Inverse problems for fractional order differential equations

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THÈSE

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présenté par

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Problèmes inverses pour des équations différentielles aux dérivées fractionnaires

sous la direction du
Pr. **Mokhtar KIRANE**

Thèse soutenue le 18 Janvier 2019 devant le jury composé de :

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To my parents

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Résumé

Cette thèse est le résultat de mes recherches durant mon doctorat à l'Université de La Rochelle. La plupart du matériel de la thèse est basée sur les quatre articles suivants publiés durant cette période :

I : B. Ahmad, A. Alsaedi, M. Kirane and **Ramiz Tapdigoglu**, An inverse problem for time fractional evolution equations with an involution perturbation, Quaestiones Mathematicae, 2017, vol. 40, no 2, p. 151-160.

II : B.T. Torebek and **Ramiz Tapdigoglu**, Some inverse problems for the nonlocal heat equation with Caputo fractional derivative, June 2017, Mathematical Methods in the Applied Sciences, 2017, vol. 40, no 18, p. 6468-6479.

III : **Ramiz Tapdigoglu** and B.T. Torebek, Inverse source problems for a wave equation with involution, Bulletin of the Karaganda University-Mathematics, 2018, vol. 91, no 3, p. 75-82.

IV : **Ramiz Tapdigoglu** and B.T. Torebek, On a nonlinear boundary-layer problem for the fractional Blasius type equation, International Journal of Nonlinear Sciences and Numerical Simulation, 2018, vol. 19, no 5, p. 493-498.

L'organisation de cette thèse est la suivante : La thèse commence par le Chapitre 1 qui contient un bref historique sur l'apparition du concept de dérivation fractionnaire et un description de quelques problèmes physiques modélisés par des équations différentielles fractionnaires. Puis, nous décrirons quelques concepts de base et donnons quelques informations nécessaires sur les problèmes directs et inverses. Ensuite, nous recueillons les résultats obtenus à partir de quatre articles ci-dessus et les présentons respectivement dans les chapitres 2, 3, 4 et 5.

Les trois chapitres suivants sont structurés sur les Problèmes Inverses comme suit : Nous commençons le Chapitre 2, par l'étude d'un problème inverse à dérivée fractionnaire. En utilisant la méthode de Fourier, nous prouvons deux théorèmes sur l'existence et l'unicité de solutions d'équations différentielles d'ordre fractionnaire avec involution. Dans le Chapitre 3, nous considérons une classe de problèmes inverses pour restaurer le terme forcing d'une équation fractionnelle de chaleur avec involution et présentons les résultats sur l'existence et l'unicité des solutions pour certaines valeurs de la condition initiale de ces problèmes. Nous discu-

tons également des équations à retards aux dérivées fractionnaires afin d'obtenir certains résultats connexes. Dans le Chapitre 4, nous étudions deux problèmes inverses concernant l'équation d'onde avec un terme perturbatif de type involution par rapport à la variable d'espace. Nous obtenons des résultats d'existence et d'unicité pour ces problèmes, basés sur la méthode de Fourier.

La dernière partie est consacrée à l'étude d'une équation de type Blasius différentielle séquentielle non linéaire avec une dérivée de Caputo. Nous réduisons le problème à une équation intégrale non linéaire et prouvons ensuite la continuité complète de l'opérateur intégral non linéaire. Nous démontrons l'existence d'une solution du problème pour l'équation de Blasius d'ordre fractionnaire. Nous le présentons au Chapitre 5.

Abstract

This thesis is the outcome of my research during my Ph.D. study at La Rochelle University. The principal materials in the thesis are based on the following articles from this period :

Paper I : B. Ahmad, A. Alsaedi, M. Kirane and **Ramiz Tapdigoglu**, An inverse problem for time fractional evolution equations with an involution perturbation, Quaestiones Mathematicae, 2017, vol. 40, no 2, p. 151-160.

Paper II : B.T. Torebek and **Ramiz Tapdigoglu**, Some inverse problems for the nonlocal heat equation with Caputo fractional derivative, June 2017, Mathematical Methods in the Applied Sciences, 2017, vol. 40, no 18, p. 6468-6479.

Paper III : **Ramiz Tapdigoglu** and B.T. Torebek, Inverse source problems for a wave equation with involution, Bulletin of the Karaganda University-Mathematics, 2018, vol. 91, no 3, p. 75-82.

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The organization of this thesis is as follows : The thesis begins in Chapter 1 that contains a brief history about the appearance of the concept of fractional derivation and a description of some physical problems that are modeled by fractional differential equations. In the sequel, we describe some basic concepts and give some information about direct and inverse problems. Afterward, we collect the results obtained from four articles above and present them respectively in Chapters 2, 3, 4 and 5.

The next three chapters are structured on Inverse Problems as follows : We start in Chapter 2, by studying an inverse problem in fractional calculus. Using the Fourier method, we prove two theorems of existence and uniqueness for the solutions of fractional order differential equations with involution. In Chapter 3, we consider a class of inverse problems for restoring the forcing term of a fractional heat equation with involution and present the results on existence and uniqueness of solutions of these problems. We also discuss delay fractional order differential equa-

tions to achieve some related results. In Chapter 4, we study two inverse problems concerning the wave equation with a perturbative term of involution type with respect to the space variable. We obtain existence and uniqueness results for these problems based on the Fourier method.

The last part is devoted to studying a nonlinear sequential differential equation of Blasius type with Caputo fractional derivative. We reduce the problem to the equivalent nonlinear integral equation and prove the complete continuity of the nonlinear integral operator. We prove also the existence of a solution of the problem for the Blasius equation of fractional order. We present it in Chapter 5.

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1 Introduction

This chapter consists of two major parts, namely "Introduction (Française)" and "Introduction (English)".

1.1 Introduction (Française)

1.1.1 Calcul Fractionnaire

Le calcul fractionnaire est une branche des mathématiques qui fait référence à l'extension du concept de dérivation classique à la dérivation d'ordre non entier. L'histoire de la dérivée d'ordre non entier remonte à la fin de 1695 quand L'Hospital a posé une question à Leibniz en s'interrogeant sur la signification de

$$\frac{d^n y}{dx^n} \textit{ lorsque } n = \frac{1}{2}.$$

Leibniz répond que cela mêne à un paradoxe dont on tirera un jour d'utiles conséquences. Cependant, pas beaucoup des progrès ont été faits dans ce domaine pendant trois siècles. Une des raisons est que les outils mathématiques du calcul fractionnaire n'étaient pas disponibles. Une autre raison est le manque d'applications pratiques de ce concept. Plus de 300 ans après, on commence seulement à surmonter les difficultés. De nombreux mathématiciens se sont penchés sur cette question, en particulier Euler (1730), Fourier (1822), Abel (1823), Liouville (1832-1837), Riemann (1847), Grünwald (1867-1872), Letnikov (1868-1872), Gelfand and Shilov (1959- 1964), Caputo (1969), etc.. Le calcul fractionnaire est devenu l'un des domaines les plus développés de l'analyse mathématique. Il a fait un développement rapide et s'est révélé comme un outil puissant dans la modélisation de certains phénomènes dans plusieurs domaines de la sciences tels que la physique, la chimie, la biologie, l'ingénierie et la finance, surtout au cours des trois dernières décennies.

Les dérivées fractionnaires et les intégrales fractionnaires représentent un comportement non local (en raison de l'intégrale impliquée dans la définition) associé aux effets de mémoire. Calculer une dérivée fractionnaire à un certain moment exige tous les processus précédents avec des propriétés de mémoire [83]. C'est l'avantage principale du calcul fractionnaire d'expliquer les processus associés aux systèmes physiques complexes qui ont une mémoire à long terme et / ou des interactions spatiales à longue distance. De plus, les équations différentielles fractionnaires peuvent nous aider à réduire les erreurs découlant de paramètres négligés dans la modélisation des phénomènes physiques [7], [67]. Ce thème de recherche peut mieux traduire la réalité de la nature des phénomènes de la vie réelle, ce qui l'a rendu plus populaire au sein de la communauté des chercheurs et des ingénieurs [66].

Les champs d'application du calcul fractionnaire s'étendent rapidement. En génie mécanique, il existe plusieurs applications du calcul fractionnaire, par exemple, l'étude des systèmes de contrôle [69], modélisation de la diffusion anormale [109]. Le comportement non-standard ou anomalie de l'équation de réaction-diffusion, équation de transport, est connu sous le nom de diffusion anormale non gaussienne, qui a des effets de mémoires longues. Une solution de modèle de diffusion anormale basée sur le calcul fractionnaire à différentes conditions d'opérations permet d'obtenir de meilleurs résultats [71]. En physique, il existe plusieurs applications potentielles de dérivées fractionnaires. Une monographie intéressante sur les applications des intégrales et des dérivées fractionnaires à la physique des polymères, à la biophysique, à la thermodynamique, à la rhéologie et aux systèmes chaotiques a été éditée par R. Hilfer [44]. De plus, en médecine, il a été déduit que les membranes des cellules d'un organisme biologique ont une conductance électrique d'ordre fractionnaire et ensuite, elles sont classées dans des groupes de modèles d'ordre non-entier. Les dérivées fractionnaires incarnent les caractéristiques essentielles du comportement rhéologique cellulaire. Ils ont un grand succès dans le domaine de la rhéologie [7].

Actuellement dans la littérature mathématique, il existe plusieurs définitions des dérivées fractionnaires. Parmi les plus populaires, citons la dérivée de Grünwald-Letnikov, Riemann-Liouville, Caputo et Riesz-Feller [83]. Même si elles sont différentes, elles sont toutes liées les unes aux autres. Grünwald-Letnikov est l'approche la plus évidente pour définir les dérivés fractionnaires. Elle est principalement utilisée pour l'approximation numérique des dérivées fractionnaires. La derivée de Riemann-Liouville a joué un rôle important dans son application en mathématiques pures tandis que Caputo a été mise en place pour répondre aux problèmes appliqués. Caputo a été le premier à appliquer le calcul fractionnaire à la mécanique, en particulier aux modèles linéaires de viscoélasticité [21], [22]. Les dérivées de Caputo permettent l'utilisation des conditions initiales physiquement interprétables, ce qui n'est pas autorisé par la derivée de Riemann-Liouville. Un bref aperçu historique du développement du calcul fractionnaire est donné par Ross [92]. Le manuel de Oldham et Spanier [78] est concerné par les définitions et les propriétés des opérateurs intégro-différentielles d'ordre fractionnaire. En 1987, un livre encyclopédique a été écrit par Samko, Kilbas, et Marichev [96]. Une présentation des nombreuses applications issues du calcul fractionnaire est présentée dans Podlubny [83]. Récemment, plusieurs mathématiciens et chercheurs ont obtenu des résultats et des généralisations importantes de la modélisation des processus réels à l'aide de calcul fractionnaire ([7], [26], [36], [81], [103]).

1.1.2 Revue historique sur les Problèmes Inverses

En science, un problème inverse est une situation dans laquelle à partir d'observations expérimentales, on cherche à déterminer les causes d'un phénomène. La théorie mathématique des problèmes inverses a été essentiellement ignorée jusqu'au milieu du vingtième siècle. Au lieu de cela, les scientifiques se sont concentrés sur des problèmes directs, c'est-à-dire la construction du modèle lui-même plutôt que le processus d'inversion. Puisque le modèle lui-même est inexact, un tel processus d'inversion entraîne généralement des problèmes d'existence et de stabilité. Au début du vingtième siècle, l'idée de problèmes directs dominait la physique mathématique. En effet, le mathématicien français Hadamard estimait qu'un problème physique important devait être bien posé, c'est-à-dire que le problème devait toujours avoir une solution unique qui dépend continûment des données. Cette idée a persisté au milieu du vingtième siècle. Cependant, l'avènement de la mécanique quantique et de nombreux problèmes dans les domaines de la physique classique tels que la conduction thermique et la géophysique ont lentement convaincu les mathématiciens et les scientifiques que les problèmes directs n'étaient pas les seuls problèmes scientifiques et que la théorie mathématique des problèmes inverses commençait à être développé par des mathématiciens de l'Union Soviétique dirigés par Tikhonov. La solution d'un problème inverse consiste à inverser le modèle pour récupérer des informations cachées sur les phénomènes physiques à partir des observations. Une étude complète de nombreux domaines des problèmes inverses et de l'imagerie peut être trouvée dans [99]. Les premières publications sur les problèmes inverses et les problèmes mal posés remontent à la première moitié du XXe siècle. Leurs sujets étaient liés à la physique, la géophysique, l'astronomie et d'autres domaines de la science. Depuis l'avènement des ordinateurs puissants, le domaine d'application de la théorie des problèmes inverses et des problèmes mal posés s'est étendu à presque tous les domaines de la science qui utilisent des méthodes mathématiques. La résolution de problèmes inverses peut également aider à déterminer la localisation, la forme et la structure des intrusions, des défauts, des sources (de chaleur, d'ondes, de différence de potentiel, de pollution), etc. Compte tenu d'une telle variété d'applications, il n'est pas surprenant que la théorie des problèmes inverses et des problèmes mal posés soit devenue l'un des domaines de la science moderne qui se développe le plus rapidement depuis son apparition.

1.1.3 Problèmes Directs

Pour définir diverses classes de problèmes inverses, nous devons d'abord définir le problème direct. En effet, quelque chose "inverse" doit être le contraire de quelque chose de "direct". En général, les problèmes directs sont bien posés [56]. Le concept mathématique de problème bien posé a été proposé par Hadamard en 1932. Il croyait que les modèles mathématiques des phénomènes physiques devraient avoir les propriétés suivantes:

·Une solution existe;

·La solution est unique;

·Elle dépend continûment de la donnée.

La première condition décrit la cohérence du modèle mathématique, la deuxième reflète la précision de la situation réelle et la troisième condition exprime la stabilité de l'équation, c'est-à-dire qu'un petit changement dans l'équation ou dans les conditions latérales entraîne un léger changement de la solution. Un problème qui ne satisfait pas à l'une des conditions précédentes est un problème mal posé.

En physique mathématique, un problème direct est généralement un problème de modélisation de certains champs, processus ou phénomènes physiques (acoustique, électromagnétique, chaleur sismique, etc.). Le but de résoudre un problème direct est de trouver une fonction qui décrit un champ ou un processus physique en n'importe quel point d'un domaine donné à tout instant (si le champ est non stationnaire).

Les problèmes directs pour les équations de diffusion fractionnaire telles que les problèmes à valeur initiale ou limite ont été étudiés en détail dans ([31], [65], [70], [95]) et les références qui y figurent.

1.1.4 Problèmes Inverses

Un problème inverse est généralement mal posé. Un problème mal posé est un problème qui ne répond pas à l'un des trois critères de Hadamard pour être bien posé, c'est-à-dire, des petits changements dans les données de mesure entraînent des changements indéfiniment importants dans la solution. La plupart des difficultés à résoudre des problèmes mal posés sont causées par l'instabilité de la solution. Par conséquent, l'expression "problème mal posé" est souvent utilisée pour des problèmes instables. A cette époque, on pensait que les problèmes naturels devaient avoir des solutions mathématiques continues; on pensait que cela faisait partie de l'ordre inhérent des choses. Depuis lors, nous avons découvert que de nombreux problèmes scientifiques et techniques importants ne sont pas, en fait, bien posés au sens traditionnel, car ils n'ont pas des solutions continues. Les problèmes inverses et mal posés ont commencé à être étudiés et appliqués systématiquement pour fournir des informations à de nombreuses applications dans différents domaines. Cela inclut les problèmes en médecine (par exemple, dans les tissus organiques en imagerie médicale par résonance magnétique), en physique (mécanique quantique, acoustique, etc.), en économie (en théorie du contrôle optimal, etc.) et tous les autres domaines où les méthodes mathématiques sont utilisées (voir, par exemple, [10], [33], [106] et [107]). L'un des premiers problèmes inverses résolus dans le passé était la découverte de Newton des forces qui font que les planètes se déplacent conformément aux lois de Kepler. Des recherches sur la structure interne de la croûte terrestre ont impliqué des champs électromagnétiques dans la théorie des problèmes inverses.

Tichonov [1963] fut le premier à traiter des problèmes mal posés, introduisant ainsi le concept de régularisation. Un problème mal posé devra souvent être régularisé ou reformulé avant de pouvoir procéder à une analyse numérique complète à l'aide d'algorithmes numériques. La régularisation demande souvent de nouvelles hypothèses pour affiner complètement le problème et le réduire. L'idée de la méthode de régularisation est de remplacer le problème mal posé par un problème bien posé, ce qui peut être fait en introduisant un opérateur régularisé qui considère les informations préalables concernant la solution exacte.

Au cours des dernières années, des avancées significatives ont été réalisées dans le domaine des problèmes inverses linéaires par Hansen [1992 a,b], [1995], Hanke et Hansen [1993], Oldenburg et al. [1991], [1993], [1994], Scales [1987], Scales et al. [1987], [1990], [1994], Parker [1994], Parker et Whaler [1981], Nolet et Snieder [1990], et d'autres.

Dans le domaine des problèmes inverses non linéaires, il y a beaucoup plus de progrès à faire.

D'autre part, les équations différentielles fractionnaires deviennent un outil important dans la modélisation de nombreux problèmes de la vie réelle et il y a eu donc un intérêt croissant pour l'étude des problèmes inverses avec des équations différentielles fractionnaires ([6], [39], [54], [113]). Les premiers résultats mathématiques pour le problème inverse de trouver un coefficient de diffusion pour une équation différentielle fractionnaire sont obtenus dans [73]. De nombreux types de problèmes aux limites, y compris les problèmes directs et inverses, ont été formulés pour les différents types d'EDP d'ordre entier et avec plusieurs opérateurs différentiels d'ordre fractionnaire. Il existe de nombreux travaux sur l'étude des problèmes directs et inverses pour les équations de diffusion fractionnaire en temps ou d'onde avec la dérivée de Caputo (voir [35], [65], [95]).

Dans cette thèse, nous nous intéressons à l'existence et l'unicité des solutions de problèmes inverses pour les équations différentielles fractionnaires en temps.

1.2 Introduction (English)

1.2.1 Fractional Calculus

The Non-Integer Order Calculus, traditionally known as Fractional Calculus is the branch of mathematics that tries to interpolate the classical derivatives and integrals and generalizes them for any orders, not necessarily integer order. But with this definition, many interesting questions will arise; for example, if the first derivative of a function gives you the slope of the function, what is the geometrical meaning of half derivative? In half order, which operator must be used twice to obtain the first derivative? The early history of this questions goes back to the birth of fractional calculus first appeared in the correspondence of Leibniz with L'Hospital (1695), Johann Bernoulli (1695), and John Wallis (1697) as a mere question or maybe even play of thoughts. Many mathematicians focused on this topic. However, nothing much has been done in the field. One of the reasons is that the mathematical tools of fractional calculus were not available. Another reason is the lack of practical applications of this field. Nevertheless, beginning with the nineteenth century, interesting developments have been made in the theory of Fractional Calculus: Laplace (1812), Lacroix (1812), Fourier (1822), Abel (1823-1826), Liouville (1832-1837), Riemann (1847), Grünwald (1867-1872), Letnikov (1868-1872), Sonin (1869), Laurent (1884), Heaviside (1892-1912), Weyl (1917), Davis (1936), Erdèlyi (1939-1965), Gelfand and Shilov (1959- 1964), Caputo (1969), and many others. Yet, it is only after the first specialized conference organized by B. Ross on "Fractional Calculus and its applications", the fractional calculus has become one of the most intensively developing areas of mathematical analysis. Many applications of various kind of fractional differential equations became a target of specialists due to both theoretical and practical reasons [9]. It has gone through a rapid development and has been revealed as a powerful tool in the modeling of certain phenomena in several sciences as Physics, Chemistry, Biology, Engineering, and Finance especially during the past three decades.

Considering a differential equation that describes a specific phenomenon, a common way to use fractional modeling is to replace the integer order derivatives by non-integer derivatives, usually with order lower than or equal to the order of the original derivatives, so that the usual solution may be recovered as a particular case [83].

A simple example of fractional derivatives of the function $f(t) = t^2$, is plotted for different values of the fractional order in Figure 1. The different values of the fractional order are obtained using the expression

$$\frac{d^m}{dt^m} t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-m)} t^{p-m},$$

where p is a real number.

Actually, when modeling real physical phenomena, fractional derivatives can provide more accurate results than integer order derivatives

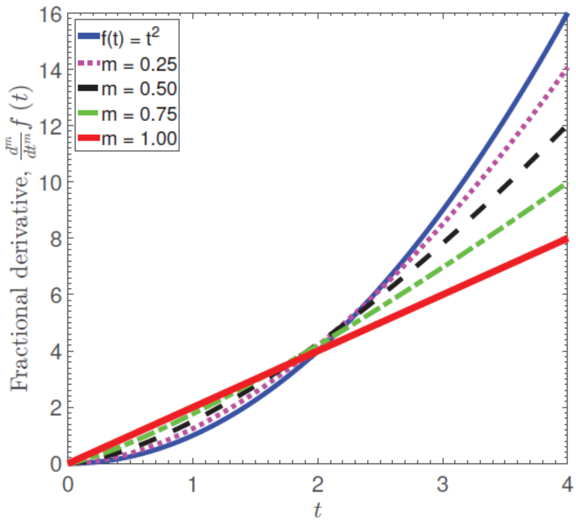


Figure 1.1 – *Fractional derivatives $\frac{d^m}{dt^m} f(t)$, of a quadratic function, $f(t) = t^2$ (blue, solid line) with the order m which has values 0.25 (magenta, dotted line), 0.50 (black, dashed line), 0.75 (green, dash-dot line), and 1 (red, thicker solid line).*

[66]. The advantages of fractional derivatives are that they have a greater degree of flexibility in the model and provide an excellent instrument for the description of the reality. This is because of the fact that the realistic modeling of a physical phenomenon does not depend only on the instant time, but also on the history of the previous time, i.e., calculating time-fractional derivative at some time requires all the previous processes with memory and hereditary properties [83]. It exists also in many biological systems ([29], [32]).

