Pseudofinite structures and counting dimensions

Tingxiang Zou

Université Claude Bernard Lyon 1
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Structures pseudo-finies et dimensions de comptage

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Tingxiang Zou

après avis des rapporteurs :

Alf Onshuus	Université de los Andes Bogota
Katrin Tent	Université de Münster

devant le jury composé de :

Itaï Ben Yaacov	(Université Lyon 1)	Examinateur
Zoé Chatzidakis	(ENS Paris)	Examinatrice
Ehud Hrushovski	(Université d'Oxford)	Examinateur
Dugald Macpherson	(Université de Leeds)	Examinateur
Katrin Tent	(Université de Münster)	Rapporteuse
Frank Wagner	(Université Lyon 1)	Directeur de thèse

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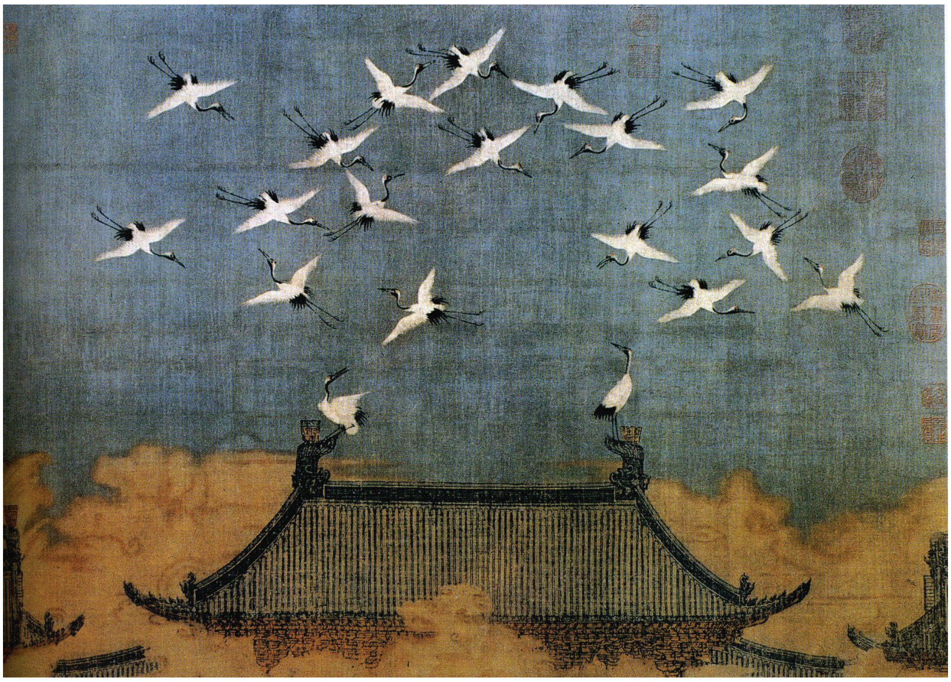

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Thèse de doctorat

Université Claude Bernard Lyon 1

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Contents

Introduction en Français

Les structures pseudo-finies sont définies comme étant des structures élémentairement équivalentes à des ultraproduits de structures finies. En théorie des modèles, il existe une littérature abondante consacrée à l'étude de cettes structures. Comme limites asymptotiques de structures finies, leurs propriétés modèle-théoriques révèlent souvent, via le théorème de Łoś, des comportements asymptotiques des classes de structures finies correspondantes.

La théorie des modèles moderne a commencé par l'étude du problème de la catégoricité: à quelle(s) condition(s) une théorie complète du premier ordre ne comporte-t-elle qu'un seul modèle d'une certaine cardinalité à isomorphisme près? Ce problème a conduit au théorème de catégoricité de Morley, qui stipule qu'une théorie complète dénombrable a exactement un modèle d'un certain cardinal non-dénombrable si et seulement si c'est le cas pour tous les cardinaux non-dénombrables. Dans l'étude des théories catégoriques non-dénombrables, Morley a développé une notion de rang: le rang de Morley. Il a également identifié une classe de théories du premier ordre, les théories totalement transcendantes, qui sont les théories avec un rang de Morley ordinal. Dès lors, les rangs ont été l'un des outils les plus importants de la théorie des modèles pour étudier le comportement d'ensembles définissables et d'espaces de types d'une théorie du premier ordre.

Les rangs définis sur des ensembles ou des types définissables jouent le rôle de dimensions. On peut souvent définir une relation d'indépendance via les rangs. En un sens, les deux directions principales de recherche en théorie des modèles pure sont les suivantes: premièrement, l'analyse des relations d'indépendance provenant des rangs (locaux), la *stabilité géométrique*; deuxièmement, l'extension de ces outils à d'autres classes de théories, la *néo-stabilité*.

Les structures pseudo-finies ne sont pas a priori une classe de structures modérées. Un ultraproduit de structures finies peut avoir une théorie très compliquée, mais on peut le munir de dimensions de comptage naturelles. L'histoire commence avec [CvdDM92], où une notion de mesure et de dimension de comptage pour les ensembles définissables dans les corps pseudo-finis a été développée à l'aide de l'estimation de Lang-Weil. Dans cet exemple la dimension de comptage coïncide avec le rang SU et avec le degré de transcendance. Inspiré par ce phénomène dans la classe des corps finis, un cadre général pour les classes de structures finies a été proposé dans [MS08] et [Elw07], ce qui donna naissance aux *classes asymptotiques undimensionnelles* et *classes asymptotiques de dimension finie*. Les ultraproduits de ces classes ont des théories modérées. En particulier, le rang SU de ces théories est majoré par la dimension, elles sont donc supersimples de rang SU fini. De nombreux exemples appartiennent à cette catégorie, y compris des familles de groupes simples finis de type de Lie et de rang de Lie borné [Ryt07]. Cette approche a été approfondie dans [HW08] et [Hru13] en toute généralité, sans hypothèse de modération. Deux dimensions pseudo-finies importantes y ont été développées: la *dimension pseudo-finie fine* qui vient avec des mesures, et la *dimension pseudo-finie grossière*. Comme il a été montré dans [GMS15], les théories avec une dimension pseudo-finie fine qui se comporte bien sont modérées et il existe un lien entre la chute de la dimension fine et la déviation (donc la chute du rang SU dans les theories supersimples).

Plus important encore, on peut étudier, avec ces dimensions de comptage, si le comportement asymptotique (en ce qui concerne le comptage) d'ensembles finis dans une structure (éventuellement infinie) révèle certaines propriétés structurelles de ces ensembles finis. Ce type de problèmes a été étudié de manière intensive en combinatoire additive depuis longtemps. Par exemple, le célèbre théorème de Szemerédi stipule que tout sous-ensemble de $\mathbb{Z}$ ayant une densité supérieure strictement positive contiendra des suites arithmétiques arbitrairement longues. Cela équivaut à affirmer que dans l'ultrapuissance $\prod_{n\in\mathbb{N}}(\mathbb{Z},+)/\mathcal{U}$, tout sous-ensemble interne $B \subseteq A := \prod_{n\in\mathbb{N}}\{1,\ldots,n\}/\mathcal{U}$ de même dimension fine que A contiendra une suite arithmétique infinie. La théorie des modèles ayant développé de puissants outils en relation avec les notions de dimension et d'indépendance, elle apporte de nouvelles méthodes pour étudier les problèmes liés à la combinatoire additive. Dans [HW08] et [Hru13], quelques liens entre la combinatoire additive et les dimensions de comptage des sous-ensembles pseudo-finis ont été étudiés, par exemple, l'inégalité de Larsen-Pink, le phénomène de produit-somme et le théorème de Szemerédi–Trotter. Récemment, des progrès importants ont été réalisés dans cette direction, par exemple une généralisation du théorème de Elekes-Szabó a été présentée en utilisant la dimension pseudo-finie grossière dans [BB18].

Le résultat le plus inspirant dans ce sens provient des travaux de Hrushovski sur les sous-groupes approximatifs dans [Hru12]. Il a découvert une surprenante généralisation du théorème du stabilisateur pour les groupes stables à la classe des sous-groupes approximatifs finis en utilisant la mesure de la dimension pseudo-finie fine. Cela a conduit à la classification complète des sous-groupes approximatifs finis dans [BGT12].

Cette thèse porte sur la théorie des modèles des structures pseudo-finies en mettant l'accent sur les groupes et les corps. Le but est d'approfondir notre compréhension des interactions entre les dimensions de comptage pseudo-finies et les propriétés algébriques de leurs structures sous-jacentes, ainsi que de la classification de certaines classes de structures en fonction de leurs dimensions. Notre approche se fait par l'étude d'exemples. Nous avons examiné trois classes de structures. La première est la classe des H-structures, qui sont des expansions génériques. Nous avons donné une construction explicite de H-structures pseudo-finies comme ultraproduits de structures finies. Le deuxième exemple est la classe des corps aux différences finis. Nous avons étudié les propriétés de la dimension pseudo-finie grossière de cette classe. Nous avons montré qu'elle est définissable et prend des valeurs entières. Le troisième exemple est la classe des groupes de permutations primitifs pseudo-finis. Nous avons généralisé le théorème classique de classification de Hrushovski pour les groupes stables de permutations d'un ensemble fortement minimal au cas où une dimension abstraite existe, cas qui inclut à la fois les rangs classiques de la théorie des modèles et les dimensions de comptage pseudo-finies. Dans cette thèse, nous avons aussi généralisé le théorème de Schlichting aux sous-groupes approximatifs, en utilisant une notion de commensurabilité.

Le chapitre 1 traite des H-structures introduites par Berenstein et Vassiliev dans [BV16]. Ce sont des expansions de structures par un ensemble algébriquement indépendant. Moralement, dans une structure où la clôture algébrique donne une dimension qui se comporte bien (les *structures géométriques*), il s'agit d'ajouter un prédicat pour un ensemble algébriquement indépendant tel que cet ensemble et son complémentaire intersectent tout ensemble définissable non-algébrique. Cette expansion conserve certaines bonnes propriétés modèle-théoriques et les ensembles définissables peuvent être compris à partir de ceux de la structure d'origine. Les expansions génériques ont été étudiées intensivement en théorie des modèles (voir par exemple [Poi83], [CP98] et [BYPV03]). Ce chapitre est motivé par la question suivante: l'expansion générique d'une structure pseudo-finie est-elle encore pseudo-finie ? Nous avons donné une réponse négative dans le cas des belles paires de corps pseudo-finis. C'est-à-dire qu'aucune belle paire de corps pseudo-finis ne peut être équivalente à un ultraproduit de paires de corps finis. Cependant, nous avons donné une réponse positive en ce qui concerne les H-expansions de corps pseudo-finis. En fait, la preuve de cette deuxième utilise uniquement le fait que la dimension fine des corps pseudo-finis a de bonnes propriétés: dans toute famille définissable d'ensembles, la dimension fine prend des valeurs finies discrètes et les mesures et les dimensions sont définissables. Par conséquent, le résultat s'étend à tout ultraproduit d'une classe asymptotique undimensionnelle, puisqu'il s'agit de structures géométriques.

Théorème A. Soit $\mathcal{C}$ une classe asymptotique undimensionnelle dans un langage dénombrable. Soit $\mathcal{M} := \prod_{i\in I} M_i/\mathcal{U}$ un ultraproduit infini d'éléments de $\mathcal{C}$. Alors, pour chaque $i \in I$, il existe $H_i \subseteq M_i$ tel que $(\mathcal{M}, H(\mathcal{M})) := \prod_{i\in I}(M_i, H_i)/\mathcal{U}$ soit une H-structure.

La deuxième partie de ce chapitre concerne les groupes définissables dans les H-structures. À l'aide du théorème de fragment de groupe (voir Fact 0.27), qui est une variante du théorème de configuration de groupe, nous avons réussi à classifier tous les groupes (type-)définissables dans les H-expansion d'une théorie supersimple de rang SU 1.

Théorème B. Soit T supersimple de rang SU 1 et $(M, H(M))$ une H-structure tel que $M \models T$. Soit G un groupe (type-)définissable dans $(M, H(M))$. Alors, G est définissablement isomorphe à un groupe (type-)interprétable dans M.

En particulier, si T élimine les imaginaires, alors tout groupe (type-)définissable dans $(M, H(M))$ est définissablement isomorphe à un groupe (type-)définissable dans M.

Le chapitre 2 étudie la théorie asymptotique des corps aux différences finis. La motivation provient d'un théorème prouvé par Mark Ryten dans [Ryt07] qui stipule que pour tout $p \in \mathbb{P}$ et $m, n > 1$ premiers entre eux,

$$\mathcal{C}_{p,m,n} := \{(\mathbb{F}_{p^{km+n}}, \mathrm{Frob}_{p^k}) : k \in \mathbb{N}\}$$

est une classe asymptotique undimensionnelle, où Frob_{p^k} est l'automorphisme de $\mathbb{F}_{p^{km+n}}$ qui à x associe x^{p^k}. Que se passe t-il si la caractéristique des corps change également ? Est-il possible d'avoir des classes asymptotiques undimensionnelles de corps aux différences finis à caractéristique non-fixée? La réponse s'est avérée négative. En fait, si la caractéristique d'un ultraproduit de corps aux différences finis est 0 et que l'automorphisme n'est pas trivial, le corps fixé par l'automorphisme sera un sous-corps infini non trivial. Alors, le rang SU de la théorie sera strictement supérieur à 1. Mais les ultraproduits d'une classe asymptotique undimensionnelle ont rang SU 1.

Cependant, puisque l'endomorphisme de Frobenius Frob_p est définissable dans le langage des anneaux $\mathcal{L}$ pour chaque nombre premier p, toute formule $\varphi(x)$ du langage des anneaux aux différences $\mathcal{L}_\sigma := \mathcal{L} \cup \{\sigma\}$ peut être traduite en une formule $\varphi_p(x)$ dans $\mathcal{L}$, en remplaçont σ par Frob_p. Comme les corps finis forment une classe asymptotique undimensionnelle, $\varphi_p(x)$ aura une dimension fine $d_p \leq |x|$ pour chaque p, et lorsque p varie, l'ultrafiltre choisira un $d \leq |x|$ qui deviendra la dimension grossière de φ lorsque le corps est suffisamment grand. En conclusion, nous avons le résultat suivant:

Théorème C. Il existe une fonction $f : \mathbb{N} \to \mathbb{N}$ telle que pour tout (F, Frob) dans

$$\mathcal{S} := \left\{ \prod_{p \in \mathbb{P}} (\mathbb{F}_{p^{k_p}}, \mathrm{Frob}_p)/\mathcal{U} : k_p \geq f(p),\ \mathcal{U} \text{ ultrafiltre non principal} \right\},$$

la dimension pseudo-finie grossière $\boldsymbol{\delta}_F$ par rapport à F prend des valeurs entières pour tout ensemble $\mathcal{L}_\sigma$-définissable. De plus, $\boldsymbol{\delta}_F$ est *définissable* dans $\mathcal{L}_\sigma$.

En fait, l'énoncé du Théorème C est aussi vrai pour les corps aux différences pseudo-finis de la forme $\prod_{i \in I} (\mathbb{F}_{{p_i}^{k_i}}, \mathrm{Frob}_{{p_i}^{t_i}})/\mathcal{U}$ tant que ${p_i}^{k_i} >> {p_i}^{t_i}$ pour presque tout i.

Cependant, comme nous demandons que le corps ambiant soit bien plus grand que le sous-corps fixé par l'automorphisme, nous pouvons adapter la preuve de Duret de la propriété d'independance pour les corps pseudo-finis [Dur80] afin de montrer que tous les sous-ensembles internes du sous-corps fixé sont uniformément définissables. Ainsi, aucune structure dans $\mathcal{S}$ n'est modérée.

Théorème D. Soit $\mathcal{S}$ défini comme dans le Théorème C. Supposons que $(F, \mathrm{Frob}) \in \mathcal{S}$ et soit T la théorie de (F, Frob) dans $\mathcal{L}_\sigma$. Alors T a la propriété de l'ordre strict et TP2. De plus, T n'est pas décidable.

Au vu de ce résultat, il sera difficile d'analyser les structures dans $\mathcal{S}$ et leurs théories à l'aide de l'indépendance de la déviation ou avec les rangs classiques de la théorie des modèles. Nous allons essayer de comprendre ces structures en utilisant la dimension grossière $\boldsymbol{\delta}_F$. L'idée est de créer un lien entre $\boldsymbol{\delta}_F$ et quelque chose de connu, ou de trouver le sens algébrique de $\boldsymbol{\delta}_F$. Un candidat naturel est le *degré de transcendance transformelle*. Il est facile de voir que $\boldsymbol{\delta}_F$ d'un uplet fini est majoré par son degré de transcendance transformelle. Nous pensons que ces deux dimensions sont identiques dans toutes les structures de $\mathcal{S}$. Comme le degré de transcendance transformelle d'un uplet est entièrement déterminé par son type sans quantificateur, si notre conjecture est vraie, tout ensemble définissable dans une structure de $\mathcal{S}$ est "équivalent en dimension grossière" à un ensemble défini par une formule sans quantificateur.

Nous donnons une application de la conjecture, qui vise à comprendre les sous-groupes définissables d'un groupe algébrique.

Théorème E. Soit $(F, \mathrm{Frob}) \in \mathcal{S}$. Supposons que $\boldsymbol{\delta}_F$ et le degré de transcendance transformelle sont identiques dans (F, Frob). Soit G un sous-groupe définissable d'un groupe algébrique $H(F) \subseteq F^n$. Alors il existe un groupe définissable sans quantificateur D tel que $G \leq D \leq H(F)$ et $\boldsymbol{\delta}_F(D) = \boldsymbol{\delta}_F(G)$.

Le chapitre 3 traite des groupes de permutations dans une théorie dimensionnelle. L'origine de cette étude peut être retracée aux groupes de rang de Morley petit. Reineke a montré dans [Rei75] qu'un groupe connexe de rang de Morley 1 est abélien. Cherlin a continué dans [Che79] et a montré qu'un groupe connexe de rang de Morley 2 est résoluble, et qu'un groupe simple connexe de rang de Morley 3 contenant un sous-groupe définissable de rang de Morley 2 est isomorphe à $\mathrm{PSL}_2(K)$ pour un corps K définissable algébriquement clos. Sur ce sujet, Hrushovski a classifié les groupes de permutations transtifs G sur un ensemble X fortement minimal dans une théorie stable en trois cas:

- Le rang de Morley de G est égal à 1, et G est connexe, et l'action de G sur X est régulière;
- Le rang de Morley de G est égal à 2, et G est isomorphe à $\mathrm{AGL}_1(K)$ pour un corps K définissable algébriquement clos, et l'action est sur $\mathrm{AG}_1(K)$ via les applications $x \mapsto ax + b$.
- Le rang de Morley de G est égal à 3, et G est isomorphe à $\mathrm{PSL}_2(K)$ pour un corps K définissable algébriquement clos, et l'action est sur $\mathrm{PG}_1(K)$.

Ces résultats ont été généralisés aux groupes pseudo-finis de rang SU 1 et 2 et aux groupes de permutations pseudo-finis définissablement primitifs sur un ensemble de rang SU 1 dans une théorie supersimple de rang SU fini dans [EJMR11].

Les résultats concernant les groupes pseudo-finis de rang SU 1 et 2 ont été généralisés par Wagner dans [Wag18], où le rang SU est remplacé par une dimension abstraite et l'hypothèse de modération de la théorie remplacée par certaines conditions de chaîne sur les centralisateurs, appelé la *condition* $\widetilde{\mathfrak{M}}_c$, tandis que l'hypothèse de pseudo-finitude est conservée. D'une part, le but de l'introduction d'une dimension abstraite est d'unifier plusieurs objets semblables à une dimension dans les théories modérées, par exemple le rang de Lascar ou le rang SU dans les théories stables ou simples, la dimension o-minimale et les dimensions de comptage pseudo-finies. Plus précisément, cette dimension abstraite sur des ensembles interprétables doit être *additive* et prendre des valeurs entières. Mais il n'est pas nécessaire que les ensembles de dimension 0 soient toujours finis, ce qui inclura les cas de rang SU ou de Lascar infini (dans ces cas, la dimension est le coefficient de ω^α pour un certain ordinal α), ainsi que les dimensions pseudo-finies grossières, comme $\boldsymbol{\delta}_F$ dans le chapitre 2. D'autre part, la condition $\widetilde{\mathfrak{M}}_c$, qui stipule qu'il n'y a pas de chaîne infinie de centralisateurs, chacun d'indice infini dans son prédécesseur, est davantage axée sur les propriétés combinatoires qu'une théorie modérée devrait avoir. Cette condition elle-même restreint la complexité des groupes et donne quelques propriétés structurelles intéressantes pour les sous-groupes définissables (voir [Hem15] pour plus de détails).

Basé sur le résultat de Wagner sur les groupes $\widetilde{\mathfrak{M}}_c$ pseudo-finis de petite dimension, le but du chapitre 3 est de généraliser la classification des groupes de permutations pseudo-finis définissablement primitifs avec une dimension additive à valeurs entières et satisfaisant certaines conditions de chaîne sur les sous-groupes.

Théorème F. Soit (G, X) un groupe pur de permutations pseudo-fini définissablement primitif, avec une dimension additive à valeurs entières dim telle que $\dim(X) = 1$, $\dim(G) < \infty$ et tel que G et ses quotients définissables vérifient la condition $\widetilde{\mathfrak{M}}_c$.

- Si $\dim(G) = 1$, alors G a un sous-groupe A abelien distingué définissable, tel que $\dim(A) = 1$ et l'action de A sur X est régulière;

- Si $\dim(G) = 2$, alors G a un sous-groupe distingué définissable H de dimension 2 et un corps K pseudo-fini interprétable de dimension 1 tel que (H, X) est définissablement isomorphe à $(K^+ \rtimes D, K^+)$, où $D \leq K^\times$ est de dimension 1.

- Supposons en outre qu'il n'existe pas de chaîne infinie descendante de stabilisateurs de G chacun d'indice infini dans son prédécesseur, et que X ne puisse pas être partitionné en une infinité de classes d'équivalence définissables de dimension 1. Si $\dim(G) \geq 3$, alors $\dim(G) = 3$, et il existe un corps K pseudo-fini interprétable de dimension 1, tel que (G, X) est définissablement isomorphe à $(H, \mathrm{PG}_1(K))$, où
$$\mathrm{PSL}_2(K) \leq H \leq \mathrm{P\Gamma L}_2(K).$$

En particulier, le résultat ci-dessus s'applique aux groupes pseudo-finis définissablement primitifs de rang SU infini. Dans le cas où la dimension du groupe de permutations est au moins deux, il existe toujours un corps pseudo-fini interprétable, avec un groupe interprétable d'automorphismes de ce corps. Cela n'est pas possible si la théorie ambiante est simple et le groupe d'automorphismes est infini. Pour cette raison, une part importante de la classification dans les cas de rangs SU infinis se réduit au cas de rang SU fini.

Théorème G. Soit (G, X) un groupe pur de permutations pseudo-fini définissablement primitif dont la théorie est supersimple. Soit $\mathrm{SU}(G) = \omega^\alpha n + \gamma$ pour certains $\gamma < \omega^\alpha$ et $n \geq 1$. Supposons que $\mathrm{SU}(X) = \omega^\alpha + \beta$ pour un certain $\beta < \omega^\alpha$. Alors on est dans l'un des cas suivants:

- $SU(G) = \omega^\alpha + \gamma$, et G a un sous-groupe A abelien distingué définissable de rang SU ω^α, et l'action de A sur X est régulière;

- $SU(G) = 2$, et il existe un corps K pseudo-fini interprétable de rang SU 1 tel que G est définissablement isomorphe à $K^+ \rtimes D$ où D est d'indice fini dans $K^\times$;

- $SU(G) = 3$ et il existe un corps K pseudo-fini interprétable de rang SU 1 tel que G est définissablement isomorphe à $\mathrm{PSL}_2(K)$ ou $\mathrm{PGL}_2(K)$.

Le dernier chapitre, Chapitre 4, traite d'un analogue du théorème de Schlichting pour les sous-groupes approximatifs. Le théorème de Schlichting pour les groupes (voir Fact 0.36) stipule que s'il existe une famille de sous-groupes uniformément commensurables, alors il existe un sous-groupe invariant commensurable avec tous. Nous prouvons qu'il en va de même pour les sous-groupes approximatifs, avec la commensurabilité définie de la façon suivante: un nombre fini de translatés de l'un recouvre l'autre.

Théorème H. Si $\mathcal{X}$ est une famille uniforme de sous-groupes approximatifs commensurables dans un groupe G, alors il existe un sous-groupe approximatif $H \subseteq G$ tel que H est commensurable avec $\mathcal{X}$ et invariant par tout automorphisme de G stabilisant $\mathcal{X}$ en tant qu'ensemble.

Ce résultat met encore en évidence les similitudes entre les groupes et les sous-groupes approximatifs. Cependant, contrairement au cas des groupes, où le sous-groupe invariant est une extension finie d'une intersection finie, nous devons ici prendre des unions infinies ou des intersections infinies pour obtenir le sous-groupe approximatif invariant.

Introduction

Pseudofinite structures are structures that are elementary equivalent to ultraproducts of finite structures. In the development of model theory, there is a rich literature devoted to the study of pseudofinite structures. Since they are asymptotic limits of finite structures, their model theoretic properties often reveal asymptotic behaviours of the corresponding finite classes via Łoś's Theorem.

Modern model theory started with the study of the categoricity problem: When does a complete first-order theory have only one model of a certain cardinality up to isomorphism? This problem led to Morley's famous categoricity theorem, which states that a complete countable theory has exactly one model of some uncountable cardinality if and only if this is the case for all uncountable cardinalities. In the study of uncountably categorical theories, Morley developed a notion of rank: Morley rank. He also identified a class of first-order theories, totally transcendental theories, which are those theories with ordinal Morley rank. From then on, ranks have been one of the main tools in model theory to study the behaviour of definable sets and type spaces of a first-order theory, among other powerful machineries such as forking calculus.

Ranks are dimension-like objects on definable sets or types. One can often define a well-behaved independence relation from ranks, where independent elements correlate with each other in a negligible way. In a sense, the two main directions in the development of pure model theory are: firstly analysing the independence relation that comes from (local) ranks, *geometric stability theory*; and secondly extending these machinery to other classes of first-order theories, *neostability theory*.

Pseudofinite structures are not a priori a tame class of structures. There can be very complicated theories that come from ultraproducts of finite structures. But they are equipped with natural dimensions from counting. The history began in [CvdDM92], where a notion of counting measure and dimension of definable sets in pseudofinite fields was developed using the Lang-Weil estimate. In fact, in this example the counting dimension coincide with both U-rank and transcendence degree. Inspired by this phenomenon in the class of finite fields, a general framework for classes of finite structures based on counting dimension and measure of definable sets was proposed in [MS08] and [Elw07]. This was called *one/finite-dimensional asymptotic classes*. The ultraproducts of these classes turned out to be model theoretic tame structures. In particular, the SU-rank of their theories are bounded above by the dimension, hence, they are supersimple of finite SU-rank. A lot of natural examples fall into this category, including families of finite simple groups of Lie type of bounded Lie rank (see [Ryt07]). This counting approach has been further investigated in [HW08] and [Hru13] in full generality without any tameness assumptions. Two important pseudofinite dimensions have been developed there: *fine pseudofinite dimension* which comes with measures (they are the dimension and measure in one-dimensional asymptotic classes) and *coarse pseudofinite dimension*. As has shown in [GMS15], theories with well-behaved fine pseudofinite dimension are tame and there is a link between the drop of fine dimension and forking (hence dropping of SU-rank if it exists) in the theory.

More importantly, regardless of model theoretic tameness, with these counting dimensions one can study whether the asymptotic behaviour of finite sets with respect to counting in a (possibly infinite) structure will imply any structural property of these finite sets. This kind of problems has been intensively studied in additive combinatorics for a long time. For example, Szemerédi's well-known theorem states that any subset of natural numbers with a positive upper-density contains arbitrarily long arithmetic progressions. It is equivalent to the statement that in the ultrapower $\prod_{n\in\mathbb{N}}(\mathbb{Z},+)/\mathcal{U}$, any internal subset $B \subseteq A := \prod_{n\in\mathbb{N}}\{1,\ldots,n\}/\mathcal{U}$ of the same fine dimension as A will contain an infinite arithmetic progression. As model theory has developed powerful tools using different notions of dimension and independence, it brings new methods to approach problems related to additive combinatorics. In [HW08] and [Hru13], connections between additive combinatorics and counting dimensions of pseudofinite subsets in ultrapowers of tame structures for example $(\mathbb{Z},+)$, linear groups or algebraic varieties over an algebraically closed fields, have been investigated, e.g. the Larsen-Pink inequality, the sum-product phenomenon, the Szemerédi–Trotter Theorem, and so on. Recently, significant progress has been made following this approach, for example, a generalization of the Elekes-Szabó Theorem has been presented using the coarse pseudofinite dimension in [BB18].

The most inspiring result along this way is Hrushovski's work on approximate subgroups in [Hru12], where he discovered a surprising generalization of the Stabilizer Theorem of groups in stable or simple theories to arbitrary finite approximate subgroups using the measure equipped with the fine pseudofinite dimension. This led to the complete classification of all finite approximate subgroups in [BGT12].

The Stabilizer Theorem is one of the most useful tools in model theory of groups. It can be seen as a generalization of Zilber's Indecomposability Theorem, where a finite product of definable sets will generate a subgroup. The Stabilizer Theorem together with the Group Configuration Theorem, which states that an interpretable group can be constructed given certain data from a generic configuration that comes from an independence notion, are often used to classify definable groups in terms of groups that are known (e.g. linear groups, algebraic groups, semialgebraic groups) in a natural structure which expands a field (e.g. differential fields, difference fields, o-minimal structures) see [HP94], [KP02], [MOS18] and others. On the other hand, the existence of stabilizers as type-definable subgroups guarantees the existence of certain connected components of these groups. As the quotient group of G by its connected component will give rise to a locally compact group with the logic topology, it is possible to use the knowledge of locally compact groups to better understand G when the connected component exists. All these explain the importance of the generalisation of Stabilizer Theorem to contexts without tameness assumptions on the global theory. It also indicates the possible power of pseudofinite dimensions in both model theory and other area of mathematics.

This thesis is about the model theory of pseudofinite structures with the focus on groups and fields. The aim is to deepen our understanding of how pseudofinite counting dimensions can interact with the algebraic properties of underlying structures and how we could classify certain classes of structures according to their counting dimensions. Our approach is by studying examples. We treat three classes of structures: The first one is the class of H-structures, which are generic expansions of existing structures. We give an explicit construction of pseudofinite H-structures as ultraproducts of finite structures. The second one is the class of finite difference fields. We study properties of coarse pseudofinite dimension in this class, show that it is definable and integer-valued. The third example is the class of pseudofinite primitive permutation groups. We generalise Hrushovski's classical classification theorem for stable permutation groups acting on a strongly minimal set to the case where there exists an abstract notion of dimension, which includes both the classical model theoretic ranks and pseudofinite counting dimensions. We hope these examples can help us to gain some intuition on possible general structural theorems for pseudofinite structures using these counting dimensions as tools. In this thesis, we also generalise Schlichting's Theorem for groups to the case of approximate subgroups with a notion of commensurability.

Chapter 1 is about H-structures introduced in [BV16]. They are expansions of structures by a generic algebraically independent set. Roughly, if in a structure where algebraic closure gives a well-behaved dimension (called *geometric structures*), we add an algebraically independent set such that this set and its complement intersect any non-algebraic definable set ("generic" or "random" in this sense), then the expanded structure preserves model theoretical tameness and the definable sets and type spaces can be understood from those of the original structure. Generic expansions have been intensively studied in model theory (see for example [Poi83],[CP98] and [BYPV03]); they often preserve nice properties and sometimes result in model complete theories. This chapter is motived by the question if we start with a pseudofinite geometric structure, do generic expansions of it preserve pseudofiniteness in general? We gave a negative answer in terms of lovely pairs of pseudofinite fields. That is, no lovely pair of pseudofinite fields can be elementary equivalent to an ultraproduct of pairs of finite fields. And we gave a positive answer in terms of H-expansions of pseudofinite fields. In fact, the prove uses only the fact that the fine dimension for pseudofinite fields is well-behaved: in any definable family of definable sets, the fine dimension takes discrete finite values and both measure and dimension are definable. Therefore, the result extends to any ultraproduct of a one-dimensional asymptotic class, since they are geometric structures.

Theorem A. Let $\mathcal{C}$ be a one-dimensional asymptotic class in a countable language. Let $\mathcal{M} := \prod_{i\in I} M_i/\mathcal{U}$ be an infinite ultraproduct of members among $\mathcal{C}$. Then for each $i \in I$ there exists $H_i \subseteq M_i$ such that $(\mathcal{M}, H(\mathcal{M})) := \prod_{i\in I}(M_i, H_i)/\mathcal{U}$ is an H-structure.

The proof uses heavily the measure that comes with the fine dimension of the original structure, and the task of constructing a generic subset reduces to the problem of finding a special set of vertices in a dense bipartite graph. Interestingly, the independent subset we construct will have coarse pseudofinite dimension 0 with respect to the full structure. It would be an interesting problem to find out the exact behaviour of both coarse and fine dimensions in these pseudofinite H-structures.

The second part of this chapter is about definable groups in H-structures. With the help of the Group Chunk Theorem (see Fact 0.27), which is a variant of the Group Configuration Theorem, we managed to classify all (type-)definable groups in H-expansions of SU-rank 1 supersimple theories.

Theorem B. Let T be supersimple of SU-rank 1 and $(M, H(M))$ an H-structure with $M \models T$. Let G be a (type-)definable group in $(M, H(M))$. Then G is definably isomorphic to some (type)-interpretable group in M.

In particular, if T eliminates imaginaries, then every (type-)definable group in $(M, H(M))$ is definably isomorphic to some (type-)definable group in M.

Chapter 2 studies the asymptotic theory of finite difference fields. The motivation comes from a theorem proved by Mark Ryten in [Ryt07] which states that for any $p \in \mathbb{P}$ and positive coprime natural numbers $m, n > 1$, the class

$$\mathcal{C}_{p,m,n} := \{(\mathbb{F}_{p^{km+n}}, \mathrm{Frob}_{p^k}) : k \in \mathbb{N}\}$$

is a one-dimensional asymptotic class, where Frob_{p^k} is the field automorphism of $\mathbb{F}_{p^{km+n}}$ which maps x to x^{p^k}. We wondered what would happen if the characteristics of the fields also change. Is it possible to have a one-dimensional asymptotic classes of finite difference fields with non-fixed characteristic? The answer turned out to be negative. In fact, if the characteristic of an ultraproduct of finite difference fields is 0 and the automorphism is non-trivial, then the fixed field will be a non-trivial infinite subfield. Thus the SU-rank of the theory will be strictly greater than 1. But ultraproducts from a one-dimensional asymptotic class will have theories of SU-rank 1.

However, since the Frobenius map Frob_p is definable in the ring language $\mathcal{L}$ for each prime p, any formula $\varphi(x)$ in the language of difference rings $\mathcal{L}_\sigma := \mathcal{L} \cup \{\sigma\}$ can be translated into a ring formula $\varphi_p(x)$ if we replace σ by Frob_p. As finite fields form a one-dimensional asymptotic class, $\varphi_p(x)$ will have a fine dimension $d_p \leq |x|$ for each p, and when p changes, the ultrafilter will pick out one $d \leq |x|$, which will become the coarse dimension of φ with respect to the full field when the field is large enough. In conclusion, we have the following result:

Theorem C. There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for any (F, Frob) in

$$\mathcal{S} := \left\{ \prod_{p \in \mathbb{P}} (\mathbb{F}_{p^{k_p}}, \mathrm{Frob}_p)/\mathcal{U} : k_p \geq f(p),\ \mathcal{U} \text{ non-principal ultrafilter} \right\},$$

the pseudofinite coarse dimension $\boldsymbol{\delta}_F$ with respect to F is integer-valued for any $\mathcal{L}_\sigma$-definable set. Moreover, $\boldsymbol{\delta}_F$ is *definable* in $\mathcal{L}_\sigma$.

In fact, the only thing that matters is that the full fields grow fast enough, and the statement holds generally for $\prod_{i \in I} (\mathbb{F}_{{p_i}^{k_i}}, \mathrm{Frob}_{{p_i}^{t_i}})/\mathcal{U}$ provided ${p_i}^{k_i} >> {p_i}^{t_i}$ for almost all i.

However, as we ask the full field to be much bigger than the fixed field, we can adapt the proof that the theory of pseudofinite fields has the independence property in [Dur80] to show that all internal subsets of the fixed field are uniformly definable. Thus, all structures in $\mathcal{S}$ are not model theoretically tame.

Theorem D. Let $\mathcal{S}$ be defined as in Theorem C. Suppose $(F, \mathrm{Frob}) \in \mathcal{S}$ and let T be the theory of (F, Frob) in the language of difference rings. Then T has the strict order property and TP2. Moreover, T is not decidable.

With this result, it would be hard to analyse these structures and their theories from classical model theoretic forking independence or ranks on types. However, we will try to understand these structures in terms of coarse dimension $\boldsymbol{\delta}_F$. The idea is to build a link between $\boldsymbol{\delta}_F$ and something we know, or, to find the algebraic meaning of $\boldsymbol{\delta}_F$. One natural candidate is the *transformal transcendence degree*. It is easy to see that $\boldsymbol{\delta}_F$ of a finite tuple is bounded above by its transformal transcendence degree. We suspect these two dimensions agree in all structures in $\mathcal{S}$. Since transformal transcendence degree is totally determined by the quantifier-free type of a tuple, if our conjecture is true, then it means that any definable set of structures in $\mathcal{S}$ is "coarse-dimensionally equivalent" to a quantifier-free definable set.

We give an application of the conjecture, which is aimed to understand definable subgroups of algebraic groups.

Theorem E. Let $(F, \mathrm{Frob}) \in \mathcal{S}$. Suppose $\boldsymbol{\delta}_F$ and transformal transcendence degree coincide in (F, Frob). Let G be a definable subgroup of some algebraic group $H(F) \subseteq F^n$. Then there is a quantifier-free definable group D such that $G \leq D \leq H(F)$ and $\boldsymbol{\delta}_F(D) = \boldsymbol{\delta}_F(G)$.

Chapter 3 is about permutation groups in a dimensional theory. The history can be traced back to the study of groups of small Morley rank. Reineke showed in [Rei75] that a connected group of Morley Rank 1 is abelian and is either elementary abelian or divisible torsion-free. Cherlin proceeded in [Che79] and showed that a connected group of Morley rank 2 is soluble, and a connected simple group of Morley rank 3 with a definable subgroup of Morley rank 2 is isomorphic to $\mathrm{PSL}_2(K)$ for some definable algebraically closed field K. Related to this, in [Hru89] Hrushovski classified permutation groups G acting transitively on a strongly minimal set X in a stable theory into the following three cases:

- The Morley rank of G is 1, and G is connected acting regularly on X;
- The Morley rank of G is 2, and G is isomorphic to $\mathrm{AGL}_1(K)$ for some definable algebraically closed field K, acting on affine line by maps $x \mapsto ax + b$.
- The Morley rank of G is 3, and G is isomorphic to $\mathrm{PSL}_2(K)$ for some definable algebraically closed field K, acting on the projective line $\mathrm{PG}_1(K)$.

These results have been generalised to pseudofinite groups of SU-rank 1 and 2, and pseudofinite definably primitive permutation groups acting on a set of SU-rank 1 in a supersimple finite SU-rank theory in [EJMR11]. There are three key ingredients in this generalization: The first one is that there is a finite integer-valued dimension, SU-rank, that plays the same role as Morley rank in the original results. The second one is the assumption of a tame ambient theory, namely a supersimple theory of finite SU-rank. There are powerful structural theories about definable groups in such theories, for example, the Indecomposability Theorem (see Fact 0.32) and the Stabilizer Theorem. And the third one is the most important one in generalising Hrushovski's result about permutation groups, pseudofiniteness. With this assumption, it is possible to use the knowledge about finite primitive permutation groups and use the *classification of finite simple groups* via the O'Nan-Scott Theorem to analyse the structure of primitive permutation groups of SU-rank at least 3.

The result about pseudofinite groups of SU-rank 1 and 2 have been generalised further in [Wag18], where SU-rank is replaced by an abstract dimension and the tameness assumption of the full theory is replaced by certain chain condition on centralizers, called the $\widetilde{\mathfrak{M}}_c$-*condition*, while the pseudofiniteness assumption is kept. The aim of introducing an abstract dimension is to unify several different dimension-like objects in tame theories, for example the Lascar or SU-rank in stable and simple theories, the o-minimal dimension and the pseudofinite counting dimensions. More precisely, this abstract dimension on interpretable sets is required to be *additive* and takes value in integers. But there is no requirement that dimension 0 sets are always finite, which will include cases of infinite Lascar or SU-rank (in this case, dimension is defined as the coefficient of ω^α for some ordinal α) and coarse pseudofinite dimensions, such as $\boldsymbol{\delta}_F$ in Chapter 2. On the other hand, the $\widetilde{\mathfrak{M}}_c$-condition, which states that there is no infinite chain of centralizers each of infinite index in its predecessor, focuses more on the combinatoric properties that a tame theory should have. This condition itself decreases the complexity of groups and gives some nice structural theorems for definable subgroups (see [Hem15] for more details). However, the powerful tools about groups in tame theories we have mentioned before, such as the Indecomposability Theorem, is no longer available.

Based on Wagner's result on small dimensional pseudofinite $\widetilde{\mathfrak{M}}_c$-groups, the aim of Chapter 3 is to generalise the classification of pseudofinite definably primitive permutation groups with similar assumptions, i.e. the existence of an additive integer-valued dimension and certain chain conditions on subgroups.

Theorem F. Let (G, X) be a pseudofinite definably primitive permutation group with an additive integer-valued dimension dim such that $\dim(X) = 1$, $\dim(G) < \infty$ and G and its definable quotients satisfy the $\widetilde{\mathfrak{M}}_c$-condition.

- If $\dim(G) = 1$, then G has a definable normal abelian subgroup A, such that $\dim(A) = 1$ and A acts regularly on X.
- If $\dim(G) = 2$, then G has a definable normal subgroup H of dimension 2, and there is an interpretable pseudofinite field K of dimension 1 such that (H, X) is definably isomorphic to $(K^+ \rtimes D, K^+)$, where $D \leq K^\times$ is of dimension 1.
- Suppose in addition that there is no infinite descending chain of stabilizers of G each of infinite index in its predecessor, and that X cannot be partitioned into infinitely many definable equivalent classes of dimension 1. If $\dim(G) \geq 3$, then $\dim(G) = 3$, and there is an interpretable pseudofinite field K of dimension 1, such that (G, X) is definably isomorphic to $(H, \mathrm{PG}_1(K))$, where
$$\mathrm{PSL}_2(K) \leq H \leq \mathrm{P\Gamma L}_2(K).$$

In particular, the above result applies to pseudofinite definably primitive groups of infinite SU-rank. In the case when the dimension of the permutation group is at least two, there is always an interpretable pseudofinite field with a group of field-automorphisms. This cannot happen if the ambient theory is simple and the group of automorphisms is infinite. For this reason, a major part of the classification in infinite SU-rank cases collapses to the finite SU-rank case.

Theorem G. Let (G, X) be a pure pseudofinite definably primitive permutation group whose theory is supersimple. Let $\mathrm{SU}(G) = \omega^\alpha n + \gamma$ for some $\gamma < \omega^\alpha$ and $n \geq 1$. Suppose $\mathrm{SU}(X) = \omega^\alpha + \beta$ for some $\beta < \omega^\alpha$. Then one of the following holds:

- $SU(G) = \omega^\alpha + \gamma$, and there is a definable normal abelian subgroup A of SU-rank ω^α which acts regularly on X.

- $SU(G) = 2$, and there is an interpretable pseudofinite field K of SU-rank 1 such that G is definably isomorphic to $K^+ \rtimes D$ where D has finite index in $K^\times$.
- $SU(G) = 3$, and there is an interpretable pseudofinite field K of SU-rank 1 such that G is definably isomorphic to $\mathrm{PSL}_2(K)$ or $\mathrm{PGL}_2(K)$.

The last chapter, Chapter 4, is about an analogue of Schlichting's Theorem for approximate subgroups. Schlichting's Theorem for groups (see Fact 0.36) states that if there is a family of subgroups uniformly commensurable with each other, then there is an invariant one commensurable with all of them. We prove that the same holds for approximate subgroups with the commensurability defined as finitely many translates of one covering the other.

Theorem H. If $\mathcal{X}$ is a uniform family of commensurable approximate subgroups in an ambient group G, then there is an approximate subgroup $H \subseteq G$ such that H is commensurable with $\mathcal{X}$ and invariant under all automorphisms of G stabilizing $\mathcal{X}$ setwise.

This result further highlights similarities between groups and approximate subgroups. However, unlike the case of groups, where the invariant object is obtained by a finite extension of a finite intersection, we need to take infinite unions or infinite intersections to get the invariant approximate subgroup.

Remark: The four main chapters of this thesis are from four corresponding preprints with slight modifications, such as shortening the introduction to avoid repetition and moving some of the facts and definitions to the chapter Preliminaries. Chapter 1 is based on [Zou18b], which is accepted by *The Journal of Symbolic Logic*. Chapter 2 is from [Zou18a]. Chapter 3 corresponds to [Zou18c], which has been submitted. Chapter 4 is based on [Zou18d], which has been submitted as well.

Preliminaries

Notations

We first list some notations and conventions.

- Throughout the thesis, when we talk about languages, we always mean first-order languages, denoted by $\mathcal{L}, \mathcal{L}', \ldots$. We write $M, N, \ldots$ and $\mathcal{M}, \mathcal{N}, \cdots$ for models, T for a first-order theory and $Th(M)$ for *the theory of* M, i.e. the collection of all sentences that are true in M.

- Let M be a κ-saturated model for a regular cardinal κ. We denote by $a, b, c, \ldots$ finite tuples of elements, $A, B, C, \ldots$ parameter sets whose size are small, that is of size at strict less than κ. We will denote by $\varphi, \psi, \phi, \cdots$ formulas (possibly with parameters), $x, y, z, \cdots$ tuples of variables, $|x|$ and $|a|$ the length of the corresponding tuple, and $|\varphi|$ the length of the formula φ.

- Suppose M is an $\mathcal{L}$-structure and $\varphi(x)$ an $\mathcal{L}$-formula with parameters in M. We write $\varphi(M^{|x|})$ to be the definable set given by $\varphi(x)$ in M, i.e.

$$\varphi(M^{|x|}) := \{a \in M^{|x|} : M \models \varphi(a)\}.$$

- $\mathbb{F}_q$ will denote the finite field with q elements, similarly, $\mathbb{F}_{p^n}$ denotes the finite field of characteristic p with p^n elements. $\tilde{\mathbb{F}}_p$ will be the algebraic closure of $\mathbb{F}_p$. If F is a field, we denote the additive group as F^+ and multiplicative group as $F^\times$.

- We denote by $\mathbb{P}$ the set of prime numbers.

- If G is a group and $g_0, \ldots, g_n \in G$, we will write $Z(G)$ for the center of G and $C_G(g_0, \cdots, g_n)$ the centralizer of $g_0, \ldots, g_n$, that is

$$C_G(g_1, \cdots, g_n) := \{h \in G : hg_i = g_i h, \text{ for all } i \leq n\}.$$

  If $H \leq G$ is a subgroup, and $h, g \in G$, we write h^g for $g^{-1}hg$ and H^g for $g^{-1}Hg$. We denote $N_G(H)$ the normalizer of H in G, i.e. $N_G(H) := \{g \in G : H^g = H\}$. We also write the index of the subgroup H in G as $[G : H]$.

Ultraproducts and pseudofinite structures

Ultraproducts and ultrapowers are fundamental constructions in model theory. They are useful tools to construct explicitly models of theories from existing ones in a way that resulting models have nicer properties, e.g. saturation.

Let $\mathcal{L}$ be a language, I an index set and $\{M_i : i \in I\}$ a family of $\mathcal{L}$-structures. Let $\mathcal{U}$ be an ultrafilter on I. We denote by $M := \prod_{i \in I} M_i/\mathcal{U}$ the ultraproduct of $\{M_i : i \in I\}$ with respect to $\mathcal{U}$. If $\{a_i \in (M_i)^n : i \in I\}$ is a family of n-tuples, we denote by $(a_i)_{i \in I}/\mathcal{U}$ the corresponding tuple in M^n.

The fundamental theorem about ultraproducts is Łoś's Theorem, which gives a transfer principle between the structures $\{M_i, i \in I\}$ and their ultraproduct $\prod_{i \in I} M_i/\mathcal{U}$.

Fact 0.1. (Jerzy Łoś, 1955) Let $M = \prod_{i \in I} M_i/\mathcal{U}$ be an ultraproduct of $\mathcal{L}$-structures $\{M_i, i \in I\}$ with respect to an ultrafilter $\mathcal{U}$ on I. Then for any $\mathcal{L}$-formula $\varphi(x)$ and $a := (a_i)_{i \in I}/\mathcal{U} \in M^{|x|}$, we have

$$M \models \varphi(a) \text{ if and only if } \{i \in I : M_i \models \varphi(a_i)\} \in \mathcal{U}.$$

As we have mentioned before, a certain saturation can be obtained by the ultraproduct construction.

Fact 0.2. (see [Gar18, Proposition 1.6]) Let $M = \prod_{i \in I} M_i/\mathcal{U}$ be an ultraproduct with respect to a non-principal ultrafilter $\mathcal{U}$ on an infinite set I. Then M is $\aleph_1$-saturated.

Definition 0.3. Let $M = \prod_{i \in I} M_i/\mathcal{U}$ be an ultraproduct. A set $A \subseteq M^n$ is called *internal* if $A = \prod_{i \in I} A_i/\mathcal{U}$ where $A_i \subseteq (M_i)^n$ for each $i \in I$.

Pseudofinite structures can be defined using ultraproducts.

Definition 0.4. An $\mathcal{L}$-structure is called *pseudofinite* if M is elementary equivalent to an ultraproduct of finite $\mathcal{L}$-structures.

The following fact states that there are several equivalent definitions of pseudofinite structures.

Fact 0.5. (see [Gar18, Proposition 1.4]) Let M be an $\mathcal{L}$-structure. Then the following are equivalent:

1. M is pseudofinite;
2. Every sentence true in M has a finite model;
3. For any sentence, if it is satisfied in all finite $\mathcal{L}$-structures, then it is satisfied in M.

