'_c\varepsilon H'_c = (H\widehat{\otimes}_\pi H)'_c$ and it is well-known that $\ell^1(\mathcal{N}) \simeq D$ is a complemented subspace (therefore a quotient) of $H\widehat{\otimes}_\pi H$ as diagonal copy (see e.g. [Ry, ex 2.10]) with the projection a symmetric map. Thus $D'_c \subset H'_c\varepsilon H'_c$ and it is easy to see it is included in the symmetric part $Sym(E'_c\varepsilon E'_c)$. As a consequence, $C^\infty_{co}(H,\mathcal{K})$ is also such a universal generator of Schwartz spaces.

Finally, consider $E = \ell^m(\mathcal{N},\mathcal{C})$ $m \in \mathcal{N}, m \geqslant 1$. The canonical multiplication map from Holder $\ell^m(\mathcal{N},\mathcal{C})^{\pi m} \longrightarrow \ell^1(\mathcal{N},\mathcal{C})$ is a metric surjection realizing the target as a quotient of the symmetric subspace generated by tensor powers (indeed $\sum a_k e_k$ is the image of $(\sum a_k^{1/m} e_k)^m$ so that $(\ell^1(\mathcal{N},\mathcal{C}))'_c \subset Sym([(\ell^m(\mathcal{N},\mathcal{C}))'_c]^{\varepsilon m})$. Thus $C^\infty_{co}(\ell^m(\mathcal{N},\mathcal{C}),\mathcal{K})$ is also such a universal generator of Schwartz spaces.

We actually checked that for any $\mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$ with $\ell^1(\mathcal{N}) \in \mathscr{C}$ or $\ell^2(\mathcal{N}) \in \mathscr{C}$ or $\ell^m(\mathcal{N},\mathcal{C}) \in \mathscr{C}$ then $\mathscr{C}^\infty = \mathbf{Sch}$ so that

$$\mathscr{S} = \mathscr{S}_{Ban} = \mathscr{S}_{Hilb} = \mathscr{S}_{\mathscr{C}} = \mathscr{S}_{\mathbf{F}\times\mathbf{DFS}}.$$

As a consequence, we can improve slightly our previous results in this context:

Lemma B.2.9. Let $\mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$ as above. For any lcs E, there is a continuous identity map: $E'_{\mathscr{C}} \longrightarrow \mathscr{S}_{\mathscr{C}}(E'_\mu) \longrightarrow \mathscr{S}_{\mathscr{C}}(E'_c)$. If moreover E is Mackey-complete, this is a topological isomorphism.

Proof. Indeed by definition $\mathscr{S}_{\mathscr{C}}(E'_\mu)$ is described by a projective limit over maps $L(E'_\mu, C^\infty_{co}(X,\mathcal{K})) = E\eta C^\infty_{co}(X,\mathcal{K}) = C^\infty_{co}(X,\mathcal{K})\varepsilon E \subset C^\infty_{co}(X,E)$ by the Schwartz property. As in lemma B.2.7, the identity map $E'_{\mathscr{C}} \longrightarrow \mathscr{S}_{\mathscr{C}}(E'_\mu)$. But by functoriality one has also a continuous identity map $\mathscr{S}_{\mathscr{C}}(E'_\mu) \longrightarrow \mathscr{S}_{\mathscr{C}}(E'_c)$ and in the Mackey-complete case $\mathscr{S}_{\mathscr{C}}(E'_c) \longrightarrow E'_{\mathscr{C}}$ by lemma B.2.7. (This uses that Mackey-complete implies $\mathscr{C}$-complete in our case by the last statement in lemma B.2.2). ☐

Example B.2.10. Note also that if D is a quotient with quotient topology of a Fréchet space C with respect to a closed subspace, then $C^\infty_{co}(D,\mathcal{K})$ is a subspace of $C^\infty_{co}(C,\mathcal{K})$ with induced topology. Indeed, the injection is obvious and derivatives agree, and since from [K, 22.3.(7)], compacts are quotients of compacts, the topology is indeed induced. Therefore if $\mathscr{D}$ is obtained from $\mathscr{C} \subset Fre$, the category of Fréchet spaces, by taking all quotients by closed subspaces, then $\mathscr{C}^\infty = \mathscr{D}^\infty$.

Example B.2.11. If $\mathscr{C} = Fin$ then $Fin^\infty = \mathbf{Nuc}$, since $C^\infty_{co}(\mathscr{R}^n, \mathcal{K}) \simeq \mathfrak{s}^{\mathcal{N}}$ [78, (7) p 383], a countable direct product of classical sequence space $\mathfrak{s}$, which is a universal generator for nuclear spaces. Thus, the associated nuclear topology functor is $\mathscr{N}(E) = \mathscr{S}_{Fin}(E)$.

We now provide several more advanced examples which will enable us to prove that we obtain different comonads in several of our models of LL. They are all based on the important approximation property of Grothendieck.

Example B.2.12. If E a Fréchet space without the approximation property (in short AP, for instance $E = B(H)$ the space of bounded operators on a Hilbert space), then from [58, Thm 7 p 293], $C^\infty_{co}(E)$ does not have the approximation property. Actually, $E'_c \subset C^\infty_{co}(E)$ is a continuously complemented subspace so that so is $((E'_c)^*_\rho)^*_\rho \subset ((C^\infty_{co}(E))^*_\rho)^*_\rho$. But for any Banach space $E'_c = \mathscr{S}(E'_\mu)$ is Mackey-complete so that $(E'_c)^*_\rho = \mathscr{S}(E)$, $((E'_c)^*_\rho)^*_\rho = E^*_\rho = E'_c = \mathscr{S}(E'_\mu)$. Thus since for a Banach space E has the approximation property if and only if $\mathscr{S}(E'_\mu)$ has it [Ja, Thm 18.3.1], one deduces that $((C^\infty_{co}(E))^*_\rho)^*_\rho$ does not have the approximation property [Ja, Prop 18.2.3].

Remark B.2.13. We will see in appendix in lemma B.2.40 that for any lcs E, $((C^\infty_{Fin}(E))^*_\rho)^*_\rho$ is Hilbertianizable, hence it has the approximation property. This implies that $\mathscr{N}(E'_\mu) \subset C^\infty_{Fin}(E)$ with induced topology is not complemented, as soon as E is Banach space without AP, since otherwise $((\mathscr{N}(E'_\mu))^*_\rho)^*_\rho \subset ((C^\infty_{Fin}(E))^*_\rho)^*_\rho$ would be complemented and $((\mathscr{N}(E'_\mu))^*_\rho)^*_\rho = ((E'_c)^*_\rho)^*_\rho = E'_c$ would have the approximation property, and this may not be the case. This points out that the change to a different class of smooth function in the next section is necessary to obtain certain models of DiLL. Otherwise, the differential that would give such a complementation cannot be continuous.

We define $E^*_{\mathscr{C}}$ for $E \in \mathbf{McSch}$ as the Mackey completion of $\mathscr{S}_{\mathscr{C}}((\widehat{E}^M)'_\mu)$, i.e. since $\mathscr{S}_{\mathscr{C}}\mathscr{S} = \mathscr{S}_{\mathscr{C}}$:

$$E^*_{\mathscr{C}} = \mathscr{S}_{\mathscr{C}}(E^*_\rho).$$

Proposition B.2.14. *Let $Fin \subset \mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$ a small and full Cartesian subcategory. The full subcategory $\mathscr{C} - \mathbf{Mc} \subset \mathbf{McSch}$ of objects satisfying $E = \mathscr{S}_{\mathscr{C}}(E)$ is reflective of reflector $\mathscr{S}_{\mathscr{C}}$. $(\mathscr{C} - \mathbf{Mc}^{op}, \varepsilon, \mathcal{K}, (\cdot)^*_{\mathscr{C}})$ is a Dialogue category.*

Proof. Since $E \longrightarrow \mathscr{S}_{\mathscr{C}}(E)$ is the continuous identity map, the first statement about the reflector is obvious. $\mathscr{C} - \mathbf{Mc}$ is stable by ε-product since $\mathscr{S}_{\mathscr{C}}(E)\varepsilon\mathscr{S}_{\mathscr{C}}(F)$ is a projective kernel of $C^\infty_{co}(X)\varepsilon C^\infty_{co}(Y) = C^\infty_{co}(X \times Y) \in \mathcal{C}^\infty$. We use Proposition B.1.66 to get $(\mathbf{McSch}, \epsilon, \mathcal{K}, (\cdot)^*_\rho)$. One can apply Lemma B.1.41 since we have $\mathscr{S}_{\mathscr{C}} \circ (\cdot)^*_\rho = (\cdot)^*_{\mathscr{C}}$ and $I : \mathscr{C} - \mathbf{Mc} \subset \mathbf{McSch}$ satisfies $\mathscr{S}_{\mathscr{C}}(I(E\varepsilon F)) = I(E\varepsilon F) = I(E)\varepsilon I(F)$. This concludes. ☐

B.2.1.3 A general construction for LL models

We used intensively Dialogue categories from [MT, T] to obtain $*$-autonomous categories, but their notion of models of tensor logic is less fit for our purposes since the Cartesian category they use need not be Cartesian closed. For us trying to check their conditions involving an adjunction at the level of the Dialogue category would imply introducing a non-natural category of smooth maps while we have already a good Cartesian closed category. Therefore we propose a variant of their definition using relative adjunctions [U].

Definition B.2.15. A linear (resp. and commutative) categorical model of λ-tensor logic is a complete and cocomplete dialogue category $(\mathcal{C}^{op}, \mathbin{⅋}_{\mathcal{C}}, I, \neg)$ with a (resp. commutative and idempotent) continuation monad $T = \neg\neg$, jointly with a Cartesian category $(\mathcal{M}, \times, 0)$, a symmetric strongly monoidal functor $NL : \mathcal{M} \longrightarrow \mathcal{C}^{op}$ having a right $\neg$-relative adjoint U. The model is said to be a *Seely* model if U is bijective on objects.

This definition is convenient for its concision, but it does not emphasize that $\mathcal{M}$ must be Cartesian closed. Since our primitive objects are functional, we will prefer an equivalent alternative based on the two relations we started to show in lemma B.2.1, namely an enriched adjointness of Cartesian closedness and a compatibility with $⅋$.

Definition B.2.16. A (resp. commutative) λ-categorical model of λ-tensor logic is a complete and cocomplete dialogue category $(\mathcal{C}^{op}, \mathbin{⅋}_{\mathcal{C}}, 1_{\mathcal{C}} = \mathcal{K}, \neg)$ with a (resp. commutative and idempotent) continuation monad $T = \neg\neg$, jointly with a Cartesian closed category $(\mathcal{M}, \times, 0, [\cdot, \cdot])$, and a functor $NL : \mathcal{M} \longrightarrow \mathcal{C}^{op}$ having a right $\neg$-relative adjoint U, which is assumed faithful, and compatibility natural isomorphisms in $\mathcal{M}, \mathcal{C}$ respectively:

$$\Xi_{E,F} : U(NL(E) \mathbin{⅋}_{\mathcal{C}} F) \longrightarrow [E, U(F)], \Lambda^{-1}_{E,F,G} : NL(E) \mathbin{⅋}_{\mathcal{C}} \Big(NL(F) \mathbin{⅋}_{\mathcal{C}} G\Big) \longrightarrow NL(E \times F) \mathbin{⅋}_{\mathcal{C}} G$$

satisfying the following six commutative diagrams (where $Ass^{⅋}, \rho, \lambda, \sigma^{⅋}$ are associator, right and left unitors and braiding in $\mathcal{C}^{op}$ and $\Lambda^{\mathcal{M}}, \sigma^{\times}, \ell, r$ are the curry map, braiding and unitors in the Cartesian closed category $\mathcal{M}$) expressing an intertwining between curry maps:

$$\begin{tikzcd}
U\Big(NL(E) ⅋_{\mathcal{C}} \Big(NL(F) ⅋_{\mathcal{C}} G\Big)\Big) \arrow[r, "\Xi_{E,NL(F)⅋_{\mathcal{C}}G}"] & {[E, U(NL(F) ⅋_{\mathcal{C}} G)]} \arrow[r, "{[id_E, \Xi_{F,G}]}"] & {[E,[F,U(G)]]} \\
U\Big(NL(E\times F) ⅋_{\mathcal{C}} G\Big) \arrow[u, "U(\Lambda_{E,F,G})"] \arrow[rr, "\Xi_{E\times F,G}"] & & {[E\times F, U(G)]} \arrow[u, "\Lambda^{\mathcal{M}}"']
\end{tikzcd}$$

compatibility of Ξ with the (relative) adjunctions (written $\simeq$ and φ, the characteristic isomorphism of the dialogue category $\mathcal{C}^{op}$):

$$\begin{tikzcd}
\mathcal{M}(0, U\Big(NL(E) ⅋_{\mathcal{C}} F\Big)) \arrow[r, "\simeq"] & \mathcal{C}(\neg(NL(0)), NL(E) ⅋_{\mathcal{C}} F) \arrow[r, "\varphi^{op}_{NL(E),F,NL(0)}"] & \mathcal{C}(\neg(F ⅋_{\mathcal{C}} NL(0)), NL(E)) \\
M(U\Big(NL(E) ⅋_{\mathcal{C}} F\Big)) \arrow[u, "M(I_{\mathcal{M}})"] \arrow[rr, "M(\Xi_{E,F})"] & & {M([E,U(F)]) = \mathcal{M}(E,U(F))} \arrow[u, "\simeq"']
\end{tikzcd}$$

compatibility with associativity:

$$\begin{tikzcd}
NL(E) ⅋_{\mathcal{C}} \Big(NL(F) ⅋_{\mathcal{C}} G\Big) \arrow[r, "\Lambda^{-1}_{E,F,G}"] \arrow[d, "Ass^{⅋}_{NL(E),NL(F),G}"'] & NL(E\times F) ⅋_{\mathcal{C}} G \arrow[r, "\rho_{NL(E\times F)⅋_{\mathcal{C}}G}"] & \Big(NL(E\times F) ⅋_{\mathcal{C}} \mathcal{K}\Big) ⅋_{\mathcal{C}} G \\
\Big(NL(E) ⅋_{\mathcal{C}} NL(F)\Big) ⅋_{\mathcal{C}} G \arrow[rr, "(NL(E)⅋\, \rho_{NL(F)})⅋\, G"] & & \Big(NL(E) ⅋_{\mathcal{C}} (NL(F) ⅋_{\mathcal{C}} \mathcal{K})\Big) ⅋_{\mathcal{C}} G \arrow[u, "\Lambda^{-1}_{E,F,\mathcal{K}}⅋_{\mathcal{C}}G"']
\end{tikzcd}$$

compatibility with symmetry,

$$\begin{tikzcd}
NL(E) ⅋_{\mathcal{C}} NL(F) \arrow[r, "NL(E)⅋\, \rho_{NL(F)}"] \arrow[d, "\sigma^{⅋}_{NL(E),NL(F)}"'] & NL(E) ⅋_{\mathcal{C}} \Big(NL(F) ⅋_{\mathcal{C}} \mathcal{K}\Big) \arrow[r, "\Lambda^{-1}_{E,F,\mathcal{K}}"] & NL(E\times F) ⅋_{\mathcal{C}} \mathcal{K} \arrow[d, "NL(\sigma^{\times}_{E,F})⅋_{\mathcal{C}}\mathcal{K}"] \\
NL(F) ⅋_{\mathcal{C}} NL(E) \arrow[r, "NL(F)⅋\, \rho_{NL(E)}"] & NL(F) ⅋_{\mathcal{C}} \Big(NL(E) ⅋_{\mathcal{C}} \mathcal{K}\Big) \arrow[r, "\Lambda^{-1}_{F,E,\mathcal{K}}"] & NL(F\times E) ⅋_{\mathcal{C}} \mathcal{K}
\end{tikzcd}$$

and compatibility with unitors for a given canonical isomorphism $\epsilon : \mathcal{K} \longrightarrow NL(0_{\mathcal{M}})$:

$$\begin{tikzcd}
NL(0_{\mathcal{M}}) ⅋_{\mathcal{C}} NL(F) \arrow[r, "NL(0_{\mathcal{M}})⅋\, \rho_{NL(F)}"] & NL(0_{\mathcal{M}}) ⅋_{\mathcal{C}} \Big(NL(F) ⅋_{\mathcal{C}} \mathcal{K}\Big) \arrow[r, "\Lambda^{-1}_{0_{\mathcal{M}},F,\mathcal{K}}"] & NL(0_{\mathcal{M}}\times F) ⅋_{\mathcal{C}} \mathcal{K} \arrow[d, "NL(\ell_F)⅋_{\mathcal{C}}\mathcal{K}"] \\
\mathcal{K} ⅋_{\mathcal{C}} NL(F) \arrow[u, "\epsilon⅋_{\mathcal{C}}NL(F)"] \arrow[r, "\lambda^{-1}_{NL(F)}"] & NL(F) \arrow[r, "\rho_{NL(F)}"] & NL(F) ⅋_{\mathcal{C}} \mathcal{K}
\end{tikzcd}$$

$$\begin{tikzcd}
NL(E) ⅋_{\mathcal{C}} NL(0_{\mathcal{M}}) \arrow[r, "NL(E)⅋\, \rho_{NL(0_{\mathcal{M}})}"] & NL(E) ⅋_{\mathcal{C}} \Big(NL(0_{\mathcal{M}}) ⅋_{\mathcal{C}} \mathcal{K}\Big) \arrow[r, "\Lambda^{-1}_{E,0_{\mathcal{M}},\mathcal{K}}"] & NL(E\times 0_{\mathcal{M}}) ⅋_{\mathcal{C}} \mathcal{K} \arrow[d, "NL(r_E)⅋_{\mathcal{C}}\mathcal{K}"] \\
NL(E) ⅋_{\mathcal{C}} \mathcal{K} \arrow[u, "NL(E)⅋_{\mathcal{C}}\epsilon"] \arrow[rr, "id"] & & NL(E) ⅋_{\mathcal{C}} \mathcal{K}
\end{tikzcd}$$

The model is said to be a *Seely* model if U is bijective on objects.

In our examples, U must be thought of as an underlying functor that forgets the linear structure of $\mathcal{C}$ and sees it as a special smooth structure in $\mathcal{M}$. Hence we could safely assume it faithful and bijective on objects.