Fractional derivatives and integer order derivatives are both linear operators. However fractional derivatives are usually nonlocal operators while integer order derivatives are local operators. As shown in Figure 2, the integer order derivative of a function at a point depends only on the local behavior of the function. However the value of the fractional derivative at a point depends on the entire behavior of the function [101].

The application areas of fractional calculus is expanding rapidly. The increasing interest in fractional differential equations are motivated not only by their application to problems from viscoelasticity, heat conduction in materials with memory, electrodynamics with memory, and also because of they can be employed to approach nonlinear conservation laws. Besides, fractional differential equations can help us to reduce the errors arising from the neglected parameters in modeling real-life phenomena [7], [67]. In mechanical engineering, there are several applications of fractional calculus, for example, the study of control and dynamical systems [69], modeling anomalous diffusion [109]. The nonstandard behavior or anomaly of the reaction diffusion equation, transport equation, is known as anomalous non Gaussian diffusion, which has long memory effects. Among several explanations for this anomalous diffu-

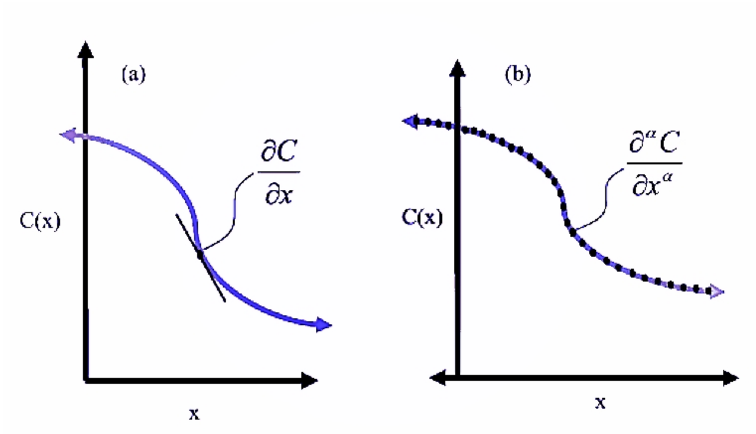


Figure 1.2 – *(a) The usual derivative is local, (b) The fractional derivative is nonlocal and depends on the behaviour of the entire functions.*

sion, one is by using fractional derivative in time or in space or both in the reaction diffusion equation, transport equation. Anomalous diffusion model solution based on fractional calculus at different operations conditions allow getting better results [71]. In physics, there are several potential applications of fractional derivative. A valuable monograph about the applications of fractional integrals and derivatives to polymer physics, biophysics, thermodynamics, rheology, chaotic systems has been edited by R. Hilfer [44]. Moreover, in medicine, it has been deduced that the membranes of cells of a biological organism have fractional order electrical conductance and then, they are classified in groups of noninteger order models. Fractional derivatives embody essential features of cell rheological behavior and have enjoyed the greatest success in the field of rheology [7].

In the literature, several definitions of the fractional derivatives have been proposed. For instance, the Grünwald-Letnikov, the Riemann-Liouville, the Caputo and the Riesz-Feller [83]. Even though they are different, they are all related to each other. The Grünwald-Letnikov is the most obvious approach to define fractional derivatives. It is mainly used for numerical approximation of fractional derivatives. However, dealing with fractional derivatives as a limit of fractional-order difference is not convenient due to the mathematical complexity. Therefore, some studies use the Grünwald Letnikov numerically, but try to solve the initial problem with other definitions. The Riemann-Liouville played an important role for its application in pure mathematics while Caputo has been introduced to respond to applied problems. Indeed, M. Caputo was the first to give applications of fractional calculus to mechanics, especially to linear models of viscoelasticity [21], [22]. Caputo derivatives allow the use of physically interpretable initial conditions, which is not permitted by the Riemann-Liouville.

Fractional calculus modeling (FCM), using Caputo derivative [93], has been recently used to generalize the logistic equation. The solution of the corresponding fractional differential equation provides a suitable description for the growth of certain types of cancer tumor [108].

A brief historical overview of the development of fractional calculus is given by Ross [92]. The textbook of Oldham and Spanier[78] is concerned with the definitions and the properties of fractional order differential/integral operators. In 1987, the huge book was written by Samko, Kilbas, and Marichev [96], referred to now as "encyclopedia" of fractional calculus. A survey of the many different applications which have emerged from fractional calculus is given in Podlubny[83]. Recently several mathematicians and applied researchers have obtained important results and generalizations from modeling real processes using FC ([7], [26], [36], [81], [103]).

1.2.2 Historical review on Inverse Problems

Inverse problems are as old as science itself. A scientific problem is the problem of constructing a model of some physical or biological phenomena that, although inexact, is accurate enough to be able to use observations or measurements to obtain information about the phenomena under investigation. The challenge is to "invert" the model to recover useful estimates of the object under investigation. Strangely enough, given the above description of the scientific method, the mathematical theory of inverse problems was essentially ignored until the middle of the twentieth century. Instead, scientists focused on direct problems, i.e. the construction of the model itself rather than the inversion process. Since the model itself is inexact, such an inversion process typically leads to problems of existence and stability.

By the beginning of the twentieth century, the idea of direct problems dominated mathematical physics. Indeed, the French mathematician Hadamard held the opinion that an important physical problem must be well-posed, i.e. the problem must always have a unique solution that depends continuously on the data. This idea persisted well into the middle of the twentieth century. However, the advent of quantum mechanics and numerous problems in areas of classical physics such as heat conduction and geophysics soon slowly convinced mathematicians and scientists that well-posed direct problems were not the only ones of scientific interest and the mathematical theory of inverse problems began to be developed by mathematicians of the Soviet Union led by Tikhonov. In particular, this theory focused on the problem of determining the parameters and data in the mathematical model of the direct problem from measurements and observations of the data that arise from the physical or biological phenomena taking place.

The solution of an inverse problem is to "invert" the model to recover hidden information about the physical phenomena from the observations.

A comprehensive survey of many areas of inverse problems and imaging can be found in 1600 pages handbook [99].

First publications on inverse and ill-posed problems date back to the first half of the 20th century. Their subjects were related to physics (inverse problems of quantum scattering theory), geophysics (inverse problems of electrical prospecting, seismology, and potential theory), astronomy, and other areas of science. Since the advent of powerful computers, the area of application for the theory of inverse and ill-posed problems has spread to almost all fields of science that uses mathematical methods.

Solving inverse problems can also help to determine the location, shape, and structure of intrusions, defects, sources (of heat, waves, potential difference, pollution), and so on. Given such a wide variety of applications, it is no surprise that the theory of inverse and ill-posed problems have become one of the most rapidly developing areas of modern science since its emergence.

1.2.3 Direct Problems

To define various classes of inverse problems, we should first define a direct (forward) problem. Indeed, something "inverse" must be the opposite of something "direct". Direct problems are based on developing a mathematical model that maps causes into effects and are typically well-posed: each cause has a unique effect and causes which are close to one another have effects which are close to each other. In general, direct problems are well-posed [56]. The well-posedness criteria was proposed by Jacques-Salomon Hadamard, a French mathematician, in 1902. He believed that mathematical models of physical phenomena should have the properties that:

·A solution exists;

·The solution is unique;

·The solution depends continuously on the data (initial conditions and source term).

The first condition describes the consistency of the mathematical model, the second reflects the definiteness of the real situation. The third condition expresses the stability of the equation, a small change in the equation or in the side conditions give rise to a small change in the solution.

In mathematical physics, a direct problem is usually a problem of modeling some physical fields, processes, or phenomena (acoustic, electromagnetic, seismic heat, etc.). The purpose of solving a direct problem is to find a function that describes a physical field or a process at any point of a given domain at any instant of time (if the field is nonstationary). The formulation of a direct problem includes:

·The domain in which the process is studied;

·The equation that describes the process;

·The initial conditions (if the process is nonstationary);

·The conditions on the boundary of the domain (existence and uniqueness involve boundary conditions).

The direct problems for fractional diffusion equations such as an initial or boundary value problems have been studied extensively in ([31], [65], [70], [95]) and references therein.

1.2.4 Inverse Problems

An inverse problem is usually ill-posed. The concept of an ill-posed problem is not new. While there is no universal formal definition for inverse problems, Hadamard [1923] defined a problem as being ill-posed if it violates the criteria of a well-posed problem, that is, either existence, uniqueness or continuous dependence on data is no longer true, i.e., arbitrarily small changes in the measurement data lead to indefinitely large changes in the solution. Hadamard did not deal with the numerics of ill-posed problems as he believed that the ill-posedness arose from an incorrect physical representation of the problem.

Most difficulties in solving ill-posed problems are caused by solution instability. Therefore, the term "ill-posed problem" is often used for unstable problems. In the majority of cases, inverse problems turn out to be ill-posed and, conversely, an ill-posed problem can usually be reduced to a problem that is inverse to some direct (well-posed) problem. Inverse and ill-posed problems began to be studied and applied systematically to provide information for many applications in various fields just like physics, geophysics, medicine, astronomy, and all other areas of knowledge where mathematical methods are used. They appear in modeling a wide variety of problems, i.e., Magnetic Resonance Imaging, Computerized Tomography, Signal Processing, and many other applications (see, for example,[10], [33], [106] and [107]). Solutions of inverse problems recover hidden information for a given system and describe important properties, such as density and velocity of wave propagation, elasticity parameters, conductivity, dielectric permittivity, and magnetic permeability, properties and location of inhomogeneities in inaccessible areas, etc [101].

One of the first inverse problems solved in the past was Newton's discovery of forces making planets move in accordance with the Kepler's laws. Researches regarding the internal structure of the Earth's crust involved electromagnetic fields in the theory of the inverse problems.

Tichonov [1963] was the first to deal numerically with ill-posedness, and in so doing introduced the concept of regularization. An ill posed problem will often need to be regularized or re-formulated before one can give it a full numerical analysis using computer algorithms or other computational methods. Regularization often involves bringing in new assumptions to fully define the problem and narrow it down. The idea of regularization method is to replace the ill-posed problem by well-posed problem, which can be done by introducing a regularized operator which considers available prior information about the exact solution.

While the concept of regularization is well understood today, the main problem is its implementation in large problems. In recent years significant advances have been made in the field of linear inverse problems by the work of Hansen [1992 a,b], [1995], Hanke and Hansen [1993], Oldenburg et al. [1991], [1993], [1994], Scales [1987], Scales et al. [1987], [1990], [1994], Parker [1994], Parker and Whaler [1981], Nolet and Snieder [1990], and others. The backward heat equation which is the model of a linear inverse problem is one of the first ill-posed problems that is systematically studied. Solving a heat equation backward in time presents the class of inverse heat conduction problems ([45], [46], [110]). However, there are still a number of unanswered questions, and more importantly, there is insufficient understanding as which method should be used for a specific problem. In the field of nonlinear inverse problems, there are far more advances to be made. Inverse scattering problem for acoustic waves is one of the best-known example of a nonlinear inverse problem and, its electromagnetic version is the mathematical basis of synthetic aperture radar [24].

Inverse problems come into various types, for example, inverse initial problems where initial data are unknown and inverse source problems where the source term is unknown. These unknown terms are to be determined using extra boundary data. Fractional differential equations, on the other hand, become an important tool in modeling many real-life problems and hence there has been growing interest in studying inverse problems of time fractional differential equations ([1], [6], [39], [54], [113]). The first mathematical results for the inverse problem of finding diffusion coefficient for a fractional differential equation is obtained in [73]. Many kinds of boundary problems, including direct and inverse problems, were formulated for the different type of PDEs of integer order and with several fractional order differential operators. For example, in [113], Zhang and Xu studied inverse source problem for a fractional diffusion equation where solutions are found based on the method of eigenfunction expansion. Yikan Liu [64] established the strong maximum principle for fractional diffusion equations with multiple Caputo derivatives and investigated the related inverse problem. We also note the work of Daftardar-Gejji and Bhalikar [25] where multi-term fractional diffusion-wave equation was considered and boundary-value problems for this equation were solved by the method of separation of variables. There are many works on studying direct and inverse problems for time-fractional diffusion or diffusion-wave equations with the Caputo derivative. Depending on the operator used in the space-variable, the existence of a classical or generalized solution is partly known, for instance, see works ([35], [65], [95]) and references therein. In [68], authors considered the initial inverse problem in heat equation with Bessel operator. They expressed the solution of the problem and the initial temperature distribution in terms of an orthogonal set of Bessel functions. These types of functions arise in the modeling of chemical engineering process including hydrodynamics, bio-processes, diffusion and heat transfer (see, for example, [68],[82]). In this thesis, we are interested in the existence and uniqueness of solutions of inverse problems for time fractional differential equations.

1.3 Presentation of the obtained results

1.3.1 Chapter 2: An inverse problem for a time nonlocal evolution equation with an involution perturbation

Statement of the problem

An inverse problem of a time fractional evolution equation interpolating the heat and wave equations with involution is considered. The goal is to determine the spectral problem associated with our problem and then determine conditions, so that the inverse problem has a unique solution. The results on the existence and uniqueness of a solution are presented by the method of separation of variables. The equation

$$D_*^{\alpha} u\left(x,t\right) - u_{xx}\left(x,t\right) + \varepsilon u_{xx}\left(-x,t\right) = f\left(x\right), \tag{1.1}$$

posed for $x \in (-\pi, \pi)$ and $t > 0$, where f and u are unknowns, $1 < \alpha < 2$, ε is a nonzero real number such that $|\varepsilon| < 1$. We equip (1.1) with the initial, final, and boundary conditions

$$u\left(x,0\right) = \phi\left(x\right), \;\; u_t\left(x,0\right) = \rho\left(x\right), \;\; u\left(x,T\right) = \psi\left(x\right), \, x \in \left[-\pi,\pi\right], \tag{1.2}$$

$$u\left(-\pi,t\right) = 0, \;\; u\left(\pi,t\right) = 0, \, t \in \left[0,T\right], \tag{1.3}$$

where $\phi\left(x\right)$ and $\psi\left(x\right)$ are given sufficiently smooth functions. The derivative D_*^{α} defined as

$$D_*^{\alpha} u(x,t) = D^{\alpha}\left(u(x,t) - u(x,0) - t u_t(x,0)\right)$$

is the Caputo derivative for a function built on the Riemann-Liouville derivative D^{α}. Caputo's derivative allows us to impose initial conditions in a natural way.
By a regular solution of problem, we mean a pair of functions $\left(u\left(x,t\right), f\left(x\right)\right)$ of the class $u\left(x,t\right) \in C_{x,t}^{2,2}\left(\Omega\right)$,(space of two times continuously differentiable functions on Ω according to both x and t), $f\left(x\right) \in C\left[-\pi,\pi\right]$,$\Omega = \left\{-\pi \le x \le \pi, \; 0 \le t \le T\right\}$.

Main result

The main result of this work is the following theorem.

Theorem 1.3.1 *Let $\phi\left(x\right), \rho\left(x\right), \psi\left(x\right) \in C^4\left[-\pi,\pi\right]$ and $\phi^{(i)}\left(\pm\pi\right) = \rho^{(i)}\left(\pm\pi\right) = \psi^{(i)}\left(\pm\pi\right) = 0, i = 0,1,2,3$. If $1 - u_0\left(.\right) \neq 0$ then, for a nonzero real number ε such that $|\varepsilon| < 1$, problem (1.1)-(1.2)-(1.3) has a unique solution* *which can be written in the form*

$$\begin{aligned} u(x,t) &= \phi(x) \\ &+ \sum_{k=0}^{\infty} \frac{\left(C^{(4)}\right)_{1k} u_0(\lambda_{k,2}^{1/\alpha} t)}{\left(k+\frac{1}{2}\right)^4} C_{k+\frac{1}{2}}(x) \\ &+ \sum_{k=0}^{\infty} \frac{\left(C^{(4)}\right)_{3k} u_1(\lambda_{k,2}^{1/\alpha} t) - \left(C^{(4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^4} C_{k+\frac{1}{2}}(x) \\ &+ \sum_{k=1}^{\infty} \frac{\left(C^{(4)}\right)_{2k} u_0(\lambda_{k,2}^{1/\alpha} t) + \left(C^{(4)}\right)_{4k} u_1(\lambda_{k,1}^{1/\alpha} t) - \left(C^{(4)}\right)_{2k}}{k^4} S_k(x), \end{aligned}$$

and

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\left(\left(\phi^{(4)}\right)_{1k} - \left(C^{(4)}\right)_{1k}\right)}{\left(k+\frac{1}{2}\right)^2} C_{k+\frac{1}{2}}(x) \\ &+ \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\left(\left(\phi^{(4)}\right)_{2k} - \left(C^{(4)}\right)_{2k}\right)}{k^2} S_k(x), \end{aligned}$$

where,

$$\begin{aligned} \left(C^{(4)}\right)_{1k} &= \frac{\left(\phi^{(4)}\right)_{1k} - \left(\psi^{(4)}\right)_{1k} + \left(C^{(4)}\right)_{3k} u_1(\lambda_{k,2}^{1/\alpha} T)}{1 - u_0\left(\lambda_{k,2}^{1/\alpha} T\right)}, \\ \left(C^{(4)}\right)_{3k} &= \frac{\left(\rho^{(4)}\right)_{1k}}{\lambda_{k,2}^{1/\alpha}}, \\ \left(C^{(4)}\right)_{2k} &= \frac{\left(\phi^{(4)}\right)_{2k} - \left(\psi^{(4)}\right)_{2k} + \left(C^{(4)}\right)_{4k} u_1(\lambda_{k,1}^{1/\alpha} T)}{1 - u_0\left(\lambda_{k,1}^{1/\alpha} T\right)}, \\ \left(C^{(4)}\right)_{4k} &= \frac{\left(\rho^{(4)}\right)_{2k}}{\lambda_{k,1}^{1/\alpha}}, \end{aligned}$$

and

$$\left(g^{(4)}\right)_{1k} = \int_{-\pi}^{\pi} g^{(4)}(x)\, C_{k+\frac{1}{2}}(x)\, dx, \quad \left(g^{(4)}\right)_{2k} = \int_{-\pi}^{\pi} g^{(4)}(x)\, S_k(x)\, dx, \quad \textit{for} \quad g = \phi, \psi, \rho.$$

1.3.2 Chapter 3: Some inverse problems for the nonlocal heat equation with Caputo fractional derivative

The purpose of this chapter is to study inverse problems for the nonlocal heat equation with involution of space variable x. Our goal is to determine the spectral problem associated with our problem and then determine conditions for which the inverse problem has a unique solution. The results on the existence and uniqueness of a solution are presented by the method of separation of variables. We consider the heat equation

$$\mathcal{D}_t^{\alpha} u(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(\pi - x, t) = f(x), \tag{1.4}$$

for $(x,t) \in \Omega = \{0 < x < \pi,\ 0 < t < T < \infty\}$, $0 < \alpha < 1$, where $\mathcal{D}_t^{\alpha}$ is the Caputo derivative (which is defined in the next section) and ε is a real number.

Statement of problems

The chapter is concerned with four inverse problems concerning the problem (1.4). We obtain existence and uniqueness results for these problems, based on the Fourier method.

Problem D. Find the couple of functions $(u(x,t), f(x))$ satisfying equation (1.4), under the conditions

$$u(x,0) = \varphi(x),\ x \in [0,\pi], \tag{1.5}$$

$$u(x,T) = \psi(x),\ x \in [0,\pi], \tag{1.6}$$

and the homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0,\ t \in [0,T], \tag{1.7}$$

where $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions.

Problem N. Find the couple of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (1.4), conditions (1.5), (1.6) and the homogeneous Neumann boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0,\ t \in [0,T]. \tag{1.8}$$

Problem P. Find the couple of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (1.4), conditions (1.5), (1.6) and the periodic boundary conditions

$$u(0,t) = u(\pi,t),\ u_x(0,t) = u_x(\pi,t),\ t \in [0,T]. \tag{1.9}$$

Problem AP. Find the couple of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (1.4), conditions (1.5), (1.6) and the anti-periodic boundary conditions

$$u(0,t) = -u(\pi,t),\ u_x(0,t) = -u_x(\pi,t),\ t \in [0,T]. \tag{1.10}$$

A regular solution of problems D, N, P and AP is the pair of functions $(u(x,t), f(x))$ where $u \in C^{2,1}_{x,t}(\bar{\Omega})$(space of two times and one time continuously differentiable functions on $\bar{\Omega}$ according to x and t respectively) and $f \in C([0,\pi])$.

Main results

For the considered problems D, N, P, AP, the following theorems hold true.

