Contrôlabilité de systèmes de réaction-diffusion non linéaires

Kévin Le Balc'H

Thèse de doctorat de l'École Normale Supérieure de Rennes

COMUE UNIVERSITE BRETAGNE LOIRE

Ecole Doctorale N°601
Mathématiques et Sciences et Technologies
de l'Information et de la Communication
Spécialité : *Mathématiques et leurs Interactions*

Kévin LE BALC'H

Contrôlabilité de systèmes de réaction-diffusion non linéaires

Thèse présentée et soutenue à l'École Normale Supérieure de Rennes, le 26 Juin 2019

Unité de recherche : Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

Rapporteurs avant soutenance :

Enrique Fernández-Cara, Professeur des Universités, Université de Séville
Marius Tucsnak, Professeur des Universités, Université de Bordeaux

Composition du jury :

Karine Beauchard, Professeur des Universités, École Normale Supérieure de Rennes - *Directrice de thèse*
Assia Benabdallah, Professeur des Universités, Université d'Aix-Marseille
Jean-Michel Coron, Professeur des Universités, Sorbonne Université
Sylvain Ervedoza, Chargé de Recherche CNRS, Institut de Mathématiques de Toulouse
Enrique Fernández-Cara, Professeur des Universités, Université de Séville - *Rapporteur*
Sergio Guerrero, Maître de Conférences, Sorbonne Université
Michel Pierre, Professeur Émérite, École Normale Supérieure de Rennes - *Directeur de thèse*
Marius Tucsnak, Professeur des Universités, Université de Bordeaux - *Rapporteur*

Remerciements

J'adresse mes premiers remerciements à Karine Beauchard et Michel Pierre, qui ont été des directeurs de thèse exceptionnels. Karine, ton encouragement sans faille, ta rigueur de pensée, ta patience, tes nombreuses questions, réponses m'ont été d'une grande aide. Tu resteras sans nul doute un modèle d'inspiration pour ma carrière future. Michel, votre générosité intellectuelle m'aura comblée pendant trois ans. Merci pour toutes vos belles idées, votre expertise en systèmes de réaction-diffusion, vos nombreux retours sur mon travail.

Je remercie vivement Enrique Fernández-Cara et Marius Tucsnak d'avoir accepté de rapporter ma thèse. Je suis très reconnaissant de leur temps consacré à la lecture attentive de mes travaux. Assia Benabdallah, Jean-Michel Coron, Sylvain Ervedoza, Sergio Guerrero me font l'honneur de participer au jury de cette soutenance. Je leur adresse toute ma gratitude.

Merci à mes professeurs qui auront su me transmettre leur passion. Je pense en particulier à Jacques Bozec (Terminale S, Lycée Brizeux à Quimper), Alain Le Boulch (MPSI 3, Lycée Chateaubriand à Rennes), Roland Louboutin (MP*, Lycée Chateaubriand à Rennes). Puis, à l'ENS et à l'université, certains cours d'analyse m'auront profondément marqué : je pense à ceux de François Castella, Arnaud Debussche, Thibaut Deheuvels, Taoufik Hmidi, Michel Pierre.

J'ai conscience de la chance d'avoir pu effectuer ma thèse et mon service d'enseignement à l'ENS Rennes. Il y règne une atmosphère conviviale. J'en profite pour remercier l'intégralité du département de mathématiques. Ces trois années ont été superbes. Merci également à Elodie Lequoc de m'avoir aidé avec patience dans toutes les démarches administratives.

J'ai également passé beaucoup de temps dans mon bureau à Beaulieu. Je remercie l'intégralité du personnel de l'IRMAR, en particulier Marie-Aude pour sa bienveillance, Olivier pour son aide informatique et Hélène pour son aide dans l'organisation du séminaire des doctorants. Merci également aux enseignants-chercheurs qui font vivre ce laboratoire !

Place aux doctorants rennais avec qui j'ai pu partager de nombreuses pauses midi, café, thé, goûter, crêperies. Merci pour tous ces moments. Je remercie ainsi Louis, Fabrice, Mercedes (merci pour l'organisation du TFJM^2), Josselin, Léo, Antoine, Chloé, Thibault, Léopold, Angelo, Thi, Paul, Zoïs et Pierre.

Je remercie mes camarades de promo de l'ENS Rennes. C'est toujours un plaisir de vous voir : Sylvain, Arnaud P. (bonne continuation à Lille), Arnaud S., Joackim (partenaire de course à pieds, de CEMRACS, de crêperies...), Léo (partenaire de tennis), Hugo, Adrien (collègue de classe, de soirées à l'ENS, de bureau), Fred (merci de m'avoir fait découvrir Strasbourg), Florian (collègue depuis la prépa), Camille L. (pour notre passion commune des belles maths), Simon (partenaire de course à pieds et de pause-thé à 16h), Laura, Camille F., Caroline, Ninon (demain, c'est ton tour...), Maylis et Maud.

Cette thèse aura également été pour moi l'occasion de voyager et de rencontrer d'autres mathématiciens, avec qui j'ai eu plaisir à discuter. Je remercie ainsi l'ensemble de la communauté des « contrôleurs », en particulier Armand pour notre projet, Frédéric pour nos échanges et Mégane (bienvenue !). Haruki, j'espère pouvoir venir te voir un jour au Japon. Lydia, Mario, j'ai hâte de vous revoir !

A tous mes amis de plus longue date, du lycée et du ping-pong, cette thèse vous doit beaucoup. C'est toujours un énorme bol d'air que de passer du bon temps avec vous. Mention spéciale à Clem', Tud', Ben', Manon (merci pour ta relecture minutieuse), Solène, Etienne, sans oublier les teams Palma, Vieux-Boucau, Boudu, QCTT et RCBTT. Ne changez rien !

Patrick, Sylvie, vous m'avez accueilli avec bienveillance dans votre famille. Cela a beaucoup compté pour moi, je vous remercie sincèrement aujourd'hui. Romane, je t'ai connu toute petite, tu es devenue aujourd'hui une personne que j'estime beaucoup. Merci également à Greg' pour tous ces bons moments.

Papa, Maman, merci de m'avoir toujours encouragé dans mes études. Je vous dois tout. Stéven, mon petit frère, merci pour cette complicité qui nous unit.

Manue, cette thèse te doit bien plus que tu ne le crois. Tu m'as soutenu au quotidien, tu as su réparer par ton amour les « Plus rien ne marche ». Je te remercie pour tout cela et pour bien d'autres choses encore...

À ma mère, pour son dévouement sans faille.

Table des matières

Introduction

Un *système dynamique* est la donnée d'un système et d'une loi décrivant l'évolution de ce système. Dans les lignes qui suivent, nous étudions deux grandes classes de systèmes dynamiques : les *équations différentielles* et les *équations aux dérivées partielles*. Un *système de contrôle* est un système dynamique sur lequel on peut agir au moyen d'un *contrôle* ou d'une *commande*. Prenez l'exemple du pilotage d'une voiture où *l'état* de votre système est la position de la voiture, la direction des roues et les moyens de contrôle sont multiples : pédales d'accélération, de frein, volant... La théorie du contrôle analyse les propriétés de ce système dans le but de l'amener d'un état initial donné à un certain état final : c'est la *contrôlabilité*, notion centrale dans cette thèse. Une fois le problème de contrôlabilité résolu, on peut de plus vouloir passer de l'état initial à l'état final, en minimisant un certain critère, on parle alors de problème de *contrôle optimal*. L'un des autres objectifs peut être de *stabiliser* le système : c'est-à-dire de le rendre insensible à de petites perturbations.

Cette thèse a pour objectif principal de répondre à des questions de *contrôlabilité d'équations aux dérivées partielles non linéaires*. Plus précisément, il s'agira surtout de *systèmes paraboliques non linéaires* ou encore de *systèmes de réaction-diffusion non linéaires*. Ces équations servent par exemple de modèle à des équations issues de la *cinétique chimique*.

Nous avons essayé de proposer une introduction didactique aux différents outils de notre travail. Ainsi, dans un premier temps, nous illustrons un certain nombre de mécanismes en dimension finie en Partie I. Puis, nous dressons un état de l'art non exhaustif de la contrôlabilité des systèmes paraboliques linéaires et non linéaires en Partie II. Nous nous sommes attachés à donner quelques démonstrations clefs de la théorie. Nous avons également essayé d'étoffer cette partie de remarques mettant en lumière les principales similitudes et différences avec la dimension finie. Pour finir, nous présentons en Partie III les résultats issus de la thèse. Ainsi, les théorèmes principaux et les idées cruciales de démonstration des articles issus de la thèse, c'est-à-dire [LB18a], [LB18b], [LB18c], [LB19] et [BKLB19] se trouvent dans la Partie III. Le lecteur désireux de connaître les détails techniques des démonstrations pourra consulter l'Annexe B, l'Annexe C, l'Annexe D, l'Annexe E et l'Annexe G.

Première partie

Contrôle d'équations différentielles

Dans cette partie, nous allons présenter certains aspects et certaines méthodes du contrôle d'équations différentielles. Bien entendu, nous serons loin d'être exhaustifs et le lecteur pourra consulter avec profit les livres [Cor07a], [Tré05], [TW09] dont nous nous sommes largement inspirés.

Un exemple physique : le ressort. Avant toute chose, nous choisissons de présenter un système contrôlé issu de la physique. Après avoir mis sous forme d'équations ce système, nous mettons en lumière les différents enjeux et principales difficultés qui tournent autour de la notion de contrôlabilité.

On considère une masse ponctuelle m qui se déplace le long d'un axe vertical (Oy) de vecteur directeur $\vec{j}$, attachée à un ressort. La position de la masse m est notée y. Cette masse est soumise à deux forces :

- son poids : $\vec{P} = mg\vec{j}$ où g est la constante universelle de gravitation,
- la force de rappel que l'on suppose égale à $\vec{F_r} = \left(-k_1(y-l) - k_2(y-l)^3\right)\vec{j}$, où l est la longueur à vide du ressort, et k_1, k_2 sont les coefficients de raideur.

De plus, on suppose que l'on peut exercer une force extérieure verticale au ressort représentée par : $\vec{F}(t) = h(t)\vec{j}$. Alors, par le principe fondamental de la dynamique, on a

$$my''(t) + k_1(y(t)-l) + k_2(y(t)-l)^3 - mg = h(t). \tag{Newton}$$

En supposant que $m = 1$, $l = 0$ (quitte à translater), on obtient alors le système différentiel suivant

$$Y'(t) = AY(t) + g(Y(t)) + Bh(t), \qquad Y(0) = Y_0, \tag{Ressort}$$

où on a posé

$$A = \begin{pmatrix} 0 & 1 \\ -k_1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ Y = \begin{pmatrix} y \\ y' \end{pmatrix},\ g(Y) = \begin{pmatrix} 0 \\ -k_2y^3 + mg \end{pmatrix},\ Y_0 = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}.$$

Le système (Ressort) est un *système différentiel contrôlé*. A chaque instant t, $Y(t)$ désigne l'*état* du système et $h(t)$ le *contrôle* agissant sur le système.

La question de *contrôlabilité* pour le système (Ressort) peut se reformuler ainsi : étant donnés un temps $T > 0$, une donnée initiale $Y_0 \in \mathbb{R}^2$, i.e. une position et une vitesse initiales, un état cible $Y_f \in \mathbb{R}^2$ i.e. une position et une vitesse cibles, peut-on trouver un contrôle $h : t \in [0,T] \to \mathbb{R}$ i.e. une force, tel que la solution de (Ressort) vérifie $Y(T) = Y_f$?

Sur ce système, on entrevoit déjà quelques difficultés mathématiques sous-jacentes à la notion de contrôlabilité :

- la réponse peut dépendre du temps T que l'on s'est donné pour contrôler l'équation. Intuitivement, plus T est petit, c'est-à-dire plus le temps imparti est court, plus il va être difficile de relier deux états. Quand on cherche à contrôler en temps arbitrairement petit, on parle de *contrôlabilité en temps petit*.

— la réponse peut dépendre de la position de l'état cible Y_f par rapport à celle de la donnée initiale Y_0. Intuitivement, plus ces deux états sont éloignés, plus il va être difficile de passer de l'un à l'autre. Quand on cherche à passer d'un état arbitraire à un autre état arbitraire, on parle de *contrôlabilité globale*.

Chapitre 1

Systèmes différentiels linéaires : la condition de Kalman

Dans ce chapitre, nous présentons la contrôlabilité des systèmes différentiels linéaires. Nous montrons en particulier qu'une certaine condition algébrique, appelée *condition de Kalman*, permet de répondre à la question de contrôlabilité.

Soit $T \in (0, +\infty)$, $n, m \in \mathbb{N}^*$, $A \in C([0,T]; \mathbb{R}^{n\times n})$ et $B \in C([0,T]; \mathbb{R}^{n\times m})$. Se donnant $y_0 \in \mathbb{R}^n$, on considère le système différentiel contrôlé

$$y'(t) = A(t)y + B(t)h, \ t \in [0,T], \qquad y(0) = y_0. \tag{1.1}$$

Dans (1.1), pour $t \in [0,T]$, $y(t) \in \mathbb{R}^n$ est l'état du système et $h(t) \in \mathbb{R}^m$ est le contrôle.

Nous donnons la définition de contrôlabilité pour le système (1.1).

Définition 1.0.1. Le système (1.1) est *contrôlable au temps* $T > 0$ si pour tout $(y_0, y_f) \in \mathbb{R}^n \times \mathbb{R}^n$, il existe $h \in C([0,T]; \mathbb{R}^m)$ telle que l'unique solution $y \in C^1([0,T]; \mathbb{R}^n)$ du problème de Cauchy (1.1) vérifie $y(T) = y_f$.

Remarque 1.0.2. L'existence et l'unicité de la solution au problème de Cauchy (1.1) sont garanties par le théorème de Cauchy-Lipschitz linéaire (voir [Gou08, Chapitre 6, Section 2, Théorème 1] par exemple).

1.1 Le cas des coefficients constants

Dans cette partie, on suppose que $A(t)$ et $B(t)$ ne dépendent pas du temps. On se donne donc $A \in \mathbb{R}^{n\times n}$ et $B \in \mathbb{R}^{n\times m}$. On considère le système contrôlé

$$y' = Ay + Bh. \tag{1.2}$$

La célèbre condition de Kalman est donnée dans le théorème suivant.

Théorème 1.1.1. *Le système $y' = Ay + Bh$ est contrôlable au temps T si et seulement si la condition suivante est vérifiée*

$$\text{Rang}\left(B, AB, \dots, A^{n-1}B\right) = n. \tag{1.3}$$

Remarque 1.1.2. Généralement, la matrice

$$K := \left(B, AB, \dots, A^{n-1}B\right) \in \mathbb{R}^{n\times nm}, \tag{1.4}$$

est appelée matrice de Kalman.

Remarque 1.1.3. Il est remarquable que le fait d'être contrôlable au temps T ne dépende pas de T mais seulement d'une condition algébrique sur les matrices A et B.

Remarque 1.1.4. Quand $\text{Rang}(B) = n$, alors la condition (1.3) est automatiquement vérifiée et donc le système est contrôlable. Cela correspond au cas où il y a au moins autant de contrôles que d'équations.

Nous allons donner deux preuves du Théorème 1.1.1. La première, qui ne démontrera que le sens indirect, passe par une mise sous forme Brunovsky. Elle consiste à transformer le système (1.2) en un *système cascade* où on voit apparaître les propriétés de contrôlabilité du système. Nous revisiterons cette méthode en dimension infinie pour comprendre les propriétés de contrôlabilité des systèmes paraboliques linéaires. La seconde est basée sur une méthode de dualité : la « *Hilbert Uniqueness Method* » (ou méthode de dualité Hilbertienne) due à Jacques-Louis Lions (voir [Lio88]). Cette dernière approche sera également très féconde en dimension infinie.

Démonstration par la mise sous forme Brunovsky. On suppose que la condition de Kalman (1.3) est vérifiée. Le but est de montrer que le système $y' = Ay + Bh$ est contrôlable au temps T. Pour simplifier, on va supposer que $m = 1$. Sans perte de généralité, par réversibilité du système (1.1), on suppose également que $y_f = 0$.

Par le théorème de Cayley-Hamilton, on sait que $\chi_A(A) = 0$, où χ_A est le polynôme caractéristique de A. On en déduit qu'il existe $c_0, \dots, c_{n-1} \in \mathbb{R}$ tels que

$$A^n = c_0 I_n + c_1 A + \dots c_{n-1}A^{n-1}.$$

Par hypothèse, la matrice $K \in \mathbb{R}^{n\times n}$ définie en (1.4) est inversible. On pose

$$\widehat{A} := \begin{pmatrix} 0 & \dots & \dots & 0 & c_0 \\ 1 & 0 & \dots & \vdots & c_1 \\ 0 & \ddots & \ddots & \vdots & c_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & c_{n-1} \end{pmatrix} \text{ et } \widehat{B} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

On vérifie qu'on a alors

$$AK = K\widehat{A} \text{ et } B = K\widehat{B}, \text{ i.e. } \widehat{A} = K^{-1}AK \text{ et } \widehat{B} = K^{-1}B.$$

Ainsi, en posant $y = Kz$, on a

$$z' = \widehat{A}z + \widehat{B}h. \tag{1.5}$$

Ainsi, nous avons réussi à transformer le système (1.2) en le système (1.5) où la matrice $\widehat{A}$ est une matrice compagnon et $\widehat{B}$ est le premier vecteur de la base canonique, c'est-à-dire

$$\begin{cases} z_1' = c_0 z_n + h \\ z_2' = z_1 + c_1 z_n \\ \vdots \\ z_n' = z_{n-1} + c_{n-1} z_n \end{cases}$$

Cette structure est généralement appelée *système cascade*. Le point important est que les termes sous diagonaux de $\widehat{A}$ ne s'annulent pas et donc créent du couplage. Intuitivement, h va contrôler la première composante z_1 grâce à la première équation, qui elle-même va contrôler z_2 grâce à la seconde équation (et au terme de couplage z_1), et ainsi de suite jusqu'à z_{n-1} qui va contrôler z_n au regard de la dernière équation.

Passons à la construction explicite du contrôle h :

— On définit $\overline{z}$ la solution libre du système (1.5) (c'est-à-dire avec $z_0 = K^{-1}y_0$ et $h = 0$).

— On se donne une fonction de troncature $\eta \in C^\infty([0,T];[0,1])$ telle que $\eta = 1$ sur $[0, T/3]$ et $\eta = 0$ sur $[2T/3, T]$.

— On commence par choisir $z_n(t) := \eta(t)\overline{z_n}(t)$, puis en utilisant la dernière équation de (1.5), on définit $z_{n-1}(t) := z_n'(t) - c_{n-1}z_n(t)$ et ainsi de suite $z_i(t) := z_{i+1}'(t) - c_i z_n(t)$ pour i entre $n-2$ et 1.

— La première équation de (1.5) nous suggère alors de poser $h(t) := z_1'(t) - c_0 z_n(t)$.

Ainsi, par construction, z est solution du système (1.5) avec le contrôle h précédemment défini.

Il suffit alors de vérifier par récurrence descendante que pour tout $k \in \{1, \dots, n\}$, on a

$$z_k = \overline{z_k} \text{ sur } [0, T/3] \text{ et } z_k = 0 \text{ sur } [2T/3, T].$$

Ce qui termine la preuve puisque $z(0) = \overline{z}(0) = K^{-1}y_0$ et $z(T) = 0$. □

Démonstration par la « HUM ». On procède en deux étapes.

Etape 1 : Contrôlabilité ⇔ Continuation unique ⇔ Inégalité d'observabilité
Se donnant $y_0 \in \mathbb{R}^n$ et $h \in C([0,T];\mathbb{R}^m)$, par la formule de Duhamel, la solution de (1.2) vaut

$$y(T) = e^{TA}y_0 + \int_0^T e^{(T-s)A}Bh(s)ds.$$

Il est facile de voir que la contrôlabilité de (1.2) au temps T est équivalente à la surjectivité de l'application linéaire

$$\Phi : h \in C([0,T];\mathbb{R}^m) \mapsto \int_0^T e^{(T-s)A}Bh(s)ds \in \mathbb{R}^n.$$

Puisque $\mathbb{R}^n$ est de dimension finie et que $C([0,T];\mathbb{R}^m)$ est dense dans $L^2(0,T;\mathbb{R}^m)$, il est également facile de voir que la contrôlabilité de (1.2) au temps T est équivalente à la surjectivité de la nouvelle application linéaire (toujours notée Φ pour simplifier)

$$\Phi : h \in L^2(0,T;\mathbb{R}^m) \mapsto \int_0^T e^{(T-s)A} Bh(s)ds \in \mathbb{R}^n. \tag{1.6}$$

Or, on a

$$\begin{aligned} \Big(\mathrm{Im}(\Phi) = \mathbb{R}^n\Big) &\Leftrightarrow \Big(\mathrm{Ker}(\Phi^*) = \{0\}\Big) & (1.7)\\ &\Leftrightarrow \Big(\exists C > 0,\ \forall \varphi_T \in \mathbb{R}^n,\ \|\varphi_T\|_{\mathbb{R}^n} \leq C \left\|\Phi^* \varphi_T\right\|_{L^2(0,T;\mathbb{R}^m)}\Big). & (1.8) \end{aligned}$$

De plus, à partir de (1.6), un calcul simple nous donne

$$\forall \varphi_T \in \mathbb{R}^n,\ \Phi^* \varphi_T = B^{\mathrm{tr}} e^{(T-\cdot)A^{\mathrm{tr}}} \varphi_T \in L^2(0,T;\mathbb{R}^m).$$

On déduit alors que la contrôlabilité de (1.2) équivaut à démontrer *le principe de continuation unique*

$$\forall \varphi_T \in \mathbb{R}^n,\ \Big(\forall s \in [0,T],\ B^{\mathrm{tr}} e^{(T-s)A^{\mathrm{tr}}} \varphi_T = 0\Big) \Rightarrow \Big(\varphi_T = 0\Big). \tag{1.9}$$

ou (toujours de manière équivalente) l'inégalité dite *inégalité d'observabilité* : il existe $C > 0$ telle que

$$\forall \varphi_T \in \mathbb{R}^n,\ \|\varphi_T\|_{\mathbb{R}^n} \leq C \left(\int_0^T \left\|B^{\mathrm{tr}} e^{(T-s)A^{\mathrm{tr}}} \varphi_T\right\|_{\mathbb{R}^m}^2 ds\right)^{1/2}. \tag{1.10}$$

Etape 2 : Continuation unique $\Leftrightarrow$ Condition de Kalman.
On notera $\varphi(s) = e^{(T-s)A^{\mathrm{tr}}} \varphi_T$ dans la suite. Par analyticité, $B^{\mathrm{tr}}\varphi(s)$ est nul si et seulement si toutes les dérivées de $\varphi(s)$ sont nulles en $s = T$. En dérivant et en évaluant en $s = T$, on trouve aisément que ceci est équivalent à pour tout $k \geq 0$, $B^{\mathrm{tr}}(A^{\mathrm{tr}})^k \varphi_T = 0$. Par le théorème de Cayley-Hamilton, ceci est encore équivalent à
$\Big(B^{\mathrm{tr}}, B^{\mathrm{tr}}A^{\mathrm{tr}}, \ldots, B^{\mathrm{tr}}(A^{\mathrm{tr}})^{n-1}\Big)\varphi_T = 0$. Ainsi, nous avons prouvé que

$$\begin{aligned} (1.9) &\Leftrightarrow \mathrm{Ker}(K^*) = \{0\} \\ &\Leftrightarrow \mathrm{Rang}(K) = n, \end{aligned}$$

où K est la matrice de Kalman définie en (1.4).

On conclut la preuve en rassemblant l'étape 1 et l'étape 2. □

Remarque 1.1.5. Dans la précédente preuve, les concepts de continuation unique et d'inégalité d'observabilité sont équivalents comme le montre l'équivalence (1.7) et (1.8). Ceci est un phénomène particulier de la dimension finie. Cela résulte de l'équivalence des normes en dimension finie.

1.2 Le cas des coefficients dépendant du temps

Dans cette partie, on suppose que A et B sont de classe C^∞ sur $[0, T]$.
On définit par récurrence sur i, la suite $B_i \in C^\infty([0, T]; \mathbb{R}^{n\times m})$ de la manière suivante :

$$\forall t \in [0, T],\ B_0(t) := B(t),\ B_i(t) := B'_{i-1}(t) - A(t)B_{i-1}(t). \tag{1.11}$$

Le résultat suivant est une condition de Kalman généralisée.

Théorème 1.2.1. *On suppose que pour* $\bar{t} \in [0, T]$,

$$\operatorname{Rang}\big(B_0(\bar{t}), B_1(\bar{t}), \ldots, B_{n-1}(\bar{t})\big) = n. \tag{1.12}$$

Alors, le système $y' = A(t)y + B(t)h$ *est contrôlable au temps* T.

Remarque 1.2.2. Quand A et B sont constants, on trouve que pour tout i, $B_i = (-1)^i A^i B$ où B_i est définie en (1.11). Ainsi, le Théorème 1.2.1 permet de retrouver la condition suffisante de contrôlabilité du Théorème 1.1.1.

Remarque 1.2.3. La condition suffisante de contrôlabilité du Théorème 1.2.1 n'est pas nécessaire (sauf si $n = 1$ ou les applications A et B sont analytiques sur $[0, T]$, voir [Tré05, Théorème 2.3.2]), comme le montre l'exemple [Cor07a, Pages 11, 12].

Pour la preuve du Théorème 1.2.1, voir [Cor07a, Théorème 1.18].

Chapitre 2

Systèmes différentiels non linéaires

Dans cette partie, nous allons considérer le système non linéaire contrôlé

$$y' = f(y, h), \tag{2.1}$$

où $y \in \mathbb{R}^n$ désigne l'état, $h \in \mathbb{R}^m$ désigne le contrôle. On suppose dans toute la suite que $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$. La contrôlabilité globale pour un système non linéaire est une question difficile en toute généralité. C'est pourquoi, nous allons d'abord nous intéresser à la *contrôlabilité locale* de (2.1) autour de points dit *points d'équilibre*. Nous formulons certains critères qui assurent la contrôlabilité locale.

Nous commençons par deux définitions.

Définition 2.0.1. Un *équilibre* du système $y' = f(y, h)$ est un couple $(y_e, h_e) \in \mathbb{R}^n \times \mathbb{R}^m$ tel que

$$f(y_e, h_e) = 0.$$

Définition 2.0.2. Soit $(y_e, h_e) \in \mathbb{R}^n \times \mathbb{R}^n$ un équilibre du système de contrôle $y' = f(y, h)$. Le système $y' = f(y, h)$ est dit *localement contrôlable en temps petit à l'équilibre* (y_e, h_e) si pour tout $\varepsilon > 0$, il existe $\eta > 0$ tel que pour tous $y_0, y_1 \in \mathbb{R}^n$ tels que $|y_0 - y_e|, |y_1 - y_e| \leq \eta$, il existe un contrôle $h \in C([0, \varepsilon]; \mathbb{R}^m)$ tel que

$$\forall t \in [0, \varepsilon],\ |h(t) - h_e| \leq \varepsilon,$$

$$\Big(y' = f(y, h(t)),\ y(0) = y_0\Big) \text{ et } \Big(y(\varepsilon) = y_1\Big).$$

2.1 Le test sur le linéarisé

Dans cette partie, nous formulons le théorème principal : Théorème 2.1.5 qui se révèle très utile en pratique pour *tester* la contrôlabilité locale du système non linéaire (2.1) sur le *linéarisé.*

Nous introduisons plusieurs définitions.