Proposition B.2.17. *A Seely λ-model of λ-tensor logic is a Seely linear model of λ-tensor logic too*

Proof. Start with a λ-model. Let

$$\mu_{E,F}^{-1} = \rho_{NL(E\times F)}^{-1} \circ \Lambda_{E,F,\mathcal{K}}^{-1} \circ (id_{NL(E)} \parr \rho_{NL(F)}) : NL(E) \parr_{\mathcal{C}} NL(F) \longrightarrow NL(E \times F)$$

using the right unitor ρ of $\mathcal{C}^{op}$, and composition in $\mathcal{C}$. The identity isomorphism ϵ is also assumed given. Since μ is an isomorphism it suffices to see it makes NL a lax symmetric monoidal functor. The symmetry condition is exactly the diagram of compatibility with symmetry that we assumed and similarly for the unitality conditions. The first assumed diagram with Λ used in conjection with U faithful enables to transport any diagram valid in the Cartesian closed category to an enriched version, and the second diagram concerning compatibility with associativity is then the only missing part needed so that μ satisfies the relation with associators of $\parr$, $\times$. □

Those models enable to recover models of linear logic. We get a linear-non-linear adjunction in the sense of [Ben] (see also [PAM, def 21 p 140]).

Theorem B.2.18. $(\mathcal{C}^{op}, \parr_{\mathcal{C}}, I, \neg, \mathcal{M}, \times, 0, NL, U)$ *a Seely linear model of λ-tensor logic. Let $\mathcal{D} \subset \mathcal{C}$ the full subcategory of objects of the form $\neg C, C \in \mathcal{C}$. Then, $\mathcal{N} = U(\mathcal{D})$ is equivalent to $\mathcal{M}$. $\neg \circ NL : \mathcal{N} \longrightarrow \mathcal{D}$ is left adjoint to $U : \mathcal{D} \longrightarrow \mathcal{N}$ and forms a linear-non-linear adjunction. Finally $! = \neg \circ NL \circ U$ gives a comonad on $\mathcal{D}$ making it a $*$-autonomous complete and cocomplete Seely category with Kleisli category for $!$ isomorphic to $\mathcal{N}$.*

Proof. This is a variant of [T, Thm 2.13]. We already saw in lemma B.1.44 that $\mathcal{D}$ is $*$-autonomous with the structure defined there. Composing the natural isomorphisms for $F \in \mathcal{D}, E \in \mathcal{M}$

$$\mathcal{M}(E, U(F)) \simeq \mathcal{C}^{op}(NL(E), \neg F) \simeq \mathcal{D}(\neg(NL(E)), F),$$

one gets the stated adjunction. The equivalence is the inclusion with inverse $\neg\neg : \mathcal{M} \longrightarrow \mathcal{N}$ which is based on the canonical map in $\mathcal{C}$, $\eta_E : \neg\neg E \longrightarrow E$ which is mapped via U to a corresponding natural transformation in $\mathcal{M}$. It is an isomorphism in $\mathcal{N}$ since any element is image of U enabling to use the $\neg$-relative adjunction for $E \in \mathcal{C}$:

$$\mathcal{M}(U(E), U(\neg\neg E)) \simeq \mathcal{C}^{op}(NL(U(E)), \neg\neg\neg E) \simeq \mathcal{C}^{op}(NL(U(E)), \neg E) \simeq \mathcal{M}(U(E), U(E)).$$

Hence the element corresponding to identity gives the inverse of η_E. Since $\mathcal{D}$ is coreflective in $\mathcal{C}$, the coreflector preserves limits enabling to compute them in $\mathcal{D}$, and by $*$-autonomy, it therefore has colimits (which must coincide with those in $\mathcal{C}$). By [PAM, Prop 25 p 149], since $U : \mathcal{D} \longrightarrow \mathcal{N}$ is still a bijection on objects, the fact that $\mathcal{D}$ is a Seely category follows and the computation of its Kleisli category too. The co-unit and co-multiplication of the co-monad $!$ come from the relative adjunction $U \dashv_{neg} NL$, and correspond respectively to the identity on E in M, and to the composition of the unit of the adjunction by $!$ on the left and U on the right. □

Remark B.2.19. In the previous situation, we checked that $U(E) \simeq U(\neg\neg E)$ in $\mathcal{M}$ and we even obtained a natural isomorphism $U \circ \neg\neg \simeq U$ and this has several consequences we will reuse. First $\neg$ is necessarily faithful on $\mathcal{C}$ since if $\neg(f) = \neg(g)$ then $U \circ \neg\neg(f) = U \circ \neg\neg(g)$ hence $U(f) = U(g)$ and U is assumed faithful hence $f = g$. Let us see that as a consequence, as for ε, $\parr_{\mathcal{C}}$ preserves monomorphisms. Indeed if $f : E \longrightarrow F$ is a monomorphism, $\neg\neg(f \parr_{\mathcal{C}} id_G)$ is the application of the $\neg\neg(\cdot) \parr G$ for the $*$-autonomous continuation category, hence a right adjoint functor, hence $\neg\neg(f \parr_{\mathcal{C}} id_G)$ is a monomorphism since right adjoints preserve monomorphisms. Since $\neg\neg$ is faithful one deduces $f \parr_{\mathcal{C}} id_G$ is a monomorphism too.

B.2.1.4 A class of examples of LL models

We now fix $Fin \subset \mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$. Recall that then $C^{\infty}_{\mathscr{C}}(E, F) = C^{\infty}(E, F)$ algebraically for any lcs E, F. The index $\mathscr{C}$ remains to point out the different topologies.

Definition B.2.20. We define $E^*_{\mathscr{C}}$ as the Mackey completion of $\mathscr{S}_{\mathscr{C}}((\widehat{E}^M)'_\mu)$. Thus we can define $\mathscr{C}$-reflexive spaces as satisfying $E = (E^*_{\mathscr{C}})^*_{\mathscr{C}}$. We denote by $\mathcal{C} - \mathbf{ref}$ the category of $\mathscr{C}$-reflexive spaces and linear maps.

The dialogue category $\mathscr{C} - \mathbf{Mc}$ enables to give a situation similar to $\rho - \mathbf{Ref}$. First for any lcs E, $(E^*_{\mathscr{C}})' = Mackey - completeE$ algebraically from lemma B.2.6.

Corollary B.2.21. For any lcs E , $E^*_{\mathscr{C}}$ is $\mathscr{C}$-reflexive, and $(E^*_{\mathscr{C}})'_{\mathscr{C}}$ is Mackey-complete, hence equal to $(E^*_{\mathscr{C}})^*_{\mathscr{C}}$

Proof. We saw $E^*_{\mathscr{C}} = \mathscr{S}_{\mathscr{C}}(E^*_\rho)$ but from lemma B.2.6 and commutation of $\mathscr{S}_{\mathscr{C}} = \mathscr{S} \circ \mathscr{S}_{\mathscr{C}}$ with Mackey completions, $[\mathscr{S}_{\mathscr{C}}(E)]^*_\rho = E^*_\rho$. Hence composing and using Theorem B.1.56, one gets the claimed reflexivity:

$$((E^*_{\mathscr{C}})^*_{\mathscr{C}})^*_{\mathscr{C}} = \mathscr{S}_{\mathscr{C}}\Big[((E^*_\rho)^*_\rho)^*_\rho\Big] = \mathscr{S}_{\mathscr{C}}\Big[E^*_\rho\Big] = E^*_{\mathscr{C}}$$

Similarly $(E^*_{\mathscr{C}})'_{\mathscr{C}} = \mathscr{S}_{\mathscr{C}}((E^*_\rho)'_{\mathbb{R}})$ which is Mackey-complete by the same result. $\square$

Theorem B.2.22. *Let $Fin \subset \mathscr{C} \subset$ $\mathbf{F} \times \mathbf{DFS}$ a full Cartesian small subcategory. $\mathcal{C} - \mathbf{ref}$ is a complete and cocomplete $*$-autonomous category with tensor product $E_{\mathscr{C}}F = (E^*_{\mathscr{C}}\varepsilon F^*_{\mathscr{C}})^*_{\mathscr{C}}$ and dual $(.)^*_{\mathscr{C}}$ and dualizing object $\mathcal{K}$. It is stable by arbitrary products. It is equivalent to the Kleisli category of $\mathscr{C} - \mathbf{Mc}$ and to ρ-**Ref** as a $*$-autonomous category via the inverse functors: $\mathscr{S}_{\mathscr{C}} : \rho\text{-}\mathbf{Ref} \longrightarrow \mathscr{C}\text{-}\mathbf{Ref}$ and $\mathscr{S}([.]_\mu) : \mathscr{C}\text{-}\mathbf{Ref} \longrightarrow \rho\text{-}\mathbf{Ref}$.*

Proof. This is a consequence of lemma B.1.44 applied to the Dialogue category $(\mathscr{C} - \mathbf{Mc}^{op}, \epsilon, \mathcal{K}, (\cdot)^*_{\mathscr{C}})$ from proposition B.2.14. Recall from the previous proof that $(\mathscr{S}_{\mathscr{C}}(E))'_\mu = E'_\mu$ and $((E)^*_{\mathscr{C}})^*_{\mathscr{C}} = \mathscr{S}_{\mathscr{C}}((E^*_\rho)^*_\rho)$. This implies the two functors are inverse of each other as stated.

We show they intertwine the other structure. We already noticed $E^*_{\mathscr{C}} = \mathscr{S}_{\mathscr{C}}(E^*_\rho)$. We computed in lemma B.1.23:

$$(E^*_\rho\varepsilon F^*_\rho)'_\mu \simeq (E^*_\rho\eta F^*_\rho)'_\mu \simeq (E^*_\rho)'^{\widehat{}M}_{\mu i}(F^*_\rho)'_\mu \simeq (E^*_{\mathscr{C}}\varepsilon F^*_{\mathscr{C}})'_\mu$$

Since ε product keeps Mackey-completeness, one can compute $(\cdot)^*_{\mathscr{C}}$ and $(\cdot)^*_\rho$ by applying respectively $\mathscr{S}_{\mathscr{C}}(\widehat{\cdot}^M)$ and $\mathscr{S}(\widehat{\cdot}^M)$, which gives the missing topological identity:

$$\mathscr{S}_{\mathscr{C}}\Big((E^*_\rho\varepsilon F^*_\rho)^*_\rho\Big) \simeq (E^*_{\mathscr{C}}\varepsilon F^*_{\mathscr{C}})^*_{\mathscr{C}}.$$

$\square$

Let $\mathcal{C} - \mathbf{ref}_\infty, \mathscr{C} - \mathbf{Mc}_\infty$ the Cartesian categories with same spaces as $\mathcal{C} - \mathbf{ref}, \mathscr{C} - \mathbf{Mc}$ and $\mathscr{C}$-smooth maps, namely conveniently smooth maps. Let $U : \mathcal{C} - \mathbf{ref} \longrightarrow \mathcal{C} - \mathbf{ref}_\infty$ the inclusion functor (forgetting linearity and continuity of the maps). Note that, for $\mathscr{C} \subset \mathscr{D}, \mathscr{C} - \mathbf{Mc}_\infty \subset \mathscr{D} - \mathbf{Mc}_\infty$ is a full subcategory.

Theorem B.2.23. *Let $Fin \subset \mathscr{C} \subset$ $\mathbf{F} \times \mathbf{DFS}$ as above. $\mathcal{C} - \mathbf{ref}$ is also a Seely category with structure extended by the comonad $!_{\mathscr{C}}(\cdot) = (\mathcal{C}^\infty_{\mathscr{C}}(\cdot))^*_{\mathscr{C}}$ associated to the adjunction with left adjoint $!_{\mathscr{C}} : \mathcal{C} - \mathbf{ref}_\infty \longrightarrow \mathcal{C} - \mathbf{ref}$ and right adjoint U.*

Proof. We apply Theorem B.2.18 to $\mathcal{C} = \mathscr{C} - \mathbf{Mc}$ so that $\mathcal{D} = \mathcal{C} - \mathbf{ref}$ and $\mathcal{N} = \mathcal{C} - \mathbf{ref}_\infty$. For that we must check the assumptions of a λ-categorical model for $\mathcal{M} = \mathscr{C} - \mathbf{Mc}_\infty$. Lemma B.2.1 shows that $\mathcal{M}$ is a Cartesian closed category since the internal hom functor $\mathcal{C}^\infty_{\mathscr{C}}(E, F)$ is almost by definition in $\mathscr{C} - \mathbf{Mc}$. Indeed it is a projective limit of $\mathcal{C}^\infty_{co}(X)\varepsilon F$ which is a projective kernel of $\mathcal{C}^\infty_{co}(X)\varepsilon\mathcal{C}^\infty_{co}(Y) = \mathcal{C}^\infty_{co}(X \times Y)$ with $X, Y \in \mathscr{C}$ as soon as $F \in \mathscr{C} - \mathbf{Mc}$. The identity in lemma B.2.2 gives the natural isomorphisms for the $(\cdot)^*_{\mathscr{C}}$-relative adjunction (the last one algebraically using $\mathcal{C}^\infty_{\mathscr{C}}(E) \in \mathscr{C} - \mathbf{Mc}$):

$$\mathcal{C}^\infty_{\mathscr{C}}(E, F) \simeq \mathcal{C}^\infty_{\mathscr{C}}(E)\varepsilon F \simeq L(F'_c, \mathcal{C}^\infty_{\mathscr{C}}(E)) = L(F^*_{\mathscr{C}}, \mathcal{C}^\infty_{\mathscr{C}}(E)) = \mathcal{C}^{op}(\mathcal{C}^\infty_{\mathscr{C}}(E), F^*_{\mathscr{C}})$$

It remains to see that $\mathcal{C}^\infty_{\mathscr{C}} : \mathcal{M} \longrightarrow \mathcal{C}$ is a symmetric unital functor satisfying the extra assumptions needed for a λ-categorical model. Note that Lemmas B.2.1 and B.2.2 also provide the definitions of the map Λ, Ξ respectively, the second diagram for Ξ. The diagram for Ξ comparing the internal hom functors is satisfied by definition of the map Λ which is given by a topological version of this diagram. Note that unitality and functoriality of $\mathcal{C}^\infty_{\mathscr{C}}$ are obvious and that $\Lambda_{E,F,G}$ is even defined for any $G \in \mathbf{Mc}$. It remains to prove symmetry and the second diagram for Λ. We first reduce it to $\mathscr{C}$ replaced by Fin. For, note that, by their definition as projective limit, there is a continuous identity map $\mathcal{C}^\infty_{\mathscr{C}}(E) \longrightarrow \mathcal{C}^\infty_{Fin}(E)$ for any lcs E, and since smooth curves only depend on the bornology, $\mathcal{C}^\infty_{Fin}(E) \simeq \mathcal{C}^\infty_{Fin}(\mathscr{N}(E))$ topologically (recall $\mathscr{N} = \mathscr{S}_{Fin}$ is the reflector of $I : Fin - \mathbf{Mc} \subset \mathscr{C} - \mathbf{Mc}$, which is a Cartesian functor [45], and thus also of $I_\infty : Fin - \mathbf{Mc}_\infty \subset \mathscr{C} - \mathbf{Mc}_\infty$ by this very remark.) Composing both, one gets easily a natural transformation $J_{\mathscr{C},Fin} : \mathcal{C}^\infty_{\mathscr{C}} \longrightarrow I \circ \mathcal{C}^\infty_{Fin} \circ \mathscr{N}$. It intertwines the Curry maps Λ as follows for $G \in \mathbf{Mc}$:

$$\begin{tikzcd}
\mathcal{C}^\infty_{\mathscr{C}}(E)\varepsilon\Big(\mathcal{C}^\infty_{\mathscr{C}}(F)\varepsilon G\Big) \arrow[r, "J_{\mathscr{C},Fin}(E)\varepsilon(J_{\mathscr{C},Fin}(F)\varepsilon G)"] & \mathcal{C}^\infty_{Fin}(\mathscr{N}(E))\varepsilon\Big(\mathcal{C}^\infty_{Fin}(\mathscr{N}(F))\varepsilon G\Big) \\
\mathcal{C}^\infty_{\mathscr{C}}(E \times F)\varepsilon G \arrow[u, "\Lambda_{E,F,G}"] \arrow[r, "J_{\mathscr{C},Fin}(E\times F)\varepsilon G"] & \mathcal{C}^\infty_{Fin}(\mathscr{N}(E) \times \mathscr{N}(F))\varepsilon G \arrow[u, "\Lambda_{\mathscr{N}(E),\mathscr{N}(F),G}"']
\end{tikzcd}$$

Now, the associativity, symmetry and unitor maps are all induced from $\mathbf{McSch}$, hence, it suffices to prove the compatibility diagrams for Λ in the case of $\mathcal{C}^\infty_{Fin}$ with $G \in \mathbf{McSch}$. In this case, we can further reduce it using that from naturality of associator, unitor and braiding, they commute with projective limits as ε does, and from its construction in lemma B.2.1 $\Lambda_{E,F,G}$ is also a projective limit of maps, hence the projective limit description of $\mathcal{C}^\infty_{Fin}$ reduces those diagrams to E, F finite dimensional. Note that for the terms with products $E \times F$ the cofinality of product maps used in the proof of lemma B.2.1) enables to rewrite the projective limit for $E \times F$ with the product of projective limits for E, F separately. The key to check the relations is to note that the target space of the diagrams is a set of multilinear maps on $(C^\infty(\mathscr{R}^n \times \mathscr{R}^m))', G'$ and to prove equality of the evaluation of both composition on an element in the source space, by linearity continuity and since $\overline{Vecte_{\mathscr{R}^{n+m}}(\mathscr{R}^{n+m})} = (C^\infty(\mathscr{R}^n \times \mathscr{R}^m))'$, it suffices to evaluate the argument in $(C^\infty(\mathscr{R}^n \times \mathscr{R}^m))'$ on Dirac masses which have a product form. Then when reduced to a tensor product argument, the associativity and braiding maps are canonical and the relation is obvious to check. □

Remark B.2.24. In $\mathcal{C} - \mathbf{ref}$ we defined $!_{\mathscr{C}}E = ((C^\infty_{\mathscr{C}}(E))^*_{\mathscr{C}})$ so that moving it back to ρ-**Ref** via the isomorphism of $*$-autonomous category of Theorem B.2.22, one gets $\mathscr{S}([!_{\mathscr{C}}E]_\mu) = ((C^\infty_{\mathscr{C}}(E))^*_\rho)$. Let us apply Lemma B.2.40 and Example B.2.12. For $\mathscr{C} = Fin$ one gets a space with its ρ-dual having the approximation property, whereas for $\mathscr{C} = Ban$, one may get one without it since $(!_{\mathscr{C}}\mathscr{S}(E))^*_\rho = ((C^\infty_{co}(E))^*_\rho)^*_\rho$ if E is a Banach space (since we have the topological identity $C^\infty_{\mathscr{C}}(E, \mathcal{K}) \simeq C^\infty_{\mathscr{C}}(\mathscr{S}(E), \mathcal{K})$ coming from the identical indexing set of curves coming from the algebraic equality $C^\infty_{co}(X, E) = C^\infty_{Ban}(X, E) = C^\infty(X, E) = C^\infty_{co}(X, \mathscr{S}(E))$). Therefore, if E is a Schwartz space associated to a Banach space in Ban without the approximation property:

$$\mathscr{S}([!_{Fin}E]_\mu) \subsetneq !_{Ban}E$$

(since both duals are algebraically equal to $C^\infty(E, \mathcal{K})$, the difference of topology implies different duals algebraically). It is natural to wonder if there are infinitely many different exponentials obtained in that way for different categories $\mathscr{C}$. It is also natural to wonder if one can characterize ρ-reflexive spaces (or even Banach spaces) for which there is equality $\mathscr{S}([!_{Fin}E]_\mu) = !_{Ban}E$.