Theorem 1.3.2 *Let $|\varepsilon| < 1$, $\varphi, \psi \in C^3[0,\pi]$ and $\varphi^{(i)}(0) = \varphi^{(i)}(\pi) = \psi^{(i)}(0) = \psi^{(i)}(\pi) = 0, i = 0,1,2$. Then the solution of problem D exists, is unique and it can be written in the form*

$$
\begin{aligned}
u(x,t) = \ & \varphi(x) \\
& + \sum_{k=0}^{\infty} \frac{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 t^\alpha\right)\right)\sin(2k+1)x}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 T^\alpha\right)\right)(2k+1)^2}\left(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}\right) \\
& + \sum_{k=1}^{\infty} \frac{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2 t^\alpha\right)\right)\sin 2kx}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2 T^\alpha\right)\right)4k^2}\left(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}\right),
\end{aligned}
$$

$$
\begin{aligned}
f(x) = \ & -\varphi''(x) + \varepsilon\, \varphi''(\pi - x) \\
& + \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\left(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}\right)}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 T^\alpha\right)\right)} \sin(2k+1)x \\
& + \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\left(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}\right)}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2 T^\alpha\right)\right)} \sin 2kx,
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_{1k}^{(2)} &= \left(\varphi''(x), y_{2k+1}^D\right), \varphi_{2k}^{(2)} = \left(\varphi''(x), y_{2k}^D\right), \\
\psi_{1k}^{(2)} &= \left(\psi''(x), y_{2k+1}^D\right), \psi_{2k}^{(2)} = \left(\psi''(x), y_{2k}^D\right),
\end{aligned}
$$

and $E_{\alpha,\beta}(\lambda t)$ is the Mittag-Leffler type function:

$$
E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}.
$$

Theorem 1.3.3 *Let $\varphi, \psi \in C^3[0,\pi]$ and $\varphi^{(i)}(0) = \varphi^{(i)}(\pi) = \psi^{(i)}(0) = \psi^{(i)}(\pi) = 0$, $i = 0,1,2$. Then the solution of problem N exists, is unique and it can be written in the form*

$$
\begin{aligned}
u(x,t) = \ & \varphi(x) + \tfrac{t}{T}\big(\psi_0 - \varphi_0\big) \\
& + \sum_{k=1}^{\infty} \frac{\big(1-E_{\alpha,1}\big(-(1-\varepsilon)4k^2 t^\alpha\big)\big)\cos 2kx}{\big(1-E_{\alpha,1}\big(-(1-\varepsilon)4k^2 T^\alpha\big)\big)4k^2}\big(\psi_{1k}^{(2)} - \varphi_{1k}^{(2)}\big) \\
& + \sum_{k=0}^{\infty} \frac{\big(1-E_{\alpha,1}\big(-(1+\varepsilon)(2k+1)^2 t^\alpha\big)\big)\cos(2k+1)x}{\big(1-E_{\alpha,1}\big(-(1+\varepsilon)(2k+1)^2 T^\alpha\big)\big)(2k+1)^2}\big(\psi_{2k}^{(2)} - \varphi_{2k}^{(2)}\big),
\end{aligned}
$$

$$
\begin{aligned}
f(x) = \ & -\varphi''(x) + \varepsilon\, \varphi''(\pi - x) \\
& + \sum_{k=1}^{\infty} \frac{(1-\varepsilon)\big(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}\big)}{\big(1-E_{\alpha,1}\big(-(1-\varepsilon)4k^2 T^\alpha\big)\big)}\cos 2kx \\
& + \sum_{k=0}^{\infty} \frac{(1+\varepsilon)\big(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}\big)}{\big(1-E_{\alpha,1}\big(-(1+\varepsilon)(2k+1)^2 T^\alpha\big)\big)}\cos(2k+1)x,
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_0 &= \big(\varphi(x), y_0^N\big), \ \varphi_{1k}^{(2)} = \big(\varphi''(x), y_{2k}^N\big), \ \varphi_{2k}^{(2)} = \big(\varphi''(x), y_{2k+1}^N\big), \\
\psi_0 &= \big(\psi(x), y_0^N\big), \ \psi_{1k}^{(2)} = \big(\psi''(x), y_{2k}^N\big), \ \psi_{2k}^{(2)} = \big(\psi''(x), y_{2k+1}^N\big).
\end{aligned}
$$

Theorem 1.3.4 *Let $\varphi, \psi \in C^3[0,\pi]$ and $\varphi^{(i)}(0) = \varphi^{(i)}(\pi)$, $\psi^{(i)}(0) = \psi^{(i)}(\pi)$, $i = 0,1,2$. Then the solution of problem P exists, is unique and it can be written in the form*

$$
\begin{aligned}
u(x,t) = \ & \varphi(x) + \tfrac{t}{T}\big(\psi_0 - \varphi_0\big) \\
& + \sum_{k=1}^{\infty} \frac{\big(1-E_{\alpha,1}\big(-(1-\varepsilon)4k^2 t^\alpha\big)\big)\cos 2kx}{\big(1-E_{\alpha,1}\big(-(1-\varepsilon)4k^2 T^\alpha\big)\big)4k^2}\big(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}\big) \\
& + \sum_{k=1}^{\infty} \frac{\big(1-E_{\alpha,1}\big(-(1+\varepsilon)4k^2 t^\alpha\big)\big)\sin 2kx}{\big(1-E_{\alpha,1}\big(-(1+\varepsilon)4k^2 T^\alpha\big)\big)4k^2}\big(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}\big),
\end{aligned}
$$

$$\begin{aligned} f(x) = & -\varphi''(x) + \varepsilon\,\varphi''(\pi - x) \\ & + \sum_{k=1}^{\infty} \frac{(1-\varepsilon)\left(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}\right)}{\left(1 - E_{\alpha,1}\left(-(1-\varepsilon)2k^2 T^\alpha\right)\right)} \cos 2kx \\ & + \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\left(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}\right)}{\left(1 - E_{\alpha,1}\left(-(1+\varepsilon)2k^2 T^\alpha\right)\right)} \sin 2kx, \end{aligned}$$

where

$$\begin{aligned} &\varphi_0 = \left(\varphi(x), y_0^P\right), \varphi_{1k}^{(2)} = \left(\varphi''(x), y_{2k}^P\right), \varphi_{2k}^{(2)} = \left(\varphi''(x), y_{2k+1}^P\right), \\ &\psi_0 = \left(\psi(x), y_0^P\right), \psi_{1k}^{(2)} = \left(\psi''(x), y_{2k}^P\right), \psi_{2k}^{(2)} = \left(\psi''(x), y_{2k+1}^P\right). \end{aligned}$$

Theorem 1.3.5 *Let $\varphi, \psi \in C^3[0,\pi]$ and $\varphi^{(i)}(0) = -\varphi^{(i)}(\pi)$, $\psi^{(i)}(0) = -\psi^{(i)}(\pi)$, $i = 0, 1, 2$. Then the solution of problem AP exists, is unique and it can be written in the form*

$$\begin{aligned} u(x,t) = & \ \varphi(x) \\ & + \sum_{k=0}^{\infty} \frac{\left(1 - E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 t^\alpha\right)\right) \cos(2k+1)x}{\left(1 - E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 T^\alpha\right)\right)(2k+1)^2} \left(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}\right) \\ & + \sum_{k=0}^{\infty} \frac{\left(1 - E_{\alpha,1}\left(-(1+\varepsilon)(2k+1)^2 t^\alpha\right)\right) \sin(2k+1)x}{\left(1 - E_{\alpha,1}\left(-(1+\varepsilon)(2k+1)^2 T^\alpha\right)\right)(2k+1)^2} \left(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}\right), \end{aligned}$$

$$\begin{aligned} f(x) = & -\varphi''(x) + \varepsilon\,\varphi''(\pi - x) \\ & + \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\left(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}\right)}{\left(1 - E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 T^\alpha\right)\right)} \cos(2k+1)x \\ & + \sum_{k=0}^{\infty} \frac{(1+\varepsilon)\left(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}\right)}{\left(1 - E_{\alpha,1}\left(-(1+\varepsilon)(2k+1)^2 T^\alpha\right)\right)} \sin(2k+1)x, \end{aligned}$$

where

$$\begin{aligned} &\varphi_{1k}^{(2)} = \left(\varphi''(x), y_{2k+1}^{AP}\right), \varphi_{2k}^{(2)} = \left(\varphi''(x), y_{2k}^{AP}\right), \\ &\psi_{1k}^{(2)} = \left(\psi''(x), y_{2k+1}^{AP}\right), \psi_{2k}^{(2)} = \left(\psi''(x), y_{2k}^{AP}\right). \end{aligned}$$

1.3.3 Chapter 4: Inverse source problems for a wave equation with involution

The purpose of this chapter is to study inverse problems for a nonlocal wave equation with involution of space variable x. We consider the nonlocal wave equation

$$u_{tt}(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(\pi - x, t) = f(x), \tag{1.11}$$

for $(x,t) \in \Omega = \{0 < x < \pi,\ 0 < t < T\}$, where ε is a real number.

Statement of problems

The chapter is devoted to two inverse problems concerning the wave equation with a perturbative term of involution type with respect to the space variable. We obtain existence and uniqueness results for these problems, based on the Fourier method.

Problem D. Find a couple of functions $(u(x,t), f(x))$ satisfying equation (1.11), under the conditions

$$u(x,0) = 0,\ x \in [0,\pi], \tag{1.12}$$

$$u(x,T) = \psi(x),\ x \in [0,\pi], \tag{1.13}$$

$$u_t(x,0) = 0,\ x \in [0,\pi], \tag{1.14}$$

and the homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0,\ t \in [0,T], \tag{1.15}$$

where $\psi(x)$ is a given sufficiently smooth function.

Problem N. Find the couple of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (1.11), conditions (1.12), (1.13), (1.14) and the homogeneous Neumann boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0,\ t \in [0,T]. \tag{1.16}$$

A regular solution of the problems D and N is the pair of functions $(u(x,t), f(x))$, where $u \in C^2(\bar{\Omega})$ and $f \in C([0,\pi])$.

Spectral properties of the perturbed Sturm-Liouville problem

Application of the Fourier method for solving problems D and N leads to a spectral problem defined by the equation

$$y''(x) - \varepsilon y''(\pi - x) + \lambda y(x) = 0, 0 < x < \pi, \tag{1.17}$$

and one of the following boundary conditions

$$y(0) = y(\pi) = 0, \tag{1.18}$$

$$y'(0) = y'(\pi) = 0. \tag{1.19}$$

Main results

For the considered problems D and N, the following theorems are valid.

Theorem 1 *Let* $|\varepsilon| < 1$, $\psi \in C^4[0,\pi]$ *and* $\psi^{(i)}(0) = \psi^{(i)}(\pi) = 0, i = 0,1,2,3,4$. *If* $\cos\sqrt{1-\varepsilon}\,(2k+1)\,T < \delta_1 < 1$ *and* $\cos\sqrt{1+\varepsilon}2kT < \delta_2 < 1$, *then the solution of problem D exists, is unique and it can be written in the form*

$$u(x,t) = \sum_{k=0}^{\infty} \frac{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,t\right)\sin(2k+1)\,x}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^4}\psi_{2k+1}^4 + \sum_{k=1}^{\infty} \frac{\left(1-\cos\sqrt{1+\varepsilon}2kt\right)\sin 2kx}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)16k^4}\psi_{2k}^4, \quad (1.20)$$

$$f(x) = \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\,\psi_{2k+1}^4}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^2}\sin(2k+1)\,x + \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\,\psi_{2k}^4}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)4k^2}\sin 2kx, \quad (1.21)$$

where $\psi_{2k+1}^{(4)} = \left(\psi^{(4)}(x), y_{2k+1}^D\right)$ *and* $\psi_{2k}^{(4)} = \left(\psi^{(4)}(x), y_{2k}^D\right)$.

Theorem 2 *Let* $|\varepsilon| < 1$, $\psi \in C^4[0,\pi]$ *and* $\psi^{(i)}(0) = \psi^{(i)}(\pi) = 0, i = 0,1,2,3,4$. *If* $\cos\sqrt{1-\varepsilon}\,(2k+1)\,T < \sigma_1 < 1$ *and* $\cos\sqrt{1+\varepsilon}2kT < \sigma_2 < 1$, *then the solution of problem N exists, is unique and it can be written in the form*

$$u(x,t) = \sum_{k=0}^{\infty} \frac{\left(1-\cos\sqrt{1+\varepsilon}\,(2k+1)\,t\right)\cos(2k+1)\,x}{\left(1-\cos\sqrt{1+\varepsilon}\,(2k+1)\,T\right)(2k+1)^4}\psi_{2k+1}^4 + \sum_{k=1}^{\infty} \frac{\left(1-\cos\sqrt{1-\varepsilon}2kt\right)\cos 2kx}{\left(1-\cos\sqrt{1-\varepsilon}2kT\right)16k^4}\psi_{2k}^4, \quad (1.22)$$

$$f(x) = \sum_{k=0}^{\infty} \frac{(1+\varepsilon)\,\psi_{2k+1}^4}{\left(1-\cos\sqrt{1+\varepsilon}\,(2k+1)\,T\right)(2k+1)^2}\cos(2k+1)\,x + \sum_{k=1}^{\infty} \frac{(1-\varepsilon)\,\psi_{2k}^4}{\left(1-\cos\sqrt{1-\varepsilon}2kT\right)4k^2}\cos 2kx, \quad (1.23)$$

where $\psi_{2k+1}^{(4)} = \left(\psi^{(4)}(x), y_{2k+1}^N\right)$ *and* $\psi_{2k}^{(4)} = \left(\psi^{(4)}(x), y_{2k}^N\right)$.

1.3.4 Chapter 5: On a nonlinear boundary-layer problem for the fractional Blasius type equation

Statement of the problem and main results

In this chapter, we consider a non-linear sequential differential equation with Caputo fractional derivative. We reduce the problem to the equivalent nonlinear integral equation and we prove the complete continuity of the nonlinear integral operator.

Consider the boundary value problem for the nonlinear fractional differential equation of Blasius type

$$\frac{d}{dt}\mathcal{D}_*^{2\alpha}x\left(t\right)+\mathcal{M}\left(x\left(t\right),t\right)\mathcal{D}_*^{2\alpha}x\left(t\right)=0,\ \ a<t<b, \tag{1.24}$$

with boundary conditions

$$x\left(a\right)=\varphi_1,\ \ D_*^{\alpha}x\left(a\right)=\varphi_2,\ \ x\left(b\right)=\varphi_3, \tag{1.25}$$

where $\alpha\in\left(\frac{1}{2},1\right)$ and φ_1, φ_2, φ_3 are given real numbers. Note that when $\alpha=1$, problem (1.24) - (1.25) is met in boundary layer theory in fluid mechanics and polymer theory. The recent surge in developing the theory of fractional differential equations has motivated the present work.

Condition (*). *Let $\mathcal{M}\left(x,t\right)$ be defined and continuous in the domain*

$$G=\left\{\left(x,t\right):\left|x\right|\leq R,R>0,a\leq t\leq b\right\},$$

where

$$R=\frac{\left|\varphi_2\right|\left|\left(b-a\right)^{\alpha}\right|}{\Gamma\left(\alpha+1\right)}+\left|\varphi_1\right|+\left|\varphi_3-\frac{\varphi_2\left(b-a\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}-\varphi_1\right|,$$

and

$$m=\min_{x,t\in G}\mathcal{M}\left(x,t\right),$$

$$M=\max\left\{\max_{x,t\in G}\mathcal{M}\left(x,t\right),0\right\}.$$

The space $C_{3-\alpha}^{3}\left(\left[a,b\right]\right)$ denotes the space:

$$C_{3-\alpha}^{3}\left(\left[a,b\right]\right)=\left\{x\in C\left(\left[a,b\right]\right):\ x'''\in C_{3-\alpha}\left(\left[a,b\right]\right)\right\}.$$

Here $C_{3-\alpha}\left(\left[a,b\right]\right)=\left\{\left(t-a\right)^{3-\alpha}x\in C\left(\left[a,b\right]\right)\right\}$.

The main result of this work is the following theorem.

Theorem 1.3.6 *If condition (*) satisfied, then problem* (1.24) - (1.25) *has a solution in $C_{3-\alpha}^{3}\left(\left[a,b\right]\right)$.*

Theorem on the existence of a unique solution of the problem for the non-linear differential equation of fractional order is formulated. In the limiting case, the considered boundary problem coincides with the boundary-layer problem for the Blasius equation.

1.4 Preliminaries

1.4.1 Basic Functions

In this section, we recall definitions of some special functions that we use later in the thesis.

Gamma Function

The Euler's gamma function $\Gamma(z)$ is one of the basic functions of fractional calculus. It generalizes the factorial $z!$ to take also non-integers and complex values and it is defined as follows.

Definition 1.4.1 *The gamma function* $\Gamma(.)$ *is defined as: for* $z \in \mathbb{C}$ *and* $\mathrm{Re}(z) > 0$

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{2.1}$$

where $t^{z-1} = e^{(z-1)\log(t)}$. *This integral is convergent for all complex* $z \in \mathbb{C}$ $(\mathrm{Re}(z) > 0)$.

For this function the reduction formula

$$\Gamma(z+1) = z\Gamma(z) \quad (\mathrm{Re}(z) > 0)$$

holds. In particular, if $z = n \in \mathbb{N}_0$, then

$$\Gamma(n+1) = n! \ (n \in \mathbb{N}_0)$$

with (as usual) $0! = 1$.

Mittag-Leffler functions

Definition 1.4.2 *The classical Mittag-Leffler function is defined by Mittag-Leffler* (1903) :

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+1)}, \ z \in \mathbb{C} \ \textit{and} \ \mathrm{Re}(\alpha) > 0. \tag{2.2}$$

Taking $\alpha = 1$ the exponential function is recovered, $E_1(z) = e^z$. A two parameter generalization has been proposed by Wiman (1905) as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n+\beta)}, \ z \in \mathbb{C} \text{ and } \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0. \tag{2.3}$$

For $\beta = 1$ the classical Mittag-Leffler function is recovered, i.e. $E_{\alpha,1}(z) = E_\alpha(z)$.

1.4.2 Fractional Derivatives

Often the easiest access to the idea of the non-integer differential and integral operators studied in the field of fractional calculus is given by Cauchy's well known representation of an n-fold integral as a convolution integral:

$$I_a^n f(t) = \int_a^t \int_a^{s_{n-1}} \dots \int_a^{s_1} f(s)\, ds ds_1 ds_{n-1}$$
$$= \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\, ds,\ n \in \mathbb{N}, t \in \mathbb{R}_+, \qquad (2.4)$$

where I_a^n is the n-fold integral operator (Cauchy formula).

The Riemann-Liouville fractional integral is a simple generalization of the Cauchy formula (2.4), the integer n is substituted by a positive real number α and the Gamma function $\Gamma(.)$ is used instead of the factorial, i.e.

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_\alpha^t (t-s)^{\alpha-1} f(s)\, ds,\ \alpha, t \in \mathbb{R}_+. \qquad (2.5)$$

The definition of fractional integral is very straightforward and there are no complications. A more difficult question is how to define a fractional derivative.

We can give the simplest definition of fractional derivative as concatenation of integer order differentiation and fractional integration, i.e.

$$D_a^\alpha f(t) = D^n I_a^{n-\alpha} f(t) \text{ or } {}^C D_a^\alpha f(t) = I_a^{n-\alpha} D^n f(t),$$

where n is the integer satisfying $\alpha \leq n < \alpha + 1$ and D^n, $n \in \mathbb{N}$, is the n-fold differential operator. The operator D_a^α is usually denoted as Riemann-Liouville differential operator, while the operator ${}^C D_a^\alpha$ is named Caputo differential operator.

Riemann-Liouville definition

The Riemann-Liouville fractional derivative of a function $f \in AC^n([a,b])$, where $-\infty < a < b < +\infty$, with $\alpha, t \in \mathbb{R}_+$ is defined as follow:

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)\, ds}{(t-s)^{\alpha-n+1}},\ (n-1 < \alpha \leq n),\ n \in \mathbb{N}. \quad (2.6)$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator proposed by Caputo ([21], [27]).

M. Caputo definition

The Caputo fractional derivative of a function $f \in C^n([a,b])$, where $-\infty < a < b < +\infty$, with $\alpha, t \in \mathbb{R}_+$ is defined by,

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)\, ds}{(t-s)^{\alpha-n+1}}, \ (n-1 < \alpha \leq n),\ n \in \mathbb{N}. \tag{2.7} $$

The Caputo derivative allows the use of physically interpretable initial conditions, which is not permitted by the Riemann-Liouville. Another difference between the two definitions appears when dealing with constant function. Indeed, for a constant, the Caputo fractional derivative is zero while Reamann-Liouville fractional derivative is not zero.

Relation between Riemann-Liouville and Caputo derivatives

Proposition 1.4.3 *If $f(t)$ is $n-1$ continuously differentiable in the interval $[a,b]$ and $f^{(n)}(t)$ is integrable in $[a,b]$, then*

$$ {}_R D_t^\alpha f(t) = \sum_{i=0}^{m-1} \frac{f^{(i)}(a)(t-a)^{i-\alpha}}{\Gamma(1+i-\alpha)} +_C D_t^\alpha f(t) \tag{2.8} $$

where $m-1 \leq \alpha \leq m < n$ with $m \in \mathbb{N}^*$.

Proof. Applying repeatedly integration by parts to the Riemann-Liouville will give us: for $\forall t \in \mathbb{R}$, and $\alpha < t$,

$$ {}_R D_t^\alpha f(t) = \sum_{i=0}^{m-1} \frac{f^{(i)}(a)(t-a)^{i-\alpha}}{\Gamma(1+i-\alpha)} + \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)} \tau d\tau}{(t-\tau)^{\alpha-m+1}}. \tag{2.9} $$

The right hand side of the above equation is equal to the Caputo derivatives. □

Proposition 1.4.4 *Let $f(t)$ be $n-1$ continuously differentiable in the interval $[a,b]$ and $f^{(n)}(t)$ be integrable in $[a,b]$. Then, if $f^{(n)}(a) = 0$, for $n = 0, 1, 2, ..., m-1$,*

$$ {}_R D_t^\alpha f(t) =_C D_t^\alpha f(t) \text{ for any } t \in \mathbb{R}, \tag{2.10} $$

where $m-1 \leq \alpha \leq m < n$ with $m \in \mathbb{N}^*$.