Définition 2.1.1. Une *trajectoire du système de contrôle* $y' = f(y,h)$ est la donnée d'un couple $(\overline{y}, \overline{h}) \in C^1([0,T];\mathbb{R}^n) \times C([0,T];\mathbb{R}^m)$ satisfaisant

$$\forall t \in [0,T],\ \overline{y}'(t) = f(\overline{y}(t), \overline{h}(t)).$$

Définition 2.1.2. Soit $(\overline{y}, \overline{h})$ une trajectoire du système $y' = f(y,h)$. Le système de contrôle $y' = f(y,h)$ est dit *localement contrôlable le long de la trajectoire* $(\overline{y}, \overline{h})$ si pour tout $\varepsilon > 0$, il existe $\eta > 0$ tel que pour tout $(a,b) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfaisant $|a - \overline{y}(0)| \leq \eta$ et $|b - \overline{y}(T)| \leq \eta$, il existe une autre trajectoire (y,h) vérifiant

$$\forall t \in [0,T],\ |h(t) - \overline{h}(t)| \leq \varepsilon,$$

$$y(0) = a,\ y(T) = b.$$

Remarque 2.1.3. Pour un point d'équilibre (y_e, h_e), on voit que la Définition 2.1.2 coïncide avec la Définition 2.0.2 pour $T = \varepsilon$ et la trajectoire constante (y_e, h_e).

Définition 2.1.4. On appelle *système linéarisé le long de la trajectoire* $(\overline{y}, \overline{h})$ le système de contrôle linéaire suivant

$$y' = \frac{\partial f}{\partial y}(\overline{y}(t), \overline{h}(t))y + \frac{\partial f}{\partial h}(\overline{y}(t), \overline{h}(t))h,\ t \in [0,T].$$

Nous avons le théorème très utile suivant.

Théorème 2.1.5. *Soit $(\overline{y}, \overline{h})$ une trajectoire de $y' = f(y,h)$. On suppose que le système linéarisé le long de cette trajectoire est contrôlable au temps T (voir Définition 1.0.1). Alors le système non linéaire $y' = f(y,h)$ est localement contrôlable le long de la trajectoire $(\overline{y}, \overline{h})$.*

La preuve du Théorème 2.1.5 est basée sur un argument d'*inversion locale* appliqué à l'application *point-cible* de classe C^1

$$\Phi\ :\ (a,h) \in \mathbb{R}^n \times C([0,T];\mathbb{R}^m) \mapsto \Phi(a,h) := (a, y(T)) \in \mathbb{R}^n \times \mathbb{R}^n,$$

où y est la solution du problème de Cauchy $y' = f(y,h),\ y(0) = a$. En effet, $D\Phi(\overline{y}(0), \overline{h})$ est l'application linéaire

$$(a,h) \in \mathbb{R}^n \times C([0,T];\mathbb{R}^m) \mapsto (a, y(T)) \in \mathbb{R}^n \times \mathbb{R}^n,$$

où y est la solution de $y' = \frac{\partial f}{\partial y}(\overline{y}(t), \overline{h}(t))y + \frac{\partial f}{\partial h}(\overline{y}(t), \overline{h}(t))h,\ y(0) = a$. Cette dernière application étant surjective par hypothèse, on en déduit que Φ est localement surjective. Pour les détails de la preuve, voir [Cor07a, Theorem 3.6].

On en déduit le corollaire suivant.

Corollaire 2.1.6. *Soit (y_e, h_e) un équilibre de $y' = f(y,h)$. On suppose que le système linéarisé autour de cet équilibre est contrôlable alors le système non linéaire $y' = f(y,h)$ est localement contrôlable en temps petit.*

La preuve du Corollaire 2.1.6 consiste à appliquer le Théorème 2.1.5 avec $T = \varepsilon$ et la trajectoire (y_e, h_e).

Exemple 2.1.7. On considère le système non linéaire contrôlé suivant :

$$\begin{cases} y_1' = h \\ y_2' = y_1 + y_1^3. \end{cases}$$

On voit aisément que $((0,0),0)$ est équilibre du système. Le linéarisé autour de cet équilibre est

$$\begin{cases} y_1' = h \\ y_2' = y_1. \end{cases}$$

Par le Théorème 1.1.1, ce dernier système est contrôlable. En effet, on a

$$\text{Rang}(K) = \text{Rang}(I_2) = 2.$$

On en déduit alors que le système non linéaire est localement contrôlable en temps petit autour de l'équilibre $((0,0),0)$ par le Corollaire 2.1.6.

Exemple 2.1.8. Pour $n \geq 2$ un entier, on considère le système non linéaire contrôlé suivant :

$$\begin{cases} y_1' = h \\ y_2' = y_1^n. \end{cases}$$

On voit aisément que $((0,0),0)$ est équilibre du système. Le linéarisé autour de cet équilibre est

$$\begin{cases} y_1' = h \\ y_2' = 0. \end{cases}$$

Ce dernier système n'est pas contrôlable. En effet, y_2 est constant. On ne peut alors rien en déduire a priori sur le système non linéaire.

Remarquons cependant que pour n pair, le système non linéaire n'est pas localement contrôlable en temps petit autour de $((0,0),0)$ puisque la seconde équation nous donne que y_2 est croissante au cours du temps. Ainsi, partant de $y_{2,0} > 0$, on a nécessairement en tout temps $t > 0$, $y_2(t) > 0$.

La Section 2.2 permet de traiter le cas n impair.

2.2 La méthode du retour

On considère toujours le système de contrôle $y' = f(y,h)$ où $y \in \mathbb{R}^n$ désigne l'état et $h \in \mathbb{R}^m$ le contrôle. On suppose que f est de classe C^∞ et que

$$f(0,0) = 0.$$

Ainsi $(0,0)$ est un point d'équilibre de $y' = f(y,h)$.

Grâce au Corollaire 2.1.6, on a vu qu'un des moyens pratiques pour tester la contrôlabilité locale en temps petit autour de l'état d'équilibre $(0,0)$ est de tester la contrôlabilité du linéarisé

$$y' = \frac{\partial f}{\partial y}(0,0)y + \frac{\partial f}{\partial h}(0,0)h.$$

Cependant, dans le cas où ce linéarisé n'est pas contrôlable, on ne peut rien conclure a priori.

L'idée de la *méthode du retour* issue de l'article [Cor92] est la suivante : au lieu de linéariser autour de l'état d'équilibre $(0,0)$, on va linéariser autour d'une trajectoire qui part de $(0,0)$ et qui *retourne* en $(0,0)$ et telle que le linéarisé le long de cette trajectoire soit contrôlable.

Plus précisément, supposons que pour tout $T>0$, tout $\varepsilon>0$, il existe $\overline{h} \in C([0,T];\mathbb{R}^m)$ tel que $\left\|\overline{h}\right\|_\infty \leq \varepsilon$, tel que la solution (maximale) associée $\overline{y}$ de $\overline{y}'(t) = f(\overline{y}(t),\overline{h}(t))$, $\overline{y}(0)=0$, vérifie :

— $\overline{y}(T)=0$,

— le système linéarisé autour de $(\overline{y},\overline{h})$ (voir Définition 2.1.4) est contrôlable sur $[0,T]$.

Alors, par le Théorème 2.1.5, il existe $\eta>0$ tel que, pour tous $y_0, y_1 \in \mathbb{R}^n$ vérifiant $|y_0|, |y_1| \leq \eta$, il existe $h \in C([0,T];\mathbb{R}^m)$ telle que $|h(t)-\overline{h}(t)| \leq \varepsilon$, $t\in[0,T]$, et telle que la solution y du problème de Cauchy

$$y' = f(y,h(t)),\ y(0)=y_0,$$

satisfait $y(T)=y_1$.

Comme $T>0$ et $\varepsilon>0$ sont arbitraires, on en déduit que $y'=f(y,h)$ est localement contrôlable en temps petit autour de l'état d'équilibre $(0,0)$.

Revenons à l'Exemple 2.1.8 avec n impair. Pour $T>0$, on construit $\overline{h}\in C^\infty([0,T];\mathbb{R})$ différent de la fonction nulle tel que

$$\int_0^{T/2} \overline{h}(t)dt = 0,\ \overline{h}(T-t) = \overline{h}(t),\ t\in[0,T].$$

On introduit la solution $(\overline{y_1},\overline{y_2}) \in C^\infty([0,T];\mathbb{R})^2$ du système

$$\begin{cases} \overline{y_1}' = \overline{h} \\ \overline{y_2}' = \overline{y_1}^n, \\ (\overline{y_1},\overline{y_2})(0) = (0,0). \end{cases}$$

On vérifie alors que

$$\overline{y_1}(T/2) = 0,\ \forall t\in[0,T],\ \overline{y_1}(T-t) = -\overline{y_1}(t),\ \overline{y_2}(T-t) = \overline{y_2}(t).$$

En particulier, on a

$$\overline{y_1}(T) = \overline{y_2}(T) = 0.$$

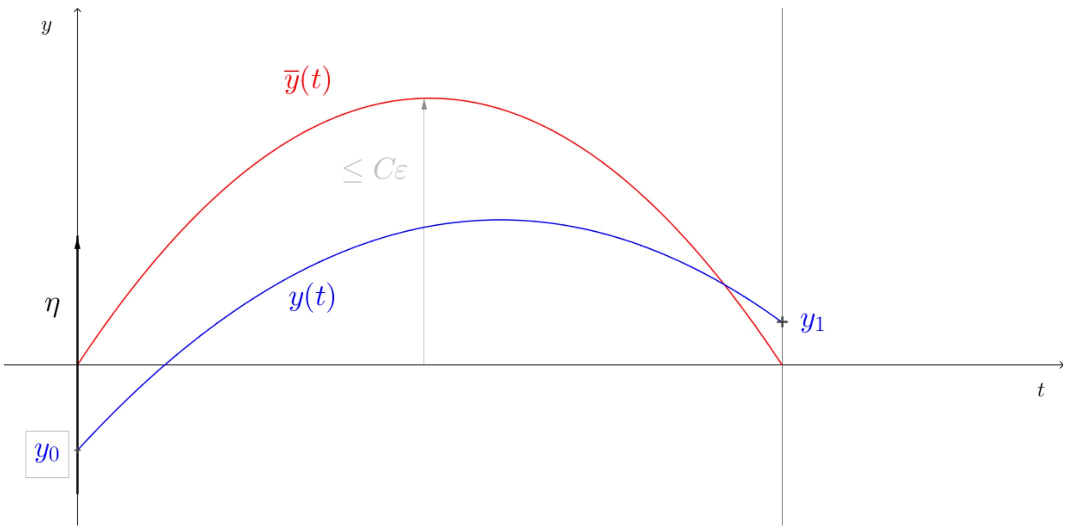

FIGURE 2.1 – Illustration de la méthode du retour

Le système linéarisé autour de $(\overline{y}, \overline{h})$ est

$$\begin{cases} y_1' = h \\ y_2' = n\overline{y_2}^{n-1}(t)y_1, \\ (\overline{y_1}, \overline{y_2})(0) = (0,0). \end{cases}$$

Et donc en posant

$$A(t) = \begin{pmatrix} 0 & 0 \\ n\overline{y_2}^{n-1}(t) & 0 \end{pmatrix}, \ B(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

on voit facilement que le condition (1.12) est satisfaite si et seulement si

$$\exists k \in \mathbb{N}, \ \frac{d^k\overline{y_2}}{dt}(\overline{t}) \neq 0 \text{ où } \overline{t} \ \in [0,T],$$

ce qui est assuré par le fait que h est différent de la fonction nulle. Par le Théorème 1.2.1, on en déduit que le linéarisé est contrôlable et donc par la méthode du retour que le système non linéaire est localement contrôlable à $((0,0),0)$ en temps petit.

Pour d'autres exemples d'applications de la méthode du retour, voir [Cor07a, Chapter 6].

Deuxième partie

Contrôlabilité d'équations et de systèmes paraboliques

Dans cette partie, nous allons aborder la contrôlabilité des systèmes paraboliques linéaires et non linéaires. Un grand nombre d'idées préalablement utilisées pour les systèmes différentiels sont revisitées dans le cadre de la dimension infinie.

Nous introduisons quelques notations qui sont utilisées dans toute la suite :

— $T \in (0, +\infty)$ est un *temps*,

— $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) est un ouvert borné connexe de classe C^2 qui représente le *domaine spatial*,

— ω est un ouvert non vide contenu dans Ω qui désigne *l'ouvert de contrôle*,

— $Q_T := (0, T) \times \Omega$ est le *cylindre parabolique*,

— $\Sigma_T := (0, T) \times \partial\Omega$ est la *frontière parabolique*,

— $q_T := (0, T) \times \omega$ est le *cylindre parabolique de contrôle*.

Avant de commencer la lecture de cette partie, il peut être utile de lire l'Annexe A qui énonce les principales propriétés des équations paraboliques dont nous allons nous servir.

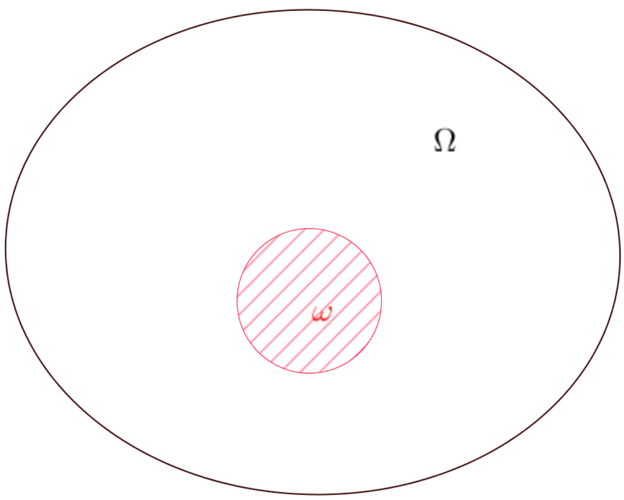

FIGURE 2.2 – L'ouvert Ω et le sous-ouvert ω

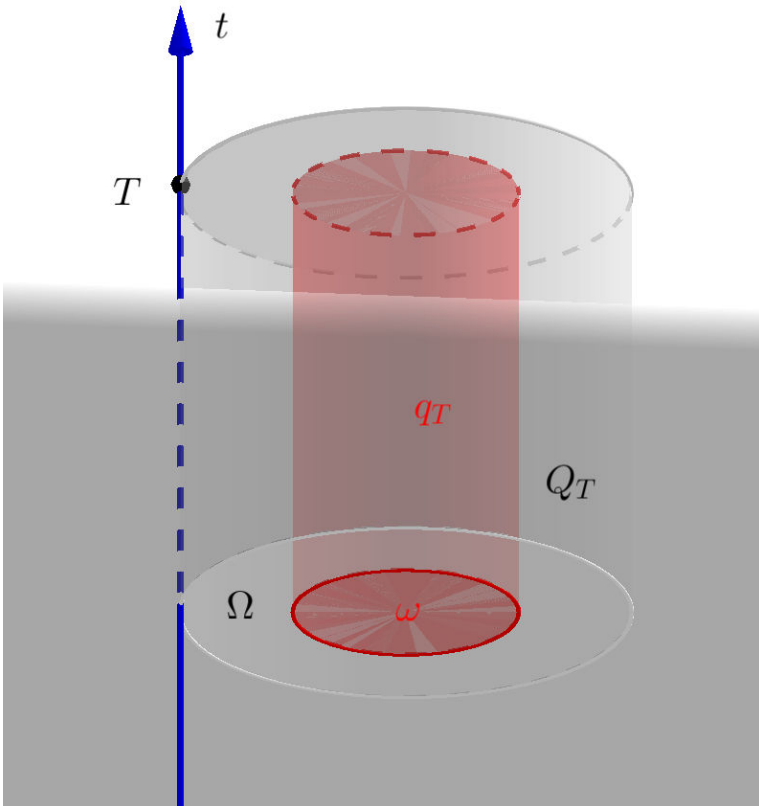

FIGURE 2.3 – Le cylindre Q_T et le sous-cylindre q_T

Chapitre 3

L'équation de la chaleur

3.1 Le problème de la contrôlabilité à zéro

L'équation de la chaleur avec conditions de Dirichlet au bord de Ω et contrôle localisé s'écrit comme suit :

$$\begin{cases} \partial_t y - \Delta y = h1_\omega & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega. \end{cases} \tag{Heat}$$

Remarque 3.1.1. Pour $(t,x) \in Q_T$, si $y(t,x)$ représente une température à l'instant t et au point x de notre domaine spatial Ω alors le terme $h1_\omega$ peut être vu comme un terme de chauffage ou de refroidissement localisé en espace.

On suppose que la donnée initiale y_0 est dans $L^2(\Omega)$ et on cherche à trouver un contrôle $h \in L^2(q_T)$ qui amène l'état $y(t,.)$ à une donnée cible au temps $t = T$, c'est-à-dire on cherche à répondre à une question de contrôlabilité pour l'équation de la chaleur au moyen d'un contrôle localisé en espace. En raison des *effets régularisants* de l'équation de la chaleur (voir par exemple l'Annexe A.6 et le Corollaire A.6.5), il n'est pas possible d'amener *exactement* les solutions de (Heat) à un état cible arbitraire dans $L^2(\Omega)$ ou $H^m(\Omega)$, à moins que nous soyons dans la situation triviale où $\omega = \Omega$. D'autre part, il semble intéressant de se demander s'il est possible d'amener les solutions de (Heat) à un état prescrit d'une trajectoire (en agissant seulement sur $\omega \subset\subset \Omega$). Cette discussion nous amène à la définition suivante.

Définition 3.1.2. On dit que (Heat) est *contrôlable à zéro* au temps T si pour toute donnée initiale $y_0 \in L^2(\Omega)$, il existe un contrôle $h \in L^2(q_T)$ telle que la solution y du problème de Cauchy (Heat) satisfait

$$y(T,.) = 0 \text{ dans } \Omega.$$

Remarque 3.1.3. Par linéarité, il est facile de voir que la contrôlabilité à zéro est équivalente à la *contrôlabilité aux trajectoires*, c'est-à-dire que pour toute donnée initiale

$y_0 \in L^2(\Omega)$, pour toute trajectoire $(\overline{y}, \overline{h})$, i.e., vérifiant

$$\begin{cases} \partial_t \overline{y} - \Delta \overline{y} = \overline{h} 1_\omega & \text{dans } Q_T, \\ \overline{y} = 0 & \text{sur } \Sigma_T, \end{cases}$$

il existe un contrôle $h \in L^2(q_T)$ tel que $y(T,.) = \overline{y}(T,.)$.

On peut également seulement demander à la solution de se rapprocher de manière arbitrairement proche d'un état prescrit : c'est la notion de *contrôlabilité approchée.*

Définition 3.1.4. On dit que (Heat) est *approximativement contrôlable* au temps T si pour toute donnée initiale $y_0 \in L^2(\Omega)$, tout état cible $y_f \in L^2(\Omega)$, tout $\varepsilon > 0$, il existe un contrôle $h \in L^2(q_T)$ tel que la solution y du problème de Cauchy (Heat) satisfait

$$\|y(T,.) - y_f(.)\|_{L^2(\Omega)} \leq \varepsilon.$$

Remarque 3.1.5. On utilisera assez peu la Définition 3.1.4 dans la suite. Cependant, ce concept nous servira à revisiter la notion de *continuation unique* déjà vue dans le cadre de la dimension finie (voir preuve du Théorème 1.1.1).

3.2 L'équation de la chaleur est contrôlable à zéro en tout temps

Le théorème suivant est le théorème central de la théorie de la contrôlabilité à zéro des équations paraboliques.

Théorème 3.2.1. *Pour tout temps $T > 0$, l'équation* (Heat) *est contrôlable à zéro au temps T.*

Remarque 3.2.2. Le Théorème 3.2.1 provient du caractère parabolique de l'équation de la chaleur et plus précisément de la *vitesse de propagation infinie* de cette équation (voir par exemple le *principe du maximum fort* énoncé en Proposition A.4.2 et la Remarque A.4.3). Une action via le contrôle h localisé sur ω va influer immédiatement sur la solution y, et ce sur Ω tout entier. A titre de comparaison, le résultat du Théorème 3.2.1 ne tient pas pour une équation de transport par exemple, qui est une équation aux dérivées partielles à *vitesse de propagation finie* (voir [Cor07a, Chapter 2, Section 2.1]).

3.3 La « Hilbert Uniqueness Method »

L'objectif de cette partie est de montrer par un argument de dualité l'équivalence entre le problème de contrôlabilité à zéro, qui est un problème de surjectivité et une inégalité d'observabilité pour le système adjoint de (Heat). Nous adaptons en dimension infinie l'argument de la seconde preuve du Théorème 1.1.1.

On introduit le système adjoint de (Heat) : pour $\varphi_T \in L^2(\Omega)$,

$$\begin{cases} -\partial_t \varphi - \Delta\varphi = 0 & \text{dans } Q_T, \\ \varphi = 0 & \text{sur } \Sigma_T, \\ \varphi(T,.) = \varphi_T & \text{dans } \Omega. \end{cases} \tag{HeatAdj}$$

On dispose du résultat suivant dû à Jacques-Louis Lions (voir [Lio88]).

Théorème 3.3.1. *L'équation* (Heat) *est contrôlable à zéro au temps T si et seulement si il existe une constante $C > 0$ telle que*

$$\forall \varphi_T \in L^2(\Omega),\ \|\varphi(0,.)\|_{L^2(\Omega)} \leq C \left(\int_0^T \int_\omega |\varphi(t,x)|^2 dtdx \right)^{1/2}. \tag{HeatObs}$$

Remarque 3.3.2. L'inégalité (HeatObs) est appelée *inégalité d'observabilité.* On dit aussi qu'on *observe* φ au temps $t = 0$ par φ sur $(0,T) \times \omega$.

La démonstration du Théorème 3.3.1 repose sur un argument de dualité.

Démonstration. On introduit deux applications :

$$\mathcal{F}_1 : y_0 \in L^2(\Omega) \mapsto y(T,.) \in L^2(\Omega),$$

où y est la solution de (Heat) associée à $(y_0, h = 0)$, et

$$\mathcal{F}_2 : h \in L^2(q_T) \mapsto y(T,.) \in L^2(\Omega),$$

où y est la solution de (Heat) associée à $(y_0 = 0, h)$.

La remarque clef est la suivante :

$$\Big(\text{Contrôlabilité à zéro}\Big) \Leftrightarrow \Big(\mathrm{Im}(\mathcal{F}_1) \subset \mathrm{Im}(\mathcal{F}_2)\Big). \tag{3.1}$$

Or, par un argument de dualité (voir [Cor07a, Lemma 2.48]), on a :

$$\begin{aligned} &\Big(\mathrm{Im}(\mathcal{F}_1) \subset \mathrm{Im}(\mathcal{F}_2)\Big) \\ &\quad \Leftrightarrow \exists C > 0,\ \forall \varphi_T \in L^2(\Omega),\ \|\mathcal{F}_1^*(\varphi_T)\|_{L^2(\Omega)} \leq C \|\mathcal{F}_2^*(\varphi_T)\|_{L^2(q_T)}. \end{aligned} \tag{3.2}$$

On calcule les adjoints de ces deux applications en utilisant l'égalité :

$$(y(T,.), \varphi_T)_{L^2(\Omega)} - (y_0, \varphi(0,.))_{L^2(\Omega)} = \int_0^T \int_\Omega h(t,x) 1_\omega(x) \varphi(t,x) dtdx.$$

On trouve ainsi

$$\forall \varphi_T \in L^2(\Omega),\ \mathcal{F}_1^*(\varphi_T) = \varphi(0,.) \text{ and } \mathcal{F}_2^*(\varphi_T) = \varphi 1_\omega. \tag{3.3}$$

En rassemblant (3.1), (3.2) et (3.3), on conclut la preuve du Théorème 3.3.1. □

Par la même preuve en remplaçant (3.1) par

$$\Big(\text{Contrôlabilité approchée}\Big) \Leftrightarrow \Big(\overline{\mathrm{Im}(\mathcal{F}_2)} = L^2(\Omega)\Big) \Leftrightarrow \Big(\mathrm{Ker}(\mathcal{F}_2^*) = \{0\}\Big), \tag{3.4}$$

on peut également montrer le théorème suivant.

Théorème 3.3.3. *L'équation* (Heat) *est approximativement contrôlable au temps T si et seulement si*

$$\forall \varphi_T \in L^2(\Omega),\ \Big(\forall (t,x) \in (0,T) \times \omega,\ \varphi(t,x) = 0\Big) \Rightarrow (\varphi_T = 0). \tag{UC}$$

Remarque 3.3.4. L'inégalité (HeatObs) est la quantification du *principe de continuation unique* (UC). En effet, si $\varphi = 0$ sur $(0,T) \times \omega$, alors par (HeatObs), on a $\varphi(0,.) = 0$ sur Ω et par *unicité rétrograde* de l'équation de la chaleur (voir la Proposition A.5.1), on a alors $\varphi_T = 0$.

Remarque 3.3.5. En dimension finie, nous avons vu que les concepts de continuation unique et d'inégalité d'observabilité sont équivalents (voir Remarque 1.1.5). Dans ce contexte, en utilisant (3.1) et (3.4), cela se traduirait par le fait qu'un sous-espace vectoriel F d'un espace vectoriel de dimension finie E est dense si et seulement si $F = E$. Bien entendu, ce n'est plus le cas en dimension infinie : pensez ici à $F = \mathrm{Im}(\mathcal{F}_2)$ et $E = L^2(\Omega)$.

Corollaire 3.3.6. *Pour tout $T > 0$, l'équation* (Heat) *est approximativement contrôlable au temps T.*

En effet, il est clair que (UC) est vérifiée puisqu'une solution φ de (HeatAdj) s'annulant sur le cylindre q_T est identiquement nulle sur le cylindre Q_T par analyticité (voir Proposition A.5.2).

3.4 Preuve de l'inégalité d'observabilité : les inégalités de Carleman

L'objectif de cette partie est de démontrer que l'inégalité (HeatObs) est vérifiée en tout temps $T > 0$. Ainsi, par le Théorème 3.3.1, on aura démontré le Théorème 3.2.1. Pour ce faire, nous allons dans un premier temps démontrer une inégalité de Carleman L^2 due à Andrei Fursikov et Oleg Imanuvilov.

3.4.1 Fonctions poids et estimations

On introduit une fonction poids dont les points critiques sont localisés à l'intérieur de la zone de contrôle.

Lemme 3.4.1. *Soit $\omega_0 \subset\subset \omega$ un ouvert non vide. Alors il existe $\eta^0 \in C^2(\overline{\Omega})$ tel que $\eta^0 > 0$ dans Ω, $\eta^0 = 0$ sur $\partial\Omega$, et $|\nabla \eta^0| > 0$ dans $\overline{\Omega \setminus \omega_0}$.*

Ce lemme est dû à Fursikov-Imanuvilov (voir [FI96]), une preuve peut être trouvée dans [Cor07a, Lemma 2.68].