B.2.1.5 A model of LL: a Seely category

We referred to [PAM] in order to produce a Seely category. Towards extensions to DiLL models it is better to make more explicit the structure we obtained. First recall the various functors. When $f : E \longrightarrow F$ is a continuous linear map with $E, F \in \mathscr{C} - \mathbf{Mc}$, we used $!_{\mathscr{C}}f : !_{\mathscr{C}}E \longrightarrow !_{\mathscr{C}}F$ defined as $(\cdot \circ f)^*_{\mathscr{C}}$. Hence $!_{\mathscr{C}}$ is indeed a functor from $\mathscr{C} - \mathbf{Mc}$ to $\mathscr{C} - -\mathbf{Ref}$.

Since $\mathcal{C}^\infty_{\mathscr{C}}$ is a functor too on $\mathscr{C} - \mathbf{Mc}_\infty$, the above functor is decomposed in a adjunction as follows. For $F : E \longrightarrow F$ $\mathscr{C}$-smooth, $\mathcal{C}^\infty_{\mathscr{C}}(F)(g) = g \circ F, g \in \mathcal{C}^\infty_{\mathscr{C}}(F, \mathscr{R})$ and for a linear map f as above, $U(f)$ is the associated smooth map, underlying the linear map. Hence we also noted $!_{\mathscr{C}}F = (\mathcal{C}^\infty_{\mathscr{C}}(F))^*_{\mathscr{C}}$ gives the functor, left adjoint to $U : \mathscr{C} - \mathbf{Mc} \longrightarrow \mathscr{C} - \mathbf{Mc}_\infty$ and our previous $!_{\mathscr{C}}$ is merely the new $!_{\mathscr{C}} \circ U$.

For any $E \in \mathcal{C} - \mathbf{ref}$, we recall the continuous isomorphism from E to $(E^*_{\mathscr{C}})^*_{\mathscr{C}} = \mathscr{S}((E'_\mu)'_\mu)$

$$ev_E : \begin{cases} E \to (E^*_{\mathscr{C}})^*_{\mathscr{C}} = E \\ x \mapsto (l \in E^*_{\mathscr{C}} \mapsto l(x)) \end{cases}$$

Note that if E is only Mackey-complete, the linear isomorphism above is still defined, in the sense that we take the extension to the Mackey-completion of $l \mapsto l(x)$, but it is only bounded/smooth algebraic isomorphism (but not continuous) by Theorem B.1.56. However, ev_E^{-1} is always linear continuous in this case too.

We may still use the notation e_E for any separated locally convex space E as the bounded linear injective map, obtained by composition of the canonical map $E \longrightarrow \widehat{E}^M$ and $ev_{\widehat{E}^M}$. We also consider the similar canonical maps:

Lemma B.2.25. For any space $E \in \mathscr{C} - \mathbf{Mc}$, there is a smooth map (the Dirac mass map):

$$\delta_E : \begin{cases} E \to (\mathcal{C}^\infty_{\mathscr{C}}(E))' \subset !_{\mathscr{C}}E \\ x \mapsto (f \in \mathcal{C}^\infty_{\mathscr{C}}(E, \mathcal{K}) \mapsto f(x) = \delta_E(x)(f)), \end{cases}$$

Proof. We could see this directly using convenient smoothness, but it is better to see it comes from our λ-categorical model structure. We have an adjunction:

$$\mathcal{C}^\infty_{\mathscr{C}}(E, !_{\mathscr{C}}E) \simeq \mathscr{C} - \mathbf{Mc}^{op}(\mathcal{C}^\infty_{\mathscr{C}}(E), (!_{\mathscr{C}}E)^*_{\mathscr{C}}) = \mathscr{C} - \mathbf{Mc}((\mathcal{C}^\infty_{\mathscr{C}}(E)^*_{\mathscr{C}})^*_{\mathscr{C}}, \mathcal{C}^\infty_{\mathscr{C}}(E))$$

and δ_E is the map in the first space, associated to $ev^{-1}_{\mathcal{C}^\infty_{\mathscr{C}}(E)}$ in the last. □

Hence, δ_E is nothing but the unit of the adjunction giving rise to $!_{\mathscr{C}}$, considered on the opposite of the continuation category.

As usual, see e.g. [PAM, section 6.7], the adjunction giving rise to $!_{\mathscr{C}}$ produces a comonad structure on this functor. The counit implementing the dereliction rule is the continuous linear map $\mathsf{d}_E :!_{\mathscr{C}}(E) \longrightarrow E$ obtained in looking at the map corresponding to identity in the adjunction:

$$\mathcal{C}^{\infty}_{\mathscr{C}}(E,E) \simeq \mathscr{C} - \mathbf{Mc}^{op}(\mathcal{C}^{\infty}_{\mathscr{C}}(E),(E)^*_{\mathscr{C}}) = \mathscr{C} - \mathbf{Mc}(E^*_{\mathscr{C}},\mathcal{C}^{\infty}_{\mathscr{C}}(E)) \simeq \mathscr{C} - \mathbf{Mc}((\mathcal{C}^{\infty}_{\mathscr{C}}(E))^*_{\mathscr{C}},E)$$

The middle map $\epsilon_E^{\mathcal{C}^{\infty}_{\mathscr{C}}} \in \mathscr{C} - \mathbf{Mc}(E^*_{\mathscr{C}},\mathcal{C}^{\infty}_{\mathscr{C}}(E))$ is the counit of the $(\cdot)^*_{\mathscr{C}}$-relative adjunction and it gives $\mathsf{d}_E = ev_E^{-1} \circ (\epsilon_E^{\mathcal{C}^{\infty}_{\mathscr{C}}})^*_{\mathscr{C}}$ when $E \in \mathcal{C} - \mathbf{ref}$. The comultiplication map implementing the promotion rule is obtained as $\mathsf{p}_E =!_{\mathscr{C}}(\delta_E) = (\mathcal{C}^{\infty}_{\mathscr{C}}(\delta_E))^*_{\mathscr{C}}$.

We can now summarize the structure. Note, that we write the usual $\top$, unit for $\times$ as 0, for the $\{0\}$ vector space.

Proposition B.2.26. *The functor $!_{\mathscr{C}}$ is an exponential modality for the Seely category of Theorem B.2.23 in the following way:*

- $(!_{\mathscr{C}},\mathsf{p},\mathsf{d})$ *is a comonad, with* $\mathsf{d}_E = ev_E^{-1} \circ (\epsilon_E^{\mathcal{C}^{\infty}_{\mathscr{C}}})^*_{\mathscr{C}}$ *and* $\mathsf{p}_E =!_{\mathscr{C}}(\delta_E) = (\mathcal{C}^{\infty}_{\mathscr{C}}(\delta_E))^*_{\mathscr{C}}$.
- $!_{\mathscr{C}} : (\mathcal{C} - \mathbf{ref}, \times, 0) \to (\mathcal{C} - \mathbf{ref}, \otimes, \mathcal{K})$ *is a strong and symmetric monoidal functor, thanks to the isomorphisms* $m^0 : \mathcal{K} \simeq !_{\mathscr{C}}(0)$ *and (the map composing tensor strengths and adjoints of* Ξ, Λ *of* λ*-tensor models):*

$$m^2_{E,F} :!_{\mathscr{C}}E\otimes!_{\mathscr{C}}!F = \Big((\mathcal{C}^{\infty}_{\mathscr{C}}(E)^*_{\mathscr{C}})^*_{\mathscr{C}} \parr_{\mathscr{C}} (\mathcal{C}^{\infty}_{\mathscr{C}}(F)^*_{\mathscr{C}})^*_{\mathscr{C}}\Big)^*_{\mathscr{C}} \simeq \Big(\mathcal{C}^{\infty}_{\mathscr{C}}(E,\mathcal{K}) \parr_{\mathscr{C}} \mathcal{C}^{\infty}_{\mathscr{C}}(F,\mathcal{K})\Big)^*_{\mathscr{C}}$$
$$\simeq \Big(\mathcal{C}^{\infty}_{\mathscr{C}}(E,\mathcal{C}^{\infty}_{\mathscr{C}}(F,\mathcal{K}))\Big)^*_{\mathscr{C}} \simeq (C^{\infty}(E \times F,\mathcal{K}))^*_{\mathscr{C}} \simeq !_{\mathscr{C}}(E \times F)$$

- *the following diagram commute:*

$$\begin{tikzcd}
!_{\mathscr{C}}E\otimes_{\mathscr{C}}!_{\mathscr{C}}F \arrow[r, "m^2_{E,F}"] \arrow[d, "\mathsf{p}_E\otimes_{\mathscr{C}}\mathsf{p}_F"'] & !(E\times F) \arrow[r, "\mathsf{p}_{E\times F}"] & !_{\mathscr{C}}!_{\mathscr{C}}(E\times F) \arrow[d, "!_{\mathscr{C}}\langle !_{\mathscr{C}}\pi_1, !_{\mathscr{C}}\pi_2\rangle"] \\
!_{\mathscr{C}}!_{\mathscr{C}}E\otimes_{\mathscr{C}}!_{\mathscr{C}}!_{\mathscr{C}}F \arrow[rr, "m^2_{!_{\mathscr{C}}E,!_{\mathscr{C}}F}"'] & & !_{\mathscr{C}}(!_{\mathscr{C}}E\times !_{\mathscr{C}}F)
\end{tikzcd}$$

Moreover, the comonad induces a structure of bialgebra on every space $!_{\mathscr{C}}E$ and this will be crucial to obtain models of DiLL [20]. We profit of this section for recalling how all the diagrams there not involving codereliction are satisfied. In general, we have maps giving a commutative comonoid structure (this is the coalgebra part of the bialgebra, but it must not be confused with the coalgebra structure from the comonad viewpoint):

- $\mathsf{c}_E : !_{\mathscr{C}}E \to !_{\mathscr{C}}(E \times E) \simeq !_{\mathscr{C}}E \otimes !_{\mathscr{C}}E$ given by $\mathsf{c}_E = (m^2_{E,E})^{-1}\circ!_{\mathscr{C}}(\Delta_E)$ with $\Delta_E(x) = (x,x)$ the canonical diagonal map of the Cartesian category.
- $\text{W}\textsc{eak}E = (m^0)^{-1}\circ!_{\mathscr{C}}(n_E) : !_{\mathscr{C}}E \to !_{\mathscr{C}}0 \simeq \mathscr{R}$ with $n_E : E \longrightarrow 0$ the constant map, hence more explicitly $\text{W}\textsc{eak}E(h) = h(1)$ for $h \in !_{\mathscr{C}}E$ and $1 \in \mathcal{C}^{\infty}_{\mathscr{C}}(E)$ the constant function equal to 1.

This is exactly the structure considered in [Bie93, Chap 4.6] giving a Seely category (in his terminology a new-Seely category) the structure of a Linear category (called $\mathcal{L}^!_{\otimes}$-model in [25]) from his Definition 35 in his Thm 25. See also [PAM, 7.4] for a recent presentation. This especially also contains the compatibility diagrams of [20, 2.6.1]. Especially, $\mathsf{p}_E : (!_{\mathscr{C}}E, \text{W}\textsc{eak}E, \mathsf{c}_E) \longrightarrow (!_{\mathscr{C}}!_{\mathscr{C}}E, \text{W}\textsc{eak}!_{\mathscr{C}}E, \mathsf{c}_{!_{\mathscr{C}}E})$ is a comonoid morphism as in [20, 2.6.3]. Also $!_{\mathscr{C}}$ is given the structure of a symmetric monoidal endofunctor on $\mathcal{C} - \mathbf{ref}$, $(!_{\mathscr{C}}, \mu^0, \mu^2)$ making $\text{W}\textsc{eak}E, \mathsf{c}_E$ coalgebra morphisms. For instance, $\mu^0 : \mathscr{R} \longrightarrow !_{\mathscr{C}}(\mathscr{R})$ (the space of distributions) is given by [Bie93, Chap 4 Prop 20] as $!_{\mathscr{C}}(v_{\mathscr{R}})\circ m^0$, i.e. $\mu^0(1) = \delta_1$ with $v_{\mathscr{R}} : 0 \longrightarrow \mathscr{R}$ the map with $u_R(0) = 1$. By [Bie93], a Linear category with products is actually the same thing as a Seely category. This is what is called in [25] a $\mathcal{L}^!_{\otimes,\times}$-model. So far, this structure is available in the setting of Theorem B.2.18, and we will use it in this setting later.

As explained in [25], the only missing piece of structure to get a bicomoid structure on every $!_{\mathscr{C}}E$ is a biproduct compatible with the symmetric monoidal structure, or equivalently a **Mon**-enriched symmetric monoidal category, where **Mon** is the category of monoids. This is what he calls a $\mathcal{L}^!_{\otimes,*}$-model.

His Theorem 3.1 then provides us with the two first compatibility diagrams in [20, 2.6.2] and the second diagram in [20, 2.6.4].

In our case $\nabla_E : E \times E \longrightarrow E$ is the sum when seeing $E \times E = E \oplus E$ as coproduct and its unit $u : 0 \longrightarrow E$ is of course the 0 map. Hence $(\mathscr{C} - Mc, 0, \times, u, \nabla; n, \Delta)$ is indeed a biproduct structure. And compatibility with the monoidal structure, which boils down to biadditivity of tensor product, is obvious. One gets cocontraction and coweakening maps:

- $\bar{\mathsf{c}}_E : !_{\mathscr{C}}E \otimes !_{\mathscr{C}}E \simeq !_{\mathscr{C}}(E \times E) \to !_{\mathscr{C}}E$ is the convolution product, namely it corresponds to $!_{\mathscr{C}}(\nabla_E)$.
- $\overline{\mathsf{w}}_E : \mathscr{R} \simeq !_{\mathscr{C}}(0) \to !_{\mathscr{C}}E$ is given by $\overline{\mathsf{w}}_E(1) = (ev_0)^*_{\mathscr{C}}$ with $ev_0 = \mathcal{C}^\infty_{\mathscr{C}}(u_E)$ i.e. $ev_0(f) = f(0)$.

From [25, Prop 3.2] $(!_{\mathscr{C}}E, \mathsf{c}_E, \mathrm{W}\textsc{eak}E, \bar{\mathsf{c}}_E, \overline{\mathsf{w}}_E)$ is a commutative bialgebra. The remaining first diagram in [20, 2.6.4] is easy and comes in our case for $f \in \mathcal{C}^\infty_{\mathscr{C}}(!E)$ from

$$[!_{\mathscr{C}}(\delta_E \circ u_E)](f) = \delta_{\delta_0}(f) = \delta_1(\lambda \mapsto f(\lambda(\delta_0)) = [!_{\mathscr{C}}\overline{\mathsf{w}}_E(\delta_1)](f) = [!_{\mathscr{C}}\overline{\mathsf{w}}_E(\mu^0(1))](f).$$

To finish checking the assumptions in [20], it remains to check the assumptions in 2.5 and 2.6.5. As [25] is a conference paper, they were not explicitly written there.

$$\begin{tikzcd} E & \\ & !E \arrow[ul, "\mathsf{d}_E"'] \\ 1 \arrow[uu, "0"] \arrow[ur, "\overline{\mathsf{w}}_E"'] & \end{tikzcd} \qquad \begin{tikzcd} E & \\ & !E \arrow[ul, "\mathsf{d}_E"'] \\ !E \otimes !E \arrow[uu, "\rho_E^{-1}\circ(\mathsf{d}_E \otimes \mathrm{W}\textsc{eak}E) + \lambda_E^{-1}\circ(\mathrm{W}\textsc{eak}E \otimes \mathsf{d}_E)"] \arrow[ur, "\bar{\mathsf{c}}_E"'] & \end{tikzcd} \tag{B.17}$$

$$\begin{tikzcd} !E \arrow[rr, "!0"] \arrow[dr, "\mathrm{W}\textsc{eak}E"'] & & !E \\ & 1 \arrow[ur, "\overline{\mathsf{w}}_E"'] & \end{tikzcd} \qquad \begin{tikzcd} !E \arrow[r, "!(f+g)"] \arrow[d, "\mathsf{c}_E"'] & !F \\ !E \otimes !E \arrow[r, "!f \otimes !g"] & !F \otimes !F \arrow[u, "\bar{\mathsf{c}}_F"'] \end{tikzcd} \tag{B.18}$$

The first is $ev_E^{-1}\circ(\epsilon_E^{\mathcal{C}^\infty_{\mathscr{C}}})^*_{\mathscr{C}}\circ(\mathcal{C}^\infty_{\mathscr{C}}(u_E))^*_{\mathscr{C}} = ev_E^{-1}\circ(\mathcal{C}^\infty_{\mathscr{C}}(u_E)\circ\epsilon_E^{\mathcal{C}^\infty_{\mathscr{C}}})^*_{\mathscr{C}} = ev_E^{-1}\circ((u_E)^*_{\mathscr{C}})^*_{\mathscr{C}} = u_E = 0$ as expected. The second is $ev_E^{-1}\circ(\epsilon_E^{\mathcal{C}^\infty_{\mathscr{C}}})^*_{\mathscr{C}}\circ(\mathcal{C}^\infty_{\mathscr{C}}(\nabla_E))^*_{\mathscr{C}} = ev_E^{-1}\circ(\mathcal{C}^\infty_{\mathscr{C}}(\nabla_E)\circ\epsilon_E^{\mathcal{C}^\infty_{\mathscr{C}}})^*_{\mathscr{C}} = ev_E^{-1}\circ((\nabla_E)^*_{\mathscr{C}})^*_{\mathscr{C}}\circ(\epsilon_E^{\mathcal{C}^\infty_{\mathscr{C}}})^*_{\mathscr{C}}\otimes(\epsilon_E^{\mathcal{C}^\infty_{\mathscr{C}}})^*_{\mathscr{C}}$ which gives the right value since $\nabla_E = r_E^{-1}\circ(id_E \times n_E) + \ell_E^{-1}\circ(n_E \times id_E)$.

The third diagram comes from $n_E u_E = 0$ and the last diagram from $\nabla_Y \circ(f \times g)\circ\Delta_X = f + g$ which is the definition of the additive structure on maps.