1.4.3 Properties of Fractional Derivatives

We recall some useful properties of fractional derivatives.

1. **Linearity**

Assuming that the fractional derivatives of f and g exists, then for $\lambda, \mu \in \mathbb{R}$ [83] :

$$D^{\alpha}\left(\lambda f\left(t\right)+\mu g\left(t\right)\right)=\lambda D^{\alpha}f\left(t\right)+\mu D^{\alpha}g\left(t\right), \tag{2.11}$$

where D^{α} denotes any of the fractional derivatives we have defined before.

2. **Laplace Transform of Fractional Derivatives**

We assume that the fractional derivative and the Laplace transform of f exists. Then, the Laplace transform of the Riemann-Liouville and Caputo fractional derivative are defined by [83] : $\forall s \in \mathbb{C}$,

— **Riemann-Liouville**

$$\mathcal{L}\left[{}_{R}D_{t}^{\alpha}f\left(t\right)\right]=s^{\alpha}\widehat{f}\left(s\right)-\sum_{i=0}^{n-1}s^{i}\left[{}_{R}D_{t}^{\alpha-i-1}f\left(t\right)\right]_{t=0},\ n-1\leq\alpha<n \tag{2.12}$$

— **Caputo**

$$\mathcal{L}\left[{}_{C}D_{t}^{\alpha}f\left(t\right)\right]=s^{\alpha}\widehat{f}\left(s\right)-\sum_{i=0}^{n-1}s^{\alpha-i-1}f^{(i)}\left(0\right),\ n-1\leq\alpha<n \tag{2.13}$$

where $\widehat{f}\left(.\right)$ denotes the Laplace transform of $f\left(.\right)$, and s is the variable in the frequency domain.

3. **Fourier Transform of Fractional Derivatives**

We assume that the fractional derivative and the Fourier transform of f exists. Then, the Fourier transform of the Grünwald-Letnikov, Riemann-Liouville, and the Caputo fractional derivatives are defined by [1] : $\forall k \in \mathbb{R}$,

$$\mathcal{F}\left[{}_{O}D_{t}^{\alpha}f\left(t\right)\right]=\left(-ik\right)^{\alpha}\widetilde{f}\left(s\right) \tag{2.14}$$

where ${}_{O}D_{t}^{\alpha}$ denotes any of the mentioned fractional differentiations, $\widetilde{f}\left(.\right)$ denotes the Fourier transform of $f\left(.\right)$, and k is the variable in the frequency domain.

4. **Derivative of the Fractional Operator with Respect to** α.

In the next proposition, we present the derivative of the fractional derivative with respect to the fractional order α. We consider the left Riemann-Liouville derivative. However, similar results can be obtained using other definitions.

Proposition 1.4.5 *[83] If the α^{th} order Riemann-Liouville derivative of f exists where $n-1\leq\alpha<n$, then the derivative of $\frac{\partial^{\alpha}f}{\partial x^{\alpha}}$ with respect to α is given by*

$$\frac{\partial}{\partial\alpha}\frac{\partial^{\alpha}f\left(x\right)}{\partial x^{\alpha}}=\psi_{0}\left(n-\alpha\right)\frac{\partial^{\alpha}f\left(x\right)}{\partial x^{\alpha}}-\frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^{n}}{dx^{n}}\int_{0}^{x}\left(x-\tau\right)^{n-\alpha-1}\ln\left(x-\tau\right)f\left(\tau\right)d\tau, \tag{2.15}$$

where $\psi_{0}\left(n-\alpha\right)=\frac{\Gamma'(n-\alpha)}{\Gamma(n-\alpha)}$.

Proof. The result can be obtained by differentiating (2.6) with respect to α. □

2 An inverse problem for time fractional evolution equations with an involution perturbation

Abstract

In this chapter, we consider an inverse problem for a time fractional evolution equation, interpolating the heat and the wave equations, with an involution. Results on the existence and uniqueness of a solution are presented via the method of separation of variables.

2.1 Introduction and statement of the problem

Differential equations with operations (equations with shift (involution)), apparently started with the work of Babbage [8], and were discussed later by Carleman [23] in 1932. In the late sixties and early seventies of the 20th century, Przewoerska-Rolewicz addressed many questions about differential equations with involutions in a series of nice papers [86], [84], [85], [87], [90], [89] and then compiled her results in form of a text [88]. For generalized solutions of functional differential equations, see the book by Wiener [111]. Recently, Kaliev et al [48], [49], Orazov and Sadybekov [80], [79],Sarsenbi [98], Sadybekov and Sarsenbi [94], Sarsenbi and Tengaeva [97], treated spectral problems and inverse problems for evolution equations with involution. In their talk [16], Cabada and Tojo mentioned an application of a parabolic equation with an involution related to heat conduction. In this chapter, we address an inverse problem for the time fractional evolution equation with involution

$$D_*^{\alpha} u(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x), \tag{2.1}$$

posed for $x \in (-\pi, \pi)$ and $t > 0$, where f and u are unknowns, $1 < \alpha < 2$, ε is a nonzero real number such that $|\varepsilon| < 1$. We equip (2.1) with the initial, final, and boundary conditions

$$u(x,0) = \phi(x), \ \ u_t(x,0) = \rho(x), \ \ u(x,T) = \psi(x), \ x \in [-\pi, \pi], \tag{2.2}$$

$$u(-\pi,t) = 0, \ \ u(\pi,t) = 0, \ t \in [0,T], \tag{2.3}$$

where $\phi(x)$ and $\psi(x)$ are given sufficiently smooth functions. The derivative D_*^{α} defined as

$$D_*^{\alpha} u(x,t) = D^{\alpha}(u(x,t) - u(x,0) - t u_t(x,0))$$

is the Caputo derivative for a regular function built on the Riemann-Liouville derivative D^{α}. Caputo's derivative allows us to impose initial conditions in a natural way.

By a regular solution of the problem, we mean a pair of functions $(u(x,t), f(x))$ of the class $u(x,t) \in C_{x,t}^{2,2}(\Omega)$(space of two times continuously differentiable functions on Ω according to both x and t), $f(x) \in C[-\pi,\pi]$, $\Omega = \{-\pi \le x \le \pi, \ 0 \le t \le T\}$.
When one uses the method of separation of variables to solve the problem, a spectral problem appears, which is mentioned in the next section.

2.2 The Spectral Problem

The spectral problem consists of the equation:

$$X''(x) - \varepsilon X''(-x) + \lambda X(x) = 0, \ -\pi \le x \le \pi, \tag{2.4}$$

where λ is the spectral parameter, equipped with the boundary conditions:

$$X(-\pi) = X(\pi) = 0, \quad X'(-\pi) = X'(\pi) = 0. \tag{2.5}$$

It is proved in [59] that expressing the solution of problem (2.4)- (2.5) in terms of the sum of even and odd functions, one finds the following eigenvalues:

$$\lambda_{k,1} = (1+\varepsilon)\, k^2,\ k \in \mathbb{Z}_+, \quad \lambda_{k,2} = (1-\varepsilon)\left(k + \frac{1}{2}\right)^2,\ k \in \mathbb{N},$$

with the corresponding normalized eigenfunctions given by

$$X_{k,1} = \frac{1}{\sqrt{\pi}} \sin kx =: S_k(x), \quad k \in \mathbb{Z}_+, \ X_{k,2} = \frac{1}{\sqrt{\pi}} \cos\left(k + \tfrac{1}{2}\right) x =: C_{k+\frac{1}{2}}(x),\ k \in \mathbb{N}. \tag{2.6}$$

Observe that the systems of functions (2.6) is complete in $L_2(-\pi, \pi)$.[59]

We will use the following result which appears in part 3 of [41].

Lemma 2.2.1 *The following differential equation of fractional order $\alpha > 0$*

$$D_*^\alpha u(t) = D^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0^+) \right) = -u(t) + q(t), \quad t > 0,$$

where m is a positive integer uniquely defined by $m-1 < \alpha \le m$, with the prescribed initial values

$$u^{(k)}(0^+) = c_k, \quad k = 0, 1, 2, ..., m-1,$$

has the solution

$$u(t) = \sum_{k=0}^{m-1} c_k u_k(t) + \int_0^t q(t-s) u_\delta(s)\, ds,$$

with

$$u_k(t) = J^k u_0(t), \quad u_k^{(h)} = \delta_{k,h},\ h, k = 0, 1, ..., m-1, \tag{2.7}$$
$$u_\delta(t) = -u_0'(t),$$

the functions $u_k(t)$ represent the fundamental solutions of the differential equation of order m.

2.3 Main results

Here we present the existence and uniqueness results for our problem.

Theorem 2.3.1 *Let* $\phi(x), \rho(x), \psi(x) \in C^4[-\pi, \pi]$ *and* $\phi^{(i)}(\pm\pi) = \rho^{(i)}(\pm\pi) = \psi^{(i)}(\pm\pi) = 0, i = 0,1,2,3$. *If* $1 - u_0(.) \neq 0$ *then, for a nonzero real number* ε *such that* $|\varepsilon| < 1$, *problem* (2.1) - (2.3) *has a unique solution which can be written in the form*

$$\begin{aligned} u(x,t) &= \phi(x) \\ &+ \sum_{k=0}^{\infty} \frac{\left(C^{(4)}\right)_{1k} u_0(\lambda_{k,2}^{1/\alpha} t)}{\left(k+\frac{1}{2}\right)^4} C_{k+\frac{1}{2}}(x) \\ &+ \sum_{k=0}^{\infty} \frac{\left(C^{(4)}\right)_{3k} u_1(\lambda_{k,2}^{1/\alpha} t) - \left(C^{(4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^4} C_{k+\frac{1}{2}}(x) \\ &+ \sum_{k=1}^{\infty} \frac{\left(C^{(4)}\right)_{2k} u_0(\lambda_{k,2}^{1/\alpha} t) + \left(C^{(4)}\right)_{4k} u_1(\lambda_{k,1}^{1/\alpha} t) - \left(C^{(4)}\right)_{2k}}{k^4} S_k(x), \end{aligned}$$

and

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\left(\left(\phi^{(4)}\right)_{1k} - \left(C^{(4)}\right)_{1k}\right)}{\left(k+\frac{1}{2}\right)^2} C_{k+\frac{1}{2}}(x) \\ &+ \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\left(\left(\phi^{(4)}\right)_{2k} - \left(C^{(4)}\right)_{2k}\right)}{k^2} S_k(x), \end{aligned}$$

where

$$\begin{aligned} \left(C^{(4)}\right)_{1k} &= \frac{\left(\phi^{(4)}\right)_{1k} - \left(\psi^{(4)}\right)_{1k} + \left(C^{(4)}\right)_{3k} u_1(\lambda_{k,2}^{1/\alpha} T)}{1 - u_0\left(\lambda_{k,2}^{1/\alpha} T\right)}, \\ \left(C^{(4)}\right)_{3k} &= \frac{\left(\rho^{(4)}\right)_{1k}}{\lambda_{k,2}^{1/\alpha}}, \\ \left(C^{(4)}\right)_{2k} &= \frac{\left(\phi^{(4)}\right)_{2k} - \left(\psi^{(4)}\right)_{2k} + \left(C^{(4)}\right)_{4k} u_1(\lambda_{k,1}^{1/\alpha} T)}{1 - u_0\left(\lambda_{k,1}^{1/\alpha} T\right)}, \\ \left(C^{(4)}\right)_{4k} &= \frac{\left(\rho^{(4)}\right)_{2k}}{\lambda_{k,1}^{1/\alpha}}, \end{aligned}$$

and

$$\begin{aligned} \left(g^{(4)}\right)_{1k} &= \int_{-\pi}^{\pi} g^{(4)}(x)\, C_{k+\frac{1}{2}}(x)\, dx, \\ \left(g^{(4)}\right)_{2k} &= \int_{-\pi}^{\pi} g^{(4)}(x)\, S_k(x)\, dx, \quad \text{for} \quad g = \phi, \psi, \rho. \end{aligned} \tag{2.8}$$

2.4 Proof of the Result

2.4.1 Existence

Here, we give the full proof of the existence of a solution of the problem as stated in Theorem 2.3.1.

As the eigenfunctions system (2.6) forms an orthonormal basis in $L_2(-\pi,\pi)$, the functions $u(x,t)$ and $f(x)$ can be represented as follows

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) C_{k+\frac{1}{2}}(x) + \sum_{k=1}^{\infty} v_k(t) S_k(x), \tag{2.9}$$

and

$$f(x) = \sum_{k=0}^{\infty} f_{1k} C_{k+\frac{1}{2}}(x) + \sum_{k=1}^{\infty} f_{2k} S_k(x), \tag{2.10}$$

where $u_k(t)$, $v_k(t)$, f_{1k} and f_{2k} are unknown. Substituting (2.9) and (2.10) into equation (2.1), we obtain the following equations for the functions $u_k(t)$, $v_k(t)$ and the constants f_{1k}, f_{2k} :

$$D_*^{\alpha} u_k(t) + \lambda_{k,2} u_k(t) = f_{1k},$$

and

$$D_*^{\alpha} v_k(t) + \lambda_{k,1} v_k(t) = f_{2k},$$

which we write

$$\frac{1}{\lambda_{k,2}} D_*^{\alpha} u_k(t) + u_k(t) = \frac{1}{\lambda_{k,2}} f_{1k}, \tag{2.11}$$

and

$$\frac{1}{\lambda_{k,1}} D_*^{\alpha} v_k(t) + v_k(t) = \frac{1}{\lambda_{k,1}} f_{2k}. \tag{2.12}$$

By a change of scale $t \mapsto \lambda_{k,2}^{\frac{1}{\alpha}} t$ in (2.11), and using Lemma 2.2.1, we obtain

$$u_k(t) = c_0 u_0(\lambda_{k,2}^{\frac{1}{\alpha}} t) + c_1 u_1(\lambda_{k,2}^{\frac{1}{\alpha}} t) - \frac{f_{1k}}{\lambda_{k,2}} (u_0(\lambda_{k,2}^{\frac{1}{\alpha}} t) - u_0(0^+)),$$

where

$$u_0(\tau) = e_{\alpha}(\tau) = E_{\alpha}(-\tau^{\alpha}) = \sum_{n=0}^{\infty} \frac{(-\tau^{\alpha})^n}{\Gamma(\alpha n + 1)},$$

$$u_1(\tau) = J e_{\alpha}(\tau) = \int_0^t e_{\alpha}(s)\, ds,$$

$$u_0(0^+) = 1, \qquad u_0'(0^+) = 0,$$

$$u_1(0^+) = 0, \qquad u_1'(0^+) = 1,$$

(e_{α} and $J e_{\alpha}$ represent the fundamental solution of the equation $D_*^{\alpha} u_k(t) = -u_k(t) + f_{1k}/\lambda_{k,2}$);

so

$$u_k(t) = (c_0 - \frac{f_{1k}}{\lambda_{k,2}})u_0(\lambda_{k,2}^{\frac{1}{\alpha}}t) + c_1 u_1(\lambda_{k,2}^{\frac{1}{\alpha}}t) + \frac{f_{1k}}{\lambda_{k,2}},$$

and

$$u_k(t) =: C_{1k}u_0(\lambda_{k,2}^{\frac{1}{\alpha}}t) + C_{3k}u_1(\lambda_{k,2}^{\frac{1}{\alpha}}t) + \frac{f_{1k}}{\lambda_{k,2}}.$$

Similarly

$$v_k(t) = (c_0 - \frac{f_{2k}}{\lambda_{k,1}})u_0(\lambda_{k,1}^{\frac{1}{\alpha}}t) + c_1 u_1(\lambda_{k,1}^{\frac{1}{\alpha}}t) + \frac{f_{1k}}{\lambda_{k,1}},$$

and

$$v_k(t) =: C_{2k}u_0(\lambda_{k,1}^{\frac{1}{\alpha}}t) + C_{4k}u_1(\lambda_{k,1}^{\frac{1}{\alpha}}t) + \frac{f_{2k}}{\lambda_{k,1}},$$

where the constants C_{1k}, C_{2k}, C_{3k}, C_{4k}, f_{1k}, and f_{2k} are to be determined using the given data.

Expanding the functions $\phi(x), \rho(x)$ and $\psi(x)$ using the eigenfunctions system (2.6), we obtain

$$C_{1k} = \phi_{1k} - \frac{f_{1k}}{\lambda_{k,2}}, \quad C_{2k} = \phi_{2k} - \frac{f_{2k}}{\lambda_{k,1}},$$

$$\lambda_{k,2}^{1/\alpha}\, C_{3k} = \rho_{1k}, \qquad \lambda_{k,1}^{1/\alpha}\, kC_{4k} = \rho_{2k},$$

$$C_{1k}u_0(\lambda_{k,2}^{1/\alpha}T) + C_{3k}u_1(\lambda_{k,2}^{1/\alpha}T) + \frac{f_{1k}}{\lambda_{k,2}} = \psi_{1k},$$

and

$$C_{2k}u_0(\lambda_{k,1}^{1/\alpha}T) + C_{4k}u_1(\lambda_{k,1}^{1/\alpha}T) + \frac{f_{2k}}{\lambda_{k,1}} = \psi_{2k},$$

where, $\phi_{ik}, \rho_{ik}, \psi_{ik}, i = 1,2$ are the coefficients of the expansions of the functions $\phi(x), \rho(x), \psi(x)$ given by

$$g_{1k} = \int_{-\pi}^{\pi} g(x)\, C_{k+\frac{1}{2}}(x)\, dx, \quad g_{2k} = \int_{-\pi}^{\pi} g(x)\, S_k(x)\, dx, \quad \text{for} \quad g = \phi, \rho, \psi.$$

Solving the above set of equations for C_{1k}, C_{2k}, C_{3k}, C_{4k}, f_{1k}, and f_{2k}, we get

$$C_{1k} = \frac{\phi_{1k} - \psi_{1k} + C_{3k}u_1(\lambda_{k,2}^{1/\alpha}T)}{1 - u_0(\lambda_{k,2}^{1/\alpha}T)}, \qquad C_{3k} = \frac{\rho_{1k}}{\lambda_{k,2}^{1/\alpha}},$$

$$C_{2k} = \frac{\phi_{2k} - \psi_{2k} + C_{4k}u_1(\lambda_{k,1}^{1/\alpha}T)}{1 - u_0(\lambda_{k,1}^{1/\alpha}T)}, \qquad C_{4k} = \frac{\rho_{2k}}{\lambda_{k,1}^{1/\alpha}},$$

and

$$f_{1k} = \lambda_{k,2}(\phi_{1k} - C_{1k}), \qquad f_{2k} = \lambda_{k,1}(\phi_{2k} - C_{2k}). \tag{2.13}$$

Now, substituting the expressions for $u_k(t)$, $v_k(t)$, f_{1k}, f_{2k} into (2.9) and (2.10), we obtain

$$\begin{aligned} u(x,t) &= \phi(x) \\ &+ \sum_{k=0}^{\infty}\left(C_{1k}u_0(\lambda_{k,2}^{1/\alpha}t)+C_{3k}u_1(\lambda_{k,2}^{1/\alpha}t)-C_{1k}\right)C_{k+\frac{1}{2}}(x) \\ &+ \sum_{k=1}^{\infty}\left(C_{2k}\,u_0(\lambda_{k,1}^{1/\alpha}t)+C_{4k}u_1(\lambda_{k,1}^{1/\alpha}t)-C_{2k}\right)S_kx, \end{aligned}$$

and

$$f(x)=\sum_{k=0}^{\infty}\lambda_{k,2}\,\left(\phi_{1k}-C_{1k}\right)C_{k+\frac{1}{2}}(x)+\sum_{k=1}^{\infty}\lambda_{k,1}\left(\phi_{2k}-C_{2k}\right)S_k(x).$$

Moreover, if $\phi^{(i)}(\pm\pi)=\rho^{(i)}(\pm\pi)=\psi^{(i)}(\pm\pi)=0$, $i=0,1,2,3$, then integrating $\phi_{ik},\rho_{ik},\psi_{ik}, i=1,2$, by parts yields

$$g_{1k}=\frac{\left(g^{(4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^4} \quad \text{and} \quad g_{2k}=\frac{\left(g^{(4)}\right)_{2k}}{k^4},$$

for $g=\phi,\rho,\psi$, where, $\left(\phi^{(4)}\right)_{ik}, \left(\rho^{(4)}\right)_{ik}, \left(\psi^{(4)}\right)_{ik}, i=1,2$ are the coefficients of the expansions of the functions $\phi^{(4)}(x)$, $\rho^{(4)}(x)$, $\psi^{(4)}(x)$ and are given by

$$(g^{(4)})_{1k}=\int_{-\pi}^{\pi}g^{(4)}(x)C_{k+\frac{1}{2}}(x)\,dx, \quad (g^{(4)})_{2k}=\int_{-\pi}^{\pi}g^{(4)}(x)S_k(x)\,dx,$$

for $g=\phi,\psi,\rho$.
Then the constants C_{1k}, C_{2k}, C_{3k}, C_{4k}, f_{1k}, and f_{2k} can be written as

$$C_{1k}=\frac{\left(C^{(4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^4}, \qquad C_{3k}=\frac{\left(C^{(4)}\right)_{3k}}{\left(k+\frac{1}{2}\right)^4},$$

$$C_{2k}=\frac{\left(C^{(4)}\right)_{2k}}{k^4}, \qquad C_{4k}=\frac{\left(C^{(4)}\right)_{4k}}{k^4},$$

and

$$\begin{aligned} f_{1k} &= \frac{(1-\varepsilon)}{\left(k+\frac{1}{2}\right)^2}\left(\left(\phi^{(4)}\right)_{1k}-\left(C^{(4)}\right)_{1k}\right), \\ f_{2k} &= \frac{(1+\varepsilon)}{k^2}\left(\left(\phi^{(4)}\right)_{2k}-\left(C^{(4)}\right)_{2k}\right), \end{aligned} \tag{2.14}$$

where,

$$\left(C^{(4)}\right)_{1k}=\frac{\left(\phi^{(4)}\right)_{1k}-\left(\psi^{(4)}\right)_{1k}+\left(C^{(4)}\right)_{3k}u_1(\lambda_{k,2}^{1/\alpha}T)}{1-u_0(\lambda_{k,2}^{1/\alpha}T)},\quad \left(C^{(4)}\right)_{3k}=\frac{(\rho^{(4)})_{1k}}{\lambda_{k,2}^{1/\alpha}},$$

$$\left(C^{(4)}\right)_{2k}=\frac{(\phi^{(4)})_{2k}-\left(\psi^{(4)}\right)_{2k}+\left(C^{(4)}\right)_{4k}u_1(\lambda_{k,1}^{1/\alpha}T)}{1-u_0(\lambda_{k,1}^{1/\alpha}T)},\quad \left(C^{(4)}\right)_{4k}=\frac{(\rho^{(4)})_{2k}}{\lambda_{k,1}^{1/\alpha}}.$$

Thus the solution of our problem takes the form

$$\begin{aligned} u\left(x,t\right) &= \phi\left(x\right) \\ &+\sum_{k=0}^{\infty}\frac{\left(C^{(4)}\right)_{1k}u_1(\lambda_{k,2}^{1/\alpha}t)}{(k+\frac{1}{2})^4}C_{k+\frac{1}{2}}(x) \\ &+\sum_{k=0}^{\infty}\frac{\left(C^{(4)}\right)_{3k}u_1(\lambda_{k,2}^{1/\alpha}t)-\left(C^{(4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^4}C_{k+\frac{1}{2}}(x) \\ &+\sum_{k=1}^{\infty}\frac{\left(C^{(4)}\right)_{2k}u_0(\lambda_{k,1}^{1/\alpha}t)+\left(C^{(4)}\right)_{4k}u_1(\lambda_{k,1}^{1/\alpha}t)-\left(C^{(4)}\right)_{2k}}{k^4}S_k(x), \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} f\left(x\right) &= \sum_{k=0}^{\infty}\frac{(1-\varepsilon)\left(\left(\phi^{(4)}\right)_{1k}-\left(C^{(4)}\right)_{1k}\right)}{\left(k+\frac{1}{2}\right)^2}C_{k+\frac{1}{2}}(x) \\ &+\sum_{k=1}^{\infty}\frac{(1+\varepsilon)\left(\left(\phi^{(4)}\right)_{2k}-\left(C^{(4)}\right)_{2k}\right)}{k^2}S_k(x). \end{aligned} \tag{2.16}$$

This completes the proof of Theorem 2.3.1.

