l-adic,p-adic and geometric invariants in families of varieties.

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Thèse de doctorat

Alas! what are you, after all, my written and painted thoughts! Not long ago you were so variegated, young and malicious, so full of thorns and secret spices, that you made me sneeze and laugh- and now? You have already doffed your novelty, and some of you, I fear, are ready to become truths, so immortal do they look, so pathetically honest, so tedious!

Beyond Good and Evil
Friedrich Nietzsche

Introduction

This thesis is divided in two parts.

Part I is devoted to give a broad picture of the works presented in this thesis. It is divided in two chapters. Chapter 1 is of preliminary nature: we recall the tools that we will use in the rest of the thesis and some previously known results. There are two copy of Chapter 1: one in French and one in English. Chapter 2 is devoted to summarize in a uniform way the new results obtained in this thesis, trying to explain how they relate to each others.

Part II consists of 6 chapters, each of them corresponding to one of my papers, and of an appendix.

Chapter 3: A uniform open image theorem for ℓ-adic representations in positive characteristic ([Amb17]);

Chapter 4: Specialization of Néron-Severi groups in positive characteristic ([Amb18a]);

Chapter 5: Maximal tori in monodromy groups of F-isocrystals and applications ([AD18], joint with Marco D'Addezio);

Chapter 6: Specialization of p-adic monodromy groups ([Amb19b]);

Chapter 7: A note on the behaviour of the Tate conjecture under finitely generated field extension ([Amb18b]);

Chapter 8: Uniform boundedness of Brauer groups of forms ([Amb19a]);

Appendix A: Results on gonality.

We kept the introductory sections of each paper, so that the reader could read it without referring back to the previous chapters.

Specialization of ℓ-adic representations and Néron-Severi groups in positive characteristic

Chapters 3 and 4 are devoted to extend to positive characteristic the results of Cadoret-Tamagawa [CT12b] and of André in [And96]. Let k be a finitely generated field of characteristic $p > 0$ and $\ell \neq p$ a prime. Let X be a smooth, geometrically connected k-variety.

In Chapter 3, we consider the following problem: given a continuous representation $\rho : \pi_1(X) \to GL_r(\mathbb{Z}_\ell)$ of the étale fundamental group of X with image Π, study how the image Π_x of the local representation $\rho_x : \pi_1(Spec(k)) \to \pi_1(X) \to GL_r(\mathbb{Z}_\ell)$ induced by a k-rational point x varies with $x \in X(k)$. The main result is that if X is a curve and every open subgroup of $\rho(\pi_1(X_{\overline{k}}))$ has finite abelianization, then the set $X_\rho^{ex}(k)$ of $x \in X(k)$ such that Π_x is not open in Π is finite and there exists an integer $N \geq 0$, depending only on ρ, such that $[\Pi : \Pi_x] \leq N$ for all

$x \in X(k) - X_{\rho}^{ex}(k)$. This result can be applied to representations arising from geometry, to obtain uniform bounds for the ℓ-primary torsion of groups theoretic invariants in one dimensional families of smooth proper varieties. For example, torsion of abelian varieties and the Galois invariant part of the geometric Brauer group. This extends to positive characteristic previous results of Cadoret-Tamagawa ([CT12b]) and Cadoret-Charles ([CC18]) in characteristic 0.

In Chapter 4, we move to the specialization of Néron-Severi groups. The ℓ-adic Tate conjecture for divisors predicts that if $Y \to X$ is a smooth proper morphism, then the variation of the Picard rank of the fibres is controlled by group theoretic invariants. We show that this is indeed the case, without assuming the ℓ-adic Tate conjecture. Combining this with an ℓ-adic variant of the Hilbert irreducibility theorem and the result of Chapter 3, we deduce that if $f : Y \to X$ is a smooth proper morphism, then there are "lots" of closed points $x \in X$ such that the fibre of f at x has the same geometric Picard rank as the generic fibre and that, if X is a curve, this is true for all but finitely many k-rational points. In characteristic zero, these results have been proved by André (existence) and Cadoret-Tamagawa (finiteness) using Hodge theoretic methods. The starting point is to try and exploit the variational form of the crystalline variational Tate conjecture ([Mor15]). To do this, the main difficulty to overcome - and this is the main contribution of this chapter- is to compare crystalline local systems (F-isocrystals) with ℓ-adic lisse sheaves. Since the F-isocrystal $R^2 f_{crys,*}\mathcal{O}_{Y/K}(1)$ has a behaviour which is quite different from $R^2 f_* \mathbb{Q}_\ell(1)$, this comparison cannot be done directly. The idea is to show that $R^2 f_{crys,*}\mathcal{O}_{Y/K}(1)$ is coming from a smaller and better behaved category of p-adic local systems: the category of overconvergent F-isocrystals. As it has been understood that overconvergent F-isocrystals share many properties with lisse sheaves, the idea is to compare first $R^2 f_{crys,*}\mathcal{O}_{Y/K}(1)$ with its overconvergent incarnation $R^2 f_* \mathcal{O}^{\dagger}_{Y/K}(1)$ via various p-adic comparison theorems and then $R^2 f_* \mathcal{O}^{\dagger}_{Y/K}(1)$ with $R^2 f_* \mathbb{Q}_\ell(1)$ relating their monodromy groups via the theory of weights.

p-adic monodromy groups

Chapters 5 and 6 are devoted to study the monodromy groups of convergent and overconvergent F-isocrystals and their specialization theory.

Chapters 5 is a joint work with Marco D'Addezio. The arguments in Chapter 4 show that the category of F-isocrystals contains interesting p-adic information and that, sometimes, its pathological behaviour can be overcome relating it with the better behaved category of overconvergent F-isocrystals. This is the central topic of Chapter 5. Let X be a smooth geometrically connected variety defined over a finite field $\mathbb{F}_q$ and let $\mathcal{M}$ be an overconvergent F-isocrystal on X. To $\mathcal{M}$ we can associate a convergent F-isocrystal $\mathcal{M}^{conv}$, a overconvergent isocrystal $\mathcal{M}^{geo}$ and a convergent isocrystal $\mathcal{M}^{conv,geo}$. Each of these objects defines a monodromy group $G(-)$ and the main technical result of Chapter 5 is that $G(\mathcal{M}^{conv,geo})$ and $G(\mathcal{M}^{geo})$ have maximal tori of the same dimension. As an application we prove a special case of a conjecture of Kedlaya on homomorphisms of F-isocrystals. Using this special case, we prove that if A is an abelian variety without isotrivial geometric isogeny factors over a function field F over $\overline{\mathbb{F}}_p$, then the group $A(F^{\text{perf}})_{tors}$ is finite, where F^{perf} is the perfect closure of F. This may be regarded as an extension of the Lang–Néron theorem and answer positively to a question of Esnault.

In Chapter 6, we define a $\overline{\mathbb{Q}}_p$-linear category of overconvergent F-isocrystals and a $\overline{\mathbb{Q}}_p$-linear category of convergent F-isocrystals for varieties defined over an infinite finitely generated field k and the monodromy groups of their objects. Using the theory of companions, we relate the specialization theory of p-adic monodromy groups of overconvergent F-isocrystals to the specialization theory of ℓ-adic monodromy groups. This allows us to transfer the results of Chapter 3 to this new p-adic setting. In particular, if X is a curve, we prove that for all but finitely many $x \in X(k)$, the neutral component of the monodromy group of a geometrically semisimple overconvergent F-isocrystal $\mathcal{M}$ over X is the same as the neutral component of the monodromy group of the fibre of $\mathcal{M}$ at x. Under stronger assumptions, similar some result is given for convergent F-isocrystals.

Further results

The last two chapters are devoted to complements and variations on the topic of Chapters 3 and 4.

In Chapter 7, we study the behaviour of the Tate conjecture under finitely generated field extension. The results in Chapter 4 can be used to show that the ℓ-adic Tate conjecture for divisors on smooth proper varieties over finitely generated fields of characteristic $p > 0$ follows from the ℓ-adic Tate conjecture for divisors on smooth proper varieties over fields of transcendence degree 1 over $\mathbb{F}_p$. In Chapter 7, we show that one can further reduce the ℓ-adic Tate conjecture for divisors to finite fields: the ℓ-adic Tate conjecture for divisors on smooth proper varieties over finitely generated fields of positive characteristic follows directly from the ℓ-adic Tate conjecture for divisors on smooth projective surfaces over finite fields.

Chapter 8 is devoted to the study of Brauer groups of forms. If X is a smooth proper variety over a finitely generated field k of characteristic $p > 0$ satisfying the ℓ-adic Tate conjecture for divisors, it is well known that the Galois invariants $Br(X_{\overline{k}})[\ell^\infty]^{\pi_1(k)}$ part of the ℓ-primary torsion of the geometric Brauer group of X is finite. The results in Chapters 3 and 4, give uniform boundedness results for $|Br(X_{\overline{k}})[\ell^\infty]^{\pi_1(k)}|$ in one dimensional families of varieties. However, recent works of Cadoret, Hui and Tamagawa show that, if X satisfies the ℓ-adic Tate conjecture for divisors for every $\ell \neq p$, the Galois invariant $Br(X_{\overline{k}})[p']^{\pi_1(k)}$ part of the prime-to-p torsion of the geometric Brauer group of X is finite. The results in Chapter 3 are not sufficient to give uniform boundedness results for $|Br(X_{\overline{k}})[p']^{\pi_1(k)}|$. In Chapter 8, we give a few evidences that such boundedness results could hold: we prove that, for every integer $d \geq 1$, there exists an integer $N \geq 1$, depending only on X and d, such that for every finite extension of fields $k \subseteq k'$ with $[k' : k] \leq d$ and every $(\overline{k}/k')$-form Y of X one has $|(Br(Y \times_{k'} \overline{k})[p']^{\pi_1(k')}| \leq N$. The theorem is a consequence of general results for forms of compatible systems of $\pi_1(k)$-representations and it extends to positive characteristic a recent result of Orr and Skorobogatov in characteristic zero.

Appendix: results on gonality

For sake of completeness, in the appendix we generalize the main technical result of Chapter 3 from genus to gonality, following arguments of Cadoret and Tamagawa. This has some application to the study of p-adic representations and of not necessarely GLP ℓ-adic representations and it could be helpful for further developments.

Contents

Part I

Chapitre 1

Préliminaires (en Français)

1.1 Cadre absolu

Soit k un corps de caractéristique $p \geq 0$ et soit X une variété propre et lisse sur k.

L'objet principal de la géométrie arithmétique est l'étude des liens entre les propriétés arithmétiques et géométriques de X. Ces liens étant extrêmement riches et complexes, la stratégie générale développée au vingtième siècle est d'associé à X des groupes abéliens ou des espaces vectoriels munis de structures supplémentaires encodant en partie les propriétés de X. Par exemple :

- Le groupe de Chow $\mathrm{CH}^i(X)$ des cycles de codimension i de X à équivalence rationelle près ([Ful98]) ;
- Si $k = \mathbb{C}$, la cohomologie de Betti $H^i(X^{an}, \mathbb{Q})$ muni d'une structure de Hodge ([GH94]) ;
- Si k est un corps quelconque, pour tout $\ell \neq p$ la cohomologie étale ℓ-adique $H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ munie d'une action continue de $\pi_1(k)$ ([SGA4]) ;
- Si k est parfait et $p > 0$, la cohomologie cristalline $H^i_{crys}(X, K)$ munie d'une action du Frobenius absolu de k ([B078]).

La théorie des motifs ([And04, Section 4]) et les conjectures de plénitude ([And04, Section 7]) (comme par exemple la conjecture de Hodge ([Hod50]) et la conjecture de Tate ([Tat65])) donnent un cadre conjectural dans lequel comparer ces invariants. Dans cette section nous faisons quelques rappels sur ce sujet.

1.1.1 Cycles algébriques et motifs

1.1.1.1 Cycles algébriques

Soit L un anneau intègre de caractéristique zéro et soit $Z^i(X)$ le groupe abélien libre engendré par les sous-variétés intègres de codimension i de X. Soit $\sim$ une relation d'équivalence adéquate sur $Z^i(X)$ (c.f. [And04, Section 3.1]). On définit $\mathrm{CH}^i_L(X)_\sim$ comme étant le quotient de $Z^i(X) \otimes L$ par cette relation d'équivalence.

Si $\sim = rat$ est la relation d'équivalence rationnelle, alors $\mathrm{CH}^i(X) := \mathrm{CH}^i_{\mathbb{Z}}(X)_{rat}$ est appelé le groupe de Chow des cycles de codimension i modulo équivalence rationnelle. Si $L \subseteq L'$ est une inclusion d'anneaux alors $\mathrm{CH}^i_L(X)_{rat} \otimes_L L' \simeq \mathrm{CH}^i_{L'}(X)_{rat}$ ([And04, 3.2.2]). En général, les groupes $\mathrm{CH}^i(X)$ sont compliqués et de rang infini. Quand $i = 1$, le groupe $\mathrm{CH}^1(X)$ s'identifie au groupe de Picard $Pic(X)$ de X, qui classifie les fibrés en droites sur X à isomorphisme près.

Si $\sim = alg$ est la relation d'équivalence algébrique, alors $\mathrm{CH}^i_{alg}(X) := \mathrm{CH}^i_{\mathbb{Z}}(X)_{alg}$ est appelé le groupe de Chow des cycles de codimension i modulo équivalence algébrique. Si $L \subseteq L'$ est une inclusion d'anneaux, alors $\mathrm{CH}^i_L(X)_{alg} \otimes_L L' \simeq \mathrm{CH}^i_{L'}(X)_{alg}$ ([And04, 3.7.2.1]). Etant donné que la relation d'équivalence rationnelle est plus fine que la relation d'équivalence algébrique, il y a un morphisme quotient naturel $q : \mathrm{CH}^i_L(X)_{rat} \twoheadrightarrow \mathrm{CH}^i_L(X)_{alg}$, que quand $i = 1$, idenfie à le quotient naturelle $Pic(X) \otimes L \to NS(X) \otimes L$, où $NS(X) := Pic_X(k)/Pic^0_X(k)$ est le quotient des k-points du schéma de Picard Pic_X de X modulo les k-points de sa composante neutre Pic^0_X.

Si $\sim = num$ est la relation d'équivalence numérique, alors $\mathrm{CH}^i_{num}(X) := \mathrm{CH}^i_{\mathbb{Z}}(X)_{num}$ est appelé le groupe de Chow des cycles de codimension i modulo équivalence numérique. En général $\mathrm{CH}^i_L(X)_{num}$ est un L-module libre de type fini et si $L \subseteq L'$ est une inclusion d'anneaux, alors $\mathrm{CH}^i_L(X)_{num} \otimes_L L' \simeq \mathrm{CH}^i_{L'}(X)_{num}$ ([And04, 3.7.2.1]). Etant donné que la relation d'équivalence algebrique est plus fine que la relation d'équivalence numérique, il y a un morphisme quotient naturel $q : \mathrm{CH}^i_L(X)_{alg} \twoheadrightarrow \mathrm{CH}^i_L(X)_{num}$, que quand L est un corps, identifie ([Mat57], [And04, Section 3.2.7]) $NS(X) \otimes L$ avec $\mathrm{CH}^i_L(X)_{num}$.

Soit maintenant H^* une théorie cohomologique de Weil à coefficients dans un corps de caractéristique nulle $F \supseteq L$ (cf. [Saa72, Appendices] et [And04, Section 3.3]). Soit $c^i_H : \mathrm{CH}^i(X) \otimes L \to H^{2i}(X)(i)$ l'application classe de cycle associée à H^*. On définit le groupe des cycles de codimension i modulo H-équivalence homologique, $\mathrm{CH}^i_L(X)_H$, comme étant l'image de c_X : $\mathrm{CH}^i(X) \otimes L \to H^{2i}(X)(i)$. Si $L = F$, puisque $H^{2i}(X)(i)$ est de dimension finie sur F, $\mathrm{CH}^i_F(X)_H$ est un F-espace vectoriel de dimension finie. Il n'est pas vrai en général que la flèche naturelle $\mathrm{CH}^i_L(X)_H \otimes F \to \mathrm{CH}^i_F(X)_H$ est injective et on ne sait pas si $\mathrm{CH}^i_L(X)_H$ est finiment engendré sur L.

La relation d'équivalence algebrique étant plus fine que la relation d'équivalence homologique qui est elle même plus fine que la relation d'équivalence numérique, l'application quotient $q : \mathrm{CH}^i_L(X)_{alg} \twoheadrightarrow \mathrm{CH}^i_L(X)_{num}$ se factorise en la composition de $q_1 : \mathrm{CH}^i_L(X)_{alg} \twoheadrightarrow \mathrm{CH}^i_L(X)_H$ et $q_2 : \mathrm{CH}^i_L(X)_H \twoheadrightarrow \mathrm{CH}^i_L(X)_{num}$. Via q_2, $\mathrm{CH}^1_F(X)_H$ s'identifie ([And04, Proposition 3.4.6.1]) à $NS(X) \otimes F$. En général, une des conjectures standard de Grothendieck ([Kle94, Conjecture D]), prédit que $\mathrm{CH}^i_F(X)_H = \mathrm{CH}^i_F(X)_{num}$.

1.1.1.2 Motifs

On suppose maintenant que $L = F$. Pour $\sim \in \{num, H\}$, on note $\mathbf{Mot}^F_\sim(k)$ la catégorie F-linéaire pseudoabélienne tensorielle rigide des motifs purs à $\sim$-équivalence près([And04, Section 4.1.3]), $\mathbf{SPV}(k)$ la catégorie des variétés propres et lisses et $H^* : \mathbf{SPV}(k) \to \mathbf{Mot}^F_\sim(k)$ le foncteur canonique. Il existe un foncteur de réalisation $R_H : \mathbf{Mot}^F_H(k) \to \mathbf{GrVect}_F$ dans la catégorie des F-espaces vectoriels gradués. Par ailleurs Jannsen a prouvé ([Jan92]) que $\mathbf{Mot}^F_{num}(k)$ est une catégorie abélienne semi-simple.

En supposant les conjectures standard de Grothendieck ([Gro69]), on devrait pouvoir modifier la contrainte de commutativité dans $\mathbf{Mot}^F_H(k)$ (cf. [And04, Section 5.1.3]) afin d'obtenir un foncteur fibre $R_H : \mathbf{Mot}^F_H(k) \to \mathbf{Vect}_F$. En combinant [Jan92], les conjectures standard de Grothendieck et la conjecture $\mathrm{CH}^i_F(X)_H = \mathrm{CH}^i_F(X)_{num}$, la catégorie $\mathbf{Mot}^F_H(k)$ devrait être une catégorie Tannakienne F-linéaire semisimple ([Saa72]) munie d'un foncteur fibre $R_H : \mathbf{Mot}^F_H(k) \to \mathbf{Vect}_F$. Ainsi, pour tout $\mathcal{M}$ in $\mathbf{Mot}^F_H(k)$ on devrait pouvoir considérer la sous-catégorie Tannakienne $\langle\mathcal{M}\rangle \subseteq \mathbf{Mot}^F_H(k)$ engendrée par $\mathcal{M}$ et son groupe Tannakien réductif $G(\mathcal{M})$ ([And04, Section 6]).

On suppose maintenant que l'image essentielle de $R_H : \mathbf{Mot}^F_H(k) \to \mathbf{Vect}_F$ se factorise à travers une catégorie F-linéaire Tannakienne enrichie ([And04, Section 7.1.1]) $\mathcal{C}$ (par exemple la catégorie de structures de Hodge polarisées ou la catégorie des représentations continus $\mathbb{Q}_\ell$-linéaires de $\pi_1(k)$). Alors les conjectures de plénitudes ([Hod50], [Tat65], [And04, Section 7.1]) prédisent que l'image essentielle de $R_H : \mathbf{Mot}^F_H(k) \to \mathcal{C}$ est une sous-catégorie Tannakienne semi-simple de $\mathcal{C}$. Les groupes réductifs étant déterminés par leurs invariants tensoriels ([DM82, Proposition 3.1]), cela impliquerait que $G(\mathcal{M})$ s'identifie au groupe de Tannaka de la sous-catégorie Tannakienne $\langle R_H(\mathcal{M})\rangle \subseteq \mathcal{C}$ engendrée par $R_H(\mathcal{M})$.

Soit H' une autre théorie de cohomologie à coefficients dans $F \subseteq F'$ telle que $H' \otimes_F F' \simeq H$ en tant que cohomologies de Weil. Alors il existe un foncteur naturel $-\otimes F' : \mathbf{Mot}_H^F(k) \to \mathbf{Mot}_{H'}^{F'}(k)$. Les conjectures de plénitudes et de semi-simplicité pour H et H' en addition à l'équivalence des relations d'équivalences homologiques et numériques impliquent que, pour tout $\mathcal{M} \in \mathbf{Mot}_H^F(k)$, on a

$$G(R_{H'}(\mathcal{M}\otimes F')) \simeq G((H')^*(\mathcal{M}\otimes F')) \simeq G(H^*(\mathcal{M})\otimes F') \simeq G(R_H(\mathcal{M}))\otimes F'.$$

1.1.2 Cohomologie ℓ-adique

Dans cette section ℓ est un nombre premier différent de p.

1.1.2.1 Cohomologie étale et la conjecture de Tate

Pour tous les entiers $i \geq 0, j \in \mathbb{Z}$, Grothendieck a défini ([SGA4]) un groupe de cohomologie étale $H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j))$. C'est un $\mathbb{Q}_\ell$-espace vectoriel de dimension finie ([SGA4, XIV, Corollaire 1.2]) muni d'une action continue de $\pi_1(k)$ et l'image CH_ℓ^i de l'application classe de cycle $c_\ell^i : \mathrm{CH}^i(X) \to H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i))$ vit dans le sous-espace

$$\bigcup_{[k':k]<+\infty} H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k')}.$$

Dans ce cadre, la conjecture de plénitude est la conjecture de Tate ([Tat65]) qui prédit la relation suivante entre cycles algébriques et cohomologie.

Conjecture 1.1.2.1.1 ($\mathrm{T}(X,i,\ell)$). Si k est finiment engendré, alors l'application classe de cycle

$$c_\ell^i : \mathrm{CH}^i(X_{\overline{k}})\otimes \mathbb{Q}_\ell \to \bigcup_{[k':k]<+\infty} H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k)}$$

est surjective

La conjecture 1.1.2.1.1 est largement ouverte en général mais lorsque $i = 1$ elle est connue pour les variétés abéliennes ([Tat66], [Zar75], [Zar77], [FW84]), les surfaces K3 ([NO85], [Tan95], [And96a], [Char13], [MP15], [KMP15]) et quelques autres classe de variétés ; on pourra par exemple consulter [MP15, Section 5.13] et [Moo17].

1.1.2.2 Groupes de monodromie

L'action de $\pi_1(k)$ sur $H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j))$ induit un morphisme continu

$$\rho_\ell^{i,j} : \pi_1(k) \to \mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$$

et on pose $\Pi_\ell^{i,j} := \rho_\ell^{i,j}(\pi_1(k))$. Comme tout sous-groupe fermé de $\mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$, $\Pi_\ell^{i,j}$ est un groupe de Lie ℓ-adique compact ([Ser65, Lie Groups, Chapter V, Section 9]) et donc un presque pro-ℓ-groupe topologiquement finiment engendré ([DdSMSeg91]). Soit $G_\ell^{i,j}$ l'adhérence de Zariski de $\Pi_\ell^{i,j}$ dans $\mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$. Du point de vue Tannakien, si on écrit $\langle \rho_\ell^{i,j}\rangle$ pour la sous-catégorie Tannakienne de $\mathbf{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ engendrée par $\rho_\ell^{i,j}$, le groupe algébrique $G_\ell^{i,j}$ est caractérisé ([And04, Section 7.1.3]) par le fait que $Rep_{\mathbb{Q}_\ell}(G_\ell^{i,j}) \simeq \langle \rho_\ell^{i,j}\rangle$. Si $\rho_\ell^{i,j}$ est semi-simple alors $G_\ell^{i,j}$ peut aussi être décrit comme étant le sous-groupe de $\mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$ fixant $(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j))^{\otimes m} \otimes (H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j))^\vee)^{\otimes n})^{\pi_1(k)}$ pour tous les entiers $n, m \geq 0$.

1.1.3 Caractéristique nulle : cohomologie de Betti et théorie de Hodge

On suppose maintenant que $p = 0$ et, pour simplifier, qu'il existe une inclusion $k \subseteq \mathbb{C}$.

1.1.3.1 Cohomologie de Betti et conjecture de Hodge

On peut associer à $X_{\mathbb{C}}$ ([Ser56, Section 2], [SGA1, Exposé XII]) un espace analytique complexe $X_{\mathbb{C}}^{an}$ et donc considérer la cohomologie de Betti $H_B^i(X, \mathbb{Q}) := H^i(X_{\mathbb{C}}^{an}, \mathbb{Q})$ de X. La décomposition de Hodge ([Hod41], [GH94, Chapter 0]) donne un isomorphisme canonique

$$H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}) \otimes \mathbb{C} \simeq \bigoplus_{p+q=i} H^p(X_{\mathbb{C}}^{an}, \Omega^q_{X_{\mathbb{C}}^{an}}).$$

On en déduit que $H^i(X_{\mathbb{C}}^{an}, \mathbb{Q})$ est une $\mathbb{Q}$-structure de Hodge polarisée ([Moo04, Section 1]) que l'on peut tordre par $\mathbb{Q}(j)$, pour tout $j \in \mathbb{Z}$, afin d'obtenir $H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}(j))$. En composant l'application classe de cycle $\mathrm{CH}^i(X_{\mathbb{C}}^{an}) \otimes \mathbb{Q} \to H^{2i}(X_{\mathbb{C}}^{an}, \mathbb{Q}(i)) \subseteq H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}(i)) \otimes \mathbb{C}$, avec l'isomorphisme $\mathrm{CH}^i(X_{\mathbb{C}}) \simeq \mathrm{CH}^i(X_{\mathbb{C}}^{an})$ induit par le foncteur d'analytification, on obtient une application classe de cycle $c_B^i : \mathrm{CH}^i(X_{\mathbb{C}}) \otimes \mathbb{Q} \to H^{2i}(X_{\mathbb{C}}^{an}, \mathbb{Q}(i)) \subseteq H_B^n(X, \mathbb{Q}(i)) \otimes \mathbb{C}$ dont l'image est contenue dans $H^i(X_{\mathbb{C}}^{an}, \Omega^i_{X_{\mathbb{C}}^{an}}) \cap H_B^{2i}(X, \mathbb{Q}(i)) =: H_B^{2i}(X, \mathbb{Q}(i))^{0,0}$. Dans ce cadre, la conjecture de plénitude est la conjecture de Hodge ([Hod50]) :

Conjecture 1.1.3.1.1 ($H(X, i)$)**.** L'application classe de cycle

$$c_B^i : \mathrm{CH}^i(X_{\mathbb{C}}) \otimes \mathbb{Q} \to H_B^{2i}(X, \mathbb{Q}(i))^{0,0}$$

est surjective.

Contrairement à $\mathrm{T}(X, 1, \ell)$ qui est largement ouverte en général, on déduit de la suite exacte exponentielle ([GH94, Pag. 163]) et de la décomposition de Hodge le théorème de Lefschetz (1,1).

Fait 1.1.3.1.2 ([Lef24][GH94, Pag. 163-164])**.** La conjecture $H(X, 1)$ est vraie.

Remarque 1.1.3.1.3. Bien que la cohomologie ℓ-adique et la cohomologie de Betti soient conjecturalement des incarnations du même motif on voit déjà qu'elles ont des propriétés bien spécifiques : la cohomologie ℓ-adique nous permet d'utiliser la théorie des groupes de Lie ℓ-adiques et l'action de $\pi_1(k)$ alors que la cohomologie de Betti nous permet d'utiliser des techniques de théorie de Hodge analytique complexe. Des résultats de comparaisons entre elles devraient être utile pour combiner ces différentes informations.

1.1.3.2 Groupes de monodromie

La structure de Hodge sur $H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}(j))$ est décrite ([Moo04, Section 3]) par un morphisme de groupes algébriques

$$h_B^{i,j} : Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{GL}(H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}(j)) \otimes \mathbb{R}),$$

et le groupe de Mumford Tate $G_B^{i,j}$ est ([Moo04, Section 4]) le plus petit sous-groupe connexe de $\mathrm{GL}(H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}(j)))$ tel que $G_B^{i,j} \otimes \mathbb{R}$ contient $\mathrm{Im}(h_B^{i,j})$. Comme dans le cadre ℓ-adique, le groupe $G_B^{i,j}$ est caractérisé comme étant l'unique groupe algébrique (à isomorphisme près) tel que $Rep_{\mathbb{Q}}(G_B^{i,j})$ est équivalente à la sous-catégorie Tannakienne $\langle H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}) \rangle$ engendrée par $H^i(X_{\mathbb{C}}^{an}, \mathbb{Q})$ dans la catégorie des structures de Hodge polarisées. La catégorie des $\mathbb{Q}$-structures de Hodge polarisées étant semi-simple ([Moo04, Proposition 4.9]), $G_B^{i,j}$ est réductif. Il peut donc être décrit comme étant le sous-groupe de $\mathrm{GL}(H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}(j)))$ fixant toutes les (0,0)-classes dans $H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}(j))^{\otimes n} \otimes (H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}(j))^{\vee})^{\otimes m}$ pour tous les entiers $m, n \geq 0$.

1.1.3.3 Comparaison entre le site singulier et le site étale

En conséquence de l'invariance de la cohomologie étale par extensions de corps algébriquement clos de caractéristique nulle ([SGA4$\frac{1}{2}$, Corollaire 5.3.3]) et du théorème de comparaison d'Artin ([SGA4, XI, Theorem 4.4]), il existe un isomorphisme canonique

$$H^i(X_{\overline{k}}, \mathbb{Q}_\ell) \simeq H^i(X_{\mathbb{C}}, \mathbb{Q}_\ell) \simeq H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}) \otimes \mathbb{Q}_\ell,$$

qui s'inscrit dans le diagramme commutatif suivant

$$\begin{tikzcd}
(\mathrm{CH}^i(X_{\mathbb{C}}) \otimes \mathbb{Q}) \otimes \mathbb{Q}_\ell \arrow[r] \arrow[d, "c_B^i \otimes \mathbb{Q}_\ell"] & \mathrm{CH}^i(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[d, "c_\ell^i"] \\
H^{2i}(X_{\mathbb{C}}^{an}, \mathbb{Q}(i)) \otimes \mathbb{Q}_\ell \arrow[r, "\simeq"] & H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i)).
\end{tikzcd}$$

Le groupe de Mumford-Tate $G_B^{i,j} \subseteq \mathrm{GL}(H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}(j)))$ est envoyé, par le théorème de comparaison d'Artin, vers un $\mathbb{Q}_\ell$-groupe algébrique $G_B^{i,j} \otimes \mathbb{Q}_\ell \subseteq \mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$. En associant la philosophie des motifs (Section 1.1.1.2) aux conjectures 1.1.2.1.1 and 1.1.3.1.1 on obtient la conjecture de Mumford-Tate :

Conjecture 1.1.3.3.1. Si k est finiment engendré, $G_B^{i,j} \otimes \mathbb{Q}_\ell = (G_\ell^{i,j})^0$ modulo le théorème de comparaison d'Artin.

Bien que les conjectures 1.1.2.1.1 et 1.1.3.1.1 soient complètement ouvertes en général la conjecture 1.1.3.3.1 est connue dans certains cas (on pourra par exemple consulter [Pin98]). Si X est une variété abélienne Deligne a montré ([DM82]) qu'il y a une inclusion $(G_\ell^{i,j})^0 \subseteq G_B^{i,j} \otimes \mathbb{Q}_\ell$.

La conjecture 1.1.3.3.1 prédit aussi le résultat suivant.

Conjecture 1.1.3.3.2. Si k est finiment engendré, il existe un groupe algébrique connexe $G^{i,j}$ sur $\mathbb{Q}$ et une représentation fidèle $G \subseteq \mathrm{GL}(V^{i,j})$ tels que pour tout $\ell \neq p$ il y a un isomorphisme $V^{i,j} \otimes \mathbb{Q}_\ell \simeq H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j))$ qui identifie $G^{i,j} \otimes \mathbb{Q}_\ell \subseteq \mathrm{GL}(V^{i,j} \otimes \mathbb{Q}_\ell) \simeq \mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$ avec $(G_\ell^{i,j})^0$.

1.1.4 Caractéristique positive : cohomologie cristalline

On suppose maintenant $p > 0$ et que k est parfait. Soit $W := W(k)$ l'anneau des vecteurs de Witt de k et K son corps des fractions K. Dans cette section on rappelle les idées principales qui rentrent en jeu dans la construction d'une théorie cohomologique de Weil p-adique. Un exemple classique de Serre (voir par exemple [Gro68, Section 1.7]) montre qu'il n'existe pas de théories cohomologiques à coefficients dans $\mathbb{Q}_p$, il faut donc définir une théorie cohomologique à coefficients dans K.

1.1.4.1 Site infinitésimal

Soient S un schéma et $f : Z \to S$ un morphisme, on note $H^i_{dr}(Z/S)$ la cohomologie de de Rham relative de Z sur S ([Gro66], [MP12, Section 4.5]). Bien que $H^i_{dr}(Z/S)$ ait une description relativement concrète en terme de formes différentielles, Grothendieck a montré dans [Gro68] que, au moins en caractéristique nulle, elle peut aussi être définie via la théorie des topos. Pour ce faire il a défini ([Gro68, Section 4]) un site infinitésimal $Inf(Z/S)$ muni d'un topos de faisceaux en groupes abéliens $(Z/S)_{inf}$ et un faisceau structural $\mathcal{O}_{Z/S}$. On note $H^i_{inf}(Z/S, \mathcal{O}_{Z/S})$ le $i^{\text{ème}}$ groupe de cohomologie de $\mathcal{O}_{Z/S}$. Grothendieck a prouvé le résultat suivant.

Fait 1.1.4.1.1 ([Gro68, Theorem 4.1 and Section 5.3]).

1. Si $f : Z \to S$ est lisse et S est de caractéristique nulle, il y a un isomorphisme naturel
$$H^i_{dr}(Z/S) \simeq H^i_{inf}(Z/S, \mathcal{O}_{Z/S});$$
2. Si $Z' \to Z$ est un épaississement nilpotent de S-schémas, il y a un isomorphisme naturel
$$H^i_{inf}(Z'/S, \mathcal{O}_{Z'/S}) \simeq H^i_{inf}(Z/S, \mathcal{O}_{Z/S}).$$

Le fait 1.1.4.1.1 peut être utilisé pour montrer que la cohomologie de de Rham de la déformation d'une variété propre et lisse ne dépend que de la variété de départ.

Remarque 1.1.4.1.2 ([B078, Pag. 1.11])**.** Notons $S := \mathrm{Spec}(\mathbb{C}[[T]])$ et $S_n := \mathrm{Spec}(\mathbb{C}[[T]]/(T^n))$. Soit $Z \to S$ un morphisme propre et lisse, on pose $Z_n := Z \times_S S_n$. Comme f est propre, on a $H_{dr}(Z/S) \simeq \varprojlim_n H_{dr}(Z_n/S_n)$. Puisque $Z_n \to S_n$ est lisse et $Z_1 \to Z_n$ est un épaississement infinitésimal, par le fait 1.1.4.1.1 on a
$$H_{dr}(Z/S) \simeq \varprojlim_n H_{dr}(Z_n/S_n) \simeq \varprojlim_n H^i_{inf}(Z_n/S_n, \mathcal{O}_{Z_n/S_n}) \simeq \varprojlim_n H^i_{inf}(Z_1/S_n, \mathcal{O}_{Z_1/S_n}).$$
Cela montre que la cohomologie de de Rham relative de $Z \to S$ ne dépend que de Z_1.

1.1.4.2 Site cristallin

Au vu de la remarque 1.1.4.1.2, pour construire une théorie cohomologique à coefficients dans K pour les variétés sur k on pourrait essayer de relever X en un schéma propre et lisse $\mathfrak{X}$ sur W et ensuite prendre la cohomologie de de Rham de $\mathfrak{X}_K := X \times_W K$. Toutefois toutes les variétés ne sont pas relevable en caractéristique nulle et même si c'était le cas il ne serait pas évident de montrer que la cohomologie obtenue serait (canoniquement) indépendante du choix de relèvement. Les arguments de la remarque 1.1.4.1.2 suggèrent que, pour montrer cette indépendance on pourrait utiliser une théorie cohomologique pour laquelle l'analogue du fait 1.1.4.1.1 est vrai. Toutefois dans le fait 1.1.4.1.1(1), l'hypothèse de caractéristique nulle est nécessaire.

Exemple 1.1.4.2.1. Si $S = k$ et $Z = \mathbb{A}^1_k$, alors on veut montrer que $d : k[x] \to k[x]dx$ est surjectif. Si $f = \sum a_i x^i dx$ and $p = 0$, alors $f = d(\sum (a_i/i+1)x^{i+1})$.

Au vu de l'exemple 1.1.4.2.1, l'idée est de remplacer le site infinitésimal par un site plus fin, pour lesquels les recouvrement possèdent une opération analogue à $1/i+1$: le site cristallin. Soit (S, I, γ) un schéma muni d'une structure de puissances divisées ([B078, Pag. 3.18]) et soit $f : Z \to S$ un S-schéma sur lequel γ s'étend ([B078, Definition 3.14]). Dans [B078, Section 5], Berthelot définit le site cristallin $Crys(Z/S)$, le topos de faisceaux en groupes abéliens $(Z/S)_{crys}$ et le faisceau structural $\mathcal{O}_{Z/S}$. Il montre ensuite :

Fait 1.1.4.2.2 ([B078, Corollary 7.4 and Theorem 5.17])**.** Si p est nilpotent sur S les assertions suivantes sont vraies.

- Si $Z \to S$ est lisse, alors il y a un isomorphisme naturel
$$H^i_{dr}(Z/S) \simeq H^i_{crys}(Z/S, \mathcal{O}_{Z/S});$$
- Si $Z' \to Z$ est un épaississement nilpotent, alors il y un isomorphisme naturel
$$H^i_{crys}(Z'/S, \mathcal{O}_{Z'/S}) \simeq H^i_{crys}(Z/S, \mathcal{O}_{Z/S}).$$

1.1.4.3 Cohomologie cristalline

Soit $W_n := W_n(k)$ l'anneau des vecteurs de Witt n-tronqués de k. La structure de puissances divisées naturelle γ sur W_n, definie par $\gamma_m(p) = p^m/(m!)$ si $m < n$ et $\gamma_m(p) = 0$ sinon, s'étend automatiquement ([B078, Proposition 3.15]) à tous les W_n-schémas $T \to W_n$. Cela permet permet de définir la cohomologie cristalline d'une k-variété propre et lisse X ([B078, Summary 7.26]) de la façon suivante

$$H^i_{crys}(X/K) := (\varprojlim_n H^i_{crys}(X/W_n, \mathcal{O}_{X/W_n})) \otimes \mathbb{Q}.$$

Alors, si $\mathfrak{X} \to \mathrm{Spec}(W(k))$ est propre et lisse et $X_n := \mathfrak{X} \times_W W_n$, par le fait 1.1.4.2.2 on a :

$$H^i_{dr}(\mathfrak{X}/K) \otimes \mathbb{Q} \simeq H^i_{dr}(\mathfrak{X}/W(k)) \otimes \mathbb{Q} \simeq H^i_{crys}(X_1/K).$$

Le foncteur $H^i_{crys}(-/K)$ donne une théorie cohomologique de Weil à coefficients dans K et le Frobenius absolu φ de k induit une action semi-linéaire sur $H^i_{crys}(X/K)$.

1.1.4.4 Conjecture de Tate cristalline

L'image de l'application classe de cycle

$$c^i_p : \mathrm{CH}^i(X) \to H^{2i}_{crys}(X/K)$$

est contenue dans $H^{2i}_{crys}(X/K)^{\varphi=p}$. Si $k = \mathbb{F}_q$ avec $q = p^s$, alors l'action de $F := \varphi^s$ sur $H^{2i}_{crys}(X/K)$ est K-linéaire et, dans ce cadre, la conjecture de plénitude est la suivante.

Conjecture 1.1.4.4.1 ($\mathrm{T}(X, i, p)$)**.** Si $k = \mathbb{F}_q$, l'application classe de cycle

$$c^i_p : \mathrm{CH}^i(X) \otimes K \to H^{2i}_{crys}(X/K)^{F=q}$$

est surjective

1.1.4.5 Comparaison

Alors qu'en caractéristique nulle on peut comparer directement les cohomologies ℓ-adiques et de Betti via l'isomorphisme de comparaison d'Artin, il n'y pas de tel isomorphisme de comparaison, en caractéristique positive, entre les cohomologies ℓ-adiques et cristallines. Lorsque $k = \mathbb{F}_q$ est un corps fini avec $q = p^s$ éléments, on peut essayer de palier le manque d'isomorphisme de comparaison en utilisant la théorie des poids de Frobenius. Pour tout $\ell \neq p$ le Frobenius arithmétique $F \in \pi_1(\mathbb{F}_q)$ agit linéairement sur l'espace vectoriel $H^i_\ell(X) := H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ et si $\ell = p$ la $s^{\text{ème}}$ puissance du Frobenius absolu F agit linéairement sur $H^i_p(X) := H^i_{crys}(X)$. Soit $\mathcal{L}$ l'ensemble de tous les nombres premiers.

Fait 1.1.4.5.1 ([Del74], [KM74])**.** Pour $? \in \mathcal{L}$, le polynôme caractéristique Φ de F agissant sur $H^i_?(X)$ est dans $\mathbb{Q}[T]$ et il est indépendant de $? \in \mathcal{L}$. De plus pour toutes les racines α de Φ et pour tous les plongements $\iota : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$, on a $|\iota(\alpha)| = q^{i/2}$.

Remarque 1.1.4.5.2.

- Par le fait 1.1.4.5.1, l'adhérence de Zariski de l'image de $\pi_1(\mathbb{F}_q)$ agissant sur la semi simplification de $H^i_?(X)$ est définie sur $\mathbb{Q}$ et est indépendante de ℓ. En particulier une version de la conjecture 1.1.3.3.2 est vraie dans ce contexte à semi-simplification près.
- Si k est un corps finiment engendré de caractéristique positive, pour définir une notion raisonnable d'indépendance et obtenir un analogue du fait 1.1.4.5.1, on doit se ramener au cas des corps finis au prix de devoir travailler dans un cadre relatif. On discutera de ce point plus en détails plus tard, voir le chapitre 6.

1.2 Cadre relatif

L'objet principal de cette thèse est l'étude des notions introduites dans la section 1.1 dans un cadre relatif et non pas absolu. C'est à dire qu'au lieu de considérer une unique variété X, on étudie des familles de variétés.

Soit k un corps de caractéristique $p \geq 0$ et soit X une variété lisse et géométriquement connexe sur k avec un point générique η. Soit $f : Y \to X$ un morphisme propre et lisse et pour tout $x \in X$, $\overline{x}$ un point géométrique au dessus de x. On note Y_x et $Y_{\overline{x}}$ la fibre de $f : Y \to X$ en x et $\overline{x}$ respectivement.

On souhaite étudier comment les invariants de Y_x et $Y_{\overline{x}}$ varient avec $x \in X$. Un premier résultat dans cette direction est le théorème de changement de base propre et lisse : la dimension des différents groupes de cohomologie $H^i(Y_x(\mathbb{C}), \mathbb{Q})$, $H^i(Y_{\overline{x}}, \mathbb{Q}_\ell)$ et $H^i_{crys}(Y_x)$ sont indépendants de $x \in X$. Ce ne sont donc pas des invariants très intéressants de la famille si on les considère seulement comme des espaces vectoriels. Toutefois il est très intéressant d'étudier les structures supplémentaires que possèdent ces espaces vectoriels : la filtration de Hodge, l'action de Galois et celle du Frobenius. Les familles $\{H^i_B(Y_x, \mathbb{Q}))\}_{x\in X}$, $\{H^i(Y_{\overline{x}}, \mathbb{Q}_\ell)\}_{x\in X}$ donnent lieu à des systèmes locaux (une variation de structures de Hodge et une représentation de $\pi_1(X)$) mais, dans le cadre p-adique, $\{H^i_{crys}(Y_x)\}_{x\in X}$ donne lieu à deux systèmes locaux très différents : un F-isocristal convergent et un F-isocristal surconvergent. Dans cette section on rappelle ce que sont ces objets et différents outils qui permettent de les étudier.

1.2.1 Motifs et cycles algébriques

1.2.1.1 Cycles algébriques

Par [SGA6, X, App 7] (voir aussi [MP12, Sections 3.2 and 9.1]), pour tout $x \in X$ il existe un morphisme de spécialisation

$$sp^i_{\eta,x} : \mathrm{CH}^i_{alg}(Y_{\overline{\eta}}) \to \mathrm{CH}^i_{alg}(Y_{\overline{x}})$$

qui s'inscrit dans un diagramme commutatif

$$\begin{tikzcd} & \mathrm{CH}^i_{alg}(Y_{\overline{k}}) \arrow[dl, "i^*_\eta"'] \arrow[dr, "i^*_x"] & \\ \mathrm{CH}^i_{alg}(Y_{\overline{\eta}}) \arrow[rr, "sp^i_{\eta,x}"] & & \mathrm{CH}^i_{alg}(Y_{\overline{x}}), \end{tikzcd}$$

où $i^*_\eta : \mathrm{CH}^i_{alg}(Y_{\overline{k}}) \to \mathrm{CH}^i_{alg}(Y_{\overline{\eta}})$ et $i_x : \mathrm{CH}^i_{alg}(Y_{\overline{k}}) \to \mathrm{CH}^i_{alg}(Y_{\overline{x}})$ sont induits par les inclusions $i_\eta : Y_{\overline{\eta}} \to Y_{\overline{k}}$ et $i_x : Y_{\overline{x}} \to Y_{\overline{k}}$. Pour tout nombre premier $\ell \neq p$ la construction est compatible à l'équivalence homologique ℓ-adique. On en déduit qu'après avoir tensorisé avec $\mathbb{Q}$, on obtient une injection

$$sp^{i,\ell}_{\eta,x} : \mathrm{CH}^i_\ell(Y_{\overline{\eta}}) \otimes \mathbb{Q} \hookrightarrow \mathrm{CH}^i_\ell(Y_{\overline{x}}) \otimes \mathbb{Q},$$

dont on pourrait espérer qu'elle soit un isomorphisme au moins pour certains $x \in X$.

Exemple 1.2.1.1.1 ([MP12, Proof of Proposition 1.13]). Soit $Y \to X$ une famille non isotriviale de courbes elliptiques et soit $f : Y \times_X Y \to X$. Alors $sp^{1,\ell}_{\eta,x}$ est un isomorphisme si et seulement si $Y_{\overline{x}}$ n'a pas de multiplication complexe.

1.2.1.2 Variations de groupes de Galois motiviques

On fixe une théorie cohomologique de Weil H^* à coefficients dans un corps de caractéristique nulle F et on suppose vraies les conjectures standards de la section 1.1.1.2. En particulier, pour tout $x \in X$, on a un groupe algébrique réductif motivique $G(H^*(Y_{\overline{x}}))$ sur F.

Alors, $\mathrm{CH}^*_F(Y_{\overline{x}})_H$ est obtenu ([And04, Section 6.3]) comme étant l'ensemble des points fixes de l'action de $G(H^*(Y_{\overline{x}}))$ sur la représentation canonique $H^*(Y_x)$ et, réciproqiement, puisque $G(H^*(Y_{\overline{x}}))$ est réductif, il existe des entiers $m, n \geq 0$ and $v_1, ...v_r \in H^*(Y_x)^{\otimes m} \otimes (H^*(Y_x)^\vee)^{\otimes n} \subseteq H^*(Y_x^{n+m})$ tels $G(H^*(Y_{\overline{x}}))$ est le sous-groupe de $\mathrm{GL}(H^*(Y_x^{n+m}))$ fixant $v_1, ..., v_r$. Autrement dit décrire les variations de cycles algébriques sur toutes les puissances Y_x^n revient à décrire les variations de $G(H^*(Y_{\overline{x}}))$.

Si le foncteur de réalisation $R_H : \mathbf{Mot}_H^F(k) \to \mathbf{Vect}_F$ se factorise à travers une catégorie Tannakienne F-linéaire enrichie $\mathcal{C}$, alors la conjecture de Grothendieck-Serre-Tate prédit que décrire les variations de $G(H^*(Y_{\overline{x}}))$ revient à décrire les variations de $G(R_H(H^*(Y_{\overline{x}})))$. Finalement, la conjecture $H = num$ suggère que la variation des différents groupes de Tannaka ne devrait pas dépendre de la théorie de cohomologie considérée et donc que l'on devrait pouvoir transférer de l'information entre les groupes de monodromie des différentes réalisations.

1.2.2 Faisceaux lisses et représentations

Dans cette section on suppose que ℓ est un nombre premier différent de p.

1.2.2.1 Faisceaux lisses motiviques

Pour tout $x \in X$ on note $\pi_1(X, \overline{x})$ le groupe fondamental étale ([SGA1, V, 7]) de X pointé en $\overline{x}$. Par le théorème de changement de base propre et lisse $R^i f_* \mathbb{Q}_\ell(j)$ est un faisceau lisse sur X ([SGA4, XVI, Corollaire 2.2], [SGA4, XII, Theorem 2.2]) et induit donc, par l'équivalence de catégorie $\mathbf{LS}(X, \mathbb{Q}_\ell) \simeq \mathbf{Rep}_{\mathbb{Q}_\ell}(\pi_1(X, \overline{\eta}))$ entre la categorie des faisceaux lisses et la categorie des representations de $\pi_1(X, \overline{\eta})$, une action de $\pi_1(X, \overline{\eta})$ sur $R^i f_* \mathbb{Q}_\ell(j)_{\overline{\eta}} \simeq H^i(Y_{\overline{\eta}}, \mathbb{Q}_\ell(j))$. Pour tout $x \in X$, le choix d'un chemin étale entre $\overline{x}$ et $\overline{\eta}$ induit un isomorphisme $\pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta})$ et, respectivement à cet isomorphisme, des isomorphismes équivariants

$$H^i(Y_{\overline{\eta}}, \mathbb{Q}_\ell(j))) \simeq R^i f_* \mathbb{Q}_\ell(j)_{\overline{\eta}} \simeq R^i f_* \mathbb{Q}_\ell(j)_{\overline{x}} \simeq H^i(Y_{\overline{x}}, \mathbb{Q}_\ell(j))).$$

L'action de $\pi_1(x, \overline{x})$ induite par restriction via $\pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta})$ sur $H^i(Y_{\overline{\eta}}, \mathbb{Q}_\ell(j)) \simeq H^i(Y_{\overline{x}}, \mathbb{Q}_\ell(j)))$ s'identifie avec l'action naturelle de $\pi_1(x, \overline{x})$ sur $H^i(Y_{\overline{x}}, \mathbb{Q}_\ell(j)))$. Cette construction rends le diagramme suivant commutatif

$$\begin{array}{ccc} \mathrm{CH}^i_\ell(Y_{\overline{\eta}}) \otimes \mathbb{Q} & \xrightarrow{sp^{i,\ell}_{\eta,x}} & \mathrm{CH}^i_\ell(Y_{\overline{x}}) \otimes \mathbb{Q} \\ \downarrow{\scriptstyle c_{Y_{\overline{\eta}}}} & & \downarrow{\scriptstyle c_{Y_{\overline{x}}}} \\ H^{2i}(Y_{\overline{\eta}}, \mathbb{Q}_\ell(i)) & \simeq & H^{2i}(Y_{\overline{x}}, \mathbb{Q}_\ell(i)) \end{array}$$

L'application $sp^{i,\ell}_{\eta,x}$ est $\pi_1(x, \overline{x})$-équivariante respectivement à l'action naturelle de $\pi_1(x, \overline{x})$ sur $\mathrm{CH}^i_\ell(Y_{\overline{\eta}})$ et celle de $\pi_1(x, \overline{x})$ sur $\mathrm{CH}^i_\ell(Y_{\overline{\eta}})$ par restriction via le morphisme $\pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta})$ ([SGA6, X, App 7]). En particulier $sp^{i,\ell}_{\eta,x}$ se restreint en une injection

$$sp^{i,\ell,ar}_{\eta,x} : \mathrm{CH}^i_\ell(Y_\eta) \otimes \mathbb{Q} = (\mathrm{CH}^i_\ell(Y_{\overline{\eta}}) \otimes \mathbb{Q})^{\pi_1(k(\eta))} \hookrightarrow (\mathrm{CH}^i_\ell(Y_{\overline{x}}) \otimes \mathbb{Q})^{\pi_1(k(x))} = \mathrm{CH}^i_\ell(Y_x) \otimes \mathbb{Q}.$$

1.2.2.2 Lieu strictement exceptionnel

Plus généralement, pour tout ρ dans $\mathbf{Rep}_{\mathbb{Z}_\ell}(\pi_1(X))$ et tout $x \in X$ le choix d'un chemin étale entre $\overline{x}$ et $\overline{\eta}$ donne lieu à une représentation

$$\rho_x : \pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta}) \to \mathrm{GL}_n(\mathbb{Z}_\ell),$$

et donc à une inclusion

$$\Pi_x := \mathrm{Im}(\rho_x) \subseteq \mathrm{Im}(\rho) =: \Pi.$$

Suivant [CK16], on donne la définition suivante.

Définition 1.2.2.2.1. On dit que $x \in |X|$ est strictement Galois générique pour ρ, si $\Pi_x = \Pi$. Si x n'est pas strictement Galois générique, on dit que x est strictement Galois exceptionnel pour ρ.

On note X_ρ^{stex} (resp. X_ρ^{sgen}) l'ensemble des points strictement Galois exceptionnel (resp. strictement Galois générique) pour ρ. On pose

$$X_\rho^{stex}(\leq d) := X_\rho^{stex} \cap X(\leq d); \qquad X_\rho^{sgen}(\leq d) := X_\rho^{sgen} \cap X(\leq d).$$

Au vu de la section 1.2.2.1, l'étude de X_ρ^{stex} est un problème important puisqu'il pourrait contrôler des invariants fins de la famille Y_x, $x \in X$. Mentionnons que si k est suffisamment riche arithmétiquement X_ρ^{sgen} est non vide. Ce fait a été observé indépendamment par Serre ([Ser89, Section 10.6]) et Terasoma ([Ter85]). Plus précisément on a :

Fait 1.2.2.2.2. Si k est Hilbertien, il existe $d \geq 1$ tel que $X_\rho^{sgen}(\leq d)$ est infini.

Cela découle du lemme 1.2.2.3.1 ci-dessous et du fait que le sous-groupe de Frattini d'un groupe de Lie ℓ-adique est ouvert ([Ser89, 148]). D'après le théorème d'irréductibilité de Hilbert ([FJ05, Chapter 13]) le fait 1.2.2.2.2 est en particulier vrai si k est infini et finiment engendré.

1.2.2.3 Dictionnaire anabélien

Pour tout sous-groupe ouvert $U \subseteq \Pi$, on note $X_U \to X$ le recouvrement étale induit par le sous-groupe ouvert $\rho^{-1}(U) \subseteq \pi_1(X)$. D'après le formalisme des catégories Galoisiennes ([SGA1, V, 3-5]), on obtient le dictionnaire anabélien suivant entre points rationnels de X_U et sous-groupes de Π :

Lemme 1.2.2.3.1 ([CT12b, Section 3.2 (2)])**.** Pour tout $x \in |X|$, les assertions suivantes sont équivalentes

- Il y a une inclusion $\Pi_x \subseteq U$.
- $x : \mathrm{Spec}(k(x)) \to X$ se relève en un point $k(x)$ rationnel de X_U.

$$\begin{tikzcd} & X_U \arrow[d] \\ \mathrm{Spec}(k(x)) \arrow[ur, dotted, "\exists"] \arrow[r, "x"] & X \end{tikzcd}$$

Le lemme 1.2.2.3.1 traduit le problème de théorie des groupes de la variation de Π_x en fonction de $x \in |X|$ en le problème Diophantien de la description de l'image de points rationnels de X_U dans X.

1.2.2.4 Argument de Frattini

On note $\Phi(\Pi) \subseteq \Pi$ le sous-groupe de Frattini de Π, c'est à dire l'intersection de tous les sous-groupes ouverts maximaux de Π. On note $\mathcal{C}$ l'ensemble des sous-groupes $U \subseteq \Pi$ tels que $\Phi(\Pi) \subseteq U$. Par [Ser89, Pag. 148] et la définition du sous-groupe de Frattini, on déduit le résultat suivant.

Lemme 1.2.2.4.1.

1. $\mathcal{C}$ est fini.
2. Si $C \subseteq \Pi$ est un groupe fermé propre, alors il existe un $U \in \mathcal{C}$ tel que $C \subseteq U$.

On en déduit que

$$\begin{aligned} x \in X_\rho^{stex} &\Leftrightarrow \text{ il existe } U \in \mathcal{C} \text{ tel que } \Pi_x \subseteq U && \text{(lemme 1.2.2.4.1(2))} \\ &\Leftrightarrow \text{ il existe } U \in \mathcal{C} \text{ tel qu } x \in \mathrm{Im}(X_U(k(x)) \to X(k(x))) && \text{(remarque 1.2.2.3.1),} \end{aligned}$$

et donc que

$$X_\rho^{stex} = \bigcup_{U \in \mathcal{C}} (\bigcup_{[k':k]<+\infty} \mathrm{Im}(X_U(k') \to X(k'))). \tag{1.2.2.4.2}$$

1.2.2.5 Propriété Hilbertienne

On rappelle ([MP12, Definition 8.1]) la définition d'un ensemble clairsemé.

Définition 1.2.2.5.1. Soit B une variété irréductible sur k et $S \subseteq |B|$ un sous-ensemble. On dit que S est clairsemé si il existe un morphisme dominant et génériquement fini $\pi : T \to B$ de variétés irréductibles sur k tel que pour tout $s \in S$, la fibre T_s de $\pi : T \to B$ en s, est soit vide soit contient plus d'un point fermé.

Puisque $X_U \to X$ est un recouvrement étale fini de degré > 1, l'ensemble

$$\bigcup_{k \subseteq k'} \mathrm{Im}(X_U(k') \to X(k')) \subseteq |X|$$

est clairsemé. L'union d'un nombre fini de sous-ensembles clairsemés étant clairsemé ([MP12, Proposition 8.5 (b)]) et puisque $\mathcal{C}$ est fini (lemme 1.2.2.4.1(1)), on déduit de (1.2.2.4.2) que X_ρ^{stex} est clairsemé. C'est suffisant, grâce au lemme suivant, pour prouver le fait 1.2.2.2.2.

Lemme 1.2.2.5.2. Si k est Hilbertien et si $S \subseteq |X|$ est un sous-ensemble clairsemé, il existe $d \geq 1$ tel que $|X| - S$ contient une infinité de points de degré $\leq d$.

Démonstration. Puisque pour tout sous-ensemble ouvert dense $U \subseteq X$, l'ensemble $U \cap S$ est clairsemé dans U ([MP12, Proposition 8.5.(a)]), on peut remplacer X par un sous-ensemble ouvert dense et donc supposer que X est affine de dimension $n \geq 1$. Par le théorème de normalisation de Noether, il existe un morphisme fini surjectif $\pi : X \to \mathbb{A}_k^n$ de degré $d \geq 1$. L'image d'un sous-ensemble clairsemé par un morphisme fini surjectif étant clairsemée ([MP12, Proposition 8.5 (c)]), l'ensemble $\pi(S) \subseteq \mathbb{A}_k^n$ est clairsemé. On en déduit, par ([MP12, Proposition 8.5 (d)]), que $\mathbb{A}_k^n(k) \cap \pi(S)$ est mince (voir [Ser89, Section 9.1] pour la définition). Puisque k est Hilbertien, l'ensemble $\mathbb{A}_k^n(k) - (\mathbb{A}_k^n(k) \cap \pi(S))$ est infini. On en conclut que $\pi^{-1}(\mathbb{A}_k^n(k) - (\mathbb{A}_k^n(k) \cap \pi(S))) \subseteq X - S$ contient une infinité de points de degré $\leq d$. □

1.2.3 Caractéristique nulle : Variations de structures de Hodge motiviques

Soit $k \subseteq \mathbb{C}$ un sous-corps finiment engendré de $\mathbb{C}$.

1.2.3.1 Systèmes locaux analytiques et image géométrique

Soit $x \in |X_\mathbb{C}|$. D'après le théorème de changement de base propre et lisse on obtient, à partir de $f^{an} : Y_\mathbb{C}^{an} \to X_\mathbb{C}^{an}$, un $\mathbb{Q}$-système local $R^i f_*^{an}\mathbb{Q}$ sur $X_\mathbb{C}^{an}$. On note Π_B l'image de l'action de $\pi_1^{top}(X_\mathbb{C}, x)$ sur $H_B^i(Y_x, \mathbb{Q})$ qui en résulte. Par l'invariance du site étale sous les extensions de corps algébriquement clos en caractéristique nulle ([SGA1, XIII]), il existe un isomorphisme naturel $\pi_1(X_{\overline{k}}, x) \simeq \pi_1(X_\mathbb{C}, x)$. Par le théorème d'existence de Riemann [SGA1, XII, Theoreme 5.1], il existe un morphisme naturel d'algébrisation $\pi_1^{top}(X_\mathbb{C}, x) \to \pi_1(X_\mathbb{C}, x)$ qui identifie $\pi_1(X_\mathbb{C}, x)$ avec la complétion profinie de $\pi_1^{top}(X_\mathbb{C}, x)$ ([SGA1, XII, Corollaire 5.2]).

L'action de $\pi_1^{top}(X_{\mathbb{C}}, x)$ sur $H^i_B(Y_x, \mathbb{Q}) \otimes \mathbb{Q}_\ell$, se factorise à travers l'application de complétion profinie $\pi_1^{top}(X_{\mathbb{C}}, x) \to \pi_1(X_{\mathbb{C}}, x) \simeq \pi_1(X_{\overline{k}}, x)$. L'action de $\pi_1(X_{\overline{k}}, x)$ sur $H^i_B(Y_x, \mathbb{Q}) \otimes \mathbb{Q}_\ell$ s'identifie, via l'isomorphisme de comparaison $H^i_B(Y_x, \mathbb{Q}) \otimes \mathbb{Q}_\ell \simeq H^i(Y_x, \mathbb{Q}_\ell)$, à l'action de $\pi_1(X, x)$ sur $H^i(Y_x, \mathbb{Q}_\ell)$ obtenue par restriction via le morphisme $\pi_1(X_{\overline{k}}, x) \to \pi_1(X, x)$.

On note $G_\ell^{i,geo}$ l'adhérence de Zariski de l'image $\Pi_\ell^{i,geo}$ de l'action de $\pi_1(X_{\overline{k}}, x)$ sur $H^i(Y_x, \mathbb{Q}_\ell)$. Puisque $\pi_1^{top}(X_{\mathbb{C}}, x) \to \pi_1(X_{\mathbb{C}}, x)$ a une image dense, la discussion précédente implique le résultat d'indépendance suivant pour $G_\ell^{i,geo}$, qui est un analogue géométrique de la conjecture 1.1.3.3.2.

Proposition 1.2.3.1.1. Il existe un $\mathbb{Q}$-groupe algébrique $G^{i,geo}$, une représentation fidèle $G^{i,geo} \subseteq \mathrm{GL}(V^i)$ et un isomorphisme $V^i \otimes \mathbb{Q}_\ell \simeq H^i(Y_{\overline{k}}, \mathbb{Q}_\ell)$ pour tout ℓ, tel que la composition $G^{i,geo} \otimes \mathbb{Q}_\ell \subseteq \mathrm{GL}(V^i) \otimes \mathbb{Q}_\ell \simeq \mathrm{GL}(H^i(Y_{\overline{x}}, \mathbb{Q}_\ell))$ identifie $G^{i,geo} \otimes \mathbb{Q}_\ell$ avec $G_\ell^{i,geo}$.

1.2.3.2 La conjecture de Hodge variationnelle

La suite spectrale de Leray associée à $f^{an} : Y^{an}_{\mathbb{C}} \to X^{an}_{\mathbb{C}}$ induit un morphisme

$$H^2_B(Y_{\mathbb{C}}, \mathbb{Q}(1)) \to H^0(X^{an}_{\mathbb{C}}, R^2 f^{an}_* \mathbb{Q}(1))$$

qui s'inscrit dans le diagramme commutatif suivant

$$\begin{tikzcd}
 & H^2_B(Y_{\mathbb{C}}, \mathbb{Q}(1)) \arrow[d] \arrow[dr, "i_x^*"] & & Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q} \arrow[ll, "c_{Y_{\mathbb{C}}}"'] \arrow[dr, "i_x^*"] & \\
 & H^0(X^{an}_{\mathbb{C}}, R^2 f^{an}_* \mathbb{Q}(1)) \otimes \mathbb{Q} \arrow[r, hook] & H^2_B(Y_x, \mathbb{Q}(1)) & & Pic(Y_x) \otimes \mathbb{Q}. \arrow[ll, "c_{Y_x}"]
\end{tikzcd}$$

La conjecture de Hodge pour les diviseurs (Fact 1.1.3.1.2) et la théorie développée dans [Del71] permettent de prouver une version variationnelle de la conjecture de Hodge pour les diviseurs (voir aussi [Char11, Section 3.1]).

Fait 1.2.3.2.1. Pour tout $x \in |X_{\mathbb{C}}|$ et tout $z_x \in Pic(Y_x) \otimes \mathbb{Q}$ les assertions suivantes sont équivalentes

1. Il existe un $z \in Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q}$ tel que $i_x^*(c_{Y_{\mathbb{C}}}(z)) = c_{Y_x}(z_x)$;
2. Il existe un $z \in H^2_B(X_{\mathbb{C}}, \mathbb{Q}(1))$ tel que $i_x^*(z) = c_{Y_x}(z)$;
3. $c_{Y_x}(z)$ est dans l'image de $H^0(X^{an}_{\mathbb{C}}, R^2 f^{an}_* \mathbb{Q}(1)) \hookrightarrow H^2_B(Y^{an}_x, \mathbb{Q}(1))$.

Démonstration. On a clairement (1) $\Rightarrow$ (2) $\Rightarrow$ (3). On montre (3) $\Rightarrow$ (1). Soit $i : Y_{\mathbb{C}} \subseteq Y^{cmp}_{\mathbb{C}}$ une compactification lisse. Le diagramme commutatif cartésien de $\mathbb{C}$-variétés suivant

$$\begin{tikzcd}
Y_x \arrow[r, "i_x"] \arrow[d] \arrow[dr, phantom, "\square"] & Y_{\mathbb{C}} \arrow[r, "i"] \arrow[d, "f"] & Y^{cmp}_{\mathbb{C}} \\
\mathrm{Spec}(\mathbb{C}) \arrow[r, "x"] & X_{\mathbb{C}}, &
\end{tikzcd}$$

induit un diagramme commutatif

$$\begin{tikzcd}
 & H^2_B(Y^{cmp}_{\mathbb{C}}, \mathbb{Q}(1)) \arrow[d, two heads, "i^*"] & & Pic(Y^{cmp}_{\mathbb{C}}) \otimes \mathbb{Q} \arrow[ll, "c_{Y^{cmp}_{\mathbb{C}}}"'] \arrow[d, "i^*"] & \\
 & H^2_B(Y_{\mathbb{C}}, \mathbb{Q}(1)) \arrow[d, two heads] \arrow[dr, "i_x^*"] & & Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q} \arrow[ll, "c_{Y_{\mathbb{C}}}"'] \arrow[dr, "i_x^*"] & \\
 & H^0(X^{an}_{\mathbb{C}}, R^2 f^{an}_* \mathbb{Q}(1)) \otimes \mathbb{Q} \arrow[r, hook] & H^2_B(Y_x, \mathbb{Q}(1)) & & Pic(Y_x) \otimes \mathbb{Q}. \arrow[ll, "c_{Y_x}"]
\end{tikzcd}$$

Par le thèorème global des cycles invariants ([Del71, 4.1.1]), l'application

$$H^2_B(Y^{cmp}_{\mathbb{C}}, \mathbb{Q}(1)) \to H^2_B(Y_{\mathbb{C}}, \mathbb{Q}(1)) \to H^0(X^{an}_{\mathbb{C}}, R^2 f^{an}_* \mathbb{Q}(1))$$

est surjective. On en déduit, par (3), que $c_{Y_x}(z_x) \in H^2_B(Y_x, \mathbb{Q}(1))$ est dans l'image de $i_x^* \circ i^* : H^2_B(Y^{cmp}_{\mathbb{C}}, \mathbb{Q}(1)) \to H^2_B(Y_x, \mathbb{Q}(1))$. Puisque $H^2_B(Y^{cmp}_{\mathbb{C}}, \mathbb{Q}(1))$ est une $\mathbb{Q}$-structure de Hodge semi-simple, l'application $H^2_B(Y^{cmp}_{\mathbb{C}}, \mathbb{Q}(1)) \to \mathrm{Im}(i_x^*)$ se scinde en tant que un morphisme de $\mathbb{Q}$-structures de Hodge. Puisque $c_{Y_x}(z_x)$ est dans $H^2_B(Y^{cmp}_{\mathbb{C}}, \mathbb{Q}(1))^{0,0}$, $c_{Y_x}(z_x)$ est l'image d'un $z' \in H^2_B(Y^{cmp}_{\mathbb{C}}, \mathbb{Q}(1))^{0,0}$ via $H^2_B(Y^{cmp}_{\mathbb{C}}, \mathbb{Q}(1))^{0,0} \to H^2_B(Y_x, \mathbb{Q}(1))$. Par la conjecture de Hodge pour les diviseurs (Fact 1.1.3.1.2) $z = c_{Y^{cmp}_{\mathbb{C}}}(z^{cmp})$ pour un $z^{cmp} \in Pic(Y^{cmp}_{\mathbb{C}}) \otimes \mathbb{Q}$. Alors $z = i^*(z^{cmp}) \in Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q}$ est tel que $i_x^*(c_Y(z)) = c_{Y_x}(z_x)$. □

1.2.4 Caractéristique positive : F-isocristaux

En caractéristique positive, il y a deux catégories différentes de systèmes locaux p-adiques : les F-isocristaux et les F-isocristaux surconvergents. Dans cette section on rappelle rapidement leurs définitions et les relations entre les deux. Soit k un corps parfait de caractéristique $p > 0$ et soit X une variété lisse et géométriquement connexe sur k.

1.2.4.1 F-isocristaux

En adaptant légèrement les arguments de 1.1.4.2, on définit un topos cristallin $(X/W)_{crys}$, un site cristallin $Crys(X/W)$ sur X au dessus de W et un faisceau structural $\mathcal{O}_{X/W}$, cf. [B078, Section 7.17] et [Mor13, Section 2]. Pour tout $(U \hookrightarrow T, \gamma)$[1] dans $(X/W)_{crys}$ et tout faisceau en $\mathcal{O}_{X/W}$-modules cohérents $\mathcal{E}$, on a un $\mathcal{O}_T$-module cohérent $\mathcal{E}_T$ et pour tout morphisme $g : (U', T', \gamma') \to (U, T, \gamma)$ dans $(X/W)_{crys}$ on a un morphisme naturel $g^*\mathcal{E}_T \to \mathcal{E}_{T'}$ de $\mathcal{O}_{T'}$-modules cohérents. Un cristal sur X est alors un faisceau $\mathcal{E}$ de $\mathcal{O}_{X/W}$-modules cohérents tel que pour tout morphisme $g : (U', T', \gamma') \to (U, T, \gamma)$ dans $Crys(X/W)$, le morphisme naturel $g^*\mathcal{E}_{T'} \to \mathcal{E}_T$ est un isogénie.

On note $\mathbf{Crys}(X|W)$ la catégorie des cristaux, $\mathbf{Crys}(X|W)_{\mathbb{Q}} := \mathbf{Crys}(X|W) \otimes \mathbb{Q}$ et $\mathcal{O}_{X/K} := \mathcal{O}_{X/W} \otimes \mathbb{Q}$. Pour tout entier $s \geq 1$, la $s^{\text{ème}}$-puissance F du Frobenius absolu φ de X agit sur $\mathbf{Crys}(X|W)_{\mathbb{Q}}$ et la catégorie $\mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$ des F-isocristaux est définie comme étant la catégorie des couples $(\mathcal{E}, \Phi)$, où $\mathcal{E}$ est dans $\mathbf{Crys}(X|W)_{\mathbb{Q}}$ et $\Phi : F^*\mathcal{E} \to \mathcal{E}$ est une isogénie. Pour tout $\mathcal{E}$ dans $\mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$ il y a un groupe de cohomologie $H^i(X, \mathcal{E})$ (un K-espace vectoriel) muni d'une action semi-linéaire de F. On pose $H^i_{crys}(X) := H^i_{crys}(X, \mathcal{O}_{X/K})$.

1.2.4.2 La conjecture de Tate variationnelle cristalline

Par [Mor13], il existe un F-isocristal image directe supérieure $R^i f_{crys,*}\mathcal{O}_{Y/K}$ et la suite spectrale de Leray pour $f : Y \to X$ induit, pour tout $x \in |X|$, un diagramme commutatif

$$\begin{tikzcd}
H^2_{crys}(Y) \arrow[d, "Leray"] \arrow[dr, "i_x^*"] & & Pic(Y) \otimes \mathbb{Q} \arrow[ll, "c_Y"'] \arrow[dr, "i_x^*"] & \\
H^0(X, R^2 f_{crys,*}\mathcal{O}_{Y/K}) \arrow[r, hook] & H^2_{crys}(Y_x) & & Pic(Y_x) \otimes \mathbb{Q}. \arrow[ll, "c_{Y_x}"]
\end{tikzcd}$$

Bien que le conjecture de Tate cristalline ne soit pas connue, Morrow en a démontré une version variationnelle, qui est un analogue du fait 1.2.3.2.1.

Fait 1.2.4.2.1 ([Mor15, Theorem 1.4]). Si $f : Y \to X$ est projectif, pour tout $z_x \in Pic(Y_x) \otimes \mathbb{Q}$ les assertions suivantes sont équivalentes :

[1] U est un ouvert de X, $U \hookrightarrow T$ est une immersion fermée nilpotente de W-schémas et γ est une structure de puissances divisées sur $\mathrm{Ker}(\mathcal{O}_T \to \mathcal{O}_U)$.

1. Il existe $z \in Pic(Y) \otimes \mathbb{Q}$ tel que $c_{Y_x}(z_x) = i_x^*(c_Y(z))$;

2. Il existe $z \in H^2_{crys}(Y)$ tel que $c_{Y_x}(z_x) = i_x^*(z)$;

3. $c_{Y_x}(z_x)$ est dans l'image de $H^0(X, R^2 f_{crys,*}O_{Y/K}) \hookrightarrow H^2_{crys}(Y_x)$.

1.2.4.3 Pentes

Une des propriétés spécifiques de $\mathbf{F}$-$\mathbf{Crys}(X|W)_{\mathbb{Q}}$, qui n'a pas d'analogue ℓ-adique, est la théorie des pentes (cf. [Kat79], [Ked17, Sections 3 and 4]). Soit $\mathcal{E}$ un élément de $\mathbf{F}$-$\mathbf{Crys}(X|W)_{\mathbb{Q}}$ de rang r. Pour tout $x \in |X|$, on considère l'ensemble de nombres rationnels $\{a_i^x(\mathcal{E})\}_{1\leq i\leq r}$ des pentes ([Ked17, Definition 3.3]) de $\mathcal{E}$ en x. On dit que $\mathcal{E}$ est isocline (de pente $a_1^{\mathfrak{t}}(\mathcal{E})$) si $a_1^{\mathfrak{t}}(\mathcal{E}) = a_r^x(\mathcal{E})$ pour tout $x \in |X|$ et unité si il est isocline de pente nulle. On note $\mathbf{F}$-$\mathbf{Crys}^{un}(X|W)_{\mathbb{Q}} \subseteq \mathbf{F}$-$\mathbf{Crys}(X|W)_{\mathbb{Q}}$ la catégorie des F-isocristaux unité. Finalement, on dit que $\mathcal{E}$ a polygone de Newton constant si la fonction

$$N_{\mathcal{E}} : |X| \to \mathbb{Q}^r$$
$$x \mapsto (a_i^x(\mathcal{E}))_{1\leq i\leq r}$$

est constante.

Fait 1.2.4.3.1. Soit $\mathcal{E}$ un élément de $\mathbf{F}$-$\mathbf{Crys}(X|W)_{\mathbb{Q}}$.

1. ([Kat79, Theorem 2.3.1], [Ked17, Theorem 3.12]) : Il existe une immersion ouverte dense $\mathrm{i} : U \to X$ telle que $\mathrm{i}^*\mathcal{E}$ a polygone de Newton constant.

2. ([Kat79, Theorem 2.6.2], [Ked17, Corollary 4.2]) : Si $\mathcal{E}$ a polygone de Newton constant, alors il existe une unique filtration
$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq ... \subseteq \mathcal{E}_{n-1} \subseteq \mathcal{E}_n = \mathcal{E} \quad \text{in } \mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$$
telle que $\mathcal{E}_i/\mathcal{E}_{i-1}$ est isocline de pente s_i avec $s_1 < s_2 < ... < s_n$.

3. ([Tsu02], [Ked17, Theorem 3.9]) : Il y a une équivalence de catégorie naturelle $\mathbf{F}$-$\mathbf{Crys}^{un}(X|W)_{\mathbb{Q}} \simeq \mathbf{Rep}_K(\pi_1(X))$.

La filtration du fait 1.2.4.3.1(2) est appelé la filtration par les pentes de $\mathcal{E}$.

1.2.4.4 Comparaison I : F-isocristaux vs représentations ℓ-adiques

Il serait naturel de penser que le F-isocristal $R^i f_{crys,*}\mathcal{O}_{Y/K}$ est l'analogue p-adique de $R^i f_*\mathbb{Q}_\ell$. Toutefois, le comportement de $\mathbf{F}$-$\mathbf{Crys}(X|W)_{\mathbb{Q}}$ est très différent de celui de $\mathbf{LS}(X, \mathbb{Q}_\ell)$.

Exemple 1.2.4.4.1 ([Gro68, Section 2.1])**.** En général le K-espace vectoriel $H^1(X, \mathcal{E})$ n'est pas de dimension finie. Considérons $X = \mathbb{A}^1_k$ et posons

$$K\langle T\rangle := \{\sum_{i=0}^{+\infty} a_i T^i \text{ telles que } \lim_{i\mapsto+\infty} |a_i| = 0\}.$$

Il y a un isomorphisme naturel

$$H^1_{crys}(X) \simeq \mathrm{Coker}(d : K\langle T\rangle \to K\langle T\rangle dT).$$

Puisque

$$\lim_{i\to+\infty} |a_i| = 0,$$

n'implique pas en général que

$$\lim_{i\to+\infty} |a_i/i+1| = 0,$$

on voit que $H^1_{crys}(X)$ est un K-espace vectoriel de dimension infinie. Toutefois, suivant [MW68], on peut remplacer $K\langle T\rangle$ par le sous-anneau

$$K\langle T\rangle^\dagger := \{\sum_{i=0}^{+\infty} a_i T^i \text{ tel qu'il existe un } c>1 \text{ avec } \lim_{n\to+\infty} c^i|a_i| = 0\},$$

et montrer que $d : \mathrm{Coker}(K\langle T\rangle^\dagger \to K\langle T\rangle^\dagger dT) = 0$. L'anneau $K\langle T\rangle$ est l'anneau des fonctions rigides analytiques sur le disque unité ouvert alors que $K\langle T\rangle^\dagger \subseteq K\langle T\rangle$ est le sous-anneau des fonctions qui convergent sur un voisinage ouvert analytique plus large.

Exemple 1.2.4.4.2 ([Ked17, Example 4.6]). Soit $f : Y \to X$ une famille non isotriviale de courbes elliptiques dont l'une des fibres est supersingulière et soit $\mathcal{E} := R^i f_* \mathcal{O}_{Y/K}$. Alors il existe un sous-schéma ouvert et dense $i : U \hookrightarrow X$ tel que pour tout $x \in U$, la courbe elliptique Y_x est ordinaire. On a les résultats suivants :

1. $\mathcal{E}$ est irréductible ;
2. $i^*\mathcal{E} \simeq R^i f_{U,*}\mathcal{O}_{Y_U/K}$ a une filtration a deux crans non scindée (la filtration par les pentes de la section 1.2.4.3) qui reflète la filtration du groupe p-divisible de la fibre générique, donnée par la suite exacte connexe-étale

$$0 \to Y_\eta[p^\infty]^0 \to Y_\eta[p^\infty] \to Y_\eta[p^\infty]^{et} \to 0.$$

On en déduit les observations suivantes :

1. Bien que dans le cadre ℓ-adique la restriction à un ouvert d'un faisceau lisse irréductible reste irréductible, dans le cas cristallin ce n'est pas le cas ;
2. Bien que $R^i f_{U,*}\mathbb{Q}_\ell$ soit semi-simple, ce n'est pas le cas de $i^*\mathcal{E}$.

En conclusion on voit que d'un côté la catégorie $\mathbf{F}\text{-}\mathbf{Crys}(X|W)_\mathbb{Q}$ a un comportement pathologique respectivement à $\mathbf{LS}(X,\mathbb{Q}_\ell)$ mais que de l'autre elle contient des informations p-adiques fines.

1.2.4.5 F-isocristaux surconvergents

Les exemples 1.2.4.4.1 et 1.2.4.4.2 suggèrent que pour obtenir une catégorie de systèmes locaux p-adiques ayant un comportement similaire à $LS(X,\mathbb{Q}_\ell)$ il faut rigidifier la catégorie $\mathbf{F}\text{-}\mathbf{Crys}(X|W)_\mathbb{Q}$. Cela mène à l'introduction des catégories d'isocristaux surconvergents $\mathbf{Isoc}^\dagger(X|K)$ et de F-iscocristaux surconvergents $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X|K)$ ainsi qu'à celle de la cohomologie rigide $H^i(X,\mathcal{E})$ pour $\mathcal{E}$ dans $\mathbf{Isoc}^\dagger(X|K)$. Les définitions de ces objets sont techniques et on renvoie le lecteur à [Ber96] pour les définitions précises. On se contente de donner un exemple.

Exemple 1.2.4.5.1. Soit $X = \mathbb{A}^1_k$. On garde les notations de l'exemple 1.2.4.4.1. Un isocristal surconvergent sur X est un $K\langle T\rangle^\dagger$-module cohérent $\mathcal{E}$, muni d'une connexion intégrable

$$d : \mathcal{E} \to \mathcal{E} \otimes_{K\langle T\rangle^\dagger} K\langle T\rangle^\dagger dT.$$

La cohomologie rigide de $\mathcal{E}$ est alors définie par

$$H^0(X,\mathcal{E}) = \mathrm{Ker}(d : \mathcal{E} \to \mathcal{E} \otimes_{K\langle T\rangle^\dagger} K\langle T\rangle^\dagger dT);$$
$$H^1(X,\mathcal{E}) = \mathrm{Coker}(d : \mathcal{E} \to \mathcal{E} \otimes_{K\langle T\rangle^\dagger} K\langle T\rangle^\dagger dT);$$
$$H^i(X,\mathcal{E}) = 0 \quad \text{si} \quad i \geq 2$$

Le morphisme naturel $F : K\langle T\rangle \to K\langle T\rangle$ qui envoie $\sum a_i T^i$ sur $\sum F(a_i)T^{pi}$, induit un morphisme $F : K\langle T\rangle^\dagger \to K\langle T\rangle^\dagger$, et on peut donc considérer l'isocristal surconvergent $F^*\mathcal{E}$. Un F-iscocristal surconvergent sur X est alors un isocristal surconvergent $\mathcal{E}$ sur $K\langle T\rangle^\dagger$, muni d'un isomorphisme $F^*\mathcal{E} \to \mathcal{E}$.

Pour comparer les F-isocristaux et les F-isocristaux surconvergents, on introduit les catégories $\mathbf{Isoc}(X|K)$ et $\mathbf{F}\text{-}\mathbf{Isoc}(X|K)$ des isocristaux convergents et F-isocristaux convergents ([Ogu84], [Ber96, 2.3.2]). Les catégories de cristaux que nous avons introduites jusqu'à présent vivent dans un diagramme commutatif ([Ber96, Section 2.4]) de foncteurs fidèles :

$$\begin{tikzcd} \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X|K) \arrow[r, "(-)^{geo}"] \arrow[d, "(-)^{conv}"] & \mathbf{Isoc}^\dagger(X|K) \arrow[d, "(-)^{conv}"] \\ \mathbf{F}\text{-}\mathbf{Isoc}(X|K) \arrow[r, "(-)^{geo}"] \arrow[d, "\Phi"] & \mathbf{Isoc}(X|K) \arrow[d] \\ \mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}} \arrow[r, "(-)^{geo}"] & \mathbf{Crys}(X|W)_{\mathbb{Q}}. \end{tikzcd} \quad (1.2.4.5.2)$$

De plus

Fait 1.2.4.5.3.

- ([Ber96, Theoreme 2.4.2]) : Le foncteur $\Phi : \mathbf{F}\text{-}\mathbf{Isoc}(X|K) \to \mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$ est une équivalence de catégorie.
- ([Ked18]) : Le foncteur $(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X|K) \to \mathbf{F}\text{-}\mathbf{Isoc}(X|K)$ est pleinement fidèle.

Un des résultats non triviaux de cette thèse dans le chapitre 4 assure que $R^i f_{crys,*}\mathcal{O}_{Y/K}$ est dans l'image essentielle de $(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X|K) \to \mathbf{F}\text{-}\mathbf{Isoc}(X|K) \simeq \mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$.

1.2.4.6 Groupes de monodromie

Si $\mathcal{E}$ est un $\mathbb{Q}_\ell$-faisceau lisse sur X, on a vu que l'on pouvait définir, de manière équivalente, le groupe de monodromie $G(\mathcal{E})$ of $\mathcal{E}$ comme étant soit l'adhérence de Zariski de l'image de $\pi_1(X, \overline{x})$ agissant sur $\mathcal{E}_{\overline{x}}$ soit le groupe des automorphismes du foncteur d'oubli $\langle\mathcal{E}\rangle \to \mathbf{Vect}_{\mathbb{Q}_\ell}$. Pour les isocristaux seule cette dernière construction est disponible. Ce fut fait en premier par Crew dans [Cre92]. A partir de maintenant, on suppose que $k = \mathbb{F}_q$, avec $q = p^s$ et, pour simplifier, que X a un point $\mathbb{F}_q$-rationnel $x : \mathrm{Spec}(\mathbb{F}_q) \to X$. Puisqu'il y a une équivalence de catégories naturelle $\mathbf{Isoc}(\mathbb{F}_q|K) \simeq \mathbf{Vect}_K$, le foncteur

$$x^* : \mathbf{Isoc}(X|K) \to \mathbf{Isoc}(\mathbb{F}_q|K) \simeq \mathbf{Vect}_K$$

induit une neutralisation des quatre catégories dans le diagramme (1.2.4.5.2). Ainsi, pour tout $\mathcal{E}$ dans $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X|K)$, on obtient un diagramme commutatif de catégories Tannakiennes :

$$\begin{tikzcd} \langle\mathcal{E}\rangle \arrow[r, "(-)^{geo}"] \arrow[d, "(-)^{conv}"] & \langle\mathcal{E}^{geo}\rangle \arrow[d, "(-)^{conv}"] \\ \langle\mathcal{E}^{conv}\rangle \arrow[r, "(-)^{geo}"] & \langle\mathcal{E}^{geo,conv}\rangle. \end{tikzcd}$$

Par la dualité Tannakienne, ce diagramme correspond à un diagramme commutatif exact d'immersions fermées de groupes algébriques

$$\begin{tikzcd} G(\mathcal{E}^{geo,conv}) \arrow[r, hook] \arrow[d, hook] & G(\mathcal{E}^{conv}) \arrow[d, hook] \\ G(\mathcal{E}^{geo}) \arrow[r, hook] & G(\mathcal{E}), \end{tikzcd}$$

dans lequel ([D'Ad17, Appendix]) les sous-groupes $G(\mathcal{E}^{geo,conv}) \subseteq G(\mathcal{E}^{conv})$ et $G(\mathcal{E}^{geo}) \subseteq G(\mathcal{E})$ sont normaux.

Exemple 1.2.4.6.1. On garde les notations de l'exemple 1.2.4.4.2. On a

$$G(\mathcal{E}^{conv}) = G(\mathcal{E}) = \mathrm{GL}_2 \quad \text{and} \quad G(\mathcal{E}^{geo,conv}) = G(\mathcal{E}^{geo}) = SL_2$$

alors que

$$B = G(i^*\mathcal{E}^{conv}) \subseteq G(i^*\mathcal{E}) = \mathrm{GL}_2 \quad \text{and} \quad B' = G(i^*\mathcal{E}^{geo,conv}) \subseteq G(i^*\mathcal{E}^{geo}) = SL_2$$

où $B \subseteq \mathrm{GL}_2$ et $B' \subseteq SL_2$ sont les sous-groupes de Borel des matrices triangulaires supérieures. Cela reflète le fait que $i^*\mathcal{E}$ admet une filtration par des F-isocristaux qui ne viennent pas de F-isocristaux surconvergents, qui correspond au drapeau stabilisé par B et B' mais par GL_2 et SL_2.

1.2.4.7 Comparaison II : F-isocristaux surconvergents vs représentations ℓ-adiques

Alors qu'en caractéristique nulle on peut essayer de comparer les différents groupes de monodromie via le théorème de comparaison entre le site singulier et le site étale, en caractéristique positive on a besoin d'outils différents. On rappelle quelques résultats dans ce cadre. Pour des raisons techniques il est plus facile de travailler avec des coefficients dans des corps algébriquement clos. Soit ℓ un nombre premier. Suivant [Ked17], on note $\mathbf{Coef}(X, \ell)$ la catégorie des $\overline{\mathbb{Q}}_\ell$-faisceaux lisses ([Del80, 1.1.1]) et $\mathbf{Coef}(X, p)$ la catégorie des $\overline{\mathbb{Q}}_p$-F-isocristaux surconvergents ([Abe18, Sections 2.4.14-2.4.18]). Soit $\mathcal{E}_\ell$ un élément de $\mathbf{Coef}(X, \ell)$. Pour tout $x \in |X|$ il existe un polynôme caractéristique $\phi_x(\mathcal{E}_\ell) \in \overline{\mathbb{Q}}_\ell[T]$ de $\mathcal{E}$ en x (cf. par exemple [D'Ad17, 2.1.4 and 2.2.10.]). On fixe une collection $\underline{\iota} := \{\iota_\ell\}_{\ell \in \mathcal{L}}$ d'isomorphismes $\iota_\ell : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. On dit que $\mathcal{E}_\ell$ est ι_ℓ-pure (de poids w), si toutes les racines de $\iota_\ell(\phi_x(\mathcal{E}_\ell))$ ont valeur absolue complexe $q^{[\mathbb{F}_q(x):\mathbb{F}_q]w/2}$. Soit $\{\mathcal{E}_\ell\}_{\ell \in \mathcal{L}}$ une collection de $\mathcal{E}_\ell$ dans $\mathbf{Coef}(X, \ell)$. On dit que $\{\mathcal{E}_\ell\}_{\ell \in \mathcal{L}}$ est un $\underline{\iota}$-système compatible si $\iota_\ell(\phi_x(\mathcal{E}_\ell)) = \iota_{\ell'}(\phi_x(\mathcal{E}_{\ell'}))$, pour tout $\ell \neq \ell'$ et tout $x \in |X|$. Via la théorie des poids ([Del80], [Ked06b], [AC13b]), les conditions de pureté et de compatibilité sont suffisamment fortes pour garantir que les différents $\mathcal{E}_\ell$ partagent plusieurs propriétés.

Exemple 1.2.4.7.1. Soit deux nombres premiers $\ell \neq \ell'$, on suppose pour simplifier que $\ell \neq p \neq \ell'$. Si $\mathcal{E}_\ell$ dans $\mathbf{Coef}(X, \ell)$ et $\mathcal{E}_{\ell'}$ dans $\mathbf{Coef}(X, \ell')$ sont pures et compatibles, alors les faits suivants découlent de la théorie des poids ([Del80]) et de la formule des traces de Grothendieck-Lefschetz ([Fu15, Theorem 10.5.1, page 603]) :

- $\mathcal{E}_\ell$ est irréductible si et seulement si $\mathcal{E}_{\ell'}$ est irréductible (cf. par exemple [D'Ad17, Corollary 3.5.6]) ;
- $Dim(H^0(X_{\mathbb{F}}, \mathcal{E}_\ell)) = Dim(H^0(X_{\mathbb{F}}, \mathcal{E}_{\ell'}))$ (cf. par exemple [D'Ad17, Corollary 3.4.11]).

On fixe $x \in |X|$ et on note $\mathcal{E}_{\ell,\overline{x}}$ la fibre de $\mathcal{E}_\ell$ en $\overline{x}$. En utilisant le foncteur x^*, pour tout $\mathcal{E}_\ell$ dans $\mathbf{Coef}(X, \ell)$ on définit un groupe de monodromie $G(\mathcal{E}_\ell) \subseteq \mathrm{GL}(\mathcal{E}_{\ell,\overline{x}})$. De plus, on peut construire un groupe de monodromie géométrique $G(\mathcal{E}_\ell^{geo}) \subseteq G(\mathcal{E}_\ell)$: si $\ell \neq p$, $G(\mathcal{E}_\ell^{geo})$ est défini comme étant le groupe de monodromie du changement de base de $\mathcal{E}_\ell$ à $X_{\overline{\mathbb{F}}_q}$ et si $\ell = p$, $G(\mathcal{E}_p^{geo})$ est défini comme étant le groupe de monodromie de l'image de $\mathcal{E}$ dans la catégorie des $\overline{\mathbb{Q}}_p$-isocristaux linéaires surconvergents sur X. Un résultat frappant récent, se basant sur la correspondance de Langlands et la théorie des compagnons pour les faisceaux ℓ-adiques et les isocristaux surconvergents ([Laf02], [Dri12], [Abe18], [AE16]), est un analogue de la conjecture 1.1.3.3.2 et de la proposition 1.2.3.1.1.

Fait 1.2.4.7.2 ([Chi03], [D'Ad17])**.** Soit $\{\mathcal{E}_\ell\}_{\ell\in\mathcal{L}}$ système compatible pure. Alors :

- Il existe un groupe algébrique connexe G^{geo} sur $\overline{\mathbb{Q}}$, une représentation fidèle $\rho : G^{geo} \subseteq \mathrm{GL}(V)$ et un isomorphisme (non canonique) $V \otimes \overline{\mathbb{Q}}_\ell \simeq \mathcal{E}_{\ell,\overline{x}}$ pour tout ℓ, tels que la composition $G^{geo} \otimes \overline{\mathbb{Q}}_\ell \subseteq \mathrm{GL}(V) \otimes \overline{\mathbb{Q}}_\ell \simeq \mathrm{GL}(\mathcal{E}_{\ell,\overline{x}})$ identifie $G^{geo} \otimes \overline{\mathbb{Q}}_\ell$ avec $G(\mathcal{E}_{\ell,\overline{x}})^0$.

- Supposons de plus que $\mathcal{E}_\ell$ est semi-simple pour tout $\ell \in \mathcal{L}$. Alors, il existe un groupe algébrique connexe G sur $\overline{Q}$, une représentation fidèle $\rho : G \subseteq \mathrm{GL}(V)$ et un isomorphisme (non canonique) $V \otimes \overline{\mathbb{Q}}_\ell \simeq \mathcal{E}_{\ell,\overline{x}}$ pour tout ℓ, tel que la composition $G \otimes \overline{\mathbb{Q}}_\ell \subseteq \mathrm{GL}(V) \otimes \overline{\mathbb{Q}}_\ell \simeq \mathrm{GL}(\mathcal{E}_{\ell,\overline{x}})$ identifie $G \otimes \overline{\mathbb{Q}}_\ell$ avec $G(\mathcal{E}_{\ell,\overline{x}})^0$.

1.3 Spécialisations de représentations ℓ-adiques et groupes de Néron-Severi en caractéristique nulle

Soit k un corps de caractéristique nulle. Soit X une variété lisse et géométriquement connexe sur k et soit η le point générique de X. Dans cette section on rappelle certains résultats de Cadoret-Tamagawa ([CT12b], [CT13]) et de André ([And96]).

1.3.1 Un théorème d'image uniforme pour les représentations ℓ-adiques

Dans cette section on discute d'un résultat de finitude de Cadoret et Tamagawa qui améliore le fait 1.2.2.2.2 quand X est une courbe.

1.3.1.1 Lieu exceptionnel

Soit X une courbe et $\rho : \pi_1(X, \overline{\eta}) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ une représentation continue d'image Π. Dans la section 1.2.2.2, on a rappelé que, pour tout $x \in |X|$, le choix d'un chemin étale entre η et x induit une représentation Galoisienne locale

$$\rho_x : \pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta}) \xrightarrow{\rho} \mathrm{GL}_r(\mathbb{Z}_\ell)$$

d'image Π_x et une inclusion $\Pi_x \subseteq \Pi$. Suivant [CK16], on donne la définition suivante.

Définition 1.3.1.1.1. On dit que $x \in |X|$ est Galois générique pour ρ si $\Pi_x \subseteq \Pi$ est un sous-groupe ouvert. Si x n'est pas Galois générique on dit que x est Galois exceptionnel pour ρ.

On note X^{ex}_ρ et X^{gen}_ρ le lieu des points fermés Galois exceptionnels et Galois génériques pour ρ et on pose

$$X^{ex}_\rho(\leq d) := X^{ex}_\rho \cap X(\leq d); \quad X^{gen}_\rho(\leq d) := X^{gen}_\rho \cap X(\leq d).$$

1.3.1.2 Enoncé

La variété X étant géométriquement connexe, on peut considérer la représentation

$$\rho^{geo} : \pi_1(X_{\overline{k}}, \overline{\eta}) \to \pi_1(X, \overline{\eta}) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$$

et son image Π^{geo}. Rappelons la définition suivante de [CT12b, Section 1]

Définition 1.3.1.2.1. On dit que ρ est géométriquement Lie parfaite (ou GLP) si [2] l'abélianisé de tout sous-groupe ouvert de Π^{geo} est fini.

Cadoret et Tamagawa montrent le résultat suivant

Fait 1.3.1.2.2 ([CT12b])**.** Supposons que k est finiment engendré et que X est une courbe. Si ρ est GLP, pour tout entier $d \geq 1$, l'ensemble $X_\rho^{ex}(\leq d)$ est fini et il existe un entier $N \geq 1$, qui ne dépend que de d et ρ, tel que, pour tout $x \in X_\rho^{gen}(\leq d)$, on a $[\Pi : \Pi_x] \leq N$.

Dans les sous-sections suivantes on rappelle les idées principales de la preuve du fait 1.3.1.2.2.

1.3.1.3 Théorie des groupes : un système projectif de sous-groupes

On rappelle que $\Phi(\Pi)$ est le sous-groupe de Frattini de Π, c'est à dire l'intersection de tous les sous-groupes ouverts maximaux de Π. Dans la preuve du fait 1.2.2.2.2, un des ingrédients principaux était de considérer l'ensemble fini $\mathcal{C}$ des sous-groupes $U \subseteq \Pi$ tels que $\Phi(\Pi) \subseteq U$. Pour prouver le fait 1.3.1.2.2, Cadoret et Tamagawa construisent dans [CT12b, Section 3] un système projectif qui raffine $\mathcal{C}$. Pour chaque sous-groupe C de Π, on note

$$C(n) := Ker(C \subseteq \Pi \subseteq \mathrm{GL}_r(\mathbb{Z}_\ell) \to \mathrm{GL}_r(\mathbb{Z}_\ell/\ell^n).$$

On définit $\mathcal{C}_0(\Pi) := \{\Pi\}$ et pour tout entier $n \geq 1$

$$\mathcal{C}_n(\Pi) := \{U \subseteq \Pi \text{ tels que } \Phi(\Pi(n-1)) \subseteq U \text{ et } \Pi(n-1) \not\subseteq U\}.$$

Par [CT12b, Lemma 3.1], les applications $\psi_n : \mathcal{C}_{n+1}(\Pi) \to \mathcal{C}_n(\Pi)$ qui envoient U sur $U\Phi(\Pi(n-1))$ sont bien définies et munissent donc la collection $\{\mathcal{C}_n(\Pi)\}_{n\in\mathbb{N}}$ d'une structure de système projectif. L'analogue du lemme 1.2.2.4.1 est alors le suivant.

Lemme 1.3.1.3.1 ([CT12b, Lemma 3.3])**.**

1. Pour tout entier $n \geq 0$, l'ensemble $\mathcal{C}_n(\Pi)$ est fini ;
2. Pour $n \gg 0$, si $C \subseteq \Pi$ est un sous-groupe fermé tel que $\Pi(n-1) \not\subseteq C$, alors il existe $U \in \mathcal{C}_n(\Pi)$ tel que $C \subseteq U$.

1.3.1.4 Dictionnaire anabélien I

Pour chaque entier $n \geq 0$ on note

$$\mathcal{X}_n := \coprod_{U \in \mathcal{C}_n(\Pi)} X_U \to X.$$

Alors, puisque la famille $\{\Pi(n)\}_{n\in\mathbb{N}}$ forme un système fondamental de voisinages ouverts de 1 dans Π, on a

$$\begin{aligned} x \in X_\rho^{ex} &\Leftrightarrow \text{ pour } n \gg 0\ \Pi(n-1) \not\subseteq \Pi_x \\ &\Leftrightarrow \text{ pour } n \gg 0 \text{ il existe } U \in \mathcal{C}_n \text{ avec } \Pi_x \subseteq U \qquad (\text{lemme 1.3.1.3.1(2)}) \\ &\Leftrightarrow \text{ pour } n \gg 0\ x \in \mathrm{Im}(\mathcal{X}_n(k(x)) \to X(k(x)) \qquad (\text{remarque 1.2.2.3.1}) \end{aligned}$$

Cela montre que

$$X_\rho^{ex}(\leq d) = \bigcap_{n\geq 1} \mathrm{Im}(\mathcal{X}_n(\leq d) \to X(\leq d))$$

[2] La terminologie vient du fait que cette condition est équivalente à ce que $(Lie(\Pi^{geo}))^{ab} = 0$.

et que, pour $n \gg 0$, on a

$$\{x \in X(\leq d) \text{ with } [\Pi : \Pi_x] \leq [\Pi : \Pi(n)]\} \subseteq X(\leq d) - \mathrm{Im}(\mathcal{X}_n(\leq d) \to X(\leq d)). \quad (1.3.1.4.1)$$

Par (1.3.1.4.1), comme Π a un nombre fini de sous-groupes ouverts d'indices bornés et $\mathcal{C}_n(\Pi)$ est fini, pour montrer le fait 1.3.1.2.2 il suffit de montrer que, pour $n \gg 0$ et pour tout $U \in \mathcal{C}_n(\Pi)$, l'ensemble $X_U(\leq d)$ est fini.

1.3.1.5 Enoncé Diophantien : genre et gonalité

La finitude du nombre de points rationnels d'une courbe lisse Y est contrôlée par le genre g_Y et la gonalité[3] γ_Y de la compactification lisse de $Y_{\overline{k}}$. Plus précisément, on a le résultat suivant :

Fait 1.3.1.5.1. Soit k un corps finiment engendré de caractéristique nulle et soit Y une courbe propre et lisse sur k.

1. ([FW84]) : Si $g_Y \geq 2$ alors $Y(k)$ est fini.

2. ([Fal91], [Fre94]) : Si $\gamma_Y \geq 2d+1$ alors $Y(\leq d)$ est fini.

Revenons aux revêtements $X_U \to X$, on veut maintenant montrer que leurs genre et leur gonalité sont grands. Pour chaque sous-groupe ouvert $U \subseteq \Pi$, on note $k \subseteq k_U$ la plus petite extension finie de k sur laquelle X_U est géométriquement connexe et on note g_U et γ_U le genre et la gonalité d'une compactification lisse de $X_U \times_{k_U} \overline{k}$. Alors, pour prouver le théorème 1.3.1.2.2, il est suffisant de montrer le fait suivant.

Fait 1.3.1.5.2. Supposons que ρ est GLP et fixons des entiers $d_1 \geq 0$, $d_2 \geq 1$. Alors :

1. ([CT12b, Corollary 3.8]) : Il existe un entier $N_g \geq 1$, dépendant uniquement de ρ, d_1, d_2, tel que pour tout entier $n \geq N_g$ et tout $U \in \mathcal{C}_n(\Pi)$ on a $g_U \geq d_1$ ou $[k_U : k] \geq d_2$.

2. ([CT13, Corollary 3.11]) : Il existe un entier $N_\gamma \geq 1$, dépendant uniquement de ρ, d_1, d_2, tel que pour tout entier $n \geq N_\gamma$ et tout $U \in \mathcal{C}_n(\Pi)$ on a $\gamma \geq d_1$ ou $[k_U : k] \geq d_2$.

Remarque 1.3.1.5.3. A posteriori, via la formule de Riemann-Hurwitz 1.3.1.5.2(2) implique 1.3.1.5.2(1) mais 1.3.1.5.2(1) est en fait utilisé dans la preuve de 1.3.1.5.2(2).

1.3.1.6 Dictionnaire anabélien II : l'hypothèse GLP

Pour illustrer l'idée de la preuve du fait 1.3.1.5.2(1), on montre dans cette section, suivant [CT12a, Section 4.1.3], que si $k = \overline{k}$, alors la représentation ρ est GLP et si Π est infini, alors $g_{\Pi(n)}$ tend vers l'infini. Soit $n_0 \geq 1$ un entier. Pour tout $n \geq n_0$, la formule de Riemann Hurwitz pour le recouvrement $X_{\Pi(n)} \to X_{\Pi(n_0)}$ implique que

$$\lim_{n\to+\infty} 2g_{\Pi(n)} - 2 \geq \lim_{n\to+\infty} (|\Pi(n_0)/\Pi(n)|)(2g_{\Pi(n_0)} - 2) \quad (1.3.1.6.1)$$

Puisque Π est infini, on a

$$\lim_{n\to+\infty} |\Pi(n_0)/\Pi(n)| = |\Pi(n_0)| = +\infty.$$

On en déduit que si $sup_n(g_{\Pi(n)}) \geq 2$ il existe un n_0 tel que $g_{\Pi(n_0)} \geq 2$ et l'équation (1.3.1.6.1) impliquent que $g_{\Pi(n)}$ tendent vers l'infini. On doit donc éliminer les deux possibilités suivantes :

[3] Rappelons que la gonalité d'une courbe propre et lisse Y sur $\overline{k}$ est le degré minimum d'un morphisme non constant $Y \to \mathbb{P}^1_{\overline{k}}$.

1. $\sup(g_{\Pi(n)}) = 1$. Alors il existe n_0 tel que pour tout $n \geq n_0$ la compactification lisse de $X_{\Pi(n)}$ est une courbe elliptique. Puisque tous les morphismes finis entre courbes elliptiques sont non ramifiés, le groupe de Galois $\Pi(n_0)/\Pi(n)$ de $X_{\Pi(n)} \to X_{\Pi(n_0)}$ est un quotient du groupe fondamental étale de la compactification lisse de $X_{\Pi(n_0)}$. En particulier il est abélien et donc $\Pi(n_0) = \varprojlim_n \Pi(n_0)/\Pi(n)$ est abélien et infini. Mais cela contredit le fait que ρ est GLP, puisque $\Pi(n_0)$ serait un sous-groupe ouvert abélien infini de Π.