Pseudofinite counting dimensions

Fix an ultraproduct of finite structures $\mathcal{M} := \prod_{i\in I} M_i/\mathcal{U}$. Let $\mathbb{R}^* := \prod_{i\in I} \mathbb{R}/\mathcal{U}$ be the non-standard reals. Then any internal set $D \subseteq M^n$ has a non-standard cardinality $|D| \in \mathbb{R}^*$, as does any internal interpretable sets $D \subseteq M^n/E$ where $E \subseteq M^n \times M^n$ is an internal equivalence relation. In the following we will define the *pseudofinite counting dimension* δ_C with respect to a convex subgroup $C \supseteq \mathbb{R}$. The fine and coarse pseudofinite dimensions are special cases of δ_C. We will specify them later.

Definition 0.6. Let C be a non-zero convex subgroup of $(\mathbb{R}^*, +)$ containing $\mathbb{R}$. The *pseudofinite counting dimension* δ_C *with respect to* C is a function from all interpretable sets in $\mathcal{M}$ to the quotient group $(\mathbb{R}^*/C, +)$, defined as

$$\delta_C(D) := \log|D| + C$$

for an interpretable set D in $\mathcal{M}$.

Remark: $\mathbb{R}^*/C$ is an ordered $\mathbb{Q}$-vector space.

Fact 0.7. ([Hru12, section 5]) Properties of δ_C:

- $\delta_C(X) = 0$ for finite X;
- $\delta_C(X \cup Y) = \max\{\delta_C(X), \delta_C(Y)\}$;
- $\delta_C(X \times Y) = \delta_C(X) + \delta_C(Y)$;
- (subadditivity) Let $f : X \to Y$ be an interpretable function. If $\delta_C(f^{-1}(y)) \leq \alpha$ for all $y \in Y$ and $\delta_C(Y) \leq \beta$, then $\delta_C(X) \leq \alpha + \beta$.
- Let X be an interpretable set. The interpretable subsets Y of X with $\delta_C(Y) < \delta_C(X)$ form an ideal.

We now define the fine and coarse dimension.

Definition 0.8. Let C_{fin} be the smallest convex subgroup in $(\mathbb{R}^*, +)$ containing $\mathbb{R}$. The *fine pseudofinite dimension* or shortly *fine dimension* is defined as $\delta_{C_{fin}}$, written as δ_{fin}.

Remark:([Hru13, section 2]) Among all δ_C, the characteristic feature of δ_{fin} is that any dimension $\alpha \in \mathbb{R}^*/C_{fin}$ comes with a *real-valued measure* μ_α (up to a scalar multiple) such that

- $\mu_\alpha(X) = 0$ iff $\delta_{\text{fin}}(X) < \alpha$;
- $\mu_\alpha(X) = \infty$ iff $\delta_{\text{fin}}(X) > \alpha$;
- if $\delta_{\text{fin}}(X) = \delta_{\text{fin}}(Y) = \alpha$, then $\mu_\alpha(X) = \text{st}\,(|X|/|Y|)\,\mu_\alpha(Y)$, in which $\text{st} : \mathbb{R}^* \to \mathbb{R} \cup \{\pm\infty\}$ is the standard part map.

If X is an internal set with $\delta_{\text{fin}}(X) = \alpha$, we can define $\mu_\alpha(D) := \text{st}(|D|/|X|)$.

While the fine dimension is the finest pseudofinite counting dimension, coarse dimension is the coarsest one if one has an internal set X in mind and the dimension does not give X value 0. Let $\alpha := \log|X| \in \mathbb{R}^*$ and $C_{<\alpha}$ be the largest convex subgroup that does not contain α. The *coarse dimension normalised by* α is defined as $\delta_{C_{<\alpha}}$ and is denoted as $\boldsymbol{\delta}_\alpha$. In fact, $C_{<\alpha}$ has an explicit definition

$$C_{<\alpha} = \bigcap_{n\in\mathbb{N}} \{\beta : -\alpha < n\beta < \alpha\}.$$

Claim 0.9. Let

$$V_\alpha := \{\beta \in \mathbb{R}^* : \text{ exists } n \in \mathbb{N} \text{ with } -n\alpha \leq \beta \leq n\alpha\}.$$

Then there is a natural isomorphism $(V_\alpha/C_{<\alpha}, +, \leq) \to (\mathbb{R}, +, \leq)$ mapping α to 1.

Proof. Let $\tau : (V_\alpha, +, \leq) \to (\mathbb{R}, +, \leq)$ be defined as $\tau(\beta) := \text{st}(\beta/\alpha)$. It is easy to see that τ is a surjective group homomorphism with kernel $C_{<\alpha}$. Thus we have the desired result. □

Remark: Instead of working in $\mathbb{R}^*/C_{<\alpha}$, people are more used to dealing with $\mathbb{R}\cup\{\pm\infty\}$ via the map τ defined before and regard elements in $(\mathbb{R}^*/C_{<\alpha})\backslash(V_\alpha/C_{<\alpha})$ as $\pm\infty$. Hence, we often use the following definition for coarse dimension $\boldsymbol{\delta}_\alpha$ instead.

Definition 0.10. Let $\mathcal{M} := \prod_{i\in I} M_i/\mathcal{U}$ be an ultraproduct of finite $\mathcal{L}$-structures. The *coarse dimension on* $\mathcal{M}$ *normalised by* α, denoted as $\boldsymbol{\delta}_\alpha$, is a function from interpretable sets of $\mathcal{M}$ to $\mathbb{R}^{\geq 0} \cup \{\infty\}$, defined as

$$\boldsymbol{\delta}_\alpha(A) := \text{st}\left(\frac{\log|A|}{\alpha}\right),$$

for $A \subseteq M^n/E$ interpretable. When $\alpha := \log|X|$ for some internal set X, we also write $\boldsymbol{\delta}_\alpha$ as $\boldsymbol{\delta}_X$ and call $\boldsymbol{\delta}_X$ the *coarse pseudofinite dimension with respect to* X.

In an ultraproduct of finite $\mathcal{L}$-structures, pseudofinite counting dimensions always exist. However, if the language is not expressive enough, there might be no link between these dimensions and the theory. In fact, δ_C could have different values for definable sets defined by $\varphi(x, a)$ and $\varphi(x, b)$ where a and b have the same type. This is not in the spirit of model theory where we take types rather than elements as the main objects of study. The following definition ensures invariance for coarse dimension.

Definition 0.11.

- We say $\boldsymbol{\delta}_\alpha$ is *continuous* if for any $\emptyset$-definable formula $\phi(x, y)$, for any $r_1 < r_2 \in \mathbb{R}$, there is some $\emptyset$-definable set D with

$$\{a \in M^{|y|} : \boldsymbol{\delta}_\alpha(\phi(M^{|x|}, a)) \leq r_1\} \subseteq D \subseteq \{a \in M^{|y|} : \boldsymbol{\delta}_\alpha(\phi(M^{|x|}, a)) < r_2\}.$$

- We say $\boldsymbol{\delta}_\alpha$ is *definable* if $\boldsymbol{\delta}_\alpha$ is continuous and the set $\{\boldsymbol{\delta}_\alpha(\phi(M^{|x|}, a)) : a \in M^{|\bar{y}|}\}$ is finite for any $\emptyset$-definable formula $\phi(x, y)$. By compactness, it is equivalent to the following: for any $\emptyset$-definable formula $\phi(x, y)$ and $a \in M^{|y|}$, there is $\xi(y) \in \text{tp}(a)$ such that

$$M \models \xi(b) \text{ if and only if } \boldsymbol{\delta}_\alpha(\phi(M^{|x|}, b)) = \boldsymbol{\delta}_\alpha(\phi(M^{|x|}, a)).$$

Remark: If X is $\emptyset$-definable, we can always make $\boldsymbol{\delta}_X$ continuous by adding the *cardinality comparison quantifier*:

$$(Cx)\varphi(y_0, y_1) \Leftrightarrow |\varphi(\mathcal{M}, y_0)| \leq |\varphi(\mathcal{M}, y_1)|.$$

This is because given $0 < a < b \in \mathbb{R}$, let $a < \frac{n}{m} < b$ with $n, m \in \mathbb{N}$, then the $\emptyset$-definable set $D := \{y : |\varphi(\mathcal{M}, y)^m| \leq |X^n|\}$ satisfies

$$\{y : \boldsymbol{\delta}_X(\varphi(\mathcal{M}, y)) \leq a\} \subseteq D \subseteq \{y : \boldsymbol{\delta}_X(\varphi(\mathcal{M}, y)) < b\}.$$

However, expanding the language might add new definable sets to the original structure, which could be an inconvenience.

Definition 0.12. Let M be a pseudofinite structure and $\alpha \in \mathbb{R}^*$. Let a be a finite tuple in M and $A \subseteq M$. Define

$$\boldsymbol{\delta}_\alpha(a/A) := \inf\left\{\boldsymbol{\delta}_\alpha(\varphi(M^{|x|})), \varphi(x) \in \mathrm{tp}(a/A)\right\}.$$

Fact 0.13. ([Hru13, Lemma 2.10]) If $\boldsymbol{\delta}_\alpha$ is continuous, then $\boldsymbol{\delta}_\alpha$ is additive, i.e. for any $a, b, A \subseteq M$ we have $\boldsymbol{\delta}_\alpha(a, b/A) = \boldsymbol{\delta}_\alpha(a/A, b) + \boldsymbol{\delta}_\alpha(b/A)$.

One-dimensional asymptotic classes

One-dimensional asymptotic classes are classes of finite structures with a nicely behaved dimension and counting measure on all families of uniformly definable sets. They are introduced in [MS08] inspired by the class of finite fields. Basically, ultraproducts of one-dimensional asymptotic classes will give rise to pseudofinite structures with well behaved fine pseudofinite dimension. Namely, for a uniformly definable family of definable sets, the fine dimensions of them take a finite set of discrete values and for any such value, if we look at the measure that comes with this fine dimension, then there are only finitely many possible values within this definable family. Moreover, both the dimension and the measure are definable.

We start with the case of finite fields.

Fact 0.14. ([CvdDM92, Main Theorem]) Let $\mathcal{L}$ be the language of rings. For every formula $\varphi(x, y) \in \mathcal{L}$ with $|x| = n$ and $|y| = m$ there are a constant $C_\varphi > 0$, a finite set $D_\varphi \subset \{0, \ldots, n\} \times \mathbb{Q}^{>0}$ and formulas $\psi_{d,\mu}(y)$ for any $(d, \mu) \in D_\varphi$ such that the following holds:

- For any finite field $\mathbb{F}_q$ and $a \in (\mathbb{F}_q)^m$, if $\varphi((\mathbb{F}_q)^n, a) \neq \emptyset$, then there is some $(d, \mu) \in D_\varphi$ such that

$$||\varphi((\mathbb{F}_q)^n, a)| - \mu \cdot q^d| \leq C_\varphi \cdot q^{d-\frac{1}{2}}. \qquad (\star)$$

- The formula $\psi_{d,\mu}(y)$ defines in each $\mathbb{F}_q$ the set of tuples a such that $(\star)$ holds.

Now we recall the definition of a one-dimensional asymptotic class and list some examples and properties of them.

Definition 0.15. Fix a language $\mathcal{L}$. A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is called a *one-dimensional asymptotic class* if the following holds: For every $m \in \mathbb{N}^{>0}$ and every formula $\varphi(x, y)$ with $|x| = 1$ and $|y| = m$:

1. There is a positive constant C and a finite set $E \subseteq \mathbb{R}^{>0}$ such that for any $M \in \mathcal{C}$ and $b \in M^m$, either $|\varphi(M, b)| < C$ or there is $\mu \in E$ with
$$||\varphi(M, b)| - \mu|M|| < C \cdot |M|^{\frac{1}{2}}.$$
2. For every $\mu \in E$ there is an $\mathcal{L}$-formula $\varphi_\mu(y)$ such that for any $M \in \mathcal{C}$ and $b \in M^m$
$$M \models \varphi_\mu(b) \text{ if and only if } ||\varphi(M, b)| - \mu|M|| < C \cdot |M|^{\frac{1}{2}}.$$

Remark: Note that the definition only requires that families of definable subsets of structures are uniformly definable. The higher dimensional families can be obtained from it.

Fact 0.16. ([MS08, Theorem 2.1]) Let $\mathcal{C}$ be a one dimensional class of finite $\mathcal{L}$-structures. For every formula $\varphi(x, y) \in \mathcal{L}$ with $|x| = n, |y| = m$ there are a constant $C_\varphi > 0$, a finite set $D_\varphi \subset \{0, \ldots, n\} \times \mathbb{R}^{>0}$ and formulas $\psi_{d,\mu}(y)$ for any $(d, \mu) \in D_\varphi$ such that the following holds:

- For any $M \in \mathcal{C}$ and $a \in M^m$, if $\varphi(M^n, a) \neq \emptyset$, then there is some $(d, \mu) \in D_\varphi$ such that
$$||\varphi(M^n, a)| - \mu \cdot |M|^d| \leq C_\varphi \cdot |M|^{d-\frac{1}{2}}. \qquad (\star\star)$$
- The formula $\psi_{d,\mu}(y)$ defines in each M the set of tuples a such that $(\star\star)$ holds.

Examples of one-dimensional asymptotic classes are:

- The class of finite fields.
- The class of finite-dimensional vector spaces over a fixed finite field.
- The class of finite cyclic groups.

The ultraproducts of one-dimensional classes give infinite structures that are model theoretically tame.

Fact 0.17. ([MS08, Lemma 4.1]) Let $\mathcal{C}$ be a one-dimensional asymptotic class and M an infinite ultraproduct of members of $\mathcal{C}$. Then $Th(M)$ is supersimple of SU-rank 1.

Shelah's dividing lines

While studying the categoricity problem, Michael Morley proposed a problem concerning the number of non-isomorphic models for a complete theory in uncountable cardinalities, which was solved by Saharon Shelah in [She90]. To do this, Shelah developed *classification theory*, where he drew several dividing lines in first-order theories through their ability to encode certain combinatorial configurations. Theories that cannot code complicated configurations are considered *tame*, while theories with too strong coding power are considered *wild*, for example Peano Arithmetic and ZFC.

We list the definitions of some of the important tame classes here.

Definition 0.18. A formula $\varphi(x, y)$ has the *order property* in T if there is a model M and $(a_i, b_i)_{i<\omega}$ such that $M \models \varphi(a_i, b_j)$ if and only if $i < j$.

T is *stable* if no formula has the order property in T.

Definition 0.19. A formula $\varphi(x, y)$ has the *independence property* in T if there is a model M and $(a_i)_{i<\omega}$ and $(b_I)_{I\subseteq\omega}$ such that $M \models \varphi(a_i, b_I)$ if and only if $i \in I$.

T is *NIP* if no formula has the independence property in T.

Definition 0.20. A formula $\varphi(x, y)$ has the *tree property* in T if there is $(b_\eta)_{\eta\in\omega^{<\omega}}$ and some $k \geq 2$ such that

- for all $\sigma \in \omega^\omega$, $\{\varphi(x, b_{\sigma\restriction n}) : n < \omega\}$ is consistent;
- for all $\eta \in \omega^{<\omega}$, $\{\varphi(x, b_{\eta^\frown n}) : n < \omega\}$ is k-inconsistent;

T is *simple* if no formula has the tree property in T.

Definition 0.21. A formula $\varphi(x, y)$ has the *tree property 2* (TP2) in T if there is $(a_{i,j})_{i,j<\omega}$ and $k \geq 2$ such that

- for all $\sigma \in \omega^\omega$, $\{\varphi(x, a_{n,\sigma(n)}) : n < \omega\}$ is consistent;
- for all $n < \omega$, $\{\varphi(x, a_{n,j}) : j < \omega\}$ is k-inconsistent;

T is *NTP2* if no formula has TP2 in T.

Definition 0.22. A formula $\varphi(x, y)$ has the *strict order property* in T if there is a model M and $(a_i)_{i<\omega}$ such that $\varphi(M^{|x|}, a_i) \subsetneq \varphi(M^{|x|}, a_j)$ for all $i < j$.

T is *NSOP* if no formula has the strict order property in T.

The following fact is easy to see, it indicates the inclusion of the tame classes.[1]

Fact 0.23. We write "property A implies property B" to denote if a formula $\varphi(x, y)$ has property A in T, then it also has property B in T.

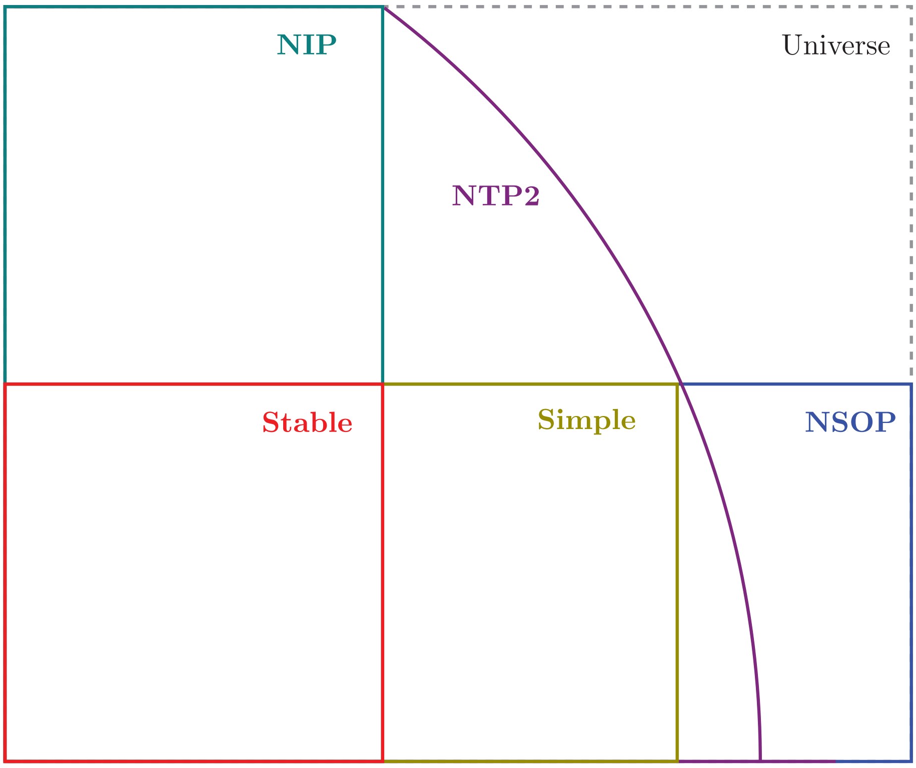
- Tree property implies order property.
- Independence property implies order property.
- TP2 implies tree property and independence property.
- Strict order property implies tree property.

[1] For detailed inclusions of the classes and more definitions according to Shelah's dividing lines, see http://www.forkinganddividing.com.

The following fact is proved by Shelah.

Fact 0.24. ([She71]) A theory T is stable if and only if it is both NSOP and NIP if only if it is both simple and NIP.

The above facts correspond to the following diagram in terms of theories.



Groups in simple and supersimple theories

Groups in simple theories

As we defined in the previous section, simple theories are theories that cannot define a "tree-like" configuration (the tree property) by a formula. There are other characterisations of simple theories, notably in terms of the local rank $D(\cdot, \varphi, k)$ and of the existence of an independence relation with some nice properties.

Groups in simple theories enjoy a lot of structural properties. Most of them can be deduced from the local rank and the well-behaved forking independence in simple theories.

To state the results about groups in simple theories in full generality, we recall the notion of *hyper-definability*.

Definition 0.25. Let M be a structure. A set X is *hyper-definable* over $A \subseteq M$ if there is a type-definable set $Y \subseteq M^n$ for some $n \in \mathbb{N}$ and a type-definable equivalence relation E on Y both defined over A such that $X = Y/E$.

Now we list some facts about groups in simple theories. The following one is a very useful tool to show certain definable sets generates a definable subgroup.

Fact 0.26. [Wag00, Lemma 4.4.8] Let G be a type-definable/hyper-definable group in a simple theory. Let X be a non-empty type-definable/hyper-definable subset of G. Suppose for independent $g, g' \in X$ we have $g^{-1} \cdot g' \in X$, and put $Y = X \cdot X$. Then Y is a type-definable/hyper-definable subgroup of G, and X is generic in Y. In fact, X contains all generic types for Y.

We also state the *Group Chunk Theorem* here. Basically, it says that in a simple theory, if there is a group-like object that are only defined partially on the "generic parts", then we can reconstruct a group from it.

Fact 0.27. [Wag00, Theorem 4.7.1] We fix an ambient simple theory. Let π be a partial type and $\star$ be a partial type-definable function defined on pairs of independent realizations of π, both over $\emptyset$ such that

1. Generic independence: for independent realizations a, b of π the product $a \star b$ realizes π and is independent from a and from b;
2. Generic associativity: for three independent realizations a, b, c of π, we have $(a \star b) \star c = a \star (b \star c)$;
3. Generic surjectivity: for any independent a, b realizing π, there are c and c' independent from a and from b, with $a \star c = b$ and $c' \star a = b$.

Then there are a hyper-definable group G and a hyper-definable bijection from π to the generic types of G, such that generically $\star$ is mapped to the group multiplication. G is unique up to definable isomorphism.

Groups in supersimple theories

Supersimple theoreis are defined in terms of a global rank on types, called the *Lascar rank or SU-rank* induced from forking extensions. We recall the definition of Lascar rank here.

Definition 0.28. Let $\mathrm{Ord} \cup \{\infty\}$ be the class of ordinals together with an extra element ∞ which is greater than any element in the ordinals. The *SU-rank* or *Lascar rank* is the least function from all types to $\mathrm{Ord} \cup \{\infty\}$ satisfying:

$$\mathrm{SU}(p) \geq \alpha + 1 \text{ if there is a forking extension } q \text{ of } p \text{ with } \mathrm{SU}(q) \geq \alpha.$$

T is called *supersimple* if $\mathrm{SU}(p) < \infty$ for any type p in T.

Let a be a tuple and A be a small set of parameters in a monster model. We denote $\mathrm{SU}(\mathrm{tp}(a/A))$ as $\mathrm{SU}(a/A)$. The following inequality is the fundamental inequality for SU-rank.

Fact 0.29. (see [Wag00, Theorem 5.1.6]) In any theory, we have the following inequality, called the *Lascar Inequality*:

$$\mathrm{SU}(a/bA) + \mathrm{SU}(b/A) \leq \mathrm{SU}(ab/A) \leq \mathrm{SU}(a/bA) \oplus \mathrm{SU}(b/A),$$

where $+$ is the ordinal sum, $\oplus$ is the natural sum (or the Hessenberg sum) and the operations with ∞ are defined as $\infty + \alpha = \alpha + \infty = \infty + \infty = \infty$ and $\infty \oplus \alpha = \alpha \oplus \infty = \infty \oplus \infty = \infty$ for any ordinal α.

Let G be an interpretable group in a theory and $H \leq G$ be an interpretable subgroup. Let G/H be the left coset space, it is an interpretable set. Then the Lascar inequality specialises to the following case for interpretable groups.

Fact 0.30. Lascar inequality for groups:

$$\mathrm{SU}(H) + \mathrm{SU}(G/H) \leq \mathrm{SU}(G) \leq \mathrm{SU}(H) \oplus \mathrm{SU}(G/H).$$

In supersimple theories, often, when we study groups we only talk about properties of them up to finite index. This gives rise to an important notion: *commensurability*.

Definition 0.31. Let H and D be two subgroups of G. We say G is *commensurable* with H if $[G : G \cap H]$ and $[H : H \cap G]$ are both finite.

One of the most powerful tool in groups of finite Morley rank is the *Indecomposability Theorem*. It has a corresponding generalization for groups in supersimple theories.

Fact 0.32. (Indecomposability Theorem, [Wag18, Theorem 5.4.5]) Let G be an interpretable group in a simple theory with $\mathrm{SU}(G) < \omega^{\alpha+1}$, and $\mathcal{X}$ a family of interpretable subsets of G. Then there exists an interpretable subgroup H of G with $H \subseteq X_0^{\pm 1} \cdots X_n^{\pm 1}$ for some $X_0, \cdots, X_n \in \mathcal{X}$ such that $\mathrm{SU}(XH) < \mathrm{SU}(H) + \omega^{\alpha}$ for all interpretable $X \subseteq \langle \mathcal{X} \rangle$ (and in particular for all $X \in \mathcal{X}$). Moreover, H is unique up to commensurability.

In particular, if $\mathrm{SU}(G) < \omega$, then X_i/H is finite for each $i \in I$.

Moreover, if the collection $\mathcal{X}$ is setwise invariant under some group Σ of definable automorphisms of G, then H can be chosen to be Σ-invariant.[2]

We list in the following three facts about groups in supersimple theories that will be used in Chapter 3.

Fact 0.33. ([Wag00, Theorem 5.4.3]) Suppose G is an interpretable group defined in a supersimple theory and $\mathrm{SU}(G) = \sum_{j \leq k} \omega^{\alpha_j} n_j$ with $\alpha_0 > \alpha_1 > \cdots > \alpha_k$ and put $\beta_i = \sum_{j \leq i} \omega^{\alpha_j} n_j$ for $i \leq k$. Then G has an interpretable normal subgroup G_i of SU-rank β_i which is unique up to commensurability.

Fact 0.34. ([Wag00, Theorem 5.4.9]) Suppose G is an interpretable, interpretably simple (G has no interpretable proper non-trivial normal subgroup) non-abelian group in a simple theory with $\mathrm{SU}(G) < \infty$. Then G is simple and $\mathrm{SU}(G) = \omega^{\alpha} n$ for some ordinal α and $n < \omega$.

Fact 0.35. ([Wag00, Lemma 5.5.3]) Suppose G is a type-definable group over $\emptyset$ in a supersimple theory with $\mathrm{SU}(G) = \omega^{\alpha} n$. Then there are a definable super group G_0 of G and definable subgroups G_i of G_0 for $i \in I$ with $G = \bigcap_{i \in I} G_i$

[2] This is because H is unique up to commensurability, so we can apply Schlichting's Theorem for all such H, see Fact 0.36.

The last fact I want to recall about groups is a general fact that does not depend on the theory. It is called Schlichting's Theorem, first discovered in [Sch80] with the focus on the existence of normal subgroups.

Fact 0.36. ([Wag00, Theorem 4.2.4]) Let G be a group and $\mathcal{F}$ be a family of subgroups of G. If there is some $n \in \mathbb{N}$ such that $[H : H \cap H'] < n$ for all H and $H' \in \mathcal{F}$, then there is a subgroup N which is commensurable with every member of $\mathcal{F}$ and invariant under all automorphisms of G which stabilize $\mathcal{F}$ set-wise.

Moreover, $\bigcap \mathcal{F} \leq N \leq \langle \mathcal{F} \rangle$, and N is a finite extension of a finite intersection of groups in $\mathcal{F}$. In particular, if $\mathcal{F}$ is a family of definable/interpretable groups, then N is also definable/interpretable.

Chapter 1

Pseudofinite H-structures

1.1 Introduction

H-structures are introduced in [BV16]. They are based on a geometric theory, where algebraic closure satisfies the exchange property and $\exists^\infty$ is eliminated. When a dense and co-dense independent subset is added to a model of this theory, the resulting structure is an H-structure. Strongly minimal theories, supersimple SU-rank one theories and superrosy thorn-rank one theories with elimination of $\exists^\infty$ are examples of geometric theories. In these cases, the corresponding H-structures preserve ω-stability, supersimplicity or superrosiness and the rank is either one or ω.

In the following, we will recall the definition of H-structures and some of their main properties.

Let T be a complete geometric theory in a language $\mathcal{L}$. Let H be a unary predicate and put $\mathcal{L}_H = \mathcal{L} \cup \{H\}$. Let $M \models T$; we say that $A \subseteq M$ is finite dimensional if $A \subseteq \operatorname{acl}_{\mathcal{L}}(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in M$. For a tuple a and a set of parameters A, we write $\dim_{\operatorname{acl}_{\mathcal{L}}}(a/A)$ as the length of a maximal $\operatorname{acl}_{\mathcal{L}}$-independent subtuple of a over A.

Definition 1.1. We say that $(M, H(M))$ is an *H-expansion of M* [1] if:

1. $M \models T$;
2. $H(M)$ is an $\operatorname{acl}_{\mathcal{L}}$-independent subset of M;
3. (Density/coheir property) If $A \subseteq M$ is finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in H(M)$ such that $a \models q$;
4. (Extension property) If $A \subseteq M$ is finite dimensional and $q \in S_1(A)$ is non-algebraic, then there is $a \in M$, $a \models q$ and $a \notin \operatorname{acl}_{\mathcal{L}}(A \cup H(M))$.

Equivalently, we can replace density and extension properties with the following more general ones:

[1] It is just called an H-structure in [BV16], we add this terminology to be more precise about the base theory or the base model.

- (Generalised density/coheir property) If $A \subseteq M$ is finite dimensional and $q \in S_n(A)$ has dimension n, then there is $a \in H(M)^n$ such that $a \models q$;
- (Generalised extension property) If $A \subseteq M$ is finite dimensional and $q \in S_n(A)$ is non-algebraic, then there is $a \in M^n$, $a \models q$ and

$$\dim_{\mathrm{acl}_{\mathcal{L}}}(a/A, H(M)) = \dim_{\mathrm{acl}_{\mathcal{L}}}(a/A).$$

A structure $\mathcal{M}$ is called an *H-structure* if it is an H-expansion of some model of a geometric theory.

H-structures are closely related to lovely pairs, where, instead of an independent subset, a dense and co-dense elementary substructure is added. We recall the definition of lovely pairs in the special case that the base theory is geometric, see [BV10].

Definition 1.2. Let T be a geometric theory in a language $\mathcal{L}$ and let $\mathcal{L}_P$ be the expansion of $\mathcal{L}$ by a unary predicate P. An $\mathcal{L}_P$-structure (M, N) is a *lovely pair of models of T*, if

1. $M \models T$;
2. N is an $\mathcal{L}$-elementary submodel of M;
3. (Density/coheir property) If $A \subseteq M$ is finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in N$ such that $a \models q$;
4. (Extension property) If $A \subseteq M$ is finite dimensional and $q \in S_1(A)$ is non-algebraic, then there is $a \in M$, $a \models q$ and $a \notin \mathrm{acl}_{\mathcal{L}}(A \cup N)$.

Fact 1.3. [BV16], [BV10]. Properties of H-structures and lovely pairs.

Let T be a complete geometric theory in a language $\mathcal{L}$.

- H-expansions of models of T exist and all of them are $\mathcal{L}_H$-elementary equivalent. Let T_H be the corresponding theory. Similarly, lovely pairs of models of T exist, and all of them are $\mathcal{L}_P$-elementary equivalent.
- If the geometry of T is nontrivial and T is strongly minimal/supersimple/superrosy of rank 1, then T_H is ω-stable/supersimple/superrosy of rank ω.
- Let $(M, H(M))$ be an H-structure. Then $(M, \mathrm{acl}_{\mathcal{L}}(H(M)))$ is a lovely pair.

Consider the theory of pseudofinite fields. It is supersimple of SU-rank one. By the fact above, H-expansions and lovely pairs of pseudofinite fields exist. However, the proof of existence uses general model theoretic techniques such as saturated models and union of chains. It is not clear whether it is possible to have H-expansions or lovely pairs of pseudofinite fields that are ultraproducts of finite structures.

The answer turns out to be negative for lovely pairs.

Lemma 1.4. *If (K, k) is a lovely pair of pseudofinite fields, then it is not pseudofinite.*[2]

[2] This was already noticed by Gareth Boxall (private communication).

Proof. Let $(K', k') = \prod_{i \in I}(K'_i, k'_i)/\mathcal{U}$ be a pair of pseudofinite fields with $\mathrm{char}(K') = \mathrm{char}(k')$ such that $k'_i \subsetneq K'_i$ are finite fields for any $i \in I$.

Suppose $\mathrm{char}(K') \neq 2$. We will show that there are $a_1, a_2 \in K'$ and $\varphi(x; y_1, y_2)$ in the language of rings such that $\varphi(x; a_1, a_2)$ is non-algebraic, but there is no $b \in k'$ such that $\varphi(b; a_1, a_2)$ holds. However, as (K, k) is a lovely pair, the following holds in (K, k):

$$\forall y_1 \forall y_2 (\exists^\infty x\ \varphi(x; y_1, y_2) \to \exists z \in k\ \varphi(z; y_1, y_2)).$$

Therefore, (K, k) is not elementary equivalent to (K', k').

As $\mathrm{char}(K') \neq 2$, we may assume that $\mathrm{char}(K_i) \neq 2$ for all $i \in I$. For any $i \in I$ take $\sigma_i \in Gal(K'_i/k'_i)$ with $\sigma_i \neq id$. Let $a_{i_1}, a_{i_2} \in K'_i$ be such that $\sigma_i(a_{i_1}) = a_{i_2}$ and $a_{i_1} \neq a_{i_2}$. Let $\sigma = (\sigma_i)_{i \in I}/\mathcal{U}$, $a_1 := (a_{i_1})_{i \in I}/\mathcal{U}$ and $a_2 := (a_{i_2})_{i \in I}/\mathcal{U}$. Then $a_1 \neq a_2$, $\sigma(a_1) = a_2$ and $k' \subseteq \mathrm{Fix}(\sigma)$. Define

$$\varphi(x; y_1, y_2) := (\exists z\ z^2 = x - y_1)\ \wedge\ \neg(\exists z\ z^2 = x - y_2).$$

We claim that $\varphi(x; a_1, a_2)$ is non-algebraic in K'. Since $\mathrm{char}(K'_i) \neq 2$ for any $i \in I$, we have $\{x^2 : x \in K'_i\} \subsetneq K'_i$. Let e_i be such that there is no $x \in K'_i$ with $x^2 = e_i$. Then by [Dur80, Proposition 4.3], the ideal generated by $\{(X_1)^2 - (X - a_{i_1});\ (X_2)^2 - e_i(X - a_{i_2})\}$ is absolutely prime and does not contain $X - a_{i_1}$ or $X - a_{i_2}$. Let V be the corresponding irreducible variety. Then V has dimension 1; by the Lang-Weil estimate $|V \cap K'_i| \approx |K'_i|$. We claim that $K_i \models \varphi(x; a_{i_1}, a_{i_2})$ for any $(x_1, x_2, x) \in V \cap K'_i$ with $x \neq a_{i_2}$. Since if not, there is some x_3 such that $x - a_{i_2} = (x_3)^2$. As $x \neq a_{i_2}$, we have $x_3 \neq 0$. Then $e_i = (\frac{x_2}{x_3})^2$, contracting that e_i is not a square-root. Therefore, we can define a function

$$\tau_i : (V \cap K'_i) \setminus \{(x_1, x_2, a_{i_2}) : x_1, x_2 \in K'_i\} \to \varphi(K'_i; a_{i_1}, a_{i_2})$$

by $\tau_i(x_1, x_2, x) := x$. As $\mathrm{char}(K'_i) \neq 2$, it is easy to see that τ_i is a four-to-one function. By that $|V \cap K'_i| \approx |K'_i|$, we conclude that

$$|\varphi(K'_i; a_{i_1}, a_{i_2})| \approx \frac{1}{4}|V \cap K'_i|.$$

Thus, $\varphi(x; a_1, a_2)$ is non-algebraic.

On the other hand, for any $b \in k'$ we have

$$\exists z(z^2 = b - a_1) \iff \exists z(\sigma(z^2) = \sigma(b - a_1)) \iff \exists z(\sigma(z)^2 = b - a_2) \iff \exists z(z^2 = b - a_2).$$

Therefore, there is no $b \in k'$ such that $\varphi(b; a_1, a_2)$ holds.

The case of $\mathrm{char}(K') = 2$ is similar, using cubes instead of squares (and possibly going to some finite extension of K'). $\square$

In view of the close connection between H-structures and lovely pairs, we might expect H-expansions of pseudofinite fields never to be pseudofinite. Luckily, this is not so. In fact, we can see from the proof above that the reason (K', k') is not a lovely pair is the existence of a nontrivial automorphism σ of K' that fixes k'. In the case of H-expansions, instead of a subfield we only need to add a subset. Intuitively, we might be able to choose a pseudofinite set large enough such that no non-trivial automorphism can fix all the points in this set.

Definition 1.5. Let T be a geometric theory in a language $\mathcal{L}$. Let $\mathcal{M} = \prod_{i\in I} M_i/\mathcal{U} \models T$ be an infinite ultraproduct of finite structures. We call an H-expansion $(\mathcal{M}, H(\mathcal{M}))$ an *exact pseudofinite H-expansion of $\mathcal{M}$* if $(\mathcal{M}, H(\mathcal{M})) = \prod_{i\in I}(M_i, H_i)/\mathcal{U}$ with $H_i \subseteq M_i$ for all $i \in I$.

Remark: Let $\mathcal{M} = \prod_{i\in I} M_i/\mathcal{U} \models T$ be pseudofinite. Then an arbitrary pseudofinite H-expansion need not to be exact, since it need not be this particular ultraproduct. For example, let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$ and $V = \prod_{i\in\mathbb{N}} V_n/\mathcal{U}$ an ultraproduct of finite vector spaces over $\mathbb{F}_2$ such that $\lim_{n\in\mathbb{N}} \dim(V_n) = \infty$. It is easy to build an exact pseudofinite H-expansion of V by choosing an independent set $H_n \subseteq V_n$ for each $n \in \mathbb{N}$ with $\lim\limits_{n\in\mathbb{N}} \dim(H_n) = \lim\limits_{n\in\mathbb{N}} \operatorname{codim}(H_n) = \infty$ and put $(V, H) = \prod_{n\in\mathbb{N}}(V_n, H_n)/\mathcal{U}$. Let $H' \subseteq V$ be a countable independent set of V. Then (V, H') is pseudofinite H-expansion of V as $(V, H') \equiv (V, H)$. But (V, H') is not $\aleph_1$-saturated, hence cannot be an ultraproduct over non-principal ultrafilters. Thus (V, H') is not exact.

Let $\mathcal{C}$ be a one-dimensional asymptotic class and $\mathcal{M}$ be an infinite ultraproduct of members of $\mathcal{C}$. In section 1.2 we show that exact pseudofinite H-expansions of $\mathcal{M}$ always exist. In particular, pseudofinite H-expansions of pseudofinite fields do exist.

Section 1.3 deals with definable groups in H-structures. Our motivation is to classify definable groups in H-expansions of pseudofinite fields. There are some results about definable groups in H-structures when the base theory is superstable in [BV16] using the group configuration theorem. The problem to generalise these results is that in simple (even in supersimple) theories, there is no nice version of the group configuration theorem available in general. However, pseudofinite fields are exceptional: the group configuration theorem for pseudofinite fields has essentially been given in [HP94]. We can easily deduce that definable groups in H-expansions of pseudofinite fields are virtually isogenous to algebraic groups.

However, this is not very satisfactory. It is of course the best one could get when one compares definable groups in H-expansions of pseudofinite fields with algebraic groups. But as has been noticed in [BV16], "since the geometry on H is trivial, we expected adding H should not introduce new definable groups". With the help of the group chunk theorem in simple theories (see Fact 0.27) we give a more satisfactory answer, namely, there are no new definable groups in H-structures when the base theory is supersimple of SU-rank one. Notably, Eleftheriou also got a same classification of definable groups in H-structures in the setting of o-minimal theories using the similar strategy, see [Ele18, Theorem 1.2].

1.2 Pseudofinite H-structures

This section deals with pseudofinite H-structures built from one-dimensional asymptotic classes.

Notation: In this section, we will distinguish elements and tuples by denoting elements as $a, b, c, \ldots$ and tuples as $\bar{a}, \bar{b}, \bar{c}, \ldots$, same for variables and tuples of variables. We will denote $\varphi(x; \bar{y})$ for formulas in variable x and parameters $\bar{y}$, where parameters have not been specified yet.[3]

[3] This notation is only kept for this section, in other sections and other chapters, we use the standard notation.

Definition 1.6. Let $\mathcal{C}$ be a one-dimensional asymptotic class in a language $\mathcal{L}$. Let $\varphi(x;\bar{y})$ ($\bar{y}$ non-empty) be an $\mathcal{L}$-formula and $E \subseteq \mathbb{R}^{>0}$ be as in Definition 0.15. Put

$$\psi_\varphi(\bar{y}) := \bigvee_{\mu \in E} \varphi_\mu(\bar{y}).$$

For a structure $M \in \mathcal{C}$ and a subset $X \subseteq M$, we say X *covers* $\psi_\varphi(\bar{y})$ *in* M if the following holds:

$$\bigcup_{x \in X} \varphi(x; M^{|\bar{y}|}) \supseteq \psi_\varphi(M^{|\bar{y}|}).$$

Let $\phi(x;\bar{y})$ be a formula. Suppose $\phi(x;\bar{y})$ is algebraic ($\bar{y}$ can be empty) over any $\bar{y}$. For a structure $M \in \mathcal{C}$ and a linearly-ordered subset $X \subseteq M$, we say that X *avoids* $\phi(x;\bar{y})$ *in* M if there is no $x, x_1, \dots, x_{|\bar{y}|} \in X^{|\bar{y}|+1}$ such that $x > \max\{x_1, \dots, x_{|\bar{y}|}\}$ and

$$M \models \phi(x; x_1, \dots, x_{|\bar{y}|}).$$

Let $\mathcal{M}$ be an infinite ultraproduct of members of $\mathcal{C}$. For any $\varphi(x,\bar{y})$ and $\bar{a} \in \mathcal{M}^{|\bar{y}|}$, if $\mathcal{M} \models \psi_\varphi(\bar{a})$, then there is $\mu \in E$ such that $|\varphi(\mathcal{M},\bar{a})| \approx \mu|\mathcal{M}|$. As $\mu > 0$ and $\mathcal{M}$ is infinite, we get $\varphi(\mathcal{M},\bar{a})$ is infinite. On the other hand, if $\mathcal{M} \models \neg\psi_\varphi(\bar{a})$, then by the definition one-dimensional asymptotic class, there must be some $C \in \mathbb{N}$ such that $|\varphi(\mathcal{M},\bar{a})| \leq C$. Therefore, $\psi_\varphi(\bar{y})$ defines the set of $\bar{a}$ such that $\varphi(x,\bar{a})$ is non-algebraic in any infinite ultraproduct of members of $\mathcal{C}$.

Lemma 1.7. *Let $\mathcal{C}$ be a one-dimensional asymptotic class, Γ be a finite set of algebraic formulas of the form $\phi(x;\bar{z})$ ($\bar{z}$ could be empty) and Δ any finite set of formulas of the form $\varphi(x;\bar{y})$ (the length of $\bar{y}$ can vary and $\bar{y}$ is non-empty). Then there are $N_{\Delta,\Gamma} \in \mathbb{N}$ and $C_{\Delta,\Gamma} \in \mathbb{R}^{>0}$ such that the following holds:*

For any $M \in \mathcal{C}$ with $|M| \geq N_{\Delta,\Gamma}$, there exists $(H_{\Delta,\Gamma}(M), \leq)$ with $H_{\Delta,\Gamma}(M) \subseteq M$ and $|H_{\Delta,\Gamma}(M)| \leq C_{\Delta,\Gamma} \cdot \log|M|$ such that for any $\varphi(x;\bar{y}) \in \Delta$ and $\phi(x;\bar{z}) \in \Gamma$, we have $H_{\Delta,\Gamma}(M)$ covers $\psi_\varphi(\bar{y})$ and avoids $\phi(x;\bar{z})$ in M.

In particular, $|M| \geq N_{\Delta,\Gamma}$ should imply the equation (1.2) and the inequality (1.3), which are defined throughout the proof.

Proof. By Definition 0.15, for each $\varphi(x;\bar{y}) \in \Delta$ there are finitely many $\mu_{0,\varphi}, \dots, \mu_{k_\varphi,\varphi} > 0$ and $C_\varphi \in \mathbb{R}$, such that for any $M \in \mathcal{C}$ and $\bar{a} \in M^{|\bar{y}|}$,

$$\psi_\varphi(\bar{a}) \implies \bigvee_{j \leq k_\varphi} (||\varphi(M;\bar{a})| - \mu_{j,\varphi} \cdot |M|| < C_\varphi \cdot |M|^{\frac{1}{2}}).$$

Take $0 < \mu < \min\{\mu_{0,\varphi}, \dots, \mu_{k_\varphi,\varphi} : \varphi \in \Delta\}$. Let

$$\mathcal{C}_\mu := \bigcap_{\varphi \in \Delta} \{M \in \mathcal{C} : \text{ for any } \bar{a},\ \psi_\varphi(\bar{a}) \text{ implies } |\varphi(M;\bar{a})| \geq \mu \cdot |M|\}.$$

We claim that there is some $N \in \mathbb{N}$ such that for any $M \in \mathcal{C}$ and $|M| > N$, we have $M \in \mathcal{C}_\mu$. Otherwise, there are $\varphi(x;\bar{y}) \in \Delta, \mu_{i_0,\varphi} > 0$ and $\{M_i \in \mathcal{C}, \bar{a}_i \in M_i^{|\bar{y}|} : i \in \mathbb{N}\}$ such that the following holds:

- $\lim_{i\to\infty} |M_i| = \infty$;

- $M_i \models \varphi_{\mu_{i_0,\varphi}}(\bar{a}_i)$ for each $i \in \mathbb{N}$;
- $|\varphi(M_i;\bar{a}_i)| < \mu \cdot |M_i| < \mu_{i_0,\varphi} \cdot |M_i|$ for each $i \in \mathbb{N}$.

Therefore,

$$\mu_{i_0,\varphi} \cdot |M_i| - |\varphi(M_i;\bar{a}_i)| > (\mu_{i_0,\varphi} - \mu) \cdot |M_i| = (\mu_{i_0,\varphi} - \mu) \cdot |M_i|^{\frac{1}{2}} \cdot |M_i|^{\frac{1}{2}}.$$

By the definition of one-dimensional asymptotic class, there is some $C_\varphi > 0$ such that

$$||\varphi(M_i;\bar{a}_i)| - \mu_{i_0,\varphi} \cdot |M_i|| < C_\varphi \cdot |M_i|^{\frac{1}{2}}.$$

Since $\lim_{i\to\infty}(\mu_{i_0,\varphi} - \mu) \cdot |M_i|^{\frac{1}{2}} = \infty$, there is clearly a contradiction.

Assume $\Delta = \{\varphi_1(x;\bar{y}_1), \ldots, \varphi_n(x;\bar{y}_n)\}$. Fix any $M \in \mathcal{C}$ with $|M| > N$, for $1 \leq i \leq n$, define inductively the following sets: $X^i_j, L^i_j, H^i_j \subseteq M$ and $Y^i_j \subseteq \psi_{\varphi_i}(M^{|\bar{y}_i|})$.

- $Y^1_0 := \psi_{\varphi_1}(M^{|\bar{y}_1|})$;
- $X^1_0 := H^1_0 := L^1_0 := \emptyset$;

Suppose $Y^i_j, X^i_j, H^i_j, L^i_j$ are defined. There are two cases.

- If $Y^i_j = \emptyset$ and $i < n$, define
  - $Y^{i+1}_0 := \psi_{\varphi_{i+1}}(M^{|\bar{y}_{i+1}|})$;
  - $X^{i+1}_0 := L^{i+1}_0 := \emptyset$;
  - $H^{i+1}_0 := H^i_j$.
- If $Y^i_j \neq \emptyset$, define
  - $L^i_{j+1} := \bigcup_{\phi(x;\bar{z})\in\Gamma}\{a \in M : \exists \bar{z} \in (H^i_j)^{|\bar{z}|}, M \models \phi(a;\bar{z})\} \cup \bigcup_{\phi'(x)\in\Gamma} \phi'(M)$.
  - $X^i_{j+1} := M \setminus (H^i_j \cup L^i_{j+1})$.
  - Choose an element h^i_{j+1} in X^i_{j+1} such that $\varphi_i(h^i_{j+1}; Y^i_j)$ has the maximal cardinality among $\{\varphi_i(a; Y^i_j) : a \in X^i_{j+1}\}$.
  - $H^i_{j+1} := H^i_j \cup \{h^i_{j+1}\}$ and $Y^i_{j+1} = Y^i_j \setminus \varphi_i(h^i_{j+1}; Y^i_j)$.

The construction stops either when Y^n_j is empty, that is H^i_j covers $\psi_{\varphi_i}(\bar{y}_i)$ for any $1 \leq i \leq n$, or when $Y^i_j \neq \emptyset$ and $X^i_{j+1} = \emptyset$ for some $1 \leq i \leq n$ and $j \in \mathbb{N}$.

Let $Y^1_0, \ldots, Y^i_j$ be a maximal sequence of the construction. Define $H_{\Delta,\Gamma}(M) := H^i_j$ if $i = n$ and $Y^i_j = \emptyset$.

Claim 1.8. There is $N_{\Delta,\Gamma} \in \mathbb{N}$ such that if $M \in \mathcal{C}$ and $|M| \geq N_{\Delta,\Gamma}$, then $H_{\Delta,\Gamma}(M)$ is always defined.

Proof. Suppose $|M| > N$ and $M \in \mathcal{C}$. We first estimate the size of Y^i_{j+1} in terms of Y^i_j when the latter is not empty during the construction of $\{H^i_j, Y^i_j, L^i_j, X^i_j : i \leq n, j \geq 0\}$.