B.2.1.6 Comparison with the convenient setting of Global analysis and Blute-Ehrhard-Tasson

In [6], the authors use the Global setting of convenient analysis [**?** KM] in order to produce a model of Intuitionistic differential Linear logic. They work on the category $\mathrm{C}\textsc{onv}$ of convenient vector spaces, i.e. bornological Mackey-complete (separated) lcs, with continuous (equivalently bounded), linear maps as morphisms. Thus, apart for the bornological requirement, the setting seems really similar to ours. It is time to compare them.

First any bornological space has its Mackey topology, let us explain why $\mathscr{S} : \mathrm{C}\textsc{onv} \longrightarrow \rho - \mathbf{Ref}$ is an embedding giving an isomorphic full subcategory (of course with inverse $(\cdot)_\mu$ on its image). Indeed, for $E \in \mathrm{C}\textsc{onv}$ we use Theorem B.1.59 in order to see that $\mathscr{S}(E) \in \rho - \mathbf{Ref}$ and it only remains to note that E'_μ is Mackey-complete.

As in Remark B.1.33, E bornological Mackey-complete, thus ultrabornological, implies E'_μ and even $\mathcal{S}(E'_\mu)$ complete hence Mackey-complete (and E'_c k-quasi-complete).

Said otherwise, the bornological requirement ensures a stronger completeness property of the dual than Mackey-completeness, the completeness of the space, our functor $(\widehat{(\cdot)'_\mu})'_\mu$ should thus be thought of as a replacement of the bornologification functor in [**?**] and $((\cdot)^*_\rho)^*_\rho$ is our analogue of their Mackey-completion functor in [53] (recall that their Mackey completion is what we would call Mackey-completion of the bornologification). Of course, we already noticed that we took the same smooth maps and $\mathscr{S} : \mathrm{C}\textsc{onv}_\infty \longrightarrow \rho - \mathbf{Ref}_\infty$ is even an equivalence of categories. Indeed, $E \longrightarrow E_{born}$ is smooth and gives the inverse for this equivalence.

Finally note that $E^*_\rho \varepsilon F = L_\beta(E, F)$ algebraically if $E \in \mathrm{C}\textsc{onv}$ since $E^*_\rho \varepsilon F = L_\epsilon((E^*_\rho)'_c, F) = L_{\mathbb{R}}(E, F)$ topologically and the space of continuous and bounded linear maps are the same in the bornological case. $L_\beta(E, F) \longrightarrow L_{\mathbb{R}}(E, F)$ is clearly continuous hence so is $\mathscr{S}(L_\beta(E, F)) \longrightarrow \mathscr{S}(L_{\mathbb{R}}(E, F)) = L_{\mathbb{R}}(E, F)$.

But the closed structure in CONV is given by $(L_\beta(E,F))_{born}$ which uses a completion of the dual and hence we only have a lax closed functor property for $\mathscr{S}$, in form (after applying $((\cdot)^*_\rho)^*_\rho$) of a continuous map:

$$\mathscr{S}((L_\beta(E,F))_{born}) \longrightarrow ((E^*_\rho \varepsilon F)^*_\rho)^*_\rho. \tag{B.19}$$

Similarly, most of the linear logical structure is not kept by the functor $\mathscr{S}$.

B.2.2 Models of DiLL

Smooth linear maps in the sense of Frlicher are bounded but not necessarily continuous. Taking the differential at 0 of functions in $\mathcal{C}^\infty(E,F)$ thus would not give us a morphisms in $\mathbf{k}-\mathbf{Ref}$, thus we have no interpretation for the codereliction $\overline{\mathsf{d}}$ of DiLL. We first introduce a general differential framework fitting Dialogue categories, and show that the variant of smooth maps introduce in section B.2.2.4 allows for a model of DiLL.

B.2.2.1 An intermediate notion: models of differential λ-Tensor logic.

We refer to [20, 20] for surveys on differential linear logic.

According to Fiore and Ehrhard [20, 25], models of differential linear logic are given by Seely $*$-autonomous complete categories $\mathcal{C}$ with a biproduct structure and either a creation operator natural transformation $\partial_E : !EE \longrightarrow !E$ or a creation map/codereliction natural transformation $\overline{\mathsf{d}}_E : E \longrightarrow !E$ satisfying proper conditions. We recalled in subsection B.2.1.5 the structure available without codereliction. Moreover, in the codereliction picture, one requires the following diagrams to commute [20, 2.5, 2.6.2,2.6.4]:

$$\begin{tikzcd} E \arrow[dr, "\overline{\mathsf{d}}_E"] \arrow[dd, "0"'] & \\ & !E \arrow[dl, "\textsc{Weak}E"] \\ 1 & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[dr, "\overline{\mathsf{d}}_E"] \arrow[dd, "(\overline{\mathsf{d}}_E\otimes\overline{\mathsf{w}}_E)\circ\rho_E+(\overline{\mathsf{w}}_E\otimes\overline{\mathsf{d}}_E)\circ\lambda_E"'] & \\ & !E \arrow[dl, "\mathsf{c}_E"] \\ !E\otimes !E & \end{tikzcd} \qquad \begin{tikzcd} & !E \arrow[dr, "\mathsf{d}_E"] & \\ E \arrow[ur, "\overline{\mathsf{d}}_E"] \arrow[rr, "id_E"'] & & E \end{tikzcd} \tag{B.20}$$

$$\begin{tikzcd} E\otimes !F \arrow[r, "\overline{\mathsf{d}}_E\otimes !F"] \arrow[dr, "E\otimes\mathsf{d}_E"'] & !E\otimes !F \arrow[r, "\mu^2_{E,F}"] & !(E\otimes F) \\ & E\otimes F \arrow[ur, "\overline{\mathsf{d}}_{E\otimes F}"'] & \end{tikzcd} \tag{B.21}$$

$$\begin{tikzcd} E \arrow[r, "\overline{\mathsf{d}}_E"] \arrow[d, "\lambda_E"'] & !E \arrow[r, "\mathsf{p}_E"] & !!E \\ 1\otimes E \arrow[r, "\overline{\mathsf{w}}_E\otimes\overline{\mathsf{d}}_E"'] & !E\otimes !E \arrow[r, "\mathsf{p}_E\otimes\overline{\mathsf{d}}_{!E}"'] & !!E\otimes !!E \arrow[u, "\overline{\mathsf{c}}_{!E}"'] \end{tikzcd} \tag{B.22}$$

Then from [25, Thm 4.1] (see also [20, section 3]) the creation operator $\partial_E = \overline{\mathsf{c}}_E \circ (!E \otimes \overline{\mathsf{d}}_E)$

We again need to extend this structure to a Dialogue category context. In order to get a natural differential extension of Cartesian closed category, we use differential λ-categories from [BEM]. This notion gathers the maybe very general Cartesian differential categories of Blute-Cockett-Seely to Cartesian closedness, via the key axiom (D-curry), relating applications of the differential operator D and the curry map Λ for $f : C \times A \longrightarrow B$ (we don't mention the symmetry of Cartesian closed category $(C \times C \times A) \times A \simeq (C \times A) \times (C \times A)$):

$$D(\Lambda(f)) = \Lambda\Big(D(f) \circ \langle(\pi_1 \times 0_A), \pi_2\rangle\Big) : (C \times C) \longrightarrow [A, B].$$

We also use $Diag(E) = E \times E$ the obvious functor. We also suppose that the Cartesian structure is a biproduct, a supposition that is equivalent to supposing a **Mon**-enriched category as shown by Fiore [25].

The idea is that while D encodes the usual rules needed for differential calculus, d encodes the fact that we want the derivatives to be smooth, that is compatible with the linear duality structure we had before.

Definition B.2.27. A (resp. commutative) model of differential λ-tensor logic is a (resp. commutative) λ-categorical model of λ-tensor logic with dialogue category $(\mathcal{C}^{op}, \parr_{\mathcal{C}}, 1_{\mathcal{C}} = \mathcal{K}, \neg)$ with a biproduct structure compatible with the symmetric monoidal structure, a Cartesian closed category $(\mathcal{M}, \times, 0, [\cdot,\cdot])$, which is a differential λ-category with operator D internalized as a natural transformation $D_{E,F} : [E,F] \longrightarrow [Diag(E), F]$ (so that D in the definition of those categories is given by $M(D_{E,F}) : \mathcal{M}(E,F) \longrightarrow \mathcal{M}(E \times E, F)$ with M the basic functor to sets of the closed category $\mathcal{M}$). We assume $U : \mathcal{C} \longrightarrow \mathcal{M}$ and $\neg : \mathcal{C}^{op} \longrightarrow \mathcal{C}$ are **Mon**-enriched functors. We also assume given an internalized differential operator, given by a natural transformation

$$d_{E,F} : NL(U(E)) \parr_{\mathcal{C}} F \longrightarrow NL(U(E)) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} F)$$

satisfying the following commutative diagrams (with the opposite of the counit of the relative adjointness relation, giving a map in $\mathcal{C}$ written: $\epsilon_E^{NL} : \neg E \longrightarrow NL(U(E)) \equiv NL_E$) expressing compatibility of the two differentials. We have a first diagram in $\mathcal{M}$:

$$\begin{tikzcd}
U\Big(NL_E \parr_{\mathcal{C}} F\Big) \arrow[r, "U(d_{E,F})"] \arrow[d, "\Xi_{E,F}"'] & U\Big(NL_E \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} F)\Big) \arrow[r, "U(NL_E) \parr_{\mathcal{C}} (\epsilon_E^{NL} \parr_{\mathcal{C}} F))"] & U\Big(NL_E \parr_{\mathcal{C}} (NL_E \parr_{\mathcal{C}} F)\Big) \arrow[d, "{[id_{U(E)}, \Xi_{E,F}] \circ \Xi_{E,NL(E) \parr_{\mathcal{C}} F}}"] \\
{[U(E), U(F)]} \arrow[r, "D_{U(E),U(F)}"] & {[U(E \times E), U(F)]} \arrow[r, "\Lambda^{\mathcal{M}}_{U(E),U(E),U(F)}"] & {[U(E), [U(E), U(F)]]}
\end{tikzcd}$$

and weak differentiation property diagram in $\mathcal{C}$:

$$\begin{tikzcd}
NL(U(E)) \parr_{\mathcal{C}} F \arrow[r, "d_{E,F}"] \arrow[d, "\rho_{NL(U(E))} \parr_{\mathcal{C}} F"'] & NL(U(E)) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} F) \arrow[r, "Ass^{\parr_{\mathcal{C}}}_{NL(U(E)),\neg E,F}"] & (NL(U(E)) \parr_{\mathcal{C}} \neg E) \parr_{\mathcal{C}} F \arrow[d, "NL(U(E)) \parr_{\mathcal{C}} \rho_{\neg E} \parr_{\mathcal{C}} F"] \\
(NL(U(E)) \parr_{\mathcal{C}} \mathcal{K}) \parr_{\mathcal{C}} F \arrow[rr, "d_{E,\mathcal{K}} \parr_{\mathcal{C}} F"] & & (NL(U(E)) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} \mathcal{K})) \parr_{\mathcal{C}} F
\end{tikzcd}$$

The idea is that while D encodes the usual rules needed for a Cartesian Differential Category, d encodes the fact that we want the derivatives to be smooth, that is compatible with the linear duality structure we had before.

Theorem B.2.28. $(\mathcal{C}^{op}, \parr_{\mathcal{C}}, I, \neg, \mathcal{M}, \times, 0, NL, U, D, d)$ *a Seely model of differential λ-tensor logic. Let $\mathcal{D} \subset \mathcal{C}$ the full subcategory of objects of the form $\neg C, C \in \mathcal{C}$, equipped with $! = \neg \circ NL \circ U$ as comonad on $\mathcal{D}$ making it a $*$-autonomous complete and cocomplete Seely category with Kleisli category for $!$ isomorphic to $\mathcal{N} = U(\mathcal{D})$. With the dereliction d_E as in subsection B.2.1.5 and the codereliction interpreted by: $\overline{\mathsf{d}}_E = \neg\Big((NL(E) \parr_{\mathcal{C}} \rho^{-1}_{\neg E}) \circ d_{E,\mathcal{K}} \circ \rho_{NL(E)}\Big) \circ (\overline{\mathsf{w}}_E \otimes Id_E) \circ \lambda_E$, this makes $\mathcal{D}$ a model of differential Linear Logic.*

Proof. The setting comes from Theorem B.2.18 giving already a model of Linear logic. Recall from subsection B.2.1.5 that we have already checked all diagrams not involving codereliction. We can and do fix $E = \neg C$ so that $\epsilon_E^{\neg\neg} : \neg\neg E \longrightarrow E$ is an isomorphism that we will ignore safely in what follows.

Step 1: Internalization of D-curry from [BEM]

Let us check:

$$\begin{tikzcd}
NL(E) \parr NL(E) \parr (\neg E \parr \mathcal{K}) & NL(E) \parr (NL(E) \parr \mathcal{K}) \arrow[l, "NL(E) \parr d_{E,\mathcal{K}}"'] & NL(E \times E) \parr \mathcal{K} \arrow[l, "\Lambda_{E,E,\mathcal{K}}"'] \arrow[d, "d_{E \times E,\mathcal{K}}"] \\
 & NL(E \times E) \parr (\neg E \parr \mathcal{K}) \arrow[ul, "\Lambda_{E,E,\neg E \parr \mathcal{K}}"] & NL(E \times E) \parr ((\neg E)^{\oplus 2}) \parr \mathcal{K}) \arrow[l, "NL(E \times E) \parr (\pi_2 \parr \mathcal{K})"']
\end{tikzcd}$$

Indeed using compatibility with symmetry from definition of λ-categorical models, it suffices to check a flipped version with the derivation acting on the first term. Then applying the faithful U, intertwining with Ξ and using the compatibility with $D_{E,F}$ the commutativity then follows easily from D-curry.

Step 2: Internalization of chain rule D5 from [BEM]

For $g \in \mathcal{M}(U(E), U(F)) = \mathcal{M}(0, [U(E), U(F)]) \simeq \mathcal{M}(0, U(NL(E) \parr_{\mathcal{C}} F) \simeq \mathcal{C}(\neg(NL(0)), NL(E) \parr_{\mathcal{C}} F \parr_{\mathcal{C}} \mathcal{K})$, which gives a map $h : \neg(\mathcal{K}) \longrightarrow NL(E) \parr_{\mathcal{C}} F \parr_{\mathcal{C}} \mathcal{K}$. One gets $d_{E,F} \circ h : \neg(\mathcal{K}) \longrightarrow NL(E) \parr_{\mathcal{C}} \neg E \parr_{\mathcal{C}} F \parr_{\mathcal{C}} \mathcal{K}$ giving by characteristic diagram of dialogue categories (for $\mathcal{C}^{op}$, recall our maps are in the opposite of this dialogue category) a map $dH : \neg F \longrightarrow NL(E) \parr_{\mathcal{C}} \neg E \parr_{\mathcal{C}} \mathcal{K}$. We leave as an exercise to the reader to check that D5 can be rewritten as before:

$$\begin{tikzcd}
NL(U(E)) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} \mathcal{K}) & NL(U(E)) \parr_{\mathcal{C}} \mathcal{K} \arrow[l, "d_{E,\mathcal{K}}"'] & NL(U(F)) \parr_{\mathcal{C}} \mathcal{K} \arrow[l, "NL(g) \parr_{\mathcal{C}} \mathcal{K}"'] \arrow[d, "d_{F,\mathcal{K}}"] \\
NL(U(E)) \parr_{\mathcal{C}} NL(U(E)) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} \mathcal{K}) \arrow[u, "\left(NL(\Delta_{U(E)})\Lambda^{-1}_{E,E}\right) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} \mathcal{K})"] & & NL(U(F)) \parr_{\mathcal{C}} (\neg(F) \parr_{\mathcal{C}} \mathcal{K}) \arrow[ll, "NL(g) \parr_{\mathcal{C}} dH"]
\end{tikzcd}$$

Note that we can see dH in an alternative way using our weak differentiation property. Composing with a minor isomorphism, if we see $h : \neg(\mathcal{K}) \longrightarrow (NL(E) \parr_{\mathcal{C}} \mathcal{K}) \parr_{\mathcal{C}} F$ then one can consider $(d_{E,\mathcal{K}} \parr F) \circ h$ and it gives $d_{E,F} \circ h$ after composition by a canonical map. But if $H : \neg(F) \longrightarrow (NL(E) \parr_{\mathcal{C}} \mathcal{K})$ is the map associated to h by the map φ of dialogue categories, the naturality of this map gives exactly $dH = d_{E,\mathcal{K}} \circ H$. Note that if $g = U(g')$, by the naturality of the isomorphisms giving H, it is not hard to see that $H = \epsilon_E^{NL} \circ \neg(g')$.

Step 3: Two first diagrams in (B.20).

For the first diagram,by functoriality, it suffices to see $d_{E,\mathcal{K}} \circ (NL(n_E) \parr \mathcal{K}) = 0$. Applying step 2 to $g = n_E$, one gets $H = u_{NL(E)\parr\mathcal{K}}$ hence $dH = d_{E,\mathcal{K}} \circ H = 0$ as expected thanks to axiom D1 of [BEM] giving $D(0) = 0$.

For the second diagram, we compute $\mathsf{c}_E\overline{\mathsf{d}}_E = (m^2_{E,E})^{-1} \circ \neg(NL(\Delta_{U(E)})) \circ \neg\Big((NL(u_E) \parr_{\mathcal{C}} \rho^{-1}_{\neg E}) \circ d_{E,\mathcal{K}} \circ \rho_{NL(E)}\Big) \circ \lambda_E$. We must compute $[NL(u_E) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} \mathcal{K})] \circ d_{E,\mathcal{K}} \circ (NL(\Delta_{U(E)}) \parr \mathcal{K})$ using step 2 again with $g = \Delta_{U(E)} = U(\Delta_E)$, hence $H = \epsilon_E^{NL} \circ \neg(\Delta_E) = \epsilon_E^{NL} \circ \nabla_{\neg E}$.