Soit ω_0 un ouvert non vide satisfaisant $\omega_0 \subset\subset \omega$. On fixe η^0 comme dans le Lemme 3.4.1 et on introduit

$$\alpha(t,x) := \frac{e^{2\lambda m \|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t(T-t)}, \tag{3.5}$$

$$\xi(t,x) := \frac{e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t(T-t)}, \tag{3.6}$$

pour $(t,x) \in Q_T$, où $\lambda \geq 1$ est un paramètre et $m > 1$.

Les estimations suivantes sont d'usage constant dans la suite.

Lemme 3.4.2. *Il existe une constante $C = C(\Omega, \omega) > 0$ telle que pour tous $T > 0$, $\lambda \geq 1$, $(t,x) \in Q_T$,*

$$\begin{aligned}
|\partial_i \alpha| = |-\partial_i \xi| = |-\lambda \partial_i \eta^0 \xi| &\leq C\lambda\xi, \\
|\Delta\alpha| = |-\Delta\xi| = |-\lambda\Delta\eta^0\xi - \lambda^2 |\nabla\eta^0|^2 \xi| &\leq C\lambda^2\xi, \\
|\partial_t \alpha| = \left| -(T-2t) \frac{e^{2\lambda m \|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t^2(T-t)^2} \right| &\leq CT\xi^2, \\
|\partial_t \xi| = \left| -(T-2t) \frac{e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t^2(T-t)^2} \right| &\leq CT\xi^2.
\end{aligned}$$

3.4.2 L'estimation de Carleman

Nous avons l'estimation L^2 suivante.

Théorème 3.4.3. *Il existe trois constantes $\lambda_1 = \lambda_1(\Omega, \omega) \geq 1$, $s_1 = C(\Omega, \omega)(T + T^2)$ et $C_1 = C_1(\Omega, \omega)$ telles que pour tous $\lambda \geq \lambda_1$, $s \geq s_1$,*

$$\begin{aligned}
&\lambda^4 \int_{Q_T} e^{-2s\alpha} (s\xi)^3 |\varphi|^2 dtdx + \lambda^2 \int_{Q_T} e^{-2s\alpha} (s\xi) |\nabla\varphi|^2 dtdx \\
&+ \int_{Q_T} e^{-2s\alpha} (s\xi)^{-1} \left(|\partial_t \varphi|^2 + |\Delta\varphi|^2 \right) dtdx \\
&\leq C_1 \left(\int_{Q_T} e^{-2s\alpha} |\partial_t \varphi + \Delta\varphi|^2 dtdx + \lambda^4 \int_{(0,T)\times\omega} e^{-2s\alpha} (s\xi)^3 |\varphi|^2 dtdx \right),
\end{aligned} \tag{Carleman}$$

où $\varphi \in C^2(\overline{Q_T})$ avec $\varphi = 0$ sur Σ_T.

Remarque 3.4.4. L'inégalité de Carleman du Théorème 3.4.3 est une estimation d'énergie à poids (de type exponentiellement décroissant en $t = 0$ et $t = T$ et de type exponentielle d'exponentielle en espace).

Remarque 3.4.5. Les paramètres λ et s jouent un double rôle. D'une part, ils sont cruciaux dans la preuve du Théorème 3.4.3 pour absorber des termes. D'autre part, lorsque nous considérerons non plus la simple équation de la chaleur mais une équation parabolique plus générale, ils joueront également un rôle (toujours d'absorption) pour montrer une inégalité de Carleman adaptée à cette équation (voir Section 3.5).

Remarque 3.4.6. L'inégalité (Carleman) est également vraie sous la forme suivante : pour $k \in \mathbb{N}$,

$$\begin{aligned}
&\lambda^4 \int_{Q_T} e^{-2s\alpha}(s\xi)^k |\varphi|^2 dtdx + \lambda^2 \int_{Q_T} e^{-2s\alpha}(s\xi)^{k-2} |\nabla\varphi|^2 dtdx \\
&+ \int_{Q_T} e^{-2s\alpha}(s\xi)^{k-4} \left(|\partial_t\varphi|^2 + |\Delta\varphi|^2\right) dtdx \\
&\leq C_1 \left(\int_{Q_T} e^{-2s\alpha}(s\xi)^{k-3} |\partial_t\varphi + \Delta\varphi|^2 dtdx + \lambda^4 \int_{(0,T)\times\omega} e^{-2s\alpha}(s\xi)^k |\varphi|^2 dtdx \right).
\end{aligned} \tag{3.7}$$

Nous utiliserons cette inégalité pour des systèmes paraboliques linéaires présentant des couplages d'ordre deux. Pour la preuve de (3.7), on renvoie à [FI96].

3.4.3 Idée de la preuve de l'inégalité de Carleman

Nous présentons les grandes lignes de la démonstration du Théorème 3.4.3. Pour les détails, voir [FCG06, Lemma 1.3]

Démonstration. La preuve suit plusieurs étapes.

Etape 1 : Calcul de l'opérateur conjugué et élévation au carré. Soit $\psi = e^{-s\alpha}\varphi$ et $g = e^{-s\alpha} f$ où on a noté $f = \partial_t\varphi + \Delta\varphi$. On regarde l'équation parabolique satisfaite par ψ et on obtient que

$$M_1\psi + M_2\psi = g_{s,\lambda}, \tag{3.8}$$

où on a défini

$$M_1\psi = -2s\lambda^2 |\nabla\eta^0|^2 \xi\psi - 2s\lambda\xi\nabla\eta^0.\nabla\psi + \partial_t\psi, \tag{3.9}$$

$$M_2\psi = s^2\lambda^2 |\nabla\eta^0|^2 \xi^2\psi + \Delta\psi + s\alpha_t\psi, \tag{3.10}$$

et

$$g_{s,\lambda} = g + s\lambda\Delta\eta^0\xi\psi - s\lambda^2|\nabla\eta^0|^2\xi\psi. \tag{3.11}$$

Pour simplifier les notations, on appelle $(M_i\psi)_j$, $(1 \leq i \leq 2, 1 \leq j \leq 3)$ le $j^{\text{ème}}$ terme de l'expression $M_i\psi$ donnée en (3.9), (3.10).

Remarque 3.4.7. Si on oublie le premier terme de M_1, on remarque que les opérateurs M_1 et M_2 sont respectivement antisymétriques et symétriques. Cela joue un rôle crucial dans la preuve. Il semblerait plus naturel à première vue de mettre le premier terme de M_1 dans le terme de droite de (3.8). Cependant, en le laissant à gauche, il nous aide à créer un terme positif en $|\nabla\psi|^2$ (voir le calcul du produit scalaire $((M_1\psi)_1, (M_2\psi)_2)_{L^2(Q_T)}$ plus loin).

On déduit de (3.8), en élevant au carré et en intégrant en temps-espace, que

$$\|M_1\psi\|^2_{L^2(Q_T)} + \|M_2\psi\|^2_{L^2(Q_T)} + 2\sum_{i,j=1}^{3}((M_1\psi)_i,(M_2\psi)_j)_{L^2(Q_T)} = \|g_{s,\lambda}\|^2_{L^2(Q_T)}. \quad (3.12)$$

Etape 2 : Estimation du double produit. La suite de la preuve consiste à montrer qu'il existe $C = C(\Omega,\omega) > 0$ tel que pour tous $\lambda \geq s_1$, $s \geq s_1$ où λ_1 et s_1 sont comme dans le Théorème 3.4.3, on a

$$2\sum_{i,j=1}^{3}((M_1\psi)_i,(M_2\psi)_j)_{L^2(Q_T)} \geq C\left(\int_{Q_T}|\psi|^2+|\nabla\psi|^2\right) - C\left(\int_{(0,T)\times\omega_0}|\psi|^2+|\nabla\psi|^2\right).$$

Par le Lemme 3.4.1, on a

$$c := \min_{x\in\overline{\Omega\setminus\omega_0}}|\nabla\eta^0(x)|^2 > 0. \quad (3.13)$$

Dans toute la suite de la preuve, C désigne une constante strictement positive dépendant de Ω et ω et qui peut changer d'une ligne à l'autre.

Par intégration par parties en espace, en utilisant $2\psi\nabla\psi = \nabla(\psi^2)$, on trouve grâce au Lemme 3.4.2 que

$$((M_1\psi)_2,(M_2\psi)_1) \geq 3s^3\lambda^4\int_{Q_T}|\nabla\eta^0|^4\xi^3|\psi|^2 - Cs^3\lambda^3\int_{Q_T}\xi^3|\psi|^2.$$

Puis en utilisant (3.13), on a

$$s^3\lambda^4\int_{Q_T}|\nabla\eta^0|^4\xi^3|\psi|^2 \geq cs^3\lambda^4\left(\int_{Q_T}\xi^3|\psi|^2 - \int_{(0,T)\times\omega_0}\xi^3|\psi|^2\right).$$

De plus, pour λ suffisamment grand : $\lambda \geq C$, on a

$$C\int_{Q_T}\xi^3|\psi|^2 \leq \frac{c}{2}s^3\lambda^4\int_{Q_T}\xi^3|\psi|^2.$$

Ainsi, en rassemblant les trois dernières estimations, on a pour $\lambda \geq C$,

$$((M_1\psi)_2,(M_2\psi)_1) \geq Cs^3\lambda^4\left(\int_{Q_T}\xi^3|\psi|^2 - \int_{(0,T)\times\omega_0}\xi^3|\psi|^2\right).$$

D'autre part, on a également, par intégration par parties en espaces,

$$((M_1\psi)_1,(M_2\psi)_2) = s\lambda^2\int_{Q_T}|\nabla\eta^0|^2\xi|\nabla\psi|^2 - Cs\lambda^3\int_{Q_T}\xi|\nabla\psi||\psi|.$$

Et donc, toujours par (3.13), on a

$$s\lambda^2\int_{Q_T}|\nabla\eta^0|^2\xi|\nabla\psi|^2 \geq cs\lambda^2\left(\int_{Q_T}\xi|\nabla\psi|^2 - \int_{(0,T)\times\omega_0}\xi|\psi|^2\right).$$

Puis, par l'inégalité de Young, on a

$$Cs\lambda^3 \int_{Q_T} \xi |\nabla\psi||\psi| \leq Cs^2\lambda^4 \int_{Q_T} \xi^2|\psi|^2 + C\lambda^2 \int_{Q_T} |\nabla\psi|^2.$$

En rassemblant les quatre dernières estimations et en prenant $s \geq CT^2$, on a alors

$$\begin{aligned}
&((M_1\psi)_2, (M_2\psi)_1) + ((M_1\psi)_1, (M_2\psi)_2) \\
&\geq C\left(\int_{Q_T} s^3\lambda^4\xi^3|\psi|^2 + s\lambda^2\xi|\nabla\psi|^2 - \left(\int_{(0,T)\times\omega_0} s^3\lambda^4\xi^3|\psi|^2 + s\lambda^2\xi|\nabla\psi|^2\right)\right).
\end{aligned}$$

Ainsi, en montrant que les autres termes du double produit sont négligeables au sens où ils peuvent être absorbés par les termes globaux $s^3\lambda^4 \int_{Q_T} \xi^3|\psi|^2$ ou $s\lambda^2 \int_{Q_T} \xi|\nabla\psi|^2$ pour $\lambda \geq \lambda_1$, $s \geq s_1$, on aboutit en utilisant (3.12) à

$$\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q_T)} + \|M_2\psi\|^2_{L^2(Q_T)} + \lambda^4 \int_{Q_T} (s\xi)^3|\psi|^2 + \lambda^2 \int_{Q_T} s\xi|\nabla\psi|^2 \\
&\leq C\left(\|g\|^2_{L^2(Q_T)} + s^2\lambda^4 \int_{Q_T} \xi^2\psi^2 + \lambda^4 \int_{(0,T)\times\omega_0} (s\xi)^3|\psi|^2 + \lambda^2 \int_{(0,T)\times\omega_0} s\xi|\nabla\psi|^2\right)
\end{aligned}$$

Et donc en absorbant le second terme du membre de droite de la précédente inégalité, on aboutit finalement à

$$\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q_T)} + \|M_2\psi\|^2_{L^2(Q_T)} + \lambda^4 \int_{Q_T} (s\xi)^3|\psi|^2 + \lambda^2 \int_{Q_T} s\xi|\nabla\psi|^2 \\
&\quad\leq C\left(\|g\|^2_{L^2(Q_T)} + \lambda^4 \int_{(0,T)\times\omega_0} (s\xi)^3|\psi|^2 + \lambda^2 \int_{(0,T)\times\omega_0} s\xi|\nabla\psi|^2\right). \qquad (3.14)
\end{aligned}$$

Etape 3 : Rajout des termes en ∂_t, Δ. On montre facilement à partir de (3.9), (3.10) et (3.14) que

$$\begin{aligned}
&\int_{Q_T} (s\xi)^{-1}\left(|\partial_t\psi|^2 + |\Delta\psi|^2\right) + \lambda^4 \int_{Q_T} (s\xi)^3|\psi|^2 + \lambda^2 \int_{Q_T} s\xi|\nabla\psi|^2 \\
&\quad\leq C\left(\|g\|^2_{L^2(Q_T)} + \lambda^4 \int_{(0,T)\times\omega_0} (s\xi)^3|\psi|^2 + \lambda^2 \int_{(0,T)\times\omega_0} s\xi|\nabla\psi|^2\right). \qquad (3.15)
\end{aligned}$$

Etape 4 : Élimination du terme local en ∇. Par un argument de troncature et une intégration par parties en espace appliquée au dernier terme du membre de droite de (3.15), on aboutit à

$$\begin{aligned}
&\int_{Q_T} (s\xi)^{-1}\left(|\partial_t\psi|^2 + |\Delta\psi|^2\right) + \lambda^4 \int_{Q_T} (s\xi)^3|\psi|^2 + \lambda^2 \int_{Q_T} s\xi|\nabla\psi|^2 \\
&\quad\leq C\left(\|g\|^2_{L^2(Q_T)} + \lambda^4 \int_{0,T)\times\omega} (s\xi)^3|\psi|^2\right). \qquad (3.16)
\end{aligned}$$

Etape 5 : Retour à la variable φ. On réécrit (3.16) à l'aide de la variable φ et on en déduit (Carleman). □

3.4.4 De l'inégalité de Carleman à l'inégalité d'observabilité

Nous montrons comment passer du Théorème 3.4.3 à (HeatObs).

Démonstration. Soit $\varphi_T \in L^2(\Omega)$ et φ la solution de (HeatAdj) associée. En utilisant que $\overline{C_c^\infty(\Omega)}^{L^2(\Omega)} = L^2(\Omega)$, on montre aisément par un argument de densité que (Carleman) est vérifiée pour φ.

On fixe $\lambda = \lambda_1$ et $s = s_1$ dans (Carleman) pour obtenir

$$\int_{Q_T} t^{-3}(T-t)^{-3}e^{-2s\alpha}|\varphi|^2 dxdt \leq C_1 \int_{(0,T)\times\omega} t^{-3}(T-t)^{-3}e^{-2s\alpha}|\varphi|^2 dxdt. \tag{3.17}$$

Premièrement, on remarque que sur $(T/4, 3T/4) \times \Omega$,

$$\begin{aligned} t^{-3}(T-t)^{-3}e^{-2s\alpha} &\geq \frac{C}{T^6} \exp\left(-\frac{C(\Omega,\omega)\left(T+T^2\right)}{T^2}\right) \\ &\geq \frac{C}{T^6} e^{-C(\Omega,\omega)\left(1+\frac{1}{T}\right)}. \end{aligned} \tag{3.18}$$

Deuxièmement, en utilisant que $x^3 e^{-Mx} \leq C/M^3$ pour $x, M \geq 0$ avec $x = t^{-1}(T-t)^{-1}$ et $M = C(\Omega,\omega)\left(T+T^2\right)$, on remarque que sur $(0,T)\times\omega$,

$$\begin{aligned} & t^{-3}(T-t)^{-3}e^{-2s\alpha} \\ & \leq t^{-3}(T-t)^{-3}\exp\left(-C(\Omega,\omega)\left(T+T^2\right)t^{-1}(T-t)^{-1}\right) \\ & \leq \frac{C}{\left(C(\Omega,\omega)\left(T+T^2\right)\right)^3} \\ & \leq \frac{C(\Omega,\omega)}{T^6}. \end{aligned} \tag{3.19}$$

Ainsi, on déduit de (3.17), (3.18) et (3.19) que

$$\int_{(T/4,3T/4)\times\Omega} |\varphi|^2 dxdt \leq e^{C(\Omega,\omega)\left(1+\frac{1}{T}\right)} \int_{(0,T)\times\omega} |\varphi|^2 dxdt. \tag{3.20}$$

D'autre part, par dissipation en temps de la norme L^2 (voir Proposition A.1.5), on a

$$\|\varphi(0,.)\|_{L^2(\Omega)} \leq \frac{2}{T}\int_{T/4}^{3T/4} \|\varphi(t,.)\|_{L^2(\Omega)}\, dt. \tag{3.21}$$

En rassemblant (3.20) et (3.21), on déduit

$$\|\varphi(0,.)\|_{L^2(\Omega)} \leq e^{C(\Omega,\omega)\left(1+\frac{1}{T}\right)} \left(\int_{(0,T)\times\omega} |\varphi|^2 dxdt\right)^{1/2},$$

ce qui conclut la preuve de (HeatObs). □

3.5 Équations paraboliques linéaires

Le but de cette partie est de généraliser le Théorème 3.2.1 à des opérateurs paraboliques plus généraux.

On s'intéresse à présent à une équation parabolique linéaire de la forme

$$\begin{cases} \partial_t y - \Delta y - \nabla.(B(t,x)y) + a(t,x)y = h1_\omega & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega, \end{cases} \tag{3.22}$$

où $y_0 \in L^2(\Omega)$, $a \in L^\infty(Q_T)$ et $B \in L^\infty(Q_T)^N$.

On dispose du théorème suivant.

Théorème 3.5.1. *Pour tout temps $T > 0$, l'équation (3.22) est contrôlable à zéro au temps T.*

La preuve du Théorème 3.5.1 repose également sur l'obtention d'une inégalité d'observabilité (HeatObs) mais cette fois-ci pour φ solution de

$$\begin{cases} -\partial_t \varphi - \Delta\varphi - B(t,x).\nabla\varphi + a(t,x)\varphi = 0 & \text{dans } Q_T, \\ \varphi = 0 & \text{sur } \Sigma_T, \\ \varphi(T,.) = \varphi_T & \text{dans } \Omega. \end{cases} \tag{3.23}$$

Démonstration. En appliquant (Carleman) à φ, on trouve pour $\lambda \geq \lambda_1$, $s \geq s_1$,

$$\begin{aligned} &\lambda^4 \int_{Q_T} e^{-2s\alpha}(s\xi)^3|\varphi|^2 dtdx + \lambda^2 \int_{Q_T} e^{-2s\alpha}(s\xi)|\nabla\varphi|^2 dtdx \\ &\quad \leq C\left(\int_{Q_T} e^{-2s\alpha}\left(|a\varphi|^2 + |B\nabla\varphi|^2\right) dtdx + \lambda^4 \int_{(0,T)\times\omega} e^{-2s\alpha}(s\xi)^3|\varphi|^2 dtdx\right). \end{aligned}$$

Fixons $\lambda = \lambda_1$, alors pour $s \geq C(\Omega,\omega)T^2\left(\|a\|_\infty^{2/3} + \|B\|_\infty^2\right)$, on obtient

$$C\int_{Q_T} e^{-2s\alpha}\left(|a\varphi|^2 + |B\nabla\varphi|^2\right) \leq \frac{1}{2}\lambda^4 \int_{Q_T} e^{-2s\alpha}(s\xi)^3|\varphi|^2 + \frac{1}{2}\lambda^2 \int_{Q_T} e^{-2s\alpha}(s\xi)|\nabla\varphi|^2.$$

D'où, par absorption (et en oubliant ensuite le terme positif à gauche en ∇), on obtient

$$\lambda^4 \int_{Q_T} e^{-2s\alpha}(s\xi)^3|\varphi|^2 dtdx \leq C\lambda^4 \int_{(0,T)\times\omega} e^{-2s\alpha}(s\xi)^3|\varphi|^2 dtdx.$$

En procédant comme dans la preuve précédente : voir (3.17), (3.18) et (3.19), on aboutit donc à

$$\int_{(T/4,3T/4)\times\Omega} |\varphi|^2 dxdt \leq e^{C(\Omega,\omega)\left(1+\frac{1}{T}+\|a\|_\infty^{2/3}+\|B\|_\infty^2\right)} \int_{(0,T)\times\omega} |\varphi|^2 dxdt.$$

Par estimation de dissipation L^2 (voir Proposition A.1.5 et Remarque A.1.6), on a également

$$\|\varphi(0,.)\|^2_{L^2(\Omega)} \leq \exp\left(CT\left(\|a\|_\infty + \|B\|^2_\infty\right)\right)\|\varphi(t,.)\|^2_{L^2(\Omega)}.$$

D'où

$$\|\varphi(0,.)\|^2_{L^2(\Omega)} \leq e^{C(\Omega,\omega)\left(1+\frac{1}{T}+\|a\|_\infty^{2/3}+T\|a\|_\infty+(1+T)\|B\|^2_\infty\right)}\int_{(0,T)\times\omega}|\varphi|^2dxdt,$$

ce qui conclut la preuve de l'inégalité d'observabilité donc du Théorème 3.5.1. □

Remarque 3.5.2. On tire en fait de la précédente preuve les inégalités d'observabilité suivantes :

$$\|\varphi(0,.)\|^2_{L^2(\Omega)} \leq C(\Omega,\omega,T,a,B)\int_{(0,T)\times\omega}|\varphi|^2dxdt, \tag{3.24}$$

$$\|\varphi(0,.)\|^2_{L^2(\Omega)} \leq C(\Omega,\omega,T,a,B)\int_{(0,T)\times\omega}e^{-2s\alpha}(s\xi)^3|\varphi|^2dxdt, \tag{3.25}$$

$$\int_{Q_T}e^{-2s\alpha}(s\xi)^3|\varphi|^2dtdx \leq C(\Omega,\omega,T,a,B)\int_{(0,T)\times\omega}e^{-2s\alpha}(s\xi)^3|\varphi|^2dxdt, \tag{3.26}$$

où

$$C(\Omega,\omega,T,a,B) = \exp\left(C(\Omega,\omega)\left(1+\frac{1}{T}+\|a\|_\infty^{2/3}+T\|a\|_\infty+(1+T)\|B\|^2_\infty\right)\right).$$

3.6 Commentaires bibliographiques

La contrôlabilité à zéro de l'équation de la chaleur en tout temps, i.e. le Théorème 3.2.1 a été démontré par Gilles Lebeau et Luc Robbiano (voir [LR95]) et de manière indépendante par Andrei Fursikov et Oleg Imanuvilov (voir [FI96]).

Ici, nous avons suivi l'approche de Fursikov et Imanuvilov qui consiste à établir des *inégalités de Carleman paraboliques*. Pour cela, nous nous sommes inspirés de la présentation de [FCG06].

L'approche de Lebeau et Robbiano consiste, quant à elle, à démontrer des *inégalités de Carleman elliptiques*. A partir de celles-ci, on peut en déduire une *inégalité spectrale* donc un *résultat de contrôle basse fréquence*. Une méthode communément appelée aujourd'hui *méthode de Lebeau-Robbiano* permet alors de passer de ce résultat de contrôle basse fréquence au Théorème 3.2.1. On pourra consulter avec profit [LRL12] pour les détails mais aussi [Mil10] et [BPS18] pour des généralisations.

Avant les travaux de Lebeau, Robbiano et Fursikov, Imanuvilov, le Théorème 3.2.1 avait été démontré dans le cas de la dimension un d'espace par la *méthode des moments* par Hector Fattorini et David Russell (voir [FR71], [TT07]). Le cas unidimensionnel a d'ailleurs été redémontré récemment par une approche de *backstepping* par Jean-Michel Coron et Hoai-Minh Nguyen (voir [CN17]) ou par une approche de type Carleman par Jérémy Dardé et Sylvain Ervedoza (voir [DE19, Section 4.4]).

Le cas multidimensionnel avec une *condition de contrôle géométrique dite « GCC »* peut être retrouvé à partir d'un résultat de contrôlabilité pour l'équation des ondes (voir [BLR92]) et de la méthode de transmutation (voir [Mil06], [EZ11b]). Pour un résultat de contrôlabilité sans « GCC » de l'équation de la chaleur à partir de l'équation des ondes, voir [EZ11a].

On peut également démontrer que la contrôlabilité à zéro de l'équation de la chaleur (Heat) a lieu en tout temps $T > 0$ et pour tout ensemble mesurable de contrôle ω de mesure de Lebesgue non nulle, contenu dans Ω (voir [AEWZ14] et les références dedans).

Le Théorème 3.5.1 est attribué à Andrei Fursikov et Oleg Imanuvilov (voir [FI96]). Les estimations d'observabilité quantifiées de la Remarque 3.5.2 sont dues à Anna Doubova, Enrique Fernández-Cara, Manuel Gonzalez Burgos et Enrique Zuazua (voir [FCZ00] et [DFCGBZ02]).

Il est également démontré dans [VZ08] et [Erv08] des résultats de contrôlabilité à zéro pour l'équation de la chaleur avec potentiel singulier.

Dans le cas unidimensionnel, mais pour des opérateurs paraboliques à coefficients peu réguliers, mentionnons les travaux [LR07], [BDLR07] et [AE08].

Concernant la contrôlabilité à zéro de la chaleur avec termes non locaux, le lecteur pourra consulter par exemple [FCLZ16] et [CSZZ17].

Chapitre 4

Les systèmes paraboliques linéaires : à la recherche du couplage

Les systèmes que nous allons étudier dans la suite prennent la forme suivante :

$$\begin{cases} \partial_t U - D\Delta U = A(t,x)U + BH1_\omega & \text{dans } Q_T, \\ U = 0 & \text{sur } \Sigma_T, \\ U(0,.) = U_0 & \text{dans } \Omega, \end{cases} \tag{4.1}$$

où $D \in \mathbb{R}^{n\times n}$ est diagonalisable à valeurs propres strictement positives, $A \in L^\infty(Q_T;\mathbb{R}^{n\times n})$ et $B \in \mathbb{R}^{n\times m}$. Dans (4.1), l'état U est à valeurs dans $\mathbb{R}^n$ et le contrôle H est à valeurs dans $\mathbb{R}^m$. Le système adjoint de (4.1) est le système

$$\begin{cases} -\partial_t\varphi - D^{\mathrm{tr}}\Delta\varphi = A(t,x)^{\mathrm{tr}}\varphi & \text{dans } Q_T, \\ \varphi = 0 & \text{sur } \Sigma_T, \\ \varphi(T,.) = \varphi_T & \text{dans } \Omega. \end{cases} \tag{4.2}$$

Par la « HUM », on a le résultat suivant.