Suppose all $\phi(x;\bar{z}) \in \Gamma$ have no more than C-many solutions over any parameter $\bar{z}$ ($\bar{z}$ can be empty). Let $C_\Gamma := C \cdot |\Gamma|$ and $k_0 := \max\{|\bar{z}| : \phi(x;\bar{z}) \in \Gamma\}$. Then $|L^i_{j+1}| \leq C_\Gamma \cdot (|H^i_j|+1)^{k_0}$.[4]

Therefore,

$$|X^i_{j+1}| \geq |M| - C_\Gamma \cdot (|H^i_j|+1)^{k_0} - |H^i_j|. \tag{1.1}$$

By construction, $Y^i_{j+1} = Y^i_j \backslash \{\varphi_i(h^i_{j+1};Y^i_j)\}$. As $\varphi_i(h^i_{j+1};Y^i_j)$ is maximal among $\{\varphi_i(a;Y^i_j) : a \in X^i_{j+1}\}$, we get

$$|\varphi_i(h^i_{j+1};Y^i_j)| \geq \frac{|\bigcup_{a\in X^i_{j+1}}\{(a,\bar{y}) : \bar{y} \in \varphi_i(a;Y^i_j)\}|}{|X^i_{j+1}|} \geq \frac{|\bigcup_{a\in X^i_{j+1}}\{(a,\bar{y}) : \bar{y} \in \varphi_i(a;Y^i_j)\}|}{|M|}.$$

Let $\mathrm{Tot} := \bigcup_{x\in(M\backslash H^i_j)}\{(x,\bar{y}) : \bar{y} \in \varphi_i(x;Y^i_j)\}$, then

$$\bigcup_{a\in X^i_{j+1}}\{(a,\bar{y}) : \bar{y} \in \varphi_i(a;Y^i_j)\} = \mathrm{Tot} \setminus \bigcup_{a\in L^i_{j+1}}\{(a,\bar{y}) : \bar{y} \in \varphi_i(a;Y^i_j)\}.$$

As $M \in \mathcal{C}_\mu$, for each $\bar{y} \in Y^i_j$ we have $|\varphi_i(M;\bar{y})| \geq \mu \cdot |M|$. And by the definition of Y^i_j, for any $\bar{y} \in Y^i_j$, if $M \models \varphi_i(a;\bar{y})$, then $a \notin H^i_j$. Hence, $|\mathrm{Tot}| \geq \mu \cdot |M| \cdot |Y^i_j|$. On the other hand,

$$|\bigcup_{a\in L^i_{j+1}}\{(a,\bar{y}) : \bar{y} \in \varphi_i(a;Y^i_j)\}| \leq \left|L^i_{j+1}\right| \cdot \left|Y^i_j\right| \leq C_\Gamma \cdot (|H^i_j|+1)^{k_0} \cdot |Y^i_j|.$$

Hence,

$$|\varphi_i(h^i_{j+1};Y^i_j)| \geq \frac{\mu \cdot |M| \cdot |Y^i_j| - C_\Gamma \cdot (|H^i_j|+1)^{k_0} \cdot |Y^i_j|}{|M|} = \left(\mu - \frac{C_\Gamma \cdot (|H^i_j|+1)^{k_0}}{|M|}\right)|Y^i_j|.$$

Let $\ell_0 := \max\{|\bar{y}_i| : 1 \leq i \leq n\}$. Define

$$h_M := \lceil \frac{\ell_0 \cdot \log|M|}{-\log(1-\mu/2)} \rceil + 1. \tag{1.2}$$

Then there is some $N_{\mu/2}$ such that whenever $|M| \geq N_{\mu/2}$, we have

$$\frac{C_\Gamma \cdot (n \cdot h_M + \ell_0)^{k_0}}{|M|} \leq \frac{\mu}{2}. \tag{1.3}$$

In particular, we have

$$\frac{C_\Gamma \cdot (n \cdot h_M + 1)^{k_0}}{|M|} \leq \frac{\mu}{2}. \tag{1.4}$$

Therefore, when $|H^i_j| \leq n \cdot h_M$, we have $|\varphi_i(h^i_{j+1};Y^i_j)| \geq \frac{\mu}{2}|Y^i_j|$, and hence,

$$|Y^i_{j+1}| = |Y^i_j| - |\varphi_i(h^i_{j+1};Y^i_j)| \leq \left(1 - \frac{\mu}{2}\right)|Y^i_j|.$$

[4] Since we need to include the algebraic elements over $\emptyset$ defined by formulas in Γ, it can be that $H^i_j = \emptyset$ but $L^i_{j+1} \neq \emptyset$, that's the reason we put $|H^i_j|+1$ instead of $|H^i_j|$.

Consequently,

$$|Y^i_{j+1}| \leq \left(1-\frac{\mu}{2}\right)|Y^i_j| \leq \left(1-\frac{\mu}{2}\right)^2|Y^i_{j-1}| \leq \cdots \leq \left(1-\frac{\mu}{2}\right)^{j+1}|Y^i_0| \leq \left(1-\frac{\mu}{2}\right)^{j+1}\cdot|M|^{\ell_0}.$$

There is some $N_{\Delta,\Gamma} > \max\{N_{\mu/2}, N\}$ such that whenever $|M| > N_{\Delta,\Gamma}$, we have $(1-\frac{\mu}{2})\cdot |M| > n \cdot h_M$. Fix some $M \in \mathcal{C}$ with $|M| > N_{\Delta,\Gamma}$ and let

$$Y^1_0, \dots, Y^1_{t_1}; \cdots ,; Y^i_0, \dots, Y^i_{t_i}$$

be a maximal sequence. We claim that for each $i' \leq i$, if $|H^{i'}_{t_{i'}}| \leq n \cdot h_M$, then $t_{i'} \leq h_M$. Otherwise, $Y^{i'}_{h_M}$ is in the sequence. By the argument above, $|Y^{i'}_{h_M}| \leq (1-\frac{\mu}{2})^{h_M}\cdot|M|^{\ell_0}$. By calculation, we have

$$k > \frac{\ell_0 \cdot \log|M|}{-\log(1-\mu/2)} \implies \left(1-\frac{\mu}{2}\right)^k \cdot |M|^{\ell_0} < 1.$$

Hence, $Y^{i_0}_{h_M} = \emptyset$. We conclude $t_{i_0} \leq h_M$. Therefore, $t_1 \leq h_M$ and by induction, for each $1 \leq i' \leq n$, we have $|H^{i'}_{t_{i'}}| = \sum_{1\leq j \leq i'} t_j \leq i' \cdot h_M$. Now we can see that $|H^i_{t_i}| \leq n \cdot h_M$.

Consider the set $X^i_{t_i+1}$. By inequality (1.1),

$$|X^i_{t_i+1}| \geq |M| - C_\Gamma \cdot (|H^i_{t_i}|+1)^{k_0} - |H^i_{t_i}| \geq |M| - C_\Gamma\cdot(n\cdot h_M+1)^{k_0} - n\cdot h_M.$$

By inequality (1.4) and $(1-\frac{\mu}{2})\cdot|M| > n \cdot h_M$, we get

$$|X^i_{t_i+1}| \geq |M| - \frac{\mu}{2}|M| - n\cdot h_M > 0.$$

Hence $X^i_{t_i+1} \neq \emptyset$. As $Y^i_{t_i}$ is the end term of a maximal sequence, it can only be the case that $Y^i_{t_i} = \emptyset$ and $i = n$.

Therefore, if $|M| > N_{\Delta,\Gamma}$ and $M \in \mathcal{C}$, then $H_{\Delta,\Gamma}(M)$ exists and

$$|H_{\Delta,\Gamma}(M)| \leq n \cdot h_M \leq C_{\Delta,\Gamma}\cdot\log|M|,$$

where $C_{\Delta,\Gamma} := n \cdot \left(\lceil \frac{\ell_0}{-\log(1-\mu/2)}\rceil + 1\right)$. □

Take any $M \in \mathcal{C}$ with $|M| \geq N_{\Delta,\Gamma}$, let $H_{\Delta,\Gamma}(M)$ as defined in Claim 1.8 and for $h^i_j, h^t_m \in H_{\Delta,\Gamma}$, define $h^i_j \leq h^t_m$ if $i < t$ or $i = t$ and $j \leq m$. By construction we have $(H_{\Delta,\Gamma}(M), \leq)$ covers $\psi_\varphi(\bar{y})$ and avoids $\phi(x,\bar{y})$ in M for any $\varphi \in \Delta$ and $\phi(x,\bar{y}) \in \Gamma$. □

Theorem 1.9. *Let $\mathcal{C}$ be a one-dimensional asymptotic class in a countable language $\mathcal{L}$. Let $\mathcal{M} := \prod_{i\in I} M_i/\mathcal{U}$ be an infinite ultraproduct of members among $\mathcal{C}$. Then exact pseudofinite H-expansions of $\mathcal{M}$ exist.*

Proof. Let $\{\varphi_i(x;\bar{y}_i), i \in \mathbb{N}\}$ be a list of all formulas in $\mathcal{L}$ such that x is in one variable and $\bar{y}_i \neq \emptyset$ is a tuple of variables. For $n \in \mathbb{N}$, let $\Delta_n := \{\varphi_i(x;\bar{y}_i) : i \leq n\}$.

Let $\{\xi_i(x;\bar{z}_i) : i \in \mathbb{N}\}$ be a list of all formulas such that $\xi_i(x;\bar{z}_i)$ is algebraic ($\bar{z}_i$ can be empty). Let $\Gamma_n := \{\xi_i(x;\bar{z}_i) : i \leq n\}$.

By Lemma 1.7, there are $N_{\Delta_n,\Gamma_n} \in \mathbb{N}$ such that for any $M \in \mathcal{C}$ with $|M| \geq N_{\Delta_n,\Gamma_n}$ there exists $(H_{\Delta_n,\Gamma_n}(M), \leq)$ with $H_{\Delta_n,\Gamma_n}(M) \subseteq M$ such that $H_{\Delta_n,\Gamma_n}(M)$ covers $\psi_\varphi(\bar{y})$ and avoids $\xi(x;\bar{z})$ in M for all $\varphi \in \Delta_n$ and $\xi(x,\bar{z}) \in \Gamma_n$.

For any $i \in I$, let $i_n := \max\{n : |M_i| \geq N_{\Delta_n,\Gamma_n}\}$ (set $\max\emptyset = -\infty$). Define $H_i := H_{\Delta_{i_n},\Gamma_{i_n}}(M_i)$ if $i_n \neq \infty$; otherwise let $H_i := \emptyset$.

Claim 1.10. $(\mathcal{M}, H(\mathcal{M})) := \prod_{i\in I}(M_i, H_i)/\mathcal{U}$ is an exact pseudofinite H-expansion of $\mathcal{M}$.

Proof. We only need to show that $(\mathcal{M}, H(\mathcal{M}))$ is an H-expansion of $\mathcal{M}$. We verify the conditions one by one.

1. $\mathcal{M} \models \mathrm{Th}_\mathcal{L}(\mathcal{M})$: clear.

2. $H(\mathcal{M})$ is an $\mathrm{acl}_\mathcal{L}$-independent subset: Suppose, towards a contradiction, that there are $\{a_0, a_1, \ldots, a_k\}$ which are not $\mathrm{acl}_\mathcal{L}$-independent. We may assume that any proper subset of $\{a_0, a_1, \ldots, a_k\}$ is an $\mathrm{acl}_\mathcal{L}$-independent set. Suppose for $0 \leq t \leq k$, each $a_t := (a^i_t)_{i\in I}/\mathcal{U}$. Let $O := (i_0 i_1 \cdots i_k)$ be an ordering of $0, 1, \ldots, k$. Define
$$I_O := \{j \in I : (a^j_{i_0}, a^j_{i_1}, \ldots, a^j_{i_k}) \text{ is increasing in } (H_j, \leq)\}.$$
Let A be the collections of all the orderings of $0, 1, \ldots, k$. Since A is finite and $I = \bigcup_{O\in A} I_O$, we have exactly one $I_O \in \mathcal{U}$. We may assume that $O = (0 \cdots k)$. Suppose $a_i \in \mathrm{acl}_\mathcal{L}(\{a_0, \ldots, a_k\} \setminus \{a_i\})$. By assumption, $a_i \notin \mathrm{acl}_\mathcal{L}(\{a_0, \ldots, a_k\} \setminus \{a_i, a_k\})$. Since $\mathrm{acl}_\mathcal{L}$ satisfies the exchange property, we have $a_k \in \mathrm{acl}_\mathcal{L}(a_0, \ldots, a_{k-1})$. Let $\varphi(x; z_0, \ldots, z_{k-1})$ witness algebraicity (i.e., $\varphi(x; z_0, \ldots, z_{k-1})$ is algebraic and $\mathcal{M} \models \varphi(a_k; a_0, \ldots, a_{k-1})$). By the list of all algebraic formulas, $\varphi(x; z_0, \ldots, z_{k-1}) = \xi_j(x; z_0, \ldots, z_{k-1}) := \xi_j(x; \bar{z}_j)$ for some j.
Let $J := \{i \in I : i_n \geq j\} = \{i \in I : |M_i| \geq N_{\Delta_j,\Gamma_j}\}$. Since $\mathcal{M}$ is infinite, $J \in \mathcal{U}$. For any $i \in J$, we have $\xi_j(x; \bar{z}_j) \in \Gamma_{i_n}$, hence H_i avoids $\xi_j(x; \bar{z}_j)$. As $a^i_k > \max\{a^i_0, \ldots, a^i_{k-1}\}$ in H_i, by construction, the set H_i avoids $\xi_j(x; \bar{z}_j)$, we get
$$M_i \models \neg\xi_j(a^i_k; a^i_0, \ldots, a^i_{k-1})$$
for any $i \in J$. We conclude $\mathcal{M} \models \neg\xi_j(a_k; a_0, \ldots, a_{k-1})$, contradiction.

3. Density/coheir property: As $(\mathcal{M}, H(\mathcal{M}))$ is pseudofinite, it is $\aleph_1$-saturated. Therefore, we only need to show that for any $a_0, \ldots, a_k \in \mathcal{M}$, if $\varphi(x; a_0, \ldots, a_k)$ is non-algebraic, then there is $h \in H(\mathcal{M})$ such that $\mathcal{M} \models \varphi(h; a_0, \ldots, a_k)$. We may assume that $\varphi(x; y_0, \ldots, y_k) = \varphi_j(x; \bar{y}_j)$.
Let $J := \{i \in I : i_n \geq j\} = \{i \in I : |M_i| \geq N_{\Delta_j,\Gamma_j}\}$. Then $J \in \mathcal{U}$. Note that $\varphi_j(x; \bar{y}_j) \in \Delta_{i_n}$ for any $i \in J$. Therefore H_i covers $\psi_{\varphi_j}(\bar{y}_j)$ in M_i for any $i \in J$.
Suppose $a_t := (a^i_t)_{i\in I}/\mathcal{U}$ for $0 \leq t \leq k$. Let
$$J' := \{i \in J : M_i \models \psi_{\varphi_j}(a^i_0, \ldots, a^i_k)\}.$$
As $\varphi_j(x; a_0, \ldots, a_k)$ is non-algebraic, $J' \in \mathcal{U}$.
For any $i \in J'$, since H_i covers $\psi_{\varphi_j}(\bar{y}_j)$ in M_i and $M_i \models \psi_{\varphi_j}(a^i_0, \ldots, a^i_k)$, there is some $h_i \in H_i$ such that $M_i \models \varphi_j(h_i; a^i_0, \ldots, a^i_k)$. For $i \notin J'$, choose $h_i \in M_i$ randomly. Let $h := (h_i)_{i\in I}/\mathcal{U}$. Then $h \in H(\mathcal{M})$ and $\mathcal{M} \models \varphi_j(h; a_0, \ldots, a_k)$, i.e., $\mathcal{M} \models \varphi(h; a_0, \ldots, a_k)$.

4. Extension Property: Suppose $A \subseteq \mathcal{F}$ is finite dimensional. Let $A' = \{a_0, \ldots, a_k\}$ be a base of A. Suppose $a_t := (a_t^i)_{i\in I}/\mathcal{U}$ for each $t \leq k$. Let $A'_i = \{a_0^i, \ldots, a_k^i\} \subseteq M_i$. Let

$$\mathrm{clos}_i(H_i \cup A'_i) := \bigcup_{j \leq i_n,\ \bar{a} \in (H_i \cup A'_i)^{|\bar{z}_j|}} \xi_j(M_i; \bar{a}),$$

and define $\mathrm{clos}(H(\mathcal{M}') \cup A') := \prod_{i\in I} \mathrm{clos}_i(H_i \cup A'_i)/\mathcal{U}$. By essentially the same argument as $\mathrm{acl}_{\mathcal{L}}$-independence of $H(\mathcal{M})$, we have

$$\mathrm{acl}_{\mathcal{L}}(H(\mathcal{M}) \cup A) \subseteq \mathrm{clos}(H(\mathcal{M}) \cup A').$$

By the fact that $(\mathcal{M}, \mathrm{clos}(H(\mathcal{M}) \cup A'))$ is pseudofinite, hence $\aleph_1$-saturated, we only need to show that for any $b_0, \ldots, b_t \in A$, if $\varphi(x; b_0, \ldots, b_t)$ is non-algebraic, then there is $a \in \mathcal{M} \setminus \mathrm{clos}(H(\mathcal{M}) \cup A')$ such that $\mathcal{M} \models \varphi(a; b_0, \ldots, b_t)$. We may assume that $\varphi(x; y_0, \ldots, y_t) = \varphi_j(x; \bar{y}_j)$. Assume $b_k = (b_k^i)_{i\in I}/\mathcal{U}$ for $k \leq t$. There is some $J \in \mathcal{U}$ and $\mu > 0$ such that for all $i \in J$, we have $|\varphi(M_i; b_0^i, \ldots, b_t^i)| \geq \mu \cdot |M_i|$.

Consider the size of $\mathrm{clos}_i(H_i \cup A')$. We have

$$|\mathrm{clos}_i(H_i \cup A')| \leq C_{\Gamma_{i_n}} \cdot (|H_i \cup A'|)^{k_0},$$

where as above $\Gamma_{i_n} := \{\xi_j(x; \bar{z}_j) : j \leq i_n\}$, $k_0 := \max\{|\bar{z}_j| : j \leq i_n\}$ and $C_{\Gamma_{i_n}} := (i_n + 1) \cdot C$ with C is the largest number of solutions of ξ_j over parameters for $j \leq i_n$.

Let $\Delta_{i_n} := \{\varphi_j(x; \bar{y}_j) : j \leq i_n\}$ and $\ell_0 := \max\{|\bar{y}_j| : j \leq i_n\}$. Note that there is some $J' \in \mathcal{U}$ such that for all $i \in J'$ we have $k \leq \ell_0$. Hence

$$|H_i \cup A'| \leq |H_i| + k \leq |\Delta_{i_n}| \cdot h_{M_i} + \ell_0,$$

where h_{M_i} is defined as the equation (1.2). By the inequality (1.3), we have

$$C_{\Gamma_{i_n}} \cdot (|\Delta_{i_n}| \cdot h_{M_i} + \ell_0)^{k_0} \leq \frac{\mu}{2} \cdot |M_i|.$$

Therefore,

$$|\mathrm{clos}_i(H_i \cup A')| \leq C_{\Gamma_{i_n}} \cdot (|H_i \cup A'|)^{k_0} \leq \frac{\mu}{2} \cdot |M_i|,$$

for all $i \in J \cap J'$.

As $|\varphi(M_i; b_0^i, \ldots, b_t^i)| \geq \mu \cdot |M_i|$, there must be some

$$a_i \in \varphi(M_i; b_0^i, \ldots, b_t^i) \setminus \mathrm{clos}_i(H_i \cup A')$$

for all $i \in J \cap J'$. Choose a_i at random for $i \notin J \cap J'$. Set $a := (a_i)_{i\in I}/\mathcal{U}$, then $a \notin \mathrm{clos}(H \cup A')$ and $\mathcal{M} \models \varphi(a; b_0, \ldots, b_t)$. □

□

Corollary 1.11. *Let $\mathcal{C}$ be a one-dimensional asymptotic class in a language $\mathcal{L}$ and $\mathcal{M}$ be an infinite ultraproduct of members of $\mathcal{C}$. Suppose $\mathrm{acl}_{\mathcal{L}}$ of $Th_{\mathcal{L}}(\mathcal{M})$ is non-trivial. Then the exact pseudofinite H-expansion $(\mathcal{M}, H(\mathcal{M}))$ is a pseudofinite structure whose theory is supersimple of SU-rank ω.*

Remark: Let $\mathcal{M} := \prod_{i\in I} M_i/\mathcal{U}$ be an infinite ultraproduct of a one-dimensional asymptotic class. We can also make the H-expansion $(\mathcal{M}, H(\mathcal{M})) := \prod_{i\in I}(M_i, H_i)/\mathcal{U}$ satisfying

$$\lim_{i\in I} \frac{\log |H_i|}{\log |M_i|} = 0 \quad \text{that is} \quad \boldsymbol{\delta}_{\mathcal{M}}(H(\mathcal{M})) = 0,$$

that is the pseudofinite coarse dimension of $H(\mathcal{M})$ with respect to $\mathcal{M}$ is zero.

This is because by Lemma 1.7 we know that $|H_i| = C_{\Delta_{i_n},\Gamma_{i_n}} \cdot \log |M_i|$ where $C_{\Delta_{i_n},\Gamma_{i_n}}$ depends only on Δ_{i_n} and Γ_{i_n}. If we redefine

$$i_n := \max\{n : |M_i| > N_{\Delta_n,\Gamma_n} \text{ and } |M_i| > (C_{\Delta_n,\Gamma_n})^n\},$$

we see that additionally $\boldsymbol{\delta}_{\mathcal{M}}(H(\mathcal{M})) = 0$.

Note that for generic element $m \in M$, we have $\mathrm{SU}_H(m) = \omega$ while $\mathrm{SU}_H(h) < \omega$ for any element $h \in H(M)$. In a following project, together with other collaborators, we found this fact generalises to all definable sets. That is, the coarse dimension of a definable set equals to the coefficient of the ω-part of the SU-rank of generic elements. We also wonder if $(M_i)_{i\in I}$ is a one-dimensional asymptotic class, then the class $(M_i, H_i)_{i\in I}$ we build in Claim 1.10 forms a multidimensional asymptotic class. We expect this should involve a more detailed treatment of definable sets in H-structures.

1.3 Groups in H-structures

This section deals with definable groups in H-structures when the base theory is supersimple of SU-rank one. We ask whether there are any new definable groups in H-structures. As we said before, in [BV16] the authors have partially solved the question by showing that in stable theories the connected component of an $\mathcal{L}_H$-definable group in an H-structure is isomorphic to some $\mathcal{L}$-definable group. We record their results here.

Fact 1.12. ([BV16, Proposition 6.5])

Let D be a group in a language $\mathcal{L}$ with $RM(D) = 1$ and assume that (D, H) is an $\aleph_0$-saturated H-structure. Let $A \subseteq D$ be finite and let $G \leq D^n$ be an $\mathcal{L}_H$-definable subgroup defined over A. Then G is $\mathcal{L}$-definable over A.

Fact 1.13. ([BV16, Proposition 6.6])

Let M be a stable structure of U-rank one in a language $\mathcal{L}$ and let H be a subset of M such that (M, H) is an $\aleph_1$-saturated H-structure. Let $A \subseteq M$ be countable and let $G \subseteq M^n$ be an $\mathcal{L}_H$-definable group over A. Let G^0 be the connected component of G. Then G^0 is definably isomorphic to an $\mathcal{L}$-definable group over A.

In this section, we will show that in supersimple theories, all $\mathcal{L}_H$-definable groups in H-structures are definably isomorphic to $\mathcal{L}$-definable groups.[5]

We first introduce some basic notions and facts about H-structures developed in [BV16].

[5] Indeed, we need to assume that the base theory has elimination of imaginaries. Fact 1.12 and 1.13 also have this assumption.

Let $(M, H(M))$ be an H-structure. To simplify the notation, we write with subscript/superscript H for notions in $T_H := Th_{\mathcal{L}_H}(M, H(M))$ and no subscript/superscript for $T = Th_{\mathcal{L}}(M)$. We also write $\mathcal{L}$-independent to denote forking independence in T ($\mathcal{L}_H$-independent for T_H respectively), and $\mathcal{L}$-generic for generic group element in T ($\mathcal{L}_H$-generic for T_H respectively).

Definition 1.14. Let A be a subset of an H-structure $(M, H(M))$. We say that A is H-independent if $A \mathop{\smile\!\!\!\!\!\mid}_{A\cap H(M)} H(M)$.

Remark: Note that this is not the same as being $\mathcal{L}_H$-independent in the sense of forking in T_H.

Definition 1.15. Let a be a tuple in an H-structure $(M, H(M))$ and let $C = \mathrm{acl}(C)$ be H-independent. Define the *H-basis of a over C*, denoted by $HB(a/C)$, as the smallest tuple h in $H(M)$ such that $a \mathop{\smile\!\!\!\!\!\mid}_{C,h} H(M)$.

By [BV16, Proposition 3.9], H-bases exist and are unique up to permutation. Here is a useful observation:

Lemma 1.16. *Let $(M, H(M))$ be an H-structure and a be a tuple. Suppose a subset $C = acl(C)$ is H-independent and $HB(a/C) = \emptyset$. Then $HB(a, C) = HB(C)$.*

Proof. Suppose not, then $a, C \not\mathop{\smile\!\!\!\!\!\mid}_{HB(C)} H(M)$. There is a finite tuple $c \subseteq C$ such that $a, c \not\mathop{\smile\!\!\!\!\!\mid}_{HB(C)} H(M)$. Denote the dimension of the underlying geometric theory as $\dim_{\mathrm{acl}}$. Let $c' \subseteq C$ be a finite tuple such that $\dim_{\mathrm{acl}}(a/C) = \dim_{\mathrm{acl}}(a/c')$. Let $c'' \subseteq C$ be a tuple containing both c and c'. Then $\dim_{\mathrm{acl}}(a, c''/HB(C)) > \dim_{\mathrm{acl}}(a, c''/H(M))$. By the choice of c'', we have

$$\dim_{\mathrm{acl}}(a/c'') \geq \dim_{\mathrm{acl}}(a/c'', HB(C)) \geq \dim_{\mathrm{acl}}(a/C) = \dim_{\mathrm{acl}}(a/c'').$$

By assumption, $\dim_{\mathrm{acl}}(a/C, H(M)) = \dim_{\mathrm{acl}}(a/C)$. Therefore,

$$\dim_{\mathrm{acl}}(a/c'') \geq \dim_{\mathrm{acl}}(a/c'', H(M)) \geq \dim_{\mathrm{acl}}(a/C, H(M)) = \dim_{\mathrm{acl}}(a/C) = \dim_{\mathrm{acl}}(a/c'').$$

We conclude that $\dim_{\mathrm{acl}}(a/c'', H(M)) = \dim_{\mathrm{acl}}(a/c'') = \dim_{\mathrm{acl}}(a/c'', HB(C))$. Since C is H-independent, we also have $\dim_{\mathrm{acl}}(c''/H(M)) = \dim_{\mathrm{acl}}(c''/HB(C))$. By additivity of $\dim_{\mathrm{acl}}$, we have

$$\begin{aligned} \dim_{\mathrm{acl}}(a, c''/H(M)) &= \dim_{\mathrm{acl}}(a/c'', H(M)) + \dim_{\mathrm{acl}}(c''/H(M)) \\ &= \dim_{\mathrm{acl}}(a/c'', HB(C)) + \dim_{\mathrm{acl}}(c''/HB(C)) = \dim_{\mathrm{acl}}(a, c''/HB(C)), \end{aligned}$$

a contradiction. ∎

Fact 1.17. [BV16, Lemma 2.8, Corollary 3.14, Proposition 6.2]

Let $(M, H(M))$ be an H-structure.

1. Let a, b be H-independent tuples such that $\mathrm{tp}(a, HB(a)) = \mathrm{tp}(b, HB(b))$. Then $\mathrm{tp}_H(a) = \mathrm{tp}_H(b)$.

2. Let A be a subset of M, then $\mathrm{acl}_H(A) = \mathrm{acl}(A, HB(A))$.

3. Suppose $Th(M)$ is superrosy of thorn-rank one and $(M, H(M))$ is $\aleph_0$-saturated. Let D be an $\mathcal{L}_H$-definable group over some finite H-independent set A. Let b be a generic element of the group. Then $HB(b/A) = \emptyset$.

Fact 1.18. [BV16, Proposition 5.6] Let $(M, H(M)) \models T_H$ be a κ-saturated H-structure and $C \subseteq D \subseteq M$ be acl_H-closed and $\max\{|C|, |D|\} < \kappa$. Suppose T is supersimple of SU-rank one and $a \in M$. Then $a \mathop{\smile\!\!\!\!\!\vert}_C^H D$ if and only if none of the following holds:

- $a \in D \setminus C$;
- $a \in \mathrm{acl}(H(M), D) \setminus \mathrm{acl}(H(M), C)$;
- $HB(a/C) \neq HB(a/D)$.

We proceed by some lemmas, most of which are about the properties of generic elements of definable groups in H-structures.

In the following we will assume κ is an cardinal with $\kappa \geq |\mathcal{L}|$.

Lemma 1.19. *Let $(M, H(M))$ be a κ-saturated H-structure such that $Th(M)$ is supersimple of SU-rank one. Let G be an $\mathcal{L}_H$-(type-)definable group over some set A with $|A| < \kappa$ and $acl_H(A) = A$. Let a, b be $\mathcal{L}_H$-independent and $\mathcal{L}_H$-generic elements in G. Then $a \cdot b \in dcl(a, b, A)$ and $a^{-1} \in dcl(a, A)$.*

Proof. By Fact 1.17 (3), $HB(a/A) = HB(b/A) = \emptyset$. That is $a \mathop{\smile\!\!\!\!\!\vert}_A H(M)$ and $b \mathop{\smile\!\!\!\!\!\vert}_A H(M)$.

By assumption, $a \mathop{\smile\!\!\!\!\!\vert}_A^H b$. Hence, $a \mathop{\smile\!\!\!\!\!\vert}_{A,H(M)} b$. Thus, $a \mathop{\smile\!\!\!\!\!\vert}_{A,H(M)} bH(M)$. Together with $a \mathop{\smile\!\!\!\!\!\vert}_A A, H(M)$, we get $a \mathop{\smile\!\!\!\!\!\vert}_A b, H(M)$. Hence, $a, b \mathop{\smile\!\!\!\!\!\vert}_{A,b} H(M)$. Again, as $b \mathop{\smile\!\!\!\!\!\vert}_A H(M)$, we have $a, b \mathop{\smile\!\!\!\!\!\vert}_A H(M)$. Since $A \mathop{\smile\!\!\!\!\!\vert}_{HB(A)} H(M)$, we conclude that $a, b, A \mathop{\smile\!\!\!\!\!\vert}_{HB(A)} H(M)$. Therefore, $HB(a, b, A) \subseteq HB(A) \subseteq A$.

As $c := a \cdot b \in \mathrm{acl}_H(a, b, A) = \mathrm{acl}(a, b, A, HB(a, b, A)) = \mathrm{acl}(a, b, A)$, we have

$$a, b, c, A \mathop{\smile\!\!\!\!\!\vert}_{HB(A)} H(M).$$

Take $c' \in M$ with $\mathrm{tp}(c'/a, b, A) = \mathrm{tp}(c/a, b, A)$. As $c' \in \mathrm{acl}(a, b, A)$, we still have $a, b, c', A \mathop{\smile\!\!\!\!\!\vert}_{HB(A)} H(M)$. Therefore, a, b, c, A and a, b, c', A are H-independent tuples of the same $\mathcal{L}$-type. By Fact 1.17 (1), $\mathrm{tp}_H(a, b, c'/A) = \mathrm{tp}_H(a, b, c/A)$. As c is in the $\mathcal{L}_H$-definable closure of a, b, A, we get $c' = c$. Hence, $c \in \mathrm{dcl}(a, b, A)$ as we have claimed.

The proof of $a^{-1} \in \mathrm{dcl}(a, A)$ is similar. □

Lemma 1.20. *Let $(M, H(M))$ be a κ-saturated model of T_H. Let $G \subseteq M^n$ be an $\mathcal{L}_H$-type-definable group over A with $acl_H(A) = A$ and $|A| < \kappa$. Then there are a partial $\mathcal{L}_H$-type $\pi_G(x)$ and a partial $\mathcal{L}$-type $\pi_{\mathcal{L}}(x)$ over A such that:*

1. *$\pi_G(M^n)$ is the set of all $\mathcal{L}_H$-generics in G.*
2. *For any complete $\mathcal{L}$-type $q(x)$ over A with $q(x) \supseteq \pi_{\mathcal{L}}(x)$, there is a complete $\mathcal{L}_H$-type $p(x)$ over A such that $p(x) \supseteq q(x) \cup \pi_G(x)$;*

3. *Let a, b, c be three realizations of $\pi_{\mathcal{L}}(x)$ over A. Then there are $a', b', c' \in G$ such that a', b', c' realise $\pi_G(x)$, $HB(a', b', c'/A) = \emptyset$ and $tp(a, b, c/A) = tp(a', b', c'/A)$. In addition, if a, b, c are $\mathcal{L}$-independent, then a', b', c' are $\mathcal{L}_H$-independent.*

Proof. Suppose G is defined by a partial type $\delta(x)$. Let $\pi_G(x)$ be the partial $\mathcal{L}_H$-type over A which contains $\delta(x)$ and is closed under implication such that for all $a \in M^n$, $a \models \pi_G(x)$ if and only if a is $\mathcal{L}_H$-generic in G. Let $\pi_{\mathcal{L}}(x) \subseteq \pi_G(x)$ be the restriction of $\pi_G(x)$ in the language $\mathcal{L}$.

Claim: Item 2 holds. If not, then there exists $\mathcal{L}$-type $q(x)$ over A extending $\pi_{\mathcal{L}}(x)$ such that $q(x) \cup \pi_G(x)$ is inconsistent. By compactness, there is some $\psi(x) \in q(x)$ such that $\pi_G(x) \vdash \neg\psi(x)$. As $\pi_G(x)$ is closed under implication, $\neg\psi(x) \in \pi_G(x)$, hence also $\neg\psi(x) \in \pi_{\mathcal{L}}(x)$, which contradicts that $q(x) \supseteq \pi_{\mathcal{L}}(x)$.

Now we prove item 3. Write $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$, where $\mathrm{SU}(a_1/A) = |a_1|$, $a_2 \in \mathrm{acl}(a_1, A)$; $\mathrm{SU}(b_1/A, a) = |b_1|$, $b_2 \in \mathrm{acl}(b_1, a, A)$ and $\mathrm{SU}(c_1/A, a, b) = |c_1|$, $c_2 \in \mathrm{acl}(c_1, a, b, A)$. (We remark that b_1, c_1 can be empty.) As $\mathrm{SU}(a_1, b_1, c_1/A) = |a_1| + |b_1| + |c_1|$ and T has SU-rank 1, we get a_1, b_1, c_1 are $\mathcal{L}$-independent. By the axioms of of T_H and κ-saturation, there are a'_1, b'_1, c'_1 in M such that $\mathrm{tp}(a_1, b_1, c_1/A) = \mathrm{tp}(a'_1, b'_1, c'_1/A)$ and

$$a'_1, b'_1, c'_1 \underset{A}{\perp} H(M).$$

Let a'_2, b'_2, c'_2 be such that

$$\mathrm{tp}(a'_1, a'_2, b'_1, b'_2, c'_1, c'_2/A) = \mathrm{tp}(a_1, a_2, b_1, b_2, c_1, c_2/A).$$

Define $a' := (a'_1, a'_2)$, $b' := (b'_1, b'_2)$ and $c' := (c'_1, c'_2)$.

Since $a'_1, b'_1, c'_1 \perp_A H(M)$ and $a', b', c' \in \mathrm{acl}(a'_1, b'_1, c'_1, A)$, we get $a', b', c' \perp_A H(M)$. Therefore, $HB(a', b', c'/A) = \emptyset$. Hence, $HB(a'/A) = HB(b'/A) = HB(c'/A) = \emptyset$.

We only need to show that a', b' and c' satisfy $\pi_G(x)$. Let $q(x) := \mathrm{tp}(a/A) \supseteq \pi_{\mathcal{L}}(x)$. By item 2, there is a complete $\mathcal{L}_H$-type $p(x)$ over A extending $q(x) \cup \pi_G(x)$. Let a'' be a realization of $p(x)$. By Fact 1.17 (3), $HB(a''/A) = \emptyset$. Therefore, both a', A and a'', A are H-independent and

$$\mathrm{tp}(a', A, HB(a', A)) = \mathrm{tp}(a, A, HB(A)) = \mathrm{tp}(a'', A, HB(a'', A)).$$

By Fact 1.17 (1), $\mathrm{tp}_H(a'/A) = \mathrm{tp}_H(a''/A)$. Hence $\mathrm{tp}_H(a'/A) \supseteq \pi_G(x)$. Similarly, b' and c' are realizations of $\pi_G(x)$.

In addition, if a, b, c are $\mathcal{L}$-independent, then $b' = (b'_1, b'_2)$ and $c' = (c'_1, c'_2)$ are such that $\mathrm{SU}(b'_1/A) = \mathrm{SU}(b'_1/A, a') = |b'_1|$, $\mathrm{SU}(c'_1/A) = \mathrm{SU}(c'_1/A, a', b') = |c'_1|$ and $b'_2 \in \mathrm{acl}(b'_1, A)$, $c'_2 \in \mathrm{acl}(c'_1, A)$. As $a'_1, b'_1, c'_1 \perp_A H(M)$ and $a'_1 \perp_A b'_1, c'_1$, we get

$$a'_1 \underset{A}{\perp} b'_1, c'_1, H(M).$$

Therefore, $a' \perp_A b', c', H(M)$, whence $a' \perp_{AH(M)} b', c', H(M)$. Together with $HB(a'/A) = HB(a'/Ab'c') = \emptyset$ we get $a' \perp^H_A b', c'$. The other $\mathcal{L}_H$-independences among a', b', c' are similar. Hence, a', b', c' are $\mathcal{L}_H$-independent. $\square$

Lemma 1.21. *Let $\mathcal{L}_0 \subseteq \mathcal{L}_1$ be two languages. Let M be an $\mathcal{L}_1$-structure. Suppose Y is $\mathcal{L}_0$-hyper-definable and G is $\mathcal{L}_1$-type-definable in M such that there is an $\mathcal{L}_1$-isomorphism from Y to G, then Y is $\mathcal{L}_0$-type-interpretable.*

Proof. Suppose $G = \bigcap_{i\in I} G_i$ is $\mathcal{L}_1$-type-definable, $Y = X/R$ where $X = \bigcap_{i\in I} X_i$ and $R = \bigcap_{i\in I} R_i$ are $\mathcal{L}_0$-type-definable and $\Phi(x, y) := \bigcap_{i\in I} \Phi_i : X_i \to G_i$ is $\mathcal{L}_1$-type-definable which induces an isomorphism between Y and G.

As Φ is the graph of a function from X to G, we have:

$$\bigwedge_{i,j,k\in I} X_i(x) \wedge G_j(y) \wedge G_j(y') \wedge \Phi_k(x, y) \wedge \Phi_k(x, y') \models y = y'.$$

By compactness, there are some $i_0, \ldots, i_k$ such that

$$f(x, y) := \bigcap_{j\leq k} \Phi_{i_j}(x, y) \ \subseteq \ \left(\bigcap_{j\leq k} X_{i_j} \times \bigcap_{j\leq k} G_{i_j}\right)$$

is an $\mathcal{L}_1$-definable graph of a partial function.

Let $R' \subseteq \left(\bigcap_{j\leq k} X_{i_j}\right) \times \left(\bigcap_{j\leq k} X_{i_j}\right)$ be the $\mathcal{L}_1$-definable equivalence relation given by $R'(x, x')$ if and only if there is some $g \in \bigcap_{j\leq k} G_{i_j}$ such that both $f(x, g)$ and $f(x', g)$ hold. We claim that

$$R' \restriction (X \times X) = R.$$

Let $x, x' \in X$. Suppose $R(x, x')$ holds. As Φ is an isomorphism between Y and G, there is some $g \in G$ with $\Phi(x, g)$ and $\Phi(x', g)$. Therefore, both $f(x, g)$ and $f(x', g)$ hold and so does $R'(x, x')$. On the other hand, if $R'(x, x')$ holds, then there is $g \in \bigcap_{j\leq k} G_{i_j}$ with $f(x, g)$ and $f(x', g)$. Let $g', g'' \in G$ such that $\Phi(x, g')$ and $\Phi(x', g'')$. Thus, we also have $f(x, g')$ and $f(x', g'')$. Since f is a partial function, $g = g' = g''$. Therefore, $R(x, x')$ holds.

As R is defined by $\bigcap_{i\in I} R_i$, by compactness, there is some $\{j_0, \ldots, j_t\} \supseteq \{i_0, \ldots, i_k\}$ such that on $\left(\bigcap_{i\leq t} X_{j_i}\right) \times \left(\bigcap_{i\leq t} X_{j_i}\right)$ we have

$$R_{\mathcal{L}_0}(x, x') := \bigcap_{i\leq t} R_{j_i}(x, x') \subseteq R'(x, x').$$

Thus, $R_{\mathcal{L}_0}$ is $\mathcal{L}_0$-definable and it agrees with R on X. We have

$$\begin{aligned}\bigwedge_{i\in I}(X_i(x_1) \wedge X_i(x_2) \wedge X_i(x_3)) \models R_{\mathcal{L}_0}(x_1, x_1) \\ \wedge(R_{\mathcal{L}_0}(x_1, x_2) \to R_{\mathcal{L}_0}(x_2, x_1)) \\ \wedge(R_{\mathcal{L}_0}(x_1, x_2) \wedge R_{\mathcal{L}_0}(x_2, x_3) \to R_{\mathcal{L}_0}(x_1, x_3)).\end{aligned}$$

By compactness, there are $\{k_0, \ldots, k_m\} \supseteq \{j_0, \ldots, j_t\}$ such that $R_{\mathcal{L}_0}$ is an equivalence relation on $\bigcap_{t\leq m} X_{k_t}$. Therefore, R is $\mathcal{L}_0$-definable. □

We first consider $\mathcal{L}_H$-(type-)definable subgroups of $\mathcal{L}$-(type-)definable groups. We generalize Fact 1.12 to supersimple theories.

Theorem 1.22. *Let T be non-trivial of SU-rank one and let $(M, H(M)) \models T_H$ be κ-saturated. Suppose D is an $\mathcal{L}$-(type-)definable group and G is an $\mathcal{L}_H$-(type-)definable subgroup of D, both defined over some set $A = acl_H(A)$ with $|A| < \kappa$. Then G is $\mathcal{L}$-(type-) definable ovear A.*

Proof. Suppose $D \subseteq M^n$. Let $\pi_G(x)$ and $\pi_{\mathcal{L}}(x)$ be defined as in Lemma 1.20 with $|x| = n$. Suppose D is defined by the partial $\mathcal{L}$-type $\chi(x)$. As $\pi_G(x)$ is closed under implication, $\pi_G(x) \supseteq \chi(x)$. Therefore, $\pi_{\mathcal{L}}(x) \supseteq \chi(x)$.

By Fact 0.26, $G = \pi_G(M^n) \cdot \pi_G(M^n)$. We will show that $\pi_{\mathcal{L}}(M^n)$ also satisfies the conditions of Fact 0.26 in T.

Let $X := \pi_{\mathcal{L}}(M^n)$. Since $\chi(x) \subseteq \pi_{\mathcal{L}}(x)$, we have $X \subseteq D$. Take two $\mathcal{L}$-independent realizations a, b of $\pi_{\mathcal{L}}(x)$. By Lemma 1.20, there are a', b' both realising $\pi_G(x)$ such that $\text{tp}(a, b/A) = \text{tp}(a', b'/A)$ and $a' \mathop{\smile\!\!\!\!\!|}^{H}_{A} b'$. Therefore, $(a')^{-1} \cdot b'$ is also generic in G, which implies

$$\pi_{\mathcal{L}}(x) \subseteq \pi_G(x) \subseteq \text{tp}_H((a')^{-1} \cdot b'/A).$$

As $\text{tp}(a, b/A) = \text{tp}(a', b'/A)$ and group operations are $\mathcal{L}$-definable, we have

$$\text{tp}(a^{-1} \cdot b/A) = \text{tp}((a')^{-1} \cdot b'/A).$$

Therefore, $\pi_{\mathcal{L}}(x) \subseteq \text{tp}(a^{-1} \cdot b/A)$, whence $a^{-1} \cdot b \in X$. By Fact 0.26 we get an $\mathcal{L}$-type-definable group $D_G := X \cdot X$ such that X contains all $\mathcal{L}$-generics in D_G.

Clearly, $G \leq D_G$. Let a be an $\mathcal{L}_H$-generic element in D_G. By Fact 1.17(3), we have $HB(a/A) = \emptyset$. Since a is also $\mathcal{L}$-generic in D_G, we get $a \in X$. By Lemma 1.20 there is an a' satisfying $\pi_G(x)$ such that $\text{tp}(a/A) = \text{tp}(a'/A)$. As a' is $\mathcal{L}_H$-generic in G, $HB(a'/A) = \emptyset = HB(a/A)$. By Fact 1.17(1), $\text{tp}_H(a'/A) = \text{tp}_H(a/A)$. Hence, a realizes $\pi_G(x)$, i.e., a is $\mathcal{L}_H$-generic in G. Therefore, every $\mathcal{L}_H$-generic element of D_G is contained in G, whence $D_G \leq G$. We conclude that $G = D_G$. □

Now we consider general $\mathcal{L}_H$-(type-)definable groups. The following is a generalization of Fact 1.13.

Theorem 1.23. *Let T be supersimple of SU-rank one and $(M, H(M)) \models T_H$ be κ-saturated. Let G be an $\mathcal{L}_H$-(type-)definable group over a set $A = acl_H(A)$ of size less than κ. Then G is $\mathcal{L}_H$-definably isomorphic to some $\mathcal{L}$-(type)-interpretable group. In particular, if T eliminates imaginaries, then every $\mathcal{L}_H$-(type-)definable group is $\mathcal{L}_H$-definably isomorphic to some $\mathcal{L}$-(type-)definable group.*

Proof. Suppose G is type-definable. Let $\pi_G(x)$ and $\pi_{\mathcal{L}}(x)$ be defined as in Lemma 1.20. In the following, we will extend $\mathcal{L}$-generically and $\mathcal{L}$-type-definably the group operation $\cdot$ of G to $\star$ on $\pi_{\mathcal{L}}(x)$.

Let $\pi^2_G(x, y) \supseteq \pi_G(x) \cup \pi_G(y)$ be the partial $\mathcal{L}_H$-type over A such that a, b are $\mathcal{L}_H$-independent and $\mathcal{L}_H$-generic in G over A if and only if $(a, b) \models \pi^2_G(x, y)$ for any $a, b \in M^n$. For $(a, b) \models \pi^2_G(x, y)$, we have $a \cdot b \in \text{dcl}(a, b)$ by Lemma 1.19. That is $a \cdot b = f_{a,b}(a, b)$ for some $\mathcal{L}$-definable function $f_{a,b}$ over A. Let $\text{dom}_{a,b}(x, y)$ be the $\mathcal{L}$-formula that defines the domain of the function $f_{a,b}$. Then define the $\mathcal{L}_H$-formula

$$\varphi_{a,b}(x, y) := \text{dom}_{a,b}(x, y) \wedge x \cdot y = f_{a,b}(x, y).$$

Then we can see that

$$\pi_G^2(x,y) \subseteq \bigcup_{(a,b)\models \pi_G^2(x,y)} \varphi_{a,b}(x,y).$$

By compactness, there are $(a_1,b_1),(a_2,b_2),\dots,(a_k,b_k)$ such that

$$\pi_G^2(x,y) \models \bigvee_{1\leq i\leq k} \varphi_{a_i,b_i}(x,y).$$

Let (a,b), (c,d) be two pairs of realizations of $\pi_G^2(x,y)$ such that $\mathrm{tp}(a,b/A)=\mathrm{tp}(c,d/A)$. Note that (a,b) is an $\mathcal{L}_H$-generic element in $G\times G$. By Fact 1.17(3), $HB(a,b/A)=\emptyset$. Similarly, $HB(c,d/A)=\emptyset$. Applying Fact 1.17(1), we get $\mathrm{tp}_H(a,b/A)=\mathrm{tp}_H(c,d/A)$. Therefore, $(M,H(M))\models \varphi_{a_i,b_i}(a,b)\leftrightarrow \varphi_{a_i,b_i}(c,d)$ for all $1\leq i\leq k$. The above argument shows:

$$\pi_G^2(x,y)\wedge \pi_G^2(x',y')\wedge \bigwedge_{\psi\in\mathcal{L}(A)} \psi(x,y)\leftrightarrow \psi(x',y') \models \bigwedge_{1\leq i\leq n} (\varphi_{a_i,b_i}(x,y)\leftrightarrow \varphi_{a_i,b_i}(x',y')).$$

By compactness, there is some finite set of $\mathcal{L}(A)$ formulas Δ such that the Δ-type of any pair $(a,b)\models \pi_G^2(x,y)$ determines $(a,b)\models \varphi_{a_i,b_i}(x,y)$ or $(a,b)\models \neg\varphi_{a_i,b_i}(x,y)$ for any $1\leq i\leq k$. Hence, there are $\mathcal{L}$-formulas $\psi_1(x,y),\dots,\psi_k(x,y)$ such that

$$\pi_G^2(x,y)\models \bigvee_{1\leq i\leq k}\psi_i(x,y)$$

and for any $1\leq i\leq k$, we have

$$\pi_G^2(x,y)\models \psi_i(x,y)\to \left(\varphi_{a_i,b_i}(x,y)\wedge \bigwedge_{1\leq j<i}\neg\varphi_{a_j,b_j}(x,y)\right).$$

Let $\pi_{\mathcal{L}}^2(x,y)\supseteq \pi_{\mathcal{L}}(x)\cup\pi_{\mathcal{L}}(y)$ be the partial $\mathcal{L}$-type over A such that $(a,b)\models \pi_{\mathcal{L}}^2(x,y)$ if and only if a,b are $\mathcal{L}$-independent over A. By Lemma 1.20, for $(a,b)\models\pi_{\mathcal{L}}^2(x,y)$, there are a',b' realizing $\pi_G(x)$ such that $a' \mathop{\smile\!\!\!\!\!\!|}\,_A^H b'$ and $\mathrm{tp}(a,b/A)=\mathrm{tp}(a',b'/A)$. Note that $(a',b')\models \pi_G^2(x,y)$. Hence,

$$(a',b')\models \psi_i(x,y)\wedge\varphi_{a_i,b_i}(x,y)\wedge\bigwedge_{1\leq j<i}\neg\varphi_{a_j,b_j}(x,y)$$

for some $1\leq i\leq k$. As $\mathrm{tp}(a,b/A)=\mathrm{tp}(a',b'/A)$, we also have

$$(a,b)\models \psi_i(x,y)\wedge \mathrm{dom}_{a_i,b_i}(x,y).$$

Define $a\star b := f_{a_i,b_i}(a,b)$. As $f_{a_i,b_i}(a',b')\models \pi_{\mathcal{L}}(x)$ and $\mathrm{tp}(a,b/A)=\mathrm{tp}(a',b'/A)$, we also have $f_{a_i,b_i}(a,b)\models\pi_{\mathcal{L}}(x)$. Note that $a\star b$ is defined by $f_{a_i,b_i}(x,y)$ if and only if $(a,b)\models\psi_i(x,y)$. Hence, $\star$ is an $\mathcal{L}$-type-definable function from $\pi_{\mathcal{L}}^2(M^n,M^n)$ to $\pi_{\mathcal{L}}(M^n)$ and $\star$ agrees with $\cdot$ on $\pi_G^2(M^n,M^n)$.

We now verify all the conditions of the group chunk theorem (Fact 0.27) in order to obtain an $\mathcal{L}$-hyper-definable group out of the generically given group operation.