Using (B.23) below, one gets $dH = (NL(n_E) \parr \nabla_{\neg E} \parr \mathcal{K}) \circ Isom$, so that, using $Isom \circ (\Delta_{U(E)} \times n_E) \circ \Delta_{U(E)} = \Delta_{U(E)}$, one obtains

$$CD_E := [NL(u_E) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} \mathcal{K})] \circ d_{E,\mathcal{K}} \circ (NL(\Delta_{U(E)}) \parr \mathcal{K}) = (NL(\Delta_{U(E)} \circ u_E) \parr \nabla_{\neg E} \parr \mathcal{K}) \circ d_{E^2,\mathcal{K}}.$$

Hence, noting that by naturality $NL(0) \parr \nabla_{\neg E} \parr \mathcal{K}) = \nabla_{NL(0)\parr\neg E\parr\mathcal{K}}$ and using the formula in step 1,

$$\begin{aligned}
CD_E\Lambda^{-1}_{E,E,\mathcal{K}} &= \nabla_{NL(0)\parr\neg E\parr\mathcal{K}} \bigoplus_{i=1,2} (NL(u_{E^2}) \parr \pi_i \parr \mathcal{K}) \circ d_{E^2,\mathcal{K}}\Lambda^{-1}_{E,E,\mathcal{K}} \\
&= \nabla_{NL(0)\parr\neg E\parr\mathcal{K}}\Big[(NL(u_E) \parr_{\mathcal{C}} \Big[(NL(u_E) \parr \neg E \parr \mathcal{K}))d_{E,\mathcal{K}}\Big]), (\Big[(NL(u_E) \parr \neg E \parr \mathcal{K}))d_{E,\mathcal{K}}\Big] \parr_{\mathcal{C}} NL(u_E))\Big].
\end{aligned}$$

On the other hand, we can compute $(\overline{\mathsf{w}}_E \otimes \overline{\mathsf{d}}_E) \circ \lambda_E = \neg\Big((NL(u_E) \parr_{\mathcal{C}} \rho^{-1}_{\neg E}) \circ \lambda^{-1} \circ (NL(u_E) \parr_{\mathcal{C}} d_{E,\mathcal{K}}) \circ (NL(E) \parr_{\mathcal{C}} \rho_{NL(E)})\Big) \circ \lambda_E$. From the symmetric computation, one sees (in using $\neg$ is additive) that our expected equation reduces to proving the formula which reformulates our previous result:

$$CD_E \circ \Lambda^{-1}_{E,E,\mathcal{K}} = \lambda^{-1} \circ (NL(u_E) \parr_{\mathcal{C}} \Big[(NL(u_E) \parr \neg E \parr \mathcal{K}))d_{E,\mathcal{K}}\Big]) + \rho^{-1} \circ (\Big[(NL(u_E) \parr \neg E \parr \mathcal{K}))d_{E,\mathcal{K}}\Big] \parr_{\mathcal{C}} NL(u_E))$$

Step 4: Final Diagrams for codereliction.

To prove (B.22),(B.21), one can use [25, Thm 4.1] (and the note added in proof making (14) redundant, but we could also check it in the same vein as below using step 2) and only check (16) and the second part of his diagram (15) on $\partial_E = \neg\Big((NL(E) \parr_{\mathcal{C}} \rho^{-1}_{\neg E}) \circ d_{E,\mathcal{K}} \circ \rho_{NL(E)}\Big)$. Indeed, our choice $\overline{\mathsf{d}}_E = (\partial_E) \circ (\overline{\mathsf{w}}_E \otimes Id_E) \circ \lambda_E$ is exactly the direction of this bijection producing the codereliction.

One must check:

$$\begin{tikzcd}
!EE \arrow[r, "\partial_E"] \arrow[d, "\mathsf{c}_E E"'] & !E \arrow[r, "\mathsf{p}_E"] & !!E \\
!E!EE \arrow[rr, "\mathsf{p}_E \otimes \partial_E"] & & !!E \otimes !E \arrow[u, "\partial_{!E}"']
\end{tikzcd}$$

and recall $\mathsf{c}_E = (m^2_{E,E})^{-1} \circ \neg(NL(\Delta_E))$, $\mathsf{p}_E = \neg(NL(\delta_E))$, and $(m^2_{E,E})^{-1} = \neg(\Lambda^{-1}_{E,E}(\epsilon^{\neg} \parr \epsilon^{\neg}))$,with $\Lambda^{-1}_{E,E} = \rho_{NL(E\times E)}\Lambda^{-1}_{E,E,\mathcal{K}}(NL(E) \parr \rho_{NL(E)})$, $\epsilon^{\neg} : \neg\neg E \longrightarrow E$ the counit of self-adjunction.

Hence our diagram will be obtained by application of $\neg$ (after intertwining with ρ) if we prove:

$$\begin{tikzcd}
NL(U(E)) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} \mathcal{K}) & NL(U(E)) \parr_{\mathcal{C}} \mathcal{K} \arrow[l, "d_{E,\mathcal{K}}"'] & NL(U(!E)) \parr_{\mathcal{C}} \mathcal{K} \arrow[l, "NL(\delta_E)\parr_{\mathcal{C}}\mathcal{K}"'] \arrow[d, "d_{!E,\mathcal{K}}"'] \\
NL(U(E)) \parr_{\mathcal{C}} NL(U(E)) \parr_{\mathcal{C}} (\neg E \parr_{\mathcal{C}} \mathcal{K}) \arrow[u, "\left(NL(\Delta_E)\Lambda^{-1}_{E,E}\right)\parr_{\mathcal{C}}(\neg E\parr_{\mathcal{C}}\mathcal{K})"'] & & NL(U(!E)) \parr_{\mathcal{C}} (\neg(!E) \parr_{\mathcal{C}} \mathcal{K}) \arrow[ll, "NL(\delta_E)\parr_{\mathcal{C}}(d_{E,\mathcal{K}}\circ\epsilon^{\neg})"]
\end{tikzcd}$$

This is the diagram in step 2 for $g = \delta_E$ if we see that $dH = (d_{E,\mathcal{K}} \circ \epsilon^{\neg})$. For, it suffices to see $H = \epsilon^{\neg}$, which is essentially the way δ_E is defined as in proposition B.2.25.

We also need to check the diagram [25, (16)]which will follow if we check the (pre)dual diagram:

$$\begin{tikzcd}
NL(E\times E) \parr (\neg E \parr \mathcal{K}) & NL(E) \parr (NL(E) \parr \mathcal{K}) \arrow[l, "NL(E)\parr d_{E,\mathcal{K}}"'] & NL(E\times E) \parr \mathcal{K} \arrow[l, "\Lambda_{E,E,\mathcal{K}}"'] \\
& NL(E) \parr (\neg E \parr \mathcal{K}) \arrow[ul, "\Lambda_{E,E,\neg E\parr\mathcal{K}}\circ(NL(U(\nabla_E))\parr(\neg E\parr\mathcal{K}))"] & NL(E) \parr \mathcal{K} \arrow[l, "d_{E,\mathcal{K}}"'] \arrow[u, "NL(U(\nabla_E))\parr\mathcal{K}"']
\end{tikzcd}$$

Using step 1 and step 2 with $g = U(\nabla_E)$, it reduces to:

$$\begin{tikzcd}
NL(E^2) \parr (\neg E \parr \mathcal{K}) & NL(E^2) \parr ((\neg E)^{\oplus 2}) \parr \mathcal{K}) \arrow[l, "NL(E^2)\parr(\pi_2\parr\mathcal{K})"'] \\
NL(E) \parr (\neg E \parr \mathcal{K}) \arrow[u, "(NL(U(\nabla_E))\parr(\neg E\parr\mathcal{K}))"] \arrow[r, "NL(U(\nabla_E))\parr dH"] & NL(E^2) \parr NL(E^2) \parr ((\neg E)^{\oplus 2}) \parr \mathcal{K}) \arrow[u, "(NL(\Delta_{E^2})\Lambda^{-1}_{E^2,E^2})\parr_{\mathcal{C}}((\neg E)^{\oplus 2}\parr_{\mathcal{C}}\mathcal{K})"']
\end{tikzcd}$$

Recall that here, from step 2, $dH = d_{E^2,\mathcal{K}} \circ H$. In our current case, we noticed that $H = \epsilon^{NL}_{E^2} \circ \neg(\nabla_E)$. Using (B.23) with E^2 instead of E, and $Isom \circ (id_{E^2} \times n_{E^2})\Delta_{E^2} = id_{E^2}$, the right hand side of the diagram we must check reduces to the map $NL(U(\nabla_E)) \parr (\pi_2 \circ \neg(\nabla_E)) \parr \mathcal{K} = NL(U(\nabla_E)) \parr \neg E \parr \mathcal{K}$ as expected, using only the defining property of ∇_E from the coproduct.

Let us turn to proving the first diagram in [25, (15)], which will give at the end $\overline{\mathsf{d}}_E \circ \mathsf{d}_E = Id_E$. Modulo applying $\neg$ and intertwining with canonical isomorphisms, it suffices to see:

$$\begin{tikzcd}
& NL(U(E)) \parr \mathcal{K} \arrow[dl, "d_{E,\mathcal{K}}"'] & \\
NL(U(E)) \parr \neg E \parr \mathcal{K} & NL(U(0)) \parr \neg E \parr \mathcal{K} \arrow[l, "NL(n_E)\parr\neg E\parr\mathcal{K}"] & \neg E \parr \mathcal{K} \arrow[l, "\simeq"'] \arrow[ul, "\epsilon^{NL}_E\parr\mathcal{K}"']
\end{tikzcd} \tag{B.23}$$

For it suffices to get the diagram after precomposition by any $h : \neg\mathcal{K} \longrightarrow \neg E$ (using the $\mathcal{D}$ is closed with unit $\neg\mathcal{K}$ for the closed structure). Since $E \in \mathcal{D}$ this is the same thing as $g = \neg h : E \simeq \neg\neg E \longrightarrow \mathcal{K}$ so that one can apply naturality in E of all the maps in the above diagram. This reduces the diagram to the case $E = \mathcal{K}$.

But from axiom D3 of [BEM], we have $D(Id_{U(E)}) = \pi_2$, projection on the second element of a pair, for $E \in \mathcal{C}$. When we apply the compatibility diagram between D and $d_{E,E}$ to $\mathsf{d}_{\neg E}$, which corresponds through Ξ to $Id_U(E)$, we have (for $\pi_2 \in M(E \times E, E)$ the second projection):

$$M\Big[Isom \circ U\big(NL_E \parr_{\mathcal{C}} (\epsilon^{NL}_E \parr_{\mathcal{C}} E)\big) \circ U(d_{E,E}) \circ U((\epsilon^{NL}_E \parr \mathcal{K}) \circ (I_E))\Big] = \pi_2$$

Here we used $I_E : \neg\mathcal{K} \longrightarrow \neg E \parr E$ used from the axiom of dialogue categories corresponding via φ to $id_{\neg E}$ and where we use $M(\Xi_{E,E} \circ (U((\epsilon^{NL}_E \parr \mathcal{K}) \circ (I_E)))) = Id_{U(E)}$. This comes via naturality for φ from the association via φ of $(\epsilon^{NL}_E \parr \mathcal{K}) \circ (I_E)$ to the map $\epsilon^{NL}_E : \neg E \longrightarrow NL(E)$, and then from the use of the compatibility of Ξ with adjunctions in definition B.2.16 jointly with the definition of ϵ^{NL} as counit of adjunction, associating it to $Id_{U(E)}$. Thus applying this to $E = \mathcal{K}$ and since we can always apply the faithful functors U, M to our relation and compose it with the monomorphism applied above after $U(d_{E,E})$ and on the other side to $U(I_{\mathcal{K}}) \simeq Id$, it is easy to see that the second composition is also π_2. $\square$

B.2.2.2 A general construction for DiLL models

Assume given the situation of Theorem B.2.18, with $\mathcal{C}$ having a biproduct structure with $U, \neg$ **Mon**-enriched and assume that $\mathcal{M}$ is actually given the structure of a differential λ-category with operator internalized as a natural transformation $D_{E,F} : [E,F] \longrightarrow [Diag(E), F]$ (so that D in the definition of those categories is given by $M(D_{E,F}) : \mathcal{M}(E,F) \longrightarrow \mathcal{M}(E \times E, F)$ with M the basic functor to sets of the closed category $\mathcal{M}$) and U bijective on objects. Assume also that there is a map $D'_{E,F} : NL(E) \parr_{\mathcal{C}} F \longrightarrow NL(E \times E) \parr_{\mathcal{C}} F$ in $\mathcal{C}$, natural in E such that

$$\Xi_{E\times E,F} \circ U(D'_{E,F}) = D_{U(E),U(F)} \circ \Xi_{E,F}.$$

and

$$D'_{E,F} = \left(\rho^{-1}_{NL(E^2)} \circ D'_{E,\mathcal{K}} \circ \rho_{NL(E)}\right) \parr_{\mathcal{C}} F. \tag{B.24}$$

Our non-linear variables are the first one after differentiation.

We assume $\parr_{\mathcal{C}}$ commutes with limits and finite coproducts in $\mathcal{C}$ and recall from remark B.2.19 that it preserves monomorphisms and that $\neg$ is faithful. Note that since $\mathcal{C}$ is assumed complete and cocomplete, it has coproducts $\oplus = \times$, by the biproduct assumption, and that $\neg(E \times F) = \neg(E) \oplus \neg(F)$ since $\neg : \mathcal{C}^{op} \longrightarrow \mathcal{C}$ is left adjoint to its opposite functor $\neg : \mathcal{C} \longrightarrow \mathcal{C}^{op}$ which therefore preserves limits. We will finally need the following:

$$\begin{tikzcd}
\neg(E \times F) \parr_{\mathcal{C}} G \arrow[r, "\epsilon^{NL}_{E\times F} \parr_{\mathcal{C}} G"] \arrow[d, "\simeq"] & NL(E \times F) \parr_{\mathcal{C}} G \arrow[r, "NL(U((id_E \times 0_F)\circ r)) \parr_{\mathcal{C}} G"] & NL(E) \parr_{\mathcal{C}} G \\
\left(\neg E \parr_{\mathcal{C}} G\right) \times \left(\neg F \parr_{\mathcal{C}} G\right) \arrow[rr, "\pi_1"] & & \neg E \parr_{\mathcal{C}} G \arrow[u, "\epsilon^{NL}_E \parr_{\mathcal{C}} G"']
\end{tikzcd} \tag{B.25}$$

This reduces to the case $G = \mathcal{K}$ by functoriality and then, this is a consequence of naturality of ϵ^{NL} since the main diagonal of the diagram taking the map via the lower left corner is nothing but $\neg((id_E \times 0_F) \circ r)$ with $r : E \longrightarrow E \times 0$ the right unitor for the Cartesian structure on $\mathcal{C}$.

We want to build from that data a new category $\mathcal{M}_{\mathcal{C}}$ giving jointly with $\mathcal{C}$ the structure of a model of differential λ-tensor logic.

$\mathcal{M}_{\mathcal{C}}$ has the same objects as $\mathcal{M}$ (and thus as $\mathcal{C}$ too) but new morphisms that will have as derivatives maps from $\mathcal{C}$, or rather from its continuation category. Consider the category $Diff_{\mathcal{N}}$ with objects $\{0\} \times \mathcal{N} \cup \{1\} \times \mathcal{N}^*$ generated by the following family of morphisms without relations: one morphism $d = d_i : (0,i) \longrightarrow (0,i+1)$ for all $i \in \mathcal{N}$ which will be mapped to a differential and one morphism $j = j_i : (1,i+1) \longrightarrow (0,i+1)$ for all $i \in \mathcal{N}$ which will give an inclusion. Hence all the morphism are given by $d^k : (0,i) \longrightarrow (0,i+k)$, $d^k \circ j : (1,i+1) \longrightarrow (0,i+k+1)$.

We must define the new Hom set. We actually define an internal Hom. Consider, for $E,F \in \mathcal{C}$ the functor $Diff_{E,F}, Diff_E : Diff_{\mathcal{N}} \longrightarrow \mathcal{C}$ on objects by

$$Diff_E((0,i)) = NL(U(E)^{i+1}) \parr_{\mathcal{C}} \mathcal{K}, \quad Diff_E((1,i+1)) = (NL(U(E)) \parr_{\mathcal{C}} ((\neg E)^{\parr_{\mathcal{C}} i+1} \parr_{\mathcal{C}} \mathcal{K}))$$

with the obvious inductive definition $(\neg E)^{\parr_{\mathcal{C}} i+1} \parr_{\mathcal{C}} \mathcal{K} = \neg E \parr_{\mathcal{C}} \left[(\neg E)^{\parr_{\mathcal{C}} i} \parr_{\mathcal{C}} \mathcal{K}\right]$. Then we define $Diff_{E,F} = Diff_E \parr_{\mathcal{C}} F$.

The images of the generating morphisms are defined as follows:

$$Diff_E(d_i) = \Lambda^{-1}_{U(E)^2,U(E)^i,\mathcal{K}} \circ (D'_{E,NL(U(E)^i)\parr_{\mathcal{C}}\mathcal{K}}) \circ \Lambda_{U(E),U(E)^i,\mathcal{K}},$$

$$Diff_E(j_{i+1}) = \left[\rho_{NL(U(E)^{i+1})} \circ \Lambda^{-1}_{U(E),U(E)^{i+1},\mathcal{K}} \circ \left(NL(U(E)) \parr_{\mathcal{C}} \left[\Lambda^{-1}_{U(E);i+1,\mathcal{K}} \circ ((\epsilon^{NL}_E)^{\parr_{\mathcal{C}} i+1} \parr_{\mathcal{C}} \mathcal{K})\right]\right)\right]$$

where we wrote

$$\Lambda^{-1}_{U(E);i+1,F} = \Lambda^{-1}_{U(E),U(E^i),F} \circ \cdots \circ (NL(U(E))^{\parr_{\mathcal{C}} i-1} \parr_{\mathcal{C}} \Lambda^{-1}_{U(E),U(E),F})$$

Since $\mathcal{M}$ has all small limits, one can consider the limit of the functor $U \circ Diff_{E,F}$ and write it $[U(E), U(F)]_{\mathcal{C}}$. Since U bijective on objects, this induces a Hom set:

$$\mathcal{M}_{\mathcal{C}}(U(E), U(F)) = M([U(E), U(F)]_{\mathcal{C}}).$$

We define $NL_{\mathcal{C}}(U(E))$ as the limit in $\mathcal{C}$ of $Diff_E$. Note that, since $\parr_{\mathcal{C}}$ commutes with limits in $\mathcal{C}$, $NL_{\mathcal{C}}(U(E))\parr_{\mathcal{C}} F$ is the limit of $Diff_{E,F} = Diff_E \parr_{\mathcal{C}} F$.

From the universal property of the limit, it comes with canonical maps

$$D^k_{E,F} : NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F \longrightarrow Diff_{E,F}((1,k)), \quad j = j_{E,F} : NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F \longrightarrow Diff_{E,F}((0,0)).$$

Note that $j_{E,F} = j_{E,\mathcal{K}} \parr_{\mathcal{C}} F$ is a monomorphism since for a pair of maps f, g with target $NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F$, using that lemma B.2.29 below implies that all $Diff_E(j_{i+1})$ are monomorphisms, one deduces that all the compositions with all maps of the diagram are equal, hence, by the uniqueness in the universal property of the projective limit, f, g must be equal.