2. $\sup(g_{\Pi(n)}) = 0$. Alors pour tout $n \geq 0$, la compactification lisse de $X_{\Pi(n)}$ est isomorphe à $\mathbb{P}^1$. Le groupe de Galois $\Pi(1)/\Pi(n)$ du recouvrement $X_{\Pi(n)} \to X_{\Pi(1)}$ est donc un sous-groupe de $\mathrm{PGL}_2(k)$. En utilisant la classification des sous-groupes finis de $\mathrm{PGL}_2(k)$ (cf. par exemple [Cad12a, Corollary 10]) on obtient une contradiction grâce à l'hypothèse GLP comme dans le cas 1.

La preuve du fait 1.3.1.5.2(1) est significativement plus difficile, car les recouvrements $X_U \to X$ ne sont pas Galoisiens en général. L'idée est de prendre un recouvrement Galoisien $X_{\widetilde{U}} \to X$ au dessus de $X_U \to X$ et proche de la clôture Galoisienne de $X_U \to X$ et alors :

- On applique d'abord l'argument précédent à $X_{\widetilde{U}}$ ([CT12b, Section 3.3.1]) ;
- On compare ensuite le genre de $X_{\widetilde{U}}$ et X_U via la formule de Riemann-Hurwitz ([CT12b, Section 3.3.2]).

On discutera plus en détails de cette stratégie dans la section 2.1.1.3.

1.3.2 Spécialisations du groupe de Néron-Severi

Soit $Y \to X$ un morphisme propre et lisse. Dans cette section on discute d'un résultat de André, qui lie les faits 1.2.2.2.2 et 1.3.1.2.2 à la spécialisation du groupe de Néron-Severi.

1.3.2.1 Points NS-génériques

On spécialise la discussion de la section 1.2.1 au cas des diviseurs. Soit Z une variété propre et lisse sur k. Dans ce cadre, puisque les équivalences algébriques et numériques coïncident rationnellement pour les diviseurs, pour tout couple de nombres premiers ℓ, ℓ' on a les égalités

$$\mathrm{CH}^1_{\ell'}(Z_{\overline{k}}) \otimes \mathbb{Q} = \mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q} = \mathrm{CH}^1_{\ell}(Z_{\overline{k}}) \otimes \mathbb{Q}.$$

De plus, puisque $H^1(\pi_1(k), Pic^0(Z))$ est de torsion, la suite exacte de k-schémas en groupes

$$0 \to Pic^0_Z \to Pic_Z \to \mathrm{NS}_Z \to 0$$

montre que $\mathrm{NS}(Z) \otimes \mathbb{Q} = (\mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q})^{\pi_1(k)}$. Donc, pour tout $x \in X$, les morphismes de spécialisation de la section 1.2.1 pour le morphisme$f : Y \to X$ donnent

$$sp_{\eta,x} : \mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \to \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q} \quad \text{et} \quad sp^{ar}_{\eta,x} : \mathrm{NS}(Y_{\eta}) \otimes \mathbb{Q} \to \mathrm{NS}(Y_x) \otimes \mathbb{Q}.$$

Définition 1.3.2.1.1. On dit que $x \in |X|$ est NS-générique (resp. arithmétiquement NS-générique) pour $f : Y \to X$ si $sp_{\eta,x}$ (resp. $sp^{ar}_{\eta,x}$) est un isomorphisme.

1.3.2.2 NS-générique vs Galois générique

Pour tout $x \in X$, le choix d'un chemin étale entre $\overline{x}$ et $\overline{\eta}$ induit des isomorphismes

$$\pi_1(X,\overline{x}) \simeq \pi_1(X,\overline{\eta}), \quad H^2(Y_{\overline{\eta}},\mathbb{Q}_\ell(1))) \simeq R^2f_*\mathbb{Q}_\ell(1)_{\overline{\eta}} \simeq R^2f_*\mathbb{Q}_\ell(1)_{\overline{x}} \simeq H^2(Y_{\overline{x}},\mathbb{Q}_\ell(1))),$$

qui identifient l'action de $\pi_1(x,\overline{x})$ induite par restriction via $\pi_1(x,\overline{x}) \to \pi_1(X,\overline{x}) \simeq \pi_1(X,\overline{\eta})$ sur $H^2(Y_{\overline{\eta}},\mathbb{Q}_\ell(1)) \simeq H^2(Y_{\overline{x}},\mathbb{Q}_\ell(1)))$ avec l'action naturelle de $\pi_1(x,\overline{x})$ sur $H^2(Y_{\overline{x}},\mathbb{Q}_\ell(1)))$. Rappelons que le morphisme de spécialisation fait commuter le diagramme suivant

$$\begin{tikzcd}
Pic(Y_\eta)\otimes\mathbb{Q} \arrow[d, "c_{Y_\eta}"] & Pic(Y)\otimes\mathbb{Q} \arrow[l, "i_\eta^*"'] \arrow[r, "i_x^*"] & Pic(Y_x)\otimes\mathbb{Q} \arrow[d, "c_{Y_x}"] \\
\mathrm{NS}(Y_{\overline{\eta}})\otimes\mathbb{Q} \arrow[rr, hook, "sp_{\eta,x}"] \arrow[d, hook] & & \mathrm{NS}(Y_{\overline{x}})\otimes\mathbb{Q} \arrow[d, hook] \\
H^2(Y_{\overline{\eta}},\mathbb{Q}_\ell(1)) & \simeq & H^2(Y_{\overline{x}},\mathbb{Q}_\ell(1)).
\end{tikzcd}$$

et que $x \in |X|$ est dit Galois générique (resp. strictement Galois générique) pour $\rho_\ell^{2,1} : \pi_1(X) \to \mathrm{GL}(H^2(Y_{\overline{x}},\mathbb{Q}_\ell(1)))$ si l'image de $\pi_1(x,\overline{x})$ agissant sur $H^2(Y_{\overline{\eta}},\mathbb{Q}_\ell(1))$ est ouverte (resp. coïncide) dans (resp. avec) l'image de $\pi_1(X,\overline{\eta})$ agissant sur $H^2(Y_{\overline{\eta}},\mathbb{Q}_\ell(1))$. La conjecture 1.1.2.1.1 prédit que tout point (strictement) Galois générique est (arithmétiquement) NS-générique. André a montré que c'est vrai sans supposer la conjecture 1.1.2.1.1.

Fait 1.3.2.2.1 ([And96]). Tout point (strictement) Galois générique pour $\rho_\ell^{2,1}$ est (arithmétiquement) Néron-Severi générique.

En combinant le fait 1.3.2.2.1 avec les faits 1.2.2.2.2 et 1.3.1.2.2, on obtient l'existence et l'abondance des points (arithmétiquement) NS-génériques. La preuve du fait 1.3.2.2.1 se décompose en deux étapes :

- On relie les cycles algébriques à la cohomologie via la conjecture de Hodge variationnelle pour les diviseurs (fait 1.2.3.2.1) ;
- On relie la théorie de Hodge à la cohomologie ℓ-adique via la comparaison entre le site étale et le site singulier.

Dans la sous-section suivante, on rappelle plus en détails la preuve du fait 1.3.2.2.1 (cf. aussi [CC18, Proposition 3.2.1]).

1.3.2.3 Preuve du fait 1.3.2.2.1

Soit $x \in |X|$ un point Galois générique pour $\rho^{2,1}$. En remplaçant X par un recouvrement fini étale on peut supposer que $\mathrm{NS}(Y_{\overline{\eta}})\otimes\mathbb{Q} = \mathrm{NS}(Y_\eta)\otimes\mathbb{Q}$, $\mathrm{NS}(Y_{\overline{x}})\otimes\mathbb{Q} = \mathrm{NS}(Y_x)\otimes\mathbb{Q}$ et que l'adhérence de Zariski $G_\ell^{2,1}$ de l'image de $\pi_1(X,\overline{\eta})$ agissant sur $H^2(Y_{\overline{\eta}},\mathbb{Q}_\ell(1))$ est connexe. Le diagramme commutatif cartésien de variétés sur k suivant

$$\begin{tikzcd}
Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \arrow[dr, phantom, "\square"] & Y_\eta \arrow[l, hook'] \arrow[d] \\
k(x) \arrow[r, "x"] & X & k(\eta). \arrow[l]
\end{tikzcd}$$

induit un diagramme commutatif

$$\begin{tikzcd}
Pic(Y_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[d, two heads] & Pic(Y_{\overline{k}}) \otimes \mathbb{Q} \arrow[l, two heads] \arrow[r] & Pic(Y_{\overline{x}}) \otimes \mathbb{Q} \arrow[d, two heads] \\
\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[rr, hook, "sp_{\eta,x}"] & & \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q}.
\end{tikzcd}$$

Il est suffisant de montrer que tout $z_x \in \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q}$ se relève en un élément de $Pic(Y_{\overline{k}}) \otimes \mathbb{Q}$. Puisque l'image de $Pic(Y_{\overline{k}}) \otimes \mathbb{Q} \to H^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1))$ s'identifie[4] à l'image de $Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q} \to H^2(Y_{\mathbb{C}}, \mathbb{Q}_\ell(1))$ via l'isomorphisme de changement de base $H^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) \simeq H^2(Y_{\mathbb{C}}, \mathbb{Q}_\ell(1))$ et puisque le groupe de Néron-Severi est invariant par extension de corps algébriquement clos, il est suffisant de montrer que tout $z_x \in \mathrm{NS}(Y_{\overline{x},\mathbb{C}}) \otimes \mathbb{Q}$ se relève en un élément de $Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q}$. Considérons le diagramme commutatif

$$\begin{tikzcd}
Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q} \arrow[r] \arrow[dd] & Pic(Y_{\overline{x},\mathbb{C}}) \otimes \mathbb{Q} \arrow[d, two heads] \\
& \mathrm{NS}(Y_{\overline{x},\mathbb{C}}) \otimes \mathbb{Q} \arrow[d, hook] \\
H^0(X^{an}_{\mathbb{C}}, R^2 f_* \mathbb{Q}) \arrow[r, hook] & H^2_B(Y_{\overline{x},\mathbb{C}}, \mathbb{Q}).
\end{tikzcd}$$

Soit $z_x \in \mathrm{NS}(Y_{\overline{x},\mathbb{C}}) \otimes \mathbb{Q}$. Par la conjecture de Hodge variationnelle pour les diviseurs (fait 1.2.3.2.1) il est suffisant de montrer que z_x est dans l'image de $H^0(X, R^2 f^{an}_* \mathbb{Q}) \hookrightarrow H^2_B(Y_{\overline{x},\mathbb{C}}, \mathbb{Q})$. Puisque z_x est fixé par $\pi_1(x, \overline{x})$, le groupe $G^{2,1}_\ell$ est connexe et x est Galois générique, z_x est fixé par $\pi_1(X, \overline{x})$, donc par $\pi_1(X_{\mathbb{C}}, \overline{x})$. Grâce à la comparaison entre le site étale et le site singulier, z_x est alors fixé par $\pi_1^{top}(X^{an}_{\mathbb{C}}, \overline{x})$, il est donc dans l'image de $H^0(X, R^2 f^{an}_* \mathbb{Q}) \simeq H^2_B(Y_{\overline{x},\mathbb{C}}, \mathbb{Q})^{\pi_1^{top}(X_{\mathbb{C}}, \overline{x})} \hookrightarrow H^2_B(Y_{\overline{x},\mathbb{C}}, \mathbb{Q})$. Cela termine la preuve du fait 1.3.2.2.1.

[4] Cela découle de l'invariance de la cohomologie étale sous les extensions de corps algébriquement clos en caractéristique nulle, la suite exacte de Kummer et le fait que la flèche $H^2(Y_{\overline{k}}, \mathbb{G}_m) \to H^2(Y_{\mathbb{C}}, \mathbb{G}_m)$ est injective.

Chapter 1

Preliminaries (in English)

1.1 Absolute setting

Let k be a field of characteristic $p \geq 0$ and let X be a smooth proper k-variety.

The main topic of arithmetic geometry is to study the interplay between the arithmetic and the geometric properties of X. Since these are extremely rich and complicated, the general strategy developed in the 20^{th} century is to associate to X abelian groups or vector spaces endowed with some additional structure, encoding part of the properties of X. For example:

- the Chow group $\mathrm{CH}^i(X)$ of co-dimensional i cycles modulo rational equivalence ([Ful98]);
- if $k = \mathbb{C}$, the Betti cohomology $H^i(X^{an}, \mathbb{Q})$ endowed with an Hodge structure ([GH94]);
- if k is any field, for every $\ell \neq p$ the étale ℓ-adic cohomology $H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ endowed with a continuous action of $\pi_1(k)$ ([SGA4]);
- if k is perfect and $p > 0$, the crystalline cohomology $H^i_{crys}(X, K)$ endowed with an action of the absolute Frobenius of k ([B078]).

The theory of motives ([And04, Section 4]) and the fullness conjectures ([And04, Section 7]) (as the Hodge conjecture ([Hod50] and Tate conjecture ([Tat65])) give a conjectural framework to compare these invariants. In this section we quickly review them.

1.1.1 Algebraic cycles and motives

1.1.1.1 Algebraic cycles

Let L be an integral ring of characteristic zero and let $Z^i(X)$ be the free abelian group generated by codimension i integral subvarieties of X. Let $\sim$ be an adequate equivalence relation on $Z^i(X)$ (see [And04, Section 3.1]) and write $\mathrm{CH}^i_L(X)_\sim$ for the quotient of $Z^i(X) \otimes L$ by this equivalence.

If $\sim = rat$ is the rational equivalence, then $\mathrm{CH}^i(X) := \mathrm{CH}^i_{\mathbb{Z}}(X)_{rat}$ is called the Chow group of co-dimensional i cycles modulo rational equivalence and if $L \subseteq L'$ is an inclusion of rings then $\mathrm{CH}^i_L(X)_{rat} \otimes_L L' \simeq \mathrm{CH}^i_{L'}(X)_{rat}$ ([And04, 3.2.2]). In general, the groups $\mathrm{CH}^i(X)$ are complicated and of infinite rank. When $i = 1$, the group $\mathrm{CH}^1(X)$ identifies with the Picard group $Pic(X)$ of X, classifying line bundles up to isomorphism.

If $\sim = alg$ is the algebraic equivalence, then $\mathrm{CH}^i_{alg}(X) := \mathrm{CH}^i_{\mathbb{Z}}(X)_{alg}$ is called the Chow group of co-dimensional i cycles modulo algebraic equivalence. If $L \subseteq L'$ is an inclusion of rings, then $\mathrm{CH}^i_L(X)_{alg} \otimes_L L' \simeq \mathrm{CH}^i_{L'}(X)_{alg}$ ([And04, 3.7.3]). Since rational equivalence is finer than algebraic equivalence, one has a canonical quotient morphism $q : \mathrm{CH}^i_L(X)_{rat} \twoheadrightarrow \mathrm{CH}^i_L(X)_{alg}$, which, when $i = 1$, identifies with the natural morphism $Pic(X) \otimes L \to NS(X) \otimes L$, where $NS(X) := Pic_X(k)/Pic^0_X(k)$ is the quotient of the k-points of the Picard scheme Pic_X of X modulo the k-points of its neutral component Pic^0_X.

If $\sim = num$ is the numerical equivalence, then $\mathrm{CH}^i_{num}(X) := \mathrm{CH}^i_{\mathbb{Z}}(X)_{num}$ is called the Chow group of co-dimensional i cycles modulo numerical equivalence. In general $\mathrm{CH}^i_L(X)_{num}$ is a free and finitely generated L-module and if $L \subseteq L'$ is an inclusion of rings, then $\mathrm{CH}^i_L(X)_{num} \otimes_L L' \simeq \mathrm{CH}^i_{L'}(X)_{num}$ ([And04, 3.7.2.1]). Since algebraic equivalence is finer than numerical equivalence, one has a canonical quotient morphism $q : \mathrm{CH}^i_L(X)_{alg} \twoheadrightarrow \mathrm{CH}^i_L(X)_{num}$ which, when $i = 1$ and L is a field, identifies ([Mat57], [And04, Section 3.2.7]) the group $\mathrm{CH}^1_{num}(X) \otimes L$ with $NS(X) \otimes L$.

Let now H^* be a Weil-cohomology theory with coefficients in a characteristic zero field $F \supseteq L$ (see [Saa72, Appendices] and [And04, Section 3.3]). Set $c^i_H : \mathrm{CH}^i(X) \otimes L \to H^{2i}(X)$ for the cycle class map for H^* and define the group of cycles of codimension i modulo (the appropriate) H-homological equivalence $\mathrm{CH}^i_L(X)_H$ as the image of $c_X : \mathrm{CH}^i(X) \otimes L \to H^{2i}(X)(i)$. If $L = F$, since $H^{2i}(X)(i)$ has finite F-dimension, $\mathrm{CH}^i_F(X)_H$ is a finite dimensional F-vector space. It is not true in general that the natural map $\mathrm{CH}^i_L(X)_H \otimes_L F \to \mathrm{CH}^i_F(X)_H$ is injective and it is still unknown whether $\mathrm{CH}^i_L(X)_H$ is finitely generated over L.

Since algebraic equivalence is finer than homological equivalence and homological equivalence is finer than numerical equivalence, the quotient $q : \mathrm{CH}^i_L(X)_{alg} \twoheadrightarrow \mathrm{CH}^i_L(X)_{num}$ factorizes as the composition of $q_1 : \mathrm{CH}^i_L(X)_{alg} \twoheadrightarrow \mathrm{CH}^i_L(X)_H$ and $q_2 : \mathrm{CH}^i_L(X)_H \twoheadrightarrow \mathrm{CH}^i_L(X)_{num}$. Under q_2, $\mathrm{CH}^1_F(X)_H$ identifies ([And04, Proposition 3.4.6.1]) with $NS(X) \otimes F$. In general, one of the standard conjecture of Grothendieck ([Kle94, Conjecture D], [And04, Section 5.4.1]), predicts that $\mathrm{CH}^i_F(X)_H = \mathrm{CH}^i_F(X)_{num}$.

1.1.1.2 Motives

Assume now that $L = F$. For $\sim \in \{num, H\}$, write $\mathbf{Mot}^F_\sim(k)$ for the F-linear pseudoabelian rigid tensor category of pure motives up to $\sim$-equivalence ([And04, Section 4.1.3]), $\mathbf{SPV}(k)$ for the category of smooth proper varieties and $H^* : \mathbf{SPV}(k) \to \mathbf{Mot}^F_\sim(k)$ for the canonical functor. On the one hand, there is a realization functor $R_H : \mathbf{Mot}^F_H(k) \to \mathbf{GrVect}_F$ into the category of graded F-vector spaces. On the other hand, Jannsen proved ([Jan92]) that $\mathbf{Mot}^F_{num}(k)$ is a semisimple abelian category.

Under the standard conjectures of Grothendieck ([Gro69]), one should be able to modify the commutativity constraint in $\mathbf{Mot}^F_H(k)$ (see [And04, Section 5.1.3]) to obtain a fibre functor $R_H : \mathbf{Mot}^F_H(k) \to \mathbf{Vect}_F$. Combining [Jan92], the standard conjectures of Grothendieck and the conjecture $\mathrm{CH}^i_F(X)_H = \mathrm{CH}^i_F(X)_{num}$, the category $\mathbf{Mot}^F_H(k)$ should be a semisimple F-linear Tannakian category ([Saa72]) endowed with a fibre functor $R_H : \mathbf{Mot}^F_H(k) \to \mathbf{Vect}_F$. So for every $\mathcal{M}$ in $\mathbf{Mot}^F_H(k)$ one would then be able to consider the Tannakian subcategory $\langle \mathcal{M} \rangle \subseteq \mathbf{Mot}^F_H(k)$ generated by $\mathcal{M}$ and its reductive Tannakian group $G(\mathcal{M})$ ([And04, Section 6]).

Assume now that the essential image of $R : \mathbf{Mot}^F_H(k) \to \mathbf{Vect}_F$ factors trough an enriched F-linear Tannakian category ([And04, Section 7.1.1]) $\mathcal{C}$ (for example the category of polarized Hodge structure, the category of continuous $\mathbb{Q}_\ell$-linear $\pi_1(k)$-representations or the category of K-vector spaces endowed with an automorphism). Then the fullness conjectures ([Hod50], [Tat65], [And04, Section 7.1]) predict that $R_H : \mathbf{Mot}^F_H(k) \to \mathcal{C}$ is fully faithful and the Grothendieck-Serre semisimplicity conjecture ([And04, Section 7.3]) predicts that the essential image of $R_H : \mathbf{Mot}^F_H(k) \to \mathcal{C}$ is a semisimple subcategory of $\mathcal{C}$. Since reductive algebraic groups are determined by their tensor invariants ([DM82, Proposition 3.1]), this would imply that $G(\mathcal{M})$ identifies with the Tannakian group $G(R_H(\mathcal{M}))$ of the Tannakian category $\langle R_H(\mathcal{M}) \rangle$ generated by $\mathcal{M}$ in $\mathcal{C}$.

Let H' be another cohomology theory with coefficients in $F \subseteq F'$ such that $H' \otimes_F F' \simeq H$

as Weil-cohomology theories. Then there is a natural functor $-\otimes F : \mathbf{Mot}_H^F(k) \to \mathbf{Mot}_{H'}^{F'}(k)$ and the fullness and the semisimplicity conjectures for H and H', together with the equivalence of homological and numerical equivalence, imply that, for every $\mathcal{M} \in \mathbf{Mot}_H^F(k)$, one should have

$$G(R_{H'}(\mathcal{M}\otimes F')) \simeq G((H')^*(\mathcal{M}\otimes F')) \simeq G(H^*(\mathcal{M})\otimes F') \simeq G(R_H(\mathcal{M}))\otimes F'.$$

1.1.2 ℓ-adic cohomology

In this section ℓ is a prime $\neq p$.

1.1.2.1 Étale cohomology and the Tate conjecture

For every integers $i \geq 0, j \in \mathbb{Z}$, Grothendieck defined ([SGA4]) an étale cohomology group $H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j))$. It is a finite dimensional $\mathbb{Q}_\ell$-vector space ([SGA4, XIV, Corollaire 1.2]) endowed with a continuous action of $\pi_1(k)$ and the image CH^i_ℓ of the cycles class map $c^i_\ell : \mathrm{CH}^i(X) \to H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i))$ lies in the subspace

$$\bigcup_{[k':k]<+\infty} H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k')}.$$

In this setting, the fullness conjecture is the Tate conjecture ([Tat65]) and predicts the following relation between algebraic cycles and cohomology.

Conjecture 1.1.2.1.1 ($\mathrm{T}(X,i,\ell)$)**.** If k is finitely generated, then the cycle class map

$$c^i_\ell : \mathrm{CH}^i(X_{\overline{k}})\otimes\mathbb{Q}_\ell \to \bigcup_{[k':k]<+\infty} H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k)}$$

is surjective.

Conjecture 1.1.2.1.1 is widely open in general, but when $i = 1$ it is known for abelian varieties ([Tat66], [Zar75], [Zar77], [FW84]), K3 surfaces ([NO85], [Tan95], [And96a], [Char13], [MP15], [KMP15]) and some other special class of k-varieties; see for example [MP15, Section 5.13] and [Moo17].

1.1.2.2 Monodromy groups

The action of $\pi_1(k)$ on $H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j))$ gives rise to a continuous homomorphism

$$\rho^{i,j}_\ell : \pi_1(k) \to \mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$$

and we set $\Pi^{i,j}_\ell := \rho^{i,j}_\ell(\pi_1(k))$. As any closed subgroup of $\mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$, $\Pi^{i,j}_\ell$ is a compact ℓ-adic Lie group ([Ser65, Lie Groups, Chapter V, Section 9]), hence a topologically finitely generated almost pro-ℓ group ([DdSMSeg91]). Write $G^{i,j}_\ell$ for the Zariski closure of $\Pi^{i,j}_\ell$ in $\mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$. From the Tannakian point of view, if we write $\langle\rho^{i,j}_\ell\rangle$ for the Tannakian subcategory in $\mathbf{Rep}_{\mathbb{Q}_\ell}(\pi_1(k))$ generated by $\rho^{i,j}_\ell$, the algebraic group $G^{i,j}_\ell$ is characterized ([And04, Section 7.1.3]) by the fact that $Rep_{\mathbb{Q}_\ell}(G^{i,j}_\ell) \simeq \langle\rho^{i,j}_\ell\rangle$. If $\rho^{i,j}_\ell$ is semisimple, then $G^{i,j}_\ell$ can be also described as the subgroup of $\mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$ fixing $(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j))^{\otimes m} \otimes (H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j))^\vee)^{\otimes n})^{\pi_1(k)}$ for all integers $n, m \geq 0$.

1.1.3 Characteristic zero: Betti cohomology and Hodge theory

Assume now that $p = 0$ and, to simplify, that there is an inclusion $k \subseteq \mathbb{C}$.

1.1.3.1 Betti cohomology and the Hodge conjecture

To $X_{\mathbb{C}}$ one can associated ([Ser56, Section 2], [SGA1, Exposé XII]) a complex analytic space $X_{\mathbb{C}}^{an}$ and hence consider the Betti cohomology $H^i_B(X,\mathbb{Q}) := H^i(X_{\mathbb{C}}^{an},\mathbb{Q})$ of X. The Hodge decomposition ([Hod41], [GH94, Chapter 0]) gives a canonical isomorphism

$$H^i(X_{\mathbb{C}}^{an},\mathbb{Q})\otimes\mathbb{C}\simeq\bigoplus_{p+q=i}H^p(X_{\mathbb{C}}^{an},\Omega^q_{X_{\mathbb{C}}^{an}}).$$

Hence $H^i(X_{\mathbb{C}}^{an},\mathbb{Q})$ is a polarized $\mathbb{Q}$-Hodge structure ([Moo04, Section 1]), which, for every $j\in\mathbb{Z}$, we can twist with $\mathbb{Q}(j)$ to obtain $H^i(X_{\mathbb{C}}^{an},\mathbb{Q}(j))$. Combining the cycle class map $\mathrm{CH}^i(X_{\mathbb{C}}^{an})\otimes\mathbb{Q}\to H^{2i}(X_{\mathbb{C}}^{an},\mathbb{Q}(i))\subseteq H^i(X_{\mathbb{C}}^{an},\mathbb{Q}(i))\otimes\mathbb{C}$, with the isomorphism ([Ser56]) $\mathrm{CH}^i(X_{\mathbb{C}})\simeq\mathrm{CH}^i(X_{\mathbb{C}}^{an})$ induced by the analytification functor, we get a cycles class map $c^i_B:\mathrm{CH}^i(X_{\mathbb{C}})\otimes\mathbb{Q}\to H^{2i}(X_{\mathbb{C}}^{an},\mathbb{Q}(i))\subseteq H^n_B(X,\mathbb{Q}(i))\otimes\mathbb{C}$ whose image is contained in $H^i(X_{\mathbb{C}}^{an},\Omega^i_{X_{\mathbb{C}}^{an}})\cap H^{2i}_B(X,\mathbb{Q}(i)) =: H^{2i}_B(X,\mathbb{Q}(i))^{0,0}$. In this setting, the fullness conjecture is the Hodge conjecture ([Hod50]):

Conjecture 1.1.3.1.1 ($H(X,i)$)**.** The cycle class map

$$c^i_B:\mathrm{CH}^i(X_{\mathbb{C}})\otimes\mathbb{Q}\to H^{2i}_B(X,\mathbb{Q}(i))^{0,0}$$

is surjective

While $\mathrm{T}(X,1,\ell)$ is widely open in general, from the exponential exact sequence ([GH94, Pag. 163]) and the Hodge decomposition, one deduces the so called Lefschetz (1,1) theorem.

Fact 1.1.3.1.2 ([Lef24][GH94, Pag. 163-164])**.** Conjecture $H(X,1)$ holds.

Remark 1.1.3.1.3. Even if ℓ-adic cohomology and Betti cohomology should be incarnations of the same motive, we already see that they have some distinct specific features: ℓ-adic cohomology enables us to use the theory of ℓ-adic Lie groups and the action of $\pi_1(k)$, while Betti cohomology enables us to use complex Hodge theoretic analytic techniques. Comparison results between them could be then helpful to combine these different information.

1.1.3.2 Monodromy groups

The Hodge structure on $H^i(X_{\mathbb{C}}^{an},\mathbb{Q}(j))$ is described ([Moo04, Section 3]) via a morphism of algebraic groups

$$h^{i,j}_B: Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m\to\mathrm{GL}(H^i(X_{\mathbb{C}}^{an},\mathbb{Q}(j))\otimes\mathbb{R}),$$

and the Mumford Tate group $G^{i,j}_B$ is ([Moo04, Section 4]) the smallest connected subgroup of $\mathrm{GL}(H^i(X_{\mathbb{C}}^{an},\mathbb{Q}(j)))$ such that $G^{i,j}_B\otimes\mathbb{R}$ contains $\mathrm{Im}(h^{i,j}_B)$. As in the ℓ-adic setting, the group $G^{i,j}_B$ can be characterized as the unique (up to isomorphism) algebraic group, such that $Rep_{\mathbb{Q}}(G^{i,j}_B)$ is equivalent to Tannakian subcategory $\langle H^i(X_{\mathbb{C}}^{an},\mathbb{Q})\rangle$ generated by $H^i(X_{\mathbb{C}}^{an},\mathbb{Q})$ in the category of polarized Hodge structures. Since the category of polarized $\mathbb{Q}$-Hodge structures is semisimple ([Moo04, Proposition 4.9]), $G^{i,j}_B$ is reductive, hence it can be described as the subgroup of $\mathrm{GL}(H^i(X_{\mathbb{C}}^{an},\mathbb{Q}(j)))$ fixing all the (0,0) classes in $H^i(X_{\mathbb{C}}^{an},\mathbb{Q}(j))^{\otimes n}\otimes(H^i(X_{\mathbb{C}}^{an},\mathbb{Q}(j))^\vee)^{\otimes m}$ for all integers $m,n\geq 0$.

1.1.3.3 Comparison of singular and étale sites

By the invariance of étale cohomology under algebraically closed field extensions in characteristic zero ([SGA4$\frac{1}{2}$, Corollaire 5.3.3]) and the Artin comparison theorem ([SGA4, XI, Theorem 4.4]), there are canonical isomorphisms

$$H^i(X_{\overline{k}},\mathbb{Q}_\ell)\simeq H^i(X_{\mathbb{C}},\mathbb{Q}_\ell)\simeq H^i(X_{\mathbb{C}}^{an},\mathbb{Q})\otimes\mathbb{Q}_\ell,$$

fitting into following diagram commutative

$$\begin{tikzcd}
(\mathrm{CH}^i(X_{\mathbb{C}}) \otimes \mathbb{Q}) \otimes \mathbb{Q}_\ell \arrow[r] \arrow[d, "c_B^i \otimes \mathbb{Q}_\ell"] & \mathrm{CH}^i(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[d, "c_\ell^i"] \\
H^{2i}(X_{\mathbb{C}}^{an}, \mathbb{Q}(i)) \otimes \mathbb{Q}_\ell \arrow[r, "\simeq"] & H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i)).
\end{tikzcd}$$

The Mumford Tate group $G_B^{i,j} \subseteq \mathrm{GL}(H^i(X_{\mathbb{C}}^{an}, \mathbb{Q}(j)))$ maps, via the Artin comparison isomorphism, to a $\mathbb{Q}_\ell$-algebraic group $G_B^{i,j} \otimes \mathbb{Q}_\ell \subseteq \mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$. The philosophy of motives (Section 1.1.1.2), Conjectures 1.1.2.1.1 and 1.1.3.1.1 then predict the Mumford-Tate conjecture:

Conjecture 1.1.3.3.1. If k is finitely generated, $G_B^{i,j} \otimes \mathbb{Q}_\ell = (G_\ell^{i,j})^0$ modulo the Artin comparison isomorphism.

While in general Conjectures 1.1.2.1.1 and 1.1.3.1.1 are widely open, Conjecture 1.1.3.3.1 is known in some cases (see for example [Pin98]) and when X is an abelian variety, Deligne proved ([DM82]) that there is an inclusion $(G_\ell^{i,j})^0 \subseteq G_B^{i,j} \otimes \mathbb{Q}_\ell$

Conjecture 1.1.3.3.1 predicts the following, which could be stated over any finitely generated fields.

Conjecture 1.1.3.3.2. If k is finitely generated, there exists a connected algebraic group $G^{i,j}$ over $\mathbb{Q}$ and a faithful representation $G \subseteq \mathrm{GL}(V^{i,j})$ such that for every $\ell \neq p$ there is an isomorphism $V^{i,j} \otimes \mathbb{Q}_\ell \simeq H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j))$ identifying $G^{i,j} \otimes \mathbb{Q}_\ell \subseteq \mathrm{GL}(V^{i,j} \otimes \mathbb{Q}_\ell) \simeq \mathrm{GL}(H^i(X_{\overline{k}}, \mathbb{Q}_\ell(j)))$ with $(G_\ell^{i,j})^0$.

1.1.4 Positive Characteristic: crystalline cohomology

Assume now that $p > 0$ and k is perfect. Write $W(k)$ (or just W) for the Witt ring of k and $K(k)$ (or just K) for the fraction field of W. In this section we recall the main ideas in the construction of a p-adic Weil cohomology theory. A classical example of Serre (see for example [Gro68, Section 1.7]) shows that there is no cohomology theory with $\mathbb{Q}_p$-coefficients, so that we will define a Weil cohomology theory with K coefficients.

1.1.4.1 Infinitesimal site

Let S be a scheme, $f : Z \to S$ a morphism and write $H^i_{dr}(Z/S)$ for relative algebraic de Rham cohomology of Z over S ([Gro66], [MP12, Section 4.5]). While $H^i_{dr}(Z/S)$ has a somehow concrete description in terms of differential forms, Grothendieck showed in [Gro68] that, at least in characteristic zero, it could be also defined via topos theory. To do this, he defined ([Gro68, Section 4]) an infinitesimal site $Inf(Z/S)$, a topos of sheaves of abelian groups $(Z/S)_{inf}$ on it and a structural sheaf $\mathcal{O}_{Z/S}$. Writing $H^i_{inf}(Z/S, \mathcal{O}_{Z/S})$ for the i^{th} cohomology group of $\mathcal{O}_{Z/S}$ Grothendieck proved the following.

Fact 1.1.4.1.1 ([Gro68, Theorem 4.1 and Section 5.3])**.**

1. If $f : Z \to S$ is smooth and S has characteristic zero, there is a canonical isomorphism
$$H^i_{dr}(Z/S) \simeq H^i_{inf}(Z/S, \mathcal{O}_{Z/S});$$

2. If $Z' \to Z$ is a nilpotent thickening of S-schemes, there is a canonical isomorphism
$$H^i_{inf}(Z'/S, \mathcal{O}_{Z'/S}) \simeq H^i_{inf}(Z/S, \mathcal{O}_{Z/S}).$$

Fact 1.1.4.1.1 can be used to show that the de Rham cohomology of a deformation of a smooth proper variety depends only on the variety.

Remark 1.1.4.1.2 ([B078, Pag. 1.11]). Write $S := \mathrm{Spec}(\mathbb{C}[[T]])$ and $S_n := \mathrm{Spec}(\mathbb{C}[[T]]/(T^n))$. Let $Z \to S$ be a smooth proper morphism and write $Z_n := Z \times_S S_n$. Since f is proper, we have $H_{dr}(Z/S) \simeq \varprojlim_n H_{dr}(Z_n/S_n)$. Since $Z_n \to S_n$ is smooth and $Z_1 \to Z_n$ is a nilpotent thickening, by Fact 1.1.4.1.1 we have

$$H_{dr}(Z/S) \simeq \varprojlim_n H_{dr}(Z_n/S_n) \simeq \varprojlim_n H^i_{inf}(Z_n/S_n, \mathcal{O}_{Z_n/S_n}) \simeq \varprojlim_n H^i_{inf}(Z_1/S_n, \mathcal{O}_{Z_1/S_n}).$$

This shows that the relative de Rham cohomology of $Z \to S$ depends only on Z_1.

1.1.4.2 Crystalline site

In light of Remark 1.1.4.1.2, to construct a cohomology theory with K-coefficients for k-varieties, one could try to lift X to a smooth proper W-scheme $\mathfrak{X}$ and then take the De Rham cohomology of $\mathfrak{X}_K := X \times_W K$. Besides the fact that not all varieties are liftable to characteristic zero, it is not clear that the obtained cohomology is (canonically) independent of the lifting. The arguments in Remark 1.1.4.1.2 suggest that, to prove this independence one could use a cohomology theory in which an analogue of Fact 1.1.4.1.1 holds. But in Fact 1.1.4.1.1(1), the characteristic zero assumption is necessary.

Example 1.1.4.2.1. If $S = k$ and $Z = \mathbb{A}^1_k$, then one wants to show that $d : k[x] \to k[x]dx$ is surjective. If $f = \sum a_i x^i dx$ and $p = 0$, then $f = d(\sum (a_i/i+1)x^{i+1})$.

In light of Example 1.1.4.2.1, the idea is then to restrict the infinitesimal site with a finer site, whose covering have an operation that looks like $1/i+1$: the crystalline site. Let (S, I, γ) be a P.D. scheme ([B078, Pag. 3.18]) and let $f : Z \to S$ be an S-scheme on which γ extends ([B078, Definition 3.14]). In [B078, Section 5], Berthelot defined a crystalline site $Crys(Z/S)$, the topos of sheaves of abelian groups $(Z/S)_{crys}$ on it, a structural sheaf $\mathcal{O}_{Z/S}$ and then proves:

Fact 1.1.4.2.2 ([B078, Corollary 7.4 and Theorem 5.17]). If p is nilpotent on S the following hold.

- If $Z \to S$ is smooth, then there is a natural isomorphism
$$H^i_{dr}(Z/S) \simeq H^i_{crys}(Z/S, \mathcal{O}_{Z/S});$$
- If $Z' \to Z$ is a nilpotent thickening, then there is a natural isomorphism
$$H^i_{crys}(Z'/S, \mathcal{O}_{Z'/S}) \simeq H^i_{crys}(Z/S, \mathcal{O}_{Z/S}).$$

1.1.4.3 Crystalline cohomology

Let $W_n := W_n(k)$ be the n-truncated ring of Witt vectors of k. The natural P.D. structure γ on W_n, sending $\gamma_m(p) = p^m/(m!)$ if $m < n$ and $\gamma_m(p) = 0$ otherwise, extends automatically ([B078, Proposition 3.15]) to every W_n-scheme $T \to W_n$, so that we can define the crystalline cohomology of the smooth proper k-variety X ([B078, Summary 7.26]) as

$$H^i_{crys}(X/K) := (\varprojlim_n H^i_{crys}(X/W_n, \mathcal{O}_{X/W_n})) \otimes \mathbb{Q}.$$

Then, if $\mathfrak{X} \to \mathrm{Spec}(W(k))$ is smooth and proper and $X_n := \mathfrak{X} \times_W W_n$, by Fact 1.1.4.2.2 we have:

$$H^i_{dr}(\mathfrak{X}/K) \otimes \mathbb{Q} \simeq H^i_{dr}(\mathfrak{X}/W(k)) \otimes \mathbb{Q} \simeq H^i_{crys}(X_1/K).$$

The functor $H^i_{crys}(-/K)$ gives a Weil cohomology theory with coefficients in K and the absolute Frobenius φ of k induces a semi linear action on $H^i_{crys}(X/K)$.

1.1.4.4 Crystalline Tate conjecture

The image of the cycle class map

$$c_p^i : \mathrm{CH}^i(X) \to H^{2i}_{crys}(X/K)$$

is contained in $H^{2i}_{crys}(X/K)^{\varphi=p}$. If $k = \mathbb{F}_q$ with $q = p^s$, the action of $F := \varphi^s$ on $H^{2i}_{crys}(X/K)$ is then K-linear and, in this setting, the fullness conjecture is the following.

Conjecture 1.1.4.4.1 ($\mathrm{T}(X, i, p)$)**.** If $k = \mathbb{F}_q$, the cycle class map

$$c_p^i : \mathrm{CH}^i(X) \otimes K \to H^{2i}_{crys}(X/K)^{F=q}$$

is surjective

1.1.4.5 Comparison

While in characteristic zero one can compare directly ℓ-adic and Betti cohomology via the Artin comparison isomorphism, in positive characteristic one there is no such a direct comparison isomorphism between ℓ-adic and crystalline cohomology. When $k = \mathbb{F}_q$ is a finite field with $q = p^s$ elements, one can try and remedy the lack of a comparison isomorphism using the theory of Frobenius weights. For every $\ell \neq p$ the arithmetic Frobenius $F \in \pi_1(\mathbb{F}_q)$ acts linearly on the finite dimensional vector spaces $H^i_\ell(X) := H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ and if $\ell = p$ the s-power of the absolute Frobenius F acts linearly on $H^i_p(X) := H^i_{crys}(X)$. Let $\mathcal{L}$ be the set of all prime numbers.

Fact 1.1.4.5.1 ([Del74], [KM74])**.** For $? \in \mathcal{L}$, the characteristic polynomial Φ of F acting on $H^i_?(X)$ is in $\mathbb{Q}[T]$ and it is independent of $? \in \mathcal{L}$. Moreover for every roots α of Φ and for every embedding $\iota : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$, one has $|\iota(\alpha)| = q^{i/2}$.

Remark 1.1.4.5.2.

- By Fact 1.1.4.5.1, the Zariski closure of the image of $\pi_1(\mathbb{F}_q)$ acting of the semi simplification of $H^i_?(X)$ is defined over $\mathbb{Q}$ and independent of ℓ. In particular a version of Conjecture 1.1.3.3.2 is true in this setting up to semisemplification.
- If k is a finitely generated field of positive characteristic, to construct a reasonable notion of independence and to get an analogue of Fact 1.1.4.5.1, one has to reduce to the finite field setting at the expense of working in a relative setting. We will discuss this in more details later on; see Chapter 6.

1.2 Relative setting

The main topic of this thesis is the study of the various notions introduced in Section 1.1, not in the absolute but in the relative setting. Instead of considering a single variety X, we will study families of varieties.

Let k be a field of characteristic $p \geq 0$ and let X be a smooth geometrically connected k-variety X with generic point η. Let $f : Y \to X$ be a smooth proper morphism and for every $x \in X$ fix a geometric point $\overline{x}$ over it and write Y_x and $Y_{\overline{x}}$ for the fibre of $f : Y \to X$ at x and $\overline{x}$, respectively.

The general question is then how the invariants of Y_x and $Y_{\overline{x}}$ vary with $x \in X$. A first result in this direction is the smooth proper base change theorem: the dimensions of the various cohomology groups $H^i(Y_x(\mathbb{C}), \mathbb{Q})$, $H^i(Y_{\overline{x}}, \mathbb{Q}_\ell)$ and $H^i_{crys}(Y_x)$ are independent of $x \in X$. Hence, regarded only as vector spaces, their are not interesting invariants of the family. On the other hand what is rich and worth studying is the extra structure that these vector spaces have: the Hodge filtration, the Galois action and the Frobenius action. While each of the collection $\{H^i_B(Y_x,\mathbb{Q}))\}_{x\in X}$, $\{H^i(Y_{\overline{x}},\mathbb{Q}_\ell)\}_{x\in X}$ gives rise to a local system (a variation of Hodge structure, a representation of $\pi_1(X)$), in the p-adic setting $\{H^i_{crys}(Y_x)\}_{x\in X}$ give rise to two very different local systems: a convergent and an overconvergent F-isocrystals. In this section we recall these objects and various tools to study them.

1.2.1 Algebraic cycles and motives

1.2.1.1 Algebraic cycles

By [SGA6, X, App 7] (see also [MP12, Sections 3.2 and 9.1]), for every $x \in X$ there is a specialization morphism

$$sp^i_{\eta,x} : \mathrm{CH}^i_{alg}(Y_{\overline{\eta}}) \to \mathrm{CH}^i_{alg}(Y_{\overline{x}})$$

fitting into a commutative diagram

$$\begin{tikzcd} & \mathrm{CH}^i_{alg}(Y_{\overline{k}}) \arrow[dl, "i^*_\eta"'] \arrow[dr, "i^*_x"] & \\ \mathrm{CH}^i_{alg}(Y_{\overline{\eta}}) \arrow[rr, "sp^i_{\eta,x}"] & & \mathrm{CH}^i_{alg}(Y_{\overline{x}}), \end{tikzcd}$$

where $i^*_\eta : \mathrm{CH}^i_{alg}(Y_{\overline{k}}) \to \mathrm{CH}^i_{alg}(Y_{\overline{\eta}})$ and $i_x : \mathrm{CH}^i_{alg}(Y_{\overline{k}}) \to \mathrm{CH}^i_{alg}(Y_{\overline{x}})$ are induced by the inclusions $i_\eta : Y_{\overline{\eta}} \to Y_{\overline{k}}$ and $i_x : Y_{\overline{x}} \to Y_{\overline{k}}$. For every prime $\ell \neq p$ the construction pass trough ℓ-adic homological equivalence and tensoring with $\mathbb{Q}$ we get an injection

$$sp^{i,\ell}_{\eta,x} : \mathrm{CH}^i_\ell(Y_{\overline{\eta}}) \otimes \mathbb{Q} \hookrightarrow \mathrm{CH}^i_\ell(Y_{\overline{x}}) \otimes \mathbb{Q},$$

that one could hope to be an isomorphism at least for some $x \in X$.

Example 1.2.1.1.1 ([MP12, Proof of Proposition 1.13])**.** Let $Y \to X$ a non isotrivial family of elliptic curves and consider $f : Y \times_X Y \to X$. Then $sp^{1,\ell}_{\eta,x}$ is an isomorphism if and only if $Y_{\overline{x}}$ has not complex multiplication.

1.2.1.2 Variation of motivic Galois groups

Fix a Weil cohomology theory H^* with coefficient in a characteristic zero field F and let us assume the standard conjectures of Section 1.1.1.2, so that for every $x \in X$ we have a motivic reductive algebraic group $G(H^*(Y_{\overline{x}}))$ over F.

Then, $\mathrm{CH}^*_F(Y_{\overline{x}})_H$ is described ([And04, Section 6.3]) as the fixed points of the action of $G(H^*(Y_{\overline{x}}))$ on the canonical representation $H^*(Y_x)$ and, the other way around, since $G(H^*(Y_{\overline{x}}))$ is reductive, there exist integers $m, n \geq 0$ and $v_1, ...v_r \in H^*(Y_x)^{\otimes m} \otimes (H^*(Y_x)^\vee)^{\otimes n} \subseteq H^*(Y_x^{n+m})$ such that $G(H^*(Y_{\overline{x}}))$ is the subgroup in $\mathrm{GL}(H^*(Y_x^{n+m}))$ fixing $v_1, ..., v_r$. So describing the variation of algebraic cycles on all the powers Y^n_x amounts to describing the variation of $G(H^*(Y_{\overline{x}}))$

If the realization functor $R_H : \mathbf{Mot}^F_H \to \mathbf{Vect}_F$ factors trough some enriched L-linear Tannakian category $\mathcal{C}$, the Grothendieck-Serre-Tate conjecture predicts that describing the variation of $G(H^*(Y_{\overline{x}}))$ amounts to describing the variation of $G(R_H(H^*(Y_{\overline{x}})))$. Finally, the conjecture $H = num$ suggests that the variation of the various Tannakian groups should not depend on the cohomology theory, hence that one should be able to transfer information between the monodromy groups of the various realizations.

1.2.2 Lisse sheaves and representations

In this section ℓ is a prime $\neq p$.

1.2.2.1 Motivic lisse sheaves

For every $x \in X$ write $\pi_1(X, \overline{x})$ for the étale fundamental group ([SGA1, V, 7]) of X pointed at $\overline{x}$. By smooth proper base change $R^i f_* \mathbb{Q}_\ell(j)$ is a lisse sheaves ([SGA4, XVI, Corollaire 2.2], [SGA4, XII, Theorem 2.2]) on X. Via the equivalence of categories $\mathbf{LS}(X, \mathbb{Q}_\ell) \simeq \mathbf{Rep}_{\mathbb{Q}_\ell}(\pi_1(X, \overline{\eta}))$ between the category of ℓ-adic lisse sheaves $\mathbf{LS}(X, \mathbb{Q}_\ell)$ and the category $\mathbf{Rep}_{\mathbb{Q}_\ell}(\pi_1(X, \overline{\eta}))$ of ℓ-adic representation of $\pi_1(X)$, $R^i f_* \mathbb{Q}_\ell(j)$ induces an action of $\pi_1(X, \overline{\eta})$ on $R^i f_* \mathbb{Q}_\ell(j)_{\overline{\eta}} \simeq H^i(Y_{\overline{\eta}}, \mathbb{Q}_\ell(j))$. For every $x \in X$, the choice of an étale path between $\overline{x}$ and $\overline{\eta}$ induces an isomorphism $\pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta})$ and, with respect to this, equivariant isomorphisms

$$H^i(Y_{\overline{\eta}}, \mathbb{Q}_\ell(j))) \simeq R^i f_* \mathbb{Q}_\ell(j)_{\overline{\eta}} \simeq R^i f_* \mathbb{Q}_\ell(j)_{\overline{x}} \simeq H^i(Y_{\overline{x}}, \mathbb{Q}_\ell(j))),$$

in the sense that the action of $\pi_1(x, \overline{x})$ induced by restriction via $\pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta})$ on $H^i(Y_{\overline{\eta}}, \mathbb{Q}_\ell(j)) \simeq H^i(Y_{\overline{x}}, \mathbb{Q}_\ell(j)))$ identifies with the natural action of $\pi_1(x, \overline{x})$ on $H^i(Y_{\overline{x}}, \mathbb{Q}_\ell(j)))$. The construction makes the diagram

$$\begin{tikzcd}
\mathrm{CH}^i_\ell(Y_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[r, "sp^{i,\ell}_{\eta,x}"] \arrow[d, "c_{Y_{\overline{\eta}}}"] & \mathrm{CH}^i_\ell(Y_{\overline{x}}) \otimes \mathbb{Q} \arrow[d, "c_{Y_{\overline{x}}}"] \\
H^{2i}(Y_{\overline{\eta}}, \mathbb{Q}_\ell(i)) \arrow[r, phantom, "\simeq"] & H^{2i}(Y_{\overline{x}}, \mathbb{Q}_\ell(i))
\end{tikzcd}$$

commutative and the map $sp^{i,\ell}_{\eta,x}$ is $\pi_1(x, \overline{x})$-equivariant with respect to the natural action of $\pi_1(x, \overline{x})$ on $\mathrm{CH}^i_\ell(Y_{\overline{\eta}})$ and the action of $\pi_1(x, \overline{x})$ on $\mathrm{CH}^i_\ell(Y_{\overline{\eta}})$ by restriction trough the morphism $\pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta})$ ([SGA6, X, App 7]). In particular $sp^{i,\ell}_{\eta,x}$ restricts to an injection

$$sp^{i,\ell,ar}_{\eta,x} : \mathrm{CH}^i_\ell(Y_\eta) \otimes \mathbb{Q} = (\mathrm{CH}^i_\ell(Y_{\overline{\eta}}) \otimes \mathbb{Q})^{\pi_1(k(\eta))} \hookrightarrow (\mathrm{CH}^i_\ell(Y_{\overline{x}}) \otimes \mathbb{Q})^{\pi_1(k(x))} = \mathrm{CH}^i_\ell(Y_x) \otimes \mathbb{Q}.$$

1.2.2.2 Strictly exceptional locus

More generally, for every ρ in $\mathbf{Rep}_{\mathbb{Z}_\ell}(\pi_1(X))$ and every $x \in X$ the choice of an étale path between $\overline{x}$ and $\overline{\eta}$ gives rise to a representation

$$\rho_x : \pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta}) \to \mathrm{GL}_n(\mathbb{Z}_\ell),$$

hence to an inclusion

$$\Pi_x := \mathrm{Im}(\rho_x) \subseteq \mathrm{Im}(\rho) =: \Pi.$$

Following [CK16], we give the following definition.

Definition 1.2.2.2.1. We say that $x \in |X|$ is strictly Galois generic for ρ, if $\Pi_x = \Pi$. If x is not strictly Galois generic, we say that x is strictly Galois exceptional for ρ.

Write X^{stex}_ρ (resp. X^{sgen}_ρ) for the set of strictly Galois exceptional (resp. strictly Galois generic) points for ρ. For any integer $d \geq 1$, let $X(\leq d)$ be the set of all $x \in |X|$ such that $[k(x) : k] \leq d$ and set

$$X^{stex}_\rho(\leq d) := X^{stex}_\rho \cap X(\leq d); \qquad X^{sgen}_\rho(\leq d) := X^{sgen}_\rho \cap X(\leq d).$$

In light of Section 1.2.2.1, the study of X^{stex}_ρ is an important problem, since it could control fine invariants of the family Y_x, $x \in X$. Let us first point out that - as soon as k is arithmetically rich enough - X^{sgen}_ρ is non empty. This was observed independently by Serre ([Ser89, Section 10.6]) and Terasoma ([Ter85]). More precisely:

Fact 1.2.2.2.2. If k is Hilbertian, there exists an integer $d \geq 1$ such that $X^{sgen}_\rho(\leq d)$ is infinite.

This follows from Lemma 1.2.2.3.1 below and the fact that the Frattini subgroup of an ℓ-adic Lie group is open ([Ser89, 148]). By Hilbert irreducibility theorem ([FJ05, Chapter 13]), Fact 1.2.2.2.2 holds in particular if k is infinite finitely generated.

1.2.2.3 Anabelian dictionary

For every open subgroup $U \subseteq \Pi$, write $X_U \to X$ for the connected étale cover induced by the open subgroup $\rho^{-1}(U) \subseteq \pi_1(X)$. From the formalism of Galois categories ([SGA1, V, 3-5]), one gets the following anabelian dictionary between rational points of X_U and subgroups of Π:

Lemma 1.2.2.3.1. [CT12b, Section 3.2 (2)] For every $x \in |X|$, the following are equivalent:

- There is an inclusion $\Pi_x \subseteq U$
- $x : \mathrm{Spec}(k(x)) \to X$ lifts to a $k(x)$-rational point of X_U.

$$\begin{tikzcd} & X_U \arrow[d] \\ \mathrm{Spec}(k(x)) \arrow[ur, dotted, "\exists"] \arrow[r, "x"] & X \end{tikzcd}$$

Lemma 1.2.2.3.1 translates the group theoretic problem of understanding how Π_x varies with $x \in |X|$ to the diophantine problem of describing the image of rational points of X_U in X.

1.2.2.4 Frattini argument

Write $\Phi(\Pi) \subseteq \Pi$ for the Frattini subgroup of Π, i.e. the intersection of all the maximal open subgroups of Π and write $\mathcal{C}(\Pi)$ for the set of open subgroups $U \subsetneq \Pi$ such that $\Phi(\Pi) \subseteq U$. From [Ser89, Pag. 148] and the definition of the Frattini subgroup, one deduces the following.

Lemma 1.2.2.4.1.

1. $\mathcal{C}(\Pi)$ is finite.
2. If $C \subsetneq \Pi$ is a proper closed subgroup, then there exists a $U \in \mathcal{C}(\Pi)$ such that $C \subseteq U$.

So

$$\begin{aligned} x \in X_\rho^{stex} &\Leftrightarrow \text{ there exists } U \in \mathcal{C}(\Pi) \text{ with } \Pi_x \subseteq U && \text{(Lemma 1.2.2.4.1(2))} \\ &\Leftrightarrow \text{ there exists } U \in \mathcal{C}(\Pi) \text{ such that } x \in \mathrm{Im}(X_U(k(x)) \to X(k(x))) && \text{(Remark 1.2.2.3.1),} \end{aligned}$$

hence

$$X_\rho^{stex} = \bigcup_{U \in \mathcal{C}(\Pi)} \Big(\bigcup_{[k':k]<+\infty} \mathrm{Im}(X_U(k') \to X(k')) \Big) \tag{1.2.2.4.2}$$

1.2.2.5 Hilbertian property

Recall ([MP12, Definition 8.1]) the definition of sparse set.

Definition 1.2.2.5.1. Let B an irreducible k-variety and $S \subseteq |B|$ a subset. We say that S is sparse if there exists a dominant and generically finite morphism $\pi : T \to B$ of irreducible k-varieties such that for each $s \in S$, the fibre T_s of $\pi : T \to B$ at s, is either empty or contains more than one closed point.

Since $X_U \to X$ is a finite étale cover of degree > 1, the set

$$\bigcup_{k \subseteq k'} \mathrm{Im}(X_U(k') \to X(k')) \subseteq |X|$$

is sparse. Since a finite union of sparse is sparse ([MP12, Proposition 8.5 (b)]) and $\mathcal{C}(\Pi)$ is finite (Lemma 1.2.2.4.1(1)), by (1.2.2.4.2) we see that X_ρ^{stex} is sparse. This is enough to prove Fact 1.2.2.2.2, thanks to the following consequence of the definition of Hilbertian field.

Lemma 1.2.2.5.2. If k is Hilbertian and $S \subseteq |X|$ is a sparse set, there exists a $d \geq 1$ such that $|X| - S$ contains infinitely many points of degree $\leq d$.

Proof. Since for every dense open subset $U \subseteq X$ the set $U \cap S$ is sparse in U ([MP12, Proposition 8.5.(a)]), we may replace X with a dense open subset and hence assume that X is affine of dimension $n \geq 1$. By Noether normalization theorem, there exists a surjective finite morphism $\pi : X \to \mathbb{A}^n_k$ of some degree $d \geq 1$. Since the image of a sparse set via a surjective finite morphism is sparse ([MP12, Proposition 8.5 (c)]), the set $\pi(S) \subseteq \mathbb{A}^n_k$ is sparse. So, by ([MP12, Proposition 8.5 (d)]) $\mathbb{A}^n_k(k) \cap \pi(S)$ is thin; see [Ser89, Section 9.1] for the definition. Since k is Hilbertian, the set $\mathbb{A}^n_k(k) - (\mathbb{A}^n_k(k) \cap \pi(S))$ is infinite. Hence $\pi^{-1}(\mathbb{A}^n_k(k) - (\mathbb{A}^n_k(k) \cap \pi(S))) \subseteq X - S$ contains infinitely many points of degree $\leq d$. $\square$

1.2.3 Characteristic zero: Variation of motivic Hodge structure

Let $k \subseteq \mathbb{C}$ be a finitely generated sub field of $\mathbb{C}$.

1.2.3.1 Analytic local systems and geometric image

Fix $x \in |X_{\mathbb{C}}|$. By smooth proper base change, from $f^{an} : Y^{an}_{\mathbb{C}} \to X^{an}_{\mathbb{C}}$ one obtains a $\mathbb{Q}$-local system $R^i f^{an}_* \mathbb{Q}$ on $X^{an}_{\mathbb{C}}$ and writes Π_B for the image of the resulting action of $\pi_1^{top}(X_{\mathbb{C}}, x)$ on $H^i_B(Y_x, \mathbb{Q})$. By the invariance of the étale site for algebraically closed field extensions in characteristic zero ([SGA1, XIII]), there is a natural isomorphism $\pi_1(X_{\overline{k}}, x) \simeq \pi_1(X_{\mathbb{C}}, x)$. By the Riemann existence theorem [SGA1, XII, Theoreme 5.1], there is a natural algebraization morphism $\pi_1^{top}(X_{\mathbb{C}}, x) \to \pi_1(X_{\mathbb{C}}, x)$ identifying $\pi_1(X_{\mathbb{C}}, x)$ with the profinite completion of $\pi_1^{top}(X_{\mathbb{C}}, x)$ ([SGA1, XII, Corollaire 5.2]).

The action of $\pi_1^{top}(X_{\mathbb{C}}, x)$ on $H^i_B(Y_x, \mathbb{Q}) \otimes \mathbb{Q}_\ell$, factors trough the profinite completion map $\pi_1^{top}(X_{\mathbb{C}}, x) \to \pi_1(X_{\mathbb{C}}, x) \simeq \pi_1(X_{\overline{k}}, x)$ and, under the comparison isomorphism $H^i_B(Y_x, \mathbb{Q}) \otimes \mathbb{Q}_\ell \simeq H^i(Y_x, \mathbb{Q}_\ell)$, the action of $\pi_1(X_{\overline{k}}, x)$ on $H^i_B(Y_x, \mathbb{Q}) \otimes \mathbb{Q}_\ell$ identifies with the restriction via $\pi_1(X_{\overline{k}}, x) \to \pi_1(X, x)$ of the action of $\pi_1(X, x)$ on $H^i(Y_x, \mathbb{Q}_\ell)$.

Write $G^{i,geo}_\ell$ for the Zariski closure of the image $\Pi^{i,geo}_\ell$ of the action of $\pi_1(X_{\overline{k}}, x)$ on $H^i(Y_x, \mathbb{Q}_\ell)$. Since $\pi_1^{top}(X_{\mathbb{C}}, x) \to \pi_1(X_{\mathbb{C}}, x)$ has dense image, the previous discussion implies the following independence result for $G^{i,geo}_\ell$, which is a geometric analogue of Conjecture 1.1.3.3.2.

Proposition 1.2.3.1.1. There exist a $\mathbb{Q}$-algebraic group $G^{i,geo}$, a faithful representation $G^i \subseteq \mathrm{GL}(V^i)$ and an isomorphism $V^i \otimes \mathbb{Q}_\ell \simeq H^i(Y_{\overline{k}}, \mathbb{Q}_\ell)$ for every ℓ, such that the composition $G^{i,geo} \otimes \mathbb{Q}_\ell \subseteq \mathrm{GL}(V^i) \otimes \mathbb{Q}_\ell \simeq \mathrm{GL}(H^i(Y_{\overline{x}}, \mathbb{Q}_\ell))$ identifies $G^{i,geo} \otimes \mathbb{Q}_\ell$ with $G^{i,geo}_\ell$.

1.2.3.2 Variational Hodge conjecture

The Leray spectral sequence for $f^{an} : Y^{an}_{\mathbb{C}} \to X^{an}_{\mathbb{C}}$ induces a morphism

$$H^2_B(Y_{\mathbb{C}}, \mathbb{Q}(1)) \to H^0(X^{an}_{\mathbb{C}}, R^2 f^{an}_* \mathbb{Q}(1))$$

fitting, for every $x \in |X_{\mathbb{C}}|$, into a commutative diagram

$$\begin{tikzcd}
H^2_B(Y_{\mathbb{C}}, \mathbb{Q}(1)) \arrow[d] \arrow[drr, "i_x^*"] & & Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q} \arrow[ll, "c_{Y_{\mathbb{C}}}"'] \arrow[drr, "i_x^*"] & & \\
H^0(X^{an}_{\mathbb{C}}, R^2 f^{an}_* \mathbb{Q}(1)) \otimes \mathbb{Q} \arrow[rr, hook] & & H^2_B(Y_x, \mathbb{Q}(1)) & & Pic(Y_x) \otimes \mathbb{Q}. \arrow[ll, "c_{Y_x}"]
\end{tikzcd}$$

The Hodge conjecture for divisors (Fact 1.1.3.1.2) and the theory developed in [Del71] enable to prove a variational version of the Hodge conjecture for divisors (see also [Char11, Section 3.1]).

Fact 1.2.3.2.1. For every $x \in |X_{\mathbb{C}}|$ and every $z_x \in Pic(Y_x) \otimes \mathbb{Q}$ the following are equivalent.

1. There exists a $z \in Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q}$ such that $i_x^*(c_{Y_{\mathbb{C}}}(z)) = c_{Y_x}(z_x)$;
2. There exists a $z \in H_B^2(X_{\mathbb{C}}, \mathbb{Q}(1))$ such that $i_x^*(z) = c_{Y_x}(z)$;
3. $c_{Y_x}(z)$ is in the image of $H^0(X_{\mathbb{C}}^{an}, R^2 f_*^{an}\mathbb{Q}(1)) \hookrightarrow H_B^2(Y_x^{an}, \mathbb{Q}(1))$.

Proof. Clearly we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). We show that (3) $\Rightarrow$ (1). Let $i : Y_{\mathbb{C}} \subseteq Y_{\mathbb{C}}^{cmp}$ be a smooth compactification. The commutative cartesian diagram of $\mathbb{C}$-varieties

$$\begin{tikzcd}
Y_x \arrow[r, "i_x"] \arrow[d] \arrow[dr, phantom, "\square"] & Y_{\mathbb{C}} \arrow[r, "i"] \arrow[d, "f"] & Y_{\mathbb{C}}^{cmp} \\
\mathrm{Spec}(\mathbb{C}) \arrow[r, "x"] & X_{\mathbb{C}},
\end{tikzcd}$$

induces a commutative diagram:

$$\begin{tikzcd}
& H_B^2(Y_{\mathbb{C}}^{cmp}, \mathbb{Q}(1)) \arrow[d, two heads, "i^*"] & Pic(Y_{\mathbb{C}}^{cmp}) \otimes \mathbb{Q} \arrow[l, "c_{Y_{\mathbb{C}}^{cmp}}"'] \arrow[d, "i^*"] & \\
& H_B^2(Y_{\mathbb{C}}, \mathbb{Q}(1)) \arrow[d] \arrow[dr, "i_x^*"] & Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q} \arrow[l, "c_{Y_{\mathbb{C}}}"'] \arrow[dr, "i_x^*"] & \\
& H^0(X_{\mathbb{C}}^{an}, R^2 f_*^{an}\mathbb{Q}(1)) \otimes \mathbb{Q} \arrow[r, hook] & H_B^2(Y_x, \mathbb{Q}(1)) & Pic(Y_x) \otimes \mathbb{Q}. \arrow[l, "c_{Y_x}"]
\end{tikzcd}$$

By the Global invariant cycles theorem ([Del71]) the map

$$H_B^2(Y_{\mathbb{C}}^{cmp}, \mathbb{Q}(1)) \to H_B^2(Y_{\mathbb{C}}, \mathbb{Q}(1)) \to H^0(X_{\mathbb{C}}^{an}, R^2 f_*^{an}\mathbb{Q}(1))$$

is surjective, hence, by (3), $c_{Y_x}(z_x) \in H_B^2(Y_x, \mathbb{Q}(1))$ is in the image of $i_x^* \circ i^* : H_B^2(Y_{\mathbb{C}}^{cmp}, \mathbb{Q}(1)) \to H_B^2(Y_x, \mathbb{Q}(1))$. Since $H_B^2(Y_{\mathbb{C}}^{cmp}, \mathbb{Q}(1))$ is a semisimple $\mathbb{Q}$-Hodge structure, the map $H_B^2(Y_{\mathbb{C}}^{cmp}, \mathbb{Q}(1)) \to \mathrm{Im}(i_x^*)$ splits as a morphism of $\mathbb{Q}$-Hodge structure. Since $c_{Y_x}(z_x)$ is in $H_B^2(Y_{\mathbb{C}}^{cmp}, \mathbb{Q}(1))^{0,0}$, then $c_{Y_x}(z_x)$ is the image of some $z' \in H_B^2(Y_{\mathbb{C}}^{cmp}, \mathbb{Q}(1))^{0,0}$ via $H_B^2(Y_{\mathbb{C}}^{cmp}, \mathbb{Q}(1))^{0,0} \to H_B^2(Y_x, \mathbb{Q}(1))$. By the Hodge conjecture for divisors (Fact 1.1.3.1.2) $z = c_{Y_{\mathbb{C}}^{cmp}}(z^{cmp})$ for some $z^{cmp} \in Pic(Y_{\mathbb{C}}^{cmp}) \otimes \mathbb{Q}$. Then $z = i^*(z^{cmp}) \in Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q}$ is such that $i_x^*(c_Y(z)) = c_{Y_x}(z_x)$. ☐

1.2.4 Positive Characteristic: F-isocrystals

In positive characteristic, there are two different categories of p-adic local systems: F-isocrystals and overconvergent F-isocrystals. In this Section we briefly recall their definitions and the relations between them. Let k be a perfect field of characteristic $p > 0$, write $W := W(k)$ for the Witt ring of k and $K := K(k)$ for the fraction field of W. Let X be a smooth geometrically connected k-variety.

1.2.4.1 F-isocrystals

Slightly adapting the arguments in 1.1.4.2, one defines a crystalline topos $(X/W)_{crys}$, a crystalline site $Crys(X/W)$ of X over W and a structural sheaf $\mathcal{O}_{X/W}$, see [B078, Section 7.17] and [Mor13, Section 2]. For every $(U \hookrightarrow T, \gamma)$[1] in $(X/W)_{crys}$ and every sheaf of coherent $\mathcal{O}_{X/W}$-modules $\mathcal{E}$, one has a coherent $\mathcal{O}_T$-module $\mathcal{E}_T$ and for every morphism $g : (U', T', \gamma') \to (U, T, \gamma)$

[1] U is a Zariski open subset of X, $U \hookrightarrow T$ is a nilpotent closed immersion of W-schemes and γ a P.D. structure on $\mathrm{Ker}(\mathcal{O}_T \to \mathcal{O}_U)$

in $(X/W)_{crys}$ a natural morphism $g^*\mathcal{E}_T \to \mathcal{E}_{T'}$ of coherent $\mathcal{O}_{T'}$-modules. A crystal over X is then a sheaf $\mathcal{E}$ of coherent $\mathcal{O}_{X/W}$-modules, such that for every morphism $g:(U',T',\gamma')\to(U,T,\gamma)$ in $Crys(X/W)$, the natural morphism $g^*\mathcal{E}_{T'}\to\mathcal{E}_T$ is an isogeny. Write $\mathbf{Crys}(X|W)$ for the category of crystals, $\mathbf{Crys}(X|W)_{\mathbb{Q}} := \mathbf{Crys}(X|W)\otimes\mathbb{Q}$ and $\mathcal{O}_{X/K} := \mathcal{O}_{X/W}\otimes\mathbb{Q}$. For every integer $s\geq 1$, the s-power F of the absolute Frobenius φ of X acts on $\mathbf{Crys}(X|W)_{\mathbb{Q}}$ and the category $\mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$ of F-isocrystals is made by the couples $(\mathcal{E},\Phi)$, where $\mathcal{E}$ is in $\mathbf{Crys}(X|W)_{\mathbb{Q}}$ and $\Phi : F^*\mathcal{E}\to\mathcal{E}$ is an isogeny. For every $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$ there is a cohomology group $H^i(X,\mathcal{E})$ (a K-vector space) endowed with a semi linear action of F. Set $H^i_{crys}(X) := H^i_{crys}(X,\mathcal{O}_{X/K})$.

1.2.4.2 Crystalline variational Tate conjecture

By [Mor13], there is an higher direct image F-isocrystal $R^if_{crys,*}\mathcal{O}_{Y/K}$ and the Leray spectral sequence for $f:Y\to X$ induces, for every $x\in|X|$, a commutative diagram

$$\begin{tikzcd}
H^2_{crys}(Y) \arrow[d, "Leray"] \arrow[dr, "i_x^*"] & Pic(Y)\otimes\mathbb{Q} \arrow[l, "c_Y"'] \arrow[dr, "i_x^*"] & \\
H^0(X,R^2f_{crys,*}O_{Y/K}) \arrow[r, hook] & H^2_{crys}(Y_x) & Pic(Y_x)\otimes\mathbb{Q}. \arrow[l, "c_{Y_x}"]
\end{tikzcd}$$

Even though the crystalline Tate conjecture for divisors is not known, Morrow proved a variational version of it, giving an analogue of Fact 1.2.3.2.1.

Fact 1.2.4.2.1 ([Mor15, Theorem 1.4])**.** If $f:Y\to X$ is projective, for every $z_x\in Pic(Y_x)\otimes\mathbb{Q}$ the following are equivalent:

1. There exists $z\in Pic(Y)\otimes\mathbb{Q}$ such that $c_{Y_x}(z_x)=i_x^*(c_Y(z))$;
2. There exists $z\in H^2_{crys}(Y)$ such that $c_{Y_x}(z_x)=i_x^*(z)$;
3. $c_{Y_x}(z_x)$ is in the image of $H^0(X,R^2f_{crys,*}O_{Y/K})\hookrightarrow H^2_{crys}(Y_x)$.

1.2.4.3 Slopes

One of the specific features of $\mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$, which has no ℓ-adic analogue, is the theory of slopes; see [Kat79], [Ked17, Sections 3 and 4]. Let $\mathcal{E}$ be in $\mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$ of rank r. For every $x\in|X|$, one considers the multi-set of rational numbers $\{a_i^x(\mathcal{E})\}_{1\leq i\leq r}$ of the slopes ([Ked17, Definition 3.3]) of $\mathcal{E}$ at x. We say that $\mathcal{E}$ is isoclinic (of slope $a_1^{\mathfrak{t}}(\mathcal{E})$) if $a_1^{\mathfrak{t}}(\mathcal{E})=a_r^x(\mathcal{E})$ for every $x\in|X|$ and unit-root if it is isoclinic of slope 0. Write $\mathbf{F}\text{-}\mathbf{Crys}^{ur}(X|W)_{\mathbb{Q}}\subseteq\mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$ for the category of unit-root F-isocrystals. Finally, we say that $\mathcal{E}$ has constant Newton polygon if the function

$$N_{\mathcal{E}}:|X|\to\mathbb{Q}^r$$
$$x\mapsto(a_i^x(\mathcal{E}))_{1\leq i\leq r}$$

is constant.

Fact 1.2.4.3.1. Let $\mathcal{E}$ be in $\mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$.

1. ([Kat79, Theorem 2.3.1], [Ked17, Theorem 3.12]): There exists a dense open immersion $\mathrm{i}^*:U\to X$ such that $\mathrm{i}^*\mathcal{E}$ has constant Newton polygon;
2. ([Kat79, Theorem 2.6.2], [Ked17, Corollary 4.2]): If $\mathcal{E}$ has constant Newton polygon, then there exists a unique filtration

$$0=\mathcal{E}_0\subseteq\mathcal{E}_1\subseteq...\subseteq\mathcal{E}_{n-1}\subseteq\mathcal{E}_n=\mathcal{E}\quad\text{in } \mathbf{F}\text{-}\mathbf{Crys}(X|W)_{\mathbb{Q}}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is isoclinic of some slope s_i with $s_1<s_2<...<s_n$.

3. ([Tsu02], [Ked17, Theorem 3.9]): There is a natural equivalence of categories $\mathbf{F\text{-}Crys}^{ur}(X|W)_{\mathbb{Q}} \simeq \mathbf{Rep}_K(\pi_1(X))$

The filtration of Fact 1.2.4.3.1(2) is called the slope filtration of $\mathcal{E}$.

1.2.4.4 Comparison I: F-isocrystals versus ℓ-adic representations

The F-isocrystal $R^i f_{crys,*}\mathcal{O}_{Y/K}$ could appear as a p-adic analogue of $R^i f_* \mathbb{Q}_\ell$. However, the behaviour of $\mathbf{F\text{-}Crys}(X|W)_{\mathbb{Q}}$ is quite different from the one of $\mathbf{LS}(X, \mathbb{Q}_\ell)$.

Example 1.2.4.4.1 ([Gro68, Section 2.1]). In general the K-vector space $H^1(X, \mathcal{E})$ is not finite dimensional. Consider $X = \mathbb{A}^1_k$ and write

$$K\langle T\rangle := \{\sum_{i=0}^{+\infty} a_i T^i \text{ such that } \lim_{i\to+\infty} |a_i| = 0\}.$$

There is a natural isomorphism

$$H^1_{crys}(X, \mathcal{O}_{X/K}) \simeq \mathrm{Coker}(d : K\langle T\rangle \to K\langle T\rangle dT).$$

Since

$$\lim_{i\to+\infty} |a_i| = 0,$$

does not imply in general that

$$\lim_{i\to+\infty} |a_i/i+1| = 0,$$

one sees that $H^1_{crys}(X/K)$ is an infinite dimensional K-vector space. However, following [MW68], one can replace $K\langle T\rangle$ with the sub ring

$$K\langle T\rangle^\dagger := \{\sum_{i=0}^{+\infty} a_i T^i \text{ such that there exists a } c > 1 \text{ with } \lim_{n\to+\infty} c^i |a_i| = 0\},$$

and then check that $d : \mathrm{Coker}(K\langle T\rangle^\dagger \to K\langle T\rangle^\dagger dT) = 0$. While $K\langle T\rangle$ is the ring of function of the rigid analytic open disc, $K\langle T\rangle^\dagger \subseteq K\langle T\rangle$ is the sub ring of functions that converge on some larger analytic open neighbourhood.

Example 1.2.4.4.2 ([Ked17, Example 4.6]). Let $f : Y \to X$ be a non isotrivial family of elliptic curves with a supersingular fibre and set $\mathcal{E} := R^i f_* \mathcal{O}_{Y/K}$. Then there is a dense open subscheme $i : U \hookrightarrow X$ such that for all $x \in U$, the elliptic curve Y_x is ordinary. We have the following:

1. $\mathcal{E}$ is irreducible;

2. $i^*\mathcal{E} \simeq R^i f_{U,*}\mathcal{O}_{Y_U/K}$ has a non split two steps filtration (the slope filtration of Section 1.2.4.3) reflecting the filtration of the p-divisible group of the generic fibre, given by the connected-étale exact sequence

$$0 \to Y_\eta[p^\infty]^0 \to Y_\eta[p^\infty] \to Y_\eta[p^\infty]^{et} \to 0.$$

Hence:

1. While in the ℓ-adic setting the restriction to an open subset of an irreducible lisse sheaf remains irreducible, in the crystalline setting this is not true;

2. While $R^i f_{U,*}\mathbb{Q}_\ell$ is semisimple, $i^*\mathcal{E}$ is not.

So, on the one hand, the category $\mathbf{F\text{-}Crys}(X|W)_{\mathbb{Q}}$ has a somehow pathological behaviour with respect to $\mathbf{LS}(X, \mathbb{Q}_\ell)$, but on the other hand, it contains fine p-adic information.

1.2.4.5 Overconvergent F-isocrystals

Examples 1.2.4.4.1 and 1.2.4.4.2 suggest that to get a category of p-adic local systems with a behaviour similar to $LS(X, \mathbb{Q}_\ell)$, one needs to rigidify the category $\mathbf{F}\text{-}\mathbf{Crys}(X|W)_\mathbb{Q}$. This leads to the introduce the category of overconvergent isocrystals $\mathbf{Isoc}^\dagger(X|K)$ and overconvergent F-isocrystals $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X|K)$ and of rigid cohomology $H^i(X, \mathcal{E})$ for $\mathcal{E}$ in $\mathbf{Isoc}^\dagger(X|K)$. The definitions of these objects are technical, so we refer the reader to [Ber96] for the precise definitions and we recall the description on a specific example.

Example 1.2.4.5.1. Set $X = \mathbb{A}^1_k$ and retain the notation of Example 1.2.4.4.1. An overconvergent isocrystal on X is a coherent $K\langle T\rangle^\dagger$-module $\mathcal{E}$, endowed with a integrable connection

$$d : \mathcal{E} \to \mathcal{E} \otimes_{K\langle T\rangle^\dagger} K\langle T\rangle^\dagger dT.$$

The rigid cohomology of $\mathcal{E}$ is then defined as

$$\begin{aligned} H^0(X, \mathcal{E}) &= \mathrm{Ker}(d : \mathcal{E} \to \mathcal{E} \otimes_{K\langle T\rangle^\dagger} K\langle T\rangle^\dagger dT); \\ H^1(X, \mathcal{E}) &= \mathrm{Coker}(d : \mathcal{E} \to \mathcal{E} \otimes_{K\langle T\rangle^\dagger} K\langle T\rangle^\dagger dT); \\ H^i(X, \mathcal{E}) &= 0 \quad \text{if} \quad i \geq 2 \end{aligned}$$

The natural morphism $F : K\langle T\rangle \to K\langle T\rangle$ sending $\sum a_i T^i$ to $\sum F(a_i) T^{pi}$, induces a morphism $F : K\langle T\rangle^\dagger \to K\langle T\rangle^\dagger$, so that one can consider the overconvergent isocrystal $F^*\mathcal{E}$. An overconvergent F-isocrystal on X is then a overconvergent isocrystal $\mathcal{E}$ on $K\langle T\rangle^\dagger$, endowed with an isomorphism $F^*\mathcal{E} \to \mathcal{E}$.

To compare F-isocrystals and overconvergent F-isocrystals, one introduces the categories $\mathbf{Isoc}(X|K)$ and $\mathbf{F}\text{-}\mathbf{Isoc}(X|K)$ of convergent isocrystals and convergent F-isocrystals ([Ogu84], [Ber96, 2.3.2]).The categories of isocrystals introduced so far, fit into a commutative diagram ([Ber96, Section 2.4]) of faithful functors:

$$\begin{tikzcd} \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X|K) \arrow[r, "(-)^{geo}"] \arrow[d, "(-)^{conv}"] & \mathbf{Isoc}^\dagger(X|K) \arrow[d, "(-)^{conv}"] \\ \mathbf{F}\text{-}\mathbf{Isoc}(X|K) \arrow[r, "(-)^{geo}"] \arrow[d, "\Phi"] & \mathbf{Isoc}(X|K) \arrow[d] \\ \mathbf{F}\text{-}\mathbf{Crys}(X|W)_\mathbb{Q} \arrow[r, "(-)^{geo}"] & \mathbf{Crys}(X|W)_\mathbb{Q}. \end{tikzcd} \tag{1.2.4.5.2}$$

Furthermore:

Fact 1.2.4.5.3.

- ([Ber96, Theoreme 2.4.2]): The functor $\Phi : \mathbf{F}\text{-}\mathbf{Isoc}(X|K) \to \mathbf{F}\text{-}\mathbf{Crys}(X|W)_\mathbb{Q}$ is an equivalence of categories.
- ([Ked18]): The functor $(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X|K) \to \mathbf{F}\text{-}\mathbf{Isoc}(X|K)$ is fully faithful.

It is a non trivial result of this thesis in Chapter 4 that $R^i f_{crys,*}\mathcal{O}_{Y/K}$ is in the essential image of $(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X|K) \to \mathbf{F}\text{-}\mathbf{Isoc}(X|K) \simeq \mathbf{F}\text{-}\mathbf{Crys}(X|W)_\mathbb{Q}$.

1.2.4.6 Monodromy groups

If $\mathcal{E}$ is a $\mathbb{Q}_\ell$-lisse sheaf on X, we could define the monodromy group $G(\mathcal{E})$ of $\mathcal{E}$ equivalently as the Zariski closure of the image of $\pi_1(X,\overline{x})$ acting on $\mathcal{E}_{\overline{x}}$ or as the automorphism group of the forgetful tensor functor $\langle\mathcal{E}\rangle \to \mathbf{Vect}_{\mathbb{Q}_\ell}$. For isocrystals, only the latter construction is available. This was first worked out by Crew in [Cre92]. From now on, assume that $k = \mathbb{F}_q$, with $q = p^s$ and, to simplify, that X has a $\mathbb{F}_q$-rational point $x : \mathrm{Spec}(\mathbb{F}_q) \to X$. Since there is a natural equivalence of categories $\mathbf{Isoc}(\mathbb{F}_q|K) \simeq \mathbf{Vect}_K$, the functor

$$x^* : \mathbf{Isoc}(X|K) \to \mathbf{Isoc}(\mathbb{F}_q|K) \simeq \mathbf{Vect}_K$$

induces a neutralization for all the four categories in Diagram (1.2.4.5.2). Hence, for each $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X|K)$, one obtains a commutative diagram of Tannakian categories:

$$\begin{tikzcd} \langle\mathcal{E}\rangle \arrow[r, "(-)^{geo}"] \arrow[d, "(-)^{conv}"] & \langle\mathcal{E}^{geo}\rangle \arrow[d, "(-)^{conv}"] \\ \langle\mathcal{E}^{conv}\rangle \arrow[r, "(-)^{geo}"] & \langle\mathcal{E}^{geo,conv}\rangle. \end{tikzcd}$$

By Tannakian duality, this diagram corresponds to a commutative exact diagram of closed immersions of algebraic groups

$$\begin{tikzcd} G(\mathcal{E}^{geo,conv}) \arrow[r, hook] \arrow[d, hook] & G(\mathcal{E}^{conv}) \arrow[d, hook] \\ G(\mathcal{E}^{geo}) \arrow[r, hook] & G(\mathcal{E}), \end{tikzcd}$$

in which ([D'Ad17, Appendix]) the subgroups $G(\mathcal{E}^{geo,conv}) \subseteq G(\mathcal{E}^{conv})$ and $G(\mathcal{E}^{geo}) \subseteq G(\mathcal{E})$ are normal .

Example 1.2.4.6.1. Retain the notation of Example 1.2.4.4.2. Then one has

$$G(\mathcal{E}^{conv}) = G(\mathcal{E}) = \mathrm{GL}_2 \quad \text{and} \quad G(\mathcal{E}^{geo,conv}) = G(\mathcal{E}^{geo}) = SL_2$$

while

$$B = G(i^*\mathcal{E}^{conv}) \subseteq G(i^*\mathcal{E}) = \mathrm{GL}_2 \quad \text{and} \quad B' = G(i^*\mathcal{E}^{geo,conv}) \subseteq G(i^*\mathcal{E}^{geo}) = SL_2$$

where $B \subseteq \mathrm{GL}_2$ and $B' \subseteq SL_2$ are the Borel subgroups of upper triangular matrices. This reflects the fact that $i^*\mathcal{E}$ admits a filtration made by convergent F-isocrystals that are not coming form overconvergent ones, which corresponds to the flag stabilized by B and B' but not by GL_2 and SL_2.

1.2.4.7 Comparison II: overconvergent F-isocrystals vs ℓ-adic representation

While in characteristic zero one can try to compare the various monodromy groups via the comparison between the singular and étale sites, in positive characteristic one needs different tools. We recall some results in this setting. For technical reason it is easier to work with coefficients in algebraically closed fields. Let ℓ be a prime. Following [Ked17], let $\mathbf{Coef}(X,\ell)$ be the category of $\overline{\mathbb{Q}}_\ell$-lisse sheaves ([Del80, 1.1.1]) and let $\mathbf{Coef}(X,p)$ (denoted also with $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$) be the category of $\overline{\mathbb{Q}}_p$-overconvergent F-isocrystals ([Abe18, Sections 2.4.14-2.4.18]). Let $\mathcal{E}_\ell$ be in $\mathbf{Coef}(X,\ell)$. For every $x \in |X|$ there is a characteristic polynomial $\phi_x(\mathcal{E}_\ell) \in \overline{\mathbb{Q}}_\ell[T]$ of $\mathcal{E}$ in x (see e.g. [D'Ad17, 2.1.4 and 2.2.10.]). Fix a collection $\underline{\iota} := \{\iota_\ell\}_{\ell\in\mathcal{L}}$ of isomorphisms $\iota_\ell : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. We say that $\mathcal{E}_\ell$ is ι_ℓ-pure (of weight w), if all the roots of $\iota_\ell(\phi_x(\mathcal{E}_\ell))$ have complex absolute value $q^{[\mathbb{F}_q(x):\mathbb{F}_q]w/2}$. Let $\{\mathcal{E}_\ell\}_{\ell\in\mathcal{L}}$ be a collection of $\mathcal{E}_\ell$ in $\mathbf{Coef}(X,\ell)$. We say that $\{\mathcal{E}_\ell\}_{\ell\in\mathcal{L}}$ is a

$\underline{\iota}$-compatible system if $\iota_\ell(\phi_x(\mathcal{E}_\ell)) = \iota_{\ell'}(\phi_x(\mathcal{E}_{\ell'}))$, for every $\ell \neq \ell'$ and every $x \in |X|$. Via the theory of weights ([Del80], [Ked06b], [AC13b]) the conditions of purity and compatibility are strong enough to guarantee that the different $\mathcal{E}_\ell$ share several properties.

Example 1.2.4.7.1. Take two primes $\ell \neq \ell'$ and assume, for simplicity, that $\ell \neq p \neq \ell'$. If $\mathcal{E}_\ell$ in $\mathbf{Coef}(X, \ell)$ and $\mathcal{E}_{\ell'}$ in $\mathbf{Coef}(X, \ell')$ are pure and compatible, then the following follows from the theory of weights ([Del80]) and the Grothendieck-Lefschetz trace formula ([Fu15, Theorem 10.5.1, page 603]):

- $\mathcal{E}_\ell$ is irreducible if and only if $\mathcal{E}_{\ell'}$ is irreducible (see e.g. [D'Ad17, Corollary 3.5.6]);
- $Dim(H^0(X_{\mathbb{F}}, \mathcal{E}_\ell)) = Dim(H^0(X_{\mathbb{F}}, \mathcal{E}_{\ell'}))$ (see e.g. [D'Ad17, Corollary 3.4.11]).

Fix $x \in |X|$ and write $\mathcal{E}_{\ell,\overline{x}}$ for the fibre of $\mathcal{E}_\ell$ at $\overline{x}$. Using the functor x^*, for every $\mathcal{E}_\ell$ in $\mathbf{Coef}(X, \ell)$ one defines a monodromy group $G(\mathcal{E}_\ell) \subseteq \mathrm{GL}(\mathcal{E}_{\ell,\overline{x}})$. Furthermore, one can construct a geometric monodromy group $G(\mathcal{E}_\ell^{geo}) \subseteq G(\mathcal{E}_\ell)$: if $\ell \neq p$, $G(\mathcal{E}_\ell^{geo})$ is defined as the monodromy group of the base change of $\mathcal{E}_\ell$ to $X_{\overline{\mathbb{F}}_q}$ and if $\ell = p$, $G(\mathcal{E}_p^{geo})$ is defined as the monodromy group of the image of $\mathcal{E}$ in the category of $\overline{\mathbb{Q}}_p$-linear overconvergent isocrystal over X. A striking recent result, building on the Langlands correspondence and the theory of companions for ℓ-adic sheaves and overconvergent F-isocrystals ([Laf02], [Dri12], [Abe18], [AE16]), is an analogue of Conjecture 1.1.3.3.2 and Proposition 1.2.3.1.1.

Fact 1.2.4.7.2 ([Chi03], [D'Ad17])**.** Let $\{\mathcal{E}_\ell\}_{\ell \in \mathcal{L}}$ be a pure compatible system. Then:

- There exist a connected $\overline{\mathbb{Q}}$-algebraic group G^{geo}, a faithful representation $\rho : G^{geo} \subseteq \mathrm{GL}(V)$ and a (non canonical) isomorphism $V \otimes \overline{\mathbb{Q}}_\ell \simeq \mathcal{E}_{\ell,\overline{x}}$ for every ℓ, such that the composition $G^{geo} \otimes \overline{\mathbb{Q}}_\ell \subseteq \mathrm{GL}(V) \otimes \overline{\mathbb{Q}}_\ell \simeq \mathrm{GL}(\mathcal{E}_{\ell,\overline{x}})$ identifies $G^{geo} \otimes \overline{\mathbb{Q}}_\ell$ with $G(\mathcal{E}_{\ell,\overline{x}})^0$.
- Assume moreover that $\mathcal{E}_\ell$ is semisimple for every $\ell \in \mathcal{L}$. There exist a $\overline{\mathbb{Q}}$-connected algebraic group G, a faithful representation $\rho : G \subseteq \mathrm{GL}(V)$ and a (non canonical) isomorphism $V \otimes \overline{\mathbb{Q}}_\ell \simeq \mathcal{E}_{\ell,\overline{x}}$ for every ℓ, such that the composition $G \otimes \overline{\mathbb{Q}}_\ell \subseteq \mathrm{GL}(V) \otimes \overline{\mathbb{Q}}_\ell \simeq \mathrm{GL}(\mathcal{E}_{\ell,\overline{x}})$ identifies $G \otimes \overline{\mathbb{Q}}_\ell$ with $G(\mathcal{E}_{\ell,\overline{x}})^0$.

1.3 Specialization of ℓ-adic representations and Néron-Severi groups in characteristic 0

Let k be a field of characteristic zero. Let X be a smooth geometrically connected k-variety and write η for the generic point of X. In this section we recall results of Cadoret-Tamagawa ([CT12b], [CT13]) and André ([And96]).

1.3.1 A uniform open image theorem for ℓ-adic representations

In this section we discuss a finiteness result of Cadoret and Tamagawa, which strengthens Fact 1.2.2.2.2 when X is a curve.

1.3.1.1 Exceptional locus

Let X be a curve and $\rho : \pi_1(X, \overline{\eta}) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ be a continuous representation with image Π. In Section 1.2.2.2, we recalled that, for every $x \in |X|$, the choice of an étale path between η and x induces a local Galois representation

$$\rho_x : \pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta}) \xrightarrow{\rho} \mathrm{GL}_r(\mathbb{Z}_\ell)$$

with image Π_x and an inclusion $\Pi_x \subseteq \Pi$. Following [CK16], we give the following definition.

Definition 1.3.1.1.1. We say that $x \in |X|$ is Galois generic for ρ if $\Pi_x \subseteq \Pi$ is an open subgroup. If x is not Galois generic, we say that x is Galois exceptional for ρ.

Write X_ρ^{ex} and X_ρ^{gen} for the set of closed Galois exceptional and Galois generic points for ρ and set,

$$X_\rho^{ex}(\leq d) := X_\rho^{ex} \cap X(\leq d); \quad X_\rho^{gen}(\leq d) := X_\rho^{gen} \cap X(\leq d).$$

1.3.1.2 Statement

Since X is geometrically connected, we can consider the representation

$$\rho^{geo} : \pi_1(X_{\overline{k}}, \overline{\eta}) \to \pi_1(X, \overline{\eta}) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$$

and its image Π^{geo}. Recall the following definition from [CT12b, Section 1]

Definition 1.3.1.2.1. We say that ρ is geometrically Lie perfect (*GLP* for short) if[2] every open subgroup of Π^{geo} has finite abelianization.

Then Cadoret-Tamagawa prove:

Fact 1.3.1.2.2 ([CT12b]). Assume that k is finitely generated and X is a curve. If ρ is *GLP*, for every integer $d \geq 1$, the set $X_\rho^{ex}(\leq d)$ is finite and there exists an integer $N \geq 1$, depending only on d and ρ, such that, for every $x \in X_\rho^{gen}(\leq d)$, one has $[\Pi : \Pi_x] \leq N$.

In the following subsections we will recall the main ideas in the proof of Fact 1.3.1.2.2.

1.3.1.3 Group theory: A projective system of subgroups

Recall that $\Phi(\Pi)$ denotes the Frattini subgroup of Π, i.e. the intersection of the maximal open subgroups of Π. In the proof of Fact 1.2.2.2.2, one of the key input was to consider the finite set $\mathcal{C}(\Pi)$ of subgroup $U \subseteq \Pi$ such that $\Phi(\Pi) \subseteq U$. To prove Fact 1.3.1.2.2, Cadoret-Tamagawa construct in [CT12b, Section 3] a projective system refining $\mathcal{C}(\Pi)$. For every subgroup C of Π, write

$$C(n) := Ker(C \subseteq \Pi \subseteq \mathrm{GL}_r(\mathbb{Z}_\ell) \to \mathrm{GL}_r(\mathbb{Z}_\ell/\ell^n)).$$

Set $\mathcal{C}_0(\Pi) := \{\Pi\}$ and for every integer $n \geq 1$

$$\mathcal{C}_n(\Pi) := \{U \subseteq \Pi \text{ such that } \Phi(\Pi(n-1)) \subseteq U \text{ and } \Pi(n-1) \not\subseteq U\}.$$

By [CT12b, Lemma 3.1], the maps $\psi_n : \mathcal{C}_{n+1}(\Pi) \to \mathcal{C}_n(\Pi)$ sending U to $U\Phi(\Pi(n-1))$ are well defined, hence they endow the collection $\{\mathcal{C}_n(\Pi)\}_{n \in \mathbb{N}}$ with the structure of a projective system. The analogue of Lemma 1.2.2.4.1 is then the following.

Lemma 1.3.1.3.1 ([CT12b, Lemma 3.3]).

1. For each integer $n \geq 0$, the set $\mathcal{C}_n(\Pi)$ is finite;

2. For $n \gg 0$, if $C \subseteq \Pi$ is a closed subgroup such that $\Pi(n-1) \not\subseteq C$, then there exists $U \in \mathcal{C}_n(\Pi)$ such that $C \subseteq U$.

[2] The terminology comes from the fact that this condition is equivalent to $(Lie(\Pi^{geo}))^{ab} = 0$.

1.3.1.4 Anabelian dictionary I

For each integer $n \geq 0$ write
$$\mathcal{X}_n := \coprod_{U \in \mathcal{C}_n(\Pi)} X_U \to X.$$

Then, since the collection $\{\Pi(n)\}_{n\in\mathbb{N}}$ is a fundamental system of open neighbourhoods of 1 in Π, one has

$$\begin{aligned} x \in X_\rho^{ex} &\Leftrightarrow \text{ for } n \gg 0\ \Pi(n-1) \not\subseteq \Pi_x \\ &\Leftrightarrow \text{ for } n \gg 0 \text{ there exists } U \in \mathcal{C}_n(\Pi) \text{ with } \Pi_x \subseteq U \qquad \text{(Lemma 1.3.1.3.1(2))} \\ &\Leftrightarrow \text{ for } n \gg 0\ x \in \mathrm{Im}(\mathcal{X}_n(k(x)) \to X(k(x)) \qquad \text{(Remark 1.2.2.3.1)} \end{aligned}$$

This shows that
$$X_\rho^{ex}(\leq d) = \bigcap_{n\geq 1} \mathrm{Im}(\mathcal{X}_n(\leq d) \to X(\leq d))$$

and that, for $n \gg 0$, one has

$$\{x \in X(\leq d) \text{ with } [\Pi : \Pi_x] \leq [\Pi : \Pi(n)]\} \subseteq X(\leq d) - \mathrm{Im}(\mathcal{X}_n(\leq d) \to X(\leq d)). \qquad (1.3.1.4.1)$$

By (1.3.1.4.1), since Π has a finite number of open subgroups of bounded index and $\mathcal{C}_n(\Pi)$ is finite, to prove Fact 1.3.1.2.2 it is enough to show that, for $n \gg 0$ and for every $U \in \mathcal{C}_n(\Pi)$, the set $X_U(\leq d)$ is finite.

1.3.1.5 Genus and gonality

The finiteness of rational points of a smooth curve Y is controlled by the genus g_Y and the gonality[3] γ_Y of the smooth compactification of $Y_{\overline{k}}$. More precisely, one has the following:

Fact 1.3.1.5.1. Let k a finitely generated fields of characteristic 0 and let Y be a smooth proper k-curve.

1. ([FW84]): Assume that $g_Y \geq 2$. Then $Y(k)$ is finite.

2. ([Fal91], [Fre94]): Assume that $\gamma_Y \geq 2d+1$. Then $Y(\leq d)$ is finite.

Coming back to $X_U \to X$, we now aim to show that they have large genus and gonality. For every open subgroup $U \subseteq \Pi$, write $k \subseteq k_U$ for the smallest finite extension of k on which X_U is geometrically connected and write g_U and γ_U for the genus and the gonality of the smooth compactification of $X_U \times_{k_U} \overline{k}$ respectively. Then, to prove Theorem 1.3.1.2.2, it is enough to show the following.

Fact 1.3.1.5.2. Assume that ρ is GLP and fix integers $d_1 \geq 0$, $d_2 \geq 1$. Then:

1. ([CT12b, Corollary 3.8]): There exists an integer $N_g \geq 1$, depending only on ρ, d_1, d_2, such that for every $n \geq N_g$ and every $U \in \mathcal{C}_n(\Pi)$ one has $g_U \geq d_1$ or $[k_U : k] \geq d_2$.

2. ([CT13, Corollary 3.11]): There exists an integer $N_\gamma \geq 1$, depending only on ρ, d_1, d_2, such that for every $n \geq N_\gamma$ and every $U \in \mathcal{C}_n(\Pi)$ one has $\gamma \geq d_1$ or $[k_U : k] \geq d_2$.

Remark 1.3.1.5.3. A posteriori, via the Riemann-Hurwitz formula Fact 1.3.1.5.2(2) implies Fact 1.3.1.5.2(1) but, actually, Fact 1.3.1.5.2(1) is used in the proof of Fact 1.3.1.5.2(2).

[3] Recall that the gonality of a smooth proper $\overline{k}$-curve Y is the minimum degree of a non constant morphism $Y \to \mathbb{P}^1_{\overline{k}}$.

1.3.1.6 Anabelian dictionary II: GLP assumption

To illustrate the idea in the proof of Fact 1.3.1.5.2(1), in this section we show, following [CT12a, Section 4.1.3], that if $k = \overline{k}$, the representation ρ is GLP and Π is infinite, then $g_{\Pi(n)}$ tends to infinity. Let $n_0 \geq 1$ be an integer. For each $n \geq n_0$, the Riemann Hurwitz formula for the cover $X_{\Pi(n)} \to X_{\Pi(n_0)}$ implies that

$$\lim_{n\to+\infty} 2g_{\Pi(n)} - 2 \geq \lim_{n\to+\infty} (|\Pi(n_0)/\Pi(n)|)(2g_{\Pi(n_0)} - 2) \tag{1.3.1.6.1}$$

Since Π in infinite, one has

$$\lim_{n\to+\infty} |\Pi(n_0)/\Pi(n)| = |\Pi(n_0)| = +\infty.$$

Hence, if $sup_n(g_{\Pi(n)}) \geq 2$ then there exists an n_0 such that $g_{\Pi(n_0)} \geq 2$ and Equation (1.3.1.6.1) implies that $g_{\Pi(n)}$ tends to infinity. So we need to rule out the following two possibilities:

1. $\sup(g_{\Pi(n)}) = 1$. Then there exists n_0 such that for all $n \geq n_0$ the smooth compactification of $X_{\Pi(n)}$ is an elliptic curve. Since all finite morphisms between elliptic curves are unramified, the Galois group $\Pi(n_0)/\Pi(n)$ of $X_{\Pi(n)} \to X_{\Pi(n_0)}$ would be a quotient of the étale fundamental group of the smooth compactification of $X_{\Pi(n_0)}$. In particular it would be abelian and hence $\Pi(n_0) = \varprojlim_n \Pi(n_0)/\Pi(n)$ would be abelian and infinite. But this contradicts the fact that ρ is GLP, since $\Pi(n_0)$ would be an infinite abelian open subgroup of Π.

2. $\sup(g_{\Pi(n)}) = 0$. Then for all $n \geq 0$, the smooth compactification of $X_{\Pi(n)}$ is isomorphic to $\mathbb{P}^1$. So the Galois group $\Pi(1)/\Pi(n)$ of the cover $X_{\Pi(n)} \to X_{\Pi(1)}$ is a subgroup of $\mathrm{PGL}_2(k)$. Using the classification of subgroups of $\mathrm{PGL}_2(k)$ (see e.g. [Cad12a, Corollary 10]) one gets a contradiction via the *GLP* assumption as in point 1.

The proof of Fact 1.3.1.5.2(1) is significantly more involved, since the covers $X_U \to X$ are not in general Galois. The idea is then to take a Galois cover $X_{\widetilde{U}} \to X$ over $X_U \to X$ and close to the Galois closure of $X_U \to X$ and then:

- First apply the previous argument to $X_{\widetilde{U}}$ ([CT12b, Section 3.3.1]);
- Then compare the genus of $X_{\widetilde{U}}$ and X_U via the Riemann-Hurwitz formula ([CT12b, Section 3.3.2]).

We will discuss in more details this strategy in Section 2.1.1.3.

1.3.2 Specialization of Neron-Severi groups

Let $Y \to X$ be a smooth proper morphism. In this section we discuss a result of André, which relates Facts 1.2.2.2.2 and 1.3.1.2.2 to the specialization of the Néron-Severi group.

1.3.2.1 NS-generic points

We specialize the discussion of Section 1.2.1 to the case of divisors. Let Z be a smooth proper k-variety. In this setting, since algebraic and numerical equivalence coincide rationally for divisors, for every couples of primes ℓ, ℓ' we have equalities

$$\mathrm{CH}^1_{\ell'}(Z_{\overline{k}}) \otimes \mathbb{Q} = \mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q} = \mathrm{CH}^1_{\ell}(Z_{\overline{k}}) \otimes \mathbb{Q}.$$

Moreover, since $H^1(\pi_1(k), Pic^0(Z))$ is torsion, the exact sequence of k-group schemes

$$0 \to Pic(Z)^0 \to Pic(Z) \to \mathrm{NS}(Z) \to 0$$

shows that $\mathrm{NS}(Z) \otimes \mathbb{Q} = (\mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q})^{\pi_1(k)}$. So, for every $x \in X$, the specialization morphisms of Section 1.2.1 for the morphism $f : Y \to X$ read:

$$sp_{\eta,x} : \mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \to \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q} \quad \text{and} \quad sp^{ar}_{\eta,x} : \mathrm{NS}(Y_{\eta}) \otimes \mathbb{Q} \to \mathrm{NS}(Y_{x}) \otimes \mathbb{Q}.$$

Definition 1.3.2.1.1. We say that $x \in |X|$ is NS-generic (resp. arithmetically NS-generic) for $f : Y \to X$ if $sp_{\eta,x}$ (resp. $sp^{ar}_{\eta,x}$) is an isomorphism.

1.3.2.2 NS-generic vs Galois generic

For every $x \in X$, the choice of an étale path between $\overline{x}$ and $\overline{\eta}$ induces isomorphisms

$$\pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta}), \quad H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))) \simeq R^2 f_* \mathbb{Q}_\ell(1)_{\overline{\eta}} \simeq R^2 f_* \mathbb{Q}_\ell(1)_{\overline{x}} \simeq H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))),$$

identifying the action of $\pi_1(x, \overline{x})$ induced by restriction via $\pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta})$ on $H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)) \simeq H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)))$ with the natural action of $\pi_1(x, \overline{x})$ on $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)))$. Recall that the specialization morphism makes the following diagram commutative:

$$\begin{tikzcd}
Pic(Y_\eta) \otimes \mathbb{Q} \arrow[d, "c_{Y_\eta}"] & Pic(Y) \otimes \mathbb{Q} \arrow[l, "i_\eta^*"'] \arrow[r, "i_x^*"] & Pic(Y_x) \otimes \mathbb{Q} \arrow[d, "c_{Y_x}"] \\
\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[rr, hook, "sp_{\eta,x}"] \arrow[d, hook] & & \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q} \arrow[d, hook] \\
H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)) & \simeq & H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)).
\end{tikzcd}$$

and that $x \in |X|$ is said to be Galois generic (resp. strictly Galois generic) for $\rho_\ell^{2,1} : \pi_1(X) \to \mathrm{GL}(H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)))$ if the image of $\pi_1(x, \overline{x})$ acting on $H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))$ is open (resp. coincide) with the image of $\pi_1(X, \overline{\eta})$ acting on $H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))$. Conjecture 1.1.2.1.1 predicts that every (strictly) Galois generic point is (arithmetically) NS-generic. André proved that this holds without assuming Conjecture 1.1.2.1.1.

Fact 1.3.2.2.1 ([And96])**.** Every (strictly) Galois generic point for $\rho_\ell^{2,1}$ is (arithmetically) Néron-Severi generic.

Combining Fact 1.3.2.2.1 with Facts 1.2.2.2.2 and 1.3.1.2.2, one gets the existence and the abundance of (arithmetically) NS-generic points. The proof of Fact 1.3.2.2.1 decomposes into two steps:

- One relates algebraic cycles to cohomology via the Variational Hodge conjecture for divisors (Fact 1.2.3.2.1);
- One relates Hodge theory to ℓ-adic cohomology via the comparison between the étale and the singular sites;

In the next subsection, we recall in more details the argument for Fact 1.3.2.2.1 (See also [CC18, Proposition 3.2.1]).