Théorème 4.0.1. *Le système (4.1) est contrôlable à zéro au temps T si et seulement si il existe une constante $C > 0$ telle que*

$$\forall\varphi_T \in L^2(\Omega)^n,\ \|\varphi(0,.)\|_{L^2(\Omega)^n} \leq C\left(\int_0^T\int_\omega |B^{\mathrm{tr}}\varphi(t,x)|^2 dtdx\right)^{1/2}. \tag{4.3}$$

4.1 Couplages d'ordre zéro : cas des systèmes 2×2

Afin d'illustrer les principaux rouages de la machinerie dans le cadre des systèmes linéaires paraboliques contrôlés, nous allons nous concentrer dans un premier temps sur des systèmes 2×2, i.e. $n = 2$. On supposera également que $D = I_2$. On s'intéresse principalement aux cas où $B = I_2$, c'est-à-dire au système

$$\begin{cases} \partial_t u_1 - \Delta u_1 = a_{11}u_1 + a_{12}u_2 + h_1 1_\omega & \text{dans } Q_T, \\ \partial_t u_2 - \Delta u_2 = a_{21}u_1 + a_{22}u_2 + h_2 1_\omega & \text{dans } Q_T, \\ u_1 = u_2 = 0 & \text{sur } \Sigma_T, \\ (u_1, u_2)(0, .) = (u_{1,0}, u_{2,0}) & \text{dans } \Omega, \end{cases} \tag{4.4}$$

et au cas où $B = (1, 0)^{\text{tr}}$, c'est-à-dire au système

$$\begin{cases} \partial_t u_1 - \Delta u_1 = a_{11}u_1 + a_{12}u_2 + h_1 1_\omega & \text{dans } Q_T, \\ \partial_t u_2 - \Delta u_2 = a_{21}u_1 + a_{22}u_2 & \text{dans } Q_T, \\ u_1 = u_2 = 0 & \text{sur } \Sigma_T, \\ (u_1, u_2)(0, .) = (u_{1,0}, u_{2,0}) & \text{dans } \Omega. \end{cases} \tag{4.5}$$

Le système adjoint s'écrit alors

$$\begin{cases} -\partial_t \varphi_1 - \Delta \varphi_1 = a_{11}\varphi_1 + a_{21}\varphi_2 & \text{dans } Q_T, \\ -\partial_t \varphi_2 - \Delta \varphi_2 = a_{12}\varphi_1 + a_{22}\varphi_2 & \text{dans } Q_T, \\ \varphi_1 = \varphi_2 = 0 & \text{sur } \Sigma_T, \\ (\varphi_1, \varphi_2)(T, .) = (\varphi_{1,T}, \varphi_{2,T}) & \text{dans } \Omega. \end{cases} \tag{4.6}$$

4.1.1 Autant de contrôles que d'équations

Théorème 4.1.1. *Le système (4.4) est contrôlable à zéro au temps T.*

Démonstration. Par le Théorème 4.0.1, il suffit de démontrer (4.3). On applique l'inégalité (Carleman) à chacune des équations du système adjoint (4.6). On trouve :

$$\begin{aligned} &\lambda^4 \int_{Q_T} e^{-2s\alpha}(s\xi)^3|\varphi_1|^2 dtdx + \lambda^4 \int_{Q_T} e^{-2s\alpha}(s\xi)^3|\varphi_2|^2 dtdx \\ &\leq C\left(\sum_{i,j=1}^{2} \int_{Q_T} e^{-2s\alpha}|a_{ij}(t,x)\varphi_j|^2 dtdx + \lambda^4 \int_{q_T} e^{-2s\alpha}(s\xi)^3|\varphi_j|^2 dtdx\right). \end{aligned}$$

Par absorption, en prenant s suffisamment grand, on aboutit à

$$\lambda^4 \int_{Q_T} e^{-2s\alpha}(s\xi)^3|\varphi|^2 dtdx \leq C\lambda^4 \int_{q_T} e^{-2s\alpha}(s\xi)^3|\varphi|^2 dtdx,$$

et donc à (4.3) en utilisant des arguments similaires à ceux précédemment employés dans la Section 3.4.4. □

4.1.2 Un seul contrôle : contrôle indirect

On a le théorème suivant dû à Luz de Teresa (voir [dT00]).

Théorème 4.1.2. *On fait l'hypothèse suivante*

$$\exists\ t_1 < t_2 \in (0,T),\ \omega_0 \subset\subset \omega,\ \varepsilon > 0,\ \forall (t,x) \in (t_1,t_2) \times \omega_0,\ a_{21}(t,x) \geq \varepsilon. \tag{4.7}$$

Alors, le système (4.5) est contrôlable à zéro au temps T.

Remarque 4.1.3. La philosophie derrière ce théorème est que pour contrôler un système de deux équations avec un seul contrôle, il va falloir un bon terme de couplage. Heuristiquement, h_1 contrôle u_1 grâce la première équation et u_1 contrôle de manière indirecte u_2 grâce au terme de couplage $a_{21}u_1$.

Remarque 4.1.4. Le Théorème 4.1.2 reste vrai si $a_{21} \leq -\varepsilon$ sur $(t_1,t_2) \times \omega_0$.

Démonstration. Par le Théorème 4.0.1, il suffit de démontrer (4.3). On procède comme précédemment : on applique l'inégalité (Carleman) en remplaçant $\omega \leftarrow \omega_0$ à chacune des deux équations de (4.6) et on absorbe pour aboutir à

$$\begin{aligned}
&\sum_{i=1}^{2} \Big(\int_{Q_T} e^{-2s\alpha}(s\xi)^3 |\varphi_i|^2 dtdx + \int_{Q_T} e^{-2s\alpha}(s\xi)|\nabla\varphi_i|^2 dtdx \\
&+ \int_{Q_T} e^{-2s\alpha}(s\xi)^{-1} \left(|\partial_t\varphi_i|^2 + |\Delta\varphi_i|^2\right) dtdx \Big) \\
&\leq C \left(\int_{(0,T)\times\omega_0} e^{-2s\alpha}(s\xi)^3|\varphi_1|^2 dtdx + \int_{(0,T)\times\omega_0} e^{-2s\alpha}(s\xi)^3|\varphi_2|^2 dtdx \right).
\end{aligned} \tag{4.8}$$

Afin d'obtenir l'inégalité d'observabilité désirée, il nous faut nous débarrasser du second terme du membre de droite de la précédente inégalité. Pour cela, nous allons utiliser (4.7). On se donne une fonction de troncature en espace χ telle que $\chi = 1$ sur ω_0 et $\mathrm{supp}(\chi) \subset \omega$. On multiplie alors la première équation de (4.2) par $\chi e^{-2s\alpha}(s\xi)^3\varphi_2$ et on intègre sur $(0,T) \times \Omega$, on trouve

$$\int_{(0,T)\times\omega_0} e^{-2s\alpha}(s\xi)^3|\varphi_2|^2 \leq \frac{C}{\varepsilon} \int_{Q_T} \chi e^{-2s\alpha}(s\xi)^3 \varphi_2 \left(-\partial_t\varphi_1 - \Delta\varphi_1 - a_{11}\varphi_1\right). \tag{4.9}$$

Par intégration par parties en temps, on a

$$\int_{Q_T} \chi e^{-2s\alpha}(s\xi)^3 \varphi_2 \partial_t\varphi_1 = \int_{Q_T} \chi \left(\partial_t(e^{-2s\alpha}(s\xi)^3)\varphi_2 + e^{-2s\alpha}(s\xi)^3 \partial_t\varphi_2\right)\varphi_1.$$

En utilisant $\partial_t(e^{-2s\alpha}(s\xi)^3) \leq Ce^{-2s\alpha}(s\xi)^5$ grâce au Lemme 3.4.2, on déduit que par l'inégalité de Young, pour tout $\delta > 0$,

$$\int_{Q_T} \chi\partial_t(e^{-2s\alpha}(s\xi)^3)\varphi_2 \leq \delta \int_{Q_T} e^{-2s\alpha}(s\xi)^3|\varphi_2|^2 + C_\delta \int_{(0,T)\times\omega} e^{-2s\alpha}(s\xi)^7|\varphi_1|^2.$$

De plus, on a

$$\int_{Q_T} \chi e^{-2s\alpha}(s\xi)^3 (\partial_t\varphi_2)\varphi_1 \leq \delta \int_{Q_T} e^{-2s\alpha}(s\xi)^{-1}|\partial_t\varphi_2|^2 + C_\delta \int_{(0,T)\times\omega} e^{-2s\alpha}(s\xi)^7|\varphi_1|^2.$$

D'où

$$\int_{Q_T} \chi e^{-2s\alpha}(s\xi)^3\varphi_2\partial_t\varphi_1$$
$$\leq \delta \int_{Q_T} e^{-2s\alpha}\left((s\xi)^3|\varphi_2|^2 + (s\xi)^{-1}|\partial_t\varphi_2|^2\right) + C_\delta \int_{(0,T)\times\omega} e^{-2s\alpha}(s\xi)^7|\varphi_1|^2. \qquad (4.10)$$

Par intégrations par parties en espace, en utilisant à nouveau le Lemme 3.4.2, on montre que

$$\int_{Q_T} \chi e^{-2s\alpha}(s\xi)^3\varphi_2\Delta\varphi_1$$
$$\leq \delta \int_{Q_T} e^{-2s\alpha}\left((s\xi)^3|\varphi_2|^2 + (s\xi)|\nabla\varphi_2|^2 + (s\xi)^{-1}|\Delta\varphi_2|^2\right)$$
$$+ C_\delta \int_{q_T} e^{-2s\alpha}(s\xi)^7|\varphi_1|^2. \qquad (4.11)$$

Puis, on a aussi

$$\int_{Q_T} \chi e^{-2s\alpha}(s\xi)^3\varphi_2 a_{11}\varphi_1 \leq \delta \int_{Q_T} e^{-2s\alpha}(s\xi)^3|\varphi_2|^2 + C_\delta \int_{q_T} e^{-2s\alpha}(s\xi)^3|\varphi_1|^2. \qquad (4.12)$$

En rassemblant (4.8), (4.9), (4.10), (4.11) et (4.12) et en prenant δ suffisamment petit (pour absorber), on aboutit à (oubliant les termes positifs à gauche en ∇, ∂_t, Δ)

$$\int_{Q_T} e^{-2s\alpha}(s\xi)^3|\varphi|^2 \leq C \int_{(0,T)\times\omega} e^{-2s\alpha}(s\xi)^7|\varphi_1|^2,$$

d'où (4.3). □

On déduit du Théorème 4.1.2 le résultat suivant.

Corollaire 4.1.5. *On suppose que A est constant en temps et en espace dans* (4.5)*. Alors, le système* (4.1) *est contrôlable à zéro au temps T si et seulement si $a_{21} \neq 0$.*

Démonstration. Si $a_{21} \neq 0$ alors le résultat découle immédiatement du Théorème 4.1.2.

Réciproquement, si $a_{21} = 0$, alors la seconde équation de (4.1) est

$$\partial_t u_2 - \Delta u_2 = a_{22}u_2,$$

qui est découplée de u_1. Ainsi, le système (4.1) n'est pas contrôlable à zéro. □

4.2 Un exemple important : le cas des systèmes cascades

Nous avons le résultat suivant dû à Manuel Gonzalez-Burgos et Luz de Teresa qui est une généralisation du Théorème 4.1.2 (voir [GBdT10]).

Théorème 4.2.1. *On suppose que $D = I_n$ et $B = (1, 0, \dots, 0)^{\mathrm{tr}}$. On suppose également qu'il existe $(t_1, t_2) \subset (0, T)$, $\omega_0 \subset\subset \omega$ et $\varepsilon > 0$ tels que*

$$\forall 1 \leq i, j \leq n,\ a_{i,j} = 0 \textit{ pour } i < j - 1,\ \forall (t, x) \in (t_1, t_2) \times \omega_0,\ a_{i+1,i}(t, x) \geq \varepsilon. \tag{4.13}$$

Alors le système (4.1) est contrôlable à zéro.

La preuve du Théorème 4.2.1 utilise sensiblement les mêmes arguments que la preuve du Théorème 4.1.2.

Remarque 4.2.2. La philosophie à retenir derrière le Théorème 4.2.1 est la suivante. A l'aide du contrôle h_1 présent dans la première équation, on contrôle u_1. Puis le terme de couplage $a_{21}u_1$ présent dans la seconde équation permet le contrôle indirect de u_2, et ainsi de suite jusqu'au terme de couplage $a_{n,n-1}u_{n-1}$ qui permet le contrôle indirect de u_n. D'où l'appellation de système cascade comme dans la preuve par l'approche de Brunovsky du Théorème 1.1.1.

Remarque 4.2.3. Le résultat du Théorème 4.2.1 reste vrai si on remplace $\Big(a_{i+1,i} \geq \varepsilon\Big)$ par $\Big(a_{i+1,i} \geq \varepsilon \text{ ou } a_{i+1,i} \leq -\varepsilon\Big)$ sur $(t_1, t_2) \times \omega_0$.

4.3 Le retour de la condition de Kalman

Introduisons $(\lambda_k)_{k\geq 0}$ la suite de valeurs propres strictement positives de l'opérateur $(-\Delta, H^2 \cap H_0^1(\Omega))$ et $(e_k)_{k\geq 0}$ la suite de fonctions propres associées. On a alors le résultat suivant de type condition de Kalman pour le système (4.1) démontré par Farid Ammar-Khodja, Assia Benabdallah, Cédric Dupaix et Manuel Gonzalez-Burgos (voir [AKBDGB09a]).

Théorème 4.3.1. *On suppose que D, A et B ne dépendent pas du temps ni de l'espace. Alors, le système (4.1) est contrôlable à zéro au temps T si et seulement si la condition de Kalman suivante est vérifiée :*

$$\forall k \geq 0,\ \mathrm{Rang}\Big(B, (-\lambda_k D + A)B, \dots, (-\lambda_k D + A)^{n-1}B\Big) = n. \tag{4.14}$$

Remarque 4.3.2. Quand $D = dI_n$ avec $d > 0$, alors (4.14) devient exactement la condition de Kalman (1.3).

On prouve seulement le sens direct du Théorème 4.3.1. Une idée de preuve du sens indirect se trouve dans [AKBGBdT11, Section 5.2] : elle repose sur une inégalité de Carleman.

Démonstration. Par la « HUM », si le système (4.1) est contrôlable à zéro au temps T alors l'inégalité d'observabilité (4.3) est satisfaite.

Soit $k \geq 0$. En appliquant (4.3) à $\varphi_T = ve_k$ où $v \in \mathbb{R}^n$ est arbitraire, on obtient que le système différentiel $y' = (-\lambda_k D + A)z + Bh$ est contrôlable au temps T. D'où, par la condition de Kalman du Théorème 1.1.1, on en déduit la condition nécessaire (4.14). □

4.4 Couplages d'ordre deux : cas des systèmes 2×2

Dans cette partie, nous allons nous intéresser à un système parabolique 2×2 avec des couplages par des opérateurs différentiels d'ordre 2. On considère

$$\begin{cases} \partial_t u_1 - \Delta u_1 = u_2\theta_1 + h_1 1_\omega & \text{dans } Q_T, \\ \partial_t u_2 - \Delta u_2 = \Delta(u_1\theta_2) & \text{dans } Q_T, \\ u_1 = u_2 = 0 & \text{sur } \Sigma_T, \\ (u_1, u_2)(0, .) = (u_{1,0}, u_{2,0}) & \text{dans } \Omega, \end{cases} \tag{4.15}$$

où $\theta_1, \theta_2 \in C^\infty(\overline{\Omega})$. On dispose du résultat suivant dû à Sergio Guerrero (voir [Gue07]).

Théorème 4.4.1. *On suppose que θ_2 n'est pas identiquement nulle sur ω. Alors le système (4.15) est contrôlable au temps T.*

Démonstration. Par hypothèse et sans perte de généralité, on sait qu'il existe $\omega_0 \subset\subset \omega$ et $\varepsilon > 0$ tel que

$$\theta_2 \geq \varepsilon \text{ sur } \omega_0.$$

Par la « HUM », il suffit de démontrer l'inégalité d'observabilité

$$\forall \varphi_T \in L^2(\Omega)^2, \quad \|\varphi(0, .)\|_{L^2(\Omega)^2} \leq C \left(\int_0^T \int_\omega |\varphi_1(t,x)|^2 dtdx \right)^{1/2},$$

où φ est solution de

$$\begin{cases} -\partial_t \varphi_1 - \Delta\varphi_1 = \theta_2 \Delta\varphi_2 & \text{dans } Q_T, \\ -\partial_t \varphi_2 - \Delta\varphi_2 = \theta_1 \varphi_1 & \text{dans } Q_T, \\ \varphi_1 = \varphi_2 = 0 & \text{sur } \Sigma_T, \\ (\varphi_1, \varphi_2)(T, .) = (\varphi_{1,T}, \varphi_{2,T}) & \text{dans } \Omega. \end{cases} \tag{4.16}$$

On peut supposer que $\varphi_T \in L^2(\Omega)^2$ et ainsi φ est régulière. En particulier, on peut montrer que φ_2 est solution de

$$\begin{cases} -\partial_t \Delta\varphi_2 - \Delta\Delta\varphi_2 = -\Delta(\theta_1\varphi_1) & \text{dans } Q_T, \\ \Delta\varphi_2 = 0 & \text{sur } \Sigma_T. \end{cases} \tag{4.17}$$

On applique l'inégalité (Carleman) à $\Delta\varphi_2$ en remplaçant $\omega \leftarrow \omega_0$, et on trouve

$$\begin{aligned} &\lambda^4 \int_{Q_T} e^{-2s\alpha}(s\xi)^3 |\Delta\varphi_2|^2 dtdx + \lambda^2 \int_{Q_T} e^{-2s\alpha}(s\xi)|\nabla\Delta\varphi_2|^2 dtdx \\ &+ \int_{Q_T} e^{-2s\alpha}(s\xi)^{-1} \left(|\partial_t \Delta\varphi_2|^2 + |\Delta\Delta\varphi_2|^2 \right) dtdx \\ &\qquad \leq C \left(\int_{Q_T} e^{-2s\alpha}(|\varphi_1|^2 + |\nabla\varphi_1|^2 + |\Delta\varphi_1|^2) + \lambda^4 \int_{(0,T)\times\omega_0} e^{-2s\alpha}(s\xi)^3 |\Delta\varphi_2|^2 \right). \end{aligned}$$

En appliquant alors l'inégalité de Carleman modifiée (3.7) avec $k = 4$ à φ_1 (en multipliant par λ), on obtient

$$\begin{aligned}
&\int_{Q_T} e^{-2s\alpha} \left(\lambda^5 (s\xi)^4 |\varphi_1|^2 + \lambda^3 (s\xi)^2 |\nabla \varphi_1|^2 + \lambda |\Delta \varphi_1|^2\right) \\
&\qquad \leq C \left(\lambda \int_{Q_T} e^{-2s\alpha} s\xi |\Delta \varphi_2|^2 + \lambda^5 \int_{(0,T)\times\omega_0} e^{-2s\alpha} (s\xi)^4 |\varphi_1|^2 \right).
\end{aligned}$$

On somme alors les deux précédentes inégalités, et on absorbe les termes globaux à droite en prenant λ, s suffisamment grand pour obtenir :

$$\begin{aligned}
&\int_{Q_T} e^{-2s\alpha} \left((s\xi)^4 |\varphi_1|^2 + (s\xi)^2 |\nabla \varphi_1|^2 + |\Delta \varphi_1|^2\right) \\
&+ \int_{Q_T} e^{-2s\alpha} (s\xi)^3 |\Delta \varphi_2|^2 + \int_{Q_T} e^{-2s\alpha} (s\xi) |\nabla \Delta \varphi_2|^2 \\
&+ \int_{Q_T} e^{-2s\alpha} (s\xi)^{-1} \left(|\partial_t \Delta \varphi_2|^2 + |\Delta\Delta \varphi_2|^2\right) \\
&\qquad \leq C \left(\int_{(0,T)\times\omega_0} e^{-2s\alpha} (s\xi)^4 |\varphi_1|^2 + \int_{(0,T)\times\omega_0} e^{-2s\alpha} (s\xi)^3 |\Delta \varphi_2|^2 \right). \qquad (4.18)
\end{aligned}$$

Il faut à présent se débarrasser de $\int_{(0,T)\times\omega_0} e^{-2s\alpha} (s\xi)^3 |\Delta \varphi_2|^2$. Pour ce faire, on introduit une fonction de troncature χ localisée sur ω telle que $\chi = 1$ sur ω_0. On multiplie alors la première équation de (4.16) par $\chi e^{-2s\alpha} (s\xi)^3 \Delta \varphi_2$ et on intègre sur $(0,T) \times \Omega$. En effectuant des intégrations par parties et en invoquant des arguments similaires à la preuve du Théorème 4.1.2, on aboutit à : pour tout $\delta > 0$,

$$\begin{aligned}
&\int_{(0,T)\times\omega_0} e^{-2s\alpha} (s\xi)^3 |\Delta \varphi_2|^2 \\
&\leq \delta \Bigg(\int_{Q_T} e^{-2s\alpha} (s\xi)^3 |\Delta \varphi_2|^2 dtdx + \int_{Q_T} e^{-2s\alpha} (s\xi) |\nabla \Delta \varphi_2|^2 dtdx \\
&\qquad + \int_{Q_T} e^{-2s\alpha} (s\xi)^{-1} \left(|\partial_t \Delta \varphi_2|^2 + |\Delta\Delta \varphi_2|^2\right) dtdx \Bigg) \\
&\qquad + C_\delta \int_{q_T} e^{-2s\alpha} (s\xi)^{k'} |\varphi_1|^2, \;\; k' \in \mathbb{N},
\end{aligned}$$

et donc (par absorption) en utilisant (4.18), on aboutit à

$$\begin{aligned}
&\int_{Q_T} e^{-2s\alpha}\left((s\xi)^4|\varphi_1|^2+(s\xi)^2|\nabla\varphi_1|^2+|\Delta\varphi_1|^2\right)\\
&+\int_{Q_T} e^{-2s\alpha}(s\xi)^3|\Delta\varphi_2|^2+\int_{Q_T} e^{-2s\alpha}(s\xi)|\nabla\Delta\varphi_2|^2\\
&+\int_{Q_T} e^{-2s\alpha}(s\xi)^{-1}\left(|\partial_t\Delta\varphi_2|^2+|\Delta\Delta\varphi_2|^2\right)\\
&\qquad\leq C\int_{q_T} e^{-2s\alpha}(s\xi)^{k'}|\varphi_1|^2.
\end{aligned}$$

On déduit alors de la précédente estimation l'inégalité d'observabilité désirée en utilisant notamment l'inégalité de Poincaré pour récupérer une estimation à gauche de φ_2 à partir de $\Delta\varphi_2$. □

Remarque 4.4.2. Pour tirer profit d'un couplage d'ordre deux dans le Théorème 4.4.1, nous avons appliqué l'inégalité (Carleman) à l'équation satisfaite par $\Delta\varphi_2$. Cela nous a coûté notamment le terme global $\int_{Q_T} e^{-2s\alpha}|\Delta\varphi_1|^2$. Ainsi, pour absorber ce terme, il a fallu employer une inégalité de Carleman modifiée (avec un exposant plus élevé) à l'équation satisfaite par φ_1. On retiendra la chose suivante : pour tirer profit d'un couplage d'ordre deux, cela nous coûte un exposant dans l'inégalité de Carleman à employer.

4.5 Commentaires bibliographiques

La condition suffisante (4.7) du Théorème 4.1.2 n'est pas une condition nécessaire de contrôlabilité à zéro pour un système de deux équations avec un seul contrôle (voir [ABL11] ou [AKBGBdT11, Section 7.2.4]).

Pour des matrices dépendant du temps, le Théorème 4.3.1 se généralise en une condition de Kalman suffisante de contrôlabilité similaire à celle établie en dimension finie dans le Théorème 1.2.1 (voir [AKBDGB09b] ou [AKBGBdT11, Section 5.1]).

On peut également mentionner qu'une autre stratégie pour démontrer un résultat de contrôle pour un système parabolique avec moins de contrôles que d'équations consiste à utiliser la *méthode de contrôle fictif*. Cela a permis notamment à Michel Duprez et Pierre Lissy dans [DL16] et [DL18] d'établir de nouveaux résultats de contrôlabilité à zéro dans le cadre de couplages d'ordre zéro et un. Les articles [Mau13], [BGBPGa04] et [GBPG06] utilisent également cette méthode.

Dans la preuve du Théorème 4.4.1, nous avons vu qu'il fallait utiliser deux inégalités de Carleman avec des poids sensiblement différents. Cela est dû aux couplages d'ordre deux et à la méthode employée. Une telle approche présente des limitations pour un système de taille plus grande, mentionnons à ce propos l'article [FCGBdT15] qui traite de la contrôlabilité à zéro de systèmes paraboliques à matrice de diffusion non diagonalisable avec une restriction sur la taille des blocs de Jordan qui ne doit pas excéder quatre. Ce problème a d'ailleurs été résolu dans le cas constant dans le preprint [LZ17] par une méthode de Lebeau-Robbiano.

Pour d'autres références, le lecteur pourra consulter l'état de l'art dressé dans [AKBGBdT11] et les introductions des thèses [Mau12], [Oli13], [Dup15].

Chapitre 5

L'équation de la chaleur semilinéaire

Dans cette section, nous allons nous intéresser à la contrôlabilité de l'équation :

$$\begin{cases} \partial_t y - \Delta y + f(y) = h 1_\omega & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega, \end{cases} \tag{HeatSL}$$

où $f \in C^\infty(\mathbb{R};\mathbb{R})$ est telle que $f(0) = 0$.

Un espace fonctionnel agréable pour travailler en non linéaire est l'espace L^∞ car c'est une algèbre de Banach. Nous commençons par deux définitions.

Définition 5.0.1. Pour $T > 0$, l'équation (HeatSL) est dite globalement contrôlable à zéro dans $L^\infty(\Omega)$ au temps T si pour toute donnée initiale $y_0 \in L^\infty(\Omega)$, il existe un contrôle $h \in L^\infty(q_T)$ tel que la solution y de (HeatSL) satisfait $y(T, \cdot) = 0$.

Définition 5.0.2. Pour $T > 0$, l'équation (HeatSL) est dite localement contrôlable à zéro dans $L^\infty(\Omega)$ au temps T s'il existe $\delta_T > 0$ tel que pour toute donnée initiale $y_0 \in L^\infty(\Omega)$ vérifiant $\|y_0\|_{L^\infty(\Omega)} \leq \delta_T$, il existe un contrôle $h \in L^\infty(q_T)$ tel que la solution y de (HeatSL) satisfait $y(T, \cdot) = 0$.

Dans un premier temps, nous allons transposer les résultats linéaires obtenus dans un cadre L^2 pour l'équation de la chaleur avec potentiel

$$\begin{cases} \partial_t y - \Delta y + a(t,x) y = h 1_\omega & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega. \end{cases} \tag{HeatPot}$$

où $a \in L^\infty(Q_T)$ (voir Théorème 3.5.1) à un cadre L^∞. Après avoir fait un bref rappel sur le *coût de contrôle* et *l'inégalité d'observabilité*, nous passons en revue quelques méthodes permettant la construction de contrôles plus réguliers que L^2. Nous en retiendrons essentiellement une pour la suite : la « *Penalized Hilbert Uniqueness Method* » car elle s'adapte à de très nombreuses situations. Nous insistons davantage qu'auparavant sur les estimations que nous pouvons faire sur les contrôles que nous construisons en fonction des données du problème : données initiales, potentiels dans les équations. En effet, ces estimations sont cruciales pour le passage au non linéaire.