Moreover, since $U : \mathcal{C} \longrightarrow \mathcal{M}$ is right adjoint to $\neg \circ NL$, it preserves limits, so that one gets an isomorphism $\Xi^{\mathcal{M}_{\mathcal{C}}}_{U(E),F} : U(NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F) \simeq U(\lim Diff_{E,F}) \simeq [U(E), U(F)]_{\mathcal{C}}$. It will remain to build $\Lambda^{\mathcal{M}_{\mathcal{C}}}$ but we can already obtain $d_{E,F}$.

We build it by the universal property of limits, consider the maps (obtained using canonical maps for the monoidal category $\mathcal{C}^{op}$)

$$D^{(1,k)}_{E,F} :\ NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F \xrightarrow{D^{k+1}_{E,F}} (NL(U(E)) \parr_{\mathcal{C}} ((\neg E)^{\parr_{\mathcal{C}} k+1} \parr_{\mathcal{C}} \mathcal{K})) \parr_{\mathcal{C}} F \xrightarrow{\simeq} Diff_{E,\neg E \parr_{\mathcal{C}} F}((1,k))$$

$$J^1 :\ NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F \xrightarrow{D^1_{E,F}} (NL(U(E)) \parr_{\mathcal{C}} ((\neg E) \parr_{\mathcal{C}} \mathcal{K})) \parr_{\mathcal{C}} F \xrightarrow{\simeq} Diff_{E,\neg E \parr_{\mathcal{C}} F}((0,0))$$

Those maps extends uniquely to a cone enabling to get by the universal properties of limits our expected map:$d_{E,F}$. This required checking the identities

$$Diff_{E,\neg E \parr_{\mathcal{C}} F}(d^k) \circ J^1 = Diff_{E,\neg E \parr_{\mathcal{C}} F}(j_k) \circ D^{(1,k)}_{E,F}$$

that comes from $Diff_{E,F}(d^k \circ j_1) \circ D^1_{E,F} = Diff_{E,F}(j_{k+1}) \circ D^{1+k}_{E,F}$ (by definition of $D^{1+k}_{E,F}$ as map coming from a limit) which is exactly the previous identity after composition with structural isomorphisms and $NL(E^{k+1}) \parr_{\mathcal{C}} \epsilon^{NL}_E \parr_{\mathcal{C}} F$ which is a monomorphism, hence the expected identity, thanks to the next:

Lemma B.2.29. In the previous situation, ϵ^{NL}_E is a monomorphism.

Proof. Since $\neg : \mathcal{C}^{op} \longrightarrow \mathcal{C}$ is faithful, it suffices to see $\neg(\epsilon^{NL}_E) : \neg(NL(U(E)) \longrightarrow \neg\neg E$ is an epimorphism. But its composition with the epimorphism $\neg\neg E \longrightarrow E$, as counit of an adjunction with faithful functors $\neg$, is also the counit of $\neg \circ NL$ with right adjoint U which is faithful too, hence the composition is an epimorphism too. But $U(\neg\neg E) \simeq U(E)$ by the proof of Theorem B.2.18, thus $U(\neg(\epsilon^{NL}_E))$ is an epimorphism and U is also faithful so reflects epimorphisms. □

Theorem B.2.30. *In the above situation, $(\mathcal{C}^{op}, \parr_{\mathcal{C}}, I, \neg, \mathcal{M}_{\mathcal{C}}, \times, 0, [.,.]_{\mathcal{C}}, NL_{\mathcal{C}}, U, D, d)$ has a structure of Seely model of differential λ-tensor logic.*

Proof. For brevity, we call $A_k = (1,k), k > 0, A_0 = (0,0) = B_0, B_k = (0,k)$

Step 1: $\mathcal{M}_{\mathcal{C}}$ is a Cartesian (not full) subcategory of $\mathcal{M}$ and $U : \mathcal{C} \longrightarrow \mathcal{M}_{\mathcal{C}}, NL_{\mathcal{C}} : \mathcal{M}_{\mathcal{C}} \longrightarrow \mathcal{C}$ are again functors, the latter being right $\neg$-relative adjoint of the former.

Fix

$$g \in \mathcal{M}_{\mathcal{C}}(U(E), U(F)) = \mathcal{C}(\neg K, NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F) \longrightarrow \mathcal{C}(\neg K, NL(U(E)) \parr_{\mathcal{C}} F) \simeq \mathcal{C}(\neg F, NL(U(E))) \ni d^0 g.$$

Similarly, composing with $D^k_{E,F}$ one obtains:

$$d^k g \in \mathcal{C}(\neg K, Diff_{E,\mathcal{K}}(A_k) \parr_{\mathcal{C}} F) \simeq \mathcal{C}(\neg F, Diff_{E,\mathcal{K}}(A_k).$$

We first show that $NL(g) = \cdot \circ g : NL(U(F)) \longrightarrow NL(U(E))$ induces via the monomorphisms j a map $NL_{\mathcal{C}}(g) : NL_{\mathcal{C}}(F) \longrightarrow NL_{\mathcal{C}}(E)$ such that $j_{E,\mathcal{K}} NL_{\mathcal{C}}(g) = NL(g) j_{F,\mathcal{K}}$. This relation already determines at most one $NL_{\mathcal{C}}(g)$, one must check such a map exists in using the universal property for $NL_{\mathcal{C}}(E)$. We must build maps:

$$NL^k_{\mathcal{C}}(g) : NL_{\mathcal{C}}(F) \longrightarrow Diff_{E,\mathcal{K}}(A_k)$$

with $NL^0_{\mathcal{C}}(g) = NL(g)j_{F,\mathcal{K}}$ satisfying the relations for $k \geqslant 0$ ($j_0 = id$):

$$Diff_{E,\mathcal{K}}(d_k \circ j_k) \circ NL^k_{\mathcal{C}}(g) = Diff_{E,\mathcal{K}}(j_{k+1}) \circ NL^{k+1}_{\mathcal{C}}(g). \tag{B.26}$$

An abstract version of Faà di Bruno's formula will imply the form of $NL^k_{\mathcal{C}}(g)$, that we will obtain it as sum of $NL^{k,\pi}_{\mathcal{C}}(g) : Diff_{F,\mathcal{K}}(A_{|\pi|}) \longrightarrow Diff_{E,\mathcal{K}}(A_k)$ for $\pi = \{\pi_1, ..., \pi_{|\pi|}\} \in P_k$ the set of partitions of $[\![1, k]\!]$. We define it as

$$NL^{k,\pi}_{\mathcal{C}}(g) = (NL(\Delta^{|\pi|+1}_E) \parr_{\mathcal{C}} Id) \circ IsomAss_{|\pi|} \circ [d^0 g \parr_{\mathcal{C}} d^{|\pi_1|} g \parr_{\mathcal{C}} \cdots \parr_{\mathcal{C}} d^{|\pi_{|\pi|}|} g \parr_{\mathcal{C}} id_{\mathcal{K}}]$$

with $IsomAss_k : NL(E) \parr_{\mathcal{C}} (NL(E) \parr_{\mathcal{C}} E_1) \parr_{\mathcal{C}} \cdots \parr_{\mathcal{C}} (NL(E) \parr_{\mathcal{C}} E_k) \simeq NL(E^{k+1}) \parr_{\mathcal{C}} (E_1 \parr_{\mathcal{C}} \cdots \parr_{\mathcal{C}} E_k)$ and $\Delta_k : E \longrightarrow E^k$ the diagonal of the Cartesian category $\mathcal{C}$. We will compose it with $d^{P_k} : NL_{\mathcal{C}}(F) \longrightarrow \prod_{\pi \in P_k} Diff_{F,\mathcal{K}}(A_{|\pi|})$ given by the universal property of product composing to $d^{|\pi|}$ in each projection.

Then using the canonical sum map $\Sigma^k_E : \prod_{i=1}^k E \simeq \bigoplus_{i=1}^k E \longrightarrow E$ obtained by universal property of coproduct corresponding to identity maps, one can finally define the map inspired by Faà di Bruno's Formula:

$$NL^k_{\mathcal{C}}(g) = \Sigma^{|P_k|}_{Diff_{E,\mathcal{K}}(A_k)} \circ \Big(\prod_{\pi \in P_k} NL^{k,\pi}_{\mathcal{C}}(g) \Big) \circ d^{P_k}.$$

Applying U and composing with Ξ, (B.26) is then obtained in using the chain rule D5 on the inductive proof of Faà di Bruno's Formula, using also that U is additive.

Considering $NL_{\mathcal{C}}(g) \parr_{\mathcal{C}} G : NL_{\mathcal{C}}(F) \parr_{\mathcal{C}} G \longrightarrow NL_{\mathcal{C}}(E) \parr_{\mathcal{C}} G$ which induces a composition on $\mathcal{M}_{\mathcal{C}}$, one gets that $\mathcal{M}_{\mathcal{C}}$ is a subcategory of $\mathcal{M}$ (from the agreement with previous composition based on intertwining with j) as soon as we see $id_{U(E)} \in \mathcal{M}_{\mathcal{C}}(U(E), U(E))$. This boils down to building a map in $\mathcal{C}$, $I_{\mathcal{M}_{\mathcal{C}}} : \neg(\mathcal{K}) \longrightarrow NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} E$ using the universal property such that $j_{E,E} \circ I_{\mathcal{M}_{\mathcal{C}}} = I_{\mathcal{M}} : \neg(\mathcal{K}) \longrightarrow NL(U(E)) \parr_{\mathcal{C}} E$ corresponds to identity map. We define it in imposing $D^k_{E,E} \circ I_{\mathcal{M}_{\mathcal{C}}} = 0$ if $k \geqslant 2$ and

$$D^1_{E,E} \circ I = NL(0_E) \parr_{\mathcal{C}} i_{\mathcal{C}} : \neg(\mathcal{K}) \simeq NL(0) \parr_{\mathcal{C}} \neg(\mathcal{K}) \longrightarrow NL(E) \parr_{\mathcal{C}} \neg E \parr_{\mathcal{C}} E$$

with $i_{\mathcal{C}} \in \mathcal{C}(\neg(\mathcal{K}), \neg E \parr_{\mathcal{C}} E) \simeq \mathcal{C}(\neg E, \neg E)$ corresponding to identity via the compatibility for the dialogue category $(\mathcal{C}^{op}, \parr_{\mathcal{C}}, \mathcal{K}, \neg)$. This satisfies the compatibility condition enabling to define a map by the universal property of limits because of axiom D3 in [BEM] implying (recall our linear variables are in the right contrary to theirs) $D(Id_{U(E)}) = \pi_2, D(\pi_2) = \pi_2\pi_2$ (giving vanishing starting at second derivative via D-curry) and of course $(\epsilon^{NL}_E \parr_{\mathcal{C}} E) \circ i_{\mathcal{C}} = I_{\mathcal{M}}$ from the adjunction defining ϵ^{NL}.

As above we can use known adjunctions to get the isomorphism

$$\begin{aligned}\mathcal{M}_{\mathcal{C}}(U(E), U(F)) = \mathcal{M}(0, [U(E), U(F)]_{\mathcal{C}}) &\simeq \mathcal{C}^{op}(NL(0), \neg(NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F)) \\ &\simeq \mathcal{C}(\neg(\mathcal{K}), NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F) \simeq \mathcal{C}^{op}(NL_{\mathcal{C}}(U(E)), \neg F)\end{aligned} \tag{B.27}$$

where the last isomorphism is the compatibility for the dialogue category $(\mathcal{C}^{op}, \parr_{\mathcal{C}}, \mathcal{K}, \neg)$. Hence the map $id_{U(E)}$ we have just shown to be in the first space gives $\epsilon^{NL_{\mathcal{C}}}_E : \neg E \longrightarrow NL_{\mathcal{C}}(U(E))$ with $j_{E,\mathcal{K}} \circ \rho_{NL_{\mathcal{C}}(U(E))} \circ \epsilon^{NL_{\mathcal{C}}}_E = \rho_{NL(U(E))} \circ \epsilon^{NL}_E$.

Let us see that U is a functor too. Indeed $\epsilon^{NL_{\mathcal{C}}}_E \parr_{\mathcal{C}} F : \neg E \parr_{\mathcal{C}} F \longrightarrow NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F$ can be composed with the adjunctions and compatibility for the dialogue category again to get:

$$\mathcal{C}(E, F) \longrightarrow \mathcal{C}(\neg\neg E, F) \simeq \mathcal{C}(\neg\mathcal{K}, \neg E \parr_{\mathcal{C}} F) \longrightarrow \mathcal{C}(\neg NL(0), NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F),$$

the last space being nothing but $\mathcal{M}_{\mathcal{C}}(U(E), U(F)) = \mathcal{M}(0, [U(E), U(F)]_{\mathcal{C}})$ giving the wanted $U(g)$ for $g \in \mathcal{C}(E, F)$ which is intertwined via j with the $\mathcal{M}$ valued one, hence U is indeed a functor too. The previous equality is natural in F via the intertwining with j and the corresponding result for $\mathcal{M}$.

Now one can see that (B.27) is natural in $U(E), F$. For it suffices to note that the first equality is natural by definition and all the following ones are already known. Hence the stated $\neg$-relative adjointeness.

This implies U preserve products as right adjoint of $\neg \circ NL_{\mathcal{C}}$, hence the previous products $U(E) \times U(F) = U(E \times F)$ are still products in the new category, and the category $\mathcal{M}_{\mathcal{C}}$ is indeed Cartesian.

Step 2: Curry map

It remains a few structures to define, most notably the internalized Curry map: $\Lambda^{\mathcal{C}}_{E,F,G} : NL_{\mathcal{C}}(E \times F) \parr_{\mathcal{C}} G \longrightarrow NL_{\mathcal{C}}(E) \parr_{\mathcal{C}} \Big(NL_{\mathcal{C}}(F) \parr_{\mathcal{C}} G\Big)$. We use freely the structure isomorphisms of the monoidal category $\mathcal{C}$.

For we use the universal property of limits as before, we need to define:

$$\Lambda^k_{E,F,G} : NL_{\mathcal{C}}(E \times F) \parr_{\mathcal{C}} G \longrightarrow Diff_{E,(NL_{\mathcal{C}}(F)\parr_{\mathcal{C}}G)}(A_k)$$

satisfying the relations for $k \geqslant 0$ ($j_0 = id$):

$$Diff_{E,(NL_{\mathcal{C}}(F)\parr_{\mathcal{C}}G)}(d_k \circ j_k) \circ \Lambda^k_{E,F,G} = Diff_{E,(NL_{\mathcal{C}}(F)\parr_{\mathcal{C}}G)}(j_{k+1}) \circ \Lambda^{k+1}_{E,F,G}. \tag{B.28}$$

Since $Diff_{E,(NL_{\mathcal{C}}(F)\parr_{\mathcal{C}}G)}(A_k) \simeq NL_{\mathcal{C}}(F) \parr_{\mathcal{C}} \Big(NL(E) \parr_{\mathcal{C}} (\neg E)^{\parr_{\mathcal{C}} k} \parr_{\mathcal{C}} \mathcal{K}\Big) \parr_{\mathcal{C}} G$ we use again the same universal property to define the map $\Lambda^k_{E,F,G}$ and we need to define:

$$\Lambda^{k,l}_{E,F,G} : NL_{\mathcal{C}}(E \times F) \parr_{\mathcal{C}} G \longrightarrow Diff_{F,\Big(NL(E)\parr_{\mathcal{C}}(\neg E)^{\parr_{\mathcal{C}} k}\parr_{\mathcal{C}}\mathcal{K}\Big)\parr_{\mathcal{C}}G}(A_l).$$

satisfying the relations:

$$Diff_{F,\Big(NL(E)\parr_{\mathcal{C}}(\neg E)^{\parr_{\mathcal{C}} k}\parr_{\mathcal{C}}\mathcal{K}\Big)\parr_{\mathcal{C}}G}(d_l \circ j_l) \circ \Lambda^{k,l}_{E,F,G} = Diff_{F,\Big(NL(E)\parr_{\mathcal{C}}(\neg E)^{\parr_{\mathcal{C}} k}\parr_{\mathcal{C}}\mathcal{K}\Big)\parr_{\mathcal{C}}G}(j_{l+1}) \circ \Lambda^{k,l+1}_{E,F,G}. \tag{B.29}$$

But we can consider the map:

$$D^{k+l}_{E\times F,G} : NL_{\mathcal{C}}(E \times F) \parr_{\mathcal{C}} G \longrightarrow (NL(U(E \times F)) \parr_{\mathcal{C}} ((\neg(E \times F))^{\parr_{\mathcal{C}} k+l} \parr_{\mathcal{C}} \mathcal{K})) \parr_{\mathcal{C}} G$$

Let us describe an obvious isomorphism of the space of value to extract the component we need. First, using the assumptions on $\parr_{\mathcal{C}}$ and $\neg$:

$$\begin{aligned}((\neg(E_1 \times E_2))^{\parr_{\mathcal{C}} k+l} \parr_{\mathcal{C}} \mathcal{K})) \parr_{\mathcal{C}} G &\simeq \bigoplus_{i:[\![1,k+l]\!] \longrightarrow \{1,2\}} (\neg E_{i_1} \parr_{\mathcal{C}} \neg E_{i_2} \parr_{\mathcal{C}} \cdots \parr_{\mathcal{C}} \neg E_{i_{k+l}}) \parr_{\mathcal{C}} G \\ &\simeq \bigoplus_{i:[\![1,k+l]\!] \longrightarrow \{1,2\}} (\neg(E_1)^{\parr_{\mathcal{C}}(\#f^{-1}(\{1\}))} \parr_{\mathcal{C}} \neg(E_2)^{\parr_{\mathcal{C}}(\#i^{-1}(\{2\}))}) \parr_{\mathcal{C}} G.\end{aligned}$$

Hence using also $\Lambda_{E,F,.}$ one gets:

$$\begin{aligned}&\Lambda : (NL(U(E \times F)) \parr_{\mathcal{C}} ((\neg(E \times F))^{\parr_{\mathcal{C}} k+l} \parr_{\mathcal{C}} \mathcal{K})) \parr_{\mathcal{C}} G \\ &\simeq \bigoplus_{i:[\![1,k+l]\!] \longrightarrow \{1,2\}} Diff_F(A_{\#i^{-1}(\{2\})}) \parr_{\mathcal{C}} \Big(Diff_E(A_{\#i^{-1}(\{1\})}) \parr_{\mathcal{C}} G\Big).\end{aligned}$$

Composing with $P_{k,l}$ a projection on a term with $\#i^{-1}(\{1\}) = k$, one gets the map $P_{k,l} \circ \Lambda \circ D^{k+l}_{E\times F,G} = \Lambda^{kl}_{E,F,G}$ we wanted. One could check this does not depend on the choice of term using axiom (D7) of Differential Cartesian categories giving an abstract Schwarz lemma, but for simplicity we choose $i(1) = \cdots = i(l) = 2$ which corresponds to differentiating all variables in E first and then all variables in F. The relations we want to check will follow from axiom (D-curry) of differential λ-categories.