1.3.2.3 Proof of Fact 1.3.2.2.1

Let $x \in |X|$ be a Galois generic point for $\rho^{2,1}$. Replacing X with a finite étale cover we can assume that $\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} = \mathrm{NS}(Y_\eta) \otimes \mathbb{Q}$, $\mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q} = \mathrm{NS}(Y_x) \otimes \mathbb{Q}$ and that the Zariski closure $G_\ell^{2,1}$ of the image of $\pi_1(X, \overline{\eta})$ acting on $H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))$ is connected. The commutative cartesian diagram of k-varieties

$$\begin{tikzcd}
Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \arrow[dr, phantom, "\square"] & Y_\eta \arrow[l, hook'] \arrow[d] \\
k(x) \arrow[r, "x"] & X & k(\eta). \arrow[l]
\end{tikzcd}$$

induces a commutative diagram:

$$\begin{tikzcd}
Pic(Y_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[d, two heads] & Pic(Y_{\overline{k}}) \otimes \mathbb{Q} \arrow[l, two heads] \arrow[r] & Pic(Y_{\overline{x}}) \otimes \mathbb{Q} \arrow[d, two heads] \\
\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[rr, hook, "sp_{\eta,x}"] & & \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q}.
\end{tikzcd}$$

Then, it is enough to show that every $z_x \in \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q}$ lifts to an element of $Pic(Y_{\overline{k}}) \otimes \mathbb{Q}$. Since the image of $Pic(Y_{\overline{k}}) \otimes \mathbb{Q} \to H^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1))$ identifies[4] with the image of $Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q} \to H^2(Y_{\mathbb{C}}, \mathbb{Q}_\ell(1))$ via the base change isomorphism $H^2(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) \simeq H^2(Y_{\mathbb{C}}, \mathbb{Q}_\ell(1))$ and the Néron-Severi group is invariant under algebraically closed fields extension, it is enough to show that every $z_x \in \mathrm{NS}(Y_{\overline{x},\mathbb{C}}) \otimes \mathbb{Q}$ lifts to an element of $Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q}$. Consider the commutative diagram:

$$\begin{tikzcd}
Pic(Y_{\mathbb{C}}) \otimes \mathbb{Q} \arrow[r] \arrow[dd] & Pic(Y_{\overline{x},\mathbb{C}}) \otimes \mathbb{Q} \arrow[d, two heads] \\
& \mathrm{NS}(Y_{\overline{x},\mathbb{C}}) \otimes \mathbb{Q} \arrow[d, hook] \\
H^0(X_{\mathbb{C}}^{an}, R^2 f_* \mathbb{Q}) \arrow[r, hook] & H_B^2(Y_{\overline{x},\mathbb{C}}, \mathbb{Q}).
\end{tikzcd}$$

Take any $z_x \in \mathrm{NS}(Y_{\overline{x},\mathbb{C}}) \otimes \mathbb{Q}$. By the Variational Hodge conjecture for divisors (Fact 1.2.3.2.1) it is enough to show that z_x is in the image of $H^0(X, R^2 f_*^{an} \mathbb{Q}) \hookrightarrow H_B^2(Y_{\overline{x},\mathbb{C}}, \mathbb{Q})$. Since z_x is fixed by $\pi_1(x, \overline{x})$, the group $G_\ell^{2,1}$ is connected and x is Galois-generic, z_x is fixed by $\pi_1(X, \overline{x})$, hence by $\pi_1(X_{\mathbb{C}}, \overline{x})$. Via the comparison between the étale and the singular sites, z_x is then fixed by $\pi_1^{top}(X_{\mathbb{C}}^{an}, \overline{x})$, hence it is in the image of $H^0(X, R^2 f_*^{an} \mathbb{Q}) \simeq H_B^2(Y_{\overline{x},\mathbb{C}}, \mathbb{Q})^{\pi_1^{top}(X_{\mathbb{C}}, \overline{x})} \hookrightarrow H_B^2(Y_x, \mathbb{Q})$. This concludes the proof of Fact 1.3.2.2.1.

[4] This follows from the invariance of étale cohomology under algebraically closed field extension in characteristic zero, the Kummer exact sequence and the fact that the map $H^2(Y_{\overline{k}}, \mathbb{G}_m) \to H^2(Y_{\mathbb{C}}, \mathbb{G}_m)$ is injective.

Chapter 2

Presentation of the work

This Chapter is devoted to summarize in a uniform way the new results obtained in this thesis, trying to explain how they relate to each others.

2.1 Specialization of ℓ-adic representations and Néron-Severi groups in positive characteristic

Chapters 3 and 4 are devoted to extend to positive characteristic of the results of Cadoret-Tamagawa (Fact 1.3.1.2.2) and André (Fact 1.3.2.2.1). Let k be a field of characteristic $p > 0$ and let X be a smooth geometrically connected k-variety. Write η for the generic point of X. Fix a prime $\ell \neq p$.

2.1.1 A uniform open image theorem in positive characteristic

2.1.1.1 Statement

We briefly recall the setting. Let

$$\rho : \pi_1(X, \overline{\eta}) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$$

be a continuous representation. For every $x \in X$ the choice of an étale path between $\overline{x}$ and $\overline{\eta}$ give rise to a representation

$$\rho_x : \pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta}) \to \mathrm{GL}_n(\mathbb{Z}_\ell),$$

hence to an inclusion $\Pi_x := \mathrm{Im}(\rho_x) \subseteq \mathrm{Im}(\rho) =: \Pi$. Set Π^{geo} for the image of

$$\rho^{geo} : \pi_1(X_{\overline{k}}, \overline{\eta}) \to \pi_1(X, \overline{\eta}) \to \mathrm{GL}_r(\mathbb{Z}_\ell).$$

Recall that $x \in |X|$ is Galois generic if $\Pi_x \subseteq \Pi$ is an open subgroup and that ρ is said to be GLP if every open subgroup of Π^{geo} has finite abelianization. Write $X_\rho^{gen}(k)$ for the set of k-rational Galois generic points and $X_\rho^{ex}(k) := X(k) - X_\rho^{gen}(k)$. The first main result is the extension of the $d = 1$ case of Fact 1.3.1.2.2 to positive characteristic.

Theorem 2.1.1.1.1. Assume that k is finitely generated, X is a curve and ρ is GLP. Then $X_\rho^{ex}(k)$ is finite and exists an integer $N \geq 1$, depending only on ρ, such that for every $x \in X_\rho^{gen}(k)$, one has $[\Pi : \Pi_x] \leq C$

Remark 2.1.1.1.2. Fact 1.3.1.2.2 holds not only for k-rational points but also for points of bounded degree. The reason why we get only the statement for k-rational points is that the analogue of Fact 1.3.1.5.1(2) does not hold in positive characteristic. See Sections 2.1.1.4 and 3.3.3 for more details.

2.1.1.2 General Strategy

To prove Theorem 2.1.1.1.1 we follow the strategy of Cadoret-Tamagawa for Fact 1.3.1.2.2. First recall (Section 1.3.1.4) that, for each integer $n \geq 0$, there is a finite set $\mathcal{C}_n$ of subgroups of Π and a (possibly disconnected) étale cover

$$\mathcal{X}_n := \coprod_{U \in \mathcal{C}_n} X_U \to X.$$

such that, for $n \gg 0$,

$$X_\rho^{ex}(k) = \bigcap_{n \geq 1} Im(\mathcal{X}_n(k) \to X(k)),$$
$$\{x \in X(k) \mid [\Pi : \Pi_x] \leq [\Pi : \Pi(n)]\} \subseteq X(k) - Im(\mathcal{X}_n(k) \to X(k)).$$

Since the set $\mathcal{C}_n$ is finite, to prove Theorem 2.1.1.1.1, it is enough to show that for $n \gg 0$ and each $U \in \mathcal{C}_n$ the scheme X_U has only finitely many k-rational points. Then by [Sam66] and an argument of Voloch (see [EElsHKo09, Theorem 3] for more details), we have the following analogue of Fact 1.3.1.5.1.

Fact 2.1.1.2.1. There exists an integer $g \geq 2$, depending only on k, such that every smooth proper curve over k with genus $\geq g$ has only finitely many k-rational points.

By Fact 2.1.1.2.1, Theorem 2.1.1.1.1 boils down to prove an analogue of Fact 1.3.1.5.2(1), which, by group theoretic arguments, one reduces to the following:

Theorem 2.1.1.2.2. If ρ is GLP then for every closed but not open subgroup $C \subseteq \Pi^{geo}$, one has

$$\lim_{n \mapsto +\infty} g_{C\Pi^{geo}(n)} = +\infty$$

The proof of Theorem 2.1.1.2.2 follows the strategy of the proof of Fact 1.3.1.5.2. However, in positive characteristic, the Riemann-Huruwitz formula, used to study the growth of genus, involves wild inertia terms. Even assuming $\ell \neq p$, which is crucial here, controlling this wild inertia terms is rather delicate. This is our main technical contribution.

2.1.1.3 Controlling the wild inertia

To simplify the notation, assume from now that $k = \overline{k}$. For a group Γ and subgroups $I, C \subseteq \Gamma$ write

$$K_C(\Gamma) := \cap_{g \in \Gamma} gCg^{-1} \quad \text{and} \quad I_C := I/(I \cap K_C(\Gamma))$$

for the largest normal subgroup of Γ contained in C and the largest quotient of I acting faithfully on Γ/C. Define $\widetilde{\Pi}(n) \subseteq \Pi$ and $\Pi_C(n) \subseteq \Pi_C$ by the following commutative exact diagram:

$$\begin{tikzcd}
1 \arrow[r] & \widetilde{\Pi}(n) \arrow[r] \arrow[d] & \Pi \arrow[r] \arrow[d] & (\Pi_n)_{C_n} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Pi_C(n) \arrow[r] & \Pi_C \arrow[r] \arrow[d] & (\Pi_n)_{C_n} \arrow[r] & 1 \\
 & & 1 & &
\end{tikzcd}$$

The diagram induces a commutative diagram of covers

$$\begin{tikzcd} X_{\widetilde{\Pi}(n)} \arrow[r] \arrow[d] & X_{C\Pi(n)} \arrow[dl] \\ X & \end{tikzcd}$$

where $X_{\widetilde{\Pi}(n)} \to X$ and $X_{\widetilde{\Pi}(n)} \to X_{C\Pi(n)}$ are Galois. After a preliminary reduction where we show that we can assume that $\widetilde{\Pi}(1)/\widetilde{\Pi}(n)$ is an ℓ-group, the proof decomposes into two steps:

1. We show that $g_{\widetilde{\Pi}(n)}$ tends to infinity. Since $\widetilde{\Pi}(n) \subseteq \Pi$ is a normal open subgroup and ρ is GLP, this will follow from the fact that Galois covers of curves of genus ≥ 1 contains large abelian subgroups, as in Section 1.3.1.6;

2. We show that (1) implies that $g_{C\Pi(n)}$ tends to infinity. To do this one has to relate $g_{\widetilde{\Pi}(n)}$ and $g_{C\Pi(n)}$ via the Riemann Huruwitz formula for the cover $X_{\widetilde{\Pi}(n)} \to X_{C\Pi(n)}$. This, in turn, boils down to study the ramification of the cover $X_{\widetilde{\Pi}(n)} \to X_{C\Pi(n)}$ and we do this in two steps:

   (a) We first consider the commutative diagram:

   $$\begin{tikzcd} X_{\widetilde{\Pi}(n)} \arrow[r] \arrow[d] & X_{C\Pi(n)} \arrow[dl] \\ X. & \end{tikzcd}$$

   The behaviour of the ramification under intermediate cover allows us to understand the ramification of $X_{\widetilde{\Pi}(n)} \to X_{C\Pi(n)}$ via the ramification of $X_{\widetilde{\Pi}(n)} \to X$.

   (b) We then study the wild ramification of $X_{\widetilde{\Pi}(n)} \to X$ via the commutative diagram:

   $$\begin{tikzcd} X_{\widetilde{\Pi}(n)} \arrow[r] \arrow[d] & X_{\widetilde{\Pi}(1)} \arrow[dl] \\ X. & \end{tikzcd}$$

   By the preliminary reduction the Galois group of $X_{\widetilde{\Pi}(n)} \to X_{\widetilde{\Pi}(1)}$ is an ℓ-group and hence the morphism between the wild inertia subgroups of the covers $X_{\widetilde{\Pi}(n)} \to X$ and $X_{\widetilde{\Pi}(1)} \to X$ is an isomorphism. This implies that the wild inertia of $X_{\widetilde{\Pi}(n)} \to X$ grows in an explicit linear way and this is enough to control it.

2.1.1.4 Gonality

We keep the notation of Section 2.1.1.1, but in this subsection we allow the prime ℓ to be equal to p.

As already mentioned in Section 2.1.1.2, Theorem 2.1.1.2.2 implies the natural extension to positive characteristic of Fact 1.3.1.5.2(1)). Moving from genus to gonality, in Appendix A, we show how to adapt the arguments of Fact 1.3.1.5.2(2) to prove its positive characteristic version. More precisely, we show the following, which can be used to extend Fact 1.3.1.5.2(2) to positive characteristic.

Theorem 2.1.1.4.1. Let $C \subseteq \Pi_{\overline{k}}$ be a closed subgroup of of codimension j. The following hold:

1. If $\ell \neq p$, the representation ρ is GLP and $j \geq 1$, then

$$\lim_{n \to +\infty} \gamma_{C\Pi_{\overline{k}}(n)} = +\infty.$$

2. If $\ell \neq p$ and $j \geq 3$, then

$$\lim_{n \to +\infty} \gamma_{C\Pi_{\overline{k}}(n)} = +\infty,$$

3. If $\ell = p$ and $j \geq 2$, then

$$\lim_{n \to +\infty} \gamma_{C\Pi_{\overline{k}}(n)} = +\infty,$$

As mentioned in Remark 2.1.1.1.2, Fact 1.3.1.5.1(2) is not true in positive characteristic. Hence one cannot use directly Theorem 2.1.1.4.1 to obtain the version of Theorem 2.1.1.1.1 for points of bounded degree (see Section 3.3.3 in Chapter 3 for a discussion around this issue). However, Theorem 2.1.1.4.1(2) and (3) can be used to obtain results on not necessarely GLP or ℓ-adic representations. We obtain the following, which extends [CT13, Theorem 1.3] to positive characteristic.

Corollary 2.1.1.4.2. Assume that X is a curve and k is finitely generated. The following hold:

1. If $\ell = p$, then for all but at most finitely many $x \in X(k)$, $\Pi_x \subseteq \Pi$ has codimension ≤ 1;
2. If $\ell \neq p$, then for all but at most finitely many $x \in X(k)$, $\Pi_x \subseteq \Pi$ has codimension ≤ 2.

2.1.2 Specialization of Néron-Severi groups in positive characteristic

Let $f : Y \to X$ be a smooth proper morphism and set

$$\rho : \pi_1(X, \overline{\eta}) \to \mathrm{GL}(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)))$$

2.1.2.1 Statement

Let us recall (Section 1.3.2.1) that, for every $x \in X$ there are specialization morphisms

$$sp_{\eta,x} : \mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \hookrightarrow \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q} \quad \text{and} \quad sp^{ar}_{\eta,x} : \mathrm{NS}(Y_\eta) \otimes \mathbb{Q} \hookrightarrow \mathrm{NS}(Y_x) \otimes \mathbb{Q}$$

and that x is said to be NS-generic (resp. arithmetically NS-generic) if $sp_{\eta,x}$ (resp. $sp^{ar}_{\eta,x}$) is an isomorphism. The main result of this section is the analogue of Fact 1.3.2.2.1 in positive characteristic.

Theorem 2.1.2.1.1. Assume that k is finitely generated. If $f : Y \to X$ is projective, every (strictly) Galois generic point for ρ is (arithmetically) Neron-Severi generic. If $f : Y \to X$ is proper, the same is true replacing X with a dense open subset.

Remark 2.1.2.1.2. The reason why, contrary to the characteristic zero case, we are not able to prove Theorem 2.1.2.1.1 for a general smooth proper morphism $f : Y \to X$ is that in Fact 1.2.4.2.1 the morphism is assumed to be projective. On the other hand, using De Jong alteration's theorem ([dJ96]) one can prove Theorem 2.1.2.1.1 for not necessary proper morphism up to replacing X with a dense open subset. For most of the applications Theorem 2.1.2.1.1 is enough.

From now on assume, for simplicity, that $f : Y \to X$ is projective.

2.1.2.2 General strategy

The starting point of our proof is to replace the use of Hodge theory used in Fact 1.3.2.2.1 with crystalline cohomology, since a variational form of the Tate conjecture (Fact 1.2.4.2.1) is known in this setting. The main difficulty to overcome is to transfer the information about the Galois invariants of the ℓ-adic lisse sheaf $R^2 f_*\mathbb{Q}_\ell(1)$ to the crystalline local system (F-isocrystal) $R^2 f_{crys,*}\mathcal{O}_{Y/K}(1)$. This is the main new contribution of Chapter 4. More precisely, since the F-isocrystal $R^2 f_{crys,*}\mathcal{O}_{Y/K}(1)$ has a behaviour which is quite different from $R^2 f_*\mathbb{Q}_\ell(1)$ (see Section 1.2.4.4), this comparison cannot be done directly. The idea is then to show that $R^2 f_{crys,*}\mathcal{O}_{Y/K}(1)$ is coming from a smaller and better behaved category of p-adic local systems: the category of overconvergent F-isocrystals. As it has been understood that overconvergent F-isocrystals share many properties with lisse sheaves (see Section 1.2.4.7), the idea is to compare first $R^2 f_{crys,*}\mathcal{O}_{Y/K}(1)$ with its overconvergent incarnation $R^2 f_*\mathcal{O}^\dagger_{Y/K}(1)$ via various p-adic comparison theorems and then $R^2 f_*\mathcal{O}^\dagger_{Y/K}(1)$ with $R^2 f_*\mathbb{Q}_\ell(1)$ via the theory of weights ([Del80], [KM74]).

In the next to subsections we explain the strategy in more details.

2.1.2.3 Spreading out

One additional difficulty in our setting is that crystalline cohomology has a good behaviour only over a perfect field, while our base field k is, in general, not perfect. To overcome this problem one uses a spreading out argument, so that our morphism $f : Y \to X$ will appear as the generic fibre of a smooth projective morphism $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$, where $\mathcal{X}$ is a smooth geometrically connected $\mathbb{F}_q$-variety. The idea is then to lift an element $\epsilon_x \in \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q}$ to $\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q}$ by specializing it first to an element $\epsilon_{\mathfrak{t}} \in \mathrm{NS}(\mathcal{Y}_{\bar{\mathfrak{t}}}) \otimes \mathbb{Q}$ of a closed fibre of $\mathcal{Y} \to \mathcal{X}$ and then to try and lift $\epsilon_{\mathfrak{t}}$ to an element $\epsilon \in Pic(\mathcal{Y}) \otimes \mathbb{Q}$, via the crystalline variational Tate conjecture (Fact 1.2.4.2.1) over $\mathbb{F}_q$.

2.1.2.4 From ℓ to p

In order to show that $\epsilon_{\mathfrak{t}} \in \mathrm{NS}(Y_{\bar{\mathfrak{t}}}) \otimes \mathbb{Q}$ satisfies the assumption of Fact 1.2.4.2.1, one has to transfer the ℓ-adic information that x is Galois generic to crystalline cohomology. Assume that $\mathcal{Z}$ is a smooth geometrically connected $\mathbb{F}_q$-variety admitting an $\mathbb{F}_q$-rational point $\mathfrak{t}$ and that there is a map $\mathfrak{g} : \mathcal{Z} \to \mathcal{X}$ (in our application $\mathfrak{g} : \mathcal{Z} \to \mathcal{X}$ is a model for $x : k(x) \to X$). The cartesian square

$$\begin{tikzcd} \mathcal{Y}_{\mathcal{Z}} \arrow[r] \arrow[d, "\mathfrak{f}_{\mathcal{Z}}"] \arrow[dr, phantom, "\square"] & \mathcal{Y} \arrow[d, "\mathfrak{f}"] \\ \mathcal{Z} \arrow[r, "\mathfrak{g}"] & \mathcal{X} \end{tikzcd}$$

induces representations

$$\pi_1(\mathcal{Z}, \bar{\mathfrak{t}}) \to \pi_1(\mathcal{X}, \bar{\mathfrak{t}}) \to \mathrm{GL}(H^i(\mathcal{Y}_{\bar{\mathfrak{t}}}, \mathbb{Q}_\ell(j)).$$

Theorem 2.1.2.4.1. Assume that the image of $\pi_1(\mathcal{Z}, \bar{\mathfrak{t}}) \to \pi_1(\mathcal{X}, \bar{\mathfrak{t}}) \to \mathrm{GL}(H^i(\mathcal{Y}_{\bar{\mathfrak{t}}}, \mathbb{Q}_\ell(j))$ is open in the image of $\pi_1(\mathcal{X}, \bar{\mathfrak{t}}) \to \mathrm{GL}(H^i(\mathcal{Y}_{\bar{\mathfrak{t}}}, \mathbb{Q}_\ell(j))$ and that the Zariski closures of the images of $\pi_1(\mathcal{X}, \bar{\mathfrak{t}})$ and $\pi_1(\mathcal{X}_{\overline{\mathbb{F}}_q}, \bar{\mathfrak{t}})$ acting on $H^i(\mathcal{Y}_{\bar{\mathfrak{t}}}, \mathbb{Q}_\ell(j))$ are connected. Then the base change map

$$H^0(\mathcal{X}, R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W})^{F=q^j} \otimes \mathbb{Q} \to H^0(\mathcal{Z}, R^i\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W})^{F=q^j} \otimes \mathbb{Q}$$

is an isomorphism.

In light of the general strategy explained in Section 2.1.2.2, the proof of Theorem 2.1.2.4.1 decomposes then as follows:

1. We prove that $R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W}\otimes\mathbb{Q}$ and $R^i\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W}\otimes\mathbb{Q}$ are overconvergent F-isocrystals (building on the work of Shiho on relative log convergent cohomology and relative rigid cohomology [Shi08a], [Shi08b]);

2. We use that one doesn't loose information passing from crystalline cohomology to overconvergent F-isocrystals (Fact 1.2.4.5.3);

3. Let G_p and $G_{\mathcal{Z},p}$ be the Tannakian groups of $R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W}\otimes\mathbb{Q}$ and $R^i\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W}\otimes\mathbb{Q}$ as overconvergent F-isocrystals. Theorem 2.1.2.4.1 amounts to showing that $G_p = G_{\mathcal{Z},p}$.

4. The assumption implies that the Zariski closures G_ℓ and $G_{\mathcal{Z},\ell}$ of the image of $\pi_1(\mathcal{X},\bar{\mathfrak{t}})$ and $\pi_1(\mathcal{Z},\bar{\mathfrak{t}})$ acting on $H^i(\mathcal{Y}_{\bar{\mathfrak{t}}},\mathbb{Q}_\ell(j))$ are equal.

5. We show that (4) implies (3), using the theory of Frobenius weights and the fact that reductive algebraic groups are essentially determined by their tensor invariants.

2.1.3 Applications

The first applications of Theorems 2.1.1.1.1 and 2.1.2.1.1 are to uniform boundedness problems.

2.1.3.1 Uniform boundedness for abelian varieties

If Z is a k-abelian variety write $Z(k)_{tors}$ and $Z[\ell^\infty](k)$ for its k-rational torsion and its ℓ-primary torsion respectively. By the arithmetic Lang-Néron theorem ([LN59]), the group $Z(k)$ is finitely generated, hence $Z(k)_{tors}$ and $Z[\ell^\infty](k)$ are finite. A folklore conjecture is the following:

Conjecture 2.1.3.1.1. Let g be an integer ≥ 1. If k is finitely generated, then there exists an integer $N \geq 1$, depending only on k and g, such that $|Z(k)_{tors}| \leq N$ for all k-abelian varieties of dimension g.

Conjecture 2.1.3.1.1 is known when $g = 1$ ([Lev68]), but it is widely open in general. As a consequence of Theorem 2.1.1.1.1 we prove the weaker form of Conjecture 2.1.3.1.1 for the ℓ-primary torsion of abelian varieties of arbitrary dimension in one-dimensional families.

Corollary 2.1.3.1.2. Assume that k is finitely generated, $f : Y \to X$ is an abelian scheme, X is a curve and $\ell \neq p$. Then there exists an integer $N \geq 1$, depending only on $Y \to X$ and ℓ, such that $|Y_x[\ell^\infty](k)| \leq N$ for every $x \in X(k)$.

2.1.3.2 Uniform boundedness for Brauer groups

Let Z be a smooth proper k-variety. As it is well known (see e.g. [CC18, Proposition 2.1.1]), Conjecture $\mathrm{T}(Z,1,\ell)$ holds if and only if the ℓ-primary torsion $\mathrm{Br}(Z_{\overline{k}})[\ell^\infty]^{\pi_1(k)}$ of the Galois invariants of the geometric Brauer group $\mathrm{Br}(Z_{\overline{k}}) := H^2(Z_{\overline{k}},\mathbb{G}_m)$ of $Z_{\overline{k}}$ is finite. In [VAV17], Várilly-Alvarado proposed an analogue of Conjecture 2.1.3.1.1 for the Brauer group of K3 surfaces in characteristic zero. Combining Theorem 2.1.2.1.1 with Theorem 2.1.1.1.1 and the arguments of [CC18], one gets the following version for the ℓ-primary part in one dimensional families:

Corollary 2.1.3.2.1. Assume that k is finitely generated, X is a curve and $\ell \neq p$. If $\mathrm{T}(Y_{\overline{x}},1,\ell)$ holds for all $x \in |X|$, then there exists an integer $N \geq 1$, depending only on $Y \to X$ and ℓ, such that $|\operatorname{Br}(Y_{\overline{x}})[\ell^\infty]^{\pi_1(x,\overline{x})}| \leq N$ for every $x \in X(k)$.

2.1.3.3 Geometric applications

Assume now that k has transcendence degree ≥ 1 over $\mathbb{F}_p$. Then Theorem 2.1.2.1.1, together with a spreading out argument, implies the following:

Corollary 2.1.3.3.1. For every smooth proper morphism $Y \to X$ there exists a NS generic point. If k if finitely generated, then there exists a arithmetically NS-generic point.

Corollary 2.1.3.3.1 implies in particular (see the proof of [MP12, Theorem 7.1]):

Corollary 2.1.3.3.2. Assume furthermore that Y_x is projective for every $x \in |X|$. Then there exists a dense open subscheme $U \subseteq X$ such that the base change $f_U : U \times_X Y \to U$ of $f : Y \to X$ is projective.

The second application of Corollary 2.1.3.3.1 is to hyperplane sections in smooth projective varieties. Assume that Z is a smooth projective k-variety of dimension ≥ 3 and let $Z \subseteq \mathbb{P}^n_k$ be a projective embedding. Every smooth hyperplane section $D \subseteq Z$ induces an injection

$$\mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q} \to \mathrm{NS}(D_{\overline{k}}) \otimes \mathbb{Q}.$$

To reduce problems on the Néron-Severi groups to the case of surface, it could helpful to know whether there exists a smooth hyperplane section $D \subseteq Z$, such that $\mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q} \hookrightarrow \mathrm{NS}(D_{\overline{k}}) \otimes \mathbb{Q}$ is an isomorphism. This is not true in general (see Example 4.4.1.1 in Chapter 4), but one can apply Corollary 2.1.3.3.1 to an appropriate pencil of hyperplane sections to obtain the following arithmetic variant.

Corollary 2.1.3.3.3. If k is finitely generated and $dim(Z) \geq 3$, then there are infinitely many smooth k-rational hyperplane sections $D \subseteq Z$ such that the canonical map

$$\mathrm{NS}(Z) \otimes \mathbb{Q} \to \mathrm{NS}(D) \otimes \mathbb{Q}$$

is an isomorphism.

2.1.3.4 Applications to the Tate conjecture

As already mentioned in Section 2.1.2.1, Conjecture 1.1.2.1.1 implies Theorem 2.1.2.1.1. Using the corollaries in Section 2.1.3.3, one can enlarge the class of varieties for which the ℓ-adic Tate conjecture holds.

Corollary 2.1.3.4.1. Assume that k is infinite finitely generated. Then:

- Let Z be a smooth projective k-variety of dimension ≥ 3 and choose a projective embedding $Z \subseteq \mathbb{P}^n_k$. If $\mathrm{T}(D, 1, \ell)$ holds for every smooth hyperplane sections $D \subseteq Z$, then $\mathrm{T}(Z, 1, \ell)$ holds.
- Let $Y \to X$ be a smooth proper morphism. If $\mathrm{T}(Y_x, 1, \ell)$ holds for every $x \in |X|$, then $\mathrm{T}(Y_\eta, 1, \ell)$ holds.

2.2 p-adic monodromy groups

The proof of Theorem 2.1.2.1.1 suggests that studying the interplay between the categories of F-isocrystals and overconvergent F-isocrystals could lead to fine p-adic information on families of varieties. The next Chapters 5 and 6 are devoted to the investigation of this problem in more details.

2.2.1 Maximal tori in monodromy groups of F-isocrystals and applications (joint with Marco D'Addezio)

In this section $k = \mathbb{F}_q$ is the finite field with $q = p^s$ elements. For simplicity we assume that X has an $\mathbb{F}_q$-rational point x.

2.2.1.1 Maximal tori in monodromy groups of F-isocrystals

Let $\mathcal{E}$ be a pure and p-plain[1] overconvergent F-isocrystal on X. As recalled in Section 1.2.4.6, once x is fixed, to $\mathcal{E}$ we can associate four algebraic groups, fitting into a commutative diagram:

$$\begin{tikzcd} G(\mathcal{E}^{geo,conv}) \arrow[r, hook] \arrow[d, hook] & G(\mathcal{E}^{conv}) \arrow[d, hook] \\ G(\mathcal{E}^{geo}) \arrow[r, hook] & G(\mathcal{E}). \end{tikzcd}$$

If $\mathcal{E}$ has constant Newton polygon, Crew asks whether $G(\mathcal{E}^{geo,conv})$ is a parabolic subgroup of $G(\mathcal{E}^{geo})$, hence whether $G(\mathcal{E}^{conv})$ is a parabolic subgroup of $G(\mathcal{E})$. If $G(\mathcal{E}^{conv}) \subseteq G(\mathcal{E})$ and $G(\mathcal{E}^{geo,conv}) \subseteq G(\mathcal{E}^{geo})$ are parabolic subgroups then, in particular, $G(\mathcal{E}^{conv})$ and $G(\mathcal{E}^{geo,conv})$ contain a maximal torus of $G(\mathcal{E})$ and $G(\mathcal{E}^{geo})$ respectively. From the results in [D'Ad17] or in [HP18], one deduces that $G(\mathcal{E}^{conv})$ contains a maximal torus of $G(\mathcal{E})$; see Corollary 5.2.3.2.1 in Chapter 5. We prove that this is also true for the inclusion $G(\mathcal{E}^{geo,conv}) \subseteq G(\mathcal{E}^{geo})$.

Theorem 2.2.1.1.1. If $\mathcal{E}$ is pure and p-plain, then $G(\mathcal{E}^{geo,conv})$ contains a maximal torus of $G(\mathcal{E}^{geo})$.

Our main motivations to prove Theorem 2.2.1.1.1 were the applications that we explain in the following two sections.

2.2.1.2 A special case of a conjecture of Kedlaya

The following corollary of Theorem 2.2.1.1.1 proves the particular case of the optimistic conjecture[2] in [Ked17, Remark 5.14] where, with the notation of [Ked17, Remark 5.14], $\mathcal{F}_1 \subseteq \mathcal{E}_1$ has minimal slope and $\mathcal{E}_2$ is the convergent isocrystal $\mathcal{O}_X$.

Corollary 2.2.1.2.1. Let $\mathcal{E}$ be an (absolutely) irreducible overconvergent F-isocrystal. If $\mathcal{E}^{conv}$ admits a subobject of minimal slope $\mathcal{F} \subseteq \mathcal{E}^{conv}$ with a non-zero morphism $\mathcal{F}^{conv,geo} \to \mathcal{O}_X^{geo}$ then $\mathcal{E}^{geo} \simeq \mathcal{O}_X^{geo}$.

To deduce Corollary 2.2.1.2.1 from Theorem 2.2.1.1.1, one first reduces to the situation where the determinant of $\mathcal{E}$ has finite order. To simplify, let us assume that $\mathcal{F} = \mathcal{E}_1^{conv}$ and that $G(\mathcal{E}^{conv})$ is connected. Then Theorem 2.2.1.1.1, together with the Global monodromy theorem for overconvergent F-isocrystals ([D'Ad17, Corollary 3.5.5]), implies that $G(\mathcal{E}^{conv,geo}) = G(\mathcal{E}^{conv})$, hence that the morphism $\mathcal{E}_1^{conv,geo} \to \mathcal{O}_X^{geo}$ induces a surjection $\mathcal{E}_1^{conv} \to \mathcal{O}_X$. In particular, $\mathcal{E}_1^{conv}$ has slope zero, so that the minimal slope of $\mathcal{E}^{conv}$ is zero. Since the determinant of $\mathcal{E}$ has finite order, this implies that $\mathcal{E}^{conv} = \mathcal{E}_1^{conv}$ and hence that $\mathcal{E}^{conv}$ admits a quotient $\mathcal{E}^{conv} \twoheadrightarrow \mathcal{O}_X$ in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$. As $(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(X) \to \mathbf{F}\text{-}\mathbf{Isoc}(X)$ is fully faithful, $\mathcal{E}$ admits a quotient $\mathcal{E} \twoheadrightarrow \mathcal{O}_X$ in $\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(X)$. Since $\mathcal{E}$ is irreducible, this implies $\mathcal{E} \simeq \mathcal{O}_X$.

[1] Recall that an algebraic number is p-plain if it is an ℓ-adic unit for every prime $\ell \neq p$ and that an F-isocrystal is said to be *p-plain* if the eigenvalues of the Frobenii at closed points are p-plain algebraic numbers.

[2] Note that the optimistic conjecture in [Ked17, Remark 5.14] turned out to be false in general

2.2.1.3 Perfect torsion points of abelian varieties

Let $\mathbb{F} \subseteq F$ be a finitely generated field extension and A a F-abelian variety without isotrivial geometric isogeny factors. Write $A^{(n)}$ for the Frobenius twist of A by the p^n-power absolute Frobenius. Since $A(F)_{\mathrm{tors}}$ is finite by the Lang-Néron Theorem ([LN59]) there is a tower of finite groups $A(F)_{\mathrm{tors}} \subseteq A^{(1)}(F)_{\mathrm{tors}} \subseteq A^{(2)}(F)_{\mathrm{tors}} \subseteq \dots$. In June 2011, in a correspondence with Langer and Rössler, Esnault asked whether this chain is eventually stationary. An equivalent way to formulate the question is to ask whether the group of F^{perf}-rational torsion points $A(F^{\mathrm{perf}})_{\mathrm{tors}}$ is a finite group, where F^{perf} is a perfect closure of F. Using Corollary 2.2.1.2.1, we can give a positive answer to her question.

Corollary 2.2.1.3.1. Let A be an abelian variety over F without isotrivial geometric isogeny factors. Then the group $A(F^{\mathrm{perf}})_{\mathrm{tors}}$ is finite.

To relate Corollaries 2.2.1.3.1 and 2.2.1.2.1 we use crystalline Dieudonné theory, as developed in [BBM82]. The proof of Theorem 2.2.1.3.1 is by contradiction. If $|A[p^\infty](F^{\mathrm{perf}})| = \infty$, there exists an injective map $\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow A[p^\infty]^{\text{ét}}$ from the trivial p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$ over F and the étale part of the p-divisible group of A. Spreading out to a "nice" model $\mathcal{A}/\mathcal{F}$ of A/F and applying the contravariant crystalline Dieudonné functor $\mathbb{D}$, one gets a surjection of F-isocrystals $\mathbb{D}(\mathcal{A}[p^\infty]^{\text{ét}}) \twoheadrightarrow \mathbb{D}((\mathbb{Q}_p/\mathbb{Z}_p)_{\mathcal{F}}) \simeq \mathcal{O}_{\mathcal{F}/K}$ over $\mathcal{F}$. By a descent argument and a careful use of Corollary 2.2.1.2.1, the quotient extends to a quotient $\mathbb{D}(\mathcal{A}[p^\infty]) \twoheadrightarrow \mathcal{O}_{\mathcal{F}}$ over $\mathcal{F}$. Going back to p-divisible groups, this gives an injective map $\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow A[p^\infty]$ over F. Therefore $A[p^\infty](F)$ would be an infinite group, contradicting Lang-Néron Theorem.

2.2.2 Specialization of p-adic monodromy groups over finitely generated fields

The main topic of Chapter 6 is the definition and the study of a category of (over)convergent F-isocrystals for varieties defined over infinite finitely generated fields.

2.2.2.1 (Over)convergent F-isocrystals over finitely generated fields

As already mentioned in Section 2.1.2.3, crystalline cohomology and in general the various category of p-adic local systems work well when k is perfect. On the other hand Fact 1.2.2.2.2 and Theorem 2.1.1.1.1 require that k is arithmetically rich enough, i.e. that k is finitely generated. So to obtain (variants of) Fact 1.2.2.2.2 and Theorem 2.1.1.1.1 in the p-adic setting, one would like to have good categories of (over)convergent F-isocrystals for varieties defined over infinite finitely generated fields. Let k be an infinite finitely generated field of characteristic $p > 0$ and X a smooth geometrically connected k-variety. To define and study a category of (over)convergent F-isocrystals we follow the main ideas in Sections 2.1.2.3 and 2.1.2.4, spreading out X over a finite field. We define (roughly) an (over)convergent F-isocrystal over X as an equivalence class $[\mathcal{E}]$ of couples $(\mathcal{X}, \mathcal{E})$, where $\mathcal{X}$ is an appropriate model of X over $\mathbb{F}_q$ and $\mathcal{E}$ is in $\mathbf{F\text{-}Isoc}^{(\dagger)}(\mathcal{X})$. Write $\widetilde{\mathbf{F\text{-}Isoc}}^\dagger(X)$ and $\widetilde{\mathbf{F\text{-}Isoc}}(X)$ for the categories of overconvergent F-isocrystals and convergent F-isocrystals over X respectively.

2.2.2.2 Monodromy groups of (Over)convergent F-isocrystals

To every couple $(\mathcal{X}, \mathcal{E})$ representing the equivalence class $[\mathcal{E}]$ of an (over)convergent F-isocrystal over X, we can associate an algebraic group $G(\mathcal{E})$ as in Section 1.2.4.6. Showing that $G(\mathcal{E})$ does not depend on the choice of a representative $(\mathcal{X}, \mathcal{E})$ of the equivalence class of $[\mathcal{E}]$, amounts to showing that every dense open immersion $\mathfrak{i} : \mathcal{U} \to \mathcal{X}$ of smooth $\mathbb{F}_q$-varieties induces an isomorphism $G(\mathfrak{i}^*\mathcal{E}) \simeq G(\mathcal{E})$. While this is true for overconvergent F-isocrystals, it does not hold in general for every convergent F-isocrystal; see Example 1.2.4.6.1. Indeed if the Newton polygon (see Section 1.2.4.3) of $\mathcal{E}$ is not constant on $\mathcal{X}$, there exists an open immersion $\mathfrak{i} : \mathcal{U} \to \mathcal{X}$ and a canonical filtration

$$\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq ... \subseteq \mathfrak{i}^*\mathcal{E} \text{ in } \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{U})$$

encoding the slopes of $\mathfrak{i}^*\mathcal{E}$. In general the sub-objects $\mathcal{E}_i$ are not in the essential image of $\mathfrak{i}^* : \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}) \to \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{U})$ and this is an obstruction to have $G(\mathfrak{i}^*\mathcal{E}) = G(\mathcal{E})$. However, we prove that this is the only obstruction, hence we get well defined monodromy groups for convergent F-isocrystals with constant Newton polygon.

2.2.2.3 Specialization of overconvergent F-isocrystals

After having settled the general formalism, one attaches to every overconvergent F-isocrystals (resp. convergent F-isocrystals with constant Newton polygon) $[\mathcal{E}]$ an exceptional locus $X^{ex}_{[\mathcal{E}]}$ and a strictly exceptional locus $X^{stex}_{[\mathcal{E}]}$. In the overconvergent setting our main result is an analogue of Fact 1.2.2.2.2 and Theorem 2.1.1.1.1.

Theorem 2.2.2.3.1. Let $[\mathcal{E}]$ be a geometrically semisimple overconvergent F-isocrystal over X (see Section 6.3.2 for the definitions). Then:

- The set $X^{ex}_{[\mathcal{E}]}$ is sparse. In particular there exists a $d \geq 1$ such that $X^{gen}_{[\mathcal{E}]}(\leq d)$ is infinite.
- If $[\mathcal{E}]$ is algebraic, then the set $X^{stex}_{[\mathcal{E}]}$ is sparse. In particular there exists a $d \geq 1$ such that $X^{sgen}_{[\mathcal{E}]}(\leq d)$ is infinite.
- If X is a curve, the set $X^{ex}_{[\mathcal{E}]}(\leq 1)$ is finite.

The proof of Fact 1.2.2.2.2 and Theorem 2.1.1.1.1 relies heavily on the fact that $\Pi_{\mathcal{F}_\ell}$ is an ℓ-adic Lie group, hence, implicitly, on the Galois-theoretic structure of $\mathbf{LS}(X, \ell)$. These features are not available in this p-adic setting. Instead, the idea is to use companions theory ([Laf02], [Dri12], [Abe18], [AE16]) for both overconvergent F-isocrystals and lisse sheaves, which associates to an overconvergent F-isocrystal $[\mathcal{E}]$ with representative $(\mathcal{X}, \mathcal{E})$ an ℓ-adic companion $[\mathcal{F}_\ell]$ with representative $(\mathcal{X}, \mathcal{E}_\ell)$ for some $\ell \neq p$. Then we show that the exceptional loci of $[\mathcal{E}]$ and $[\mathcal{F}_\ell]$ coincide, so that we can deduce Theorem 2.2.2.3.1 from Fact 1.2.2.2.2 and Theorem 2.1.1.1.1.

2.2.2.4 Specialization of convergent F-isocrystals

In the convergent setting, we get somehow weaker results. The fully faithful functor $(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}) \to \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ induces a fully faithful functor

$$(-)^{conv} : \widetilde{\mathbf{F}\text{-}\mathbf{Isoc}^\dagger}(X) \to \widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}(X).$$

Let $[\mathcal{E}]$ be in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X})$ represented by $(\mathcal{X}, \mathcal{E})$ and assume that $\mathcal{E}^{conv}$ has constant Newton polygon over $\mathcal{X}$. Then there the slope filtration of $\mathcal{E}$ over $\mathcal{X}$ induces a canonical filtration

$$[\mathcal{E}]^{conv}_1 \subseteq [\mathcal{E}]^{conv}_2 \subseteq ... \subseteq [\mathcal{E}]^{conv}$$

and morphisms of algebraic groups

$$G([\mathcal{E}]^{conv}_1) \twoheadleftarrow G([\mathcal{E}]^{conv}) \subseteq G([\mathcal{E}]).$$

For every algebraic group G write $\mathrm{rk}(G)$ for its reductive rank and recall that for any subgroup $H \subseteq G$ one has $\mathrm{rk}(G) \geq \mathrm{rk}(H)$.

Theorem 2.2.2.4.1. Let $[\mathcal{E}]$ be a pure and p-plain overconvergent F-isocrystal over X with constant Newton polygon (see Section 6.5.2 for the definitions).

- The set of $x \in |X|$ such that $\mathrm{rk}(G([\mathcal{E}]^{conv})) > \mathrm{rk}(G(x^*[\mathcal{E}]^{conv}))$ is sparse. In particular, if k is infinite there exists an integer $d \geq 1$ and infinitely many $x \in X(\leq d)$ such $\mathrm{rk}(G(x^*[\mathcal{E}]^{conv})) = \mathrm{rk}(G(x^*[\mathcal{E}]^{conv}))$.

- If X is a curve for all but at most finitely many k-rational points x one has $\mathrm{rk}(G([\mathcal{E}]^{conv})) = \mathrm{rk}(G(x^*[\mathcal{E}]^{conv}))$ and $\mathrm{rk}(G([\mathcal{E}]_1^{conv})) = \mathrm{rk}(G(x^*[\mathcal{E}]_1^{conv}))$.

- If X is a curve and $G([\mathcal{E}]_1^{conv,geo})^0$ is abelian, then $X^{ex}_{[\mathcal{E}]_1^{conv}}(\leq 1)$ is finite.

Remark 2.2.2.4.2. If $[\mathcal{E}]_1^{conv}$ has slope zero, the fact that $X^{stex}_{[\mathcal{E}]_1^{conv}}$ is sparse follows directly from Fact 1.2.2.2.2 and 1.2.4.3.1.

Via Theorem 2.2.2.3.1, Theorem 2.2.2.4.1 amounts to compare $X^{stex}_{[\mathcal{E}]_1^{conv}}$, $X^{stex}_{[\mathcal{E}]^{conv}}$ and $X^{stex}_{[\mathcal{E}]}$. To do this, one uses that for every $x \in |X|$ there is a canonical diagram of algebraic groups

$$\begin{tikzcd}
G(x^*[\mathcal{E}]) \arrow[r, hook] & G([\mathcal{E}]) \\
G(x^*[\mathcal{E}]^{conv}) \arrow[r, hook] \arrow[u, hook] \arrow[d, two heads] & G([\mathcal{E}]^{conv}) \arrow[u, hook] \arrow[d, two heads] \\
G(x^*[\mathcal{E}]_1^{conv}) \arrow[r, hook] & G([\mathcal{E}]_1^{conv}),
\end{tikzcd}$$

so that one can try and obtain information on $X^{gen}_{[\mathcal{E}]_1^{conv}}$ and $X^{gen}_{[\mathcal{E}]^{conv}}$ from $X^{gen}_{[\mathcal{E}]}$, via the results in Chapter 5.

2.3 Further results

Chapter 7 and 8 give complements the results of the previous Chapters.

2.3.1 Reduction to the Tate conjecture for divisors to finite fields

2.3.1.1 Statements

Corollary 2.1.3.3.1 and a spreading out argument show that $\mathrm{T}(Z, 1, \ell)$ for all smooth proper varieties over finitely generated fields of transcendence degree 1 over $\mathbb{F}_p$ implies $\mathrm{T}(Z, 1, \ell)$ for all smooth proper varieties over finitely generated fields of characteristic p. While Corollary 2.1.3.3.1 is false when k is finite, mimicking the arguments in the proof of Fact 1.2.3.2.1, we can further reduce $\mathrm{T}(Z, 1, \ell)$ to the case of finite field.

Theorem 2.3.1.1.1. Assume $p > 0$. Then $\mathrm{T}(Z, 1, \ell)$ for every finite field k of characteristic p and every smooth projective k-variety Z implies $\mathrm{T}(Z, 1, \ell)$ for every finitely generated field k of characteristic p and every smooth proper k-variety Z.

By an unpublished result ([dJ]) of De Jong (whose proof has been simplified in [Mor15, Theorem 4.3]), over finite fields the ℓ-adic Tate conjecture for divisors for smooth projective varieties follows from the ℓ-adic Tate conjecture for divisors for smooth projective surfaces and hence Theorem 2.3.1.1.1 implies the following:

Corollary 2.3.1.1.2. Assume $p > 0$. Then $\mathrm{T}(Z, 1, \ell)$ for every finite field k of characteristic p and every smooth projective k-surface Z implies $\mathrm{T}(Z, 1, \ell)$ for every finitely generated field k of characteristic p and every smooth proper k-variety Z.

2.3.1.2 Sketch of the proof

The idea is to try and transpose the Hodge theoretic arguments in the proof of Fact 1.2.3.2.1 to the ℓ-adic setting. We spread out Z to a smooth proper morphism $\mathcal{Z} \to \mathcal{K}$ of $\mathbb{F}_q$-varieties such that $\mathcal{Z}$ embeds as a dense open subset into a smooth proper $\mathbb{F}_q$-variety $\mathcal{Z}^{cmp}$. By smooth proper base change and the global invariant cycles theorem ([Del80]; see [And06, Theorem 1.1.1]), a class in $H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k)}$ arises from a class in $H^2(\mathcal{Z}^{cmp}_{\overline{\mathbb{F}}}, \mathbb{Q}_\ell(1))^{\pi_1(\mathbb{F}_q)}$, hence, by $\mathrm{T}(\mathcal{Z}^{cmp}, 1, \ell)$, from a divisor on $\mathcal{Z}^{cmp}$. Compared to [And96, Section 5.1], the extra difficulties come from the fact that resolution of singularities in positive characteristic and the semisimplicity of the Galois action in ℓ-adic cohomology are not known. The first issue can be overcome using De Jong's alteration theorem and the second adjusting an argument of Tate ([Tat94, Proposition 2.6.]). Applying De Jong's alteration theorem, we find a generically étale alteration $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ such that $\widetilde{\mathcal{Z}}$ embeds as a dense open subset into a smooth proper $\mathbb{F}_q$-variety. The problem is that the resulting morphism $\widetilde{\mathcal{Z}} \to \mathcal{Z} \to \mathcal{K}$ is not, in general, generically smooth, so that we cannot apply directly the global invariant cycles theorem. To solve this issue, we use the main ingredients in its proof: Hard Lefschetz theorem [Del80, Theorem 4.1.1] and the theory of weights for $\mathbb{F}_q$-schemes of finite type [Del80, Theorem 1].

2.3.2 Uniform boundedness of Brauer groups of forms in positive characteristic

Let k be a field of characteristic $p \geq 0$ and let X be a smooth proper k-variety .

2.3.2.1 Finiteness of Brauer groups

As already mentioned in Section 2.1.3.2, it is classically known that Conjecture $\mathrm{T}(X, 1, \ell)$ holds if and only if $\mathrm{Br}(X_{\overline{k}})[\ell^\infty]^{\pi_1(k)}$ is a finite group. The results in Chapters 3 and 4 (Corollary 2.1.3.2.1) give uniform boundedness results for $|Br(X_{\overline{k}})[\ell^\infty]^{\pi_1(k)}|$ in one dimensional families of varieties. However, recent results show that one can expect stronger finiteness statements. Write $\mathrm{Br}(X_{\overline{k}})[p']^{\pi_1(k)}$ for the prime-to-p torsion of $Br(X_{\overline{k}})^{\pi_1(k)}$.

Fact 2.3.2.1.1. Assume that k is finitely generated and X is a smooth, proper k-variety. Then:

1. ([OS18, Theorem 5.1]): If $p = 0$ and the integral Mumford Tate conjecture for X holds ([Ser77, Conjecture C.3]), then $\mathrm{Br}(X_{\overline{k}})^{\pi_1(k)}$ is finite.

2. ([CHT17, Corollary 1.2]): If $p > 0$ and $\mathrm{T}(X, 1, \ell)$ holds for every $\ell \neq p$, then $\mathrm{Br}(X_{\overline{k}})[p']^{\pi_1(k)}$ is finite.

The results in Chapter 3 are not sufficient to give uniform boundedness results for $|Br(X_{\overline{k}})[p']^{\pi_1(k)}|$. In Chapter 8, we give a few evidences that such boundedness results could hold.

2.3.2.2 Uniform boundedness in forms

Recall that for a field extension $k \subseteq k' \subseteq \overline{k}$, a $(\overline{k}/k')$-form of X is a k'-variety Y such that $Y_{\overline{k}} := Y \times_{k'} \overline{k} \simeq X_{\overline{k}}$. Let $k \subseteq k'$ be a finite field extension and Y a $(\overline{k}/k')$-form of X. If $p = 0$ and X satisfies the integral Mumford Tate conjecture (resp. if $p > 0$ and X satisfies the Tate conjecture for divisors for every $\ell \neq p$), then the same is true for Y, hence $\mathrm{Br}(Y_{\overline{k}})^{\pi_1(k)}$ (resp. $\mathrm{Br}(Y_{\overline{k}})[p']^{\pi_1(k')}$) is a finite group. But, for an integer $d \geq 1$, it is not clear whether one can find a uniform bound (depending only on X and d) for $|\,\mathrm{Br}(Y_{\overline{k}})^{\pi_1(k')}|$ (resp. $|\,\mathrm{Br}(Y_{\overline{k}})[p']^{\pi_1(k')}|$), while k' is varying on the finite field extensions $k \subseteq k'$ with $[k' : k] \leq d$ and Y among all the $(\overline{k}/k')$-forms of X. If $p = 0$, this is proven by Orr-Skorobogatov in [OS18, Theorem 5.1]. If $p > 0$, this is the main result of this chapter.

Theorem 2.3.2.2.1. Assume that k is finitely generated, X is a smooth proper k-variety and $p > 0$. If $\mathrm{T}(X, 1, \ell)$ holds for every $\ell \neq p$, then for every integer $d \geq 1$, there exists an integer $N \geq 1$, depending only on X and d, such that for every finite field extension $k \subseteq k'$ of degree $\leq d$ and every $(\overline{k}/k')$-form Y of X one has

$$|\operatorname{Br}(Y \times_{k'} \overline{k})[p']|^{\pi_1(k')} \leq N.$$

2.3.2.3 Strategy

The proof of Theorem 2.3.2.2.1 is a combination of Tannakian, ℓ-adic and ultrafilter techniques and it is a consequence of a general theorem on forms of compatible systems of representations over finitely generated fields of positive characteristic. First one reduces to prove Theorem 2.3.2.2.1 replacing $\operatorname{Br}(Y_{\overline{k}})[p']$ with

$$\underline{M}(\operatorname{Br}(Y_{\overline{k}})) := \prod_{\ell \neq p} T_\ell(\operatorname{Br}(Y_{\overline{k}})) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

Then the Kummer exact sequence and the theory of weights ([Del80]) imply that the collection $\{T_\ell(\operatorname{Br}(Y_{\overline{k}}))\}_{\ell \neq p}$ is a compatible system of $\pi_1(k)$-representations. The key point are the following to steps:

1. for every $(\overline{k}/k)$-forms Y of X, there exists a finite field extension $k \subseteq k_Y$ of degree depending only on the $Rank(T_\ell(Br(X_{\overline{k}}))$ and an isomorphism of $\pi_1(k_Y)$-modules $\underline{M}(\operatorname{Br}(Y_{\overline{k}})) \simeq \underline{M}(\operatorname{Br}(X_{\overline{k}}))$;

2. Up to replacing k with a finite extension, for every integer $d \geq 1$ there exists an integer $N \geq 1$, depending only on X and d, such that, for every field extension $k' \subseteq k''$ with $[k'' : k'] \leq d$, one has

$$\prod_{\ell \neq p} [M_\ell(\operatorname{Br}(X_{\overline{k}}))^{\pi_1(k'')} : M_\ell(\operatorname{Br}(X_{\overline{k}}))^{\pi_1(k')}] \leq N.$$

Statements (1) and (2) hold more generally for every compatible system of $\mathbb{Z}_\ell$-representations $\{T_\ell\}_{\ell \neq p}$ and, with Statements (1) and (2) in hands one can adjust the arguments in [OS18, Section 5] to obtain Theorem 2.3.2.2.1.

To prove (1), first we bound the number of connected components of the Zariski closure of the image of an ℓ-adic representation of a profinite group, only in terms of ℓ and of the rank of the representation. To get (1), one has to get rid of the dependency on ℓ. This follows formally from the compatibility assumption and the fact that the connectedness of the ℓ-adic monodromy group can be read off the L-function of the various compatible systems $\{T_\ell^{\otimes n} \otimes (T_\ell^\vee)^{\otimes m}\}_{\ell \neq p}$. For (2), the key point is to show that, if the Zariski closure of the image of $\pi_1(k)$ acting on V_ℓ is connected, then for each integer $d \geq 0$ there exists an integer D, depending only on d and $\{T_\ell\}_{\ell \neq p}$, such that, for every finite field extension $k \subseteq k'$ of degree $\leq d$, one has $(T_\ell/\ell)^{\pi_1(k)} = (T_\ell/\ell)^{\pi_1(k')}$ for every $\ell \geq D$. To prove this, one exploits again independence results, not in the ℓ-adic setting but in the ultrafilter setting, recently obtained by Cadoret-Hui-Tamagawa in [CHT17] and by Cadoret in [Cad19a].

Part II

Chapter 3

A uniform open image for ℓ-adic representations in positive characteristic

3.1 Introduction

3.1.1 Notation

In this Chapter k is a field of characteristic $p > 0$ with algebraic closure $k \subseteq \overline{k}$. For a k-variety X, write $|X|$ for the set of closed points and for every integer $d \geq 1$, $X(\leq d)$ for the set of all $x \in |X|$ with residue field $k(x)$ of degree $\leq d$ over k. If $d = 1$ we often write $X(\leq 1) = X(k)$. Let ℓ be a prime always $\neq p$.

3.1.2 Exceptional Locus

From now on, let X be a smooth geometrically connected k-variety. Let $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ be a continuous representation of the étale fundamental group[1] of X. By functoriality of the étale fundamental group, every $x \in |X|$ induces a continuous group homomorphism $\pi_1(x) \to \pi_1(X)$, hence a "local" Galois[2] representation $\rho_x : \pi_1(x) \to \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$. Set

$$\Pi = \rho(\pi_1(X)) \quad \Pi_{\overline{k}} = \rho(\pi_1(X_{\overline{k}})) \quad \Pi_x = \rho_x(\pi_1(x)).$$

Write X_ρ^{gen} for the set of all $x \in |X|$ such that $\Pi_x \subseteq \Pi$ is an open subgroup of Π and set

$$X_\rho^{ex} := |X| - X_\rho^{gen}; \quad X_\rho^{gen}(\leq d) := X_\rho^{gen} \cap X(\leq d); \quad X_\rho^{ex}(\leq d) := X_\rho^{ex} \cap X(\leq d).$$

We call X_ρ^{ex} the exceptional locus of ρ. The study of $X_\rho^{gen}(\leq d)$ is an important problem especially when the representation comes from a smooth proper morphism $f : Y \to X$ (see Subsection 3.1.5), so that Π_x controls fine arithmetic and geometric invariants of the family Y_x, $x \in |X|$. Since the Frattini subgroup of Π is open ([Ser89, Pag. 148]), a classical argument (see Fact 1.2.2.2.2) shows that if k is Hilbertian (in particular if k is finitely generated) there exists a $d \geq 1$ such that $X_\rho^{gen}(\leq d)$ is infinite.

3.1.3 Uniform open image theorem

When X is a curve and k is finitely generated, one can go further, under a mild assumption on $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$.

[1] As the choice of fibre functors will play no part in the following we will omit them for the notation for étale fundamental group.

[2] Recall that $\pi_1(x) \simeq \pi_1(Spec(k(x)))$ identifies with the absolute Galois group of $k(x)$.

Definition 3.1.3.1. A topological group Π is Lie perfect[3] (or *LP* for short) if every open subgroup of Π has finite abelianization. We say that $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ is Lie perfect (or *LP* for short) if Π is *LP* and that ρ is geometrically Lie perfect (or *GLP* for short) if $\Pi_{\overline{k}}$ is *LP*.

With this terminology, we can state our main result, which is an extension of [CT12b, Theorem 1.1] to positive characteristic.

Theorem 3.1.3.2. Assume that X is a curve and k is finitely generated. If ρ is *GLP*, then $X_\rho^{ex}(\leq 1)$ is finite and there exists an integer $N \geq 1$, depending only on ρ, such that $[\Pi : \Pi_x] \leq N$ for all $x \in X_\rho^{gen}(\leq 1)$.

When $X(k)$ is infinite, Theorem 3.1.3.2 gives us uniform boundedness results that are impossible to achieve using Fact 1.2.2.2.2; see for example Corollaries 3.1.5.1.1 and 3.1.5.2.1.

3.1.4 Strategy

While the general strategy of the proof of Theorem 3.1.4.2.2 is similar to the one of [CT12b, Theorem 1.1], the technical details are more complicated in positive characteristic. Indeed, the proof of Theorem 3.1.3.2 is based on the genus computations, via the Riemann-Hurwitz formula, of careful chosen abstract modular curves. In positive characteristic, the Riemann-Hurwitz formula involves wild inertia terms and - even assuming $\ell \neq p$ - controlling those wild inertia terms is rather delicate. To deal with them, we generalize the computations made in [CT12a].

3.1.4.1 Abstract modular scheme

For every open subgroup $U \subseteq \Pi$ write $f_U : X_U \to X$ for the connected étale cover corresponding to the open subgroup $\rho^{-1}(U) \subseteq \pi_1(X)$ and k_U for the smallest separable field extension of k over which X_U is geometrically connected. Write $U_{\overline{k}} = U \cap \Pi_{\overline{k}}$ and recall the following anabelian dictionary.

Fact 3.1.4.1.1. For every open subgroup $U \subseteq \Pi$ the following hold:

1. For every $x \in |X|$, we have that $\Pi_x \subseteq U$ if and only if x lifts to a $k(x)$-rational point on X_U;
2. The cover $X_{U_{\overline{k}}} \to X_{\overline{k}}$ corresponding to the open subgroup $U_{\overline{k}} \subseteq \Pi_{\overline{k}}$ is $X \times_{k_U} \overline{k} \to X_{\overline{k}}$.

In view of Fact 3.1.4.1.1, we call X_U the connected abstract modular scheme associated to U. Fact 3.1.4.1.1 enabled Cadoret-Tamagawa in [CT12b] to construct a projective system of abstract modular schemes (whose definition is recalled in Section 3.3.1.2):

$$f_n : \mathcal{X}_n := \coprod_{U \in \mathcal{C}_n(\Pi)} X_U \to X.$$

This system has the property that if $x \in |X|$ does not lift to a $k(x)$-rational point of $\mathcal{X}_n$ for some $n \geq 1$, then $\Pi_x \subseteq \Pi$ is not an open subgroup; see Lemma 3.3.1.2.1. The finiteness of $X_\rho(\leq d)$ can be then formulated in diophantine terms as follows:

$$(1)\text{: The image of } \varprojlim \mathcal{X}_n(\leq d) \to X(\leq d) \text{ is finite.}$$

To prove (1) it is enough to show

$$(2)\text{: The set } \mathcal{X}_n(\leq d) \text{ is finite for } n \gg 0.$$

[3] The terminology comes from the fact that if Π is an ℓ adic Lie group this condition is equivalent to $Lie(\Pi)^{ab} = 0$.

3.1.4.2 Growth of Genus

If X is a curve, k is finitely generated and $d = 1$, by [Sam66] and an argument of Voloch (see [EElsHKo09, Theorem 3] for more details), the finiteness of $\mathcal{X}_n(k)$ is controlled by the genus g_U of the smooth compactification of $X_{U_{\overline{k}}}$ for $U \in \mathcal{C}_n(\Pi)$.

Fact 3.1.4.2.1. If k is finitely generated of positive characteristic, there exists an integer $g \geq 2$, depending only on k, such that for every smooth proper k-curve Y with genus $\geq g$, the set $Y(k)$ is finite.

Fact 3.1.4.2.1 reduces (2) to the geometric Theorem 3.1.4.2.2 below, which extends [CT12b, Theorem 3.4] to positive characteristic. Write $\Pi_{\overline{k}}(n) := \mathrm{Ker}(\Pi_{\overline{k}} \to \mathrm{GL}_r(\mathbb{Z}_\ell/\ell^n))$.

Theorem 3.1.4.2.2. Assume that X is a curve, ρ is *GLP* and $\ell \neq p$. Then for every closed but not open subgroup $C \subseteq \Pi_{\overline{k}}$ we have

$$\lim_{n\to+\infty} g_{C\Pi_{\overline{k}}(n)} = +\infty.$$

To prove Theorem 3.1.4.2.2, one may assume $k = \overline{k}$, hence that $\Pi = \Pi_{\overline{k}}$. We first replace $X_{C\Pi(n)} \to X$ with a Galois cover $X_{\widetilde{\Pi}_C(n)} \to X$, closely related to the Galois closure of $X_{C\Pi(n)} \to X$, and we use the *GLP* hypothesis to show that the genus of $X_{\widetilde{\Pi}_C(n)}$ goes to infinity. Then we translate into group theoretical terms the Riemann-Hurwitz formula for $X_{\widetilde{\Pi}_C(n)} \to X_{C\Pi(n)}$ to show that the genus of $X_{\widetilde{\Pi}_C(n)}$ tends to infinity (if and) only if the genus of $X_{C\Pi(n)}$ does. Here, we use crucially that $\ell \neq p$ to control the wild inertia terms appearing in the Riemann-Hurwitz formula for $X_{\widetilde{\Pi}_C(n)} \to X_{C\Pi(n)}$. This part of the argument is significantly more difficult than in the proof of [CT12b, Theorem 3.4].

3.1.5 Applications to motivic representations

Let $f : Y \to X$ be a smooth proper morphism and let $\ell \neq p$ be a prime. For $x \in X$, choose a geometric point $\overline{x}$ over x and set Y_x (resp. $Y_{\overline{x}}$) for the fibre of f at x (resp. $\overline{x}$). By smooth proper base change $R^i f_* \mathbb{Z}_\ell(j)$ is a lisse sheaf hence, for every $x \in |X|$, gives rise to a continuous representation

$$\rho_\ell : \pi_1(X) \to \mathrm{GL}(H^i(Y_{\overline{x}}, \mathbb{Z}_\ell(j)))$$

such that $\rho_{\ell,x} : \pi_1(x) \to \mathrm{GL}(H^i(Y_{\overline{x}}, \mathbb{Z}_\ell(j)))$ identifies with the natural Galois action of $\pi_1(x)$ on $H^i(Y_{\overline{x}}, \mathbb{Z}_\ell(j))$. By [CT12b, Theorem 5.8], the representation ρ_ℓ is *GLP*, so that we can apply Theorem 3.1.3.2 to it.

3.1.5.1 Uniform boundedness ℓ-primary torsion of abelian schemes

Let $f : Y \to X$ be a g-dimensional abelian scheme. For $x \in X$ and any integer $n \geq 1$, write $Y_{\overline{x}}[\ell^n] := Y_{\overline{x}}[\ell^n](\overline{k(x)})$ for the ℓ^n-torsion of $Y_{\overline{x}}$ and set

$$Y_{\overline{x}}[\ell^\infty] := \bigcup_n Y_{\overline{x}}[\ell^n]; \quad T_\ell(Y_{\overline{x}}) := \varprojlim_n Y_{\overline{x}}[\ell^n].$$

Since k is finitely generated, $Y_x[\ell^\infty](k(x)) (= Y_{\overline{x}}[\ell^\infty]^{\pi_1(x)})$ is finite by the Mordell-Weil theorem. From the $\pi_1(x)$-equivariant isomorphisms

$$T_\ell(Y_{\overline{x}}) \simeq H^{2g-1}(Y_{\overline{x}}, \mathbb{Z}_\ell(g)); \quad T_\ell(Y_{\overline{x}}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \simeq Y_x[\ell^\infty](\overline{k(x)})$$

and Theorem 3.1.3.2, we obtain the following uniform bound for $Y_x[\ell^\infty](k)$, $x \in X(k)$.

Corollary 3.1.5.1.1. Assume that k is finitely generated, X is a curve and $f : Y \to X$ is an abelian scheme. There exists an integer $N \geq 1$, depending only on $f : Y \to X$ and ℓ, such that $|Y_x[\ell^\infty](k)| \leq N$ for every $x \in X(k)$.

Proof. Since $\Pi_\ell := \rho_\ell(\pi_1(X))$ is a compact ℓ-adic Lie group, it is topologically finitely generated hence it has only finitely many open subgroups of bounded index. So, by Theorem 3.1.3.2, the set of subgroups $\Pi_{\ell,x} \subseteq \Pi_\ell$ appearing as $\rho_{\ell,x}(\pi_1(x))$ for $x \in X(k)$ is finite. In particular, the set of abelian groups $\{Y_{\overline{x}}[\ell^\infty]^{\pi_1(x)} \simeq Y_x[\ell^\infty](k) \mid x \in X(k)\}$ is finite. □

3.1.5.2 Further applications

In Chapter 4, Theorem 3.1.3.2 is used to prove the following results. For $x \in |X|$, let $\mathrm{Br}(Y_{\overline{x}})^{\pi_1(x)}[\ell^\infty]$ denote the Galois invariants of the ℓ-primary torsion of the geometric Brauer group $\mathrm{Br}(Y_{\overline{x}}) := H^2(Y_{\overline{x}}, \mathbb{G}_m)$ of Y_x.

Corollary 3.1.5.2.1. Assume that k is finitely generated and that X is a curve with generic point η. Then

- Corollary 4.1.7.3.1: Assume that all the closed fibres of $f : Y \to X$ satisfy[4] the ℓ-adic Tate conjecture for divisors ([Tat65]). Then there exists an integer $N \geq 1$, depending only on $f : Y \to X$ and ℓ, such that $|\mathrm{Br}(Y_{\overline{x}})^{\pi_1(x)}[\ell^\infty]| \leq N$ for every $x \in X(k)$.

- Corollary 4.1.7.1.2: For all but at most finitely many $x \in X(k)$, the rank of the Néron-Severi group of $Y_{\overline{x}}$ is the same as the one of the Néron-Severi group of $Y_{\overline{\eta}}$

Corollaries 3.1.5.1.1 and 3.1.5.2.1 are extensions to positive characteristic of previous results obtained in [CT12b], [VAV17, Thm. 1.6, Cor. 1.7] and [CC18].

3.1.6 Organization of the chapter

In Section 3.2 we prove Theorem 3.1.4.2.2. In Section 3.3 we recall the construction of a projective system of abstract modular schemes $\mathcal{X}_n \to X$, parametrizing points with small image and some facts about them. After this, we prove Theorem 3.1.3.2. In Subsection 3.3.3, we discuss possible extensions of Theorem 3.1.3.2 to points of bounded degree. All the results and the proofs in this Chapter work in the characteristic zero setting but, since this situation is already treated in [CT12b], we will assume that $p > 0$ to simplify the exposition.

3.2 Proof of Theorem 3.1.4.2.2

3.2.1 Notation

3.2.1.1

For a group Γ and subgroups $I, H \subseteq \Gamma$ write

$$K_H(\Gamma) := \cap_{g\in\Gamma} gHg^{-1} \quad \text{and} \quad I_H := I/(I \cap K_H(\Gamma))$$

for the largest normal subgroup of Γ contained in H and the largest quotient of I that acts faithfully on Γ/H. For every closed subgroup $\Gamma \subseteq \mathrm{GL}_r(\mathbb{Z}_\ell)$, write $\Gamma(n) := \mathrm{Ker}(\Gamma \to \mathrm{GL}_r(\mathbb{Z}_\ell/\ell^n))$ and $\Gamma_n := \mathrm{Im}(\Gamma \to \mathrm{GL}_r(\mathbb{Z}_\ell/\ell^n))$. We use $\twoheadrightarrow$ and $\hookrightarrow$ to denote surjective and injective maps respectively.

[4] This holds, for example, if $f : Y \to X$ is a family of abelian varieties or of K3 surfaces.

3.2.1.2

From now on we retain the notation of Theorem 3.1.4.2.2. By Fact 3.1.4.1.1(2), we may assume $k = \overline{k}$, hence that $\Pi = \Pi_{\overline{k}}$ is LP. Set

$$\widetilde{\Pi}_C(n) := \mathrm{Ker}(\Pi_C \twoheadrightarrow (\Pi_n)_{C_n}) \quad \text{and} \quad \Pi_C(n) := \mathrm{Ker}(\Pi \twoheadrightarrow (\Pi_n)_{C_n}).$$

The following exact diagram summarizes the situation:

$$\begin{tikzcd}
 & & & 1 \arrow[d] & & & \\
 & 1 \arrow[dr] & & \widetilde{\Pi}_C(n) \arrow[dd] \arrow[ddll, bend right, two heads] \arrow[ddrr, bend left, two heads] & & 1 \arrow[dl] & \\
1 \arrow[dr] & & K_C(\Pi) \arrow[ur, hook] \arrow[dl] \arrow[dr] & & \Pi(n) \arrow[ul, hook'] \arrow[dl] \arrow[dr, two heads] & & 1 \arrow[dl] \\
 & K_{C_n}(\Pi_n) \arrow[dr] & & \Pi \arrow[dl] \arrow[dr] \arrow[dd] & & \Pi_C(n) \arrow[dl] & \\
 & & \Pi_n \arrow[dl] \arrow[dr] & & \Pi_C \arrow[dl] \arrow[dr] & & \\
 & 1 & & (\Pi_n)_{C_n} \arrow[dl] \arrow[d] \arrow[dr] & & 1 & \\
 & & 1 & 1 & 1 & &
\end{tikzcd}$$

After some preliminary reduction (Section 3.2.2), the proof of Theorem 3.1.3.2 decomposes as follows:

1. We first show that $g_{\widetilde{\Pi}_C(n)} \to +\infty$ using that Π is *LP* (Section 3.2.3)
2. Then, we use that $\ell \neq p$ to show that $g_{\widetilde{\Pi}_C(n)} \to +\infty$ implies $g_{C\Pi(n)} \to +\infty$ (Section 3.2.4).

3.2.2 Preliminary reductions

In this section we show that we can assume that for every integer $n \geq 1$:

1. $K_C(\Pi) = K_C(C\Pi(n))$;
2. $\widetilde{\Pi}_C(1)/\widetilde{\Pi}_C(n)$ is an ℓ-group.

Since we are interested in the asymptotic behaviour of $g_{C\Pi(n)}$ we can freely replace Π with $C\Pi(n_0)$ for some integer $n_0 \geq 1$. So:

1. Follows from the fact the increasing sequence $K_C(\Pi) \subseteq K_C(C\Pi(1)) \subseteq ... \subseteq K_C(C\Pi(n)) \subseteq ...$ of closed subgroups of Π stabilizes ([CT12b, Theorem 6.1]);
2. Follows if we prove that $\widetilde{\Pi}_C(n_0)/\widetilde{\Pi}_C(n)$ is an ℓ-group for some integer $n_0 \geq 1$ and any $n > n_0$. Write $A_n := \widetilde{\Pi}_C(n)/\Pi(n)$. Using the commutative exact diagram

$$\begin{tikzcd}
1 \arrow[r] & \Pi(n) \arrow[r] \arrow[d, hook] & \widetilde{\Pi}_C(n) \arrow[r] \arrow[d, hook] & A_n \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \Pi(1) \arrow[r] & \widetilde{\Pi}_C(1) \arrow[r] & A_1 \arrow[r] & 1
\end{tikzcd}$$

we find an exact sequence

$$1 \to B_{\ell,n} \to \widetilde{\Pi}_C(1)/\widetilde{\Pi}_C(n) \to A_1/A_n \to 1,$$

where $B_{\ell,n}$ is a quotient of $\Pi(1)/\Pi(n)$, hence an ℓ-group. Since A_1 is finite, for some $n_0 \gg 0$ and any $n \geq n_0$ the surjection $A_1/A_n \twoheadrightarrow A_1/A_{n-1}$ is an isomorphism. The (non abelian) snake lemma applied to the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & B_{\ell,n} & \longrightarrow & \widetilde{\Pi}_C(1)/\widetilde{\Pi}_C(n) & \longrightarrow & A_1/A_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow\!\!\downarrow & & \downarrow{\scriptstyle\simeq} & & \\
1 & \longrightarrow & B_{\ell,n-1} & \longrightarrow & \widetilde{\Pi}_C(1)/\widetilde{\Pi}_C(n-1) & \longrightarrow & A_1/A_{n-1} & \longrightarrow & 1,
\end{array}$$

shows that

$$\widetilde{\Pi}_C(n-1)/\widetilde{\Pi}_C(n) = \mathrm{Ker}(\widetilde{\Pi}_C(1)/\widetilde{\Pi}_C(n) \to \widetilde{\Pi}_C(1)/\widetilde{\Pi}_C(n-1)) \subseteq B_{\ell,n},$$

hence that $\widetilde{\Pi}_C(n-1)/\widetilde{\Pi}_C(n)$ is an ℓ-group. We conclude by induction on $n \geq n_0$ using the exact sequence

$$1 \to \widetilde{\Pi}_C(n-1)/\widetilde{\Pi}_C(n) \to \widetilde{\Pi}_C(n_0)/\widetilde{\Pi}_C(n) \to \widetilde{\Pi}_C(n_0)/\widetilde{\Pi}_C(n-1) \to 1.$$

So, from now on we may and do assume that (1) and (2) hold.

3.2.3 $\mathbf{g}_{\widetilde{\Pi}_C(n)} \to +\infty$

We use that Π is Lie perfect and $X_{\widetilde{\Pi}_C(n)} \to X$ is Galois. Since C is closed but not open in Π, $|(\Pi_n)_{C_n}| \to +\infty$ hence $g_{\widetilde{\Pi}_C(n)} \to \infty$ as soon as $\sup g_{\widetilde{\Pi}_C(n)} > 1$. Indeed, assume that $g_{\widetilde{\Pi}_C(n_0)} > 1$ for some $n_0 \geq 1$. Then, for every $n > n_0$, the Riemann Hurwitz formula for $X_{\widetilde{\Pi}_C(n)} \to X_{\widetilde{\Pi}_C(n_0)}$ yields

$$\lim_{n\to+\infty} 2g_{\widetilde{\Pi}_C(n)} - 2 \geq \lim_{n\to+\infty} \frac{|(\Pi_n)_{C_n}|}{|(\Pi_{n_0})_{C_{n_0}}|}(2g_{\widetilde{\Pi}_C(n_0)} - 2) = +\infty.$$

So it remains to show that $\sup g_{\widetilde{\Pi}_C(n)} = 1$ and $\sup g_{\widetilde{\Pi}_C(n)} = 0$ are not possible.

3.2.3.1 $\sup g_{\widetilde{\Pi}_C(n)} = 1$

Assume $\sup g_{\widetilde{\Pi}_C(n)} = 1$. Then there exists n_0 such that for all $n \geq n_0$ the smooth compactification of $X_{\widetilde{\Pi}_C(n)}$ is an elliptic curve. Since finite morphisms between elliptic curves are unramified, the Galois group $\widetilde{\Pi}(n_0)/\widetilde{\Pi}(n) \simeq \Pi_C(n_0)/\Pi_C(n)$ of $X_{\widetilde{\Pi}_C(n)} \to X_{\widetilde{\Pi}_C(n_0)}$ would be a quotient of the étale fundamental group of the smooth compactification of $X_{\widetilde{\Pi}_C(n_0)}$. In particular it would be abelian, hence

$$\Pi_C(n_0) = \varprojlim_n \Pi_C(n_0)/\Pi_C(n)$$

would be abelian and infinite. But this contradicts the fact that Π is Lie perfect, since $\Pi_C(n_0)$ would be an infinite abelian quotient of the open subgroup $\widetilde{\Pi}_C(n_0)$ of Π.

3.2.3.2 $\sup g_{\widetilde{\Pi}_C(n)} = 0$

Assume $\sup g_{\widetilde{\Pi}_C(n)} = 0$. This means that for all $n \geq 0$, the smooth compactification of $X_{\widetilde{\Pi}_C(n)}$ is isomorphic to $\mathbb{P}^1$. So the Galois group $\widetilde{\Pi}_C(1)/\widetilde{\Pi}_C(n) \simeq \Pi_C(1)/\Pi_C(n)$ of $X_{\widetilde{\Pi}_C(n)} \to X_{\widetilde{\Pi}_C(1)}$ is a subgroup of $\mathrm{PGL}_2(k)$. We use the following:

Fact 3.2.3.2.1 ([Cad12a, Corollary 10]). Suppose that k is an algebraically closed field of characteristic $p > 0$. A finite subgroup of $\mathrm{PGL}_2(k)$ is isomorphic to one of the following groups:

- A cyclic group;
- A dihedral group D_{2m} of order $2m$, for some $m > 0$;
- A_4, A_5, S_4;

- An extension $1 \to A \to \Pi \to Q \to 1$, with A an elementary abelian p-group and Q a cyclic group of prime-to-p order;
- $\mathrm{PSL}_2(\mathbb{F}_{p^r})$, for some $r > 0$;
- $\mathrm{PGL}_2(\mathbb{F}_{p^r})$, for some $r > 0$;

where $\mathbb{F}_{p^r}$ denotes the finite field with p^r elements.

From Fact 3.2.3.2.1 and the fact that $\Pi_C(1)/\Pi_C(n)$ is an ℓ-groups by Section 3.2.2(2), the only possibility is that $\Pi_C(1)/\Pi_C(n)$ is a cyclic group or $\ell = 2$ and $\Pi_C(1)/\Pi_C(n) \simeq D_{2^m}$. If the groups $\Pi_C(1)/\Pi_C(n)$ are abelian for $n \gg 0$ we can conclude as in 3.2.3.1. So assume $\ell = 2$ and $\Pi_C(1)/\Pi_C(n) \simeq D_{2^m}$. Since D_{2^m} fits into an exact sequence

$$0 \to \mathbb{Z}/2^{m-1} \to D_{2^m} \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 0,$$

the exactness of $\varprojlim$ on finite groups yields an infinite abelian open subgroup $\mathbb{Z}_2 \subseteq \varprojlim_n \Pi_C(1)/\Pi_C(n) \simeq \Pi_C(1)$, and we conclude as in 3.2.3.1.

3.2.4 $g_{C\Pi(n)} \to +\infty$

3.2.4.1 Definition of λ

If $f : Y \to X$ is a cover we define

$$\lambda_{Y/X} := \frac{2g_Y - 2}{deg(f)}.$$

The following directly follows from the Riemann-Hurwitz formula.

Lemma 3.2.4.1.1. Let $... \to X_{n+1} \to X_n \to ... \to ... \to X$ be a sequence of finite covers of smooth proper connected curves over an algebraically closed field k. Then $\lambda_{X_{n+1}/X} \geq \lambda_{X_n/X}$. Assume furthermore that $Deg(X_n \to X) \to +\infty$. Then $g_{X_n} \to +\infty$ if and only if $\lim\limits_{n\to+\infty} \lambda_{X_n/X} >$ 0

For an open subgroup $U \subseteq \Pi$ write $\lambda_U := \lambda_{X_U/X}$. With this notation, applying Lemma 3.2.4.1.1 to

$$\begin{array}{ccccccccc}
... & \longrightarrow & X_{\widetilde{\Pi}_C(n+1)} & \longrightarrow & X_{\widetilde{\Pi}_C(n)} & \longrightarrow ... \longrightarrow & X_{\widetilde{\Pi}_C(1)} & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \downarrow & & \| \\
... & \longrightarrow & X_{C\Pi(n+1)} & \longrightarrow & X_{C\Pi(n)} & \longrightarrow ... \longrightarrow & X_{C\Pi(1)} & \longrightarrow & X
\end{array}$$

one gets inequalities

$$\begin{array}{ccc}
\lambda_{C\Pi(n+1)} & \geq & \lambda_{C\Pi(n)} \\
\wedge\!\mid & & \wedge\!\mid \\
\lambda_{\widetilde{\Pi}_C(n+1)} & \geq & \lambda_{\widetilde{\Pi}_C(n)}
\end{array}$$

hence $\lambda_{\widetilde{\Pi}_C} := \lim\limits_{n\to+\infty} \widetilde{\Pi}_C(n)$ and $\lambda_C := \lim\limits_{n\to+\infty} \lambda_{C\Pi(n)}$ exist with $\lambda_{\widetilde{\Pi}_C} \geq \lambda_C$. Also, since $C \subseteq \Pi$ is closed but not open

1. $|(\Pi_n)_{C_n}| \to +\infty$, hence $g_{\widetilde{\Pi}_C(n)} \to +\infty$ if and only if $\lambda_{\widetilde{\Pi}_C} > 0$;
2. $|\Pi_n/C_n| \to +\infty$, hence $g_{C\Pi(n)} \to +\infty$ if and only if $\lambda_C > 0$.

By 3.2.3, $\lambda_{\widetilde{\Pi}_C} > 0$ hence it is enough to show that $\lambda_{\widetilde{\Pi}_C} = \lambda_C$. The remaining part of this section is devoted to the proof of this fact.

3.2.4.2 Inertia subgroups

Consider the commutative diagram:

$$\begin{tikzcd}
X_{\widetilde{\Pi}_C(n)} \arrow[rr, "e_Q^n, d_Q^n"] \arrow[rr, bend left=30, dash, "C_n/K_{C_n}(\Pi_n)"] \arrow[dr, "e_{i,n}, d_{i,n}"] \arrow[dr, bend right=60, dash, "(\Pi_n)_{C_n}"'] & & X_{C\Pi(n)} \arrow[dl, "e_{Q,n}, d_{Q,n}"'] \arrow[dl, bend left=60, dash, "\Pi_n/C_n"] \\
& X &
\end{tikzcd}$$

Suppose that $X^{cpt} - X = \{P_1, ..., P_r\}$ and denote with $I_i \subseteq \Pi$ the image via $\pi_1(X) \twoheadrightarrow \Pi$ of the inertia group of the point P_i. The situation is then the following.

- $X_{\widetilde{\Pi}_C(n)} \to X$ is a Galois cover with Galois group $(\Pi_n)_{C_n}$. The inertia group and the ramification index of any point of $X_{\widetilde{\Pi}_C(n)}$ over P_i are given[5] by $(I_{i,n})_{C_n} \subseteq (\Pi_n)_{C_n}$ and $e_{i,n} := |(I_{i,n})_{C_n}|$. Write $((I_{i,n})_{C_n})_j \subseteq (\Pi_n)_{C_n}$ for the j^{th}-ramification group in lower numbering (see [Ser68, Section 1, IV]) over the point P_i and $(e_{i,n})_j$ for its cardinality. Finally set $d_{i,n}$ for the exponent of the different of any point of $X_{\widetilde{\Pi}_C(n)}$ over P_i.

- $X_{C\Pi(n)} \to X$ is the cover corresponding to the open subgroup $C\Pi(n) \subseteq \Pi$. If $Q \in X_{C\Pi(n)}$ is over P_i we denote with $e_{Q,n}, d_{Q,n}$ the ramification index and the exponent of the different of Q over P_i.

- $X_{\widetilde{\Pi}_C(n)} \to X_{C\Pi(n)}$ is a Galois cover with Galois group $C_n/K_{C_n}(\Pi_n) \subseteq (\Pi_n)_{C_n}$ and there is a natural bijection of sets

$$\{Q \in X_{C\Pi(n)} \mid Q|P_i\} \simeq (I_{i,n})_{C_n} \backslash \Pi_n / C_n.$$

  If Q correspond to the element $(I_{i,n})_{C_n} x \in (I_{i,n})_{C_n} \backslash \Pi_n / C_n$, then the inertia group and the ramification index at Q are given by $Stab_{(I_{i,n})_{C_n}}((I_{i,n})_{C_n} x)$ and $|Stab_{(I_{i,n})_{C_n}}((I_{i,n})_{C_n} x)| := e_Q^n$. The j^{th}-ramification group is given by $((I_{i,n})_{C_n})_j \cap Stab_{(I_{i,n})_{C_n}}(x)$. Write $|((I_{i,n})_{C_n})_j \cap Stab_{(I_{i,n})_{C_n}}(x)| = (e_Q^n)_j$.

By [Ser68, Section 4, III, Pag. 51] we have the following relations:

$$e_{i,n} = e_Q^n e_{Q,n}; \quad d_{i,n} = d_Q^n + e_Q^n d_{Q,n}; \quad \sum_{Q|P_i} e_{Q,n} = |\Pi_n/C_n|.$$

3.2.4.3 Comparison

Using the Riemann-Hurwitz formula we get

$$\lambda_{C\Pi(n)} = 2g_X - 2 + \frac{1}{|\Pi_n/C_n|} \sum_{1 \le i \le r} \sum_{Q|P_i} d_{Q,n} \quad \text{and} \quad \lambda_{\widetilde{\Pi}_C(n)} = 2g_X - 2 + \sum_{1 \le i \le r} \frac{d_{i,n}}{e_{i,n}},$$

hence

$$\lambda_{\widetilde{\Pi}_C(n)} - \lambda_{C\Pi(n)} = \frac{1}{|\Pi_n/C_n|} \left(\sum_{1 \le i \le r} \frac{d_{i,n}|\Pi_n/C_n|}{e_{i,n}} - \sum_{Q|P_i} \frac{d_{Q,n} e_{i,n}}{e_{i,n}} \right)$$

[5] Since the cover is Galois the conjugacy class of the ramification group does not depend on the choice of the point over P_i

$$= \frac{1}{|\Pi_n/C_n|}\Bigg(\sum_{1\le i\le r}\frac{d_{i,n}\sum_{Q|P_i}e_{Q,n}}{e_{i,n}} - \sum_{Q|P_i}\frac{d_{Q,n}e_{i,n}}{e_{i,n}}\Bigg) = \frac{1}{|\Pi_n/C_n|}\Bigg(\sum_{1\le i\le r}\sum_{Q|P_i}\frac{d_{i,n}e_{Q,n} - d_{Q,n}e_{i,n}}{e_{i,n}}\Bigg)$$
$$= \frac{1}{|\Pi_n/C_n|}\Bigg(\sum_{1\le i\le r}\sum_{Q|P_i}\frac{d_{i,n} - d_{n,Q}e_Q^n}{e_Q^n}\Bigg) = \frac{1}{|\Pi_n/C_n|}\Bigg(\sum_{1\le i\le r}\sum_{Q|P_i}\frac{d_Q^n}{e_Q^n}\Bigg).$$

So it is enough to show that for every integer $1 \le i \le r$ one has

$$\lim_{n\to\infty}\frac{1}{|\Pi_n/C_n|}\sum_{Q|P_i}\frac{d_Q^n}{e_Q^n} = 0.$$

By [Ser68, Proposition 4, IV, Pag. 64] we have

$$\frac{1}{|\Pi_n/C_n|}\sum_{Q|P_i}\frac{d_Q^n}{e_Q^n} = \frac{1}{|\Pi_n/C_n|}\sum_{j\ge 0}\sum_{Q|P_i}\frac{(e_Q^n)_j - 1}{e_Q^n}.$$

3.2.4.4 Galois formalism

Consider the surjection

$$\phi_j : ((I_{i,n})_{C_n})_j\backslash\Pi_n/C_n \to (I_{i,n})_{C_n}\backslash\Pi_n/C_n$$

and recall the following elementary lemma.

Lemma 3.2.4.4.1 ([CT12a, Lemma 4.3])**.** Let G be a finite group and $H \subseteq G$ a normal subgroup. Let X be a finite set on which G acts and consider the natural surjection $q : H\backslash X \twoheadrightarrow G\backslash X$. If $Gx \in G\backslash X$ then

$$|q^{-1}(Gx)| = \frac{|G||Stab_G(Gx) \cap H|}{|H||Stab_G(Gx)|}.$$

If, under the bijection

$$(I_{i,n})_{C_n}\backslash\Pi_n/C_n \simeq \{Q \in X_{C\Pi(n)} \mid Q|P_i\},$$

the element $(I_{i,n})_{C_n}x \in (I_{i,n})_{C_n}\backslash\Pi_n/C_n$ corresponds to the point $Q \in X_{C\Pi(n)}$ above P_i, by Lemma 3.2.4.4.1 we have

$$|\phi_j^{-1}((I_{i,n})_{C_n}x)| = \frac{|(I_{i,n})_{C_n}|}{|((I_{i,n})_{C_n})_j|}\frac{|((I_{i,n})_{C_n})_j \cap Stab_{(I_{i,n})_{C_n}}((I_{i,n})_{C_n}x)|}{|Stab_{(I_{i,n})_{C_n}}((I_{i,n})_{C_n}x)|} = \frac{e_{i,n}}{(e_{i,n})_j}\frac{(e_Q^n)_j}{e_Q^n}.$$

Summing over all the $Q \in X_{C\Pi(n)}$ above P_i, we get

$$\sum_{Q|P_i}\frac{e_{i,n}}{(e_{i,n})_j}\frac{(e_Q^n)_j}{(e_Q^n)} = |((I_{i,n})_{C_n})_j\backslash\Pi_n/C_n|.$$

A similar reasoning gives

$$\sum_{Q|P_i}\frac{e_{i,n}}{e_Q^n} = |\Pi_n/C_n|,$$

hence

$$\frac{1}{|\Pi_n/C_n|}\sum_{j\ge 0}\sum_{Q|P_i}\frac{(e_Q^n)_j - 1}{e_Q^n} = \frac{1}{|\Pi_n/C_n|}\Bigg(\sum_{Q|P_i}\frac{e_Q^n - 1}{e_Q^n}\frac{e_{i,n}}{e_{i,n}}\Bigg) + \frac{1}{|\Pi_n/C_n|}\Bigg(\sum_{j\ge 1}\sum_{Q|P_i}\frac{(e_Q^n)_j - 1}{e_Q^n}\frac{(e_{i,n})_j e_{i,n}}{(e_{i,n})_j e_{i,n}}\Bigg).$$

The first term is

$$\frac{1}{|\Pi_n/C_n|}\sum_{Q|P_i}1 - \frac{1}{|\Pi_n/C_n|}\frac{1}{e_{i,n}}\sum_{Q|P_i}\frac{e_{i,n}}{e_Q^n} = \frac{|(I_{i,n})_{C_n}\backslash\Pi_n/C_n|}{|\Pi_n/C_n|} - \frac{1}{|(I_{i,n})_{C_n}|}.$$

Recall the following:

Fact 3.2.4.4.2 ([CT12b, Theorem 2.1]). Let $\Pi \subseteq \mathrm{GL}_r(\mathbb{Z}_\ell)$ be a closed subgroup and $C \subseteq \Pi$ a closed but not open subgroup. If $K_C(\Pi) = K_C(C\Pi(n))$ for every integer $n \geq 0$, then for every closed subgroup $I \subseteq \Pi$ one has

$$\lim_{n\to+\infty} \frac{|I_n\backslash \Pi_n/C_n|}{|\Pi_n/C_n|} = \frac{1}{|I_C|}.$$

Since $\varprojlim (I_{i,n})_{C_n} = (I_i)_C$, Fact 3.2.4.4.2 and Section 3.2.2(1) show that

$$\lim_{n\to+\infty} \frac{|(I_{i,n})_{C_n}\backslash \Pi_n/C_n|}{|\Pi_n/C_n|} - \frac{1}{|(I_{i,n})_{C_n}|} = 0.$$

The second term is

$$\frac{1}{|\Pi_n/C_n|}\left(\sum_{j\geq 1}\frac{(e_{i,n})_j}{e_{i,n}}\Big(\sum_{Q|P_i}\frac{(e_Q^n)_j}{e_Q^n}\frac{e_{i,n}}{(e_{i,n})_j}\Big)-\frac{1}{e_{i,n}}\sum_{Q|P_i}\frac{e_{i,n}}{e_Q^n}\right) = \sum_{j\geq 1}\frac{(e_{i,n})_j}{e_{i,n}}\frac{|((I_{i,n})_{C_n})_j\backslash \Pi_n/C_n|}{|\Pi_n/C_n|}-\frac{1}{|(I_{i,n})_{C_n}|}.$$

3.2.4.5 Stabilization of the wild inertia

Assume from now on that $j \geq 1$. We compute $(e_{i,n})_j$ using the diagram

$$\begin{tikzcd}
X_{\widetilde{\Pi}_C(n)} \arrow[rr] \arrow[rr, bend left, "\widetilde{\Pi}_C(1)/\widetilde{\Pi}_C(n)", no head] \arrow[dr] \arrow[dr, bend right, "(\Pi_n)_{C_n}"', no head] & & X_{\widetilde{\Pi}_C(1)} \arrow[dl] \arrow[dl, bend left, "(\Pi_1)_{C_1}", no head] \\
& X &
\end{tikzcd}$$

Write $((I_{i,n})_{C_n})_+$ for the wild inertia subgroup of $(I_{i,n})_{C_n}$ and

$$(I_{i,n})_{C_n}(1) := \mathrm{Ker}((I_{i,n})_{C_n} \to (I_{i,1})_{C_1}), \quad e_{i,n}(1) := |(I_{i,n})_{C_n}(1)|.$$

Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (I_{i,n})_{C_n}(1) \arrow[r] \arrow[d, hook] & (I_{i,n})_{C_n} \arrow[r] \arrow[d, hook] & (I_{i,1})_{C_1} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \widetilde{\Pi}_C(1)/\widetilde{\Pi}_C(n) \arrow[r] & (\Pi_n)_{C_n} \arrow[r] & (\Pi_1)_{C_1} \arrow[r] & 0.
\end{tikzcd}$$

Since $(I_{i,n})_{C_n}(1) \subseteq \widetilde{\Pi}_C(1)/\widetilde{\Pi}_C(n)$ are ℓ-groups by Section 3.2.2(1) and $((I_{i,n})_{C_n})_j \subseteq ((I_{i,n})_{C_n})_+$ are p-groups by definition, we see that

1. the map $(I_{i,n})_{C_n} \twoheadrightarrow (I_{i,1})_{C_1}$ induces an isomorphism
$$\phi_{i,n} : ((I_{i,n})_{C_n})_+ \simeq ((I_{i,1})_{C_1})_+;$$
2. $((I_{i,n})_{C_n})_j \cap \widetilde{\Pi}_C(1)/\widetilde{\Pi}_C(n) = 1$, so that Fact 3.2.4.5.1 below yields
$$\phi_{i,n}(((I_{i,n})_{C_n})_j) = ((I_{i,1})_{C_1})_{\lceil j/e_{i,n}(1)\rceil}.$$

Write $j_{i,0}$ for smallest integer ≥ 0 such that $(e_{i,1})_{j_{i,0}} = 0$ and

$$((\widetilde{I}_{i,n})_{C_n})_j := \phi_{i,n}^{-1}((I_{i,1})_j) \subseteq (I_{i,n})_{C_n}, \quad ((\widetilde{I}_i)_C)_j := \varprojlim_i ((\widetilde{I}_{i,n})_{C_n})_j \subseteq \Pi_C.$$

Combining (1) and (2), we get

$$((I_{i,n})_{C_n})_j = ((\widetilde{I}_{i,n})_{C_n})_{\lceil j/e_{i,n}(1)\rceil} =$$

$$\begin{cases} ((\widetilde{I}_{i,n})_{C_n})_k = ((\widetilde{I}_i)_C)_k & \text{if } \exists\ 1 \leq k \leq j_{i,0} \text{ such that } e_{i,n}(1)(k-1) < j \leq e_{i,n}(1)k \\ 1 & \text{if } j > e_{i,n}(1)j_{i,0}. \end{cases}.$$

Fact 3.2.4.5.1 ([Ser68, Lemma 5, IV, Pag. 75])**.** Let $K \subseteq L$ a finite Galois extension of local fields with group G. For $-1 \leq u \in \mathbb{R}$, write G_u for the $\lceil u \rceil^{th}$ ramification group in lower numbering and consider the function:

$$\psi_{L/K}(u) = \int_0^u \frac{1}{[G_0 : G_u]}\,\mathrm{dt}.$$

If $N \subseteq G$ if a normal subgroup corresponding to a Galois extension $K \subseteq K'$, then

$$G_u N/N = (G/N)_{\psi_{L/K'}(u)}$$

3.2.4.6 End of proof

We can continue the computation

$$\sum_{j\geq 1} \frac{|((I_{i,n})_{C_n})_j|}{|(I_{i,n})_{C_n}|}\frac{|((I_{i,n})_{C_n})_j\backslash \Pi_n/C_n|}{|\Pi_n/C_n|} - \frac{1}{|(I_{i,n})_{C_n}|} =$$

$$e_{i,n}(1)\sum_{1\leq k\leq j_{i,0}} \frac{|((\widetilde{I}_{i,n})_{C_n})_k||((\widetilde{I}_{i,n})_{C_n})_k\backslash \Pi_n/C_n|}{|(I_{i,n})_{C_n}||\Pi_n/C_n|} - \frac{1}{|(I_{i,n})_{C_n}|} =$$

$$\frac{e_{i,n}(1)}{|(I_{i,n})_{C_n}|}\sum_{1\leq k\leq j_{i,0}} |((\widetilde{I}_i)_C)_k|\Big(\frac{|((\widetilde{I}_{i,n})_{C_n})_k\backslash \Pi_n/C_n|}{|\Pi_n/C_n|} - \frac{1}{|((\widetilde{I}_i)_C)_k|}\Big) =$$

$$\frac{1}{|(I_{i,1})_{C_1}|}\sum_{1\leq k\leq j_{i,0}} |((\widetilde{I}_i)_C)_k|\Big(\frac{|((\widetilde{I}_{i,n})_{C_n})_k\backslash \Pi_n/C_n|}{|\Pi_n/C_n|} - \frac{1}{|((\widetilde{I}_i)_C)_k|}\Big).$$

Setting $(\widetilde{I}_i)_k$ for the preimage of $((\widetilde{I}_i)_C)_k$ under the map $\Pi \twoheadrightarrow \Pi_C$ and observing that $((\widetilde{I}_i)_k)_C = ((\widetilde{I}_i)_C)_k$, we conclude the proof since

$$\lim_{n\to+\infty} \frac{|((\widetilde{I}_{i,n})_{C_n})_k\backslash \Pi_n/C_n|}{|\Pi_n/C_n|} - \frac{1}{|((\widetilde{I}_i)_C)_k|} = \lim_{n\to+\infty} \frac{|((\widetilde{I}_{i,n})_{C_n})_k\backslash \Pi_n/C_n|}{|\Pi_n/C_n|} - \frac{1}{|((\widetilde{I}_i)_k)_C|} = 0$$

by Fact 3.2.4.4.2 and Section 3.2.2(1).

3.3 Proof of Theorem 3.1.3.2

3.3.1 Projective systems of abstract modular scheme

3.3.1.1 Group theory

Fix a closed subgroup Π of $\mathrm{GL}_r(\mathbb{Z}_\ell)$, write $\Phi(\Pi)$ for the Frattini subgroup of Π, i.e. the intersection of the maximal open subgroups of Π. Set $\mathcal{C}_0(\Pi) := \{\Pi\}$ and for every integer $n \geq 1$ define $\mathcal{C}_n(\Pi)$ as the set of open subgroups $U \subseteq \Pi$ such that $\Phi(\Pi(n-1)) \subseteq U$ but $\Pi(n-1) \not\subseteq U$. By [CT12b, Lemma 3.1], the maps $\psi_n : \mathcal{C}_{n+1}(\Pi) \to \mathcal{C}_n(\Pi)$ $\psi_n : U \mapsto U\Phi(\Pi(n-1))$ are well defined and they endow to the collection $\{\mathcal{C}_n(\Pi)\}_{n\in\mathbb{N}}$ with a structure of a projective system. For any $\underline{C} := (C[n])_{n\geq 0} \in \varprojlim \mathcal{C}_n(\Pi)$ write

$$C[\infty] := \varprojlim C[n] = \cap C[n] \subseteq \Pi.$$

By [CT12b, Lemma 3.3], one has the following.

Lemma 3.3.1.1.1.

1. $\mathcal{C}_n(\Pi)$ is finite and, for $n \gg 0$ (depending only on Π), it coincide with set of open subgroups $U \subseteq \Pi$ such that $\Pi(n) \subseteq U$ but $\Pi(n-1) \nsubseteq U$

2. For any $\underline{C} := (C[n])_{n\geq 0} \in \varprojlim_n \mathcal{C}_n(\Pi)$, the subgroup $C[\infty]$ is a closed but not open subgroup of Π.

3. For any closed subgroup $C \subseteq \Pi$ such that $\Pi(n-1) \nsubseteq C$ there exists $U \in \mathcal{C}_n(\Pi)$ such that $C \subseteq U$.

3.3.1.2 Anabelian dictionary

Let X be a smooth geometrically connected k-variety and assume now that Π is the image of a continuous representation $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$. Consider the following (possibly disconnected) étale covers:

$$f_n : \mathcal{X}_n := \coprod_{U\in\mathcal{C}_n(\Pi)} X_U \to X.$$

Proposition 3.3.1.2.1. Let n be an integer $\gg 0$ (depending only on Π). If $x \in X(k) - f_n(\mathcal{X}_n(k))$, then $\Pi(n-1) \subseteq \Pi_x$, hence $[\Pi : \Pi_x] \leq [\Pi : \Pi(n-1)]$.

Proof. This follows from Fact 3.1.4.1.1 and Lemma 3.3.1.1.1(3). □

Assume from now on that X is a curve. From Theorem 3.1.4.2.2 we deduce:

Corollary 3.3.1.2.2. Assume that ρ is GLP, $\ell \neq p$ and fix two integers $d_1, d_2 \geq 1$. Then there exists an integer $N \geq 1$, depending only on ρ, d_1, d_2, such that for every $n \geq N$ and every $U \in \mathcal{C}_n(\Pi)$ we have $[k_U : k] > d_1$ or $g_U > d_2$.

Proof. This follows from Theorem 3.1.4.2.2 arguing as in [CT12b, Corollaries 3.7 and 3.8]. □

3.3.2 Proof of Theorem 3.1.3.2 and a corollary

3.3.2.1 Proof of Theorem 3.1.3.2

Assume that X is a curve and ρ is GLP. Consider the projective system of covers constructed in 3.3.1.2

$$f_n : \mathcal{X}_n := \coprod_{U\in\mathcal{C}_n(\Pi)} X_U \to X.$$

By Corollary 3.3.1.2.2 we can choose an n_0 such that each connected component of $\mathcal{X}_{n_0}$ has genus lager then the constant g of Fact 3.1.4.2.1 or is defined over a non trivial extension of k. By the choice of n_0, the image X_{n_0} of $f_{n_0} : \mathcal{X}_{n_0}(k) \to X(k)$ has a finite number of points. Up to replace n_0 with some integer $n_0' \geq n_0$, by Lemma 3.3.1.2.1 for all $x \in X(k) - X_{n_0}$ we have $\Pi(n_0) \subseteq \Pi_x$. Hence $X^{ex}(k) \subseteq X_{n_0}$ is finite and one can take

$$N := \max_{x\in X_{n_0} - X_\rho^{ex}(k)} \Big\{[\Pi : \Pi(n_0)], [\Pi : \Pi_x]\Big\}.$$

This concludes the proof of Theorem 3.1.3.2.

3.3.2.2 Uniform boundedness of ℓ-primary torsion

For further use, we state a generalization of Corollary 3.1.5.1.1 for arbitrary GLP representations. We recall the notation and the terminology of [CT12b, Section 4]. Given a finitely generated free $\mathbb{Z}_\ell$ module $T \simeq \mathbb{Z}_\ell^r$ with a continuous action of $\pi_1(X)$ write $V := T \otimes \mathbb{Q}_\ell$ and $M := V/T$. For a character $\chi : \pi_1(X) \to \mathbb{Z}_\ell^*$, a field extension $k \subseteq L$ and a morphism $\xi : Spec(L) \to X$, let χ_ξ (resp. ρ_ξ) denote the composition of χ (resp. ρ) with the morphism $\pi_1(L) \to \pi_1(X)$. Consider the following $\pi_1(L)$-sets

$$\overline{M}_\xi := \{v \in M \mid \rho_\xi(\sigma)v \in < v >\}, \quad \overline{T}_\xi := \{v \in T \mid \rho_\xi(\sigma)v \in < v >\},$$

and $\pi_1(L)$-modules

$$M_\xi(\chi) := \{v \in M \mid \rho_\xi(\sigma)v = \chi_\xi(\sigma)v\}, \quad T_\xi(\chi) := \{v \in T \mid \rho_\xi(\sigma)v = \chi_\xi(\sigma)v\}.$$

Recall that χ is said to be non-sub-ρ if χ_x is not isomorphic to a sub representation of ρ_x for any $x \in X(k)$. Finally denote with $T_{(0)}$ the maximal isotrivial submodule of T, i.e. the maximal submodule of T on which $\pi_1(X_{\overline{k}})$ acts via a finite quotient.