5.1 Coût de contrôle et inégalité d'observabilité

Le but de cette partie est de montrer le lien entre le *coût de contrôle* et *l'inégalité d'observabilité*. Introduisons d'abord le système adjoint :

$$\begin{cases} -\partial_t \varphi - \Delta \varphi + a(t,x)\varphi = 0 & \text{dans } Q_T, \\ \varphi = 0 & \text{sur } \Sigma_T, \\ \varphi(T,.) = \varphi_T & \text{dans } \Omega. \end{cases} \tag{HeatPotAdj}$$

Nous avons le résultat suivant (voir [Cor07a, Theorem 2.44]).

Théorème 5.1.1. *Soit $C > 0$. On a équivalence entre les deux propriétés suivantes :*

— *Pour tout $y_0 \in L^2(\Omega)$, il existe $h \in L^2(q_T)$ vérifiant*

$$\|h\|_{L^2(q_T)} \leq C \|y_0\|_{L^2(\Omega)}, \tag{5.1}$$

tel que la solution de (HeatPot) vérifie $y(T,.) = 0$.

— *Pour tout $\varphi_T \in L^2(\Omega)$, la solution φ de (HeatPotAdj) vérifie*

$$\|\varphi(0,.)\|_{L^2(\Omega)^n} \leq C \left(\int_0^T \int_\omega |\varphi(t,x)|^2 dt dx \right)^{1/2} \tag{5.2}$$

De la preuve du Théorème 3.5.1, voir en particulier (3.24), on déduit le résultat suivant.

Théorème 5.1.2. *Il existe $C > 0$ de la forme*

$$C(\Omega, \omega, T, a) = \exp\left(C(\Omega,\omega) \left(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3} \right) \right), \tag{5.3}$$

tel que (5.2) est vérifiée pour tout $\varphi_T \in L^2(\Omega)$.

D'où, on déduit du Théorème 5.1.1 et du Théorème 5.1.2 le résultat suivant.

Théorème 5.1.3. *Pour tout $y_0 \in L^2(\Omega)$, il existe $h \in L^2(q_T)$ vérifiant*

$$\|h\|_{L^2(q_T)} \leq C \|y_0\|_{L^2(\Omega)}, \tag{5.4}$$

où C est de la forme (5.3), tel que la solution de (HeatPot) vérifie $y(T,.) = 0$.

5.2 Construction de contrôles réguliers : deux méthodes

Le but de cette partie est de montrer le résultat suivant.

Théorème 5.2.1. *L'équation (HeatPot) est contrôlable dans $L^\infty(\Omega)$ à zéro au temps T. Plus précisément, pour tout $y_0 \in L^\infty(\Omega)$, il existe $h \in L^\infty(q_T)$ satisfaisant*

$$\|h\|_{L^\infty(q_T)} \leq C(\Omega, \omega, T, a) \|y_0\|_{L^2(\Omega)}, \tag{5.5}$$

où $C(\Omega,\omega,T,a)$ est de la forme (5.3), tel que la solution y de (HeatPot) satisfait $y(T,.) = 0$.

Nous allons donner deux démonstrations de ce résultat, l'une est basée sur un argument de troncature adéquat tandis que la seconde est basée sur la « PHUM ».

Démonstration par argument de troncature. Soit $y_0 \in L^2(\Omega)$. Nous allons seulement montrer l'existence d'un contrôle L^∞ qui amène la solution à zéro sans chercher à démontrer l'estimation (5.5), même s'il est possible en travaillant un peu plus de prouver (5.5) avec cette approche (voir par exemple [FCGBGP06a, Section 2] dans le cas Neumann).

On introduit deux ouverts ω' and ω'', tels que $\omega'' \subset\subset \omega' \subset\subset \omega$. En appliquant le Théorème 3.5.1 avec $\omega \leftarrow \omega''$, il existe un contrôle $\widetilde{h} \in L^2(q_T)$ tel que la solution $\widetilde{y}$ de (HeatPot) avec $\omega \leftarrow \omega''$ satisfait $\widetilde{y}(T,.) = 0$.

On introduit à présent une fonction de troncature en temps $\eta \in C^\infty([0,T])$ telle que $\eta = 1$ sur $[0, T/3]$, $\eta = 0$ sur $[2T/3, T]$.

Posons $\overline{y}$ la solution libre de l'équation de la chaleur

$$\begin{cases} \partial_t \overline{y} - \Delta \overline{y} + a(t,x)\overline{y} = 0 & \text{dans } Q_T, \\ \overline{y} = 0 & \text{sur } \Sigma_T, \\ \overline{y}(0,.) = y_0 & \text{dans } \Omega. \end{cases} \tag{5.6}$$

Alors la fonction $\widetilde{z} := \widetilde{y} - \eta\overline{y}$ satisfait

$$\begin{cases} \partial_t \widetilde{z} - \Delta \widetilde{z} + a(t,x)\widetilde{z} = -\eta'\overline{y} + \widetilde{h}1_{\omega''} & \text{dans } Q_T, \\ \widetilde{z} = 0 & \text{sur } \Sigma_T, \\ (\widetilde{z}(0,.), \widetilde{z}(T,.)) = (0,0) & \text{dans } \Omega. \end{cases} \tag{5.7}$$

Posons ω_0 tel que $\omega' \subset\subset \omega_0 \subset\subset \omega$ et une fonction de troncature en espace Θ, avec $\Theta \in C_c^\infty(\omega_0)$ et $\Theta = 1$ sur ω'. On pose $z := (1-\Theta)\widetilde{z}$. On déduit de (5.7),

$$\begin{cases} \partial_t z - \Delta z + a(t,x)z = -\eta'\overline{y} + \widetilde{h}1_{\omega} & \text{dans } Q_T, \\ z = 0 & \text{sur } \Sigma_T, \\ (z(0,.), z(T,.)) = (0,0) & \text{dans } \Omega. \end{cases} \tag{5.8}$$

avec

$$h := \Theta\eta'\overline{y} + 2\nabla\Theta.\nabla\widetilde{z} + (\Delta\Theta)\widetilde{z} + \underbrace{(1-\Theta)\widetilde{h}1_{\omega''}}_{0}. \tag{5.9}$$

Grâce à (5.9), on remarque que supp $h \subset [0,T] \times \omega$. De plus, la fonction $y := z + \eta\overline{y}$ résout (HeatPot) (avec contrôle h) et $y(T,.) = 0$ en utilisant (5.8) et (5.6). Puis grâce aux effets régularisants de l'équation de la chaleur avec potentiel, on montre aisément que $h \in L^\infty(Q_T)$. $\square$

Remarque 5.2.2. L'idée de la précédente preuve est d'une part d'éliminer l'irrégularité de la donnée initiale y_0 par un argument de troncature en temps : c'est le rôle de $\widetilde{z}$ et d'autre part d'éliminer l'irrégularité du contrôle $\widetilde{h}$ par un argument de troncature en espace : c'est le rôle de z.

La seconde méthode est due à Viorel Barbu, cette méthode très robuste est basée sur l'inégalité de Carleman, une méthode de type « HUM » pénalisée et sur un argument de bootstrap. Nous allons tirer profit des inégalités d'observabilité avec poids établies précédemment, i.e. (3.25) et (3.26).

Démonstration par la PHUM. Dans toute la preuve, les constantes $C > 0$ sont de la forme (5.3) mais peuvent changer d'une ligne à l'autre.

On se donne $y_0 \in L^2(\Omega)$ et $\varepsilon > 0$. On introduit la fonctionnelle :

$$\forall h \in L^2(q_T),\ J_\varepsilon(h) = \frac{1}{2}\int_{(0,T)\times\omega} e^{2s\alpha}(s\xi)^{-3}|h|^2 dtdx + \frac{1}{2\varepsilon}\|y(T,.)\|^2_{L^2(\Omega)},$$

où y est la solution de (HeatPot) avec donnée initiale y_0 et contrôle h.

On voit aisément que J_ε est une fonctionnelle de classe C^1, strictement convexe et coercive sur $L^2(q_T)$ donc elle possède un unique minimum h_ε. Appelons y_ε la solution de (HeatPot) associée à ce contrôle h_ε. Par l'équation d'Euler-Lagrange, on a que pour tout $h \in L^2(q_T)$,

$$\int_{(0,T)\times\omega} e^{2s\alpha}(s\xi)^{-3}h_\varepsilon h dtdx + \frac{1}{\varepsilon}\int_\Omega y_\varepsilon(T,.).y(T,.) = 0, \tag{5.10}$$

où y est la solution de (HeatPot) avec donnée initiale nulle et contrôle h.

Introduisons φ_ε la solution de (HeatPotAdj) avec donnée initiale $-y_\varepsilon(T,.)/\varepsilon$. Par un argument de dualité entre y et φ_ε, on a que

$$\begin{aligned}\int_\Omega y(T,.).\varphi_\varepsilon(T,.) &= \int_{(0,T)\times\omega} h\varphi_\varepsilon,\\ -\frac{1}{\varepsilon}\int_\Omega y_\varepsilon(T,.).y(T,.) &= \int_{(0,T)\times\omega} h\varphi_\varepsilon,\end{aligned}$$

ce qui donne grâce à (5.10) :

$$\int_{(0,T)\times\omega}\varphi_\varepsilon h = \int_{(0,T)\times\omega} e^{2s\alpha}(s\xi)^{-3}h_\varepsilon h,\ \forall h \in L^2(q_T).$$

D'où, on trouve

$$h_\varepsilon = e^{-2s\alpha}(s\xi)^3\varphi_\varepsilon 1_\omega. \tag{5.11}$$

En utilisant (5.11) et un nouvel argument de dualité entre y_ε et φ_ε, on a

$$\begin{aligned}\int_\Omega y(T,x).\varphi_\varepsilon(T,x)dx &= \int_\Omega y_0(x)\varphi_\varepsilon(0,x)dx + \int_{(0,T)\times\omega} h_\varepsilon\varphi_\varepsilon\\ -\frac{1}{\varepsilon}\|y_\varepsilon(T,.)\|^2_{L^2(\Omega)} &= \int_\Omega y_0(x)\varphi_\varepsilon(0,x)dx + \int_{(0,T)\times\omega} e^{-2s\alpha}(s\xi)^3|\varphi_\varepsilon|^2.\end{aligned} \tag{5.12}$$

Par l'inégalité (3.25) appliquée à φ_ε, on trouve

$$\|\varphi_\varepsilon(0,.)\|^2_{L^2(\Omega)} \leq C\int_{(0,T)\times\omega} e^{-2s\alpha}(s\xi)^3|\varphi_\varepsilon|^2. \tag{5.13}$$

D'où, en utilisant (5.12), (5.13) et l'inégalité de Young, on a

$$\frac{1}{\varepsilon}\|y_\varepsilon(T,.)\|^2_{L^2(\Omega)} + \left\|(e^{2s\alpha}(s\xi)^{-3})^{1/2}h_\varepsilon\right\|^2_{L^2(q_T)} \leq C\|y_0\|^2_{L^2(\Omega)}. \tag{5.14}$$

Remarque 5.2.3. En passant à la limite dans (5.14), on peut déjà déduire l'existence d'un contrôle h tel que $(e^{2s\alpha}(s\xi)^{-3})^{1/2}h \in L^2(q_T)$ qui amène la solution y à 0 au temps $t = T$. Afin de borner la norme L^∞ du contrôle h_ε, nous allons utiliser un argument de bootstrap.

On introduit

$$\forall k \geq 0,\ \psi_{\varepsilon,k} := e^{-s\alpha}(s\xi)^{3/2-2k}\varphi_\varepsilon,$$

et

$$p_0 := 2 \text{ et } \forall k \geq 1,\ p_k := \begin{cases} \frac{(N+2)p_{k-1}}{N+2-p_{k-1}} & \text{si } p_{k-1} < N+2, \\ 2p_{k-1} & \text{si } p_{k-1} = N+2, \\ +\infty & \text{si } p_{k-1} > N+2. \end{cases}$$

On voit aisément qu'il existe $l \in \mathbb{N}$, tel que $p_l = +\infty$.

Par un calcul assez fastidieux et en utilisant notamment le Lemme 3.4.2 (voir par exemple [CGR10, Section 3.1.2]), on montre par récurrence sur $k \in \mathbb{N}^*$ que

$$\begin{cases} -\partial_t \psi_{\varepsilon,k} - \Delta\varepsilon_k = a_k(t,x)\psi_{\varepsilon,k-1} + (s\xi)^{-1}B_k(t,x).\nabla\psi_{\varepsilon,k-1} & \text{dans } Q_T, \\ \psi_{\varepsilon,k} = 0 & \text{sur } \Sigma_T, \\ \psi_{\varepsilon,k}(T,.) = 0 & \text{dans } \Omega, \end{cases}$$

où $a_k \in L^\infty(Q_T)$ et $B_k \in L^\infty(Q_T)^N$.

Or, par l'inégalité d'observabilité (3.26), (5.11) et (5.14), on sait que

$$\left\| A_1\psi_{\varepsilon,0} + (s\xi)^{-1}B_1.\nabla\psi_{\varepsilon,0} \right\|_{L^2(Q_T)} \leq C \left\| y_0 \right\|_{L^2(\Omega)},$$

donc par régularité parabolique dans L^2 (voir Proposition A.6.2 avec $p = 2$), on en déduit que

$$\psi_{\varepsilon,1} \in X_{p_0} \text{ et } \left\| \psi_{\varepsilon,1} \right\|_{X_{p_0}} \leq C \left\| y_0 \right\|_{L^2(\Omega)}.$$

Or, par injection de Sobolev (voir Lemme A.6.3), on a alors

$$\psi_{\varepsilon,1} \in L^{p_1}(0,T;W_0^{1,p_1}(\Omega)) \text{ et } \left\| \psi_{\varepsilon,1} \right\|_{L^{p_1}(0,T;W_0^{1,p_1}(\Omega))} \leq C \left\| y_0 \right\|_{L^2(\Omega)}.$$

En itérant cet argument pour k entre 1 et l à l'aide d'arguments de régularité parabolique dans L^{p_k} (voir Proposition A.6.2), on obtient

$$\psi_{\varepsilon,l} \in L^{p_l}(0,T;W_0^{1,p_l}(\Omega)) \text{ et } \left\| \psi_{\varepsilon,l} \right\|_{L^{p_l}(0,T;W_0^{1,p_l}(\Omega))} \leq C \left\| y_0 \right\|_{L^2(\Omega)}.$$

D'où en reprenant l'expression de $\psi_{\varepsilon,l}$, (5.11) et en utilisant à nouveau une injection de Sobolev (voir Lemme A.6.3),

$$\left\| h_\varepsilon \right\|_{L^\infty(q_T)} \leq C \left\| y_0 \right\|_{L^2(\Omega)}. \tag{5.15}$$

Ainsi, en utilisant (5.15), (5.14), les estimations L^2 et L^∞ sur l'équation de la chaleur avec potentiel énoncées en Proposition A.1.3 et Proposition A.1.4, on déduit qu'il existe $h \in L^\infty(Q_T)$ vérifiant (5.15) et $y \in W_T \cap L^\infty(Q_T)$ tels que

$$h_\varepsilon \rightharpoonup^* h \text{ dans } L^\infty(Q_T) \text{ quand } \varepsilon \to 0,$$

$$y_\varepsilon \rightharpoonup y \text{ dans } W_T \;\Rightarrow y_\varepsilon(0,.) \rightharpoonup y(0,.) = y_0,\; y_\varepsilon(T,.) \rightharpoonup y(T,.) = 0 \text{ dans } L^2(\Omega) \quad \text{quand } \varepsilon \to 0.$$

Ainsi, y est solution de (HeatPot) associée à la donnée initiale y_0 et au contrôle h vérifiant l'estimation (5.5). De plus, $y(T,.) = 0$. Ceci conclut la preuve. □

5.3 Contrôlabilité locale à zéro de l'équation de la chaleur semilinéaire

Le but de cette partie est de démontrer le théorème suivant.

Théorème 5.3.1. *Pour tout $T > 0$, l'équation de la chaleur* (HeatSL) *est localement contrôlable à zéro dans $L^\infty(\Omega)$.*

Démonstration. Soit $T > 0$ et $y_0 \in L^\infty(\Omega)$ tel que $\|y_0\|_{L^\infty(\Omega)} \leq \delta_T$ où $\delta_T > 0$ sera fixé plus tard.

On introduit la fonction $g \in C^\infty(\mathbb{R};\mathbb{R})$ définie de la manière suivante

$$g(s) = \begin{cases} f(s)/s & \text{si } s \neq 0, \\ f'(0) & \text{si } s = 0. \end{cases} \tag{5.16}$$

Soit $r > 0$ (quelconque) et $B_r := \{z \in L^\infty(Q_T)\ ;\ \|z\| \leq r\}$. Par le Théorème 5.2.1, on sait qu'il existe $C > 0$ tel que pour tout $z \in B_r$, il existe $h \in L^\infty(q_T)$ satisfaisant :

$$\|h\|_{L^\infty(q_T)} \leq C\, \|y_0\|_{L^2(\Omega)}, \tag{5.17}$$

tel que la solution y du problème linéarisé

$$\begin{cases} \partial_t y - \Delta y + g(z) y = h 1_\omega & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega, \end{cases} \tag{5.18}$$

satisfait $y(T,.) = 0$.

La constante C dans (5.17) dépend de Ω, ω, T, g et r mais pas de y_0.

On peut alors définir l'application Φ de la manière suivante : pour tout $z \in B_r$, $\Phi(z)$ est l'ensemble des $y \in L^\infty(Q_T)$ tels qu'il existe $h \in L^\infty(q_T)$ satisfaisant (5.17) et tels que y soit solution de (5.18) vérifiant $y(T,.) = 0$.

Si on démontre que Φ possède un point fixe, i.e. il existe $y \in B_r$ tel que $y \in \Phi(y)$, alors y est solution de

$$\partial_t y - \Delta y + g(y) y = \partial_t y - \Delta y + f(y) = h 1_\omega, \qquad (y(0,.), y(T,.)) = (y_0, 0),$$

alors on a prouvé le Théorème 5.3.1. Afin de prouver l'existence d'un point fixe, nous allons utiliser le théorème de point fixe de Kakutani (voir [Zei86, Theorem 9.B., page 452]).

Théorème 5.3.2. *On suppose que Φ vérifie les trois hypothèses suivantes :*

1. *pour tout $z \in B_r$, $\Phi(z)$ est un ensemble non vide, fermé, convexe de E,*
2. *il existe un compact convexe $K \subset B_r$ tel que*

$$\forall z \in B_r,\ \Phi(z) \subset K,$$

3. *Φ est semicontinue supérieurement sur E, c'est-à-dire que pour tout ensemble fermé $\mathcal{A} \subset B_r$,*

$$\Phi^{-1}(\mathcal{A}) := \{z \in B_r\ ;\ \Phi(z) \cap \mathcal{A} \neq \emptyset\} \text{ est fermé.}$$

Alors, Φ possède un point fixe.

Rappelons que le caractère non vide de $\Phi(z)$ pour $z \in B_r$ découle de l'existence d'un contrôle qui amène la solution y de (5.18) à zéro. L'hypothèse difficile à vérifier du Théorème 5.3.2 est l'hypothèse 2, qui implique notamment que Φ stabilise B_r. Démontrons ce point. En utilisant (5.17), la définition de l'application Φ et des estimations standards de type L^∞ appliquées au système (5.18) (voir Proposition A.1.4), on en déduit que

$$\forall z \in B_r,\ \forall y \in \Phi(z),\ \|y\|_{L^\infty(Q_T)} \leq C\,\|y_0\|_{L^\infty(\Omega)}\,.$$

D'où, en posant $\delta_T > 0$ tel que

$$C\delta_T \leq r,$$

on a alors

$$\forall z \in B_r,\ \Phi(z) \subset B_r.$$

En admettant qu'on vérifie alors aisément les autres points du Théorème 5.3.2, on en déduit que Φ admet un point fixe y, ce qui conclut la preuve. □

5.4 Contrôlabilité globale à zéro de l'équation de la chaleur semilinéaire

5.4.1 Cas global Lipschitz

Dans cette partie, on suppose que f est globalement Lipschitzienne. On a alors le résultat suivant.

Théorème 5.4.1. *Pour tout $T > 0$, l'équation de la chaleur* (HeatSL) *est globalement contrôlable à zéro dans $L^\infty(\Omega)$.*

Démonstration. Soit $T > 0$ et $y_0 \in L^\infty(\Omega)$. On procède alors comme dans la preuve précédente.

Comme f est globalement Lipschitzienne, on a alors que g définie en (5.16) est bornée.

Soit $R > 0$ à fixer plus tard et $B_R := \{z \in L^\infty(Q_T)\ ;\ \|z\| \leq R\}$. Par le Théorème 5.2.1, on sait qu'il existe $C > 0$ tel que pour tout $z \in B_R$, il existe $h \in L^\infty(q_T)$ satisfaisant (5.17) tel que la solution y du problème linéarisé (5.18) satisfait $y(T, .) = 0$.

A présent, la constante C dans (5.17) dépend de Ω, ω, T, g mais pas de y_0 ni de R. Le fait que g soit bornée parce que f est globalement Lipschitzienne joue ici un rôle crucial.

On définit Φ de la même manière que précédemment et on cherche à montrer que Φ stabilise B_R. Par des estimations standards (voir Proposition A.1.4), on montre qu'il existe $C > 0$ indépendante de y_0 et R telle que

$$\forall z \in B_R,\ \forall y \in \Phi(z),\ \|y\|_{L^\infty(Q_T)} \leq C \|y_0\|_{L^\infty(\Omega)}.$$

Donc, quitte à prendre R assez grand, par exemple $R = 2C \|y_0\|_{L^\infty(\Omega)}$, on montre que Φ stabilise B_R. □

5.4.2 Cas des non linéarités du type $y \log^\alpha(1 + |y|)$ pour $0 < \alpha < 3/2$

Dans cette partie, on suppose que

$$\frac{f(s)}{|s| \log^{3/2}(1 + |s|)} \to 0 \text{ quand } |s| \to +\infty. \tag{5.19}$$

On a alors le théorème suivant.

Théorème 5.4.2. *Pour tout $T > 0$, l'équation de la chaleur (HeatSL) est globalement contrôlable à zéro dans $L^\infty(\Omega)$.*

Remarque 5.4.3. Le Théorème 5.4.2 implique en particulier le Théorème 5.4.1.

Démonstration. Soit $T > 0$ et $y_0 \in L^\infty(\Omega)$.

En utilisant (5.19), la fonction $g \in C^\infty(\mathbb{R}; \mathbb{R})$ définie en (5.16) satisfait

$$\forall \varepsilon > 0,\ \exists C_\varepsilon > 0,\ \forall s \in \mathbb{R},\ |g(s)|^{2/3} \leq \varepsilon \log(1 + |s|) + C_\varepsilon \tag{5.20}$$

Soit $R > 0$ à fixer plus tard et $B_R := \{z \in L^\infty(Q_T)\ ;\ \|z\| \leq R\}$. Par le Théorème 5.2.1, on sait que pour tout $z \in B_R$, tout $\widetilde{T} > 0$, il existe $h \in L^\infty(q_{\widetilde{T}})$ satisfaisant (5.17) avec C de la forme

$$C(\Omega, \omega, \widetilde{T}, g(z)) = \exp\left(C(1 + \frac{1}{T} + T \|g(z)\|_{L^\infty(Q_T)} + \|g(z)\|_{L^\infty(Q_T)}^{2/3} \right). \tag{5.21}$$

tel que la solution y de (5.18) satisfait $y(\widetilde{T}, .) = 0$.

Se donnant $z \in B_R$, on applique ce qui précède avec

$$\widetilde{T} = T_z := \min\left(T, \|g(z)\|_{L^\infty(Q_T)}^{-1/3}\right). \tag{5.22}$$

Grâce à (5.21) et (5.22), il existe alors un contrôle $h_z \in L^\infty(q_{T_z})$ satisfaisant :

$$\|h\|_{L^\infty(q_{T_z})} \leq \exp\left(C(\Omega, \omega, T) \|g(z)\|_{L^\infty(Q_T)}^{2/3} \right) \|y_0\|_{L^2(\Omega)}, \tag{5.23}$$

tel que $y(T_z, .) = 0$. Prolongeant le contrôle h_z par 0 au-delà de T_z et ce jusqu'à T, on a alors aussi $y(T, .) = 0$.

On définit alors l'application Φ de la manière suivante : pour tout $z \in B_R$, $\Phi(z)$ est l'ensemble des $y \in L^\infty(Q_T)$ telle qu'il existe $h \in L^\infty(q_T)$ satisfaisant (5.23) et tel que y soit solution de (5.18) vérifiant $y(T,.) = 0$.

On cherche à montrer que Φ stabilise B_R. Par des estimations L^∞ (voir Proposition A.1.4), on montre que

$$\forall z \in B_R,\ \forall y \in \Phi(z),\ \|y\|_{L^\infty(Q_T)} \leq \exp\left(C(\Omega,\omega,T)\,\|g(z)\|_{L^\infty(Q_T)}^{2/3}\right)\|y_0\|_{L^\infty(\Omega)}.$$

En invoquant l'estimation sur g : (5.20), on déduit donc :

$$\begin{aligned}\forall z \in B_R,\ \forall y \in \Phi(z),\ \|y\|_{L^\infty(Q_T)} &\leq \exp\left(C\varepsilon\log(1+R)+C_\varepsilon\right)\|y_0\|_{L^\infty(\Omega)}\\ &\leq C_\varepsilon(1+R)^{\varepsilon C}\|y_0\|_{L^\infty(\Omega)},\end{aligned}$$

Ainsi, en prenant ε suffisamment petit, par exemple $\varepsilon C = 1/2$, alors pour R suffisamment grand, on a bien

$$\forall z \in B_R,\ \forall y \in \Phi(z),\ \|y\|_{L^\infty(Q_T)} \leq R,$$

et donc Φ stabilise B_R. □

Remarque 5.4.4. Un des points clefs de la précédente preuve est de chercher à minimiser la fonction

$$t \in (0,T] \mapsto \frac{1}{t} + t\,\|g(z)\|_{L^\infty(Q_T)} + \|g(z)\|_{L^\infty(Q_T)}^{2/3},$$

dans le but de sentir le moins possible l'influence de la non linéarité, c'est-à-dire celle de $\|g(z)\|_{L^\infty(Q_T)}$ ici. Heuristiquement, contrôler au temps $\widetilde{T}$ défini en (5.22), c'est-à-dire agir dans l'équation en temps très court, permet de lutter contre l'explosion et d'amener la solution à zéro. On voit ici le rôle fondamental de l'exposant $2/3$ du coût de contrôle du Théorème 5.2.1 pour contrôler à zéro des non linéarités satisfaisant (5.19).

5.5 Commentaires bibliographiques

Le Théorème 5.1.3 et le Théorème 5.2.1 sont dûs à Enrique Fernández-Cara et Enrique Zuazua (voir [FCZ00]).