2.4.2 Convergence of the series

To establish that the formal solution is indeed a true solution, we will show that all operations performed in the proof are valid.

The convergence of the series in (2.15) and (2.16) are based on the following estimates for $u\left(x,t\right)$ and $f\left(x\right)$:

$$\begin{aligned} |u(x,t)|\leq|\phi(x)| &+ \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\sqrt{2}\left|\phi_{1k}^{(4)}\right|+\sqrt{2}\left|\psi_{1k}^{(4)}\right|+4\left|\rho_{1k}^{(4)}\right|}{\sqrt{1-\varepsilon}\left(1-u_0(\lambda_{k,2}^{1/\alpha}T)\right)\left(k+\frac{1}{2}\right)^4} \\ &+ \frac{2}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{\sqrt{2}\left|\phi_{2k}^{(4)}\right|+\sqrt{2}\left|\psi_{2k}^{(4)}\right|+2\left|\rho_{2k}^{(4)}\right|}{\sqrt{1+\varepsilon}\left(1-u_0\left(\lambda_{k,1}^{1/\alpha}T\right)\right)k^4} \end{aligned} \tag{2.17}$$

and

$$|f(x)| \leq \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{6\left|\phi_{1k}^{(4)}\right| + 2\left|\psi_{1k}^{(4)}\right| + 2\sqrt{2}\left|\rho_{1k}^{(4)}\right|}{\left(1 - u_0(\lambda_{k,2}^{1/\alpha}T)\right)\left(k+\frac{1}{2}\right)^2}$$
$$+\frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{6\left|\phi_{2k}^{(4)}\right| + 2\left|\psi_{2k}^{(4)}\right| + \sqrt{2}\left|\rho_{2k}^{(4)}\right|}{\left(1 - u_0(\lambda_{k,1}^{1/\alpha}T)\right)k^2} \tag{2.18}$$

As $\phi(x), \rho(x), \psi(x) \in C^4[-\pi,\pi]$, by the Bessel inequality for trigonometric series, the following series converge:

$$\sum_{k=0}^{\infty} \left|g_{1k}^{(4)}\right|^2 \leq C \left\|g^{(4)}(x)\right\|_{L_2(-\pi,\pi)}^2, \quad \textit{for} \quad g = \phi, \rho, \psi. \tag{2.19}$$

and

$$\sum_{k=1}^{\infty} \left|g_{2k}^{(4)}\right|^2 \leq C \left\|g^{(4)}(x)\right\|_{L_2(-\pi,\pi)}^2, \quad \textit{for} \quad g = \phi, \rho, \psi, \tag{2.20}$$

which implies that the set

$$\left\{\varphi_{ik}^{(4)}, \rho_{ik}^{(4)}, \psi_{ik}^{(4)}\right\}, \; k = 1, 2.$$

is bounded.

Theorem 2.4.1 (Weierstrass M test) *Let $\{u_n\}$ be a sequence of real or complex-valued functions defined on a set X and that there is a sequence of positive numbers $\{M_n\}$ satisfying*

$\forall n \geq 1$, $\forall x \in X$: $|f_n(x)| \leq M_n$, such that

$$\sum_{n=1}^{\infty} M_n < \infty$$

Then the series $\sum_{k=1}^{\infty} f_n(x)$ converges absolutely and uniformly on X.

Therefore, by the Weierstrass M-test, series (2.17) and (2.18) converge absolutely and uniformly in the region Ω.

Now, using termwise differentiation of the series (2.15) twice with respect to the variables x and t, we get the following estimates for $u_{xx}(x,t)$ and $u_{tt}(x,t)$,

$$|u_{xx}(x,t)| \leq |\phi''(x)| + \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\sqrt{2}\left|\phi_{1k}^{(4)}\right| + \sqrt{2}\left|\psi_{1k}^{(4)}\right| + 4\left|\rho_{1k}^{(4)}\right|}{\sqrt{1-\varepsilon}\left(1 - u_0(\lambda_{k,2}^{1/\alpha}T)\right)\left(k+\frac{1}{2}\right)^2}$$
$$+ \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\sqrt{2}\left|\phi_{2k}^{(4)}\right| + \sqrt{2}\left|\psi_{2k}^{(4)}\right| + 2\left|\rho_{2k}^{(4)}\right|}{\sqrt{1+\varepsilon}\left(1 - u_0(\lambda_{k,1}^{1/\alpha}T)\right)k^2}$$

and

$$|u_{tt}(x,t)| \leq \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2\left|\phi_{1k}^{(4)}\right| + 2\left|\psi_{1k}^{(4)}\right| + 6\sqrt{2}\left|\rho_{1k}^{(4)}\right|}{\left(1 - u_0(\lambda_{k,2}^{1/\alpha}T)\right)\left(k+\frac{1}{2}\right)^2}$$
$$+\frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{2\left|\phi_{2k}^{(4)}\right| + 2\left|\psi_{2k}^{(4)}\right| + 3\sqrt{2}\left|\rho_{2k}^{(4)}\right|}{\left(1 - u_0(\lambda_{k,1}^{1/\alpha}T)\right)k^2}$$

which, on using (2.19), (2.20) and the Weierstrass M-test, also converge absolutely and uniformly on Ω.

2.4.3 Uniqueness

The uniqueness of the solution follows from representation of the solution given in the theorem, and from the completeness of the system (2.6).

Suppose that there are two solutions $\{u_1(x,t), f_1(x)\}$ and $\{u_2(x,t), f_2(x)\}$ of problem. Denote

$$u(x,t) = u_1(x,t) - u_2(x,t)$$

and

$$f(x) = f_1(x) - f_2(x).$$

Then the functions $u(x,t)$ and $f(x)$ satisfy (2.1), conditions of (2.2) and homogeneous boundary conditions(2.3).

Let

$$u_k(t) = \int_{-\pi}^{\pi} u(x,t)\, S_k(x)dx, k \in \mathbb{Z}_+, \tag{2.21}$$

$$v_k(t) = \int_{-\pi}^{\pi} u(x,t)\, C_{k+\frac{1}{2}}(x)dx, k \in \mathbb{N}, \tag{2.22}$$

$$f_{1k} = \int_{-\pi}^{\pi} f(x)\, S_k(x)dx, k \in \mathbb{Z}_+, \tag{2.23}$$

$$f_{2k} = \int_{-\pi}^{\pi} f(x)\, C_{k+\frac{1}{2}}(x)dx, k \in \mathbb{N}. \tag{2.24}$$

Applying the operator $\mathcal{D}^{\alpha}$ to the equation (2.21) we have

$$\mathcal{D}^{\alpha}u_k(t) = \int_{-\pi}^{\pi} \mathcal{D}_t^{\alpha}u(x,t)\, S_k(x)dx = \int_{-\pi}^{\pi} \left(u_{xx}(x,t) - \varepsilon u_{xx}(-x,t)\right) S_k(x)dx + f_{1k}.$$

Integrating by parts and taking into account the homogeneous conditions (2.2) and (2.3) , we obtain

$$\mathcal{D}^{\alpha} u_{k,1}(t) = f_{1k}\ ,\ u(0) = 0\ ,\ u(T) = 0.$$

Consequently, $f_{1k} = 0, u_k(t) \equiv 0$.

In a similar way for the functions (2.22), (2.23), (2.24) one proves that $f_{2k} = 0, v_k(t) \equiv 0$.

Further, by the completeness of the system (2.6) in $L^2(-\pi, \pi)$ we obtain $f(t) \equiv 0, u(x,t) \equiv 0, 0 \leq t \leq T, -\pi \leq x \leq \pi$.

Uniqueness of the solution of problem is proved.

Some inverse problems for the nonlocal heat equation with Caputo fractional derivative

3

Abstract

In this chapter, a class of inverse problems for restoring the right-hand side of a fractional heat equation with involution is considered. The results on existence and uniqueness of solutions of these problems are presented.

3.1 Introduction

The purpose of this chapter is to study inverse problems for the nonlocal heat equation with involution of space variable x. We consider the heat equation

$$\mathcal{D}_t^{\alpha} u(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(\pi - x, t) = f(x), \tag{3.1}$$

for $(x,t) \in \Omega = \{0 < x < \pi,\ 0 < t < T < \infty\}$, $0 < \alpha < 1$, where $\mathcal{D}_t^{\alpha}$ is the Caputo derivative (which is defined in the next section) and ε is a nonzero real number such that $|\varepsilon| < 1$.

Before describing our results, let us dwell a while on the existing literature concerning differential equation with delay either in time or in space.

Differential equations with time deviating arguments have been treated in a sizable number of articles and monographs, to cite but a few: [47], [60], [112]. For example in [47], the authors considered an example of parabolic functional differential equation with time delay of the following form:

$$u_t = u_{xx}(t-h, x), \quad t > 0,\ 0 < x < \pi,\ h > 0,$$

to study the spectrum distribution of its symbols (characteristic quasipolinomials). Delay differential equations occur in a variety of real world applications: biological modelling, automatic control systems, economics, epidemiology, feedback problems, the theory of climate models, etc.

Ample opportunities of applying equations with deviating argument in mathematical models have increased the interest of the study of new problems for partial differential equations [15], [43], [91]. Among differential equations with deviating arguments, special place is occupied by equations with a deviation of arguments of alternating character. Such deviations include the so-called deviation of involution type [18]. To describe them, let Γ be an interval in $\mathbb{R}$ and let $X \in \Gamma$ be a real variable. The homeomorphism

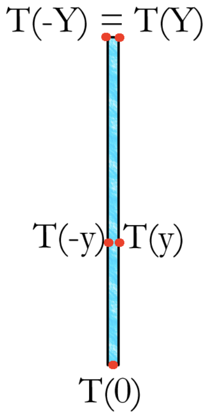
$$\alpha^2(X) = \alpha(\alpha(X)) = X,$$

is called a Carleman shift (deviation of involution) [23].

Equations containing Carleman shift are equations with an alternating deviation (at $X^* < X$ being equations with advanced, and at $X^* > X$ being equations with delay, where X^* is a fixed point of the mapping $\alpha(X)$).

However, some interesting works contain equations with modifications of the spatial variable in the unknown function that are motivated by the nonlinear optics, studied in a number of papers (see [57], [74]); for instance, in [74], the author studied the Cauchy problem for the difference-differential parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{h \in \mathcal{M}} a_h u(x-h, t),$$



(1).png

Figure 3.1

where $\mathcal{M}$ is a finite set of vectors in $\mathbb{R}^n$ parallel to coordinate axes (or any other orthogonal vector system) and the coefficients a_h are real. It should be noted that parabolic functional-differential equation arise in the investigation of nonlinear optic systems with two-dimensional feedback(see [4] and the references therein). **Skubachevskii** introduced these operators for nonlocal problems in heat conduction problems [102]. In contrast to classical parabolic differential equations, these equations have a number of new properties. For instance, the smoothness of generalized solution can be violated inside the cylindrical domain even for an infinitely smooth right-hand side of the equation.

Furthermore, for the equations containing transformation of the spatial variable in the diffusion term , we can cite the **talk of Cabada and Tojo** [20], where an example that describes a concrete situation in physics is given: Consider a metal wire around a thin sheet of insulating material in a way that some parts overlap some others as shown in Figure 1.

Assuming that the position y = 0 is the lowest of the wire, and the insulation goes up to the left at -Y and to the right up to Y.

For the proximity of two sections of wires they added the third term with modifications on the spatial variable to the right-hand side of the heat equation with respect to the wire:

$$\frac{\partial T}{\partial t}(y,t) = \alpha\frac{\partial^2 T}{\partial y^2}(y,t) + \beta\frac{\partial^2 T}{\partial y^2}(-y,t),$$

where T is the temperature at (y, t). Such equations have also a purely theoretical value.

Concerning the inverse problems and spectral problems for equations with involutions, some recent works have been done by Kaliev [48], [50], Kirane [52], [5], Sadybekov [80], [79], Sarsenbi [59], [98], [97].

3.2 Statement of problems

The chapter is concerned with four inverse problems concerning the time fractional heat equation with a perturbative term of involution type in the space variable. We obtain existence and uniqueness results for these problems, based on the Fourier method.

Problem D. Find the couple of functions $(u(x,t), f(x))$ satisfying the equation (3.1), under the conditions

$$u(x,0) = \varphi(x), \ x \in [0,\pi], \tag{3.2}$$

$$u(x,T) = \psi(x), \ x \in [0,\pi], \tag{3.3}$$

and the homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0 \ , \ t \in [0,T], \tag{3.4}$$

where $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions.

Problem N. Find the couple of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (3.1), conditions (3.2), (3.3) and the homogeneous Neumann boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0 \ , \ t \in [0,T]. \tag{3.5}$$

Problem P. Find the couple of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (3.1), conditions (3.2), (3.3) and the periodic boundary conditions

$$u(0,t) = u(\pi,t), \ u_x(0,t) = u_x(\pi,t), \ t \in [0,T]. \tag{3.6}$$

Problem AP. Find the couple of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (3.1), conditions (3.2), (3.3) and the anti-periodic boundary conditions

$$u(0,t) = -u(\pi,t), \ u_x(0,t) = -u_x(\pi,t), \ t \in [0,T]. \tag{3.7}$$

A regular solution of problems D, N, P and AP is the pair of functions $(u(x,t), f(x))$ where $u \in C^{2,1}_{x,t}(\bar{\Omega})$(space of two times and one time continuously differentiable functions on $\bar{\Omega}$ according to x and t respectively)and $f \in C([0,\pi])$.

Note that similar problems for the heat equation and their fractional analogues have been considered in [39], [53], [76].

3.3 Spectral properties of the perturbed Sturm-Liouville problem

Application of the Fourier method for solving problems D, N, P, AP leads to the spectral problem defined by the equation

$$y''(x) - \varepsilon y''(\pi - x) + \lambda y(x) = 0, \ \ 0 < x < \pi, \tag{3.8}$$

and one of the following boundary conditions

$$y(0) = y(\pi) = 0, \tag{3.9}$$

$$y'(0) = y'(\pi) = 0, \tag{3.10}$$

$$y(0) = y(\pi), \; y'(0) = y'(\pi), \tag{3.11}$$

$$y(0) = -y(\pi), \; y'(0) = -y'(\pi). \tag{3.12}$$

It is easy to see that the Sturm-Liouville problem for the equation (3.8) with one of the boundary conditions (3.9) - (3.12) is self-adjoint. It is known that the self-adjoint problem has real eigenvalues and their eigenfunctions form a complete orthonormal basis in $L^2(0,\pi)$ [75]. To further investigate the problems under consideration, we need to calculate the explicit form of the eigenvalues and eigenfunctions.

For $|\varepsilon| < 1$ the problem (3.8), (3.9) has the following eigenvalues

$$\lambda_{2k}^D = (1+\varepsilon)\,4k^2, \; k \in \mathbb{N},$$

$$\lambda_{2k+1}^D = (1-\varepsilon)\,(2k+1)^2, \; k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

and eigenfunctions

$$\begin{cases} y_{2k}^D = \sqrt{\frac{2}{\pi}}\sin(2kx), \; k \in \mathbb{N}, \\ y_{2k+1}^D = \sqrt{\frac{2}{\pi}}\sin(2k+1)\,x, \; k \in \mathbb{N}_0. \end{cases} \tag{3.13}$$

Similarly, the problem (3.8), (3.10) has the eigenvalues

$$\lambda_{2k+1}^N = (1+\varepsilon)\,(2k+1)^2, \; k \in \mathbb{N}_0,$$

$$\lambda_{2k}^N = (1-\varepsilon)\,4k^2, \; k \in \mathbb{N}_0,$$

and corresponding eigenfunctions

$$\begin{cases} y_0^N = \frac{1}{\sqrt{\pi}}, \\ y_{2k+1}^N = \sqrt{\frac{2}{\pi}}\cos(2k+1)\,x, \; k \in \mathbb{N}_0, \\ y_{2k}^N = \sqrt{\frac{2}{\pi}}\cos(2kx), \; k \in \mathbb{N}. \end{cases} \tag{3.14}$$

The eigenvalues of the problem (3.8), (3.11) are

$$\lambda_{2k+1}^P = (1+\varepsilon)\,4k^2, \; k \in \mathbb{N},$$

$$\lambda_{2k}^P = (1-\varepsilon)\,4k^2, \; k \in \mathbb{N}_0,$$

with the corresponding eigenfunctions

$$\begin{cases} y_0^P = \frac{1}{\sqrt{\pi}}, \\ y_{2k+1}^P = \sqrt{\frac{2}{\pi}} \sin(2kx),\ k \in \mathbb{N}, \\ y_{2k}^P = \sqrt{\frac{2}{\pi}} \cos(2kx),\ k \in \mathbb{N}. \end{cases} \tag{3.15}$$

Finally, the problem (3.8), (3.12) has the following eigenvalues

$$\lambda_{2k+1}^{AP} = (1+\varepsilon)(2k+1)^2,\ k \in \mathbb{N}_0,$$

$$\lambda_{2k}^{AP} = (1-\varepsilon)(2k+1)^2,\ k \in \mathbb{N}_0,$$

and corresponding eigenfunctions

$$\begin{cases} y_{2k+1}^{AP} = \sqrt{\frac{2}{\pi}} \cos(2k+1)x,\ k \in \mathbb{N}_0, \\ y_{2k}^{AP} = \sqrt{\frac{2}{\pi}} \sin(2k+1)x,\ k \in \mathbb{N}_0. \end{cases} \tag{3.16}$$

Lemma 3.3.1 *The systems of functions (3.13), (3.14), (3.15), (3.16) are complete and orthonormal in $L^2(0,\pi)$.*

Proof It is known (see. [72]) that the systems of (3.15) and (3.16) form a complete orthonormal system in $L^2(0,\pi)$.

It remains to prove the completeness of systems (3.13) and (3.14). We prove the completeness of the system (3.13).

The system (3.13) is complete in $L^2(0,\pi)$ if the equalities

$$\int_0^\pi f(x)\sin(2kx)dx = 0,\ k \in \mathbb{N},$$

$$\int_0^\pi f(x)\sin(2k+1)xdx = 0,\ k \in \mathbb{N}_0,$$

for $f \in L^2(0,\pi)$ lead to $f(x) = 0$ in $L^2(0,\pi)$.

We have

$$\begin{aligned} &\int_0^\pi f(x)\sin(2k+1)xdx \\ &\quad = \int_0^{\frac{\pi}{2}} f(x)\sin(2k+1)xdx + \int_{\frac{\pi}{2}}^\pi f(x)\sin(2k+1)xdx \\ &\quad = \int_0^{\frac{\pi}{2}} (f(x) - f(\pi - x))\sin(2k+1)xdx = 0. \end{aligned} \tag{3.17}$$

Then by the completeness of the system $\{\sin(2k+1)x\}_{k\in\mathbb{N}_0}$ in $L^2\left(0,\frac{\pi}{2}\right)$ [72] , we obtain $f(x) = f(\pi - x), 0 < x < \frac{\pi}{2}$.