Corollary 3.3.2.2.1. Assume that k is finitely generated, X is a curve, $\ell \neq p$ and that $\rho : \pi_1(X) \to \mathrm{GL}(T)$ is *GLP*. Then

1. For every non-sub-ρ character $\chi : \pi_1(X) \to \mathbb{Z}_\ell^*$, there exists an integer $N \geq 1$, depending only on ρ and χ, such that, for any $x \in X(k)$ the $\pi_1(x)$-module $M_x(\chi)$ is contained in $M[\ell^N]$.

2. Assume furthermore that $T_{(0)} = 0$. Then there exists an integer $N \geq 1$, depending only on ρ, such that for every $x \in X(k) - X_\rho^{ex}(k)$, the $\pi_1(k)$-set $\overline{M}_x$ is contained in $M[\ell^N]$.

Proof. This follows from Theorem 3.1.3.2 as in the proof of [CT12b, Corollary 4.3]. □

3.3.3 Further remarks

Let k be a finitely generated field of characteristic $p \geq 0$, let X be a smooth geometrically connected k-curve. Let $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ be a continuous representation and retain the notation of Section 3.1.2.

3.3.3.1 Points of bounded degree

As already mentioned in Section 3.1.3, Theorem 3.1.3.2 is the natural extension to positive characteristic of the main result of [CT12b]. In the subsequent paper [CT13], Cadoret-Tamagawa show ([CT13, Theorem 1.1]) that if $p = 0$ and ρ is GLP, then for every $d \geq 1$, the set $X_\rho^{ex}(\leq d)$ is finite and there exists an integer $N(\rho, d) := N \geq 1$, depending only on ρ and d, such that $[\Pi : \Pi_x] \leq N$ for all $x \in X_\rho^{gen}(\leq d)$. To prove this they study the gonality of the connected components of the abstract modular curves $\mathcal{X}_n$.

3.3.3.2 Gonality

For a smooth proper k-curve Y, the (geometric) gonality γ_Y of Y is the minimum degree of a non constant map $Y_{\overline{k}} \to \mathbb{P}^1_{\overline{k}}$. While the genus g_Y controls the finiteness of k-rational points, the gonality, in characteristic zero, controls the finiteness of points of bounded degree.

Fact 3.3.3.2.1 ([Fal91], [Fre94])**.** If k is a finitely generated field of characteristic zeros and $d \geq 1$ is an integer, for every smooth proper k-curve Y such that $\gamma_Y \geq 2d + 1$, the set $Y(\leq d)$ is finite.

In light of Fact 3.3.3.2.1 and of the strategy described in Section 3.1.4.1, to prove [CT13, Theorem 1.1] when $p = 0$, Cadoret-Tamagawa show ([CT13, Theorem 3.3]) that, for every $C \subseteq \Pi$ closed but not open subgroup, the gonality $\gamma_{C\Pi_{\overline{k}}(n)}$ of the smooth compactification of $X_{C\Pi_{\overline{k}}(n)}$ tends to infinity with n. While one can adapt (see Appendix A) the arguments of [CT13, Theorem 3.3] to prove that $\gamma_{C\Pi_{\overline{k}}(n)}$ tends to infinity also when $p > 0$, the positive characteristic variant[6] of Fact 3.3.3.2.1 is not true, so that one cannot deduces directly from the growth of the gonality the positive characteristic analogue of [CT13, Theorem 1.1].

3.3.3.3 Isogonality

However, in [CT15b, Appendix] Cadoret-Tamagawa have introduced a new invariant, the isogonality, that could be used to study points of bounded degree is positive characteristic.

Definition 3.3.3.3.1. Let k a field of characteristic $p > 0$ and Y a smooth proper geometrically connected k-curve. The $\overline{k}$-isogonality γ_Y^{iso} of Y is defined as $d+1$, where d is the smallest integer which satisfies the following condition:

- There is no diagram $Y_{\overline{k}} \leftarrow Y' \rightarrow B$ of non constant morphisms of smooth proper curves over $\overline{k}$, with B an isotrivial[7] curve and $deg(Y' \rightarrow B) \leq d$.

Their result is the following:

Fact 3.3.3.3.2 ([CT15b, Corollary A.7])**.** If k is a finitely generated field of positive characteristic and $d \geq 1$ is an integer, and $d \geq 1$ is an integer, then for every smooth proper k-curve Y such that $\gamma_Y \geq 2d + 1$ and $\gamma_Y^{iso} \geq d + 1$, the set $Y(\leq d)$ is finite.

Since, by the results in Appendix A, we know that $\gamma_{C\Pi_{\overline{k}}(n)}$ tends to infinity, to extend Theorem 3.1.3.2 to points of bounded degree it would be enough to show the following.

Conjecture 3.3.3.3.3. Assume that ρ is a GLP, $p > 0$ and $\ell \neq p$. Then for every closed but not open subgroup $C \subseteq \Pi_{\overline{k}}$ one has

$$\lim_{n \to +\infty} \gamma^{iso}_{X_{C\Pi_{\overline{k}}(n)}} = +\infty.$$

[6] This is due to isotriviality issues in the positive characteristic variant of the Mordell-Lang conjecture; see [CT15b, Appendix].

[7] If k is a field of characteristic $p > 0$, a k-scheme S is said to be isotrivial, if there exists a finite field $\mathbb{F}_q \subseteq k$ and a $\mathbb{F}_q$-scheme S_0 such that $S_0 \times_{\mathbb{F}_q} \overline{k} \simeq S_{\overline{k}}$.

Chapter 4

Specialization of Néron-Severi groups in positive characteristic

4.1 Introduction

4.1.1 Conventions

For a field k and a k-variety X, write $|X|$ for the set of closed points. If $x \in X$, write $k(x)$ for its residue field and $\overline{x}$ for a geometric point over x. If $Y \to X$ is a morphism and $x \in X$ write $i_x : Y_x \to Y$ for the natural inclusion of the fibre Y_x at x in Y. We use $\twoheadrightarrow$ and $\hookrightarrow$ to denote surjective and injective maps respectively. If $\mathbb{F}_q$ is a finite field, write $\mathbb{F}$ for its algebraic closure. If $\mathcal{C}$ is an abelian category write $\mathcal{C} \otimes \mathbb{Q}$ for its isogeny category and $\otimes\mathbb{Q} : \mathcal{C} \to \mathcal{C} \otimes \mathbb{Q}$ for the canonical functor.

4.1.2 Summary

Let k be a finitely generated field of characteristic $p > 0$, $\ell \neq p$ a prime and $f : Y \to X$ a smooth proper morphism. In first approximation, the main result of this chapter is a version of the variational Tate conjecture for divisors in the generic case: for $x \in |X|$, if $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$ has no more Galois invariants than the generic fibre, then $Y_{\overline{x}}$ has no more divisors than the generic fibre. When k is a field of characteristic zero, this has been proved by André as a consequence of Lefschetz (1,1)-theorem and the Hodge theory in [Del71]; see Section 4.1.5 for more details.

The starting point of our proof is to replace Hodge theory with crystalline cohomology, since a variational form of the Tate conjecture (Fact 4.1.6.1.1) is known in this setting. The main difficulty to overcome is to transfer the information about the Galois invariants of the ℓ-adic lisse sheaf $R^2f_*\mathbb{Q}_\ell(1)$ to the crystalline local system (F-isocrystal) $R^2f_{crys,*}\mathcal{O}_{Y/K}(1)$. This is the main new contribution of this chapter (Theorem 4.1.6.3.1). More precisely, since the F-isocrystal $R^2f_{crys,*}\mathcal{O}_{Y/K}(1)$ has a behaviour which is quite different from $R^2f_*\mathbb{Q}_\ell(1)$ (for example, in general its cohomology is not finite dimensional), this comparison cannot be done directly. The idea is then to show (Theorem 4.6.5.4.1) that $R^2f_{crys,*}\mathcal{O}_{Y/K}(1)$ is coming from a smaller and better behaved category of p-adic local systems: the category of overconvergent F-isocrystals. As it has been understood that overconvergent F-isocrystals share many properties with lisse sheaves ([Cre92], [Ked06a], [AC13b]), the idea is to compare first $R^2f_{crys,*}\mathcal{O}_{Y/K}(1)$ with its overconvergent incarnation $R^2f_*\mathcal{O}^\dagger_{Y/K}(1)$ via various p-adic comparison theorems and then $R^2f_*\mathcal{O}^\dagger_{Y/K}(1)$ with $R^2f_*\mathbb{Q}_\ell(1)$ via the theory of weights ([Del80], [KM74]).

However, the theory of weights allows us to transfer only information readable on characteristic polynomials of the Frobenii, that is to compare $R^2f_*\mathcal{O}^\dagger_{Y/K}(1)$ and $R^2f_*\mathbb{Q}_\ell(1)$ only up to semi-simplification. The way to grasp the missing information is Tannakian: instead of considering only $R^2 f_* \mathcal{O}^\dagger_{Y/K}(1)$ and $R^2 f_* \mathbb{Q}_\ell(1)$, we consider all the possible tensor constructions and sub quotients arising from them, obtaining, via the Tannakian formalism, two algebraic groups G_p and G_ℓ. Since G_ℓ identifies with the Zariski closure of the image of $\pi_1(X, \overline{x})$ acting on $(R^2 f_* \mathbb{Q}_\ell(1))_{\overline{x}} \simeq H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$, instead of asking that $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$ has no more Galois invariants than the generic fibre, we ask that the Zariski closure $G_{\ell,x}$ of the image of $\pi_1(x, \overline{x})$ acting on $H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$ identifies with G_ℓ. Then, the theory of weights, combined now with some algebraic groups theory, allows us to relate the inclusion of the local p-adic monodromy group $G_{p,x} \subseteq G_p$ at x with the inclusion $G_{\ell,x} \subseteq G_\ell$.

Behind this is the idea that, while $R^2 f_* \mathcal{O}^\dagger_{Y/K}(1)$ and $R^2 f_* \mathbb{Q}_\ell(1)$ should be different incarnations of the same motives, each of them contains some specific feature: $R^2 f_* \mathbb{Q}_\ell(1)$ can be studied via ℓ-adic Lie groups theory, while $R^2 f_* \mathcal{O}^\dagger_{Y/K}(1)$ is an overconvergent incarnation of $R^2 f_{crys,*} \mathcal{O}_{Y/K}(1)$, which, in turn, contains information on the deformations of cycles.

4.1.3 Galois generic points

Let k be a field of characteristic $p > 0$ with algebraic closure $\overline{k}$, X a smooth and geometrically connected k-variety with generic point η and $f : Y \to X$ a smooth proper morphism of k-varieties. For $x \in X$, fix an étale path from $\overline{x}$ to $\overline{\eta}$. For every $\ell \neq p$, by smooth proper base change $R^2 f_* \mathbb{Q}_\ell(1)$ is a lisse sheaf on X and the choice of the étale path gives equivariant isomorphisms

$$\begin{array}{ccccccc} H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)) & \simeq & R^2 f_* \mathbb{Q}_\ell(1)_{\overline{\eta}} & \simeq & R^2 f_* \mathbb{Q}_\ell(1)_{\overline{x}} & \simeq & H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)) \\ \circlearrowright & & & & & & \circlearrowright \\ \pi_1(X, \overline{\eta}) & \simeq & \pi_1(X, \overline{x}) & \longleftarrow & & & \pi_1(x, \overline{x}). \end{array}$$

Definition 4.1.3.1. A point $x \in X$ is ℓ-Galois generic (resp. strictly ℓ-Galois generic) for $f : Y \to X$ if the image of $\pi_1(x, \overline{x}) \to \pi_1(X, \overline{\eta}) \to \mathrm{GL}(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)))$ is open (resp. coincides with) in the image of $\pi_1(X, \overline{\eta}) \to \mathrm{GL}(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)))$.

By [Cad17, Theorem 1.1], x is ℓ-Galois generic for one $\ell \neq p$ if and only if x is ℓ-Galois generic for every $\ell \neq p$. So one simply says that x is Galois generic for f. This is not true for strictly Galois generic points, and one says that x is strictly Galois generic if there exists an $\ell \neq p$ such that x is strictly ℓ-Galois generic.

4.1.4 Neron-severi generic points

4.1.4.1 Tate conjecture for divisors

The geometric Néron-Severi group $\mathrm{NS}(Z_{\overline{k}})$ of a smooth proper k-variety Z is a finitely generated abelian group such that $\mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q}$ identifies with the image of the cycle class map for ℓ-adic cohomology

$$c_{Z_{\overline{k}}} : \mathrm{Pic}(Z_{\overline{k}}) \otimes \mathbb{Q} \to H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1)).$$

Since $\mathrm{NS}(Z_{\overline{k}})$ is a finitely generated abelian group, $\pi_1(k)$ acts on it trough a finite quotient and hence $\mathrm{NS}(Z_{\overline{k}}) \subseteq H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))$ is fixed under the action of the connected component G^0_ℓ of the Zariski closure of the image G_ℓ of $\pi_1(k)$ acting on $H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))$. Recall that the ℓ-adic Tate conjecture for divisors ([Tat65]) predicts the following:

Conjecture 4.1.4.1.1 ($T(Z, \ell)$)**.** Let k be a finitely generated field and Z a smooth proper k-variety. Then the map $c_{Z_{\overline{k}}} : \mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \to H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{G^0_\ell}$ is an isomorphism

4.1.4.2 Specialization morphisms

Retain the notation and the assumptions of Section 4.1.3. For every $x \in X$, there is an injective specialization homomorphism (see e.g. [MP12, Proposition 3.6.])

$$sp_{\eta,x} : \mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \to \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q}$$

compatible with the cycle class map, in the sense that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Pic}(Y_\eta) \otimes \mathbb{Q} \arrow[d, "c_{Y_\eta}"] & \mathrm{Pic}(Y) \otimes \mathbb{Q} \arrow[l, "i_\eta^*"'] \arrow[r, "i_x^*"] & \mathrm{Pic}(Y_x) \otimes \mathbb{Q} \arrow[d, "c_{Y_x}"] \\
\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[d, hook] \arrow[rr, hook, "sp_{\eta,x}"] & & \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q} \arrow[d, hook] \\
H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)) & \simeq & H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))
\end{tikzcd}$$

Since the Néron -Severi group is invariant under extensions of algebraically closed fields (see e.g. [MP12, Proposition 3.1]), the map $sp_{\eta,x}$ is well defined, independently of the choice of the geometric points $\overline{\eta}$ over η and $\overline{x}$ over x.

The abelian group $\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q}$ is a $\pi_1(X, \overline{\eta})$-module and hence the group $\pi_1(x, \overline{x})$ acts on $\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q}$ by restriction trough the morphism $\pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta})$. Since the map $sp_{\eta,x}$ is $\pi_1(x, \overline{x})$-equivariant with respect to the natural action of $\pi_1(x, \overline{x})$ on $\mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q}$, one constructs an injective specialization map

$$sp^{ar}_{\eta,x} : \mathrm{NS}(Y_\eta) \otimes \mathbb{Q} \subseteq (\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q})^{\pi_1(x,\overline{x})} \xrightarrow{sp_{\eta,x}} \mathrm{NS}(Y_x) \otimes \mathbb{Q},$$

where for a smooth proper k-variety Z one writes $\mathrm{NS}(Z) \otimes \mathbb{Q} := (\mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q})^{\pi_1(k)}$.

Definition 4.1.4.2.1. One says that x is NS-generic (resp. arithmetically NS-generic) for $f : Y \to X$ if $sp_{\eta,x}$ (resp. $sp^{ar}_{\eta,x}$) is an isomorphism.

Conjecture 4.1.4.1.1 predicts that every (strictly) Galois generic point is (arithmetically) NS-generic. Our main result is that this holds (without assuming Conjecture 4.1.4.1.1), at least when $f : Y \to X$ is projective.

Theorem 4.1.4.2.2. Let k be a finitely generated field and $f : Y \to X$ a smooth projective morphism. If $x \in X$ is Galois-generic (resp. strictly Galois generic) for $f : Y \to X$ then it is NS-generic (resp. arithmetically NS-generic) for $f : Y \to X$. If $f : Y \to X$ is smooth and proper, the same is true for all x in a dense open subset of X.

4.1.5 Proof in characteristic zero

When k is a field of characteristic zero Theorem 4.1.4.2.2 is due to André ([And96]; see also [Cad12b, Corollary 5.4] and [CC18, Proposition 3.2.1]) and it holds for $f : Y \to X$ proper. Since it is the starting point for our proof we briefly recall the argument when $k \subseteq \mathbb{C}$ and x is a closed point. Fix a smooth compactification $Y \subseteq \overline{Y}$ of Y. The commutative diagram of k-varieties

$$\begin{tikzcd}
Y_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[r, hook] \arrow[d] & \overline{Y} \\
k(x) \arrow[r, "x"] & X &
\end{tikzcd}$$

induces a commutative diagram:

$$\begin{tikzcd}
H^0(X_{\mathbb{C}}, R^2 f_* \mathbb{Q}(1)) \arrow[d, "\simeq"] & H^2(Y_{\mathbb{C}}, \mathbb{Q}(1)) \arrow[l, "\mathrm{Ler}"'] & \mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[dd, bend left=60, "sp_{\eta,x}"] \\
H^2(Y_{\overline{x}}, \mathbb{Q}(1))^{\pi_1^{top}(X_{\mathbb{C}})} \arrow[d, hook] & H^2(\overline{Y}_{\mathbb{C}}, \mathbb{Q}(1)) \arrow[l] \arrow[u] & \mathrm{NS}(\overline{Y}_{\mathbb{C}}) \otimes \mathbb{Q} \arrow[l, hook'] \arrow[u] \arrow[d] \\
H^2(Y_{\overline{x}}, \mathbb{Q}(1)) & & \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q} \arrow[ll, hook', "c_{Y_{\overline{x}}}"]
\end{tikzcd}$$

where Ler is the edge map in the Leray spectral sequence attached to $f : Y \to X$. Take any $z_x \in \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q}$. Since z is fixed by an open subgroup of $\pi_1(x)$ and x is Galois-generic, up to replacing X with a finite étale cover one can assume that z_x is fixed by $\pi_1(X_{\mathbb{C}})$. By the comparison between the étale and the singular sites, z_x is fixed by $\pi_1^{top}(X_{\mathbb{C}})$. By Deligne's fixed part theorem ([Del71, Theoreme 4.1.1]) the map

$$H^2(\overline{Y}_{\mathbb{C}}, \mathbb{Q}(1)) \to H^2(Y_{\overline{x}}, \mathbb{Q}(1))^{\pi_1^{top}(X_{\mathbb{C}})}$$

is surjective. By semisemplicity, the map $H^2(\overline{Y}_{\mathbb{C}}, \mathbb{Q}(1)) \to H^2(Y_{\overline{x}}, \mathbb{Q}(1))^{\pi_1^{top}(X_{\mathbb{C}})}$ splits in the category of polarized $\mathbb{Q}$-Hodge structures, so that z_x is the image of a $z \in H^{0,0}(\overline{Y}_{\mathbb{C}}, \mathbb{Q}(1))$. By the Lefschetz (1,1) theorem, z lies in $\mathrm{NS}(\overline{Y}_{\mathbb{C}}) \otimes \mathbb{Q}$. One concludes the proof observing that, by construction, the restriction z_η of z to $\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q}$ is such that $sp_{\eta,x}(z_\eta) = z_x$.

4.1.6 Strategy in positive characteristic

In characteristic zero the main ingredients are the combination of Deligne's fixed part theorem and the Lefschetz-(1,1) theorem (what is called the variational Hodge conjecture for divisors; see e.g. [MP12, Conjecture 9.6, Remark 9.7]) and the comparison between the étale and the singular sites. To try and make the argument of Section 4.1.5 works in positive characteristic the idea is to replace Betti cohomology with crystalline cohomology. The main reason for this is that the variational Tate conjecture for projective morphisms (Fact 4.1.6.1.1), that we now recall, is known in this setting.

4.1.6.1 Crystalline variational Tate conjecture

Let $\mathbb{F}_q$ be the finite field with $q = p^s$ elements, $\mathcal{X}$ a connected smooth $\mathbb{F}_q$-variety and $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ a smooth proper morphism of $\mathbb{F}_q$-varieties (in our application $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ is a model for $f : Y \to X$). Write respectively $Mod(\mathcal{X}|W)$, $Mod(\mathcal{Y}|W)$ for the categories of $\mathcal{O}_{\mathcal{X}|W}$, $\mathcal{O}_{\mathcal{Y}|W}$ modules in the crystalline site of $\mathcal{X}$, $\mathcal{Y}$ over $W := W(\mathbb{F}_q)$ ([Mor13, Section 2]). Then there is a higher direct image functor

$$R^i \mathfrak{f}_{crys,*} : Mod(\mathcal{Y}|W) \to Mod(\mathcal{X}|W)$$

and, for every $\mathfrak{t} \in \mathcal{X}(\mathbb{F}_q)$, a commutative diagram

$$\begin{tikzcd}
H^2_{crys}(\mathcal{Y}) \arrow[d, "\mathrm{Ler}"] \arrow[dr, "i_{\mathfrak{t}}^*"] & \mathrm{Pic}(\mathcal{Y}) \otimes \mathbb{Q} \arrow[l, "c_{\mathcal{Y}}"'] \arrow[dr, "i_{\mathfrak{t}}^*"] & \\
H^0(\mathcal{X}, R^2 \mathfrak{f}_{crys,*} \mathcal{O}_{\mathcal{Y}/W}) \otimes \mathbb{Q} \arrow[r] & H^2_{crys}(\mathcal{Y}_{\mathfrak{t}}) & \mathrm{Pic}(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q} \arrow[l, "c_{\mathcal{Y}_{\mathfrak{t}}}"]
\end{tikzcd}$$

where $H^2_{crys}(\mathcal{Y})$ and $H^2_{crys}(\mathcal{Y}_{\mathfrak{t}})$ are the (rational) crystalline cohomology of $\mathcal{Y}$ and $\mathcal{Y}_{\mathfrak{t}}$ respectively, Leray is the edge map in the Leray spectral sequence attached to $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ and $c_{\mathcal{Y}}, c_{\mathcal{Y}_{\mathfrak{t}}}$ are the crystalline cycle class maps. Write F for the s-power of the absolute Frobenius of $\mathcal{X}$ and recall that the images of $c_{\mathcal{Y}}$ and $c_{\mathcal{Y}_{\mathfrak{t}}}$ lie in $H^2_{crys}(\mathcal{Y})^{F=q}$ and $H^2_{crys}(\mathcal{Y}_{\mathfrak{t}})^{F=q}$, respectively. Then we have the variational Tate conjecture in crystalline cohomology:

Fact 4.1.6.1.1 ([Mor15, Theorem 1.4])**.** If $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ is projective, for every $z_{\mathfrak{t}} \in \mathrm{Pic}(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q}$ the following are equivalent:

1. There exists $z \in \mathrm{Pic}(\mathcal{Y}) \otimes \mathbb{Q}$ such that $c_{\mathcal{Y}_{\mathfrak{t}}}(z_{\mathfrak{t}}) = i_{\mathfrak{t}}^*(c_{\mathcal{Y}}(z))$;
2. $c_{\mathcal{Y}_{\mathfrak{t}}}(z_{\mathfrak{t}})$ lies in $H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W}) \otimes \mathbb{Q}$;
3. $c_{\mathcal{Y}_{\mathfrak{t}}}(z_{\mathfrak{t}})$ lies in $H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W})^{F=q} \otimes \mathbb{Q}$.

However, to apply Fact 4.1.6.1.1 in our setting, there are two difficulties to overcome:

1. Crystalline cohomology works well only over a perfect field, while our base field k is not perfect;
2. There is no direct way to compare the ℓ-adic and the crystalline sites, so that one has to find a different way to transfer the Galois generic assumption to the crystalline setting.

4.1.6.2 Spreading out

To overcome (1) one uses a spreading out argument, so that our morphism $f : Y \to X$ will appear as the generic fibre of a smooth projective morphism $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$, where $\mathcal{X}$ is a smooth geometrically connected $\mathbb{F}_q$-variety. The idea is then to lift an element $\epsilon_x \in \mathrm{NS}(Y_{\bar{x}}) \otimes \mathbb{Q}$ to $\mathrm{NS}(Y_{\bar{\eta}}) \otimes \mathbb{Q}$ by specializing it first to an element $\epsilon_{\mathfrak{t}} \in \mathrm{NS}(\mathcal{Y}_{\bar{\mathfrak{t}}}) \otimes \mathbb{Q}$ of a closed fibre of $\mathcal{Y} \to \mathcal{X}$ and then to try and lift $\epsilon_{\mathfrak{t}}$ to an element $\epsilon \in \mathrm{Pic}(\mathcal{Y}) \otimes \mathbb{Q}$, via the crystalline variational Tate conjecture over $\mathbb{F}_q$.

4.1.6.3 From ℓ to p

In order to show that $\epsilon_{\mathfrak{t}} \in \mathrm{NS}(Y_{\bar{\mathfrak{t}}}) \otimes \mathbb{Q}$ satisfies the assumption of Fact 4.1.6.1.1, one has to transfer the ℓ-adic information that x is Galois generic to crystalline cohomology. For this the key ingredient is Theorem 4.1.6.3.1 below. Assume that $\mathcal{Z}$ is a smooth geometrically connected $\mathbb{F}_q$-variety admitting an $\mathbb{F}_q$-rational point $\mathfrak{t}$ and that there is a map $\mathfrak{g} : \mathcal{Z} \to \mathcal{X}$ (in our application $\mathfrak{g} : \mathcal{Z} \to \mathcal{X}$ is a model for $x : k(x) \to X$). The cartesian square

$$\begin{tikzcd} \mathcal{Y}_{\mathcal{Z}} \arrow[r] \arrow[d, "\mathfrak{f}_{\mathcal{Z}}"] \arrow[dr, phantom, "\square"] & \mathcal{Y} \arrow[d, "\mathfrak{f}"] \\ \mathcal{Z} \arrow[r, "\mathfrak{g}"] & \mathcal{X} \end{tikzcd}$$

induces representations

$$\pi_1(\mathcal{Z}, \bar{\mathfrak{t}}) \to \pi_1(\mathcal{X}, \bar{\mathfrak{t}}) \to \mathrm{GL}(H^i(\mathcal{Y}_{\bar{\mathfrak{t}}}, \mathbb{Q}_\ell(j))).$$

Theorem 4.1.6.3.1. Assume that the image of $\pi_1(\mathcal{Z}, \bar{\mathfrak{t}}) \to \pi_1(\mathcal{X}, \bar{\mathfrak{t}}) \to \mathrm{GL}(H^i(\mathcal{Y}_{\bar{\mathfrak{t}}}, \mathbb{Q}_\ell(j))$ is open in the image of $\pi_1(\mathcal{X}, \bar{\mathfrak{t}}) \to \mathrm{GL}(H^i(\mathcal{Y}_{\bar{\mathfrak{t}}}, \mathbb{Q}_\ell(j))$ and that the Zariski closures of the images of $\pi_1(\mathcal{X}, \bar{\mathfrak{t}})$ and $\pi_1(\mathcal{X}_{\mathbb{F}}, \bar{\mathfrak{t}})$ acting on $H^i(\mathcal{Y}_{\bar{\mathfrak{t}}}, \mathbb{Q}_\ell(j))$ are connected. Then the base change map

$$H^0(\mathcal{X}, R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W})^{F=q^j} \otimes \mathbb{Q} \to H^0(\mathcal{Z}, R^i\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W})^{F=q^j} \otimes \mathbb{Q}$$

is an isomorphism.

As mentioned in Section 4.1.2, the subtle point in the proof of Theorem 4.1.6.3.1 is to compare the category of F-isocrystals, where the crystalline variational Tate conjecture holds, with the category of ℓ-adic lisse sheaves. These categories behaves differently. For example, if $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ is a non-isotrivial family of ordinary elliptic curves, $R^1\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W} \otimes \mathbb{Q}$ carries a two steps filtrations, reflecting the decomposition of the p-divisible groups of the generic fibre of

$\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ into étale and connected parts, while $R^1\mathfrak{f}_*\mathbb{Q}_\ell$ is irreducible. This leads to consider the smaller category of overconvergent F-isocrystals, whose behaviour is closer to the one of ℓ-adic lisse sheaves. Then the proof of Theorem 4.1.6.3.1 decomposes as follows:

1. We prove that $R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W}\otimes\mathbb{Q}$ and $R^i\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W}\otimes\mathbb{Q}$ are overconvergent F-isocrystals (Theorem 4.2.1.1.2, which uses a technical result proved in the Part 3, building on the work of Shiho on relative log convergent cohomology and relative rigid cohomology [Shi08a], [Shi08b]);

2. We use that one doesn't loose information passing from crystalline cohomology to overconvergent F-isocrystals (Fact 4.2.1.1.1);

3. Let G_p and $G_{\mathcal{Z},p}$ be the Tannakian groups of $R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W}\otimes\mathbb{Q}$ and $R^i\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W}\otimes\mathbb{Q}$ as overconvergent F-isocrystals. Theorem 4.1.6.3.1 amounts to showing that $G_p = G_{\mathcal{Z},p}$.

4. The assumption implies that the Zariski closures G_ℓ and $G_{\mathcal{Z},\ell}$ of the image of $\pi_1(\mathcal{X},\bar{\mathfrak{t}})$ and $\pi_1(\mathcal{Z},\bar{\mathfrak{t}})$ acting on $H^i(\mathcal{Y}_{\bar{\mathfrak{t}}},\mathbb{Q}_\ell(j))$ are equal.

5. To show that (4) implies (3), one uses the theory of Frobenius weights and that to compare reductive algebraic groups it is enough to compare their tensor invariants.

Remark 4.1.6.3.2. In Theorem 4.1.6.3.1, the assumptions that $\mathcal{Z}$ has a $\mathbb{F}_q$ rational point and that the Zariski closure of the image of $\pi_1(\mathcal{X}_{\mathbb{F}},\bar{\mathfrak{t}})$ is connected are not necessary, but the proof without these assumptions requires the more elaborated formalism of $\overline{\mathbb{Q}}_p$-overconvergent isocrystals. In our application to Theorem 4.1.4.2.2 one can reduce to the case where these assumptions are satisfied, so that we did not include the proof of the general form of Theorem 4.1.6.3.1.

Remark 4.1.6.3.3. In characteristic 0, the proof sketched in 4.1.5 shows that the variational Hodge conjecture implies the ℓ-adic variational Tate conjecture over fields of characteristic zero. In positive characteristic, our method does not show that the crystalline variational Tate conjecture implies the ℓ-adic one. The issue comes from the fact that one does not know how to compare the ℓ-adic and the crystalline cycle class maps.

4.1.7 Applications

4.1.7.1 Existence of NS-generic points

Let k be a field of transcendence degree ≥ 1 over $\mathbb{F}_p$ and X a smooth geometrically connected k-variety with generic point η. Let $f : Y \to X$ be a smooth proper morphism of k-varieties. Recall the following:

Fact 4.1.7.1.1. Assume that k is finitely generated. Then:

- ([Ser89, Section 10.6], Fact 1.2.2.2.2): The subset of non strictly ℓ-Galois-generic points for $f : Y \to X$ is sparse. In particular there exists an integer $d \geq 1$ such that there are infinitely many $x \in |X|$ with $[k(x) : k] \leq d$ that are strictly ℓ-Galois-generic for $f : Y \to X$.

- (Theorem 3.1.3.2): If X is a curve, all but finitely many $x \in X(k)$ are Galois-generic for $f : Y \to X$.

Theorem 4.1.4.2.2, together with Fact 4.1.7.1.1 and the fact that if $S \subseteq |X|$ is a subset and $U \subseteq X$ is a dense open subscheme such that $U \cap S \subseteq U$ is sparse then $S \subseteq |X|$ is again sparse ([MP12, Proposition 8.5 (a)]), implies:

Corollary 4.1.7.1.2. Assume that k is finitely generated. Then:

- The subset of closed non arithmetically NS-generic points for $f : Y \to X$ is sparse. In particular there exists an integer $d \geq 1$ such that there are infinitely many $x \in |X|$ with $[k(x) : k] \leq d$ that are arithmetically NS-generic for $f : Y \to X$.
- If X is a curve, all but finitely many $x \in X(k)$ are NS-generic for $f : Y \to X$.

Via a spreading out argument one has the following extension of the main result of [MP12] to positive characteristic:

Corollary 4.1.7.1.3. If k is a field of transcendence degree ≥ 1 over $\mathbb{F}_p$, then X has a closed NS-generic point.

Remark 4.1.7.1.4. Atticus Christensen ([Chr18, Theorem 1.0.1]) has independently proved Corollary 4.1.7.1.3. His proof is very different from ours, since his approach is inspired from the analytic approach in [MP12], while ours is inspired from the Hodge theoretic approach in [And96]. On the other hand, it seems that Corollary 4.1.7.1.2 (that will be used to prove Corollaries 4.1.7.2.1,4.1.7.2.2 and 4.1.7.3.1) can not be obtained via his method, that gives different information on the set of NS generic points ([Chr18, Theorems 1.0.3, 1.0.4.]).

From Corollary 4.1.7.1.3 one easily deduces the following results on the behaviour of the Tate conjecture in families:

Corollary 4.1.7.1.5. If $T(Y_x, \ell)$ holds for all $x \in |X|$, then $T(Y_\eta, \ell)$ holds.

Remark 4.1.7.1.6. Corollary 4.1.7.1.5 together with a spreading out argument can be used to reduce the Tate conjecture for smooth proper varieties over arbitrary finitely generated fields of characteristic p, to fields of transcendence degree one over $\mathbb{F}_p$, extending results from [Mor77], specific to abelian schemes, to arbitrary families of varieties.

The argument in [MP12, Theorem 7.1.] shows that Corollary 4.1.7.1.3 is enough to prove the following:

Corollary 4.1.7.1.7. Assume furthermore that Y_x is projective for every $x \in |X|$. Then there exists a dense open subscheme $U \subseteq X$ such that the base change $f_U : U \times_X Y \to U$ of $f : Y \to X$ trough $U \subseteq X$ is projective.

Remark 4.1.7.1.8. Whether the analogue of Corollary 4.1.7.1.7 holds over fields algebraic over $\mathbb{F}_p$ is not known. The problem over this kind of fields is that it is not true in general that there exists a NS-generic closed point (as the example of a family of abelian surfaces such that the generic fibre has not complex multiplication shows).

4.1.7.2 Hyperplane sections

From now on, assume that k is finitely generated. Assume that Z is a smooth projective k-variety of dimension ≥ 3 and let $Z \subseteq \mathbb{P}^n_k$ be a projective embedding. One can ask whether there exists a smooth hyperplane section D of Z such that the canonical map

$$\mathrm{NS}(Z_{\bar{k}}) \otimes \mathbb{Q} \to \mathrm{NS}(D_{\bar{k}}) \otimes \mathbb{Q}$$

is an isomorphism. This is not true in general (see Example 4.4.1.1), but one can apply Theorem 4.1.4.2.2 to obtain the following arithmetic variant:

Corollary 4.1.7.2.1. If $dim(Z) \geq 3$ there are infinitely many smooth k-rational hyperplane sections $D \subseteq Z$ such that the canonical map

$$\mathrm{NS}(Z) \otimes \mathbb{Q} \to \mathrm{NS}(D) \otimes \mathbb{Q}$$

is an isomorphism.

As already mentioned in Section 4.1.4, Conjecture 4.1.4.1.1 implies Theorem 4.1.4.2.2. The ℓ-adic Tate conjecture for divisors conjecture is still widely open except for some special classes of varieties like Abelian varieties and K3 surfaces. Using Corollary 4.1.7.2.1, one can enlarge the class of varieties for which it holds:

Corollary 4.1.7.2.2. Let Z be a smooth projective k-variety of dimension ≥ 3 and choose a projective embedding $Z \subseteq \mathbb{P}^n_k$. If $T(D, \ell)$ holds for the smooth hyperplane sections $D \subseteq Z$, then $T(Z, \ell)$ holds.

Remark 4.1.7.2.3. Corollary 4.1.7.2.2 can be used to reduce the ℓ-adic Tate conjecture for divisors on smooth proper k-varieties to smooth projective k-surfaces, extending an unpublished result ([dJ]) of De Jong (whose proof has been simplified in [Mor15, Theorem 4.3]) to infinite finitely generated fields.

4.1.7.3 Uniform boundedness of Brauer groups

Combining Theorem 4.1.4.2.2 with the main result of Chapter 3 (Theorem 3.1.3.2) and the arguments of [CC18], one gets the following application to uniform boundedness for the ℓ-primary torsion of the cohomological Brauer group in smooth proper families of k-varieties.

Corollary 4.1.7.3.1. Let X be a smooth geometrically connected k-curve and let $f : Y \to X$ be a smooth proper morphism of k-varieties. If $T(Y_{\overline{x}}, \ell)$ holds for all $x \in |X|$, then there exists a constant $C := C(Y \to X, \ell)$ such that

$$|\,\mathrm{Br}(Y_{\overline{x}})[\ell^\infty]^{\pi_1(x,\overline{x})}| \leq C$$

for all $x \in X(k)$.

Corollary 4.1.7.3.1 extends to positive characteristic the main result of [CC18] and gives some evidence for a positive charateristic version of the conjectures on the uniform boundedness of Brauer group in [VAV17]. Elaborating the argument in the proof of Corollary 4.1.7.3.1, one gets also an unconditional variant of Corollary 4.1.7.3.1 (Corollary 4.5.2.2) and a result on the specialization of the p-adic Tate module of the Brauer group (Corollary 4.5.3.1).

4.1.8 Organization of the chapter

In the first two sections we prove of Theorem 4.1.4.2.2: in Section 4.2 we prove Theorem 4.1.6.3.1 and in Section 4.3 we show Theorem 4.1.4.2.2. Sections 4.4 and 4.5 are devoted to applications: in Section 4.4 we prove Corollary 4.1.7.2.1, and in Section 4.5 we give the proof of Corollary 4.1.7.3.1. In Sections 4.6 and 4.7 we prove the overconvergence of the higher direct image in crystalline cohomology (Theorem 4.6.5.4.1), which is used in the proof of Theorem 4.1.6.3.1.

4.1.9 Acknowledgements

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4.2 Proof of Theorem 4.1.6.3.1

This section is devoted to the proof of Theorem 4.1.6.3.1. In Section 4.2.1, after recalling the various categories of isocrystals needed in our argument, we reformulate Theorem 4.1.6.3.1 in terms of monodromy groups of F-overconvergent isocrystals. In Section 4.2.2, we use independence techniques to prove Theorem 4.1.6.3.1.

4.2.1 Tannakian reformulation of Theorem 4.1.6.3.1

4.2.1.1 Overconvergent isocrystals

Let $\mathcal{X}$ be a smooth geometrically connected $\mathbb{F}_q$-variety with $q = p^s$ and write F for s-power of the absolute Frobenius on $\mathcal{X}$. Write $W := W(\mathbb{F}_q)$ for the Witt ring of $\mathbb{F}_q$, K for its fraction field and $Mod(\mathcal{X}|W)$ for the category of $\mathcal{O}_{\mathcal{X}|W}$-modules in the crystalline site of $\mathcal{X}$. Consider the following categories:

Notation	Name	Reference
$\mathbf{Crys}(\mathcal{X}\|W)_{\mathbb{Q}}$	Isocrystals	[Mor13, Section 2]
$\mathbf{Isoc}^{\dagger}(\mathcal{X}\|K)$	Overconvergent Isocrystals	[Ber96, Definition 2.3.6]
$\mathbf{Isoc}(\mathcal{X}\|K)$	Convergent isocrystals	[Ber96, Definition 2.3.2]

and their enriched version with Frobenius structure: $\mathbf{F\text{-}Crys}(\mathcal{X}|W)_{\mathbb{Q}}$, $\mathbf{F\text{-}Isoc}^{\dagger}(\mathcal{X}|K)$ and $\mathbf{F\text{-}Isoc}(\mathcal{X}|K)$. They fit into the following commutative diagram:

$$\begin{tikzcd}
\mathbf{F\text{-}Crys}(\mathcal{X}|W)_{\mathbb{Q}} \arrow[r, "(1)", "\simeq"'] \arrow[d, "(-)^{geo}"'] & \mathbf{F\text{-}Isoc}(\mathcal{X}|K) & \mathbf{F\text{-}Isoc}^{\dagger}(\mathcal{X}|K) \arrow[l, "(2)"'] \arrow[d, "(-)^{geo}"] \arrow[ll, bend right, "(3)"'] \arrow[ddl, "(4)"] \\
\mathbf{Crys}(\mathcal{X}|W)_{\mathbb{Q}} \arrow[dr, "(5)"'] & & \mathbf{Isoc}^{\dagger}(\mathcal{X}|K) \\
 & Mod(\mathcal{X}|W) \otimes \mathbb{Q} &
\end{tikzcd}$$

where $(-)^{geo}$ are the forgetful functors, (1) is the equivalence of categories constructed in [Ber96, Theoreme 2.4.2] and (2) is the obvious functor. Write

$$R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K} := R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W} \otimes \mathbb{Q} \in Mod(\mathcal{X}|W) \otimes \mathbb{Q}$$

and recall the following fact:

Fact 4.2.1.1.1 ([Ked04, Theorem 1.1])**.** The functor (3) is fully faithful.

The following result, which gives us an overconvergent incarnation of $R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K}$, is a consequence of the main result (Theorem 4.6.5.4.1) of Sections 4.6 and 4.7, building on the work of Shiho on relative rigid cohomology ([Shi08a], [Shi08b]).

Theorem 4.2.1.1.2. Let $\mathfrak{f} : \mathcal{Y} \to X$ be a smooth proper morphism. Then $R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K} \in Mod(\mathcal{X}|W) \otimes \mathbb{Q}$ lies in the essential image of (4).

Proof. By [Mor13, Proposition 3.2], $R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K}$ is in the essential image of $(5) \circ (-)^{geo}$. Under the equivalence (1), $R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K}$ is sent to the Ogus higher direct image $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$, see [Ogu84, Section 3, Theorem 3.1] and [Mor13, Corollary 6.2]. One concludes by Theorem 4.6.5.4.1, which says that $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$ is in the image of an F-overconvergent isocrystal. $\square$

Write $R^i\mathfrak{f}_*\mathcal{O}^\dagger_{\mathcal{Y}|K}$ for the (unique up to isomorphism) object of $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)$ lifting $R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K}$.

4.2.1.2 Tannakian formalism

Since $\mathcal{X}$ is a geometrically connected $\mathbb{F}_q$ variety, $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)$ and $\mathbf{Isoc}^\dagger(\mathcal{X}|K)$ are K-linear Tannakian categories over K[1], see [D'Ad17, Section 3.2] for more details. If one furthermore assumes that $\mathcal{X}$ has a $\mathbb{F}_q$-rational point $\mathfrak{t}$, the categories $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)$, $\mathbf{Isoc}^\dagger(\mathcal{X}|K)$ are neutralized by the fibre functors

$$\begin{tikzcd} \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K) \arrow[r, "(-)^{geo}"] \arrow[dr, "\mathfrak{t}^*"'] & \mathbf{Isoc}^\dagger(\mathcal{X}|K) \arrow[d, "\mathfrak{t}^*"] \\ & \mathbf{Isoc}^\dagger(\mathbb{F}_q) \simeq Vect_K. \end{tikzcd}$$

Write $\pi_1^\dagger(\mathcal{X},\mathfrak{t})$ and $\pi_1^{\dagger,geo}(\mathcal{X},\mathfrak{t})$ for the Tannakian groups of $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)$ and $\mathbf{Isoc}^\dagger(\mathcal{X}|K)$ respectively. For $\mathcal{F} \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)$, let $G(\mathcal{F},\mathfrak{t})$, $G^{geo}(\mathcal{F},\mathfrak{t})$ denote the Tannakian groups of $\langle\mathcal{F}\rangle^\otimes \subseteq \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)$ and $\langle\mathcal{F}^{geo}\rangle^\otimes \subseteq \mathbf{Isoc}^\dagger(\mathcal{X}|K)$ respectively. By the general Tannakian formalism the forgetful functor $(-)^{geo} : \langle\mathcal{F}\rangle^\otimes \to \langle\mathcal{F}^{geo}\rangle^\otimes$ corresponds to a closed immersion

$$G^{geo}(\mathcal{F},\mathfrak{t}) \subseteq G(\mathcal{F},\mathfrak{t}).$$

Alternatively $G^{geo}(\mathcal{F},\mathfrak{t})$, $G(\mathcal{F},\mathfrak{t})$ can be described as the images of

$$\pi_1^{\dagger,geo}(\mathcal{X},\mathfrak{t}) \to \pi_1^\dagger(\mathcal{X},\mathfrak{t}) \to \mathrm{GL}(\mathcal{F}_\mathfrak{t})$$

where $\mathcal{F}_\mathfrak{t} := \mathfrak{t}^*\mathcal{F}$. If $\mathfrak{g} : \mathcal{Z} \to \mathcal{X}$ is a morphism of geometrically connected $\mathbb{F}_q$-varieties and $\mathfrak{t} \in \mathcal{Z}(\mathbb{F}_q)$ the canonical functors

$$\begin{tikzcd} \langle\mathcal{F}\rangle^\otimes \arrow[r, "g^*"] \arrow[d, "(-)^{geo}"'] & \langle\mathfrak{g}^*\mathcal{F}\rangle^\otimes \arrow[d, "(-)^{geo}"] \\ \langle\mathcal{F}^{geo}\rangle^\otimes \arrow[r, "g^*"] & \langle\mathfrak{g}^*\mathcal{F}^{geo}\rangle \end{tikzcd}$$

[1] Recall that F is the s-power Frobenius, so that its action on $\mathbf{Isoc}^\dagger(\mathcal{X}|K)$ is K-linear.

correspond to a commutative diagram of closed immersions

$$\begin{tikzcd} G^{geo}(\mathfrak{g}^*\mathcal{F},\mathfrak{t}) \arrow[r, hook] \arrow[d, hook] & G^{geo}(\mathcal{F},\mathfrak{t}) \arrow[d, hook] \\ G(\mathfrak{g}^*\mathcal{F},\mathfrak{t}) \arrow[r, hook] & G(\mathcal{F},\mathfrak{t}). \end{tikzcd}$$

Let $\mathfrak{f}:\mathcal{Y}\to\mathcal{X}$ be a smooth proper morphism of $\mathbb{F}_q$-varieties and write $\mathcal{F}_p := R^i\mathfrak{f}_*\mathcal{O}^\dagger_{\mathcal{Y}|K}(j)$ for the j^{th}-twist of the F-overconvergent isocrystals provided by Theorem 4.2.1.1.2.

Proposition 4.2.1.2.1. There is a natural isomorphism

$$H^0(\mathcal{X}, R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K})^{F=q^j} \simeq \mathcal{F}_{p,\mathfrak{t}}^{G(\mathcal{F},\mathfrak{t})}.$$

Proof. Write $\mathcal{O}_{\mathcal{X}/K}$ and $\mathcal{O}^\dagger_{\mathcal{X}/K}$ for the structural sheaves in $\mathbf{F}\text{-}\mathbf{Crys}(\mathcal{X}|W)_\mathbb{Q}$ and $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)$ respectively. One has

$$H^0(\mathcal{X}, R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K}) \simeq Hom_{\mathbf{Crys}(\mathcal{X}|W)_\mathbb{Q}}(\mathcal{O}_{\mathcal{X}/K}, R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K})$$

and so an isomorphism

$$H^0(\mathcal{X}, R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K})^{F=q^j} \simeq Hom_{\mathbf{F}\text{-}\mathbf{Crys}(\mathcal{X}|W)_\mathbb{Q}}(\mathcal{O}_{\mathcal{X}/K}(-j), R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K}).$$

By Fact 4.2.1.1.1

$$Hom_{\mathbf{F}\text{-}\mathbf{Crys}(\mathcal{X}|W)_\mathbb{Q}}(\mathcal{O}_{\mathcal{X}/K}(-j), R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K}) \simeq Hom_{\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)}(\mathcal{O}^\dagger_{\mathcal{X}/K}(-j), R^i\mathfrak{f}_*\mathcal{O}^\dagger_{\mathcal{Y}/K}) \simeq$$

$$\simeq Hom_{\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)}(\mathcal{O}^\dagger_{\mathcal{X}/K}, R^i\mathfrak{f}_*\mathcal{O}^\dagger_{\mathcal{Y}/K}(j)) \simeq Hom_{\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)}(\mathcal{O}^\dagger_{\mathcal{X}/K}, \mathcal{F}_p).$$

Since $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)$ is a Tannakian category with fibre functor $\mathfrak{t}^*$ and $\mathfrak{t}^*\mathcal{O}^\dagger_{\mathcal{X}/K}$ corresponds to the trivial one dimensional representation K of $\pi_1^\dagger(\mathcal{X},\mathfrak{t})$, one deduces

$$Hom_{\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}|K)}(\mathcal{O}^\dagger_{\mathcal{X}/K}, \mathcal{F}_p) \simeq Hom_{Rep_K(\pi_1^\dagger(\mathcal{X},\mathfrak{t}))}(\mathfrak{t}^*\mathcal{O}^\dagger_{\mathcal{X}/K}, \mathfrak{t}^*\mathcal{F}_p) \simeq Hom_{Rep_K(\pi_1^\dagger(\mathcal{X},\mathfrak{t}))}(K, \mathcal{F}_{p,\mathfrak{t}}) \simeq \mathcal{F}_{p,\mathfrak{t}}^{\pi_1^\dagger(\mathcal{X},\mathfrak{t})}.$$

Since the image of the action of $\pi_1^\dagger(\mathcal{X},\mathfrak{t})$ on $\mathcal{F}_{p,\mathfrak{t}}$ is $G_p(\mathcal{F},\mathfrak{t})$, one sees that

$$\mathcal{F}_{p,\mathfrak{t}}^{\pi_1^\dagger(\mathcal{X},\mathfrak{t})} \simeq \mathcal{F}_{p,\mathfrak{t}}^{G_p(\mathcal{F},\mathfrak{t})}$$

and this concludes the proof. □

4.2.1.3 Tannakian reinterpretation of Theorem 4.1.6.3.1

We now retain the notation and assumption of Theorem 4.1.6.3.1. With the notation of Theorem 4.2.1.1.2, write

$$\mathcal{F}_p := R^i\mathfrak{f}_*\mathcal{O}^\dagger_{\mathcal{Y}/K}(j)$$

$$\mathcal{F}_{\mathcal{Z},p} := R^i\mathfrak{f}_{\mathcal{Z},*}\mathcal{O}^\dagger_{\mathcal{Y}_\mathcal{Z}|K}(j) \simeq \mathfrak{g}^*\mathcal{F}_p$$

where the isomorphism comes from smooth proper base change in crystalline cohomology (e.g. [Mor13, Proposition 3.2]) and Fact 4.2.1.1.1. By Proposition 4.2.1.2.1 one has a commutative diagram

$$\begin{tikzcd} H^0(\mathcal{X}, R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K})^{F=q} \arrow[r, "\simeq"] \arrow[d] & \mathcal{F}_{p,\mathfrak{t}}^{G(\mathcal{F}_p,\mathfrak{t})} \arrow[d, hook] \\ H^0(\mathcal{Z}, R^i\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_\mathcal{Z}/K})^{F=q} \arrow[r, "\simeq"] & \mathcal{F}_{p,\mathfrak{t}}^{G(\mathcal{F}_{\mathcal{Z},p},\mathfrak{t})} \end{tikzcd}$$

Hence it is enough to show that the natural inclusion $G(\mathcal{F}_{\mathcal{Z},p},\mathfrak{t}) \subseteq G(\mathcal{F}_p,\mathfrak{t})$ is an isomorphism.

4.2.2 End of the proof

4.2.2.1 Compatibility

For $\ell \neq p$ write

$$\mathcal{F}_\ell := R^i\mathfrak{f}_*\mathbb{Q}_\ell(j) \quad \mathcal{F}_{\mathcal{Z},\ell} := R^i\mathfrak{f}_{\mathcal{Z},*}\mathbb{Q}_\ell(j)$$

and let $G(\mathcal{F}_{\mathcal{Z},\ell},\mathfrak{t})$ (resp. $G(\mathcal{F}_\ell,\mathfrak{t})$) denotes the Zariski closure of the image of $\pi_1(\mathcal{Z},\mathfrak{t})$ (resp. $\pi_1(\mathcal{X},\mathfrak{t})$) acting on $H^i(\mathcal{Y}_{\bar{\mathfrak{t}}},\mathbb{Q}_\ell(j))$. Since $\pi_1(\mathcal{Z},\bar{\mathfrak{t}}) \to \pi_1(\mathcal{X},\bar{\mathfrak{t}}) \to \mathrm{GL}(H^i(\mathcal{Y}_{\bar{\mathfrak{t}}},\mathbb{Q}_\ell(j))$ is open in the image of $\pi_1(\mathcal{X},\bar{\mathfrak{t}}) \to \mathrm{GL}(H^i(\mathcal{Y}_{\bar{\mathfrak{t}}},\mathbb{Q}_\ell(j))$ and $G(\mathcal{F}_\ell,\mathfrak{t})$ is connected, one has $G(\mathcal{F}_\ell,\mathfrak{t}) = G(\mathcal{F}_{\mathcal{Z},\ell},\mathfrak{t})$. To prove that the natural inclusion $G(\mathcal{F}_{\mathcal{Z},p},\mathfrak{t}) \subseteq G(\mathcal{F}_p,\mathfrak{t})$ is an isomorphism, the idea is to compare $G(\mathcal{F}_{\mathcal{Z},p},\mathfrak{t})$ and $G(\mathcal{F}_p,\mathfrak{t})$ with $G(\mathcal{F}_{\mathcal{Z},\ell},\mathfrak{t})$ and $G(\mathcal{F}_\ell,\mathfrak{t})$. To do this, the main ingredient is the following.

Fact 4.2.2.1 ([Del80], [KM74]). $\mathcal{F}_p$,$\mathcal{F}_\ell$ (resp. $\mathcal{F}_{\mathcal{Z},p}$,$\mathcal{F}_{\mathcal{Z},\ell}$) is a $\mathbb{Q}$-rational compatible system on $\mathcal{X}$ (resp. $\mathcal{Z}$) pure of weight $i+2j$.

4.2.2.2 Geometric monodromy

Write $\mathcal{F}_\ell^{geo}$ (resp. $\mathcal{F}_{\mathcal{Z},\ell}^{geo}$) for the restriction of $\mathcal{F}_\ell$ (resp. $\mathcal{F}_{\mathcal{Z},\ell}$) to $\mathcal{X}_{\mathbb{F}}$ (resp. $\mathcal{Z}_{\mathbb{F}}$) and $G^{geo}(\mathcal{F}_\ell,\mathfrak{t})$ (resp. $G^{geo}(\mathcal{F}_{\mathcal{Z},\ell},\mathfrak{t})$) for the Zariski closure of the image of $\pi_1(\mathcal{X}_{\mathbb{F}},\mathfrak{t})$ (resp. $\pi_1(\mathcal{Z}_{\mathbb{F}},\mathfrak{t})$) acting on $H^i(\mathcal{Y}_{\bar{\mathfrak{t}}},\mathbb{Q}_\ell(j))$. Recall the following:

Fact 4.2.2.2.1. For $? \in \{\ell,p\}$ one has:

1. The groups $G^{geo}(\mathcal{F}_?,\mathfrak{t})$ and $G^{geo}(\mathcal{F}_{\mathcal{Z},?},\mathfrak{t})$ are reductive algebraic groups.
2. $G(\mathcal{F}_?,\mathfrak{t})^0 = G(\mathcal{F}_{\mathcal{Z},?},\mathfrak{t})^0$ if and only if $G^{geo}(\mathcal{F}_?,\mathfrak{t})^0 = G^{geo}(\mathcal{F}_{\mathcal{Z},?},\mathfrak{t})^0$

Proof.

1. Since $\mathcal{F}_?^{geo}$ and $\mathcal{F}_{\mathcal{Z},?}^{geo}$ are pure, this follows from [Del80, Theorem 3.4.1] if $?=\ell$ and from [Ked17, Remark 10.6] if $?=p$.
2. One implication follows from the fact that $G^{geo}(\mathcal{F}_?,\mathfrak{t})^0$ and $G^{geo}(\mathcal{F}_{\mathcal{Z},?},\mathfrak{t})^0$ are the derived subgroup of $G(\mathcal{F}_?,\mathfrak{t})^0$ and $G(\mathcal{F}_{\mathcal{Z},?},\mathfrak{t})^0$ (a consequence of the global monodromy theorem, see e.g. [D'Ad17, Corollary 3.4.10]). The other one follows from [D'Ad17, Corollary 3.2.7]. □

By assumption $G(\mathcal{F}_\ell,\mathfrak{t}) = G(\mathcal{F}_{\mathcal{Z},\ell},\mathfrak{t})$ and $G^{geo}(\mathcal{F}_\ell,\mathfrak{t})$ are connected, so by Fact 4.2.2.2.1 for $\ell \neq p$, one gets that $G^{geo}(\mathcal{F}_\ell,\mathfrak{t}) = G^{geo}(\mathcal{F}_{\mathcal{Z},\ell},\mathfrak{t})$ is connected. By [D'Ad17, Theorem 4.1.1.], being connected is independent from $? \in \{\ell,p\}$, so that $G(\mathcal{F}_p,\mathfrak{t}), G(\mathcal{F}_{\mathcal{Z},p},\mathfrak{t}), G^{geo}(\mathcal{F}_p,\mathfrak{t})$ and $G^{geo}(\mathcal{F}_{\mathcal{Z},p},\mathfrak{t})$ are connected. So, by Fact 4.2.2.2.1 for $\ell = p$, it is enough to show that $G^{geo}(\mathcal{F}_{\mathcal{Z},p},\mathfrak{t}) = G^{geo}(\mathcal{F}_p,\mathfrak{t})$.

4.2.2.3 Purity

For every integers $n,m \geq 0$, write

$$T^{m,n}(\mathcal{F}_{?,\mathfrak{t}}) := \mathcal{F}_{?,\mathfrak{t}}^{\otimes m} \otimes (\mathcal{F}_{?,\mathfrak{t}}^\vee)^{\otimes n} \quad \text{and} \quad T^{m,n}(\mathcal{F}_?) := \mathcal{F}_?^{\otimes m} \otimes (\mathcal{F}_?^\vee)^{\otimes n}.$$

Since $G^{geo}(\mathcal{F}_p,\mathfrak{t})$ and $G^{geo}(\mathcal{F}_{\mathcal{Z},p},\mathfrak{t})$ are reductive (4.2.2.2.1), by Chevalley theorem it is enough to show that

$$T^{m,n}(\mathcal{F}_{p,\mathfrak{t}})^{G^{geo}(\mathcal{F}_p,\mathfrak{t})} = T^{m,n}(\mathcal{F}_{p,\mathfrak{t}})^{G^{geo}(\mathcal{F}_{\mathcal{Z},p},\mathfrak{t})}.$$

Since $G^{geo}(\mathcal{F}_\ell, \mathfrak{t}) = G^{geo}(\mathcal{F}_{\mathcal{Z},\ell}, \mathfrak{t})$, it is enough to show that

$$Dim(T^{m,n}(\mathcal{F}_{\ell,\mathfrak{t}})^{G^{geo}(\mathcal{F}_\ell,\mathfrak{t})}) = Dim(T^{m,n}(\mathcal{F}_{p,\mathfrak{t}})^{G^{geo}(\mathcal{F}_p,\mathfrak{t})}) \quad \text{and}$$
$$Dim(T^{m,n}(\mathcal{F}_{\ell,\mathfrak{t}})^{G^{geo}(\mathcal{F}_{\mathcal{Z},\ell},\mathfrak{t})}) = Dim(T^{m,n}(\mathcal{F}_{p,\mathfrak{t}})^{G^{geo}(\mathcal{F}_{\mathcal{Z},p},\mathfrak{t})}).$$

We prove the first equality, the proof of second being analogue. As in Proposition 4.2.1.2.1, one has

$$T^{m,n}(\mathcal{F}_{\ell,\mathfrak{t}})^{G^{geo}(\mathcal{F}_\ell,\mathfrak{t})} = H^0(\mathcal{X}_{\mathbb{F}}, T^{m,n}(\mathcal{F}_\ell^{geo})) \quad \text{and} \quad T^{m,n}(\mathcal{F}_{p,\mathfrak{t}})^{G^{geo}(\mathcal{F}_p,\mathfrak{t})} = H^0(\mathcal{X}, T^{m,n}(\mathcal{F}_p^{geo})).$$

So it is enough to show that

$$Dim(H^0(\mathcal{X}_{\mathbb{F}}, T^{m,n}(\mathcal{F}_\ell)^{geo})) = Dim(H^0(\mathcal{X}, T^{m,n}(\mathcal{F}_p)^{geo})).$$

Since $\mathcal{F}_\ell$ and $\mathcal{F}_p$ are pure, the same is true for $T^{m,n}(\mathcal{F}_\ell)^\vee(d)$ and $T^{m,n}(\mathcal{F}_p)^\vee(d)$, where d is the dimension of $\mathcal{X}$. Hence, by Grothendieck-Lefschetz fixed point formula ([Fu15, Theorem 10.5.1, page 603] if $? = \ell$ and [ES93, Theorem 6.3] if $? = p$) the left and the right hand sides are the number of poles, counted with multiplicity, with absolute value $q^{w/2}$ in the L-function of $T^{m,n}(\mathcal{F}_\ell)^\vee(d)$ and $T^{m,n}(\mathcal{F}_p)^\vee(d)$ (see e.g. [D'Ad17, Proposition 3.4.11] for more details). Since $\mathcal{F}_\ell$ and $\mathcal{F}_p$ are compatible, the same is true for $T^{m,n}(\mathcal{F}_\ell)^\vee(d)$ and $T^{m,n}(\mathcal{F}_p)^\vee(d)$, hence the L-function of $T^{m,n}(\mathcal{F}_?)^\vee(d)$ does not depend on $? \in \{\ell, p\}$. This concludes the proof of Theorem 4.1.6.3.1

4.3 Proof of Theorem 4.1.4.2.2

In Section 4.3.1, we collect some preliminary remarks. The proof when $f : Y \to X$ is proper is a technical elaboration (involving alteration and the trace formalism) of the proof when $f : Y \to X$ is projective. To clarify the exposition we carry out the proof when $f : Y \to X$ is projective in Section 4.3.2 and turn to the general case in Section 4.3.3.

4.3.1 Preliminary remarks

4.3.1.1 Strictly generic vs generic

Observe that the assertion for Galois generic points implies the assertion for strictly Galois generic points. Indeed, strictly Galois generic implies Galois generic, hence for a strictly Galois generic point $x \in X$ the specialization morphism

$$sp_{\eta,x} : \mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \to \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q}$$

is an isomorphism. Recall that, as explained in 4.1.4, the map $sp_{\eta,x}$ is $\pi_1(x, \overline{x})$-equivariant. Since $\pi_1(x, \overline{x})$ and $\pi_1(X, \overline{\eta}) \simeq \pi_1(X, \overline{x})$ acting on $H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)) \simeq H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$ have the same image Π_ℓ (since x is strictly Galois generic), taking Π_ℓ-invariants in $sp_{\eta,x}$, one deduces the statement for strictly Galois generic points. So, from on, we focus on the assertion for Galois generic points. To simplify, in this section, we omit base points in our notation for the étale fundamental group.

4.3.1.2 Finite cover

If $X' \to X$ is a surjective finite morphism of smooth connected k-varieties, the map $\pi_1(X') \to \pi_1(X)$ has open image. So $x \in X$ is Galois generic (resp. NS generic) for $f : Y \to X$ if and only if any lifting $x' \in X'$ of x if Galois generic (resp. NS generic) for the base change $f_{X'} : Y' \times_X X' \to X'$ of $f : Y \to X$ along $X' \to X$. As a consequence we can freely replace X with X' during the proof.

4.3.2 Proof when f is projective

Let $f : Y \to X$ be smooth projective. For the general strategy of the proof see Section 4.1.6.2.

4.3.2.1 Step 1: Spreading out

Replacing k with a finite field extension (4.3.1.2), one can assume that there exists a finite field $\mathbb{F}_q$, smooth and geometrically connected $\mathbb{F}_q$-varieties $\mathcal{K}$, $\mathcal{Z}$ with generic points $\zeta : k \to \mathcal{K}$, $\beta : k(x) \to \mathcal{Z}$ and a commutative cartesian diagram

$$\begin{tikzcd}
\mathcal{Y}_{\mathfrak{t}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}_{\mathcal{Z}} \arrow[r] \arrow[d, "\mathfrak{f}_{\mathcal{Z}}"] \arrow[dr, phantom, "\square"] & \mathcal{Y} \arrow[d, "\mathfrak{f}"] \arrow[dr, phantom, "\square"] & Y \arrow[l] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Y_x \arrow[l] \arrow[d, "f_x"] \arrow[lll, bend right] \\
\mathbb{F}_q \arrow[r, "\mathfrak{t}"] \arrow[dr, equal] & \mathcal{Z} \arrow[r] \arrow[d] \arrow[dr] & \mathcal{X} \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[l] \arrow[d] & k(x) \arrow[l, "x"'] \arrow[lll, bend left, "\beta"] \\
 & \mathbb{F}_q & \mathcal{K} \arrow[l] & k \arrow[l, "\zeta"] &
\end{tikzcd}$$

where $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ is a smooth projective morphism and the base change of $f_{\mathcal{Z}} : \mathcal{Y}_{\mathcal{Z}} \to \mathcal{Z}$ along $\beta : k(x) \to \mathcal{Z}$ identifies with $f_x : Y_x \to k(x)$. Replacing X with a finite étale cover (4.3.1.2) one can also assume that

1. $\mathrm{NS}(Y_{\bar{x}}) \otimes \mathbb{Q} = \mathrm{NS}(Y_x) \otimes \mathbb{Q}$ and $\mathrm{NS}(\mathcal{Y}_{\bar{\mathfrak{t}}}) \otimes \mathbb{Q} = \mathrm{NS}(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q}$;

2. the Zariski closures of the images of $\pi_1(\mathcal{X}) \to \mathrm{GL}(H^2(Y_{\bar{\eta}}, \mathbb{Q}_\ell(1)))$ and $\pi_1(\mathcal{X}_{\mathbb{F}}) \to \mathrm{GL}(H^2(Y_{\bar{\eta}}, \mathbb{Q}_\ell(1)))$ are connected.

Note that, by smooth proper base change, one has the following factorization

$$\begin{tikzcd}
\pi_1(x) \arrow[r] \arrow[d, two heads] & \pi_1(X) \arrow[r] \arrow[d, two heads] & \mathrm{GL}(H^2(Y_{\bar{\eta}}, \mathbb{Q}_\ell(1))) \simeq \mathrm{GL}(H^2(\mathcal{Y}_{\bar{t}}, \mathbb{Q}_\ell(1))) \\
\pi_1(\mathcal{Z}) \arrow[r] & \pi_1(\mathcal{X}). \arrow[ur] &
\end{tikzcd}$$

In particular, since x is Galois generic with respect to $f : Y \to X$, the image of $\pi_1(\mathcal{Z}) \to \pi_1(\mathcal{X}) \to \mathrm{GL}(H^2(\mathcal{Y}_{\bar{t}}, \mathbb{Q}_\ell(1)))$ is open in the image of $\pi_1(\mathcal{X}) \to \mathrm{GL}(H^2(\mathcal{Y}_{\bar{t}}, \mathbb{Q}_\ell(1)))$. Hence by 4.3.2.1(2) and Theorem 4.1.6.3.1 the base change map

$$H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W})^{F=q} \otimes \mathbb{Q} \to H^0(\mathcal{Z}, R^2\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W})^{F=q} \otimes \mathbb{Q}$$

is an isomorphism.

4.3.2.2 Step 2: Using the variational Tate conjecture

Since $\mathfrak{t}$ is a specialization of x (in $\mathcal{Z}$) and x is a specialization of η (in $\mathcal{X}$), there is a canonical commutative diagram

$$
\begin{tikzcd}[column sep=small]
z \in \mathrm{Pic}(\mathcal{Y}) \otimes \mathbb{Q} \arrow[rr] \arrow[d] \arrow[dr, "i_{\mathfrak{t}}^*"] \arrow[dddd, bend right, "c_{\mathcal{Y}}"'] & & \mathrm{Pic}(\mathcal{Y}_{\mathcal{Z}}) \otimes \mathbb{Q} \ni z_x \arrow[d, two heads, "(i)"] \arrow[dl, "i_{\mathfrak{t}}^*"'] \arrow[dddd, bend left, "c_{\mathcal{Y}_{\mathcal{Z}}}"] \\
\mathrm{Pic}(Y_\eta) \otimes \mathbb{Q} \arrow[d] & \mathrm{Pic}(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q} \ni z_{\mathfrak{t}} \arrow[dd, two heads, "(ii)"] \arrow[dddd, bend left, "c_{\mathcal{Y}_{\mathfrak{t}}}"] & \mathrm{Pic}(Y_x) \otimes \mathbb{Q} \arrow[d, two heads, "(ii)"] \\
\epsilon_\eta \in \mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[rr, hook, "sp_{\eta,x}"] \arrow[dr, hook, "sp_{\eta,\mathfrak{t}}"'] & & \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q} \ni \epsilon_x \arrow[dl, hook', "sp_{x,\mathfrak{t}}"] \\
& \mathrm{NS}(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q} = \mathrm{NS}(\mathcal{Y}_{\overline{\mathfrak{t}}}) \otimes \mathbb{Q} \ni \epsilon_{\mathfrak{t}} \arrow[dd, hook] & \\
H^2_{crys}(\mathcal{Y}) \arrow[dr, "i_{\mathfrak{t}}^*"] \arrow[dd, "\mathrm{Ler}"'] & & H^2_{crys}(\mathcal{Y}_{\mathcal{Z}}) \arrow[dl, "i_{\mathfrak{t}}^*"'] \arrow[dd, "\mathrm{Ler}"] \\
& H^2_{crys}(\mathcal{Y}_{\mathfrak{t}}) & \\
H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W}) \otimes \mathbb{Q} \arrow[rr, hook] \arrow[ur, hook] & & H^0(\mathcal{Z}, R^2\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W}) \otimes \mathbb{Q} \arrow[ul, hook'] \\
H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W})^{F=q} \otimes \mathbb{Q} \arrow[rr, "(iii)"] \arrow[u, hook] & & H^0(\mathcal{Z}, R^2\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W})^{F=q} \otimes \mathbb{Q} \arrow[u, hook]
\end{tikzcd}
$$

where the arrow (i) is surjective, since an open immersion of smooth varieties induces a surjection on the Picard groups, the arrows (ii) are surjective by 4.3.2.1(1) and the arrow (iii) is an isomorphism by Theorem 4.1.6.3.1.

The images of

$$\mathrm{Pic}(\mathcal{Y}) \otimes \mathbb{Q} \to H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W}) \otimes \mathbb{Q} \quad \text{and} \quad \mathrm{Pic}(\mathcal{Y}_{\mathcal{Z}}) \otimes \mathbb{Q} \to H^0(\mathcal{Z}, R^2\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W}) \otimes \mathbb{Q}$$

lie in $H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W})^{F=q} \otimes \mathbb{Q}$ and $H^0(\mathcal{Z}, R^2\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{W}}/K})^{F=q} \otimes \mathbb{Q}$ respectively. Take an ϵ_x in $\mathrm{NS}(Y_x) \otimes \mathbb{Q}$ with lifting $z_x \in \mathrm{Pic}(\mathcal{Y}_{\mathcal{Z}}) \otimes \mathbb{Q}$ and write

$$z_{\mathfrak{t}} := i_{\mathfrak{t}}^*(z_x) \in \mathrm{Pic}(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q} \quad \text{and} \quad \epsilon_{\mathfrak{t}} = sp_{x,\mathfrak{t}}(\epsilon_x) = c_{\mathcal{Y}_{\mathfrak{t}}} i_{\mathfrak{t}}^*(z_x) \in \mathrm{NS}(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q}.$$

By construction $\epsilon_{\mathfrak{t}}$ is in the image of

$$(iii) : H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W})^{F=q} \otimes \mathbb{Q} \xrightarrow{\sim} H^0(\mathcal{Z}, R^2\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W})^{F=q} \otimes \mathbb{Q}.$$

Moreover $\epsilon_{\mathfrak{t}} = c_{\mathcal{Y}_{\mathfrak{t}}} i_{\mathfrak{t}}^*(z_x)$ and so, by Fact 4.1.6.1.1 applied to $\mathfrak{f} : \mathcal{Y} \to \mathcal{Z}$ and $\mathfrak{t}$, there exists $z \in \mathrm{Pic}(\mathcal{Y}) \otimes \mathbb{Q}$ such that $i_{\mathfrak{t}}^* c_{\mathcal{Y}}(z) = \epsilon_{\mathfrak{t}}$. Let ϵ_η be the image of z in $\mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q}$. By construction and the commutativity of the diagram one has:

$$sp_{x,\mathfrak{t}}(sp_{\eta,x}(\epsilon_\eta)) = sp_{\eta,\mathfrak{t}}(\epsilon_\eta) = \epsilon_{\mathfrak{t}} = sp_{x,\mathfrak{t}}(\epsilon_x)$$

Since $sp_{x,\mathfrak{t}}$ is injective, this concludes the proof of Theorem 4.1.4.2.2 when f is projective.

4.3.3 Proof when f is proper

Assume now that $f : Y \to X$ is only proper. Since Fact 4.1.6.1.1 is only available when f is projective, we cannot longer apply it directly to $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$. To overcome this difficulty we proceed as follows. Using De Jong's alteration theorem and replacing X with a dense open subset, one first constructs a commutative diagram

$$
\begin{tikzcd}
\widetilde{Y} \arrow[rr, "g"] \arrow[dr, "\widetilde{f}"'] & & Y \arrow[dl, "f"] \\
& X &
\end{tikzcd}
$$

with $\widetilde{f}$ smooth projective and g dominant and generally finite. While every $x \in X$ which is NS-generic for $\widetilde{f}$ is NS-generic for f (as the argument in 4.3.3.4 shows), the hypothesis of being Galois generic for f does not transfer to $\widetilde{f}$ in general, so that one cannot reduce directly the assertion for the (proper) morphism $f : Y \to X$ to the assertion for the (projective) morphism $\widetilde{f} : \widetilde{Y} \to X$. However, the trace formalism is functorial enough to allow us to transfer information from $f : Y \to X$ to $\widetilde{f} : \widetilde{Y} \to X$ for cohomology classes coming from Y.

4.3.3.1 Step 1: De Jong's alterations theorem

First one reduces to the situation where$f$ has geometrically connected fibres (this hypothesis is used in 4.3.3.4 to apply Poincare duality). By [SGA1, X, Proposition 1.2] and replacing X with a finite étale cover (4.3.1.2), one can assume that $f : Y \to X$ decomposes in a disjoint union of morphisms $f_i : Y_i \to X$ with geometrically connected fibres. Since for every (not necessarily closed) point $x \in X$ there are natural decompositions

$$\begin{tikzcd}
\mathrm{NS}(Y_{\bar{x}}) \otimes \mathbb{Q} \arrow[r, "c_{Y_{\bar{x}}}"] \arrow[d, "\simeq"] & H^2(Y_{\bar{x}}, \mathbb{Q}_\ell(i)) \arrow[d, "\simeq"] \\
\oplus_i \mathrm{NS}(Y_{i,\bar{x}}) \otimes \mathbb{Q} \arrow[r, "c_{Y_{\bar{x}}}"] & \oplus_i H^2(Y_{i,\bar{x}}, \mathbb{Q}_\ell(i))
\end{tikzcd}$$

one may work with each $f_i : Y_i \to X$ separately and hence assume that $f : Y \to X$ has geometrically connected fibres.

By De Jong's alterations theorem ([dJ96]) for $Y_{\bar{\eta}}$ over $\overline{k(\eta)}$, there exists a proper, surjective and generically finite morphism $\widetilde{Y}_{\bar{\eta}} \to Y_{\bar{\eta}}$, where $\widetilde{Y}_{\bar{\eta}}$ is a connected, smooth and projective $\overline{k(\eta)}$-variety. By descent and spreading out, there exists a commutative diagram of connected smooth k-varieties:

$$\begin{tikzcd}
\widetilde{Y}_{\bar{\eta}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Y}_{\eta'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Y} \arrow[d, "g"] \arrow[dd, bend right, "\widetilde{f}"'] & & \\
Y_{\bar{\eta}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y_{\eta'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y_{U'} \arrow[r] \arrow[d, "f_{U'}"] \arrow[dr, phantom, "\square"] & Y_U \arrow[r] \arrow[d, "f_U"] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\
\overline{k(\eta)} \arrow[r] & k(\eta') \arrow[r, "\eta'"] & U' \arrow[r, "j"] & U \arrow[r, "i"] & X
\end{tikzcd}$$

where $\eta' : k(\eta') \to U'$ is the generic point of U', $i : U \to X$ is a open immersion with dense image, $j : U' \to U$ is a finite surjective morphism, $\widetilde{f} : \widetilde{Y} \to U'$ is smooth, projective with geometrically connected fibres and $g : \widetilde{Y} \to Y_{U'}$ is proper, surjective and generically finite. In conclusion, replacing X with U' (4.3.1.2), one can assume that there exists a diagram

$$\begin{tikzcd}
\widetilde{Y} \arrow[rr, "g"] \arrow[dr, "\widetilde{f}"'] & & Y \arrow[dl, "f"] \\
& X &
\end{tikzcd}$$

where $\widetilde{f} : \widetilde{Y} \to X$ is smooth projective with geometrically connected fibres, $f : Y \to X$ is smooth proper with geometrically connected fibres and $g : \widetilde{Y} \to Y$ is generically finite and dominant.

4.3.3.2 Step 2: Spreading out

Now one spreads out to finite fields. Up to replacing k with a finite field extension (4.3.1.2), there exists a finite field $\mathbb{F}_q$, smooth and geometrically connected $\mathbb{F}_q$-varieties $\mathcal{K}$,$\mathcal{Z}$ with generic points $\zeta : k \to \mathcal{K}$, $\beta : k(x) \to \mathcal{Z}$ and a commutative cartesian diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{Y}}_{\mathfrak{t}} \arrow[r] \arrow[d, "\mathfrak{g}_{\mathfrak{t}}"] \arrow[dd, bend right, "\widetilde{\mathfrak{f}}_{\mathfrak{t}}"'] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{Y}}_{\mathcal{Z}} \arrow[r] \arrow[d, "\mathfrak{g}_{\mathcal{Z}}"] \arrow[dd, bend right, "\widetilde{\mathfrak{f}}_{\mathcal{Z}}"'] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{Y}} \arrow[d, "\mathfrak{g}"] \arrow[dd, bend right, "\widetilde{\mathfrak{f}}"'] \arrow[dr, phantom, "\square"] & \widetilde{Y} \arrow[l] \arrow[d, "g"] \arrow[dd, bend right, "\widetilde{f}"'] \arrow[dr, phantom, "\square"] & \widetilde{Y}_x \arrow[l] \arrow[d, "g_x"] \\
\mathcal{Y}_{\mathfrak{t}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}_{\mathcal{Z}} \arrow[r] \arrow[d, "\mathfrak{f}_{\mathcal{Z}}"] \arrow[dr, phantom, "\square"] & \mathcal{Y} \arrow[d, "\mathfrak{f}"] \arrow[dr, phantom, "\square"] & Y \arrow[l] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Y_x \arrow[l] \arrow[d, "f_x"] \\
\mathbb{F}_q \arrow[r, "\mathfrak{t}"] \arrow[dr, equal] & \mathcal{Z} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[l] \arrow[d] & k(x) \arrow[l, "x"'] \arrow[lll, bend left, "\beta"] \\
 & \mathbb{F}_q & \mathcal{K} \arrow[l] & k \arrow[l, "\zeta"]
\end{tikzcd}$$

where $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ is smooth proper with geometrically connected fibres, $\widetilde{\mathfrak{f}} : \widetilde{\mathcal{Y}} \to \mathcal{X}$ is smooth projective with geometrically connected fibres, $\mathfrak{g} : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ is a dominant generically finite morphism and the base change of $\widetilde{\mathcal{Y}}_{\mathcal{Z}} \to \mathcal{Y}_{\mathcal{Z}} \to \mathcal{Z}$ along $k(x) \to \mathcal{Z}$ identifies with $\widetilde{Y}_x \to Y_x \to k(x)$. Replacing X with a finite étale cover (4.3.1.2) one can also assume that

1. $\mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Q} = \mathrm{NS}(Y_x) \otimes \mathbb{Q}$, $\mathrm{NS}(\mathcal{Y}_{\overline{\mathfrak{t}}}) \otimes \mathbb{Q} = \mathrm{NS}(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q}$, $\mathrm{NS}(\widetilde{\mathcal{Y}}_{\mathfrak{t}}) \otimes \mathbb{Q} = \mathrm{NS}(\widetilde{\mathcal{Y}}_{\overline{\mathfrak{t}}}) \otimes \mathbb{Q}$;
2. the Zariski closures of the images of $\pi_1(\mathcal{X}) \to \mathrm{GL}(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)))$ and $\pi_1(\mathcal{X}_{\mathbb{F}}) \to \mathrm{GL}(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)))$ are connected.

Note that, by smooth proper base change, one has the following factorization

$$\begin{tikzcd}
\pi_1(x) \arrow[r] \arrow[d, two heads] & \pi_1(X) \arrow[r] \arrow[d, two heads] & \mathrm{GL}(H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))) \simeq \mathrm{GL}(H^2(\mathcal{Y}_{\overline{t}}, \mathbb{Q}_\ell(1))) \\
\pi_1(\mathcal{Z}) \arrow[r] & \pi_1(\mathcal{X}) \arrow[ur]
\end{tikzcd}$$

In particular, since x is Galois generic for $f : Y \to X$, the image of $\pi_1(\mathcal{Z}) \to \pi_1(\mathcal{X}) \to \mathrm{GL}(H^2(\mathcal{Y}_{\overline{t}}, \mathbb{Q}_\ell(1)))$ is open in the image of $\pi_1(\mathcal{X}) \to \mathrm{GL}(H^2(\mathcal{Y}_{\overline{t}}, \mathbb{Q}_\ell(1)))$. Hence by (2) and Theorem 4.1.6.3.1 the base change map

$$H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W})^{F=q} \otimes \mathbb{Q} \to H^0(\mathcal{Z}, R^2\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W})^{F=q} \otimes \mathbb{Q}$$

is an isomorphism.

4.3.3.3 Step 3: Using the Variational Tate conjecture

Take an ϵ_x in $\mathrm{NS}(Y_x) \otimes \mathbb{Q}$. The goal of this subsection is to prove that there exists a $\widetilde{\epsilon}_\eta \in \mathrm{NS}(\widetilde{Y}_{\overline{\eta}}) \otimes \mathbb{Q}$ such that $\widetilde{sp}_{\eta,x}(\widetilde{\epsilon}_\eta) = g^*(\epsilon_x)$, where

$$\widetilde{sp}_{\eta,x} : \mathrm{NS}(\widetilde{Y}_{\overline{\eta}}) \otimes \mathbb{Q} \to \mathrm{NS}(\widetilde{Y}_{\overline{x}}) \otimes \mathbb{Q}$$

is the specialization map for $\widetilde{f} : \widetilde{Y} \to X$. Consider the commutative diagram in 4.3.2.2. Let $z_x \in \mathrm{Pic}(\mathcal{Y}_{\mathcal{Z}}) \otimes \mathbb{Q}$ be a lift of ϵ_x and write

$$z_{\mathfrak{t}} := i_{\mathfrak{t}}^*(z_x) \in \mathrm{Pic}(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q} \quad \text{and} \quad \epsilon_{\mathfrak{t}} = sp_{x,\mathfrak{t}}(\epsilon_x) = c_{\mathcal{Y}_{\mathfrak{t}}} i_{\mathfrak{t}}^*(z_x) \in \mathrm{NS}(\mathcal{Y}_{\mathfrak{t}}) \otimes \mathbb{Q}.$$

By construction $\epsilon_{\mathfrak{t}}$ is in the image of

$$(iii) : H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W})^{F=q} \otimes \mathbb{Q} \xrightarrow{\sim} H^0(\mathcal{Z}, R^2\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W})^{F=q} \otimes \mathbb{Q}.$$

Moreover $\epsilon_{\mathfrak{t}} = c_{\mathcal{Y}_{\mathfrak{t}}} i_{\mathfrak{t}}^*(z_x)$. Since $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ is only assumed to be proper, one cannot apply directly Fact 4.1.6.1.1 to it. However the previous reasoning shows that

$$H^0(\mathcal{X}, R^2\widetilde{\mathfrak{f}}_{*,crys}\mathcal{O}_{\widetilde{\mathcal{Y}}/W})\otimes\mathbb{Q} \supseteq \mathfrak{g}^*(H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W})\otimes\mathbb{Q}) \ni \mathfrak{g}_{\mathfrak{t}}^*(\epsilon_{\mathfrak{t}}) = \mathfrak{g}_{\mathfrak{t}}^*(c_{\mathcal{Y}_{\mathfrak{t}}} i_{\mathfrak{t}}^*(z_x)) = c_{\widetilde{\mathcal{Y}}_{\mathfrak{t}}} \widetilde{i}_{\mathfrak{t}}^* \mathfrak{g}_{\mathcal{Z}}^*(z_x),$$

where the notation is as in the canonical commutative diagram:

$$\begin{tikzcd}
\widetilde{z} \in \mathrm{Pic}(\widetilde{\mathcal{Y}}) \otimes \mathbb{Q} \arrow[rr] \arrow[d, "\widetilde{i}_\eta"] \arrow[dr, "\widetilde{i}_{\mathfrak{t}}^*"] \arrow[dddd, bend right, "c_{\widetilde{\mathcal{Y}}}"'] & & \mathrm{Pic}(\widetilde{\mathcal{Y}}_{\mathcal{Z}}) \otimes \mathbb{Q} \ni \mathfrak{g}_{\mathcal{Z}}^*(z_x) \arrow[dl, "\widetilde{i}_{\mathfrak{t}}^*"'] \arrow[d, two heads, "(i)"] \arrow[dddd, bend left, "c_{\widetilde{\mathcal{Y}}_{\mathcal{Z}}}"] \\
\mathrm{Pic}(\widetilde{Y}_\eta) \otimes \mathbb{Q} \arrow[d] & \mathrm{Pic}(\widetilde{\mathcal{Y}}_{\mathfrak{t}}) \otimes \mathbb{Q} \ni z_t \arrow[dd, two heads] \arrow[dddd, bend left, "c_{\widetilde{\mathcal{Y}}_{\mathfrak{t}}}"] & \mathrm{Pic}(\widetilde{Y}_x) \otimes \mathbb{Q} \arrow[d, two heads, "(ii)"] \\
\widetilde{\epsilon}_\eta \in \mathrm{NS}(\widetilde{Y}_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[rr, hook, "\widetilde{sp}_{\eta,x}"] \arrow[dr, hook, "\widetilde{sp}_{\eta,\mathfrak{t}}"] & & \mathrm{NS}(\widetilde{Y}_{\overline{x}}) \otimes \mathbb{Q} \ni g^*(\epsilon_x) \arrow[dl, hook', "\widetilde{sp}_{x,\mathfrak{t}}"'] \\
& \mathrm{NS}(\widetilde{\mathcal{Y}}_{\overline{\mathfrak{t}}}) \otimes \mathbb{Q} \ni \mathfrak{g}_{\mathfrak{t}}(\epsilon_{\mathfrak{t}}) \arrow[dd, hook] & \\
H^2_{crys}(\widetilde{\mathcal{Y}}) \arrow[dr, "\widetilde{i}_{\mathfrak{t}}^*"] \arrow[dd, "\mathrm{Ler}"'] & & H^2_{crys}(\widetilde{\mathcal{Y}}_{\mathcal{Z}}) \arrow[dl, "\widetilde{i}_{\mathfrak{t}}^*"'] \arrow[dd, "\mathrm{Ler}"] \\
& H^2_{crys}(\widetilde{\mathcal{Y}}_{\mathfrak{t}}) & \\
\mathfrak{g}^*(\epsilon_{\mathfrak{t}}) \in H^0(\mathcal{X}, R^2\widetilde{\mathfrak{f}}_{crys,*}\mathcal{O}_{\widetilde{\mathcal{Y}}/W}) \otimes \mathbb{Q} \arrow[ur, hook] \arrow[rr, hook] & & H^0(\mathcal{Z}, R^2\widetilde{\mathfrak{f}}_{\mathcal{Z},crys,*}\mathcal{O}_{\widetilde{\mathcal{Y}}_{\mathcal{Z}}/W}) \otimes \mathbb{Q} \ni \mathfrak{g}_{\mathcal{Z}}^*(\epsilon_{\mathfrak{t}}) \arrow[ul, hook'] \\
\epsilon_{\mathfrak{t}} \in H^0(\mathcal{X}, R^2\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/W}) \otimes \mathbb{Q} \arrow[u, "\mathfrak{g}^*"] \arrow[rr] & & H^0(\mathcal{Z}, R^2\mathfrak{f}_{\mathcal{Z},crys,*}\mathcal{O}_{\mathcal{Y}_{\mathcal{Z}}/W}) \otimes \mathbb{Q} \ni \epsilon_{\mathfrak{t}} \arrow[u, "\mathfrak{g}_{\mathcal{Z}}^*"']
\end{tikzcd}$$

So, by Fact 4.1.6.1.1 applied to $\widetilde{\mathfrak{f}} : \widetilde{\mathcal{Y}} \to \mathcal{X}$ and $\mathfrak{t}$, there exists $\widetilde{z} \in \mathrm{Pic}(\widetilde{\mathcal{Y}}) \otimes \mathbb{Q}$ such that $\mathfrak{g}_{\mathfrak{t}}^*(\epsilon_{\mathfrak{t}}) = \widetilde{i}_{\mathfrak{t}}^* c_{\widetilde{\mathcal{Y}}}(\widetilde{z})$. Write $\widetilde{\epsilon}_\eta := \widetilde{i}_\eta^* c_{\widetilde{\mathcal{Y}}}(\widetilde{z})$. From the commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(\widetilde{\mathcal{Y}}) \otimes \mathbb{Q} \arrow[r, "\widetilde{i}_\eta^*"] \arrow[dd] & \mathrm{Pic}(\widetilde{Y}_\eta) \otimes \mathbb{Q} \arrow[r, "c_{\widetilde{Y}_\eta}"] & \mathrm{NS}(\widetilde{Y}_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[dr, "\widetilde{sp}_{\eta,x}"] \arrow[dd, "\widetilde{sp}_{\eta,\mathfrak{t}}"] & \\
& & & \mathrm{NS}(\widetilde{Y}_{\overline{x}}) \arrow[dl, "\widetilde{sp}_{x,\mathfrak{t}}"] \\
\mathrm{Pic}(\widetilde{\mathcal{Y}}_{\mathfrak{t}}) \arrow[rr, "c_{\widetilde{\mathcal{Y}}_{\mathfrak{t}}}"] & & \mathrm{NS}(\widetilde{\mathcal{Y}}_{\overline{\mathfrak{t}}}) \otimes \mathbb{Q} &
\end{tikzcd}$$

one deduces

$$\widetilde{sp}_{x,\mathfrak{t}}(\widetilde{sp}_{\eta,x}(\widetilde{\epsilon}_\eta)) = \widetilde{sp}_{x,\mathfrak{t}}(g^*\epsilon_x).$$

Since $\widetilde{sp}_{x,\mathfrak{t}}$ is injective this implies

$$\widetilde{sp}_{\eta,x}(\widetilde{\epsilon}_\eta) = g^*(\epsilon_x).$$

4.3.3.4 Step 4: Trace argument

To conclude the proof one has to descend from $\widetilde{Y}$ to Y. For this we use the trace formalism. Since $f : Y \to X$ and $\widetilde{f} : \widetilde{Y} \to X$ are smooth proper morphisms with geometrically connected fibres, by the relative Poincaré duality ([SGA4, Exposé XVIII]), there are canonical isomorphisms

$$R^2 f_*\mathbb{Q}_\ell \simeq (R^{2d-2} f_*\mathbb{Q}_\ell(d))^\vee \quad \text{and} \quad R^2 \widetilde{f}_*\mathbb{Q}_\ell \simeq (R^{2d-2}\widetilde{f}_*\mathbb{Q}_\ell(d))^\vee,$$

where $d = Dim(Y_x) = Dim(\widetilde{Y}_x)$. Dualizing and twisting the base change map

$$R^{2d-2} f_*\mathbb{Q}_\ell(d) \to R^{2d-2}\widetilde{f}_*\mathbb{Q}_\ell(d),$$

one gets a morphism

$$g_* : R^2\widetilde{f}_*\mathbb{Q}_\ell(1) \simeq (R^{2d-2}\widetilde{f}_*\mathbb{Q}_\ell(d))^\vee(1) \to (R^{2d-2} f_*\mathbb{Q}_\ell(d))^\vee(1) \simeq R^2 f_*\mathbb{Q}_\ell(1).$$

By the compatibility of Poincaré duality with base change, for every (not necessarily closed) $x \in X$, the fibre of g_* at $\overline{x}$ is the usual push forward map $g_{x,*} : H^2(\widetilde{Y}_{\overline{x}}, \mathbb{Q}_\ell(1)) \to H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1))$ in étale cohomology. In particular it is compatible with the push forward of algebraic cycles $g_{x,*} : \mathrm{Pic}(\widetilde{Y}_{\overline{x}}) \otimes \mathbb{Q} \to \mathrm{Pic}(Y_{\overline{x}}) \otimes \mathbb{Q}$. Since g^* and g_* are maps of sheaves, they are compatible with the specialization isomorphisms and hence the following canonical diagram commutes:

$$\begin{tikzcd}
\mathrm{Pic}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[r, "g^*"] \arrow[d, "c_{Y_{\overline{\eta}}}"] & \mathrm{Pic}(\widetilde{Y}_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[r, "g_*"] \arrow[d, "c_{\widetilde{Y}_{\overline{\eta}}}"] & \mathrm{Pic}(Y_{\overline{\eta}}) \otimes \mathbb{Q} \arrow[d, "c_{Y_{\overline{\eta}}}"] \\
H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)) \arrow[r, "g^*"] \arrow[d, "sp_{\eta,x}"] & H^2(\widetilde{Y}_{\overline{\eta}}, \mathbb{Q}_\ell(1)) \arrow[r, "g_*"] \arrow[d, "sp'_{\eta,x}"] & H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1)) \arrow[d, "sp_{\eta,x}"] \\
H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)) \arrow[r, "g^*"] & H^2(\widetilde{Y}_{\overline{x}}, \mathbb{Q}_\ell(1)) \arrow[r, "g_*"] & H^2(Y_{\overline{x}}, \mathbb{Q}_\ell(1)) \\
\mathrm{Pic}(Y_{\overline{x}}) \otimes \mathbb{Q} \arrow[r, "g^*"] \arrow[u, "c_{Y_{\overline{x}}}"] & \mathrm{Pic}(\widetilde{Y}_{\overline{x}}) \otimes \mathbb{Q} \arrow[r, "g_*"] \arrow[u, "c_{\widetilde{Y}_{\overline{x}}}"] & \mathrm{Pic}(Y_{\overline{x}}) \otimes \mathbb{Q} \arrow[u, "c_{Y_{\overline{x}}}"]
\end{tikzcd}$$

Since the the horizontal arrows are the multiplication by $n := deg(g)$, one concludes observing that $g_* c_{\widetilde{Y}_{\overline{\eta}}}(\widetilde{\epsilon}_\eta) \in \mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Q}$ is such that $sp_{\eta,x}(\frac{1}{n} g_* c_{\widetilde{Y}_{\overline{\eta}}}(\widetilde{\epsilon}_\eta)) = \epsilon_x$.

4.4 Hyperplane sections

Let k be an infinite finitely generated field of characteristic $p > 0$. In this section we apply Theorem 4.1.4.2.2 to Lefschetz pencils of hyperplane sections. The main result is Corollary 4.1.7.2.1.

4.4.1 Geometric versus arithmetic hyperplane sections

Let Z be a smooth projective k-variety and fix a closed embedding $Z \subseteq \mathbb{P}^n_k$. One can ask whether there exists a smooth hyperplane section D of Z such that the canonical map

$$i_{D_{\overline{k}}} := \mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q} \to \mathrm{NS}(D_{\overline{k}}) \otimes \mathbb{Q}$$

is an isomorphism. If $Dim(Z) = 2$, then D is a curve so that $\mathrm{NS}(D_{\overline{k}}) \otimes \mathbb{Q} = \mathbb{Q}$ hence $i_{D_{\overline{k}}}$ is not injective as soon as $\mathrm{NS}(Z_{\overline{k}}) \otimes \mathbb{Q}$ has rank ≥ 2. Weak Lefschetz ([Mil80, Thm. 7.1, p. 318]) and Grothendieck–Lefschetz ([SGA2, Exp. XI]) theorems ensure that $i_{D_{\overline{k}}}$ is injective if $Dim(Z) \geq 3$, and an isomorphism if $Dim(Z) \geq 4$. There are smooth projective varieties of dimension 3 such that the surjectivity of $i_{D_{\overline{k}}}$ fails for all smooth hyperplane sections.

Example 4.4.1.1. Take $Z = \mathbb{P}^3_k$ embedded in $\mathbb{P}^9_k$ via the Veronese embedding:

$$\begin{aligned} \mathbb{P}^3_k &\to \mathbb{P}^9_k \\ [x:y:z:w] &\mapsto [x^2:y^2:z^2:w^2:xy:xz:xw:yz:yw:zw]. \end{aligned}$$

Then a smooth hyperplane section $D \subseteq Z$ in $\mathbb{P}^9_k$ is a smooth quadric surface in $\mathbb{P}^3_k$, so that $D_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$. Hence $\mathrm{NS}(D_{\overline{k}}) \simeq \mathbb{Z} \times \mathbb{Z}$, while $\mathrm{NS}(Z_{\overline{k}}) = \mathbb{Z}$.

But things change if one replaces the geometric Néron-Severi groups with their arithmetic counterparts.

Example 4.4.1.2. Assume $p \geq 17$ and consider the pencil of hyperplane sections of Z in $\mathbb{P}^9_k$ given by the hyperplanes $a(x_1+x_2+x_3+x_4)+b(x_1+4x_2+9x_3+16x_4)$, where $[a:b] \in \mathbb{P}^1(k)$ and $x_1, ..., x_{10}$ are the coordinates in $\mathbb{P}^9_k$. This corresponds in $\mathbb{P}^3$ to the pencil of quadric surfaces

$$Q_{[a,b]} : a(x^2+y^2+z^2+w^2) + b(x^2+4y^2+9z^2+16w^2) = 0.$$

When $Q_{[a:b]}$ is smooth, it a is quadric surface and, for a $[a:b]$ in an open subset of $\mathbb{P}^1_k$

$$\mathrm{NS}(D) \otimes \mathbb{Q} \simeq (\mathrm{NS}(D_{\overline{k}}) \otimes \mathbb{Q})^{\pi_1(k)} = (\mathbb{Q}^2)^{\pi_1(k)} = \mathbb{Q}$$

is generated by the sum of the two families of $\mathbb{P}^1_{\overline{k}}$. So there are "lots" of $[a:b] \in \mathbb{P}^1(k)$ such that the canonical map

$$\mathrm{NS}(\mathbb{P}^3_k) \otimes \mathbb{Q} \to \mathrm{NS}(Q_{[a:b]}) \otimes \mathbb{Q}$$

is an isomorphism.

The main result is of this subsection is Corollary 4.1.7.2.1 that we now recall:

Corollary. *If $\dim(Z) > 2$ there are infinitely many k-rational hyperplane sections D such that the canonical map*

$$\mathrm{NS}(Z) \otimes \mathbb{Q} \to \mathrm{NS}(D) \otimes \mathbb{Q}$$

is an isomorphism.

4.4.2 Proof of Corollary 4.1.7.2.1

By ([SGA7, Exp. XVII]) there exists a pencil of hyperplanes $L := \{H_x\}_{x \in \check{\mathbb{P}}^1_k}$ such that:

- For all x in an dense open subscheme $U \subseteq \check{\mathbb{P}}^1_k$, the intersection $H_x \cap Z$ is smooth;
- The base locus $B := \cap_{x \in L} Z \cap H_x \subseteq Z$ is smooth.

Then one gets a diagram

$$Z \xleftarrow{\pi} \widetilde{Z} \xrightarrow{f} \check{\mathbb{P}}^1_k$$

where $\pi : \widetilde{Z} \to Z$ is the blow up of Z along B, $f : \widetilde{Z} \to \check{\mathbb{P}}^1_k$ is a projective flat morphism, smooth over U and for each $x \in \check{\mathbb{P}}^1_k$ the fibre $\widetilde{Z}_x$ of $f : \widetilde{Z} \to \check{\mathbb{P}}^1_k$ at x identifies via $\pi : \widetilde{Z} \to Z$ with the hyperplane section $Z \cap H_x \subseteq Z$. Write $E := \pi^{-1}(B)$ for the exceptional divisor. Explicitly $\widetilde{Z}$ is the closed subscheme of $Z \times \check{\mathbb{P}}^1_k$ defined by

$$\widetilde{Z} := \{(z, x) \in Z \times \check{\mathbb{P}}^1_k \text{ with } z \in Z \cap H_x\} \hookrightarrow Z \times \check{\mathbb{P}}^1_k$$

$\pi : \widetilde{Z} \to Z$, $f : \widetilde{Z} \to \check{\mathbb{P}}^1_k$ identify with the canonical projections onto Z, $\check{\mathbb{P}}^1_k$ respectively and E with $B \times \check{\mathbb{P}}^1_k$. Write η for the generic point of $\check{\mathbb{P}}^1_k$. Combing Lemma 4.4.2.1 below with Corollary 4.1.7.1.2 one gets Corollary 4.1.7.2.1.

Lemma 4.4.2.1. If $Dim(Z) > 2$, the canonical map

$$i^*_{\widetilde{Z}_\eta} : \mathrm{NS}(Z) \otimes \mathbb{Q} \to \mathrm{NS}(\widetilde{Z}_\eta) \otimes \mathbb{Q}$$

is an isomorphism.

Proof. This is inspired from [Mor15, Corollary 1.5]. Fix $x \in U$. The natural commutative diagram

$$\begin{tikzcd}
 & \widetilde{Z}_\eta \arrow[dl, "i_{\widetilde{Z}_\eta}"'] \arrow[d] \arrow[dd, bend left, "i_\eta"] \arrow[r] \arrow[dr, phantom, "\square"] & k(\eta) \arrow[d, "\eta"] \arrow[dd, bend left] \\
Z & \widetilde{Z} \arrow[l, "\pi"'] \arrow[r, "f"] & \check{\mathbb{P}}^1_k \\
 & \widetilde{Z}_U \arrow[u, "i_U"] \arrow[r, "f_U"] & U \arrow[u] \\
 & \widetilde{Z}_x = Z \cap H_x \arrow[uuul, "i_{\widetilde{Z}_x}"] \arrow[u, "i_x"] \arrow[r] \arrow[ur, phantom, "\square"] & k(x) \arrow[u, "x"']
\end{tikzcd}$$

induces a commutative diagram:

$$\begin{tikzcd}
\mathrm{Pic}(Z) \otimes \mathbb{Q} \arrow[r, "\pi^*"] \arrow[d, two heads, "c_Z"] & \mathrm{Pic}(\widetilde{Z}) \otimes \mathbb{Q} \arrow[r, two heads, "i_U^*"] \arrow[d, two heads, "c_{\widetilde{Z}}"] & \mathrm{Pic}(\widetilde{Z}_U) \otimes \mathbb{Q} \arrow[r, "i_x^*"] \arrow[rr, bend left, two heads, "i_\eta^*"] & \mathrm{Pic}(\widetilde{Z}_x) \otimes \mathbb{Q} \arrow[d, two heads, "c_{\widetilde{Z}_x}"] & \mathrm{Pic}(\widetilde{Z}_\eta) \otimes \mathbb{Q} \arrow[d, two heads, "c_{\widetilde{Z}_\eta}"] \\
\mathrm{NS}(Z) \otimes \mathbb{Q} \arrow[r, "\pi^*"] \arrow[rrr, bend right, "i^*_{\widetilde{Z}_x}"'] & \mathrm{NS}(\widetilde{Z}) \otimes \mathbb{Q} \arrow[rr] & & \mathrm{NS}(\widetilde{Z}_x) \otimes \mathbb{Q} & \mathrm{NS}(\widetilde{Z}_\eta) \otimes \mathbb{Q} \arrow[l, hook', "sp^{ar}_{\eta,x}"]
\end{tikzcd}$$

Since $Dim(Z) > 2$, the map

$$i^*_{\widetilde{Z}_\eta} : \mathrm{NS}(Z) \otimes \mathbb{Q} \to \mathrm{NS}(\widetilde{Z}_\eta) \otimes \mathbb{Q}$$

is injective. To prove the surjectivity, let $\epsilon \in \mathrm{NS}(\widetilde{Z}_\eta) \otimes \mathbb{Q}$ with lift $z \in \mathrm{Pic}(\widetilde{Z}_\eta) \otimes \mathbb{Q}$. Since $sp^{ar}_{\eta,x}$ is injective, it is enough to show that $\epsilon_x := sp^{ar}_{\eta,x}(\epsilon)$ is in the image of $i^*_{\widetilde{Z}_x}$. Since the maps i^*_U and i^*_η are surjective, $z \in \mathrm{Pic}(\widetilde{Z}_\eta)$ lifts to a $\widetilde{z} \in \mathrm{Pic}(\widetilde{Z}) \otimes \mathbb{Q}$ and, by the commutativity of the diagram, $\widetilde{z}$ maps to ϵ_x in $\mathrm{NS}(\widetilde{Z}_x) \otimes \mathbb{Q}$. Now, since π is the blow up of Z along B, $c_{\widetilde{Z}}(\widetilde{z})$ can be written as $\pi^* c_Z(z') + b c_{\widetilde{Z}}(E)$, where $z' \in \mathrm{Pic}(Z) \otimes \mathbb{Q}$ and $b \in \mathbb{Q}$. The conclusion follows from the following claim, since it implies that ϵ_x is the image of $c_Z(z') + b c_Z(Z \cap H_x) \in \mathrm{NS}(Z) \otimes \mathbb{Q}$.
Claim: *The restrictions of $c_{\widetilde{Z}}(E)$ and $\pi^* c_Z(Z \cap H_x)$ to $\widetilde{Z}_x$ coincide.*

Proof of the claim. By direct computations, one sees that $E = B \times \check{\mathbb{P}}^1_k$ intersects transversally $\widetilde{Z}_x$ and that $E \cap \widetilde{Z}_x = B$, so that the restriction of $c_{\widetilde{Z}}(E)$ to $\widetilde{Z}_x$ is given by $c_{\widetilde{Z}_x}(B) \in \mathrm{Pic}(\widetilde{Z}_x) \otimes \mathbb{Q}$. To compute the restriction $\pi^* c_Z(Z \cap H_x)$ to $\widetilde{Z}_x$ observe that it is equal to $i^*_{\widetilde{Z}_x}(c_Z(Z \cap H_x))$. Then take any $y \neq x \in L$ and compute $i^*_{\widetilde{Z}_x}(c_Z(Z \cap H_x))$ as $c_{\widetilde{Z}_x}(Z \cap H_x \cap H_y) = c_{\widetilde{Z}_x}(B)$, since $B = Z \cap H_x \cap H_y$ for any $x \neq y \in L$. □

Remark 4.4.2.2. The key fact that $\mathrm{Pic}(\widetilde{Z}) \otimes \mathbb{Q} \to \mathrm{Pic}(\widetilde{Z}_\eta) \otimes \mathbb{Q}$ is surjective, does not hold for $\mathrm{Pic}(\widetilde{Z}_{\overline{k}}) \otimes \mathbb{Q} \to \mathrm{Pic}(\widetilde{Z}_{\overline{\eta}}) \otimes \mathbb{Q}$ (see Example 4.4.1.2). This is why it is not true in general that for a point $x \in |U|$, which is Galois-generic for $\widetilde{Z}_U \to U$ the canonical map

$$i_{\widetilde{Z}_x} : \mathrm{NS}(\widetilde{Z}_x) \otimes \mathbb{Q} \to \mathrm{NS}(Z) \otimes \mathbb{Q}$$

is an isomorphism and one really needs to restrict to strictly Galois generic points: during the proof one cannot replace U with a finite étale cover, since any base change destroys the geometry of the pencil.