Lemma 1.24. *The $\mathcal{L}$-type-definable function $\star : \pi^2_{\mathcal{L}}(M^n, M^n) \to \pi_{\mathcal{L}}(M^n)$ satisfies all the conditions in Fact 0.27.*

Proof. Generic independence: Let a, b be $\mathcal{L}$-independent realizations of $\pi_{\mathcal{L}}(x)$ and $c := a \star b$. Then there are $\mathcal{L}_H$-independent and $\mathcal{L}_H$-generic elements a', b' over A such that $\mathrm{tp}(a', b'/A) = \mathrm{tp}(a, b/A)$. Let $c' := a' \cdot b'$. Since $\star$ is $\mathcal{L}$-definable and agrees with $\cdot$ on $\pi^2_G(M^n, M^n)$, we get $c' = a' \star b'$. Therefore, $\mathrm{tp}(a', b', c'/A) = \mathrm{tp}(a, b, c/A)$. As $c' \mathop{\smile\!\!\!\!\!\!|}^H_A a'$, we have $c' \mathop{\smile\!\!\!\!\!\!|}_A a'$. Hence, we also have $c \mathop{\smile\!\!\!\!\!\!|}_A a$. Similarly, $c \mathop{\smile\!\!\!\!\!\!|}_A b$.

Generic associativity: Let a, b, c be $\mathcal{L}$-independent realizations of $\pi_{\mathcal{L}}(x)$. By Lemma 1.20, there are $\mathcal{L}_H$-generic and $\mathcal{L}_H$-independent realizations a', b', c' such that

$$\mathrm{tp}(a, b, c/A) = \mathrm{tp}(a', b', c'/A).$$

Now we have

$$\mathrm{tp}((a \star b) \star c), a \star (b \star c)) = \mathrm{tp}((a' \star b') \star c', a' \star (b' \star c')) = \mathrm{tp}((a' \cdot b') \cdot c', a' \cdot (b' \cdot c')).$$

Since $(a' \cdot b') \cdot c' = a' \cdot (b' \cdot c')$ we get $(a \star b) \star c = a \star (b \star c)$.

Generic surjectivity: for any $\mathcal{L}$-independent realizations a, b of $\pi_{\mathcal{L}}(x)$, there are $\mathcal{L}_H$-independent realizations a', b' of $\pi_G(x)$ such that $\mathrm{tp}(a, b/A) = \mathrm{tp}(a', b'/A)$. Let $c' := (a')^{-1} \cdot b'$. Then c' is $\mathcal{L}_H$-independent from a' and from b'. By Lemma 1.19, $c' \in \mathrm{dcl}((a')^{-1}, b', A) = \mathrm{dcl}(a', b', A)$. Let c be the element with $\mathrm{tp}(a, b, c/A) = \mathrm{tp}(a', b', c'/A)$. Clearly, c realizes $\pi_{\mathcal{L}}(x)$ and is $\mathcal{L}$-independent from a and from b. Since $a' \cdot c' = a' \star c' = b'$ and $\mathrm{tp}(a, b, c/A) = \mathrm{tp}(a', b', c'/A)$, we have $a \star c = b$. Similarly, we can find c'' realizing $\pi_{\mathcal{L}}(x)$, $\mathcal{L}$-independent from a and from b such that $c'' \star a = b$. □

By Fact 0.27, there are an $\mathcal{L}$-hyper-definable group D over A, and an $\mathcal{L}$-type-definable embedding $f : \pi_{\mathcal{L}}(M^n) \to D$ over A such that $f(\pi_{\mathcal{L}}(M^n))$ contains all $\mathcal{L}$-generics of D.

Consider $f(\pi_G(M^n)) \subseteq D$. Take g, g' $\mathcal{L}_H$-independent elements in $f(\pi_G(M^n))$. Suppose $g = f(a)$ and $g' = f(b)$. As f is an $\mathcal{L}_H$-definable injection, we get $a \mathop{\smile\!\!\!\!\!\!|}^H_A b$. Hence, $a^{-1} \star b \models \pi_G(x)$ and $a \mathop{\smile\!\!\!\!\!\!|}^H_A a^{-1} \star b$. Since f preserves $\star$ generically and $a, a^{-1}, b \in G$, we have

$$f(a) \cdot f(a^{-1} \star b) = f(a \star (a^{-1} \star b)) = f(a \cdot (a^{-1} \cdot b)) = f(b).$$

Hence, $f(a)^{-1} \cdot f(b) = f(a^{-1} \star b) \in f(\pi_G(M^n))$. By Fact 0.26,

$$G_f := f(\pi_G(M^n)) \cdot f(\pi_G(M^n))$$

is an $\mathcal{L}_H$-hyper-definable group, and $f(\pi_G(x))$ contains all $\mathcal{L}_H$-generics in G_f.

Let $X := \{(g, f(g)) : g \models \pi_G(x)\} \subseteq G \times G_f$. Let $(g_1, f(g_1))$ and $(g_2, f(g_2))$ be $\mathcal{L}_H$-independent tuples in X. Consider

$$x_{g_1, g_2} := (g_1, f(g_1))^{-1} \cdot (g_2, f(g_2)) = (g_1^{-1}, f(g_1^{-1})) \cdot (g_2, f(g_2)) = (g_1^{-1} \star g_2, f(g_1^{-1} \star g_2)).$$

As $g_1 \mathop{\smile\!\!\!\!\!\!|}^H_A g_2$ in $\pi_G(x)$ we get $g_1^{-1} \star g_2 = g_1^{-1} \cdot g_2 \in \pi_G(x)$. Therefore, $x_{g_1, g_2} \in X$. By Fact 0.26, $C := X \cdot X$ is a subgroup of $G \times G_f$. Consider the projection $\rho_1(C) \leq G$. It contains $\pi_G(M^n)$, hence contains all $\mathcal{L}_H$-generics of G. Thus $\rho_1(C) = G$. Similarly, $\rho_2(C) = G_f$. Let $I := \{g : (g, 1) \in C\}$ and $I' := \{g : (1, g) \in C\}$. If $g \in I$, then there are $g_1, g_2 \in \pi_G(M^n)$ such that $g = g_1 \star g_2$ and $f(g_1) \cdot f(g_2) = f(g_1 \star g_2) = 1$. As f is an embedding, we get $g_1 \star g_2 = 1$. Therefore, $I = \{1\}$. Similarly, $I' = \{1\}$. Hence, C is the graph a group isomorphism between G and G_f.

Let a be an $\mathcal{L}_H$-generic in D. Then $HB(a/A) = \emptyset$. Since a is also $\mathcal{L}$-generic in D, we get that $f^{-1}(a)$ satisfies $\pi_{\mathcal{L}}(x)$. As f is an $\mathcal{L}_H$-definable embedding, we have $HB(f^{-1}(a)/A) = \emptyset$. Since $f^{-1}(a) \models \pi_{\mathcal{L}}(x)$, by Lemma 1.20 there is a' realizing $\pi_G(x)$ such that a' and $f^{-1}(a)$ have the same $\mathcal{L}$-type over A. Note that $HB(a'/A) = \emptyset$. By Fact 1.17 (1), $\mathrm{tp}_H(a'/A) = \mathrm{tp}_H(f^{-1}(a)/A)$. Hence, $f^{-1}(a)$ realizes $\pi_G(x)$, and $a = f(f^{-1}(a))$ is $\mathcal{L}_H$-generic in G_f. Therefore, the set of $\mathcal{L}_H$-generics of D is contained in G_f, whence $D \leq G_f$. Together with $G_f \leq D$, we get $G_f = D$ and G is $\mathcal{L}_H$-type-definably isomorphic to D.

Now Lemma 1.21 implies that D is $\mathcal{L}$-type-interpretable.

Suppose $D = D_G/E$ where E is an $\mathcal{L}$-definable equivalence relation and D_G is $\mathcal{L}$-type-definable. If G is definable, then D_G is the image of an $\mathcal{L}_H$-definable function, hence $\mathcal{L}_H$-definable. By compactness D_G is $\mathcal{L}$-definable. Therefore, G is $\mathcal{L}_H$-definably isomorphic to an $\mathcal{L}$-interpretable group D. □

Remark: Given an $\mathcal{L}_H$-definable group G, without the assumption that G lives inside an $\mathcal{L}$-definable group, we cannot generally have that G is $\mathcal{L}$-definable. Here is an example.

Example 1.1. *Let $D = (D, \cdot, ^{-1})$ be a group without involutions of SU-rank one in the language $\mathcal{L} = \{\cdot, ^{-1}\}$. Let $(D, H(D))$ be an H-structure.*

Define $\sigma : D \to D$ as $\sigma(x) = x$ if $x \notin H(D) \cup (H(D))^{-1}$; and $\sigma(x) = x^{-1}$ if $x \in H(D) \cup (H(D))^{-1}$. Let $\star : G \times G \to G$ be defined as $a \star b := \sigma^{-1}(\sigma(a) \cdot \sigma(b))$. Then the group $(D, \star, ^{-1})$ is $\mathcal{L}_H$-isomorphic to $(D, \cdot, ^{-1})$ via σ, but not $\mathcal{L}$-definable.

Chapter 2

Pseudofinite Difference Fields

2.1 Introduction

The class of various expansions of fields is one of the key objects of study in model theory. Examples are differentially closed fields, Henselian valued fields, algebraically closed fields with a generic automorphism, etc. There are lots of natural examples of such structures that are intensively investigated in other areas of mathematics, while the model theories of them often extends well-known results to a wider context and sometimes, model theoretic techniques can help to discover new phenomenons. For example, the theory of differentially closed fields plays an important role in Hrushovski's proof of the Mordell-Lang conjecture [Hru96].

In this chapter, we will consider expansions of pseudofinite fields with a distinguished automorphism. The model theory of pseudofinite fields has been initiated by J. Ax in [Ax68] and subsequently developed in [Dur80], [CvdDM92], [HP94]. On the other hand, the model theory of fields with a distinguished automorphism has also been investigated. The best understood one is possibly ACFA: the theory of algebraically closed fields with a generic automorphism, developed notably in [CH99], [CHP02]. It is the model companion of the theory of difference fields and, interestingly, the fixed field of any model of ACFA is a pseudofinite field. Based on these, one might expect a theory of pseudofinite difference fields which is a mixture of PSF (the theory of pseudofinite fields) and ACFA.

M. Ryten studied a specific class of pseudofinite difference fields with the motivation of understanding the asymptotic behaviour of Suzuki groups and Ree groups. As we have mentioned in the introduction, he showed that given any prime p and a pair of coprime numbers $m, n > 1$, the class $\{(\mathbb{F}_{p^{k\cdot m+n}}, \mathrm{Frob}_{p^k}) : k \in \mathbb{N}\}$ is a one-dimensional asymptotic class in [Ryt07]. He also gave a recursive axiomatization of asymptotic theories of such structures: $\mathrm{PSF}_{(m,n,p)}$. In a sense, $\mathrm{PSF}_{(m,n,p)}$ is a mixture of PSF and ACFA. In fact, any model of $\mathrm{PSF}_{(m,n,p)}$ can be obtained as a definable substructure of some model of ACFA[1], and the one-dimensional asymptotic class result is based on the uniform estimate of the number of solutions of definable sets of finite σ-degree in some model of ACFA in [RT06].

[1]See [Ryt07, Lemma 3.3.6].

However, $\mathrm{PSF}_{(m,n,p)}$ is a bit restricted in the sense that in models of $\mathrm{PSF}_{(m,n,p)}$ there are no transformally transcendental elements, i.e. elements that satisfy no non-trivial difference polynomial. And most of the nice model theoretic properties of $\mathrm{PSF}_{(m,n,p)}$ come from the tameness of ACFA. Our aim in this chapter is to study a class of pseudofinite difference fields with transformally transcendental elements.

Another class of closely related structures is the class of pairs of pseudofinite fields, as the fixed field of a pseudofinite difference field is finite or pseudofinite. As noticed by Macintyre and Cherlin, there are pairs of pseudofinite fields whose theory is not decidable. This wild phenomenon also occurs in the structures that we study. In fact, we will show that in some ultraproduct of finite difference fields there is a definable set such that the family of all internal subsets of it is uniformly definable, see Theorem 2.17. This means in particular that the fine pseudofinite dimension behaves badly and the theory fails to possess tame model theoretic properties either in the sense of Shelah's classification theory or being decidable, see Corollary 2.21.[2] However, if we allow the size of the underlying field to grow rapidly enough, then the coarse pseudofinite dimension with respect to the full field behaves extremely well. It takes values in the integers and given a family of uniformly definable sets and an integer n, the set of parameters such that the coarse dimension of the corresponding definable sets have value n is definable, see Corollary 2.9. This coarse dimension of a definable set in difference fields essentially comes from the fine dimension in pseudofinite fields, which is the Zariski-dimension. Along the line of studying the interaction between counting dimensions and algebraic properties of the underlying structures, we investigate the relation between the integer-valued coarse dimension in our classes of pseudofinite difference fields and the transformal transcendence degree in the algebraic closure. We prove that coarse dimension is always bounded by transformal transcendental degree. And if they agree then it is possible to classify existentially definable subgroups of algebraic groups, see Theorem 2.14.

We remark there that we aim to study the theory of pseudofinite difference fields, which is different with, though closely related to, the theory of pseudofinite fields with a distinguished automorphism. Since there is the concern that the latter may not have a model companion,[3] neither of these two theories has been carefully studied.

The rest of this chapter is organized as the following. Section 2.2 starts with a quick recap of coarse pseudofinite dimension, followed by the definition of a class of ultraproducts of finite difference fields $\mathcal{S}$. The main result is Theorem 2.5 and Corollary 2.9 which states that for any pseudofinite difference field in $\mathcal{S}$, the coarse dimension with respect to the full field $\boldsymbol{\delta}_F$ is integer-valued and definable. Section 2.3 studies the relation between $\boldsymbol{\delta}_F$ and the transformal transcendence degree and its application to definable groups. The main result is Theorem 2.14. Section 2.4 studies the negative model theoretic aspects of structures of $\mathcal{S}$. They do not belong to any well-studied tame class, is not decidable (Corollary 2.21) and the model theoretic algebraic closure is different from the algebraic closure in the sense of difference algebra (Theorem 2.22).

[2] This does not mean that any theory of pseudofinite difference fields with transformally transcendental elements is not tame. We think it is possible that some of them have a decidable theory. But it is not clear which classes and what kind of theories they should be.

[3] It was claimed that it does not have a model companion in for example [CP98, section 3], but there are some obstacles see [Cha15, 1.12].

2.2 Coarse pseudofinite dimension

We will study the coarse pseudofinite dimension of a class of ultraproducts of finite difference fields in this section. We will show that their coarse dimension with respect to the full field behaves well. The main tool is that the fine dimension of pseudofinite fields is integer-valued and there are only finitely many possible values of the measure for a uniformly definable family of sets of a fixed dimension (see Fact 0.14). This allows us to estimate the size of sets defined by difference formulas in certain finite difference fields. We show further that the coarse dimension is definable, with only the assumptions that the dimension is integer-valued and a field structure is included in the language.

We begin with some preliminaries on difference fields.

Definition 2.1. A *difference field* is a field $(F, +, \cdot, 0, 1)$ together with a field automorphism σ (in particular σ is surjective).

The *language of difference rings* $\mathcal{L}_\sigma$ is the language of rings augmented by a unary function symbol σ.

Definition 2.2. We fix an ambient difference field L.

- Let A be a subset. We denote by A_σ the smallest difference subfield containing A and closed under σ and σ^{-1}.
- Let E be a difference subfield and a be a tuple. The *σ-degree*, $\mathsf{deg}_\sigma(a/E)$, is the transcendence degree of $(E, a)_\sigma$ over E.
- Let E be a difference subfield. If there is no non-zero difference polynomial over E vanishing on a, then we say a is *transformally transcendental* over E if a is an element in L and a is *transformally independent* over E if a is a tuple in L.
- Let E be a difference subfield and a be a tuple. The *transformal transcendence degree* of a over E is defined as the maximal length of a transformally independent subtuple of a over E.

Now we start to define a special class of ultraproducts of finite difference fields and study their coarse pseudofinite dimension with respect to the full field. The main observation is that given a difference formula $\varphi(x)$ and we want to estimate the size of the set that $\varphi(x)$ defines in a finite difference field $(\mathbb{F}_{p^k}, \mathrm{Frob}_{p^m})$. If we allow k grow while keep p and m fixed, then the set defined by $\varphi(x)$ has a dimension which comes from the fine pseudofinite dimension in the classes of pseudofinite fields. The trick is that we translate the difference formula $\varphi(x)$ into a ring formula $\varphi_{p^m}(x)$ by replacing terms $\sigma(t)$ with t^{p^m}. If k is big enough compared to p and m, then the set defined by $\varphi(x)$ in $(\mathbb{F}_{p^k}, \mathrm{Frob}_{p^m})$ will be roughly propositional to $(p^k)^d$, where $d \leq |x|$ is the fine dimension of φ_{p^m}, which depends on φ, p and m. If we take an ultraproduct of $\{(\mathbb{F}_{p^k}, \mathrm{Frob}_{p^m}) : p \in \mathbb{P}, k, m \geq 1\}$ over some non-principal ultrafilter $\mathcal{U}$, then $\mathcal{U}$ will pick one of the dimension $d \leq |x|$. Suppose almost all k in $(\mathbb{F}_{p^k}, \mathrm{Frob}_{p^m})$ are big enough compared to p and m, then d will be the coarse pseudofinite dimension with respect to the full field of the set defined by φ in the ultraproduct.

Definition 2.3. Let $\mathcal{L}_\sigma$ be the language of difference rings. Let $\varphi(x, y)$ be a formula defined in $\mathcal{L}_\sigma$ without parameters. For any prime p, define $\varphi_p(x, y)$ as the result of replacing each occurrence of $\sigma(t)$ in $\varphi(x, y)$ by t^p. Clearly, $\varphi_p(x, y)$ is a formula in the language of rings $\mathcal{L}$.

Recall that we denote by $\mathbb{P}$ the set of all primes. For any formula $\varphi(x,y)$ in $\mathcal{L}_\sigma$ and $p \in \mathbb{P}$, consider $\varphi_p(x,y) \in \mathcal{L}$. There are C_{φ_p} and the finite set D_{φ_p} as stated in Fact 0.14. Let

$$E_{\varphi_p} := \bigcup_{0 \le d \le |x|} \{\mu : (d,\mu) \in D_{\varphi_p}\}.$$

Define

$$N^p_{\varphi(x,y)} := \max\left\{\mu, \frac{1}{\mu}, 2\log_p\left(\frac{2C_{\varphi_p}}{\mu}\right) : \mu \in E_{\varphi_p}\right\}.$$

Let

$$f(\ell,p) := \max\{N^p_{\varphi(x,y)} : |\varphi(x,y)| \le \ell\}. \tag{2.1}$$

Definition 2.4. Define the family $\mathcal{S}$ of pseudofinite difference fields as

$$\mathcal{S} := \left\{\prod_{p\in\mathbb{P}} (\mathbb{F}_{p^{k_p}}, \mathrm{Frob}_p)/\mathcal{U} : k_p \ge f(p,p) \text{ for all } p \in \mathbb{P},\ \mathcal{U} \text{ a non-principal ultrafilter}\right\}.$$

Theorem 2.5. *Let $(F, Frob) := \prod_{p\in\mathbb{P}}(\mathbb{F}_{p^{k_p}}, Frob_p)/\mathcal{U} \in \mathcal{S}$. Then the coarse pseudofinite dimension with respect to F is integer-valued on all $\mathcal{L}_\sigma$-definable sets.*

Proof. Let $\varphi(x,y)$ be an $\mathcal{L}_\sigma$-formula. Consider a parameter $a = (a_p)_{p\in\mathbb{P}}/\mathcal{U} \in F^{|y|}$. For any $p \in \mathbb{P}$, we know that there are $(d_{k_p}, \mu_{k_p}) \in \{0,\dots,|x|\} \times \mathbb{R}^{>0}$ and $C_{\varphi_p} \ge 0$ such that for $a_p \in (\mathbb{F}_{p^{k_p}})^{|y|}$, we have

$$\left||\varphi_p((\mathbb{F}_{p^{k_p}})^{|x|}, a_p)| - \mu_{k_p} \cdot p^{k_p \cdot d_{k_p}}\right| \le C_{\varphi_p} \cdot p^{k_p(d_{k_p} - \frac{1}{2})}.$$

We say that $\varphi_p(x,a_p)$ has dimension d_{k_p} in $\mathbb{F}_{p^{k_p}}$. As $d_{k_p} \le |x|$, there is exactly one $d \in \{0,\dots,|x|\}$ with $\{p \in \mathbb{P} : \varphi_p(x,a_p) \text{ has dimension } d \text{ in } \mathbb{F}_{p^{k_p}}\} \in \mathcal{U}$. We claim that $\boldsymbol{\delta}_F(\varphi(F^{|x|},a)) = d$.

Proof of the claim: Note that for any $p \in \mathbb{P}$ and $c \in (\mathbb{F}_{p^{k_p}})^{|x|}$, we have

$$\mathbb{F}_{p^{k_p}} \models \varphi_p(c,a_p) \text{ if and only if } (\mathbb{F}_{p^{k_p}}, \mathrm{Frob}_p) \models \varphi(c,a_p).$$

Let $I = \{p \in \mathbb{P} : p > |\varphi(x,y)| \text{ and } \varphi_p(x,a_p) \text{ has dimension } d \text{ in } \mathbb{F}_{p^{k_p}}\}$. Clearly, $I \in \mathcal{U}$. Then for any $p \in I$,

$$\left||\varphi_p((\mathbb{F}_{p^{k_p}})^{|x|}, a_p)| - \mu_{k_p} \cdot p^{k_p \cdot d}\right| \le C_{\varphi_p} \cdot p^{k_p(d - \frac{1}{2})},$$

and $k_p \ge f(p,p) \ge \max\{\mu_{k_p}, \frac{1}{\mu_{k_p}}, 2\log_p\left(\frac{2C_{\varphi_p}}{\mu_{k_p}}\right)\}$.

As $k_p \ge 2\log_p\left(\frac{2C_{\varphi_p}}{\mu_{k_p}}\right)$, we get

$$C_{\varphi_p} \cdot p^{k_p(d-\frac{1}{2})} \le \frac{1}{2}\mu_{k_p} \cdot p^{k_p \cdot d}.$$

Therefore,

$$\frac{1}{2}\mu_{k_p} \cdot p^{k_p \cdot d} \le |\varphi_p((\mathbb{F}_{p^{k_p}})^{|x|}, a_p)| \le \frac{3}{2}\mu_{k_p} \cdot p^{k_p \cdot d}.$$

Furthermore, by the definition of k_p, we have $\frac{1}{k_p} < \mu_{k_p} < k_p$. Hence,

$$\frac{1}{2k_p} \cdot p^{k_p \cdot d} \leq |\varphi_p((\mathbb{F}_{p^{k_p}})^{|x|}, a_p)| \leq 2k_p \cdot p^{k_p \cdot d}.$$

This implies

$$d - \frac{\log(2k_p)}{k_p \cdot \log p} \leq \frac{\log |\varphi_p((\mathbb{F}_{p^{k_p}})^{|x|}, a_p)|}{\log(p^{k_p})} \leq d + \frac{\log(2k_p)}{k_p \cdot \log p}.$$

Since $\lim_{p\to\infty} \dfrac{\log(2k_p)}{k_p \cdot \log p} = 0$, we have

$$\lim_{p\to\infty,\ p\in I} \frac{\log |\varphi_p((\mathbb{F}_{p^{k_p}})^{|x|}, a_p)|}{\log(p^{k_p})} = d.$$

Therefore, $\boldsymbol{\delta}_F(\varphi(F^{|x|}, a)) = d$. $\square$

Remark: This proof works also for pseudofinite difference fields of characteristic $p > 0$, that is, for $\prod_{i\in I}(\mathbb{F}_{p^{k_i}}, \mathrm{Frob}_{p^{m_i}})/\mathcal{U}$ provided $k_i >> m_i$ for almost all i. More precisely, in the proof of Theorem 2.5, instead of translating φ to φ_p for each prime p, we translate it to $\varphi_{p^{m_i}}$ for each $i \in I$. That is, given a difference formula $\varphi(x, y)$ we consider the following ring formula $\varphi_{p^{m_i}}(x, y)$ obtained by replacing each occurrence of $\sigma(t)$ in $\varphi(x, y)$ by $t^{p^{m_i}}$. Then we use Fact 0.14 and the same strategy to get the desired result.

In the following, we will show that the coarse dimension $\boldsymbol{\delta}_F$ is definable using the field structure. To prove this, we first need a lemma.

Lemma 2.6. *Let M be an ultraproduct of finite structures in the language $\mathcal{L}'$ and X be an internal subset of M. Let $\varphi(x, y)$ be an $\mathcal{L}'$-formula with $|x| = m$ and $|y| = n$. Suppose there is some $r \in \mathbb{R}^{\geq 0}$ such that for all $b \in M^m$ we have $\boldsymbol{\delta}_X(\varphi(M^n, b)) = r$ whenever $\varphi(M^n, b) \neq \emptyset$. Then*

$$\boldsymbol{\delta}_X(\varphi(M^{n+m})) = r + \boldsymbol{\delta}_X(\exists x\varphi(x, M^m)).$$

Proof. Suppose $(M, X) = \prod_{i\in I}(M_i, X_i)/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on an index set I and $X_i \subseteq M_i$ finite sets. For each $i \in I$ pick b_i^{max} and b_i^{min} in $(M_i)^m$ such that $|\varphi((M_i)^n, b_i^{max})|$ is maximal and $|\varphi((M_i)^n, b_i^{min})|$ is minimal non-zero respectively. Clearly, we have

$$|\varphi((M_i)^n, b_i^{min})| \cdot |\exists x\varphi(x, (M_i)^m)| \leq |\varphi((M_i)^{n+m})| \leq |\varphi((M_i)^n, b_i^{max})| \cdot |\exists x\varphi(x, (M_i)^m)|.$$

Let $b^{max} := (b_i^{\max})_{i\in I}/\mathcal{U} \in M$ and $b^{min} := (b_i^{min})_{i\in I}/\mathcal{U} \in M$ respectively. By assumption, $\boldsymbol{\delta}_X(\varphi(M^n, b^{max})) = \boldsymbol{\delta}_X(\varphi(M^n, b^{min})) = r$. Therefore, for any $\epsilon > 0$, there is some $J \in \mathcal{U}$ such that for all $i \in J$, we have

$$|X_i|^{r-\epsilon} \leq |\varphi((M_i)^n, b_i^{min})| \leq |\varphi((M_i)^n, b_i^{max})| \leq |X_i|^{r+\epsilon}.$$

Multiplying each term by $|\exists x\varphi(x, (M_i)^m)|$ and combining the inequality before, we get

$$|X_i|^{r-\epsilon} \cdot |\exists x\varphi(x, (M_i)^m)| \leq \varphi((M_i)^{n+m}) \leq |X_i|^{r+\epsilon} \cdot |\exists x\varphi(x, (M_i)^m)|.$$

Therefore,

$$r-\epsilon+\frac{\log|\exists x\varphi(x,(M_i)^m)|}{\log|X_i|}\leq\frac{\log|\varphi((M_i)^{n+m})|}{\log|X_i|}\leq r+\epsilon+\frac{\log|\exists x\varphi(x,(M_i)^m)|}{\log|X_i|}.$$

By the definition of $\boldsymbol{\delta}_X$ we conclude that

$$r+\epsilon+\boldsymbol{\delta}_X(\exists x\varphi(x,M^m))\leq\boldsymbol{\delta}_X(\varphi(M^{n+m}))\leq r-\epsilon+\boldsymbol{\delta}_X(\exists x\varphi(x,M^m)).$$

Since ϵ is arbitrary, we get the desired result. □

Corollary 2.7. *Let M be a pseudofinite structure in the language $\mathcal{L}$ and let $X\subseteq M^n$ be an internal set. Suppose there is some $r\in\mathbb{N}$ such that for any $\mathcal{L}$-formula $\varphi(x,y)$ with $|x|=1$ over $\emptyset$ and any $b\in M^{|y|}$, we have $\boldsymbol{\delta}_X(\varphi(M,b))\in\{0,1,\ldots,r\}$ and for each $i\leq r$, the set*

$$\{b\in M^{|y|}:\boldsymbol{\delta}_X(\varphi(M,b))=i\}$$

is $\emptyset$-definable. Then for any formula $\psi(x,y)$ and any tuple $c\in M^{|y|}$, we have

$$\boldsymbol{\delta}_X(\psi(M^{|x|},c))\in\{0,\ldots,|x|\cdot r\}.$$

Moreover, $\boldsymbol{\delta}_X$ is definable.

Proof. We use induction on the length of $|x|$. The case $|x|=1$ is given by assumption.

Suppose the conclusion holds for $|x|=n$, we prove it for $|x|=n+1$. Let $\psi(x_0,\ldots,x_n,y)$ be a formula with $|x_i|=1$ for $0\leq i\leq n$. We know that there are formulas without parameters $\theta_\ell(x_1,\ldots,x_n,y)$ for $\ell\in\{0,1,\ldots,r\}$ which define respectively the sets

$$\{(x_1,\ldots,x_n,y)\in M^{n+|y|}:\boldsymbol{\delta}_M(\psi(M,x_1,\ldots,x_n,y))=\ell\text{ and }\psi(M,x_1,\ldots,x_n,y)\neq\emptyset\}.$$

For any $c\in M^{|y|}$, note that $\psi(M^{n+1},c)$ is the disjoint union of

$$\{\psi(M^{n+1},c)\wedge\theta_\ell(M^n,c):\ell\in\{0,1,\ldots,r\}\},$$

and Lemma 2.6 applies to each of these formulas. Hence,

$$\boldsymbol{\delta}_X(\psi(M^{n+1},c)\wedge\theta_\ell(M^n,c))=\ell+\boldsymbol{\delta}_X(\exists x_0(\psi(x_0,M^n,c)\wedge\theta_\ell(M^n,c))=\ell+\boldsymbol{\delta}_X(\theta_\ell(M^n,c)).$$

By induction hypothesis, $\boldsymbol{\delta}_X(\theta_\ell(M^n,c))\in\{0,\ldots,r\cdot n\}$. Therefore,

$$\boldsymbol{\delta}_X(\psi(M^{n+1},c))=\max\{\ell+\boldsymbol{\delta}_X(\theta_\ell(M^n,c)):0\leq\ell\leq r\}\in\{0,\ldots,r\cdot(n+1)\}.$$

Again by induction hypotheses, for any $k\in\{0,\ldots,r\cdot n\}$ there are $\emptyset$-definable $\xi_\ell^k(y)$ with $\ell\in\{0,\ldots,r\}$, which define the corresponding sets

$$\{y\in F^{|y|}:\boldsymbol{\delta}_X(\theta_\ell(M^n,y))=k\text{ and }\theta_\ell(M^n,y)\neq\emptyset\}.$$

Then the formula $\bigvee\limits_{0\leq\ell\leq r,\ 0\leq j\leq r\cdot n,\ \ell+j=t}\xi_\ell^j(y)$ defines the set

$$\{y\in M^{n+1}:\boldsymbol{\delta}_M(\psi(M^{n+1},y))=t\text{ and }\psi(M^{n+1},y)\neq\emptyset\}$$

for any $t \in \{0, \dots, r \cdot (n+1)\}$. $\square$

Lemma 2.8. *Let $\mathcal{M} = (F, +, \cdot, 0, 1, \dots)$ be a pseudofinite field with some extra structures. Let $\boldsymbol{\delta}_F$ be the coarse pseudofinite dimension normalised by $|F|$. Suppose for any formula $\varphi(x, y)$ with $|x| = 1$ we have $\boldsymbol{\delta}_F(\varphi(F, b)) \in \{0, 1\}$ for any tuple $b \in F^{|y|}$. Then $\boldsymbol{\delta}_F$ is definable and for any formula $\psi(x, y)$ and any tuple $c \in F^{|y|}$, we have $\boldsymbol{\delta}_F(\psi(F^{|x|}, c)) \in \{0, \dots, |x|\}$.*

Proof. By Corollary 2.7, we only need to show definability when $|x| = 1$.

For each $\psi(x, y)$, consider the formula

$$\theta_\psi(y) := \forall z \exists x_1 \exists x_2 \exists x_3 \exists x_4 \left(\bigwedge_{1 \le i \le 4} \psi(x_i, y) \ \wedge \ x_3 \ne x_4 \ \wedge \ z = (x_1 - x_2) \cdot (x_3 - x_4)^{-1} \right).$$

We claim that $\theta_\psi(c)$ holds if and only if $\boldsymbol{\delta}_F(\psi(F, c)) = 1$ for all $c \in F^{|y|}$. Suppose $\theta_\psi(c)$ hold. Then there is a map from $(\psi(F, c))^4$ to F defined by sending (x_1, x_2, x_3, x_4) to $(x_1 - x_2)(x_3 - x_4)^{-1}$ if $x_3 \ne x_4$, otherwise we map (x_1, x_2, x_3, x_4) to 0. The formula $\theta_\psi(c)$ holds means exactly that the map is surjective. Therefore, $\boldsymbol{\delta}_F(\psi(F, c)) \ge \frac{1}{4}\boldsymbol{\delta}_F(F) = \frac{1}{4}$. By assumption, $\boldsymbol{\delta}_F(\psi(F, c)) \in \{0, 1\}$. Hence, $\boldsymbol{\delta}_F(\psi(F, c)) = 1$. On the other hand, if $\neg\theta_\psi(c)$ holds, there is $a \in F$ such that for any $x_1, x_2, x_3, x_4 \in \psi(F, c)$ we have $a \ne (x_1 - x_2)(x_3 - x_4)^{-1}$ whenever $x_3 \ne x_4$. Let $f : (\psi(F, c))^2 \to F$ be defined as $f(x_1, x_2) := x_1 + a x_2$. Then f is an injection. Therefore, $\boldsymbol{\delta}_F(\psi(F, c)) \le \frac{1}{2}$. We conclude that $\boldsymbol{\delta}_F(\psi(F, c)) = 0$.

Hence, the set $\{c \in F^{|y|} : \boldsymbol{\delta}_F(\psi(F, c)) = 0 \text{ and } \psi(F, c) \ne \emptyset\}$ is defined by $\neg\theta_\psi(y) \wedge \exists x \psi(x, y)$, and $\theta_\psi(y)$ defines the set $\{c \in F^{|y|} : \boldsymbol{\delta}_F(\psi(F, y)) = 1\}$. $\square$

Corollary 2.9. *For any pseudofinite difference field $(F, Frob) \in \mathcal{S}$, the coarse dimension $\boldsymbol{\delta}_F$ is definable and integer-valued for all $\mathcal{L}_\sigma$-definable sets. Moreover, $\boldsymbol{\delta}_F$ is additive in the language $\mathcal{L}_\sigma$.*

Proof. By Theorem 2.5, for any $\mathcal{L}_\sigma$-formula $\psi(x, y)$ with $|x| = 1$, any $b \in F^{|y|}$ we have

$$\boldsymbol{\delta}_F(\psi(F, b)) \in \{0, 1\}.$$

Applying Lemma 2.8 we get the desired result. $\square$

Remark: In general, the coarse dimension does not have the property that a definable set has dimension 0 if only if it is finite. Similarly, in a pseudofinite group, a subgroup of infinite index does not necessarily have smaller dimension, as we show in the next example.

Example 2.1. *Let $(F, Frob) = \prod_{p \in \mathbb{P}} (\mathbb{F}_{p^{k_p}}, Frob_p)/\mathcal{U} \in \mathcal{S}$. Define a function $f : F^\times \to F^\times$ as*

$$f(x) := x^{-1} \cdot Frob(x).$$

It is easy to see that f is a group homomorphism. Therefore, the image $T := f(F^\times)$ is a definable subgroup of $F^\times$. There is a corresponding $f_p : (\mathbb{F}_{p^{k_p}})^\times \to (\mathbb{F}_{p^{k_p}})^\times$ and $T_p := f_p((\mathbb{F}_{p^{k_p}})^\times)$ for any $p \in \mathbb{P}$. Since the kernel of f_p is $(\mathbb{F}_p)^\times$, we get $[(\mathbb{F}_{p^{k_p}})^\times : T_p] = p - 1$. Hence, T has infinite index in $F^\times$, though $\boldsymbol{\delta}_F(T) = \boldsymbol{\delta}_F(F^\times)$.

2.3 Coarse dimension and transformal transcendence degree

In the following, we will study some algebraic properties of difference fields that are intrinsic to the coarse dimension $\boldsymbol{\delta}_F$. Our aim is to understand the theory of difference fields in $\mathcal{S}$ in terms of $\boldsymbol{\delta}_F$.

Let us start with an observation. Given $(F, \mathrm{Frob}) = (\mathbb{F}_{p^{k_p}}, \mathrm{Frob}_p)/\mathcal{U} \in \mathcal{S}$. Let

$$(\tilde{F}, \mathrm{Frob}) := \prod_{p \in \mathbb{P}} (\tilde{\mathbb{F}}_p, \mathrm{Frob}_p)/\mathcal{U},$$

then by [Hru04, Theorem 1.4] we have $(\tilde{F}, \mathrm{Frob})$ is a model of ACFA, which contains (F, Frob) as a substructure.

In ACFA, there is a notion of dimension which is also integer-valued, and it is induced by SU-rank.

Let $\mathbf{k}$ be a saturated model of ACFA.

Definition 2.10. Let a be a finite tuple in $\mathbf{k}$ and $A \subseteq \mathbf{k}$. Then $\mathrm{SU}(a/A) = \omega \cdot k + n$ for some $0 \leq k \leq |a|$. Define the *rank-dimension* $\dim_{rk}$ of $\mathrm{tp}(a/A)$ as $\dim_{rk}(a/A) := k$.

Remark: $\dim_{rk}(a/A)$ coincides with the transformal transcendence degree of a over A_σ (the difference field generated by A).

Now we have two integer-valued additive dimensions on types: the rank-dimension $\dim_{rk}$ and the coarse dimension $\boldsymbol{\delta}_F$. It is natural to ask whether they coincide. One of the inequalities is obvious.

Lemma 2.11. *Let $(F, Frob) \in \mathcal{S}$. For any tuple $a \in F$ and subset $A \subseteq F$ we have $\boldsymbol{\delta}_F(a/A) \leq dim_{rk}(a/A)$.*

Proof. Note that by the additivity of both $\dim_{rk}$ and $\boldsymbol{\delta}_F$, we only need to prove the inequality when a is a single element. We may assume that $A = A_\sigma$. By [CH99], we know that $SU(a/A) = \omega$ if and only if a is transformally transcendental over A if and only if $deg_\sigma(a/A) = \infty$. Therefore, we need to show that if $deg_\sigma(a/A) < \infty$ then $\boldsymbol{\delta}_F(a/A) = 0$.

Suppose $deg_\sigma(a/A) < \infty$. Then there is some m and a non-trivial polynomial $f(x; y_1, \ldots, y_m)$ with coefficients in A, such that $f(\sigma^m(a); \sigma^{m-1}(a), \ldots, a) = 0$. Take any prime $p \in \mathbb{P}$ and let $g_p(x) := f(x^{p^m}; x^{p^{m-1}}, \ldots, x)$. Then

$$|\{a' \in \mathbb{F}_{p^{k_p}} : g_p(a') = 0\}| \leq p^{C \cdot m}$$

for some constant C depending on f. Let $\varphi(x) := f(\sigma^m(x); \sigma^{m-1}(x), \ldots, x) = 0$. Then $\varphi(x)$ defines exactly the set of zeros of g_p in $(\mathbb{F}_{p^{k_p}}, \mathrm{Frob}_p)$. Therefore, $\boldsymbol{\delta}_F(\varphi(F)) = 0$. As $a \in \varphi(F)$, we get $\boldsymbol{\delta}_F(a/A) = 0$. $\square$

We conjecture that in general the two dimensions coincide.

In the following we will demonstrate an application with the assumption that $\dim_{rk}$ is controlled by $\boldsymbol{\delta}_F$. The strategy is the following: we start with a definable object in (F, Frob). If we have the control over $\dim_{rk}$ of elements in it, then we work in $(\tilde{F}, \mathrm{Frob})$. As it is a model of ACFA, we can use all the model theoretic tools there. Finally, we transfer the results from $(\tilde{F}, \mathrm{Frob})$ back to (F, Frob).

Fact 2.12. [Cha05, Section 6.5] Let (k, σ) be a model of ACFA. Let G be a definable subgroup of some algebraic group $H(k)$. Let acl_σ denote the algebraic closure in ACFA. Suppose G is definable over $E = \mathrm{acl}_\sigma(E)$. Then G is contained in a group $\tilde{G}$ which is quantifier-free definable over E and has the same SU-rank as G.

Notation: For a difference formula $\varphi(x)$ with parameters $A \subseteq (\tilde{F}, \mathrm{Frob})$. Let

$$\begin{aligned} d &= \max\{\dim_{rk}(a/A) : a \in \varphi(\tilde{F}^{|\bar{x}|})\} \\ &= \max\{n \leq |x| : \mathrm{SU}(a/A) = \omega \cdot n + m, \text{ for some } a \in \varphi(\tilde{F}^{|x|})\}. \end{aligned}$$

We define $\dim_{rk}(\varphi(x)) := d$.

Lemma 2.13. *Let $(F, Frob) \in \mathcal{S}$, $a \in F^n$ and $A \subseteq F$. Suppose $dim_{rk}(a/A) = k$. Then there is a finite set $\{P_1(x), \ldots, P_m(x)\}$ of difference polynomials with parameters in A such that $(F, Frob) \models \bigwedge_{i \leq m} P_i(a) = 0$ and $dim_{rk}(\bigwedge_{i \leq m} P_i(x) = 0) = k$.*

Proof. We may write a into two parts a_1 and a_2 where $\dim_{rk}(a_1/A) = |a_1| = k$, and $\dim_{rk}(a_2/Aa_1) = 0$. Let $(Aa_1)_\sigma$ be the difference field generated by $A \cup \{a_1\}$. Suppose $a_2 := a_2^1 \cdots a_2^m$ with each $|a_2^i| = 1$. Since $\dim_{rk}(a_2^i/Aa_1) = 0$ for each $i \leq m$, we get $deg_\sigma(a_2^i/(Aa_1)_\sigma) < \infty$. Therefore, there is a difference polynomial $P_i(y_i, b_i)$ with $b_i \subseteq (Aa_1)_\sigma$ such that a_2^i vanishes on it. Write $b_i = f_i(a_1)$ where f_i is a difference polynomial with parameters in A. We should rearrange the order of variables such that $x_0, \ldots, x_{|a|-1}$ corresponds to the order of a. Suppose $a_1 = a^{\ell_1} \cdots a^{\ell_{|a_1|}}$ and $a_2 = a^{t_1} \cdots a^{t_{|a_2|}}$ where a^j is the j^{th} component of the tuple a. Now it is easy to see that a satisfies the formula

$$\varphi(x) := \bigwedge_{i \leq m} P_i(x_{t_i}, f_i(x_{\ell_1}, \ldots, x_{\ell_{|a_1|}})) = 0,$$

and $\dim_{rk}(\varphi(x)) = k$. □

Theorem 2.14. *Let $(F, Frob) \in \mathcal{S}$. Suppose G is a definable over a finite set $A \subseteq F$ subgroup of some algebraic group $H(F) \subseteq F^n$. If for any $g \in G$ we have $dim_{rk}(g/A) \leq \boldsymbol{\delta}_F(G)$, then there is a quantifier-free definable group $\bar{G} \geq G$ (defined with parameters in F), such that $\boldsymbol{\delta}_F(\bar{G}) = \boldsymbol{\delta}_F(G)$.*

Proof. Suppose G is defined by the formula φ_G. Let $k := \boldsymbol{\delta}_F(G)$.

Let Π_A denote the set of difference polynomials in n-variables with coefficients in A.

By Lemma 2.13, for any element $a \in G$, there are some $\{P_{a,i}(x) : 1 \leq i \leq m_a\} \subset \Pi_A$ such that $(F, \mathrm{Frob}) \models \bigwedge_{i \leq m_a} P_{a,i}(a) = 0$ and $\dim_{rk}(\bigwedge_{i \leq m_a} P_{a,i}(x) = 0) = \dim_{rk}(a/A)$. By assumption, $\dim_{rk}(a/A) \leq \boldsymbol{\delta}_F(G) = k$. Therefore, $\varphi_G(x)$ is covered by the collection of formulas $\{\bigwedge_{i \leq m_a} P_{a,i}(x) = 0 : a \in G\}$. Since $[\varphi_G]$ is closed in the compact space $S_n(F)$, we have by compactness, there is some finite set $a_0, \ldots, a_\ell$ such that

$\varphi_G(x) \models \bigvee_{j\leq \ell}\left(\bigwedge_{i\leq m_{a_j}} P_{a_j,i}(x)=0\right)$. Let $\Phi(x) := \bigvee_{j\leq \ell}\left(\bigwedge_{i\leq m_{a_j}} P_{a_j,i}(x)=0\right)$. As $\dim_{rk}(\bigwedge_{i\leq m_{a_j}} P_{a_j,i}(x)=0)\leq k$ for each $j\leq \ell$, we get $\dim_{rk}(\Phi(x))\leq k$.

Write $\Phi(x)$ into the conjunctive normal form $\bigwedge\limits_{u\leq N}\bigvee\limits_{v\leq M_u}(P_{u,v}(x)=0)$ for some natural numbers N, M_u, and each $P_{u,v}(x)\in\{P_{a_j,i}(x): j\leq \ell, i\leq m_{a_j}\}$. Hence, for each $u\leq N$, we have $\varphi_G(x)\models(\prod_{v\leq M_u} P_{u,v}(x))=0$.

Let $G_{\tilde F}$ be the σ-Zariski closure of G in $H(\tilde F)$, that is, if we define $I_{\tilde F}(G)=\{p\in \tilde F[x]_\sigma : p(g)=0 \text{ for all } g\in G\}$, then

$$G_{\tilde F} := \{h\in H(\tilde F) : p(h)=0 \text{ for all } p\in I_{\tilde F}(G)\}.$$

As prime σ-ideals are finitely generated, $G_{\tilde F}$ is quantifier-free definable. Note that $\prod_{v\leq M_u} P_{u,v}(x)\in I_{\tilde F}(G)$ for each $u\leq N$. Since

$$\dim_{rk}\left(\bigwedge_{u\leq N}\left(\prod_{v\leq M_u} P_{u,v}(x)\right)=0\right)=\dim_{rk}\left(\bigvee_{j\leq \ell}\bigwedge_{i\leq m_{a_j}} P_{a_j,i}(x)=0\right)\leq k,$$

we get $\dim_{rk}(G_{\tilde F})\leq k$.

Take an automorphism α of $(\tilde F, \mathrm{Frob})$ fixing F. Then $G=\alpha(G)\subseteq\alpha(G_{\tilde F})$. As $\alpha(G_{\tilde F})$ is also closed under the σ-Zariski topology in $(\tilde F,\mathrm{Frob})$, we get $G_{\tilde F}\subseteq\alpha(G_{\tilde F})$ which implies $G_{\tilde F}=\alpha(G_{\tilde F})$. Therefore, $G_{\tilde F}$ is invariant under automorphisms fixing F, hence it is definable over F. Let $E=\mathrm{acl}_\sigma(F)=F^{alg}$, then by Fact 2.12, there is G_E which contains $G_{\tilde F}$, has the same SU-rank as G_E and is quantifier-free definable over E. In fact, G_E is the smallest closed set containing $G_{\tilde F}$ in the σ-Zariski topology in $(F^{alg}, \mathrm{Frob}\restriction_{F^{alg}})$.

Suppose G_E is defined by

$$\bigwedge_{0\leq j\leq \ell'} P'_j(x,\sigma(x),\ldots,\sigma^m(x),c_j)=0,$$

where P'_j are polynomials in the language of rings and $c_j\subseteq F^{alg}$. For any $0\leq j\leq \ell'$, let $\{c_j^0,\ldots,c_j^{N_j}\}\subseteq (F^{alg})^{|c_j|}$ be the set of all field conjugates of c_j over F. Note that for any $g\in G$ we have $g,\sigma(g),\ldots,\sigma^m(g)\subseteq F$. Hence, $P'_j(g,\sigma(g),\ldots,\sigma^m(g),c_j)=0$ if and only if $P'_j(g,\sigma(g),\ldots,\sigma^m(g),c_j^i)=0$ for any $g\in G$ and $0\leq i\leq N_j$.

Let B_j be the set in $H(\tilde F)$ vanishing on $\{P'_j(x,\sigma(x),\ldots,\sigma^m(x),c_j^i): 0\leq i\leq N_j\}$. Then from the above argument, we know $B_j\supseteq G$. As B_j is closed under the σ-Zariski topology in $(\tilde F,\mathrm{Frob})$, we get $B_j\supseteq G_{\tilde F}$. Similarly, by B_j being closed under the σ-Zariski topology in $(F^{alg},\mathrm{Frob}\restriction_{F^{alg}})$, we get $B_j\supseteq G_E$.

Now consider the formula

$$\bigwedge_{0\leq j\leq \ell'}\ \bigwedge_{0\leq i\leq N_j} P'_j(x,\sigma(x),\ldots,\sigma^m(x),c_j^i)=0.$$

It defines $\bigcap_{j\leq \ell'} B_j$. As before, we know that $\bigcap_{j\leq\ell'} B_j\supseteq G_E$. Clearly, we also have $\bigcap_{j\leq \ell'} B_j\subseteq G_E$. Hence, the formula above also defines G_E in $H(\tilde F)$. Now we show that G_E can be made quantifier-free definable over F.

Fix $0 \leq j \leq \ell'$ and consider the formula

$$\bigwedge_{0 \leq i \leq N_j} P'_j(x, x_1, \ldots, x_m, c^i_j) = 0,$$

where $x_1, \ldots, x_m$ are distinct tuples of variables all have the same length as x. For $1 \leq k \leq N_j + 1$, let $e_k(t_0, \ldots, t_{N_j})$ be the k-elementary symmetric polynomials in $N_j + 1$-variables, i.e.

$$e_k(t_0, \ldots, t_{N_j}) := \sum_{0 \leq i_1 < \cdots < i_k \leq N_j} t_{i_1} \cdots t_{i_k}.$$

Then we have $\bigwedge_{0 \leq i \leq N_j} P'_j(x, x_1, \ldots, x_m, c^i_j) = 0$ if and only if

$$\bigwedge_{1 \leq k \leq N_j+1} e_k(P'_j(x, x_1, \ldots, x_m, c^0_j), \ldots, P'_j(x, x_1, \ldots, x_m, c^{N_j}_j)) = 0.$$

For each $1 \leq k \leq N_j + 1$, as $\{c^i_j : 0 \leq j \leq N_j\}$ is the set of all field conjugates of c_j in F^{alg} over F and that e_k is symmetric, we get

$$Q^k_j(x, \ldots, x_m, b^k_j) := e_k(P'_j(x, x_1, \ldots, x_m, c^0_j), \ldots, P'_j(x, x_1, \ldots, x_m, c^{N_j}_j))$$

is invariant under field automorphisms in $\mathrm{Gal}(F^{alg}/F)$. Therefore, since F is a pseudofinite field, F is perfect and we have $b^k_j \subseteq F$ for all $1 \leq j \leq \ell'$ and $1 \leq k \leq N_j + 1$.