Similarly

$$\int_0^\pi f(x)\sin(2kx)dx = \int_0^{\frac{\pi}{2}} f(x)\sin(2kx)dx + \int_{\frac{\pi}{2}}^\pi f(x)\sin(2kx)dx$$

$$= \int_0^{\frac{\pi}{2}} (f(x) + f(\pi - x))\sin(2kx)dx = 0.$$

Then by the completeness of the system $\{\sin(2kx)\}_{k\in\mathbb{N}}$ in $L^2\left(0,\frac{\pi}{2}\right)$ [72], we have $f(x) = -f(\pi - x), 0 < x < \frac{\pi}{2}$. Whereupon , $f(x) = 0$ in $L^2\left(0,\frac{\pi}{2}\right)$, and consequently $f(x) = 0$ in $L^2(0,\pi)$.

The completeness of the system (3.14) is proved similarly. □

3.4 Main results

For the considered problems D, N, P, AP, the following theorems hold true.

Theorem 3.4.1 *Let* $|\varepsilon| < 1$, $\varphi, \psi \in C^3[0,\pi]$ *and* $\varphi^{(i)}(0) = \varphi^{(i)}(\pi) = \psi^{(i)}(0) = \psi^{(i)}(\pi) = 0, i = 0, 1, 2$. *Then the solution of the problem D exists, is unique and it can be written in the form*

$$\begin{aligned} u(x,t) = {} & \varphi(x) \\ & + \sum_{k=0}^{\infty} \frac{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 t^{\alpha}\right)\right)\sin(2k+1)x}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 T^{\alpha}\right)\right)(2k+1)^2}(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}) \\ & + \sum_{k=1}^{\infty} \frac{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2 t^{\alpha}\right)\right)\sin(2kx)}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2 T^{\alpha}\right)\right)4k^2}(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}), \end{aligned}$$

$$\begin{aligned} f(x) = {} & -\varphi''(x) + \varepsilon\, \varphi''(\pi - x) \\ & + \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\left(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}\right)}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 T^{\alpha}\right)\right)}\sin(2k+1)x \\ & + \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\left(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}\right)}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2 T^{\alpha}\right)\right)}\sin(2kx), \end{aligned}$$

where

$$\varphi_{1k}^{(2)} = \left(\varphi''(x), y_{2k+1}^D\right), \quad \varphi_{2k}^{(2)} = \left(\varphi''(x), y_{2k}^D\right), \quad \psi_{1k}^{(2)} = \left(\psi''(x), y_{2k+1}^D\right), \quad \psi_{2k}^{(2)} = \left(\psi''(x), y_{2k}^D\right),$$

and $E_{\alpha,\beta}(\lambda t)$ *is the Mittag-Leffler type function:*

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}.$$

Theorem 3.4.2 *Let $\varphi, \psi \in C^3[0,\pi]$ and $\varphi^{(i)}(0) = \varphi^{(i)}(\pi) = \psi^{(i)}(0) = \psi^{(i)}(\pi) = 0$, $i = 0,1,2$. Then the solution of problem N exists, is unique and it can be written in the form*

$$
\begin{aligned}
u(x,t) = {} & \varphi(x) + \frac{t}{T}\left(\psi_0 - \varphi_0\right) \\
& + \sum_{k=1}^{\infty} \frac{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)4k^2t^{\alpha}\right)\right)\cos(2kx)}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)4k^2T^{\alpha}\right)\right)4k^2}(\psi_{1k}^{(2)} - \varphi_{1k}^{(2)}) \\
& + \sum_{k=0}^{\infty} \frac{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)(2k+1)^2t^{\alpha}\right)\right)\cos(2k+1)x}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)(2k+1)^2T^{\alpha}\right)\right)(2k+1)^2}(\psi_{2k}^{(2)} - \varphi_{2k}^{(2)}),
\end{aligned}
$$

$$
\begin{aligned}
f(x) = {} & -\varphi''(x) + \varepsilon\, \varphi''(\pi - x) \\
& + \sum_{k=1}^{\infty} \frac{(1-\varepsilon)\left(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}\right)}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)4k^2T^{\alpha}\right)\right)}\cos(2kx) \\
& + \sum_{k=0}^{\infty} \frac{(1+\varepsilon)\left(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}\right)}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)(2k+1)^2T^{\alpha}\right)\right)}\cos(2k+1)x,
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_0 &= \left(\varphi(x), y_0^N\right), \varphi_{1k}^{(2)} = \left(\varphi''(x), y_{2k}^N\right), \varphi_{2k}^{(2)} = \left(\varphi''(x), y_{2k+1}^N\right), \\
\psi_0 &= \left(\psi(x), y_0^N\right), \psi_{1k}^{(2)} = \left(\psi''(x), y_{2k}^N\right), \psi_{2k}^{(2)} = \left(\psi''(x), y_{2k+1}^N\right).
\end{aligned}
$$

Theorem 3.4.3 *Let $\varphi, \psi \in C^3[0,\pi]$ and $\varphi^{(i)}(0) = \varphi^{(i)}(\pi)$, $\psi^{(i)}(0) = \psi^{(i)}(\pi)$, $i = 0,1,2$. Then the solution of problem P exists, is unique and it can be written in the form*

$$
\begin{aligned}
u(x,t) = {} & \varphi(x) + \frac{t}{T}\left(\psi_0 - \varphi_0\right) \\
& + \sum_{k=1}^{\infty} \frac{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)4k^2t^{\alpha}\right)\right)\cos(2kx)}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)4k^2T^{\alpha}\right)\right)4k^2}(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}) \\
& + \sum_{k=1}^{\infty} \frac{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2t^{\alpha}\right)\right)\sin(2kx)}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2T^{\alpha}\right)\right)4k^2}(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}),
\end{aligned}
$$

$$\begin{aligned} f(x) = & -\varphi''(x) + \varepsilon\,\varphi''(\pi - x) \\ & + \sum_{k=1}^{\infty} \frac{(1-\varepsilon)\left(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}\right)}{\left(1 - E_{\alpha,1}\left(-(1-\varepsilon)2k^2T^\alpha\right)\right)} \cos(2kx) \\ & + \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\left(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}\right)}{\left(1 - E_{\alpha,1}\left(-(1+\varepsilon)2k^2T^\alpha\right)\right)} \sin(2kx), \end{aligned}$$

where

$$\begin{aligned} \varphi_0 = \left(\varphi(x), y_0^P\right), \varphi_{1k}^{(2)} = \left(\varphi''(x), y_{2k}^P\right), \varphi_{2k}^{(2)} = \left(\varphi''(x), y_{2k+1}^P\right), \\ \psi_0 = \left(\psi(x), y_0^P\right), \psi_{1k}^{(2)} = \left(\psi''(x), y_{2k}^P\right), \psi_{2k}^{(2)} = \left(\psi''(x), y_{2k+1}^P\right). \end{aligned}$$

Theorem 3.4.4 *Let $\varphi, \psi \in C^3[0,\pi]$ and $\varphi^{(i)}(0) = -\varphi^{(i)}(\pi)$, $\psi^{(i)}(0) = -\psi^{(i)}(\pi)$, $i = 0, 1, 2$. Then the solution of problem AP exists, is unique and it can be written in the form*

$$\begin{aligned} u(x,t) = & \varphi(x) \\ & + \sum_{k=0}^{\infty} \frac{\left(1 - E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2t^\alpha\right)\right)\cos(2k+1)x}{\left(1 - E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2T^\alpha\right)\right)(2k+1)^2} (\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}) \\ & + \sum_{k=0}^{\infty} \frac{\left(1 - E_{\alpha,1}\left(-(1+\varepsilon)(2k+1)^2t^\alpha\right)\right)\sin(2k+1)x}{\left(1 - E_{\alpha,1}\left(-(1+\varepsilon)(2k+1)^2T^\alpha\right)\right)(2k+1)^2} (\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}), \end{aligned}$$

$$\begin{aligned} f(x) = & -\varphi''(x) + \varepsilon\,\varphi''(\pi - x) \\ & + \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\left(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}\right)}{\left(1 - E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2T^\alpha\right)\right)} \cos(2k+1)x \\ & + \sum_{k=0}^{\infty} \frac{(1+\varepsilon)\left(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}\right)}{\left(1 - E_{\alpha,1}\left(-(1+\varepsilon)(2k+1)^2T^\alpha\right)\right)} \sin(2k+1)x, \end{aligned}$$

where

$$\begin{aligned} \varphi_{1k}^{(2)} = \left(\varphi''(x), y_{2k+1}^{AP}\right), \varphi_{2k}^{(2)} = \left(\varphi''(x), y_{2k}^{AP}\right), \\ \psi_{1k}^{(2)} = \left(\psi''(x), y_{2k+1}^{AP}\right), \psi_{2k}^{(2)} = \left(\psi''(x), y_{2k}^{AP}\right). \end{aligned}$$

3.5 Proof of the existence of the solution of problem D

We give the full proof for problem D. The existence of the solution of problems P, N and AP are proved analogously.

As the eigenfunctions system (3.13) of problem D forms an orthonormal basis in $L^2(0,\pi)$ (this follows from the self-adjoint problem (3.8), (3.9)), the functions $u(x,t)$ and $f(x)$ can be expanded as follows

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)\sin(2k+1)x + \sum_{k=1}^{\infty} v_k(t)\sin(2kx), \tag{3.18}$$

$$f(x) = \sum_{k=0}^{\infty} f_k^1 \sin(2k+1)x + \sum_{k=1}^{\infty} f_k^2 \sin(2kx), \tag{3.19}$$

where $f_k^1, f_k^2, u_k(t), v_k(t)$ are unknown. Substituting (3.18) and (3.19) into (3.1), we obtain the following equation for the functions $u_k(t), v_k(t)$ and the constants f_k^1, f_k^2 :

$$\mathcal{D}^{\alpha} u_k(t) + (1-\varepsilon)(2k+1)^2 u_k(t) = f_k^1,$$

$$\mathcal{D}^{\alpha} v_k(t) + (1+\varepsilon)\,4k^2 v_k(t) = f_k^2.$$

Solving these equations [51] we obtain

$$u_k(t) = \frac{f_k^1}{(1-\varepsilon)(2k+1)^2} + C_{1k}E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 t^{\alpha}\right),$$

$$v_k(t) = \frac{f_k^2}{(1+\varepsilon)k^2} + C_{2k}E_{\alpha,1}\left(-(1+\varepsilon)\,4k^2 t^{\alpha}\right),$$

where the constants $C_{1k}, C_{2k}, f_k^1, f_k^2$ are unknown. To find these constants, we use conditions (3.2). Let $\varphi_{ik}, \psi_{ik}, i = 1,2$ be the coefficients of the expansions of $\varphi(x)$ and $\psi(x)$

$$\varphi_{1k} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \varphi(x)\sin(2k+1)x\,dx,$$

$$\varphi_{2k} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \varphi(x)\sin(2kx)\,dx,$$

$$\psi_{1k} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \psi(x)\sin(2k+1)x\,dx,$$

$$\psi_{2k} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \psi(x)\sin(2kx)\,dx.$$

We first find C_{1k} :

$$u_k(0) = \frac{f_k^1}{(1-\varepsilon)(2k+1)^2} + C_{1k} = \varphi_{1k},$$

$$u_k(T) = \frac{f_k^1}{(1-\varepsilon)(2k+1)^2} + C_{1k}E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 T^\alpha\right) = \psi_{1k},$$

$$\varphi_{1k} - C_{1k} + C_{1k}E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 T^\alpha\right) = \psi_{1k}.$$

Then

$$C_{1k} = \frac{\varphi_{1k} - \psi_{1k}}{1 - E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 T^\alpha\right)}.$$

The constant f_k^1 is represented as

$$f_k^1 = (1-\varepsilon)(2k+1)^2 \varphi_{1k} - (1-\varepsilon)(2k+1)^2 C_{1k}.$$

Now we find C_{2k} :

$$v_k(0) = \frac{f_k^2}{(1+\varepsilon)4k^2} + C_{2k} = \varphi_{2k},$$

$$v_k(T) = \frac{f_k^2}{(1+\varepsilon)4k^2} + C_{2k}e^{-(1+\varepsilon)4k^2T} = \psi_{2k},$$

$$\varphi_{2k} - C_{2k} + C_{2k}e^{-(1+\varepsilon)4k^2T} = \psi_{2k}.$$

Then

$$C_{2k} = \frac{\varphi_{2k} - \psi_{2k}}{1 - E_{\alpha,1}\left(-(1+\varepsilon)4k^2T^\alpha\right)}.$$

For the constant f_k^2 , we found :

$$f_k^2 = (1+\varepsilon)4k^2\varphi_{2k} - (1+\varepsilon)4k^2C_{2k}.$$

Substituting $u_k(t)$, $v_k(t)$, f_k^1, f_k^2 into (3.18) and (3.19) we find

$$\begin{aligned} u(x,t) = \; & \varphi(x) \\ & + \sum_{k=0}^{\infty} C_{1k}\left(E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2 t^\alpha\right) - 1\right)\sin(2k+1)x \\ & + \sum_{k=1}^{\infty} C_{2k}\left(E_{\alpha,1}\left(-(1+\varepsilon)4k^2t^\alpha\right) - 1\right)\sin(2kx). \end{aligned}$$

Suppose that

$$\varphi^{(i)}(0) = 0\,,\; \varphi^{(i)}(\pi) = 0\,,\; i = 0,1,2,$$

$$\psi^{(i)}(0) = 0\,,\; \psi^{(i)}(\pi) = 0\,,\; i = 0,1,2,$$

then

$$C_{1k} = \frac{\varphi_{1k}-\psi_{1k}}{1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2T^\alpha\right)} = -\frac{\varphi_{1k}^{(2)}-\psi_{1k}^{(2)}}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2T^\alpha\right)\right)(2k+1)^2}.$$

Similarly,

$$C_{2k} = -\frac{\varphi_{2k}^{(2)}-\psi_{2k}^{(2)}}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2T^\alpha\right)\right)4k^2}.$$

Then

$$\begin{aligned} u(x,t) = \; & \varphi(x) \\ & + \sum_{k=0}^{\infty} \frac{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2t^\alpha\right)\right)\sin(2k+1)x}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2T^\alpha\right)\right)(2k+1)^2}(\varphi_{1k}^{(2)} - \psi_{1k}^{(2)}) \\ & + \sum_{k=1}^{\infty} \frac{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2t^\alpha\right)\right)\sin(2kx)}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2T^\alpha\right)\right)4k^2}(\varphi_{2k}^{(2)} - \psi_{2k}^{(2)}). \end{aligned}$$

Similarly,

$$\begin{aligned} f(x) = \; & -\varphi''(x) + \varepsilon\, \varphi''(\pi - x) \\ & + \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\left(\varphi_{1k}^{(2)}-\psi_{1k}^{(2)}\right)}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2T^\alpha\right)\right)}\sin(2k+1)x \\ & + \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\left(\varphi_{2k}^{(2)}-\psi_{2k}^{(2)}\right)}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2T^\alpha\right)\right)}\sin(2kx). \end{aligned}$$

Now for the convergence of the series, we have the following estimate

$$\begin{aligned} |u(x,t)| \leq \; & C\,|\varphi(x)| \\ & + C\sum_{k=0}^{\infty} \frac{\left|\varphi_{1k}^{(2)}\right|+\left|\psi_{1k}^{(2)}\right|}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2T^\alpha\right)\right)(2k+1)^2} \\ & + C\sum_{k=1}^{\infty} \frac{\left|\varphi_{1k}^{(2)}\right|+\left|\psi_{1k}^{(2)}\right|}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2T^\alpha\right)\right)k^2}, \quad C = \text{const}. \end{aligned} \tag{3.20}$$

Similarly for $f(x)$ we obtain the estimate

$$\begin{aligned} |f(x)| \leq \; & C\,|\varphi(x)| + C\,|\varphi(-x)| \\ & + C\sum_{k=0}^{\infty} \frac{\left|\varphi_{1k}^{(2)}\right|+\left|\psi_{1k}^{(2)}\right|}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2T^\alpha\right)\right)} \\ & + C\sum_{k=1}^{\infty} \frac{\left|\varphi_{1k}^{(2)}\right|+\left|\psi_{1k}^{(2)}\right|}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2T^\alpha\right)\right)}, \quad C = \text{const}. \end{aligned} \tag{3.21}$$

Since by hypotheses of Theorem 3.4.1, the functions $\varphi^{(2)}$, $\psi^{(2)}$ are continuous on $[0,\pi]$, then by the Bessel inequality for the trigonometric series the following series converge:

$$\sum_{k=0}^{\infty}\left|\varphi_{ik}^{(2)}\right|^{2}\leq C\left\|\varphi^{(2)}(x)\right\|_{L_2(-\pi,\pi)}^{2},i=1,2, \tag{3.22}$$

$$\sum_{k=1}^{\infty}\left|\psi_{ik}^{(2)}\right|^{2}\leq C\left\|\psi^{(2)}(x)\right\|_{L_2(-\pi,\pi)}^{2},i=1,2, \tag{3.23}$$

which implies the boundedness of the set

$$\left\{\varphi_{1k}^{(2)},\psi_{1k}^{(2)},\varphi_{2k}^{(2)},\psi_{2k}^{(2)},k=1,2,...\right\}.$$

Therefore, by the Weierstrass M-test (see[58]), series (3.20) and (3.21) converge absolutely and uniformly in the region $\bar{\Omega}$.

Now we show the possibility of termwise differentiation of the series (3.20) twice in the variable x and once in the variable t. For this purpose, we prove that the obtained term by term differentiation of the series converge absolutely and uniformly on $\bar{\Omega}$. Given the estimates (3.22) and (3.23) we have

$$\begin{aligned}|u_{xx}(x,t)|\leq\ & C\left|\varphi''(x)\right| \\ & + C\sum_{k=0}^{\infty}\frac{\left|\varphi_{1k}^{(2)}\right|+\left|\psi_{1k}^{(2)}\right|}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2T^{\alpha}\right)\right)} \\ & + C\sum_{k=1}^{\infty}\frac{\left|\varphi_{1k}^{(2)}\right|+\left|\psi_{1k}^{(2)}\right|}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2T^{\alpha}\right)\right)}<\infty,\end{aligned}$$

$$\begin{aligned}\left|\mathcal{D}_t^{\alpha}u(x,t)\right|\leq\ & C\sum_{k=0}^{\infty}\frac{\left|\varphi_{1k}^{(2)}\right|+\left|\psi_{1k}^{(2)}\right|}{\left(1-E_{\alpha,1}\left(-(1-\varepsilon)(2k+1)^2T^{\alpha}\right)\right)} \\ & + C\sum_{k=1}^{\infty}\frac{\left|\varphi_{1k}^{(2)}\right|+\left|\psi_{1k}^{(2)}\right|}{\left(1-E_{\alpha,1}\left(-(1+\varepsilon)4k^2T^{\alpha}\right)\right)}<\infty.\end{aligned}$$

Hence the obtained solution satisfies the equation (3.1) point-wise; by construction, it satisfies the conditions (3.2)-(3.4).

3.6 Proof of the uniqueness of the solution of problem P

Suppose that there are two solutions $\{u_1(x,t),f_1(x)\}$ and $\{u_2(x,t),f_2(x)\}$ of problem P. Denote

$$u(x,t)=u_1(x,t)-u_2(x,t)$$

and

$$f(x) = f_1(x) - f_2(x).$$

Then the functions $u(x,t)$ and $f(x)$ satisfy (3.1)- (3.3) and periodic conditions(3.6).

Let

$$u_0(t) = \frac{1}{\sqrt{\pi}} \int_0^{\pi} u(x,t)\, dx, \tag{3.24}$$

$$u_{1k}(t) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} u(x,t) \cos(2kx) dx, k \in \mathbb{N}, \tag{3.25}$$

$$u_{2k}(t) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} u(x,t) \sin(2kx) dx, k \in \mathbb{N}, \tag{3.26}$$

$$f_0 = \frac{1}{\sqrt{\pi}} \int_0^{\pi} f(x) dx, \tag{3.27}$$

$$f_{1k} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x) \cos(2kx) dx, k \in \mathbb{N}, \tag{3.28}$$

$$f_{2k} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x) \sin(2kx) dx, k \in \mathbb{N}. \tag{3.29}$$

Applying the operator $\mathcal{D}^{\alpha}$ to the equation (3.24) we have

$$\mathcal{D}^{\alpha} u_0(t) = \frac{1}{\sqrt{\pi}} \int_0^{\pi} \mathcal{D}_t^{\alpha} u(x,t)\, dx = \frac{1}{\sqrt{\pi}} \int_0^{\pi} \left(u_{xx}(x,t) - \varepsilon u_{xx}(\pi - x, t)\right) dx + f_0.$$

Integrating by parts and taking into account the homogeneous conditions (3.2) and (3.6) , we obtain

$$\mathcal{D}^{\alpha} u_0(t) = f_0\,,\ u(0) = 0\,,\ u(T) = 0.$$

Consequently, $f_0 = 0, u_0(t) \equiv 0$.

In a similar way for the functions (3.25), (3.26), (3.27), (3.28), (3.29) one proves that $f_{1k} = 0, f_{2k} = 0, u_{1k}(t) \equiv 0, u_{2k}(t) \equiv 0$.

Further, by the completeness of the system (3.15) in $L^2(0,\pi)$ we obtain $f(t) \equiv 0, u(x,t) \equiv 0, 0 \le t \le T, 0 \le x \le \pi$.

Uniqueness of the solution of problem P is proved.

The uniqueness of the solution of problems D, N and AP can be proved similarly.