4.4.3 Proof of Corollary 4.1.7.2.2

Replacing k with a finite extension, it is enough to show that the map

$$\mathrm{NS}(Z) \otimes \mathbb{Q}_\ell \to H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k)}$$

is an isomorphism. By Corollary 4.1.7.2.1, there exists k-rational hyperplane section $D \to Z$ such that the canonical map $\mathrm{NS}(Z) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{NS}(D) \otimes \mathbb{Q}_\ell$ is an isomorphism. The conclusion follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{NS}(Z) \otimes \mathbb{Q}_\ell \arrow[r, hook] \arrow[d, "i_D^*"] & H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k)} \arrow[d, hook, "i_D^*"] \\
\mathrm{NS}(D) \otimes \mathbb{Q}_\ell \arrow[r, hook, "(2)"] & H^2(D_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k)}
\end{tikzcd}$$

since i_D^* is an isomorphism by the choice of D and (2) is an isomorphism by $T(D, \ell)$. □

4.5 Brauer groups in families

Let k be a field of transcendence degree ≥ 1 over $\mathbb{F}_p$. We assume that k is finitely generated except in Subsection 4.5.3.

4.5.1 Specialization of Brauer groups

4.5.1.1 Brauer group

For a smooth proper k-variety Z write $H^2(Z_{\overline{k}}, \mathbb{G}_m) := \mathrm{Br}(Z_{\overline{k}})$ for the (cohomological) Brauer group of $Z_{\overline{k}}$, $\mathrm{Br}(Z_{\overline{k}})[n]$ for its n-torsion subgroup and

$$T_\ell(\mathrm{Br}(Z_{\overline{k}})) := \varprojlim_n \mathrm{Br}(Z_{\overline{k}})[\ell^n], \quad \mathrm{Br}(Z_{\overline{k}})[\ell^\infty] := \varinjlim_n \mathrm{Br}(Z_{\overline{k}})[\ell^n], \quad \mathrm{Br}(Z_{\overline{k}})[p'] := \varinjlim_{n \nmid p} \mathrm{Br}(Z_{\overline{k}})[n].$$

Recall that $\mathrm{Br}(Z_{\overline{k}})$ is a torsion group and that Kummer theory induces, for every $p \nmid n \in \mathbb{N}$, an exact sequence:

$$0 \to \mathrm{NS}(Z_{\overline{k}})/n \to H^2(Z_{\overline{k}}, \mu_n) \to \mathrm{Br}(Z_{\overline{k}})[n] \to 0.$$

It is classically known that if $T(Z, \ell)$ holds, then $\mathrm{Br}(Z_{\overline{k}})[\ell^\infty]^{\pi_1(k)}$ is finite (see e.g. [CC18, Proposition 2.1.1]).

4.5.1.2 Brauer generic points

Let X be a smooth geometrically connected k-variety with generic point η and $f : Y \to X$ a smooth proper morphism of k-varieties. Taking the direct limit over $p \nmid n$ on the Kummer exact sequence, one gets a commutative specialization exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_{n \nmid p} \mathrm{NS}(Y_{\overline{\eta}})/n \arrow[r] \arrow[d, hook, "sp_{\eta,x}"] & \varinjlim_{n \nmid p} H^2(Y_{\overline{\eta}}, \mu_n) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Br}(Y_{\overline{\eta}})[p'] \arrow[r] \arrow[d, two heads, "sp^{Br}_{\eta,x}"] & 0 \\
0 \arrow[r] & \varinjlim_{n \nmid p} \mathrm{NS}(Y_{\overline{x}})/n \arrow[r] & \varinjlim_{n \nmid p} H^2(Y_{\overline{x}}, \mu_n) \arrow[r] & \mathrm{Br}(Y_{\overline{x}})[p'] \arrow[r] & 0
\end{tikzcd}$$

Since the group $Ker(sp_{\eta,x})$ is of p-torsion and $Coker(sp_{\eta,x})_{tors} = Coker(sp_{\eta,x})[p^\infty]$ (see [MP12, Proposition 3.6]), one sees that a $x \in |X|$ is NS-generic if and only if the map

$$sp^{Br}_{\eta,x} : \mathrm{Br}(Y_{\overline{\eta}})[p'] \to \mathrm{Br}(Y_{\overline{x}})[p']$$

is an isomorphism. In particular (Corollary 4.1.7.1.2) the set of $x \in |X|$ such that $sp^{Br}_{\eta,x}$ is not an isomorphism is sparse and if X is a curve it contains at most finitely many of k-rational points.

4.5.2 Uniform boundedness

Retain the notation and the assumptions as in the previous Section 4.5.1.2. Taking inverse limit in the Kummer exact sequence, one gets a commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{NS}(Y_{\overline{\eta}}) \otimes \mathbb{Z}_\ell \arrow[r] \arrow[d, hook, "sp_{\eta,x}"] & H^2(Y_{\overline{\eta}}, \mathbb{Z}_\ell(1)) \arrow[r] \arrow[d, "\simeq"] & T_\ell(\mathrm{Br}(Y_{\overline{\eta}})) \arrow[r] \arrow[d, two heads, "sp^{Br}_{\eta,x}"] & 0 \\
0 \arrow[r] & \mathrm{NS}(Y_{\overline{x}}) \otimes \mathbb{Z}_\ell \arrow[r] & H^2(Y_{\overline{x}}, \mathbb{Z}_\ell(1)) \arrow[r] & T_\ell(\mathrm{Br}(Y_{\overline{x}})) \arrow[r] & 0
\end{tikzcd}$$

The group $\pi_1(x, \overline{x})$ acts on $T_\ell(\mathrm{Br}(Y_{\overline{\eta}}))$ by restriction trough the map $\pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}) \simeq \pi_1(X, \overline{\eta})$ and $sp^{Br}_{\eta,x}$ is $\pi_1(x, \overline{x})$-equivariant with respect to the natural action of $\pi_1(x, \overline{x})$ on $T_\ell(\mathrm{Br}(Y_{\overline{x}}))$. Hence the arguments in Section 4.5.1 combined with Theorem 4.1.4.2.2 show the following

Lemma 4.5.2.1. Up to replacing X with an open subset, for every Galois generic $x \in |X|$ and every $\ell \neq p$, the $\pi_1(x, \overline{x})$-equivariant specialization morphism

$$sp^{Br}_{\eta,x} : T_\ell(\mathrm{Br}(Y_{\overline{\eta}})) \to T_\ell(\mathrm{Br}(Y_{\overline{x}}))$$

is an isomorphism

Replacing [CC18, Proposition 3.2.1] with Lemma 4.5.2.1 and [CC18, Fact 3.4.1] with the main result of Chapter 3 (Theorem 3.1.3.2), one can make the arguments in the proof of [CC18, Theorem 1.2.1] work in positive characteristic and prove Corollary 4.1.7.3.1. In the same way, using the arguments in the proof of [CC18, Theorem 3.5.1], one gets the following unconditional variant:

Corollary 4.5.2.2. Let X be a curve and assume that the Zariski closure of the image of $\pi_1(X)$ acting on $H^2(Y_{\overline{\eta}}, \mathbb{Q}_\ell(1))$ is connected. Then there exists an integer $C := C(Y \to X, \ell)$ such that

$$[\mathrm{Br}(Y_{\overline{x}})^{\pi_1(x,\overline{x})}[\ell^\infty] : \mathrm{Br}(Y_{\overline{\eta}})^{\pi_1(X,\overline{\eta})}[\ell^\infty]] \leq C$$

for all but finitely many $x \in X(k)$.

4.5.3 p-adic Tate module

Assume that X is a smooth connected k-variety with generic point η, where k is an algebraically closed field of characteristic p and that $Y \to X$ is a smooth projective morphism.

Corollary 4.5.3.1. There exists an $x \in |X|$ such that $Rank(T_p(\mathrm{Br}(Y_{\overline{x}}))) = Rank(T_p(\mathrm{Br}(Y_{\overline{\eta}})))$

Proof. For every geometric point $\overline{t} \in X$, one has ([Ill79, Proposition 5.12]):

$$Dim(\mathrm{NS}(Y_{\overline{t}}) \otimes \mathbb{Q}_p) = Dim(H^2_{crys}(Y_{\overline{t}})) - 2Dim(H^2_{crys}(Y_{\overline{t}})_{[1]}) - Rank(T_p(\mathrm{Br}(Y_{\overline{t}})))$$

where $H^2_{crys}(Y_{\overline{t}})_{[1]}$ is the slope one part of the crystalline cohomology of $Y_{\overline{t}}$ (see e.g. [Ked17, Section 3] for the definition). By [Ked17, Theorem 3.12, Corollary 4.2] there exists a dense open subset U of X such that for all $x \in |U|$ one has

$$Dim(H^2_{crys}(Y_{\overline{x}})_{[1]}) = Dim(H^2_{crys}(Y_{\overline{\eta}})_{[1]})$$

Since $Dim(H^2_{crys}(Y_{\overline{x}}))$ is independent of $x \in X$ (smooth proper base change in crystalline cohomology), one concludes applying Corollary 4.1.7.1.2 to $Y_U \to U$. □

4.6 Preliminaries for Theorem 4.6.5.4.1

In the next two sections, we use the work of Shiho on relative rigid cohomology to prove Theorem 4.6.5.4.1, which is a key ingredient in the proof of Theorem 4.2.1.1.2. In Section 4.6, we recall the definitions of various categories of isocrystals, the relations between them and we state Theorem 4.6.5.4.1. In Section 4.7, we prove Theorem 4.6.5.4.1.

4.6.1 Notation

Let k be a perfect field of characteristic $p > 0$. Write K for the fraction field of the Witt ring $W := W(k)$ of k and $|-| : K \to \mathbb{R}$ for the norm induced by the ideal $pW \subseteq W$. For any k-variety X, write F_X for a power of the absolute Frobenius on X and, if there is no danger of confusion, one often drops the lower index and writes just F.

Gothic letters ($\mathfrak{T}, \mathfrak{X}, \mathfrak{U}$...) denote separated, p-adic formal schemes over W. Write $\mathfrak{X}_1$ for the special fibre of $\mathfrak{X}$, $\mathfrak{X}_K$ for its rigid analytic generic fibre and $sp : \mathfrak{X}_K \to \mathfrak{X}_1$ for the specialization map. There is an equivalence between the isogeny category $Coh(\mathfrak{X}) \otimes \mathbb{Q}$ of the category $Coh(\mathfrak{X})$ of coherent sheaves on $\mathfrak{X}$ and the category $Coh(\mathfrak{X}_K)$ of coherent sheaves on $\mathfrak{X}_K$ (see [Ogu84, Remark 1.5]).

If $f : X \to \mathfrak{X}_1$ is a closed immersion, one can consider the open tube $]X[_{\mathfrak{X}} := sp^{-1}(X)$ and the closed tube of radius $|p|$, $[X]_{\mathfrak{X},|p|}$ of X in $\mathfrak{X}$ (see [Ber96, Definition 1.1.2, Section 1.1.8]). They are admissible open subsets of $\mathfrak{X}_K$ and there is an inclusion $[X]_{\mathfrak{X},|p|} \subseteq]X[_{\mathfrak{X}}$.

A couple $(X, \overline{X})$ is an open immersion of k-varieties $X \to \overline{X}$ and a frame $(X, \overline{X}, \mathfrak{X})$ is a couple $(X, \overline{X})$ together with a closed immersion of $\overline{X}$ into a p-adic formal schemes $\mathfrak{X}$. Morphisms of couples and frames are defined in the obvious way. A couple $(Y, \overline{Y})$ over a frame $(X, \overline{X}, \mathfrak{X})$ is a morphism of couples $(Y, \overline{Y}) \to (X, \overline{X})$ and a frame $(X, \overline{X}, \mathfrak{X})$ over a couple $(Y, \overline{Y})$ is a morphism of couples $(X, \overline{X}) \to (Y, \overline{Y})$. If $(X, \overline{X}, \mathfrak{X})$ is a frame, for any sheaf $\mathcal{F}$ over $]\overline{X}[_{\mathfrak{X}}$ one writes

$$j_X^\dagger \mathcal{F} := \varinjlim_V j_{V*} j_V^* \mathcal{F}$$

where the limit runs over all the strict neighbourhoods V of X in $\overline{X}$ (see [Ber96, Definition 1.2.1]) and $j_V : V \to]\overline{X}[_{\mathfrak{X}}$ in the inclusion map.

If $f : Y \to X$ is a morphism of k-varieties, for every morphism $Z \to X$ write:

$$\begin{tikzcd} Y_Z \arrow[r] \arrow[d, "f_Z"] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd}$$

4.6.2 Categories of isocrystals

To a k-variety X one can associate the following categories of isocrystals:

- $\mathbf{Isoc}^{(p)}(X|K)$, the p-adically convergent isocrystals (see [Ogu84, Definition 2.1]);

- $\mathbf{Isoc}^{(1)}(X|K)$, the convergent isocrystals (see [Ogu84, Definition 2.1]);

If $(X, \overline{X})$ is a couple there is a category $\mathbf{Isoc}^{\dagger}(X, \overline{X}|K)$ of isocrystal on X overconvergent along $\overline{X} - X$ see [Ber96, Definition 2.3.2]. If $\overline{X}$ is a compactification of X, one writes $\mathbf{Isoc}^{\dagger}(X, \overline{X}|K) := \mathbf{Isoc}^{\dagger}(X|K)$ and calls the object in there overconvergent isocrystals on X. It is known that $\mathbf{Isoc}^{\dagger}(X|K)$ does not depend on the choice of the compactification, so that $\mathbf{Isoc}^{\dagger}(X|K)$ is well defined (see [Ber96, 2.3.6]).

- $\mathbf{Isoc}^{(p)}(X|K)$ (resp. $\mathbf{Isoc}^{(1)}(X|K)$). Write $I_X^{(p)}$ (resp. $I_X^{(1)}$) for the category of p-adic enlargements (resp. enlargements). This is the category of pairs $(\mathfrak{T}, z_{\mathfrak{T}})$ such that $\mathfrak{T}$ is a flat p-adic formal W-scheme and $z_{\mathfrak{T}}$ is a morphism $\mathfrak{T}_1 \to X$ (resp. $(\mathfrak{T}_1)_{red} \to X$). A morphism $g : (\mathfrak{Z}, z_{\mathfrak{Z}}) \to (\mathfrak{T}, z_{\mathfrak{T}})$ between p-adic enlargements (resp. enlargements) is a morphism $g : \mathfrak{Z} \to \mathfrak{T}$ such that $z_{\mathfrak{Z}} \circ g_1 = z_{\mathfrak{T}}$ (resp. $z_{\mathfrak{Z}} \circ (g_1)_{red} = z_{\mathfrak{T}}$), where $g_1 : \mathfrak{Z}_1 \to \mathfrak{T}_1$ (resp. $(g_1)_{red} : (\mathfrak{Z}_1)_{red} \to (\mathfrak{T}_1)_{red}$) are the natural morphisms induced by g. A p-adically convergent isocrystals (resp. a convergent isocrystal) is the following set of data:

  - For every $(\mathfrak{T}, z_{\mathfrak{T}}) \in Ob(I_X^{(p)})$ (resp. $\in Ob(I_X^{(1)})$), a $\mathcal{M}_{(\mathfrak{T}, z_{\mathfrak{T}})} \in Coh(\mathfrak{T}_K)$;
  - For every morphism $g : (\mathfrak{Z}, z_{\mathfrak{Z}}) \to (\mathfrak{T}, z_{\mathfrak{T}})$ in $I_X^{(p)}$ (resp. $I_X^{(1)}$) an isomorphism

$$\phi_g : g^* \mathcal{M}_{(\mathfrak{T}, z_{\mathfrak{T}})} \to \mathcal{M}_{(\mathfrak{Z}, z_{\mathfrak{Z}})}$$

    in $Coh(\mathfrak{Z}_K)$ such that $\phi_{Id} = Id$ and for every other morphism $h : (\mathfrak{T}, z_{\mathfrak{T}}) \to (\mathfrak{U}, z_{\mathfrak{U}})$ one has $g^* \phi_h = \phi_{h \circ g}$.

  A morphism of p-adically convergent isocrystals (resp. convergent isocrystals) $\mathcal{M} \to \mathcal{N}$ is a collection of morphisms $\{\mathcal{M}_{(\mathfrak{T}, z_{\mathfrak{T}})} \to \mathcal{N}_{(\mathfrak{T}, z_{\mathfrak{T}})}\}_{(\mathfrak{T}, z_{\mathfrak{T}}) \in Ob(I_X^{(p)})}$ (resp. ${}_{(\mathfrak{T}, z_{\mathfrak{T}}) \in Ob(I_X^{(1)})}$) compatible with the isomorphism ϕ_g for all morphisms g.

- $\mathbf{Isoc}^{\dagger}(X, \overline{X}|K)$. Write $I_{(X, \overline{X}|K)}$ for the category of frames over $(X, \overline{X})$. Then an isocrystals on X overconvergent along $\overline{X} - X$ is the following set of data:

  - For every $(T, \overline{T}, \mathfrak{T}) \in Ob(I_{(X, \overline{X})})$ a coherent $j_T^{\dagger} \mathcal{O}_{]\overline{T}[_{\mathfrak{T}}}$ module $\mathcal{M}_{(T, \overline{T}, \mathfrak{T})}$;
  - For every morphism $g : (Z, \overline{Z}, \mathfrak{Z}) \to (T, \overline{T}, \mathfrak{T})$ in $I_{(X, \overline{X})}$ an isomorphism

$$\phi_g : g^* \mathcal{M}_{(T, \overline{T}, \mathfrak{T})} \to \mathcal{M}_{(Z, \overline{Z}, \mathfrak{Z})}$$

    of coherent $j_Z^{\dagger} \mathcal{O}_{]\overline{Z}[_{\mathfrak{Z}}}$ modules such that $\phi_{Id} = Id$ and for every other morphism $h : (T, \overline{T}, \mathfrak{T}) \to (U, \overline{U}, \mathfrak{U})$ one has $g^* \phi_h = \phi_{h \circ g}$.

  A morphism $\mathcal{M} \to \mathcal{N}$ in $\mathbf{Isoc}^{\dagger}(X, \overline{X}|K)$ is a collection of morphisms $\{\mathcal{M}_{(T, \overline{T}, \mathfrak{T})} \to \mathcal{N}_{(T, \overline{T}, \mathfrak{T})}\}_{(T, \overline{T}, \mathfrak{T}) \in Ob(I_{(X, \overline{X})})}$ compatible with the isomorphism ϕ_g for all morphisms g.

There are also enriched versions of the previous categories with Frobenius structure, which we denote $\mathbf{F}\text{-}\mathbf{Isoc}^{(p)}(X|K)$, $\mathbf{F}\text{-}\mathbf{Isoc}^{(1)}(X|K)$ and $\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(X, \overline{X}|K)$. For example, the absolute Frobenius F_X induces an endofunctor

$$F_X^* : \mathbf{Isoc}^{(p)}(X|K) \to \mathbf{Isoc}^{(p)}(X|K)$$

and $\mathbf{F}\text{-}\mathbf{Isoc}^{(p)}(X|K)$ is the category of pairs $(\mathcal{M}, \Phi)$, where $\mathcal{M} \in \mathbf{Isoc}^{(p)}(X|K)$ and Φ is a Frobenius structure on $\mathcal{M}$, i.e. an isomorphism $F_X^* \mathcal{M} \to \mathcal{M}$. A morphism in $\mathbf{F}\text{-}\mathbf{Isoc}^{(p)}(X|K)$ is a morphism in $\mathbf{Isoc}^{(p)}(X|K)$ compatible with the Frobenius structures. The constructions of $\mathbf{F}\text{-}\mathbf{Isoc}^{(1)}(X|K)$ and $\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(X, \overline{X}|K)$ are similar.

4.6.3 Functors between the categories

For every couple $(X, \overline{X})$ there is a canonical commutative diagram of functors:

$$\begin{tikzcd}
\mathbf{F\text{-}Isoc}^{(p)}(X|K) \arrow[d] & \mathbf{F\text{-}Isoc}^{(1)}(X|K) \arrow[l, "F1-Fp"'] \arrow[d] & \mathbf{F\text{-}Isoc}^{\dagger}(X,X|K) \arrow[l, "Fconv-F1"'] \arrow[d] & \mathbf{F\text{-}Isoc}^{\dagger}(X,\overline{X}|K) \arrow[l, "Fov-Fconv"'] \arrow[d] \\
\mathbf{Isoc}^{(p)}(X|K) & \mathbf{Isoc}^{(1)}(X|K) \arrow[l, "1-p"'] & \mathbf{Isoc}^{\dagger}(X,X|K) \arrow[l, "conv-1"'] & \mathbf{Isoc}^{\dagger}(X,\overline{X}|K) \arrow[l, "ov-conv"']
\end{tikzcd}$$

and

- $F1-Fp, conv-1, Fconv-F1$ are equivalences of categories ([Ogu84, Proposition 2.18], [Ber96, 2.3.4]) ;
- $Fov-Fconv$ is fully faithful if X is smooth ([Ked04, Theorem 1.1]).

All the functors are easy to construct from the definitions. For example, to construct $conv-1$, to an enlargement $(\mathfrak{T}, z_{\mathfrak{T}})$ one associates the frame $((\mathfrak{T}_1)_{red}, (\mathfrak{T}_1)_{red}, \mathfrak{T})$ over (X, X) and so, for every $\mathcal{M} \in \mathbf{F\text{-}Isoc}^{\dagger}(X, X|K)$, one defines

$$conv-1(\mathcal{M})_{(\mathfrak{T},z_{\mathfrak{T}})} := \mathcal{M}_{((\mathfrak{T}_1)_{red},(\mathfrak{T}_1)_{red},\mathfrak{T})}.$$

The constructions of $1-p, ov-conv$ are similar. In view of these functors, if $\mathcal{M}$ is in $\mathbf{F\text{-}Isoc}^{\dagger}(X, \overline{X}|K)$ and $(\mathfrak{T}, z_{\mathfrak{T}})$ is a p-adic enlargement of X, write

$$\mathcal{M}_{(\mathfrak{T},z_{\mathfrak{T}})} := \mathcal{M}_{(\mathfrak{T}_1,\mathfrak{T}_1,\mathfrak{T})}.$$

4.6.4 Stratification

Assume that X admits a closed immersion into a p-adic formal scheme $\mathfrak{X}$ formally smooth over W. Then the categories of isocrystals on X admit a more concrete description in term of modules with a stratification. We now recall the notion of universal p-adic enlargement and we use it to define modules with a stratification.

4.6.4.1 Universal p-adic enlargements

By [Ogu84, Proposition 2.3], there exists a universal p-adic enlargement $(\mathfrak{T}(X), z_{\mathfrak{T}(X)})$ of X in $\mathfrak{X}$. The p-adic enlargement $(\mathfrak{T}(X), z_{\mathfrak{T}(X)})$ of X is endowed with a map $g : \mathfrak{T}(X) \to \mathfrak{X}$ making the following diagram commutative:

$$\begin{tikzcd}
 & \mathfrak{T}(X)_1 \arrow[r, hook] \arrow[dl, "z_{\mathfrak{T}(X)}"'] \arrow[d, "g_1"] & \mathfrak{T}(X) \arrow[d, "g"] \\
X \arrow[r, hook] & \mathfrak{X}_1 \arrow[r] & \mathfrak{X}
\end{tikzcd}$$

which is universal for all the p-adic enlargements $(\mathfrak{Y}, z_{\mathfrak{Y}})$ of X in $\mathfrak{X}$, i.e for all the p-adic enlargements $(\mathfrak{Y}, z_{\mathfrak{Y}})$ admitting a map $g : \mathfrak{Y} \to \mathfrak{X}$ making the previous diagram commutative.

Write $\mathfrak{T}(X)(1)$ for the universal p-adic enlargements of X in $\mathfrak{X} \times \mathfrak{X}$, where one considers X embedded in $\mathfrak{X}_1 \times \mathfrak{X}_1$ via the diagonal immersion. The p-adic formal schemes $\mathfrak{T}(X)$ and $\mathfrak{T}(X)(1)$ are such that such that $\mathfrak{T}(X)_K = [X]_{\mathfrak{X},|p|}$ and $\mathfrak{T}(X)(1)_K = [X]_{\mathfrak{X}\times\mathfrak{X},|p|}$ (see [Ber96, 1.1.10]).

4.6.4.2 Stratifications

Let $\mathbf{Strat}^{(p)}(X, \mathfrak{X}|K)$ be the category of modules with a p-adically convergent stratifications ([Ogu84, Proposition 2.11]). An object $(\mathcal{M}, \epsilon)$ in $\mathbf{Strat}^{(p)}(X, \mathfrak{X}|K)$ is a coherent module $\mathcal{M}$ over $[X]_{\mathfrak{X},|p|} = \mathfrak{T}(X)_K$ together with an isomorphism $\epsilon : p_1^*\mathcal{M} \to p_2^*\mathcal{M}$ satisfying a natural cocycle condition, where the $p_i's$ are the two projections $\mathfrak{T}(X)(1)_K = [X]_{\mathfrak{X}\times\mathfrak{X},|p|} \to \mathfrak{T}(X)_K = [X]_{\mathfrak{X},|p|}$. The projections $p_1, p_2 : \mathfrak{X}\times\mathfrak{X} \to \mathfrak{X}$ give morphisms of enlargements $p_1, p_2 : (\mathfrak{T}(X)(1), z_{\mathfrak{T}(X)(1)}) \to (\mathfrak{T}(X), z_{\mathfrak{T}(X)})$. Hence, if $\mathcal{M}$ is in $\mathbf{Isoc}^{(p)}(X|K)$, there is an isomorphism

$$\epsilon_{\mathcal{M},\mathfrak{X}} : p_1^*(\mathcal{M}_{(\mathfrak{T}(X),z_{\mathfrak{T}(X)})}) \simeq \mathcal{M}_{(\mathfrak{T}(X)(1),z_{\mathfrak{T}(X)(1)})} \simeq p_2^*(\mathcal{M}_{(\mathfrak{T}(X),z_{\mathfrak{T}(X)})}).$$

This gives a functor

$$(-_{(\mathfrak{T}(X),z_{\mathfrak{T}(X)})}, \epsilon_{-,\mathfrak{X}}) : \mathbf{Isoc}^{(p)}(X|K) \to \mathbf{Strat}^{(p)}(X, \mathfrak{X}|K)$$

that sends $\mathcal{M}$ to $(\mathcal{M}_{(\mathfrak{T}(X),z_{\mathfrak{T}(X)})}, \epsilon_{\mathcal{M},\mathfrak{X}})$. By the universal property of $\mathfrak{T}(X)$, this functor is an equivalence of categories ([Ogu84, Proposition 2.11]).

Given a frame $(X, \overline{X}, \mathfrak{X})$, one can define the category $\mathbf{Strat}(X, \overline{X}, \mathfrak{X}|K)$ of modules with a stratification on X overconvergent along $\overline{X} - X$, see [Shi08a, P. 50] where it is denoted by $I^\dagger((X, \overline{X})/W|\mathfrak{X})$. An object $(\mathcal{M}, \epsilon)$ in $\mathbf{Strat}(X, \overline{X}, \mathfrak{X}|K)$ is a coherent $j_X^\dagger\mathcal{O}_{]\overline{X}[_\mathfrak{X}}$ module together with a $j_X^\dagger\mathcal{O}_{]\overline{X}[_{\mathfrak{X}\times\mathfrak{X}}}$-linear isomorphism $\epsilon : p_1^*\mathcal{M} \simeq p_2^*\mathcal{M}$ satisfying a natural cocycle condition, where the $p_i's$ are the two projection maps $]\overline{X}[_{\mathfrak{X}\times\mathfrak{X}} \to]\overline{X}[_\mathfrak{X}$. As in the p-adically convergent situation, one constructs a functor

$$(-_{(X,\overline{X},\mathfrak{X})}, \epsilon_{-,\mathfrak{X}}) : \mathbf{Isoc}^\dagger(X, \overline{X}|K) \to \mathbf{Strat}(X, \overline{X}, \mathfrak{X}|K)$$

which is an equivalence of categories (see [LeS07, Propositions 7.2.2 and 7.3.11]).

4.6.5 Relative p-adic cohomology theories

4.6.5.1 Relative p-adic cohomology theories

Fix a smooth proper morphism of k-varieties $f : Y \to X$ and a closed immersion $i : X \to \mathfrak{X}$, where $\mathfrak{X}$ is a flat p-adic formal scheme. Assume that $f : Y \to X$ has (log-) smooth parameter in the sense of [Shi08a, Definition 3.4].

Remark 4.6.5.1.1. If $f : Y \to X$ has (log-) smooth parameter, for every morphism of k-varieties $Z \to X$, the base change $Y_Z \to Z$ has (log-) smooth parameter ([Shi08a, Remark 3.5]). Moreover, if X is smooth, every smooth proper morphism $f : Y \to X$ of k-varieties has (log-) smooth proper parameter.

Depending on the nature of $i : X \to \mathfrak{X}$ one defines different p-adic cohomology theories:

- If $X = \mathfrak{X}_1$ and $i : X \to \mathfrak{X}$ is the canonical inclusion, then one can define the crystalline higher direct image $R^i f_{\mathfrak{X},crys,*}\mathcal{O}_{Y/\mathfrak{X}}$, that is the higher direct image in the relative crystalline site of X in $\mathfrak{X}$, well defined since $X \subseteq \mathfrak{X}$ is defined by the ideal (p). It lives in $Coh(\mathfrak{X}_K)$, see e.g [Shi08a, Section 1].

- If $i : X \to \mathfrak{X}$ is an homeomorphism, then one can define the convergent higher direct image $R^i f_{\mathfrak{X},conv,*}\mathcal{O}_{Y/\mathfrak{X}}$, that is the higher direct image in the relative convergent site of X in $\mathfrak{X}$. It lives in $Coh(\mathfrak{X}_K)$. See e.g [Shi08a, Sections 2-3].

- If $i : X \to \mathfrak{X}$ is an arbitrary closed immersion, one can define the analytic higher direct image $R^i f_{\mathfrak{X},an,*}\mathcal{O}_{Y/\mathfrak{X}}$. It is defined via descent using De Rham cohomology, and it lives in $Coh(]X[_\mathfrak{X})$. For details see [Shi08a, Section 4].

We complete the picture discussing higher direct images for couples and frames, in the context of overconvergent isocrystals. If $(Y,\overline{Y})$ is a couple, write $\mathcal{O}^{\dagger}_{(Y,\overline{Y})} \in \mathbf{Isoc}^{\dagger}(Y,\overline{Y})$ for the unique overconvergent isocrystal such that, for every frame $(Z,\overline{Z},\mathfrak{Z})$ over $(Y,\overline{Y})$, the restriction of $\mathcal{O}^{\dagger}_{(Y,\overline{Y})}$ to $(Z,\overline{Z},\mathfrak{Z})$ is given by $j^{\dagger}_{Z}\mathcal{O}_{]\overline{Z}[_{\mathfrak{Z}}}$. If $(Y,\overline{Y})$ is a couple over a frame $(X,\overline{X},\mathfrak{X})$ and the first arrow $f : Y \to X$ is smooth and proper, one can define the overconvergent higher direct image $R^i f_{(Y,\overline{Y})/\mathfrak{X},rig,*}\mathcal{O}^{\dagger}_{(Y,\overline{Y})}$. It is again defined using De Rham cohomology and descent. See [Shi08b, Section 5] for the definition, it is a $j^{\dagger}_{X}\mathcal{O}_{]\overline{X}[_{\mathfrak{X}}}$ module. It is still an open question whether $R^i f_{(Y,\overline{Y})/\mathfrak{X},rig,*}\mathcal{O}^{\dagger}_{(Y,\overline{Y})}$ is a coherent $j^{\dagger}_{X}\mathcal{O}_{]\overline{X}[_{\mathfrak{X}}}$- module.

4.6.5.2 Comparison

In some particular situation one can compare the various higher direct images defined in Section 4.6.5.1. Assume that $i : X \to \mathfrak{X}$ is a closed immersion, $\mathfrak{X}$ is formally smooth over W and $f : Y \to X$ is smooth proper with (log-) smooth parameter. Using f one considers (Y,Y) as a couple over the frame $(X,X,\mathfrak{X})$. The universal p-adic enlargement $\mathfrak{T}(X)$ of X in $\mathfrak{X}$ induces a commutative diagram:

$$\begin{tikzcd}
Y_{\mathfrak{T}(X)_1} \arrow[r, "f_{\mathfrak{T}(X)_1}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{T}(X)_1 \arrow[rr] \arrow[d, "u"] \arrow[dr] & & \mathfrak{T}(X) \arrow[d, "u"] \\
 & & \mathfrak{X}_1 \arrow[dr] & \\
Y \arrow[r, "f"] & X \arrow[rr] \arrow[ur] & & \mathfrak{X}
\end{tikzcd}$$

By remark 4.6.5.1.1, the morphism $f_{\mathfrak{T}(X)_1} : Y_{\mathfrak{T}(X)_1} \to \mathfrak{T}(X)_1$ has (log-) smooth parameter. In this situation one has $j^{\dagger}_{X}\mathcal{O}_{]X[_{\mathfrak{X}}} = \mathcal{O}_{]X[_{\mathfrak{X}}}$, so that $R^i f_{(Y,Y)/\mathfrak{X},rig,*}\mathcal{O}^{\dagger}_{(Y,Y)}$ and $R^i f_{\mathfrak{X},an,*}\mathcal{O}_{Y/\mathfrak{X}}$ are coherent $\mathcal{O}_{]X[_{\mathfrak{X}}}$ modules (see [Shi08a, Theorem 5.13]), while $R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),an,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)}$, $R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),conv,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)}$ and $R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)}$ are coherent modules over $[X]_{\mathfrak{X},|p|} = \mathfrak{T}(X)_K$. Write

$$u : [X]_{\mathfrak{X},|p|} \to]X[_{\mathfrak{X}}$$

for the natural inclusion. By [Shi08a, Theorem 5.13], one has a canonical isomorphism of $\mathcal{O}_{]X[_{\mathfrak{X}}}$-modules

$$R^i f_{(Y,Y)/\mathfrak{X},rig,*}\mathcal{O}^{\dagger}_{(Y,Y)} \simeq R^i f_{\mathfrak{X},an,*}\mathcal{O}_{Y/\mathfrak{X}}.$$

Pulling back along u, one finds canonical isomorphisms of coherent $[X]_{\mathfrak{X},|p|}$-modules

$$\begin{aligned} u^* R^i f_{(Y,Y)/\mathfrak{X},rig,*}\mathcal{O}^{\dagger}_{(Y,Y)} &\simeq u^* R^i f_{\mathfrak{X},an,*}\mathcal{O}_{Y/\mathfrak{X}} \simeq R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),an,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)} \\ &\simeq R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),conv,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)} \simeq R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)}. \end{aligned} \tag{4.6.5.2.1}$$

where the second isomorphism comes from [Shi08a, Remark 4.2], the third from [Shi08a, Theorem 4.6] and the last one from [Shi08a, Theorem 2.3.6].

These isomorphisms are functorial in the following sense. Assume that there is a k-variety Z, a closed embedding $Z \to \mathfrak{Z}$ into a p-adic formal scheme $\mathfrak{Z}$ formally smooth over W and a commutative diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "g"] & \mathfrak{Z} \arrow[d, "g"] \\
X \arrow[r] & \mathfrak{X}
\end{tikzcd}$$

By the universal property of $\mathfrak{T}(X)$, there is an induced map $\mathfrak{T}(Z) \to \mathfrak{T}(X)$ that fits into a commutative diagram

$$\begin{tikzcd}
& & Y_Z \arrow[r] \arrow[d, "f_Z"] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\
Y_{\mathfrak{T}(Z)_1} \arrow[r] \arrow[urr] \arrow[d, "f_{\mathfrak{T}(Z)_1}"] \arrow[dr, phantom, "\square"] & Y_{\mathfrak{T}(X)_1} \arrow[urr] \arrow[d, "f_{\mathfrak{T}(X)_1}"'] & Z \arrow[r, "g"] \arrow[d] & X \arrow[d] \\
\mathfrak{T}(Z)_1 \arrow[r] \arrow[urr] \arrow[d] & \mathfrak{T}(X)_1 \arrow[urr] \arrow[d] & \mathfrak{Z} \arrow[r, "g"] & \mathfrak{X} \\
\mathfrak{T}(Z) \arrow[r, "g"] \arrow[urr, "u"] & \mathfrak{T}(X) \arrow[urr, "u"] & &
\end{tikzcd}$$

Then the following diagram is commutative

$$\begin{tikzcd}
u^*g^*R^i f_{(Y,Y)/\mathfrak{X},rig,*}\mathcal{O}^\dagger_{(Y,Y)} \arrow[r] \arrow[d, "\simeq"] & u^*R^i f_{(Y_Z,Y_Z)/\mathfrak{Z},rig,*}\mathcal{O}^\dagger_{(Y_Z,Y_Z)} \arrow[d, "\simeq"] \\
g^*u^*R^i f_{(Y,Y)/\mathfrak{X},rig,*}\mathcal{O}^\dagger_{(Y,Y)} \arrow[r] \arrow[d, "\simeq"] & u^*R^i f_{(Y_Z,Y_Z)/\mathfrak{Z},rig,*}\mathcal{O}^\dagger_{(Y_Z,Y_Z)} \arrow[d, "\simeq"] \\
g^*u^*R^i f_{\mathfrak{X},an,*}\mathcal{O}_{Y/\mathfrak{X}} \arrow[r] \arrow[d, "\simeq"] & u^*R^i f_{\mathfrak{Z},an,*}\mathcal{O}_{Y_Z/\mathfrak{Z}} \arrow[d, "\simeq"] \\
g^*R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),an,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)} \arrow[r] \arrow[d, "\simeq"] & R^i f_{\mathfrak{T}(Z)_1,\mathfrak{T}(Z),an,*}\mathcal{O}_{Y_{\mathfrak{T}(Z)_1}/\mathfrak{T}(Z)} \arrow[d, "\simeq"] \\
g^*R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),conv,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)} \arrow[r] \arrow[d, "\simeq"] & R^i f_{\mathfrak{T}(Z)_1,\mathfrak{T}(Z),conv,*}\mathcal{O}_{Y_{\mathfrak{T}(Z)_1}/\mathfrak{T}(Z)} \arrow[d, "\simeq"] \\
g^*R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)} \arrow[r] & R^i f_{\mathfrak{T}(Z)_1,\mathfrak{T}(Z),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(Z)_1}/\mathfrak{T}(Z)}
\end{tikzcd} \tag{4.6.5.2.2}$$

where the vertical arrows are the isomorphisms in (4.6.5.2.1) and the horizontal arrows are the base change maps.

4.6.5.3 Ogus higher direct image

Fix a smooth proper morphism $f : Y \to X$ of k-varieties. Write $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$ in $\mathbf{F}\text{-}\mathbf{Isoc}^{(p)}(X)$ for the Ogus higher direct image ([Ogu84, Section 3, Theorem 3.1]) and recall that its formation is compatible with base change ([Ogu84, Proposition 3.5]).

As object in $\mathbf{Isoc}^{(p)}(X)$, $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$ is characterized by the property that for every p-adic enlargement $(\mathfrak{T}, z_{\mathfrak{T}})$ one has

$$(R^i f_{Ogus,*}\mathcal{O}_{Y/K})_{(\mathfrak{T},z_{\mathfrak{T}})} = R^i f_{\mathfrak{T}_1,\mathfrak{T},crys,*}\mathcal{O}_{Y_{\mathfrak{T}_1}/\mathfrak{T}}$$

and if $g : (\mathfrak{T}, z_{\mathfrak{T}}) \to (\mathfrak{Z}, z_{\mathfrak{Z}})$ if a morphism of p-adic enlargements, the map

$$\begin{tikzcd}
g^*(R^i f_{Ogus,*}\mathcal{O}_{Y/K})_{(\mathfrak{Z},z_{\mathfrak{Z}})} \arrow[r, "\phi_g"] \arrow[d, equal] & (R^i f_{Ogus,*}\mathcal{O}_{Y/K})_{(\mathfrak{T},z_{\mathfrak{T}})} \arrow[d, equal] \\
g^*R^i f_{\mathfrak{Z}_1,\mathfrak{Z},crys,*}\mathcal{O}_{Y_{\mathfrak{Z}_1}/\mathfrak{Z}} \arrow[r, "\phi_g"] & R^i f_{\mathfrak{T}_1,\mathfrak{T},crys,*}\mathcal{O}_{Y_{\mathfrak{T}_1}/\mathfrak{T}}
\end{tikzcd}$$

is the base change morphism induced by g (see the proof of [Ogu84, Theorem 3.1]). In particular if X admits a closed immersion into a p-adic formal scheme $\mathfrak{X}$ formally smooth over W, the image of $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$ in $\mathbf{Strat}^{(p)}(X,\mathfrak{X}|K)$ is given by the couple

$$(R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)}, \epsilon_{R^i f_{Ogus,*}\mathcal{O}_{Y/K},\mathfrak{X}}),$$

where $\epsilon_{R^i f_{Ogus,*}\mathcal{O}_{Y/K},\mathfrak{X}}$ is induced by the base change morphisms

$$p_1^* R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)} \xrightarrow{\simeq} R^i f_{\mathfrak{T}(X)(1)_1,\mathfrak{T}(X)(1),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(X)(1)_1}/\mathfrak{T}(X)(1)} \xleftarrow{\simeq} p_2^* R^i f_{\mathfrak{T}(X)_1,\mathfrak{T}(X),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(X)_1}/\mathfrak{T}(X)}.$$

The Frobenius structure

$$F_X^* R^i f_{Ogus,*}\mathcal{O}_{Y/K} \to R^i f_{Ogus,*}\mathcal{O}_{Y/K}$$

is constructed in the following way (see the proof of [Ogu84, Theorem 3.7]). Consider the commutative cartesian diagram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ X \arrow[r, "F_X"] & X \end{tikzcd}$$

and for every p-adic enlargement $(\mathfrak{T}, z_{\mathfrak{T}})$ of X, consider the following diagram

$$\begin{tikzcd} Y_{\mathfrak{T}_1} \arrow[r, "Fr_{Y_{\mathfrak{T}_1}/\mathfrak{T}_1}"'] \arrow[rr, bend left, "F_{Y_{\mathfrak{T}_1}}"] \arrow[d, "f_{\mathfrak{T}_1}"] & Y'_{\mathfrak{T}_1} \arrow[r] \arrow[d, "f'_{\mathfrak{T}_1}"] \arrow[dr, phantom, "\square"] & Y_{\mathfrak{T}_1} \arrow[d, "f_{\mathfrak{T}_1}"] \\ \mathfrak{T}_1 \arrow[r, equal] \arrow[d] & \mathfrak{T}_1 \arrow[r, "F_{\mathfrak{T}_1}"] \arrow[d] & \mathfrak{T}_1 \\ \mathfrak{T} \arrow[r, equal] & \mathfrak{T} & \end{tikzcd}$$

where $Fr_{Y_{\mathfrak{T}_1}/\mathfrak{T}_1}$ is the relative Frobenius morphism. By the compatibility of $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$ with base change, there is a canonical isomorphism

$$F_X^* R^i f_{Ogus,*}\mathcal{O}_{Y/K} \simeq R^i f'_{Ogus,*}\mathcal{O}_{Y'/K}$$

and hence

$$(F_X^* R^i f_{Ogus,*}\mathcal{O}_{Y/K})_{(\mathfrak{T},z_{\mathfrak{T}})} \simeq (R^i f'_{Ogus,*}\mathcal{O}_{Y'/K})_{(\mathfrak{T},z_{\mathfrak{T}})} = R^i f'_{\mathfrak{T}_1,\mathfrak{T},crys,*}\mathcal{O}_{Y'_{\mathfrak{T}_1}/\mathfrak{T}}.$$

Then the Frobenius structure is constructed as the base change map

$$\begin{tikzcd} (F_X^* R^i f_{Ogus,*}\mathcal{O}_{Y/K})_{(\mathfrak{T},z_{\mathfrak{T}})} \arrow[r] \arrow[d, "\simeq"] & (R^i f_{Ogus,*}\mathcal{O}_{Y/K})_{(\mathfrak{T},z_{\mathfrak{T}})} \arrow[d, "\simeq"] \\ R^i f'_{\mathfrak{T}_1,\mathfrak{T},crys,*}\mathcal{O}_{Y'_{\mathfrak{T}_1}/\mathfrak{T}} \arrow[r] & R^i f_{\mathfrak{T}_1,\mathfrak{T},crys,*}\mathcal{O}_{Y_{\mathfrak{T}_1}/\mathfrak{T}} \end{tikzcd}$$

induced by $Fr_{Y_{\mathfrak{T}_1}/\mathfrak{T}_1}$.

4.6.5.4 Statement of Theorem 4.6.5.4.1

The aim of the following Section 4.7 is to prove the following theorem.

Theorem 4.6.5.4.1. Assume that X is a smooth k-variety and $f : Y \to X$ a smooth proper morphism. Then $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$ is in the essential image of $\mathbf{F\text{-}Isoc}^\dagger(X|K) \to \mathbf{F\text{-}Isoc}^\dagger(X, X|K) \simeq \mathbf{F\text{-}Isoc}^{(p)}(X|K)$.

Remark 4.6.5.4.2. Theorem 4.6.5.4.1 already appears in the literature as [Laz16, Corollary 6.2], but, as pointed out to us by T.Abe, there might be a gap in the proof. The problem is in the gluing process in [Laz16, Corollary 6.1]. The author uses the theory of arithmetic D-modules and he tries to compare the higher direct image in that world with $R^i\mathfrak{f}_{crys,*}\mathcal{O}_{\mathcal{Y}/K}(j)$. Locally they coincide, but it is not so clear that the gluing data are compatible, since the isomorphism is defined not at level of complex but only on the level of the derived category. So, following a suggestion of T.Abe, we give another proof of Theorem 4.6.5.4.1, using the work of Shiho on the relative log crystalline cohomology ([Shi08b]).

Remark 4.6.5.4.3. The proof actually works more generally for every $E \in \mathbf{F\text{-}Isoc}^{(p)}(Y)$. The construction of $R^i f_{Ogus,*}E$ does not appear in the literature, so we decided to restrict ourself to $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$.

4.7 Proof of Theorem 4.6.5.4.1

4.7.1 Construction of an overconvergent F-isocrystal

Fix compactifications $Y \subseteq \overline{Y}$ and $X \subseteq \overline{X}$ such that the morphism $f : Y \to X$ extends to a map of couples $(Y, \overline{Y}) \to (X, \overline{X})$ and X (resp. Y) is dense in $\overline{X}$ (resp. $\overline{Y}$). We start recalling the main result of [Shi08b]. This gives a $\mathcal{M}$ in $\mathbf{F\text{-}Isoc}^\dagger(X, \overline{X}|K)$ which, after a base change and on appropriate frames, looks like $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$. To recall the statement, it is helpful to give the following definition.

Definition 4.7.1.1. If $(Z, \overline{Z}, \mathfrak{Z}) \to (X', \overline{X}', \mathfrak{X}')$ is a morphism of frames over $(X, \overline{X})$ we say that $(Z, \overline{Z}, \mathfrak{Z})$ has $(P_{(X', \overline{X}', \mathfrak{X}')})$ if $Z = X \times_{\overline{X}} \overline{Z}$ and $\mathfrak{Z} \to \mathfrak{X}'$ is formally smooth.

By [Shi08b, Theorem 7.9] (and its proof) there exists a frame $(X', \overline{X}', \mathfrak{X}')$ over $(X, \overline{X})$ such that

- $X' := X \times_{\overline{X}} \overline{X}'$;
- $\mathfrak{X}'$ is formally smooth over W;
- the map $\overline{X}' \to \overline{X}$ is a composition of a surjective proper map followed by a surjective étale map;

and an $\mathcal{M}$ in $\mathbf{F\text{-}Isoc}^\dagger(X, \overline{X}|K)$ with the following properties:

1. Let $(Z, \overline{Z}, \mathfrak{Z})$ be a frame over $(X', \overline{X}', \mathfrak{X}')$ that has $(P_{(X', \overline{X}', \mathfrak{X}')})$, so that there is a commutative diagram

$$\begin{array}{ccccc}
(Y, \overline{Y}) & \longleftarrow & (Y_{X'}, \overline{Y}_{\overline{X}'}) & \longleftarrow & (Y_Z, \overline{Y}_{\overline{Z}}) \\
\downarrow & \square & \downarrow & \square & \downarrow \\
(X, \overline{X}) & \longleftarrow & (X', \overline{X}', \mathfrak{X}') & \longleftarrow & (Z, \overline{Z}, \mathfrak{Z}).
\end{array}$$

Then, the image of $\mathcal{M}$ in $\mathbf{Strat}(Z, \overline{Z}, \mathfrak{Z}|K)$ is given by

$$(R^i f_{(Y_Z,\overline{Y}_{\overline{Z}})/\mathfrak{Z},rig,*}\mathcal{O}^\dagger_{(Y_Z,\overline{Y}_{\overline{Z}})}, \epsilon)$$

where ϵ is an isomorphism:

$$p_1^* R^i f_{(Y_Z,\overline{Y}_{\overline{Z}})/\mathfrak{Z},rig,*}\mathcal{O}^\dagger_{(Y_Z,\overline{Y}_{\overline{Z}})} \to R^i f_{(Y_Z,\overline{Y}_{\overline{Z}})/\mathfrak{Z}\times\mathfrak{Z},rig,*}\mathcal{O}^\dagger_{(Y_Z,\overline{Y}_{\overline{Z}})} \leftarrow p_2^* R^i f_{(Y_Z,\overline{Y}_{\overline{Z}})/\mathfrak{Z},rig,*}\mathcal{O}^\dagger_{(Y_Z,\overline{Y}_{\overline{Z}})}$$

and $p_1, p_2 :]\overline{Z}[_{\mathfrak{Z}\times_W\mathfrak{Z}} \to]\overline{Z}[_{\mathfrak{Z}}$ are the projection maps. If moreover $Z = \overline{Z}$, then ϵ is induced by the base change morphisms ([Shi08b, Last paragraph of page 74] and [Shi08a, Theorem 5.14]);

2. Let $h : (Z, \overline{Z}, \mathfrak{Z}) \to (T, \overline{T}, \mathfrak{T})$ be a morphism of frames over $(X', \overline{X}', \mathfrak{X}')$ that have $(P_{(X',\overline{X}',\mathfrak{X}')})$, so that there is a commutative diagram

$$\begin{array}{ccc} (Y_Z, \overline{Y}_{\overline{Z}}) & \longrightarrow & (Y_T, \overline{Y}_{\overline{T}}) \\ \downarrow & & \downarrow \\ (Z, \overline{Z}, \mathfrak{Z}) & \longrightarrow & (T, \overline{T}, \mathfrak{T}). \end{array}$$

Then, the isomorphism

$$\begin{array}{ccc} h^*\mathcal{M}_{(Z,\overline{Z},\mathfrak{Z})} & \xrightarrow{\phi_h} & \mathcal{M}_{(T,\overline{T},\mathfrak{T})} \\ \downarrow\simeq & & \downarrow\simeq \\ h^* R^i f_{(Y_Z,\overline{Y}_{\overline{Z}})/\mathfrak{Z},rig,*}\mathcal{O}^\dagger_{(Y_Z,\overline{Y}_{\overline{Z}})} & \longrightarrow & R^i f_{(Y_T,\overline{Y}_{\overline{T}})/\mathfrak{T},rig,*}\mathcal{O}^\dagger_{(Y_T,\overline{Y}_{\overline{T}})} \end{array}$$

given by the isocrystals structure is the base change morphism (This is the functoriality in the statement of [Shi08b, Theorem 7.9], see [Shi08a, Proof of Theorem 4.8]);

3. Let $(Z, \overline{Z}, \mathfrak{Z})$ be a frame over $(X, \overline{X})$ that has $(P_{(X',\overline{X}',\mathfrak{X}')})$ and assume that $\mathfrak{Z}$ admits a lifting $\sigma_{\mathfrak{Z}}$ of $F_{\mathfrak{Z}_1}$, so that there is a commutative diagram

$$\begin{array}{ccc} (Y_Z, \overline{Y}_{\overline{Z}}) & \xrightarrow{(F_{Y_Z}, F_{\overline{Y}_{\overline{Z}}})} & (Y_Z, \overline{Y}_{\overline{Z}}) \\ \downarrow & & \downarrow \\ (Z, \overline{Z}, \mathfrak{Z}) & \xrightarrow{(F_Z, F_{\overline{Z}}, \sigma_{\mathfrak{Z}})} & (Z, \overline{Z}, \mathfrak{Z}). \end{array}$$

Then, the isomorphism induced by the Frobenius structure

$$\begin{array}{ccc} \sigma_{\mathfrak{Z}}^*\mathcal{M}_{(Z,\overline{Z},\mathfrak{Z})} \simeq (F_X^*\mathcal{M})_{(Z,\overline{Z},\mathfrak{Z})} & \longrightarrow & \mathcal{M}_{(Z,\overline{Z},\mathfrak{Z})} \\ \downarrow\simeq & & \downarrow\simeq \\ \sigma_{\mathfrak{Z}}^* R^i f_{(Y_Z,\overline{Y}_{\overline{Z}})/\mathfrak{Z},rig,*}\mathcal{O}^\dagger_{(Y_Z,\overline{Y}_{\overline{Z}})} & \longrightarrow & R^i f_{(Y_Z,\overline{Y}_{\overline{Z}})/\mathfrak{Z},rig,*}\mathcal{O}^\dagger_{(Y_Z,\overline{Y}_{\overline{Z}})} \end{array}$$

is given by the base change morphism induced by $\sigma_{\mathfrak{Z}}$ and F_{Y_Z} ([Shi08b, Proof of Theorem 7.9]).

4.7.2 Strategy

To prove Theorem 4.6.5.4.1, it is enough to show that the image of $\mathcal{M}$ in $\mathbf{F}\text{-}\mathbf{Isoc}^{(p)}(X|K)$ is isomorphic to $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$. Since X' does not admit a closed immersion directly in $\mathfrak{X}'$ and $\mathfrak{X}'$ does not admits a lifting of the absolute Frobenius of $\mathfrak{X}'_1$, one can't use directly the description of $\mathcal{M}$ given in previous Section 4.7.1. But there exists[2] an étale surjective morphism $U \to X'$ such that U admits a closed immersion into a p-adic formal scheme $\mathfrak{U}$ which is formally smooth over $\mathfrak{X}'$ and it is endowed with a lifting $\sigma_{\mathfrak{U}}$ of $F_{\mathfrak{U}_1}$. Write g for the composition $U \to X' \to X$.

To prove Theorem 4.6.5.4.1, first one constructs an isomorphism

$$\psi : g^*\mathcal{M} \simeq g^* R^i f_{Ogus,*}\mathcal{O}_{Y/K} \simeq R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K} \text{ in } \mathbf{F}\text{-}\mathbf{Isoc}^{(1)}(U|K)$$

where the isomorphism on the right comes from the fact that the formation of $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$ is compatible with base change, see Section 4.6.5.3. Then one uses étale and proper descent for convergent isocrystals to deduce that ψ descent to $\mathbf{F}\text{-}\mathbf{Isoc}^{(p)}(X|K)$. More precisely the proof decomposes as follows:

1. One constructs an isomorphism

$$\psi : g^*\mathcal{M} \simeq R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K}$$

in $\mathbf{Isoc}^{(p)}(U|K) \simeq \mathbf{Strat}^{(p)}(U, \mathfrak{U}|K)$. This is done in Section 4.7.3, using that $(U, U, \mathfrak{U}|K)$ has $(P_{(X',\overline{X}',\mathfrak{X}')})$ (so that one can apply the property (1) of $\mathcal{M}$) and the comparison isomorphisms in 4.6.5.2;

2. One verifies that the ψ commutes with the Frobenius structures i.e. that ψ makes the following diagram commutative

$$\begin{tikzcd} F_U^* g^*\mathcal{M} \arrow[r] \arrow[d, "F_U^*\psi"] & g^*\mathcal{M} \arrow[d, "\psi"] \\ F_U^* R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K} \arrow[r] & R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K} \end{tikzcd}$$

in $\mathbf{Isoc}^{(p)}(U|K) \simeq \mathbf{Strat}^{(p)}(U, \mathfrak{U}|K)$. This is done in Section 4.7.4, using that $\mathfrak{U}$ has a lifting of $F_{\mathfrak{U}_1}$ (so that one can apply the property (3) of $\mathcal{M}$) and the comparison isomorphisms in 4.6.5.2;

3. By the equivalence $F1 - Fp$ in Section 4.6.3, the first two steps imply that there is an isomorphism

$$\psi : g^*\mathcal{M} \simeq R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K}$$

in $\mathbf{F}\text{-}\mathbf{Isoc}^{(1)}(U|K)$;

4. To apply descent for convergent isocrystals, one has to check that ψ makes the following diagram in $\mathbf{F}\text{-}\mathbf{Isoc}^{(1)}(U \times_X U|K)$ commutative:

[2] To construct it, consider a finite covering $\{Spf(A_i)\}$ of $\mathfrak{X}'$ by formal affine open sub schemes. Then $\{Spec(A_{i,1})\}$ is a covering of $\mathfrak{X}'_1$ by affine open sub schemes and $\{V_i := Spec(A_{i,1}) \times_{\mathfrak{X}'_1} X'\}$ is a Zariski open covering of X'. Consider a finite covering $\{U_{i,j}\}$ of V_i by affine open sub schemes. Then the maps $U_{i,j} \to Spec(A_{i,1})$ are affine and of finite type, so that there are closed immersions $U_{i,j} \to \mathbb{A}^{n_{i,j}}_{\mathrm{Spec}(A_{i,1})}$. Write $\mathfrak{U}_{i,j}$ for the formal affine space of dimension $n_{i,j}$ over $Spf(A_i)$. Then $U := \coprod_{i,j} U_{i,j}$ admits a closed immersion into $\mathfrak{U} := \coprod_{i,j} \mathfrak{U}_{i,j}$ and $\mathfrak{U}$ is formally smooth over $\mathfrak{X}'$. To show that $\mathfrak{U}$ admits a lifting of $F_{\mathfrak{U}_1}$ it is enough to show that each $\mathfrak{U}_{i,j}$ admits a lifting of $F_{\mathfrak{U}_{i,j,1}}$. This follows from the fact that $\mathfrak{U}_{i,j}$ is formally affine and formally smooth over W.

$$\begin{array}{ccc} q_1^*g^*\mathcal{M} & \longrightarrow & q_2^*g^*\mathcal{M} \\ \downarrow{\scriptstyle q_1^*\psi} & & \downarrow{\scriptstyle q_2^*\psi} \\ q_1^*g^*R^if_{Ogus,*}\mathcal{O}_{Y/K} & \longrightarrow & q_2^*g^*R^if_{Ogus,*}\mathcal{O}_{Y/K} \end{array}$$

where $q_1, q_2 : U \times_X U \to U$ are the projections. To check this, by the equivalence $F1-Fp$, it is enough to show that it is commutative in $\mathbf{F}\text{-}\mathbf{Isoc}^{(p)}(U \times_X U|K)$ or equivalently in $\mathbf{Isoc}^{(p)}(U \times_X U|K) \simeq \mathbf{Strat}^{(p)}(U \times_X U, \mathfrak{U} \times_W \mathfrak{U}|K)$. This is done in Section 4.7.5, using that $q_1, q_2 : (U \times_X U, U \times_X U, \mathfrak{U} \times_W \mathfrak{U}) \to (U, U, \mathfrak{U})$ are morphisms of frames that have $(P_{(X',\overline{X}',\mathfrak{X}')})$ (so that one can apply the property (2) of $\mathcal{M}$) and the comparison isomorphisms in 4.6.5.2.

Remark 4.7.2.1. The reason why one needs to pass back and forth between $\mathbf{F}\text{-}\mathbf{Isoc}^{(p)}(U|K)$ and $\mathbf{F}\text{-}\mathbf{Isoc}^{(1)}(U|K)$ is that proper descent is not known for the category $\mathbf{Isoc}^{(p)}(U|K)$, while proper descent for the category $\mathbf{Isoc}^{(1)}(U|K)$ (and hence for $\mathbf{F}\text{-}\mathbf{Isoc}^{(1)}(U|K)$) is proved in [Ogu84]. On the other hand one knows the value of $R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K}$ only on p-adic enlargements. The equivalences of categories in Section 4.6.3 allow to combine these informations.

4.7.3 Comparison of isocrystals

In this section we construct an isomorphism

$$\psi : g^*\mathcal{M} \simeq R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K} \text{ in } \mathbf{Isoc}^{(p)}(U|K).$$

Consider the universal p-adic enlargements $\mathfrak{T}(U)$ and $\mathfrak{T}(U)(1)$ of U in $\mathfrak{U}$ and $\mathfrak{U}\times\mathfrak{U}$ and write u, p_1, p_2 for the natural morphisms

$$\begin{array}{ccc} (\mathfrak{T}(U)(1)_1, \mathfrak{T}(U)(1)_1, \mathfrak{T}(U)(1)) & \xrightarrow{u} & (U, U, \mathfrak{U}\times\mathfrak{U}) \\ {\scriptstyle p_2}\downarrow\downarrow{\scriptstyle p_1} & & {\scriptstyle p_2}\downarrow\downarrow{\scriptstyle p_1} \\ (\mathfrak{T}(U)_1, \mathfrak{T}(U)_1, \mathfrak{T}(U)) & \xrightarrow{u} & (U, U, \mathfrak{U}) \end{array}$$

Since $(U, U, \mathfrak{U})$ has $(P_{(X',\overline{X}',\mathfrak{X}')})$, by the property (1) of $\mathcal{M}$ in 4.7.1, one has:

$$\mathcal{M}_{(U,U,\mathfrak{U})} = R^if_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)} \text{ in } Coh(]U[_\mathfrak{U}).$$

Since $\mathcal{M}$ is an isocrystal, one gets $\mathcal{M}_{(\mathfrak{T}(U)_1,\mathfrak{T}(U)_1,\mathfrak{T}(U))} \simeq u^*\mathcal{M}_{(U,U,\mathfrak{U})}$ in $Coh(\mathfrak{T}(U)_K)$. Then as in (4.6.5.2.1):

$$\mathcal{M}_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \simeq u^*R^if_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)} \simeq u^*R^if_{U,\mathfrak{U},an,*}\mathcal{O}_{Y_U/\mathfrak{U}} \simeq R^if_{\mathfrak{T}(U)_1,\mathfrak{T}(U),an,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)}$$
$$\simeq R^if_{\mathfrak{T}(U)_1,\mathfrak{T}(U),conv,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \simeq R^if_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \text{ in } Coh(\mathfrak{T}(U)_K).$$

Since, by construction (4.6.5.3),

$$R^if_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} = (R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K})_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})}$$

one has an isomorphism

$$\psi : \mathcal{M}_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \simeq (R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K})_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \text{ in } Coh(\mathfrak{T}(U)_K).$$

To promote ψ to an isomorphism in $\mathbf{Strat}^{(p)}(U,\mathfrak{U}|K) \simeq \mathbf{Isoc}^{(p)}(U|K)$ one has to check that ψ is compatible with the stratifications $\epsilon_{g^*\mathcal{M},\mathfrak{U}}$ on $g^*\mathcal{M}$ and $\epsilon_{R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K},\mathfrak{U}}$ on $(R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K})_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})}$.

Since $(U, U, \mathfrak{U})$ has $(P_{(X',\overline{X}',\mathfrak{X}')})$, by the property (1) in 4.7.1, the stratification $\epsilon_{g^*\mathcal{M},\mathfrak{U}}$ is given by the base change morphisms:

$$p_1^*R^if_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)} \to R^if_{(Y_U,Y_U)/\mathfrak{U}\times\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)} \leftarrow p_2^*R^if_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)}.$$

As in (4.6.5.2.2) pulling back to u^*, one has a commutative diagram

$$\begin{tikzcd}
u^*p_1^*R^if_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & u^*R^if_{(Y_U,Y_U)/\mathfrak{U}\times\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)} \arrow[d, "\simeq"] & u^*p_2^*R^if_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)} \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\
p_1^*u^*R^if_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & u^*R^if_{(Y_U,Y_U)/\mathfrak{U}\times\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)} \arrow[d, "\simeq"] & p_2^*u^*R^if_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)} \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\
p_1^*u^*R^if_{U,\mathfrak{U},an,*}\mathcal{O}_{Y_U/\mathfrak{U}} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & u^*R^if_{U,\mathfrak{U}\times\mathfrak{U},an,*}\mathcal{O}_{Y_U/\mathfrak{U}\times\mathfrak{U}} \arrow[d, "\simeq"] & p_2^*u^*R^if_{U,\mathfrak{U},an,*}\mathcal{O}_{Y_U/\mathfrak{U}} \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\
p_1^*R^if_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \arrow[r, "\simeq"] & R^if_{\mathfrak{T}(U)(1)_1,\mathfrak{T}(U)(1),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)(1)_1}/\mathfrak{T}(U)(1)} & p_2^*R^if_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \arrow[l, "\simeq"']
\end{tikzcd}$$

where the horizontal maps are the natural base change maps. So, the stratification $\epsilon_{g^*\mathfrak{M},\mathfrak{U}}$ on $g^*\mathcal{M}$

$$\begin{tikzcd}
p_1^*\mathcal{M}_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \arrow[r] \arrow[d, "\simeq"] & \mathcal{M}_{(\mathfrak{T}(U)(1),z_{\mathfrak{T}(U)(1)})} \arrow[d, "\simeq"] & p_2^*\mathcal{M}_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \arrow[l] \arrow[d, "\simeq"] \\
p_1^*R^if_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \arrow[r, "\simeq"] & R^if_{\mathfrak{T}(U)(1)_1,\mathfrak{T}(U)(1),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)(1)_1}/\mathfrak{T}(U)(1)} & p_2^*R^if_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \arrow[l, "\simeq"']
\end{tikzcd}$$

is induced by the base change morphisms. Since $\epsilon_{R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K},\mathfrak{U}}$ is induced by the base change morphisms by construction (4.6.5.3), one concludes that

$$\psi : \mathcal{M}_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \simeq (R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K})_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \text{ in } Coh(\mathfrak{T}(U)_K)$$

is compatible with the stratifications and hence induces an isomorphism

$$\psi : g^*\mathcal{M} \simeq R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K} \text{ in } \mathbf{Strat}^{(p)}(U,\mathfrak{U}|K) \simeq \mathbf{Isoc}^{(p)}(U|K).$$

4.7.4 Comparison of Frobenius structures

We now check that ψ is compatible with the Frobenius structures, i.e. that the following diagram in $\mathbf{Isoc}^{(p)}(U|K)$ is commutative:

$$\begin{tikzcd}
F_U^*g^*\mathcal{M} \arrow[r] \arrow[d, "F_U^*\psi"] & g^*\mathcal{M} \arrow[d, "\psi"] \\
F_U^*R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K} \arrow[r] & R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K}
\end{tikzcd}$$

Since

$$(-_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})}, \epsilon_{-,\mathfrak{U}}) : \mathbf{Isoc}^{(p)}(U|K) \to \mathbf{Strat}^{(p)}(U,\mathfrak{U}|K)$$

is an equivalence of categories, it is enough to show that

$$\begin{tikzcd}
(F_U^*g^*\mathcal{M})_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \arrow[r] \arrow[d] & g^*\mathcal{M}_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \arrow[d] \\
(F_U^*R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K})_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \arrow[r] & (R^if_{U,Ogus,*}\mathcal{O}_{Y_U/K})_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})}
\end{tikzcd}$$

is commutative. Since $(U,U,\mathfrak{U})$ has $(P_{(X',\overline{X}',\mathfrak{X}')})$ and it is endowed with a morphism $\sigma_{\mathfrak{U}}$ lifting $F_{\mathfrak{U}_1}$, by the property (3) of $\mathcal{M}$ in 4.7.1, the Frobenius structure on $\mathcal{M}_{(U,U,\mathfrak{U})}$ is given by the base change map induced by $\sigma_{\mathfrak{U}}$ and F_{Y_U} :

$$\sigma_{\mathfrak{U}}^*R^if_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)} \to R^if_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^\dagger_{(Y_U,Y_U)}.$$

By the universal property of the universal p-adic enlargement one gets a commutative diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[d, "F_U"] & \mathfrak{T}(U) \arrow[r, "u"] \arrow[d, "\sigma_{\mathfrak{T}(U)}"] & \mathfrak{U} \arrow[d, "\sigma_{\mathfrak{U}}"] \\ U \arrow[r] & \mathfrak{T}(U) \arrow[r, "u"] & \mathfrak{U} \end{tikzcd}$$

Pulling back via u, there is a commutative diagram (4.6.5.2.2):

$$\begin{tikzcd} u^*\sigma_{\mathfrak{U}}^* R^i f_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^{\dagger}_{(Y_U,Y_U)} \arrow[r] \arrow[d, "\simeq"] & u^* R^i f_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^{\dagger}_{(Y_U,Y_U)} \arrow[d, "\simeq"] \\ \sigma_{\mathfrak{T}(U)}^* u^* R^i f_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^{\dagger}_{(Y_U,Y_U)} \arrow[r] \arrow[d, "\simeq"] & u^* R^i f_{(Y_U,Y_U)/\mathfrak{U},rig,*}\mathcal{O}^{\dagger}_{(Y_U,Y_U)} \arrow[d, "\simeq"] \\ \sigma_{\mathfrak{T}(U)}^* R^i f_{\mathfrak{T}(U)_1,\mathfrak{T}(U),an,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \arrow[r] \arrow[d, "\simeq"] & R^i f_{\mathfrak{T}(U)_1\mathfrak{T}(U),an,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \arrow[d, "\simeq"] \\ \sigma_{\mathfrak{T}(U)}^* R^i f_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \arrow[r] & R^i f_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \end{tikzcd}$$

where the horizontal morphisms are the base change morphisms. So the morphism

$$\begin{tikzcd} \sigma_{\mathfrak{T}}^*\mathcal{M}_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \simeq (F_U^* g^*\mathcal{M})_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \arrow[r] \arrow[d, "\simeq"] & g^*\mathcal{M}_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \arrow[d, "\simeq"] \\ \sigma_{\mathfrak{T}(U)}^* R^i f_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \arrow[r] & R^i f_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \end{tikzcd}$$

is induced by the base change morphism for $F_{Y_{\mathfrak{T}(U)_1}}$ and $\sigma_{\mathfrak{T}(U)}$.

We check that the same is true for $R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K}$. Consider the commutative diagram

$$\begin{tikzcd} Y_{\mathfrak{T}(U)_1} \arrow[r, "Fr_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)_1}"'] \arrow[d, "f_{\mathfrak{T}(U)_1}"] \arrow[rr, bend left, "F_{Y_{\mathfrak{T}(U)_1}}"] & Y'_{\mathfrak{T}(U)_1} \arrow[r] \arrow[d, "f'_{\mathfrak{T}(U)_1}"] \arrow[dr, phantom, "\square"] & Y_{\mathfrak{T}(U)_1} \arrow[d, "f_{\mathfrak{T}(U)_1}"] \\ \mathfrak{T}(U)_1 \arrow[r, equal] \arrow[d] & \mathfrak{T}(U)_1 \arrow[r, "F_{\mathfrak{T}(U)_1}"] \arrow[d] & \mathfrak{T}(U)_1 \arrow[d] \\ \mathfrak{T}(U) \arrow[r, equal] & \mathfrak{T}(U) \arrow[r, "\sigma_{\mathfrak{T}(U)}"] & \mathfrak{T}(U) \end{tikzcd}$$

and recall (4.6.5.3) that the Frobenius structure is defined by the base change map induced by $Fr_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)_1}$:

$$(F_U^* R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K})_{(\mathfrak{T}(U)_1,\mathfrak{T}(U)_1,\mathfrak{T}(U))} \simeq R^i f'_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y'_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \to R^i f_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)}$$

Since there is a lifting $\sigma_{\mathfrak{T}(U)}$ of $F_{\mathfrak{T}(U)_1}$, there is a morphism of enlargements

$$\sigma_{\mathfrak{T}(U)} : (\mathfrak{T}(U), z_{\mathfrak{T}(U)}) \to (\mathfrak{T}(U), z_{\mathfrak{T}(U)} \circ F_{\mathfrak{T}(U)_1}).$$

Since $F_U^* R^i f_{Ogus,*}\mathcal{O}_{\mathcal{Y}_U/K}$ is a crystal, there is an isomorphism

$$\sigma_{\mathfrak{T}(U)}^* R^i f_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \to R^i f'_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y'_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)}.$$

that, as recalled in section 4.6.5.3, identifies with the base change map induced by $\sigma_{\mathfrak{T}(U)}$ and $F_{\mathfrak{T}(U)_1}$. So the Frobenius structure

$$\begin{array}{ccc} \sigma^*_{\mathfrak{T}(U)}(R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K})_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \simeq (F_U^* R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K})_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} & \longrightarrow & (R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K})_{(\mathfrak{T}(U),z_{\mathfrak{T}(U)})} \\ \downarrow \simeq & & \downarrow \simeq \\ \sigma^*_{\mathfrak{T}(U)} R^i f_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} & \longrightarrow & R^i f_{\mathfrak{T}(U)_1,\mathfrak{T}(U),crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} \end{array}$$

is given by the composition of the base change maps induced by $Fr_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)_1}$ followed by the base change map induced by $\sigma_{\mathfrak{T}(U)}$ and $F_{\mathfrak{T}(U)_1}$, hence it is given by the base change map induced by $F_{Y_{\mathfrak{T}(U)_1}}$ and $\sigma_{\mathfrak{T}(U)}$.

In conclusion, ψ is compatible with the Frobenius structures of $\mathcal{M}$ and $R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K}$, so that ψ is gives an isomorphism

$$\psi : R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K} \simeq g^*\mathcal{M} \text{ in } \mathbf{F}\text{-}\mathbf{Isoc}^{(p)}(U|K)$$

and hence an isomorphism

$$\psi : R^i f_{U,Ogus,*}\mathcal{O}_{Y_U/K} \simeq g^*\mathcal{M} \text{ in } \mathbf{F}\text{-}\mathbf{Isoc}^{(1)}(U|K).$$

4.7.5 Descent

Now one has to descend from U to X. To do this, consider the closed immersion

$$U \times_X U \to U \times_k U \to \mathfrak{U}_1 \times_k \mathfrak{U}_1 \to \mathfrak{U} \times_W \mathfrak{U}$$

where the first map is a closed immersion by [SP, Tag 01KR] since X is separated. Write $\mathfrak{T}(U \times_X U)$ for the universal p-adic enlargement of $U \times_X U$ in $\mathfrak{U} \times_W \mathfrak{U}$ and q_1, q_2 for the projections

$$U \times_X U \to U \quad \text{and} \quad \mathfrak{U} \times_W \mathfrak{U} \to \mathfrak{U}.$$

Finally write $u_{\mathfrak{T}(U\times_X U)}, q_{\mathfrak{T}(U\times_X U),1}$ and $q_{\mathfrak{T}(U\times_X U),2}$ for the natural morphisms:

$$\begin{array}{ccc} (\mathfrak{T}(U\times_X U)_1, \mathfrak{T}(U\times_X U)_1, \mathfrak{T}(U\times_X U)) & \xrightarrow{u_{\mathfrak{T}(U\times_X U)}} & (U\times_X U, U\times_X U, \mathfrak{U}\times_W \mathfrak{U}) \\ {\scriptstyle q_{\mathfrak{T}(U\times_X U),2}}\downarrow\downarrow{\scriptstyle q_{\mathfrak{T}(U\times_X U),1}} & & {\scriptstyle q_2}\downarrow\downarrow{\scriptstyle q_1} \\ (\mathfrak{T}(U)_1, \mathfrak{T}(U)_1, \mathfrak{T}(U)) & \xrightarrow{\quad u \quad} & (U, U, \mathfrak{U}) \end{array}$$

and g' for $U \times_X U \to X$. By the equivalence $F1 - Fp$ in Section 4.6.3, to show that the descent diagram in $\mathbf{F}\text{-}\mathbf{Isoc}^{(1)}(U \times_X U|K)$

$$\begin{array}{ccc} q_1^* g^*\mathcal{M} & \longrightarrow & q_2^* g^*\mathcal{M} \\ \downarrow & & \downarrow \\ q_1^* g^* R^i f_{Ogus,*}\mathcal{O}_{Y/K} & \longrightarrow & q_2^* g^* R^i f_{Ogus,*}\mathcal{O}_{Y/K} \end{array} \tag{4.7.5.1}$$

is commutative, it is enough to show that it is commutative in $\mathbf{F}\text{-}\mathbf{Isoc}^{(p)}(U \times_X U|K)$. Then one decomposes 4.7.5.1 as follows:

$$\begin{array}{ccccc} q_1^* g^*\mathcal{M} & \longrightarrow & g'^{,*}\mathcal{M} & \longleftarrow & q_2^* g^*\mathcal{M} \\ \downarrow & & \downarrow & & \downarrow \\ q_1^* R^i f_{Ogus,*}\mathcal{O}_{Y/K} & \longrightarrow & g'^{,*} R^i f_{Ogus,*}\mathcal{O}_{Y/K} & \longleftrightarrow & q_2^* R^i f_{Ogus,*}\mathcal{O}_{Y/K} \end{array}$$

So it is enough to show that, for $? \in \{1, 2\}$, the following diagram is commutative

$$\begin{array}{ccc} q_?^* g^* \mathcal{M} & \longrightarrow & g^{',*}\mathcal{M} \\ \downarrow & & \downarrow \\ q_?^* g^* R^i f_{Ogus,*}\mathcal{O}_{Y/K} & \longrightarrow & g^{',*} R^i f_{Ogus,*}\mathcal{O}_{Y/K} \end{array}$$

in $\mathbf{F}$-$\mathbf{Isoc}^{(p)}(U \times_X U|K)$ or, equivalently, in $\mathbf{Isoc}^{(p)}(U \times_X U|K)$. Since

$$(-_{(\mathfrak{T}(U\times_X U), z_{\mathfrak{T}(U\times_X U)})}, \epsilon_{-,\mathfrak{U}\times_W \mathfrak{U}}) : \mathbf{Isoc}^{(p)}(U \times_X U|K) \to \mathbf{Strat}^{(p)}(U \times_X U, \mathfrak{U}\times_W \mathfrak{U}|K)$$

is an equivalence of categories, it is enough to show that

$$\begin{array}{ccc} (q_?^* g^* \mathcal{M})_{(\mathfrak{T}(U\times_X U), z_{\mathfrak{T}(U\times_X U)})} & \longrightarrow & (g^{',*}\mathcal{M})_{(\mathfrak{T}(U\times_X U), z_{\mathfrak{T}(U\times_X U)})} \\ \downarrow & & \downarrow \\ (q_?^* g^* R^i f_{Ogus,*}\mathcal{O}_{Y/K})_{(\mathfrak{T}(U\times_X U), z_{\mathfrak{T}(U\times_X U)})} & \longrightarrow & (g^{',*} R^i f_{Ogus,*}\mathcal{O}_{Y/K})_{(\mathfrak{T}(U\times_X U), z_{\mathfrak{T}(U\times_X U)})} \end{array}$$

commutes. Since $q_? : (U \times_X U, U \times_X U, \mathfrak{U}\times_W \mathfrak{U}) \to (U, U, \mathfrak{U})$ is a morphism of frame over $(X', \overline{X}', \mathfrak{X}')$ that have $(P_{(X', \overline{X}', \mathfrak{X}')})$, by the property (2) of $\mathcal{M}$ in 4.7.1, the morphism given by the isocrystals structure

$$q_?^* R^i f_{(Y_U, Y_U)/\mathfrak{U}, rig,*}\mathcal{O}^\dagger_{(Y_U, Y_U)} \to R^i f_{(Y_{U\times_X U}, Y_{U\times_X U})/\mathfrak{U}\times\mathfrak{U}, rig,*}\mathcal{O}^\dagger_{(Y_{U\times_X U}, Y_{U\times_X U})}$$

is the natural base change map. Pulling back via $u_{\mathfrak{T}(U\times_X U)}$ one finds a commutative diagram (4.6.5.2.2)

$$\begin{array}{ccc} u^*_{\mathfrak{T}(U\times_X U)} q_?^* R^i f_{(Y_U, Y_U)/\mathfrak{U}, rig,*}\mathcal{O}^\dagger_{(Y_U, Y_U)} & \longrightarrow & u^*_{\mathfrak{T}(U\times_X U)} R^i f_{(Y_{U\times_X U}, Y_{U\times_X U})/\mathfrak{U}\times_W\mathfrak{U}, rig,*}\mathcal{O}^\dagger_{(Y_{U\times_X U}, Y_{U\times_X U})} \\ \downarrow \simeq & & \downarrow \simeq \\ q^*_{\mathfrak{T}(U\times_X U),?} u^* R^i f_{(Y_U, Y_U)/\mathfrak{U}, rig,*}\mathcal{O}^\dagger_{(Y_U, Y_U)} & \longrightarrow & u^*_{\mathfrak{T}(U\times_X U)} R^i f_{(Y_{U\times_X U}, Y_{U\times_X U})/\mathfrak{U}\times_W\mathfrak{U}, rig,*}\mathcal{O}^\dagger_{(Y_{U\times_X U}, Y_{U\times_X U})} \\ \downarrow \simeq & & \downarrow \simeq \\ q^*_{\mathfrak{T}(U\times_X U),?} R^i f_{\mathfrak{T}(U)_1, \mathfrak{T}(U), crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} & \longrightarrow & R^i f_{\mathfrak{T}(U\times_X U)_1, \mathfrak{T}(U\times_X U), crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U\times_X U)_1}/\mathfrak{T}(U\times_X U)} \end{array}$$

such that the horizontal morphism are the base change morphism. So the morphism

$$\begin{array}{ccc} (q_?^* g^* \mathcal{M})_{(\mathfrak{T}(U\times_X U), z_{\mathfrak{T}(U\times_X U)})} & \longrightarrow & (g^{',*}\mathcal{M})_{(\mathfrak{T}(U\times_X U), z_{\mathfrak{T}(U\times_X U)})} \\ \downarrow \simeq & & \downarrow \simeq \\ q^*_{\mathfrak{T}(U\times_X U),?} R^i f_{\mathfrak{T}(U)_1, \mathfrak{T}(U), crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U)_1}/\mathfrak{T}(U)} & \longrightarrow & R^i f_{\mathfrak{T}(U\times_X U)_1, \mathfrak{T}(U\times_X U), crys,*}\mathcal{O}_{Y_{\mathfrak{T}(U\times_X U)_1}/\mathfrak{T}(U\times_X U)} \end{array}$$

is induced by the base change morphism. Since the same is true for $R^i f_{Ogus,*}\mathcal{O}_{Y/K}$ by construction (4.6.5.3), this shows that the descent diagram 4.7.5.1 is commutative, hence

$$\psi : g^*\mathcal{M} \simeq g^* R^i f_{Ogus,*}\mathcal{O}_{Y/K}$$

gives an isomorphism in the category of descent data for the category $\mathbf{F}$-$\mathbf{Isoc}^{(1)}(U|K)$ of U over X.

By étale and proper descent for convergent isocrystals ([Ogu84, Theorems 4.5 and 4.6]), this implies that ψ descends to an isomorphism

$$\mathcal{M} \simeq R^i f_{Ogus,*}\mathcal{O}_{Y/K} \text{ in } \mathbf{F}\text{-}\mathbf{Isoc}^{(1)}(X|K)$$

and concludes the proof of Theorem 4.6.5.4.1.

Chapter 5

Maximal tori in monodromy groups of F-isocrystals and applications (joint with Marco D'Addezio)

5.1 Introduction

Let p be a prime, let $\mathbb{F}_q$ be the finite field with $q = p^s$ elements and write $\mathbb{F}_q \subseteq \mathbb{F}$ for an algebraic closure. If X_0 is a $\mathbb{F}_q$-variety, set $X := X_0 \times_{\mathbb{F}_q} \mathbb{F}$ and write F for the s-power of the absolute Frobenius on X_0.

5.1.1 Convergent vs overconvergent F-isocrystals

From now on let X_0 be a smooth geometrically connected $\mathbb{F}_q$-variety.

5.1.1.1 Convergent and overconvergent F-isocrystals

The first Weil cohomology which has been introduced to study X_0 is the ℓ-adic étale cohomology, where $\ell \neq p$ is a prime. Its associated category of local systems $\mathbf{Weil}(X_0, \overline{\mathbb{Q}_\ell})$ is the category of Weil lisse $\overline{\mathbb{Q}_\ell}$-sheaves. While p-adic étale cohomology is not a Weil cohomology theory, moving from ℓ to p one encounters two main p-adic cohomology theories: crystalline and rigid cohomology. These two give rise to different categories of p-adic "local systems": the category $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ of $\overline{\mathbb{Q}_p}$-linear convergent F-isocrystals over X_0 and the category $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$ of $\overline{\mathbb{Q}_p}$-linear overconvergent F-isocrystals over X_0. By [Ked04], the two categories are related by a natural fully faithful functor $(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0) \to \mathbf{F}\text{-}\mathbf{Isoc}(X_0)$. When X_0 is proper, $(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0) \to \mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ is an equivalence, but in general the two categories have different behaviours. While $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$ shares with $\mathbf{Weil}(X_0, \overline{\mathbb{Q}_\ell})$ many properties (see [D'Ad17] or [Ked18]), the category $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ has some exceptional p-adic features.

5.1.1.2 Slopes

One of the exceptional p-adic features of $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ is the theory of slopes; see [Ked17, Sections 3 and 4]. For every $\mathcal{E}_0$ in $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ of rank r and every $x_0 \in X_0$, let $\{a_i^{x_0}(\mathcal{E}_0)\}_{1\leq i\leq r}$ be the set of slopes of $\mathcal{E}_0$ at x_0 (with the convention that $a_1^{x_0}(\mathcal{E}_0) \leq \cdots \leq a_r^{x_0}(\mathcal{E}_0)$). If $\eta_0 \in X_0$ is the generic point, we call $a_i^{\eta_0}(\mathcal{E}_0)$ the generic slopes of $\mathcal{E}_0$. A subobject $\mathcal{F}_0 \subseteq \mathcal{E}_0$ is said to be of minimal generic slope if all its slopes are equal to $a_1^{\eta_0}(\mathcal{E}_0)$.

5.1.1.3 Main result

Let $\mathcal{E}_0^\dagger$ be in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$. Write $\mathbf{Isoc}^\dagger(X_0)$ and $\mathbf{Isoc}(X_0)$ for the $\overline{\mathbb{Q}_p}$-linear categories of overconvergent isocrystals and convergent isocrystals respectively and consider the natural commutative diagram of functors

$$\begin{array}{ccc} \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0) & \xrightarrow{(-)^{geo}} & \mathbf{Isoc}^\dagger(X_0) \\ \downarrow{\scriptstyle (-)^{conv}} & & \downarrow{\scriptstyle (-)^{conv}} \\ \mathbf{F}\text{-}\mathbf{Isoc}(X_0) & \xrightarrow{(-)^{geo}} & \mathbf{Isoc}(X_0). \end{array} \tag{5.1.1.3.1}$$

To simplify the notation we set:

$$\mathcal{E}^\dagger := (\mathcal{E}_0^\dagger)^{geo}; \quad \mathcal{E}_0 := (\mathcal{E}_0^\dagger)^{conv}; \quad \mathcal{E} := ((\mathcal{E}_0^\dagger)^{conv})^{geo}.$$

Our main result highlights a new relationship between $\mathcal{E}_0^\dagger$ and the subobjects $\mathcal{F}_0 \subseteq \mathcal{E}_0$ in $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ of minimal generic slope. Write $\mathbb{1}_0^\dagger$ for the overconvergent F-isocrystals $\mathcal{O}_{X_0}^\dagger$ endowed with the trivial Frobenius structure. Our main result is the following.

Theorem 5.1.1.3.2. If $\mathcal{E}_0^\dagger$ is irreducible in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$ and there exists a subobject $\mathcal{F}_0 \subseteq \mathcal{E}_0$ of minimal generic slope in $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ such that $\mathrm{Hom}_{\mathbf{Isoc}(X_0)}(\mathcal{F}, \mathbb{1}) \neq 0$, then $\mathcal{E}^\dagger \simeq \mathbb{1}^\dagger$ in $\mathbf{Isoc}^\dagger(X_0)$.

Remark 5.1.1.3.3. Theorem 5.1.1.3.2 proves a particular case of the conjecture in [Ked17, Remark 5.14]. Even if the conjecture turned out to be false in general, Theorem 5.1.1.3.2 corresponds, with the notation of [Ked17, Remark 5.14], to the case when $\mathcal{F}_1 \subseteq \mathcal{E}_1$ has minimal slope and $\mathcal{E}_2$ is the convergent isocrystal $\mathcal{O}_{X_0}$ endowed with some Frobenius structure.

5.1.1.4 Torsion points of abelian varieties

Before explaining the main ingredients in the proof of Theorem 5.1.1.3.2, let us describe an application to torsion points of abelian varieties. Let $\mathbb{F} \subseteq k$ be a finitely generated field extension. Let A be a k-abelian variety and recall the Lang-Néron Theorem.

Fact 5.1.1.4.1 ([LN59]). If $\mathrm{Tr}_{k/\mathbb{F}}(A) = 0$, then $A(k)$ is a finitely generated abelian group.

By Fact 5.1.1.4.1, denoting by $A^{(n)}$ the Frobenius twist of A by the p^n-power of the absolute Frobenius, we have a tower of finite groups $A(k)_{\mathrm{tors}} \subseteq A^{(1)}(k)_{\mathrm{tors}} \subseteq A^{(2)}(k)_{\mathrm{tors}} \subseteq \dots$. In June 2011, in a correspondence with Langer and Rössler, Esnault asked whether this chain is eventually stationary. An equivalent way to formulate the question is to ask whether the group of k^{perf}-rational torsion points $A(k^{\mathrm{perf}})_{\mathrm{tors}}$ is a finite group, where k^{perf} is a perfect closure of k. As an application of Theorem 5.1.1.3.2 we give a positive answer to her question.

Theorem 5.1.1.4.2. If $\mathrm{Tr}_{k/\mathbb{F}}(A) = 0$, then $A(k^{\mathrm{perf}})_{\mathrm{tors}}$ is a finite abelian group.

Remark 5.1.1.4.3. Theorem 5.1.1.4.2 was already known for elliptic curves ([Lev68]) and ordinary abelian varieties ([Rös17, Theorem 1.4]).

When ℓ is a prime $\neq p$, one has $A[\ell^\infty](k^{\mathrm{perf}}) = A[\ell^\infty](k)$, hence Theorem 5.1.1.4.2 amounts to show that $A[p^\infty](k^{\mathrm{perf}})$ is finite. To relate Theorem 5.1.1.4.2 with Theorem 5.1.1.3.2 we use then crystalline Dieudonné theory ([BBM82]). The proof of Theorem 5.1.1.4.2 is by contradiction. If $|A[p^\infty](k^{\mathrm{perf}})| = \infty$, there exists a monomorphism $\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow A[p^\infty]^{\text{ét}}$ from the trivial p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$ to the étale part $A[p^\infty]^{\text{ét}}$ of the p-divisible group $A[p^\infty]$ of A. Spreading out to a "nice" model $\mathcal{A}/\mathcal{X}$ of A/k and applying the contravariant crystalline Dieudonné functor $\mathbb{D}$, one gets an epimorphism of F-isocrystals $\mathbb{D}(\mathcal{A}[p^\infty]^{\text{ét}}) \twoheadrightarrow \mathbb{D}((\mathbb{Q}_p/\mathbb{Z}_p)_{\mathcal{X}}) \simeq \mathcal{O}_{\mathcal{X}}$ over $\mathcal{X}$. By a descent argument and a careful use of Theorem 5.1.1.3.2, the quotient extends to a quotient $\mathbb{D}(\mathcal{A}[p^\infty]) \twoheadrightarrow \mathcal{O}_{\mathcal{X}}$ over $\mathcal{X}$. Going back to p-divisible groups, this gives an injective map $\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow A[p^\infty]$ over k. Therefore $A[p^\infty](k)$ would be an infinite group, contradicting Fact 5.1.1.4.1.

5.1.2 Maximal tori of monodromy groups

To prove Theorem 5.1.1.3.2, we study the monodromy groups associated to the objects involved.

5.1.2.1 Monodromy groups

The categories in (5.1.1.3.1) are neutral Tannakian categories and the choice of an $\mathbb{F}$-point x of X_0 induces fibre functors for all of them. Hence, via the Tannakian formalism, one obtains an algebraic group $G(-)$ for each of $\mathcal{E}_0^\dagger$, $\mathcal{E}^\dagger$, $\mathcal{E}_0$ and $\mathcal{E}$ and a commutative diagram of closed immersions:

$$\begin{tikzcd} G(\mathcal{E}) \arrow[r, hook] \arrow[d, hook] & G(\mathcal{E}_0) \arrow[d, hook] \\ G(\mathcal{E}^\dagger) \arrow[r, hook] & G(\mathcal{E}_0^\dagger). \end{tikzcd}$$

5.1.2.2 Maximal tori

By the global monodromy theorem for overconvergent F-isocrystals ([D'Ad17, Corollary 3.5.5]), if $\mathcal{E}_0^\dagger$ is irreducible in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$ with finite order determinant, the group $G(\mathcal{E}_0^\dagger)/G(\mathcal{E}^\dagger)$ is finite. Even though $\mathcal{E}_0$ is not irreducible in general in $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ and the global monodromy theorem does not hold in $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$, we show that $G(\mathcal{E}_0)/G(\mathcal{E})$ is still finite (Proposition 5.3.1.1).

Recall ([D'Ad17, Definition 3.1.11]) that $\mathcal{E}_0^\dagger$ is said to be p-plain, if the eigenvalues of the Frobenii at closed points are algebraic number which are ℓ-adic unit for every prime $\ell \neq p$. To prove that $G(\mathcal{E})$ is "big enough", we prove the following.

Theorem 5.1.2.2.1. If $\mathcal{E}_0^\dagger$ in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$ is pure and p-plain, then $G(\mathcal{E})$ contains a maximal torus of $G(\mathcal{E}^\dagger)$.

The key input in the proof of Theorem 5.1.2.2.1 is the existence of Frobenius tori ([D'Ad17, Theorem 4.2.6]) of overconvergent F-isocrystals realizing a maximal torus of $G(\mathcal{E}_0^\dagger)$.

Remark 5.1.2.2.2. In [Cre92, page 460] Crew asks whether, under the assumptions of Theorem 5.1.2.2.1, $G(\mathcal{E})$ is a parabolic subgroup of $G(\mathcal{E}^\dagger)$. In the subsequent articles [Cre92b] and [Cre94], he gives a positive answer to his question in some particular cases. Since parabolic subgroups of reductive groups always contain a maximal torus, Theorem 5.1.2.2.1 is an evidence for Crew's expectation.

To deduce Theorem 5.1.1.3.2 from Proposition 5.3.1.1, one first reduces to the situation where $\mathcal{E}_0$ has finite order determinant. To simplify, let us assume that $\mathcal{F}_0 = \mathcal{E}_0^1$ is the maximal subobject of minimal generic slope and that $G(\mathcal{E}_0)$ is connected. Then Proposition 5.3.1.1 implies that $G(\mathcal{E}) = G(\mathcal{E}_0)$, hence that the quotient $\mathcal{E}^1 \twoheadrightarrow \mathbb{1}$ in $\mathbf{Isoc}(X_0)$ promotes to a quotient $\mathcal{E}_0^1 \twoheadrightarrow \mathbb{1}_0$ in $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$. In particular, the minimal slope of $\mathcal{E}_0$ is zero. Since the determinant of $\mathcal{E}_0$ has finite order, this implies that $\mathcal{E}_0 = \mathcal{E}_0^1$ hence that $\mathcal{E}_0$ admits a quotient $\mathcal{E}_0 \twoheadrightarrow \mathbb{1}_0$ in $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$. As $(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0) \to \mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ is fully faithful, $\mathcal{E}_0^\dagger$ admits a quotient $\mathcal{E}_0^\dagger \twoheadrightarrow \mathbb{1}_0^\dagger$ in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$ hence, since $\mathcal{E}_0^\dagger$ is irreducible, $\mathcal{E}_0^\dagger \simeq \mathbb{1}_0^\dagger$.

5.1.2.3 Weak (weak) semi-simplicity

As an additional outcome of Theorem 5.1.2.2.1, we get a semi-simplicity result for extensions of constant F-isocrystals. For us, a constant convergent F-isocrystal will be an object $\mathcal{E}_0 \in \mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ such that $\mathcal{E} \simeq \mathbb{1}^r$ for some integer $r \geq 0$.

Write $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{pure}^\dagger}(X_0)$ for the Tannakian subcategory of $\mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ generated by the essential image via $(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0) \to \mathbf{F}\text{-}\mathbf{Isoc}(X_0)$ of pure objects in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$. The category $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{pure}^\dagger}(X_0)$ is large enough to contain all the convergent F-isocrystals "coming from geometry": for every smooth and proper morphism $f_0 : Y_0 \to X_0$ and every $i \in \mathbb{N}$, the subquotients of the higher direct image $R^i f_{0,\mathrm{crys}*}\mathcal{O}_{Y_0}$ are in $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{pure}^\dagger}(X_0)$ by [KM74] and [Shi08b] (see Theorem 4.2.1.1.2 and Fact 4.2.2.1 in Chapter 4). Thanks to a group-theoretic argument (Lemma 5.2.3.2.3), Theorem 5.1.2.2.1 implies the following.

Corollary 5.1.2.3.1. A convergent F-isocrystal in $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{pure}^\dagger}(X_0)$ which is an extension of constant convergent F-isocrystals is constant.

Remark 5.1.2.3.2. In [Chai13, Conjecture 7.4 and Remark 7.4.1], Chai conjectures that if $\mathcal{E}_0^\dagger$ is the higher direct image of a family of ordinary abelian varieties, and $\mathcal{E}_0^1 \subseteq \mathcal{E}_0$ is the maximal subobject of minimal generic slope, then the monodromy group $G(\mathcal{E}^1)$ of $\mathcal{E}^1$ is reductive. Since $G(\mathcal{E}^1)$ is a quotient of $G(\mathcal{E})$, Corollary 5.1.2.3.1 implies that $G(\mathcal{E}^1)$ has no unipotent quotients, hence it may be thought as a first step towards his conjecture.

5.1.3 Organization of the chapter

In Section 5.2 we study the monodromy groups of (over)convergent F-isocrystals and we prove Theorem 5.1.2.2.1. The remain part of the Chapter is devoted to applications: in Section 5.3 we prove Theorem 5.1.1.3.2 and in Section 5.4 we prove Theorem 5.1.1.4.2.

5.1.4 Acknowledgements

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5.2 Proof of Theorem 5.1.2.2.1

Let X_0 be a smooth geometrically connected $\mathbb{F}_q$-variety. Let $\mathcal{E}_0^\dagger$ be in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$.

5.2.1 Monodromy groups of (over)convergent F-isocrystal

For $? \in \{\emptyset, \dagger\}$, write $\langle \mathcal{E}_0^? \rangle$ (resp. $\langle \mathcal{E}^? \rangle$) for the smallest Tannakian subcategory of $\mathbf{F}\text{-}\mathbf{Isoc}^?(X_0)$ (resp. $\mathbf{Isoc}^?(X_0)$) containing $\mathcal{E}_0^?$ (resp. $\mathcal{E}^?$). The choice of a geometric closed point x_1 of X_0 induces fibre functors x_1^* for all these categories, hence via Tannakian duality, we get algebraic groups $G(\mathcal{E}_0^?)$ and $G(\mathcal{E}^?)$. The natural commutative diagram of faithful tensor functors

$$\begin{tikzcd} \langle \mathcal{E}_0^\dagger \rangle \arrow[r, "(-)^{geo}"] \arrow[d, "(-)^{conv}"] & \langle \mathcal{E}^\dagger \rangle \arrow[d, "(-)^{conv}"] \\ \langle \mathcal{E}_0 \rangle \arrow[r, "(-)^{geo}"] & \langle \mathcal{E} \rangle. \end{tikzcd}$$

induces a commutative diagram of closed immersions of algebraic groups

$$\begin{tikzcd} G(\mathcal{E}) \arrow[r, hook] \arrow[d, hook] & G(\mathcal{E}_0) \arrow[d, hook] \\ G(\mathcal{E}^\dagger) \arrow[r, hook] & G(\mathcal{E}_0^\dagger). \end{tikzcd}$$

5.2.2 Constant F-isocrystals and the fundamental exact sequence

Recall ([D'Ad17, Definition A.2.3]) that a $\mathcal{F}_0^?$ in $\mathbf{F}\text{-}\mathbf{Isoc}^?(X_0)$ is said to be constant if $\mathcal{F}^? \simeq (\mathbb{1}^?)^r$ for some integer $r \geq 1$. Then a $\mathcal{F}_0^?$ in $\mathbf{F}\text{-}\mathbf{Isoc}^?(X_0)$ is constant if and only if there exists $\mathcal{M}_0$ in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathrm{Spec}(\mathbb{F}_q)) \simeq \mathbf{F}\text{-}\mathbf{Isoc}(\mathrm{Spec}(\mathbb{F}_q))$ such that $g^*\mathcal{M}_0 \simeq \mathcal{F}_0^?$, where $g : X_0 \to \mathrm{Spec}(\mathbb{F}_q)$ is the structural morphism.

By [D'Ad17, Appendix], the subgroup $G(\mathcal{E}^?) \subseteq G(\mathcal{E}_0^?)$ is normal. The quotient $G(\mathcal{E}_0^?)/G(\mathcal{E}^?)$ is abelian and identifies canonically with the Tannakian group $G(\mathcal{E}_0^?)$ of the full Tannakian subcategory $\langle\mathcal{E}_0^?\rangle_{\mathrm{cst}} \subseteq \langle\mathcal{E}_0^?\rangle$ made by the $\mathcal{F}_0^?$ in $\langle\mathcal{E}_0^?\rangle$ which are constant.

The natural functor $\langle\mathcal{E}_0^\dagger\rangle_{\mathrm{cst}} \to \langle\mathcal{E}_0\rangle_{\mathrm{cst}}$ is fully faithful and its essential image is closed under subquotients. Hence we have the following natural commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & G(\mathcal{E}) \arrow[r] \arrow[d, hook] & G(\mathcal{E}_0) \arrow[r] \arrow[d, hook] & G(\mathcal{E}_0)^{\mathrm{cst}} \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & G(\mathcal{E}^\dagger) \arrow[r] & G(\mathcal{E}_0^\dagger) \arrow[r] & G(\mathcal{E}_0^\dagger)^{\mathrm{cst}} \arrow[r] & 0, \end{tikzcd} \tag{5.2.2.1}$$

in which the left and the central vertical arrows are injective and the right one is surjective.

Remark 5.2.2.2. It is not clear, a priori, whether the surjection $\varphi : G(\mathcal{E}_0)^{\mathrm{cst}} \twoheadrightarrow G(\mathcal{E}_0^\dagger)^{\mathrm{cst}}$ is an isomorphism. Via the Tannakian formalism, to prove the injectivity of φ, one has to show that the functor $\langle\mathcal{E}_0^\dagger\rangle_{\mathrm{cst}} \to \langle\mathcal{E}_0\rangle_{\mathrm{cst}}$ is essentially surjective. While every $\mathcal{F}_0 \in \langle\mathcal{E}_0\rangle_{\mathrm{cst}}$ comes from an object $\mathcal{F}_0^\dagger$ in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$, it is not clear whether $\mathcal{F}_0^\dagger$ is in $\langle\mathcal{E}_0^\dagger\rangle$. One can use Theorem 5.1.2.2.1 to show that, if $\mathcal{E}_0^\dagger$ is pure and p-plain, then $\varphi : G(\mathcal{E}_0)^{\mathrm{cst}} \twoheadrightarrow G(\mathcal{E}_0^\dagger)^{\mathrm{cst}}$ is an isogeny.

5.2.3 Maximal tori of (over)convergent F-isocrystals

For every algebraic group G, write $\mathrm{rk}(G)$ for the reductive rank of G (i.e. the dimension of a maximal torus of G) and recall that a subgroup $H \subseteq G$ is said to be of maximal rank if $\mathrm{rk}(H) = \mathrm{rk}(G)$.

5.2.3.1 Maximal tori of overconvergent F-isocrystals

For every $x_0 \in |X_0|$, the natural functor

$$\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathrm{Spec}(\mathbb{F}_q(x_0))) \to \mathbf{F}\text{-}\mathbf{Isoc}(\mathrm{Spec}(\mathbb{F}_q(x_0)))$$

is an equivalence of categories. Hence there is a commutative diagram of closed immersions

$$\begin{tikzcd} G(x_0^*\mathcal{E}_0) \arrow[r, hook] \arrow[d, "\simeq"] & G(\mathcal{E}_0) \arrow[d, hook] \\ G(x_0^*\mathcal{E}_0^\dagger) \arrow[r, hook] & G(\mathcal{E}_0^\dagger) \end{tikzcd} \tag{5.2.3.1.1}$$

where the map $G(x_0^*\mathcal{E}_0) \to G(x_0^*\mathcal{E}_0^\dagger)$ is an isomorphism. Recall the following result from [D'Ad17].

Fact 5.2.3.1.2. If $\mathcal{E}_0^\dagger$ is and pure and p-plain, there exist infinitely many $x_0 \in |X_0|$ such that $G(x_0^*\mathcal{E}_0^\dagger) \subseteq G(\mathcal{E}_0^\dagger)$ is a subgroup of maximal rank.