Let $\varphi_H(x)$ be the quantifier-free formula with parameters in A that defines the algebraic group H. Now consider

$$\psi(x) := \varphi_H(x) \wedge \left(\bigwedge_{0 \leq j \leq \ell'} \bigwedge_{1 \leq k \leq N_j+1} Q^k_j(x, \sigma(x), \ldots, \sigma^m(x), b^k_j) = 0 \right).$$

It is easy to see that $\psi(x)$ defines G_E in $(\tilde{F}, \mathrm{Frob})$. Note that $\psi(x)$ is quantifier-free and defined over F, so we can consider $\bar{G} := \{g \in F^t : (F, \mathrm{Frob}) \models \psi(g)\}$. Since $H(F)$ is an algebraic group and F is definably closed in $\tilde{F}$ in the language of rings, $\bar{G}$ is a quantifier-free definable group in (F, Frob) and contains G. Note that $\dim_{rk}(G_E) = \dim_{rk}(G_{\tilde{F}}) \leq k$. Hence, $\boldsymbol{\delta}_F(\bar{G}) \leq \dim_{rk}(\psi(x)) = \dim_{rk}(G_E) \leq k$. On the other hand, since $\bar{G} \supseteq G$ and $\boldsymbol{\delta}_F(G) = k$, we get $\boldsymbol{\delta}_F(\bar{G}) \geq k$. Therefore, $\boldsymbol{\delta}_F(\bar{G}) = \boldsymbol{\delta}_F(G) = k$, which concludes the proof of Corollary 2.14. □

2.4 Wildness of $\mathcal{S}$

This section will be some discussions about negative model theoretic properties of the class $\mathcal{S}$ defined in Section 2.2. We will first investigate whether this family $\mathcal{S}$ is *tame* in terms of the properties in Shelah's classification theory [She90]. It turns out that the answer is negative. As we have mentioned before, we will show that if a structure expands a pseudofinite field with a "logarithmically small" definable subset, then all the internal subsets of this definable set will be uniformly definable.[4] Therefore, theories of

[4] This result is known among experts. As we could not find a proof in the literature, we include it here for completeness.

structures in $\mathcal{S}$ have TP2 and the strict order property and is not decidable. We proceed by an example in $\mathcal{S}$ where the model theoretic algebraic closure does not coincide with the algebraic closure in the sense of difference algebra. We conclude with some general remarks and questions.

2.4.1 Non-tameness

In this subsection we will show that the theory of any member of $\mathcal{S}$ has TP2 and the strict order property and is not decidable.

The proof is based on the result that the theory of pseudofinite fields has the independence property in [Dur80]. The strategy is to modify Duret's proof to show that when an internal set is very small compared to the size of the field, then every internal subset of it can also be coded uniformly.

Fact 2.15. ([Dur80, Proposition 4.3]) Let k be a field and p a prime different from $\mathrm{char}(k)$ such that k contains a p^{th}-root of unity. Let $\tilde{k}$ be the algebraic closure of k. Suppose $f_i \in k[Y_1, \ldots, Y_m]$ and $F_i = X^p - f_i \in k[Y_1, \ldots, Y_m, X]$ for $1 \leq i \leq n$. If there exist $g_i, h_i \in \tilde{k}[Y_1, \ldots, Y_m]$ and $q_i \in \mathbb{N}$ such that:

- for all $i,\ f_i = g_i^{q_i} h_i$;
- for all $i,\ g_i$ is prime in $\tilde{k}[Y_1, \ldots, Y_m]$
- for all $i \neq j,\ g_i \neq g_j$
- for all i and $j,\ g_i$ does not divide h_j
- for all $i,\ p$ does not divide q_i.

Then the ideal J in $k[Y_1, \ldots, Y_m, X_1, \ldots, X_n]$ generated by $\{F_i(X_i) :\ 1 \leq i \leq n\}$ is absolutely prime, and does not contain any non-zero element in $k[Y_1, \ldots, Y_m]$.

Fact 2.16. ([CM06, Theorem 7.1]) Let $V \subseteq (\tilde{\mathbb{F}}_q)^n$ be an absolutely irreducible $\mathbb{F}_q$-variety of dimension $r > 0$ and degree ℓ. If $q > 2(r+1)\ell^2$, then the following estimate holds:

$$||(V \cap (\mathbb{F}_q)^n)| - q^r| \leq (\ell - 1)(\ell - 2)q^{r-\frac{1}{2}} + 5\ell^{\frac{13}{3}} q^{r-1}.$$

Theorem 2.17. *Let $F = \prod_{i \in I} \mathbb{F}_{q_i}/\mathcal{U}$ be a pseudofinite field and $A = \prod_{i \in I} A_i/\mathcal{U}$ an infinite internal subset of F. Suppose there is a positive constant C such that $\{i \in I : |A_i| \leq C \log_2 q_i\} \in \mathcal{U}$. Then all internal subsets of A are uniformly definable.*

Proof. Consider the finite algebraic extension F' of F of degree $14\lceil C \rceil$. As F is pseudofinite, there is only one such extension and is definable. To see the definability, suppose $F' = F(\alpha)$. Let f be the minimal polynomial of α over F. Then we can define F' as the $14\lceil C \rceil$-dimensional vector space over F with multiplication defined according to the minimal polynomial f.

We distinguish two cases according to $p_i := \mathrm{char}(\mathbb{F}_{q_i})$. First, let us suppose $p_i \neq 2$ and $q_i = p_i^{n_i}$. Since $x^{p_i^{14\lceil C \rceil n_i} - 1} = 1$ for all $x \in \mathbb{F}_{p_i^{14\lceil C \rceil n_i}}$, the square root of unity exists in $\mathbb{F}_{p_i^{14\lceil C \rceil n_i}}$. As the multiplicative group of $\mathbb{F}_{p_i^{14\lceil C \rceil n_i}}$ is cyclic, take $\alpha_i \in \mathbb{F}_{p_i^{14\lceil C \rceil n_i}}$ a generator, then α_i is not a square in $\mathbb{F}_{p_i^{14\lceil C \rceil n_i}}$.

Claim 2.18. Let $\varphi(y,u)$ be the formula $\exists x(x^2 = y+u)$. Then for all $i \in I$ with $p_i \neq 2$ and for all $E_i \subseteq A_i$, there is $y_i \in F_{p_i^{14\lceil C\rceil n_i}}$ such that

$$E_i = \varphi(y_i, \mathbb{F}_{p_i^{14\lceil C\rceil n_i}}) \cap A_i.$$

Proof. Let $i \in I$ with $p_i \neq 2$, $E_i \subseteq A_i$ and $t_i := |A_i| \leq Cn_i \log_2 p_i$. Let J be the ideal in $\mathbb{F}_{p_i^{14\lceil C\rceil n_i}}[X_1, \ldots, X_{t_i}, Y]$ generated by

$$\{X_j^2 - (Y + c_j) : c_j \in E_i\} \cup \{X_j^2 - \alpha_i(Y + d_j) : d_j \in A_i \setminus E_i\},$$

where α_i is a generator of $\mathbb{F}^{\times}_{p_i^{14\lceil C\rceil n_i}}$ as defined before. Let $V(J)$ be the corresponding $\mathbb{F}_{p_i^{14\lceil C\rceil n_i}}$-variety. Then $V(J)$ is absolutely irreducible by Fact 2.15,

Suppose $V(J) \cap (\mathbb{F}_{p_i^{14\lceil C\rceil n_i}})^{t_i+1} \neq \emptyset$. Let $(x_1, \ldots, x_{t_i}, y_i)$ be a solution. Then clearly $E_i \subseteq \varphi(y_i, \mathbb{F}_{p_i^{14\lceil C\rceil n_i}})$. On the other hand, if there is $d \in A_i \setminus E_i$, such that $\varphi(y_i, d)$. Then there are $x_j, x \in \mathbb{F}_{p_i^{14\lceil C\rceil n_i}}$ such that:

$$\begin{aligned} x_j^2 &= \alpha_i(y_i + d); \\ x^2 &= y_i + d; \\ y_i - d &\neq 0, \end{aligned}$$

where the last inequality follows from Fact 2.15, as $Y - d \notin J$. Hence, $\alpha_i = \left(\frac{x_j}{x}\right)^2$, contradicting that α_i is not a square root. Therefore, $E_i = \varphi(y_i, \mathbb{F}_{p_i^{14\lceil C\rceil n_i}}) \cap A_i$.

So we only need to show $V(J) \cap \mathbb{F}_{p_i^{14\lceil C\rceil n_i}} \neq \emptyset$.

Let $|A_i| = t_i \leq Cn_i \log_2 p_i$. We calculate the dimension and the degree of $V(J)$. It is clear that the dimension of $V(J)$ is 1, as all X_j are algebraic over Y. Let $c_1, \ldots, c_{t_i}$ be a list of all elements in A_i, and for $1 \leq j \leq t_i$, let V_j be the variety defined by either the set of solutions of $X_j^2 - (Y + c_j)$ if $c_j \in E_i$, or $X_j^2 - \alpha_i(Y + c_j)$ if $c_j \notin E_i$. Then $V(J) = \bigcap_{1 \leq j \leq t_i} V_j$ and each V_j has degree 2. Therefore, by the Bézout inequality, the degree of $V(J)$ is less than or equal to 2^{t_i}.

Suppose, towards a contradiction, that $V(J) \cap (\mathbb{F}_{p_i^{14\lceil C\rceil n_i}})^{t_i+1} = \emptyset$. Then by Fact 2.16,

$$\begin{aligned} p_i^{14\lceil C\rceil n_i} &\leq (2^{t_i} - 1)(2^{t_i} - 2)p_i^{7\lceil C\rceil n_i} + 5 \times 2^{\frac{13}{3}t_i} \\ &\leq (p_i^{Cn_i} - 1)(p_i^{Cn_i} - 2)p_i^{7\lceil C\rceil n_i} + 5 \times p_i^{\frac{13}{3}Cn_i} \\ &< p_i^{2Cn_i} p_i^{7\lceil C\rceil n_i} + p_i^{8Cn_i} = p_i^{9\lceil C\rceil n_i} + p_i^{8Cn_i} \\ &< p_i^{14\lceil C\rceil n_i}, \end{aligned}$$

contradiction. □

The case $\mathrm{char}(q_i) = 2$ is similar. Suppose $q_i = 2^{n_i}$. Since 3 divides $2^{14\lceil C\rceil n_i} - 1$ for each i, there exists $x \in \mathbb{F}_{2^{14\lceil C\rceil n_i}}$ such that $x^3 = 1$. Take β_i to be the generator of the multiplicative group of $\mathbb{F}_{2^{14\lceil C\rceil n_i}}$. Then there is no $y \in \mathbb{F}_{2^{14\lceil C\rceil n_i}}$ such that $y^3 = \beta_i$.

Claim 2.19. Let $\psi(y,u)$ be the formula $\exists x(x^3 = y+u)$. Then for all $i \in I$ and $E_i \subseteq A_i$, there is $y_i \in \mathbb{F}_{2^{14\lceil C\rceil n_i}}$ such that $E_i = \psi(y_i, \mathbb{F}_{2^{14\lceil C\rceil n_i}}) \cap A_i$.

Proof. Fix some i and $E_i \subseteq A_i$. Let J be the ideal in $\mathbb{F}_{2^{14\lceil C\rceil n_i}}[X_1,\ldots,X_{t_i},Y]$ generated by

$$\{X_j^3 - (Y+c_j) : c_j \in E_i\} \cup \{X_j^3 - \beta_i(Y+d_j) : d_j \in A_i \setminus E_i\}.$$

As in the previous argument, the variety $V(J)$ is absolutely irreducible of dimension 1 and of degree less than or equal to 3^{t_i}. To prove the claim, we only need to show that $V(J) \cap (\mathbb{F}_{2^{14\lceil C\rceil n_i}})^{t_i+1} \neq \emptyset$. Suppose not, then by Fact 2.16,

$$2^{14\lceil C\rceil n_i} \leq (3^{t_i}-1)(3^{t_i}-2)2^{7\lceil C\rceil n_i} + 5\times 3^{\frac{13}{3}t_i} \leq 3^{2Cn_i}2^{7\lceil C\rceil n_i} + 3^{7Cn_i} < 2^{14\lceil C\rceil n_i},$$

contradiction. □

Let $A = \prod_{i\in I} A_i/\mathcal{U}$. Assume A is defined by $\chi(x)$. Define $\phi(x,y) := \psi(y,x)\wedge\chi(x)$ if the characteristic of F' is 2, and $\phi(x,y) := \varphi(y,x)\wedge\chi(x)$ otherwise. Let $E = \prod_{i\in I} E_i/\mathcal{U} \subseteq A$ be any internal subset. By the previous two claims, there is $y_E \in F'$ such that $E = \phi(F', y_E)$ in F'. Remember that we regard F' as $14\lceil C\rceil$-dimensional vector space over F and $A \subseteq F$. So as F' is definable in F, let $\phi'(\bar{x},\bar{y})$ be the corresponding translation of $\phi(x,y)$ in F and put $\theta(x,\bar{y}) := \phi'(x,0,\ldots,0,\bar{y})$. We see that $\theta(x,\bar{y})$ codes uniformly all internal subsets of A. □

Remark:

- From the proof we know that if $\mathrm{char}(F) \neq 2$ and $q_i \geq 2^{14|A_i|}$ for all large enough i, then we can take $\theta(x,\bar{y}) := \exists z^2(z^2 = x+y) \wedge \chi(x)$ where x, y are single variables and $\chi(x)$ is the formula defining A.

- The above proof of Theorem 2.17 is purely algebraic. However, it is possible to use the Paley graphs (P_q, R) constructed from $\mathbb{F}_q$ and the Bollobás-Thomason inequalities to give a combinatoric and more neat proof when $q \equiv 1 \pmod 4$.[5] The idea is that suppose we have a small subset $A \subseteq \mathbb{F}_q$ with $|A| = m$ and $E \subseteq A$. Let $V(E, A\setminus E)$ be set of vertices in $\mathbb{F}_q$ not in A which connect to everything in E and nothing in $A \setminus E$. Then the Bollobás-Thomason inequality will give

$$\left||V(E,A\setminus E)| - 2^{-m}q\right| \leq \frac{1}{2}\left(m-2+2^{-m+1}\right)q^{\frac{1}{2}} + \frac{m}{2}.$$

  Hence, when $q >> 2^m$, then $V(E, A\setminus E) \neq \emptyset$. And any element in $V(E, A\setminus E)$ will code the subset E inside A, and the coding is uniform by the formula $\varphi(x,y) := x \in A \wedge xRy$.

Corollary 2.20. *Let $F = \prod_{i\in I}\mathbb{F}_{q_i}/\mathcal{U}$ be a pseudofinite field and $B = \prod_{i\in I} B_i/\mathcal{U}$ an infinite internal subset of F. Suppose there is a positive constant C such that $\{i \in I : |B_i| \leq C\log_2 q_i\} \in \mathcal{U}$. Then (F,B) interprets the structure $N = \prod_{i\in I}(N_i,+,\times)/\mathcal{U}$, where $N_i = \{j \in \mathbb{N} : 0 \leq j \leq m_i\}$ for some $m_i \in \mathbb{N}$, and $+,\times$ are the addition and multiplication truncated on N_i respectively.*

[5]We would like to thank the referee to point out this observation. In fact, the Bollobás-Thomason inequality will give a better bound than the bound we use for the Lang-Weil estimate in Fact 2.16. But the author has not yet found the equivalent Bollobás-Thomason inequality in the characteristic 2 case.

Proof. For each $i \in I$, pick $Y_i \subseteq B_i$ such that $|B_i|^{\frac{1}{4}} \leq |Y_i| \leq |B_i|^{\frac{1}{3}}$. Let $Y = \prod_{i \in I} Y_i/\mathcal{U}$. By Theorem 2.17, Y is definable and all subsets of Y_i are uniformly definable by some $\psi_1(y, u)$. For each $i \in I$, consider the set $W_i := \left\{\frac{y_1 - y_2}{y_3 - y_4} : y_1, y_2, y_3, y_4 \in Y_i, y_3 \neq y_4\right\}$. The set W_i has size at most $|Y_i|^4 << |\mathbb{F}_{q_i}|$. Take any $a \notin W_i \cup \{0\}$. Then the set $T_i := \{y_1 + ay_2 : y_1, y_2 \in Y_i\}$ is in definable bijection with $Y_i \times Y_i$ and of size less than $\log_2 q_i$. By Theorem 2.17, all subsets of T_i, hence of $Y_i \times Y_i$, are uniformly definable by some $\psi_2(y, u)$. Similarly, we can show that all subsets of $Y_i \times Y_i \times Y_i$ are uniformly definable by some $\psi_3(y, u)$.

For $a \in \mathbb{F}_{q_i}$, we write $S_a^1 \subseteq Y_i$ for the set $\psi_1(a, \mathbb{F}_{q_i})$ and $S_a^2 \subseteq Y_i \times Y_i$, $S_a^3 \subseteq Y_i \times Y_i \times Y_i$ for $\psi_2(a, \mathbb{F}_{q_i}), \psi_3(a, \mathbb{F}_{q_i})$ respectively.

Now define a relation $R_+ \subseteq (\mathbb{F}_{q_i})^3$ by: $R_+(a, b, c)$ if there exist $g \in \mathbb{F}_{q_i}$ and $y \neq y' \in Y_i$ such that

- either S_g^3 is the graph of a bijective function from $(S_a^1 \times \{y\}) \cup (S_b^1 \times \{y'\})$ to S_c^1;
- or $S_c^1 = Y_i$ and S_g^3 is the graph of a surjective function from $(S_a^1 \times \{y\}) \cup (S_b^1 \times \{y'\})$ to Y_i;

Similarly, we define $R_\times \subseteq (\mathbb{F}_{q_i})^3$ by: $R_\times(a, b, c)$ if there exists $g \in \mathbb{F}_{q_i}$ such that

- either S_g^3 is the graph of a bijective function from $S_a^1 \times S_b^1$ to S_c^1;
- or $S_c^1 = Y_i$ and S_g^3 is the graph of a surjective function from $S_a^1 \times S_b^1$ to Y_i;

We also define an equivalence relation $E \subseteq (\mathbb{F}_{q_i})^2$ by: $E(a, b)$ if and only if there exists $g \in \mathbb{F}_{q_i}$ such that S_g^2 is the graph of a bijective function from S_a^1 to S_b^1.

It is easy to see then that $R^+, R^\times$ respect the equivalence relation E and

$$(|Y_i|, +, \times) \simeq ((\mathbb{F}_{q_i})^2/E, R^+/E, R^\times/E).$$

$\square$

Corollary 2.21. *Let $(F, Frob) \in \mathcal{S}$ and $T := Th(F, Frob)$. Then T has the strict order property and TP2. Moreover, T is not decidable.*

Proof. As the fixed field $\mathrm{Fix}(F) := \{x \in F : \sigma(x) = x\}$ is definable and satisfies the condition in Theorem 2.17, every internal subset of $\mathrm{Fix}(F)$ can be coded uniformly by some formula $\varphi(x, t)$. In particular, it will code some infinite strictly increasing chain $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots$ of subsets of $\mathrm{Fix}(F)$. Therefore, T has the strict order property.

Let $\varphi(x, t)$ be the same formula. To see that T has TP2, by compactness, we only need to show that given any $n \in \mathbb{N}$, there is some $(a_{ij})_{1 \leq i,j \leq n}$ such that for any $1 \leq i \leq n$, we have $\{\varphi(x, a_{ij}) : 1 \leq j \leq n\}$ is 2-inconsistent and $\{\varphi(x, a_{if(i)}) : 1 \leq i \leq n\}$ is consistent for any $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

Given $n \in \mathbb{N}$, let $A_n \subseteq \mathrm{Fix}(F)$ be a set with n^n-many elements. Fix a bijection $\eta : A_n \to \{1, \ldots, n\}^{\{1,\ldots,n\}}$ where $\{1, \ldots, n\}^{\{1,\ldots,n\}}$ is the set of all functions from $\{1, \ldots, n\}$ to itself. Let $(a_{ij})_{1 \leq i,j \leq n}$ be such that $\varphi(x, a_{ij})$ codes the set

$$B_{ij} := \{a \in A_n : \eta(a)(i) = j\} \subseteq A_n.$$

For any $1 \leq i \leq n$, as $B_{i1}, \ldots, B_{in}$ form a complete partition of A_n, we get $\{\varphi(x, a_{ij}) : 1 \leq j \leq n\}$ is 2-inconsistent. On the other hand, for any $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ the element $\eta^{-1}(f) \in A_n$ witnesses that $\{\varphi(x, a_{if(i)}) : 1 \leq i \leq n\}$ is consistent.

Finally, as (F, Frob) interprets ultraproducts of initial segments of natural numbers with truncated addition and multiplication by Corollary 2.21, the undecidability follows from [KZ05, Section 4]. □

2.4.2 Algebraic closure

We now turn out attention to the study of the algebraic closure for a structure $(F, \mathrm{Frob}) \in \mathcal{S}$. Let F be a pseudofinite field and F^{alg} be the smallest algebraically closed field containing F. Take a tuple $a \in F$. Then the algebraic closure in the pseudofinite field $\mathrm{acl}_F(a)$ is simply the algebraic closure in F^{alg} intersected with F, i.e. $\mathrm{acl}_F(a) = \mathrm{acl}_{F^{alg}}(a) \cap F$.

As ACFA is the model companion of the theory of difference fields, we can embed (F, Frob) into some $(K, \sigma) \models$ ACFA. We might wonder if similarly, the algebraic closure in the theory of (F, Frob) is the same as the algebraic closure in (K, σ) intersected with F, i.e. the algebraic elements are defined by difference polynomials. The following results provide a negative answer to this.

Theorem 2.22. *For any $n > 0$, there is some $(F, Frob) \in \mathcal{S}$, an element $a_n \in F$and a tuple b_n such that a_n belongs to the definable closure of b_n in $(F, Frob)$, but $deg_\sigma(a_n/b_n) = n$.*

We need a lemma first.

Lemma 2.23. *Let $\varphi(x; y_1, \ldots, y_n) := \exists z(z^2 = x + y_1) \wedge \bigwedge_{2 \leq i \leq n} \forall z \neg(z^2 = x + y_i)$. There is $C_n \in \mathbb{R}^{>0}$ such that for any $\mathbb{F}_q$ with $char(\mathbb{F}_q) \neq 2$ and $b_1, \ldots, b_n$ distinct n-elements in $\mathbb{F}_q$, we have*

$$\left| |\varphi(\mathbb{F}_q, b_1, \ldots, b_n)| - \frac{q}{2^n} \right| \leq C_n \cdot q^{\frac{1}{2}}.$$

Proof. Given distinct elements $b_1, \ldots, b_n \in \mathbb{F}_q$. Take an element $a \in \mathbb{F}_q$ such that a is not a square. Let J be the ideal in $\mathbb{F}_q[X, X_1, \ldots, X_n]$ generated by

$$\{X_1^2 - (X + b_1)\} \cup \{X_i^2 - a(X + b_i) : 2 \leq i \leq n\}.$$

By Fact 2.15, J is absolutely prime, whence $V(J)$ is an absolutely irreducible variety of dimension 1. By the Lang-Weil estimate

$$||V(J) \cap (\mathbb{F}_q)^{n+1}| - q| \leq N_n \cdot q^{\frac{1}{2}},$$

where N_n is a constant only depends on the degree and dimension of the variety, which in our case is independent from $b_1, \ldots, b_n, a$ and $\mathbb{F}_q$ and only depends on n. Let

$$\pi : V(J) \cap (\mathbb{F}_q)^{n+1} \to \mathbb{F}_q$$

be the projection on the the first coordinate. Clearly, π is a 2^n-to-one function. Therefore,

$$|\varphi(\mathbb{F}_q, b_1, \ldots, b_n)| = |\pi(V(J) \cap (\mathbb{F}_q)^{n+1})| = \frac{1}{2^n} \cdot |V(J) \cap (\mathbb{F}_q)^{n+1}|.$$

Let $C_n := \frac{N_n}{2^n}$. We conclude that

$$\left| |\varphi(\mathbb{F}_q, b_1, \ldots, b_n)| - \frac{q}{2^n} \right| \leq C_n \cdot q^{\frac{1}{2}}.$$

□

Now we prove Theorem 2.22.

Proof. Given $n \in \mathbb{N}$, for each $p \in \mathbb{P}$, let $k_p \in \mathbb{N}$ be such that

- $k_p > \max\{f(p,p), 14p^n\}$ where $f(p,p)$ is given by Equation 2.1 in Definition 2.3;
- $n!$ divides k_p;
- $\frac{p^{k_p}}{2^{p^n}} > 2C_{p^n} \cdot p^{\frac{k_p}{2}}$.

Let $(F, \mathrm{Frob}) := \prod_{p \in \mathbb{P}} (\mathbb{F}_{p^{k_p}}, \mathrm{Frob}_p)/\mathcal{U}$ where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{P}$. Clearly, $(F, \mathrm{Frob}) \in \mathcal{S}$ and $\mathrm{Fix}(\sigma^n) := \{x \in F : \sigma^n(x) = x\} \neq \mathrm{Fix}(\sigma^k)$ for any $k < n$.

Take an element $a_n \in \mathrm{Fix}(\sigma^n)$ such that $deg_\sigma(a_n) = n$. Let

$$\xi(x, a_n) := \exists z(z^2 = a_n + x) \wedge \forall y(\sigma^n(y) = y \wedge (y \neq a_n \to \neg\exists z(z^2 = y + x))).$$

As $k_p > 14p^n$, for each prime $p \in \mathbb{N}$ we know by Theorem 2.17 and the subsequent remark that $Y_n := \xi((F, \mathrm{Frob}), a_n) \neq \emptyset$. We claim that $\boldsymbol{\delta}_F(Y_n) = 1$. Suppose $a_n = (a_p)_{p \in \mathbb{P}}/\mathcal{U}$. For each $p \in \mathbb{P}$, let $a_p, b_1, \ldots, b_{p^n-1}$ be a list of all elements in $\mathbb{F}_{p^n} \subseteq \mathbb{F}_{p^{k_p}}$. Let

$$\varphi(x, y_1, \ldots, y_{p^n}) := \exists z(z^2 = x + y_1) \wedge \bigwedge_{2 \leq i \leq p^n} \forall z \neg(z^2 = x + y_i).$$

Note that for any $b \in \mathbb{F}_{p^{k_p}}$ we have

$$\xi((\mathbb{F}_{p^{k_p}}, \mathrm{Frob}_p), a_p) = \varphi(\mathbb{F}_{p^{k_p}}, a_p, b_1, \ldots, b_{p^n-1}).$$

By Lemma 2.23,

$$||\varphi(\mathbb{F}_{p^{k_p}}, a_p, b_1, \ldots, b_{p^n-1})| - \frac{p^{k_p}}{2^{p^n}}| \leq C_{p^n} \cdot p^{\frac{p^{k_p}}{2}},$$

for all $p > 2$. Therefore,

$$|Y_n| \geq \frac{p^{k_p}}{2^{p^n}} - C_{p^n} \cdot p^{\frac{p^{k_p}}{2}} > \frac{1}{2} \cdot \frac{p^{k_p}}{2^{p^n}}.$$

Since

$$\lim_{p \to \infty} \frac{\log(p^{k_p}/2 \cdot 2^{p^n})}{\log p^{k_p}} = 1,$$

we get $\boldsymbol{\delta}_F(Y_n) = 1$.

Take an element $b_n \in Y_n$ such that $\boldsymbol{\delta}_F(b_n) > 0$. Note that $a_n \in \mathrm{dcl}(b_n)$ and $\boldsymbol{\delta}_F(a_n) = 0$. Thus, using additivity of $\boldsymbol{\delta}_F$,

$$\boldsymbol{\delta}_F(b_n/a_n) = \boldsymbol{\delta}_F(a_n, b_n) - \boldsymbol{\delta}_F(a_n) = \boldsymbol{\delta}_F(b_n) + \boldsymbol{\delta}_F(a_n/b_n) - \boldsymbol{\delta}_F(a_n) = \boldsymbol{\delta}_F(b_n) > 0.$$

Therefore, $\mathrm{SU}_{\mathrm{ACFA}}(b_n/a_n) = \omega$. By our choice, we also have $\mathrm{SU}_{\mathrm{ACFA}}(b_n) = \omega$. Hence, a_n is independent from b_n in $(\tilde{F}, \mathrm{Frob})$. Again, by our choice, $deg_\sigma(a_n) = n$. But if $deg_\sigma(a_n/b_n) < n$, then a_n and b_n will not be independent in $(\tilde{F}, \mathrm{Frob})$ in the theory of ACFA. We conclude that $deg_\sigma(a_n/b_n) = n$ and a_n is in the definable closure of b_n. $\square$

2.4.3 Further remarks:

We conclude this chapter with some remarks.

1. As we have mentioned in the remark after Theorem 2.5, we can easily generalise the results of this chapter to other classes, provided the fields grow fast enough. Let $(F, \sigma) := \prod_{i \in I} (\mathbb{F}_{{p_i}^{k_i}}, \mathrm{Frob}_{{p_i}^{m_i}})/\mathcal{U}$, with $p_i^{k_i} >> p_i^{m_i}$ for all $i \in I$, then all the results in Section 2.2 and Section 2.3 are true for (F, σ) as well. Corollary 2.21 will also be true if the fixed field of (F, σ) is infinite. However, if $(F, \sigma) := \prod_{i \in I} (\mathbb{F}_{p^{k_i}}, \mathrm{Frob}_{p^{m_i}})$ with k_i and p_i coprime for all $i \in I$, then it is not clear whether its theory will always be wild.

2. One of the main open problems of this chapter is that whether the coarse dimension $\boldsymbol{\delta}_F$ for structures in $\mathcal{S}$ coincide with the transformal transcendence degree. The first step towards proving this is to prove this holds for quantifier-free formulas. We might need tools from other areas of mathematics to prove this.

Chapter 3

Pseudofinite Primitive Permutation Groups

3.1 Introduction

Finite primitive permutation groups have been classified into several types by the well-known O'Nan-Scott Theorem. This classification reduces most problems concerning finite primitive permutation groups to problems of finite simple groups. Together with the *classification of finite simple groups (CFSG)*, it gives a good understanding of finite primitive permutation groups. As pseudofinite groups can be seen as limits of finite groups, we might wonder if it is also possible to give a nice description of pseudofinite permutation groups. There have been some attempts. In [LMT10], pseudofinite definably primitive permutation groups have been extensively studied via the O'Nan-Scott Theorem. In [EJMR11], under the additional assumption that (G, X) lives in a supersimple theory of finite SU-rank and that the SU-rank of X is one, Elwes, Jaligot, Macpherson and Ryten managed to get a complete classification, which is analogous to the well-known classification of stable permutation groups acting on strongly minimal sets in [Hru89].

We recall the classification in [EJMR11].

Fact 3.1. ([EJMR11, Theorem 1.3])

Let (G, X) be a pseudofinite definably primitive permutation group. Let T be the theory of (G, X) in the language $\mathcal{L}$. Suppose T is supersimple of finite SU-rank such that T^{eq} eliminates $\exists^\infty$ and $\mathrm{SU}(X) = 1$. Then the socle of G (the subgroup generated by all minimal non-trivial normal subgroups), $soc(G)$, exists and is definable, and one of the following holds:

1. $\mathrm{SU}(G) = 1$, and $soc(G)$ is abelian of finite index in G and acts regularly on X;
2. $\mathrm{SU}(G) = 2$, and there is an interpretable pseudofinite field F of SU-rank 1 such that (G, X) is definably isomorphic to $(F^+ \rtimes H, F^+)$, where $H \leq F^\times$ is of finite index.

3. $\mathrm{SU}(G) = 3$, and there is an interpretable pseudofinite field F of SU-rank 1 such that (G, X) is definably isomorphic to $(H, \mathrm{PG}_1(F))$, where $\mathrm{PSL}_2(F) \leq H \leq \mathrm{P\Gamma L}_2(F)$.[1] Moreover, $soc(G)$ is definably isomorphic to $\mathrm{PSL}_2(F)$.

This result is based on the investigation of pseudofinite groups of small SU-rank in the same paper [EJMR11]. Basically, they showed that pseudofinite groups of SU-rank 1 are finite-by-abelian-by-finite, and those of SU-rank 2 are soluble-by-finite. We list them here.

Fact 3.2. ([EJMR11, Lemma 3.1(i)]) Let G be an infinite group definable in a supersimple theory T such that T^{eq} eliminates $\exists^\infty$. Let $H \leq G$ be an infinite finite-by-abelian subgroup. Then H is contained in an infinite definable finite-by-abelian subgroup $K \leq G$.

Fact 3.3. ([EJMR11, Theorem 1.2]) Let G be a pseudofinite group definable in a supersimple theory T such that T^{eq} eliminates $\exists^\infty$. Suppose $\mathrm{SU}(G) = 2$. Then G is soluble-by-finite.

The analysis of pseudofinite groups of small SU-rank has been generalised in [Wag18] to a wider context which includes the pseudofinite supersimple and superrosy groups of infinite rank. Basically, Wagner replaces finite SU-rank by an abstract dimension which satisfies some nice properties, together with some chain condition on centralizers.

In this chapter, we generalize Fact 3.1 to the same context as in [Wag18], which in particular includes the pseudofinite definably primitive permutation groups in supersimple or superrosy theories of infinite rank. Interestingly, as we do not assume supersimplicity of the ambient theory, the Indecomposability Theorem is not available. However, in one main step of the proof, we go to a subgroup of the permutation group, whose theory in the pure group language is supersimple. Via this, we use the powerful structural theorems in supersimple theories to get the desired result.

Let us introduce the general context that we will work with and state our main theorem.

Definition 3.4. A *dimension* on a theory T is a function dim from all interpretable subsets of a monster model to $\mathbb{R}^{\geq 0} \cup \{\infty\}$, satisfying:

1. Invariance: If $a \equiv a'$, then $\mathsf{dim}(\varphi(x, a)) = \mathsf{dim}(\varphi(x, a'))$;
2. Algebraicity: If X is finite, then $\mathsf{dim}(X) = 0$;
3. Union: $\mathsf{dim}(X \cup Y) = \max\{\mathsf{dim}(X), \mathsf{dim}(Y)\}$;
4. Fibration: If $f : X \to Y$ is an interpretable surjection and $\mathsf{dim}(f^{-1}(y)) = r$ for all $y \in Y$, then $\mathsf{dim}(X) = \mathsf{dim}(Y) + r$;

We define the dimension of a tuple of elements a over a set B as

$$\mathsf{dim}(a/B) := \inf\{\mathsf{dim}(\varphi(x)) : \varphi \in \mathrm{tp}(a/B)\}.$$

[1] In fact, we think H should be contained in $\mathrm{PGL}_2(F)$, there shouldn't be any non-trivial automorphism of F induced by G, see Lemma 3.36 and Corollary 3.57.

When the equation $\mathsf{dim}(a, b/C) = \mathsf{dim}(\mathsf{a/b, C}) + \mathsf{dim}(\mathsf{b/C})$ holds for any tuples a, b and any set C, we say that the dimension dim is *additive*.

When dim has its range in $\mathbb{N}$ then we say that the dimension dim is *integer-valued*.

Example 3.1. *In ultraproducts of finite structures the coarse pseudofinite dimension satisfies all the conditions for the dimension we defined above and is additive (in a certain expansion of the language, see the remark after Definition 0.11). But it is not necessarily integer-valued.*

Another family of examples of dimensions is the following. Take a superstable (or supersimple, or superrosy) theory, suppose $rk(T) = \omega^\alpha \cdot n + \beta$ for some ordinals α, β with $\beta < \omega^\alpha$ and some integer n, where rk is lascar, SU or thorn-rank. Then for any interpretable set X, define $\mathsf{dim}(X) := k$ if $rk(X) = \omega^\alpha \cdot k + \gamma$ for some $k \in \mathbb{N}$ and $\gamma < \omega^\alpha$. With this definition, dim is an additive integer-valued dimension.

Remark: Note that in the definition of a dimension, it is not required that dimensional 0 sets are finite. In fact, in the examples above where the dimension comes from the coefficient of ω^α of lascar/SU/thorn-rank with $\alpha \neq 0$, we will always have infinite definable sets of dimension 0. This is one of the major difficulties in generalizing Fact 3.1, 3.2 and 3.3.

Definition 3.5. Let G be a group. We say that G satisfies the $\widetilde{\mathfrak{M}}_c$*-condition* or G is an $\widetilde{\mathfrak{M}}_c$*-group* if the following holds:

$$\exists d \in \mathbb{N}, \forall g_0, \cdots, g_d \in G, \bigvee_{i<d} \left([C_G(g_0, \cdots, g_i) : C_G(g_0, \cdots, g_{i+1})] \leq d \right).$$

Fact 3.6. ([Wag00, Theorem 4.2.12, Proposition 4.4.3]) All interpretable groups in simple theories satisfy the $\widetilde{\mathfrak{M}}_c$-condition.

Here is the generalization of Fact 3.2 and 3.3 in [Wag18].

Fact 3.7. ([Wag18, Theorem 4.11, Corollary 4.14]) Let G be a pseudofinite $\widetilde{\mathfrak{M}}_c$-group with an additive dimension dim such that $\mathsf{dim}(G) > 0$.

1. Then G has a definable finite-by-abelian subgroup C with $\mathsf{dim}(C) > 0$.
2. If dim is integer-valued and $\mathsf{dim}(G) = 1$, then G has a definable characteristic finite-by-abelian subgroup C such that $\mathsf{dim}(C) = 1$.

Fact 3.8. ([Wag18, Theorem 5.1, Corollary 5.2]) Let G be a pseudofinite $\widetilde{\mathfrak{M}}_c$-group with an additive integer-valued dimension dim such that $\mathsf{dim}(G) = 2$.

1. Then G has a definable finite-by-abelian subgroup C such that $\mathsf{dim}(C) \geq 1$ and $\mathsf{dim}(N_G(C)) = 2$.
2. If definable sections of G also satisfy the $\widetilde{\mathfrak{M}}_c$-condition, then G has a definable soluble subgroup D with $\mathsf{dim}(D) = 2$.

Remark: The proof of Fact 3.8, more precisely, of Theorem 5.1 in [Wag18] uses the CFSG. But the assumption of Theorem 5.1 in [Wag18] is slightly weaker than the one we stated. We refer to an earlier version of this proof, [Wag15, Theorem 13, Corollary 14], which does not use the CFSG.

We specify the language for permutation groups: $\mathcal{L}$ contains two sorts G and X, with the group language $\{\cdot, (-)^{-1}, id\}$ on G and a function $(-)^{(-)} : X \times G \to X$ which represents the action of G on X. For $x \in X$ and $g \in G$, we denote x^g the value of the action of g on x. We will also denote the conjugation $g^{-1}hg$ inside a group G as h^g.

We recall the definition of a (definably) primitive permutation group.

Definition 3.9. A permutation group G acting on a non-empty set X is called *primitive* if G acts transitively on X and preserves no non-trivial partition of X. If G is transitive and preserves no non-trivial definable partition of X, then G is called *definably primitive*.

Remark: A transitive permutation group G is primitive if and only if any point stabilizer $\mathsf{Stab}_G(x) := \{g \in G : x^g = x\}$ is a maximal proper subgroup of G. Similarly, G is definably primitive if and only if any $\mathsf{Stab}_G(x)$ is a definably maximal proper subgroup of G, that is there is no definable subgroup $D \leq G$ such that $\mathsf{Stab}_G(x) \lneq D \lneq G$.

Definition 3.10. We define $\mathcal{S}$ to be the class of all pseudofinite definably primitive permutation groups (G, X) with an additive integer-valued dimension dim such that $\mathsf{dim}(X) = 1$, and such that G satisfies the $\widetilde{\mathfrak{M}}_c$-condition.

By Example 3.1 and Fact 3.6, $\mathcal{S}$ contains all pseudofinite definably primitive permutation groups (G, X) in supersimple finite SU-rank theories such that $\mathrm{SU}(X) = 1$. The aim of this chapter is to get a classification of $\mathcal{S}$ similar to Fact 3.1. It turned out that the restrictions on $\mathcal{S}$ are enough for us to classify members of $\mathcal{S}$ of dimension 1 and 2. However, we need more combinatorial assumptions for dimension greater or equal to 3, one of which is similar to the $\widetilde{\mathfrak{M}}_c$-condition but for stabilizers, and the other one is a minimality condition on X. We list them here.

Notation: Let G be a group acting on some structure X, for $x \in X$ we write $\mathsf{Stab}_G(x)$ for the point-stabilizer $\{g \in G : x^g = x\}$, and for $B \subseteq X$ we write

$$\mathsf{PStab}_G(B) := \bigcap_{x \in B} \mathsf{Stab}_G(x)$$

as the point-wise stabilizer.

1. $\widetilde{\mathfrak{M}}_s$-condition on (G, X):

   $$\exists d \in \mathbb{N}, \forall g_0, \ldots, g_d \in G, \bigvee_{i<d} \left([\mathsf{PStab}_G(g_0, \ldots, g_i) : \mathsf{PStab}_G(g_0, \ldots, g_{i+1})] \leq d \right).$$

2. (EX)-condition on X:

   X contains no infinite set of 1-dimensional equivalence classes for any definable equivalence relation on X.

Fact 3.11. ([Wag00, Theorem 4.2.12, Proposition 4.4.3]) All interpretable groups in simple theories satisfy the $\widetilde{\mathfrak{M}}_s$-condition.

Now we are able to state our main result.

Theorem 3.12. *Let $(G, X) \in \mathcal{S}$.*

1. *If $\mathsf{dim}(G) = 1$, then G has a definable normal abelian subgroup A, such that $\mathsf{dim}(A) = 1$ and A acts regularly on X.*
2. *If $\mathsf{dim}(G) = 2$ and definable sections of G satisfy the $\widetilde{\mathfrak{M}}_c$-condition. Then there is a definable subgroup $H \trianglelefteq G$ of dimension 2, and an interpretable pseudofinite field F of dimension 1, such that (H, X) is definably isomorphic to $(F^+ \rtimes D, F^+)$ for some $D \leq F^\times$ of dimension 1.*
3. *If $\mathsf{dim}(G) \geq 3$. Suppose definable sections of G satisfy the $\widetilde{\mathfrak{M}}_c$-condition, G satisfies the $\widetilde{\mathfrak{M}}_s$-condition and X satisfies the (EX)-condition. Then $\mathsf{dim}(G) = 3$ and there is a definable subgroup $D \leq G$ of dimension 3 and an interpretable pseudofinite field F of dimension 1 such that D is definably isomorphic to $\mathrm{PSL}_2(F)$ and (G, X) is definably isomorphic to $(H, \mathrm{PG}_1(F))$, where $\mathrm{PSL}_2(F) \leq H \leq \mathrm{P\Gamma L}_2(F)$.*

This theorem enables us to analyse the pseudofinite definably primitive permutation groups of infinite SU-rank, which is an immediate generalization of Fact 3.1.

Corollary 3.13. *Let (G, X) be a pseudofinite definably primitive permutation group in a supersimple theory. Suppose $SU(G) = \omega^\alpha n + \gamma$ and $SU(X) = \omega^\alpha + \beta$ for some $\gamma, \beta < \omega^\alpha$ and $n \in \mathbb{N}$. Then one of the following holds:*

1. *$SU(G) = \omega^\alpha + \gamma$, and there is a definable abelian subgroup A of SU-rank ω^α acting regularly on X.*
2. *$SU(G) = 2$, and there is an interpretable pseudofinite field F of SU-rank 1 with (G, X) definably isomorphic to $(F^+ \rtimes H, F^+)$, where H is a subgroup of $F^\times$ of finite index.*
3. *$SU(G) = 3$, and there is an interpretable pseudofinite field F of SU-rank 1 such that (G, X) is definably isomorphic to $(\mathrm{PSL}_2(F), \mathrm{PG}_1(F))$ or $(\mathrm{PGL}_2(F), \mathrm{PG}_1(F))$.*

Remark: Fact 3.1 uses the CFSG for SU-rank greater or equal to 3, so do our results for dimension greater or equal to 3, in particular Section 3.4 and Section 3.5 uses the CFSG without mentioning it explicitly.

The rest of this chapter is organised as the following. Section 3.2 gives some general analysis of the basic properties of $\widetilde{\mathfrak{M}}_c$-groups with an additive integer-valued dimension. Section 3.3 deals with pseudofinite definably primitive permutation groups of dimensions 1 and 2. The main results are Theorem 3.27 and Theorem 3.35. Section 3.4 handles the rest, i.e. permutation groups of dimension greater or equal to 3. The corresponding result is obtained in Theorem 3.53. The last part, Section 3.5 studies the special case of pseudofinite definably primitive permutation groups in supersimple theories of infinite rank. Theorem 3.58 concludes this section.

3.2 $\widetilde{\mathfrak{M}}_c$-groups with a dimension

In this section we will first establish some general results about $\widetilde{\mathfrak{M}}_c$-groups with an additive integer-valued dimension.

In the following lemmas, we assume that dim is an additive integer-valued dimension on a group G.

Definition 3.14. We say a subgroup $H \leq G$ is *broad* if $\mathsf{dim}(H) > 0$. And we say H is *wide in* G if $\mathsf{dim}(H) = \mathsf{dim}(G)$.

Lemma 3.15. *Let $H_0, \ldots, H_n$ be a finite family of wide definable subgroups of G. Then $\bigcap_{i \leq n} H_i$ is also wide in G.*

Proof. It suffices to prove the claim when $n = 1$, the rest follows by induction. By the properties of dimension, we have that $\mathsf{dim}(G/H_0) = \mathsf{dim}(G) - \mathsf{dim}(H_0) = 0$. Similarly, $\mathsf{dim}(G/H_1) = 0$.

Note that there is a definable injection from $G/(H_0 \cap H_1)$ to $G/H_0 \times G/H_1$ sending $g(H_0 \cap H_1)$ to (gH_0, gH_1). Hence $\mathsf{dim}(G/(H_0 \cap H_1)) \leq \mathsf{dim}(G/H_0) + \mathsf{dim}(G/H_1) = 0$. We obtain

$$\mathsf{dim}(H_0 \cap H_1) = \mathsf{dim}(G) - \mathsf{dim}(G/(H_0 \cap H_1)) = \mathsf{dim}(G). \quad \square$$

Lemma 3.16. *Suppose G is finite-by-abelian. Then for any $g_0, \ldots, g_n \in G$, the centralizer $C_G(g_0, \ldots, g_n)$ is wide in G.*

Proof. Since G is finite-by-abelian, the derived subgroup G' is finite. For any $g \in G$, the set $g^{-1}g^G = \{g^{-1}h^{-1}gh : h \in G\}$ is a subset of G', hence is finite. Therefore, g^G is finite and is of dimension 0. Note that there is a definable bijection between g^G and $G/C_G(g)$. Thus, $\mathsf{dim}(C_G(g)) = \mathsf{dim}(G) - \mathsf{dim}(g^G) = \mathsf{dim}(G)$.

As $C_G(g_i)$ is definable and wide in G for each $i \leq n$, so is $C_G(g_0, \ldots, g_n)$ by Lemma 3.15. $\square$

Lemma 3.17. *Let $B_1 \trianglelefteq A_1$ and $B_2 \trianglelefteq A_2$ be subgroups of G. If both A_1/B_1 and A_2/B_2 are finite-by-abelian, then so is $(A_1 \cap A_2)/(B_1 \cap B_2)$.*

Proof. For the derived subgroups, we have

$$((A_1 \cap A_2)/(B_1 \cap B_2))' = ((A_1 \cap A_2)'(B_1 \cap B_2))/(B_1 \cap B_2) \subseteq ((A_1' \cap A_2')(B_1 \cap B_2))/(B_1 \cap B_2).$$

Since both $A_1'B_1/B_1 = (A_1/B_1)'$ and $A_2'B_2/B_2 = (A_2/B_2)'$ are finite, so is the product $(A_1'B_1/B_1) \times (A_2'B_2/B_2)$. Define a function

$$f : ((A_1' \cap A_2')(B_1 \cap B_2))/(B_1 \cap B_2) \longrightarrow (A_1'B_1/B_1) \times (A_2'B_2/B_2)$$

by sending $a(B_1 \cap B_2)$ to (aB_1, aB_2). It is easy to check that f is injective. Therefore, $((A_1' \cap A_2')(B_1 \cap B_2))/(B_1 \cap B_2)$ is finite. We conclude that $((A_1 \cap A_2)/(B_1 \cap B_2))'$ is finite and $(A_1 \cap A_2)/(B_1 \cap B_2)$ is finite-by-abelian. $\square$

From now on, we assume further that G is $\widetilde{\mathfrak{M}}_c$.

Definition 3.18. Let H_1 and H_2 be two subgroups of G. We say H_1 is *almost contained* in H_2, denoted as $H_1 \lesssim H_2$, if $[H_1 : H_2 \cap H_1] < \infty$. If both $H_1 \lesssim H_2$ and $H_2 \lesssim H_1$ hold, then H_1 and H_2 are called *commensurable*.

For two subgroups $H, K \leq G$, the *almost centralizer* of K in H is defined as

$$\widetilde{C}_H(K) := \{h \in H : [K : C_K(h)] < \infty\}.$$

The *almost center* is defined as $\widetilde{Z}(H) := \widetilde{C}_H(H)$.

Let $\mathcal{D}$ be an infinite family of subgroups of G. We say $\mathcal{D}$ is *uniformly commensurable* if there is some $N \in \mathbb{N}$ such that $[D : D \cap D'] \leq N$ for all $D, D' \in \mathcal{D}$.

Fact 3.19. ([Hem15, Proposition 3.3]) When G is $\widetilde{\mathfrak{M}}_c$ and H, K are definable subgroups of G, then $\widetilde{C}_H(K)$ is also definable.

We list a useful fact for almost centralizers here.

Fact 3.20. [Hem15, Theorem 2.10] Let H and K be two definable subgroups of G. Then $H \lesssim \widetilde{C}_G(K)$ if and only if $K \lesssim \widetilde{C}_G(H)$.