La première démonstration de l'existence de contrôles L^∞ à partir de contrôles L^2 (voir Théorème 5.2.1) que nous avons donnée est due à Olivier Bodart, Manuel Gonzalez-Burgos et Rosario Perez-Garcia (voir [BGBPGa04]) tandis que la seconde est essentiellement due à Viorel Barbu (voir [Bar00]). Une troisième preuve de ce résultat se trouve dans [FCZ00], elle s'appuie sur une inégalité d'observabilité L^2-L^1 démontrée à partir d'une inégalité d'observabilité L^2-L^2 et des effets régularisants de l'équation de la chaleur. En plus de démontrer l'existence de contrôles L^∞, ce type d'inégalité permet en fait de prouver l'existence de *contrôles bang-bang* c'est-à-dire des contrôles sous la forme d'une superposition de fonctions de Heaviside.

Le Théorème 5.3.1 et le Théorème 5.4.1 sont dûs à Andrei Fursikov et Oleg Imanuvilov (voir [FI96, Chapter 1, Section 3 et 4]) même si l'utilisation du théorème de point fixe de Kakutani est plutôt inspirée de [FPZ95] (voir Proposition A.6.1).

Le Théorème 5.4.2 est dû à Enrique Fernández-Cara et Enrique Zuazua (voir [FCZ00]) et de manière indépendante (mais sous une condition de signe) à Viorel Barbu (voir [Bar00]).

Une généralisation du Théorème 5.4.2 à des semilinéarités dépendant de l'état y et de son gradient ∇y se trouve dans [DFCGBZ02] (voir aussi [AB00]). Pour des non linéarités non locales, voir par exemple [FCLdM12].

Pour une autre stratégie de passage du linéaire au non linéaire s'appuyant sur la contrôlabilité d'un seul linéarisé communément appelée *méthode du terme source*, le lecteur pourra consulter l'article [LTT13].

Pour un résultat de contrôlabilité en temps long entre états stationnaires d'une équation de la chaleur semilinéaire s'appuyant sur une *méthode de déformation quasi-statique*, on peut consulter [CT04].

Pour d'autres commentaires bibliographiques, voir [Bar18].

Chapitre 6

Systèmes de réaction-diffusion non linéaires

Dans cette partie, on se contente d'énoncer des résultats pour des systèmes 2×2 pour simplifier. La majeure partie de ces résultats s'adapte sans grande difficulté aux systèmes de taille $n \times n$ avec n quelconque. On s'intéresse donc à

$$\begin{cases} \partial_t U - D\Delta U = F(U) + BH1_\omega & \text{dans } Q_T, \\ U = 0 & \text{sur } \Sigma_T, \\ U(0,.) = U_0 & \text{dans } \Omega, \end{cases} \tag{6.1}$$

où $D \in \mathbb{R}^{2\times 2}$ est diagonalisable à valeurs propres strictement positives, $B \in \mathbb{R}^{2\times m}$ avec $1 \leq m \leq 2$ et $F \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ est tel que $F(0) = 0$.

Le linéarisé de (6.1) autour de $(0, 0)$ est

$$\begin{cases} \partial_t U - D\Delta U = F'(0)U + BH1_\omega & \text{dans } Q_T, \\ U = 0 & \text{sur } \Sigma_T, \\ U(0,.) = U_0 & \text{dans } \Omega, \end{cases} \tag{6.2}$$

où $F'(0) = (\partial_j F_i(0, 0))$ est la Jacobienne de F en 0. Les Définition 5.0.1 et Définition 5.0.2 s'adaptent aisément au cas des systèmes.

6.1 Cas favorable : le linéarisé est contrôlable

Nous avons les deux résultats suivants qui découlent de la théorie linéaire.

Théorème 6.1.1. *On suppose que $B = I_2$. Alors, pour tout $T > 0$, le système* (6.1) *est localement contrôlable à zéro dans $L^\infty(\Omega)^2$.*

La preuve est basée sur le Théorème 4.1.1 qui démontre la contrôlabilité du linéarisé et sur une adaptation des arguments de la preuve de la contrôlabilité locale à zéro de l'équation de la chaleur semilinéaire, c'est-à-dire du Théorème 5.3.1.

Théorème 6.1.2. *On suppose que $B = (1,0)^{\mathrm{tr}}$. On suppose également que l'on a*

$$\frac{\partial f_2}{\partial u_1}(0,0) \neq 0.$$

Alors, pour tout $T > 0$, le système (6.1) est localement contrôlable à zéro dans $L^\infty(\Omega)^2$.

La preuve est basée sur le Théorème 4.1.2 qui démontre la contrôlabilité du linéarisé et sur une adaptation des arguments de la preuve du Théorème 5.3.1.

6.2 Cas défavorable : le linéarisé n'est pas contrôlable

On supposera dans cette partie que $B = (1,0)^{\mathrm{tr}}$ et $\frac{\partial f_2}{\partial u_1}(0,0) = 0$, si bien que le système linéarisé (6.2) n'est pas contrôlable. Nous disposons néanmoins du résultat suivant.

Théorème 6.2.1. *On suppose qu'il existe une trajectoire régulière $(\overline{U}, \overline{h})$ de (6.1) satisfaisant*

$$(\overline{U}, \overline{h})(0,.) = (\overline{U}, \overline{h})(T,.) = 0,$$

et qu'il existe $(t_1, t_2) \times \omega_0 \subset\subset (0,T) \times \omega$, $\varepsilon > 0$ tel que

$$\forall (t,x) \in (t_1,t_2) \times \omega_0,\ \frac{\partial f_2}{\partial u_1}(\overline{u_1}(t,x), \overline{u_2}(t,x)) \geq \varepsilon.$$

Alors le système (6.1) est localement contrôlable à zéro dans $L^\infty(\Omega)^2$.

La preuve du Théorème 6.2.1 se trouve par exemple dans [CGR10]. Elle est essentiellement basée sur le Théorème 4.1.2 et sur une adaptation des arguments de la preuve du Théorème 5.3.1.

Corollaire 6.2.2. *On suppose que $f_2(u_1, u_2) = u_1^3 + Rv$, où $R \in \mathbb{R}$. Alors, pour tout $T > 0$, le système (6.1) est localement contrôlable à zéro dans $L^{\infty\cdot}(\Omega)^2$.*

La preuve du Corollaire 6.2.2, due à Jean-Michel Coron, Sergio Guerrero et Lionel Rosier, est basée sur le Théorème 6.2.1 et sur l'existence d'une trajectoire non triviale qui part de $((0,0),0)$ et qui *retourne* à $((0,0),0)$ pour le système (6.1) (voir [CGR10]) : c'est la méthode du retour déjà rencontrée en dimension finie (voir Section 2.2).

6.3 Commentaires bibliographiques

Dans le cas d'un linéarisé contrôlable, des théorèmes proches du Théorème 6.1.2 ont été démontrés dans [Bar02], [AKBD06], [WZ06], [LCM+16], [CSB15], [GZ16] et [CFCLM13].

Le Théorème 5.4.2 a été étendu à un système 2×2 de réaction-diffusion dans [AKBDK03] et [TGY12].

Le Corollaire 6.2.2 est étendu à des couplages plus généraux par un *argument d'homogénéité* dans [CGMR15]. Citons également l'article de Jean-Michel Coron et Jean-Philippe Guilleron [CG17] qui traite d'un système cascade de taille trois avec couplages cubiques en utilisant également la méthode du retour.

Un mot sur les conditions de Neumann

Tous les résultats énoncés précédemment dans le cas Dirichlet hormis le Théorème 4.4.1 restent vrais dans le cas Neumann. Pour adapter les preuves, le lecteur pourra consulter les articles [FCGBGP06b] et [FCGBGP06a].

Le Théorème 4.4.1 s'adapte en prenant des conditions initiales $(u_{1,0}, u_{2,0}) \in L^2(\Omega)^2$ telles que $\int_\Omega u_{2,0}(x)dx = 0$. Ceci est dû au fait que dans le cas de conditions de Neumann, la masse de u_2 est conservée au cours du temps. En effet, cette propriété se vérifie en intégrant en espace l'équation vérifiée par u_2 dans (4.15).

De la même manière, pour le système de Keller-Segel

$$\begin{cases} \partial_t u_1 - \Delta u_1 = -\nabla.(u_1 \nabla u_2) & \text{dans } Q_T, \\ \partial_t u_2 - \Delta u_2 = a u_1 - b u_2 + h 1_\omega & \text{dans } Q_T, \\ \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0 & \text{sur } \Sigma_T, \\ (u_1, u_2)(0,.) = (u_{1,0}, u_{2,0}) & \text{dans } \Omega, \end{cases}$$

où $a, b \in \mathbb{R}$, la masse de u_1 est conservée au cours du temps. C'est pourquoi le résultat de contrôlabilité locale à états stationnaires strictement positifs constants (M_1, M_2) obtenus par Felipe Walison Chaves-Silva et Sergio Guerrero dans [CSG15] nécessite

$$\frac{1}{|\Omega|} \int_\Omega u_{2,0}(x)dx = M_2.$$

La preuve de ce résultat est basée sur une linéarisation autour de (M_1, M_2) et sur une adaptation du Théorème 4.4.1.

Troisième partie

Principaux résultats de la thèse

Chapitre 7

Contrôlabilité de systèmes de réaction-diffusion non linéaires issus de la cinétique chimique

L'objectif de cette partie est de présenter les résultats obtenus dans les articles [LB19] et [LB18b]. Ces deux papiers répondent à des questions de contrôlabilité pour des systèmes de réaction-diffusion non linéaires modélisant des réactions chimiques réversibles.

Dans un premier temps, nous présentons quelques aspects de modélisation de ces équations réversibles en Section 7.1. Le but de cette partie est d'aboutir à la forme générale des systèmes de réaction-diffusion que nous allons étudier par la suite : c'est-à-dire au système (7.6).

Dans un second temps, nous dressons en Section 7.2 un bref aperçu des questions d'existence globale associées à ces systèmes. Ces questions ont été et sont toujours beaucoup étudiées. En particulier, il n'était pas connu jusqu'à récemment s'il y avait existence globale de solutions classiques pour un système à non linéarité quadratique.

Nous présentons en Section 7.3 le système contrôlé qui va nous intéresser. La principale question à laquelle nous allons répondre par la suite est une question de contrôlabilité locale en temps petit à états stationnaires positifs (constants) pour ces systèmes.

En Section 7.4, nous menons une étude approfondie d'un système 4×4 de réaction-diffusion à non linéarité quadratique et présentons ainsi les résultats issus de l'article [LB19]. Ce système provient de la modélisation d'une équation réversible faisant intervenir quatre espèces chimiques. Plusieurs aspects ont motivé cette étude :

— il n'était pas connu jusqu'à récemment s'il y avait existence globale de solutions classiques pour ce système,

— ce système offre de multiples difficultés du point de vue de la contrôlabilité : le linéarisé peut ne pas être contrôlable, des quantités invariantes apparaissent quand il n'y a pas assez de contrôles dans les équations : ce qui empêche la contrôlabilité d'avoir lieu dans tout l'espace,

— ce système permet de dégager une stratégie robuste pour comprendre les propriétés de contrôlabilité d'un système de réaction-diffusion plus général.

Nous établissons ainsi dans la Section 7.4 des résultats de contrôlabilité locale en temps petit (voir Théorème 7.4.1, Théorème 7.4.5) qui nous permettent de déduire des résultats de contrôlabilité globale en temps long en petite dimension (voir Théorème 7.4.6) grâce à l'asymptotique déjà connue du système libre.

Nous retournons au cas général en Section 7.5, ce qui nous permet de présenter les résultats de l'article [LB18b]. Sous une hypothèse qui assure la contrôlabilité du linéarisé, nous démontrons un résultat de contrôlabilité locale en temps petit (voir Théorème 7.5.3). Alors que la stratégie générale de preuve suit les mêmes mécanismes que pour le système de quatre espèces, la méthode de preuve se doit d'être adaptée en raison du trop grand nombre d'équations que ne peut pas gérer la stratégie de Carleman habituelle. Nous dégageons également de cette étude un nouvel apport méthodologique de l'article [LB18b] pour démontrer un résultat de contrôlabilité locale en temps petit dans L^∞ pour un système de réaction-diffusion non linéaire.

7.1 Modélisation des équations réversibles

Soit $n \geq 2$ un entier. On considère la réaction chimique réversible

$$\alpha_1 A_1 + \cdots + \alpha_n A_n \rightleftharpoons \beta_1 A_1 + \cdots + \beta_n A_n, \tag{7.1}$$

où $A_1, \ldots, A_n$ désignent n espèces chimiques et $(\alpha_1, \ldots, \alpha_n)$, $(\beta_1, \ldots, \beta_n) \in (\mathbb{N})^n$ vérifient

$$\forall i \in \{1, \ldots, n\}, \ \alpha_i \neq \beta_i.$$

La réaction directe $\rightharpoonup$ de (7.1) est gouvernée par le principe suivant : quand α_i molécules de A_i disparaissent $(1 \leq i \leq n)$, dans ce cas les A_i disparaissant sont appelés *réactifs*, alors β_i molécules de A_i apparaissent $(1 \leq i \leq n)$, dans ce cas les A_i apparaissant sont appelés *produits*. La réaction inverse $\leftharpoondown$ de (7.1) suit la même loi : quand β_i molécules de A_i disparaissent $(1 \leq i \leq n)$, alors α_i molécules de A_i apparaissent $(1 \leq i \leq n)$.

Soit $i \in \{1, \ldots, n\}$ et $(t, x) \in [0, +\infty) \times \Omega$. On appelle $u_i(t, x) \in \mathbb{R}$ (respectivement $\vec{v_i}(t, x) \in \mathbb{R}^N$) la concentration (respectivement la vitesse de la particule) de l'espèce chimique A_i à l'instant t et à la position x. La *loi d'action de masse* stipule que la vitesse de réaction est proportionnelle à la concentration des réactifs. Ainsi, en écrivant la variation instantanée de la concentration u_i, on obtient

$$\partial_t u_i + div(u_i \vec{v_i}) + \underbrace{\alpha_i \prod_{k=1}^n u_k^{\alpha_k}}_{\text{réactifs de la réaction directe}} + \underbrace{\beta_i \prod_{k=1}^n u_k^{\beta_k}}_{\text{réactifs de la réaction inverse}} \tag{7.2}$$

$$= \underbrace{\beta_i \prod_{k=1}^n u_k^{\alpha_k}}_{\text{produits de la réaction directe}} + \underbrace{\alpha_i \prod_{k=1}^n u_k^{\beta_k}}_{\text{produits de la réaction inverse}} . \tag{7.3}$$

De plus, la *loi de Fick* régit le comportement sur le *flux de matière* $u_i\vec{v_i}$ de la manière suivante :

$$u_i\vec{v_i} = -d_i \nabla u_i,$$

où $d_i \in (0, +\infty)$ est le *coefficient de diffusion* de l'espèce A_i. Autrement dit, cette loi exprime que le déplacement des particules se fait dans le sens opposé du gradient de u_i, c'est-à-dire des régions où la concentration de A_i est la plus dense vers celles où elle l'est le moins. Ainsi, (7.2) devient

$$\partial_t u_i - d_i \Delta u_i = (\beta_i - \alpha_i) \left(\prod_{k=1}^{n} u_k^{\alpha_k} - \prod_{k=1}^{n} u_k^{\beta_k} \right). \tag{7.4}$$

Pour fermer le système, il est indispensable de décrire ce qui se passe au bord du domaine spatial Ω. Une condition classique consiste à écrire que le milieu est isolé et donc qu'aucune des substances ne traverse la frontière du domaine. Nous obtenons alors les conditions dites de *Neumann* au bord :

$$\frac{\partial u_i}{\partial \nu} = 0. \tag{7.5}$$

Ainsi, en rassemblant (7.4), (7.5), et en posant

$$U := (u_1, \dots, u_n)^{\mathrm{tr}},$$

on obtient que U satisfait le système de *réaction-diffusion non linéaire* suivant

$$\begin{cases} \partial_t U - D\Delta U = F(U) & \text{dans } Q_T, \\ \frac{\partial U}{\partial \nu} = 0 & \text{sur } \Sigma_T, \\ U(0,.) = u_0 & \text{dans } \Omega, \end{cases} \tag{7.6}$$

où

$$D := \mathrm{diag}(d_1, \dots, d_n), \tag{7.7}$$

$$F(U) := (f_i(u_1, \dots, u_n))_{1 \le i \le n}^{\mathrm{tr}}, \tag{7.8}$$

avec

$$\forall i \in \{1, \dots, n\},\ f_i(u_1, \dots, u_n) := (\beta_i - \alpha_i) \left(\prod_{k=1}^{n} u_k^{\alpha_k} - \prod_{k=1}^{n} u_k^{\beta_k} \right). \tag{7.9}$$

De manière implicite, quand nous parlerons du système (7.6) dans toute la suite, nous ferons référence au système (7.6) pour D définie comme dans (7.7) et F définie comme dans (7.8), (7.9).

Exemple 7.1.1. L'hémoglobine Hb peut réagir avec le dioxygène O_2 pour former de l'oxyhémoglobine HbO_2. La réaction chimique réversible associée s'écrit

$$Hb + O_2 \rightleftharpoons HbO_2,$$

et le système de réaction-diffusion non linéaire est, pour $u_1 = [Hb]$, $u_2 = [O_2]$, $u_3 = [HbO_2]$,

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2 + u_3 & \text{dans } Q_T, \\ \partial_t u_1 - d_2 \Delta u_2 = -u_1 u_2 + u_3 & \text{dans } Q_T, \\ \partial_t u_3 - d_3 \Delta u_3 = u_1 u_2 - u_3 & \text{dans } Q_T, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial u_3}{\partial \nu} = 0 & \text{sur } \Sigma_T, \\ (u_1, u_2, u_3)(0, .) = (u_{1,0}, u_{2,0}, u_{3,0}) & \text{dans } \Omega. \end{cases}$$

Pour d'autres exemples de réactions chimiques réversibles, on pourra consulter [Per15, Chapter 1].

7.2 Un petit aperçu des questions d'existence globale

Cette partie est largement inspirée de l'état de l'art dressé dans [Pie10]. On trouvera dans [Pie10] beaucoup d'autres résultats et de références sur le sujet.

Nous introduisons la notation

$$d := \max \left(\sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n} \beta_i \right), \tag{7.10}$$

qui désigne le degré du polynôme associé à la non linéarité F.

Définition 7.2.1. On dit que U est une *solution classique* de (7.6) sur $[0, T)$ si U est une solution de (7.6) au sens de la Définition A.2.1 sur $[0, T - \tau]$ pour tout $\tau > 0$.

Remarque 7.2.2. La Définition A.2.1 est à adapter aux conditions de Neumann et au cas des systèmes (voir [LB18b, Definition 1.1]).

Rappelons le résultat d'existence locale d'une solution classique à (7.6) (voir [Pie10, Lemma 1.1]).

Théorème 7.2.3. *Soit $U_0 \in L^\infty(\Omega)^n$. Alors, il existe une unique solution classique maximale définie sur $[0, T^*)$ et on a l'implication suivante :*

$$\left(\sup_{t \in [0, T^*)} \|U(t)\|_{L^\infty(\Omega)^n} < +\infty \right) \Rightarrow \left(T^* = +\infty \right).$$

De plus,

$$\left(\forall i \in \{1, \ldots, n\},\ u_{i,0} \geq 0 \right) \Rightarrow \left(\forall i \in \{1, \ldots, n\},\ \forall t \in [0, T^*),\ u_i(t) \geq 0 \right).$$

Quand la non linéarité n'est pas trop grosse (typiquement $d < (N + 2)/2$) et qu'elle est bornée dans $L^1(Q_T)$, on peut démontrer un résultat d'existence globale par un argument de bootstrap couplé aux effets régularisants du semi-groupe de la chaleur (voir Proposition A.6.1). On peut consulter par exemple [Ali79].

Mentionnons le résultat suivant d'existence de solutions globales de solutions classiques pour (7.6), énoncé avec $n = 2$ pour simplifier (voir [Pie10, Theorem 3.1]).

Théorème 7.2.4. *On suppose que $U_0 \in L^\infty(\Omega; \mathbb{R}^+)$ et qu'il existe $M, C \geq 0$ tels que*

$$\forall u_1 \geq M,\ \forall u_2 \geq 0,\ f_1(u_1, u_2) \leq C(1 + u_1 + u_2), \tag{7.11}$$
$$\exists r \geq 1,\ \forall u_1, u_2 \geq 0,\ f_2(u_1, u_2) \leq C(1 + u_1^r + u_2^r). \tag{7.12}$$

Alors, la solution maximale est globale : $T^* = +\infty$.

Remarque 7.2.5. La première condition (7.11) est une bonne estimation au sens où on demande à la première non linéarité de l'équation d'être au plus linéaire. Tandis que la seconde condition (7.12) impose une croissance au plus polynomiale à la seconde linéarité.

Exemple 7.2.6. Considérons la réaction chimique réversible

$$A_1 \rightleftharpoons 2A_2.$$

Dans ce cas, on a

$$f_1(u_1, u_2) = -f_2(u_1, u_2) = -u_1 + 2u_2^2,$$

qui satisfait les conditions du Théorème 7.2.4.

Quand $d \leq 2$, c'est-à-dire quand la non linéarité est au plus quadratique, l'existence de *solutions faibles* globales est garantie par le [Pie10, Proposition 5.12]. Pour la définition de solutions faibles, voir [Pie10, Section 5.2]. Des résultats récents (voir [CGV17], [Sou18] et [FMT18]) montrent l'existence globale de solutions classiques dans le cas quadratique avec des hypothèses supplémentaires. Ces articles s'appuient notamment sur des idées issues de [Kan90]. Pour des systèmes généraux du type (7.6) où f_i est définie en (7.9), l'existence de *solutions renormalisées* globales est obtenue dans [Fis15].

7.3 Le système contrôlé

On suppose que l'on peut agir sur le système au moyen de contrôles localisés dans un ouvert ω contenu dans Ω. D'un point de vue chimique, cela veut dire qu'on peut ajouter ou retirer une espèce chimique à un endroit spécifique (dans ω) du domaine spatial Ω. Plus précisément, posons

$$J \subset \{1, \ldots, n\} \text{ et } m := \#J \leq n \text{ le nombre de contrôles.} \tag{7.13}$$

Quitte à renuméroter $(u_i)_{1 \leq i \leq n}$, on peut supposer que $J = \{1, \ldots, m\}$ où J est défini en (7.13). Ainsi, on peut définir le contrôle vectoriel

$$H^J := (h_1, \ldots, h_m, 0, \ldots, 0)^{\mathrm{tr}}. \tag{7.14}$$

Le problème de contrôle se met alors sous la forme

$$\begin{cases} \partial_t U - D\Delta U = F(U) + H^J 1_\omega & \text{dans } Q_T, \\ \frac{\partial U}{\partial \nu} = 0 & \text{sur } \Sigma_T, \\ U(0, .) = U_0 & \text{dans } \Omega, \end{cases} \tag{7.15}$$

où D et F sont définies respectivement en (7.7), (7.8), (7.9).

Pour $t \in [0,T]$, $U(t,.) : \Omega \to \mathbb{R}^n$ est l'état de notre système sur lequel on veut agir au moyen du contrôle $H^J(t,.) : \Omega \to \mathbb{R}^m$ localisé dans ω.

Soit

$$U^* := (u_1^*, \dots, u_n^*)^{\mathrm{tr}}, \tag{7.16}$$

un état stationnaire positif de (7.6), c'est-à-dire,

$$\forall 1 \leq i \leq n,\ u_i^* \in [0,+\infty) \text{ et } \prod_{k=1}^{n} {u_k^*}^{\alpha_k} = \prod_{k=1}^{n} {u_k^*}^{\beta_k}. \tag{7.17}$$

Les solutions stationnaires positives de (7.6) ne dépendent pas de la variable spatiale (voir [LB18b, Section 1.8.2]). Ainsi, ce n'est pas restrictif de supposer $U^* \in [0,+\infty)^n$.

On introduit la définition suivante de contrôlabilité locale à un état stationnaire U^*.

Définition 7.3.1. Pour $T > 0$, $U^* \in [0,+\infty)^n$ vérifiant (7.17) et $X \subset L^\infty(\Omega)$, on dit que le système (7.15) est localement contrôlable à U^* dans X au temps T s'il existe $\delta > 0$ tel que pour tout $U_0 \in X$ satisfaisant $\|U_0 - U^*\|_{L^\infty(\Omega)^n} \leq \delta$, il existe $H \in L^\infty(q_T)^m$ tel que la solution U de (7.15) vérifie $U(T,.) = U^*$.

7.4 Étude approfondie du système de quatre espèces chimiques : nouveaux résultats de contrôlabilité

Nous allons d'abord nous concentrer sur un cas particulier d'équation réversible faisant intervenir quatre espèces chimiques. En effet, l'étude de ce système nous permettra de dégager les principaux mécanismes de contrôlabilité des systèmes (7.15). Pour $n = 4$ et $\alpha_1 = \alpha_3 = \beta_2 = \beta_4 = 1$ et $\alpha_2 = \alpha_4 = \beta_1 = \beta_3 = 0$, (7.1) devient

$$A_1 + A_3 \rightleftharpoons A_2 + A_4.$$

Le système contrôlé de réaction-diffusion non linéaire (7.15) associé s'écrit alors : pour tout $i \in \{1,2,3,4\}$,

$$\begin{cases} \forall 1 \leq i \leq 4, & \\ \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) + h_i 1_J(i) 1_\omega & \text{dans } Q_T, \\ \frac{\partial u_i}{\partial n} = 0 & \text{sur } \Sigma_T, \\ u_i(0,.) = u_{i,0} & \text{dans } \Omega. \end{cases} \tag{7.18}$$

La linéarisation autour de U^* de chacune des équations constituant (7.18) donne : pour tout $1 \leq i \leq 4$,

$$\partial_t u_i - d_i \Delta u_i = (-1)^i (u_3^* u_1 - u_4^* u_2 + u_1^* u_3 - u_2^* u_4) + h_i 1_J(i) 1_\omega. \tag{7.19}$$

Dans la suite, on exclura le cas $J = \{1,2,3,4\}$ qui sera en fait contenu dans le cas $J = \{1,2,3\}$.

7.4.1 Nouveau résultat de contrôlabilité locale en temps petit avec trois contrôles

On suppose que $J = \{1, 2, 3\}$. Ainsi, le contrôle H^J agit sur les trois premières équations de (7.18).

En utilisant (7.19), les composantes u_1, u_2 et u_3 sont facilement contrôlables en raison de la présence de contrôles dans les trois premières équations. La difficulté est donc le contrôle de la dernière composante, c'est-à-dire u_4.

S'il y a du couplage dans la quatrième équation, c'est-à-dire si $(u_1^*, u_3^*, u_4^*) \neq (0, 0, 0)$ alors, nous sommes dans un cadre favorable. En effet, heuristiquement, u_4 est contrôlée de manière indirecte grâce à $u_3^* u_1$ si $u_3^* \neq 0$, grâce à $u_4^* u_2$ si $u_4^* \neq 0$, grâce à $u_1^* u_3$ si $u_1^* \neq 0$.