Proof. In [D'Ad17, Theorem 4.2.6] the result is proven for subquotients of an E-rational, p-plain and pure overconvergent F-isocrystal which admits an E-compatible lisse sheaf (cf. [D'Ad17]). Thanks to [D'Ad17, Theorem 3.4.3], this condition is always satisfied by $\mathcal{E}_0^\dagger$. □

5.2.3.2 Maximal tori of convergent F-isocrystals

From Fact 5.2.3.1.2 and (5.2.3.1.1) we get:

Corollary 5.2.3.2.1. If $\mathcal{E}_0^\dagger$ is pure and p-plain, then $G(\mathcal{E}_0) \hookrightarrow G(\mathcal{E}_0^\dagger)$ is a subgroup of maximal rank.

Corollary 5.2.3.2.2. If $\mathcal{E}_0^\dagger$ is semi-simple, then $G(\mathcal{E}_0)^{\mathrm{cst}}$ and $G(\mathcal{E}_0^\dagger)^{\mathrm{cst}}$ are groups of multiplicative type.

Proof. Since the groups $G(\mathcal{E}_0^\dagger)^{\mathrm{cst}}$ and $G(\mathcal{E}_0)^{\mathrm{cst}}$ are commutative, it is enough to show that they are also reductive. The former is a quotient of $G(\mathcal{E}_0^\dagger)$, which is reductive because $\mathcal{E}_0^\dagger$ is semi-simple. The latter is a quotient of $G(\mathcal{E}_0)$, which by Corollary 5.2.3.2.1 is a subgroup of $G(\mathcal{E}_0^\dagger)$ of maximal rank. Since $G(\mathcal{E}_0)^{\mathrm{cst}}$ is commutative, $R_u(G(\mathcal{E}_0)^{\mathrm{cst}})$ is a quotient of $G(\mathcal{E}_0)$. Thus $R_u(G(\mathcal{E}_0)^{\mathrm{cst}})$ is trivial by the (group theoretic) Lemma 5.2.3.2.3 below. □

Lemma 5.2.3.2.3. Let L be an algebraically closed field of characteristic 0, let G be a reductive group over L and H a subgroup of G of maximal rank. There does not exist any non-trivial morphism from H to a unipotent group. Equivalently, the group $\mathrm{Ext}^1_H(L, L)$ vanishes.

Proof. It is enough to show that there are no non-trivial morphisms from H to $\mathbb{G}_a$. Assume by contradiction that there exists a normal subgroup $K \subseteq H$ such that $H/K \simeq \mathbb{G}_a$. Since every map from a torus to $\mathbb{G}_a$ is trivial, K would be a subgroup of G of maximal rank, so that, by [Mil15, Lemma 18.52], $N_G(K^\circ)^\circ = K^\circ$. Since $K \subseteq H$ is normal, $H \subseteq N_G(K^\circ)$. Hence $H^\circ = K^\circ$ so that $H/K \simeq \mathbb{G}_a$ would be finite, a contradiction. □

5.2.4 Proof of Theorem 5.1.2.2.1

Retain the notation and the assumptions as in Theorem 5.1.2.2.1. Since it is enough to prove Theorem 5.1.2.2.1 after twist, we may assume that $\mathcal{E}_0^\dagger$ is pure of weight 0. For every algebraic group G, write $X^*(G)$ for its group of characters.

Choose a set of rank 1 convergent F-isocrystals $\chi_{1,0}, \dots, \chi_{n,0}$ in $\langle \mathcal{E}_0 \rangle_{\mathrm{cst}}$ that generates $X^*(G(\mathcal{E}_0)^{\mathrm{cst}})$. As every constant F-isocrystal comes from an overconvergent F-isocrystal, for every i, the character $\chi_{i,0}$ is the essential image of a $\chi_{i,0}^\dagger$ in $\mathbf{F\text{-}Isoc}^\dagger(X_0)$ via $\mathbf{F\text{-}Isoc}^\dagger(X_0) \to \mathbf{F\text{-}Isoc}(X_0)$. Write

$$\widetilde{\mathcal{E}}_0^\dagger := \mathcal{E}_0^\dagger \oplus \bigoplus_{i=1}^n \chi_{i,0}^\dagger \text{ in } \mathbf{F\text{-}Isoc}^\dagger(X_0).$$

By construction, the groups $X^*(G(\widetilde{\mathcal{E}}_0)^{\mathrm{cst}})$ and $X^*(G(\mathcal{E}_0)^{\mathrm{cst}})$ are isomorphic. Moreover, since $\widetilde{\mathcal{E}}^\dagger \simeq \mathcal{E}^\dagger \oplus \overline{\mathbb{Q}}_p^{\oplus n}$ and $\widetilde{\mathcal{E}} \simeq \mathcal{E} \oplus \overline{\mathbb{Q}}_p^{\oplus n}$, we get isomorphisms $G(\widetilde{\mathcal{E}}^\dagger) \simeq G(\mathcal{E}^\dagger)$ and $G(\widetilde{\mathcal{E}}) \simeq G(\mathcal{E})$. Hence it is enough to show that $\mathrm{rk}(G(\widetilde{\mathcal{E}}^\dagger)) = \mathrm{rk}(G(\widetilde{\mathcal{E}}))$. Consider the commutative exact diagram (5.2.2.1)

$$\begin{tikzcd}
0 \arrow[r] & G(\widetilde{\mathcal{E}}) \arrow[r] \arrow[d, hook] & G(\widetilde{\mathcal{E}}_0) \arrow[r] \arrow[d, hook] & G(\widetilde{\mathcal{E}}_0)^{\mathrm{cst}} \arrow[r] \arrow[d, two heads, "h"] & 0 \\
0 \arrow[r] & G(\widetilde{\mathcal{E}}^\dagger) \arrow[r] & G(\widetilde{\mathcal{E}}_0^\dagger) \arrow[r] & G(\widetilde{\mathcal{E}}_0^\dagger)^{\mathrm{cst}} \arrow[r] & 0.
\end{tikzcd}$$

As $\widetilde{\mathcal{E}}_0^\dagger$ is still p-plain and pure of weight 0, by Corollary 5.2.3.2.1, $\mathrm{rk}(G(\widetilde{\mathcal{E}}_0)) = \mathrm{rk}(G(\widetilde{\mathcal{E}}_0^\dagger))$. Since the reductive rank is additive in exact sequences, it is enough to show that $h : G(\widetilde{\mathcal{E}}_0)^{\mathrm{cst}} \twoheadrightarrow G(\widetilde{\mathcal{E}}_0^\dagger)^{\mathrm{cst}}$ is an isomorphism. Since $h : G(\widetilde{\mathcal{E}}_0)^{\mathrm{cst}} \twoheadrightarrow G(\widetilde{\mathcal{E}}_0^\dagger)^{\mathrm{cst}}$ is surjective and $G(\widetilde{\mathcal{E}}_0)^{\mathrm{cst}}$ and $G(\widetilde{\mathcal{E}}_0^\dagger)^{\mathrm{cst}}$ are groups of multiplicative type by Corollary 5.2.3.2.2, it is enough to show that the map $h^* : X^*(G(\widetilde{\mathcal{E}}_0^\dagger)^{\mathrm{cst}}) \to X^*(G(\widetilde{\mathcal{E}}_0)^{\mathrm{cst}})$ is surjective. Recall that $X^*(G(\widetilde{\mathcal{E}}_0)^{\mathrm{cst}}) = X^*(G(\mathcal{E}_0)^{\mathrm{cst}})$

is generated by $\chi_{1,0}, \ldots, \chi_{n,0}$. Since, by construction, the character $\chi_{i,0}^{\dagger} \in X^*(G(\widetilde{\mathcal{E}}_0^{\dagger})^{\mathrm{cst}})$ is sent by h^* to $\chi_{i,0}$ the morphism $h^* : X^*(G(\widetilde{\mathcal{E}}_0^{\dagger})^{\mathrm{cst}}) \to X^*(G(\widetilde{\mathcal{E}}_0)^{\mathrm{cst}})$ is surjective. This concludes the proof of Theorem 5.1.2.2.1.

5.2.5 Corollaries

From Fact 5.2.3.1.2, Theorem 5.1.2.2.1, diagram (5.2.2.1) and the additivity of the reductive ranks with respect to exact sequences we deduce:

Corollary 5.2.5.1. The reductive rank of $G(\mathcal{E}_0^{\dagger})^{\mathrm{cst}}$ is the same as the one of $G(\mathcal{E}_0)^{\mathrm{cst}}$.

Another consequence of Theorem 5.2.2.1 is the following result that we will not use, but which has its own interest. We have already discussed it in §5.1.2.3.

Corollary 5.2.5.2. If $\mathcal{E}_0^{\dagger}$ is pure, then every $\mathcal{F}_0 \in \langle \mathcal{E}_0 \rangle$ which is an extension of constant F-isocrystals is constant.

Proof. The statement is equivalent to the fact that the group $\mathrm{Ext}^1_{G(\mathcal{E})}(\overline{\mathbb{Q}_p}, \overline{\mathbb{Q}_p})$ vanishes. Since $\mathcal{E}_0^{\dagger}$ is pure, by [D'Ad17, Corollary 3.4.9], $\mathcal{E}^{\dagger}$ is semi-simple. Therefore, we may take the semi-simplification of $\mathcal{E}_0^{\dagger}$ without changing $G(\mathcal{E})$. Moreover, by [D'Ad17, Proposition 3.4.2], we may twist each irreducible summand of $\mathcal{E}_0^{\dagger}$ in order to get a p-plain overconvergent F-isocrystal. Even this operation does not change $G(\mathcal{E})$. The result then follows from Theorem 5.1.2.2.1 by Lemma 5.2.3.2.3. □

5.3 Proof of Theorem 5.1.1.3.2

Let X_0 be a smooth geometrically connected $\mathbb{F}_q$-variety and let $\mathcal{E}_0^{\dagger}$ be in $\mathbf{F\text{-}Isoc}^{\dagger}(X_0)$.

5.3.1 Proof of Theorem 5.1.1.3.2

Before proving Theorem 5.1.1.3.2, let us show the following consequence of Theorem 5.1.2.2.1 and of the global monodromy theorem for overconvergent F-isocrystals.

Proposition 5.3.1.1. If $\mathcal{E}_0^{\dagger}$ is irreducible with finite order determinant, then $G(\mathcal{E}_0)^{\mathrm{cst}}$ is finite. In particular, every constant subquotient of the F-isocrystal $\mathcal{E}_0$ is finite.

Proof. By the Langlands correspondence for lisse sheaves and overconvergent F-isocrystals ([Laf02], [Abe18]) every irreducible overconvergent F-isocrystal with finite order determinant is pure and p-plain (see for example [D'Ad17, Theorem 3.6.6]). Since $G(\mathcal{E}_0^{\dagger})^{\mathrm{cst}}$ is finite by the global monodromy theorem for overconvergent F-isocrystals (see e.g. [D'Ad17, Corollary 3.4.5]), it is enough to show that $G(\mathcal{E}_0^{\dagger})^{\mathrm{cst}}$ and $G(\mathcal{E}_0)^{\mathrm{cst}}$ have the same dimension. Since, by Corollary 5.2.3.2.2, the groups $G(\mathcal{E}_0)^{\mathrm{cst}}$ and $G(\mathcal{E}_0^{\dagger})^{\mathrm{cst}}$ are of multiplicative type, we conclude by Corollary 5.2.5.1. □

Proof of Theorem 5.1.1.3.2. Retain the notation and the assumptions as in Theorem 5.1.1.3.2. Since both the hypothesis and the conclusion are invariant under twist, by [Abe15, Lemma 6.1], we can then assume that $\mathcal{E}_0^{\dagger}$ has finite order determinant, so that $Det(\mathcal{E}_0^{\dagger})$ is unit-root. We first prove that $\mathcal{E}_0^{\dagger}$ is unit-root as well. If r is the rank of $\mathcal{E}_0^{\dagger}$, since

$$\sum_{i=1}^{r} a_i^{\eta}(\mathcal{E}_0^{\dagger}) = a_1^{\eta}(\det(\mathcal{E}_0^{\dagger})) = 0 \quad \text{and} \quad a_1^{\eta}(\mathcal{E}_0^{\dagger}) \leq \cdots \leq a_r^{\eta}(\mathcal{E}_0^{\dagger}),$$

it suffices to show that $a_1^\eta(\mathcal{E}_0^\dagger) = 0$. Let $\mathcal{F} \twoheadrightarrow \mathcal{T}$ be the maximal trivial quotient of $\mathcal{F}$. By maximality, it lifts to a quotient $\mathcal{F}_0 \twoheadrightarrow \mathcal{T}_0$, where $\mathcal{T}_0$ is a constant F-isocrystal. Since $\mathcal{E}_0^\dagger$ in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_0)$ satisfies the assumptions of Proposition 5.3.1.1, $\mathcal{T}_0$ is finite. As the F-isocrystal $\mathcal{F}_0$ is of minimal generic slope and it admits a non-zero quotient which is finite, it is unit-root. This implies that $a_1^\eta(\mathcal{E}_0^\dagger) = 0$, as we wanted.

We now prove that $\mathcal{E}_0^\dagger$ has rank 1. Since $\mathcal{E}_0^\dagger$ is unit-root, by [Ked17, Theorem 3.9], the functor $\langle \mathcal{E}_0^\dagger \rangle \to \langle \mathcal{E}_0 \rangle$ is an equivalence of categories. Therefore, if $\mathcal{E}_0$ has a constant subquotient, the same is true for $\mathcal{E}_0^\dagger$. But $\mathcal{E}_0^\dagger$ is irreducible by assumption, thus $\mathcal{E}_0^\dagger$ has to be itself a constant F-isocrystal. Since irreducible constant F-isocrystals have rank 1, this ends the proof. □

Remark 5.3.1.2. Theorem 5.1.1.3.2 is false in general if we do not assume that $\mathcal{F}_0 \subseteq \mathcal{E}_0$ is of minimal generic slope. A counterexample is provided in [Ked17, Example 5.15].

5.3.2 A corollary

Corollary 5.3.2.1. If $\mathcal{E}^\dagger$ is semi-simple and $\mathcal{F}_0 \subseteq \mathcal{E}_0$ is of minimal generic slope, then the restriction morphism $\mathrm{Hom}_{\mathbf{Isoc}(X_0)}(\mathcal{E}, \mathbb{1}) \to \mathrm{Hom}_{\mathbf{Isoc}(X_0)}(\mathcal{F}, \mathbb{1})$ is surjective.

Proof. As $\mathcal{E}^\dagger$ is semi-simple, replacing $\mathcal{E}_0^\dagger$ with its semi-simplification we do not change the isomorphism class of $\mathcal{E}^\dagger$. Thus we may assume that $\mathcal{E}_0^\dagger$ is semi-simple. The proof is then an induction on the number n of summands of some decomposition of $\mathcal{E}_0^\dagger$ in irreducible overconvergent F-isocrystals. If $n = 1$ this follows from Theorem 5.1.1.3.2. Suppose now that the result is known for every positive integer $m < n$ and take an irreducible subobject $\mathcal{G}_0^\dagger$ of $\mathcal{E}_0^\dagger$. Write $\mathcal{H}_0 := \mathcal{G}_0 \times_{\mathcal{E}_0} \mathcal{F}_0$ and consider the following commutative diagram with exact rows and injective vertical arrows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H} \arrow[r] \arrow[d, hook] & \mathcal{F} \arrow[r] \arrow[d, hook] & \mathcal{F}/\mathcal{H} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}/\mathcal{G} \arrow[r] & 0. \end{tikzcd}$$

As $\mathcal{E}_0^\dagger$ is semi-simple, the quotient $\mathcal{E}_0^\dagger \twoheadrightarrow \mathcal{E}_0^\dagger/\mathcal{G}_0^\dagger$ admits a splitting. So, applying $\mathrm{Hom}_{\mathbf{Isoc}(X_0)}(-, \mathbb{1})$, we get the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathbf{Isoc}(X_0)}(\mathcal{E}/\mathcal{G}, \mathbb{1}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{Isoc}(X_0)}(\mathcal{E}, \mathbb{1}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{Isoc}(X_0)}(\mathcal{G}, \mathbb{1}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_{\mathbf{Isoc}(X_0)}(\mathcal{F}/\mathcal{H}, \mathbb{1}) \arrow[r] & \mathrm{Hom}_{\mathbf{Isoc}(X_0)}(\mathcal{F}, \mathbb{1}) \arrow[r] & \mathrm{Hom}_{\mathbf{Isoc}(X_0)}(\mathcal{H}, \mathbb{1}). \end{tikzcd}$$

Since $\mathcal{H}_0$ and $\mathcal{F}_0/\mathcal{H}_0$ are subobjects of minimal slope of $\mathcal{G}_0$ and $\mathcal{E}_0/\mathcal{G}_0$ respectively, by the induction hypothesis, the left and the right vertical arrows are surjective. By diagram chasing, this implies that the central vertical arrow is also surjective, as we wanted. □

Remark 5.3.2.2. Corollary 5.3.2.1 is false in general if $\mathcal{E}^\dagger$ is not semi-simple, as any extension of two rank 1 constant F-isocrystals with different slopes which does not split in $\mathbf{Isoc}(X_0)$ shows.

5.4 Proof of Theorem 5.1.1.4.2

Let $\mathbb{F} \subseteq k$ be a finitely generated extension and A a k-abelian variety. As already mentioned in Section 5.1.1.4, Theorem 5.1.1.4.2 amounts to show that if $\mathrm{Tr}_{k/\mathbb{F}}(A) = 0$ then $|A[p^\infty](k^{\mathrm{perf}})|$ is finite.

5.4.1 Notation

If $\mathcal{C}$ is an additive category write $\mathcal{C}_{\mathbb{Q}}$ for its isogeny category. For every $\mathbb{F}_p$-scheme S, write $\mathbf{p\text{-}div}(S)$ for the category of p-divisible groups over S. For every perfect field L of characteristic $p > 0$, write $W(L)$ for the ring of Witt vectors of L and $K(L)$ for its fraction field. For every smooth L-variety X write $\mathbf{Isoc}(X/K(L))$ and (resp. $\mathbf{Crys}(X/W(L))$) for the category of convergent isocrystal (resp. of crystals of finite $\mathcal{O}_{X,\mathrm{crys}}$-modules) and $\mathbf{F\text{-}Isoc}(X/K(L))$ (resp. $\mathbf{F\text{-}Crys}(X/W(L))$) for the category of objects in $\mathbf{Isoc}(X/K(L))$ (resp. in $\mathbf{Crys}(X/W(L))$) endowed with a Frobenius structure. By [Ber96, Theoreme 2.4.2], there exists a natural equivalence of categories

$$\mathbf{F\text{-}Crys}(X/W(L))_{\mathbb{Q}} \xrightarrow{\sim} \mathbf{F\text{-}Isoc}(X/K(L)). \tag{5.4.1.1}$$

5.4.2 p-torsion and p-divisible groups

Consider the exact sequence in $\mathbf{p\text{-}div}(k)$:

$$0 \to A[p^\infty]^0 \to A[p^\infty] \to A[p^\infty]^{\text{ét}} \to 0. \tag{5.4.2.1}$$

Since (5.4.2.1) splits over k^{perf} and $A[p^\infty]^{\text{ét}}$ is étale, one has $A[p^\infty](k^{perf}) = A[p^\infty]^{\text{ét}}(k)$. Since $A[p^\infty](k)$ is infinite if and only if $\mathrm{Hom}_{\mathbf{p\text{-}div}(k)_{\mathbb{Q}}}(\mathbb{Q}_p/\mathbb{Z}_p, A[p^\infty]) \neq 0$, Fact 5.1.1.4.1 implies that $\mathrm{Tr}_{k/\mathbb{F}}(A) = 0$ if and only if $\mathrm{Hom}_{\mathbf{p\text{-}div}(k)_{\mathbb{Q}}}(\mathbb{Q}_p/\mathbb{Z}_p, A[p^\infty]) \neq 0$. So, since $A[p^\infty](k^{perf}) = A[p^\infty]^{\text{ét}}(k)$ is infinite if and only if $\mathrm{Hom}_{\mathbf{p\text{-}div}(k)_{\mathbb{Q}}}(\mathbb{Q}_p/\mathbb{Z}_p, A[p^\infty]^{\text{ét}})$, Theorem 5.1.1.4.2 amounts to show that

$$\mathrm{Hom}_{\mathbf{p\text{-}div}(k)_{\mathbb{Q}}}(\mathbb{Q}_p/\mathbb{Z}_p, A[p^\infty]) = 0 \text{ implies } \mathrm{Hom}_{\mathbf{p\text{-}div}(k)_{\mathbb{Q}}}(\mathbb{Q}_p/\mathbb{Z}_p, A[p^\infty]^{\text{ét}}) = 0. \tag{5.4.2.2}$$

5.4.3 Spreading out

Let $k_0 \subseteq k$ be a finitely generated field with $k = \mathbb{F}k_0$ and such that there exists an abelian variety A_0/k_0 with $A \simeq A_0 \times_{k_0} k$. Let $\mathbb{F}_q$ be the algebraic closure of $\mathbb{F}_p$ in k_0. We choose a smooth geometrically connected $\mathbb{F}_q$-variety $\mathcal{X}_0$ with generic point $\eta_0 : \mathrm{Spec}(k_0) \to \mathcal{X}_0$ and an abelian scheme $f_0 : \mathcal{A}_0 \to \mathcal{X}_0$ with constant Newton polygon fitting into a commutative cartesian diagram:

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{A}_0 \arrow[d, "f_0"] \\ \mathrm{Spec}(k_0) \arrow[r, "\eta_0"] & \mathcal{X}_0. \end{tikzcd}$$

Since $f_0 : \mathcal{A}_0 \to \mathcal{X}_0$ has constant Newton polygon, the exact sequence

$$0 \to A_0[p^\infty]^0 \to A_0[p^\infty] \to A_0[p^\infty]^{\text{ét}} \to 0$$

in $\mathbf{p\text{-}div}(k_0)_{\mathbb{Q}}$, extends to an exact sequence

$$0 \to \mathcal{A}_0[p^\infty]^0 \to \mathcal{A}_0[p^\infty] \to \mathcal{A}_0[p^\infty]^{\text{ét}} \to 0$$

in $\mathbf{p\text{-}div}(\mathcal{X}_0)_{\mathbb{Q}}$. By [deJ98, Corollary 1.2], the horizontal arrows in the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{p\text{-}div}(\mathcal{X})_{\mathbb{Q}}}((\mathbb{Q}_p/\mathbb{Z}_p)_{\mathcal{X}}, \mathcal{A}[p^\infty]) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}_{\mathbf{p\text{-}div}(k)_{\mathbb{Q}}}(\mathbb{Q}_p/\mathbb{Z}_p, A[p^\infty]) \arrow[d] \\ \mathrm{Hom}_{\mathbf{p\text{-}div}(\mathcal{X})_{\mathbb{Q}}}((\mathbb{Q}_p/\mathbb{Z}_p)_{\mathcal{X}}, \mathcal{A}[p^\infty]^{\text{ét}}) \arrow[r, "\simeq"] & \mathrm{Hom}_{\mathbf{p\text{-}div}(k)_{\mathbb{Q}}}(\mathbb{Q}_p/\mathbb{Z}_p, A[p^\infty]^{\text{ét}}) \end{tikzcd}$$

are isomorphism. So, by (5.4.2.2), Theorem 5.1.1.4.2 amounts to show that

$$\mathrm{Hom}_{\mathbf{p\text{-}div}(\mathcal{X})_{\mathbb{Q}}}((\mathbb{Q}_p/\mathbb{Z}_p)_{\mathcal{X}}, \mathcal{A}[p^\infty]) = 0 \text{ implies } \mathrm{Hom}_{\mathbf{p\text{-}div}(\mathcal{X})_{\mathbb{Q}}}((\mathbb{Q}_p/\mathbb{Z}_p)_{\mathcal{X}}, \mathcal{A}[p^\infty]^{\text{ét}}) = 0. \tag{5.4.3.1}$$

5.4.4 Reformulation in terms of convergent F-isocrystals

Let

$$\mathbb{D}_{\mathcal{X}} : \mathbf{p\text{-}div}(\mathcal{X})_{\mathbb{Q}} \to \mathbf{F\text{-}Crys}(\mathcal{X}/W(\mathbb{F}))_{\mathbb{Q}} \overset{(5.4.1.1)}{\simeq} \mathbf{F\text{-}Isoc}(\mathcal{X}/K(\mathbb{F}))$$

be the crystalline Dieudonné module (contravariant) functor, cf. [BBM82]. In [BBM82], it is proven that this functor is fully faithful and that $\mathbb{D}(\mathcal{A}[p^\infty]) \simeq R^1 f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}$. Since $\mathcal{A}_0 \to \mathcal{X}_0$ has constant Newton polygon, $R^1 f_{0,\mathrm{crys}*}\mathcal{O}_{\mathcal{A}_0}$ has a maximal subobject $\mathcal{F}_0$ of slope zero. Write

$$(-)_{\mathcal{X}} : \mathbf{F\text{-}Isoc}(\mathcal{X}_0/K(\mathbb{F}_q)) \to \mathbf{F\text{-}Isoc}(\mathcal{X}/K(\mathbb{F}))$$

for the natural functor.

Lemma 5.4.4.1. $\mathbb{D}_{\mathcal{X}}(\mathcal{A}[p^\infty]^{\text{ét}}) = (\mathcal{F}_0)_{\mathcal{X}}$.

Proof. For every $\mathfrak{t} \in \mathcal{X}$, $\mathbb{D}_{\mathrm{Spec}(\mathbb{F}(\mathfrak{t}))}$ induces an anti-equivalence

$$\mathbb{D}_{\mathrm{Spec}(\mathbb{F}(\mathfrak{t}))} : \mathbf{p\text{-}div}(\mathrm{Spec}(\mathbb{F}(\mathfrak{t})))_{\mathbb{Q}} \xrightarrow{\sim} \mathbf{F\text{-}Isoc}_{[0,1]}(\mathrm{Spec}(\mathbb{F}(\mathfrak{t}))/K(\mathbb{F})),$$

where $\mathbf{F\text{-}Isoc}_{[0,1]}(\mathrm{Spec}(\mathbb{F}(\mathfrak{t}))/K(\mathbb{F}))$ is the category of F-isocrystals with slopes between 0 and 1. Since $\mathbb{D}_{\mathcal{X}}$ is compatible with base change, this implies that $\mathbb{D}_{\mathcal{X}}$ sends epimorphism to monomorphism, it preserves the heights/ranks and it sends étale p-divisible groups to unit-root F-isocrystals. In particular, the quotient $\mathcal{A}[p^\infty] \twoheadrightarrow \mathcal{A}[p^\infty]^{\text{ét}}$ is sent to the maximal unit-root subobject

$$\mathbb{D}_{\mathcal{X}}(\mathcal{A}[p^\infty]^{\text{ét}}) \hookrightarrow \mathbb{D}_{\mathcal{X}}(\mathcal{A}[p^\infty]) \simeq R^1 f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}.$$

Since $\mathcal{F}_0 \subseteq R^1 f_{0,\mathrm{crys}*}\mathcal{O}_{\mathcal{A}_0}$ is the maximal unit-root subobject, $(\mathcal{F}_0)_{\mathcal{X}}$ is also the maximal unit-root subobject of $(R^1 f_{0,\mathrm{crys}*}\mathcal{O}_{\mathcal{A}_0})_{\mathcal{X}} \simeq R^1 f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}$. $\square$

Since $\mathbb{D}_{\mathcal{X}}$ is fully faithful, by Lemma 5.4.4.1 and (5.4.3.1), Theorem 5.1.1.4.2 amounts to show that

$$\mathrm{Hom}_{\mathbf{F\text{-}Isoc}(\mathcal{X}/K(\mathbb{F}))}(R^1 f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}, \mathcal{O}_{\mathcal{X}}) = 0 \text{ implies } \mathrm{Hom}_{\mathbf{F\text{-}Isoc}(\mathcal{X}/K(\mathbb{F}))}((\mathcal{F}_0)_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) = 0. \quad (5.4.4.2)$$

5.4.5 Using Corollary 5.3.2.1

Now we show that

$$\mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))}(R^1 f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}, \mathcal{O}_{\mathcal{X}}) \to \mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))}((\mathcal{F}_0)_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$$

is surjective using Corollary 5.3.2.1. By a descent argument (see the proof of [Kat79, Proposition 1.3.2]), the natural functor

$$(-)_{\mathcal{X}} : \mathbf{Isoc}(\mathcal{X}_0/K(\mathbb{F}_q)) \to \mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))$$

satisfies

$$\mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}_0/K(\mathbb{F}_q))}(A, B) \otimes K(\mathbb{F}) = \mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))}(A_{\mathcal{X}}, B_{\mathcal{X}}).$$

Hence it is enough to show that

$$\mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}_0/K(\mathbb{F}_q))}(R^1 f_{0,\mathrm{crys}*}\mathcal{O}_{\mathcal{A}_0}, \mathcal{O}_{\mathcal{X}_0}) \to \mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}_0/K(\mathbb{F}_q))}(\mathcal{F}_0, \mathcal{O}_{\mathcal{X}_0})$$

is surjective. Since the extension of scalar functor

$$- \otimes \overline{\mathbb{Q}}_p : \mathbf{Isoc}(\mathcal{X}_0/K(\mathbb{F}_q)) \to \mathbf{Isoc}(\mathcal{X}_0)$$

satisfies ([AK02, Proposition 5.3.1])

$$\mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}_0/K(\mathbb{F}_q))}(A,B)\otimes\overline{\mathbb{Q}}_p=\mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}_0)}(A\otimes\overline{\mathbb{Q}}_p,B\otimes\overline{\mathbb{Q}}_p),$$

it is enough to show that

$$\mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}_0)}(R^1f_{0,\mathrm{crys}*}\mathcal{O}_{\mathcal{A}_0}\otimes\overline{\mathbb{Q}}_p,\mathbb{1})\to\mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}_0)}(\mathcal{F}_0\otimes\overline{\mathbb{Q}}_p,\mathbb{1})\tag{5.4.5.1}$$

is surjective. By [Ete02, Théorème 7], $R^1f_{0,\mathrm{crys}*}\mathcal{O}_{\mathcal{A}_0}\otimes\overline{\mathbb{Q}}_p$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}_0)$ is the image via $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}_0)\to\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}_0)$ of a $\mathcal{E}_0^\dagger$ in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X}_0)$. Let $\mathcal{E}_0^1$ be the maximal unit-root subobject of $\mathcal{E}_0$, so that $\mathcal{E}_0^1\simeq\mathcal{F}_0\otimes\overline{\mathbb{Q}}_p$. Since $\mathcal{E}_0^\dagger$ is pure by [Del80] and [KM74], $\mathcal{E}^\dagger$ is semisimple by [Ked17, Remark 10.6.]. So the surjectivity of (5.4.5.1) follows from Corollary 5.3.2.1.

5.4.6 End of the proof

Now we conclude the proof proving (5.4.4.2). Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))}(R^1f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}},\mathcal{O}_{\mathcal{X}}) \arrow[r] \arrow[d, hook] & \mathrm{Hom}_{\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))}((\mathcal{F}_0)_{\mathcal{X}},\mathcal{O}_{\mathcal{X}}) \arrow[d, hook] \\
\mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))}(R^1f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}},\mathcal{O}_{\mathcal{X}}) \arrow[r, two heads, "(1)"] & \mathrm{Hom}_{\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))}((\mathcal{F}_0)_{\mathcal{X}},\mathcal{O}_{\mathcal{X}})
\end{tikzcd}$$

in which (1) is surjective by Section 5.4.5. Assume now that $\mathrm{Hom}_{\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))}(R^1f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}},\mathcal{O}_{\mathcal{X}})=0$ and suppose by contradiction that $\mathrm{Hom}_{\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))}((\mathcal{F}_0)_{\mathcal{X}},\mathcal{O}_{\mathcal{X}})\neq 0$. Then, by the surjectivity of (1), there exists a morphism $g:R^1f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}\to\mathcal{O}_{\mathcal{X}}$ in $\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))$ which, once restricted to $(\mathcal{F}_0)_{\mathcal{X}}$ is not trivial and compatible with the Frobenius structure. Let $R^1f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}\to\mathcal{T}$ be the maximal trivial quotient in $\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))$. By maximality, $R^1f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}\to\mathcal{T}$ descend to a quotient $R^1f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}\twoheadrightarrow\mathcal{T}_0$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))$, where $\mathcal{T}_0$ is an F-isocrystals coming from $\mathrm{Spec}(\mathbb{F})$. Since $\mathbb{F}$ is algebraically closed, by [Ked17, Theorem 3.5] the category $\mathbf{F}\text{-}\mathbf{Isoc}(\mathrm{Spec}(\mathbb{F})/K(\mathbb{F}))$ is semisimple and every unit-root object in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathrm{Spec}(\mathbb{F})/K(\mathbb{F}))$ is trivial as F-isocrystal. So, $\mathcal{T}_0$ decomposes in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))$ as

$$\mathcal{T}_0=(\mathcal{T}_0')_{\mathcal{X}}\oplus\mathcal{O}_{\mathcal{X}}^{\oplus n}$$

where $\mathcal{O}_{\mathcal{X}}^{\oplus n}$ is the maximal unit-root subobject of $\mathcal{T}_0$ and n is an integer ≥ 0. Since $(\mathcal{F}_0)_{\mathcal{X}}$ is unit-root and the restriction of $g:R^1f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}\to\mathcal{T}_0$ to $(\mathcal{F}_0)_{\mathcal{X}}\subseteq R^1f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}$ is non trivial and compatible with the Frobenius structure, we see that $n>0$. Thus there exists a quotient $R^1f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}}\twoheadrightarrow\mathcal{T}_0\twoheadrightarrow\mathcal{O}_{\mathcal{X}}$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))$ in contradiction with $\mathrm{Hom}_{\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}/K(\mathbb{F}))}(R^1f_{\mathrm{crys}*}\mathcal{O}_{\mathcal{A}},\mathcal{O}_{\mathcal{X}})=0$.

Remark 5.4.6.1. In Fact 5.2.3.1.2, quoting [D'Ad17], we are using the Langlands correspondence for lisse sheaves and overconvergent F-isocrystals ([Laf02], [Abe18]). However, the proof of Theorem 5.1.1.4.2 can be obtained without using it. Indeed the overconvergent F-isocrystal $\mathcal{E}_0^\dagger$ provided by [Ete02, Théorème 7] it admits an explicit ℓ-adic companion: $R^1f_{0,*}\overline{\mathbb{Q}_\ell}$ (see also [D'Ad17, Remark 4.2.7]).

Chapter 6

Specialization of p-adic monodromy groups

6.1 Introduction

6.1.1 Notation

In this chapter k is a finitely generated field of characteristic $p > 0$ with algebraic closure $k \subseteq \overline{k}$. Set $\mathbb{F}_q$ for the algebraic closure of $\mathbb{F}_p$ in k, so that $\mathbb{F}_q$ a finite field with $q = p^s$ elements and write $\mathbb{F}$ for the algebraic closure of $\mathbb{F}_q$ in $\overline{k}$. For a k-variety X and for every integer $d \geq 1$, let $X(\leq d)$ denote the set of all $x \in |X|$ with residue field $k(x)$ of degree $\leq d$ over k. If $d = 1$ we often write $X(\leq 1) = X(k)$. Write φ_X for the absolute Frobenius of X (or just φ if there is no danger of confusion) and F_X (or just F) for its s-power.

6.1.2 ℓ–adic exceptional locus

From now on, let X be a smooth geometrically connected k-variety. For every prime ℓ, consider the category $\mathbf{LS}(X, \ell)$ of étale lisse $\overline{\mathbb{Q}}_\ell$-sheaves over X. For a geometric point $\overline{x} \in X$ and a $\mathcal{F}_\ell$ in $\mathbf{LS}(X, \ell)$, write $\mathcal{F}_{\ell,\overline{x}}$ for the fibre of $\mathcal{F}_\ell$ at $\overline{x}$. Then $\mathcal{F}_{\ell,\overline{x}}$ is endowed with a continuous action $\rho_{\mathcal{F}_\ell} : \pi_1(X) \to GL(\mathcal{F}_{\ell,\overline{x}})$ of[1] $\pi_1(X)$ and the functor $\mathcal{F}_\ell \mapsto \mathcal{F}_{\ell,\overline{x}}$ induces an equivalence of categories

$$\mathbf{LS}(X, \overline{\mathbb{Q}}_\ell) \simeq \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(\pi_1(X))$$

onto the category $\mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(\pi_1(X))$ of $\overline{\mathbb{Q}}_\ell$-linear continuous representations of $\pi_1(X)$ factorizing trough a finite extension of $\mathbb{Q}_\ell$

By functoriality of the étale fundamental group, every $x \in |X|$ induces a continuous group homomorphism $\pi_1(x) \to \pi_1(X)$, hence a "local" Galois[2] representation $\rho_{\mathcal{F}_\ell,x} : \pi_1(x) \to \pi_1(X) \to GL(\mathcal{F}_{\ell,\overline{x}})$. Write

$$\rho_{\mathcal{F}_\ell,x}(\pi_1(x)) =: \Pi_{\mathcal{F}_\ell,x} \subseteq \Pi_{\mathcal{F}_\ell} := \rho_{\mathcal{F}_\ell}(\pi_1(X)), \quad \Pi_{\mathcal{F}_\ell,\overline{k}} := \rho_{\mathcal{F}_\ell}(\pi_1(X_{\overline{k}})).$$

Write

$$X^{gen}_{\mathcal{F}_\ell} := \{x \in |X| \text{ with } [\Pi_{\mathcal{F}_\ell} : \Pi_{\mathcal{F}_\ell,x}] < +\infty\}; \quad X^{sgen}_{\mathcal{F}_\ell} := \{x \in |X| \text{ with } \Pi_{\mathcal{F}_\ell} = \Pi_{\mathcal{F}_\ell,x}\}$$

and define the following sets:

$$X^{ex}_{\mathcal{F}_\ell} := |X| - X^{gen}_{\mathcal{F}_\ell}; \quad X^{ex}_{\mathcal{F}_\ell}(\leq d) := X^{ex}_{\mathcal{F}_\ell} \cap X(\leq d); \quad X^{gen}_{\mathcal{F}_\ell}(\leq d) := X^{gen}_{\mathcal{F}_\ell} \cap X(\leq d);$$

[1] As the choice of fibre functors will play no part in the following we will omit them for the notation for étale fundamental group.

[2] Recall that $\pi_1(x) \simeq \pi_1(\mathrm{Spec}(k(x)))$ identifies with the absolute Galois group of $k(x)$.

$$X^{stex}_{\mathcal{F}_\ell} := |X| - X^{sgen}_{\mathcal{F}_\ell}; \quad X^{stex}_{\mathcal{F}_\ell}(\leq d) := X^{stex}_{\mathcal{F}_\ell} \cap X(\leq d); \quad X^{sgen}_{\mathcal{F}_\ell}(\leq d) := X^{sgen}_{\mathcal{F}_\ell} \cap X(\leq d).$$

Following [CK16], we call $X^{ex}_{\mathcal{F}_\ell}$ the exceptional locus of $\rho_{\mathcal{F}_\ell}$ and $X^{stex}_{\mathcal{F}_\ell}$ the strict exceptional locus of $\rho_{\mathcal{F}_\ell}$.

An important problem in arithmetic geometry, especially when $\mathcal{F}_\ell \simeq R^i f_* \overline{\mathbb{Q}}_\ell$ for some smooth proper morphism $f : Y \to X$, is to understand how $\Pi_{\mathcal{F}_\ell,x}$ varies with $x \in |X|$; see for example [CT12b], [CT13], [Cad12b], [CC18], Chapters 3 and 4. When k is arithmetically rich enough one expects that there are lots of $x \in |X|$ such that $\Pi_{\mathcal{F}_\ell,x}$ is open in $\Pi_{\mathcal{F}_\ell}$. More precisely we have:

Fact 6.1.2.1.

1. ([Ser89, Section 10.6], Fact 1.2.2.2.2): The set $X^{stex}_{\mathcal{F}_\ell}$ is sparse. In particular, if k is infinite there exists a $d \geq 1$ such that $X^{sgen}_{\mathcal{F}_\ell}(\leq d)$ is infinite.

2. (Theorem 3.1.3.2): Assume that $\ell \neq p$. If X is a curve and $(Lie(\Pi_{\mathcal{F}_\ell,\bar{k}}))^{ab} = 0$, the set $X^{ex}_{\mathcal{F}_\ell}(\leq 1)$ is finite.

In [Cad17], Cadoret extends Fact 6.1.2.1 to adelic representations. The goal of this paper is the extension of Fact 6.1.2.1 to various p-adic settings.

6.1.3 (Over)convergent F-Isocrystals over X

Even though Fact 6.1.2.1(1) holds for $\mathbf{LS}(X,p)$, the category of p-adic representations presents some pathologies which makes it very different from $\mathbf{LS}(X,\ell)$ for $\ell \neq p$. A first problem is then to find an analogue of the category $\mathbf{LS}(X,\ell)$ when $\ell = p$.

If $k = \mathbb{F}_q$ is a finite field, there are at least two possible Tannakian categories of p-adic "local systems": the category $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ of $\overline{\mathbb{Q}}_p$-convergent F-isocrystals and the category $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)$ of $\overline{\mathbb{Q}}_p$-overconvergent F-isocrystals. One has a fully faithful functor

$$(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X) \to \mathbf{F}\text{-}\mathbf{Isoc}(X)$$

which is an equivalence only if X is proper. The category $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)$ behaves very much like $\mathbf{LS}(X,\ell)$. For example, we have the finiteness of cohomology [Ked06a], a theory of weights [AC13b] and a trace formula [ES93]. On the other hand, the category $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ has a somehow pathological p-adic behaviour but it contains fine p-adic information; see for example [DK17] and Chapter 5.

However, to have results on the existence of (strictly) generic points, like Fact 6.1.2.1, one needs that k is arithmetically rich enough. So, to study the specialization theory of p-adic invariants, one would like to have categories of p-adic "local systems" for varieties defined over infinite finitely generated fields. The construction of $\overline{\mathbb{Q}}_p$-linear categories of (over)convergent F-isocrystals for variety over infinite finitely generated fields of positive characteristic is then the first topic of this chapter. Roughly, an (over)convergent F-isocrystals $[\mathcal{E}]$ over X is defined as an equivalence class $[\mathcal{E}]$ of couples $(\mathcal{X}, \mathcal{E})$, where $\mathcal{X}$ is an appropriate model of X over $\mathbb{F}_q$ and $\mathcal{E}$ is an (over)convergent F-isocrystal over $\mathcal{X}$; see Section (6.3.2.1) 6.5.2.1 for the precise definitions. Write $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}^{(\dagger)}(X)$ for the category of (over)convergent F-isocrystals over X.

6.1.4 Monodromy groups of (over)convergent F-Isocrystals over X

Since (over)convergent F-isocrystals do not correspond directly to representations, to define their exceptional loci, one has to use the Tannakian formalism.

If $\mathcal{E}$ is a $\mathbb{Q}_\ell$-lisse sheaf on X, we could define the monodromy group $G(\mathcal{F}_\ell)$ of $\mathcal{F}_\ell$ equivalently as the Zariski closure of the image of $\pi_1(X)$ acting on $\mathcal{F}_{\ell,\bar{x}}$ or as the automorphism group of the forgetful tensor functor $\langle \mathcal{F}_\ell \rangle \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$. For F-isocrystals, only the latter construction is available. If $[\mathcal{E}]$ is an (over)convergent F-isocrystals over X represented by $(\mathcal{X}, \mathcal{E})$, the choice of a point geometric point $\bar{\mathfrak{t}}$ of $\mathcal{X}$ defines a fibre functor $\langle \mathcal{E} \rangle \to \mathbf{Vect}_{\overline{\mathbb{Q}}_p}$, hence a monodromy group $G(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{X}$. Showing that $G(\mathcal{E})$ does not depend on the choice of a representative $(\mathcal{X}, \mathcal{E})$ of the equivalence class of $[\mathcal{E}]$, amounts to showing that every dense open immersion $\mathrm{j} : \mathcal{U} \to \mathcal{X}$ of smooth $\mathbb{F}_q$-varieties induces an isomorphism $G(\mathrm{j}^*\mathcal{E}) \simeq G(\mathcal{E})$. While this is true for overconvergent F-isocrystals (Fact 6.3.1.5.1), it does not hold in general for every convergent F-isocrystals; see Example 6.5.1.4.2. Indeed if the Newton polygon (see Section 6.5.1.2.2) of $\mathcal{E}$ is not constant on $\mathcal{X}$, there exists an open immersion $\mathrm{j} : \mathcal{U} \to \mathcal{X}$ and a canonical filtration

$$\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \ldots \subseteq \mathrm{j}^*\mathcal{E} \text{ in } \mathbf{F\text{-}Isoc}(\mathcal{U}) \tag{6.1.4.1}$$

encoding the slopes of $\mathrm{j}^*\mathcal{E}$. In general the sub-objects $\mathcal{E}_i$ are not in the essential image of $\mathrm{j}^* : \mathbf{F\text{-}Isoc}(\mathcal{X}) \to \mathbf{F\text{-}Isoc}(\mathcal{U})$ and this is an obstruction to have $G(\mathrm{j}^*\mathcal{E}) = G(\mathcal{E})$. However, we prove (Proposition 6.5.1.4.3) that this is the only obstruction, hence we get well defined monodromy groups for convergent F-isocrystals with constant Newton polygon.

6.1.5 Exceptional loci

After having settled the general formalism, one attaches to every overconvergent F-isocrystals (resp. convergent F-isocrystals with constant Newton polygon) $[\mathcal{E}]$ an exceptional locus $X^{ex}_{[\mathcal{E}]}$ and a strictly exceptional locus $X^{stex}_{[\mathcal{E}]}$.

6.1.5.1 Overconvergent F-isocrystals

In the overconvergent setting our main result is an analogue of Fact 6.1.2.1.

Theorem 6.1.5.1.1. Let $[\mathcal{E}]$ be a geometrically semisimple overconvergent F-isocrystal over X (see Section 6.3.2 for the definitions). Then:

- The set $X^{ex}_{[\mathcal{E}]}$ is sparse. In particular, if k is infinite there exists a $d \geq 1$ such that $X^{gen}_{[\mathcal{E}]}(\leq d)$ is infinite.
- If $[\mathcal{E}]$ is algebraic, then the set $X^{stex}_{[\mathcal{E}]}$ is sparse. In particular, if k is infinite there exists a $d \geq 1$ such that $X^{sgen}_{[\mathcal{E}]}(\leq d)$ is infinite.
- If X is a curve, the set $X^{ex}_{[\mathcal{E}]}(\leq 1)$ is finite.

The proof of Fact 6.1.2.1 relies heavily on the fact that $\Pi_{\mathcal{F}_\ell}$ is an ℓ-adic Lie group, hence, implicitly, on the Galois-theoretic structure of $\mathbf{LS}(X, \ell)$. These features are not available in this p-adic setting. Instead, the idea is to use companions theory (Fact 6.4.2.3.1) for both overconvergent F-isocrystals and lisse sheaves, which associates to an overconvergent F-isocrystal $[\mathcal{E}]$ with representative $(\mathcal{X}, \mathcal{E})$ an ℓ-adic companion $[\mathcal{F}_\ell]$ with representative $(\mathcal{X}, \mathcal{F}_\ell)$ for some $\ell \neq p$. Then we show that the exceptional loci of $[\mathcal{E}]$ and $[\mathcal{F}_\ell]$ coincide, so that we can deduce Theorem 6.1.5.1.1 from Fact 6.1.2.1.

6.1.5.2 Convergent F-isocrystals

In the convergent setting, we get somehow weaker results. The fully faithful functor $(-)^{conv}$: $\mathbf{F\text{-}Isoc}^\dagger(\mathcal{X}) \to \mathbf{F\text{-}Isoc}(\mathcal{X})$ induces a fully faithful functor

$$(-)^{conv} : \widetilde{\mathbf{F\text{-}Isoc}^\dagger}(X) \to \widetilde{\mathbf{F\text{-}Isoc}}(X).$$

Let $[\mathcal{E}]$ be in $\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(\mathcal{X})$ with representative $(\mathcal{X}, \mathcal{E})$ and assume that $\mathcal{E}^{conv}$ has constant Newton polygon over $\mathcal{X}$. Then the filtration (6.1.4.1) induces a canonical filtration

$$[\mathcal{E}]_1^{conv} \subseteq [\mathcal{E}]_2^{conv} \subseteq ... \subseteq [\mathcal{E}]^{conv}$$

and morphisms of algebraic groups

$$G([\mathcal{E}]_1^{conv}) \twoheadleftarrow G([\mathcal{E}]^{conv}) \subseteq G([\mathcal{E}]).$$

For every algebraic group G write $\mathrm{rk}(G)$ for its reductive rank and recall that for any subgroup $H \subseteq G$ one has $\mathrm{rk}(G) \geq \mathrm{rk}(H)$.

Theorem 6.1.5.2.1. Let $[\mathcal{E}]$ be a pure and p-plain overconvergent F-isocrystal over X with constant Newton polygon (see Section 6.5.2 for the definitions).

- The set of $x \in |X|$ such that $\mathrm{rk}(G([\mathcal{E}]^{conv})) > \mathrm{rk}(G(x^*[\mathcal{E}]^{conv}))$ is sparse. In particular, if k is infinite there exists an integer $d \geq 1$ and infinitely many $x \in X(\leq d)$ such $\mathrm{rk}(G(x^*[\mathcal{E}]^{conv})) = \mathrm{rk}(G(x^*[\mathcal{E}]^{conv}))$.
- If X is a curve for all but at most finitely many k-rational points x one has $\mathrm{rk}(G([\mathcal{E}]^{conv})) = \mathrm{rk}(G(x^*[\mathcal{E}]^{conv}))$ and $\mathrm{rk}(G([\mathcal{E}]_1^{conv})) = \mathrm{rk}(G(x^*[\mathcal{E}]_1^{conv}))$.
- If X is a curve and $G([\mathcal{E}]_1^{conv,geo})^0$ is abelian, then $X^{ex}_{[\mathcal{E}]_1^{conv}}(\leq 1)$ is finite.

Remark 6.1.5.2.2. If $[\mathcal{E}]_1^{conv}$ has slope zero, the fact that $X^{stex}_{[\mathcal{E}]_1^{conv}}$ is sparse follows directly from Facts 6.1.2.1(1) and 6.5.1.2.1.

Via Theorem 6.1.5.1.1, Theorem 6.1.5.2.1(3) amounts to compare $X^{stex}_{[\mathcal{E}]_1^{conv}}$, $X^{stex}_{[\mathcal{E}]^{conv}}$ and $X^{stex}_{[\mathcal{E}]}$. To do this, one uses that for every $x \in |X|$ there is a canonical diagram of algebraic groups

$$\begin{tikzcd}
G(x^*[\mathcal{E}]) \arrow[r, hook] & G([\mathcal{E}]) \\
G(x^*[\mathcal{E}]^{conv}) \arrow[r, hook] \arrow[u, hook] \arrow[d, two heads] & G([\mathcal{E}]^{conv}) \arrow[u, hook] \arrow[d, two heads] \\
G(x^*[\mathcal{E}]_1^{conv}) \arrow[r, hook] & G([\mathcal{E}]_1^{conv}),
\end{tikzcd}$$

so that one can try and obtain information on $X^{gen}_{[\mathcal{E}]_1^{conv}}$ and $X^{gen}_{[\mathcal{E}]^{conv}}$ from $X^{gen}_{[\mathcal{E}]}$, via the results in Chapter 5.

6.1.6 An application to motivic p-adic representations

Let $f : Y \to X$ be a smooth proper morphism of k-varieties. Up to replacing X with a dense open subset, the constructible sheaf $\mathcal{F}_p := R^i f_* \overline{\mathbb{Q}}_p$ is a p-adic lisse sheaf. Hence, for every $x \in |X|$, it corresponds to a representation $\rho_{\mathcal{F}_p}$ such that $\rho_{\mathcal{F}_p,x}$ identifies with the natural action of $\pi_1(x)$ on $GL(H^i(Y_{\overline{x}}, \overline{\mathbb{Q}}_p))$. Write $G(\mathcal{F}_p)$ and $G(x^*\mathcal{F}_p)$ for the Zariski closure of $\Pi_{\mathcal{F}_p}$ and $\Pi_{\mathcal{F}_p,x}$ respectively and $G(\mathcal{F}_p)^0$ and $G(x^*\mathcal{F}_p)^0$ for their neutral components. By Fact 6.1.2.1, we know that $X^{stex}_{\mathcal{F}_p}$ is thin. As a consequence of Theorem 6.1.5.2.1 we obtain some finiteness results when X is a curve.

Corollary 6.1.6.1. Assume that X is a curve.

- For all but at most finitely many k-rational points one has $\mathrm{rk}(G(\mathcal{F}_p)) = \mathrm{rk}(G(x^*\mathcal{F}_p))$.
- If $G(\mathcal{F}_p)^0$ is abelian, for all but finitely many $x \in X(k)$ we have $G(\mathcal{F}_p)^0 = G(x^*\mathcal{F}_p)^0$.

To prove Corollary 6.1.6.1 one observes that there is is a pure and p-plain $[\mathcal{E}]$ in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}}(X)$ such that $G(\mathcal{F}_p)$ identifies naturally with $G([\mathcal{E}]_1^{conv})$, so that one can deduce Corollary 6.1.6.1 from Theorem 6.1.5.2.1.

6.1.7 A conjecture

While the results in Theorem 6.1.5.2.1 are limited to a small class of overconvergent F-isocrystals, we conjecture (Conjecture 6.6.2.2.1) that for every pure p-plain overconvergent F-isocrystals one should have

$$X^{gen}_{[\mathcal{E}]^{conv}} = X^{gen}_{[\mathcal{E}]} \subseteq X^{gen}_{[\mathcal{E}]_1^{conv}}.$$

We end the chapter discussing this conjecture and giving some evidences for it.

6.1.8 Organization of the chapter

In Section 6.2 we introduce some notation for algebraic groups and models. Sections 6.3 and 6.4 are devoted to the study of coefficient objects over finitely generated fields. In Section 6.3 we first recall various properties of overconvergent F-isocrystals and lisse sheaves over finite fields and then we use them to extend the definitions to finitely generated fields. In Section 6.4 we define their exceptional loci and we prove Theorem 6.1.5.1.1. Sections 6.5 and 6.6 are devoted to the study of convergent F-isocrystals over finitely generated fields. In Section 6.5 we first recall various properties of convergent F-isocrystals over finite fields and then we use them to extend the definitions to finitely generated fields. In Section 6.6 we define their exceptional loci, we prove Theorem 6.1.5.2.1 and its Corollary 6.1.6.1. We end Section 6.6 proposing a conjecture relating the various exceptional loci associated to an overconvergent F-isocrystal. In Appendix 6.A, we prove some easy lemma on epimorphic subgroups used in the paper.

6.2 Preliminaries

6.2.1 Notation for groups and representations

If G is an algebraic group over a field L of characteristic zero, we write G^0 for its neutral component, $\pi_0(G) := G/G^0$ for the group of connected components, $R_u(G)$ for its unipotent radical and $X^*(G)$ for its group of characters. Set $\mathbf{Rep}_L(G)$ for the category of finite dimensional L-representations of G. Write $\mathrm{rk}(G)$ for the reductive rank of G and recall that a subgroup of $H \subseteq G$ is of maximal rank if $\mathrm{rk}(H) = \mathrm{rk}(G)$. If V is in $\mathbf{Rep}_L(G)$ we write V^{ss} for its semisemplification and $V^{\vee}$ for its dual. Let $f : H \to G$ be a morphism of algebraic groups over L. We say that $f : H \to G$ is epimorphic if the induced functor $f^* : \mathbf{Rep}_L(G) \to \mathbf{Rep}_L(H)$ is fully faithful. If $f : H \to G$ is an epimorphic closed immersion, we say that H is an epimorphic subgroup of G. See Appendix 6.A for more details and basic properties of epimorphic morphism.

If Γ is a profinite group and ℓ is a prime, we write $\mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(\Gamma)$ for the category of finite dimensional, continuous $\overline{\mathbb{Q}}_\ell$-linear representations of Γ that factors trough a finite extension of $\mathbb{Q}_\ell$.

6.2.2 Models

To define (over)convergent F-isocrystals and their monodromy groups, we need to work with models of k-varieties and morphisms over $\mathbb{F}_q$. We collect here some notation and preliminaries on these models.

6.2.2.1 Models of varieties

An $\mathbb{F}_q$-model $\mathcal{K}$ of k is a smooth geometrically connected $\mathbb{F}_q$-variety with generic point η : $\mathrm{Spec}(k) \to \mathcal{K}$. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are two models of k, there are two dense open subsets $\mathcal{U}_1 \to \mathcal{K}_1$ and $\mathcal{U}_2 \to \mathcal{K}_2$ and an isomorphism of $\mathbb{F}_q$-varieties $\mathcal{U}_1 \simeq \mathcal{U}_2$.

If $X \to k$ is a smooth connected k-variety, an $\mathbb{F}_q$-model $\mathcal{X}$ of X is a smooth morphism $\mathcal{X} \to \mathcal{K}$ of smooth connected $\mathbb{F}_q$ varieties, where $\mathcal{K}$ is an $\mathbb{F}_q$-model of k, such that the base change of $\mathcal{X} \to \mathcal{K}$ along $\eta : \mathrm{Spec}(k) \to \mathcal{K}$ identifies with $X \to k$. If X is geometrically connected over k, then every $\mathbb{F}_q$-model of X is geometrically connected over $\mathbb{F}_q$.

If $\mathcal{X}_1 \to \mathcal{K}_1$ and $\mathcal{X}_2 \to \mathcal{K}_2$ are two models of $X \to k$, we write $\mathcal{X}_2 \succ \mathcal{X}_1$ if there exists a commutative cartesian diagram of $\mathbb{F}_q$-varieties

$$\begin{tikzcd} \mathcal{X}_2 \arrow[r, "\mathfrak{j}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}_1 \arrow[d] \\ \mathcal{K}_2 \arrow[r, "\mathfrak{i}"] & \mathcal{K}_1 \end{tikzcd}$$

in which $\mathfrak{i}$ (hence $\mathfrak{j}$) is an open immersion. If we want to specify the map $\mathfrak{j} : \mathcal{X}_2 \to \mathcal{X}_1$ in the diagram we write $\mathcal{X}_2 \succ_{\mathfrak{j}} \mathcal{X}_1$. For every model $\mathcal{X}$ write $\mathfrak{j}_{\mathcal{X}}$ for the morphism $\mathfrak{j}_{\mathcal{X}} : X \to \mathcal{X}$. Every X admits a model and given two models $\mathcal{X}_1$ and $\mathcal{X}_2$ of X, there exists always a model $\mathcal{X}_3$ such that $\mathcal{X}_3 \succ \mathcal{X}_i$, $i = 1, 2$. We write $\mathbf{Model}(X)$ for the set of model of X.

6.2.2.2 Models of morphisms

If $f : Y \to X$ is a morphism of smooth connected k-varieties, an $\mathbb{F}_q$-model $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ of $f : Y \to X$ is a commutative diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "\mathfrak{f}"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & \mathcal{K} & \end{tikzcd}$$

where $\mathcal{X} \to \mathcal{K}$ and $\mathcal{Y} \to \mathcal{K}$ are $\mathbb{F}_q$-models of $X \to k$ and $Y \to k$ and $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ identifies with $f : Y \to X$ after base change along $\eta : \mathrm{Spec}(k) \to \mathcal{K}$. If $f : Y \to X$ is smooth (resp. proper, resp. an open immersion, resp. a closed immersion) we require that $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ is smooth (resp. proper, resp. an open immersion, resp. a closed immersion).

If $\mathfrak{f}_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ and $\mathfrak{f}_2 : \mathcal{Y}_2 \to \mathcal{X}_2$ are two models of $f : Y \to X$, we write $(\mathfrak{f}_2 : \mathcal{Y}_2 \to \mathcal{X}_2) \succ (\mathfrak{f}_1 : \mathcal{Y}_1 \to \mathcal{X}_1)$ if $\mathcal{X}_1 \succ \mathcal{X}_2$ and there exists a commutative cartesian diagram

$$\begin{tikzcd} \mathcal{Y}_2 \arrow[r] \arrow[d, "\mathfrak{f}_2"] \arrow[dr, phantom, "\square"] & \mathcal{Y}_1 \arrow[d, "\mathfrak{f}_1"] \\ \mathcal{X}_2 \arrow[r] & \mathcal{X}_1 \end{tikzcd}$$

Any $f : Y \to X$ admits a model and if $\mathfrak{f}_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ and $\mathfrak{f}_2 : \mathcal{Y}_2 \to \mathcal{X}_2$ are two models there exists a model $\mathfrak{f}_3 : \mathcal{Y}_3 \to \mathcal{X}_3$ of $f : Y \to X$ such that $(\mathfrak{f}_3 : \mathcal{Y}_3 \to \mathcal{X}_3) \succ (\mathfrak{f}_i : \mathcal{Y}_i \to \mathcal{X}_i)$, $i = 1, 2$.

6.3 Coefficient objects over finitely generated fields

For every prime ℓ, fix an isomorphism $\iota_\ell : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ and write $\underline{\iota} := \{\iota_\ell\}_\ell$.

6.3.1 Coefficient objects over finite fields

Let $\mathcal{X}$ be a connected smooth $\mathbb{F}_q$-variety. We quickly review the theory of overconvergent F-isocrystals and lisse sheaves over $\mathcal{X}$. For more details see e.g. [D'Ad17, Section 2], [Abe18], [Ked18].

6.3.1.1 Coefficient objects

Write $\mathbf{Coef}(\mathcal{X}, p)$ for the category of $\overline{\mathbb{Q}}_p$-linear overconvergent F-isocrystals $\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(\mathcal{X}, \overline{\mathbb{Q}}_p)$ and $\mathbf{Coef}^{geo}(\mathcal{X}, p)$ for the category of $\overline{\mathbb{Q}}_p$-linear overconvergent isocrystals $\mathbf{Isoc}^{\dagger}(\mathcal{X}, \overline{\mathbb{Q}}_p)$. See [Abe18, 1.4.11, 2.2.14] for the definitions and [D'Ad17, Section 2] for a careful discussion.

If $\ell \neq p$ is a prime, write $\mathbf{Coef}(\mathcal{X}, \ell)$ for the category $\mathbf{LS}(X, \ell)$ of étale lisse $\overline{\mathbb{Q}}_\ell$-sheaves over $\mathcal{X}$ and $\mathbf{Coef}^{geo}(\mathcal{X}, \ell)$ for the category of étale lisse $\overline{\mathbb{Q}}_\ell$-sheaves over $\mathcal{X}_{\mathbb{F}}$; see [Del80, Section 1.1]. The category $\mathbf{Coef}(\mathcal{X}, \ell)$ is equivalent to $\mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(\pi_1(\mathcal{X}))$ and if X is geometrically connected $\mathbf{Coef}^{geo}(\mathcal{X}, \ell)$ is equivalent to $\mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(\pi_1(\mathcal{X}_{\mathbb{F}}))$.

For every prime ℓ, including $\ell = p$, write $(-)^{geo} : \mathbf{Coef}(\mathcal{X}, \ell) \to \mathbf{Coef}^{geo}(\mathcal{X}, \ell)$ for the canonical functor. We say that $\mathcal{E}$ is geometrically semisimple if $\mathcal{E}^{geo}$ is semisimple. Write $\mathcal{E}^{ss}$ and $\mathcal{E}^{geo,ss}$ for the semisimplification of $\mathcal{E}$ and $\mathcal{E}^{geo}$ in $\mathbf{Coef}(\mathcal{X}, \ell)$ and $\mathbf{Coef}^{geo}(\mathcal{X}, \ell)$ respectively.

For every $j \in \overline{\mathbb{Q}}_p$ and every $\mathcal{E}$ in $\mathbf{Coef}(\mathcal{X}, p)$ write $\mathcal{E}^{(j)}$ for the j^{th} twist of $\mathcal{E}$ (see [D'Ad17, Section 3.18]).

If $\ell \neq p$, for every algebraic number j such that j is an λ-adic unit[3] for every place λ of $\overline{\mathbb{Q}}$ over ℓ and every $\mathcal{E}$ in $\mathbf{Coef}(\mathcal{X}, \ell)$ write $\mathcal{E}^{(j)}$ for the j^{th} twist of $\mathcal{E}$.

If $\ell \neq p$, for every $\mathcal{F}$ in $\mathbf{Coef}(\mathcal{X}, \ell)$ write $\rho_{\mathcal{F}}$, $\rho_{\mathcal{F}^{geo}}$ for the associated representations and $\Pi_{\mathcal{F}}$, $\Pi_{\mathcal{F}^{geo}}$ for their images.

6.3.1.2 Monodromy groups

For every prime ℓ, including $\ell = p$, $\mathbf{Coef}(\mathcal{X}, \ell)$ is a neutral Tannakian category and the choice of an $\mathbb{F}$-point of $\mathcal{X}$ induces a fibre functor

$$\mathbf{Coef}(\mathcal{X}, \ell) \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}.$$

Since $\overline{\mathbb{Q}}_\ell$ is algebraically closed, any two fibre functors are, non canonically, isomorphic. So, to simplify the notation, we omit the base points.

(Resp. if $\mathcal{X}$ is geometrically connected) For every $\mathcal{E}$ in $\mathbf{Coef}(\mathcal{X}, \ell)$ (resp. $\mathcal{E}^{geo}$ in $\mathbf{Coef}^{geo}(\mathcal{X}, \ell)$), write $G(\mathcal{E})$ (resp. $G(\mathcal{E}^{geo})$) for the Tannaka group of the Tannakian subcategory $\langle \mathcal{E} \rangle \subseteq \mathbf{Coef}(\mathcal{X}, \ell)$ (resp. $\langle \mathcal{E}^{geo} \rangle \subseteq \mathbf{Coef}^{geo}(\mathcal{X}, \ell)$) generated by $\mathcal{E}$ (resp. $\mathcal{E}^{geo}$).

The faithful functor $(-)^{geo} : \mathbf{Coef}(\mathcal{X}, \ell) \to \mathbf{Coef}^{geo}(\mathcal{X}, \ell)$ induces a faithful functor $(-)^{geo} : \langle \mathcal{E} \rangle \to \langle \mathcal{E}^{geo} \rangle$, hence, if $\mathcal{X}$ is geometrically connected, a closed immersion $G(\mathcal{E}^{geo}) \subseteq G(\mathcal{E})$. Furthermore, by [D'Ad17, Appendix] the subgroup $G(\mathcal{E}^{geo}) \subseteq G(\mathcal{E})$ is normal. The algebraic group $G(\mathcal{E})^{cst} := G(\mathcal{E})/G(\mathcal{E}^{geo})$ is then abelian and identifies with the Tannakian group of the full Tannakian subcategory $\langle \mathcal{E} \rangle^{cst} \subseteq \langle \mathcal{E} \rangle$ of objects isomorphic to an object of the form $\mathfrak{q}^*\mathcal{E}'$, where $\mathfrak{q} : \mathcal{X} \to \mathrm{Spec}(\mathbb{F}_q)$ is the structural morphism and $\mathcal{E}'$ is in $\mathbf{Coef}(\mathrm{Spec}(\mathbb{F}_q), \ell)$.

Every morphism $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ of smooth connected $\mathbb{F}_q$-varieties, induces a faithful tensor functor $\mathfrak{f}^* : \mathbf{Coef}(\mathcal{X}, \ell) \to \mathbf{Coef}(\mathcal{Y}, \ell)$, hence for every $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ a natural closed immersion $G(\mathfrak{f}^*\mathcal{E}) \subseteq G(\mathcal{E})$. If moreover $\mathcal{Y}$ and $\mathcal{X}$ are geometrically connected, $\mathfrak{f}^*$ induces a faithful tensor functor $\mathfrak{f}^* : \mathbf{Coef}^{geo}(\mathcal{X}, \ell) \to \mathbf{Coef}^{geo}(\mathcal{Y}, \ell)$ fitting into a commutative diagram

[3] We need this condition to guarantee that $\mathcal{E}^{(j)}$ is still a étale lisse $\overline{\mathbb{Q}}_\ell$-sheaf and not only a Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaf.

$$\begin{tikzcd}
\mathbf{Coef}(\mathcal{X},\ell) \arrow[r] \arrow[d] & \mathbf{Coef}(\mathcal{Y},\ell) \arrow[d] \\
\mathbf{Coef}^{geo}(\mathcal{X},\ell) \arrow[r] & \mathbf{Coef}^{geo}(\mathcal{Y},\ell),
\end{tikzcd}$$

hence for every $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ a commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & G(\mathfrak{f}^*\mathcal{E}^{geo}) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*\mathcal{E}) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*\mathcal{E})^{cst} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & G(\mathcal{E}^{geo}) \arrow[r] & G(\mathcal{E}) \arrow[r] & G(\mathcal{E})^{cst} \arrow[r] & 0.
\end{tikzcd} \tag{6.3.1.2.1}$$

where, by [D'Ad17, Corollary 3.2.7], the left and the middle vertical arrows are closed immersions and the right vertical arrow is surjective.

If $\ell \neq p$, the groups $G(\mathcal{E})$ and $G(\mathcal{E}^{geo})$ identifies with the Zariski closure of $\Pi_{\mathcal{E}}$ and $\Pi_{\mathcal{E}^{geo}}$ respectively.

6.3.1.3 Independence

Let $\mathcal{E}$ be in $\mathbf{Coef}(\mathcal{X},\ell)$. For every $\mathfrak{t} \in |\mathcal{X}|$ there is a characteristic polynomial $\phi_{\mathfrak{t}}(\mathcal{E}) \in \overline{\mathbb{Q}}_\ell[T]$ of $\mathcal{E}$ in $\mathfrak{t}$ (see e.g. [D'Ad17, 2.1.4 and 2.2.10.]). One says that $\mathcal{E}$ is algebraic if for every $\mathfrak{t} \in |\mathcal{X}|$, the polynomial $\phi_{\mathfrak{t}}(\mathcal{E})$ lies in $\overline{\mathbb{Q}}[T]$. Recall that $\mathcal{E}$ is called ι_ℓ-pure (of weight $w \in \mathbb{Z}$) if $i_\ell(\phi_{\mathfrak{t}}(\mathcal{E}))$ has all the roots of complex absolute value $q^{[k(\mathfrak{t}):\mathbb{F}_q]w/2}$. Moreover we say that $\mathcal{E}$ is p-plain if it algebraic and the roots of $\phi_{\mathfrak{t}}(\mathcal{E})$ are λ-adic units for every place λ of $\overline{\mathbb{Q}}$ over every $\ell \neq p$. Take another prime $\ell' \neq \ell$ and fix $\mathcal{E}_\ell$ in $\mathbf{Coef}(\mathcal{X},\ell)$ and $\mathcal{E}_{\ell'}$ in $\mathbf{Coef}(\mathcal{X},\ell')$. We say that $\mathcal{E}_\ell$ and $\mathcal{E}_{\ell'}$ are $\underline{\iota}$-compatible (or that $\mathcal{E}_\ell$ is an ℓ-adic companion of $\mathcal{E}_{\ell'}$) if $\iota_\ell(\phi_{\mathfrak{t}}(\mathcal{E}_\ell)) = \iota_{\ell'}(\phi_{\mathfrak{t}}(\mathcal{E}_{\ell'}))$ for all $\mathfrak{t} \in |\mathcal{X}|$.

6.3.1.4 Properties

We recall the following properties:

Fact 6.3.1.4.1. Assume that $\mathcal{X}$ is geometrically connected and let $\mathcal{E}$ be in $\mathbf{Coef}(\mathcal{X},\ell)$.

1. If $\mathcal{E}$ is ι_ℓ-pure, then it is geometrically semisimple.
2. If $\mathcal{E}$ is geometrically semisimple then $G(\mathcal{E}^{geo})^0$ is a semisimple algebraic group.
3. Take another $\mathcal{E}'$ in $\mathbf{Coef}(\mathcal{X},\ell)$ such that $\phi_{\mathfrak{t}}(\mathcal{E}) = \phi_{\mathfrak{t}}(\mathcal{E}')$ for every $\mathfrak{t} \in |\mathcal{X}|$. Then $\mathcal{E}^{ss} \simeq \mathcal{E}'^{ss}$.
4. The category of geometrically semisimple objects is stable by pull-back.
5. Every semisimple object is geometrically semisimple.
6. Assume that $\mathcal{E}$ is geometrically semisimple and that $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ is a morphism of smooth geometrically connected $\mathbb{F}_q$-varieties. Then $G(\mathfrak{f}^*\mathcal{E})^0 = G(\mathcal{E})^0$ if and only if $G(\mathfrak{f}^*\mathcal{E}^{geo})^0 = G(\mathcal{E}^{geo})^0$

Proof.

1. This is [Del80, Theorem 3.4.1] if $\ell \neq p$ and [Ked17, Remark 10.6.] if $\ell = p$.
2. This is the Global monodromy theorem: [Del80, Corollarie 1.3.9] if $\ell \neq p$ and the proof of [Cre92, Corollary 4.10] if $\ell = p$. See [D'Ad17, Theorem 3.4.3] for more details.
3. If $\ell \neq p$ this is Chebotarev's theorem and if $\ell = p$ it is [Abe18, Proposition A.4.1].

4. This follows from the fact that being geometrically semisimple is equivalent to being a direct sum in $\mathbf{Coef}^{geo}(\mathcal{X},\ell)$ of ι_ℓ-pure coefficient objects (this uses the companion conjecture ([Abe18], [Laf02])) and this condition is stable by pull-back; see [D'Ad17, Corollary 3.5.8.] for more details.

5. This follows from the fact that $G(\mathcal{E}^{geo}) \subseteq G(\mathcal{E})$ is a normal subgroup.

6. One implication follows from the fact that $G(\mathcal{E}^{geo})^0$ and $G(x^*\mathcal{E}^{geo})^0$ are the derived subgroups of $G(\mathcal{E})^0$ and $G(x^*\mathcal{E})^0$ respectively; this is a consequence of the Global monodromy theorem see e.g. the proof of [D'Ad17, Corollary 3.4.10]. The other implication follows from [D'Ad17, Proposition 3.2.6]. □

6.3.1.5 Behaviour under open immersion

Fact 6.3.1.5.1. Let $\mathrm{j}:\mathcal{U}\to\mathcal{X}$ be a dense open immersion of smooth connected $\mathbb{F}_q$-varieties. Then the following hold:

1. The functor $\mathrm{j}^*:\mathbf{Coef}(\mathcal{X},\ell)\to\mathbf{Coef}(\mathcal{U},\ell)$ is fully faithful;

2. For every $\mathcal{E}$ in $\mathbf{Coef}(\mathcal{X},\ell)$, the natural inclusion $G(\mathrm{j}^*\mathcal{E})\subseteq G(\mathcal{E})$ is an isomorphism;

3. If $\mathcal{X}$ and $\mathcal{U}$ are geometrically connected, for every $\mathcal{E}$ in $\mathbf{Coef}(\mathcal{X},\ell)$ the natural inclusion $G(\mathrm{j}^*\mathcal{E}^{geo})\subseteq G(\mathcal{E}^{geo})$ is an isomorphism.

is fully faithful.

Proof.

1. If $\ell\neq p$, this follows from the fact that $\mathrm{j}:\mathcal{U}\to\mathcal{X}$ induces a surjection $\pi_1(\mathcal{U})\to\pi_1(\mathcal{X})$. If $\ell=p$ this is [Ked17, Theorem 5.3].

2. By the general Tannakian formalism it is enough to show that the functor $\mathrm{j}^*:\mathbf{Coef}(\mathcal{X},\ell)\to\mathbf{Coef}(\mathcal{U},\ell)$ is fully faithful and that the essential image is closed under sub-objects. The first condition is point (1). If $\ell=p$, the second condition is [AC13b, Lemma 1.4.6] and, if $\ell\neq p$, the second condition follows from the normality of $\mathcal{X}$.

3. The proof is the same as the one of (2), replacing, when $\ell=p$, [Ked17, Theorem 5.3] and [AC13b, Lemma 1.4.6] with [Ked07, Theorem 5.2.1.] and [Ked07, Proposition 5.3.1] respectively. □

Fact 6.3.1.5.2. Let $\mathrm{j}:\mathcal{U}\to\mathcal{X}$ be a dense open immersion of smooth connected $\mathbb{F}_q$-varieties. Then $\mathcal{E}$ in $\mathbf{Coef}(\mathcal{X},\ell)$ is algebraic (resp. ι_ℓ-pure, resp. p-plain) if and only $\mathrm{j}^*\mathcal{E}$ in $\mathbf{Coef}(\mathcal{U},\ell)$ is algebraic (resp. ι_ℓ-pure, resp. p-plain).

Proof. This follows from [Ked18, Theorem 3.3.1] (resp. [Del80, Corollaire 1.8.10] if $\ell\neq p$ and [AC13b, Remark 2.1.11] if $\ell=p$, resp. [D'Ad17, Proposition 3.1.12]) □

6.3.2 Coefficient objects over finitely generated fields

Let k be an infinitely generated field of characteristic $p>0$ and let X be a smooth connected k-variety. In this section we define and study coefficient objects over X.

6.3.2.1 Definitions

For every couples $(\mathcal{X}_i, \mathcal{E}_i)$, $i = 1, 2$, with $\mathcal{X}_i \in \mathbf{Model}(X)$ and $\mathcal{E}_i$ is in $\mathbf{Coef}(\mathcal{X}_i, \ell)$, write $(\mathcal{X}_2, \mathcal{E}_2) \succ_{\mathrm{j}} (\mathcal{X}_1, \mathcal{E}_1)$ (or simply $(\mathcal{X}_2, \mathcal{E}_2) \succ (\mathcal{X}_1, \mathcal{E}_1)$) if $\mathcal{X}_2 \succ_{\mathrm{j}} \mathcal{X}_1$ and $\mathcal{E}_2 \simeq \mathrm{j}^*\mathcal{E}_1$.

Definition 6.3.2.1.1. The category of ℓ-adic coefficient objects $\widetilde{\mathbf{Coef}}(X, \ell)$ over X is the following category:

- The objects are equivalence classes $[\mathcal{E}]$ of couples $(\mathcal{X}, \mathcal{E})$ where $\mathcal{X} \in \mathbf{Model}(X)$ and $\mathcal{E}$ is in $\mathbf{Coef}(\mathcal{X}, \ell)$. The equivalence relation is given by the relations $(\mathcal{X}_1, \mathcal{E}_1) \sim (\mathcal{X}_2, \mathcal{E}_2)$ if there exists a couple $(\mathcal{X}_3, \mathcal{E}_3)$ such that $(\mathcal{X}_3, \mathcal{E}_3) \succ (\mathcal{X}_i, \mathcal{E}_i)$, $i = 1, 2$.

- A morphism $[\mathbf{g}] : [\mathcal{E}] \to [\mathcal{E}']$ between $[\mathcal{E}]$ and $[\mathcal{E}']$ in $\widetilde{\mathbf{Coef}}(X, \ell)$ with representatives $(\mathcal{X}, \mathcal{E})$ and $(\mathcal{X}', \mathcal{E}')$ is an equivalence class of couples $(\mathcal{U}, \mathbf{g})$ where $\mathcal{U}$ in $\mathbf{Model}(X)$ is such that $\mathcal{U} \succ_{\mathrm{j}} \mathcal{X}$ and $\mathcal{U} \succ_{\mathrm{j}'} \mathcal{X}'$ and $\mathbf{g}$ is a morphism $\mathrm{j}^*\mathcal{E} \to \mathrm{j}'^{,*}\mathcal{E}'$. The equivalence relation is given by the relations $(\mathcal{U}_1, \mathbf{g}_1) \sim (\mathcal{U}_2, \mathbf{g}_2)$ if there exists $\mathcal{U}_3$ in $\mathbf{Model}(X)$ with $\mathcal{U}_3 \succ_{\mathrm{j}_i} \mathcal{U}_i$, $i = 1, 2$, and $\mathrm{j}_1^*\mathbf{g_1} = \mathrm{j}_2^*\mathbf{g_2}$.

If $\ell = p$ we write also $\widetilde{\mathbf{Coef}}(X, p) := \widetilde{\mathbf{F\text{-}Isoc}}^\dagger(X)$ and we call them overconvergent F-isocrystals over X.

We write $[0]$ for the equivalence class of $(\mathcal{X}, 0)$ where 0 is the trivial coefficient object over $\mathcal{X}$. The equivalence class of $[0]$ does not depend on the choice of the model $\mathcal{X}$ of X.

6.3.2.2 Operations and properties

For $[\mathcal{E}]$ in $\widetilde{\mathbf{Coef}}(X, \ell)$ with representatives $(\mathcal{X}_i, \mathcal{E}_i)$, $i = 1, 2$, there exists a representative $(\mathcal{X}_3, \mathcal{E}_3)$ with $(\mathcal{X}_3, \mathcal{E}_3) \succ_{\mathrm{j}_i} (\mathcal{X}_i, \mathcal{E}_i)$, $i = 1, 2$. Then:

- Since, by Fact 6.3.1.5.1, $\mathcal{E}_i$ is irreducible (resp. semisimple, resp. geometrically semisimple) if and only if $\mathrm{j}_i^*\mathcal{E}_i$ is irreducible (resp. semisimple, resp. geometrically semisimple), we say that $[\mathcal{E}]$ is irreducible (resp. semisimple, resp. geometrically semisimple) if for any representative $(\mathcal{X}, \mathcal{E})$ of $[\mathcal{E}]$, $\mathcal{E}$ is irreducible (resp. semisimple, resp. geometrically semisimple) over $\mathcal{X}$. The equivalence class $[\mathcal{E}^{ss}]$ of $(\mathcal{X}, \mathcal{E}^{ss})$ is then semisimple and it does not depend on the choice of the representative of $(\mathcal{X}, \mathcal{E})$ of $[\mathcal{E}]$.

- Since, by Fact 6.3.1.5.2, $\mathcal{E}_i$ is algebraic (resp. ι_ℓ-pure, p-plain) if and only if $\mathrm{j}_i^*\mathcal{E}_i$ is algebraic (resp. ι_ℓ-pure, p-plain) we say that if for any of the representative $(\mathcal{X}, \mathcal{E})$ of $[\mathcal{E}]$, $\mathcal{E}$ is algebraic (resp. ι_ℓ-pure, p-plain) over $\mathcal{X}$.

6.3.2.3 Tensor products and direct sums

If $[\mathcal{E}]$ and $[\mathcal{E}']$ are in $\widetilde{\mathbf{Coef}}(X, \ell)$ with representatives $(\mathcal{X}, \mathcal{E})$ and $(\mathcal{X}', \mathcal{E}')$, there exists always a model $\widetilde{\mathcal{X}}$ of X with $\widetilde{\mathcal{X}} \succ_{\mathrm{j}} \mathcal{X}$ and $\widetilde{\mathcal{X}} \succ_{\mathrm{j}'} \mathcal{X}'$. Since the functors $\mathrm{j}^* : \mathbf{Coef}(\widetilde{\mathcal{X}}, \ell) \to \mathbf{Coef}(\mathcal{X}, \ell)$ and $\mathrm{j}'^{,*} : \mathbf{Coef}(\widetilde{\mathcal{X}}, \ell) \to \mathbf{Coef}(\mathcal{X}', \ell)$ preserve the operation $\oplus$ (resp. $\otimes$), the equivalence class $[\mathcal{E} \oplus \mathcal{F}]$, (resp. $[\mathcal{E} \otimes \mathcal{F}]$) of $(\widetilde{\mathcal{X}}, \mathrm{j}^*\mathcal{E} \oplus \mathrm{j}'^{,*}\mathcal{E}')$ (resp. $(\widetilde{\mathcal{X}}, \mathrm{j}^*\mathcal{E} \otimes \mathrm{j}'^{,*}\mathcal{E}')$) does not depend on the choice of the representatives $(\mathcal{X}, \mathcal{E})$ and $(\mathcal{X}', \mathcal{E}')$ of $[\mathcal{E}]$ and $[\mathcal{E}']$. Then every semisimple $[\mathcal{E}]$ is direct sum of irreducible objects.

6.3.2.4 Kernels and cokernels

If $[\mathbf{g}] : [\mathcal{E}] \to [\mathcal{E}']$ is a morphism in $\widetilde{\mathbf{Coef}}(X, \ell)$ represented by $(\mathcal{U}_i, [\mathbf{g}_i])$, $i = 1, 2$, there exists a model $\mathcal{U}_3$ of X with $\mathcal{U}_3 \succ \mathcal{U}_i$ and $\mathrm{j}_1^*\mathbf{g_1} = \mathrm{j}_2^*\mathbf{g_2}$. Since the functors $\mathrm{j}_i^* : \mathbf{Coef}(\mathcal{U}_i, \ell) \to \mathbf{Coef}(\mathcal{U}_3, \ell)$ are exact, the equivalence class $\mathrm{Ker}([\mathbf{g}])$ (resp. $[\mathrm{Coker}([\mathbf{g}])]$) of $(\mathcal{U}, \mathrm{Ker}(\mathbf{g}))$ (resp. $(\mathcal{U}, \mathrm{Coker}([\mathbf{g}]))$

does not depend on the choice of the representative $(\mathcal{U}, \mathbf{g})$ of $[\mathbf{g}] : [\mathcal{E}] \to [\mathcal{E}']$. We say that $[\mathbf{g}]$ is a monomorphism (resp. epimorphism) if $[\mathrm{Ker}(\mathbf{g})] = 0$ (resp. $[\mathrm{Coker}([\mathbf{g}])]$).

6.3.2.5 Compatibility

Take an $\ell' \neq \ell$. If $[\mathcal{E}_\ell]$ is in $\widetilde{\mathbf{Coef}}(X, \ell)$ with representative $(\mathcal{X}_\ell, \mathcal{E}_\ell)$ and $[\mathcal{E}_{\ell'}]$ is in $\widetilde{\mathbf{Coef}}(X, \ell')$ with representative $(\mathcal{X}_{\ell'}, \mathcal{E}_{\ell'})$, we say that $[\mathcal{E}_{\ell'}]$ and $[\mathcal{E}_\ell]$ are compatible (or that $[\mathcal{E}_\ell]$ is an ℓ'-adic companion of $[\mathcal{E}_{\ell'}]$) if there exists $\mathcal{X}$ in $\mathbf{Model}(X)$ with $\mathcal{X} \succ_{\mathfrak{j}_\ell} \mathcal{X}_\ell$ and $\mathcal{X} \succ_{\mathfrak{j}_{\ell'}} \mathcal{X}_{\ell'}$ such that $\mathfrak{j}_\ell^* \mathcal{E}_\ell$ and $\mathfrak{j}_{\ell'}^* \mathcal{E}_{\ell'}$ are compatible over $\mathcal{X}$.

6.3.2.6 Functoriality

Every morphism $f : Y \to X$ of smooth connected k-varieties induces a functor

$$f^* : \widetilde{\mathbf{Coef}}(X, \ell) \to \widetilde{\mathbf{Coef}}(Y, \ell)$$

as follow.

For every morphism $f : Y \to X$ of smooth connected k-varieties and every $[\mathcal{E}]$ in $\widetilde{\mathbf{Coef}}(X, \ell)$, with representative $(\mathcal{X}, \mathcal{E})$, there is always a model $\mathfrak{f} : \mathcal{Y}_1 \to \mathcal{X}_1$ of f such that $\mathcal{X}_1 \succ_{\mathfrak{j}} \mathcal{X}$. Since for every couple of such models $\mathfrak{f}_i : \mathcal{Y}_i \to \mathcal{X}_i$, $i = 1, 2$, there exists always a model $\mathfrak{f}_3 : \mathcal{Y}_3 \to \mathcal{X}_3$ with $(\mathfrak{f}_3 : \mathcal{Y}_3 \to \mathcal{X}_3) \succ (\mathfrak{f}_i : \mathcal{Y}_i \to \mathcal{X}_i)$, the equivalence class $f^*[\mathcal{E}]$ of $(\mathcal{Y}_1, \mathfrak{f}^* \mathfrak{j}^* \mathcal{E})$ does not depend on the choice of the model $\mathfrak{f} : \mathcal{Y}_1 \to \mathcal{X}_1$ of $f : Y \to X$ such that $\mathcal{X}_1 \succ_{\mathfrak{j}} \mathcal{X}$.

Similarly, if $[\mathbf{g}] : [\mathcal{E}_1] \to [\mathcal{E}_2]$ is a morphism in $\widetilde{\mathbf{Coef}}(X, \ell)$ represented by $(\mathcal{U}, \mathbf{g})$, there is always a model $\mathfrak{f} : \mathcal{Y}_1 \to \mathcal{U}_1$ of $\mathfrak{f}$ such that $\mathcal{U}_1 \succ_{\mathfrak{j}} \mathcal{U}$ and the equivalence class $[f^*(\mathbf{g})] : [\mathcal{E}_1] \to [\mathcal{E}_2]$ of $(\mathcal{Y}_1, \mathfrak{f}^* \mathfrak{j}^* \mathbf{g})$ does not depend on the choice of the model $\mathfrak{f} : \mathcal{Y}_1 \to \mathcal{U}_1$ of $\mathfrak{f}$ such that $\mathcal{U}_1 \succ_{\mathfrak{j}} \mathcal{U}$.

6.3.2.7 Monodromy groups

For $[\mathcal{E}]$ in $\widetilde{\mathbf{Coef}}(X, \ell)$ with representatives $(\mathcal{X}_i, \mathcal{E}_i)$, $i = 1, 2$, there exists a representative $(\mathcal{X}_3, \mathcal{E}_3)$ with $(\mathcal{X}_3, \mathcal{E}_3) \succ_{\mathfrak{j}_i} (\mathcal{X}_i, \mathcal{E}_i)$. If $G(\mathcal{E}_i)$ denotes the monodromy group of $\mathcal{E}_i$ over $\mathcal{X}_i$ (see Section 6.3.1.5), by Fact 6.3.1.5.1(2) we have $G(\mathcal{E}_1) \simeq G(\mathfrak{j}_1^* \mathcal{E}_1) \simeq G(\mathfrak{j}_2^* \mathcal{E}_2) \simeq G(\mathcal{E}_2)$. Hence $G([\mathcal{E}]) := G(\mathcal{E})$ is well defined independently on the choice of $(\mathcal{X}, \mathcal{E})$ of $[\mathcal{E}]$. We call it the arithmetic monodromy group of $[\mathcal{E}]$. Similarly, using Fact 6.3.1.5.1(3), if X is also geometrically connected, $G([\mathcal{E}]^{geo}) := G(\mathcal{E}^{geo})$ is independent from the choice of $(\mathcal{X}, \mathcal{E})$. We call it the geometric monodromy group of $[\mathcal{E}]$.

6.3.3 Comparison with the category of lisse sheaves

Let X be a smooth connected k-variety. In this subsection we assume $\ell \neq p$. In this case, we have another candidate for a category of ℓ-adic local systems: $\mathbf{LS}(X, \ell)$. In this section we compare these two options.

6.3.3.1 Comparison functor

We construct a functor:

$$\Phi : \widetilde{\mathbf{Coef}}(X, \ell) \to \mathbf{LS}(X, \ell).$$

Let $[\mathcal{F}]$ be in $\widetilde{\mathbf{Coef}}(X, \ell)$ with representative $(\mathcal{X}_i, \mathcal{F}_i)$, $i = 1, 2$, then there exists a $(\mathcal{X}_3, \mathcal{F}_3)$ such that $(\mathcal{X}_3, \mathcal{F}_3) \succ (\mathcal{X}_i, \mathcal{F}_i)$. Then the commutative diagram

$$\begin{tikzcd}
 & \mathcal{X}_1 \arrow[dl] \\
\mathcal{X}_3 & X \arrow[l, "j_{\mathcal{X}_3}"] \arrow[u, "j_{\mathcal{X}_1}"'] \arrow[d, "j_{\mathcal{X}_2}"] \\
 & \mathcal{X}_2 \arrow[ul]
\end{tikzcd}$$

shows that $\Phi([\mathcal{F}]) := j_{\mathcal{X}}^{*}\mathcal{F}$ does not depend on the choice of the representative $(\mathcal{X}, \mathcal{F})$ of the equivalence class of $[\mathcal{F}]$.

Similarly, if $[\mathbf{g}] : [\mathcal{F}] \to [\mathcal{G}]$ is a morphism in $\widetilde{\mathbf{Coef}}(X, \ell)$ represented by $(\mathcal{U}, \mathbf{g})$, the morphism $\Phi([\mathbf{g}]) : \Phi([\mathcal{F}]) \to \Phi([\mathcal{G}])$ defined by $j_{\mathcal{U}}^{*}(\mathbf{g})$ does not depend on the choice of the representative $(\mathcal{U}, \mathbf{g})$ of $[\mathbf{g}] : [\mathcal{F}] \to [\mathcal{G}]$

Since for every $\mathcal{X} \in \mathbf{Model}(X)$ the natural morphism $\pi_1(X) \to \pi_1(\mathcal{X})$ is surjective, the image $\Pi_{\Phi([\mathcal{F}])}$ of $\rho_{\Phi([\mathcal{F}])}$ coincide with $\Pi_{\mathcal{F}}$ and $\Phi : \widetilde{\mathbf{Coef}}(X, \ell) \to \mathbf{LS}(X, \ell)$ is fully faithful. The functor $\Phi : \widetilde{\mathbf{Coef}}(X, \ell) \to \mathbf{LS}(X, \ell)$ is not essentially surjective in general.

Remark 6.3.3.1.1. If $X = k = \mathbb{F}_p(T)$, any $\mathbb{F}_p$-model of X is a dense open subscheme $\mathcal{U} \subseteq \mathbb{P}^1_{\mathbb{F}_p}$ and there is an exact sequence

$$0 \to I_{\mathcal{U}} \to \pi_1(k) \to \pi_1(\mathcal{U}) \to 0$$

where $I_{\mathcal{U}}$ is the subgroup generated by the inertia groups of the points in $\mathbb{P}^1_{\mathbb{F}_p} - \mathcal{U}$. Hence a representation of $\pi_1(k)$ is in the essential image of $\Phi : \widetilde{\mathbf{Coef}}(X, \ell) \to \mathbf{LS}(X, \ell)$ if and only if it is unramified outside finitely many places.

However, its essential image is big enough to contains all the representations coming from geometry. More precisely, if the $f : Y \to X$ is a smooth proper morphism, choose a smooth proper model $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ of $f : Y \to X$. By smooth proper base change, $R^i f_* \mathbb{Q}_\ell$ is $\mathbf{LS}(X, \ell)$ and $R^i \mathfrak{f}_* \mathbb{Q}_\ell$ is in $\mathbf{LS}(\mathcal{X}, \ell)$. If we write $[R^i f_* \mathbb{Q}_\ell]$ for the equivalence class of $(\mathcal{X}, R^i \mathfrak{f}_* \mathbb{Q}_\ell)$, then we have $\Phi([R^i f_* \mathbb{Q}_\ell]) = R^i f_* \mathbb{Q}_\ell$.

6.3.3.2 Geometric image

Assume now that X is geometrically connected. Then, for every $[\mathcal{F}]$ in $\widetilde{\mathbf{Coef}}(X, \ell)$, we can restrict $\rho_{\Phi([\mathcal{F}])}$ to $\pi_1(X_{\overline{k}})$ obtaining a representation

$$\rho_{\Phi([\mathcal{F}])_{\overline{k}}} : \pi_1(X_{\overline{k}}) \to GL_r(\overline{\mathbb{Q}}_\ell),$$

with image $\Pi_{\Phi([\mathcal{F}])_{\overline{k}}}$. If $(\mathcal{X}, \mathcal{F})$ is a representative for $[\mathcal{F}]$, in general $\Pi_{\Phi([\mathcal{F}])_{\overline{k}}}$ is very different from $\Pi_{\mathcal{F}^{geo}}$.

Example 6.3.3.2.1. If $X = \mathbb{P}^1_k$, then $\pi_1(X_{\overline{k}}) = 0$ so that $\Pi_{\Phi([\mathcal{F}])_{\overline{k}}} = 0$ for all $[\mathcal{F}]$ in $\widetilde{\mathbf{Coef}}(X, \ell)$. However, if $\mathcal{K}$ is any model of k, then $\mathbb{P}^1_{\mathcal{K}} \to \mathcal{K}$ is a model of $\mathbb{P}^1_k$, hence it is not true in general that $\Pi_{\mathcal{F}^{geo}} = 0$

However, there is a commutative exact diagram of groups

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\overline{k}}) \arrow[r] \arrow[d] & \pi_1(X) \arrow[r] \arrow[d, two heads] & \pi_1(k) \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \pi_1(\mathcal{X}_{\mathbb{F}}) \arrow[r] & \pi_1(\mathcal{X}) \arrow[r] & \pi_1(\mathbb{F}_q) \arrow[r] & 1
\end{tikzcd} \tag{6.3.3.2.2}$$

where the central and right arrows are surjective. Since the image of a normal subgroup trough a surjection is a normal subgroup, $\pi_1(X_{\overline{k}}) \to \pi_1(\mathcal{X}_{\mathbb{F}})$ has normal image, hence $\Pi_{\Phi([\mathcal{F}])_{\overline{k}}} \subseteq \Pi_{\mathcal{F}^{geo}}$ is a normal subgroup. As a consequence, we get the following (a mild generalization of [CT12b, Theorem 5.7]), which is needed to apply Fact 6.1.2.1(2) in our setting.

Lemma 6.3.3.2.3. If $[\mathcal{F}]$ is geometrically semisimple, then:

$$Lie(\Pi_{\Phi([\mathcal{F}])_{\overline{k}}})^{ab} = 0.$$

Proof. Let $(\mathcal{X}, \mathcal{F})$ be a representative for $[\mathcal{F}]$. Since $\Pi_{\Phi([\mathcal{F}])_{\overline{k}}}$ is a normal subgroup of $\Pi_{\mathcal{F}^{geo}}$, the Zariski closure $G(\Phi([\mathcal{F}])_{\overline{k}})$ of $\Pi_{\Phi([\mathcal{F}])_{\overline{k}}}$ is a normal subgroup of $G(\mathcal{F}^{geo})$. Since $\mathcal{F}^{geo}$ is geometrically semisimple, by Fact 6.3.1.4.1(2) $G(\mathcal{F}^{geo})$ is a semisimple algebraic group, hence $G(\Phi([\mathcal{F}])_{\overline{k}})$ is a semisimple algebraic group. By [Ser66, §1, Corollaire], this implies that $Lie(\Pi_{\Phi([\mathcal{F}])_{\overline{k}}})^{ab} = Lie(G(\Phi([\mathcal{F}])_{\overline{k}}))^{ab} = 0$. □

6.4 Exceptional loci of coefficient objects

Let X be a smooth geometrically connected k-variety.

6.4.1 Definitions and first properties

6.4.1.1 Definitions

For every morphism $f : Y \to X$ of smooth connected k-varieties and every $[\mathcal{E}]$ in $\widetilde{\mathbf{Coef}}(X, \ell)$, with representative $(\mathcal{X}, \mathcal{E})$, there is always a model $\mathfrak{f}^* : \mathcal{Y}_1 \to \mathcal{X}_1$ of $\mathfrak{f}$ such that $\mathcal{X}_1 \succ_{\mathfrak{j}} \mathcal{X}$. Hence, as in Section 6.3.2.7, we get a well defined closed immersion $G(f^*[\mathcal{E}]) = G(\mathfrak{f}^*\mathfrak{j}^*\mathcal{E}) \subseteq G(\mathfrak{j}^*\mathcal{E}) \simeq G(\mathcal{E}) = G([\mathcal{E}])$. So we can define:

Definition 6.4.1.1.1. We say that $x \in |X|$ is algebraically generic (resp. algebraically strictly generic) for $[\mathcal{E}]$ if $G(x^*[\mathcal{E}])^0 = G([\mathcal{E}])^0$ (resp. $G(x^*[\mathcal{E}]) = G([\mathcal{E}])$).

Write $X^{gen}_{[\mathcal{E}]}$ (resp. $X^{sgen}_{[\mathcal{E}]}$) for the set of $x \in |X|$ that are algebraically generic (resp. algebraically strictly generic) for $[\mathcal{E}]$. Define the following sets:

$$X^{ex}_{[\mathcal{E}]} := |X| - X^{gen}_{[\mathcal{E}]}; \quad X^{ex}_{[\mathcal{E}]}(\leq d) := X^{ex}_{[\mathcal{E}]} \cap X(\leq d); \quad X^{gen}_{[\mathcal{E}]}(\leq d) := X^{gen}_{[\mathcal{E}]} \cap X(\leq d)$$

$$X^{stex}_{[\mathcal{E}]} := |X| - X^{sgen}_{[\mathcal{E}]}; \quad X^{stex}_{[\mathcal{E}]}(\leq d) := X^{stex}_{[\mathcal{E}]} \cap X(\leq d); \quad X^{sgen}_{[\mathcal{E}]}(\leq d) := X^{sgen}_{[\mathcal{E}]} \cap X(\leq d).$$

We call $X^{ex}_{[\mathcal{E}]}$ the algebraic exceptional locus of $[\mathcal{E}]$ and $X^{stex}_{[\mathcal{E}]}$ its algebraic strictly-exceptional locus. For further use we gather an easy lemma on the behaviour of the exceptional locus under finite étale cover.

Lemma 6.4.1.1.2. Let $f : Y \to X$ be a connected finite étale cover. Then we have:

$$f(Y^{ex}_{f^*[\mathcal{E}]}) = X^{ex}_{[\mathcal{E}]}$$

Proof. Choose a model of $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ such that $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ is finite étale and a representative class of $(\mathcal{X}, \mathcal{E})$ is defined over $\mathcal{X}$. Then Lemma 6.4.1.1.2 follows from the fact that $G(f^*[\mathcal{E}]) = G(\mathfrak{f}^*\mathcal{E}) \subseteq G(\mathcal{E}) = G([\mathcal{E}])$ is an open subgroup by [D'Ad17, Proposition 3.3.4]. □

6.4.1.2 Comparison with the exceptional locus of $\Phi([\mathcal{F}])$

Assume $\ell \neq p$. In the introduction, for any $\mathcal{F}$ in $\mathbf{LS}(X, \mathbb{Q}_\ell)$, we defined the exceptional locus $X^{ex}_{\mathcal{F}}$ and the strictly exceptional locus $X^{stex}_{\mathcal{F}}$. So, if we have a $[\mathcal{F}]$ in $\widetilde{\mathbf{Coef}}(X, \ell)$, we have four subsets of X:

$$X^{gen}_{[\mathcal{F}]}; \quad X^{sgen}_{[\mathcal{F}]}; \quad X^{gen}_{\Phi([\mathcal{F}])}; \quad X^{sgen}_{\Phi([\mathcal{F}])}.$$

To apply Fact 6.1.2.1 in our setting, we need to compare them.

Lemma 6.4.1.2.1. The following hold:

1. $X^{sgen}_{\Phi([\mathcal{F}])} \subseteq X^{sgen}_{[\mathcal{F}]}$ and $X^{gen}_{\Phi([\mathcal{F}])} \subseteq X^{gen}_{[\mathcal{F}]}$
2. If $\mathcal{F}$ is geometrically semisimple then $X^{gen}_{\Phi([\mathcal{F}])} = X^{gen}_{[\mathcal{F}]}$.

Proof. Choose a model $\mathfrak{f} : \mathcal{K}' \to \mathcal{X}$ of $x \to X$ such that a representative $(\mathcal{X}, \mathcal{F})$ of $[\mathcal{F}]$ is defined over $\mathcal{X}$. Since $\Pi_{\mathcal{F}} = \Pi_{\Phi([\mathcal{F}])}$ and $\Pi_{\mathfrak{f}^*\mathcal{F}} = \Pi_{\Phi([\mathcal{F}]),x}$, (1) amounts to show that if $\Pi_{\mathfrak{f}^*\mathcal{F}} = \Pi_{\mathcal{F}}$ (resp. $\Pi_{\mathfrak{f}^*\mathcal{F}} \subseteq \Pi_{\mathcal{F}}$ is an open subgroup) then $G(\mathfrak{f}^*\mathcal{F}) = G(\mathcal{F})$ (resp. $G(\mathfrak{f}^*\mathcal{F})^0 = G(\mathcal{F})^0$). So (1) follow from the fact that $G(\mathcal{F})$ and $G(\mathfrak{f}^*\mathcal{F})$ are the Zariski closures of $\Pi_{\mathcal{F}}$ and $\Pi_{\mathfrak{f}^*\mathcal{F}}$ respectively.

Then (2) amounts to show that if $G(\mathfrak{f}^*\mathcal{F})^0 = G(\mathcal{F})^0$ then $\Pi_{\mathfrak{f}^*\mathcal{F}} \subseteq \Pi_{\mathcal{F}}$ is an open subgroup. So assume that $G(\mathfrak{f}^*\mathcal{F})^0 = G(\mathcal{F})^0$. Replacing $\mathbb{F}_q$ with a finite field extension, we can assume that $\mathcal{K}'$ is geometrically connected over $\mathbb{F}_q$. Since $G(\mathfrak{f}^*\mathcal{F})^0 = G(\mathcal{F})^0$, Fact 6.3.1.4.1(6) implies that $G(\mathfrak{f}^*\mathcal{F}^{geo})^0 = G(\mathcal{F}^{geo})^0$. Since $G(\mathfrak{f}^*\mathcal{F}^{geo})^0 = G(\mathcal{F}^{geo})^0$ is a semisimple algebraic group (6.3.1.4.1(2)), by [Ser66, §1, Corollaire] we deduce that $\Pi_{\mathfrak{f}^*\mathcal{F}^{geo}}$ is open in $\Pi_{\mathcal{F}^{geo}}$. There is a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Pi_{\mathfrak{f}^*\mathcal{F}^{geo}} \arrow[r] \arrow[d] & \Pi_{\mathfrak{f}^*\mathcal{F}} \arrow[r] \arrow[d] & \frac{\Pi_{\mathfrak{f}^*\mathcal{F}}}{\Pi_{\mathfrak{f}^*\mathcal{F}^{geo}}} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \Pi_{\mathcal{F}^{geo}} \arrow[r] & \Pi_{\mathcal{F}} \arrow[r] & \frac{\Pi_{\mathcal{F}}}{\Pi_{\mathcal{F}^{geo}}} \arrow[r] & 0,
\end{tikzcd}$$

where the right vertically arrow is surjective, since $\mathcal{K}'$ is geometrically connected over $\mathbb{F}_q$. So there is a surjection

$$\frac{\Pi_{\mathcal{F}^{geo}}}{\Pi_{\mathfrak{f}^*\mathcal{F}^{geo}}} \to \frac{\Pi_{\mathcal{F}}}{\Pi_{\mathfrak{f}^*\mathcal{F}}}$$

In particular $\frac{\Pi_{\mathcal{F}}}{\Pi_{\mathfrak{f}^*\mathcal{F}}}$ is finite hence $\Pi_{\mathfrak{f}^*\mathcal{F}}$ is open in $\Pi_{\mathcal{F}}$. □

6.4.2 Proof of Theorem 6.1.5.1.1

Now we are ready to prove Theorem 6.1.5.1.1. Let $[\mathcal{E}]$ be a geometrically semisimple overconvergent F-isocrystal over X. For every $x \in |X|$, choose a model $\mathfrak{f} : \mathcal{K}' \to \mathcal{X}$ of $x \to X$ such that a representative $(\mathcal{X}, \mathcal{E})$ of $[\mathcal{E}]$ is defined over $\mathcal{X}$.

6.4.2.1 Reducing to the semisimple case

We first reduce to the semisimple situation.