Lemma 3.21. *Let $D := C_G(\bar{g})$ be the centralizer of some finite tuple $\bar{g} \in G^n$. Suppose D is wide in G. Then there is a wide definable normal subgroup N of G such that N is commensurable with $E := \bigcap_{i \leq k} D^{t_i}$ for some $k \in \mathbb{N}$ and $t_0, \dots, t_k \in G$.*

Proof. By the $\widetilde{\mathfrak{M}}_c$-condition, there are $t_0, \dots, t_k \in G$ and $d \in \mathbb{N}$ such that for any $t \in G$ we have $[\bigcap_{i \leq k} D^{t_i} : \bigcap_{i \leq k} D^{t_i} \cap D^t] \leq d$. Let $E := \bigcap_{i \leq k} D^{t_i}$. Since E is a finite intersection of wide subgroups, E is also wide by Lemma 3.15. For any $h_1, h_2 \in G$,

$$[E^{h_1} : E^{h_1} \cap E^{h_2}] = [E : E \cap E^{h_2 h_1^{-1}}] \leq \prod_{i \leq k} [E : E \cap D^{t_i h_2 h_1^{-1}}] \leq d^{k+1}.$$

Therefore $\mathcal{E} := \{E^t : t \in G\}$ is a family of uniformly commensurable definable subgroups of G. By Schlichting's Theorem (Fact 0.36), there is a definable subgroup N of G, which is invariant under all automorphisms of G stabilizing $\mathcal{E}$ setwise, and is commensurable with all members of $\mathcal{E}$. In particular, N is normal in G and is commensurable with E, hence is also wide. $\square$

Lemma 3.22. *Let M, N be subgroups of G. Then*

$$\widetilde{Z}(M) \cap \widetilde{Z}(N) \leq \widetilde{Z}(M) \cap N \leq \widetilde{Z}(M \cap N).$$

Proof. Clearly, we have $\widetilde{Z}(M) \cap \widetilde{Z}(N) \leq \widetilde{Z}(M) \cap N$ for any $M, N \leq G$.

If $g \in \widetilde{Z}(M) \cap N$, then $g \in M \cap N$ and $[M : C_M(g)] < \infty$. Hence,

$$[M \cap N : C_{M \cap N}(g)] = [M \cap N : C_M(g) \cap N] \leq [M : C_M(g)] < \infty,$$

and we get $g \in \widetilde{Z}(M \cap N)$. Therefore, $\widetilde{Z}(M) \cap N \leq \widetilde{Z}(M \cap N)$. $\square$

Lemma 3.23. *Let M, N be subgroups of G. If M is commensurable with N, then $\widetilde{Z}(M)$ is commensurable with $\widetilde{Z}(N)$.*

Proof. If $g \in \widetilde{Z}(M \cap N)$, then

$$[M : C_M(g)] \leq [M : C_{M\cap N}(g)] \leq [M : M \cap N][M \cap N : C_{M\cap N}(g)] < \infty,$$

hence, $g \in \widetilde{Z}(M)$. Similarly, $\widetilde{Z}(M\cap N) \leq \widetilde{Z}(N)$. Therefore, $\widetilde{Z}(M\cap N) \leq \widetilde{Z}(M)\cap\widetilde{Z}(N)$. Together with Lemma 3.22, we have

$$\widetilde{Z}(M \cap N) = \widetilde{Z}(M) \cap \widetilde{Z}(N) = \widetilde{Z}(M) \cap N = \widetilde{Z}(N) \cap M.$$

Since M, N are commensurable,

$$[\widetilde{Z}(M) : \widetilde{Z}(M) \cap \widetilde{Z}(N)] = [\widetilde{Z}(M) : \widetilde{Z}(M) \cap N] \leq [M : M \cap N] < \infty.$$

Similarly, $\widetilde{Z}(N)$ and $\widetilde{Z}(M) \cap \widetilde{Z}(N)$ are commensurable. □

Lemma 3.24. *Let H, D be definable subgroups of G. Define*

$$H_0^D := \{h \in H,\ \mathsf{dim}(h^D) = 0\}.$$

Then there are $d \in \mathbb{N}$ and a definable group $T \leq D$ such that

$$H_0^D = \{h \in H,\ [T : C_T(h)] \leq d\}.$$

In particular, H_0^D is a definable subgroup of H.

Proof. It is easy to see that $1 \in H_0^D$ and that it is closed under inverse. Note that $(h_1h_2)^D \subseteq h_1^Dh_2^D$. Therefore, if $h_1, h_2 \in H_0^D$, then

$$\mathsf{dim}((h_1h_2)^D) \leq \mathsf{dim}(h_1^D) + \mathsf{dim}(h_2^D) = 0.$$

Hence, $h_1h_2 \in H_0^D$.

By the $\widetilde{\mathfrak{M}}_c$-condition, there are $h_0, \cdots, h_n \in H_0^D$ and $d \in \mathbb{N}$ such that $[T : C_T(h)] \leq d$ for all $h \in H_0^D$, where $T := C_D(h_0, \cdots, h_n)$. Since for each h_i, $\mathsf{dim}(C_D(h_i)) = \mathsf{dim}(D)$, we have $\mathsf{dim}(T) = \mathsf{dim}(C_D(h_0, \cdots, h_n)) = \mathsf{dim}(D)$. Let

$$M := \{h \in H,\ [T : C_T(h)] \leq d\}.$$

Then M is definable. We claim that $M = H_0^D$. By definition, $H_0^D \subseteq M$. On the other hand, if $h \in M$, then $\mathsf{dim}(C_D(h)) \geq \mathsf{dim}(C_T(h)) = \mathsf{dim}(T) = \mathsf{dim}(D)$. Hence, $\mathsf{dim}(h^D) = 0$ and $h \in H_0^D$. □

3.3 Permutation groups of dimension 1 and 2

In this section, we analyse the permutation groups in $\mathcal{S}$ of dimension 1 or 2.

Here is a useful lemma for (definably) primitive permutation groups that we will use a lot without referring to it explicitly.

Lemma 3.25. *Let (G, X) be a (definably) primitive permutation group and A a (definable) normal subgroup of G. Then A is either trivial or acts transitively on X.*

Proof. Fix $x \in X$. If $x^A \neq X$, then by normality of A, the set of orbits of A forms a (definable) G-invariant partition of X. By (definable) primitivity, $x^A = \{x\}$. As the action is transitive, for any $y \in X$, there is some $g \in G$ such that $y = x^g$. Thus, $y^A = x^{gA} = x^{Ag} = \{x\}^g = \{y\}$. Therefore, $A = \{\mathrm{id}\}$. $\square$

Lemma 3.26. *Let (G, X) be a definably primitive permutation group. If G has a definable non-trivial normal abelian subgroup A, then A acts regularly on X and A is either divisible torsion free or elementary abelian.*

Moreover, $G = A \rtimes G_x$ where $G_x = \mathsf{Stab}_G(x)$ for some $x \in X$, and G_x acts on $X = x^A \simeq A$ by conjugation.

In particular if $(G, X) \in \mathcal{S}$, then we have in addition $\mathsf{dim}(A) = 1$.

Proof. As G acts definably primitively on X and $A \trianglelefteq G$ is non-trivial, A acts transitively on X. If $x^a = x^b$ for some $x \in X$ and $a, b \in A$, then for any $y \in X$, by transitivity, $y = x^c$ for some $c \in A$. As A is abelian, we get

$$y^a = x^{ca} = x^{ac} = x^{bc} = x^{cb} = y^b.$$

Hence, $a = b$. Therefore, A acts regularly on X. Fix some $x \in X$. Then $a \mapsto x^a$ is a definable bijection from A to X. Thus, if $(G, X) \in \mathcal{S}$, then $\mathsf{dim}(A) = \mathsf{dim}(X) = 1$.

For any $n \in \omega$ let $nA := \{a^n : a \in A\}$. Then nA is a definable characteristic subgroup of A, hence definable abelian normal in G. If $\mathsf{dim}(nA) = 1$, then nA also acts regularly on X, whence $nA = A$. Otherwise, $\mathsf{dim}(nA) = 0$, and nA is trivial by definable primitivity of G. Therefore, A is either divisible torsion free or elementary abelian.

Let $G_x := \mathsf{Stab}_G(x)$. As A acts regularly on X, we have $A \cap G_x = \{1\}$. For any $g \in G$ there is a unique element $a \in A$ such that $x^a = x^g$. Hence, $x = x^{ga^{-1}}$, so $ga^{-1} \in G_x$ and $g \in AG_x$. As $A \cap G_x = \{1\}$, we obtain $G = A \rtimes G_x$.

Note that for any $g \in G_x$ and any $a \in A$, we have $(x^a)^g = x^{g^{-1}ag}$. Therefore, if we identify A with X via $a \mapsto x^a$, then G_x acts on A by conjugation. $\square$

Combining the two lemmas above, we get the first part of our main result.

Theorem 3.27. *Let $(G, X) \in \mathcal{S}$. If $\mathsf{dim}(G) = 1$, then G has a definable wide abelian normal subgroup A such that A acts regularly on X. Moreover, A is either divisible torsion-free or elementary abelian.*

Proof. By Fact 3.7(2), G has a definable wide normal finite-by-abelian subgroup A. Consider the derived subgroup A'. It is finite and characteristic in A, hence is a definable normal subgroup of G. Since G acts definably primitively on X, either A' is trivial or A' acts transitively on X. If A' acts transitively on X, then $\mathsf{dim}(A') \geq \mathsf{dim}(X) = 1$, contradicting that A' is finite. Hence A' is trivial and A is a definable wide abelian normal subgroup of G. By Lemma 3.26, A acts regularly on X and is either divisible torsion free or elementary abelian. $\square$

We now proceed to analyse the groups in $\mathcal{S}$ of dimension greater than 1. The following lemma gives a key property of them.

Lemma 3.28. *Let $(G, X) \in \mathcal{S}$ with $\mathsf{dim}(G) \geq 2$. If $K \trianglelefteq G$ and $\mathsf{dim}(K) \geq 2$, then there is no element $a \in K \setminus \{1\}$, such that $C_K(a)$ is wide in K.*

Proof. Suppose, towards a contradiction, that there is $a \in K \setminus \{1\}$ and $\mathsf{dim}(C_K(a)) = \mathsf{dim}(K) \geq 2$. By the $\widetilde{\mathfrak{M}}_c$-condition, there are $g_0, \cdots, g_n \in G$ such that

$$\{(\bigcap_{i \leq n} C_K(a^{g_i}))^g : g \in G\}$$

is a uniformly commensurable family. Since $K \trianglelefteq G$, we have $a^{g_i} \in K$ and $(\bigcap_{i \leq n} C_K(a^{g_i}))^g$ is a subgroup of K for any $g \in G$. Note that $C_K(a^{g_i}) = (C_K(a))^{g_i}$ is wide in K for each g_i. Thus, $\mathsf{dim}(\bigcap_{i \leq n} C_K(a^{g_i})) = \mathsf{dim}(K) \geq 2$.

By Schlichting's Theorem there is a definable subgroup N of K such that $N \trianglelefteq G$ and is commensurable with $\bigcap_{i \leq n} C_K(a^{g_i})$, whence wide in K. Consider the group $\widetilde{Z}(N)$. We claim that $\mathsf{dim}(\widetilde{Z}(N)) \geq 1$. Since N is commensurable with $\bigcap_{i \leq n} C_K(a^{g_i})$, we have $a^{g_i} \in \widetilde{C}_K(N)$ and $a^{g_i} \neq 1$. As $\widetilde{C}_K(N)$ is definable normal in G, by definable primitivity of G, it is of dimension at least 1 (otherwise, it would be trivial). Note that $\widetilde{Z}(N) = N \cap \widetilde{C}_K(N)$. Then

$$\begin{aligned} \mathsf{dim}(\widetilde{Z}(N)) &= \mathsf{dim}(K) - \mathsf{dim}(K/\widetilde{Z}(N)) \geq \mathsf{dim}(K) - (\mathsf{dim}(K/N) + \mathsf{dim}(K/\widetilde{C}_K(N))) \\ &\geq \mathsf{dim}(K) - 0 - \mathsf{dim}(K) + \mathsf{dim}(\widetilde{C}_K(N)) = \mathsf{dim}(\widetilde{C}_K(N)) \geq 1. \end{aligned}$$

Therefore $\widetilde{Z}(N)$ acts transitively on X.

By [Hem15, Proposition 3.28], the commutator group $E := [\widetilde{Z}(N), \widetilde{C}_N(\widetilde{Z}(N))]$ is finite. Since N is normal in G and E is characteristic in N and definable of dimension zero, E is trivial. Therefore, $\widetilde{C}_N(\widetilde{Z}(N)) \subseteq C_N(\widetilde{Z}(N))$.

We claim that $\widetilde{C}_N(\widetilde{Z}(N))$ is wide in K. Indeed, by Fact 3.20, we have $N \lesssim \widetilde{C}_N(\widetilde{Z}(N))$ if and only if $\widetilde{Z}(N) \lesssim \widetilde{C}_N(N) = \widetilde{Z}(N)$. Thus, N is commensurable with $\widetilde{C}_N(\widetilde{Z}(N))$.

Let $H := C_N(\widetilde{Z}(N))$. Then H is a definable wide subgroup of K and is normal in G. Fix $x \in X$. For all $h \in \widetilde{Z}(N)$,

$$\mathsf{Stab}_H(x^h) = (\mathsf{Stab}_H(x))^h = \mathsf{Stab}_H(x).$$

Since $\widetilde{Z}(N)$ acts transitively on X, we get $\mathsf{Stab}_H(x) = \{1\}$. However, as $|x^H| = [H : \mathsf{Stab}_H(x)]$ (the Orbit-Stabilizer Theorem) we have

$$\mathsf{dim}(\mathsf{Stab}_H(x)) = \mathsf{dim}(H) - \mathsf{dim}(\mathsf{Orb_H(x)}) = \mathsf{dim}(K) - \mathsf{dim}(X) \geq 2 - 1 = 1,$$

contradicting that $\mathsf{Stab}_H(x) = \{1\}$. $\square$

In the following, we will show that if we have a finite-by-abelian group acting on a one-dimensional abelian group, then under certain conditions, we can define a pseudofinite field.

Theorem 3.29. *Let A be an abelian group of dimension 1 and D a broad definable group of automorphisms of A. Suppose that $A_0 \leq A$ is definable of dimension 0 and D acts on A/A_0. Let $D_0 := \{d \in D : \forall a \in A,\ a^d \in a + A_0\}$, a definable normal subgroup* *of D. Write $a + A_0 \in A/A_0$ as $[a]$ and $dD_0 \in D/D_0$ as $[d]$. Suppose D satisfies the following condition:*

($\clubsuit$) *If $[a] \neq [0]$ then $\mathsf{dim}([a]^{C_{D/D_0}([d_1],\ldots,[d_n])}) = 1$ for any $n \in \mathbb{N}, d_1, \ldots, d_n \in D$.*

Then there is an interpretable pseudofinite field F such that F^+ is isomorphic to A/A_0 and D/D_0 embeds into $F^\times$ with $\mathsf{dim}(D/D_0) = 1$.

Remark: If D is finite-by-abelian and $A_0 := \{a \in A : \mathsf{dim}(a^D) = 0\}$ is of dimension 0, then condition ($\clubsuit$) is satisfied. Indeed, $C_D(d_1, \ldots, d_n)$ has finite index in D when D is finite-by-abelian. As $a \notin A_0$ by assumption, $\mathsf{dim}(a^D) = 1$. Hence, $\mathsf{dim}(a^{C_D(d_1,\ldots,d_n)}) = \mathsf{dim}(a^D) = 1$ and

$$\mathsf{dim}([a]^{[C_D(d_1,\ldots,d_n)]}) = \mathsf{dim}([a]^{C_{D/D_0}([d_1],\ldots,[d_n])}) = 1.$$

Also note that condition ($\clubsuit$) implies that $\mathsf{dim}(a^D) = 1$ for $a \notin A_0$.

Let $\mathcal{R}_D(A/A_0)$ be the ring of endomorphisms of A/A_0 generated by D, with addition being the component-wise addition on A and multiplication being composition. Then any $r \in \mathcal{R}_D(A/A_0)$ is equal to some $\sum_{i \leq n} (-1)^{\epsilon_i} d_i$, but this representation need not be unique.

Lemma 3.30. *For all $r \in \mathcal{R}_D(A/A_0)$, either r is the constant $[0]$ function $\mathbf{0}$, or r is an automorphism of A/A_0.*

Proof. We first prove the following claim: if there is some $[a] \in A/A_0$ such that $[a] \neq [0]$ and $[a]^r = [0]$, then $\mathsf{dim}(\ker(r)) = 1$. Indeed, let $d_1, \ldots, d_n$ be the elements of D which appear in a representation of r. Then $([a]^{[h]})^r = ([a]^r)^{[h]} = [0]$ for any $[h] \in C_{D/D_0}([d_1], \ldots, [d_n])$. As a consequence, $[a]^{C_{D/D_0}([d_1],\ldots,[d_n])} \subseteq \ker(r)$. We have $\mathsf{dim}([a]^{C_{D/D_0}([d_1],\ldots,[d_n])}) = 1$ by condition ($\clubsuit$). Therefore, $\ker(r)$ has dimension 1.

Now we prove a similar assertion for the dimension of the image: if there is some $[a] \neq [0]$ such that $[a]^r \neq [0]$, then $\mathsf{dim}(\mathrm{im}(r)) = 1$. Let $d_1, \ldots, d_n$ be all the elements in D which appear in a representation of r. For any $[d] \in C_{D/D_0}([d_1], \ldots, [d_n])$, we have $([a]^{[d]})^r = ([a]^r)^{[d]}$, i.e. $([a]^r)^{[d]} \in \mathrm{im}(r)$. Hence, $([a]^r)^{C_{D/D_0}([d_1],\ldots,[d_n])} \subseteq \mathrm{im}(r)$. Then

$$1 \geq \mathsf{dim}(\mathrm{im}(r)) \geq \mathsf{dim}(([a]^r)^{C_{D/D_0}([d_1],\ldots,[d_n])}) = 1.$$

Since $\mathsf{dim}(\ker(r)) + \mathsf{dim}(\mathrm{im}(r)) = \mathsf{dim}(A/A_0) = 1$, we can conclude that either $\ker(r) = \{[0]\}$ or $\mathrm{im}(r) = \{[0]\}$. If $\mathrm{im}(r) = \{[0]\}$, then $r = \mathbf{0}$. Otherwise r is injective. As (G, X) is a pseudofinite structure, r must also be surjective, hence an automorphism. $\square$

We can now see that $\mathcal{R}_D(A/A_0)$ is a division ring. To get an interpretable pseudofinite field, we need to define another ring. Let $\widetilde{\mathcal{R}}_D(A/A_0)$ be the ring of endomorphisms of A/A_0 generated by D and the definable set

$$\{(d - d')^{-1} : d, d' \in D, d - d' \neq \mathbf{0}\}$$

(the existence of $(d - d')^{-1}$ as automorphisms of A/A_0 is guaranteed by Lemma 3.30).

By exactly the same proof, we can show that every non-zero element of $\widetilde{\mathcal{R}}_D(A/A_0)$ is an automorphism of A/A_0.

Lemma 3.31. *The division ring $\widetilde{\mathcal{R}}_D(A/A_0)$ is interpretable.*

Proof. Pick some $[a] \neq [0]$. For any $r \in \widetilde{\mathcal{R}}_D(A/A_0)$ with $r \neq \mathbf{0}$, consider the set $[a]^{Dr}$ which is the image of $[a]^D$ under r. Since $\mathsf{dim}([a]^D) = \mathsf{dim}(a^D) = 1$ and $\ker(r)$ is of dimension 0 (as $r \neq \mathbf{0}$), we have that $[a]^{Dr}$ is of dimension 1. We claim that

$$([a]^D - [a]^D) \cap ([a]^{Dr} - [a]^{Dr}) \neq \{[0]\}.$$

Indeed, if $([a]^D - [a]^D) \cap ([a]^{Dr} - [a]^{Dr}) = \{[0]\}$, then $[a]^{d_1} + [a]^{d_2 r} = [a]^{d_3} + [a]^{d_4 r}$ if and only if $[a]^{d_1} = [a]^{d_3}$ and $[a]^{d_2 r} = [a]^{d_4 r}$ for any $d_1, d_2, d_3, d_4 \in D$. Hence any element in $[a]^D + [a]^{Dr}$ can be uniquely written as the sum. Therefore,

$$\mathsf{dim}([a]^D + [a]^{Dr}) = \mathsf{dim}([a]^D) + \mathsf{dim}([a]^{Dr}) = 2,$$

which contradicts the fact that $[a]^D + [a]^{Dr}$ is a subset of A/A_0 and A/A_0 is of dimension 1. Hence, there is some $d_1, d_2, d_3, d_4 \in D$ such that $[a]^{d_1 - d_2} = [a]^{(d_3 - d_4)r} \neq [0]$, i.e. $[a]^{(d_3-d_4)(d_3-d_4)^{-1}(d_1-d_2)} = [a]^{(d_3-d_4)r}$. Since $[a] \neq [0]$ and $d_3 - d_4$ is an automorphism, $[a]^{d_3 - d_4} \neq [0]$. Thus, $r = (d_3 - d_4)^{-1}(d_1 - d_2)$.

Therefore, $\widetilde{\mathcal{R}}_D(A/A_0)$ is a subset of

$$E/\sim := \{(d_3 - d_4)^{-1}(d_1 - d_2) : d_1, d_2, d_3, d_4 \in D, d_3 - d_4 \neq \mathbf{0}\}/\sim,$$

where $r \sim r'$ if r and r' induces the same endomorphism on A/A_0 for $r, r' \in E$. On the other hand, $E/\sim$ is clearly a subset of $\widetilde{\mathcal{R}}_D(A/A_0)$. Since E is definable, $\widetilde{\mathcal{R}}_D(A/A_0)$ is interpretable. □

Now we prove Theorem 3.29.

Proof. By Lemma 3.31, $\widetilde{\mathcal{R}}_D(A/A_0)$ is a pseudofinite interpretable domain. Any finite domain is a field (Wedderburn's Little Theorem). Therefore, it is also true for all pseudofinite domain and we get $F := \widetilde{\mathcal{R}}_D(A/A_0)$ is a field. It is an interpretable pseudofinite field.

Consider $D_0 = \{d \in D : \forall a \in A, a^d \in a + A_0\}$. Take any $a \notin A_0$, we know the set $[a]^D \subseteq A/A_0$ has dimension 1. Hence, D/D_0 has dimension at least 1.

By definition of $F = \widetilde{\mathcal{R}}_D(A/A_0)$ we know that D/D_0 embeds into $F^\times$. Hence $\mathsf{dim}(F) \geq 1$ and D/D_0 is commutative.

For any $[a] \neq [0]$, let $[a]^F := \{[a]^r : r \in F\}$. Define a map $i_a : F^+ \to [a]^F$ by sending r to $[a]^r$. It is clearly well-defined, surjective and is a group homomorphism. It is also injective. Indeed, if $[a]^r = [a]^{r'}$ for some $r, r' \in F$, then $[a]^{(r-r')} = [0]$. Hence $r - r' = \mathbf{0}$, and we get $r = r'$. Therefore, F^+ is isomorphic to $[a]^F$. Note that $[a]^F$ is a definable subgroup of A/A_0. Moreover, it is of dimension 1, since $\mathsf{dim}(F) \geq 1$. We claim that $a^F = A/A_0$. If there is $[b] \in (A/A_0) \setminus [a]^F$, then $[b]^F$ is also isomorphic to F^+ and of dimension 1. As $[a]^F$ and $[b]^F$ are wide subgroups of A, we have $[a]^F \cap [b]^F$ is of dimension 1. In particular, there is $[c] \neq [0]$. such that $[c] = [b]^{r_1} = [a]^{r_2}$ for some $r_1, r_2 \neq \mathbf{0}$. Therefore, $[b] = [a]^{r_2 r_1^{-1}}$ and $[b] \in [a]^F$, a contradiction.

Finally, we check that $\mathsf{dim}(D/D_0) = 1$. By the proof before, we know that D/D_0 is of dimension at least 1. On the other hand, we also have $\mathsf{dim}(D/D_0) \leq \mathsf{dim}(F^\times) = \mathsf{dim}(F^+) = \mathsf{dim}(A) = 1$. Hence, $\mathsf{dim}(D/D_0) = 1$ as we have claimed. □

Lemma 3.32. *Suppose A is an abelian group of dimension 1 and M is a group of automorphisms of A. Let $D \trianglelefteq M$ be a broad definable finite-by-abelian subgroup such that $A_0 := \{a \in A : \mathsf{dim}(a^D) = 0\}$ is of dimension 0. Then D satisfies the condition (♣). Let $F := \widetilde{\mathcal{R}}_D(A/A_0)$ be the interpretable pseudofinite field defined as in Theorem 3.29. Then M acts naturally by automorphisms on F and $\mathsf{PStab}_M(F)/M_0$ embeds into $F^\times$ with $\mathsf{dim}(\mathsf{PStab}_M(F)/M_0) = 1$, where $\mathsf{PStab}_M(F)$ is the point-wise stabilizer of F and*

$$M_0 := \{m \in \mathsf{PStab}_M(F) : \forall a \in A,\ a^m \in a + A_0\}.$$

Proof. Note that A_0 is definable by Lemma 3.24. And clearly, it is a D-invariant subgroup of A, so the induced action of D on A/A_0 is well-defined. By the remark following Theorem 3.29, we have that D satisfies the condition (♣).

Note that for any $a \in A$ and $m \in M$, if $\mathsf{dim}(a^D) = 0$, then $\mathsf{dim}((a^m)^D) = \mathsf{dim}((a^D)^m) = 0$. Therefore, M also acts by automorphisms on A/A_0.

We define an action of M on $F = \widetilde{\mathcal{R}}_D(A/A_0)$ by conjugation, i.e. for any $h \in M$ and $r \in F$, define $r^h := h^{-1}rh$ (as the composition of automorphisms of A/A_0). We claim that $r^h \in F$ for any $r \in F$ and $h \in M$.

We prove by induction on the construction of $r \in F$:

1. If $r = d \in D$, then $d^h = h^{-1}dh \in D$, as D is normal in M.
2. If $r = (d_1 - d_2)^{-1}$ for some $d_1 d_2^{-1} \notin D_0$, then for any $[x], [y] \in A/A_0$, we have
$$\begin{aligned} [x]^{r^h} = [y] \quad & \text{if and only if} \quad [x]^{h^{-1}(d_1-d_2)^{-1}h} = [y] \\ & \text{if and only if} \quad [x] = [y]^{h^{-1}(d_1-d_2)h} \\ & \text{if and only if} \quad [x] = [y]^{(d_1)^h-(d_2)^h} \\ & \text{if and only if} \quad [x]^{((d_1)^h-(d_2)^h)^{-1}} = [y]. \end{aligned}$$
   Thus, $r^h = ((d_1)^h - (d_2)^h)^{-1} \in F$.
3. If $r = r_1 + r_2$, then $r^h = h(r_1 + r_2)h^{-1} = (r_1)^h + (r_2)^h$. By induction hypothesis $(r_1)^h, (r_2)^h \in F$, hence $r^h \in F$.
4. If $r = r_1 r_2$, then $r^h = hr_1r_2h^{-1} = (r_1)^h(r_2)^h$. Again by induction hypothesis $(r_1)^h, (r_2)^h \in F$, hence $r^h \in F$.

Clearly, for any $h \in M$ the map $(\cdot)^h$ is a field endomorphism, whence by pseudofiniteness, $(\cdot)^h$ is surjective, whence a field automorphism of F.

Consider the group $T := \mathsf{PStab}_M(F)$. Let $T_0 := \{t \in T : \forall a \in A,\ a^t \in a + A_0\}$. Note that T_0 is normal in T as T acts on A_0. Since D/D_0 is abelian and $D_0 \subseteq T_0$, we have $DT_0/T_0 \leq Z(T/T_0)$. For any $m_1, \ldots, m_n \in T$ and $a \notin A_0$, we have $[a]^{C_{T/T_0}([m_1],\ldots,[m_n])} \supseteq [a^D]$, thus $\mathsf{dim}([a]^{C_{T/T_0}([m_1],\ldots,[m_n])}) = 1$. Therefore, we may apply Theorem 3.29 with

A, A_0 and T and get an interpretable pseudofinite field $\bar{F}$ such that $A/A_0 \simeq \bar{F}^+$, T/T_0 embeds into $\bar{F}^\times$ and $\mathsf{dim}(T/T_0) = 1$. Note that $F \subseteq \bar{F}$ and $F^+ \simeq A/A_0 \simeq \bar{F}^+$, by pseudofiniteness $\bar{F} = F$. $\square$

We now specify the case for $(G, X) \in \mathcal{S}$ with $\mathsf{dim}(G) = 2$. Basically, we will apply Theorem 3.29 to get the interpretable field. However, we still need to find a definable normal abelian subgroup in G. This is the aim of the following two lemmas.

Lemma 3.33. *Let $(G, X) \in \mathcal{S}$ with $\mathsf{dim}(G) = 2$. Then G has no definable wide finite-by-abelian subgroup.*

Proof. Suppose G has such a subgroup A. By the $\widetilde{\mathfrak{M}}_c$-condition, we can take $D := C_G(\bar{g})$ minimal up to finite index for some finite tuple $\bar{g}$ in G such that $[A : A \cap D] < \infty$.

We claim that $A \cap D \leq \widetilde{Z}(D)$. As A is finite-by-abelian, we have $[A : C_A(a)] < \infty$ for any $a \in A \cap D$. Together with $[A : A \cap D] < \infty$, we get $[A : C_A(a) \cap D] < \infty$. Since $C_A(a) \cap D \leq C_D(a)$, also $[A : A \cap C_D(a)] < \infty$. By minimality of D we have $[D : C_D(a)] < \infty$. Hence, $a \in \widetilde{Z}(D)$ and $A \cap D \leq \widetilde{Z}(D)$ as claimed. Since $A \cap D$ has finite index in A and A is wide, $\widetilde{Z}(D)$ is also wide in G.

By Lemma 3.21, there is a definable wide normal subgroup $N \trianglelefteq G$ such that N is commensurable with $\bigcap_{i \leq k} D^{g_i}$ for some $g_0, \ldots, g_k \in G$. By Lemma 3.22, we have $\bigcap_{i \leq k} \widetilde{Z}(D)^{g_i} \leq \widetilde{Z}(\bigcap_{i \leq k} D^{g_i})$. Since $\widetilde{Z}(D)$ is wide, so is $\bigcap_{i \leq k} \widetilde{Z}(D)^{g_i}$, hence also $\widetilde{Z}(\bigcap_{i \leq k} D^{g_i})$. Since N is commensurable with $\bigcap_{i \leq k} D^{g_i}$, we get $\mathsf{dim}(\widetilde{Z}(N)) = \mathsf{dim}(\widetilde{Z}(\bigcap_{i \leq k} D^{g_i})) = 2$ by Lemma 3.23. Thus, $\widetilde{Z}(N)$ is a definable normal finite-by-abelian subgroup of G. Since $\widetilde{Z}(N)'$ is finite and normal in G, it is trivial by definably primitivity. Thus, $\widetilde{Z}(N)$ is a definable normal abelian subgroup of G. By Lemma 3.26, $\mathsf{dim}(\widetilde{Z}(N)) = 1$, contradicting that $\mathsf{dim}(\widetilde{Z}(N)) = 2$. $\square$

Lemma 3.34. *Let $(G, X) \in \mathcal{S}$ with $\mathsf{dim}(G) = 2$. Assume that the definable sections of G also satisfy the $\widetilde{\mathfrak{M}}_c$-condition. Then G has a definable normal abelian subgroup A of dimension 1.*

Proof. By Fact 3.8(1), G has a broad definable finite-by-abelian subgroup C whose normalizer is wide. We refer to the proof in [Wag15, Theorem 13], see also Appendix A. From the construction of C in the proof, there are two cases. The first case is that C is normal in G. Then C is not wide by Lemma 3.33, so $\mathsf{dim}(C) = 1$. Since C' is definable normal in G of dimension 0, it is trivial. Therefore, $A := C$ is a definable normal abelian group of dimension 1.

The second case is that $C := \widetilde{Z}(D)$ where D is commensurable with $E = C_G(\bar{b})$ for some $\bar{b} \in G^n$ and $\mathsf{dim}(D) \geq 1$. By the $\widetilde{\mathfrak{M}}_c$-condition and Schlichting's Theorem, there is a definable normal subgroup H of G, such that H is commensurable with $\bigcap_{i \leq k} E^{g_i}$, for some $g_0, \ldots, g_k \in G$. We may assume that $\mathsf{dim}(\widetilde{Z}(H)) = \mathsf{dim}(\widetilde{Z}(\bigcap_{i \leq k} E^{g_i}) = 0$, for otherwise, we are in the previous case. Since H is normal in G and $\widetilde{Z}(H)$ is characteristic in H, $\widetilde{Z}(H)$ is a definable normal subgroup of G of dimension 0. Hence $\widetilde{Z}(H)$ cannot act transitively on X and is trivial by Lemma 3.25. By Lemma 3.22 and Lemma 3.23, we get $\bigcap_{i \leq k} \widetilde{Z}(E^{g_i}) \leq \widetilde{Z}(\bigcap_{i \leq k} E^{g_i})$ and $\widetilde{Z}(\bigcap_{i \leq k} E^{g_i})$ is commensurable with $\widetilde{Z}(H)$. Hence $\bigcap_{i \leq k} \widetilde{Z}(E^{g_i}) = \bigcap_{i \leq k} \widetilde{Z}(E)^{g_i}$ is finite. As D is commensurable with E, we have $\widetilde{Z}(D)$ is commensrable with $\widetilde{Z}(E)$. We may assume $[\widetilde{Z}(D) : \widetilde{Z}(D) \cap \widetilde{Z}(E)] \leq \ell$ for some $\ell \in \mathbb{N}$. Then

$$\begin{aligned}
&\left[\bigcap_{i\leq k} \widetilde{Z}(D)^{g_i} : \left(\bigcap_{i\leq k} \widetilde{Z}(D)^{g_i}\right) \cap \left(\bigcap_{i\leq k} \widetilde{Z}(E)^{g_i}\right)\right] \\
\leq &\prod_{j\leq k} \left[\bigcap_{i\leq k} \widetilde{Z}(D)^{g_i} : \left(\bigcap_{i\leq k} \widetilde{Z}(D)^{g_i}\right) \cap \widetilde{Z}(E)^{g_j}\right] \\
= &\prod_{j\leq k} \left[\bigcap_{i\leq k} \widetilde{Z}(D)^{g_i} : \left(\bigcap_{i\neq j} \widetilde{Z}(D)^{g_i}\right) \cap \left(\widetilde{Z}(D)^{g_j} \cap \widetilde{Z}(E)^{g_j}\right)\right] \\
= &\prod_{j\leq k} \left[\widetilde{Z}(D)^{g_j} : \widetilde{Z}(D)^{g_j} \cap \widetilde{Z}(E)^{g_j}\right] \leq \ell^{k+1}.
\end{aligned}$$

As $\bigcap_{i\leq k} \widetilde{Z}(E)^{g_i}$ is finite, we get $\bigcap_{i\leq k} \widetilde{Z}(D)^{g_i}$ is also finite.

By assumption, $N_G(\widetilde{Z}(D))$ is wide, hence $\mathsf{dim}(N_G(\widetilde{Z}(D))/\widetilde{Z}(D)) = 1$. By Fact 3.7, there is a definable $B \leq N_G(\widetilde{Z}(D))$ such that $B/\widetilde{Z}(D)$ is broad finite-by-abelian. Hence, B is wide in G. Clearly, $B^{g_i}/\widetilde{Z}(D)^{g_i}$ is also broad finite-by-abelian for any g_i. By Lemma 3.17, the group $\bigcap_{i\leq k} B^{g_i}/\bigcap_{i\leq k} \widetilde{Z}(D)^{g_i}$ is finite-by-abelian. Since $\bigcap_{i\leq k} \widetilde{Z}(D)^{g_i}$ is finite, $\bigcap_{i\leq k} B^{g_i}$ is finite-by-abelian. However, $\bigcap_{i\leq k} B^{g_i}$ is definable and wide in G, contradicting Lemma 3.33. □

Now we can conclude the dimension 2 case.

Theorem 3.35. *Let $(G, X) \in \mathcal{S}$ with $\mathsf{dim}(G) = 2$. Suppose the definable sections of G satisfy the $\widetilde{\mathfrak{M}}_c$-condition. Then $G = A \rtimes G_x$ and there is an interpretable pseudofinite field F such that $A \simeq F^+$ and D embeds into $F^\times$ for some wide definable subgroup $D \trianglelefteq G_x$.*

Moreover, G_x induces a group of automorphisms on F.

Proof. By Lemma 3.34, G has a definable normal abelian subgroup A. By Lemma 3.26 we have $G = A \rtimes G_x$ and G_x acts on A by conjugation, where G_x is the point-stabilizer $\mathsf{Stab}_G(x)$. By Fact 3.7(2), G_x has a definable finite-by-abelian normal subgroup D. For any $a \in A$, if $\mathsf{dim}(a^D) = 0$, then $\mathsf{dim}(C_D(a)) = \mathsf{dim}(D) = 1$. Since $A \times C_D(a) \subseteq C_G(a)$, we get $\mathsf{dim}(C_G(a)) \geq \mathsf{dim}(A \rtimes C_D(a)) = 2 = \mathsf{dim}(G)$. So $a = \mathsf{0}$ by Lemma 3.28. Therefore, $A_0 := \{a \in A : \mathsf{dim}(a^D) = 0\} = \{\mathsf{0}\}$. Applying Theorem 3.29 and Lemma 3.32 with $A_0 = \{\mathsf{0}\}$ and $D_0 = \{1\}$, we get the desired result. □

If we add some extra condition on sets of dimension 0, we can also make the full stabilizer G_x embeds into $F^\times$ as in Fact 3.1.

Lemma 3.36. *Suppose an infinite field F and a group B of field-automorphisms of F are interpretable in a theory with an additive integer-valued dimension dim such that $\mathsf{dim}(F) = 1$. Then B is either trivial or infinite.*

Proof. If B is finite, then any $\sigma \in B$ must have finite order. Thus, the fixed field $fix(\sigma)$ is of finite index in F. As $1 = \mathsf{dim}(F) = [F : fix(\sigma)] \cdot \mathsf{dim}(F)$, we get $fix(\sigma) = F$. Thus, B is trivial. □

Corollary 3.37. *Let $(G, X) \in \mathcal{S}$ with $\mathsf{dim}(G) = 2$. Suppose the definable sections of G satisfy $\widetilde{\mathfrak{M}}_c$-condition, and that the dimension-0 group $E_F := G_x/\mathsf{PStab}_{G_x}(F)$ is finite. Then G_x embeds into $F^\times$.*

Proof. By the argument before, (G, X) interprets a pseudofinite field F of dimension 1 and a group of field automorphisms $E_F := G_x/\mathsf{PStab}_{G_x}(F)$. By assumption, the group E_F is finite, hence is trivial by Lemma 3.36. By Lemma 3.32, $G_x = \mathsf{PStab}_{G_x}(F)$ embeds into $F^\times$. □

3.4 Permutation groups of dimension ≥ 3

This section deals with permutation groups in $\mathcal{S}$ of dimension greater or equal to 3. The general strategy will be different from the previous sections. All the proofs before rely mostly on the $\widetilde{\mathfrak{M}}_c$-condition and properties of dimensions. From now on we will use pseudofiniteness to go directly to finite structures, and then use the well-established results of finite groups, such as CFSG.

Remark: From now on we will often assume that we work in an ultraproduct of finite permutation group $(G, X) = \prod_{i \in I}(G_i, X_i)/\mathcal{U}$ for some non-principal ultrafilter $\mathcal{U}$ on an infinite set I. Since our main results (Theorem 3.53 and Theorem 3.58) are about interpretable properties of (G, X), any permutation group with the same theory will share these properties. And by the definition of pseudofinite structures, the main results hold for any pseudofinite permutation group satisfying the corresponding requirements.

As mentioned in the introduction, we need two extra assumptions: the $\widetilde{\mathfrak{M}}_s$-condition on (G, X), and the (EX)-condition on X.

While we need these two additional assumptions in the main result, we still make our statements as general as possible.

The following lemma only assume pseudofiniteness and the $\widetilde{\mathfrak{M}}_c$-condition.

Lemma 3.38. *Let $G = \prod_{i \in I} G_i/\mathcal{U}$ be an ultraproduct of finite groups. Suppose G satisfies the $\widetilde{\mathfrak{M}}_c$-condition. Then there is some $n < \omega$ and $J \in \mathcal{U}$ such that for all $i \in J$ we cannot find subgroups $D_0^i, \ldots, D_{n-1}^i$ of G_i which are center-less and commute with each other.*

Proof. This is standard. Fix any $d \in \mathbb{N}$. Let $n = (d + 1) \cdot m$ such that $2^m > d$. If the claim is not true, then for all $J \in \mathcal{U}$ there is $i \in J$ such that there are subgroups $D_0^i, \ldots, D_{n-1}^i$ in G_i as claimed. Let

$J_0 := \{i \in I : G_i$ has centerless subgroups $D_0^i, \ldots, D_{n-1}^i$ which commute with each other.$\}$

Then $J_0 \in \mathcal{U}$, since otherwise the complement would be in the ultrafilter which contradicts our assumption.

For $i \in J_0$, choose $1 \neq g_j^i \in D_j^i$ for each $j < n$, and put $h_k^i = \prod_{j<m}(g_{km+j}^i)$ for $k \leq d$. Clearly, for each $i \in J_0$ and for any $1 \leq k \leq d$ we have

$$\begin{aligned}
&[C_{G_i}(h_0^i, \dots, h_{k-1}^i) : C_{G_i}(h_0^i, \dots, h_k^i)] \\
&\geq [\prod_{j<m} D_{km+j}^i : C_{D_{km}^i}(g_{km}^i) C_{D_{km+1}^i}(g_{km+1}^i) \cdots C_{D_{km+m-1}^i}(g_{km+m-1}^i)] \\
&\geq \prod_{j<m} [D_{km+j}^i : C_{D_{km+j}^i}(g_{km+j}^i)] \geq 2^m > d.
\end{aligned}$$

Hence, G does not satisfy the $\widetilde{\mathfrak{M}}_c$-condition, a contradiction. □

Suppose $G = \prod_{i \in I} G_i/\mathcal{U}$. Let H_i be a non-trivial minimal normal subgroup in G_i for $i \in I$. Then H_i is a direct product of isomorphic simple groups. Suppose $H_i = T_i \odot T_i^{g_{i_1}} \odot \cdots \odot T_i^{g_{in_i}}$ with $g_{i_1}, \dots, g_{in_i} \in G_i$ and T_i simple. If H_i is not abelian, then neither is T_i. Let $H := \prod_{i \in I} H_i/\mathcal{U}$ and $T = \prod_{i \in I} T_i/\mathcal{U}$.

Lemma 3.39. *Let $(G, X) \in \mathcal{S}$. In particular, G is a pseudofinite $\widetilde{\mathfrak{M}}_c$-group. Let H be defined as above. If H is not abelian, then T is infinite and there is $m \in \mathbb{N}$ such that $H = T \odot T^{g_1} \odot \cdots \odot T^{g_m}$ for some $g_1, \dots, g_m \in G$.*

Moreover, T and H are definable, and T is a simple pseudofinite group.

Proof. By Lemma 3.38, there is $m \in \mathbb{N}$ and $J \in \mathcal{U}$ such that H_i is $m+1$-fold product of conjugates of T_i for all $i \in J$. Hence, $H = T \odot T^{g_1} \odot \cdots \odot T^{g_m}$ for some $g_1 \dots, g_m \in G$. We claim that T is infinite. Otherwise, if T is finite, then H is finite, hence definable. Since H is non-trivial, it acts transitively on X. Hence, $\mathsf{dim}(X) \leq \mathsf{dim}(H) = 0$, a contradiction.

For each $i \in I$, since T_i is non-abelian, we may assume it is either an alternating group Alt_{n_i} or a classical group of Lie type of rank n_i over some field $\mathbb{F}_{q_i}$, denoted as $\mathsf{cl}_{n_i}(q_i)$. We claim that n_i is bounded. If not, then for any n, for all large enough n_i, the group Alt_{n_i} will contain at least n commuting copies of Alt_5, and $\mathsf{cl}_{n_i}(q_i)$ will contain at least n commuting copies of $\mathrm{PSL}_2(\mathbb{F}_{p_i})$, where p_i is the characteristic of $\mathbb{F}_{q_i}$. Both cases contradict Lemma 3.38. Thus, we may assume $\{T_i : i \in I\}$ are classical groups of Lie type of bounded Lie rank.

By [Wil95], T is a simple pseudofinite group. Hence, the theory of T in the language of pure group is supersimple of finite SU-rank by [Ryt07]. As T is infinite nonabelian simple, there is some $x \in T$ such that the set x^T is infinite. By the Indecomposability Theorem (Fact 0.32), there is some infinite definable group $D \leq x^T \cdots x^T$ which is normal in T, where $x^T \cdots x^T$ is a k-fold product for some $k \in \mathbb{N}$. Denote the k-fold product of X as $X \cdot (k) \cdot X$. Since T is simple, $D = T$. Therefore, $x^T \cdot (k) \cdot x^T = T$. As H is normal and $x \in H$, we have

$$H \supseteq (x^G \cdot (k) \cdot x^G) \odot (x^G \cdot (k) \cdot x^G)^{g_1} \odot \cdots \odot (x^G \cdot (k) \cdot x^G)^{g_m} \supseteq T \odot T^{g_1} \odot \cdots \odot T^{g_m} = H.$$

Consequently, H is definable. Moreover, since $x^H \cdot (k) \cdot x^H = x^T \cdot (k) \cdot x^T = T$, we also get T definable. □

Lemma 3.40. *Let $(G, X) \in \mathcal{S}$. Suppose G satisfies the $\widetilde{\mathfrak{M}}_s$-condition. Let H be a normal definable subgroup of G. Suppose $\mathsf{dim}(H) = n$. Then there are $x_1, \dots, x_n \in X$*

such that for all $1 \le i \le n$ *we have*

$$\mathsf{dim}(\mathsf{PStab}_H(x_1,\dots,x_i)) = n - i.$$

Moreover, there are $x_1,\dots,x_t \in X$ *such that* $\mathsf{PStab}_H(x_1,\dots,x_t) = \{1\}$.

Proof. We only need to show there are $x_1,\dots,x_n \in X$ with $\mathsf{dim}(\mathsf{PStab}_H(x_1,\cdots,x_n)) = 0$. Since (G, X) satisfies $\widetilde{\mathfrak{M}_s}$-condition, so does (H, X). By the $\widetilde{\mathfrak{M}_s}$-condition, there are $x_1,\dots,x_m \in X$ and $d \in \mathbb{N}$ such that

$$[\mathsf{PStab}_H(x_1,\dots,x_m) : \mathsf{PStab}_H(x_1,\dots,x_m,x)] \le d,$$

for any $x \in X$. As H is normal in G, we get $\{(\mathsf{PStab}_H(x_1,\dots,x_m))^g : g \in G\}$ is a uniformly commensurable family of definable subgroups. By Schlichting's Theorem, there is definable $H_0 \trianglelefteq G$ such that H_0 is commensurable with $\mathsf{PStab}_H(x_1,\dots,x_m)$. By Lemma 3.25, either $x^{H_0} = X$ or H_0 is trivial. If $x^{H_0} = X$, then

$$\mathsf{dim}(x^{\mathsf{PStab}_H(x_1,\dots,x_m)}) = \mathsf{dim}(x^{H_0}) = 1.$$

By the Orbit-Stabilizer Theorem

$$|x^{\mathsf{PStab}_H(x_1,\dots,x_m)}| = [\mathsf{PStab}_H(x_1,\dots,x_m) : \mathsf{PStab}_H(x_1,\dots,x_m,x)] \le d,$$

a contradiction. Therefore, H_0 is trivial. As $\mathsf{PStab}_H(x_1,\dots,x_m)$ is commensurable with H_0, we deduce $\mathsf{PStab}_H(x_1,\dots,x_m)$ is finite. So we only need finitely many more points, say $x_{m+1},\dots,x_t \in X$, to distinguish 1 from other elements in $\mathsf{PStab}_H(x_1,\dots,x_m)$. Therefore, $\mathsf{PStab}_H(x_1,\dots,x_t) = \{1\}$.

To finish the proof we show that there is a subsequence $x_{i_1},\dots,x_{i_n}$ of $x_1,\dots,x_m$ with $\mathsf{dim}(\mathsf{PStab}_H(x_{i_1},\dots,x_{i_n})) = 0$. Consider the dimensions of the following sequence

$$\mathsf{PStab}_H(x_1), \mathsf{PStab}_H(x_1,x_2),\dots,\mathsf{PStab}_H(x_1,\dots,x_m).$$

By the Orbit-Stabilizer Theorem, the dimension can drop at most 1 in each step. Hence, $m \ge n$. Take n elements, say $x_{i_1},\dots,x_{i_n}$ with $i_1 < i_2 < \cdots < i_n$, such that each of the corresponding dimension drops. By our choice,

$$1 \ge \mathsf{dim}((x_{i_j})^{\mathsf{PStab}_H(x_{i_1},\dots,x_{i_{j-1}})}) \ge \mathsf{dim}((x_{i_j})^{\mathsf{PStab}_H(x_1,x_2,\dots,x_{i_j-1})}) = 1,$$

for each $1 \le j \le n$. Therefore, $\mathsf{dim}(\mathsf{PStab}_H(x_{i_1},\dots,x_{i_n})) = 0$. □

Lemma 3.41. *Suppose* $(G, X) \in \mathcal{S}$ *satisfies the* $\widetilde{\mathfrak{M}_s}$*-condition. Let* H *be a non-trivial normal definable subgroup of* G. *For any* $x \in X$, *define* $L_x := \{y \in X : \mathsf{dim}(x^{H_y}) = 0\}$. *Then* L_x *is uniformly definable with respect to* x.

Proof. First note that since H is a definable subgroup of G, we have (H, X) also satisfies $\widetilde{\mathfrak{M}_s}$-condition. Assume $\mathsf{dim}(H) = n$. Note that since H is non-trivial, definable and normal, it acts transitively on X. Thus, $\mathsf{dim}(\mathsf{Stab}_H(x)) = n - 1$ for any $x \in X$. By the $\widetilde{\mathfrak{M}_s}$-condition, there are $x_1,\dots,x_k \in X$ and $d \in \mathbb{N}$ such that $\mathsf{dim}(\mathsf{PStab}_H(x_1,\dots,x_k)) = n - 1$ and for any $y \in X$, we have either $\mathsf{dim}(\mathsf{PStab}_H(x_1,\dots,x_k,y)) = n - 2$ or

$$[\mathsf{PStab}_H(x_1,\dots,x_k) : \mathsf{PStab}_H(x_1,\dots,x_k,y)] \le d.$$

As $\mathsf{dim}(H_{x_1}) = \mathsf{dim}(\mathsf{PStab}_H(x_1,\ldots,x_k)) = n-1$, we get $\mathsf{dim}(z^{H_{x_1}}) = 0$ if and only if $[\mathsf{PStab}_H(x_1,\ldots,x_k) : \mathsf{PStab}_H(x_1,\ldots,x_k,z)] \leq d$ for any $z \in X$.