4 Inverse source problems for a wave equation with involution

Abstract

In this chapter, a class of inverse problems for a wave equation with involution is considered for cases of two different boundary conditions, namely, Dirichlet and Neumann boundary conditions. The existence and uniqueness of solutions of these problems are proved. The solutions are obtained in the form of series expansion using a set of appropriate orthogonal basises for each problem. Convergence of the obtained solutions is also justified.

4.1 Introduction

In many physical problems, determination of coefficients or right-hand side according to some available information (the source term, in case of a wave equation) in a differential equation is required; these problems are known as inverse problems. These kinds of problems are ill-posed in the sense of Hadamard.

The purpose of this chapter is to study inverse problems for a nonlocal wave equation with involution of space variable x. We consider the nonlocal wave equation

$$u_{tt}(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(\pi - x, t) = f(x), \tag{4.1}$$

for $(x,t) \in \Omega = \{0 < x < \pi,\ 0 < t < T\}$, where ε is a real number.

Wide opportunities for applying equations with deviating argument in mathematical models have increased the interest of the study of new problems for partial differential equations [77], [43], [91].

Equations with a deviation of arguments of alternating character has special interest in differential equations with deviating arguments. Such deviations include the so-called deviation of involution type [17]. To describe them, let Γ be an interval in $\mathbb{R}$ and let $X \in \Gamma$ be a real variable.

The homeomorphism

$$\alpha^2(X) = \alpha(\alpha(X)) = X$$

is called a Carleman shift (deviation of involution) [23].

Equations containing Carleman shift are equations with an alternating deviation (at $X^* < X$ being equations with advanced, and at $X^* > X$ being equations with delay, where X^* is a fixed point of the mapping $\alpha(X)$).

Concerning the inverse problems for partial differential equations with involutions, some recent works have been implemented in [3, 5, 52, 55, 104].

4.2 Statement of problems

The chapter is devoted to two inverse problems concerning the wave equation with a perturbative term of involution type with respect to the space variable. We obtain existence and uniqueness results for these problems, based on the Fourier method.

Problem D. Find a couple of functions $(u(x,t), f(x))$ satisfying equation (4.1), under the conditions

$$u(x,0) = 0,\ x \in [0,\pi], \tag{4.2}$$

$$u(x,T) = \psi(x),\ x \in [0,\pi], \tag{4.3}$$

$$u_t(x,0) = 0,\ x \in [0,\pi], \tag{4.4}$$

and the homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0\,,\ t \in [0,T]\,, \tag{4.5}$$

where $\psi(x)$ is a given sufficiently smooth function.

Problem N. Find the couple of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (4.1), conditions (4.2), (4.3), (4.4) and the homogeneous Neumann boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0\,,\ t \in [0,T]\,. \tag{4.6}$$

A regular solution of the problems D and N is the pair of functions $(u(x,t), f(x))$, where $u \in C^2(\bar{\Omega})$ and $f \in C([0,\pi])$.

4.3 Spectral properties of the perturbed Sturm-Liouville problem

Application of the Fourier method for solving problems D and N leads to a spectral problem defined by the equation

$$y''(x) - \varepsilon y''(\pi - x) + \lambda y(x) = 0, 0 < x < \pi, \tag{4.7}$$

and one of the following boundary conditions

$$y(0) = y(\pi) = 0, \tag{4.8}$$

$$y'(0) = y'(\pi) = 0. \tag{4.9}$$

It is easy to see that the Sturm-Liouville problem for the equation (4.7) with one of the boundary conditions (4.8) and (4.9) is self-adjoint. It is known that the self-adjoint problem has real eigenvalues and their eigenfunctions form a complete orthonormal basis in $L^2(0,\pi)$ [75]. To further investigate the problems under consideration, we need to calculate the explicit form of the eigenvalues and eigenfunctions.

It is easy to show that for $|\varepsilon| < 1$ the problem (4.7), (4.8) has the following eigenvalues

$$\lambda_{2k}^D = (1+\varepsilon)\, 4k^2,\ k \in \mathbb{N},$$
$$\lambda_{2k+1}^D = (1-\varepsilon)(2k+1)^2\,,\ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

and eigenfunctions

$$\begin{cases} y_{2k}^D = \sqrt{\frac{2}{\pi}} \sin 2kx,\ k \in \mathbb{N}, \\ \\ y_{2k+1}^D = \sqrt{\frac{2}{\pi}} \sin(2k+1)\,x,\ k \in \mathbb{N}_0. \end{cases} \tag{4.10}$$

Similarly, the problem (4.7), (4.9) has the eigenvalues

$$\lambda_{2k+1}^{N} = (1+\varepsilon)\,(2k+1)^2\,,\ k \in \mathbb{N}_0,$$
$$\lambda_{2k}^{N} = (1-\varepsilon)\,4k^2,\ k \in \mathbb{N}_0,$$

and corresponding eigenfunctions

$$\begin{cases} y_0^N = \frac{1}{\sqrt{\pi}}, \\ y_{2k+1}^N = \sqrt{\frac{2}{\pi}} \cos\,(2k+1)\,x,\ k \in \mathbb{N}_0, \\ y_{2k}^N = \sqrt{\frac{2}{\pi}} \cos 2kx,\ k \in \mathbb{N}. \end{cases} \tag{4.11}$$

The following lemma is proved in [104]

Lemma 1 *The systems of functions (4.10) and (4.11) are complete and orthonormal in* $L^2\,(0,\pi)$.

4.4 Main results

For the considered problems D and N, the following theorems are valid.

Theorem 1 *Let* $|\varepsilon| < 1$, $\psi \in C^4\,[0,\pi]$ *and* $\psi^{(i)}\,(0) = \psi^{(i)}\,(\pi) = 0, i = 0,1,2,3,4$. *If* $\cos\sqrt{1-\varepsilon}\,(2k+1)\,T < \delta_1 < 1$ *and* $\cos\sqrt{1+\varepsilon}2kT < \delta_2 < 1$, *then the solution of problem D exists, is unique and it can be written in the form*

$$u\,(x,t) = \sum_{k=0}^{\infty} \frac{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,t\right)\sin\,(2k+1)\,x}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)\,(2k+1)^4}\psi_{2k+1}^4 + \sum_{k=1}^{\infty} \frac{\left(1-\cos\sqrt{1+\varepsilon}2kt\right)\sin 2kx}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)16k^4}\psi_{2k}^4, \tag{4.12}$$

$$f\,(x) = \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\,\psi_{2k+1}^4}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)\,(2k+1)^2}\sin\,(2k+1)\,x + \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\,\psi_{2k}^4}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)4k^2}\sin 2kx, \tag{4.13}$$

where $\psi_{2k+1}^{(4)} = \left(\psi^{(4)}\,(x)\,,\, y_{2k+1}^D\right)$ *and* $\psi_{2k}^{(4)} = \left(\psi^{(4)}\,(x)\,,\, y_{2k}^D\right)$.

Theorem 2 *Let* $|\varepsilon| < 1$, $\psi \in C^4\,[0,\pi]$ *and* $\psi^{(i)}\,(0) = \psi^{(i)}\,(\pi) = 0, i = 0,1,2,3,4$. *If* $\cos\sqrt{1-\varepsilon}\,(2k+1)\,T < \sigma_1 < 1$ *and* $\cos\sqrt{1+\varepsilon}2kT < \sigma_2 < 1$, *then the solution of problem N exists, is unique and it can be written in the form*

$$u(x,t) = \sum_{k=0}^{\infty} \frac{\left(1-\cos\sqrt{1+\varepsilon}\,(2k+1)\,t\right)\cos(2k+1)\,x}{\left(1-\cos\sqrt{1+\varepsilon}\,(2k+1)\,T\right)(2k+1)^4}\psi_{2k+1}^4 + \sum_{k=1}^{\infty} \frac{\left(1-\cos\sqrt{1-\varepsilon}2kt\right)\cos 2kx}{\left(1-\cos\sqrt{1-\varepsilon}2kT\right)16k^4}\psi_{2k}^4, \quad (4.14)$$

$$f(x) = \sum_{k=0}^{\infty} \frac{(1+\varepsilon)\,\psi_{2k+1}^4}{\left(1-\cos\sqrt{1+\varepsilon}\,(2k+1)\,T\right)(2k+1)^2}\cos(2k+1)\,x + \sum_{k=1}^{\infty} \frac{(1-\varepsilon)\,\psi_{2k}^4}{\left(1-\cos\sqrt{1-\varepsilon}2kT\right)4k^2}\cos 2kx, \quad (4.15)$$

where $\psi_{2k+1}^{(4)} = \left(\psi^{(4)}(x), y_{2k+1}^N\right)$ *and* $\psi_{2k}^{(4)} = \left(\psi^{(4)}(x), y_{2k}^N\right)$.

4.5 Proof of the uniqueness of the solution

Suppose that there are two solutions $\{u_1(x,t), f_1(x)\}$ and $\{u_2(x,t), f_2(x)\}$ of the problem N. Denote

$$u(x,t) = u_1(x,t) - u_2(x,t)$$

and

$$f(x) = f_1(x) - f_2(x).$$

Then the functions $u(x,t)$ and $f(x)$ satisfy (4.1)- (4.4)and homogenious conditions (4.6).

Let

$$u_0(t) = \frac{1}{\sqrt{\pi}} \int_0^{\pi} u(x,t)\,dx, \quad (4.16)$$

$$u_{2k}(t) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} u(x,t)\cos 2kxdx, k \in \mathbb{N}, \quad (4.17)$$

$$u_{2k+1}(t) = \sqrt{\frac{2}{\pi}} \int_0^{\pi} u(x,t)\cos(2k+1)xdx, k \in \mathbb{N}_0, \quad (4.18)$$

$$f_0 = \frac{1}{\sqrt{\pi}} \int_0^{\pi} f(x)dx, \quad (4.19)$$

$$f_{2k} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x)\cos 2kxdx, k \in \mathbb{N}, \quad (4.20)$$

$$f_{2k+1} = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x) \cos(2k+1)x dx, k \in \mathbb{N}. \tag{4.21}$$

Applying the operator $\frac{\partial^2}{\partial t^2}$ to the equation (4.16) we have

$$u_0''(t) = \frac{1}{\sqrt{\pi}} \int_0^{\pi} u_{tt}(x,t)\, dx = \frac{1}{\sqrt{\pi}} \int_0^{\pi} \left(u_{xx}(x,t) - \varepsilon u_{xx}(\pi - x, t)\right) dx + f_0.$$

Integrating by parts and taking into account the homogeneous conditions (4.2) and (4.6) , we obtain

$$u_0''(t) = f_0,$$
$$u(0) = 0\,,\ u(T) = 0,\ u'(0) = 0.$$

Hence it is easy to get $f_0 = 0,\ \ u_0(t) \equiv 0.$

In a similar way for the functions (4.17), (4.18), (4.19), (4.20), (4.21) it is easy to prove that

$$f_{2k} = 0,\ \ f_{2k+1} = 0,\ \ u_{2k}(t) \equiv 0,\ \ u_{2k+1}(t) \equiv 0.$$

Further, by the completeness of the system (4.10) in $L^2(0,\pi)$ we obtain

$$f(x) \equiv 0, u(x,t) \equiv 0,\ \ 0 \le t \le T,\ \ 0 \le x \le \pi.$$

The uniqueness of the solution of the problem N is proved.

The uniqueness of the solution of the problem D can be proved similarly.

4.6 Proof of the existence of the solution

We give the full proof for the problem D. The existence of the solution of the problem N is proved analogously.

As the eigenfunctions system (4.10) of the problem D forms an orthonormal basis in $L^2(0,\pi)$ (this follows from the self-adjoint problem (4.7), (4.8)), the functions $u(x,t)$ and $f(x)$ can be expanded as follows

$$u(x,t) = \sum_{k=0}^{\infty} u_{2k+1}(t) \sin(2k+1)x + \sum_{k=1}^{\infty} u_{2k}(t) \sin 2kx, \tag{4.22}$$

$$f(x) = \sum_{k=0}^{\infty} f_{2k+1} \sin(2k+1)x + \sum_{k=1}^{\infty} f_{2k} \sin 2kx, \tag{4.23}$$

where $f_{2k+1},\ \ f_{2k},\ \ u_{2k+1}(t),\ \ u_{2k}(t)$ are unknown. Substituting (4.22) and (4.23) into (4.1), we obtain the following equation for the functions $u_{2k+1}(t),\ \ u_{2k}(t)$ and the constants $f_{2k+1},\ f_{2k}$:

$$u_{2k+1}''(t) + (1-\varepsilon)(2k+1)^2 u_{2k+1}(t) = f_{2k+1},$$
$$u_{2k}''(t) + (1+\varepsilon) 4k^2 u_{2k}(t) = f_{2k}.$$

Solving these equations [51], we obtain

$$u_{2k+1}(t) = \frac{f_{2k+1}}{(1-\varepsilon)(2k+1)^2} + C_{1k}\cos\sqrt{1-\varepsilon}\,(2k+1)\,t + C_{2k}\sin\sqrt{1-\varepsilon}\,(2k+1)\,t,$$

$$u_{2k}(t) = \frac{f_{2k}}{(1+\varepsilon)\,4k^2} + D_{1k}\cos\sqrt{1+\varepsilon}2kt + D_{2k}\sin\sqrt{1+\varepsilon}2kt,$$

where the constants C_{1k}, C_{2k}, D_{1k}, D_{2k}, f_{2k+1}, f_{2k} are unknown. To find these constants, we use the conditions (4.2). Let ψ_{2k}, ψ_{2k+1} be the coefficients of the expansions of $\psi(x)$

$$\psi_{2k+1} = \sqrt{\frac{2}{\pi}}\int_0^\pi \psi(x)\sin(2k+1)\,xdx,$$

$$\psi_{2k} = \sqrt{\frac{2}{\pi}}\int_0^\pi \psi(x)\sin 2kxdx.$$

We first find C_{1k}, C_{2k} :

$$u_{2k+1}(0) = \frac{f_{2k+1}}{(1-\varepsilon)(2k+1)^2} + C_{1k} = 0,$$
$$u'_{2k+1}(0) = C_{2k} = 0,$$
$$u_{2k+1}(T) = \frac{f_{2k+1}}{(1-\varepsilon)(2k+1)^2}\left(1 - \cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right) = \psi_{2k+1}.$$

The constant f_{2k+1} is represented as

$$f_{2k+1} = \frac{(1-\varepsilon)(2k+1)^2\,\psi_{2k+1}}{1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T}.$$

Now we find D_{1k}, D_{2k} :

$$u_{2k}(0) = \frac{f_{2k}}{(1+\varepsilon)\,4k^2} + D_{1k} = 0,$$
$$u'_{2k}(0) = D_{2k} = 0,$$
$$u_{2k}(T) = \frac{f_{2k}}{(1+\varepsilon)\,4k^2}\left(1-\cos\sqrt{1+\varepsilon}2kT\right) = \psi_{2k}.$$

For the constant f_{2k}, we find:

$$f_{2k} = \frac{(1+\varepsilon)\,4k^2\psi_{2k}}{1-\cos\sqrt{1+\varepsilon}2kT}.$$

Substituting $u_{2k}(t)$, $u_{2k+1}(t)$, f_{2k}, f_{2k+1} into (4.22) and (4.23), we find

$$\begin{aligned} u(x,t) = &\sum_{k=0}^{\infty} \frac{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,t\right)\sin(2k+1)\,x}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)} \psi_{2k+1} \\ &+ \sum_{k=1}^{\infty} \frac{\left(1-\cos\sqrt{1+\varepsilon}2kt\right)\sin 2kx}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)} \psi_{2k}. \end{aligned}$$

Suppose that

$$\psi^{(i)}(0)=0\,,\ \psi^{(i)}(\pi)=0\,,\ i=0,1,2,3,4,$$

then

$$\begin{aligned} \psi_{2k+1} &= \frac{1}{(2k+1)^4}\psi_{2k+1}^{(4)}, \\ \psi_{2k} &= \frac{1}{16k^4}\psi_{2k}^{(4)}. \end{aligned}$$

Then we have (4.12).

Similarly,

$$\begin{aligned} f(x) = &\sum_{k=0}^{\infty} \frac{(1-\varepsilon)\,\psi_{2k+1}^{4}}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^2} \sin(2k+1)\,x \\ &+ \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\,\psi_{2k}^{4}}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)4k^2} \sin 2kx. \end{aligned}$$

Now for the convergence of the series, we have the following estimate

$$\begin{aligned} |u(x,t)| \le &\sum_{k=0}^{\infty} \frac{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,t\right)}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^4} |\psi_{2k+1}^{(4)}| \\ &+ \sum_{k=1}^{\infty} \frac{\left(1-\cos\sqrt{1+\varepsilon}2kt\right)}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)16k^4} |\psi_{2k}^{(4)}| \\ &\le C\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}|\psi_{2k+1}^{(4)}| + C\sum_{k=1}^{\infty}\frac{1}{16k^4}|\psi_{2k}^{(4)}| < \infty. \end{aligned} \tag{4.24}$$

Similarly for $f(x)$ we obtain the estimate

$$\begin{aligned} |f(x)| \le \sum_{k=0}^{\infty} &\frac{(1-\varepsilon)\,|\psi_{2k+1}^{(4)}|}{\left(1-\cos\sqrt{1-\varepsilon}\,(2k+1)\,T\right)(2k+1)^2} + \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\,|\psi_{2k}^{(4)}|}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)4k^2} \\ &\le C\sum_{k=0}^{\infty} \frac{|\psi_{2k+1}^{(4)}|}{(2k+1)^2} + C\sum_{k=1}^{\infty} \frac{|\psi_{2k}^{(4)}|}{4k^2}. \end{aligned} \tag{4.25}$$

Since by hypotheses of Theorem 1, the function $\psi^{(4)}$ is continuous on $[0,\pi]$, then by the Bessel inequality for the trigonometric series the following series converge:

$$\sum_{k=1}^{\infty} \left|\psi_{2k}^{(4)}\right|^2 \le C\left\|\psi^{(4)}(x)\right\|_{L_2(0,\pi)}^2, \tag{4.26}$$

$$\sum_{k=0}^{\infty}\left|\psi_{2k+1}^{(4)}\right|^{2} \leq C\left\|\psi^{(4)}\left(x\right)\right\|_{L_2(0,\pi)}^{2}, \tag{4.27}$$

which implies the boundedness of the set

$$\left\{\psi_{2k}^{(4)}\right\}_{k=1}^{\infty},\left\{\psi_{2k+1}^{(4)}\right\}_{k=0}^{\infty}.$$

Therefore, by the Weierstrass M-test (see[58]), the series (4.24) and (4.25) converge absolutely and uniformly in the domain $\bar{\Omega}$.

Now we show the possibility of termwise differentiation of the series (4.24) twice in the variable x and twice in the variable t. For this purpose, we prove that the series obtained by means of term by term differentiation converge absolutely and uniformly on $\bar{\Omega}$. Given the estimates (4.26) and (4.27) we have

$$\begin{aligned}
|u_{xx}\left(x,t\right)| &\leq \sum_{k=0}^{\infty}\frac{\left(1-\cos\sqrt{1-\varepsilon}\left(2k+1\right)t\right)}{\left(1-\cos\sqrt{1-\varepsilon}\left(2k+1\right)T\right)\left(2k+1\right)^{2}}|\psi_{2k+1}^{(4)}| \\
&+\sum_{k=1}^{\infty}\frac{\left(1-\cos\sqrt{1+\varepsilon}2kt\right)}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)4k^{2}}|\psi_{2k}^{(4)}| \\
&\leq C\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2}}|\psi_{2k+1}^{(4)}|+C\sum_{k=1}^{\infty}\frac{1}{4k^{2}}|\psi_{2k}^{(4)}|<\infty,
\end{aligned}$$

$$\begin{aligned}
|u_{tt}\left(x,t\right)| &\leq \sqrt{1-\varepsilon}\sum_{k=0}^{\infty}\frac{\left(|\sin\sqrt{1-\varepsilon}\left(2k+1\right)t|\right)}{\left(1-\cos\sqrt{1-\varepsilon}\left(2k+1\right)T\right)\left(2k+1\right)^{2}}\left|\psi_{2k+1}^{(4)}\right| \\
&+\sqrt{1+\varepsilon}\sum_{k=1}^{\infty}\frac{\left|\sin\sqrt{1+\varepsilon}2kt\right|}{\left(1-\cos\sqrt{1+\varepsilon}2kT\right)4k^{2}}|\psi_{2k}^{(4)}| \\
&\leq C\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2}}|\psi_{2k+1}^{(4)}|+C\sum_{k=1}^{\infty}\frac{1}{4k^{2}}|\psi_{2k}^{(4)}|<\infty.
\end{aligned}$$

5 On a nonlinear boundary-layer problem for the fractional Blasius type equation

Abstract

In this chapter, we consider a non-linear sequential differential equation with Caputo fractional derivative of Blasius type and we reduce the problem to the equivalent nonlinear integral equation. We prove the complete continuity of the nonlinear integral operator. The theorem on the existence of a solution of the problem for the Blasius equation of fractional order is also proved.

5.1 Introduction

Various fields of science and engineering deal with dynamical systems, which can be described by fractional-order equations. Recently, many authors have studied fractional-order differential equations from two aspects: the theoretical aspects of existence and uniqueness of solutions and the analytic and numerical methods for finding solutions. The interest in the study of fractional order differential equations lies in the fact that fractional-order models in some situations are found to be more accurate than the classical integer-order models, that is, there are more degrees of freedom in the fractional-order models.