Then to prove the relation (B.29) we can prove it after composition by a Λ (hence the left hand side ends with application of $D'_{F,NL(U(F)^l)\parr_{\mathcal{C}}\mathcal{K}} \parr_{\mathcal{C}} \Big(Diff_E(A_k) \parr_{\mathcal{C}} G\Big)$). We can then apply $Diff_F(B_l) \parr_{\mathcal{C}} \Big(Diff_E(j_k) \parr_{\mathcal{C}} G\Big)$ which is a monomorphism and obtain, after decurryfying and applying U and various Ξ, maps in $[U(F)^2 \times U(F)^l \times U(E)^{k+1}, U(G)]$, and finally only prove equality there, the first variable F being a non-linear one.

Of course, we start from $Diff_{E\times F,G}(d_{k+l} \circ j_{k+l}) \circ D^{k+l}_{E\times F,G} = Diff_{E\times F,G}(j_{k+l+1}) \circ D^{k+l+1}_{E\times F,G}$ and use an application of (B.25):

$$\begin{aligned}&\Big[Diff_F(B_l) \parr_{\mathcal{C}} \Big(Diff_E(j_k) \parr_{\mathcal{C}} G\Big)\Big] \circ Diff_{F,Diff_E(A_k)\parr_{\mathcal{C}}G}(j_l) \circ P_{k,l} \circ \Lambda \\ &= Isom \circ \Lambda_{E,F,Diff_F(B_l)\parr_{\mathcal{C}}\Big(Diff_E(B_k)\parr_{\mathcal{C}}G\Big)} \circ NL(0_{l,k}) \circ Diff_{E\times F,G}(j_{k+l})\end{aligned}$$

with $0_{l,k} : U(E \times F) \times U(F)^l \times U(E)^k \simeq U(E \times F) \times U(0 \times F)^l \times U(E \times 0)^k \longrightarrow U(E \times F)^{k+l+1}$ the map corresponding to $id_{E\times F} \times (0_E \times id_F)^l \times (id_E \times 0_F)^k$. We thus need the following commutation relation:

$$NL(0_{l+1,k}) \circ Diff_{E\times F,G}(d_{k+l}) = Isom \circ NL(0_{1,0}) \circ (D'_{E\times F,\left(Diff_F(B_l)\parr_{\mathcal{C}} Diff_E(B_k)\parr_{\mathcal{C}} G\right)}) \circ NL(0_{l,k})$$

This composition $NL(0_{1,0}) \circ D'_{E\times F,.}$ gives exactly after composition with some Ξ the right hand side of (D-curry), hence composing all our identities, and using canonical isomorphisms of λ-models of λ-tensor logic, and this relation gives the expected (B.29) at the level of $[U(F)^2 \times U(F)^l \times U(E)^{k+1}, U(G)]$.

Let us turn to checking (B.28). It suffices to check it after composition with the monomorphism $Diff_E(B_k)\parr_{\mathcal{C}} j_{F,G}$. Then the argument is the same as for (B.29) in the case $k = 0$ and with E and F exchanged. The inverse of the Curry map is obtained similarly.

Step 3: $\mathcal{M}_{\mathcal{C}}$ is a differential λ-category.

We first need to check that $\mathcal{M}_{\mathcal{C}}$ is Cartesian closed, and we already know it is Cartesian. Since we defined the internalized curry map and Ξ one can use the first compatibility diagram in the definition B.2.16 to define $\Lambda^{\mathcal{M}_{\mathcal{C}}}$. To prove the defining adjunction of exponential objects for Cartesian closed categories, it suffices to see naturality after applying the basic functor to sets M. From the defining diagram, naturality in E, F of $\Lambda^{\mathcal{M}_{\mathcal{C}}} : [E \times F, U(G)]_{\mathcal{C}} \longrightarrow [E, [F, U(G)]_{\mathcal{C}}]_{\mathcal{C}}$ will follow if one checks the naturality of $\Xi^{\mathcal{C}}_{E,F}$ and $\Lambda^{\mathcal{C}}_{E,F,G}$ that we must check anyway while naturality in $U(G)$ and not only G will have to be considered separately.

For $\Lambda^{\mathcal{C}-1}_{E,F,G}$, take $e : E \longrightarrow E', f : F \longrightarrow F', g : G' \longrightarrow G$ the first two in $\mathcal{M}_{\mathcal{C}}$ the last one in $\mathcal{C}$. We must see $\Lambda^{\mathcal{C}-1}_{E,F,G} \circ [NL_{\mathcal{C}}(e) \parr_{\mathcal{C}} (NL_{\mathcal{C}}(f) \parr_{\mathcal{C}} g)] = [NL_{\mathcal{C}}(e \times f) \parr_{\mathcal{C}} g] \circ \Lambda^{\mathcal{C}-1}_{E',F',G'}$ and it suffices to see equality after composition with the monomorphism $j_{U^{-1}(E\times F),G} : NL_{\mathcal{C}}(E\times F)\parr_{\mathcal{C}} G \longrightarrow NL(E\times F)\parr_{\mathcal{C}} G$. But by definition, $j_{U^{-1}(E\times F),G}\Lambda^{\mathcal{C}-1}_{E,F,G} = \Lambda^{-1}_{E,F,G}(NL(E) \parr j_{U^{-1}(F),G}) j_{U^{-1}(E),NL_{\mathcal{C}}(F)\parr G}$ and similarly for $NL_{\mathcal{C}}$ functors which are also induced from NL, hence the relation comes from the one for Λ of the original model of λ-tensor logic we started with. The reasoning is similar with Ξ. Let us finally see that $M(\Lambda^{\mathcal{M}_{\mathcal{C}}})$ is natural in $U(G)$, but again from step 1 composition with a map $g \in \mathcal{M}_{\mathcal{C}}(U(G), U(G')) \subset \mathcal{M}(U(G), U(G'))$ is induced by the one from $\mathcal{M}$ and so is $\Lambda^{\mathcal{M}_{\mathcal{C}}}$ from $\Lambda^{\mathcal{M}}$ in using the corresponding diagram for the original model of λ-tensor logic we started with and all the previous induced maps for $\Xi, \Lambda^{\mathcal{C}}$. Hence also this final naturality in $U(G)$ is induced.

Having obtained the adjunction for a Cartesian closed category, we finally see that all the axioms D1–D7 of Cartesian differential categories in [BEM] and D-Curry is also induced. Indeed, our new operator D is also obtained by restriction as well as the left additive structure. Note that as a consequence the new U is still a **Mon**-enriched functor.

Step 4: $(\mathcal{M}_{\mathcal{C}}, \mathcal{C})$ form a λ-categorical model of λ-tensor logic and Conclusion.

We have already built all the data for definition B.2.16, and shown $\neg$-relative adjointness in step 1. It remains to see the four last compatibility diagrams.

But from all the naturality conditions for canonical maps of the monoidal category, one can see them after composing with monomorphisms $NL_{\mathcal{C}} \longrightarrow NL$ and induce them from the diagrams for NL.

Among all the data needed in definition B.2.27, it remains to build the internalized differential $D^{\mathcal{C}}_{E,F}$ for D in $\mathcal{M}^{\lrcorner}$ and see the two compatibility diagrams there. From the various invertible maps, one can take the first diagram as definition of $D^{\mathcal{C}}_{U(E),U(F)}$ and must see that, then $M(D^{\mathcal{C}}_{U(E),U(F)})$ is indeed the expected restriction of D. Let $j^{E,F}_{\mathcal{M}} : [U(E), U(F)]_{\mathcal{C}} \longrightarrow [U(E), U(F)]$ the monomorphism. It suffices to see $j^{E\times E,F}_{\mathcal{M}} \circ D^{\mathcal{C}}_{U(E),U(F)} \circ \Xi^{\mathcal{C}}_{E,F} = \Xi_{E\times E,F} \circ U(D'_{E,F} \circ j_{E,F})$ (note that this also gives the naturality in E, F of d from the one of D'). Hence from the definition of $D^{\mathcal{C}}$, it suffices to see the following diagram:

$$\begin{tikzcd}
U\Big(NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F\Big) \arrow[rr, "U(NL_{\mathcal{C}}(U(E)))\parr_{\mathcal{C}}(\epsilon^{NL_{\mathcal{C}}}_E \parr_{\mathcal{C}} F))\circ U(d_{E,F})"] \arrow[d, "\Xi_{E\times E,F}\circ U(D'_{E,F}\circ j_{E,F})"'] & & U\Big(NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} (NL_{\mathcal{C}}(U(E)) \parr_{\mathcal{C}} F)\Big) \arrow[d, "{[id_{U(E)},\Xi^{\mathcal{C}}_{E,F}]_{\mathcal{C}}\circ\Xi^{\mathcal{C}}_{E,NL_{\mathcal{C}}(U(E))\parr_{\mathcal{C}} F}}"'] \\
{[U(E\times E), U(F)]} & {[U(E\times E), U(F)]_{\mathcal{C}}} \arrow[l, "j^{E\times E,F}_{\mathcal{M}}"'] & {[U(E), [U(E), U(F)]_{\mathcal{C}}]_{\mathcal{C}}.} \arrow[l, "(\Lambda^{\mathcal{M}_{\mathcal{C}}}_{U(E),U(E),U(F)})^{-1}"']
\end{tikzcd}$$

First we saw from induction of our various maps that the right hand side of the diagram can be written without maps with index $\mathcal{C}$:

$$(\Lambda^{\mathcal{M}}_{U(E),U(E),U(F)})^{-1} \circ [id_{U(E)}, \Xi_{E,F}] \circ \Xi_{E,NL(E)\parr_{\mathcal{C}} F} \circ U(NL(U(E))) \parr_{\mathcal{C}} (\epsilon^{NL}_E \parr_{\mathcal{C}} F)) \circ U(j_{E,\neg E \parr_{\mathcal{C}} F} \circ d_{E,F}).$$

The expected diagram now comes the definition of $d_{E,F}$ by universal property which gives $j_{E,\neg E\parr_{\mathcal{C}} F} \circ d_{E,F} = J^1 = Isom \circ D^1_{E,F}$ and similarly $Diff_{E,F}(j_1) \circ D^1_{E,F} = D'_{E,F} \circ j_{E,F}$ so that composing the above diagrams (and an obvious commutation of the map involving ϵ^{NL} through various natural isomorphisms) gives the result.

For the last diagram in definition B.2.27, since $j = j_{E,\neg E\parr_{\mathcal{C}} F}$ is a monomorphism, it suffices to compose $d_{E,F}$ and the equivalent map stated in the diagram by j and see equality, and from the recalled formula above reducing it to $D'_{E,F}$, this reduces to (B.24). □

B.2.2.3 ρ-smooth maps as model of DiLL

Our previous categories from Theorem B.2.23 cannot give a model of DiLL with $\mathcal{C} - \mathbf{ref}_{\infty}$ as category with smooth maps. If one wants to obtain a differential map since the map won't be with value in $E \multimap_{\mathscr{C}} F$ but in spaces of bounded linear maps $L_{bd}(E, F)$. We will have to restrict to maps with iterated differential valued in $E^{\otimes_{\mathscr{C}} k} \multimap_{\mathscr{C}} F := E \multimap_{\mathscr{C}} (\cdots (E \multimap_{\mathscr{C}} F) \cdots)$. This is what we did abstractly in the previous subsection that will enable us to obtain efficiently a model.

Lemma B.2.31. The categories of Theorem B.2.23 satisfy the assumptions of subsection B.2.2.2.

Proof. We already saw in Theorem B.2.23 that the situation of Theorem B.2.18 is satisfied with dialogue category $\mathcal{C} = (\mathscr{C} - \mathbf{Mc}^{op}, \epsilon, \mathcal{K}, (\cdot)^*_{\mathscr{C}})$, and $\mathcal{M} = \mathscr{C} - \mathbf{Mc}_{\infty}$. We already know that ε-product commutes with limits and monomorphisms and the biproduct property is easy. The key is to check that we have an internalized derivative. From [53] we know that we have a derivative $d : \mathcal{C}^{\infty}_{\mathscr{C}}(E, F) \longrightarrow \mathcal{C}^{\infty}_{\mathscr{C}}(E, L_b(E, F))$ and the space of bounded linear maps $L_{bd}(E, F) \subset \mathcal{C}^{\infty}_{\mathscr{C}}(E, F)$ the set of conveniently smooth maps. Clearly, the inclusion is continuous since all the images by curves of compact sets appearing in the projective kernel definition of $\mathcal{C}^{\infty}_{\mathscr{C}}(E, F)$ are bounded. Thus one gets $d : \mathcal{C}^{\infty}_{\mathscr{C}}(E, F) \longrightarrow \mathcal{C}^{\infty}_{\mathscr{C}}(E, \mathcal{C}^{\infty}_{\mathscr{C}}(E, F)) \simeq \mathcal{C}^{\infty}_{\mathscr{C}}(E \times E, F)$. It remains to see continuity. For by the projective kernel definition, one must check that for $c = (c_1, c_2) \in \mathcal{C}^{\infty}_{co}(X, E \times E)$, $f \mapsto df \circ (c_1, c_2)$ is continuous $\mathcal{C}^{\infty}_{\mathscr{C}}(E, F) \longrightarrow \mathcal{C}^{\infty}_{co}(X, F)$. But consider the curve $c_3 : X \times X \times \mathscr{R} \longrightarrow E$ given by $c_3(x, y, t) = c_1(x) + c_2(y)t$, since $X \times X \times \mathscr{R} \in \mathscr{C}$, we know that $f \circ c_3$ is smooth and $\partial_t (f \circ c_3)(x, y, 0) = df(c_1(x))(c_2(y))$ and its derivatives in x, y are controlled by the seminorms for $\mathcal{C}^{\infty}_{\mathscr{C}}(E, F)$, hence the stated continuity. It remains to note that $\mathcal{M}$ is a differential λ-category since we already know it is Cartesian closed and all the properties of derivatives are well-known for conveniently smooth maps. For instance, the chain rule D7 is [KM, Thm 3.18]. □

Concretely, one can make explicit the stronger notion of smooth maps considered in this case.

We thus consider d^k the iterated (convenient) differential giving $d^k : \mathcal{C}^{\infty}_{\mathscr{C}}(E, F) \longrightarrow \mathcal{C}^{\infty}_{\mathscr{C}}(E, L_{bd}(E^{\otimes_{\beta} k}, F))$. Since $E^{\otimes_{\mathscr{C}} k} \multimap_{\mathscr{C}} F$ is a subspace of $L_{bd}(E^{\otimes_{\beta} k}, F)$ (unfortunately this does not seem to be in general boundedly embedded), we can consider:

$$C^{\infty}_{\mathcal{C}-\mathbf{ref}}(E, F) := \left\{ u \in \mathcal{C}^{\infty}_{\mathscr{C}}(E, F) : \forall k \geqslant 1 : d^k(u) \in \mathcal{C}^{\infty}_{\mathscr{C}}(E, E^{\otimes_{\mathscr{C}} k} \multimap_{\mathscr{C}} F) \right\}.$$

Remark B.2.32. A map $f \in C^{\infty}_{\mathcal{C}-\mathbf{ref}}(E, F)$ will be called $\mathcal{C} - \mathbf{ref}$*-smooth*. In the case $\mathscr{C} = Ban$, we say ρ-smooth maps, associated to the category ρ-Ref, and write $C^{\infty}_{\rho} = C^{\infty}_{Ban-\mathbf{Ref}}$. Actually, for $Fin \subset \mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$, from the equivalence of $*$-autonomous categories in Theorem B.2.22, and since the inverse functors keep the bornology of objects, hence don't change the notion of conveniently smooth maps, we have algebraically

$$C^{\infty}_{\mathcal{C}-\mathbf{ref}}(E, F) = C^{\infty}_{\rho}(\mathscr{S}(E_{\mu}), \mathscr{S}(F_{\mu})).$$

Hence, we only really introduced one new notion of smooth maps, namely, ρ-smooth maps. Of course, the topologies of the different spaces differ.

Thus d^k induces a map $C^{\infty}_{\mathcal{C}-\mathbf{ref}}(E, F) \longrightarrow \mathcal{C}^{\infty}_{\mathscr{C}}(E, E^{\otimes_{\mathscr{C}} k} \multimap_{\mathscr{C}} F)$ $(d^0 = id)$ and we equip $C^{\infty}_{\mathcal{C}-\mathbf{ref}}(E, F)$ with the corresponding locally convex kernel topology $\mathrm{K}_{n, \geqslant 0}(d^n)^{-1}(C^{\infty}(E, E^{\otimes n} \multimap F))$ with the notation of [K] and the previous topology given on any $C^{\infty}(E, E^{\otimes k} \multimap F)$.[2]

[2] This definition is quite similar to one definition (for the corresponding space of value $E^{\otimes k} \multimap F$ which can be interpreted as a space of hypocontinuous multilinear maps for an appropriate bornology) in [58] except that instead of requiring continuity of all derivatives, we require their smoothness in the sense of Kriegl-Michor.

We call $\mathcal{C}-\mathbf{ref}_{\infty\mathcal{C}-\mathbf{ref}}$ the category of $\mathscr{C}$-reflexive spaces with $C^{\infty}_{\mathcal{C}-\mathbf{ref}}$ as spaces of maps. Then from section B.2.2.2 we even have an induced $d: C^{\infty}_{\mathcal{C}-\mathbf{ref}}(E,F) \longrightarrow C^{\infty}_{\mathcal{C}-\mathbf{ref}}(E, E^{\otimes_{\mathscr{C}} k} \multimap_{\mathscr{C}} F)$.

Let us call $d_0(f) = df(0)$ so that $d_0 : C^{\infty}_{\mathcal{C}-\mathbf{ref}}(E,F) \longrightarrow C^{\infty}_{\mathcal{C}-\mathbf{ref}}(E,F)$ is continuous. Recall also that we introduced $\partial_E = \overline{\mathsf{c}}_E \circ (!E \otimes \overline{\mathsf{d}}_E)$ and dually $\overline{\partial}_E = (!E \otimes \mathsf{d}_E)\mathsf{c}_E : !E \longrightarrow !EE$. We conclude to our model:

Theorem B.2.33. *Let $Fin \subset \mathscr{C} \subset \mathbf{F} \times \mathbf{DFS}$ as above. $\mathcal{C}-\mathbf{ref}$ is also a Seely category with biproducts with structure extended by the comonad $!_{\mathcal{C}-\mathbf{ref}}(\cdot) = (C^{\infty}_{\mathcal{C}-\mathbf{ref}}(\cdot))^{*}_{\mathscr{C}}$ associated to the adjunction with left adjoint $!_{\mathcal{C}-\mathbf{ref}} : \mathcal{C}-\mathbf{ref}_{\infty-\mathcal{C}-\mathbf{ref}} \longrightarrow \mathcal{C}-\mathbf{ref}$ and right adjoint U. It gives a model of DiLL with codereliction $(d_0)^{*}_{\mathscr{C}}$.*

Proof. This is a combination of Theorem B.2.30, B.2.28 and the previous lemma. □

Remark B.2.34. One can check that

1. for any $E \in \mathcal{C}-\mathbf{ref}$, $\partial_E \overline{\partial}_E + id_E$ is invertible,
2. The model is Taylor in the sense of [20, 3.1], i.e. for any $f_1, f_2 : !_{\mathcal{C}-\mathbf{ref}} E \longrightarrow F$ if $f_1 \partial_E = f_2 \partial_E$ then $f_1 + f_2 \overline{\mathsf{w}}_E \mathrm{W}\textsc{eak} E = f_2 + f_1 \overline{\mathsf{w}}_E \mathrm{W}\textsc{eak} E$.