Lemma 6.4.2.1.1. Let $[\mathcal{E}]$ be in $\widetilde{\mathbf{Coef}}(X, \ell)$. If $[\mathcal{E}]$ is geometrically semisimple then

$$X^{ex}_{[\mathcal{E}]} = X^{ex}_{[\mathcal{E}^{ss}]} \quad \text{and} \quad X^{stex}_{[\mathcal{E}]} = X^{stex}_{[\mathcal{E}^{ss}]}$$

Proof. The first (resp. the second) equality amounts to show that $G(\mathfrak{f}^*\mathcal{E})^0 = G(\mathcal{E})^0$ if and only if $G(\mathfrak{f}^*\mathcal{E}^{ss})^0 = G(\mathcal{E}^{ss})^0$ (resp. $G(\mathfrak{f}^*\mathcal{E}) = G(\mathcal{E})$ if and only if $G(\mathfrak{f}^*\mathcal{E}^{ss}) = G(\mathcal{E}^{ss})$).

1. To prove the first equality, we can replace $\mathbb{F}_q$ with a finite field extension, hence assume that $\mathcal{K}'$ is geometrically connected over $\mathbb{F}_q$. Using Fact 6.3.1.4.1(4) we see that $\mathfrak{f}^*\mathcal{E}$ is again geometrically semisimple. Since semisimple implies geometrically semisimple (6.3.1.4.1(5)) we get isomorphisms

$$\mathcal{E}^{geo} \simeq (\mathcal{E}^{geo})^{ss} \simeq (\mathcal{E}^{ss})^{geo}$$

$$\mathfrak{f}^*(\mathcal{E}^{geo}) \simeq (\mathfrak{f}^*\mathcal{E})^{geo} \simeq ((\mathfrak{f}^*\mathcal{E})^{ss})^{geo} \overset{(\Delta)}{\simeq} ((\mathfrak{f}^*(\mathcal{E}^{ss}))^{ss})^{geo} \simeq \mathfrak{f}^*(\mathcal{E}^{ss})^{geo}$$

   where the equality (Δ) follows from Fact 6.3.1.4.1(3) since it implies that $(\mathfrak{f}^*\mathcal{E})^{ss} \simeq (\mathfrak{f}^*(\mathcal{E}^{ss}))^{ss}$. By Fact 6.3.1.4.1(6) we see that $G(\mathcal{E})^0 = G(\mathfrak{f}^*\mathcal{E})^0$ if and only $G(\mathcal{E}^{geo})^0 = G(\mathfrak{f}^*\mathcal{E}^{geo})^0$. So $G(\mathcal{E}^{geo})^0 = G(\mathfrak{f}^*\mathcal{E}^{geo})^0$ if and only if $G((\mathcal{E}^{ss})^{geo})^0 = G(\mathfrak{f}^*(\mathcal{E}^{ss})^{geo})^0$. But $\mathcal{E}^{ss}$ is again geometrically semisimple so that we can apply again Fact 6.3.1.4.1(6) to get that $G((\mathcal{E}^{ss})^{geo})^0 = G(\mathfrak{f}^*(\mathcal{E}^{ss})^{geo})^0$ if and only if $G(\mathcal{E}^{ss})^0 = G(\mathfrak{f}^*(\mathcal{E}^{ss}))^0$. This concludes the proof of the first equality.

2. Now we deduce the second equality from the first one, via a purely group theoretic argument. For $? \in \{\emptyset, ss\}$ we have an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G(\mathfrak{f}^*(\mathcal{E}^?))^0 \arrow[r] \arrow[d, hook, "g^{?,0}"] & G(\mathfrak{f}^*(\mathcal{E}^?)) \arrow[r] \arrow[d, hook, "g^?"] & \pi_0(G(\mathfrak{f}^*(\mathcal{E}^?))) \arrow[r] \arrow[d, "\pi_0(g^?)"] & 0 \\
0 \arrow[r] & G(\mathcal{E}^?)^0 \arrow[r] & G(\mathcal{E}^?) \arrow[r] & \pi_0(G(\mathcal{E}^?)) \arrow[r] & 0
\end{tikzcd}$$

   By (1), $g^{ss,0}$ is an isomorphism if and only if g^0 is an isomorphism. Assume this is the case, so that $\pi_0(g^?)$ is injective. Then $g^?$ is an isomorphism if and only if $\pi_0(g^?)$ is an isomorphism if and only if $|\pi_0(G(\mathcal{E}^?))| = |\pi_0(G(\mathfrak{f}^*(\mathcal{E}^?)))|$. We conclude observing that, since taking semisimplification does not change the group of connected components, we have:

$$|\pi_0(G(\mathcal{E}))| \simeq |\pi_0(G(\mathcal{E}^{ss}))|$$

$$|\pi_0(G(\mathfrak{f}^*\mathcal{E}))| \simeq |\pi_0(G((\mathfrak{f}^*\mathcal{E})^{ss}))| \simeq |\pi_0(G((\mathfrak{f}^*(\mathcal{E}^{ss}))^{ss}))| \simeq |\pi_0(G(\mathfrak{f}^*(\mathcal{E}^{ss})))|$$

   This concludes the proof □

So from now assume that $[\mathcal{E}]$ is semisimple, hence that $[\mathcal{E}] \simeq \oplus_i[\mathcal{E}_i]$ with each $[\mathcal{E}_i]$ irreducible. By [Abe15, Lemma 6.1], there exists a twist $\mathcal{E}_i^{(j_i)}$ of $\mathcal{E}_i$ such that $\mathcal{E}_i^{(j_i)}$ has determinant of finite order under tensor.

6.4.2.2 Reducing to the algebraic case

Assume first that $\mathcal{E}$ is not algebraic. We reduce to the situation in which $\mathcal{E}$ is algebraic using the following, which is a consequence of the companion conjecture ([Laf02], [Dri12], [Abe18], [AE16], see [Ked18, Corollary 3.3.3])

Fact 6.4.2.2.1. Let $\mathcal{X}$ be a smooth geometrically connected $\mathbb{F}_q$-variety and let $\mathcal{E}$ be in $\mathbf{Coef}(\mathcal{X}, \ell)$ irreducible with finite order determinant. Then $\mathcal{E}$ is algebraic.

By Fact 6.4.2.2.1 $\mathcal{E}_i^{(j_i)}$ is algebraic, hence $[\mathcal{E}'] := \oplus_i[\mathcal{E}_i^{(j_i)}]$ is algebraic. Then we have:

Lemma 6.4.2.2.2. There is an equality

$$X^{ex}_{[\mathcal{E}]} = X^{ex}_{[\mathcal{E}']}$$

Proof. We need to show that $G(\mathfrak{f}^*\mathcal{E})^0 = G(\mathcal{E})^0$ if and only if $G(\mathfrak{f}^*\mathcal{E}')^0 = G(\mathcal{E}')^0$. Replacing $\mathbb{F}_q$ with a finite field extension, we can assume that $\mathcal{K}'$ is geometrically connected over $\mathbb{F}_q$. By Fact 6.3.1.4.1(6) $G(\mathfrak{f}^*\mathcal{E})^0 = G(\mathfrak{f}^*\mathcal{E})^0$ if and only if $G(\mathfrak{f}^*\mathcal{E}^{geo})^0 = G(\mathfrak{f}^*\mathcal{E}^{geo})^0$ and $G(\mathfrak{f}^*\mathcal{E}')^0 = G(\mathcal{E}')^0$ if and only if $G(\mathfrak{f}^*\mathcal{E}'^{geo})^0 = G(\mathcal{E}'^{geo})^0$. Since $\mathcal{E}'^{geo} = \mathcal{E}^{geo}$ this concludes the proof. □

So, using Lemma 6.4.2.2.2, we may and do assume that $\mathcal{E}$ is algebraic.

6.4.2.3 Companions and independence

The idea is then to take, for some $\ell \neq p$, an ℓ-adic companion $\mathcal{F}_\ell$ of $\mathcal{E}$, to which we can apply Fact 6.1.2.1(2) thanks to Proposition 6.3.3.2.3, and to prove that the exceptional loci of $\mathcal{F}$ and $\mathcal{E}$ coincide. To do this, we exploit the companions conjecture ([Laf02], [Abe18], [AE16]).

Fact 6.4.2.3.1 ([AE16, Theorem 4.2])**.** Let $\mathcal{X}$ be a smooth geometrically connected $\mathbb{F}_q$-variety and let $\mathcal{E}$ be in $\mathbf{Coef}(\mathcal{X}, p)$ irreducible with finite order determinant. Then, for every $\ell \neq p$ there exists a (unique) $\mathcal{F}$ in $\mathbf{Coef}(\mathcal{X}, \ell)$ which is $\underline{\iota}$-compatible with $\mathcal{E}$.

Since $\mathcal{E}_i$ and $\mathcal{E}_i^{(j_i)}$ are algebraic, for every $i \geq 0$, $j_i \in \overline{\mathbb{Q}}_p$ is an algebraic number. In particular there exists an ℓ such that for every $i \geq 0$ and every place λ in $\overline{\mathbb{Q}}$ over ℓ, j_i is a λ-adic unit. Fix such ℓ and write $\mathcal{F}_{\ell,i}$ for the ℓ-adic companion of $\mathcal{E}_i$ over $\mathcal{X}$ given by Fact 6.4.2.3.1. Since for every $i \geq 0$ and every place λ in $\overline{\mathbb{Q}}$ over ℓ, j_i is a λ-adic unit, $\mathcal{F}_{\ell,i}^{(1/j_i)}$ is in $\mathbf{Coef}(\mathcal{X}, \ell)$ (see Section 6.3.1.1) and it is an ℓ-adic companion of $\mathcal{E}_i$. Consider $[\mathcal{F}_\ell]$ in $\mathbf{Coef}(X, \ell)$ represented by $(\mathcal{X}, \oplus_i \mathcal{F}_\ell^{(1/j_i)})$, so that $[\mathcal{F}_\ell]$ is an ℓ-adic companion of $\mathcal{E}$. Then from Lemmas 6.4.1.2.1, 6.3.3.2.3 and Fact 6.1.2.1 it is enough to prove the following.

Fact 6.4.2.3.2. For $? \in \{\ell, \ell'\}$ fix a $[\mathcal{E}_?]$ in $\widetilde{\mathbf{Coef}}(X, ?)$. Assume that $[\mathcal{E}_\ell]$ and $[\mathcal{E}_{\ell'}]$ are geometrically semisimple and $\underline{\iota}$-compatible. Then

$$X^{gen}_{[\mathcal{E}_\ell]} = X^{gen}_{[\mathcal{E}_{\ell'}]} \quad \text{and} \quad X^{sgen}_{[\mathcal{E}_\ell]} = X^{sgen}_{[\mathcal{E}_{\ell'}]}$$

Proof. By Lemma 6.4.2.1.1 we can assume that $[\mathcal{E}_\ell]$ and $[\mathcal{E}_{\ell'}]$ are semisimple. Choose a model $\mathfrak{f} : \mathcal{K}' \to \mathcal{X}$ of $x \to X$ such that there are representatives $(\mathcal{X}, \mathcal{E}_\ell)$ and $(\mathcal{X}, \mathcal{E}_{\ell'})$ of $[\mathcal{E}_\ell]$ and $[\mathcal{E}_{\ell'}]$ are defined over $\mathcal{X}$. Then the first (resp. the second) equality amounts to show that $G(\mathfrak{f}^*\mathcal{E}_\ell)^0 = G(\mathcal{E}_\ell^0)$ if and only if $G(\mathfrak{f}^*\mathcal{E}_{\ell'})^0 = G(\mathcal{E}_{\ell'})^0$ (resp. $G(\mathfrak{f}^*\mathcal{E}_\ell) = G(\mathcal{E}_\ell)$ if and only if $G(\mathfrak{f}^*\mathcal{E}_{\ell'}) = G(\mathcal{E}_{\ell'})$). They both follows from the proof of [Cad19b, Corollaire 8.7]. □

6.5 Convergent F-isocrystals over finitely generated fields

6.5.1 Convergent F-isocrystals over finite fields

Let $\mathcal{X}$ be a smooth connected $\mathbb{F}_q$-variety. We first quickly review the theory of convergent F-isocrystals over $\mathcal{X}$ and its relation with the theories of overconvergent F-isocrystals and p-adic representations. Then we study the behaviour of convergent F-isocrystals under open immersions of smooth $\mathbb{F}_q$-varieties.

6.5.1.1 Convergent and overconvergent F-isocrystals

Write $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ (resp. $\mathbf{Isoc}(\mathcal{X})$) for the category of $\overline{\mathbb{Q}}_p$-convergent F-isocrystals on $\mathcal{X}$ (resp. convergent F-isocrystals) and consider the canonical diagram of functors:

$$\begin{array}{ccc} \mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(\mathcal{X}) & \xrightarrow{(-)^{geo}} & \mathbf{Isoc}^{\dagger}(\mathcal{X}) \\ \big\downarrow{\scriptstyle (-)^{conv}} & & \big\downarrow{\scriptstyle (-)^{conv}} \\ \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}) & \xrightarrow{(-)^{geo}} & \mathbf{Isoc}(\mathcal{X}). \end{array}$$

Recall the following.

Fact 6.5.1.1.1. [Ked04] The functor $(-)^{conv} : \mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(\mathcal{X}) \to \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ is fully faithful.

6.5.1.2 Slopes

Let $\mathcal{E}$ be in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ of rank r. For every $\mathfrak{t} \in |\mathcal{X}|$, one considers the multi-set $\{a_i^{\mathfrak{t}}(\mathcal{E})\}_{1\leq i\leq r}$ of slopes of $\mathcal{E}$ at $\mathfrak{t}$. These are rational numbers that we assume to be ordered as $a_1^{\mathfrak{t}}(\mathcal{E}) \leq \cdots \leq a_r^{\mathfrak{t}}(\mathcal{E})$. See [Ked17, Sections 3 and 4] for more details on the theory of slopes. We say that $\mathcal{E}$ is isoclinic (of slope $a_1^{\mathfrak{t}}(\mathcal{E})$) if $a_1^{\mathfrak{t}}(\mathcal{E}) = a_r^{\mathfrak{t}}(\mathcal{E})$ for every $\mathfrak{t} \in |\mathcal{X}|$ and that $\mathcal{E}$ is unit-root if it is isoclinic of slope 0. Write $\mathbf{F}\text{-}\mathbf{Isoc}_{ur}(\mathcal{X}) \subseteq \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ for the Tannakian subcategory of unit-root convergent F-isocrystals.

Fact 6.5.1.2.1. [Tsu02] There is a natural equivalence of categories

$$\Phi_{\mathcal{X}} : \mathbf{F}\text{-}\mathbf{Isoc}_{ur}(\mathcal{X}) \simeq \mathbf{LS}(\mathcal{X}, p).$$

We say that $\mathcal{E}$ has constant Newton polygon if the function

$$\begin{aligned} N_{\mathcal{E}} : |\mathcal{X}| &\to \mathbb{Q}^r \\ \mathfrak{t} &\mapsto (a_i^{\mathfrak{t}}(\mathcal{E}))_{1\leq i\leq r} \end{aligned}$$

is constant. Write $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{CNP}}(\mathcal{X}) \subseteq \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ for the Tannakian subcategory of convergent F-isocrystals with constant Newton Polygon.

Fact 6.5.1.2.2 ([Ked17, Theorem 3.12, Corollary 4.2])**.** Let $\mathcal{E}$ be in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$.

1. There exists a dense open immersion $\mathfrak{j} : \mathcal{U} \to \mathcal{X}$ such that $\mathfrak{j}^*\mathcal{E}$ is in $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{CNP}}(\mathcal{U})$;
2. If $\mathcal{E}$ is in $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{CNP}}(\mathcal{X})$, then there exists a unique filtration

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq ... \subseteq \mathcal{E}_{n-1} \subseteq \mathcal{E}_n = \mathcal{E} \quad \text{in } \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$$

   such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is isoclinic of some slope s_i with $s_1 < s_2 < ... < s_n$.

In general, if (P) is a property of convergent F-isocrystals, we say that a $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(\mathcal{X})$ has (P) if $\mathcal{E}^{conv}$ has (P).

6.5.1.3 Monodromy groups

$\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ is a Tannakian $\overline{\mathbb{Q}}_p$-linear category and the choice of a geometric point $\bar{\mathfrak{t}}$ of $\mathcal{X}$, defines a fibre functor $\mathfrak{t}^* : \mathbf{F}\text{-}\mathbf{Isoc}(X) \to \mathbf{Vect}_{\overline{\mathbb{Q}}_p}$. For $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ write $G(\mathcal{E})$ for the Tannaka group of the Tannakian subcategory $\langle\mathcal{E}\rangle \subseteq \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$. If $\mathcal{X}$ is geometrically connected, define similarly $G(\mathcal{E}^{geo})$. The fully faithful functor $\langle\mathcal{E}\rangle \to \langle\mathcal{E}^{geo}\rangle$ induces a closed immersion $G(\mathcal{E}^{geo}) \subseteq G(\mathcal{E})$, with $G(\mathcal{E}^{geo})$ normal in $G(\mathcal{E})$; [D'Ad17, Appendix] and [AD18, Propositon 2.2.4]. The algebraic group $G(\mathcal{E})^{cst} := G(\mathcal{E})/G(\mathcal{E}^{geo})$ is abelian and identifies with the Tannakian group of the full Tannakian subcategory $\langle\mathcal{E}\rangle^{cst} \subseteq \langle\mathcal{E}\rangle$ of objects isomorphic to an object of the form $\mathfrak{q}^*\mathcal{E}'$, where $\mathfrak{q} : \mathcal{X} \to Spec(\mathbb{F}_q)$ is the structural morphism and $\mathcal{E}'$ is in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathbb{F}_q)$.

If $\mathcal{E}$ is in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X})$ the fully faithful functor $(-)^{conv} : \langle\mathcal{E}\rangle \to \langle\mathcal{E}^{conv}\rangle$ of Section 6.5.1.1, induces a closed immersion $G(\mathcal{E}^{conv}) \subseteq G(\mathcal{E})$. Similarly, if $\mathcal{X}$ is geometrically connected there is a natural closed immersion $G(\mathcal{E}^{geo,conv}) \subseteq G(\mathcal{E}^{geo})$ fitting into a commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & G(\mathcal{E}^{conv,geo}) \arrow[r, hook] \arrow[d, hook] & G(\mathcal{E}^{conv}) \arrow[r] \arrow[d, hook] & G(\mathcal{E}^{conv})^{cst} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & G(\mathcal{E}^{geo}) \arrow[r, hook] & G(\mathcal{E}) \arrow[r] & G(\mathcal{E})^{cst} \arrow[r] & 0,
\end{tikzcd}$$

where the right vertical arrow is surjective (see Section 5.2.2 in Chapter 5).

Fact 6.5.1.3.1. Let $\mathcal{E}$ be in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X})$ and assume that it is pure and p-plain. Then $G(\mathcal{E}^{conv}) \subseteq G(\mathcal{E})$ is an epimorphic subgroup of maximal rank. If moreover $\mathcal{X}$ is geometrically connected the following hold:

1. $G(\mathcal{E}^{geo,conv}) \subseteq G(\mathcal{E}^{geo})$ is a subgroup of maximal rank;
2. The abelianization of $G(\mathcal{E}^{geo,conv})$ is reductive;
3. If $\mathcal{E}$ is semisimple the natural map
$$G(\mathcal{E}^{conv})/G(\mathcal{E}^{geo,conv}) \to G(\mathcal{E})/G(\mathcal{E}^{geo})$$
is an isogeny.

Proof. All the statement follow from Fact 6.5.1.1.1 and the results in Chapter 5. More precisely, the first statement follows from Fact 6.5.1.1.1 and Corollary 5.2.3.2.1. Assume that $\mathcal{X}$ is geometrically connected. Then (1) is Theorem 5.1.2.2.1, (2) follows from (1), Fact 6.3.1.4.1(1) and Lemma 5.2.3.2.3. Finally (3) follows froms from the first statement, (1) and Corollary 5.2.3.2.2. □

Every morphism $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ of smooth connected $\mathbb{F}_q$-varieties, induces a faithful tensor functor $\mathfrak{f}^* : \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}) \to \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{Y})$, hence for every $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ a natural closed immersion $G(\mathfrak{f}^*\mathcal{E}) \subseteq G(\mathcal{E})$. If moreover $\mathcal{Y}$ and $\mathcal{X}$ are geometrically connected, $\mathfrak{f}^*$ induces a faithful tensor functor $\mathfrak{f}^* : \mathbf{Isoc}(\mathcal{X}) \to \mathbf{Isoc}(\mathcal{Y})$ fitting into a commutative diagram

$$\begin{tikzcd}
\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}) \arrow[r] \arrow[d] & \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{Y}) \arrow[d] \\
\mathbf{Isoc}(\mathcal{X}) \arrow[r] & \mathbf{Isoc}(\mathcal{Y}),
\end{tikzcd}$$

hence for every $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ a commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & G(\mathfrak{f}^*\mathcal{E}^{geo}) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*\mathcal{E}) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*\mathcal{E})^{cst} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & G(\mathcal{E}^{geo}) \arrow[r] & G(\mathcal{E}) \arrow[r] & G(\mathcal{E})^{cst} \arrow[r] & 0
\end{tikzcd} \tag{6.5.1.3.2}$$

where the left and the vertical arrows are closed immersion and the right vertical arrow is surjective.

6.5.1.4 Behaviour under open immersion

The analogue of Fact 6.3.1.5.1(1) holds in the setting of convergent F-isocrystals.

Fact 6.5.1.4.1 ([DK17, Theorem 2.2.3])**.** Let $\mathfrak{j} : \mathcal{U} \to \mathcal{X}$ be a dense open immersion of connected smooth $\mathbb{F}_q$-varieties. The functor $\mathfrak{j}^* : \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{U}) \to \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ is fully faithful.

However the analogues of Fact 6.3.1.5.1(2-3) do not hold in the setting of convergent F-isocrystals.

Example 6.5.1.4.2. Let $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ be a non isotrivial family of elliptic curves with at least one supersingular fibre. Write $\mathfrak{j} : \mathcal{U} \subseteq \mathcal{X}$ for the dense open subset with ordinary fibres and $\mathcal{E} := R^1 f_* \mathcal{O}_{\mathcal{Y}/\overline{\mathbb{Q}}_p}$ for the first convergent higher direct image. Then one has $G(\mathcal{E}) = GL_2$, while $G(\mathfrak{j}^*\mathcal{E}) \subseteq GL_2$ is the Borel subgroup of upper triangular matrices. So, while $\mathcal{E}$ is irreducible and doesn't have constant Newton polygon, $\mathfrak{j}^*\mathcal{E}$ has constant Newton polygon hence it acquires a two steps filtration, reflecting the filtration of the p-divisible group of the generic fibre of $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ into étale and multiplicative part.

In Example 6.5.1.4.2, we see that the obstruction on $G(\mathfrak{j}^*\mathcal{E}) \subseteq G(\mathcal{E})$ to be an isomorphism is the presence of new subobjects of $\mathfrak{j}^*\mathcal{E}$ arising from the slope filtration on $\mathcal{U}$. We show that this is the only obstruction, obtaining an analogue of Fact 6.3.1.5.1(2-3) for convergent F-isocrystal with constant Newton polygon.

Proposition 6.5.1.4.3. Let $\mathfrak{j} : \mathcal{U} \to \mathcal{X}$ be a dense open immersion of connected smooth $\mathbb{F}_q$-varieties and let $\mathcal{E}$ be in $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{CNP}}(\mathcal{X})$. Then:

1. The natural closed immersion $G(\mathfrak{j}^*\mathcal{E}) \subseteq G(\mathcal{E})$ is an isomorphism.
2. If $\mathcal{U}$ and $\mathcal{X}$ are geometrically connected, the natural closed immersion $G(\mathfrak{j}^*\mathcal{E}^{geo}) \subseteq G(\mathcal{E}^{geo})$ is an isomorphism.

Proof.

1. By Fact 6.5.1.4.1, the functor $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}) \to \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{U})$ is fully faithful, so that $G(\mathcal{E}^{conv}) \subseteq G(\mathcal{E})$ is an epimorphic subgroup. Since any $\mathcal{E}'$ in $\langle \mathcal{E} \rangle$ is in $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{CNP}}(\mathcal{X})$, by the group theoretic Lemma 6.5.1.4.4 below, it is enough to show that that if $\mathcal{E}'$ is in $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{CNP}}(\mathcal{X})$ and semisimple, then $\mathfrak{j}^*\mathcal{E}'$ is semisimple. Since every semisimple convergent F-isocrystal in $\mathbf{F}\text{-}\mathbf{Isoc}_{\mathrm{CNP}}(\mathcal{X})$ is a direct sum of isoclinic semisimple F-isocrystals, we may and do assume that $\mathcal{E}'$ is isoclinic and semisimple. As twisting is an equivalence of categories, we can assume that $\mathcal{E}'$ is in $\mathbf{F}\text{-}\mathbf{Isoc}_{ur}(\mathcal{X})$. By Fact 6.5.1.2.1, it is enough to show that that any semisimple ρ in $\mathbf{Rep}_{\overline{\mathbb{Q}}_p}(\pi_1(\mathcal{X}))$ stays semisimple after restriction via the map $\pi_1(\mathcal{U}) \to \pi_1(\mathcal{X})$. We conclude observing that, since $\mathcal{U} \to \mathcal{X}$ is an open immersion of connected normal schemes, the map $\pi_1(\mathcal{U}) \to \pi_1(\mathcal{X})$ is surjective.
2. We deduce (2) from (1) and Fact 6.5.1.4.1. There is a commutative exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & G(\mathfrak{j}^*\mathcal{E}^{geo}) \arrow[r] \arrow[d, hook] & G(\mathfrak{j}^*\mathcal{E}) \arrow[r] \arrow[d, hook, "\simeq"] & G(\mathfrak{j}^*\mathcal{E})^{cst} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & G(\mathcal{E}^{geo}) \arrow[r] & G(\mathcal{E}) \arrow[r] & G(\mathcal{E})^{cst} \arrow[r] & 0.
\end{tikzcd}$$

   Since the middle vertical arrow is an isomorphism by point (1), it is enough to show that the right vertical surjection is an isomorphism. By the Tannakian formalism, one needs to prove that the fully faithful functor

$$\mathbf{Rep}_{\overline{\mathbb{Q}}_p}(G(\mathcal{E})^{cst}) \to \mathbf{Rep}_{\overline{\mathbb{Q}}_p}(G(\mathfrak{j}^*\mathcal{E})^{cst}).$$

is essentially surjective. Write $\mathfrak{q}_{\mathcal{X}} : \mathcal{X} \to \mathbb{F}_q$ and $\mathfrak{q}_{\mathcal{U}} : \mathcal{U} \to \mathbb{F}_q$ for the structural morphisms. By Section 6.5.1.3, the categories $\mathbf{Rep}_{\overline{\mathbb{Q}}_p}(G(\mathcal{E})^{cst})$ and $\mathbf{Rep}_{\overline{\mathbb{Q}}_p}(G(\mathfrak{j}^*\mathcal{E})^{cst})$ are canonically equivalent to the Tannakian subcategories $\langle\mathcal{E}\rangle^{cst} \subseteq \langle\mathcal{E}\rangle$ and $\langle \mathfrak{j}^*\mathcal{E}\rangle^{cst} \subseteq \langle \mathfrak{j}^*\mathcal{E}\rangle$ made by convergent F-isocrystals of the form $\mathfrak{q}_{\mathcal{X}}^*\mathcal{E}'$ and $\mathfrak{q}_{\mathcal{U}}^*\mathcal{E}'$ for some $\mathcal{E}'$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathbb{F}_q)$, respectively. Take any $\mathfrak{q}_{\mathcal{U}}^*\mathcal{E}'$ in $\langle \mathfrak{j}^*\mathcal{E}\rangle^{cst}$. Since $\mathfrak{q}_{\mathcal{X}}^*\mathcal{E}'$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$ is such that $\mathfrak{j}^*\mathfrak{q}_{\mathcal{X}}^*\mathcal{E}' = \mathfrak{q}_{\mathcal{U}}^*\mathcal{E}'$, it is enough to show that $\mathfrak{q}_{\mathcal{X}}^*\mathcal{E}'$ is in $\langle\mathcal{E}\rangle^{cst}$. Since it is of the form $\mathfrak{q}_{\mathcal{X}}^*\mathcal{E}'$, it is enough to show that $\mathfrak{q}_{\mathcal{X}}^*\mathcal{E}'$ is in $\langle\mathcal{E}\rangle$. By the point (1), the natural functor $\mathfrak{j}^* : \langle\mathcal{E}\rangle \to \langle \mathfrak{j}^*\mathcal{E}\rangle$ is an equivalence of categories, so that there exists a $\mathcal{T}$ in $\langle\mathcal{E}\rangle$ such that $\mathfrak{j}^*\mathcal{T} \simeq \mathfrak{q}_{\mathcal{U}}^*\mathcal{E}'$ and it is enough to show that $\mathfrak{q}_{\mathcal{X}}^*\mathcal{E}' \simeq \mathcal{T}$. Since, by Fact 6.5.1.4.1, the functor $\mathfrak{j}^* : \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X}) \to \mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{U})$ is fully faithfully, hence conservative, we conclude observing that $\mathfrak{j}^*\mathfrak{q}_{\mathcal{X}}^*\mathcal{E}' \simeq \mathfrak{q}_{\mathcal{U}}^*\mathcal{E}' \simeq \mathfrak{j}^*\mathcal{T}$. □

Lemma 6.5.1.4.4. *Let $H \subseteq G$ be a closed immersion of algebraic groups over an algebraically closed field L of characteristic zero. Assume that H is an epimorphic subgroup and that $Rep_L(G) \to Rep_L(H)$ sends semisimple representations to semisimple representations. Then $H = G$.*

Proof.

- Assume first that G and H are connected. Since $H \subseteq G$ is epimorphic, by [AE16, Lemma 1.6] it is enough to show that the the map $X^*(G) \to X^*(H)$ induced at the level of the groups of characters is an isogeny. Since $\mathbf{Rep}(G) \to \mathbf{Rep}(H)$ sends semisimple representations to semisimple representations, the inclusion $H \subseteq G$ restricts to an inclusion $R_u(H) \subseteq R_u(G)$. So there is a commutative exact diagram:

$$\begin{tikzcd} 1 \arrow[r] & R_u(H) \arrow[r] \arrow[d, hook] & H \arrow[r] \arrow[d, hook] & H/R_u(H) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & R_u(G) \arrow[r] & G \arrow[r] & G/R_u(G) \arrow[r] & 1. \end{tikzcd}$$

  Hence it is enough to show that the induced morphism $H/R_u(H) \to G/R_u(G)$ is an isogeny Since $H \to G$ is epimorphic, by Lemma 6.A.3(2) also $H/R_u(H) \to G/R_u(G)$ is epimorphic. Since $H/R_u(H)$ is reductive, by Lemma 6.A.2(1), we see that the right vertical arrow is surjective. So it is enough to show that

$$Dim(G/R_u(G)) \geq Dim(H/R_u(H)).$$

  By the Levi decomposition, the surjection $H \to H/R_u(H)$ admits a splitting j. Write $L := j(H/R_u(H))$ for the corresponding Levi factor. To conclude, we have to show that L injects into $G/R_u(G)$, i.e. that $L \cap R_u(G) = 1$. Since $R_u(G)$ is normal in G, the group $R_u(G) \cap L$ is normal in L. Since $R_u(G) \cap L$ is unipotent it is also connected. But L is reductive, so that all the connected normal unipotent subgroups are trivial.

- We reduce to the case in which G and H are connected. There is a commutative diagram with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & H^0 \arrow[r] \arrow[d, hook] & H \arrow[r] \arrow[d, hook] & \pi_0(H) \arrow[r] \arrow[d, two heads] & 1 \\ 1 \arrow[r] & G^0 \arrow[r] & G \arrow[r] & \pi_0(G) \arrow[r] & 1. \end{tikzcd}$$

  Since $H \subseteq G$ is epimorphic, by Lemma 6.A.3(2), the morphism $\pi_0(H) \to \pi_0(G)$ is epimorphic as well. Since $\pi_0(H)$ is finite (hence reductive), this implies that $\pi_0(H) \to$

$\pi_0(G)$ is surjective. By diagram chasing, it is enough to show that the left vertical arrow is an isomorphism. Since $R_u(H^0) = R_u(H) \subseteq R_u(G) = R_u(G^0)$, the functor $Rep_L(G^0) \to Rep_L(H^0)$ sends semisimple representations to semisimple representations. So, by the previous point it is enough to show that $H^0 \subseteq G^0$ is an epimorphic subgroup. This follows from the fact that $H \subseteq G$ is an epimorphic subgroup and Lemma 6.A.3(1). $\square$

6.5.2 Convergent F-isocrystals over finitely generated fields

Let k be a finitely generated field of characteristic $p > 0$ and let X be a connected smooth k-variety. In this section, mimicking Section 6.3.2, we construct and study a $\overline{\mathbb{Q}}_p$-linear category $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}(X)$ of convergent F-isocrystals over X.

6.5.2.1 Definitions

For every couples $(\mathcal{X}_i, \mathcal{E}_i)$, $i = 1, 2$, with $\mathcal{X}_i \in \mathbf{Model}(X)$ and $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$, write $(\mathcal{X}_2, \mathcal{E}_2) \succ_{\mathrm{j}} (\mathcal{X}_1, \mathcal{E}_1)$ (or simply $(\mathcal{X}_2, \mathcal{E}_2) \succ (\mathcal{X}_1, \mathcal{E}_1)$) if $\mathcal{X}_2 \succ_{\mathrm{j}} \mathcal{X}_1$ and $\mathcal{E}_2 \simeq \mathrm{j}^*\mathcal{E}_1$.

Definition 6.5.2.1.1. The category $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}(X)$ of convergent F-isocrystals over X is the following category:

- The objects are equivalence classes $[\mathcal{E}]$ of couples $(\mathcal{X}, \mathcal{E})$ where $\mathcal{X} \in \mathbf{Model}(X)$ and $\mathcal{E}$ is in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$. The equivalence relation is given by the relations $(\mathcal{X}_1, \mathcal{E}_1) \sim (\mathcal{X}_2, \mathcal{E}_2)$ if there exists a couple $(\mathcal{X}_3, \mathcal{E}_3)$ such that $(\mathcal{X}_3, \mathcal{E}_3) \succ (\mathcal{X}_i, \mathcal{E}_i)$, $i = 1, 2$.

- A morphism $[\mathbf{g}] : [\mathcal{E}] \to [\mathcal{E}']$ between $[\mathcal{E}]$ and $[\mathcal{E}']$ in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}(X)$ with representatives $(\mathcal{X}, \mathcal{E})$ and $(\mathcal{X}', \mathcal{E}')$ is an equivalence class of couples $(\mathcal{U}, \mathbf{g})$ where $\mathcal{U} \in \mathbf{Model}(X)$ is such that $\mathcal{U} \succ_{\mathrm{j}} \mathcal{X}$ and $\mathcal{U} \succ_{\mathrm{j}'} \mathcal{X}'$ and $\mathbf{g}$ is a map $\mathrm{j}^*\mathcal{E} \to \mathrm{j}'^*\mathcal{E}'$. The equivalence relation is given by $(\mathcal{U}_1, \mathbf{g}_1) \sim (\mathcal{U}_2, \mathbf{g}_2)$ if there exists $\mathcal{U}_3$ in $\mathbf{Model}(X)$ with $\mathcal{U}_3 \succ_{\mathrm{j}_i} \mathcal{U}_i$, $i = 1, 2$ and $\mathrm{j}_1^*\mathbf{g}_1 = \mathrm{j}_2^*\mathbf{g}_2$.

If $[\mathcal{E}]$ and $[\mathcal{E}']$ are in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}(\mathcal{X})$, we can always choose representatives of the forms $(\mathcal{X}, \mathcal{E})$ and $(\mathcal{X}, \mathcal{E}')$. By Fact 6.3.1.5.1, once such representatives are chosen, one has $\mathrm{Hom}_{\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}(X)}([\mathcal{E}], [\mathcal{E}']) = \mathrm{Hom}_{\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})}(\mathcal{E}, \mathcal{E}')$.

6.5.2.2 Operations

Most of the constructions of Sections 6.3.2.2, 6.3.2.3, 6.3.2.4 go trough without any change. For example:

- For every $[\mathcal{E}']$ and $[\mathcal{E}]$ in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}(X)$ there are well defined tensor product $[\mathcal{E}] \otimes [\mathcal{E}']$ and direct sum $[\mathcal{E}] \oplus [\mathcal{E}']$;

- For every morphism $[\mathbf{g}] : [\mathcal{E}] \to [\mathcal{E}']$, there are well defined notion of kernel $[\mathrm{Ker}[\mathbf{g}]]$, of cokernel $[\mathrm{Coker}([\mathbf{g}])]$, of monomorphism and of epimorphism. If there exists a monomorphism $[\mathbf{g}] : [\mathcal{E}'] \to [\mathcal{E}]$, we write $[\mathcal{E}'] \subseteq [\mathcal{E}]$ and $[\mathcal{E}/\mathcal{E}'] := [\mathrm{Coker}([\mathbf{g}])]$

However, since Facts 6.3.1.5.1 fails for convergent F-isocrystals, the notions of irreducibility and semisimplicity behave differently in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}(X)$ (see Example 6.5.1.4.2).

6.5.2.3 Slopes and slope filtration

Also the notion of having constant Newton polygon (resp. being isoclinic (of slope s), resp. being unit-root) is not stable under open immersion. So we give the following definition.

Definition 6.5.2.3.1. We say that $[\mathcal{E}]$ has constant Newton polygon (resp. is isoclinic (of slope s), resp. is unit-root) if there exists a representative class of $(\mathcal{X}, \mathcal{E})$ such that $\mathcal{E}$ has constant Newton polygon (resp. is isoclinic (of slope s), resp. is unit-root) over $\mathcal{X}$.

If $[\mathcal{E}]$ has constant Newton polygon (resp. is unit-root) we call any representative $(\mathcal{X}, \mathcal{E})$ of the equivalence class of $[\mathcal{E}]$ such that $\mathcal{E}$ has constant Newton polygon (resp. is unit-root) over $\mathcal{X}$, a representative with constant Newton polygon (resp. a unit-root representative). Let $\widetilde{\textbf{F-Isoc}}_{\mathrm{CNP}}(X)$ (resp. $\widetilde{\textbf{F-Isoc}}_{ur}(X)$) be the subcategory of $\widetilde{\textbf{F-Isoc}}(X)$ of convergent F-isocrystals that have constant Newton polygon (resp. that are unit-root).

If $[\mathcal{E}]$ in $\widetilde{\textbf{F-Isoc}}_{\mathrm{CNP}}(X)$ has representatives $(\mathcal{X}_i, \mathcal{E}_i)$, $i = 1, 2$, with constant Newton polygon, then there exists a representative $(\mathcal{X}_3, \mathcal{E}_3)$ with $(\mathcal{X}_3, \mathcal{E}_3) \succ_{\mathfrak{j}_i} (\mathcal{X}_i, \mathcal{E}_i)$. If

$$0 = \mathcal{E}_{i,0} \subseteq \mathcal{E}_{i,1} \subseteq \dots \subseteq \mathcal{E}_{i,r-1} \subseteq \mathcal{E}_{i,r} = \mathcal{E}_i$$

is the slope filtration of $\mathcal{E}_i$ over $\mathcal{X}_i$ of Fact 6.5.1.2.2, then

$$0 = \mathfrak{j}_i^* \mathcal{E}_{i,0} \subseteq \mathfrak{j}_i^* \mathcal{E}_{i,1} \subseteq \dots \subseteq \mathfrak{j}_i^* \mathcal{E}_{i,r-1} \subseteq \mathfrak{j}_i^* \mathcal{E}_{i,r} = j_i^* \mathcal{E}_i$$

identifies with the slope filtration of $\mathfrak{j}_i^* \mathcal{E}$ over $\mathcal{X}_3$. Hence, by the unicity of the slope filtration, the equivalence class $[\mathcal{E}_i]$ of $(\mathcal{X}, \mathcal{E}_i)$ does not depend on the choice of the representative $(\mathcal{X}, \mathcal{E})$ with constant Newton polygon of $[\mathcal{E}]$.

In particular, every $[\mathcal{E}]$ in $\widetilde{\textbf{F-Isoc}}_{\mathrm{CNP}}(X)$ has a canonical filtration

$$0 = [\mathcal{E}_0] \subseteq [\mathcal{E}_1] \subseteq [\mathcal{E}_2] \subseteq \dots \subseteq [\mathcal{E}],$$

such that $[\mathcal{E}_i / \mathcal{E}_{i-1}]$ is isoclinic of slope s_i with $s_1 < s_2 < \dots < s_r$.

6.5.2.4 Functoriality

As in Section 6.3.2.6, every morphism $f : Y \to X$ of smooth connected k-varieties induces a functor

$$f^* : \widetilde{\textbf{F-Isoc}}(X) \to \widetilde{\textbf{F-Isoc}}(Y)$$

as follow.

For every morphism $f : Y \to X$ of smooth connected k-varieties and every $[\mathcal{E}]$ in $\widetilde{\textbf{F-Isoc}}(X)$, with representative $(\mathcal{X}, \mathcal{E})$, there is always a model $\mathfrak{f} : \mathcal{Y}_1 \to \mathcal{X}_1$ of f such that $\mathcal{X}_1 \succ_{\mathfrak{j}} \mathcal{X}$. Since for every couple of such models $\mathfrak{f}_i : \mathcal{Y}_i \to \mathcal{X}_i$, $i = 1, 2$, there exists always a model $\mathfrak{f}_3 : \mathcal{Y}_3 \to \mathcal{X}_3$ with $(\mathfrak{f}_i : \mathcal{Y}_3 \to \mathcal{X}_3) \succ (\mathfrak{f}_i : \mathcal{Y}_i \to \mathcal{X}_i)$, the equivalence class $f^*[\mathcal{E}]$ of $(\mathcal{Y}_1, \mathfrak{f}^* \mathfrak{j}^* \mathcal{E})$ does not depend on the choice of the model $\mathfrak{f} : \mathcal{Y}_1 \to \mathcal{X}_1$ of f such that $\mathcal{X}_1 \succ_{\mathfrak{j}} \mathcal{X}$.

Similarly, if $[\mathbf{g}] : [\mathcal{E}_1] \to [\mathcal{E}_2]$ is a morphism in $\widetilde{\textbf{F-Isoc}}(X)$ represented by $(\mathcal{U}, \mathbf{g})$, there is always a model $\mathfrak{f} : \mathcal{Y}_1 \to \mathcal{X}_1$ of f such that $\mathcal{U} \succ_{\mathfrak{j}} \mathcal{X}$ and the equivalence class $[f^*(\mathbf{g})] : [\mathcal{E}_1] \to [\mathcal{E}_2]$ of $(\mathcal{Y}_1, \mathfrak{f}^* \mathfrak{j}^* \mathbf{g})$ does not does not depend on the choice of the model $\mathfrak{f} : \mathcal{Y}_1 \to \mathcal{X}_1$ of f such that $\mathcal{X}_1 \succ_{\mathfrak{j}} \mathcal{X}$.

If $[\mathcal{E}]$ is in $\widetilde{\textbf{F-Isoc}}(X)^{CST}$ (resp. $\widetilde{\textbf{F-Isoc}}(X)_{ur}$), we can always choose a model $\mathfrak{f} : \mathcal{Y}_1 \to \mathcal{X}_1$ of f such that $\mathcal{X}_1 \succ_{\mathfrak{j}} \mathcal{X}$ and such that $\mathfrak{j}^* \mathcal{E}$ has constant Newton polygon (resp. is unit-root) over $\mathcal{X}_1$. This shows that if $[\mathcal{E}]$ has constant Newton Polygon (resp. is unit-root) then $f^*[\mathcal{E}]$ has constant Newton Polygon (resp. is unit-root).

6.5.2.5 Monodromy groups

For $[\mathcal{E}]$ in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}_{\mathrm{CNP}}(X)$ with representatives $(\mathcal{X}_i, \mathcal{E}_i)$, $i = 1, 2$, with constant Newton polygon, there exists a representative $(\mathcal{X}_3, \mathcal{E}_3)$ with $(\mathcal{X}_3, \mathcal{E}_3) \succ_{\mathrm{j}_i} (\mathcal{X}_i, \mathcal{E}_i)$. If $G(\mathcal{E}_i)$ denotes the monodromy group of $\mathcal{E}_i$ over $\mathcal{X}_i$ (see Section 6.3.1.5), by Fact 6.5.1.4.3(1) we have $G(\mathcal{E}_1) \simeq G(\mathrm{j}_1^*\mathcal{E}_1) \simeq G(\mathrm{j}_2^*\mathcal{E}_2) \simeq G(\mathcal{E}_2)$. Hence $G([\mathcal{E}]) := G(\mathcal{E})$ is well defined independently on the choice of the representative class $(\mathcal{X}, \mathcal{E})$ with constant Newton polygon of $[\mathcal{E}]$. We call it the arithmetic monodromy group of $[\mathcal{E}]$.

Similarly, using Fact 6.3.1.5.1(2), if X is also geometrically connected, $G([\mathcal{E}]^{geo}) := G(\mathcal{E}^{geo})$ is independent from the choice of $(\mathcal{X}, \mathcal{E})$ with constant Newton polygon. We call it the geometric monodromy group of $[\mathcal{E}]$.

6.5.3 Comparisons

Let X be a smooth connected k-variety. In the p-adic setting, we have other two candidate for a category of p-adic local systems over X: the category $\mathbf{LS}(X, p)$ of étale lisse $\overline{\mathbb{Q}}_p$-sheaves and the category $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}}(X)$ of overconvergent F-isocrystals introduced in Section 6.3. In this section we compare these options.

6.5.3.1 Comparison with the category of p-adic lisse sheaves

We use the equivalence of categories $\Phi_{\mathcal{X}}$ of Fact 6.5.1.2.1 to construct a functor:

$$\Phi : \widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}(X)_{ur} \to \mathbf{LS}(X, p).$$

For every $[\mathcal{E}]$ in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}(X)_{ur}$ with unit-root representative $(\mathcal{X}, \mathcal{E})$, the equivalence of category in Fact 6.5.1.2.1 induces a $\Phi_{\mathcal{X}}(\mathcal{E})$ in $\mathbf{LS}(\mathcal{X}, p)$. Let $[\mathcal{E}]$ be in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}_{\mathrm{ur}}(X)$ with unit-root representatives $(\mathcal{X}_i, \mathcal{E}_i)$, $i = 1, 2$. Then there exists a $(\mathcal{X}_3, \mathcal{E}_3)$ such that $(\mathcal{X}_3, \mathcal{E}_3) \succ (\mathcal{X}_i, \mathcal{E}_i)$. Then the commutative diagram

$$\begin{tikzcd}
 & \mathcal{X}_1 \arrow[dl] \\
\mathcal{X}_3 & X \arrow[l, "\mathrm{j}_{\mathcal{X}_3}"'] \arrow[u, "\mathrm{j}_{\mathcal{X}_1}"'] \arrow[d, "\mathrm{j}_{\mathcal{X}_2}"] \\
 & \mathcal{X}_2 \arrow[ul]
\end{tikzcd}$$

shows that $\Phi([\mathcal{E}]) := j_{\mathcal{X}}^*\Phi_{\mathcal{X}}(\mathcal{E})$ does not depend on the choice of the representative $(\mathcal{X}, \mathcal{E})$ of the equivalence class of $[\mathcal{E}]$.

Similarly, if $[\mathbf{g}] : [\mathcal{E}] \to [\mathcal{E}']$ is a morphism in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}_{ur}(X)$ represented by $(\mathcal{U}, \mathbf{g})$, the morphism $\Phi([\mathbf{g}]) : \Phi([\mathcal{E}]) \to \Phi([\mathcal{E}'])$ defined by $j_{\mathcal{U}}^*(\Phi_U(\mathbf{g}))$ does not depend on the choice of the representative $(\mathcal{U}, \mathbf{g})$ of $[\mathbf{g}] : [\mathcal{E}] \to [\mathcal{E}']$.

Since the morphism $\pi_1(X) \to \pi_1(\mathcal{X})$ is surjective, the functor $\Phi : \widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}_{ur}(X) \to \mathbf{LS}(X, p)$. is fully faithful and the monodromy group $G([\mathcal{E}])$ of $[\mathcal{E}]$ identifies with the Zariski closure of the image $\Pi_{\Phi([\mathcal{E}])}$ of $\rho_{\Phi([\mathcal{E}])}$.

6.5.3.2 Comparison with the category of overconvergent F-isocrystals

Similarly, we use the functor $(-)^{conv}$ in Section 6.5.1.1 to construct a functor:

$$(-)^{conv} : \widetilde{\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}}(X) \to \mathbf{F}\text{-}\mathbf{Isoc}(X, p).$$

Let $[\mathcal{E}]$ be in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}^{\dagger}(X)_{ur}$ with representative $(\mathcal{X}_i, \mathcal{E}_i)$, $i = 1, 2$, then there exists a $(\mathcal{X}_3, \mathcal{E}_3)$ such that $(\mathcal{X}_3, \mathcal{E}_3) \succ (\mathcal{X}_i, \mathcal{E}_i)$. Then the commutative diagram

$$\begin{tikzcd}
 & \mathcal{X}_1 \arrow[dl] \\
\mathcal{X}_3 & X \arrow[l, "j_{\mathcal{X}_3}"'] \arrow[u, "j_{\mathcal{X}_1}"] \arrow[d, "j_{\mathcal{X}_2}"] \\
 & \mathcal{X}_2 \arrow[ul]
\end{tikzcd}$$

shows that the equivalence class $[\mathcal{E}]^{conv}$ of $(\mathcal{X}, \mathcal{E}^{conv})$ does not depend on the choice of the representative $(\mathcal{X}, \mathcal{E})$ of the equivalence class of $[\mathcal{E}]$.

Similarly, if $[\mathbf{g}] : [\mathcal{E}] \to [\mathcal{E}']$ is a morphism in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}^{\dagger}(X)$ represented by $(\mathcal{U}, \mathbf{g})$, the morphism $[\mathbf{g}]^{conv} : [\mathcal{E}]^{conv} \to [\mathcal{E}']^{conv}$ represented by $(\mathcal{U}, (\mathbf{g})^{conv})$ does not depend on the choice of the representative $(\mathcal{U}, \mathbf{g})$ of $[\mathbf{g}] : [\mathcal{E}] \to [\mathcal{E}']$.

We say that $[\mathcal{E}]$ in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}^{\dagger}(X)$ has constant Newton polygon if $[\mathcal{E}]^{conv}$ has constant Newton polygon. For every $[\mathcal{E}]$ in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}^{\dagger}(X)$ with constant Newton Polygon, there is well defined closed immersion $G([\mathcal{E}]^{conv}) \subseteq G([\mathcal{E}])$. If moreover X is geometrically connected, there is well defined closed immersion $G([\mathcal{E}]^{conv,geo}) \subseteq G([\mathcal{E}]^{geo})$ fitting into a commutative diagram closed immersions:

$$\begin{tikzcd}
G([\mathcal{E}]^{conv,geo}) \arrow[r, hook] \arrow[d, hook] & G([\mathcal{E}]^{conv}) \arrow[d, hook] \\
G([\mathcal{E}]^{geo}) \arrow[r, hook] & G([\mathcal{E}]).
\end{tikzcd}$$

6.6 Exceptional loci of convergent F-isocrystals

Let X be a smooth geometrically connected k-variety. In this section we define the exceptional loci of convergent F-isocrystals, we prove Theorem 6.1.5.2.1 and Corollary 6.1.6.1 and finally we discuss the relation between various exceptional loci associated to an overconvergent F-isocrystal.

6.6.1 Exceptional loci and Theorem 6.1.5.2.1

6.6.1.1 Definitions

As mentioned in Section 6.5.2.4 every morphism $f : Y \to X$ of smooth connected k-varieties induces a functor $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}_{\mathrm{CNP}}(X) \to \widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}_{\mathrm{CNP}}(Y)$. Hence, arguing as in Section 6.4.1.1, for every $[\mathcal{E}]$ be in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}_{\mathrm{CNP}}(X)$, there is a well defined closed immersion $G(f^*[\mathcal{E}]) \subseteq G([\mathcal{E}])$.

Definition 6.6.1.1.1. We say that $x \in |X|$ is algebraically generic (resp. algebraically strictly generic) for $[\mathcal{E}]$ if $G(x^*[\mathcal{E}])^0 \simeq G([\mathcal{E}])^0$ (resp. $G(x^*[\mathcal{E}]) \simeq G([\mathcal{E}])$).

Write $X^{gen}_{[\mathcal{E}]}$ (resp. $X^{sgen}_{[\mathcal{E}]}$) for the set of $x \in |X|$ that are algebraically generic (resp. algebraically strictly generic) for $[\mathcal{E}]$. Define the following sets:

$$X^{ex}_{[\mathcal{E}]} := |X| - X^{gen}_{[\mathcal{E}]}; \quad X^{ex}_{[\mathcal{E}]}(\leq d) := X^{ex}_{[\mathcal{E}]} \cap X(\leq d); \quad X^{gen}_{[\mathcal{E}]}(\leq d) := X^{gen}_{[\mathcal{E}]} \cap X(\leq d);$$

$$X^{stex}_{[\mathcal{E}]} := |X| - X^{sgen}_{[\mathcal{E}]}; \quad X^{stex}_{[\mathcal{E}]}(\leq d) := X^{stex}_{[\mathcal{E}]} \cap X(\leq d); \quad X^{sgen}_{[\mathcal{E}]}(\leq d) := X^{sgen}_{[\mathcal{E}]} \cap X(\leq d).$$

We call $X^{ex}_{[\mathcal{E}]}$ the algebraic exceptional locus of $[\mathcal{E}]$ and $X^{stex}_{[\mathcal{E}]}$ its algebraic strictly-exceptional locus.

Lemma 6.6.1.1.2. Let $f : Y \to X$ be a connected finite étale cover. Then we have:

$$f(Y^{ex}_{f^*[\mathcal{E}]}) = X^{ex}_{[\mathcal{E}]}$$

Proof. Choose a model of $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ of $f : Y \to X$ such that $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ is finite étale and a representative class of $(\mathcal{X}, \mathcal{E})$ is defined over $\mathcal{X}$. Then Lemma 6.6.1.1.2 follows from the fact that $G(f^*[\mathcal{E}]) = G(\mathfrak{f}^*\mathcal{E}) \subseteq G(\mathcal{E}) = G([\mathcal{E}])$ is an open subgroup by [HP18, Lemma 6.2]. $\square$

6.6.1.2 Proof of Theorem 6.1.5.2.1

Let $[\mathcal{E}]$ be in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}^\dagger(X)$ pure and p-plain with constant Newton polygon. Points (1) and (2) of Theorem 6.1.5.2.1, follow from Theorem 6.1.5.1.1 and Lemma 6.6.1.2.1 below.

Lemma 6.6.1.2.1. If $x \in X^{gen}_{\mathcal{E}}$ then

$$\mathrm{rk}(G(x^*[\mathcal{E}]^{conv})) = \mathrm{rk}(G([\mathcal{E}]^{conv})) \quad \text{and} \quad \mathrm{rk}(G(x^*[\mathcal{E}]^{conv}_1)) = \mathrm{rk}(G([\mathcal{E}]^{conv}_1)).$$

Proof. Choose a model $\mathfrak{f} : \mathcal{K}' \to \mathcal{X}$ of $x \to X$ such that a representative $(\mathcal{X}, \mathcal{E})$ of $[\mathcal{E}]$ is defined over $\mathcal{X}$ and $\mathcal{E}$ has constant Newton polygon over $\mathcal{X}$. Then Lemma 6.6.1.2.1 amounts to show that if $G(\mathfrak{f}^*\mathcal{E})^0 = G(\mathcal{E})^0$ then $\mathrm{rk}(G(\mathfrak{f}^*\mathcal{E}^{conv})) = \mathrm{rk}(G(\mathcal{E}^{conv}))$ and $\mathrm{rk}(G(\mathfrak{f}^*\mathcal{E}^{conv}_1)) = \mathrm{rk}(G(\mathfrak{f}^*\mathcal{E}^{conv}_1))$. Then the first equality follows from Fact 6.5.1.3.1. The second equality follows from the first and the commutative diagram with surjective vertical arrows:

$$\begin{tikzcd}
G(\mathfrak{f}^*\mathcal{E}^{conv})^0 \arrow[r, hook] \arrow[d, two heads] & G(\mathcal{E}^{conv})^0 \arrow[d, two heads] \\
G(\mathfrak{f}^*\mathcal{E}^{conv}_1)^0 \arrow[r, hook] & G(\mathcal{E}^{conv}_1)^0.
\end{tikzcd}$$ $\square$

Theorem 6.1.5.2.1(3) follows from Theorem 6.1.5.1.1 and Lemma 6.6.1.2.2 below.

Lemma 6.6.1.2.2. If $G([\mathcal{E}]^{conv,geo}_1)^0$ is abelian then

$$X^{gen}_{[\mathcal{E}]} \subseteq X^{gen}_{[\mathcal{E}]^{conv}_1}$$

Proof. Choose a model $\mathfrak{f} : \mathcal{K}' \to \mathcal{X}$ of $x \to X$ such that a representative $(\mathcal{X}, \mathcal{E})$ of $[\mathcal{E}]$ is defined over $\mathcal{X}$ and $\mathcal{E}$ has constant Newton polygon over $\mathcal{X}$. Then Lemma 6.6.1.2.1 amounts to show that if $G(\mathfrak{f}^*\mathcal{E})^0 = G(\mathcal{E})^0$ and $G(\mathcal{E}^{conv,geo}_1)^0$ is abelian then $G(\mathfrak{f}^*\mathcal{E}^{conv}_1)^0 = G(\mathcal{E}^{conv}_1)^0$

Replacing $\mathbb{F}_q$ with a finite field extension, we can assume that $\mathcal{K}'$ is geometrically connected over $\mathbb{F}_q$. Replacing $\mathcal{X}$ with a finite étale cover we can assume that $G(\mathcal{E}^{conv})$ is connected. Then there is a exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G(\mathfrak{f}^*\mathcal{E}^{geo,conv}_1) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*\mathcal{E}^{conv}_1) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*\mathcal{E}^{conv}_1)^{cst} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & G(\mathcal{E}^{geo,conv}_1) \arrow[r] & G(\mathcal{E}^{conv}_1) \arrow[r] & G(\mathcal{E}^{conv}_1)^{cst} \arrow[r] & 0
\end{tikzcd}$$

on which the right vertical arrow is surjective. So it is enough to show that $Dim(G(\mathfrak{f}^*\mathcal{E}^{geo,conv}_1)) \geq Dim(G(\mathcal{E}^{geo,conv}_1))$. Since $G(\mathcal{E}^{geo}_1)$ is abelian by assumption, by Fact 6.5.1.3.1(2) $G(\mathfrak{f}^*\mathcal{E}^{geo,conv}_1)^0$ and $G(\mathcal{E}^{geo,conv}_1)^0$ are tori and so, by the commutative diagram

$$\begin{tikzcd}
G(\mathfrak{f}^*\mathcal{E}^{conv}) \arrow[r, hook] \arrow[d, two heads] & G(\mathcal{E}^{conv}) \arrow[d, two heads] & G(\mathfrak{f}^*\mathcal{E}^{conv})^0 \arrow[r, hook] \arrow[d, two heads] & G(\mathcal{E}^{conv})^0 \arrow[d, two heads] \\
G(\mathfrak{f}^*\mathcal{E}^{conv}_1) \arrow[r, hook] & G(\mathcal{E}^{conv}_1) & G(\mathfrak{f}^*\mathcal{E}^{conv}_1)^0 \arrow[r, hook] & G(\mathcal{E}^{conv}_1)^0,
\end{tikzcd}$$

it is enough to show that $\mathrm{rk}(G(\mathfrak{f}^*\mathcal{E}^{geo,conv})) = \mathrm{rk}(G(\mathcal{E}^{geo,conv}))$. But since $x \in X^{gen}_{[\mathcal{E}]}$, by Fact 6.3.1.4.1(6) we have $G(\mathcal{E}^{geo})^0 = G(\mathcal{E}^{geo})^0$ and we conclude by Fact 6.5.1.3.1(1). $\square$

6.6.1.3 Proof of Corollary 6.1.6.1

Before proving Corollary 6.1.6.1, let us observe that if we have a $[\mathcal{E}]$ in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}_{\mathrm{ur}}(X)$, we have four subset of X:

$$X^{gen}_{[\mathcal{E}]}; \quad X^{sgen}_{[\mathcal{E}]}; \quad X^{gen}_{\Phi([\mathcal{E}])}; \quad X^{sgen}_{\Phi([\mathcal{E}])}.$$

and, as in Lemma 6.4.1.2.1(1), one has inclusions

$$X^{sgen}_{\Phi([\mathcal{E}])} \subseteq X^{sgen}_{[\mathcal{E}]} \quad \text{and} \quad X^{gen}_{\Phi([\mathcal{E}])} \subseteq X^{gen}_{\mathcal{E}}.$$

Proof of Corollary 6.1.6.1. Let $f : Y \to X$ be a smooth proper morphism of k-varieties. Up to replace X with a dense open subset, the constructible sheaf $\mathcal{F}_p := R^i f_* \overline{\mathbb{Q}}_p$ is a p-adic lisse sheaf. Hence, for every $x \in |X|$, it corresponds to a representation

$$\rho_{\mathcal{F}_p} : \pi_1(X) \to GL(H^i(Y_{\overline{x}}, \overline{\mathbb{Q}}_p))$$

such that $\rho_{\mathcal{F}_p,x}$ identifies with the natural action of $\pi_1(x)$ on $GL(H^i(Y_{\overline{x}}, \overline{\mathbb{Q}}_p))$. By spreading out we find a smooth connected $\mathbb{F}_q$-variety $\mathcal{K}$ with generic point $\eta : \mathrm{Spec}(k) \to \mathcal{K}$ and a commutative cartesian diagram

$$\begin{tikzcd}
\mathcal{Y} \arrow[d, "\mathfrak{f}"] \arrow[dr, phantom, "\square"] & Y \arrow[l] \arrow[d, "f"] \\
\mathcal{X} \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[l] \arrow[d] \\
\mathcal{K} & k. \arrow[l, "\eta"]
\end{tikzcd}$$

By [Mor13], the higher direct image in crystalline cohomology $R^i\mathfrak{f}_*\mathcal{O}_{\mathcal{Y}/\overline{\mathbb{Q}}_p}$ is in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathcal{X})$. Let $[R^i f_* \mathcal{O}_{Y/\overline{\mathbb{Q}}_p}]$ be the object in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}(X)$ represented by $(\mathcal{X}, R^i\mathfrak{f}_*\mathcal{O}_{\mathcal{Y}/\overline{\mathbb{Q}}_p})$. By Theorem 4.6.5.4.1 in Chapter 4, there exists a $R^i\mathfrak{f}_*\mathcal{O}^\dagger_{\mathcal{Y}/\overline{\mathbb{Q}}_p}$ in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(\mathcal{X})$ such that

$$(R^i\mathfrak{f}_*\mathcal{O}^\dagger_{\mathcal{Y}/\overline{\mathbb{Q}}_p})^{conv} = R^i\mathfrak{f}_*\mathcal{O}_{\mathcal{Y}/\overline{\mathbb{Q}}_p}.$$

Let $[R^i f_* \mathcal{O}^\dagger_{Y/\overline{\mathbb{Q}}_p}]$ be the object in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}^\dagger(X)$ represented by $(\mathcal{X}, R^i\mathfrak{f}_*\mathcal{O}_{\mathcal{Y}/\overline{\mathbb{Q}}_p})$, so that $[R^i f_* \mathcal{O}^\dagger_{Y/\overline{\mathbb{Q}}_p}]^{conv} = [R^i f_* \mathcal{O}_{Y/\overline{\mathbb{Q}}_p}]$. Upon replacing X with a dense open subset, $[\mathcal{E}] := [R^i f_* \mathcal{O}^\dagger_{Y/\overline{\mathbb{Q}}_p}]$ has constant Newton polygon and its minimal slope is 0, hence $[\mathcal{E}]^{conv}_1$ is in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}_{\mathrm{ur}}(X)$. By [Ill79, II, 5.4] and proper base change, the functor $\Phi : \widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}_{\mathrm{ur}}(X) \to \mathbf{LS}(X,p)$ constructed in Section 6.5.3.1, identifies $\Phi([\mathcal{E}^{conv}_1])$ with $\rho_{\mathcal{F}_p}$ and the groups $G([\mathcal{E}]^{conv}_1)$ and $G(x^*[\mathcal{E}]^{conv}_1)$ with $G(\mathcal{F}_p)$ and $G(\mathcal{F}_{p,x})$ respectively. Since by [Del80] and [KM74] the overconvergent F-isocrystal $R^i f_* \mathcal{O}^\dagger_{Y/\overline{\mathbb{Q}}_p}$ is pure and p-plain, Corollary 6.1.6.1 follows then from Theorem 6.1.5.2.1. $\square$

6.6.2 Comparison with the overconvergent exceptional locus

Let $[\mathcal{E}]$ be in $\widetilde{\mathbf{F}\text{-}\mathbf{Isoc}}^\dagger(X)$ pure and p-plain with constant Newton polygon. To prove Theorem 6.1.5.2.1 we related in some case the exceptional loci of $[\mathcal{E}]$, $[\mathcal{E}]^{conv}$ and $[\mathcal{E}]^{conv}_1$. We conclude the chapter discussing further the comparison between these exceptional loci and proposing a conjecture.

6.6.2.1 A few inclusions

In the following two Lemmas 6.6.2.1.1 and 6.6.2.1.2 we prove that the exceptional locus of $[\mathcal{E}]^{conv}$ is always larger than the others.

Lemma 6.6.2.1.1. There are inclusions $X^{gen}_{[\mathcal{E}]^{conv}} \subseteq X^{gen}_{[\mathcal{E}]_1^{conv}}$ and $X^{sgen}_{[\mathcal{E}]^{conv}} \subseteq X^{sgen}_{[\mathcal{E}]_1^{conv}}$.

Proof. Let $x \in |X|$ and choose a model $\mathfrak{f} : \mathcal{K}' \to \mathcal{X}$ of $x \to X$ such that a representative $(\mathcal{X}, \mathcal{E})$ of $[\mathcal{E}]$ is defined over $\mathcal{X}$ and $\mathcal{E}$ has constant Newton polygon over $\mathcal{X}$. Then Lemma 6.6.2.1.1 amounts to show that that if $G(\mathfrak{f}^*\mathcal{E}^{conv})^0 = G(\mathcal{E}^{conv})^0$ (resp. $G(\mathfrak{f}^*\mathcal{E}^{conv}) = G(\mathcal{E}^{conv})$) then $G(\mathfrak{f}^*\mathcal{E}_1)^0 = G(\mathcal{E}_1)^0$ (resp. $G(\mathfrak{f}^*\mathcal{E}_1) = G(\mathcal{E}_1)$). This follows from the commutative diagram with surjective vertical arrows

$$\begin{tikzcd} G(\mathfrak{f}^*\mathcal{E}^{conv}) \arrow[r, hook] \arrow[d, two heads] & G(\mathcal{E}^{conv}) \arrow[d, two heads] & G(\mathfrak{f}^*\mathcal{E}^{conv})^0 \arrow[r, hook] \arrow[d, two heads] & G(\mathcal{E}^{conv})^0 \arrow[d, two heads] \\ G(\mathfrak{f}^*\mathcal{E}_1^{conv}) \arrow[r, hook] & G(\mathcal{E}_1^{conv}) & G(\mathfrak{f}^*\mathcal{E}_1^{conv})^0 \arrow[r, hook] & G(\mathcal{E}_1^{conv})^0. \end{tikzcd}$$ □

Lemma 6.6.2.1.2. There are inclusions $X^{gen}_{[\mathcal{E}]^{conv}} \subseteq X^{gen}_{[\mathcal{E}]}$ and $X^{sgen}_{[\mathcal{E}]^{conv}} \subseteq X^{sgen}_{[\mathcal{E}]}$.

Proof. By Proposition 6.4.2.1.1, $X^{gen}_{[\mathcal{E}]} = X^{gen}_{[\mathcal{E}^{ss}]}$ (resp. $X^{sgen}_{[\mathcal{E}]} = X^{sgen}_{[\mathcal{E}^{ss}]}$). Since there is also an inclusion $X^{gen}_{[\mathcal{E}]^{conv}} \subseteq X^{gen}_{([\mathcal{E}^{ss}])^{conv}}$ (resp. $X^{sgen}_{[\mathcal{E}]^{conv}} \subseteq X^{sgen}_{[\mathcal{E}^{ss}]^{conv}}$) we can assume that $[\mathcal{E}]$ is semisimple.

Let $x \in |X|$ and choose a model $\mathfrak{f} : \mathcal{K}' \to \mathcal{X}$ of $x \to X$ such that a representative $(\mathcal{X}, \mathcal{E})$ of $[\mathcal{E}]$ is defined over $\mathcal{X}$ and $\mathcal{E}$ has constant Newton polygon over $\mathcal{X}$. Then Lemma 6.6.2.1.2 amounts to show that if $G(\mathfrak{f}^*\mathcal{E}^{conv})^0 = G(\mathcal{E}^{conv})^0$ (resp. $G(\mathfrak{f}^*\mathcal{E}^{conv}) = G(\mathcal{E}^{conv})$) then $G(\mathfrak{f}^*\mathcal{E})^0 = G(\mathcal{E})^0$ (resp. $G(\mathfrak{f}^*\mathcal{E}) = G(\mathcal{E})$).

1. We show first that if $G(\mathfrak{f}^*\mathcal{E}^{conv})^0 = G(\mathcal{E}^{conv})^0$ then $G(\mathfrak{f}^*\mathcal{E})^0 = G(\mathcal{E})^0$.

   Replacing $\mathbb{F}_q$ with a finite field extension, we can assume that $\mathcal{K}'$ is geometrically connected over $\mathbb{F}_q$ and replacing $\mathcal{X}$ with a finite étale cover we can assume that $G(\mathcal{E}^{conv})$ and $G(\mathcal{E})$ are connected. Since $G(\mathfrak{f}^*\mathcal{E}^{conv}) = G(\mathcal{E}^{conv})$ and

   $$G(\mathfrak{f}^*\mathcal{E}^{conv}) \subseteq G(\mathfrak{f}^*\mathcal{E}) \quad \text{and} \quad G(\mathfrak{f}^*\mathcal{E}^{conv}) = G(\mathcal{E}^{conv}) \subseteq G(\mathcal{E})$$

   are epimorphic subgroups, by Lemma 6.A.2(2) also $G(\mathfrak{f}^*\mathcal{E}) \subseteq G(\mathcal{E})$ is an epimorphic subgroup. By Lemma 6.A.2(1), it is enough to show that $G(\mathfrak{f}^*\mathcal{E})$ is reductive. Consider the commutative diagram with exact rows:

   $$\begin{tikzcd} 0 \arrow[r] & G(\mathfrak{f}^*\mathcal{E}^{geo}) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*\mathcal{E}) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*\mathcal{E})^{cst} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G(\mathcal{E}^{geo}) \arrow[r] & G(\mathcal{E}) \arrow[r] & G(\mathcal{E})^{cst} \arrow[r] & 0. \end{tikzcd}$$

   Since $\mathcal{E}^{geo}$ is semisimple, by Fact 6.3.1.4.1(4) also $\mathfrak{f}^*\mathcal{E}^{geo}$ is semisimple. Hence it is enough to show that $G(\mathfrak{f}^*\mathcal{E})^{cst}$ is reductive. Since $\mathcal{E}$ is semisimple, $G(\mathcal{E})$ is reductive. By Fact 6.5.1.3.1, $G(\mathfrak{f}^*\mathcal{E}^{conv}) = G(\mathcal{E}^{conv})$ is a subgroup of maximal rank of both $G(\mathcal{E})$ and $G(\mathfrak{f}^*\mathcal{E})$. In particular $G(\mathfrak{f}^*\mathcal{E})$ is a subgroup of maximal rank of the reductive group $G(\mathcal{E})$. Hence, by the group theoretic Lemma 5.2.3.2.3 in Chapter 5, $G(\mathfrak{f}^*\mathcal{E})$ has no unipotent quotient so that all the abelian quotient of $G(\mathfrak{f}^*\mathcal{E})$ are reductive. Since $G(\mathfrak{f}^*\mathcal{E})^{cst}$ is such a quotient, we conclude the proof.

2. We deduce from point (1) via a group theoretic argument, that if $G(\mathfrak{f}^*\mathcal{E}^{conv}) = G(\mathcal{E}^{conv})$ then $G(\mathfrak{f}^*\mathcal{E}) = G(\mathcal{E})$.

   By point (1), $G(\mathfrak{f}^*\mathcal{E})^0 = G(\mathcal{E})^0$. Considering the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & G(\mathfrak{f}^*\mathcal{E})^0 \arrow[r] \arrow[d, hook, "\simeq"] & G(\mathfrak{f}^*\mathcal{E}) \arrow[r] \arrow[d, hook] & \pi_0(G(\mathfrak{f}^*\mathcal{E})) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G(\mathcal{E})^0 \arrow[r] & G(\mathcal{E}) \arrow[r] & \pi_0(G(\mathcal{E})) \arrow[r] & 0
\end{tikzcd}$$

one sees that it is enough to show that the natural map $\pi_0(G(\mathfrak{f}^*\mathcal{E})) \to \pi_0(G(\mathcal{E})))$ is surjective. Since $G(\mathcal{E}^{conv}) = G(\mathfrak{f}^*\mathcal{E}^{conv}) \subseteq G(\mathfrak{f}^*\mathcal{E}) \subseteq G(\mathcal{E})$ is an epimorphic subgroup of both $G(\mathfrak{f}^*\mathcal{E})$ and $G(\mathcal{E})$, also $G(\mathfrak{f}^*\mathcal{E}) \subseteq G(\mathcal{E})$ is an epimorphic subgroup. Hence also the map $\pi_0(G(\mathfrak{f}^*\mathcal{E})) \to \pi_0(G(\mathcal{E}))$ is epimorphic. Since $\pi_0(G(\mathfrak{f}^*\mathcal{E}))$ is finite (hence reductive), we conclude by Lemma 6.A.2(1). □

6.6.2.2 A conjecture

Let us formulate a conjecture

Conjecture 6.6.2.2.1. Assume that $[\mathcal{E}]$ is pure and p-plain with constant Newton polygon. Then:

$$X^{gen}_{[\mathcal{E}]^{conv}} = X^{gen}_{[\mathcal{E}]}; \quad X^{gen}_{[\mathcal{E}]} \subseteq X^{gen}_{[\mathcal{E}]_1^{conv}}$$

To explain the conjecture, let us choose a representative $(\mathcal{X}, \mathcal{E})$ of $[\mathcal{E}]$ with constant Newton polygon and let us recall a question of Crew.

In [Cre92, page 460] Crew asks whether $G(\mathcal{E}^{geo,conv})$ is a parabolic subgroup of $G(\mathcal{E}^{geo})$ hence, by (6.5.1.3.2) whether $G(\mathcal{E}^{conv})$ is a parabolic subgroup of $G(\mathcal{E})$. Since, by [Saa72, Pag 223, Proposition 2.2.5], the stabilizer of a Tannakian filtration is a parabolic subgroup, it is natural to wonder if $G(\mathcal{E}^{conv})$ is the stabilizer $Stab_{sl}(G(\mathcal{E}))$ inside $G(\mathcal{E})$ of the slope filtration. Clearly we have an inclusion $G(\mathcal{E}^{conv}) \subseteq Stab_{sl}(G(\mathcal{E}))$, but it not know whether this is an equality.

Assume that $G(\mathcal{E}^{conv}) = Stab_{sl}(G(\mathcal{E}))$ for all pure p-plain overconvergent F-isocrystals with constant Newton polygon. Then $G(\mathcal{E}^{conv})$ is uniquely determined by $G(\mathcal{E})$ and the slope filtration hence one would get the inclusion $X^{gen}_{[\mathcal{E}]} \subseteq X^{gen}_{[\mathcal{E}]^{conv}}$. Since the inclusion $X^{gen}_{[\mathcal{E}]^{conv}} \subseteq X^{gen}_{[\mathcal{E}]}$ has been proved in Lemma 6.6.2.1.2, the equality $X^{gen}_{[\mathcal{E}]} = X^{gen}_{[\mathcal{E}]^{conv}}$ is then predicted by a positive answer to a variant of Crew's question. Since the inclusion $X^{gen}_{[\mathcal{E}]^{conv}} \subseteq X^{gen}_{[\mathcal{E}]_1^{conv}}$ has been proved in Lemma 6.6.2.1.1, one would get also the inclusion $X^{gen}_{[\mathcal{E}]} \subseteq X^{gen}_{[\mathcal{E}]_1^{conv}}$.

In Lemma 6.6.1.2.2 the inclusion $X^{gen}_{[\mathcal{E}]} \subseteq X^{gen}_{[\mathcal{E}_1]}$ is proved when $G([\mathcal{E}_1^{geo}])$ is abelian. We end the chapter giving a further evidence for Conjecture 6.6.2.2.1. If $[\mathcal{E}]$ be pure and p-plain by Lemma 6.4.2.1.1 one has $X^{gen}_{[\mathcal{E}]^{ss}} = X^{gen}_{[\mathcal{E}]}$. If Conjecture 6.5.1.1 holds for $[\mathcal{E}]$ and $[\mathcal{E}]^{ss}$, then one should have

$$X^{gen}_{[\mathcal{E}^{ss}]^{conv}} = X^{gen}_{[\mathcal{E}^{ss}]} = X^{gen}_{[\mathcal{E}]} = X^{gen}_{[\mathcal{E}]^{conv}}.$$

We prove this equality, without assuming Conjecture 6.6.2.2.1.

Lemma 6.6.2.2.2. Let $[\mathcal{E}]$ be pure and p-plain. Then

$$X^{gen}_{[\mathcal{E}^{ss}]^{conv}} = X^{gen}_{[\mathcal{E}]^{conv}} \quad \text{and} \quad X^{sgen}_{[\mathcal{E}^{ss}]^{conv}} = X^{sgen}_{[\mathcal{E}]^{conv}}$$

Proof. Since the inclusions

$$X^{gen}_{[\mathcal{E}]^{conv}} \subseteq X^{gen}_{[\mathcal{E}^{ss}]^{conv}} \quad \text{and} \quad X^{sgen}_{[\mathcal{E}]^{conv}} \subseteq X^{sgen}_{[\mathcal{E}^{ss}]^{conv}}$$

follows from the definitions, we focus on the other inclusions.

Let $x \in |X|$ and choose a model $\mathfrak{f}: \mathcal{K}' \to \mathcal{X}$ of $x \to X$ such that a representative $(\mathcal{X}, \mathcal{E})$ of $[\mathcal{E}]$ is defined over $\mathcal{X}$ and $\mathcal{E}$ has constant Newton polygon over $\mathcal{X}$. Then Lemma 6.6.2.2.2 amounts to show that if $G(\mathfrak{f}^*(\mathcal{E}^{ss})^{conv})^0 = G((\mathcal{E}^{ss})^{conv})^0$ (resp. $G(\mathfrak{f}^*(\mathcal{E}^{ss})^{conv}) = G((\mathcal{E}^{ss})^{conv})$) then $G(\mathfrak{f}^*\mathcal{E})^0 = G(\mathcal{E})^0$ (resp. $G(\mathfrak{f}^*\mathcal{E}) = G(\mathcal{E})$).

1. Assume first $G(\mathfrak{f}^*(\mathcal{E}^{ss})^{conv})^0 = G((\mathcal{E}^{ss})^{conv})^0$.

   Replacing $\mathbb{F}_q$ with a finite field extension, we can assume that $\mathcal{K}'$ is geometrically connected over $\mathbb{F}_q$ and replacing $\mathcal{X}$ with a finite étale cover we can assume that $G(\mathcal{E}^{conv})$ is connected. By counting dimension in the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & G(\mathfrak{f}^*\mathcal{E}^{geo,conv}) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*\mathcal{E}^{conv}) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*\mathcal{E}^{conv})^{cst} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & G(\mathcal{E}^{geo,conv}) \arrow[r] & G(\mathcal{E}^{conv}) \arrow[r] & G(\mathcal{E}^{conv})^{cst} \arrow[r] & 0
\end{tikzcd}$$

   it is enough to show that $G(\mathcal{E}^{geo,conv})^0 = G(\mathfrak{f}^*\mathcal{E}^{geo,conv})^0$. By Lemma 6.6.2.1.1, one has $G(\mathfrak{f}^*\mathcal{E}^{ss})^0 = G(\mathcal{E}^{ss})^0$, hence, by Fact 6.3.1.4.1(6), we see that $G(\mathfrak{f}^*\mathcal{E}^{ss,geo})^0 = G(\mathcal{E}^{ss,geo})^0$. Then, the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & G(\mathfrak{f}^*(\mathcal{E}^{ss,geo})) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*(\mathcal{E}^{ss})) \arrow[r] \arrow[d, hook] & G(\mathfrak{f}^*(\mathcal{E}^{ss}))^{cst} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & G(\mathcal{E}^{ss,geo}) \arrow[r] & G(\mathcal{E}^{ss}) \arrow[r] & G(\mathcal{E}^{ss})^{cst} \arrow[r] & 0
\end{tikzcd}$$

   shows that
$$G(\mathfrak{f}^*(\mathcal{E}^{ss}))^{cst} \to G(\mathcal{E}^{ss})^{cst}$$
   is an isogeny. Hence, by Fact 6.5.1.3.1(3), also the map
$$G((\mathfrak{f}^*(\mathcal{E}^{ss}))^{conv})^{cst} \to G((\mathcal{E}^{ss})^{conv})^{cst}$$
   is an isogeny. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & G((\mathfrak{f}^*(\mathcal{E}^{ss,geo}))^{conv}) \arrow[r] \arrow[d, hook] & G((\mathfrak{f}^*(\mathcal{E}^{ss}))^{conv}) \arrow[r] \arrow[d, hook, "\simeq"] & G((\mathfrak{f}^*(\mathcal{E}^{ss}))^{conv})^{cst} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & G((\mathcal{E}^{ss,geo})^{conv}) \arrow[r] & G((\mathcal{E}^{ss})^{conv}) \arrow[r] & G((\mathcal{E}^{ss})^{conv})^{cst} \arrow[r] & 0.
\end{tikzcd}$$

   Since the right vertical arrow is an isogeny, dimension counting implies that $G((\mathfrak{f}^*(\mathcal{E}^{ss,geo}))^{conv})^0 = G((\mathcal{E}^{ss,geo})^{conv})^0$. Since $\mathcal{E}$ is geometrically semisimple, by Fact 6.3.1.4.1(4), also $\mathfrak{f}^*\mathcal{E}$ is geometrically semisimple. Hence
$$\mathfrak{f}^*(\mathcal{E}^{ss,geo}) \simeq \mathfrak{f}^*\mathcal{E}^{geo} \quad \text{and} \quad \mathcal{E}^{ss,geo} \simeq \mathcal{E}^{geo},$$
   so that
$$(\mathfrak{f}^*(\mathcal{E}^{ss,geo}))^{conv} \simeq \mathfrak{f}^*\mathcal{E}^{geo,conv} \quad \text{and} \quad (\mathcal{E}^{ss,geo})^{conv} \simeq \mathcal{E}^{geo,conv},$$
   hence
$$G((\mathfrak{f}^*(\mathcal{E}^{ss,geo}))^{conv}) \simeq G(\mathfrak{f}^*\mathcal{E}^{geo,conv}) \quad \text{and} \quad G((\mathcal{E}^{ss,geo})^{conv}) \simeq G(\mathcal{E}^{geo,conv}).$$
   Since $G((\mathfrak{f}^*(\mathcal{E}^{ss,geo}))^{conv})^0 = G((\mathcal{E}^{ss,geo})^{conv})^0$, also $G(\mathfrak{f}^*\mathcal{E}^{geo,conv})^0 = G(\mathcal{E}^{geo,conv})^0$ and this concludes the proof.

2. We deduce for point (1) and a group theoretic argument that if $G(\mathfrak{f}^*(\mathcal{E}^{ss})^{conv}) = G((\mathcal{E}^{ss})^{conv})$ then $G(\mathfrak{f}^*\mathcal{E}^{conv}) = G(\mathcal{E}^{conv})$.

   Assume that $G(\mathfrak{f}^*(\mathcal{E}^{ss})^{conv}) = G((\mathcal{E}^{ss})^{conv})$. By point (1), $G(\mathfrak{f}^*\mathcal{E}^{conv})^0 = G(\mathcal{E}^{conv})^0$. Thanks to the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(\mathfrak{f}^*(\mathcal{E}^{conv}))^0 & \longrightarrow & G(\mathfrak{f}^*(\mathcal{E}^{conv})) & \longrightarrow & \pi_0(G(\mathfrak{f}^*(\mathcal{E}^{conv}))) & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G(\mathcal{E}^{conv})^0 & \longrightarrow & G(\mathcal{E}^{conv}) & \longrightarrow & \pi_0(G(\mathcal{E}^{conv})) & \longrightarrow & 0, \end{array}$$

it is enough to show that $|\pi_0(G(\mathfrak{f}^*\mathcal{E}^{conv}))| \geq |\pi_0(G(\mathcal{E}^{conv}))|$. One has

$$\pi_0(G(\mathfrak{f}^*(\mathcal{E}^{ss})^{conv})) \simeq \pi_0(G((\mathcal{E}^{ss})^{conv}))$$

hence the surjection

$$\pi_0(G(\mathfrak{f}^*\mathcal{E}^{conv})) \twoheadrightarrow \pi_0(G(\mathfrak{f}^*(\mathcal{E}^{ss})^{conv}))$$

shows that it is enough to prove that

$$\pi_0(G(\mathcal{E}^{conv})) \to \pi_0(G((\mathcal{E}^{ss})^{conv}))$$

is an isomorphism. Since the functor π_0 is right exact, there is an exact sequence

$$\pi_0(K) \to \pi_0(G(\mathcal{E}^{conv})) \to \pi_0(G((\mathcal{E}^{ss})^{conv})) \to 0$$

where K is the kernel of $G(\mathcal{E}^{conv}) \to G((\mathcal{E}^{ss})^{conv})$. But K is contained in $R_u(G(\mathcal{E})) = \mathrm{Ker}(G(\mathcal{E}) \to G(\mathcal{E}^{ss}))$, so that it is unipotent hence connected. Hence $\pi_0(K) = 0$ and this concludes the proof. □

6.A Epimorphic morphisms

To control the exceptional locus of convergent F-isocrystals we used the notion of epimorphic morphism. In this section, L is algebraically closed field of characteristic zero.

Definition 6.A.1. Let $f : H \to G$ be a morphism of algebraic groups over L. We say that $f : H \to G$ is epimorphic if the induced functor $\mathbf{Rep}_L(G) \to \mathbf{Rep}_L(H)$ is fully faithful. If $f : H \to G$ is a closed immersion, we say that H is an epimorphic subgroup of G.

Epimorphic subgroup have been studies in details in [BB92], [BB92b] and [Bri16]. For the lack of a reference we prove a couple of easy lemmas that have been used several times in this paper.

Lemma 6.A.2. Let $f : H \to G$ be a morphism of algebraic groups over L.

1. If H is reductive, then $f : H \to G$ is epimorphic if and only if is surjective.

2. Let $g : L \to H$ be another morphism of algebraic groups. If g and $f \circ g$ are epimorphic, then f is epimorphic.

Proof.

1. If $f : H \to G$ is surjective then it is clearly epimorphic. Assume now that H is reductive. By the Tannakian formalism, it is enough to show that the essential image of $f^* : \mathbf{Rep}_L(G) \to \mathbf{Rep}_L(H)$ is closed under sub quotient. Let V be in $\mathbf{Rep}_L(G)$. Since H is reductive, every H-invariant subquotient of V is an H-invariant sub object, so that it is enough to show that every H-invariant sub object $W \subseteq V$ in also G-invariant. Since H is reductive, there exist an $\psi \in End_H(V)$ such that $Ker(\psi) = W$. Since $f : H \to G$ is epimorphic ψ is also G-invariant, hence $W = Ker(\psi)$ is also G-invariant.

2. Let V a representation of G. Then, since $f \circ g$ is epimorphic, we have $V^G = V^L$ and, since g is epimorphic, we have $V^L = V^H$. Hence $V^H = V^L = V^G$ and f is epimorphic. □

Lemma 6.A.3. Let

$$\begin{tikzcd} 0 \arrow[r] & H' \arrow[r] \arrow[d, hook] & H \arrow[r] \arrow[d, hook] & H/H' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G' \arrow[r] & G \arrow[r] & G/G' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of algebraic groups over L with exact rows.

1. If H is an epimorphic subgroup of G, G' is connected and the right vertical arrow is an isogeny, then H' is an epimoprhic subgroup of G'.

2. If H if an epimorphic subgroup of G, then $H/H' \to G/G'$ is epimorphic

3. If H' is an epimorphic subgroup of G' and right vertical arrow is surjective then H is an epimorphic subgroup of G

Proof.

1. Applying [DK17, Lemma B.6.1] to the epimorphic inclusion $H \subseteq G$ we see that $H^0(G/H, \mathcal{O}_{G/H}) = k$. Applying it to the inclusion $H' \subseteq G'$, it is enough to show that $H^0(G'/H', \mathcal{O}_{G'/H'}) = k$. The morphism $G'/H' \to G/H$ is finite étale so that the map $k = H^0(G/H, \mathcal{O}_{G/H}) \to H^0(G'/H', \mathcal{O}_{G'/H'})$ makes $H^0(G'/H', \mathcal{O}_{G'/H'})$ into a finite étale k-algebra. We conclude observing that, since G'/H' is connected, the étale k-algebra $H^0(G'/H', \mathcal{O}_{G'/H'})$ has no idempotent elements.

2. Since $H \to H/H'$ and $G \to G/G'$ are surjective, for every representation V of G/G' we have
$$V^{G/G'} = V^G = V^H = V^{H/H'}.$$

3. Let V be representation of G. By assumption we know that $V^{G'} = V^{H'}$. Since the right vertical arrow is surjective
$$V^H = (V^{H'})^{H/H'} = (V^{G'})^{H/H'} = (V^{G'})^{G/G'} = V^G.$$

□

Chapter 7

A note on the behaviour of the Tate conjecture under finitely generated field extension

7.1 Introduction

7.1.1 Statement

Let k be a field of characteristic $p \geq 0$ with algebraic closure $\overline{k}$ and write $\pi_1(k)$ for the absolute Galois group of k. For a k-variety Z write $Z_{\overline{k}} := Z \times_k \overline{k}$ and $Pic(Z_{\overline{k}})$ for its geometric Picard group. If $\ell \neq p$ is a prime, consider the ℓ-adic cycle class map

$$c_{Z_{\overline{k}}} : Pic(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \to H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))$$

and write $NS(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell$ for its image. Recall the ℓ-adic Tate conjecture for divisors [Tat65]:

Conjecture 7.1.1.1 ($T(Z, \ell)$)**.** Assume that k is finitely generated and Z is a smooth and proper k-variety. Then the map

$$c_{Z_{\overline{k}}} : NS(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \to \bigcup_{[k':k]<+\infty} H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k')}$$

is an isomorphism.

While Conjecture 7.1.1.1 is widely open in general, by the works of many people it is known for abelian varieties ([Tat66], [Zar75], [Zar77], [FW84]), K3 surfaces ([NO85], [Tan95], [And96a], [Char13], [MP15], [KMP15]) and some other special class of k-varieties; see for example [MP15, Section 5.13] and [Moo17]. For abelian varieties and K3 surfaces, Conjecture 7.1.1.1 is closely related to the finiteness of rational points on their moduli spaces; see [Tat66, Proposition 2] and [LMS14]. This may suggest that Conjecture 7.1.1.1 could be easier to prove when k is a finite field. The main result of this note is that, to prove Conjecture 7.1.1.1 for varieties over finitely generated fields of positive characteristic, it is actually enough to prove it for varieties over finite fields.

Theorem 7.1.1.2. Assume $p > 0$. Then $T(Z, \ell)$ for every finite field k of characteristic p and every smooth projective k-variety Z implies $T(Z, \ell)$ for every finitely generated field k of characteristic p and every smooth proper k-variety Z.

See Section 7.3 for a discussion on results for cycles of higher codimension and different fields.

7.1.2 Remarks

By an unpublished result ([dJ]) of De Jong (whose proof has been simplified in [Mor15, Theorem 4.3]), over finite fields the ℓ-adic Tate conjecture for divisors for smooth projective varieties follows from the ℓ-adic Tate conjecture for divisors for smooth projective surfaces. Hence Theorem 7.1.1.2 implies the following:

Corollary 7.1.2.1. Assume $p > 0$. Then $T(Z, \ell)$ for every finite field k of characteristic p and every smooth projective k-surface Z implies $T(Z, \ell)$ for every finitely generated field k of characteristic p and every smooth proper k-variety Z.

Let us mention that if k is **infinite** and finitely generated, one can use the results of [And96] (see Fact 1.3.2.2.1) if $p = 0$ or the results of Chapter 4 (see Corollary 4.1.7.1.2) if $p > 0$, together with a spreading out argument to deduce that $T(Z, \ell)$ for all smooth proper k-varieties Z implies $T(Z, \ell)$ for all smooth proper varieties Z over all fields that are finitely generated over k.

7.2 Proof of Theorem 7.1.1.2

Fix an infinite finitely generated field k of characteristic $p > 0$ inside a fixed algebraic closure $\overline{k}$ and a smooth proper k-variety Z. Let $\mathbb{F}_q$ (resp. $\mathbb{F}$) the algebraic closure of $\mathbb{F}_p$ in k (resp. $\overline{k}$)

7.2.1 Strategy

The idea is to try and transpose the Hodge theoretic arguments of [And96, Section 5.1] to the ℓ-adic setting. We spread out Z to a smooth proper morphism $\mathcal{Z} \to \mathcal{K}$ of $\mathbb{F}_q$-varieties such that $\mathcal{Z}$ embeds as a dense open subset into a smooth proper $\mathbb{F}_q$-variety $\mathcal{Z}^{cmp}$. By smooth proper base change and the global invariant cycles theorem ([Del80]; see [And06, Theoreme 1.1.1]), a class in $H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k)}$ arises from a class in $H^2(\mathcal{Z}^{cmp}_{\mathbb{F}}, \mathbb{Q}_\ell(1))^{\pi_1(\mathbb{F}_q)}$, hence, by $T(\mathcal{Z}^{cmp}, \ell)$, from a divisor on $\mathcal{Z}^{cmp}$. Compared to [And96, Section 5.1], the extra difficulties come from the fact that resolution of singularities and the semisimplicity of the Galois action in ℓ-adic cohomology are not known. The first issue can be overcome using De Jong's alteration theorem and the second adjusting an argument of Tate ([Tat94, Proposition 2.6.]). Applying De Jong's alteration theorem, we find a generically étale alteration $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ such that $\widetilde{\mathcal{Z}}$ embeds as a dense open subset into a smooth proper $\mathbb{F}_q$-variety. However, the resulting morphism $\widetilde{\mathcal{Z}} \to \mathcal{Z} \to \mathcal{K}$ is not, in general, generically smooth, so that we cannot apply directly the global invariant cycles theorem. To solve this issue, we use the main ingredients of its proof: the Hard Lefschetz theorem [Del80, Theorem 4.1.1] and the theory of weights for $\mathbb{F}_q$-schemes of finite type [Del80, Theorem 1].

7.2.2 Preliminary reductions

To prove $T(Z, \ell)$, one may freely replace k with a finite field extension. In particular we may assume that all the connected components of $Z_{\overline{k}}$ are defined over k and so, working with each component separately, that Z is geometrically connected over k. The following well known lemma, a slight variant of [Tat94, Theorem 5.2], will be used twice.

Lemma 7.2.2.1. Let W be a smooth proper k-variety and $g : W \to Z$ a generically finite dominant morphism. Then the following hold:

- The map $g^* : H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1)) \to H^2(W_{\overline{k}}, \mathbb{Q}_\ell(1))$ is injective.
- For any $z \in H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))$, if $g^*(z)$ is in the image of $c_{W_{\overline{k}}} : Pic(W_{\overline{k}}) \otimes \mathbb{Q}_\ell \to H^2(W_{\overline{k}}, \mathbb{Q}_\ell(1))$ then z is in the image of $c_{Z_{\overline{k}}} : Pic(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \to H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))$.

In particular $T(W,\ell)$ implies $T(Z,\ell)$.

Proof. Assume first that W is geometrically connected. Then, by Poincaré duality, there is a morphism $g_*: H^2(W_{\overline{k}}, \mathbb{Q}_\ell(1)) \to H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))$ which is compatible with the push forward of cycles $g_*: Pic(W_{\overline{k}}) \otimes \mathbb{Q}_\ell \to Pic(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell$ and such that $g_* g^*$ is equal to the multiplication by the generic degree of $g: W \to Z$. All the assertions then follow from the commutative diagram:

$$\begin{tikzcd}
Pic(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[r, "g^*"] \arrow[d, "c_{Z_{\overline{k}}}"] & Pic(W_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[r, "g_*"] \arrow[d, "c_{W_{\overline{k}}}"] & Pic(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[d, "c_{Z_{\overline{k}}}"] \\
H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1)) \arrow[r, "g^*"] & H^2(W_{\overline{k}}, \mathbb{Q}_\ell(1)) \arrow[r, "g_*"] & H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1)).
\end{tikzcd}$$

In general, we reduce to the situation where W is geometrically connected. To prove Lemma 7.2.2.1, we can freely replace k with a finite field extension and hence assume that all the connected components $W_{i,\overline{k}}$ of $W_{\overline{k}}$ are defined over k. Since $g: W \to Z$ is dominant and generically finite and Z is connected, there is at least one connected component (say W_1) mapping surjectively onto Z. Since Z and W_1 are smooth proper k-varieties of the same dimension, the morphism $g_1: W_1 \to W \to Z$ is still dominant and generically finite. The general case follows then from the geometrically connected case and the diagram:

$$\begin{tikzcd}
Pic(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[r] \arrow[d] & Pic(W_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[r] \arrow[d] & Pic(W_{1,\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[d] \\
H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1)) \arrow[r] & H^2(W_{\overline{k}}, \mathbb{Q}_\ell(1)) \arrow[r] & H^2(W_{1,\overline{k}}, \mathbb{Q}_\ell(1)).
\end{tikzcd}$$

□

By De Jong's alteration theorem ([dJ96]) applied to $Z_{\overline{k}}$, there exists a smooth projective $\overline{k}$-variety W' and a dominant generically finite morphism $g': W' \to Z_{\overline{k}}$. By descent and replacing k with a finite field extension, there exist a smooth projective k-variety W and a dominant generically finite morphism $g: W \to Z$ which, after base change along $Spec(\overline{k}) \to Spec(k)$, identifies with $g': W' \to Z_{\overline{k}}$. By Lemma 7.2.2.1, we may replace Z with W and hence we may assume that Z is a smooth projective k-variety. Moreover one may assume that the Zariski closure G_ℓ of the image of $\pi_1(k)$ acting on $H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))$ is connected and hence, since the action of $\pi_1(k)$ on $NS(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell$ factors through a finite quotient, that $NS(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell = NS(Z) \otimes \mathbb{Q}_\ell$. The core of the proof is the following proposition.

Proposition 7.2.2.2. Let Z be a geometrically connected smooth projective k-variety such that $NS(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell = NS(Z) \otimes \mathbb{Q}_\ell$. Assume that $T(V,\ell)$ holds for every finite field extensions $\mathbb{F}_q \subseteq \mathbb{F}_{q'}$ and every smooth proper $\mathbb{F}_{q'}$-varieties V. Up to replacing k with a finite field extension, there exist a projective k-scheme $\widetilde{Z}$ and a dominant generically finite morphism $h: \widetilde{Z} \to Z$, such that for every $z \in H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k)}$ the element $h^*(z)$ is in the image of $c_{\widetilde{Z}_{\overline{k}}}: Pic(\widetilde{Z}_{\overline{k}}) \otimes \mathbb{Q}_\ell \to H^2(\widetilde{Z}_{\overline{k}}, \mathbb{Q}_\ell(1))$.

Before proving Proposition 7.2.2.2, let us show that it implies Theorem 7.1.1.2. Replacing k with a finite field extension we can take $h: \widetilde{Z} \to Z$ as in the statement of Proposition 7.2.2.2. Write $\widetilde{Z}_{\overline{k},red}$ for the reduced closed subscheme of $\widetilde{Z}_{\overline{k}}$. Then $h_{red}: \widetilde{Z}_{\overline{k},red} \to \widetilde{Z}_{\overline{k}} \to Z_{\overline{k}}$ is still dominant and generically finite and for every $z \in H^2(\widetilde{Z}_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k)}$ the element $h^*_{red}(z) \in H^2(\widetilde{Z}_{\overline{k},red}, \mathbb{Q}_\ell(1))$ is in the image of $c_{\widetilde{Z}_{\overline{k},red}}: Pic(\widetilde{Z}_{\overline{k},red}) \otimes \mathbb{Q}_\ell \to H^2(\widetilde{Z}_{\overline{k},red}, \mathbb{Q}_\ell(1))$. So, by descent and replacing k with a finite extension we can assume that $\widetilde{Z}$ is geometrically reduced and that all the irreducible components of $\widetilde{Z}_{\overline{k}}$ are defined over k. Then, by De Jong alteration's theorem applied to $\widetilde{Z}_{\overline{k}}$ and descent, up to replacing k with a finite field extension, there exists a generically finite dominant morphism $W \to \widetilde{Z}$ with W a smooth projective k-variety. The morphism $g : W \to \widetilde{Z} \to Z$ is still generically finite and dominant and for every $z \in H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k)}$ there exists a cycles $w \in Pic(W_{\overline{k}})$ such that $c_W(w) = g^*(z)$. So Theorem 7.1.1.2 follows from Lemma 7.2.2.1.

The next subsection is devoted to the proof of Proposition 7.2.2.2.

7.2.3 Proof of Proposition 7.2.2.2

Let Z be a geometrically connected smooth projective k-variety such that $NS(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell = NS(Z) \otimes \mathbb{Q}_\ell$.

7.2.3.1 Spreading out and alterations

Spreading out to $\mathbb{F}_q$, there exist a geometrically connected, smooth $\mathbb{F}_q$-variety $\mathcal{K}$ with generic point $\eta : k \to \mathcal{K}$ and a smooth projective morphism $\mathfrak{f} : \mathcal{Z} \to \mathcal{K}$ fitting into a cartesian diagram:

$$\begin{tikzcd} Z \arrow[r, "i_\eta"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Z} \arrow[d, "\mathfrak{f}"] \\ k \arrow[r, "\eta"] & \mathcal{K}. \end{tikzcd}$$

By De Jong alteration's theorem, there exist an integral smooth $\mathbb{F}_q$-variety $\widetilde{\mathcal{Z}}$, an open embedding $\widetilde{\mathfrak{i}} : \widetilde{\mathcal{Z}} \to \widetilde{\mathcal{Z}}^{cmp}$ with dense image into a smooth projective $\mathbb{F}_q$-variety $\widetilde{\mathcal{Z}}^{cmp}$ and a generically étale, proper, dominant morphism $\mathfrak{h} : \widetilde{\mathcal{Z}} \to \mathcal{Z}$. Then $\widetilde{\mathcal{Z}}^{cmp}$ is geometrically connected over some finite field extension $\mathbb{F}_q \subseteq \mathbb{F}_{q'}$. Replacing $\mathbb{F}_q$ with $\mathbb{F}_{q'}$ amounts to replacing k with the finite field extension $k' := k\mathbb{F}_{q'}$, so we can assume that $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{Z}}^{cmp}$ are geometrically connected over $\mathbb{F}_q$.

Since $\widetilde{\mathcal{Z}} \to \mathcal{Z} \to \mathbb{F}_q$ is quasi-projective, the morphism $\mathfrak{h} : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ is quasi-projective as well ([SP, Tag 0C4N]). Since $\mathfrak{f} : \mathcal{Z} \to \mathcal{K}$ is projective, this implies that $\widetilde{\mathcal{Z}} \to \mathcal{K}$ is quasi-projective. Since $\mathfrak{h} : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ and $\mathfrak{f} : \mathcal{Z} \to \mathcal{K}$ are proper, the morphism $\widetilde{\mathcal{Z}} \to \mathcal{K}$ is proper as well. So $\widetilde{\mathcal{Z}} \to \mathcal{K}$ is proper and quasi-projective hence projective. The generic fibre $\widetilde{Z} \to k$ of $\widetilde{\mathcal{Z}} \to \mathcal{K}$ is then a projective k-scheme endowed with a generically finite dominant morphism $h : \widetilde{Z} \to Z$. The situation is summarized in the following diagram of $\mathbb{F}_q$-schemes:

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "\widetilde{i}_\eta"] \arrow[d, "h"] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{Z}} \arrow[r, hook, "\widetilde{\mathfrak{i}}"] \arrow[d, "\mathfrak{h}"] & \widetilde{\mathcal{Z}}^{cmp} \\ Z \arrow[r, "i_\eta"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Z} \arrow[d, "\mathfrak{f}"] & \\ k \arrow[r, "\eta"] & \mathcal{K}. & \end{tikzcd}$$

The Leray spectral sequence for the morphism $\mathfrak{f} : \mathcal{Z} \to \mathcal{K}$ induces a map

$$\mathrm{Ler} : H^2(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(1)) \to H^0(\mathcal{K}_{\mathbb{F}}, R^2\mathfrak{f}_*\mathbb{Q}_\ell(1)),$$

fitting into a commutative diagram:

$$
\begin{tikzcd}
 & & & Pic(\widetilde{Z}_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[d, "c_{\widetilde{Z}}"'] & Pic(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \arrow[l, "h^*"'] \arrow[d, "c_Z"] \\
Pic(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}) \otimes \mathbb{Q}_\ell \arrow[r, "\widetilde{\mathfrak{i}}^*"] \arrow[d, "c_{\widetilde{\mathcal{Z}}^{cmp}}"] & Pic(\widetilde{\mathcal{Z}}_{\mathbb{F}}) \otimes \mathbb{Q}_\ell \arrow[d, "c_{\widetilde{\mathcal{Z}}}"] \arrow[urr, "\widetilde{i}_\eta^*"] & Pic(\mathcal{Z}_{\mathbb{F}}) \otimes \mathbb{Q}_\ell \arrow[l, "\mathfrak{h}^*"] \arrow[d, "c_{\mathcal{Z}}"] \arrow[urr, "i_\eta^*"] & H^2(\widetilde{Z}_{\overline{k}}, \mathbb{Q}_\ell(1)) & H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1)) \arrow[l, "h^*"'] \\
H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1)) \arrow[r, "\widetilde{\mathfrak{i}}^*"] & H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(1)) \arrow[urr, "\widetilde{i}_\eta^*"] & H^2(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(1)) \arrow[l, "\mathfrak{h}^*"] \arrow[urr, "i_\eta^*"] \arrow[rr, "\mathrm{Ler}"] & & H^0(\mathcal{K}_{\mathbb{F}}, R^2\mathfrak{f}_*\mathbb{Q}_\ell(1)). \arrow[u]
\end{tikzcd}
$$

7.2.3.2 Hard Lefschetz Theorem

Write $\varphi \in \pi_1(\mathbb{F}_q)$ for the arithmetic Frobenius of $\mathbb{F}_q$ and, for every $\pi_1(\mathbb{F}_q)$-module V, write V_{gen}^{φ} for the generalized eigenspace on which φ acts with generalized eigenvalue 1.