It is well known that in fluid mechanics, the problems are mostly described by systems of partial differential equations (PDEs). If somehow, a system can be reduced to a single ordinary differential equation (ODE), this constitutes a considerable mathematical simplification of the problem. For this goal, one of the approaches is based on the introduction of new variables having the form of dimensionless combinations of the initially given physical variables. Therefore, if the number of independent variables can be reduced, then PDEs can be replaced by ODEs. In the problem of the modelling of boundary layer, this is sometimes possible, and in some cases, the system of PDEs reduces to a system involving a third order differential equation of the form

$$f''' + ff'' + g\left(f'\right) = 0, \tag{5.1}$$

where $g : (-\infty, \infty) \longrightarrow (-\infty, \infty)$ is some function. Notice that equation (5.1) with $g(x) = \beta\left(x^2 - 1\right)$ was first introduced by Falkner and Skan in their classical work [34] and for this reason is called the Falkner-Skan equation.

The most famous example for these types of equations is the Blasius equation:

$$f''' + ff'' = 0, \tag{5.2}$$

which corresponds to $g(x) = 0$ and arises in the study of the laminar boundary layer on a flat plate. For more information see Brighi [13] and the references therein.

It is well known [12] that the Blasius equation (5.2) with conditions:

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1$$

has a unique solution. Note also that in [34] the author proves some important results concerning to the so-called subsolutions and supersolutions of the Blasius equation (5.2).

In this chapter, we consider a non-linear sequential differential equation with Caputo fractional derivative. We reduce the problem to the equivalent nonlinear integral equation and we prove the complete continuity of the nonlinear integral operator. A theorem on the existence of a unique solution of the problem for the non-linear differential equation of fractional order is formulated.

5.2 Some properties of fractional operators

In this section, we state some basic properties of fractional differential operators.

Various properties of fractional sequential operators were studied in [51, 83, 38, 11].

Property 5.2.1 *([51], P.73) If $f \in L^1([a,b])$ and $\alpha > 0, \beta > 0$, then the following equality holds*

$$I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t).$$

Property 5.2.2 *([51], P.96) If $f \in L^1([a,b])$ and $f' * K_{1-\alpha} \in L^1([a,b])$, then*

$$I^{\alpha} D_*^{\alpha} f(t) = f(t) - f(a),$$

almost everywhere on $[a,b]$.

5.3 Statement of the problem and main results

Consider the boundary value problem for the nonlinear fractional differential equation of Blasius type

$$\frac{d}{dt}\mathcal{D}_*^{2\alpha} x\left(t\right) + \mathcal{M}\left(x\left(t\right), t\right) \mathcal{D}_*^{2\alpha} x\left(t\right) = 0, \;\; a < t < b, \tag{5.3}$$

with boundary conditions

$$x\left(a\right) = \varphi_1, \;\; D_*^{\alpha} x\left(a\right) = \varphi_2, \;\; x\left(b\right) = \varphi_3, \tag{5.4}$$

where $\alpha \in \left(\frac{1}{2}, 1\right)$ and φ_1, φ_2, φ_3 are given real numbers. Note that when $\alpha = 1$, problem (5.3) - (5.4) is met in boundary layer theory in fluid mechanics and polymer theory [40], [62], [30], [42], [100], [105]. Note also that various problems for nonlinear differential equations of fractional order are investigated in [2], [28], [61], [19], [63]. The recent surge in developing the theory of fractional differential equations has motivated the present work.

Condition (*). *Let $\mathcal{M}\left(x,t\right)$ be defined and continuous in the domain*

$$G = \left\{ (x,t) : |x| \le R, R > 0, a \le t \le b \right\},$$

where

$$R = \frac{|\varphi_2| \left|(b-a)^{\alpha}\right|}{\Gamma\left(\alpha+1\right)} + |\varphi_1| + \left| \varphi_3 - \frac{\varphi_2 \left(b-a\right)^{\alpha}}{\Gamma\left(\alpha+1\right)} - \varphi_1 \right|,$$

and

$$m = \min_{(x,t)\in G} \mathcal{M}\left(x,t\right),$$

$$M = \max\left\{ \max_{(x,t)\in G} \mathcal{M}\left(x,t\right), 0 \right\}.$$

$C^3_{3-\alpha}([a,b])$ denotes the space:

$$C^3_{3-\alpha}([a,b]) = \left\{x \in C([a,b]) : \ x''' \in C_{3-\alpha}([a,b])\right\}.$$

Here $C_{3-\alpha}([a,b]) = \left\{(t-a)^{3-\alpha} x \in C([a,b])\right\}$.

The main result of this chapter is the following theorem.

Theorem 5.3.1 *If condition (*) satisfied, then problem* (5.3) - (5.4) *has a solution in* $C^3_{3-\alpha}([a,b])$.

5.4 Auxiliary statements

In this section, we give some auxiliary statements for further investigation.

Theorem 5.4.1 *[14](Schauder fixed-point theorem.) If a completely-continuous operator A maps a bounded closed convex set K of a Banach space X into itself, then there exists at least one point $x \in K$ such that $Ax = x$.*

Consider the following operator:

$$\mathcal{B}_t(x) = \frac{\varphi_2 (t-a)^\alpha}{\Gamma(\alpha+1)} + \varphi_1 + \left(\varphi_3 - \frac{\varphi_2 (b-a)^\alpha}{\Gamma(\alpha+1)} - \varphi_1\right) \Im_x(t), \tag{5.5}$$

where

$$\Im_x(t) \equiv \frac{I^{2\alpha}\left[\exp\left(-\int\limits_a^t \mathcal{M}(x(s),s)\,ds\right)\right]}{I^{2\alpha}\left[\exp\left(-\int\limits_a^b \mathcal{M}(x(s),s)\,ds\right)\right]}. \tag{5.6}$$

Lemma 5.4.2 *Let $x(t)$ be a continuous function on $[a,b]$. Then the operator $\mathcal{B}_t(x)$ is equicontinuity on $[a,b]$.*

Proof. For $t_1, t_2 \in [a,b]$, $t_1 > t_2$, we get

$$\begin{aligned} |\mathcal{B}_{t_1}(x) - \mathcal{B}_{t_2}(x)| \leq \frac{|\varphi_2|}{\Gamma(\alpha+1)} |(t_1-a)^\alpha - (t_2-a)^\alpha| \\ + \left|\varphi_3 - \frac{\varphi_2 (b-a)^\alpha}{\Gamma(\alpha+1)} - \varphi_1\right| |\Im_x(t_1) - \Im_x(t_2)|. \end{aligned} \tag{5.7}$$

For some positive constant κ we get

$$\frac{|\varphi_2|}{\Gamma(\alpha+1)} |(t_1-a)^\alpha - (t_2-a)^\alpha| \leq \kappa.$$

For the operator $\Im_x(t)$ we have

$$\begin{aligned}
&\left|\Im_x(t_1) - \Im_x(t_2)\right| \\
&= \frac{\left| I^{2\alpha}\left[\exp\left(-\int\limits_a^{t_1} \mathcal{M}(x(s),s)\,ds\right)\right] - I^{2\alpha}\left[\exp\left(-\int\limits_a^{t_2} \mathcal{M}(x(s),s)\,ds\right)\right]\right|}{I^{2\alpha}\left[\exp\left(-\int\limits_a^{b} \mathcal{M}(x(s),s)\,ds\right)\right]}.
\end{aligned}$$

Further,

$$\begin{aligned}
&\left| I^{2\alpha}\left[\exp\left(-\int\limits_a^{t_1} \mathcal{M}(x(s),s)\,ds\right)\right] - I^{2\alpha}\left[\exp\left(-\int\limits_a^{t_2} \mathcal{M}(x(s),s)\,ds\right)\right]\right| \\
&\leq \left| I^{2\alpha}\left[\exp\left(-\int\limits_a^{t_1} \mathcal{M}(x(s),s)\,ds\right)\right]\right| + \left| I^{2\alpha}\left[\exp\left(-\int\limits_a^{t_2} \mathcal{M}(x(s),s)\,ds\right)\right]\right| \\
&\leq \frac{1}{\Gamma(2\alpha)}\int\limits_a^{t_1} (t_1-s)^{2\alpha-1}\exp\left(-\int\limits_a^{s} \mathcal{M}(x(\tau),\tau)\,d\tau\right)ds \\
&+ \frac{1}{\Gamma(2\alpha)}\int\limits_{t_2}^{a} (t_1-s)^{2\alpha-1}\exp\left(-\int\limits_a^{s} \mathcal{M}(x(\tau),\tau)\,d\tau\right)ds \\
&\leq \left| I^{2\alpha}\left[\exp\left(-\int\limits_{t_2}^{t_1} \mathcal{M}(x(s),s)\,ds\right)\right]\right|.
\end{aligned}$$

If $m = 0$, then we have

$$\left|\frac{1}{\Gamma(2\alpha)}\int_{t_2}^{t_1}(t_1-s)^{2\alpha-1}ds\right| = \frac{1}{\Gamma(2\alpha+1)}\left|t_1-t_2\right|^{2\alpha} \leq \beta_1.$$

where $0 < \beta_1 = const$.

If $m > 0$, then

$$\begin{aligned}
&\frac{1}{\Gamma(2\alpha)}\left|\int\limits_{t_2}^{t_1}(t_1-s)^{2\alpha-1}\exp\left(-m(s-a)\right)ds\right| \\
&\leq \frac{1}{\Gamma(2\alpha)}\left|\int\limits_{t_2}^{t_1}(t_1-s)^{2\alpha-1}\exp\left(-m(s-t_2)\right)ds\right|
\end{aligned}$$

where

$$\begin{aligned}
&\int_{t_2}^{t_1} (t_1 - s)^{2\alpha-1} \exp\left(-m\left(s - t_2\right)\right) ds \\
&= \int_{t_2}^{t_1} (t_1 - s)^{2\alpha-1} \sum_{k=0}^{\infty} (-m)^k \frac{(s - t_2)^k}{k!} ds \\
&= \int_{0}^{1} (t_1 - (t_1 - t_2)\,\tau - t_2)^{2\alpha-1} \sum_{k=0}^{\infty} \frac{(-m)^k}{k!} \left((t_1 - t_2)\,\tau + t_2 - t_2\right)^k (t_1 - t_2)\, d\tau \\
&= \sum_{k=0}^{\infty} \frac{(-m)^k}{k!} \int_{0}^{1} (t_1 - t_2)^{2\alpha-1} (1 - \tau)^{2\alpha-1} (t_1 - t_2)^k \tau^k (t_1 - t_2)\, d\tau \\
&= \sum_{k=0}^{\infty} \frac{(-m)^k}{k!} (t_1 - t_2)^{k+2\alpha} \int_{0}^{1} (1 - \tau)^{2\alpha-1} \tau^k d\tau \\
&= \sum_{k=0}^{\infty} \frac{(-m)^k}{k!} (t_1 - t_2)^{k+2\alpha} B\left(2\alpha, k+1\right) \\
&= \sum_{k=0}^{\infty} \frac{(-m)^k}{k!} (t_1 - t_2)^{k+2\alpha} \frac{\Gamma(2\alpha)\,\Gamma(k+1)}{\Gamma(k+2\alpha+1)} \\
&= \sum_{k=0}^{\infty} \frac{(-m)^k}{k!} (t_1 - t_2)^{k+2\alpha} \frac{\Gamma(2\alpha)\, k!}{\Gamma(k+2\alpha+1)} \\
&= \Gamma(2\alpha)\,(t_1 - t_2)^{2\alpha} \sum_{k=0}^{\infty} \frac{(-m)^k (t_1 - t_2)^k}{\Gamma(k+2\alpha+1)} \\
&= \Gamma(2\alpha)\,(t_1 - t_2)^{2\alpha} E_{1,2\alpha+1}\left(-m\left(t_1 - t_2\right)\right)
\end{aligned}$$

then

$$\begin{aligned}
&\frac{1}{\Gamma(2\alpha)} \left| \int_{t_2}^{t_1} (t_1 - s)^{2\alpha-1} \exp\left(-m\left(s - a\right)\right) ds \right| \\
&\leq \frac{1}{\Gamma(2\alpha)} \left| \int_{t_2}^{t_1} (t_1 - s)^{2\alpha-1} \exp\left(-m\left(s - t_2\right)\right) ds \right| \\
&= \left| (t_1 - t_2)^{2\alpha} E_{1,2\alpha+1}\left(-m\left(t_1 - t_2\right)\right) \right| < \beta_2,\ 0 < \beta_2 = \text{const.}
\end{aligned}$$

Here $E_{\lambda,\mu}(z)$ is a Mittag-Leffler type function.

If $m = -\tilde{m} < 0$, then

$$\frac{1}{\Gamma(2\alpha)}\left|\int_{t_2}^{t_1}(t_1-s)^{2\alpha-1}\exp\left(\tilde{m}(s-a)\right)ds\right| \le$$
$$\frac{1}{\Gamma(2\alpha)}\left|\int_{t_2}^{t_1}(t_1-s)^{2\alpha-1}\exp\left(\tilde{m}(s-t_2)\right)ds\right|$$
$$= \left|(t_1-t_2)^{2\alpha}E_{1,2\alpha+1}\left(\tilde{m}\left(t_1-t_2\right)\right)\right| < \beta_3,\ 0<\beta_3 = const.$$

Now for the estimation of $I^{2\alpha}\left[\exp\left(-\int\limits_a^b \mathcal{M}\left(x\left(s\right),s\right)ds\right)\right]$, we consider the following cases:

Case 1: Let $M > 0$, then taking into account the condition (*) and using formula (1.101) from [83], we obtain

$$I^{2\alpha}\left[\exp\left(-\int\limits_a^b \mathcal{M}\left(x\left(s\right),s\right)ds\right)\right]$$
$$= \frac{1}{\Gamma\left(2\alpha\right)}\int\limits_a^b\left(b-s\right)^{2\alpha-1}\exp\left(-\int\limits_a^s \mathcal{M}\left(x\left(\tau\right),\tau\right)d\tau\right)ds$$
$$\ge \frac{1}{\Gamma\left(2\alpha\right)}\int\limits_a^b\left(b-s\right)^{2\alpha-1}\exp\left(-M\left(s-a\right)\right)ds$$
$$= \left(b-a\right)^{2\alpha}E_{1,2\alpha+1}\left(-M\left(b-a\right)\right) \ge \gamma_1 > 0,\ \gamma_1 = const. \quad (5.8)$$

Case 2: Let $M = 0$, then

$$I^{2\alpha}\left[\exp\left(-\int\limits_a^b \mathcal{M}\left(x\left(s\right),s\right)ds\right)\right]$$
$$= \frac{1}{\Gamma\left(2\alpha\right)}\int\limits_a^b\left(b-s\right)^{2\alpha-1}ds = \frac{\left(b-a\right)^{2\alpha}}{\Gamma\left(2\alpha+1\right)} \ge \gamma_2 > 0,\ \gamma_2 = const. \quad (5.9)$$

Inequalities (5.7) - (5.9) follow the estimate:

$$\left|\mathcal{B}_{t_1}\left(x\right) - \mathcal{B}_{t_2}\left(x\right)\right| \le \varepsilon,$$

where the positive constant ε does not depend on $\left(x,t_1,t_2\right) \in G$. This ends the proof. □

5.5 Proof of the main results

We reduce the problem (5.3) - (5.4) to a nonlinear integral equation. To do this, we introduce the notation $\mathcal{D}_*^{2\alpha}x\left(t\right) = y\left(t\right)$. Then, equation (5.3) can be rewritten as

$$y'\left(t\right) + \mathcal{M}\left(x\left(t\right),t\right)y\left(t\right) = 0, a < t < b. \quad (5.10)$$

Then, the general solution of equation (5.10) has the form

$$y(t) = y(a)\exp\left(-\int_a^t \mathcal{M}(x(s), s)\, ds\right), a < t < b,$$

or equivalently,

$$\mathcal{D}_*^{2\alpha} x(t) = \mathcal{D}_*^{2\alpha} x(a)\exp\left(-\int_a^t \mathcal{M}(x(s), s)\, ds\right), a < t < b. \tag{5.11}$$

Applying the operator I^α to both sides of the equation (5.11) and using property 5.2.2 and conditions (5.4), we get

$$D_*^\alpha x(t) = \varphi_2 + \mathcal{D}_*^{2\alpha} x(a)\, I^\alpha\left[\exp\left(-\int_a^t \mathcal{M}(x(s), s)\, ds\right)\right],\ a < t < b. \tag{5.12}$$

Next, by applying to the equation (5.12) the operator I^α, and using properties 5.2.1 and 5.2.2, based on condition (5.4), we obtain

$$\begin{aligned} x(t) &= \frac{\varphi_2}{\Gamma(\alpha+1)}(t-a)^\alpha + \varphi_1 \\ &+ \mathcal{D}_*^{2\alpha} x(a)\, I^{2\alpha}\left[\exp\left(-\int_a^t \mathcal{M}(x(s), s)\, ds\right)\right],\ a < t < b. \end{aligned} \tag{5.13}$$

In (5.13), if we put $t = b$, and as $x(b) = \varphi_3$, we have

$$\mathcal{D}_*^{2\alpha} x(a) = \frac{\varphi_3 - \frac{\varphi_2}{\Gamma(\alpha+1)}(b-a)^\alpha - \varphi_1}{I^{2\alpha}\left[\exp\left(-\int_a^b \mathcal{M}(x(s), s)\, ds\right)\right]},$$

and the problem (5.3) - (5.4) is reduced to the following nonlinear integral equation

$$\begin{aligned} x(t) &= \frac{\varphi_2 (t-a)^\alpha}{\Gamma(\alpha+1)} + \varphi_1 \\ &+ \left(\varphi_3 - \frac{\varphi_2 (b-a)^\alpha}{\Gamma(\alpha+1)} - \varphi_1\right) \frac{I^{2\alpha}\left[\exp\left(-\int_a^t \mathcal{M}(x(s), s)\, ds\right)\right]}{I^{2\alpha}\left[\exp\left(-\int_a^b \mathcal{M}(x(s), s)\, ds\right)\right]},\ a < t < b. \end{aligned} \tag{5.14}$$

Let us separately investigate operator (5.6). It is obvious that for any $x \in C([a, b])$ when $t \in [a, b]$, we have the inequality for the operator (5.6)

$$0 \le \Im_x(t) \le 1.$$

In fact, $\Im_x(b) = 1$, $\Im_x(a) = 0$ and for other case $0 < \Im_x(t) < 1$. It is also easy to see that

$$I^{2\alpha}\left[\exp\left(-\int_a^t \mathcal{M}(x(s), s)\, ds\right)\right] > 0, \; t > a. \tag{5.15}$$

Consequently, for any $x \in C([a,b])$ we have the inequality

$$\begin{aligned}\|\mathcal{B}_t(x)\| &= \left|\frac{\varphi_2 (t-a)^\alpha}{\Gamma(\alpha+1)} + \varphi_1 + \left(\varphi_3 - \frac{\varphi_2 (b-a)^\alpha}{\Gamma(\alpha+1)} - \varphi_1\right)\Im_x(t)\right| \\ &\leq \frac{|\varphi_2|\left|(b-a)^\alpha\right|}{\Gamma(\alpha+1)} + |\varphi_1| + \left|\varphi_3 - \frac{\varphi_2 (b-a)^\alpha}{\Gamma(\alpha+1)} - \varphi_1\right| \equiv R. \end{aligned} \tag{5.16}$$

This means that the integral operator $\mathcal{B}_t$ maps a ball $\|x\| \leq R$ into itself.

Let us prove that operator $\mathcal{B}_t(x)$ is completely continuous. We use Arzela's theorem [105] on precompact sets in $C([a,b])$. (5.16) shows that image of set $\|x\| \leq R$ by mapping $\mathcal{B}_t(x)$ is uniformly bounded by R.

Now, we show the equicontinuity of the operator $\mathcal{B}_t(x)$. By Lemma 5.4.2, the operator $\mathcal{B}_t(x)$ satisfies the condition

$$|\mathcal{B}_{t_1}(x) - \mathcal{B}_{t_2}(x)| \leq \varepsilon.$$

As in the proof of Lemma 5.4.2, the positive constant ε does not depend on $(x, t_1, t_2) \in G$. We have then proved that the operator $\mathcal{B}_t$ is uniformly bounded and equicontinuous in $C([a,b])$. Consequently, by Arzelà-Ascoli' Theorem , the image $\mathcal{B}_t(x)$ on the ball $\mathcal{S}(0,R)$ is precompact in $C([a,b])$. By consequence the operator $\mathcal{B}_t$ is completely continuous in $C([a,b])$, and we conclude that the operator $\mathcal{B}_t$ satisfies the Schauder's conditions in Theorem 5.4.1. Then, according to the Schauder's principle, the nonlinear integral equation (5.14) has solution in the class $C([a,b])$.

For any $x(t)$ from the class $C([a,b])$, from the structures of operator $\mathcal{B}_t$ in (5.5), it is easy to verify that all derivatives up to the third order of (5.5) are continuous in the weighted class $C_{3-\alpha}([a,b])$. Therefore, the solution of equation (5.14) will belong to $C^3_{3-\alpha}([a,b])$.

Conclusion

In this thesis, we first dealt with some inverse problems in fractional calculus by using Caputo derivative. Caputo's derivative allows us to impose natural initial conditions. By using the Fourier method we proved the existence and uniqueness of each solution of related inverse problems. Convergences of the obtained solutions are also justified in order to establish that the formal solutions are indeed true solutions.

Afterwards, we have examined a non-linear sequential differential (fractional analog of the Blasius equation) equation with Caputo fractional derivative. Considered problem reduced to the equivalent nonlinear integral equation and we proved the complete continuity of the nonlinear integral operator. The result is formulated on the existence of a unique solution of the problem for the non-linear differential equation of fractional order. In the limiting case, the considered boundary problem coincides with the boundary-layer problem for the Blasius equation.

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