Indeed, the Taylor property is obvious since $df_1 = df_2$ in the convenient setting implies the same Gâteaux derivatives, hence $f_1 + f_2(0) = f_2 + f_1(0)$ on each line hence everywhere.

For (1), we define the inverse by $(I_E)^{*}_{\mathscr{C}}$ with $I_E : \mathcal{C}^{\infty}_{\mathscr{C}}(E) \longrightarrow \mathcal{C}^{\infty}_{\mathscr{C}}(E)$ as in [20, 3.2.1] by $I_E(f)(x) = \int_0^1 f(tx)dt$, which is a well-defined weak Riemann integral by Mackey-completeness of the space [53].

By [20], the two conditions reformulate the two fundamental theorems of calculus. See also [CL] for a further developments on the two conditions above.

Remark B.2.35. Let us continue our comparison of subsection B.2.1.6. Let us see that if $E, F \in \textsc{Conv}$, $C^{\infty}_{\mathcal{C}-\mathbf{ref}}(E,F) = \mathcal{C}^{\infty}_{\mathscr{C}}(E,F)$ so that we didn't introduce a new class of smooth maps for convenient vector spaces. Our notion of smoothness turning our model into a model of DiLL is only crucial on the extra-spaces we added to get a $*$-autonomous category in $\rho-\mathbf{Ref}$. For, it suffices to see that $f \in \mathcal{C}^{\infty}_{\mathscr{C}}(E,F)$ is ρ-smooth. But (B.19) gives that the derivative automatically smooth with value $L_\beta(E,F)$ by convenient smoothness is also smooth by composition with value $E^{*}_{\rho}\varepsilon F$ as expected. Since this equation only depends on the source space E to be bornological, it extends to spaces for higher derivatives, hence the conclusion.

Hence we have a functor $\mathscr{S} : \textsc{Conv}_\infty \longrightarrow \mathcal{C}-\mathbf{ref}_{\infty-\mathcal{C}-\mathbf{ref}}$ for any $\mathscr{C}$ as above. We don't think this is an equivalence of category any more, as was the corresponding functor in B.2.1.6. But finding a counterexample to essential surjectivity may be difficult, even thought we didn't really try.

B.2.2.4 k-smooth maps as model of DiLL

We now turn to improve the $*$-autonomous category $k-\mathbf{Ref}$ of section B.1.3 into a model of DiLL using the much stronger notion of k-smooth map considered in subsection B.1.3.2. For $X, Y \in k-\mathbf{Ref}$, $C^{\infty}_{co}(X,Y) \subset \mathcal{C}^{\infty}(X,Y)$, hence there is a differential map $d : C^{\infty}_{co}(X,Y) \longrightarrow C^{\infty}_{co}(X, L_\beta(X,Y))$ but it is by definition valued in $C^{0}_{co}(X, L_{co}(X,Y))$. But actually since the derivatives of these map are also known, it is easy to use the universal property of projective limits to induce a continuous map: $d : C^{\infty}_{co}(X,Y) \longrightarrow C^{\infty}_{co}(X, L_{co}(X,Y))$. Finally, note that $L_{co}(X,Y) = X^{*}_{k}\varepsilon Y$, hence the space of value is the one expected for the dialogue category Kc^{op} from Theorem B.1.43.

For simplicity, in this section we slightly change $k-\mathbf{Ref}$ to be the category of k-reflexive spaces of density character smaller than a fixed inaccessible cardinal κ, in order to have a small category $\mathscr{C} = k-\mathbf{Ref}$ and in order to define without change $\mathcal{C}^{\infty}_{\mathscr{C}}(X,Y)$

We call $k-\mathbf{Ref}_\infty$ the category of k-reflexive spaces with maps $C^{0}_{co}(X,Y)$ as obtained in subsection . We call $\mathbf{Kc}_\infty$ the category of k-quasi-complete spaces (with density character smaller than the same κ) with maps $\mathcal{C}^{\infty}_{\mathscr{C}}(X,Y)$. This is easy to see that this forms a category by definition of $\mathcal{C}^{\infty}_{\mathscr{C}}$. We first check our assumptions to produce models of LL. We call $\mathcal{C}^{\infty}_{\mathscr{C}} : \mathbf{Kc}_\infty \longrightarrow \mathbf{Kc}^{op}$ the functor associating $\mathcal{C}^{\infty}_{\mathscr{C}}(X) = \mathcal{C}^{\infty}_{\mathscr{C}}(X, \mathscr{R})$ to a space X.

Lemma B.2.36. $(\mathbf{Kc}^{op}, \varepsilon, \mathcal{K}, (\cdot)^{*}_{\rho}, \mathbf{Kc}_\infty, \times, 0, \mathcal{C}^{\infty}_{\mathscr{C}}, U)$ is a Seely linear model of λ-tensor logic.

Proof. We checked in Theorem B.1.43 that $\mathcal{C} = (\mathbf{Kc}^{op}, \varepsilon, \mathcal{K}, (\cdot)^{*}_{\rho})$ is a dialogue category. Completeness and cocompleteness are obvious using the k-quasicompletion functor to complete colimits in $\mathbf{LCS}$. Lemma B.2.1 gives the maps Ξ, Λ and taking the first diagram as definition of $\Lambda^{\mathcal{M}}$ one gets Cartesian closedness of $\mathcal{M} = (\mathbf{Kc}_\infty, \times, 0, \mathcal{C}^{\infty}_{\mathscr{C}}(\cdot,\cdot))$, and this result also gives the relative adjunction. The other compatibility diagrams are reduced to conveniently smooth maps $\mathcal{C}^{\infty}_{Fin}$ as in the proof of Theorem B.2.23. □

Note that since $k - \mathbf{Ref}^{op}$ is already $*$-autonomous and isomorphic to its continuation category

Lemma B.2.37. The categories of the previous lemma satisfy the assumptions of subsection B.2.2.2.

Proof. The differential λ-category part reduces to convenient smoothness case. The above construction of d make everything else easy. $\square$

Theorem B.2.38. $k-\mathbf{Ref}$ *is also a complete Seely category with biproducts with $*$-autonomous structure extended by the comonad $!_{co}(\cdot) = (C^{\infty}_{co}(\cdot))^*_k$ associated to the adjunction with left adjoint $!_{co} : k - \mathbf{Ref}_{co} \longrightarrow k - \mathbf{Ref}$ and right adjoint U. It gives a model of DiLL with codereliction $(d_0)^*_k$.*

Proof. Note that on $k - \mathbf{Ref}$ which corresponds to $\mathcal{D}$ in the setting of subsection B.2.2.2, we know that $C^{\infty}_{co} = \mathcal{C}^{\infty}_{\mathscr{C}}$ by the last statement in lemma B.2.1. But our previous construction of d implies that the new class of smooth maps obtained by the construction of subsection B.2.2.2 is again C^{∞}_{co}. The result is a combination of Theorem B.2.30, B.2.28 and the previous lemmas. $\square$

B.2.3 Conclusion

This work is a strong point for the validity of the classical setting of Differential Linear Logic. Indeed, if the proof-theory of Differential Linear Logic is classical, we present here the first smooth models of Differential Linear Logic which comprehend the classical structure. Our axiomatization of the rules for differential categories within the setting of Dialogue categories can be seen as a first step towards a computational classical understanding of Differential Linear Logic. We plan to explore the categorical content of our construction for new models of Smooth Linear Logic, and the diversity of models which can be constructed this way. Our results also argue for an exploration of a classical differential term calculus, as initiated by Vaux [Vaux], and inspired by works on the computational signification of classical logic [**?**] and involutive linear negation [Mu].

The clarification of a natural way to obtain $*$-autonomous categories in an analytic setting suggests to reconsider known models such a [31] from a more analytic viewpoint, and should lead the way to exploit the flourishing operator space theory in logic, following the inspiration of the tract [33]. An obvious notion of coherent operator space should enable this.

This interplay between functional analysis, physics and logic is also strongly needed as seen the more and more extensive use of convenient analysis in some algebraic quantum field theory approaches to quantum gravity [BFR]. Here the main need would be to improve the infinite dimensional manifold theory of diffeomorphism groups on non-compact manifolds. From that geometric viewpoint, differential linear logic went only half the way in considering smooth maps on linear spaces, rather that smooth maps on a kind of smooth manifold. By providing nice $?$-monads, our work suggests to try using $?$-algebras for instance in k-reflexive or ρ-reflexive spaces as a starting point (giving a base site of a Grothendieck topos) to capture better infinite dimensional features than the usual Cahier topos. Logically, this probably means getting a better interplay between intuitionist dependent type theory and linear logic. Physically, this would be useful to compare recent homotopic approaches [BSS] with applications of the BV formalism [FR, FR2]. Mathematically this probably means merging recent advances in derived geometry (see e.g. [To]) with infinite dimensional analysis. Since we tried to advocate the way linear logic nicely captures (for instance with two different tensor products) infinite dimensional features, this finally strongly suggests for an interplay of parametrized analysis in homotopy theory and parametrized versions of linear logic [**?**].

B.2.4 Appendix

We conclude with two technical lemmas only used to show we have built two different examples of ! on the same category ρ-Ref.

Lemma B.2.39. For any ultrabornological spaces E_i, any topological locally convex hull $E = \Sigma_{i\in I} A_i(E_i)$, then we have the topological identity:

$$\mathscr{S}(E'_\mu) = \mathrm{K}_{i\in I}(A_i^t)^{-1}(\mathscr{S}((E_i)'_\mu).$$

Proof. We start with case where E_i are Banach spaces. By functoriality one gets a map between two topologies on the same space (see for Mackey duals [K, p 293]):

$$\mathscr{S}(E'_\mu) \longrightarrow \mathrm{K}_{i\in I}(A_i^t)^{-1}(\mathscr{S}((E_i)'_\mu)) =: F.$$

In order to identify the topologies, it suffices to identify the duals and the equicontinuous sets on them. From [K, 22.6.(3)], the dual of the right hand side is $F' = \Sigma_{i\in I}(A_i^{tt})(\mathscr{S}((E_i)'_\mu)' = \Sigma_{i\in I}(B_i)(E_i) \longrightarrow E$ where the injective continuous map to E is obtained by duality of the previous surjective map (and the maps called B_i again are in fact compositions of A_i^{tt} and the isomorphism between $[\mathscr{S}((E_i)'_\mu)]' = E_i$). From the description of E the map above is surjective and thus we must have $F' = E$ as vector spaces.

Let us now identify equicontinuous sets. From continuity of $\mathscr{S}(E'_\mu) \longrightarrow F$ every equicontinuous set in F' is also equicontinuous in $E = (\mathscr{S}(E'_\mu))'$. Conversely an equicontinuous set in $E = (\mathscr{S}(E'_\mu))'$ is contained in the absolutely convex cover of a null-sequence $(x_n)_{n\geqslant 0}$ for the bornology of absolutely convex weakly-compact sets, (thus also for the bornology of Banach disks [Ja, Th 8.4.4 b]). By a standard argument, there is $(y_n)_{n\geqslant 0}$ null sequence of the same type such that $(x_n)_{n\geqslant 0}$ is a null sequence for the bornology of absolutely convex compact sets in a Banach space E_B with B the closed absolutely convex cover of $(y_n)_{n\geqslant 0}$.

Of course $(y_n)_{n\leqslant m}$ can be seen inside a minimal finite sum $G_m = \Sigma_{i\in I_m}(B_i)(E_i)'$ and G_m is increasing in F so that one gets a continuous map $I : \text{ind lim}_{m\in\mathcal{N}} G_m \longrightarrow F'$. Moreover each G_m being a finite hull of Banach space, it is again a Banach space thus one gets a linear map $j : E_B \longrightarrow \text{ind lim}_{m\in\mathcal{N}} G_m = G$. Since $I \circ j$ is continuous, j is a sequentially closed map, E_B is Banach space, G a (LB) space therefore a webbed space, by De Wilde's closed graph theorem [K2, 35.2.(1)], one deduces j is continuous. Therefore by Grothendieck's Theorem [K, 19.6.(4)], there is a G_m such that j is valued in G_m and continuous again with value in G_m. Therefore $(j(x_n))_{n\geqslant 0}$ is a null sequence for the bornology of absolutely convex compact sets in G_m. We want to note it is equicontinuous there, which means it is contained in a sum of equicontinuous sets.

By [K, 19.2.(3)], G_m is topologically a quotient by a closed linear subspace $\bigoplus_{i\in I_m}(B_i)(E_i)'/H$. By [K, 22.2.(7)] every compact subset of the quotient space $\bigoplus_{i\in I_m}(B_i)(E_i)'/H$ of a Banach space by a closed subspace H is a canonical image of a compact subset of the direct sum, which can be taken a product of absolutely convex covers of null sequences.Therefore our sequence $(j(x_n))_{n\geqslant 0}$ is contained in such a product which is exactly an equicontinuous set in $G_m = \left(\mathrm{K}_{i\in I_m}(A_i^t)^{-1}(\mathscr{S}((E_i)'_\mu)\right)'$ [K, 22.7.(5)] (recall also that for a Banach space $\mathscr{S}((E_i)'_\mu) = (E_i)'_c$). Therefore it is also equicontinuous in F' (by continuity of $F \longrightarrow G'_m$). This concludes to the Banach space case.

For the ultrabornological case decompose E_i as an inductive limit of Banach spaces. Get in this way a three terms sequence of continuous maps with middle term $\mathrm{K}_{i\in I}(A_i^t)^{-1}(\mathscr{S}((E_i)'_\mu)$ and end point the corresponding iterated kernel coming from duals of Banach spaces by transitivity of Kernels/hulls. Conclude by the previous case of equality of topologies between the first and third term of the sequence, and this concludes to the topological equality with the middle term too. $\square$

Lemma B.2.40. For any lcs E, $((C^\infty_{Fin}(E))^*_\rho)^*_\rho$ is Hilbertianizable, hence it has the approximation property.

Proof. We actually show that $F = ((C^\infty_{Fin}(E))^*_\rho)^*_\rho = \mathscr{S}[\left(\mathscr{C}_M\left[(C^\infty_{Fin}(E))'_\mu\right]\right)'_\mu]$ is Hilbertianizable (also called a (gH)-space) [H, Rmk 1.5.(4)].

Note that $G = C^\infty_{Fin}(E)$ is a complete nuclear space. It suffices to show that for any complete nuclear space G, $\mathscr{S}[\left(\mathscr{C}_M\left[G'_\mu\right]\right)'_\mu]$ is a complete (gH) space. Of course, we use lemma B.1.31 but we need another description of the Mackey completion $\mathscr{C}^\lambda_M(G'_\mu)$. We let $E_0 = G'_\mu, E_{\lambda+1} = \cup_{\{x_n\}\in RMC(E_\lambda)}\overline{\gamma(x_n, n\in\mathcal{N})}E_\lambda = \cup_{\mu<\lambda}E_\mu$ for limit ordinals.

Here $RMC(E_\lambda)$ is the set of sequences $(x_n) \in E_\lambda^{\mathcal{N}}$ which are rapidly Mackey-Cauchy in the sense that if x is their limit in the completion there is a bounded disk $B \subset E_{\lambda+1}$ such that for all k, $(x_n - x) \in n^{-k}B$ for n large enough. For λ_0 large enough, $E_{\lambda_0+1} = E_{\lambda_0}$ and any Mackey-Cauchy sequence x_n in E_{λ_0}, let us take its limit x in the completion and B a closed bounded disk in E_{λ_0} such that $||x_n - x||_B \longrightarrow 0$ one can extract x_{n_k} such that $||x_{n_k} - x||_B \leqslant k^{-k}$ so that $(x_{n_k} - x) \in k^{-l}B$ for k large enough (for any l) thus $(x_{n_k}) \in RMC(E_{\lambda_0})$ thus its limit is in $E_{\lambda_0+1} = E_{\lambda_0}$ which is thus Mackey-complete. To apply lemma B.1.31 with $D = \mathscr{N}((\cdot)'_\mu))$ one needs to see that $\{x_n, n \in \mathcal{N}\}$ is equicontinuous in $D(E_{\lambda_0})'$. But since E_{λ_0} is Mackey-complete, one can assume the bounded disk B is a Banach disk and $||x_n - x||_B = O(n^{-k})$ so that x_n is rapidly convergent. From [Ja, Prop 21.9.1] $\{(x_n - x), n \in \mathcal{N}\}$ is equicontinuous for the strongly nuclear topology associated to the topology of convergence on Banach disks and a fortiori equicontinuous for $D(E_{\lambda_0})'$. By translation, so is $\{x_n, n \in \mathcal{N}\}$ as expected. From application of lemma B.1.31, $H^{\lambda_0} := \mathscr{N}[(\mathscr{C}_M(G'_\mu))'_\mu]$ is complete since $\mathscr{N}[(G'_\mu)'_\mu]$ is already complete (G is complete nuclear so that $\mathscr{N}[(G'_\mu)'_\mu] \longrightarrow G$ continuous and use again [Bo2, IV.5 Rmq 2]).

H^{λ_0} is nuclear thus a (gH)-space. Since H^{λ_0} is a complete (gH) space, it is a reduced projective limit of Hilbert spaces [H, Prop 1.4] and semi-reflexive [H, Rmk 1.5 (5)]. Therefore its Mackey=strong dual [K, 22.7.(9)] is an inductive limit of the Mackey duals, thus Hilbert spaces.

One can apply lemma B.2.39 to get $\mathscr{S}([H^{\lambda_0}]_\mu)$ as a projective kernel of $\mathscr{S}(H)$ with H Hilbert spaces. But from [Bel, Thm 4.2] this is the universal generator of Schwartz (gH) spaces, therefore the projective kernel is still of (gH) space.

For λ_0 as above, this concludes to $\mathscr{S}[(\mathscr{C}_M((C^\infty_{Fin}(E))'_\mu))'_\mu]$ (gH) space, as expected. $\square$

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