For any $y \in X$, let $g \in H$ be such that $(x_1)^g = y$. Then $y \in L_x$ if and only if $\mathsf{dim}(x^{H_{(x_1)^g}}) = 0$ if and only if $\mathsf{dim}((x^{g^{-1}})^{H_{x_1}}) = 0$ if and only if there is $g \in H$ such that $(x_1)^g = y$ and

$$[\mathsf{PStab}_H(x_1,\ldots,x_k) : \mathsf{PStab}_H(x_1,\ldots,x_k,x^{g^{-1}})] \leq d. \quad \square$$

Theorem 3.42. *Suppose $(G,X) \in \mathcal{S}$ with $\mathsf{dim}(G) \geq 3$ satisfies the $\widetilde{\mathfrak{M}}_s$-condition and X satisfies the (EX)-condition. Then G does not contain any nontrivial abelian normal subgroup.*

Proof. The theorem follows from the claims below.

Claim 3.43. If G has a nontrivial normal abelian subgroup H, then G has a definable nontrivial normal abelian subgroup A.

Proof. If G has a non-trivial normal abelian subgroup, then G has a definable finite-by-abelian subgroup A, which is normal in G and contains H, by [Hem15, Theorem 3.3(1)]. Since A' is definable and of dimension 0, by definable primitivity, A' is trivial, hence A is abelian. Since A contains H, we get A is nontrivial. $\square$

Suppose the conclusion of Theorem 3.42 fails, then G has a nontrivial definable normal abelian subgroup A. By Lemma 3.26, $G = A \rtimes G_x$ where $G_x := \mathsf{Stab}_G(x)$ for some $x \in X$. We identify A with X. Then G_x acts on A by conjugation, while A acts on itself by addition. Our aim is to derive a contradiction.

Claim 3.44. Suppose $(G,X) \in \mathcal{S}$ and $\mathsf{dim}(G) \geq 2$. Assume $G = A \rtimes G_x$. Let $C \trianglelefteq G_x$ with C definable and $\mathsf{dim}(C) \geq 1$. Then $A \rtimes C$ also acts definably primitively on X.

Proof. We may assume that (G,X) is an ultraproduct of finite permutation groups and $A \rtimes G_x = \prod_{i\in I} A_i \rtimes (G_x)_i/\mathcal{U}$ for some ultrafilter $\mathcal{U}$ on I. The formula defining C also defines $C_i \trianglelefteq (G_x)_i$ for each $i \in I$. Let $W_i \leq A_i$ be a nontrivial C_i-irreducible subgroup, that is a minimal nontrivial C_i-invariant subgroup. Consider $W := \prod_{i\in I} W_i/\mathcal{U}$. Then W is nontrivial and C-invariant. If there is $V := \prod_{i\in I} V_i/\mathcal{U}$ with each $V_i \neq W_i$ nontrivial and C_i-irreducible, then $W \cap V = \emptyset$. Take $a \in W \setminus \{0\}$ and $b \in V \setminus \{0\}$. Note that $A \rtimes C \trianglelefteq G$ and $\mathsf{dim}(A \rtimes C) \geq 2$. By Lemma 3.28, we have $C_{A\rtimes C}(a)$ and $C_{A\rtimes C}(b)$ are not wide in $A \rtimes C$. Therefore, $\mathsf{dim}(a^C) = \mathsf{dim}(b^C) = 1$. Moreover, we have $(a^C - a^C) \cap (b^C - b^C) \subseteq W \cap V = \emptyset$. Hence, $\mathsf{dim}(a^C + b^C) = \mathsf{dim}(a^C) + \mathsf{dim}(b^C) = 2$, contradiction. Hence, we may assume that there is only one nontrivial C_i-irreducible subgroup in any A_i.

Let H be any non-trivial definable C-invariant subgroup of A. Then each H_i is non-trivial and C-invariant. Thus, $W_i \subseteq H_i$ and we get $W \subseteq H$. Since C is normal in G_x, H^g is also C-invariant for any $g \in G_x$. By the same argument, $W \subseteq H^g$. Therefore, $W \subseteq \bigcap_{g\in G_x} H^g$. The group $M := \bigcap_{g\in G_x} H^g \leq A$ is non-trivial, definable and G_x invariant. As $M \leq A$ is G_x invariant and $G = A \rtimes G_x$, we have M is normal in G. Since M is nontrivial, it must act transitively on X by Lemma 3.25. As A acts on X regularly by Lemma 3.26, we deduce $M = H = A$. Therefore, A is the minimal non-trivial definable C-invariant subgroup of A.

Clearly, $\mathsf{Stab}_{A\rtimes C}(x) = C$. Suppose there is a definable group $C \leq D \leq A \rtimes C$, then $D \cap A \leq A$. Moreover, as $(D\cap A)^C \leq D^C \cap A^C = D \cap A$, we have $(D\cap A)^C = D\cap A$. As A is the minimal non-trivial definable C-invariant subgroup of A, we conclude either $D\cap A = A$ or $D \cap A = \{0\}$. Therefore, either $D = C$ or $D = A \rtimes C$. □

By Lemma 3.40, we can find $\bar{x} = (x_1, \dots, x_{n-2})$ such that $\mathsf{dim}(\mathsf{PStab}_G(\bar{x})) = 2$. We may assume $\mathsf{PStab}_G(\bar{x}) \subseteq G_x$ and we write $\mathsf{PStab}_G(\bar{x})$ as $G_{\bar{x}}$. By Fact 3.8(1), $G_{\bar{x}}$ has a broad definable finite-by-abelian subgroup D such that $N_{G_{\bar{x}}}(D)$ has dimension 2.

Consider the group $A_0^D := \{a \in A : \mathsf{dim}(a^D) = 0\}$. The dimension of A_0^D is either 0 or 1. We will show that neither of them holds.

Claim 3.45. The dimension of A_0^D is not 1.

Proof. Suppose $\mathsf{dim}(A_0^D) = 1$. By Lemma 3.24, there are $d \in \mathbb{N}$ and a definable group $T \leq D$ such that $A_0^D = \{a \in A : [T : C_T(a)] \leq d\}$ and $\mathsf{dim}(T) = \mathsf{dim}(D)$. Therefore $A_0^D \leq \widetilde{C}_G(T)$. Since A is in definable bijection with X, by the (EX)-condition, A_0^D has finite index in A. Hence, $A \lesssim \widetilde{C}_G(T)$. By Fact 3.20, $T \lesssim \widetilde{C}_G(A)$.

Let $M := \widetilde{C}_G(A) \cap G_x$. Then $\mathsf{dim}(M) \geq \mathsf{dim}(T) \geq 1$. Note that $\widetilde{C}_G(A)$ is normal in G, hence, M is normal in G_x. By Lemma 3.44, $A \rtimes M = \widetilde{C}_G(A)$ also acts definably primitively on X.

As $\widetilde{C}_G(A) \lesssim \widetilde{C}_G(A)$, we have $A \lesssim \widetilde{C}_G(\widetilde{C}_G(A))$ by Fact 3.20. Thus, there is $0 \neq a \in A$ such that $[\widetilde{C}_G(A) : C_{\widetilde{C}_G(A)}(a)] < \infty$, which means $C_{\widetilde{C}_G(A)}(a)$ is wide in $\widetilde{C}_G(A)$, contradicting Lemma 3.28. □

Claim 3.46. The dimension of A_0^D is not 0.

Proof. Let $M := N_{G_{\bar{x}}}(D)$. As the normalizer of D is wide in $G_{\bar{x}}$, we have $\mathsf{dim}(M) = 2$. Suppose $\mathsf{dim}(A_0^D) = 0$. We can apply Theorem 3.29 and Lemma 3.32 to get an interpretable pseudofinite field F such that $A/A_0^D \simeq F^+$ and M extends to a group of automorphisms of F. Consider the point-wise stabilizer $\mathsf{PStab}_M(F)$. Let

$$M_0 := \{m \in \mathsf{PStab}_M(F) : \forall a \in A,\ a^m \in a + A_0^D\}.$$

By Lemma 3.32, $\mathsf{dim}(\mathsf{PStab}_M(F)/M_0) = 1$. By the second part of Lemma 3.40, the value of $m \in M_0$ is determined by its value on some $a_1, \dots, a_t \in A$. Hence,

$$\mathsf{dim}(M_0) \leq t\mathsf{dim}(A_0^D) = 0.$$

Thus, $\mathsf{dim}(\mathsf{PStab}_M(F)) = 1$.

Therefore, $T := M/\mathsf{PStab}_M(F)$ is a group of automorphisms of F such that the action is faithful and $\mathsf{dim}(T) = \mathsf{dim}(M) - \mathsf{dim}(\mathsf{PStab}_M(F)) = 2 - 1 = 1$.

Consider $F_0^T := \{k \in F : \mathsf{dim}(k^T) = 0\}$. By the fact that T is a group of automorphisms of F, we can check easily that F_0^T is a subfield of F. Note that F_0^T is definable (apply Lemma 3.24 to the group $(F^+ \rtimes T)$). We claim that either $F_0^T = F$ or $\mathsf{dim}(F_0^T) = 0$. Indeed, if $\mathsf{dim}(F_0^T) = 1$, then

$$1 = \mathsf{dim}(F) = [F : F_0^T] \cdot \mathsf{dim}(F_0^T) = [F : F_0^T],$$

and we get $F = F_0^T$.

If $F_0^T = F$, then by the $\widetilde{\mathfrak{M}}_c$-condition of the interpretable group $F^+ \rtimes T$, there are $k_0, \cdots, k_t \in F$ and $n \in \mathbb{N}$ such that if we define $H := C_T(k_0, \cdots, k_t)$, then for all $k \in F$ we have $[H : C_H(k)] \leq n$, that is $|k^H| \leq n$. Consider the group $F^+ \rtimes H$. From the above argument we know that $F^+ \lesssim \widetilde{C}_{F^+ \rtimes H}(H)$. By Fact 3.20, we have $H \lesssim \widetilde{C}_{F^+ \rtimes H}(F^+)$. Therefore, there is $h \neq id$ such that $[F^+ : C_{F^+}(h)] < \infty$. Since $C_{F^+}(h)$ is a definable subfield of F and $\mathsf{dim}(F) = 1$, we have $C_{F^+}(h) = F^+$, contradicting $h \neq id$.

Thus F_0^T is of dimension 0. We may assume we are working in an ultraproduct of finite structures. Suppose $F := \prod_{i \in I} F_i/\mathcal{U}$. Let $Y := F \setminus F_0^T$. Clearly, there is $J \in \mathcal{U}$ such that $|Y_i| \geq |F_i|/2$ for all $i \in J$. If $F_i = \mathbb{F}_{p_i^{n_i}}$, then $|T_i| \leq n_i$. Therefore, there are infinitely many T-orbits on Y and each of them has dimension 1. Note that X is in definable bijection with F^+, contradicting the (EX)-condition. $\square$

This finishes the proof of Theorem 3.42. $\square$

Corollary 3.47. *Suppose $(G, X) \in \mathcal{S}$ with $\mathsf{dim}(G) \geq 3$ satisfies the $\widetilde{\mathfrak{M}}_s$-condition and X satisfies the (EX)-condition. Then G has a definable subgroup T and a definable normal subgroup H such that $H = T \times T^{g_1} \times \cdots \times T^{g_m}$ for some $g_1, \ldots, g_m \in G$ and T is definably simple and non-abelian.*

Proof. We first take an ultraproduct of finite permutation groups (G^*, X^*) such that $(G, X) \equiv (G^*, X^*)$. By Lemma 3.39 and Theorem 3.42, we deduce that G^* has a definable normal subgroup $H^* = T^* \times (T^*)^{g_1'} \times \cdots \times (T^*)^{g_m'}$ with T^* definable and simple non-abelian. Hence, G also has definable subgroups H, T and elements $g_1, \ldots, g_m \in G$ such that T is definably simple and $H = T \times T^{g_1} \times \cdots \times T^{g_m}$ is normal in G. $\square$

In the following we will show that actually T is normal in G, hence $H = T$.

The following three lemmas all assume that $(G, X) \in \mathcal{S}$ satisfies the $\widetilde{\mathfrak{M}}_s$-condition and the (EX)-condition.

Lemma 3.48. *Let H be a non-trivial definable normal subgroup of G. Suppose $\mathsf{dim}(H) \geq 2$. Then for any $x \in X$, the group $H_x := \mathsf{Stab}_H(x)$ has only finitely many orbits on X.*

Proof. Note that H is definable normal and non-trivial. It acts transitively on X. Therefore, $\mathsf{dim}(H) \geq \mathsf{dim}(x^H) = 1$ and $\mathsf{dim}(H_x) = \mathsf{dim}(H) - \mathsf{dim}(x^H) = \mathsf{dim}(H) - 1 \geq 1$ for any $x \in X$.

Define a relation $\sim$ on X as: $x \sim y$ if $\mathsf{dim}(x^{H_y}) = 0$. Clearly, $\sim$ is reflexive. It is symmetric. If $\mathsf{dim}(x^{H_y}) = 0$, then $\mathsf{dim}(H_y/H_{yx}) = 0$. Therefore, $\mathsf{dim}(H_{yx}) = \mathsf{dim}(H_y) = \mathsf{dim}(H_x)$, and y^{H_x} has dimension 0. It is also transitive. If both x^{H_y} and y^{H_z} have dimension 0, then $\mathsf{dim}(H_x) = \mathsf{dim}(H_{xy}) = \mathsf{dim}(H_y) = \mathsf{dim}(H_{yz})$. That is, both H_{xy} and H_{yz} are wide in H_y. Therefore, $H_{xyz} = H_{xy} \cap H_{yz}$ is also wide in H_y. Hence $\mathsf{dim}(H_{xyz}) = \mathsf{dim}(H_y) = \mathsf{dim}(H_z)$. We get $\mathsf{dim}(x^{H_z}) = \mathsf{dim}(H_z/H_{xz}) \leq \mathsf{dim}(H_z/H_{xyz}) = 0$.

Moreover, $\sim$ is G-invariant and definable. It is definable by Lemma 3.41. For G-invariance, if $x \sim y$, then for any $g \in G$, we have $(x^g)^{H_{y^g}} = (x^g)^{(H_y)^g} = (x^{H_y})^g$. Thus, $\mathsf{dim}((x^g)^{H_{y^g}}) = \mathsf{dim}(x^{H_y}) = 0$. Consequently, $x^g \sim y^g$.

By definable primitivity, $\sim$ is either trivial or the universal congruence. By Lemma 3.40, there is $y \in X$ such that $\mathsf{dim}(\mathsf{PStab}_H(x, y)) = \mathsf{dim}(H_x) - 1$. Thus, $\sim$ is not the universal congruence. Therefore, every H_x orbit on $X \setminus \{x\}$ has dimension 1. By the (EX)-condition, there can be only finitely-many such orbits. $\square$

Lemma 3.49. *Let H be a normal definable subgroup of G with $\mathsf{dim}(H) \geq 2$. Suppose there is a definable subgroup E such that $\mathsf{Stab}_H(x) \leq E \leq H$ and $\mathsf{dim}(E) = \mathsf{dim}(\mathsf{Stab}_H(x))$. Then $E = \mathsf{Stab}_H(x)$.*

Proof. Let $H_x := \mathsf{Stab}_H(x)$. As $\mathsf{dim}(E) = \mathsf{dim}(H_x)$, we have $\mathsf{dim}((H_x)^m \cap H_x) = \mathsf{dim}(H_x)$ for any $m \in E$. Note that $\mathsf{dim}((H_x)^m \cap H_x) = \mathsf{dim}(H_x)$ if and only if $\mathsf{dim}(H_{x^m} \cap H_x) = \mathsf{dim}(H_x)$ if and only if $\mathsf{dim}(x^{H_{x^m}}) = 0$ if and only if $x \sim x^m$. By Lemma 3.48, $x \sim y$ if only if $x = y$. Therefore, $x^m = x$ and $m \in H_x$. We conclude that $E = H_x$. $\square$

Lemma 3.50. *If D is a definable normal subgroup of G of finite index and that $\mathsf{dim}(D) \geq 2$, then D also acts definably primitively on X.*

Proof. Let M be a definable subgroup of D such that $D_x \leq M \leq D$, where $D_x := \mathsf{Stab}_D(x)$. Then either $\mathsf{dim}(M) = \mathsf{dim}(D_x) = n - 1$ or $\mathsf{dim}(M) = \mathsf{dim}(G)$.

If $\mathsf{dim}(M) = \mathsf{dim}(D) = \mathsf{dim}(G)$, then

$$\mathsf{dim}(x^M) = \mathsf{dim}(M/M_x) = \mathsf{dim}(M/M \cap D_x) \geq \mathsf{dim}(D/D_x) = 1.$$

Consider the right coset space of M in D. Assume $D = \bigcup_{i \in I} Md_i$ with $Md_i \neq Md_j$ for $i \neq j$. Let $\mathcal{E} := \{x^{Md_i} : i \in I\}$. We claim that $x^{Md_i} \cap x^{Md_j} = \emptyset$ for any $i \neq j$. Suppose $x^{Md_i} \cap x^{Md_j} \neq \emptyset$, then there are $m_i, m_j \in M$ with $x^{m_i d_i} = x^{m_j d_j}$. Therefore, $m_i d_i (d_j)^{-1} (m_j)^{-1} \in D_x$. As $D_x \leq M$, we get $d_i (d_j)^{-1} \in M$, hence $i = j$. Note that $\mathsf{dim}(x^{Md_i}) = \mathsf{dim}(x^M) = 1$ for all $i \in I$. By the (EX)-condition, I must be finite. Consequently, M has finite index in D, hence $[G : M] < \infty$. By Poincaré's Theorem, M contains a definable normal subgroup S of G which also has finite index in G. Therefore, $x^S = X$ and $x^M \supseteq x^S = X$. For any $d \in D$, there is $m \in M$ such that $x^d = x^m$. Thus, $dm^{-1} \in D_x \leq M$ and $d \in M$. Therefore, $D = M$.

Suppose $\mathsf{dim}(M) = \mathsf{dim}(D_x)$, then by Lemma 3.49, we get $M = D_x$. Therefore, D acts definably primitively on X. $\square$

Lemma 3.51. *Let $H = T \times T^{g_1} \times \cdots \times T^{g_m}$ be as above. Then $H = T$ and $C_G(H)$ is trivial. If (G, X) is an ultraproduct of finite structures, say $(G, X) = \prod_{i \in I} (G_i, X_i)/\mathcal{U}$, then $H = \prod_{i \in I} soc(G_i)/\mathcal{U}$ where $soc(G_i)$ is the socle of G_i.*

Proof. Consider $G_T := \{g \in G : T^g = T\}$. As $\{T, T^{g_1}, \ldots, T^{g_m}\}$ is permuted by G, the index of G_T in G is finite. By Schlichting's Theorem, there is a definable normal subgroup $G_0 := \bigcap_{g \in G} (G_T)^g$, which also has finite index in G. By definition, $H \leq G_0$. By Lemma 3.50, G_0 also acts definably primitively on X.

Note that T is normal in G_0. Consider $S := C_{G_0}(T)$. It is definable and normal in G_0. If S is non-trivial, then T and S centralize each other and both act transitively on X. Fix $x \in X$. For any $h \in T$, we have $\mathsf{Stab}_S(x^h) = (\mathsf{Stab}_S(x))^h = \mathsf{Stab}_S(x)$. Since $x^T = X$, we get $\mathsf{Stab}_S(x) = \{1\}$. Similarly, $\mathsf{Stab}_T(x) = \{1\}$. We conclude that both S and T act regularly on X. Therefore, T has dimension 1. By Fact 3.7(2), T has a definable broad finite-by-abelian normal subgroup. As T is definably simple, it is abelian, which contradicts Theorem 3.42.

Therefore, $C_{G_0}(T)$ is trivial and $H = T$. By the same reason, $C_G(H) = C_G(T)$ is also trivial. Suppose $(G, X) = \prod_{i\in I}(G_i, X_i)/\mathcal{U}$ then $H = \prod_{i\in I} H_i/\mathcal{U}$ where each H_i is a minimal normal subgroup in the finite group G_i. Suppose $\{D_i : i \in I\}$ is another collection of minimal normal subgroups of G_i such that $\{i \in I : D_i \neq H_i\} \in \mathcal{U}$. Then D_i and H_i centralize each other for all $D_i \neq H_i$. Therefore, $\prod_{i\in I} D_i/\mathcal{U} \leq C_G(H)$, which entails that $\prod_{i\in I} D_i/\mathcal{U}$ is trivial. Hence, $H = \prod_{i\in I} soc(G_i)/\mathcal{U}$. □

Now, we can finish our analysis of higher dimensional cases. We state here a result concerning finite simple groups.

Fact 3.52. ([EJMR11, the Claim in Lemma 5.15]) Let $G(q)$ be a group of Lie type (possibly twisted) over a finite field $\mathbb{F}_q$, with $G \neq \mathrm{PSL}_2(\mathbb{F}_q)$, and let $P(q)$ be a parabolic subgroup of $G(q)$. Then $|G(q) : P(q)| > O(q)$.

Theorem 3.53. *Let (G, X) be a pseudofinite definably primitive permutation group satisfies the following conditions:*

1. *there is an additive integer-valued dimension on (G, X) with $\mathsf{dim}(X) = 1$ and $\mathsf{dim}(G) \geq 3$;*
2. *G and its definable sections satisfy the $\widetilde{\mathfrak{M}}_c$-condition;*
3. *X satisfies the (EX)-condition;*
4. *(G, X) satisfies the $\widetilde{\mathfrak{M}}_s$-condition.*

Then $\mathsf{dim}(G) = 3$, there is a definable subgroup $s(G)$ and an interpretable pseudofinite field F of dimension 1 such that we can identify $X \cong \mathrm{PG}_1(F)$, $s(G) \cong \mathrm{PSL}_2(F)$ and $\mathrm{PSL}_2(F) \leq G \leq \mathrm{P\Gamma L}_2(F)$. Moreover, if $(G, X) = \prod_{i\in I}(G_i, X_i)$ is an ultraproduct of finite structures, then $s(G) := \prod_{i\in I} soc(G_i)/\mathcal{U}$.

Proof. Let $H_i := soc(G_i)$ and $H := \prod_{i\in I} soc(G_i)/\mathcal{U}$. By the lemmas above, we know that $H = s(G)$ is definable and H is a pseudofinite simple group. By the main theorem of [Wil95], there is $J \in \mathcal{U}$ such that H_j is a finite Chevalley group of a fixed Lie type and of fixed Lie rank n for all $j \in J$. Take $x = \prod_{i\in I} x_i/\mathcal{U} \in X$. By Lemma 3.48, the number of orbits of $(H_i)_{x_i}$ is bounded. Hence, we may apply [Sei74, Theorem 2]. It follows that there is $J' \in \mathcal{U}$ such that $J' \subseteq J$ and for all $i \in J'$ the following holds: there is a parabolic subgroup P_i of H_i and $x_i \in X_i$ such that $(H_i)_{x_i} \leq P_i$. Let P'_i be the maximal parabolic subgroup which contains P_i. Let $P := \prod_{i\in I} P'_i/\mathcal{U}$. By [DS11, Lemma 6.2], $P \lneq H$ is definable in the language of pure groups with parameters in H. Note that P is infinite as H is. Also note that $[H : P] = \infty$, since otherwise, H would have a definable normal subgroup of finite index, contradicting that H is a pseudofinite simple group.

By [Ryt07, Chapter 5], H is uniformly bi-interpretable with a pseudofinite field F or a pseudofinite difference field (F, σ). More precisely, there is $J \in \mathcal{U}$ such that the following holds:

- For all $j \in J$, we have H_j bi-interprets a finite field $\mathbb{F}_j$, and the bi-interpretation is uniform in j;
- For all $j \in J$, we have H_j bi-interprets a finite difference field $(\mathbb{F}_{2^{2k_i+1}}, \mathrm{Frob}_{2^{k_i}})$ for some k_i, where $\mathrm{Frob}_{2^{k_i}}$ is the map $x \mapsto x^{2^{k_i}}$, and the bi-interpretation is uniform in j;
- For all $j \in J$, we have H_j bi-interprets a finite difference field $(\mathbb{F}_{3^{2k_i+1}}, \mathrm{Frob}_{3^{k_i}})$ for some k_i, where $\mathrm{Frob}_{3^{k_i}}$ is the map $x \mapsto x^{3^{k_i}}$, and the bi-interpretation is uniform in j.

We may assume $F := \prod_{i\in I} \mathbb{F}_i/\mathcal{U}$ and $(F,\sigma) := \prod_{i\in I}(\mathbb{F}_{2^{2k_i+1}}, \mathrm{Frob}_{2^{k_i}})/\mathcal{U}$ or $(F,\sigma) := \prod_{i\in I}(\mathbb{F}_{3^{2k_i+1}}, \mathrm{Frob}_{3^{k_i}})/\mathcal{U}$.

By [Hru91, Corollary 3.1] and [Ryt07, Proposition 3.3.19], the theory of F or (F,σ) eliminates imaginaries after adding parameters for an elementary submodel. Since both P and H are interpretable in F or in (F,σ), so does the right-coset space $P\backslash H$. By elimination of imaginaries, we may suppose that $P\backslash H$ is a definable subset of F^m for some m.

Now we work in F or (F,σ). We denote the SU-rank in F or (F,σ) as SU_F. And we call a definable set defined in the language of (difference) rings with parameters in F as F-definable. Note that F is an ultraproduct of a one-dimensional asymptotic class by [CvdDM92] for a pure field, and so is (F,σ) by [Ryt07, Theorem 3.5.8]. Thus, $SU_F(F) = 1$.

We claim that for any infinite F-definable set $Y \subseteq F^n$, we have Y has positive dimension in (G,X).

Indeed, since Y is infinite, $SU_F(Y) \geq 1$. For $1 \leq i \leq m$, consider the projection π_i of F^n onto the i^{th} co-ordinate. There must be some i such that $\pi_i(Y)$ is an infinite set, i.e., $SU_F(\pi_i(Y)) \geq 1$. Since $SU_F(F) = 1$ and $\pi_i(Y) \subseteq F$, we get $SU_F(\pi_i(Y)) = 1$. By the Indecomposability Theorem, there is a definable subgroup B of F^+ such that $B \subseteq (\pm\pi_i(Y))^k$ for some k-fold sum of $\pm\pi_i(Y)$, and finitely many translates of B cover $\pi_i(Y)$. Hence, $SU_F(B) = SU_F(F^+) = 1$, and B has finite index in F^+. As $B \subseteq (\pm\pi_i(Y))^k$ we get $\mathsf{dim}(B) \leq k\mathsf{dim}(\pi_i(Y))$. Therefore,

$$\mathsf{dim}(Y) \geq \mathsf{dim}(\pi_i(Y)) \geq \frac{1}{k}\mathsf{dim}(B) = \frac{1}{k}\mathsf{dim}(F^+) \geq \frac{1}{k} > 0,$$

where the penultimate inequality is by the fact that $H \subseteq F^m$ for some $m \geq 1$ and $\mathsf{dim}(H) \neq 0$, hence $\mathsf{dim}(F) \geq 1$.

Therefore, $\mathsf{dim}(P\backslash H) \geq 1$ and $\mathsf{dim}(P) \geq 1$. Note that

$$\mathsf{dim}(P\backslash H) \leq \mathsf{dim}(H_x\backslash H) = \mathsf{dim}(x^H) = 1.$$

Hence, $1 \leq \mathsf{dim}(P) = \mathsf{dim}(H_x)$. And we get $\mathsf{dim}(H) \geq 2$. Since H is a definable normal subgroup of G, by Lemma 3.49, we get $P = H_x$.

Note that X is in definable bijection with $H_x\backslash H = P\backslash H$. As P is definable in the language of pure groups with parameters in H, the action of H on X is interpretable in H itself, hence also interpretable in F or (F,σ).

By elimination of imaginaries, we may assume X is definable subset of F^m. Consider $SU_F(X)$, i.e., $SU_F(P\backslash H)$. We claim that $SU_F(X) = 1$.

Recall that any infinite F-definable set has positive dimension. Therefore, any non-algebraic F-type can be completed to a (G, X)-type of positive dimension. Take a generic element $\bar{a} = (a_1, \ldots, a_m) \in F^m$ in X. Then there is some i such that $tp_F(a_i)$ is non-algebraic. Suppose towards a contradiction that $SU_F(X) \geq 2$. Then

$$2 \leq SU_F(\bar{a}) = SU_F(\bar{a}/a_i) + SU_F(a_i) = SU_F(\bar{a}/a_i) + 1.$$

We get $SU_F(\bar{a}/a_i) \geq 1$. By the claim above, we have $\mathsf{dim}(\bar{a}/a_i) \geq 1$ and $\mathsf{dim}(a_i) \geq 1$. By the additivity of dimension, $\mathsf{dim}(X) \geq \mathsf{dim}(\bar{a}) = \mathsf{dim}(\bar{a}/a_i) + \mathsf{dim}(a_i) \geq 2$, a contradiction. Therefore, $SU_F(X) = 1$.

We conclude that

$$SU_F(P\backslash H) = SU_F(X) = 1 = SU_F(F).$$

Recall that both F and (F, σ) is an ultraproduct of a one-dimensional asymptotic class. There is a nature notion of dimension that comes from counting for all definable sets, we denote this dimension as $\dim_F$. By the fact that $1 = SU_F(F) = \dim_F(F)$, we must have that SU_F and $\dim_F$ coincide for all definable sets. Therefore, by the definition of one dimensional asymptotic class, there is $r \in \mathbb{R}^{>0}$ such that

$$st.\left(\frac{|P\backslash H|}{|F|}\right) = r.$$

By Fact 3.52, we must have $H \cong PSL_2(F)$, and X is definably isomorphic to the projective space $PG_1(F)$.

Consider $C_G(H) \trianglelefteq G$. It is trivial by Lemma 3.51. Therefore, the action of G on H by conjugation is faithful.

As $H \cong \prod_{i \in I} PSL_2(\mathbb{F}_{q_i})/\mathcal{U}$ and the largest automorphism group of $PSL_2(\mathbb{F}_{q_i})$ is $P\Gamma L_2(\mathbb{F}_{q_i})$, we get $PSL_2(F) \leq G \leq P\Gamma L_2(F)$ where $P\Gamma L_2(F) = PGL_2(F) \rtimes Aut(F)$. $\square$

3.5 Permutation groups of infinite SU-rank

In this section, we treat the special case when (G, X) is supersimple of infinite SU-rank. It is a natural candidate where our classification can be applied. However, the main result of this section is negative. More precisely, we will show that all these groups of dimension greater or equal to 2 will have SU-rank 2 or 3. Hence, there are no interesting infinite SU-rank case.

By Example 3.1, Fact 3.6 and Fact 3.11, we can take the dimension as the coefficient of some term ω^α of the SU-rank and the $\widetilde{\mathfrak{M}}_c$ and $\widetilde{\mathfrak{M}}_s$-conditions always hold in supersimple theories. To apply our classification, it remains to show that when the dimension is greater or equal to 3, X satisfies the (EX)-condition with the assumption of supersimplicity.

Lemma 3.54. *Suppose $(G, X) \in \mathcal{S}$ and its theory is supersimple. Let A be a definable abelian normal subgroup of G and* $\mathrm{SU}(A) = \omega^\alpha + \beta$ *with $\beta < \omega^\alpha$. Then* $\mathrm{SU}(A) = \omega^\alpha$.

Proof. By Fact 0.33, A has a type-definable subgroup C of SU-rank ω^α unique up to commensurability. Since A is normal in G, for any $g \in G$ we have $C^g \leq A$. Then C and C^g are commensurable, as $\mathrm{SU}(C^g) = \omega^\alpha$ and $C^g \leq A$. By Fact 0.35, there is a definable group D with $C \leq D \leq A$ such that $\mathrm{SU}(D) = \omega^\alpha$. Since $C \cap C^g \leq D \cap D^g$ and $\mathrm{SU}(C \cap C^g) = \omega^\alpha = \mathrm{SU}(D) = \mathrm{SU}(D^g)$ for any $g \in G$, we get D and D^g are commensurable. By Schlichting's Theorem, we may assume D is normal in G. By definably primitivity $D = A$. Therefore, $\mathrm{SU}(A) = \mathrm{SU}(D) = \omega^\alpha$. □

Corollary 3.55. *Let (G, X) be a pseudofinite definably primitive permutation group whose theory is supersimple. Let $SU(G) = \omega^\alpha n + \gamma$ for some $\gamma < \omega^\alpha$. Suppose $n \geq 3$ and $SU(X) = \omega^\alpha + \beta$ for some $\beta < \omega^\alpha$. Then all the conditions in Theorem 3.53 are satisfied. Hence, there is an interpretable pseudofinite field F such that $X \cong GL_1(F)$ and*

$$\mathrm{PSL}_2(F) \leq G \leq \mathrm{P\Gamma L}_2(F).$$

Moreover, G is bi-interpretable with (F, B) where B is a group of automorphisms of F.

Proof. For any interpretable set S with $\mathrm{SU}(S) = \omega^\alpha k + \beta$ for some $\beta < \omega^\alpha$ and $k \geq 0$, we put $\mathsf{dim}(S) := k$. By Example 3.1, this is an additive integer-valued dimension. Moreover, by supersimplicity G and its definable sections satisfy the $\widetilde{\mathfrak{M}}_c$ and $\widetilde{\mathfrak{M}}_s$-conditions. We only need to check the (EX)-condition. Indeed, we claim that $\mathrm{SU}(X) = \omega^\alpha$. Hence, by the Lascar Inequality, X satisfies the (EX)-condition.

Claim 3.56. $\mathrm{SU}(X) = \omega^\alpha$.

Proof. We may assume $(G, X) = \prod_{i\in I}(G_i, X_i)/\mathcal{U}$ is an ultraproduct of finite structures. Let $H := \prod_{i\in I} H_i/\mathcal{U}$, where H_i is a nontrivial minimal normal subgroup of G_i. We distinguish two cases: H is abelian and H is non-abelian.

If H is abelian. Then by [Hem15, Theorem 3.3(1)] G has a definable finite-by-abelian normal subgroup $A \geq H$. By definably primitivity, A is abelian. By Lemma 3.26, A acts regularly on X. Since $\mathsf{dim}(X) = 1$, we know that $\mathrm{SU}(A) = \mathrm{SU}(X) = \omega^\alpha + \beta$ for some $\beta < \omega^\alpha$. By Lemma 3.54, $\mathrm{SU}(A) = \omega^\alpha$. Thus, $\mathrm{SU}(X) = \omega^\alpha$.

If H is non-abelian. Then H is definable and $H = T \times T^{g_1} \times \cdots \times T^{g_m}$ for some $m \geq 0$ by Lemma 3.39. As T is definable and simple, by Fact 0.34, $\mathrm{SU}(T) = \omega^\alpha k$, for some $k \geq 1$. Therefore, $\mathrm{SU}(H) = \omega^\alpha k(m+1)$. Suppose $\mathrm{SU}(X) = \omega^\alpha + \beta$ with $\beta < \omega^\alpha$. By the Lascar Inequality (Fact 0.30), for any $x \in X$, we have

$$\mathrm{SU}(\mathsf{Stab}_H(x)) + \mathrm{SU}(x^H) \leq \mathrm{SU}(H) \leq \mathrm{SU}(\mathsf{Stab}_H(x)) \oplus \mathrm{SU}(x^H).$$

As $x^H = X$, we must have $\mathrm{SU}(\mathsf{Stab}_H(x)) = \omega^\alpha(km + k - 1) + \gamma$ for some $\gamma < \omega^\alpha$. Then

$$\omega^\alpha k(m+1) = \mathrm{SU}(H) \geq \mathrm{SU}(\mathsf{Stab}_H(x)) + \mathrm{SU}(x^H) = \omega^\alpha k(m+1) + \beta.$$

We deduce $\beta = 0$ and $\mathrm{SU}(X) = \omega^\alpha$. □

By Theorem 3.53 there is an interpretable pseudofinite field F such that $\mathrm{PSL}_2(F) \leq G \leq \mathrm{P\Gamma L}_2(F)$.

Now we prove that G is bi-interpretable with (F, B) where B is a group of automorphisms of F. We identify G with a group between $\mathrm{PSL}_2(F)$ and $\mathrm{P\Gamma L}_2(F)$ through definable isomorphism. Suppose (F, B) is given and $F = \prod_{i\in I} \mathbb{F}_{q_i}/\mathcal{U}$. As

$$\mathrm{P\Gamma L}_2(\mathbb{F}_{q_i}) = \mathrm{PGL}_2(\mathbb{F}_{q_i}) \rtimes Gal(\mathbb{F}_{q_i}/\mathbb{F}_{p_i})$$

where $p_i = \mathrm{char}(\mathbb{F}_{q_i})$ and $[\mathrm{PGL}_2(\mathbb{F}_{q_i}) : \mathrm{PSL}_2(\mathbb{F}_{q_i}))] \leq 2$ for any $i \in I$, we have either $G := \big(\prod_{i\in I} \mathrm{PSL}_2(\mathbb{F}_{q_i})/\mathcal{U}\big) \rtimes B$ or $G := \big(\prod_{i\in I} \mathrm{PGL}_2(\mathbb{F}_{q_i})/\mathcal{U}\big) \rtimes B$. Clearly G is interpretable in (F, B) in both cases.

Suppose $G = H \rtimes B$ is given, where $B \leq Aut(F)$. By the argument before, G interprets F. Let $\varphi(g, x, y)$ be the formula expressing: $x, y \in F$ and

$$\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right]^g = \left[\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right],$$

where $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$ denotes the coset $\begin{pmatrix} a & b \\ c & d \end{pmatrix} F^\times$ in $PGL_2(F)$. Then $\varphi(g, F, F)$ is the graph of a partial function. Let $\xi(g)$ be the formula expressing that $\varphi(g, F, F)$ is the graph of a field automorphism of F. Define $\phi(g, x, y) := \varphi(g, x, y) \wedge \xi(g)$ and $\sim$ be the equivalence relation on $G\times F\times F$ defined as $(g, x, y) \sim (g', x', y')$ if and only if $x = x', y = y'$ and $\varphi(g, F, F) = \varphi(g', F, F)$. Then $\phi(G, F, F)/\sim$ is a group of automorphisms of F containing B. We need to show that $\phi(G, F, F)/\sim$ contains no other automorphisms. Note that $\xi(G)$ defines a subgroup of G. Then $\xi(G) \cap H = \xi(H) \leq G$. Let $\sim_H$ be the equivalence relation such that $g \sim_H g'$ if and only if $\varphi(g, F, F) = \varphi(g', F, F)$. Then $\xi(H)/\sim_H$ is a group of automorphism of F. As H and $\xi(H)$ are interpretable in F, so does $\xi(H)/\sim_H$. We conclude $\xi(H)/\sim_H$ is trivial by the fact that a pure field can only interpret the trivial group of field-automorphisms of itself. Therefore $B = \phi(G, F, F)/\sim$.

□

In the following, we will exclude the possibility that B is infinite. This is due to Theorem 2.17.

Corollary 3.57. *Suppose $(F, B) = \prod_{i\in I}(\mathbb{F}_{p_i^{n_i}}, B_i)/\mathcal{U}$ is a pseudofinite structure with F a field and B an infinite set of automorphisms of F. Then the theory of (F, B) is not simple.*

Proof. Take a_i a generator of the multiplicative group of $\mathbb{F}_{p_i^{n_i}}$. Define $A_i = a_i^{B_i}$. As a_i is the generator and all B_i are powers of the Frobenius, we have $|A_i| = |B_i| \leq n_i$. Let $A = \prod_{i\in I} A_i/\mathcal{U}$. Then we can apply Theorem 2.17 to (F, A) and get the desired result. □

Combing the results above, we have the following conclusion.

Theorem 3.58. *Let (G, X) be a pseudofinite definably primitive permutation group whose theory is supersimple. Let $\mathrm{SU}(G) = \omega^\alpha n + \gamma$ for some $\gamma < \omega^\alpha$ and $n \geq 1$. Suppose $\mathrm{SU}(X) = \omega^\alpha + \beta$ for some $\beta < \omega^\alpha$. Then one of the following holds:*

1. $\mathrm{SU}(G) = \omega^\alpha + \gamma$, *and there is a definable, divisible torsion-free or elementary abelian subgroup A of* SU*-rank* ω^α *which acts regularly on X.*

2. $\mathrm{SU}(G) = 2$, *and there is an interpretable pseudofinite field* F *of* SU-*rank* 1 *such that* $G \cong F^+ \rtimes D$ *where* D *has finite index in* $F^\times$.

3. $\mathrm{SU}(G) = 3$, *and there is an interpretable pseudofinite field* F *of* SU-*rank* 1 *such that* $G \cong \mathrm{PSL}_2(F)$ *or* $G \cong \mathrm{PGL}_2(F)$.

Proof. Let dim be defined as the coefficient of ω^α.

When $n = 1$, we apply Theorem 3.27 and get a definable normal abelian subgroup A of SU-rank greater than or equal to ω^α. By Lemma 3.54, we have $\mathrm{SU}(A) = \omega^\alpha$.

If $n = 2$, then by Theorem 3.35, there is an interpretable pseudofinite field F of dimension 1 such that G_x induces a group of automorphisms B on F. By Corollary 3.57, we know that B must be finite. Then by Corollary 3.37, G_x embeds into $F^\times$ and B is trivial. Since the SU-rank of $F^\times$ is a monomial, and $\mathsf{dim}(F) = \mathsf{dim}(G_x) = 1$, we get $\mathrm{SU}(G_x) = \mathrm{SU}(F^\times) = \omega^\alpha$. Therefore, G_x has finite index in $F^\times$. Suppose $[F^\times : G_x] = k$. Consider $(F^\times)^k = \{g^k : g \in F^\times\}$. As $F^\times = \prod_{i\in I} F_i^\times/\mathcal{U}$, there is $J \in \mathcal{U}$ such that F_i is cyclic for all $i \in J$ and $(F_i^\times)^k$ is the unique subgroup of index k. Therefore, $(F^\times)^k$ is also the unique definable subgroup of index k of $F^\times$. Thus, $G_x = (F^\times)^k$. Now (G, X) is definable in F, so (G, X) is supersimple of SU-rank 2.

If $n \geq 3$, then by Corollary 3.55, (G, X) is bi-interpretable with a pseudofinite field F together with a group of automorphisms B. By Corollary 3.57, B is finite, hence is trivial by Lemma 3.36. Therefore, $\mathrm{PSL}_2(F) \leq G \leq \mathrm{PGL}_2(F)$. For any finite field $\mathbb{F}_q$, we have $[\mathrm{PGL}_2(\mathbb{F}_q) : \mathrm{PSL}_2(\mathbb{F}_q)] \leq 2$. Hence, either $G \cong \mathrm{PSL}_2(F)$ or $G \cong \mathrm{PGL}_2(F)$. $\square$

Chapter 4

Schlichting's Theorem for Approximate Subgroups

4.1 Introduction

Schlichting's Theorem was first introduced in [Sch80] with the focus on the existence of normal subgroups.

Fact 4.1. (Schlichting's Theorem) Let G be a group and H be a subgroup. If there is some $n \in \mathbb{N}$ such that $[H : H \cap H^g] \leq n$ for all $g \in G$, then there is a normal subgroup N of G such that N is *commensurable* with H, that is, there is $n' \in \mathbb{N}$ with

$$\max\{[N : N \cap H], [H : H \cap N]\} < n'.$$

This theorem was rediscovered and generalized to commensurable subgroups permutated by some group of automorphisms by Bergman and Lenstra in [BL89]. It was further generalized to a wide class of structures including vector spaces, fields and sets by Wagner in [Wag98] with the right notion of commensurability in each case. The group case is the Fact 0.36.

Approximate subgroups are subsets in an ambient group which are almost stable under products. They have a certain subgroup-like behaviour. Although the formal definition was given in [Tao08] around 2008, approximate subgroups have been studied for more than fifty years, especially the case of sets of integers with small doubling in additive combinatorics. The study of general finite approximate subgroups has gained more attention since the work of Breuillard, Green and Tao around 2010 who gave a complete classification of finite approximate subgroups in [BGT12].

We recall the definition of an approximate subgroups.

Definition 4.2. Let $K \in \mathbb{N}$ be a parameter, G be a group and $A \subseteq G$. We say that A is a *K-approximate subgroup*, if

- $1 \in A$,
- A is symmetric: $A = A^{-1}$; and

- there is a set $X \subseteq G$ with $|X| \leq K$ such that $AA \subseteq XA$.

We can also consider a family of K-approximate subgroups which are uniformly "close" to each other and wonder if there is an invariant object. Here closeness is defined similar to the last requirement in the definition of approximate subgroups. More precisely:

Definition 4.3. Let G be an ambient group, X, Y approximate subgroups and $N \in \mathbb{N}$. We say X is N-*commensurable* with Y if there are $Z_0, Z_1 \subseteq G$ with $\max\{|Z_0|, |Z_1|\} \leq N$ such that $X \subseteq Z_0 Y$ and $Y \subseteq Z_1 X$.

A family $\mathcal{X}$ of approximate subgroups of G is called *uniformly* N-*commensurable* if X is N-commensurable with Y for all $X, Y \in \mathcal{X}$.

We call $\mathcal{X}$ a *uniform family of commensurable approximate subgroups* if there are $K, N \in \mathbb{N}$ such that $\mathcal{X}$ is a family of uniformly N-commensurable K-approximate subgroups.

Let $\mathcal{X}, \mathcal{Y}$ be uniform families of commensurable approximate subgroups and H be an approximate subgroup. We say $\mathcal{X}$ (or H) is commensurable with $\mathcal{Y}$, if one/any member of $\mathcal{X}$ (or H respectively) is commensurable with one/any member of $\mathcal{Y}$.

Thus, Schlichting's Theorem for approximate subgroups would state:

Theorem 4.4. *If $\mathcal{X}$ is a uniform family of commensurable approximate subgroups in an ambient group G, then there is an approximate subgroup $H \subseteq G$ such that H is commensurable with $\mathcal{X}$ and invariant under all automorphisms of G stabilizing $\mathcal{X}$ set-wise.*

We will prove this theorem in this chapter. Indeed, suppose $\mathcal{X}$ is a family of uniformly N-commensurable K-approximate subgroups. We will give an explicit construction of H which is a K_H-approximate subgroup N_H-commensurable with $\mathcal{X}$. Moreover, K_H and N_H only depends on K and N but not on $\mathcal{X}$. However, we cannot get an explicit bound on K_H and N_H based on K and N. In conclusion, we have the following:

Corollary 4.5. *Let K and N be two positive natural numbers, then there is $L \in \mathbb{N}$ such that for any family $\mathcal{X}$ of uniformly N-commensurable K-approximate subgroups, there is an L-approximate subgroup H which is L-commensurable with $\mathcal{X}$ and invariant under all automorphisms of G stabilizing $\mathcal{X}$ set-wise.*

4.2 Examples and preliminaries

Let us first look at some examples.

Example 4.1.
- *Consider rational numbers with addition $(\mathbb{Q}, +)$. Let*

$$X_m := [-m-1, -m] \cup \{0\} \cup [m, m+1] \subseteq \mathbb{Q}$$

for $m \in \mathbb{N}_{\geq 0}$. Put $\mathcal{X} := \{X_m : m \in \mathbb{N}_{\geq 0}\}$. It is easy to check that $\mathcal{X}$ is a family of uniformly 3-commensurable 5-approximate subgroups. Note that the group of automorphisms of $(\mathbb{Q}, +)$ is isomorphic to $\mathbb{Q}^\times$, and the only automorphism that stabilizes $\mathcal{X}$ set-wise is ± 1. Therefore, any $X_m \in \mathcal{X}$ is an approximate subgroup as required in Theorem 4.4. In particular the interval $[-1, 1]$ is.

- *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $(\mathbb{Q}^*, \leq^*) := \prod_{n\in\mathbb{N}}(\mathbb{Q}, \leq)/\mathcal{U}$ be the ultrapower. Let $\mathcal{E}$ be the set of infinitesimals together with 0, i.e.*

$$\mathcal{E} := \{\epsilon \in \mathbb{Q}^* : -\frac{1}{n} <^* \epsilon <^* \frac{1}{n},\ \textit{for all } n \in \mathbb{N}\}.$$

As $\mathcal{U}$ is non-principal, $\mathcal{E}$ is an infinite set. For $m, \epsilon, \eta \in \mathbb{Q}^*$ let

$$X_{m,\epsilon,\eta} := [-m-\epsilon-1, -m-\eta] \cup \{0\} \cup [m+\eta, m+\epsilon+1] \subseteq \mathbb{Q}^*.$$

Let $\mathcal{X} := \{X_{m,\epsilon,\eta} : m \in \mathbb{N}_{\geq 0}, \epsilon, \eta \in \mathcal{E}\}$. Then $\mathcal{X}$ is a family of uniformly 5-commensurable 5-approximate subgroups. Now for any $\epsilon \in \mathcal{E}$, the group automorphism σ_ϵ which maps x to $(1+\epsilon)\cdot x$ stabilizes $\mathcal{X}$ set-wise, however if $\epsilon \neq 0$, then no $X \in \mathcal{X}$ is invariant.

Claim 4.6. $I := \bigcup\{[-1-\epsilon, 1+\epsilon] : \epsilon \in \mathcal{E}\}$ is an approximate subgroup commensurable with $\mathcal{X}$ and is invariant under all automorphisms of $(\mathbb{Q}^*, +)$ which stabilise $\mathcal{X}$ set-wise.