S'il n'y a pas de couplage dans la dernière équation, i.e., $(u_1^*, u_3^*, u_4^*) = (0, 0, 0)$ alors le linéarisé de (7.15) autour de U^* n'est pas contrôlable puisque la dernière équation est alors découplée des autres. Pour remédier à cela, nous allons utiliser la méthode du retour de Jean-Michel Coron qui consiste à linéariser le long d'une trajectoire de référence non triviale. Pour ce faire, on se donne une fonction $g \in C_c^\infty(\overline{Q_T}; \mathbb{R})$ non nulle telle que

$$\operatorname{supp}(g) \subset\subset (0, T) \times \omega,$$

et on pose

$$\overline{u_3} := g \text{ et } \overline{h_3} := \partial_t g - \Delta g.$$

Ainsi, la quatrième équation du linéarisé autour de la trajectoire $\big((0, u_2^*, \overline{u_3}, 0), (0, 0, \overline{h_3})\big)$ devient

$$\partial_t u_4 - d_4 \Delta u_4 = \overline{u_3} u_1 - u_2^* u_4.$$

Heuristiquement, comme $\overline{u_3}$ n'est pas identiquement nulle dans la zone de contrôle, u_4 va être indirectement contrôlé par u_1 à l'aide du terme de couplage $\overline{u_3} u_1$.

On démontre alors le nouveau résultat suivant (voir [LB19, Theorem 3.2]).

Théorème 7.4.1. *Pour $J = \{1, 2, 3\}$, $T > 0$, $U^* \in [0, +\infty)^4$ vérifiant $u_1^* u_3^* = u_2^* u_4^*$, le système de quatre espèces (7.18) est localement contrôlable à U^* dans $L^\infty(\Omega)^4$ au temps T.*

La preuve du Théorème 7.4.1 est une adaptation du théorème de contrôlabilité locale d'un système 2×2 de réaction-diffusion avec un seul contrôle : c'est-à-dire du Théorème 6.2.1.

7.4.2 Nouveau résultat de contrôlabilité locale en temps petit avec deux contrôles

On suppose que $J = \{1, 2\}$. Ainsi, le contrôle H^J agit sur les deux premières équations de (7.18).

L'une des différences entre le cas de trois contrôles et celui de deux contrôles est l'apparition de *quantités invariantes* qui vont empêcher la contrôlabilité d'avoir lieu dans tout l'espace $L^\infty(\Omega)^4$. On a le résultat suivant.

Proposition 7.4.2. *Soit (U, H^J) une trajectoire de (7.18) telle que $U(T,.) = U^*$. Alors, pour tout $t \in [0,T]$,*

$$\frac{1}{|\Omega|}\left(\int_\Omega u_3(t,x) + u_4(t,x)dx\right) = u_3^* + u_4^*, \tag{7.20}$$

$$\Big(d_3 = d_4\Big) \Rightarrow \left(u_3(t,.) + u_4(t,.) = u_3^* + u_4^*\right). \tag{7.21}$$

Démonstration. La preuve de (7.20) consiste à sommer les deux dernières équations qui ne contiennent pas de contrôle et à intégrer en espace : on trouve en utilisant les conditions de Neumann

$$\frac{d}{dt}\left(\int_\Omega u_3(t,x) + u_4(t,x)dx\right) = 0.$$

Ainsi, comme $u_3(T,.) + u_4(T,.) = u_3^* + u_4^*$, on en déduit immédiatement (7.20).

Pour démontrer (7.21), on procède de la même manière, on somme les deux dernières équations et on trouve que

$$\partial_t(u_3 + u_4) - d_3\Delta(u_3 + u_4) = 0.$$

Ainsi, $u_3 + u_4$ est solution de l'équation de la chaleur libre et comme $(u_3 + u_4)(T,.) = u_3^* + u_4^*$, alors par unicité rétrograde (voir Proposition A.5.1), on obtient (7.21). □

Remarque 7.4.3. Quand $d_3 = d_4$, on peut donc exprimer u_4 en fonction de u_3 et on est donc ramené à un système de trois équations avec deux contrôles. Ce dernier cas n'apportant alors pas de réelle difficulté par rapport à celui de trois contrôles, nous l'exclurons dans la suite.

On notera dans la suite

$$X := \left\{U_0 \in L^\infty(\Omega)^4 \ ; \ \frac{1}{|\Omega|}\left(\int_\Omega u_{3,0}(x) + u_{4,0}(x)dx\right) = u_3^* + u_4^*\right\}. \tag{7.22}$$

Quand certaines composantes de l'état stationnaire s'annulent, on montre dans le résultat suivant que cela contraint la dynamique de (7.18).

Proposition 7.4.4. *Soit (U, H^J) une trajectoire de (7.18) telle que $U(T,.) = U^*$. Alors, on a*

$$\Big((u_3^*, u_4^*) = (0,0)\Big) \Rightarrow \Big((u_{3,0}, u_{4,0}) = (0,0)\Big). \tag{7.23}$$

Réciproquement, pour tout $u_0 \in L^\infty(\Omega)^4$ tel que $(u_{3,0}, u_{4,0}) = (0,0)$, on peut trouver $H^J \in L^\infty(q_T)^2$ tel que la solution associée $U \in L^\infty(Q_T)^4$ satisfait

$$(u_1, u_2, u_3, u_4)(T,.) = (u_1^*, u_2^*, 0, 0).$$

Démonstration. Si $(u_3^*, u_4^*) = (0,0)$, on a alors

$$\begin{cases} \partial_t u_3 - d_3 \Delta u_3 = -u_1 u_3 + u_2 u_4 & \text{dans } Q_T, \\ \partial_t u_4 - d_4 \Delta u_4 = u_1 u_3 - u_2 u_4 & \text{dans } Q_T, \\ \frac{\partial u_3}{\partial n} = \frac{\partial u_4}{\partial n} = 0 & \text{sur } \Sigma_T. \end{cases} \tag{7.24}$$

Or, $(u_1, u_2) \in L^\infty(Q_T)^2$, donc par unicité rétrograde pour le système (7.24) (voir [BT73, Théorème II.1]), on a

$$\forall t \in [0, T],\ (u_3, u_4)(t, .) = (0, 0),$$

et donc (7.23).

Réciproquement, soit $U_0 \in L^\infty(\Omega)^4$ telle que $(u_{3,0}, u_{4,0}) = (0,0)$. Alors, en posant $u_3 = u_4 = 0$, on réduit le système à deux équations de la chaleur découplées, chacune d'elle étant contrôlée par un contrôle (différent), donc par contrôlabilité (globale) à zéro de l'équation de la chaleur (voir Théorème 5.2.1), on déduit qu'il existe bien un contrôle H^J amenant la solution à $(u_1^*, u_2^*, 0, 0)$. $\square$

Au regard de nos précédentes discussions (voir en particulier la Proposition 7.4.2 et la Proposition 7.4.4) on supposera que $d_3 \neq d_4$ et $(u_3^*, u_4^*) \neq (0,0)$.

Nous allons effectuer une transformation linéaire du système (7.18) qui va nous permettre de mieux comprendre les propriétés de contrôlabilité de ce système. En effet, cette transformation va réduire le système à un système *cascade* avec couplages d'ordres zéro et deux. On multiplie la troisième équation de (7.18) par $-(d_3 - d_4)^{-1}$ et la quatrième équation de (7.18) par $(d_4 - d_3)^{-1}$, et on additionne :

$$\partial_t v_4 - d_4 \Delta v_4 = -\Delta u_3, \text{ où } v_4 = \frac{u_3}{-(d_3 - d_4)} + \frac{u_4}{(d_4 - d_3)}.$$

Cette combinaison linéaire nous permet d'éliminer le terme de réaction $f_4(U)$ défini en (7.9) dans la quatrième équation et de créer un couplage du second ordre. Dans le nouveau jeu de variables (u_1, u_2, u_3, v_4), (7.19) pour $i = 3$ devient :

$$\partial_t u_3 - d_3 \Delta u_3 = -u_3^* u_1 + u_4^* u_2 + (-u_1^* - u_2^*) u_3 + (d_4 - d_3) u_2^* v_4.$$

Ainsi, ayant supposé que $(u_3^*, u_4^*) \neq (0,0)$, on en déduit heuristiquement les propriétés de contrôlabilité du système (7.18). En effet, les composantes u_1 et u_2 sont contrôlées respectivement par h_1 et h_2, puis le couplage $-u_3^* u_1$ ou $u_4^* u_2$ dans la troisième équation permet le contrôle indirect de la composante u_3, qui elle-même, va contrôler indirectement u_4 à travers le terme de couplage $-\Delta u_3$.

On aboutit au nouveau résultat suivant de contrôlabilité locale dans X défini en (7.22) (voir [LB19, Theorem 3.2]).

Théorème 7.4.5. *Pour $J = \{1, 2\}$, $d_3 \neq d_4$, $T > 0$, $U^* \in [0, +\infty)^4$ vérifiant $u_1^* u_3^* = u_2^* u_4^*$ et $(u_3^*, u_4^*) \neq (0,0)$, le système de quatre espèces (7.18) est localement contrôlable à U^* dans X au temps T.*

Pour démontrer le Théorème 7.4.5, on montre dans un premier temps que le linéarisé autour de U^* est contrôlable par des arguments similaires à ceux employés pour la preuve des théorèmes de contrôlabilité de systèmes linéaires avec couplages d'ordre zéro et d'ordre deux, c'est-à-dire le Théorème 4.1.2 et le Théorème 4.4.1. Puis, pour le passage au non linéaire, il suffit d'adapter la preuve de la contrôlabilité locale à zéro de l'équation de la chaleur, voir Théorème 5.3.1.

7.4.3 Cas d'un seul contrôle

On peut également montrer un résultat de contrôlabilité locale en temps petit à état stationnaire positif constant dans le cas où $J = \{1\}$ (voir [LB19, Theorem 3.2]). Ce dernier résultat n'offrant cependant pas de différence majeure par rapport à celui de deux contrôles, nous ne le présentons pas ici. A noter cependant que la contrôlabilité du linéarisé dans ce dernier cas est assez technique car elle doit prendre en compte un système cascade avec un couplage d'ordre 0 et deux couplages d'ordre 2. La linéarisation étant faite dans $L^\infty(Q_T)$, les coefficients du linéarisé ne sont alors plus aussi réguliers que dans le Théorème 4.4.1. Ainsi, la méthode de preuve du Théorème 4.4.1, qui consiste à appliquer l'opérateur Δ à certaines équations sous hypothèse de régularité pour tirer profit du couplage d'ordre deux, doit être adaptée. C'est pourquoi, démontrer l'inégalité d'observabilité sur le linéarisé est plus difficile et requiert notamment la régularisation des coefficients des équations, des arguments de densité et l'utilisation d'inégalités de Carleman ayant servi à Felipe Walison Chaves Silva et Sergio Guerrero dans le contexte du système de Keller-Segel (voir [CSG15]).

7.4.4 Nouveau résultat de contrôlabilité globale en temps long et en petite dimension

Rappelons le théorème suivant concernant le comportement asymptotique d'une solution libre ($H^J = 0$) du système de quatre espèces (7.18).

Théorème 7.4.6. *Soit $N \leq 2$ et $U_0 \in L^\infty(\Omega; [0, +\infty))^4$ satisfaisant*

$$\forall (i,j) \in \Big\{(1,2),(1,4),(2,3),(3,4)\Big\}, \ \frac{1}{|\Omega|}\int_\Omega (u_{i,0} + u_{j,0})(x)dx > 0. \tag{7.25}$$

Alors l'unique solution (classique) de (7.18) avec $H^J = 0$ converge dans $L^\infty(\Omega)^4$ quand $t \to +\infty$ vers un unique état stationnaire $U^* \in (0, +\infty)^4$.

La preuve du Théorème 7.4.6 repose sur le [PSZ17, Theorem 3] et sur le [PSY18, Theorem 3] (voir aussi [DF06]).

On démontre grâce au Théorème 7.4.6 et au Théorème 7.4.1 le nouveau résultat suivant (voir [LB19, Theorem 3.6]).

Théorème 7.4.7. *On suppose que $N \leq 2$ et $J = \{1, 2, 3\}$. Soit $U^* \in [0, +\infty)^4$ vérifiant $u_1^* u_3^* = u_2^* u_4^*$. Alors pour tout $U_0 \in L^\infty(\Omega; [0, +\infty))^4$ satisfaisant (7.25), il existe $T > 0$ suffisamment grand et $H^J \in L^\infty(q_T)^3$ telle que la solution u de (7.18) vérifie $U(T, .) = U^*$.*

Remarque 7.4.8. Le Théorème 7.4.7 s'adapte également au cas où $J = \{1, 2\}$, $J = \{1\}$. On peut étendre le résultat du Théorème 7.4.7 pour $N \geq 3$ quand on dispose de la convergence asymptotique dans L^∞ du système libre (c'est-à-dire avec $H^J = 0$) de (7.18). C'est par exemple le cas pour des d_i suffisamment proches (voir [CDF14, Proposition 1.3]).

On présente les grandes lignes de la preuve du Théorème 7.4.7.

Démonstration. La preuve du Théorème 7.4.7 se fait en trois étapes de contrôle.

Etape 1 : Utiliser l'asymptotique sans contrôle. Tout d'abord, en utilisant le Théorème 7.4.6, on sait que la solution libre (c'est-à-dire avec $H^J = 0$) de (7.18) converge dans $L^\infty(\Omega)^4$ quand $t \to +\infty$ vers un état stationnaire $Z \in (0, +\infty)^4$.

Etape 2 : Contrôlabilité locale. Par contrôlabilité locale dans $L^\infty(\Omega)^4$ autour de cet état stationnaire Z (voir Théorème 7.4.1), il existe donc un contrôle H_0^J qui amène exactement la solution de (7.18) à Z.

Etape 3 : Argument de connexité par arcs. Il existe un chemin continu $\gamma : [0, 1] \to (\mathbb{R}^+)^4$ reliant Z à U^* tel que pour tout $\theta \in [0, 1]$, $\gamma(\theta)$ vérifie (7.17). On montre ensuite qu'on peut trouver un rayon $r > 0$ uniforme tel que pour tout $\theta \in [0, 1]$, la contrôlabilité locale à $\gamma(\theta)$ dans $L^\infty(\Omega)^4$, a lieu dans une boule de L^∞ de taille au moins r (voir Théorème 7.4.1). Alors l'image de γ est contenue dans la réunion d'un nombre fini de boules $B(\gamma_i, r)$ pour i allant de 0 à K avec $\gamma_0 = Z$, $|\gamma_i - \gamma_{i+1}| \leq r$ et $\gamma_K = U^*$. Par contrôlabilité locale à chacun de ces γ_i, on construit alors une suite de contrôles H_i^J permettant de passer successivement de γ_i à γ_{i+1} jusqu'à atteindre $\gamma_K = U^*$. Ce qui conclut la preuve. $\square$

7.5 Retour au cas général d'un système de taille arbitraire : nouveau résultat de contrôlabilité locale en temps petit

Dans cette partie, nous établissons un résultat de contrôlabilité locale à état stationnaire positif pour le système général (7.15).

En procédant comme dans la preuve de la Proposition 7.4.2, on peut montrer que certaines quantités du système (7.15) sont conservées. Nous introduisons ainsi l'espace de conditions initiales suivant

$$\begin{aligned} X := \Big\{ & U_0 \in L^\infty(\Omega)^n \ ; \ \forall k, l \in \{m+1, \ldots, n\}, \\ & \int_\Omega \frac{u_{k,0}(x) - u_k^*}{\beta_k - \alpha_k} dx = \int_\Omega \frac{u_{l,0}(x) - u_l^*}{\beta_l - \alpha_l} dx \Big\}. \end{aligned} \tag{7.26}$$

Dans le cas où des coefficients de diffusion sont égaux, on peut montrer que la dynamique est trop contrainte comme déjà observé dans la Proposition 7.4.2. C'est pourquoi nous travaillerons sous l'hypothèse suivante.

Hypothèse 7.5.1. Pour $m \leq n-2$, on suppose que pour tous $k \neq l \in \{m+1, \dots, n\}$, $d_k \neq d_l$.

De plus, nous ferons également l'hypothèse suivante qui assure la contrôlabilité du linéarisé autour de l'état stationnaire U^*.

Hypothèse 7.5.2. Pour $m \leq n-1$, on suppose que

$$\partial_m f_{m+1}\Big(u_1^*, \dots, u_n^*\Big) \neq 0, \tag{7.27}$$

où f_{m+1} est définie en (7.9).

On montre alors le résultat suivant (voir [LB18b, Theorem 1.7]).

Théorème 7.5.3. *Soit $T > 0$ et $U^* \in [0, +\infty)^n$ un état stationnaire, i.e. vérifiant* (7.17)*. Sous l'Hypothèse 7.5.1 et l'Hypothèse 7.5.2, le système* (7.15) *est localement contrôlable à U^* dans X au temps T.*

La preuve du Théorème 7.5.3 passe par la mise sous forme cascade du système (7.15) : avec un couplage d'ordre zéro et $n-m-1$ couplages d'ordre deux. Plus précisément, après changement de variable et linéarisation, on peut montrer que le linéarisé prend la

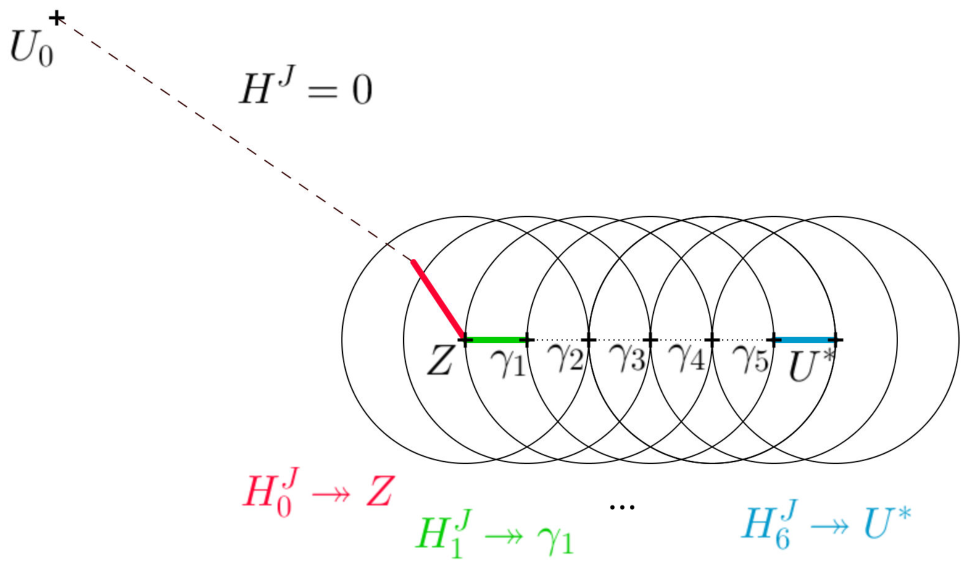

FIGURE 7.1 – Illustration de la preuve du Théorème 7.4.7

forme suivante :

$$\begin{cases} \partial_t Z - D_J \Delta Z = A_J Z + H^J 1_\omega & \text{dans } Q_T, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{sur } \Sigma_T, \\ Z(0,.) = Z_0 & \text{dans } \Omega, \end{cases}$$

où

$$D_J := \left(\begin{array}{c|c} diag(d_1, \dots, d_m) & (0) \\ \hline (0) & D_\sharp \end{array} \right), \ D_\sharp := \begin{pmatrix} d_{m+1} & 0 & \dots & \dots & 0 \\ 1 & d_{m+2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & d_n \end{pmatrix},$$

$$A_J = (a_{ik})_{1 \leq i,k \leq n} \in \mathbb{R}^{n \times n}, \qquad a_{ik} = 0 \text{ si } m+2 \leq i \leq n.$$

De plus, l'Hypothèse 7.5.2 assure le fait que $a_{m+1,m} \neq 0$. Ainsi, ces $n-m$ couplages contribuent au contrôle indirect des composantes qui ne sont pas contrôlées directement par un contrôle localisé. On a le diagramme heuristique suivant

$$\begin{aligned} & h_1 \to z_1, \ h_2 \to z_2, \ \dots \ , \ h_{m-1} \to z_{m-1}, \\ & h_m \to z_m \xrightarrow[a_{m+1m} z_m]{} z_{m+1} \xrightarrow[\Delta z_{m+1}]{} z_{m+2} \xrightarrow[\Delta z_{m+2}]{} \dots \xrightarrow[\Delta z_{n-1}]{} z_n. \end{aligned}$$

La forme cascade découle d'une transformation linéaire adéquate justifiée par un lemme purement algébrique (voir [LB18b, Section 2]). La contrôlabilité du linéarisé se montre par des arguments sensiblement différents de ceux employés précédemment puisque la stratégie de Carleman présente dans ce cas des limitations : il y a trop de couplages d'ordre deux (voir à ce titre la Remarque 4.4.2). Nous montrons par une stratégie similaire à celle de Lebeau-Robbiano qu'*un seul linéarisé* est contrôlable et nous en déduisons par une adaptation de la méthode de Liu-Takahashi-Tucsnak (voir [LTT13]) le passage au non linéaire.

Ainsi, un autre apport de l'article [LB18b] est méthodologique. Pour démontrer un résultat de contrôlabilité locale en temps petit T à un état constant U^* dans L^∞ pour un système parabolique non linéaire, on procède de la façon suivante.

- On commence par démontrer que le linéarisé autour de U^* est contrôlable dans L^2 avec une estimation du coût de contrôle en fonction de T (voir [LB18b, Section 3]). Insistons sur le fait que le linéarisé est ici à coefficients constants ici : c'est pourquoi les méthodes pour démontrer ce type de résultat sont multiples : méthode de Lebeau-Robbiano, inégalités de Carleman paraboliques, méthode des moments...
- La méthode du terme source dans L^2 nous permet de déduire de ce résultat une inégalité d'observabilité forte (voir [LB18b, Corollary 4.4]). Cette estimation ressemble à une inégalité de Carleman (avec poids exponentiellement décroissants en $t=0$ et $t=T$).
- En utilisant la « PHUM », on construit des contrôles L^∞ (voir [LB18b, Theorem 5.1]).

— On réapplique ensuite la méthode du terme source dans L^∞ pour déduire un résultat de contrôle dans L^∞ malgré un terme source (voir [LB19, Proposition 5.3]).

— On conclut par un argument d'inversion locale (voir [LB18b, Section 6]).

Remarque 7.5.4. Il est à noter que le Théorème 7.5.3 permet de retrouver les théorèmes de contrôlabilité locale pour le système de quatre espèces dans le cas où le linéarisé est contrôlable. Néanmoins, la stratégie de preuve développée dans [LB19] traite des systèmes paraboliques à coefficients dans $L^\infty(Q_T)$ avec des couplages d'ordre zéro et deux (quand le nombre d'équations n est petit). Ainsi, quand le linéarisé n'est pas contrôlable, cette dernière approche couplée à la méthode du retour est très féconde.

7.6 Perspectives et problèmes ouverts

7.6.1 Linéarisé non contrôlable pour un système de grande taille

Pour le système général (7.15), dans le cas où l'Hypothèse 7.5.2 n'est pas satisfaite, le linéarisé autour de U^* n'est pas contrôlable. Une stratégie qui permet de remédier à cela est de linéariser le long d'une trajectoire non triviale : c'est la méthode du retour que nous avons employée dans la Section 7.4.1. Mais dans ce cas, le linéarisé fait intervenir des termes de couplages qui dépendent de la variable temporelle et de la variable spatiale. Il est alors plus difficile de démontrer la contrôlabilité du linéarisé, ou de manière équivalente, de démontrer une inégalité d'observabilité pour le système adjoint. La seule méthode connue à ce jour pour gérer de tels couplages est l'emploi d'inégalités de Carleman paraboliques (voir Théorème 3.4.3). Malheureusement, cette stratégie présente une obstruction technique quand le nombre d'équations est trop élevé par rapport au nombre de contrôles. C'est l'objet de l'exemple suivant.

Exemple 7.6.1. Pour $n = 10$, soit $\alpha_1 = \alpha_3 = \alpha_5 = \alpha_7 = \alpha_9 = \beta_2 = \beta_4 = \beta_6 = \beta_8 = \beta_{10} = 1$, $\alpha_2 = \alpha_4 = \alpha_6 = \alpha_8 = \alpha_{10} = \beta_1 = \beta_3 = \beta_5 = \beta_7 = \beta_9 = 0$ et $J = \{1, 2, 3, 4, 5\}$. Le système de contrôle est alors :

$$\begin{cases} \forall 1 \leq i \leq 10, & \\ \partial_t u_i - d_i \Delta u_i = & \\ (-1)^i (u_1 u_3 u_5 u_7 u_9 - u_2 u_4 u_6 u_8 u_{10)} + h_i 1_\omega 1_J(i) & \text{dans } Q_T, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{sur } \Sigma_T, \\ u_i(0,.) = u_{i,0} & \text{dans } \Omega \end{cases} \tag{7.28}$$

L'état stationnaire

$$(0, 0, 0, 0, u_5^*, u_6^*, u_7^*, u_8^*, u_9^*, u_{10}^*),$$

où $(u_5^*, u_7^*, u_9^*) \in (0, +\infty)^3$ et $(u_6^*, u_8^*, u_{10}^*) \in [0, +\infty)^3$ ne satisfait pas l'Hypothèse 7.5.2. Dans ce cas, le système linéarisé autour de

$$\Big((0, 0, 0, 0, u_5^*, u_6^*, u_7^*, u_8^*, u_9^*, u_{10}^*), (0, 0, 0, 0, 0)\Big),$$

n'est pas contrôlable puisque la sixième équation est découplée des autres :

$$\partial_t u_6 - d_6 \Delta u_6 = 0.$$

Mais comme dans la Section 7.4.1, on peut aisément construire une trajectoire de référence :

$$\Big((0,0,g,0,u_5^*,u_6^*,u_7^*,u_8^*,u_9^*,u_{10}^*),(0,0,\partial_t g - d_3\Delta g,0,0)\Big),$$

où g est non nulle, régulière et à support dans $(0,T)\times\omega$. En utilisant le même changement de variables que dans la [LB18b, Section 2]) et en linéarisant autour de cette trajectoire, on trouve le même système que [LB18b, Section 2.2, System (L-Z)] avec $n = 10$, $m = 5$ où les coefficients de A peuvent dépendre de (t,x) et $a_{61}(t,x) \geq \varepsilon > 0$ sur $(t_1,t_2)\times\omega_0 \subset (0,T)\times\omega$. Le système linéarisé semble être contrôlable à zéro grâce au diagramme heuristique suivant :

$$h_1 \xrightarrow{controls} z_1,\ h_2 \xrightarrow{controls} z_2,\ h_3 \xrightarrow{controls} z_3,\ h_4 \xrightarrow{controls} z_4,\ h_5 \xrightarrow{controls} z_5,$$
$$z_1 \xrightarrow[a_{61}(t,x)z_1]{controls} z_6 \xrightarrow[\Delta z_6]{controls} z_7 \xrightarrow[\Delta z_7]{controls} z_8 \xrightarrow[\Delta z_8]{controls} z_9 \xrightarrow[\Delta z_9]{controls} z_{10}.$$

Malheureusement, nous ne savons pas comment montrer la contrôlabilité à zéro du linéarisé pour des raisons techniques probablement. Cela vient du fait que $m = 5 < n-4 = 6$. Avec la stratégie de preuve présente dans [LB19], on devrait tirer profit d'un couplage d'ordre zéro et de quatre couplage d'ordre deux. Cela crée un problème pour absorber certains termes globaux après l'emploi d'inégalités de Carleman avec des exposants différents (voir à ce titre la Remarque 4.4.2). Cela nous amène à formuler le problème suivant.