Let z be in $H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k)}$. In this section we lift $h^*(z) \in H^2(\widetilde{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(1))$ to $H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(1))_{gen}^{\varphi}$. By smooth proper base change, the action of $\pi_1(k)$ on $H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))$ factors trough the canonical surjection $\pi_1(\mathcal{K}) \to \pi_1(k)$, hence $H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k)} \simeq H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(\mathcal{K})}$. Since $\mathfrak{f} : \mathcal{Z} \to \mathcal{K}$ is smooth and projective, by the Hard Lefschetz Theorem [Del80, Theorem 4.1.1] and [Del68, Proposition 2.1], the map $\mathrm{Ler} : H^2(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(1)) \to H^0(\mathcal{K}_{\mathbb{F}}, R^2\mathfrak{f}_*\mathbb{Q}_\ell(1))$ is surjective. Consider the diagram:

$$
H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1)) \xrightarrow{\widetilde{\mathfrak{i}}^*} H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(1)) \xleftarrow{\mathfrak{h}^*} H^2(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(1)) \xrightarrow{\mathrm{Ler}} H^0(\mathcal{K}_{\mathbb{F}}, R^2\mathfrak{f}_*\mathbb{Q}_\ell(1)).
$$

Since

$$
z \in H^2(Z_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k)} \simeq H^0(\mathcal{K}_{\mathbb{F}}, R^2\mathfrak{f}_*\mathbb{Q}_\ell(1))^{\pi_1(\mathbb{F}_q)} \subseteq H^0(\mathcal{K}_{\mathbb{F}}, R^2\mathfrak{f}_*\mathbb{Q}_\ell(1)),
$$

the element z is in $H^0(\mathcal{K}_{\mathbb{F}}, R^2\mathfrak{f}_*\mathbb{Q}_\ell(1))_{gen}^{\varphi}$. In particular, since $\mathrm{Ler} : H^2(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(1)) \to H^0(\mathcal{K}_{\mathbb{F}}, R^2\mathfrak{f}_*\mathbb{Q}_\ell(1))$ is surjective, z is the image of some $z' \in H^2(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(1))_{gen}^{\varphi}$, so that $\mathfrak{h}^*(z') \in H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(1))_{gen}^{\varphi}$.

7.2.3.3 Theory of weights

We now prove that $\mathfrak{h}^*(z')$ is the image of some $\widetilde{z} \in H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1))_{gen}^{\varphi}$. Write d for the common dimension of $\mathcal{Z}$, $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{Z}}^{cmp}$. The localization exact sequence for the dense open immersion $\widetilde{\mathcal{Z}} \to \widetilde{\mathcal{Z}}^{cmp}$ with complement $\mathcal{D} := \widetilde{\mathcal{Z}}^{cmp} - \widetilde{\mathcal{Z}}$, gives an exact sequence

$$
H_c^{2d-3}(\mathcal{D}_{\mathbb{F}}, \mathbb{Q}_\ell(-1))(d) \to H_c^{2d-2}(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(-1))(d) \to H_c^{2d-2}(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(-1))(d).
$$

Combining this sequence with Poincaré duality for the smooth varieties $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{Z}}^{cmp}$, one sees that the cokernel of $\widetilde{\mathfrak{i}}^* : H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1)) \to H^2(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(1))$ injects into $(H_c^{2d-3}(\mathcal{D}_{\mathbb{F}}, \mathbb{Q}_\ell(-1))(d))^{\vee}$. By [Del80, Corollaire 3.3.9], the group $H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1))$ is pure of weight 0, while by [Del80, Theorem 3.3.1] the group $(H_c^{2d-3}(\mathcal{D}_{\mathbb{F}}, \mathbb{Q}_\ell(-1))(d))^{\vee}$ is mixed of weights ≥ 1. Hence, the image of $\widetilde{\mathfrak{i}}^* : H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1)) \to H^2(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(1))$ consists exactly of the generalized eigenspace on which φ acts with generalized eigenvalues of weight 0. So $\mathfrak{h}^*(z') \in H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}, \mathbb{Q}_\ell(1))_{gen}^{\varphi}$ is the image of some $\widetilde{z} \in H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1))_{gen}^{\varphi}$ by $\widetilde{\mathfrak{i}}^* : H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1)) \to H^2(\mathcal{Z}_{\mathbb{F}}, \mathbb{Q}_\ell(1))$.

7.2.3.4 Using the Tate conjecture

Since $T(\widetilde{\mathcal{Z}}^{cmp}, \ell)$ holds by assumption, it follows from [Tat94, Proposition 2.6.] that the injection

$$
H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1))^{\pi_1(\mathbb{F}_q)} \hookrightarrow H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1))
$$

has a $\pi_1(\mathbb{F}_q)$-equivariant splitting, so that $H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1))_{gen}^{\varphi} = H^2(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}, \mathbb{Q}_\ell(1))^{\varphi}$. Hence, by $T(\widetilde{\mathcal{Z}}^{cmp}, \ell)$, there exists a $\widetilde{w} \in Pic(\widetilde{\mathcal{Z}}_{\mathbb{F}}^{cmp}) \otimes \mathbb{Q}_\ell$ such that $c_{\widetilde{\mathcal{Z}}^{cmp}}(\widetilde{w}) = \widetilde{z}$. We conclude the proof observing that, thanks to the commutative diagram at the end of 7.2.3.1, $h^*(z)$ is the image of $\widetilde{i}_\eta^*\widetilde{\mathrm{i}}^*(\widetilde{w})$ via $c_{\widetilde{Z}} : Pic(\widetilde{Z}_{\overline{k}}) \otimes \mathbb{Q}_\ell \to H^2(\widetilde{Z}_{\overline{k}}, \mathbb{Q}_\ell(1))$.

7.3 Higher codimensional cycles

In this section we discuss generalizations of Theorem 7.1.1.2 to cycles of higher codimension. Compared with the case of divisors, the main issue is that [Tat94, Proposition 2.6] is no longer available, so that we have to consider also conjectures about the semisimplicity of the Galois action on étale cohomology.

7.3.1 Conjectures

Fix an $i \geq 1$, a k-variety Z and write $CH^i(Z_{\overline{k}})$ for the group of algebraic cycles of codimension i modulo rational equivalence. Recall the following conjectures ([Tat65]):

Conjecture 7.3.1.1. If k is finitely generated and Z is a smooth proper k-variety, then:

- $T(Z, i, \ell)$: The map $c_{Z_{\overline{k}}} : CH^i(Z_{\overline{k}}) \otimes \mathbb{Q}_\ell \to \bigcup_{[k':k]<+\infty} H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k')}$ is surjective;
- $S(Z, i, \ell)$: The action of $\pi_1(k)$ on $H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))$ is semisimple;
- $WS(Z, i, \ell)$: The inclusion $H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))^{\pi_1(k)} \subseteq H^{2i}(Z_{\overline{k}}, \mathbb{Q}_\ell(i))$ admits a $\pi_1(k)$-equivariant splitting.

For a field K, one says that $T(K, i, \ell, r, d)$ holds if for every finitely generated field extension $K \subseteq k$ of transcendence degree $\leq r$ and for every smooth proper k-variety Z of dimension d, $T(Z, i, \ell)$ holds. One defines similarly the conditions $S(K, i, \ell, r, d)$ and $WS(K, i, \ell, r, d)$.

7.3.2 Known results and an extension of Theorem 7.1.1.2

Clearly, for each smooth proper variety Z the condition $S(Z, i, \ell)$ implies $WS(Z, i, \ell)$. A recent result [Moo18, Theorem 1] of Moonen shows that $T(\mathbb{Q}, i, \ell, 0, d)$ for all integers $i, d \geq 1$ implies $S(\mathbb{Q}, i, \ell, r, d)$ for all integers $r \geq 0$ and it is classically known that $T(\mathbb{F}_p, i, \ell, 0, d)$ together with the equivalence of the homological and numerical equivalence relations for codimensional i cycles implies $S(\mathbb{F}_p, i, \ell, r, d)$ for all integer $r \geq 0$; see [Moo18, Theorem 2]. If K is finite (resp. K is infinite finitely generated), it follows from [Fu99, Theorem] and its proof (resp. a classical argument of Serre ([Ser89, Section 10.6])) that $S(K, i, \ell, 0, d)$ implies $S(K, i, \ell, r, d)$ for all integers $r \geq 1$.

The arguments in [And96, Section 5.1], sketched at the beginning of Section 7.2, shows that if K is of characteristic zero, then $S(K, i, \ell, 0, d+r)$ and $T(K, i, \ell, 0, d+r)$ imply $T(K, i, \ell, r, d)$. Similarly, Theorem 7.1.1.2 and its proof show that $T(\mathbb{F}_p, 1, \ell, 0, d+r)$ imply $T(\mathbb{F}_p, 1, \ell, r, d)$. To conclude, let us point out that, in the proof of Theorem 7.1.1.2, the only place where we used the hypothesis that $i = 1$ is in Section 7.2.3.4, to show that $T(\widetilde{Z}^{cmp}, 1, \ell)$ implies $WS(\widetilde{Z}^{cmp}, 1, \ell)$ (which is the content of [Tat94, Proposition 2.6]). So, the proof of Theorem 7.1.1.2 shows the following more general proposition.

Proposition 7.3.2.1. If $p > 0$, then $T(\mathbb{F}_p, i, \ell, 0, d+r)$ and $WS(\mathbb{F}_p, i, \ell, 0, d+r)$ imply $T(\mathbb{F}_p, i, \ell, r, d)$.

Chapter 8

Uniform boundedness for Brauer group of forms in positive characteristic

8.1 Introduction

Let k be a field of characteristic $p \geq 0$ with algebraic closure $\overline{k}$ and write $\pi_1(k)$ for the absolute Galois group of k. The letter ℓ will always denote a prime $\neq p$.

8.1.1 Brauer groups

8.1.1.1 Finiteness of Brauer groups

Let X be a k-variety. Write $\mathrm{Br}(X_{\overline{k}})[p']$ for the prime-to-p torsion of the (cohomological) Brauer group $\mathrm{Br}(X_{\overline{k}}) := H^2(X_{\overline{k}}, \mathbb{G}_m)$ of $X_{\overline{k}}$ and recall that if X is smooth over k then $\mathrm{Br}(X_{\overline{k}})$ is a torsion group. If k is finitely generated and X is smooth and proper over k, one expects $\mathrm{Br}(X_{\overline{k}})[p']^{\pi_1(k)}$ to be small. This is predicted by (variants of) the ℓ-adic Tate conjecture for divisors ([Tat65]):

Conjecture 8.1.1.1.1 ($T(X,\ell)$)**.** Assume that k is finitely generated and X is a smooth and proper k-variety. Then the ℓ-adic cycle class map

$$c_{X_{\overline{k}}} : Pic(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \to \bigcup_{[k':k]<+\infty} H^2(X_{\overline{k}}, \mathbb{Q}_\ell(1))^{\pi_1(k')}$$

is surjective.

As it is well known (see e.g. [CC18, Proposition 2.1.1]), Conjecture $T(X,\ell)$ holds if and only if, for any finite field extension $k \subseteq k'$, the ℓ-primary torsion $\mathrm{Br}(X_{\overline{k}})[\ell^\infty]^{\pi_1(k')}$ of $\mathrm{Br}(X_{\overline{k}})^{\pi_1(k')}$ is finite. But one can expect stronger finiteness results.

Fact 8.1.1.1.2. Assume that k is finitely generated and X is a smooth and proper k-variety. Then:

1. ([OS18, Theorem 5.5]): If $p = 0$ and the integral Mumford Tate conjecture for X holds ([Ser77, Conjecture C.3]), then $\mathrm{Br}(X_{\overline{k}})^{\pi_1(k)}$ is finite;

2. ([CHT17, Corollary 1.5]): If $p > 0$ and $T(X,\ell)$ holds for every prime $\ell \neq p$ (or equivalently for one prime $\ell \neq p$), then $\mathrm{Br}(X_{\overline{k}})[p']^{\pi_1(k)}$ is finite.

8.1.1.2 Uniform boundedness in forms

Let X be a smooth proper variety over a finitely generated field k. Recall that for a field extension $k \subseteq k' \subseteq \overline{k}$, a $(\overline{k}/k')$-form of X is a k'-variety Y such that $Y_{\overline{k}} := Y \times_{k'} \overline{k} \simeq X_{\overline{k}}$. Let $k \subseteq k'$ be a finite field extension and let Y be a $(\overline{k}/k')$-form of X. If $p = 0$ and X satisfies the integral Mumford Tate conjecture (resp. if $p > 0$ and $T(X, \ell)$ holds for every prime $\ell \neq p$), then the same is true for Y, hence $\mathrm{Br}(Y_{\overline{k}})^{\pi_1(k)}$ (resp. $\mathrm{Br}(Y_{\overline{k}})[p']^{\pi_1(k')}$) is a finite group. But, for an integer $d \geq 1$, it is not clear whether one can find a uniform bound (depending only on X and d) for $|\mathrm{Br}(Y_{\overline{k}})^{\pi_1(k')}|$ (resp. $|\mathrm{Br}(Y_{\overline{k}})[p']^{\pi_1(k')}|$), while k' is varying among the finite field extensions $k \subseteq k'$ with $[k' : k] \leq d$ and Y among the $(\overline{k}/k')$-forms of X. If $p = 0$, this is proved by Orr-Skorobogatov in [OS18, Theorem 5.1]. If $p > 0$, this is the first main result of this note.

Theorem 8.1.1.2.1. Assume that k is finitely generated, X is a smooth proper k-variety and $p > 0$. If $T(X, \ell)$ holds for every prime $\ell \neq p$ (or equivalently for one prime $\ell \neq p$), then for every integer $d \geq 1$, there exists an integer $N \geq 1$, depending only on X and d, such that for every finite field extension $k \subseteq k'$ of degree $\leq d$ and every $(\overline{k}/k')$-form Y of X one has

$$|\,\mathrm{Br}(Y_{\overline{k}})[p']|^{\pi_1(k')} \leq N.$$

8.1.2 Forms of representations

Theorem 8.1.1.2.1 is a consequence of two general results (Propositions 8.1.2.2.1 and 8.1.2.2.2) on compatible system of $\pi_1(k)$-representations. Before stating them, we introduce some definitions and notation. In the following, k is a finitely generated field of characteristic $p > 0$, $\mathbb{F}_q$ (resp. $\mathbb{F}$) is the algebraic closure of $\mathbb{F}_p$ in k (resp. in $\overline{k}$) and we write $k_{\mathbb{F}} := k \otimes_{\mathbb{F}_q} \mathbb{F} \simeq k\mathbb{F} \subseteq \overline{k}$. Set $\ell_0 = 3$ (resp. $\ell_0 = 2$) if $p \neq 3$ (resp. $p = 3$) and $s_\ell = \ell$ (resp. $s_\ell = 4$) if $\ell \neq 2$ (resp. $\ell = 2$). Fix a collection $\underline{T} := \{T_\ell\}_{\ell \neq p}$ of rank r finitely generated $\mathbb{Z}_\ell$-modules endowed with a continuous action of $\pi_1(k)$.

8.1.2.1 Definitions

We say that $\underline{T}$ is a compatible system of $\pi_1(k)$-modules if there exists a smooth geometrically connected $\mathbb{F}_q$-variety $\mathcal{K}$ with generic point $Spec(k) \to \mathcal{K}$ such that, for every prime $\ell \neq p$, the action of $\pi_1(k)$ on T_ℓ factors trough the canonical surjective morphism $\pi_1(k) \twoheadrightarrow \pi_1(\mathcal{K})$ and the collection $\{V_\ell := T_\ell \otimes \mathbb{Q}_\ell\}_{\ell \neq p}$ give rise to a $\mathbb{Q}$-rational compatible system on $\mathcal{K}$ in the sense of Serre: for each closed point $\mathfrak{t} \in \mathcal{K}$, the characteristic polynomial of the arithmetic Frobenius at $\mathfrak{t}$ acting on V_ℓ is in $\mathbb{Q}[T]$ and independent of ℓ.

Remark 8.1.2.1.1. The notion of compatible system is stable under subquotients and the usual operations $\oplus$, $\otimes$, $^\vee$.

Definition 8.1.2.1.2. Let $k \subseteq k'$ be a finite field extension. A $(\overline{k}/k')$-form of $\underline{T}$ is a compatible system of $\pi_1(k')$-representations $\underline{U}$ such that, for each $\ell \neq p$, there exists a finite field extension $k' \subseteq k_\ell$ and an isomorphism of $\pi_1(k_\ell)$-modules $T_\ell \simeq U_\ell$.

8.1.2.2 Results

In Definition 8.1.2.1.2, the extension $k \subseteq k_\ell$ is allowed to depend on ℓ. Our first main result in this setting produces an extension of (explicitly) bounded degree that works for every prime $\ell \neq p$. Let $? \in \{\emptyset, \mathbb{F}\}$.

Proposition 8.1.2.2.1. Let $\underline{U}$ be a $(\overline{k}/k)$-form of $\underline{T}$. Then, there exists a finite field extension $k_? \subseteq k_{\underline{U}}$ of degree $\leq |\mathrm{GL}_r(\mathbb{Z}/s_{\ell_0})|^2$ and a $\pi_1(k_{\underline{U}})$-equivariant isomorphism $T_\ell/(T_\ell)_{tors} \simeq U_\ell/(U_\ell)_{tors}$ for every prime $\ell \neq p$.

Proposition 8.1.2.2.1 reduces the problem of bounding uniformly the invariants of forms of $\underline{T}$ to studying the action of $\pi_1(k')$ on $\underline{T}$, when $k \subseteq k'$ is varying among the finite field extensions of bounded degree. In this setting we prove:

Proposition 8.1.2.2.2. Suppose that T_ℓ is torsion free for $\ell \gg 0$. Then there exists a finite field extension $k_? \subseteq k'$ of degree $\leq |\mathrm{GL}_r(\mathbb{Z}/s_{\ell_0})|$ with the following property: For every integer $d \geq 1$ there exists an integer $N \geq 1$, depending only on $\underline{T}$ and d, such that, for every finite field extension $k' \subseteq k''$ of degree $\leq d$, one has

$$\prod_{\ell \neq p} [(T_\ell \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\pi_1(k'')} : (T_\ell \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\pi_1(k')}] \leq N.$$

Remark 8.1.2.2.3. In the proof of Theorem 8.1.1.2.1 we only use the version of Propositions 8.1.2.2.1 and 8.1.2.2.2 where $? = \emptyset$. On the other hand, the proofs of the two versions are very similar and we believe that both versions are of independent interest.

8.1.3 Motivic representation

The main motivation to state Theorems 8.1.2.2.1 and 8.1.2.2.2 in this generality is that they apply directly to representations associated to ℓ-adic étale cohomology of smooth proper k-varieties; see Subsections 8.3.1.2 and 8.3.2.1. Since Theorems 8.1.2.2.1 and 8.1.2.2.2 require only the compatibility of the compatible system and not further assumptions as purity, one could apply them also to representations arising from the cohomology of some not necessarily smooth and proper k-varieties (e.g. semi-abelian schemes).

8.1.4 Strategy

To prove Proposition 8.1.2.2.1, first we prove a group theoretic proposition (Proposition 8.2.1.1.1) that bounds the number of connected components of the Zariski closure of the image of an ℓ-adic representation of a profinite group, only in terms of ℓ and of the rank of the representation. To get Proposition 8.1.2.2.1, one has to get rid of the dependency on ℓ. This follows formally from the fact that the connectedness of the ℓ-adic monodromy group can be read on the L-function of the various compatible systems $\{T_\ell^{\otimes n} \otimes (T_\ell^\vee)^{\otimes m}\}_{\ell \neq p}$.

For the proof of Proposition 8.1.2.2.2, the key point is to show that, if the Zariski closure of the image of $\pi_1(k)$ acting on V_ℓ is connected, then for every integer $d \geq 0$ there exists an integer $D \geq 1$, depending only on d and $\underline{T}$, such that, for every finite field extension $k \subseteq k'$ of degree $\leq d$, one has $(T_\ell/\ell)^{\pi_1(k)} = (T_\ell/\ell)^{\pi_1(k')}$ for every prime $\ell \geq D$. To prove this, one exploits again independence results, not in the ℓ-adic setting but in the ultrafilter setting, recently obtained by Cadoret-Hui-Tamagawa in [CHT17] and by Cadoret in [Cad19a, Section 15].

Smooth proper base change theorem, the Weil conjectures ([Del80]) and the independence of ℓ of homological equivalence for divisors show that $\{T_\ell(\mathrm{Br}(Y_{\overline{k}})) := \varprojlim_n \mathrm{Br}(Y_{\overline{k}})[\ell^n]\}_{\ell \neq p}$ is a compatible system. In this setting, Propositions 8.1.2.2.1 and 8.1.2.2.2 are the positive characteristic analogues of [OS18, Propositions 5.4 and 5.5], hence we can conclude the proof of Theorem 8.1.1.2.1 adjusting the arguments in [OS18, Section 5.4].

8.1.5 Organization of the chapter

In Section 8.2 we prove Theorems 8.1.2.2.1 and 8.1.2.2.2. In Section 8.3 we apply Theorems 8.1.2.2.1 and 8.1.2.2.2 to representations coming from geometry and we prove Theorem 8.1.1.2.1. We end the chapter in Section 8.3.2 discussing applications to abelian varieties.

8.1.6 Conventions and notation

For the rest of the chapter k is a finitely generated field of characteristic $p > 0$ with algebraic closure $k \subseteq \overline{k}$. We write $\mathbb{F}_q$ (resp. $\mathbb{F}$) for the algebraic closure of $\mathbb{F}_p$ in k (resp. $\overline{k}$) and $k_{\mathbb{F}} := k \otimes_{\mathbb{F}_q} \mathbb{F} \simeq k\mathbb{F} \subseteq \overline{k}$. If R is a commutative ring, A an R-module and n, m integers ≥ 0, set

$$T^{n,m}(A) := \underbrace{A \otimes_R \ldots \otimes_R A}_{n \text{ times}} \otimes_R \underbrace{A^\vee \otimes_R \ldots \otimes_R A^\vee}_{m \text{ times}}.$$

If G is an algebraic group over a field, write G^0 for its neutral component and $\pi_0(G)$ for the group of connected components. Write $\ell_0 = 3$ (resp. $\ell_0 = 2$) if $p \neq 3$ (resp. $p = 3$) and $s_\ell = \ell$ (resp. $s_\ell = 4$) if $\ell \neq 2$ (resp. $\ell = 2$).

8.2 Forms of representations

8.2.1 Proof of Proposition 8.1.2.2.1

Before proving Proposition 8.1.2.2.1, we collect a couple of preliminary propositions.

8.2.1.1 A group theoretical proposition

Let T be a free $\mathbb{Z}_\ell$-module of rank r and let $\Pi \subseteq \mathrm{GL}(T)$ be a closed subgroup. Write $V := T \otimes \mathbb{Q}_\ell$ and let $G \subseteq \mathrm{GL}(V)$ be the Zariski closure of Π. Then:

Proposition 8.2.1.1.1. $|\pi_0(G)| \leq |\mathrm{GL}_r(\mathbb{Z}/s_\ell)|$

Proof. Write G^{red} for the Zariski closure of the image of Π acting on the Π-semisimplification of V. Since the kernel of the natural surjection $G \to G^{red}$ is unipotent hence connected, it induces an isomorphism $\pi_0(G) \simeq \pi_0(G^{red})$. So, one may assume that G is reductive. Write $H := Ker(\Pi \to \mathrm{GL}(T/s_\ell))$. Since $[\Pi : H] \leq |\mathrm{GL}_r(\mathbb{Z}/s_\ell)|$ and H acts trivially on $\mathrm{GL}(T/s_\ell)$, Lemma 8.2.1.1.2 below concludes the proof. □

Lemma 8.2.1.1.2. If G is reductive and the action of Π on T/s_ℓ is trivial, then G is connected.

Proof. By [LP95, Lemma 2.3], it is enough to show that, for every irreducible representation W of $\mathrm{GL}(V)$ one has $W^G = W^{G^0}$. Since $\mathrm{GL}(V)$ is reductive, by [DM82, Proposition 3.1] every irreducible representation of $\mathrm{GL}(V)$ is a sub module of $T^{n,m}(V)$ and hence it is enough to show that for every integers $n, m \geq 0$

$$T^{n,m}(V)^G = T^{n,m}(V)^{G^0}.$$

The $\mathbb{Z}_\ell$-module $T^{n,m}(T)$ is a Π-invariant $\mathbb{Z}_\ell$-lattice in $T^{n,m}(V)$ and Π acts trivially on $T^{n,m}(T)/s_\ell = T^{n,m}(T/s_\ell)$, so that, by [CT18, Lemma 2.1], for every open subgroup $U \subseteq \Pi$ one has

$$Hom_\Pi(\mathbb{Q}_\ell, T^{n,m}(V)) = Hom_U(\mathbb{Q}_\ell, T^{n,m}(V)).$$

Applying this to $U := Ker(\Pi \twoheadrightarrow \pi_0(G))$, one gets

$$T^{n,m}(V)^G = Hom_\Pi(\mathbb{Q}_\ell, T^{n,m}(V)) = Hom_U(\mathbb{Q}_\ell, T^{n,m}(V)) = T^{n,m}(V)^{G^0}.$$

□

8.2.1.2 Independence

Let $? \in \{\emptyset, \mathbb{F}\}$. Let $\underline{T}$ be a $\pi_1(k)$-compatible system of finitely generated $\mathbb{Z}_\ell$-modules of rank r and write $G_{\ell,?}$ for the Zariski closure of the image of $\pi_1(k_?)$ acting on $V_\ell := T_\ell \otimes \mathbb{Q}_\ell$.

Corollary 8.2.1.2.1. For every prime $\ell \neq p$ one has $|\pi_0(G_{\ell,?})| \leq |\mathrm{GL}_r(\mathbb{Z}/\ell_0)|$.

Proof. By Lemma 8.2.1.1.1, it is enough to show that if $G_{\ell_0,?}$ is connected then $G_{\ell,?}$ is connected for every prime $\ell \neq \ell_0$. By definition of a compatible system, there exists a smooth geometrically connected $\mathbb{F}_q$-variety $\mathcal{K}$ with generic point $Spec(k) \to \mathcal{K}$ such that, for every prime $\ell \neq p$, the action of $\pi_1(k)$ on T_ℓ factors trough the surjection $\pi_1(k) \twoheadrightarrow \pi_1(\mathcal{K})$. So it is enough to show the corresponding statement for the actions of $\pi_1(\mathcal{K})$ and $\pi_1(\mathcal{K}_{\mathbb{F}})$ on V_ℓ. This follows from Fact 8.2.1.2.2 below. $\square$

Fact 8.2.1.2.2. $G_{\ell_0,?}$ is connected if and only if $G_{\ell,?}$ is connected.

Proof. To prove Fact 8.2.1.2.2 one can replace V_ℓ with its $\pi_1(\mathcal{K})$-semisimplification. So we may and do assume that V_ℓ is semisimple as $\pi_1(\mathcal{K})$-module, hence as $\pi_1(\mathcal{K}_{\mathbb{F}})$-module. Then, arguing as in Lemma 8.2.1.1.2, it is enough to show that for every integers $n, m \geq 0$ one has

$$T^{n,m}(V_\ell)^{G_{\ell,?}} = T^{n,m}(V_{\ell_0})^{G_{\ell_0,?}}.$$

By [Laf02] and [Dri12] every semisimple $\pi_1(\mathcal{K})$-modules is direct sum of its pure components (see [D'Ad17, Theorem 3.5.5] for more details) so that one reduces to the situation in which V_{ℓ_0} and V_{ℓ_0} are pure. Then, by the theory of weights ([Del80]), the dimensions of $T^{n,m}(V_\ell)^{G_{\ell,?}}$ and $T^{n,m}(V_{\ell_0})^{G_{\ell_0,?}}$, can be read on the L-functions of $T^{n,m}(V_\ell)$ and $T^{n,m}(V_{\ell_0})$ (see [D'Ad17, Proposition 3.4.11] for more details). Since $T^{n,m}(V_\ell)$ and $T^{n,m}(V_{\ell_0})$ are compatible, this concludes the proof. $\square$

Remark 8.2.1.2.3. Fact 8.2.1.2.2 is proved in [Ser81] if $? = \emptyset$ and in [LP95, Theorem 2.2] if $? = \mathbb{F}$ and V_ℓ is pure.

8.2.1.3 Proof of Theorem 8.1.2.2.1

Keep the notation as in the statement of Proposition 8.1.2.2.1 and fix $? \in \{\emptyset, \mathbb{F}\}$. We can replace T_ℓ with $T_\ell/(T_\ell)_{tors}$ and U_ℓ with $U_\ell/(U_\ell)_{tors}$, hence assume that T_ℓ and U_ℓ are torsion free. Since $\underline{T}$ and $\underline{U}$ are compatible systems, $\{H_\ell := T_\ell^\vee \otimes U_\ell\}_{\ell \neq p}$ is a compatible system as well. By Corollary 8.2.1.2.1, there exists a finite field extension $k_? \subseteq k_{\underline{U}}$ of degree $\leq |\mathrm{GL}_{r^2}(\mathbb{Z}/s_{\ell_0})|$ such that the Zariski closure G_ℓ of the image of $\pi_1(k_{\underline{U}})$ acting on $H_\ell \otimes \mathbb{Q}_\ell$ is connected for every prime $\ell \neq p$. We claim that $k_{\underline{U}}$ satisfies the conclusion of Proposition 8.1.2.2.1. By assumption, there exists a finite extension $k \subseteq k_\ell$ and an isomorphism

$$\psi_\ell \in H_\ell^{\pi_1(k_\ell)} \subseteq H_\ell^{\pi_1(k_\ell k_{\underline{U}})},$$

hence it is enough to show that $H_\ell^{\pi_1(k_{\underline{U}})} = H_\ell^{\pi_1(k_\ell k_{\underline{U}})}$. Since $H_\ell^{\pi_1(k_\ell k_{\underline{U}})}/H_\ell^{\pi_1(k_{\underline{U}})}$ is torsion free, it is enough to show that $(H_\ell \otimes \mathbb{Q}_\ell)^{\pi_1(k_{\underline{U}})} = (H_\ell \otimes \mathbb{Q}_\ell)^{\pi_1(k_\ell k_{\underline{U}})}$ and this follows from the facts that $k_{\underline{U}} \subseteq k_\ell k_{\underline{U}}$ is a finite field extension and G_ℓ is connected. This concludes the proof.

8.2.2 Proof of Proposition 8.1.2.2.2

Keep the notation as in the statement of Proposition 8.1.2.2.1 and fix $? \in \{\emptyset, \mathbb{F}\}$. Write

$$V_\ell := T_\ell \otimes \mathbb{Q}_\ell; \quad M_\ell := T_\ell \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell; \quad \overline{T}_\ell := T_\ell/\ell.$$

8.2.2.1 Preliminary reduction

Write $G_{\ell,?}$ for the Zariski closure of the image $\Pi_{\ell,?}$ acting on V_ℓ. By Corollary 8.2.1.2.1 and replacing $k_?$ with a finite field extension of degree $\leq |\mathrm{GL}_r(\mathbb{Z}/s_{\ell_0})|$, one may assume that $G_{\ell,?}$ is connected for every prime $\ell \neq p$. Since by assumption there are at most finitely many T_ℓ with torsion and these are finitely generated $\mathbb{Z}_\ell$-modules, we may replace T_ℓ with $T_\ell/(T_\ell)_{tors}$ hence assume that T_ℓ is torsion free for every prime $\ell \neq p$. The proof of Proposition 8.1.2.2.2 is the combination of the following two claims and the arguments in Section 8.2.2.4.
Claim 1: For every integer $d \geq 1$ and for every prime $\ell \neq p$, there exists an integer $A_\ell \geq 1$, depending only on d, ℓ and $\underline{T}$, such that, for every finite field extension $k \subseteq k'$ of degree $\leq d$, one has $[M_\ell^{\pi_1(k')} : M_\ell^{\pi_1(k_?)}] \leq A_\ell$.
Claim 2: For every integer $d \geq 1$, there exists an integer $D \geq 1$ such that, for every prime $\ell \geq D$ and for every finite field extension $k \subseteq k'$ of degree $\leq d$, one has $\overline{T}_\ell^{\pi_1(k')} = \overline{T}_\ell^{\pi_1(k_?)}$.

8.2.2.2 Proof of Claim 1

Since $\Pi_{\ell,?}$ is a compact ℓ-adic Lie group, it is topologically finitely generated and hence it has finitely many open subgroups of bounded index. So it is enough to show that if $U \subseteq \Pi_{\ell,?}$ is an open subgroup then $[M_\ell^U : M_\ell^{\Pi_{\ell,?}}] < +\infty$. This follows from [CC18, Lemma 3.3.2] and the connectedness of $G_{\ell,?}$. To the reader convenience, we briefly recall the argument.

Since $G_{\ell,?}$ is connected, one has $V_\ell^{\Pi_{\ell,?}} = V_\ell^U$ and $T_\ell^{\Pi_{\ell,?}} = T_\ell^U$. The exact sequence

$$0 \to T_\ell \to V_\ell \to M_\ell \to 0$$

induces a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & V_\ell^{\Pi_{\ell,?}}/T_\ell^{\Pi_{\ell,?}} \arrow[r] \arrow[d, equal] & M_\ell^{\Pi_{\ell,?}} \arrow[r] \arrow[d, hook] & H^1(\Pi_{\ell,?}, T_\ell) \arrow[d] \\
0 \arrow[r] & V_\ell^U/T_\ell^U \arrow[r] & M_\ell^U \arrow[r, "\Delta"] & H^1(U, T_\ell)
\end{tikzcd}$$

So $M_\ell^U/M_\ell^{\Pi_{\ell,?}}$ is a quotient of the image of Δ. But Δ has finite image since M_ℓ^U is torsion and $H^1(U, T_\ell)$ is a finitely generated $\mathbb{Z}_\ell$-module by [Ser64, Proposition 9].

8.2.2.3 Proof of Claim 2

For any finite field extension $k_? \subseteq k'$, consider the images $\Pi_{k'} \subseteq \Pi_?$ of $\pi_1(k') \subseteq \pi_1(k_?)$ acting on $\overline{T} := \prod_{\ell \neq p} \overline{T}_\ell$. By definition of a compatible system, there exists a smooth geometrically connected $\mathbb{F}_q$-variety $\mathcal{K}$ with generic point $Spec(k) \twoheadrightarrow \mathcal{K}$ such that, for every prime $\ell \neq p$, the action of $\pi_1(k)$ on T_ℓ factors trough the canonical surjection $\pi_1(k) \to \pi_1(\mathcal{K})$. By the Grothendieck-Ogg-Shafarevich formula, there exists a connected étale cover $\mathcal{K}' \to \mathcal{K}$ such that the action of $\pi_1(\mathcal{K}') \subseteq \pi_1(\mathcal{K})$ on $\overline{T}$ factors trough the tame fundamental group of $\mathcal{K}'$; see the proof of [Cad19a, Lemma 12.3.1]. Since the tame fundamental groups of $\mathcal{K}'$ and of every connected component of $\mathcal{K}'_{\mathbb{F}}$ are topologically finitely generated, this implies that $\Pi_?$ is topologically finitely generated. Hence the group $\Pi_?$ has finitely many open subgroups of index $\leq d$. So there are only finitely many possibilities for the inclusions $\Pi_{k'} \subseteq \Pi_?$, while $k_? \subseteq k'$ is varying among the finite field extensions of degree $\leq d$. So, to prove Claim 2, it is enough to show[1] that, for every finite field extension $k_? \subseteq k'$ of degree $\leq d$, there exists an integer $D \geq 1$, depending only on $\underline{T}$ and k', such that for $\ell \geq D'$ one has $\overline{T}_\ell^{\pi_1(k')} = \overline{T}_\ell^{\pi_1(k_?)}$.

[1]This is not a formal consequence of $[\pi_1(k_?) : \pi_1(k')]$ being finite, as the example $\{1\} \subseteq \{1, -1\} \subseteq \prod_{\ell \neq p} \mathrm{GL}(\overline{T}_\ell)$ shows.

Let $\mathcal{L}$ be the set of prime $\neq p$ and write $F := \prod_{\ell\in\mathcal{L}} \mathbb{F}_\ell$. We use the formalism of ultrafilters[2] on $\mathcal{L}$; see [CHT17, Appendix]. To every ultrafilter $\mathfrak{u}$ on $\mathcal{L}$ one associates a maximal ideal $\mathfrak{m}_\mathfrak{u}$ of F and writes $F_\mathfrak{u} := F/\mathfrak{m}_\mathfrak{u}$ for the characteristic zero residue field. The actions of $\pi_1(k_?)$ and $\pi_1(k')$ on $\overline{T}$ induces actions on $T_\mathfrak{u} := T \otimes_F F_\mathfrak{u}$. Since $\Pi_?$ and $\Pi_{k'}$ are topologically finitely generated groups, by [CHT17, Lemma 4.3.3] and [CHT17, Lemma 4.4.2] it is enough to show that $T_\mathfrak{u}^{\pi_1(k_?)} = T_\mathfrak{u}^{\pi_1(k')}$ for every ultrafilter $\mathfrak{u}$. Write $G_{\mathfrak{u},?}$ and $G_{\mathfrak{u},k'}$ for the Zariski closures of the images of $\pi_1(k_?)$ and $\pi_1(k')$ acting on $T_\mathfrak{u}$. Since $T_\mathfrak{u}^{\pi_1(k_?)} = T_\mathfrak{u}^{G_{\mathfrak{u},?}}$ and $T_\mathfrak{u}^{G_{\mathfrak{u},k'}} = T_\mathfrak{u}^{\pi_1(k')}$, it is enough to show that the natural inclusion $G_{\mathfrak{u},k'} \subseteq G_{\mathfrak{u},?}$ is an equality. Since $\pi_1(k') \subseteq \pi_1(k_?)$ has finite index, one has $G^0_{\mathfrak{u},k'} = G^0_{\mathfrak{u},?}$ hence it is enough to show that $G_{\mathfrak{u},?}$ is connected. This follows from the fact that G_ℓ is connected by preliminary reduction and Fact 8.2.2.3.1 below.

Fact 8.2.2.3.1. The group $G_{\ell,?}$ is connected if and only if $G_{\mathfrak{u},?}$ is connected.

Proof. If $? = \emptyset$ this is proved in [CHT17, Theorem 1.3.1] and if $? = \mathbb{F}$ this is proved in [Cad19a, Corollary 15.1.2]. □

8.2.2.4 End of the proof

To conclude the proof of Proposition 8.1.2.2.2, fix a finite field extension $k_? \subseteq k'$ of degree $\leq d$. Up to replacing d with $d!$ we may restrict to finite Galois extensions $k_? \subseteq k'$, so that $\pi_1(k') \subseteq \pi_1(k_?)$ is a normal subgroup. By Claim 1, it is enough to show that there exists an integer $A \geq 1$, depending only on $\underline{T}$ and d, such that for $\ell \geq A$ one has $M_\ell^{\pi_1(k_?)} = M_\ell^{\pi_1(k')}$ and, by Claim 2, there exists an integer $D \geq 1$, depending only on $\underline{T}$ and d, such that for $\ell \geq D$ one has $\overline{T}_\ell^{\pi_1(k_?)} = \overline{T}_\ell^{\pi_1(k')}$. We claim that $A := \max(D, d+1)$ has the desired property.

Since $M_\ell = \varinjlim_n M_\ell[\ell^n]$, it is enough to show that for $\ell \geq A$ and every $n \geq 1$ one has $M_\ell[\ell^n]^{\pi_1(k_?)} = M_\ell[\ell^n]^{\pi_1(k')}$. For this, one argues by induction on n, the case $n = 1$ being the definition of D. For $n > 1$, since T_ℓ is torsion free, there is a $\pi_1(k_?)$-invariant identification $M_\ell[\ell^n] \simeq T_\ell/\ell^n$ and a $\pi_1(k_?)$-equivariant exact sequence

$$0 \to \overline{T}_\ell \to T_\ell/\ell^n \to T_\ell/\ell^{n-1} \to 0.$$

Combined with the inflation-restriction exact sequence for the normal inclusion $\pi_1(k') \subseteq \pi_1(k_?)$, this induces a commutative exact diagram

$$\begin{tikzcd}
 & & & H^1(\pi_1(k_?)/\pi_1(k'), \overline{T}_\ell^{\pi_1(k')}) \arrow[d] \\
0 \arrow[r] & \overline{T}_\ell^{\pi_1(k_?)} \arrow[r] \arrow[d, "\simeq"] & (T_\ell/\ell^n)^{\pi_1(k_?)} \arrow[r] \arrow[d, hook] & (T_\ell/\ell^{n-1})^{\pi_1(k_?)} \arrow[r] \arrow[d, "\simeq"] & H^1(\pi_1(k_?), \overline{T}_\ell) \arrow[d] \\
0 \arrow[r] & \overline{T}_\ell^{\pi_1(k')} \arrow[r] & (T_\ell/\ell^n)^{\pi_1(k')} \arrow[r] & (T_\ell/\ell^{n-1})^{\pi_1(k')} \arrow[r] & H^1(\pi_1(k'), \overline{T}_\ell).
\end{tikzcd}$$

By the induction hypothesis the first and the third vertical arrows are isomorphisms for $\ell \geq A$. By elementary diagram chasing it is enough to show that $H^1(\pi_1(k_?)/\pi_1(k'), \overline{T}_\ell^{\pi_1(k')}) = 0$. But since $\overline{T}_\ell^{\pi_1(k_?)} = \overline{T}_\ell^{\pi_1(k')}$ one has

$$H^1(\pi_1(k_?)/\pi_1(k'), \overline{T}_\ell^{\pi_1(k')}) = H^1(\pi_1(k_?)/\pi_1(k'), \overline{T}_\ell^{\pi_1(k_?)}) = Hom(\pi_1(k_?)/\pi_1(k'), (\mathbb{Z}/\ell)^r) = 0$$

where the last equality follows from the fact that $\ell > d = |\pi_1(k_?)/\pi_1(k')|$.

[2] In this note an ultrafilter will always mean a non-principal ultrafilter.

8.3 Proof of Theorem 8.1.1.2.1

8.3.1 Proof of Theorem 8.1.1.2.1

Retain the notation and the assumption of Proposition 8.1.1.2.1. For every finite field extension $k \subseteq k'$ and every $(\overline{k}/k')$-form Y of X, write $Y_{\overline{k}} := Y \times_{k'} \overline{k}$ and

$$T_\ell(Y) := \varprojlim_n \mathrm{Br}(Y_{\overline{k}})[\ell^n]; \quad M_\ell(Y) := T_\ell(Y) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell; \quad \underline{M}(Y) := \prod_{\ell \neq p} M_\ell(\mathrm{Br}(Y_{\overline{k}}));$$
$$H^2_\ell(Y) := H^2(Y_{\overline{k}}, \mathbb{Z}_\ell(1)); \quad \underline{H}^i(Y) := \{H^2_\ell(Y)\}.$$

8.3.1.1 Reducing to the Tate module of the Brauer group

Recall (see e.g. the proof of [CC18, Proposition 2.1.1]) that there is a $\pi_1(k')$-equivariant exact sequence

$$0 \to M_\ell(\mathrm{Br}(Y_{\overline{k}})) \to \mathrm{Br}(Y_{\overline{k}})[\ell^\infty] \to H^3(Y_{\overline{k}}, \mathbb{Z}_\ell(1))[\ell^\infty] \to 0.$$

Since

- for every prime $\ell \neq p$, the group $H^3(Y_{\overline{k}}, \mathbb{Z}_\ell(1))[\ell^\infty] = H^3(X_{\overline{k}}, \mathbb{Z}_\ell(1))[\ell^\infty]$ is finite (of cardinality depending only on X) and
- for $\ell \gg 0$ (depending only on X) one has $H^3(Y_{\overline{k}}, \mathbb{Z}_\ell(1))[\ell^\infty] = H^3(X_{\overline{k}}, \mathbb{Z}_\ell(1))[\ell^\infty] = 0$ ([Gab83]);

it is enough to prove Theorem 8.1.1.2.1 replacing $\mathrm{Br}(Y_{\overline{k}})[p']$ with $\underline{M}(Y)$.

8.3.1.2 Compatibility

We now prove that $\underline{T}(Y)$ is a compatible system of $\pi_1(k')$-modules. Write $\mathrm{NS}(Y_{\overline{k}})$ for the Néron-Severi group of $Y_{\overline{k}}$. By the Kummer exact sequence

$$0 \to \mathrm{NS}(Y_{\overline{k}}) \otimes \mathbb{Z}_\ell \to H^2_\ell(Y) \to T_\ell(Y) \to 0,$$

it is enough to show that $\underline{H}^2$ and $\underline{\mathrm{NS}}(Y) := \{\mathrm{NS}(Y_{\overline{k}}) \otimes \mathbb{Z}_\ell\}_{\ell \neq p}$ are compatible systems of $\pi_1(k')$-modules. Write $\mathbb{F}_{q'}$ for the algebraic closure of $\mathbb{F}_q$ in k'. By spreading out, there exists a geometrically connected smooth $\mathbb{F}_{q'}$-variety $\mathcal{K}'$, with generic point $\eta' : Spec(k') \to \mathcal{K}'$, and a smooth proper morphism $\mathfrak{f} : \mathcal{Y} \to \mathcal{K}'$ fitting into a commutative cartesian diagram:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y} \arrow[d, "\mathfrak{f}"] \\ Spec(k') \arrow[r, "\eta'"] & \mathcal{K}'. \end{tikzcd}$$

By smooth proper base change, the action of $\pi_1(k')$ on $H^2_\ell(Y)$ factors trough the surjection $\pi_1(k') \twoheadrightarrow \pi_1(\mathcal{K}')$ and by [Del80] the collection $\underline{H}^2(Y)$ is a $\mathbb{Q}$-rational compatible system. Since homological and algebraic equivalences coincide rationally for divisors, $\mathrm{NS}(Y_{\overline{k}}) \otimes \mathbb{Q}$ identifies with the image of the cycle class map $c_{Y_{\overline{k}}} : Pic(Y_{\overline{k}}) \otimes \mathbb{Q} \to H^2_\ell(Y) \otimes \mathbb{Q}_\ell$. So $\underline{\mathrm{NS}}(Y)$ is a compatible system of $\pi_1(k')$-modules, hence $\underline{T}(Y)$ is a compatible system of $\pi_1(k')$-modules as well.

8.3.1.3 End of the proof

So we can apply Propositions 8.1.2.2.1 and 8.1.2.2.2 to $\underline{T}(Y)$. Hence, to conclude the proof, we have just to adjust the arguments in [OS18, Section 5.4], replacing [OS18, Propositions 5.4 and 5.5] with Propositions 8.1.2.2.1 and 8.1.2.2.2. Write $r := Rank_{\mathbb{Z}_\ell}(T_\ell(X))^2$ and set $B_X := |\mathrm{GL}_r(\mathbb{Z}/\ell_0)|$. By Proposition 8.1.2.2.1 for $X_{k'}$ there exists a finite field extension $k' \subseteq k_Y$ of degree $\leq B_X$ such that there is an $\pi_1(k_Y)$-equivariant isomorphism $\underline{M}(Y) \simeq \underline{M}(X)$. Then one has:

$$\underline{M}(X)^{\pi_1(k)} \subseteq \underline{M}(X)^{\pi_1(k_Y)} \simeq \underline{M}(Y)^{\pi_1(k_Y)} \supseteq \underline{M}(Y)^{\pi_1(k')}.$$

Since $T(X, \ell)$ holds for every prime $\ell \neq p$, by Fact 8.1.1.1.2 the group $\underline{M}(X)^{\pi_1(k_Y)}$ is finite. Hence it is enough to show that, for every integer $d \geq 1$, there exists an integer $C \geq 1$, depending only on X and d, such that for every finite field extension $k \subseteq k''$ of degree $\leq d$ one has $\underline{M}(X)^{\pi_1(k'')} \leq C$. To prove this, one may replace k with a finite extension and then apply Proposition 8.1.2.2.2 to conclude.

8.3.2 Further remarks

Let k be an infinite finitely generated field of characteristic $p \geq 0$.

8.3.2.1 Torsion of abelian varieties

Let X be a k-abelian variety of dimension g. By the Lang-Néron theorem [LN59], the group $X(k')_{tors}$ is finite for every finite field extension $k \subseteq k'$ and, if X has no isotrivial geometric isogeny factors, then the same is true for every field extension of $k_{\mathbb{F}}$. One can use Theorems 8.1.2.2.1 and 8.1.2.2.2 with the techniques in Section 8.3.1 to prove uniform boundedness results for the torsion of the forms of abelian varieties. More precisely, one can prove that for every integer $d \geq 1$, (resp. if X has no isotrivial geometric isogeny factors) there exists an integer $C := C(X, d)$ such that $|Y(k')| \leq C$ for every finite extension of fields $k \subseteq k'$ (resp. $k_{\mathbb{F}} \subseteq k'$) of degree $\leq d$ and every k'-abelian variety Y that is a $(\overline{k}/k')$ form of X. We conclude pointing out that the statement for abelian varieties over k follows also from the Lang-Weil bound and the specialization theory for torsion of abelian varieties.

8.3.2.2 Abelian varieties with CM

Recall that a k-abelian variety X has complex multiplication (or CM for short) if the image of the representation $\pi_1(k) \to \mathrm{GL}(T_\ell(X))$ contains an abelian open subgroup. In characteristic zero, Orr-Skorobogatov ([OS18, Corollary C.2]) prove that there is an integer $C \geq 1$, depending only on d and g, such that $|\operatorname{Br}(X_{\overline{k}})^{\pi_1(k)}| \leq C$ for every g-dimensional abelian variety with CM defined over a number field k of degree $\leq d$. This result is a consequence of the characteristic zero analogue [OS18, Theorem 5.1] of Theorem 8.1.1.2.1 and of the fact ([OS18, Theorem A]) that there are only finitely many $\overline{\mathbb{Q}}$-isomorphism classes of g-dimensional abelian varieties with CM defined over a number field of degree $\leq d$. Unfortunately, as Akio Tamagawa pointed out to us, the positive characteristic analogue of [OS18, Theorem A] is false: if X is the product of $g > 1$ supersingular elliptic curves, the k-isogeny class of X contains infinitely many[3] k-abelian varieties that are not isomorphic over $\overline{k}$. So there is no hope to deduce directly from Theorem

[3]Indeed, there is an inclusion $\alpha_p^2 \subseteq X$. Since k is infinite, the set $I := Hom_k(\alpha_p, \alpha_p \times \alpha_p)/Aut_k(\alpha_p) \simeq \mathbb{P}^1(k)$ is infinite. For each $i \in I$ define $f_i : X \to X_i := X/i(\alpha_p)$. Assume by contradiction that the $X_{i,\overline{k}}$ fall into finitely isomorphism many classes. Then there exist i_0 and an infinite subset $J \subseteq I$ such that, for every $j \in J$, there is an isomorphism $g_j : X_{j,\overline{k}} \to X_{i_0,\overline{k}}$. Then, $g_j \circ f_j : X_{\overline{k}} \to X_{i_0,\overline{k}}$ is a map of degree p. Since there are only finitely many maps $X_{\overline{k}} \to X_{i_0,\overline{k}}$ of degree p , there exists an infinite subset $J' \subseteq J$ such that $g_j \circ f_j = g_{j'} \circ f_{j'}$ for every $j, j' \in J'$. But this implies $j(\alpha_p) = j'(\alpha_p)$ and this is a contradiction.

8.1.1.2.1 the analogue of [OS18, Corollary C.2] in positive characteristic. However, a positive characteristic version of [OS18, Corollary C.2], via different techniques, has been announced by Marco D'Addezio.

Appendix A

Results on gonality

A.1 Introduction

In this chapter k is a field of characteristic $p > 0$ with algebraic closure $k \subseteq \overline{k}$. For a k-variety X, write $|X|$ and $X(k)$ for the set of closed and k-rational points, respectively.

A.1.1 Abstract modular schemes

Let $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ be a continuous representation of the étale fundamental group[1] of X. By functoriality of the étale fundamental group, every $x \in |X|$ induces a continuous group homomorphism $\pi_1(x) \to \pi_1(X)$, hence a "local" Galois[2] representation $\rho_x : \pi_1(x) \to \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$. Set

$$\Pi = \rho(\pi_1(X)) \quad \Pi_{\overline{k}} = \rho(\pi_1(X_{\overline{k}})) \quad \Pi_x = \rho_x(\pi_1(x)).$$

For every open subgroup $U \subseteq \Pi$ write $f_U : X_U \to X$ for the connected étale cover corresponding to the open subgroup $\rho^{-1}(U) \subseteq \pi_1(X)$ and k_U for the smallest separable field extension of k over which X_U is geometrically connected. Set $U_{\overline{k}} = U \cap \Pi_{\overline{k}}$.

Fact A.1.1.1. For every open subgroup $U \subseteq \Pi$ the following hold:

1. For every $x \in |X|$, we have that $\Pi_x \subseteq U$ if and only if x lifts to a $k(x)$-rational point on X_U;
2. The cover $X_{U_{\overline{k}}} \to X_{\overline{k}}$ corresponding to the open subgroup $U_{\overline{k}} \subseteq \Pi_{\overline{k}}$ is $X \times_{k_U} \overline{k} \to X_{\overline{k}}$.

In view of Fact A.1.1.1, we call X_U the connected abstract modular scheme associated to U.

A.1.2 Genus and gonality

Assume from now on that X is a curve. Write g_U and γ_U for the genus and the gonality[3] of the smooth compactification of $X_{U_{\overline{k}}}$. The representation $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ is said to be GLP (geometrically Lie perfect) if every open subgroup of $\Pi_{\overline{k}}$ has finite abelianization. Write $\Pi_{\overline{k}}(n) = Ker(\Pi_{\overline{k}} \to \mathrm{GL}_r(\mathbb{Z}_\ell/\ell^n))$. From Theorem 3.1.4.2.2 in Chapter 3 one has the following.

[1] As the choice of fibre functors will play no part in the following we will omit them for the notation for the étale fundamental group.

[2] Recall that $\pi_1(x) \simeq \pi(Spec(k(x))$ identifies with the absolute Galois group of $k(x)$.

[3] Recall that the gonality of smooth proper $\overline{k}$-curve Y is defined as the minimum of the degrees of a non constant morphism $Y \to \mathbb{P}^1_{\overline{k}}$

Fact A.1.2.1. Assume that ρ is GLP and $\ell \neq p$. Then for every closed subgroup $C \subseteq \Pi_{\overline{k}}$ of codimension ≥ 1 we have

$$\lim_{n \to +\infty} g_{C\Pi_{\overline{k}}(n)} = +\infty.$$

By the Riemann-Hurwitz formula one has (see [Poo06, Proposition 1.1(iv)])

$$g_U \geq \gamma_U - 1, \tag{A.1.2.2}$$

hence it is natural to wonder whether not only the genus but also the gonality of $X_{C\Pi_{\overline{k}}(n)}$ tends to infinity. The answer not only is yes, but one can use gonality to obtain fine results on the image of non necessarily GLP representations. The main result of this chapter is the following extension of [CT13, Theorem 3.3] to positive characteristic.

Theorem A.1.2.3. Let $C \subseteq \Pi_{\overline{k}}$ be a closed subgroup of of codimension j. The following hold:

1. If $\ell \neq p$, the representation ρ is GLP and $j \geq 1$, then
$$\lim_{n \to +\infty} \gamma_{C\Pi_{\overline{k}}(n)} = +\infty;$$
2. If $\ell \neq p$ and $j \geq 3$, then
$$\lim_{n \to +\infty} \gamma_{C\Pi_{\overline{k}}(n)} = +\infty;$$
3. If $\ell = p$ and $j \geq 2$, then
$$\lim_{n \to +\infty} \gamma_{C\Pi_{\overline{k}}(n)} = +\infty.$$

Remark A.1.2.4. By (A.1.2.2), Theorem A.1.2.3 (2)-(3) implies that $g_{C\Pi_{\overline{k}}(n)}$ tends to infinity. It is not clear to us if it is possible to prove this directly. Note that, a posteriori, Theorem A.1.2.3 (1) implies Fact A.1.2.1, but, actually, Fact A.1.2.1 is used in the proof of Theorem A.1.2.3(1) (see right after Fact A.2.2.3.1).

A.1.3 Exceptional loci

Assume from now on that k is finitely generated and X is a curve. The main motivation for proving Fact A.1.2.1 was to show (Theorem 3.1.3.2) that if $\ell \neq p$ and ρ is GLP, then for all but at most finitely many $x \in X(k)$, the closed subgroup $\Pi_x \subseteq \Pi$ is open and there exists an integer $N \geq 1$, depending only on ρ, such that for all such x one has $[\Pi : \Pi_x] \leq N$. This was a consequence of Fact A.1.2.1 and the following result of Samuel ([Sam66]) completed by an argument of Voloch (see [EElsHKo09, Theorem 3] for more details).

Fact A.1.3.1. Assume that k is finitely generated of positive characteristic. There exists an integer $g \geq 2$, depending only on k, such that for every smooth proper k-curve Y with genus $\geq g$, the set $Y(k)$ is finite.

Using Remark A.1.2.4, from Theorem A.1.2.3 we get the following, which extends [CT13, Theorem 1.3] to positive characteristic.

Corollary A.1.3.2. Assume that X is a curve and k is finitely generated. The following hold:

1. If $\ell = p$, then for all but at most finitely many $x \in X(k)$, $\Pi_x \subseteq \Pi$ has codimension ≤ 1;
2. If $\ell \neq p$, then for all but at most finitely many $x \in X(k)$, $\Pi_x \subseteq \Pi$ has codimension ≤ 2.

A.1.4 Strategy

The general strategy is similar to the one of [CT13], but the technical details are more involved, due to pathological phenomena arising from specific features of the geometry of curves in positive characteristic. In particular, the major problems to overcome are the following:

1. Morphisms between smooth proper curves are not necessarely separable (see for example Section A.2.2.3(3) and Lemma A.2.3.2.3);

2. The kernel of a morphism between abelian varieties is not necessarely reduced (see Lemma A.2.3.2.5).

A.1.5 Organization of the chapter

In Section A.2 we construct auxiliary systems of curves of genus ≤ 1 and we use this prove Theorem A.1.2.3. In Section A.3 recall the construction of a projective system of abstract modular schemes $\mathcal{X}_n \to X$, parametrizing points with small image and some facts about them. After this, we prove Corollary A.1.3.2. All the results and the proofs in this paper work in the characteristic zero setting but, since this situation is already treated in [CT13], we will assume that $p > 0$ to simplify the exposition.

A.2 Proof of Theorem A.1.2.3

This section is devoted to the proof of Theorem A.1.2.3, following the strategy of [CT13] which we first recall.

A.2.1 Strategy

Assume k algebraically closed. We start with a projective system of smooth proper curves:

$$\dots . \longrightarrow Y_N \xrightarrow{\pi_N} Y_{N-1} \xrightarrow{\pi_{N-1}} \dots . \qquad Y_1 \xrightarrow{\pi_1} Y_0$$

such that $Y_n \to Y_{n-1}$ is a (possibly ramified) Galois cover with group G_n cyclic of prime-to-p order (to simplify). Assume that γ_{Y_n} is bounded when n goes to infinity. Then we construct (Proposition A.2.1.4) a commutative and cartesian diagram of smooth proper curves

$$\begin{tikzcd}
\dots \arrow[r] & Y_n \arrow[r, "\pi_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\widetilde{\square}"] & Y_{n-1} \arrow[r, "\pi_{n-1}"] \arrow[d, "f_{n-1}"] & \dots \arrow[r, "\pi_{N-1}"] & Y_N \arrow[r, "\pi_N"] \arrow[d, "f_N"] & Y_{N-1} \arrow[r, "\pi_{N-1}"] & \dots \\
\dots \arrow[r] & B_n \arrow[r, "\pi'_n"] & B_{n-1} \arrow[r, "\pi'_{n-1}"] & \dots \arrow[r, "\pi'_{N-1}"] & B_N & &
\end{tikzcd}$$

where each B_n has genus ≤ 1.

We apply this construction to a projective system

$$\dots . \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \dots . \qquad Y_1 \longrightarrow Y_0$$

closely related to

$$\dots . \longrightarrow X_{C\Pi_{\overline{k}}(n)} \longrightarrow X_{C\Pi_{\overline{k}}(n-1)} \longrightarrow \dots . \qquad X_{C\Pi_{\overline{k}}(1)} \longrightarrow X_{C\Pi_{\overline{k}}(0)} = X \tag{A.2.1.1}$$

attached to a (GLP) representation $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ (in fact Y_0 is the smooth compactification of X). In that case (assuming moreover $N = 0$ to simplify), the auxiliary projective system

$$
\ldots. \longrightarrow B_N \longrightarrow B_{N-1} \longrightarrow \ldots. \qquad B_1 \longrightarrow B_0
$$

is closely related to a projective system of the forms (A.2.1.1) but for the induced representation $Ind_{\pi_1(X)}^{\pi_1(B)}(\rho)$, where B is an open curve in B_0 and $X \subseteq Y_0$ maps to B. The contradiction then arises from the constraints imposed on the groups G_n (automorphism groups of genus ≤ 1 curves) or on the curves B_n ($g_{B_n} \leq 1$) by the Fact that γ_{Y_n} are bounded. For instance if ρ is GLP, one can always assume that $Ind_{\pi_1(X)}^{\pi_1(B)}(\rho)$ is also GLP (Fact A.2.2.3.1), hence $g_{B_n} \leq 1$ contradicts Fact A.1.2.1.

For the construction of the projective system we use the method of E-P decomposition introduced in [CT13, Section 2]. This method allows us to construct for $n \gg 0$ cartesian diagrams:

$$
\begin{tikzcd}
Y_n \arrow[r, "\pi_n"] \arrow[dd, "f_n"] \arrow[ddr, phantom, "\widetilde{\square}"] & Y_{n-1} \arrow[r, "\pi_{n-1}"] \arrow[dd, "f_{n-1}"] & \ldots \arrow[r, "\pi_{N_0+1}"] & Y_{N_0} \arrow[r, "\pi_{N_0}"] \arrow[d] \arrow[dd, bend right, "f_{N_0}"'] \arrow[dr, phantom, "\widetilde{\square}"] & Y_{N_0-1} \arrow[r] \arrow[d, "f_{N_0-1}"] & \ldots. \arrow[r, "\pi_{N_1+1}"] & Y_{N_1} \arrow[r, "\pi_N"] \arrow[d, "f_{N_1}"] & Y_{N_1-1} \arrow[r, "\pi_{N_1-1}"] & \ldots \\
 & & & C_{N_0} \arrow[r, "\pi'_{N_0}"] \arrow[d] & B_{N_0-1} \arrow[r, "\pi'_{N_0-1}"] & \ldots. \arrow[r, "\pi'_{N_1+1}"] & B_{N_1} & & \\
B_n \arrow[r, "\pi'_n"] & B_{n-1} \arrow[r, "\pi'_{n-1}"] & \ldots \arrow[r, "\pi'_{N_0+1}"] & B_{N_0} & & & & &
\end{tikzcd}
$$

with the desired properties for some $N_0, N_1 \geq 0$. So for each $n \gg 0$ the set $\mathcal{F}_n$ of these diagrams is not empty (Proposition A.2.3.2.2). Furthermore, deleting the last arrow we get maps $\mathcal{F}_n \to \mathcal{F}_{n-1}$, so we endow the collection $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ with the structure of a projective system. To obtain the result about the existence of a "limit" diagram we have to show (Proposition A.2.1.4) that, for each n, $\mathcal{F}_n$ is finite. This requires some extra technical conditions and that G_n is cyclic of prime-to-p order.

To state more precisely the results of Section A.2.3, let us recall the following definition.

Definition A.2.1.2. A finite group G is said to be k-*exceptional* if it appears as Galois group of a Galois cover of smooth proper k-curves $X \to Y$ with $g_X = g_Y \leq 1$.

Then, in Section A.2.3 we prove:

Proposition A.2.1.3. Assume that

$$
\ldots \to Y_n \to Y_{n-1} \to \ldots \to Y_1 \to Y_0
$$

is a projective system of non constant morphisms between smooth proper k-curves such that $Y_n \to Y_{n-1}$ is a (possibly ramified) Galois cover with group G_n. Assume that

$$
\lim_{n \to +\infty} \gamma_{Y_n} = \gamma
$$

is finite. Then all but finitely many G_n are k-exceptional.

Proposition A.2.1.4. Assume furthermore that G_n is cyclic of a fixed prime-to-p order ≥ 3, for all but finitely many n. Then there exists an $N \geq 0$ such that we can construct a diagram

$$
\begin{tikzcd}
\ldots \arrow[r] & Y_n \arrow[r, "\pi_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\widetilde{\square}"] & Y_{n-1} \arrow[r, "\pi_{n-1}"] \arrow[d, "f_{n-1}"] & \ldots \arrow[r, "\pi_{N-1}"] & Y_N \arrow[r, "\pi_N"] \arrow[d, "f_N"] & Y_{N-1} \arrow[r, "\pi_{N-1}"] & \ldots \\
\ldots \arrow[r] & B_n \arrow[r, "\pi'_n"] & B_{n-1} \arrow[r, "\pi'_{n-1}"] & \ldots \arrow[r, "\pi'_{N-1}"] & B_N & &
\end{tikzcd}
$$

with the following properties:

1. $\pi'_n : B_n \to B_{n-1}$ is a (possibly ramified) Galois cover of smooth proper curves with group G_n;
2. One of the following holds:
   (a) $g_{B_n} = 0$ and $deg(f_n) = \gamma$,
   (b) $g_{B_n} = 1$ and $deg(f_n) = \frac{\gamma}{2}$;
3. the square

$$\begin{tikzcd} Y_n \arrow[r, "\pi_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\widetilde{\square}"] & Y_{n-1} \arrow[d, "f_{n-1}"] \\ B_n \arrow[r, "\pi'_n"] & B_{n-1} \end{tikzcd}$$

   is cartesian up to normalization.

A.2.2 Proof of Theorem A.1.2.3 assuming Propositions A.2.1.3 and A.2.1.4

Retain the notation and the assumptions of Theorem A.1.2.3. By Fact A.1.1.1(2), to prove Theorem A.1.2.3 we can assume k is algebraically closed, hence that $\Pi = \Pi_{\overline{k}}$.

A.2.2.1 Preliminary reduction

By [Poo06, Proposition 1.1(vi-vii)], for every non constant morphism $Z \to Y$ of degree d between smooth proper k-curves one has

$$\gamma_Y \leq \gamma_Z \leq d\gamma_Y. \tag{A.2.2.1.1}$$

So, if

$$\begin{tikzcd} \dots \arrow[r] & X'_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\widetilde{\square}"] & X'_{n-1} \arrow[r] \arrow[d] & \dots \arrow[r] & X'_0 \arrow[d] \\ \dots \arrow[r] & X_n \arrow[r] & X_{n-1} \arrow[r] & \dots \arrow[r] & X_0 \end{tikzcd}$$

is a cartesian diagram of smooth proper curves, then γ_{X_n} tends to infinity if and only if $\gamma_{X'_n}$ tends to infinity. Hence to prove to prove Theorem A.1.2.3, we can freely replace replace X with a finite étale cover.

Fix an integer $n_0 \geq 2$. Replacing X with a finite étale cover, we can assume $\Pi = \Pi(n_0)$. By [CT13, Lemma 3.5] and replacing X by $X_{\Pi(n)}$ for some $n \gg 0$ we can assume $C\Pi(n + n_0)$ is normal in $C\Pi(n)$ and $C\Pi(n+n_0)/C\Pi(n) \simeq (\mathbb{Z}/\ell^{n_0})^j$, where j is the codimension of C in Π. By (A.2.2.1.1) it is enough to prove that

$$\lim_{n\to+\infty} \gamma_{C\Pi(nn_0)} = +\infty,$$

so that we have to show that the gonality is not bounded in the tower of covers

$$\dots X_{n+1} := X_{C\Pi((nn_0+n_0)} \overset{(\mathbb{Z}/\ell^{n_0})^j}{\frown} \to X_n := X_{C\Pi(nn_0)} \dots \to X_3 := X_{C\Pi(3n_0)} \overset{(\mathbb{Z}/\ell^{n_0})^j}{\frown} \to X_2 := X_{C\Pi(2n_0)} \overset{(\mathbb{Z}/\ell^{n_0})^j}{\frown} \to X_1 = X$$

where $X_{C\Pi(nn_0+n_0)} \to X_{C\Pi(nn_0)}$ is a Galois cover with group $(\mathbb{Z}/\ell^{n_0})^j$. Assume by contradiction that

$$\lim_{n\to+\infty} \gamma_{X_n} = \gamma_0 < +\infty.$$

Then, upon replacing X with $X_{C\Pi(n)}$ for some $n \gg 0$, we can assume that $\gamma_{X_n} = \gamma_0$ for all $n \gg 0$. By Proposition A.2.1.3 almost all the Galois groups $X_{C\Pi(n_0+n)} \to X_{C\Pi(n)}$ are k-exceptional.

A.2.2.2 Proof of Theorem A.1.2.3 (2) and (3)

Assume first that $j \geq 3$ if $\ell \neq p$ and $j = 2$ if $\ell = p$. To obtain a contradiction, we use the following:

Fact A.2.2.2.1. [Cad12a, Corollary 10] Suppose that k is an algebraically closed field of characteristic $p > 0$. A finite subgroup of $\mathrm{PGL}_2(k)$ is isomorphic to one of the following groups:

- A cyclic group;
- A dihedral group D_{2m} of order $2m$, for some $m > 0$;
- A_4, A_5, S_4;
- An extension $1 \to A \to \Pi \to Q \to 1$, with A an elementary abelian p-group and Q a cyclic group of prime-to-p order;
- $\mathrm{PSL}_2(\mathbb{F}_{p^r})$, for some $r > 0$;
- $\mathrm{PGL}_2(\mathbb{F}_{p^r})$, for some $r > 0$;

where $\mathbb{F}_{p^r}$ denotes the finite field with p^r elements.

By Fact A.2.2.2.1, $(\mathbb{Z}/\ell^{n_0})^j$ does not appear as a Galois group of a cover of genus zero curves as soon as $j > 1$. So all the X_n must have genus 1. But then, since all finite morphisms between elliptic curves are unramified, $(\mathbb{Z}/\ell^{n_0})^j$ must be a quotient of the fundamental group $(\prod_{\ell\neq p} \mathbb{Z}_\ell^2) \times \mathbb{Z}_p$ of an elliptic curve and this is not possible by the choice of j.

A.2.2.3 Proof of Theorem A.1.2.3 (1)

Assume now that $\ell \neq p$, $j \geq 1$ and that ρ is GLP. The proof is similar to the proof contained in [CT13, Subsection 3.2.2]. The only difficulties come from inseparability phenomena. So we just give a sketch to show how to overcome these new problems.

1. Preliminary reduction.
   Since k is algebraically closed, $\Pi = \Pi_{\overline{k}}$, hence every open subgroup of Π has finite abelianization. Reasoning as in [CT13, Page 15] we can reduce to a situation in which $Lie(\Pi)$ has abelian solvable radical (this is used in the last step to apply Fact A.2.2.3.1). If $j \geq 3$, then Theorem A.1.2.3 (1) follows from Theorem A.1.2.3 (2) just proved. So we need to deal with $j = 2$ and $j = 1$.

2. $j = 2$.
   Assume first that $j = 2$. By Fact A.2.2.2.1, $(\mathbb{Z}/\ell^{n_0})^2$ does not appear as a Galois group of a cover of genus zero curves, hence $X_{C\Pi(n)}$ has genus 1 for $n \geq 1$. But then, since all finite morphisms between elliptic curves are étale, Galois with abelian Galois group, $C\Pi(n) \subseteq \Pi$ is normal and $\Pi/C\Pi(n)$ is abelian. By the exactness of inverse limit on profinite group

$$\Pi \twoheadrightarrow \varprojlim \Pi/C\Pi(n)$$

is an abelian quotient of Π. Since $C \subseteq \Pi$ is not an open subgroup, $\varprojlim \Pi/C\Pi(n)$ is infinite and this contradicts the fact that ρ is GLP. So, from now on we can assume that $j = 1$.

3. Use of Proposition A.2.1.4.
   By Proposition A.2.1.4 we can construct a cartesian diagram:

$$\begin{tikzcd}
\dots \arrow[r] & (X_n)^{cpt} \arrow[r, "\pi_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\widetilde{\square}"] & (X_{n-1})^{cpt} \arrow[r, "\pi_{n-1}"] \arrow[d, "f_{n-1}"] & \dots \arrow[r, "\pi_3"] & (X_2)^{cpt} \arrow[r, "\pi_2"] \arrow[d, "f_2"] \arrow[dr, phantom, "\widetilde{\square}"] & (X_1)^{cpt} = X^{cpt} \arrow[d, "f_1 = f"] \\
\dots \arrow[r] & B_n \arrow[r, "\pi'_n"] & B_{n-1} \arrow[r, "\pi'_{n-1}"] & \dots \arrow[r, "\pi'_2"] & B_2 \arrow[r, "\pi'_2"] & B_1 = B
\end{tikzcd}$$

   such that

   (a) $\pi'_n : B_n \to B_{n-1}$ is a (possibly ramified) Galois cover of smooth proper curves with group G_n
   (b) One of the following holds:
      i. $g_{B_n} = 0$ and $deg(f_n) = \gamma_0$
      ii. $g_{B_n} = 1$ and $deg(f_n) = \frac{\gamma_0}{2}$
   (c) the square

$$\begin{tikzcd}
X_n \arrow[r, "\pi_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\widetilde{\square}"] & X_{n-1} \arrow[d, "f_{n-1}"] \\
B_n \arrow[r, "\pi'_n"] & B_{n-1}
\end{tikzcd}$$

   is cartesian up to normalization.

   We will show that if ρ is GLP, then the genus of B_n should tend to infinity, contraddicting (b) above.

4. Reduction to the separable situation.
   There is a factorization $X^{cpt} \to (X^{cpt})' \to B$ with the first morphism purely inseparable and the second separable of degree $d \leq \gamma$. Write X' for the scheme theoretic image of X in $(X^{cpt})'$ and denote with X'_n the base change of X' along $B_n \to B$. Then we get another system:

$$\begin{tikzcd}
\dots \arrow[r] & (X'_n)^{cpt} \arrow[r, "\pi_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\widetilde{\square}"] & (X'_{n-1})^{cpt} \arrow[r, "\pi_{n-1}"] \arrow[d, "f_{n-1}"] & \dots \arrow[r, "\pi_3"] & (X'_2)^{cpt} \arrow[r, "\pi_2"] \arrow[d, "f_2"] \arrow[dr, phantom, "\widetilde{\square}"] & (X'_1)^{cpt} = (X^{cpt})' = (X')^{cpt} \arrow[d, "f_1 = f"] \\
\dots \arrow[r] & B_n \arrow[r, "\pi'_n"] & B_{n-1} \arrow[r, "\pi'_{n-1}"] & \dots \arrow[r, "\pi'_2"] & B_2 \arrow[r, "\pi'_2"] & B_1 = B
\end{tikzcd}$$

   such that the maps $(X'_n)^{cpt} \to B_n$ are all separable and of degree d. Since the map $X \to X'$ is a universal homeomorphism it induces an isomorphism $\pi_1(X) \to \pi_1(X')$ and so a Lie perfect representation ρ' of $\pi_1(X')$. So, to prove that g_{B_n} is not bounded, we can assume that the maps $(X_n)^{cpt} \to B_n$ are separable after replacing γ_0 with some integer $d \leq \gamma_0$

5. Reduction to the étale Galois situation.
   Since the maps are separable, the ramification locus S of $X^{cpt} \to B$ is finite, so we get an open curve $Y := B - (S \cup f(X^{cpt} - X))$. Writing Y_n for the base change of Y along the map $B_n \to B$ we get étale maps $X_n \to Y_n$ and $Y_n \to Y_{n-1}$. Then one reduces to the situation in which the morphisms $X_n \to Y_n$ are finite étale Galois after replacing γ_0 with some $d \leq \gamma_0!$.

6. Use of Fact A.1.2.1 and end of proof.
We use the following:

Fact A.2.2.3.1 ([CT13, Proof of Lemma 3.6])**.** Let $X \to Y$ a finite étale Galois cover of degree m between smooth curves over an algebraically closed field and $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ a GLP representation such that $Lie(\Pi)$ has abelian solvable radical. Then the induced representation $Ind_{\pi_1(Y)}^{\pi_1(X)}(\rho) : \pi_1(Y) \to \mathrm{GL}_{mr}(\mathbb{Z}_\ell)$ is GLP.

Since, by the preliminary reduction at the point 1 of Section A.2.2.3, the solvable radical of $Lie(\Pi)$ is abelian, by Fact A.2.2.3.1 the representation $\rho_0 := Ind_{\pi_1(Y)}^{\pi_1(X)}(\rho)$ is still GLP. Defining $U_i := \rho_0(\pi_1(Y_i))$ we can consider the system of covers $Y_{U_i} \to Y$. We prove, as in [CT13, Lemma 3.7], that $\cap_i U_i$ is closed of codimension ≥ 1. So by Fact A.1.2.1 we get that the genus of g_{U_i} tends to infinity. But we have inclusions $\pi_1(Y_i) \to \pi_1(Y_{U_i}) \to \pi_1(Y)$ and hence non constant morphisms $Y_i \to Y_{U_i} \to Y$. By assumption $g_{Y_i} \leq 1$ and this is a contradiction.

Corollary A.2.2.3.2. Let k be an algebraically closed field of characteristic p, fix a prime $\ell \neq p$ and K/k a function field of transcendence degree 1. Assume that L/K is a Galois extension ramified only at finitely many places such that $\Pi := Gal(L|K)$ is an ℓ-adic Lie group with $Lie(\Pi)^{ab} = 0$. Then there exists only finitely many extensions $K \subseteq K' \subseteq L$ with bounded gonality.

Proof. See (the proof of) [CT12b, Corollary 3.9]. □

A.2.3 Construction of curves of low genus

The main results of this section are Propositions A.2.1.3 and A.2.1.4. They have been used in the proof of Theorem A.1.2.3. In this subsection k is an algebraically closed field of positive characteristic p.

A.2.3.1 E-P decomposition

We recall the technique of E-P decomposition from [CT13, Section 2]. Let

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi"] & B \\ Y' & \end{tikzcd}$$

be a diagram of non constant morphisms between smooth proper curves, where $\pi : Y \to Y'$ is a (possibly ramified) Galois cover with group G. We say that the diagram is G-*equivariant* if for any $\sigma \in G$ there exists $\sigma_B \in Aut_k(B)$ such that $f \circ \sigma = \sigma_B \circ f$. We say that the diagram is G-*primitive* if for any commutative diagram of smooth proper curves

$$\begin{tikzcd} Y \arrow[r, "f'"] \arrow[d, "\pi"] \arrow[rr, bend left, "f"] & B' \arrow[r, "j"] & B \\ Y' & & \end{tikzcd}$$

with $deg(f') \geq 2$ the diagram

$$\begin{tikzcd} Y \arrow[r, "f'"] \arrow[d, "\pi"] & B' \\ Y' & \end{tikzcd}$$

is not G-equivariant.

For any diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi"] & B \\ Y' & \end{tikzcd}$$

we can construct a decomposition

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\pi"] \arrow[rr, bend left, "f"] & C \arrow[r] & B \\ Y' & & \end{tikzcd}$$

with

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\pi"] & C \\ Y' & \end{tikzcd}$$

G-equivariant and $Y \to C$ of degree maximal with this property. Since, by definition, we have a morphism of group $G \to Aut_k(C)$ with kernel K, we can construct a *equivariant-primitive* (*E-P* for short) decomposition of f, i.e. a diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[rrr, bend left, "f"] & Y/K := Z \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & B \\ & Y' \arrow[r] & C/(G/K) := B' & \end{tikzcd}$$

with

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & C \\ Y' & \end{tikzcd} \qquad \text{G/K-equivariant and} \qquad \begin{tikzcd} C \arrow[r] \arrow[d] & B \\ B' & \end{tikzcd} \qquad \text{G/K-primitive.}$$

Now, if we have a commutative diagram of non constant morphisms of smooth proper k-curves

$$\begin{tikzcd} Y_N \arrow[r, "\pi_N"] \arrow[d] & Y_{N-1} \arrow[r, "\pi_{N-1}"] & \dots \qquad Y_1 \arrow[r, "\pi_1"] & Y_0 \\ B_n & & & \end{tikzcd}$$

with $\pi_n : Y_n \to Y_{n-1}$ a (possibly ramified) Galois cover with group G_n, we can apply the previous construction several times to obtain the following diagram :

$$\begin{tikzcd}
Y_N \arrow[r] \arrow[rrrrrrrrr, bend left, "f_N"] & Z_N \arrow[rrrrrrr] \arrow[d] & & & & & & & C_N \arrow[r] \arrow[d] & B_N \\
& Y_{N_1} \arrow[r] \arrow[rrrrrrr, bend left, "f_{N-1}"] & Z_{N-1} \arrow[rrrrr] \arrow[d] & & & & & C_{N-1} \arrow[r] \arrow[d] & B_{N-1} & \\
& & Y_{N-2} \arrow[rrrrr, bend left, "f_{N-2}"] & & \dots & \dots & & B_{N-2} & & \\
& & & & \dots & \dots & & & & \\
& & & Y_1 \arrow[r] \arrow[rrr, bend left, "f_1"] & Z_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & B_1 & & & \\
& & & & Y_0 \arrow[r, "f_0"] & B_0 & & & &
\end{tikzcd}$$

where

$$\begin{tikzcd}
Y_k \arrow[r] \arrow[rrr, bend left, "f_k"] & Z_k \arrow[r] \arrow[d] & C_k \arrow[r] \arrow[d] & B_k \\
& Y_{k-1} \arrow[r, "f_{k-1}"] & B_{k-1} &
\end{tikzcd}$$

is an E-P decomposition of f_k.

A.2.3.2 Construction

We start with a projective system of smooth proper curves of fixed gonality $\gamma = \gamma_{Y_n}$

$$.... \longrightarrow Y_N \xrightarrow{\pi_N} Y_{N-1} \xrightarrow{\pi_{N-1}} .... \qquad Y_1 \xrightarrow{\pi_1} Y_0$$

with $Y_n \to Y_{n-1}$ a (possibly ramified) Galois cover with group G_n. We want to construct a related system of smooth proper curves with bounded genus. The construction is similar to the one of [CT13, Section 2], with some complications arising from the existence of non separable morphisms of curves in positive characteristic. Write

$$v(\gamma) = \frac{log(\gamma\sqrt{2})}{log(\sqrt{\frac{3}{2}})}$$

and for every $n > v(\gamma)$ define $\mathcal{F}_n$ as the set of (isomorphism classes of) diagrams of non constant morphism of smooth proper curves

$$\begin{tikzcd}
Y_n \arrow[r, "\pi_n"] \arrow[dd, "f_n"] \arrow[rdd, phantom, "\widetilde{\square}"] & Y_{n-1} \arrow[r, "\pi_{n-1}"] \arrow[dd, "f_{n-1}"] & \ldots \arrow[r, "\pi_{N_0+1}"] & Y_{N_0} \arrow[r, "\pi_{N_0}"] \arrow[d] \arrow[dd, bend right, "f_{N_0}"'] \arrow[rd, phantom, "\widetilde{\square}"] & Y_{N_0-1} \arrow[r] \arrow[d, "f_{N_0-1}"] & .... \arrow[r, "\pi_{N_1+1}"] & Y_{N_1} \arrow[r, "\pi_N"] \arrow[d, "f_{N_1}"] & Y_{N_1-1} \arrow[r, "\pi_{N_1-1}"] & \ldots \\
& & & C_{N_0} \arrow[r, "\pi'_{N_0}"] \arrow[d] & B_{N_0-1} \arrow[r, "\pi'_{N_0-1}"] & .... \arrow[r, "\pi'_{N_1+1}"] & B_{N_1} & & \\
B_n \arrow[r, "\pi'_n"] & B_{n-1} \arrow[r, "\pi'_{n-1}"] & \ldots \arrow[r, "\pi'_{N_0+1}"] & B_{N_0} & & & & &
\end{tikzcd}$$

that satisfy the following properties:

1. $0 \leq N_1 \leq N_0 \leq n$.
2. $N_1 \leq v(\gamma)$.
3. $g_{B_k} = 1$, $deg(f_k) = \frac{\gamma}{2}$ for $N_1 \leq k < N_0$.
4. $g_{C_{N_0}} = 1$ if $N_1 < n$.
5. $g_{B_k} = 0$, $deg(f_k) = \gamma$ for $N_0 \leq k \leq n$.
6. the square

$$\begin{tikzcd}
Y_k \arrow[r, "\pi_k"] \arrow[d, "f_k"] \arrow[rd, phantom, "\widetilde{\square}"] & Y_{k-1} \arrow[d, "f_{k-1}"] \\
B_k \arrow[r, "\pi'_k"] & B_{k-1}
\end{tikzcd}$$

is cartesian up to normalization and $G_k = Aut(Y_k \to Y_{k-1})$-equivariant for $N_1 < k \leq N_0$.

7. the square

$$\begin{tikzcd} Y_{N_0} \arrow[r, "\pi_{N_0}"] \arrow[d] \arrow[dr, phantom, "\widetilde{\square}"] & Y_{N_0-1} \arrow[d, "f_{N_0-1}"] \\ C_{N_0} \arrow[r, "\pi'_{N_0}"] & B_{N_0-1} \end{tikzcd}$$

is cartesian up to normalization and $G_{N_0} = Aut(Y_{N_0} \to Y_{N_0-1})$-equivariant if $N_0 > N_1$.

We can endow the family of $\mathcal{F}_n$ with the structure of a projective system $\mathcal{F}_n \to \mathcal{F}_{n-1}$ forgetting the last arrow on the left. The following lemma will be used several times:

Lemma A.2.3.2.1. Let $f : Y \to Z$ a prime degree cover of smooth proper curves over k with $g_Y \neq g_Z$. Then $Y \to Z$ is separable.

Proof. First observe that f is separable or purely inseparable, since the degree is prime. Assume by contradiction that f is purely inseparable. Then by [Liu06, Corollary 4.21, Chapter 7] we get $g_Y = g_Z$, contradicting the assumption. □

We first prove:

Proposition A.2.3.2.2. The set $\mathcal{F}_n$ is non empty.

Proof. The proof is the same of [CT13, Lemma 2.4] using the E-P decomposition and Lemma A.2.3.2.3 instead of [CT13, Lemma 2.5]. □

Lemma A.2.3.2.3. Let

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"] & B \\ B' & \end{tikzcd}$$

be a diagram of smooth proper k-curves with $deg(C \to B) = 2$ and $\pi : C \to B'$ a (possibly ramified) Galois cover with group G.

1. If $g_B = 0$ and $g_C \geq 2$ then the diagram is G-equivariant.

2. If $g_B = 0$ and $g_C = g_{B'} = 1$ then there exists a smooth proper k-curve B'' with $g_{B''} = 0$ and a cartesian square (up to normalization)

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\widetilde{\square}"] & B \arrow[d] \\ B' \arrow[r] & B'' \end{tikzcd}$$

Proof.

1. The morphism $C \to B$ is separable by Lemma A.2.3.2.1. Then the generator of the Galois group of $C \to B$ is an hyperelliptic involution and so we can apply [Liu06, Corollary 4.31, Chapter 7]

2. Again the morphism $C \to B$ is separable by Lemma A.2.3.2.1 and so we can write $B = C/ < i >$ where i an hyperelliptic involution. Then the proof goes exactly as in [CT13, Lemma 2.5 (2)].

□

Proof of Proposition A.2.1.3. For $n \geq v(\gamma)$ the group G_n is the Galois group of a Galois cover of curves of genus ≤ 1 by Proposition A.2.3.2.2. $\square$

To get Proposition A.2.1.4 we need the following two finiteness Lemmas:

Lemma A.2.3.2.4. Let c be a prime-to-p integer ≥ 3. Given a (possibly ramified) Galois cover $Y \to Z$ of smooth proper k-curves with Galois group $G \simeq \mathbb{Z}/c$ and such that $\gamma_Y = \gamma_Z = \gamma$, there are only finitely many isomorphism classes of G-equivariant cartesian diagrams:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\frac{\mathbb{Z}}{c\mathbb{Z}}"'] \arrow[dr, phantom, "\widetilde{\square}"] & \mathbb{P}^1 \arrow[d, "\frac{\mathbb{Z}}{c\mathbb{Z}}"] \\ Z \arrow[r, "g"] & \mathbb{P}^1 \end{tikzcd}$$

with $deg(f) = deg(g) = \gamma$

Proof. See (the proof of) [CT13, Lemma 2.6]. $\square$

Lemma A.2.3.2.5. Let Y be a smooth proper k-curve and let d be an integer ≥ 1. Denote by $E_{Y,d}$ the set of all pairs (E, f), where E is a smooth proper k-curve of genus 1 and $f : Y \to E$ is a non constant morphism of degree d, up to automorphisms of E. Then $E_{Y,d}$ is finite.

Proof. This is similar to [CT13, Lemma 2.7] but in positive characteristic there is the new problem that if $A \to B$ is a morphism of abelian varieties the kernel is not necessarily an abelian variety. We can assume that $g_Y \geq 1$. Fix $y \in Y(k)$ and consider the closed immersion $i : Y \hookrightarrow J_{Y|k}$ induced by y of Y into its Jacobian. The reasoning in [CT13, Lemma 2.7] shows that

(a) $E_{Y,d}$ is in bijection with the set $E_{J_{Y/k},d}$ of surjective morphisms $\phi : J_{Y|k} \to E$ where E is an elliptic curve such that the composition of $Y \hookrightarrow J_{Y|k} \to E$ is of degree d.

(b) It is enough to show that for every d the set $E^0_{J_{Y/k},d}$ of $\phi \in E_{J_{Y/k},d}$ such that $ker(\phi)$ is connected is finite.

(c) The subset $E^{red,0}_{J_{Y/k},d}$ of $\phi \in E^0_{J_{Y/k},d}$ such that $ker(\phi)$ is reduced is finite.

To conclude we construct a map

$$h_d : E^0_{J_{Y/k},d} \to \cup_{d' \leq d} E^{red,0}_{J_{Y/k},d'}$$

with finite fibres. For any $f \in E^0_{J_{Y/k},d}$ consider the following commutative exact diagram, where for every scheme K we denote with K_{red} the associated reduced subscheme:

$$\begin{tikzcd} 0 \arrow[r] & K_{red} \arrow[r] \arrow[d] & J_{Y|k} \arrow[r] \arrow[d, equal] & J_{Y|k}/K_{red} := E' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] & J_{Y|k} \arrow[r] & E \arrow[r] & 0. \end{tikzcd}$$

We have a factorization $Y \to E' \to E$, so that the degrees of $Y \to E'$ and of $E' \to E$ are bounded by d. So we can define h_d as

$$(J_{Y|k} \to E) \mapsto (J_{Y|k} \to E').$$

Given any $(\phi : J_{Y|k} \to E') \in \cup_{d' \leq d} E^{red,0}_{J_{Y/k},d'}$, the preimage $h_d^{-1}(J_{Y|k} \to E')$ in contained in the set of isogenies $E' \to E$ of degree $\leq d$, which is finite. So the conclusion follows from (c) above. $\square$

Proof of Proposition A.2.1.4. It is enough to show that $\varprojlim_n \mathcal{F}_n \neq \emptyset$. Since $\mathcal{F}_n \neq \emptyset$ by Proposition A.2.3.2.2 and a projective system of non empty finite sets has not empty projective limit, it is enough to show that, for each $n \gg 0$, the set $\mathcal{F}_n$ is finite. Fix such n, write

$$\mathcal{E}_n := \{f_n : Y_n \to B_n \text{ such that } f_n \text{ is contained in some diagram in } \mathcal{F}_n\}$$

and consider the obvious surjective map

$$\psi_n : \mathcal{F}_n \to \mathcal{E}_n.$$

From Lemma A.2.3.2.4 if $N_0 \neq n$ and from Lemma A.2.3.2.5 if $N_0 = n$, $\mathcal{E}_n$ is finite. So it is enough to show that ψ_n has finite fibres. We have to prove that for any arrow $Y_n \to B_n$ there are finitely many diagrams in $\mathcal{F}_n$ that contain that arrow. We first remark that a diagram of smooth proper k-curves

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

that is cartesian up to normalization is uniquely determined by

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

So any diagram in $\mathcal{F}_n$ containing $Y_n \to B_n$ is uniquely determined by $Y_n \to B_n$ and (if $N_0 \neq n$) by $Y_{N_0} \to C_{N_0}$. Hence it is enough to show that there are only finitely many possibilities for $Y_{N_0} \to C_{N_0}$ once we fix $Y_n \to B_n$. By definition the map $C_{N_0} \to B_{N_0}$ is of degree 2 and so it is separable by Lemma A.2.3.2.1. The conclusion follows from the fact that there are only finitely many intermediate separable covers for the morphism of curves $Y_{N_0} \to B_{N_0}$ and the observation that $Y_{N_0} \to B_{N_0}$ in uniquely determined by $Y_n \to B_n$ (and $Y_n \to Y_{N_0}$ that is part of the input datum). □

A.3 Proof of Corollary A.1.3.2

A.3.1 Construction of the abstract modular curves

A.3.1.1 Group theory

Let $\Pi \subseteq \mathrm{GL}_r(\mathbb{Z}_\ell)$ be a closed subgroup, write $\Phi(\Pi)$ for the Frattini subgroup of Π (the intersection of all maximal open subgroups of Π) and write

$$\Pi(n) := Ker(\Pi \to \mathrm{GL}_r(\mathbb{Z}_\ell/\ell^n)) \quad \text{and} \quad \Pi_n := Im(\Pi \to \mathrm{GL}_r(\mathbb{Z}_\ell/\ell^n)).$$

By [Ser89, Pag. 148], $\Phi(\Pi) \subseteq \Pi$ is an open subgroup. For every integers $j, n \geq 0$ consider the following sets:

- for $j \geq 1$, $\mathcal{C}_{j,n}(\Pi)$ is the set of open subgroups $U \subseteq \Pi$ of the form $U = C\Pi(n)$ for some closed subgroup C of Π of codimension $\geq j$
- $\mathcal{C}_{0,n}(\Pi)$ is the set of open subgroups $U \subseteq \Pi$ such that $\Phi(\Pi(n-1)) \subseteq U$ but $\Pi(n-1) \not\subseteq U$.

The family $\mathcal{C}_{j,n}$, $n \geq 0$, is endowed with a natural structure of projective system given by the maps $\psi_{j,n} : \mathcal{C}_{j,n+1}(\Pi) \to \mathcal{C}_{j,n}(\Pi)$:

$$\psi_{j,n} : U \mapsto U\Pi(n) \text{ if j } \geq 1$$

$$\psi_{0,n} : U \mapsto U\Phi(\Pi(n-1))$$

Observe that $\psi_{0,n}$ is well defined by [CT12b, Lemma 3.1]. For any $\underline{C} := (C[n])_{n\geq 0} \in \varprojlim \mathcal{C}_{j,n}(\Pi)$, write

$$C[\infty] := \varprojlim C[n] = \bigcap_{n\geq 1} C[n] \subseteq \Pi.$$

Lemma A.3.1.1.1.

1. $\mathcal{C}_{j,n}(\Pi)$ is finite.
2. Let $C \subseteq \Pi$ be a closed subgroup.
   (a) If $C \subseteq \Pi$ is of codimension $\geq j > 0$, for every integer $n \geq 1$ there exists a $U \in \mathcal{C}_{j,n}(\Pi)$ such that $C \subseteq U$.
   (b) For $n \gg 0$ (depending only on Π), if $\Pi(n-1) \not\subseteq C$ there exists a $U \in \mathcal{C}_{0,n}(\Pi)$ such that $C \subseteq U$.
3. Let $\underline{C} := (C[n])_{n\geq 0} \in \varprojlim \mathcal{C}_{j,n}(\Pi)$. Then:
   (a) If $j \geq 1$, the closed subgroup $C[\infty] \subseteq \Pi$ has codimension $\geq j$.
   (b) If $j = 0$, the closed subgroup $C[\infty] \subseteq \Pi$ has codimension ≥ 1.

Proof.

1. If $j > 0$ (resp. $j = 0$), every $U \in \mathcal{C}_{j,n}(\Pi)$ contains $\Pi(n)$ (resp. $\Phi(\Pi(n+1))$), hence $\mathcal{C}_{j,n}(\Pi)$ is in bijection with a subset of the set of subgroups of the finite group $\Pi/\Pi(n)$ (resp. $\Pi/\Phi(\Pi(n+1))$).
2. (a) Define $U := C\Pi(n)$.
   (b) Define $U := C\Phi(\Pi(n-1))$ and use [CT12b, Lemma 3.2].
3. By definition of the projective system and induction (and [CT12b, Lemma 3.2] if $j = 0$), for every $n \gg 0$ and $N \geq n$:

$$C[n] = C[n+1]\Pi(n) = C[N]\Pi(n).$$

Hence we get:

$$C[n] = \bigcap_{N\geq n} (C[N]\Pi(n)) = \bigcap_{N\geq n} (C[N])\Pi(n) = C[\infty]\Pi(n).$$

So:

   (a) follows from [CT13, Corollary 5.3] since

$$C[\infty]\Pi(n) = C[n] = C[n]\Pi(n)$$

   (b) follows from the fact that $\Pi(n) \not\subseteq C[\infty]$ for every n and the set of $\Pi(n)$ is a fundamental system of neighbourhoods of 1. $\square$

A.3.1.2 Anabelian dictionary

Let X be a smooth geometrically connected k-variety and assume now that Π is the image of a continuous representation $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{Z}_\ell)$. Consider the following (possibly disconnected) étale covers:

$$f_{j,n} : \mathcal{X}_{j,n} := \coprod_{U \in \mathcal{C}_{j,n}(\Pi)} X_U \to X$$

Lemma A.3.1.2.1. Let $x \in X(k)$ and assume that there exists an n such that $x \notin f_{j,n}(\mathcal{X}_{j,n}(k))$. Then:

1. If $j \geq 1$, $\Pi_x \subseteq \Pi$ is of codimension $< j$.
2. If $j = 0$ and $n \gg 0$, $\Pi_x \subseteq \Pi$ is open of index $\leq [\Pi : \Pi(n)]$.

Proof. This follows from Fact A.1.1.1 and Lemma A.3.1.1.1 (2). □

Assume from now on that X is a curve.

Corollary A.3.1.2.2. Fix 3 integers $j, c_1, c_2 \geq 0$ and consider the following conditions:

1. ρ is GLP, $\ell \neq p$ and $j \geq 0$;
2. $\ell \neq p$ and $j \geq 3$.
3. $\ell = p$ and $j \geq 2$;

If one of the previous conditions holds, then there exists an integer $N_\rho(c_1, c_2)$ such that for every $U \in \mathcal{C}_{j,n}(\Pi)$ we have $[k_U : k] > c_2$ or $\gamma_U > c_1$.

Proof. If condition (1) holds, this follows from Theorem A.1.2.3(1) as in the proof of [CT13, Corollaries 3.9-3.10-3.11]. If one of the conditions (2) or (3) holds, this follows from Theorem A.1.2.3(2-3), as in [CT13, Subsection 5.1.2]. □

A.3.2 Proof of Corollary A.1.3.2

Set $j = 3$ if $\ell \neq p$ and $j = 2$ if $\ell = p$. Consider the projective system of covers constructed in Section A.3.1.2

$$f_{j,n} : \mathcal{X}_{j,n} := \coprod_{U \in \mathcal{C}_{j,n}} X_U \to X.$$

By Corollary A.3.1.2.2 and (A.1.2.2) we can choose an n_0 such that all the connected components of $\mathcal{X}_{j,n_0}$ have genus larger then the constant g of Fact A.1.3.1 or are defined over a non trivial extension of k. By Lemma A.3.1.2.1, the set of $x \in X(k)$ such that $\Pi_x \subseteq \Pi$ has codimension $\geq j$ lies in the image of $f_{j,n} : \mathcal{X}_{j,n_0}(k) \to X(k)$, which is finite by the choice of n_0. Hence it is finite and this conclude the proof of Corollary A.1.3.2.

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Index of definitions and notations

Conventions

We collect here some standard conventions and notation used in this thesis.

- A k-variety is a reduced scheme of finite type over k. If X is a k-variety, and $k \subseteq k'$ is a field extension, set $X_{k'} := X \times_k k'$. For $x \in X$ write $k(x)$ its residue field and $\overline{x}$ for a geometric point over x. Set $|X|$ for the set of closed points. A curve is a k-variety of dimension 1.

- If G is an algebraic group over a field L of characteristic zero, we write G^0 for its neutral component, $\pi_0(G) := G/G^0$, $R_u(G)$ for its unipotent radical and $G^{ss} := G/R_u(G)$. Set $\mathbf{Rep}_L(G)$ for the category of finite dimensional L-linear representations of G and $X^*(G)$ its group of characters.

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Titre : Invariants ℓ-adiques, p-adiques et géométriques en familles de variétés

Mots clés : Géométrie arithmétique, groupe fondamental étale, caractéristique positive, familles de variétés, F-isocristaux (sur)convergents, cycles algébriques

Résumé : Cette thèse est divisée en huit chapitres. Dans le chapitre 1 on présente des résultats et des outils déjà connus qu'on utilisera dans la suite de la thèse. Le chapitre 2 est consacré à résumer de manière uniforme les nouveaux résultats présentés dans ce manuscrit.
Les six chapitres restants sont originaux. Dans les chapitres 3 et 4 on démontre le théorème suivant: soit $f : Y \to X$ un morphisme lisse et prope sur une base X lisse et géométriquament connexe sur un corps infini, finiment engendré et de caractéristique positive. Alors il y a beaucoup de points fermées $x \in |X|$ tels que le rang du groupe de Néron-Severi de la fibre géométrique de f en x est le même du groupe de Néron-Severi de la fibre géométrique générique. On preuve ça de la façon suivant: on étudie la spécialisation du faisceau lisse ℓ-adique $R^2 f_* \mathbb{Q}_\ell(1)$ ($\ell \neq p$); ensuite, on le relie avec la spécialisation du F-isocristal $R^2 f_{*,cris} \mathcal{O}_{Y/K}(1)$ en passant par la catégorie des F-isocristaux surconvergents. Au final, la conjecture de Tate varationelle dans la cohomologie cristalline, nous permet de déduire le résultat sur les groupes de Néron-Severi depuis le résultat sur $R^2 f_{*,cris} \mathcal{O}_{Y/K}(1)$. Cela étend en caractéristique positive des résultats de Cadoret-Tamagawa et André en caractéristique zero.
Les chapitres 5 et 6 sont consacrés à l'étude des groupes de monodromie des F-isocristaux (sur)convergents. En particulier, les résultats dans le chapitre 5 sont un travail en commun avec Marco D'Addezio. On étude les tores maximaux des groupes de monodromie des F-isocristaux (sur)convergents et on utilise ça pour démontrer un cas particulier d'une conjecture de Kedlaya sur les homomorphismes de F-isocristeaux convergents. En utilisant ce cas particulier, on démontre que si A est une variété abélienne sans facteurs d'isogénie isotriviaux sur un corps de fonctions F sur $\overline{\mathbb{F}}_p$, alors le groupe $A(F^{\text{perf}})_{tors}$ est fini. Cela peut être cosidéré comme une extension du théorème de Lang-Néron et donne une réponse positive à une question d'Esnault. Dans le chapitre 6, on définit une catégorie $\overline{\mathbb{Q}}_p$-linéaire des F-isocristeaux surconvergents et les groupes de monodromie de ces objets. En exploitant la théorie des companions pour les F-isocristeaux surconvergents et les faisceaux lisses, on étudie la théorie de spécialisation de ces groupes de monodromie en transférant les résultats du chapitre 3 dans ce contexte.
Les derniers deux chapitres complètent et affinent les résultats des chapitres précédents. Dans le chapitre 7, on démontre que la conjecture de Tate pour les diviseurs sur les corps finiment engendrés et de caractéristique $p > 0$ est une conséquence de la conjecture de Tate pour les diviseurs sur les corps finis de caractéristique $p > 0$. Dans le chapitre 8, on démontre des résultats de borne uniforme en caractéristique positive pour le groupes de Brauer des formes des variétés qui satisfont la conjecture de Tate ℓ-adique pour les diviseurs. Cela étend en caractéristique positive un résultat de Orr-Skorobogatov en caractéristique zero.

Title : ℓ-adic, p-adic and geometric invariants in families of varieties

Keywords : Arithmetic geometry, positive characteristic, families of varieties, étale fundamental group, (over)convergent F-isocrystals, algebraic cycles.

Abstract : This thesis is divided in 8 chapters. Chapter 1 is of preliminary nature: we recall the tools that we will use in the rest of the thesis and some previously known results. Chapter 2 is devoted to summarize in a uniform way the new results obtained in this thesis.
The other six chapters are original. In Chapters 3 and 4, we prove the following: given a smooth proper morphism $f : Y \to X$ over a smooth geometrically connected base X over an infinite finitely generated field of characteristic $p > 0$, there are lots of closed points $x \in |X|$ such that the rank of the Néron-Severi group of the geometric fibre of f at x is the same of the rank of the Néron-Severi group of the geometric generic fibre. To prove this, we first study the specialization of the ℓ-adic lisse sheaf $R^2 f_* \mathbb{Q}_\ell(1)$ ($\ell \neq p$), then we relate it with the specialization of the F-isocrystal $R^2 f_{*,crys} \mathcal{O}_{Y/K}(1)$ passing trough the category of overconvergent F-isocrystals. Then, the variational Tate conjecture in crystalline cohomology allows us to deduce the result on the Néron-Severi groups from the results on $R^2 f_{*,crys} \mathcal{O}_{Y/K}(1)$. These extend to positive characteristic results of Cadoret-Tamagawa and André in characteristic zero.
Chapters 5 and 6 are devoted to the study of the monodromy groups of (over)convergent F-isocrystals. Chapter 5 is a joint work with Marco D'Addezio. We study the maximal tori in the monodromy groups of (over)convergent F-isocrystals and using them we prove a special case of a conjecture of Kedlaya on homomorphism of convergent F-isocrystals. Using this special case, we prove that if A is an abelian variety without isotrivial geometric isogeny factors over a function field F over $\overline{\mathbb{F}}_p$, then the group $A(F^{\text{perf}})_{tors}$ is finite. This may be regarded as an extension of the Lang-Néron theorem and answer positively to a question of Esnault. In Chapter 6, we define a $\overline{\mathbb{Q}}_p$-linear category of (over)convergent F-isocrystals and the monodromy groups of their objects. Using the theory of companion for overconvergent F-isocrystals and lisse sheaves, we study the specialization theory of these monodromy groups, transferring the result of Chapter 3 to this setting via the theory of companions.
The last two chapters are devoted to complements and refinement of the results in the previous chapters. In Chapter 7, we show that the Tate conjecture for divisors over finitely generated fields of characteristic $p > 0$ follows from the Tate conjecture for divisors over finite fields of characteristic $p > 0$. In Chapter 8, we prove uniform boundedness results for the Brauer groups of forms of varieties in positive characteristic, satisfying the ℓ-adic Tate conjecture for divisors. This extends to positive characteristic a result of Orr-Skorobogatov in characteristic zero.

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