Proof. It is easy to see that I is an approximate subgroup of $(\mathbb{Q}^*, +)$ commensurable with $\mathcal{X}$. Let σ be an automorphism of $(\mathbb{Q}^*, +)$ stabilizing $\mathcal{X}$. We claim that for any $\epsilon \in \mathcal{E}$, there is $\eta \in \mathcal{E}$ such that $\sigma([-1-\epsilon, 1+\epsilon]) = [-1-\eta, 1+\eta]$. Suppose not, then there are $m \in \mathbb{N}_{\geq 0}$ and $\eta', \epsilon' \in \mathcal{E}$ such that $m + \eta' > 0$ and $\sigma([-1-\epsilon, 1+\epsilon]) = X_{m,\epsilon',\eta'}$. Let $r \in [-1-\epsilon, 1+\epsilon]$ such that $\sigma(r) = m+\eta'$. Note that $\frac{r}{2} \in [-1-\epsilon, 1+\epsilon]$ and $\sigma(\frac{r}{2}) \in X_{m,\epsilon',\eta'}$. However, $\sigma(\frac{r}{2}) = \frac{\sigma(r)}{2} = \frac{m+\eta'}{2} \notin X_{m,\epsilon',\eta'}$, a contradiction. □

Before we go to the technical details, we want to explain briefly the idea of the proof of Theorem 4.4 first. Basically, we will follow the strategy of the group case, see [Wag98] or [Wag00, Theorem 4.2.4]. Given a uniform family of commensurable approximate subgroups $\mathcal{X}$, we will first build a semi-lattice by taking finite unions. We will associate each finite union with a commensurable approximate subgroup where we reverse the order of the lattice. Let $\mathcal{I}$ the family of approximate subgroups associated to finite unions. In the group case, one can find a unique minimal object in the lattice $\mathcal{I}$, hence get an invariant object. However, in the case of approximate subgroups, it is possible that the minimal object is the infimum of the whole lattice $\mathcal{I}$ and it is not clear that we have the control of the size of the infimum. It can be shown that $\mathcal{I}$ is also a uniform family of approximate subgroups and moreover, unlike $\mathcal{X}$, elements in $\mathcal{I}$ have large finite intersections. We therefore do a dual construction. Starting from $\mathcal{I}$, we build another family of approximate subgroups $\mathcal{Y}$ which is closed under finite unions. It turns out that $\mathcal{Y}$ is uniformly upper-bounded, thus $\bigcup \mathcal{Y}$ is the invariant object that we are looking for.

In the following, we will present some lemmas that are repeatedly used in the proof of Theorem 4.4. They are straightforward generalisations of classical results from additive combinatorics (for example Lemma 4.9 is from Rusza's covering lemma).

Lemma 4.7. *Let $\mathcal{X}$ be a family of uniformly N-commensurable K-approximate subgroups in an ambient group G. Let $T := \prod_{0\leq i<n} X_i$ with $X_i \in \mathcal{X}$ and $n \geq 1$. Then T is at most $(NK)^{n-1}N$-commensurable with X for any $X \in \mathcal{X}$.*

Proof. Fix $X \in \mathcal{X}$. By assumption, there are $N_0, K_0 \subseteq G$ with $|N_0| \leq N$ and $|K_0| \leq K$ such that $X_0 \subseteq N_0 X_1$ and $X_1 X_1 \subseteq K_0 X_1$, thus

$$\prod_{0 \leq i < n} X_i \subseteq N_0 K_0 \prod_{1 \leq i < n} X_i.$$

Similarly, there are $N_1, K_1, \ldots, N_{n-2}, K_{n-2} \subseteq G$ such that

$$\prod_{0 \leq i < n} X_i \subseteq \left(\prod_{0 \leq i < n-1} N_i K_i \right) X_{n-1}.$$

By assumption $X_{n-1} \leq N_{n-1} X$ for some $|N_{n-1}| \leq N$. Therefore,

$$T = \prod_{0 \leq i < n} X_i \subseteq \left(\prod_{0 \leq i < n-1} N_i K_i \right) N_{n-1} X.$$

We have $\left| \left(\prod_{0 \leq i < n-1} N_i K_i \right) N_{n-1} \right| \leq (NK)^{n-1} N$.

On the other hand, as X is N-commensurable with $X_0 \subseteq T$, there is some Z with $|Z| \leq N$ such that $X \subseteq Z X_0 \subseteq ZT$. Hence, T is $(NK)^{n-1}N$-commensurable with X. □

Lemma 4.8. *Let G be a group and $X, Y \subseteq G$. Suppose $Y^{-1} = Y$ and there is a finite set $Z \subseteq G$ such that $X \subseteq ZY$. Let $X_0 \subseteq X$ be maximal such that $\{x_0 Y : x_0 \in X_0\}$ are disjoint. Then $|X_0| \leq |Z|$.*

Proof. Suppose, towards a contradiction, that $|X_0| > |Z|$. Then there are $x_i, x_j \in X_0$ and $z \in Z$ such that $x_i \in zY$ and $x_j \in zY$. Now we can see that $z \in x_i Y^{-1} = x_i Y$ and $z \in x_j Y^{-1} = x_j Y$, contradicting that $x_i Y \cap x_j Y = \emptyset$. □

Lemma 4.9. *Let G be a group and X, Y be N-commensurable K-approximate subgroups. Then there is some $E \subseteq G$ such that $|E| \leq KN$ and $XX \subseteq E(XX \cap YY)$.*

Proof. By definition, there is $Z_0 \subseteq G$ with $|Z_0| \leq N$ such that $X \subseteq Z_0 Y$. Let $X_0 \subseteq X$ be maximal such that $\{x_0 Y : x_0 \in X_0\}$ are disjoint. Then by Lemma 4.8 we have $|X_0| \leq |Z_0| \leq N$.

As $\{x_0 Y : x_0 \in X_0\}$ is maximal disjoint, for any $x \in X$ we have $xY \cap X_0 Y \neq \emptyset$, whence $x \in X_0 Y Y^{-1} = X_0 Y Y$. Therefore, $X \subseteq X_0 Y Y$. Note that

$$\begin{aligned} X = X_0 YY \cap X = \bigcup_{x \in X_0} (xYY \cap X) = \bigcup_{x \in X_0} (xYY \cap xx^{-1}X) \\ \subseteq \bigcup_{x \in X_0} (xYY \cap xXX) = \bigcup_{x \in X_0} x(YY \cap XX) = X_0(XX \cap YY). \end{aligned}$$

By assumption, there is some $X_1 \in G$ with $|X_1| \leq K$ and $XX \subseteq X_1 X$. Therefore, $XX \subseteq X_1 X \subseteq X_1 X_0 (XX \cap YY)$. Let $E := X_1 X_0$. Then $|E| \leq KN$ and $XX \subseteq E(XX \cap YY)$. □

4.3 Proof of the main theorem

We now proceed to the proof of Theorem 4.4.

Let G and $\mathcal{X}$ be given as in Theorem 4.4. We may assume that $\mathcal{X}$ is a family of uniformly N-commensurable K-approximate subgroups.

We define two new families. Let $\mathcal{X}^2 := \{XX : X \in \mathcal{X}\}$ and

$$\mathcal{Z} := \left\{ \bigcup_{i \in I} X_i : X_i \in \mathcal{X}^2, I \text{ finite.} \right\}$$

Remark: It is easy to see that $\mathcal{X}^2$ is a family of uniformly NK-commensurable family of K^3-approximate subgroups. Moreover, $\mathcal{X}^2$ is commensurable with $\mathcal{X}$.

Definition 4.10. Let $X, Y \subseteq G$. Define

$$[X : Y] := \max\{|X_0| : 1 \in X_0 \subseteq X \text{ and } \{xY : x \in X_0\} \text{ are disjoint.}\}$$

Notation: for $X \subseteq G$, we write X^k for the k-fold product of X.

Fix k and $Z = \bigcup_{i \in I} X_i \in \mathcal{Z}$. Let $X \in \mathcal{X}^2$. By Lemma 4.9 we have

$$X \subseteq E(X \cap X_i) \subseteq E(X \cap Z) \subseteq E(X \cap Z)^{2^k},$$

for some $i \in I$ and $|E| \leq KN$. Hence $[X : (X \cap Z)^{2^k}] \leq KN$ by Lemma 4.8. Therefore, $\max\{[X : (X \cap Z)^{2^k}] : X \in \mathcal{X}^2\}$ exists. Note that $\max\{[X : (X \cap Z)^{2^k}] : X \in \mathcal{X}^2\}$ decreases when k increases. Therefore, $\min_{k \in \mathbb{N}} \max\{[X : (X \cap Z)^{2^k}] : X \in \mathcal{X}^2\}$ exists and there is a minimal k_Z such that $\max\{[X : (X \cap Z)^{2^{k_Z}}] : X \in \mathcal{X}^2\}$ reaches this value. Let

$$m := \min_{Z \in \mathcal{Z}} \min_{k \in \mathbb{N}} \max\{[X : (X \cap Z)^{2^k}] : X \in \mathcal{X}^2\}.$$

Let

$$\mathcal{Z}_m := \{Z \in \mathcal{Z} : \min_{k \in \mathbb{N}} \max\{[X : (X \cap Z)^{2^k}] : X \in \mathcal{X}^2\} = m\}.$$

Then $\mathcal{Z}_m$ is non-empty. Moreover, for any $Z \subseteq Z' \in \mathcal{Z}$ if $Z \in \mathcal{Z}_m$, then

$$\max\{[X : (X \cap Z')^{2^{k_Z}}] : X \in \mathcal{X}^2\} \leq \max\{[X : (X \cap Z)^{2^{k_Z}}] : X \in \mathcal{X}^2\} = m. \tag{4.1}$$

Hence, $\min_{k \in \mathbb{N}} \max\{[X : (X \cap Z')^{2^k}] : X \in \mathcal{X}^2\} \leq m$, and they are equal by minimality of m. Thus, $Z' \in \mathcal{Z}_m$. We can also see from inequality (1) that $k_{Z'} \leq k_Z$.

Let $k_0 := \min\{k_Z : Z \in \mathcal{Z}_m\}$. We call $Z \in \mathcal{Z}_m$ *strong* if $k_Z = k_0$. It is easy to see that for Z and $Z' \in \mathcal{Z}$, if $Z' \supseteq Z$ and $Z \in \mathcal{Z}_m$ is strong, then so is Z'.

For strong Z, define

$$\eta(Z) := \{X \in \mathcal{X}^2 : [X : (X \cap Z)^{2^{k_0+1}}] = m\}$$

and

$$N(Z) := \bigcup_{X \in \eta(Z)} X \cap (X \cap Z)^{2^{k_0+1}}.$$

Lemma 4.11. *If $Z \subseteq Z'$ are both strong, then $N(Z) \supseteq N(Z')$.*

Proof. If $Z \subseteq Z'$ are both strong then $\eta(Z') \subseteq \eta(Z)$. Let $X \in \eta(Z')$ and $x_1 = 1, x_2, \ldots, x_m \in X$ be such that $\{x_i(X \cap Z')^{2^{k_0+1}} : i \leq m\}$ are disjoint. Note that $\{x_i(X \cap Z)^{2^{k_0}} : i \leq m\}$ are also disjoint. As $\max\{[X' : (X' \cap Z)^{2^{k_0}}] : X' \in \mathcal{X}^2\} = m$ by definition of k_0, we get $\{x_i(X \cap Z)^{2^{k_0}} : i \leq m\}$ is a maximal disjoint family in $\{x(X \cap Z)^{2^{k_0}} : x \in X\}$. Therefore,

$$X \subseteq \bigcup_{1 \leq i \leq m} x_i(X \cap Z)^{2^{k_0+1}} \subseteq \bigcup_{1 \leq i \leq m} x_i(X \cap Z')^{2^{k_0+1}}.$$

As $x_i(X \cap Z)^{2^{k_0+1}} \subseteq x_i(X \cap Z')^{2^{k_0+1}}$ for each $1 \leq i \leq m$ and $\{x_i(X \cap Z')^{2^{k_0+1}} : i \leq m\}$ are disjoint, we get

$$X \cap x_i(X \cap Z')^{2^{k_0+1}} = X \cap x_i(X \cap Z)^{2^{k_0+1}},$$

for each $i \leq m$. In particular, we have

$$X \cap (X \cap Z')^{2^{k_0+1}} = X \cap (X \cap Z)^{2^{k_0+1}}.$$

Therefore, $N(Z) \supseteq N(Z')$. □

Lemma 4.12. *Let $Z \in \mathcal{Z}$ be strong. Then $N(Z)$ covers any $X' \in \mathcal{X}^2$ with at most $(KN)^2$-translates.*

Proof. Suppose $Z = \bigcup_{i \leq n_Z} X_i$ where $X_i \in \mathcal{X}^2$. Note that $X \cap (X \cap Z)^{2^{k_0+1}} \supseteq X \cap X_0$ covers X by KN-translates for any $X \in \eta(Z)$. As $\mathcal{X}^2$ is KN-uniformly commensurable, $N(Z)$ covers any $X' \in \mathcal{X}^2$ with at most $(KN)^2$-translates. □

Lemma 4.13. *Let $Z_0, \ldots, Z_n$ be strong. Then $\bigcap_{i \leq n} N(Z_i) \supseteq N(\bigcup_{i \leq n} Z_i)$.*

Proof. By Lemma 4.11, $N(Z_i) \supseteq N(\bigcup_{i \leq n} Z_i)$ for each $i \leq n$. Thus the conclusion holds. □

For any $Z = \bigcup_{i \in I} Z_i \in \mathcal{Z}$, define $n(Z) = |I|$ (we regard $\mathcal{Z}$ as a formal family of finite unions of members in $\mathcal{X}^2$). Let $n_0 := \min\{n(Z) : Z \text{ strong.}\}$

Lemma 4.14. *Let Z_0 be strong and $n(Z_0) = n_0$. Then there is $N_Z \in \mathbb{N}$ depending on n_0, k_0, K and N such that $(Z_0)^{2^{k_0+1}}$ is N_Z-commensurable with any $X \in \mathcal{X}^2$, and $(Z_0)^{2^{k_0+2}}$ is $(N_Z)^2$-commensurable with any $X \in \mathcal{X}^2$.*

Proof. Suppose $Z_0 = \bigcup_{i \in I} X_i$ with $X_i \in \mathcal{X}^2$. Then

$$(Z_0)^{2^{k_0+1}} = \bigcup_{f\colon\, 2^{k_0+1} \to I} \ \prod_{i < 2^{k_0+1}} X_{f(i)}.$$

X is at most $(K^4N)^{2^{k_0+1}-1}KN$-commensurable with each $\prod_{i<2^{k_0+1}} X_{f(i)}$ by Lemma 4.7 and the remark before Definition 4.10. Therefore, X covers $(Z_0)^{2^{k_0+1}}$ with at most

$$N_Z := (n_0)^{2^{k_0+1}} \cdot K^{2^{k_0+3}+1} \cdot N^{2^{k_0+1}}$$

translates. As any $X_i \subseteq Z_0$ covers X with at most KN-translates, so does $(Z_0)^{2^{k_0+1}}$. Similarly, $(Z_0)^{2^{k_0+2}}$ is at most $(N_Z)^2$-commensurable with any $X \in \mathcal{X}^2$. □

Define

$$\mathcal{I} := \{N(Z) : Z \text{ strong and there is } Z' \subseteq Z \text{ with } Z' \text{ strong and } n(Z') = n_0\},$$

and define a subclass

$$\mathcal{I}' := \{N(Z) : Z \text{ strong and } n(Z) = n_0\}.$$

Lemma 4.15. *$\mathcal{I}$ is a uniform family of commensurable approximate subgroups and is commensurable with $\mathcal{X}$.*

Proof. Note that any $N(Z) \in \mathcal{I}$ is symmetric and contains the identity. Moreover, as $Z \supseteq Z_0$ for some Z_0 strong and $n(Z_0) = n_0$, we get $N(Z) \subseteq N(Z_0) \subseteq (Z_0)^{2^{k_0+1}}$ is N_Z-commensurable with any $X \in \mathcal{X}^2$ by Lemma 4.14. Since $(Z_0)^{2^{k_0+2}}$ is $(N_Z)^2$-commensurable with any $X \in \mathcal{X}^2$ and $N(Z)$ covers X with at most $(KN)^2$-translates by Lemma 4.14 and Lemma 4.12, we get

$$N(Z)^2 \subseteq N(Z_0)^2 \subseteq (Z_0)^{2^{k_0+2}} \subseteq T_0 X \subseteq T_0 T_1 N(Z),$$

where $T_0, T_1 \subseteq G$ with $|T_0| \leq (N_Z)^2$ and $|T_1| \leq (KN)^2$. Therefore, $N(Z)$ are $(N_Z KN)^2$-approximate subgroups.

If $N(Z') \in \mathcal{I}$, then by $(Z_0)^{2^{k_0+1}}$ is N_Z-commensurable with any $X \in \mathcal{X}^2$ and $N(Z')$ covers X by $(KN)^2$-translates, we get

$$N(Z) \subseteq N(Z_0) \subseteq (Z_0)^{2^{k_0+1}} \subseteq T_0' X \subseteq T_0' T_1' N(Z')$$

for some $|T_0'| \leq N_Z$ and $|T_1'| \leq (KN)^2$.

We conclude that $\mathcal{I}$ is a family of uniformly $N_Z(KN)^2$-commensurable $(N_Z KN)^2$-approximate subgroups.

By the above argument, we know that $N(Z_0)$ is N_Z-commensurable with any $X \in \mathcal{X}^2$. Hence $\mathcal{I}$ is commensurable with $\mathcal{X}^2$. As $\mathcal{X}^2$ is commensurable with $\mathcal{X}$, we get $\mathcal{I}$ is commensurable with $\mathcal{X}$. □

Note that $\mathcal{I}$ is also invariant under all automorphisms of G stabilizing $\mathcal{X}$ set-wise.

If $\mathcal{I}$ has a unique minimal element H, then H is commensurable with any $X \in \mathcal{X}$ and invariant under all automorphisms stabilizing $\mathcal{X}$ set-wise. And the proof is done.

Otherwise, we do a dual construction with the family $\mathcal{I}$ to get another family of uniformly commensurable approximate subgroups which is closed under finite unions.

As $\mathcal{I}$ is uniformly $N_Z(KN)^2$-commensurable, we get $[I : J] \leq N_Z(KN)^2$ for all $I, J \in \mathcal{I}$ by Lemma 4.8. Define

$$m' := \min_{I \in \mathcal{I}} \max\{[I : J] : J \in \mathcal{I}'\},$$

and

$$\mathcal{I}_{m'} := \{I \in \mathcal{I} : \max\{[I : J] : J \in \mathcal{I}'\} = m'\}.$$

If $I \subseteq I'$ with $I' \in \mathcal{I}_{m'}$ and $I \in \mathcal{I}$, then

$$\max\{[I : J] : J \in \mathcal{I}'\} \leq \max\{[I' : J] : J \in \mathcal{I}'\} = m'.$$

By minimality of m', we get $\max\{[I : J] : J \in \mathcal{I}'\} = m'$. Hence, $I \in \mathcal{I}_{m'}$.

Fix $I \in \mathcal{I}_{m'}$. Let $T \in \mathcal{I}'$ such that $[I : T] = m'$. Let $\{x_1T, \ldots, x_{m'}T\}$ be a maximal disjoint family in $\{iT : i \in I\}$. For any $J \supseteq I$ and $J \in \mathcal{I}_{m'}$, we have $\{x_1T, \ldots, x_{m'}T\}$ must also be maximal disjoint in $\{jT : j \in J\}$. Therefore, $J \subseteq \bigcup_{1\leq i\leq m'} x_iT^2$ and

$$\bigcup\{J \supseteq I,\ J \in \mathcal{I}_{m'}\} \subseteq \bigcup_{1\leq i\leq m'} x_iT^2.$$

Let

$$\mathcal{Y} := \{\bigcup_{i\leq n} J_i : J_i \in \mathcal{I}_{m'} \text{ and } n \in \mathbb{N}\}.$$

For any $n \in \mathbb{N}$ and $J_0, \ldots, J_n \in \mathcal{I}_{m'}$, there is some $I \in \mathcal{I}$ such that $\bigcap_{i\leq n} J_i \supseteq I$ by Lemma 4.13. As $J_i \in \mathcal{I}_{m'}$ we have $I \in \mathcal{I}_{m'}$. Therefore, $\bigcup_{i\leq n} J_i \subseteq \bigcup\{J \supseteq I,\ J \in \mathcal{I}_{m'}\}$.

Lemma 4.16. *$\mathcal{Y}$ is a uniformly commensurable family and any $Y \in \mathcal{Y}$ is commensurable with $\mathcal{X}$.*

Proof. Let $Y, Y' \in \mathcal{Y}$. Suppose $Y = \bigcup_{i\leq n} J_i$ and $Y' = \bigcup_{i\leq n'} J'_i$. By the argument before, there are $I \in \mathcal{I}_{m'}$, $T \in \mathcal{I}'$ and $M \subseteq G$ with $|M| \leq m'$ such that

$$Y \subseteq \bigcup\{J \supseteq I,\ J \in \mathcal{I}_{m'}\} \subseteq MT^2.$$

As $\mathcal{I}$ is a family of uniformly $N_Z(KN)^2$-commensurable $(N_ZKN)^2$-approximate subgroups, $T \in \mathcal{I}' \subseteq \mathcal{I}$ and $J'_0 \in \mathcal{I}$, there are M_1, M_2 with $|M_1| \leq (N_ZKN)^2$ and $|M_2| \leq N_Z(KN)^2$ such that $T^2 \subseteq M_1T$ and $T \subseteq M_2J'_0$. Thus,

$$Y \subseteq MT^2 \subseteq MM_1T \subseteq MM_1M_2J'_0 \subseteq MM_1M_2(\bigcup_{i\leq n'} J'_i) = MM_1M_2Y'.$$

Let $N_Y := m'(N_Z)^3(NY)^4$. Then $\mathcal{Y}$ is uniformly N_Y-commensurable.

By the above argument, for any $\bigcup_{i\leq n} J_i = Y \in \mathcal{Y}$ there is $T \in \mathcal{I}' \subseteq \mathcal{I}$ such that Y is contained in $m'(N_ZKN)^2$-translates of T. As $J_i \in \mathcal{I}'$ is commensurable with T and $J_i \subseteq Y$, we get Y is commensurable with T. Hence, Y is commensurable with I. As $\mathcal{I}$ is commensurable with $\mathcal{X}$ by Lemma 4.15, we get Y is commensurable with $\mathcal{X}$. $\square$

Note that any $Y = \bigcup_{i\leq n} J_i \in \mathcal{Y}$ is symmetric and contains the identity. Moreover, as $\mathcal{I}$ is a family of uniformly $N_Z(KN)^2$-commensurable $(N_ZKN)^2$-approximate subgroups, we get

$$Y^2 = \bigcup_{i,j\leq n} J_iJ_j \subseteq \bigcup_{i,j\leq n} T_{ij}(J_j)^2 \subseteq \bigcup_{i,j\leq n} T_{ij}T_jJ_j \subseteq (\bigcup_{i,j\leq n} T_{ij}T_j)Y$$

where $|T_{ij}| \leq N_Z(KN)^2$ and $|T_j| \leq (N_Z KN)^2$ for $i, j \leq n$. Therefore, Y is an approximate subgroup. But we cannot deduce a uniform bound for any $Y \in \mathcal{Y}$ from the above argument.

We conclude that $\mathcal{Y}$ is a family of approximate subgroups which are uniformly commensurable and closed under finite unions.

For any $X = X^{-1} \subseteq G$ define $\langle X \rangle := \bigvee_{k \in \mathbb{N}} X^k$, the group generated by X.

Lemma 4.17. *There is no $N_Y + 1$-chain $\langle Y_0 \rangle \lneq \langle Y_1 \rangle \lneq \cdots \lneq \langle Y_{N_Y} \rangle$ with $Y_i \in \mathcal{Y}$.*

Proof. Suppose, towards a contradiction, that there is such a chain. Then for each $i < N_Y$, there is some $y_i \in Y_{i+1} \setminus \langle Y_i \rangle$. Therefore, $y_i \langle Y_i \rangle \cap \langle Y_i \rangle = \emptyset$. Let $y_{-1} := \mathrm{id}$. We claim that $\{y_i Y_0 : -1 \leq i < N_Y\}$ is a disjoint family. Indeed, for any $i < j$, we have $y_j \langle Y_j \rangle \cap \langle Y_j \rangle = \emptyset$ and $y_i Y_0 \subseteq \langle Y_{i+1} \rangle \subseteq \langle Y_j \rangle$. Therefore, $y_j Y_0 \cap y_i Y_0 = \emptyset$. By assumption, Y_0 should be N_Y-commensurable with $\bigcup_{i \leq N_Y} Y_i \in \mathcal{Y}$. This contradicts Lemma 4.8. □

By Lemma 4.17, the family $\{\langle Y \rangle : Y \in \mathcal{Y}\}$ has a maximal element $G_{max} := \langle Y_{max} \rangle$ for some $Y_{max} \in \mathcal{Y}$. By maximality, $G_{max} \supseteq \bigcup_{Y \in \mathcal{Y}} Y$.

Lemma 4.18. *There is some $n_1 \in \mathbb{N}$ such that $Y \subseteq (Y_{max})^{n_1}$ for all $Y \in \mathcal{Y}$.*

Proof. Suppose not, then there is some $Y_0 \in \mathcal{Y}$ and $a_0 \in Y_0$ such that $a_0 \notin Y_{max}$. As $G_{max} = \langle Y_{max} \rangle \supseteq Y_0$, there is ℓ_0 with $a_0 \in (Y_{max})^{\ell_0}$. By assumption, there is some $Y_1 \in \mathcal{Y}$ and $a_1 \in Y_1$ with $a_1 \notin (Y_{max})^{\ell_0 + 2}$. Since $Y_1 \subseteq \langle Y_{max} \rangle$, we have $a_1 \in (Y_{max})^{\ell_1}$ for some $\ell_1 > \ell_0 + 2$. Repeating this procedure, we get $(Y_i)_{0 \leq i \leq N_Y}$, $(a_i)_{0 \leq i \leq N_Y}$ and $\ell_0 < \ell_1 < \cdots < \ell_{N_Y}$ such that $Y_i \in \mathcal{Y}$ and $a_i \in Y_i$, and moreover: $a_i \in (Y_{max})^{\ell_i}$ and $a_i \notin (Y_{max})^{\ell_{i-1}+2}$.

Consider $\{a_i Y_{max} : 0 \leq i \leq N_Y\}$. For any $i < j$, if $a_i Y_{max} \cap a_j Y_{max} \neq \emptyset$, then $a_j \in a_i (Y_{max})^2$ since Y_{max} is closed under inverses. As $a_i \in (Y_{max})^{\ell_i}$, we get $a_j \in (Y_{max})^{\ell_i + 2} \subseteq (Y_{max})^{\ell_{j-1}+2}$, a contradiction. Therefore, $\{a_i Y_{max} : 0 \leq i \leq N_Y\}$ are disjoint. Let $Y' := \bigcup_{0 \leq i \leq N_Y} Y_i$, then $Y' \in \mathcal{Y}$ but is not N_Y-commensurable with Y_{max}, which contradicts our assumption. □

From now on we will consider a subfamily of $\mathcal{I}_{m'}$ which is invariant under all automorphisms of G stabilizing $\mathcal{X}$ set-wise.

Let

$$n_2 := \min\{n(Z) : N(Z) \in \mathcal{I}_{m'}\},$$

and

$$\mathcal{Y}' := \{N(Z) \in \mathcal{I}_{m'} : n(Z) = n_2\}.$$

Note that $\mathcal{Y}' \subseteq \mathcal{Y}$.

Let $H := \bigcup \mathcal{Y}' \subseteq \bigcup \mathcal{Y} \subseteq (Y_{max})^{n_1}$. Then H is invariant under all automorphisms stabilizing $\mathcal{X}$, since $\mathcal{Y}'$ is. Moreover, as Y_{max} is an approximate subgroup commensurable with any $X \in \mathcal{X}$, we get H is commensurable with $\mathcal{X}$. It is also an approximate subgroup as Y_{max} is. This ends the proof of Theorem 4.4.

4.4 Uniform bound

The aim of this section is to prove Corollary 4.5. The strategy is that if we assume the bound does not exist, then we can build a counter-example using ultraproducts. To do this, we need that the approximate subgroup H constructed from $\mathcal{X}$ in Theorem 4.4 is definable.

Lemma 4.19. *Let $\mathcal{L}$ be a first-order language contains the group language. Let $\mathcal{M}$ be an $\mathcal{L}$-structure expanding a group G. Suppose $\mathcal{X}$ is a uniform family of commensurable approximate subgroups in G and that $\mathcal{X}$ is uniformly definable in $\mathcal{M}$ by a formula $\phi(x;\bar{y})$. That is, $\mathcal{X} = \{\phi(G,\bar{b}) : \bar{b} \in \mathcal{M}^{|\bar{y}|}\}$. Let H be the invariant approximate subgroup obtained by Theorem 4.4. Then H is also definable by a formula $\psi_{\mathcal{X},\phi}(x)$.*

Proof. By assumption $\mathcal{X}$ is uniformly definable. Hence, so is $\mathcal{X}^2$, but neither are $\mathcal{Z}$ or $\mathcal{Z}_m$. However, knowing m, k_0 and n_0, the family of strong Z with $n(Z) = n_0$ is uniformly definable. Given m, k_0 and a strong Z, we have that $\eta(Z)$ is definable, hence $N(Z)$ is also definable. Therefore, $\mathcal{I}'$ is uniformly definable. Similarly, knowing m' and n_2 additionally, $\mathcal{Y}'$ is uniformly definable, thus H is definable by a formula $\varphi_{\mathcal{X},\phi}(x)$.

□

Remark:

- Unlike the case of groups, H is not obtained by finite operations, the defining formula for H should involve additional existential and universal quantifiers.
- By the same reason, if $\mathcal{X}$ is a type-definable family of (type-)definable approximate subgroups, then H is also type-definable.

Now we can prove the corollary.

Proof. (Proof of Corollary 4.5) Fix K and N. Suppose Corollary 4.5 fails. Then for any $n \in \mathbb{N}$, there is a group G_n and a family of uniformly N-commensurable K-approximate subgroups $\mathcal{X}_n$ such that there is no H which is n-approximate subgroup n-commensurable with $\mathcal{X}_n$ invariant under all automorphisms stabilizing $\mathcal{X}_n$ set-wise.

Let $\mathcal{L}$ be the language $((G, 1, \cdot), I, R)$ which contains two sorts G and I and a relation $R \subseteq G \times I$ where G is equipped with a group language. We interpret $(G_n, \mathcal{X}_n)$ as $\mathcal{L}$-structures by:

- Interpret the first sort as G_n with the group operation;
- Let I_n be an index set such that there is a bijection $\tau : I_n \to \mathcal{X}_n$. Interpret the second sort as I_n and $R : G_n \times I_n$ as $R(g,i)$ if and only if $g \in \tau(i)$.

Let $\mathcal{U}$ be a non-principal ultrafilter over $\mathbb{N}$ and let $(G, \mathcal{X}) := \prod_{n\in\mathbb{N}}(G_n, \mathcal{X}_n)/\mathcal{U}$ be the ultraproduct of $\{(G_n, \mathcal{X}_n) : n \in \mathbb{N}\}$ (seen as $\mathcal{L}$-structures) along $\mathcal{U}$. Now it is easy to check that $\mathcal{X}$ is a family of uniformly N-commensurable K-approximate subgroups in G, and $\mathcal{X}$ is uniformly definable by $R(x,i)$. By Theorem 4.4, there is an L-approximate subgroup H that is N'-commensurable with $\mathcal{X}$ and invariant under all automorphisms stabilising $\mathcal{X}$ set-wise. By Lemma 4.19, H is definable. By Łos's Theorem H is an ultraproduct of $\{H_n : n \in \mathbb{N}\}$ along $\mathcal{U}$, and the set J defined as:

$$\{n \in \mathbb{N} : n > \max\{N', L\}, H_n \text{ is an } L\text{-approximate subgroup } N'\text{-commensurable with } \mathcal{X}_n\}$$

is in the ultrafilter $\mathcal{U}$. For any $n \in J$, as $n > \max\{N', L\}$, we have H_n is also an n-approximate subgroup n-commensurable with $\mathcal{X}_n$. Therefore, there is σ_n an automorphism of G_n which fixes $\mathcal{X}_n$ set-wise, but $\sigma_n(H_n) \neq H_n$. For $n \in \mathbb{N} \setminus J$ define $\sigma_n := id$, that is the identity automorphism on G_n. Let σ be the ultraproduct of $\{\sigma_n : n \in \mathbb{N}\}$ along $\mathcal{U}$. Then σ is an automorphism of G fixing $\mathcal{X}$ set-wise, but $\sigma(H) \neq H$, contradiction.

$\square$

Remark: If $\mathcal{X}$ is a family of uniformly N-commensurable finite K-approximate subgroups, then Theorem 4.4 holds as the trivial subgroup $\{\mathrm{id}\}$ is a witness. However, if the size of $X \in \mathcal{X}$ is large compared to N and K, then H we construct will also be of size comparable with $X \in \mathcal{X}$, and in particular non-trivial.

Appendix A

Pseudofinite $\widetilde{\mathfrak{M}}_C$-groups of dimension 2

In the following, we will present a proof that a pseudofinite $\widetilde{\mathfrak{M}}_c$-group of dimension 2, where the dimension is additive and integer-valued, has a finite-by-abelian subgroup of positive dimension whose normalizer is of dimension 2. The proof we present here is from [Wag15, Theorem 13], which does not use the CFSG.

Let dim be a dimension on a theory T and X a definable/interpretable set. Recall that we say X is *broad* if $\dim(X) > 0$. If $Y \subseteq X$ is definable/interpretable, we say Y is *wide in* X if $\dim(Y) = \dim(X)$.

Definition A.1. Let dim be an additive dimension on T. We say that tuple a is *independent* of b over a small set A, written as $a \mathop{\smile\!\!\!\!\!\mid}^{d}_{A} b$, if $\dim(a/A) = \dim(a/Ab)$.

Remark: If both $\dim(a/A)$ and $\dim(b/A)$ are finite, then additivity of dim will imply symmetry of $\mathop{\smile\!\!\!\!\!\mid}^{d}$, that is $a \mathop{\smile\!\!\!\!\!\mid}^{d}_{A} b \Leftrightarrow b \mathop{\smile\!\!\!\!\!\mid}^{d} a$.

Fact A.2. [Wag18, Theorem 4.9] Let G be a pseudofinite $\widetilde{\mathfrak{M}}_c$-group and dim an additive dimension on G. Then G has a definable broad finite-by-abelian subgroup. In fact, let C be any minimal broad centralizer (up to finite index) of a finite tuple. Then $\widetilde{Z}(C)$ is broad and finite-by-abelian.

Lemma A.3. *Let G be a pseudofinite $\widetilde{\mathfrak{M}}_c$-group and* dim *an additive integer-valued dimension on G with* $\dim(G) = 2$. *Suppose there is $b \in G$ with* $\dim(b) \geq 1$ *and* $\dim(C_G(b)) = 2$. *Then G has a normal definable finite-by-abelian subgroup D, and dim$(D) \geq 1$.*

Proof. Let $G_0 := \{g \in G : \dim(g^G) = 0\}$. Then G_0 is a definable characteristic subgroup of G by Lemma 3.24. Since $b \in G_0$, we get $\dim(G_0) \geq 1$. By the $\widetilde{\mathfrak{M}}_c$-condition, there are $b_0, \ldots, b_n \in G_0$ and $d \in \mathbb{N}$ such that if we define $T := C_{G_0}(b_0, \ldots, b_n)$, then $[T : C_T(g)] \leq d$ for all $g \in G_0$. Therefore $T = \widetilde{Z}(T)$. As $b_i \in G_0$ for all $0 \leq i \leq n$, we have $C_{G_0}(b_i)$ is wide in G_0. Thus, $\dim(T) = \dim(G_0) \geq 1$. Since $\{(C_{G_0}(b_0, \ldots, b_n, g) : g \in G_0\}$ is a uniformly commensurable definable family of subgroups of G_0, by Schlichting's theorem, there is a definable characteristic subgroup $N \leq G_0$, such that N is commensurable with T. Thus $D := \widetilde{Z}(N)$ is commensurable with $\widetilde{Z}(T) = T$. Note that D is normal in G and definable and finite-by-abelian as required. □

Theorem A.4. *Let G be a pseudofinite $\widetilde{\mathfrak{M}}_c$-group and* dim *an additive integer-valued dimension on G. If $\dim(G) = 2$ then G has a broad definable finite-by-abelian subgroup whose normalizer is wide.*

Proof. Let $A = \widetilde{Z}(C)$ be a broad finite-by-abelian subgroup of G, where C is a minimal broad centralizer up to finite index as in Fact A.2. If $\dim(C) = 2$, then $C \leq N_G(\widetilde{Z}(C))$, thus $A := \widetilde{Z}(C)$ is the group we are searching for. Hence, we may suppose that $\dim(C) = 1$, thus $1 \leq \dim(A) \leq \dim(C) = 1$.

We distinguish two cases. The first case is that A is commensurable with A^g for all $g \in G$. Then by Schlichting's theorem, there is a normal subgroup B of G such that B is commensurable with A. By Lemma 3.23, $\widetilde{Z}(B)$ is commensurable with $\widetilde{Z}(A) = \widetilde{Z}(\widetilde{Z}(C)) = \widetilde{Z}(C) = A$. Thus, $N := \widetilde{Z}(B)$ is a definable broad finite-by-abelian subgroup. Note that since B is normal in G and N is characteristic in B, we get that B is normal in G and we are done.

The second case is that there is some $g \in G$ such that A^g is not commensurable with A. Thus, C is not commensurable with C^g. As C is a minimal broad centralizer up to finite index, we get $\dim(C \cap C^g) = 0$. Therefore, $\dim(A \cap A^g) \leq \dim(C \cap C^g) = 0$. We conclude

$$\dim(A^g A) = \dim(A^g A/A) + \dim(A) = \dim(A^g/(A \cap A^g)) + \dim(A) = 2.$$

Take elements a, b_0 in A such that $\dim(a^g b_0/g) = 2$. Then we have

$$2 = \dim(a^g b_0/g) \leq \dim(a, b_0/g) \leq \dim(a/g, b_0) + \dim(b_0/g) \leq 1 + 1 = 2.$$

Thus, all inequalities are indeed equalities in the above equation and a, b_0 are wide in A and d-independent with each other over g. Let c_0 be wide in A and d-independent with a, b_0 over g. Then

$$2 \geq \dim(c_0 a^g b_0/g) \geq \dim(c_0 a^g b_0/g, c_0) = \dim(a^g b_0/g, c_0) = \dim(a^g b_0/g) = 2,$$

and $c_0 a^g b_0 \mathop{\smile\kern-0.6em\vert}^d_g c_0$. Similarly $c_0 a^g b_0 \mathop{\smile\kern-0.6em\vert}^d_g b_0$. Choose $d, b_1, c_1 \equiv_{g, c_0 a^g b_0} a, b_0, c_0$ such that $d, b_1, c_1 \mathop{\smile\kern-0.6em\vert}^d_{g, c_0 a^g b_0} a, b_0, c_0$. Then $c_1 d^g b_1 = c_0 a^g b_0$. Therefore, $c_0^{-1} c_1 d^g = a^g b_0 b_1^{-1}$. Let $b := b_0 b_1^{-1}$ and $c := c_0^{-1} c_1$. Then $b, c \in A$ and

$$\dim(b/a, g) \geq \dim(b/a, g, b_0, c_0) = \dim(b_1/a, g, b_0, c_0) = \dim(b_1/g, c_0 a^g b_0) = 1.$$

Therefore, b is wide in A over a, g and similarly, c is wide in A over d, g.

Since A is finite by abelian, t^A is finite for any $t \in A$. Thus, $\dim(C_A(t)) = \dim(A) - \dim(t^A) = \dim(A)$. We conclude that $E := C_A(a, b, c, d)$ is wide in A. Note that $\dim(E \cap E^g) \leq \dim(A \cap A^g) = 0$. Thus, we also have $\dim(E^g E) = 2$. Let x, y be in E such that $\dim(x^g y/a, b, c, d, g) = 2$. Then x and y are d-independent wide elements in E over a, b, c, d, g. Let $z := xgy$. Then

$$\dim(xgy/a, b, c, d, g) = \dim(x^g y/a, b, c, d, g) = 2$$

and

$$a^z b = a^{xgy} b = a^{gy} b = a^{gy} b^y = (a^g b)^y = (cd^g)^y = cd^{xgy} = cd^z.$$

Choose $z' \equiv_{a,b,c,d,g} z$ and $z' \mathop{\smile\!\!\!\!\!\!\!\;\;|}^{d}_{a,b,c,d,g} z$ and let $r := z'^{-1}z$. Then r is wide in G over a, b, c, d, g, z and

$$a^z b^r = a^{z'r} b^r = (a^{z'} b)^r = (cd^{z'})^r = c^r d^z.$$

We conclude that $c^{-1}a^z b = d^z = c^{-r}a^z b^r$ and $a^z bb^{-r} = cc^{-r}a^z$. Let $b' := bb^{-r}$ and $c' := cc^{-r}$. Then

$$a^z b' = c' a^z.$$

As $r \mathop{\smile\!\!\!\!\!\!\!\;\;|}^{d}_{g,a,b,c,d} z$ we get

$$\dim(z/a, b', c') \geq \dim(z/a, b, c, d, r, g) = \dim(z/a, b, c, d) = 2.$$

Take $z'' \equiv_{a,b',c'} z$ with $z'' \mathop{\smile\!\!\!\!\!\!\!\;\;|}^{d}_{a,b',c'} z$. Then $a^{z''} b' = c' a^{z''}$. Hence, $c' = (b')^{a^{-z}} = (b')^{a^{-z''}}$. Thus $a^{-z}a^{z''}$ commutes with b'. Let $a' := a^{-1}a^{z''z^{-1}}$. Then $(a')^z$ commutes with b'.

Claim A.5. Suppose t is a wide element in G over h and $\bar{c}$, where $h \in G$ and $\bar{c}$ is a finite tuple of elements in G. Then we may assume

$$\dim(h^t/h, \bar{c}) \geq 1.$$

If in addition $h \in A$, then we may assume $\dim(h^t/h, \bar{c}) = 1$.

Proof. Suppose $\dim(h^t/h, \bar{c}) = 0$, then

$$\begin{aligned}\dim(t/h^t, h, \bar{c}) &= \dim(t, h^t/h, \bar{c}) - \dim(h^t/h, \bar{c}) = \dim(t, h^t/h, \bar{c}) \\ &= \dim(t/h, \bar{c}) + \dim(h^t/h, t, \bar{c}) = \dim(t/h, \bar{c}) = 2\end{aligned}$$

Take $t' \equiv_{h^t,h,\bar{c}} t$ and $t' \mathop{\smile\!\!\!\!\!\!\!\;\;|}^{d}_{h^t,h,\bar{c}} t$, then $h^t = h^{t'}$. Thus $t't^{-1} \in C_G(h)$. Since

$$\begin{aligned}\dim(t't^{-1}/h, \bar{c}) &\geq \dim(t't^{-1}/h, \bar{c}, t, h^t) = \dim(t'/h, \bar{c}, t, h^t) \\ &= \dim(t'/h, \bar{c}, h^t) = \dim(t/h, \bar{c}, h^t) = 2,\end{aligned}$$

we get $C_G(h) = 2$. By Lemma A.3, G has a normal finite-by-abelian subgroup and we are done. Hence, we may suppose $\dim(h^t/h, \bar{c}) \geq 1$. If $h \in A$ and $\dim(h^t/h, \bar{c}) = 2$ then $\dim(h^G) \geq \dim(h^t/h) = 2$. Hence, $\mathsf{dim}(G/C_G(h)) = 2$ and $\mathsf{dim}(C_G(h)) = 0$. However, since $h \in A$ and A is broad finite-by-abelian, we have $\dim(C_G(h)) \geq \dim(C_A(h)) = 1$, a contradiction. $\square$

Thus, we may assume $\dim(b'/b) = \dim(b^{-r}/b) = 1$ and $\dim(a^{z''}/a, z, b', c') = 1$. Thus,

$$\dim(a'/a, z, b', c') = \dim(a^{z''}/a, z, b', c') = 1$$

and

$$\dim(a'/b', a, c') = \dim(a^{z''z^{-1}}/b', a, c') = 1$$

where the last equality comes from Claim A.5 since $\dim(z''z^{-1}/a, b', c') = 2$. We conclude,

$$\begin{aligned}\dim(z/a', b') &\geq \dim(z/a', b', a, c') = \dim(z, a'/b', a, c') - \dim(a'/b', a, c') \\ &= \dim(a'/z, b', a, c') + \dim(z/b', a, c') - \dim(a'/b', a, c') \\ &= \dim(z/b', a, c') = 2.\end{aligned}$$

Again by Claim A.5, as $\dim(z/a', b') = 2$, we get $\dim((a')^z/b', a') \geq 1$.

Note that $(a')^z \in C_G(b')$, hence $\dim(C_G(b')) \geq \dim((a')^z/b') \geq 1$. If $\dim(C_G(b')) = 2$, then we are done by Lemma A.3. Otherwise, $\dim(C_G(b')) = 1$. Choose $z^* \in G$ with $z^* \equiv_{a',b'} z$ and $z^* \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^d_{a',b'} z$. Since $(a')^z \in C_G(b')$, we have $(a')^{z^*} \in C_G(b')$. Let $h := (z^*)^{-1}z$. Then $(a')^z = (a')^{z^*h} \in C_G((b')^h)$, hence $(a')^z \in C_G(b', (b')^h)$. Since $z^* \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^d_{a',b'} z$ we have $(z^*)^{-1}z \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^d_{a',b'} z$ and $h \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^d_{a',b'} (a')^z$. Thus,

$$\dim(C_G(b', (b')^h)) \geq \dim((a')^z/a', b', h) = \dim((a')^z/a', b') \geq 1$$

and $\dim(C_G(b')/C_G(b', (b')^h)) = 0$.

By the $\widetilde{\mathfrak{M}}_c$-condition, there is a minimal broad centralizer (up to finite index) $C_G(b', \bar{c}) \leq C_G(b')$ with $\dim(C_G(b', \bar{c})) = \dim(C_G(b')) = 1$. Choose $\bar{c}' \equiv_{b'} \bar{c}$ such that $\bar{c}' \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^d_{b'} z, z^*$. Thus, $\bar{c}' \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^d_{b'} h$. Let $D := C_G(b', \bar{c}')$. Then D is also a minimal broad centralizer up to finite index by invariance of dim.

Since $\dim(C_G(b')) = \dim(D) = 1$ and $\dim(C_G(b')/C_G(b', (b')^h)) = 0$, we get

$$\begin{aligned}\dim(D/D \cap D^h) &= \dim(D/(D \cap C_G((b')^h))) + \dim((D \cap C_G((b')^h))/(D \cap D^h)) \\ &\leq \dim(C_G(b)/(C_G(b) \cap C_G((b')^h))) + \dim(C_G((b')^h)/D^h) = 0.\end{aligned}$$

We conclude that $\dim(D \cap D^h) = 1$ and D is commensurable with D^h as it is a minimal broad centralizer up to finite index. Note that since D is a minimal broad centralizer up to finite index, we have $\widetilde{N}_G(D) := \{g \in G : [D : D \cap D^g] < \infty\}$ is a definable subgroup of G and $h \in \widetilde{N}_G(D)$. As $h \mathop{\smile\hskip-0.9em\vert\hskip0.5em}^d_{b'} \bar{c}'$, we have

$$\dim(\widetilde{N}_G(D)) \geq \dim(h/b', \bar{c}') = \dim(h/b') \geq \dim(z^*/a', b', z) = \dim(z^*/a', b') = 2.$$

By definition and the $\widetilde{\mathfrak{M}}_c$-condition, the family $\{g \in \widetilde{N}_G(D) : D^g\}$ is a uniformly commensurable family. By Schlichting's theorem, there is a definable T characteristic in $\widetilde{N}_G(D)$ such that T is commensurable with D. Since $\widetilde{Z}(D)$ is broad and $\widetilde{Z}(T)$ is commensurable with $\widetilde{Z}(D)$ by Lemma 3.23, we get that $\widetilde{Z}(T)$ is a definable broad finite-by-abelian subgroup which is normal in $\widetilde{N}_G(D)$, and $\widetilde{Z}(T)$ is the group we are looking for. □

Remark: Throughout the proof, there are two cases for the finite-by-abelian group E whose normaliser is wide. The first one is that $E := \widetilde{Z}(C)$ where C is commensurable with a minimal broad centralizer up to finite index. And the second case is that E is normal in G.

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Structures pseudo-finies et dimensions de comptage

Résumé. Cette thèse porte sur la théorie des modèles des structures pseudo-finies en mettant l'accent sur les groupes et les corps. Le but est d'approfondir notre compréhension des interactions entre les dimensions de comptage pseudo-finies et les propriétés algébriques de leurs structures sous-jacentes, ainsi que de la classification de certaines classes de structures en fonction de leurs dimensions. Notre approche se fait par l'étude d'exemples. Nous avons examiné trois classes de structures. La première est la classe des H-structures, qui sont des expansions génériques. Nous avons donné une construction explicite de H-structures pseudo-finies comme ultraproduits de structures finies. Le deuxième exemple est la classe des corps aux différences finis. Nous avons étudié les propriétés de la dimension pseudo-finie grossière de cette classe. Nous avons montré qu'elle est définissable et prend des valeurs entières, et nous avons trouvé un lien partiel entre cette dimension et le degré de transcendance transformelle. Le troisième exemple est la classe des groupes de permutations primitifs pseudo-finis. Nous avons généralisé le théorème classique de classification de Hrushovski pour les groupes stables de permutations d'un ensemble fortement minimal au cas où une dimension abstraite existe, cas qui inclut à la fois les rangs classiques de la théorie des modèles et les dimensions de comptage pseudo-finies. Dans cette thèse, nous avons aussi généralisé le théorème de Schlichting aux sous-groupes approximatifs, en utilisant une notion de commensurabilité.

Mots-clés : structure pseudo-finie, dimension de comptage pseudo-finie, H-structure, corps aux différences pseudo-fini, groupe de permutations primitif, sous-groupe approximatif.

Pseudofinite structures and counting dimensions

Abstract. This thesis is about the model theory of pseudofinite structures with the focus on groups and fields. The aim is to deepen our understanding of how pseudofinite counting dimensions can interact with the algebraic properties of underlying structures and how we could classify certain classes of structures according to their counting dimensions. Our approach is by studying examples. We treat three classes of structures: The first one is the class of H-structures, which are generic expansions of existing structures. We give an explicit construction of pseudofinite H-structures as ultraproducts of finite structures. The second one is the class of finite difference fields. We study properties of coarse pseudofinite dimension in this class, show that it is definable and integer-valued and build a partial connection between this dimension and transformal transcendence degree. The third example is the class of pseudofinite primitive permutation groups. We generalise Hrushovski's classical classification theorem for stable permutation groups acting on a strongly minimal set to the case where there exists an abstract notion of dimension, which includes both the classical model theoretic ranks and pseudofinite counting dimensions. In this thesis, we also generalise Schlichting's theorem for groups to the case of approximate subgroups with a notion of commensurability.

Keywords: pseudofinite structure, pseudofinite counting dimension, H-structure, pseudofinite difference field, primitive permutation group, approximate subgroup.

Image en couverture : Grues de bon augure (Auspicious cranes)
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