Problème ouvert 7.6.2. Soit n, m deux entiers tels que $n \geq 6$, $m < n-4$ and $(d_i)_{1\leq i\leq n} \in (0,+\infty)^n$. Soit $A \in C_b^\infty(\overline{Q_T})^{(m+1)\times(m+1)}$. On suppose qu'il existe $(t_1,t_2) \subset (0,T)$, un ouvert non vide ω_0 tel que $\omega_0 \subset\subset \omega$ et $\varepsilon > 0$ tels que $A_{m+1,m}(t,x) \geq \varepsilon$ pour $(t,x) \in (t_1,t_2)\times\omega_0$. Soit $y_0 \in L^2(\Omega)^n$, $H^J \in L^2(q_T)^m$, on considère le problème de contrôle

$$\begin{cases} \partial_t y_i - d_i \Delta y_i = \sum\limits_{j=1}^{m+1} A_{i,j}(t,x) y_j + h_i 1_J(i) 1_\omega & \text{dans } Q_T,\ 1 \leq i \leq m+1 \\ \partial_t y_i - d_i \Delta y_i = \Delta y_{i-1} & \text{dans } Q_T,\ m+2 \leq i \leq n \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega. \end{cases} \tag{7.29}$$

Le système (7.29) est-il contrôlable à zéro dans $L^2(\Omega)^n$?

Remarque 7.6.3. Le Problème ouvert 7.6.2 est intimement relié à la généralisation du [FCGBdT15, Theorem 1.1] aux systèmes paraboliques avec matrices de diffusion qui contiennent des blocs de Jordan de taille plus grande que 5.

Pour se rapprocher de la modélisation faite en Section 7.1, une autre perspective serait de contrôler tout en *préservant la positivité* de la solution U. Il est alors probable qu'un *temps minimal de contrôle* puisse apparaître. Pour cela, on pourrait s'inspirer des travaux récents [LTZ17], [PZ17], [PTZ19], [HMT18], [HT17] et [MTZ18].

7.6.2 Contrôlabilité locale du système de Keller-Segel à des états stationnaires non constants

Revenons au système de chimiotaxie de Keller-Segel (voir [KS71]) déjà rencontré au Chapitre 6 :

$$\begin{cases} \partial_t u_1 - \Delta u_1 = -\nabla.(u_1 \nabla u_2) & \text{dans } Q_T, \\ \partial_t u_2 - \Delta u_2 = a u_1 - b u_2 + h 1_\omega & \text{dans } Q_T, \\ \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0 & \text{sur } \Sigma_T, \\ (u_1, u_2)(0,.) = (u_{1,0}, u_{2,0}) & \text{dans } \Omega, \end{cases} \tag{7.30}$$

où $a, b \in \mathbb{R}$. Il est démontré dans [CSG15] un résultat de contrôlabilité locale (en temps petit) pour (7.30) aux états stationnaires constants strictement positifs $(M_1, M_2) \in (0, +\infty)^2$, i.e. vérifiant $aM_1 - bM_2 = 0$.

On peut également se demander si le résultat demeure vrai pour des états stationnaires positifs (non constants en espace). Plus précisément, soit (u_1^*, u_2^*) régulier, positif, solution de

$$\begin{cases} -\Delta u_1^* = -\nabla.(u_1^* \nabla u_2^*) & \text{dans } \Omega, \\ -\Delta u_2 = a u_1^* - b u_2^* & \text{dans } \Omega, \\ \frac{\partial u_1^*}{\partial n} = \frac{\partial u_2^*}{\partial n} = 0 & \text{sur } \partial\Omega. \end{cases} \tag{7.31}$$

Concernant l'existence de tels états stationnaires, on peut consulter [Hor03] et les références associées. Le linéarisé de (7.30) autour de $((u_1^*, u_2^*), h = 0)$ est

$$\begin{cases} \partial_t u_1 - \Delta u_1 = -\nabla.(u_1^* \nabla u_2) - \nabla.(u_1 \nabla u_2^*) & \text{dans } Q_T, \\ \partial_t u_2 - \Delta u_2 = a u_1 - b u_2 + h 1_\omega & \text{dans } Q_T, \\ \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0 & \text{sur } \Sigma_T, \\ (u_1, u_2)(0,.) = (u_{1,0}, u_{2,0}) & \text{dans } \Omega. \end{cases} \tag{7.32}$$

Comme $u_1^* \geq \varepsilon > 0$ dans Ω, la composante u_1 semble contrôlée de manière indirecte par la composante u_2 à travers le terme de couplage d'ordre deux $-\nabla.(u_1^* \nabla u_2)$. Mais, on ne sait pas à l'heure actuelle démontrer la contrôlabilité à zéro du système (7.32). En effet, la difficulté réside dans l'obtention d'une inégalité d'observabilité adhoc pour le système adjoint :

$$\begin{cases} -\partial_t \varphi_1 - \Delta \varphi_1 = \nabla u_2^*.\nabla \varphi_1 + a \varphi_2 & \text{dans } Q_T, \\ -\partial_t \varphi_2 - \Delta \varphi_2 = -\nabla.(u_1^* \nabla \varphi_1) - b \varphi_2 & \text{dans } Q_T, \\ \frac{\partial \varphi_1}{\partial n} = \frac{\partial \varphi_2}{\partial n} = 0 & \text{sur } \Sigma_T, \\ (\varphi_1, \varphi_2)(T,.) = (\varphi_{1,T}, \varphi_{2,T}) & \text{dans } \Omega. \end{cases} \tag{7.33}$$

La stratégie usuelle suggère d'appliquer une inégalité de Carleman à l'équation parabolique satisfaite par $\nabla.(u_1^* \nabla \varphi_1)$ pour ensuite bénéficier du couplage $-\nabla.(u_1^* \nabla \varphi_1)$ de la seconde équation de (7.33) pour enlever le terme local

$$\int_{q_T} e^{-2s\alpha} (s\xi)^3 |\nabla.(u_1^* \nabla \varphi_1)|^2 dt dx.$$

Cependant, ce type d'inégalité fait apparaître dans le terme de droite des termes globaux du type $\int_{Q_T} |\nabla \varphi_1|^2$ qu'on ne peut pas absorber a priori.

Chapitre 8

Contrôlabilité à zéro d'un système 2×2 de réaction-diffusion avec couplage non linéaire : une nouvelle méthode de dualité

Le but de cette partie est de présenter les résultats obtenus dans l'article [LB18c]. Ce papier répond à des questions de contrôlabilité globale et locale à zéro en temps petit pour des systèmes de réaction-diffusion non linéaires du type

$$\begin{cases} \partial_t u - \Delta u = f_1(u,v) + h 1_\omega & \text{dans } Q_T, \\ \partial_t v - \Delta v = f_2(u,v) & \text{dans } Q_T, \\ u, v = 0 & \text{sur } \Sigma_T, \\ (u,v)(0,.) = (u_0, v_0) & \text{dans } \Omega, \end{cases} \tag{8.1}$$

avec $f_1, f_2 \in C^\infty(\mathbb{R}^2;\mathbb{R})$ telles que $f_1(0,0) = f_2(0,0) = 0$ dont le *linéarisé* autour de $((u,v) = (0,0), h = 0)$ *n'est pas contrôlable* (voir Section 6.2). Dans (8.1), T est un temps strictement positif (arbitrairement petit), Ω est un ouvert suffisamment régulier (typiquement de classe C^∞) et ω un ouvert non vide contenu dans Ω.

Parmi les systèmes (8.1), un cas typique de système de réaction-diffusion au linéarisé non contrôlable (voir Section 6.2) est le suivant

$$\begin{cases} \partial_t u - \Delta u = h 1_\omega & \text{dans } Q_T, \\ \partial_t v - \Delta v = u^n & \text{dans } Q_T, \\ u, v = 0 & \text{sur } \Sigma_T, \\ (u,v)(0,.) = (u_0, v_0) & \text{dans } \Omega, \end{cases} \tag{Power}$$

où $n \geq 2$ est un entier.

Pour n impair, nous démontrons en Section 8.2 la contrôlabilité globale à zéro du système (Power) (voir Théorème 8.2.2) par une nouvelle méthode non linéaire directe. Pour mieux appréhender la stratégie de preuve, nous choisissons de la présenter dans le cadre de la dimension finie en Section 8.1. Le point clef de la démonstration du Théorème 8.2.2 est la construction de contrôles pour l'équation de la chaleur (amenant à zéro) dont la racine $n^{\text{ième}}$ est régulière (voir Proposition 8.2.4). Ce dernier point est prouvé par une nouvelle méthode de dualité qui est une généralisation de la méthode de dualité Hilbertienne dans L^2 au cas réflexif dans L^p, $1 < p < \infty$. Nous nous servons de ce type de contrôles pour démontrer un résultat de contrôlabilité locale à zéro pour des systèmes généraux du type (8.1) grâce à la méthode du retour en Section 8.3.

8.1 Retour au modèle jouet de la dimension finie

L'analogue en dimension finie du système (Power) est

$$\begin{cases} y_1' = h, \\ y_2' = y_1^n, \end{cases} \tag{Jouet}$$

que nous avons déjà rencontré dans l'Exemple 2.1.8. On a montré que pour n pair, le système n'est pas localement contrôlable à zéro et que pour n impair, il est localement contrôlable à zéro grâce à la méthode du retour. On peut en fait démontrer le résultat suivant.

Proposition 8.1.1. *Pour n impair, le système* (Jouet) *est globalement contrôlable à zéro.*

Preuve par la méthode du retour et un argument d'homogénéité. On sait que le système (Jouet) est localement contrôlable à zéro (méthode du retour). Remarquons également que ce système est homogène au sens suivant : si $((y_1, y_2), h)$ est une trajectoire de (Jouet) alors pour tout $\lambda > 0$, $((\lambda y_1, \lambda^n y_2), \lambda h)$ est également une trajectoire de (Jouet). On en déduit donc que le système (Jouet) est globalement contrôlable à zéro. □

Nous allons à présent proposer une autre preuve (directe) de la Proposition 8.1.1. Nous utiliserons ensuite la même stratégie pour déduire un résultat analogue pour le système (Power) (voir Théorème 8.2.2 plus bas).

Preuve directe dans le cas $n = 3$. On se donne $(y_{1,0}, y_{2,0}) \in \mathbb{R}^2$. On cherche à construire un contrôle h qui amène (y_1, y_2) à $(0, 0)$ au temps T. La preuve s'effectue en deux étapes.

Etape 1 : Amener y_1 à 0 au temps $T/2$. On pose

$$\forall t \in [0, T/2),\ y_1(t) := C \exp\left(-\frac{1}{T/2 - t}\right),\ y_1(T/2) = 0,$$

où C est tel que $y_1(0) = y_{1,0}$. On définit ensuite

$$\forall t \in [0, T/2],\ h_{T/2}(t) := y_1'(t).$$

Posons y_2 la solution du problème de Cauchy

$$y_2' = y_1^3,\ y_2(0) = y_{2,0}.$$

Bien sûr, on n'a pas $y_2^{T/2} := y_2(T/2) = 0$ a priori. Sinon, c'est gagné.

Etape 2 : Amener y_2 à 0 au temps T au moyen d'un contrôle cubique. Cette étape est basée sur le lemme clef suivant.

Lemme 8.1.2. *Soit $\tau > 0$. Pour tout $y_0 \in \mathbb{R}$, il existe $h \in C^1([0,\tau];\mathbb{R})$ tel que*

$$h^{1/3} \in C^1([0,\tau];\mathbb{R}) \text{ vérifiant } h(0) = (h^{1/3})'(0) = h(\tau) = 0, \tag{8.2}$$

tel que la solution de

$$y' = h, \ y(0) = y_0,$$

vérifie $y(\tau) = 0$.

La preuve de ce lemme est élémentaire. En effet, il suffit de poser

$$\forall t \in [0,\tau), \ y(t) := C \exp\left(-\frac{1}{\tau^7 - t^7}\right), \ y(\tau) = 0,$$

où C est tel que $y(0) = y_0$. On a alors $y(0) = y_0$, $y(\tau) = 0$ et en posant

$$\forall t \in [0,\tau], \ h(t) := y'(t),$$

on a bien (8.2) car

$$\forall t \in [0,\tau), \ h(t) = -7\frac{t^6}{(\tau^7 - t^7)^2} \exp\left(-\frac{1}{\tau^7 - t^7}\right), \ h(\tau) = 0.$$

Ce qui conclut la preuve du lemme.

Revenons à la fin de la preuve de la Proposition 8.1.1. Par le Lemme 8.1.2 appliqué au temps $\tau = T/2$, on peut trouver un contrôle H vérifiant (8.2) tel que

$$y_2' = H, \ (y_2(T/2), y_2(T)) = (y_2^{T/2}, 0).$$

On pose alors

$$\forall t \in [T/2, T], \ y_1(t) := H(t)^{1/3},$$

ce qui équivaut à

$$\forall t \in [T/2, T], \ y_1(t)^3 = H(t).$$

Remarquons en particulier qu'on a

$$y_1(T/2) = y_1(T) = 0.$$

Puis, on pose

$$\forall t \in [T/2, T], \ h_T(t) := y_1'(t) = (H^{1/3})'(t).$$

On montre ainsi en rassemblant l'étape 1 et l'étape 2 que l'unique solution $y = (y_1, y_2) \in C^1([0,T];\mathbb{R})$ de (Jouet) associée au contrôle

$$h = h_{T/2} \text{ sur } [0, T/2], \ h_T \text{ sur } [T/2, T] \ \in C([0,T];\mathbb{R}),$$

s'annule au temps $t = T$. $\square$

8.2 Nouveau résultat de contrôlabilité globale à zéro en temps petit pour des systèmes de réaction-diffusion à couplage impair

Comme dans le cas de (Jouet), on peut également montrer le résultat (négatif) suivant.

Proposition 8.2.1. *Si n est un entier pair, alors le système* (Power) *n'est pas localement contrôlable à* 0.

En effet, en utilisant $u^n \geq 0$ et le *principe du maximum* (voir Proposition A.3.1), pour toute solution (u, v) de (Power) associée à une donnée initiale (u_0, v_0) telle que $v_0 \geq 0$ et $v_0 \neq 0$, on a

$$v(T,.) \geq \widetilde{v}(T,.) \geq 0 \qquad \text{et} \qquad \widetilde{v}(T,.) \neq 0,$$

où $\widetilde{v}$ est la solution de l'équation de la chaleur

$$\begin{cases} \partial_t \widetilde{v} - \Delta \widetilde{v} = 0 & \text{dans } Q_T, \\ \widetilde{v} = 0 & \text{sur } \Sigma_T, \\ \widetilde{v}(0,.) = v_0 & \text{dans } \Omega. \end{cases}$$

Nous avons déjà vu au Corollaire 6.2.2 que le système (Power) est localement contrôlable à zéro pour $n = 3$. Le même argument d'homogénéité qu'employé pour (Jouet) montre que si (Power) est localement contrôlable à zéro alors (Power) est globalement contrôlable à zéro.

Le but de cette partie est de présenter le nouveau résultat suivant (voir [LB18c, Theorem 2.5]).

Théorème 8.2.2. *Pour n impair, le système* (Power) *est globalement contrôlable à zéro au temps T. Plus précisément, il existe une constante $C > 0$ telle que pour tout $(u_0, v_0) \in L^\infty(\Omega)^2$, il existe $h \in L^\infty(q_T)$ vérifiant*

$$\|h\|_{L^\infty(q_T)} \leq C \left(\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)}^{1/n} \right), \tag{8.3}$$

tel que la solution de (Power) *satisfait* $(u, v)(T,.) = (0, 0)$.

Remarque 8.2.3. En plus de démontrer la contrôlabilité globale à zéro du système (Power), un autre avantage du Théorème 8.2.2 est de fournir une estimation sur la norme du contrôle en fonction de la norme de la donnée initiale. Ce que ne permet pas toujours la méthode du retour.

La preuve du Théorème 8.2.2 suit les idées de la preuve directe de la Section 8.1. Pour simplifier la présentation, nous allons supposer que $n = 3$ et nous allons seulement montrer la propriété de contrôlabilité globale sans l'estimation (8.3) sur le contrôle.

Démonstration. Soit $(u_0, v_0) \in L^\infty(\Omega)^2$. La preuve se fait en deux étapes.

Etape 1 : Amener u_1 à 0 au temps $T/2$. Par contrôlabilité globale de l'équation de la chaleur appliqué au temps $T/2$ (voir Théorème 5.2.1), on sait qu'il existe $h_1 \in L^\infty((0,T/2) \times \omega)$ tel que la solution associée $u_1 \in L^\infty((0,T/2) \times \Omega)$ vérifie

$$\begin{cases} \partial_t u_1 - \Delta u_1 = h_1 1_\omega & \text{dans } (0,T/2) \times \Omega, \\ u_1 = 0 & \text{sur } (0,T/2) \times \partial\Omega, \\ (u_1(0,.), u_1(T/2,.)) = (u_0, 0) & \text{dans } \Omega. \end{cases}$$

On pose alors $v_1 \in L^\infty((0,T/2) \times \Omega)$ comme la solution du problème de Cauchy

$$\begin{cases} \partial_t v_1 - \Delta v_1 = u_1^{2k+1} & \text{dans } (0,T/2) \times \Omega, \\ v_1 = 0 & \text{sur } (0,T/2) \times \partial\Omega, \\ v_1(0,.) = v_0 & \text{dans } \Omega. \end{cases}$$

Bien sûr, on n'a pas $v_1(T/2,.) = 0$ a priori. Sinon, c'est gagné.

Etape 2 : Amener v_2 à 0 au temps T au moyen d'un contrôle cubique. Nous allons utiliser le nouveau résultat clef suivant (voir [LB18c, Proposition 3.7]) qui est un résultat de contrôlabilité à zéro de l'équation de la chaleur avec contrôle cubique régulier.

Proposition 8.2.4. *Pour tous $\tau > 0$, $y_0 \in L^\infty(\Omega)$, il existe $h_\tau \in C^{1,2}(\overline{Q_\tau})$ vérifiant*

$$h_\tau^{1/3} \in C^{1,2}(\overline{Q_\tau}),\ h_\tau(0,.) = h_\tau(\tau,.) = 0,\ \forall t \in [0,\tau],\ supp(h_\tau(t,.)) \subset\subset \omega \tag{8.4}$$

tel que la solution $y \in L^\infty(Q_\tau)$ de l'équation de la chaleur avec donnée initiale y_0 et contrôle h_τ, satisfait $y(\tau,.) = 0$.

Admettons dans un premier temps cette proposition et revenons à notre preuve.

On applique la Proposition 8.2.4 avec $(0,\tau) \leftarrow (T/2,T)$, $y_0 \leftarrow v_1(T/2,.) \in L^\infty(\Omega)$. Il existe alors un contrôle H vérifiant (8.4) tel que la solution v_2 associée vérifie

$$\begin{cases} \partial_t v_2 - \Delta v_2 = H & \text{dans } (T/2,T) \times \Omega, \\ v_2 = 0 & \text{sur } (T/2,T) \times \partial\Omega, \\ (v_2(T/2,.), v_2(T,.)) = (v_1(T/2,.), 0) & \text{dans } \Omega. \end{cases}$$

On pose alors

$$u_2 := H^{\frac{1}{3}} \in C^{1,2}(\overline{((T/2,T) \times \Omega)}),$$

qui vérifie

$$u_2(T/2,.) = u_2(T,.) = 0.$$

Puis, nous posons

$$h_2 := \partial_t u_2 - \Delta u_2 \in L^\infty(((T/2,T) \times \Omega)), \tag{8.5}$$

à support dans $(T/2,T)\times\omega$ par (8.4).

On montre ainsi en rassemblant l'étape 1 et l'étape 2 que l'unique solution $(u,v)\in L^\infty(Q_T)$ de (Power) associée au contrôle

$$h = h_1 \text{ sur } [0,T/2],\ h_2 \text{ sur } [T/2,T]\ \in L^\infty(q_T),$$

est identiquement nulle au temps $t=T$. □

La preuve de l'existence de contrôles cubiques réguliers à l'équation de la chaleur (voir Proposition 8.2.4) est basée sur une nouvelle méthode de dualité. Nous avons choisi d'appeler cette méthode la « Reflexive Uniqueness Method » puisque c'est une adaptation de la « Hilbert Uniqueness Method » au cadre réflexif.

Pour comprendre la preuve de la Proposition 8.2.4, il faut avoir à l'esprit la construction de contrôles réguliers pour l'équation de la chaleur par la « PHUM » (voir Section 5.2). Au lieu de minimiser une fonctionnelle dans L^2 à poids, nous allons minimiser une fonctionnelle dans $L^{4/3}$ à poids. En effet, l'idée sous-jacente est la suivante : la dérivée de la fonction strictement convexe $x\mapsto |x|^{4/3}$ est $x\mapsto x^{1/3}$.

Idée de la démonstration de la Proposition 8.2.4. Se donnant un poids ρ (à choisir plus tard), $y_0\in L^{4/3}(\Omega)$ et $\varepsilon>0$, introduisons la fonctionnelle :

$$\forall h\in L^{4/3}(q_T), J_\varepsilon(h) = \frac{3}{4}\int_{(0,T)\times\omega}\rho(t,x)^{-4/3}|h|^{4/3}dtdx + \frac{3}{4\varepsilon}\left\|y(T,.)\right\|_{L^{4/3}(\Omega)}^{4/3},$$

où y est la solution de (Heat) avec donnée initiale y_0 et contrôle h.

On voit aisément que J_ε est une fonctionnelle de classe C^1, strictement convexe et coercive sur l'espace réflexif $L^{4/3}(q_T)$ donc elle possède un unique minimum h_ε. Appelons y_ε la solution de (Heat) associée à ce contrôle h_ε. Par l'équation d'Euler-Lagrange, on a que pour tout $h\in L^{4/3}(q_T)$,

$$\int_{(0,T)\times\omega}\rho(t,x)^{-4/3}h_\varepsilon^{1/3}hdtdx + \frac{1}{\varepsilon}\int_\Omega y_\varepsilon(T,.)^{1/3}.y(T,.) = 0, \tag{8.6}$$

où y est la solution de (Heat) avec donnée initiale nulle et contrôle h.

Par un argument de dualité entre y et φ_ε définie comme la solution de (HeatAdj) avec donnée initiale au temps $t=T$, $\varphi_\varepsilon(T,.) = -y_\varepsilon(T,.)^{1/3}/\varepsilon$, on a

$$\int_\Omega y(T,.).\varphi_\varepsilon(T,.) = \int_{(0,T)\times\omega} h\varphi_\varepsilon,$$
$$-\frac{1}{\varepsilon}\int_\Omega y_\varepsilon(T,.)^{1/3}.y(T,.) = \int_{(0,T)\times\omega} h\varphi_\varepsilon,$$

ce qui donne grâce à (8.6) :

$$\int_{(0,T)\times\omega}\varphi_\varepsilon h = \int_{(0,T)\times\omega}\rho(t,x)^{-4/3}h_\varepsilon^{1/3}h,\ \forall h\in L^{4/3}(q_T).$$

D'où, on trouve

$$h_\varepsilon^{1/3} = \rho(t,x)^{4/3}\varphi_\varepsilon 1_\omega. \tag{8.7}$$

En utilisant (8.7) et un nouvel argument de dualité entre y_ε et φ_ε, on a

$$\int_\Omega y(T,x).\varphi_\varepsilon(T,x)dx = \int_\Omega y_0(x)\varphi_\varepsilon(0,x)dx + \int_{(0,T)\times\omega} h_\varepsilon\varphi_\varepsilon$$

$$-\frac{1}{\varepsilon}\|y_\varepsilon(T,.)\|_{L^{4/3}(\Omega)}^{4/3} = \int_\Omega y_0(x)\varphi_\varepsilon(0,x)dx + \int_{(0,T)\times\omega} \rho(t,x)^4|\varphi_\varepsilon|^4. \tag{8.8}$$

Par une inégalité de type Carleman L^4 (celle-ci restant à établir) appliquée à φ_ε, on trouve

$$\|\varphi_\varepsilon(0,.)\|_{L^4(\Omega)}^4 \leq C\int_{(0,T)\times\omega} \rho^4|\varphi_\varepsilon|^4, \tag{8.9}$$

D'où, en utilisant (8.8), (8.9) et l'inégalité de Young, on a

$$\frac{1}{\varepsilon}\|y_\varepsilon(T,.)\|_{L^{4/3}(\Omega)}^{4/3} + \left\|\rho^{-1}h_\varepsilon\right\|_{L^{4/3}((0,T)\times\omega)}^{4/3} \leq C\|y_0\|_{L^{4/3}(\Omega)}^{4/3}. \tag{8.10}$$

Grâce à (8.7) et à l'équation satisfaite par φ_ε, on montre, par des arguments de type : régularité maximale dans les espaces L^p (voir Proposition A.6.2), régularité maximale dans les espaces de Hölder (voir Proposition A.6.4) et injections de Sobolev (voir Lemme A.6.3), que pour $\alpha > 0$,

$$\left\|h_\varepsilon^{1/3}\right\|_{C^{1+\alpha,2+2\alpha}(\overline{Q_T})} \leq C\|y_0\|_{L^{4/3}(\Omega)}^{1/3}. \tag{8.11}$$

Il reste enfin à passer à la limite dans (8.10) et (8.11) en invoquant le théorème d'Ascoli pour conclure la preuve de la Proposition 8.2.4. □

Ainsi, la preuve de la Proposition 8.2.4 est valide si l'on arrive à démontrer une inégalité de Carleman L^4 qui mène en particulier à (8.9). En fait, à partir de l'inégalité de Carleman L^2 préalablement établie, c'est-à-dire du Théorème 3.4.3, et un argument de bootstrap, on peut déduire une inégalité de Carleman L^4 (voir [LB18c, Theorem 4.4]).

Remarque 8.2.5. La fonction 1_ω dans (8.7) semble poser des problèmes de régularité a priori, c'est pourquoi il faut en réalité adapter le raisonnement précédent en cherchant des contrôles sous la forme $h\chi$ où χ est une fonction régulière de troncature en espace à support localisé dans ω.

8.3 Nouveau résultat de contrôlabilité locale à zéro en temps petit pour des systèmes de réaction-diffusion à couplage impair

Le but de cette partie est de généraliser le Théorème 8.2.2 à des systèmes de réaction-diffusion plus généraux (voir [LB18c, Theorem 2.8]).

Théorème 8.3.1. *Soit* $k \in \mathbb{N}$. *Soit* $(g_1, g_2) \in C^\infty(\mathbb{R}; \mathbb{R})^2$ *tels que*

$$\begin{aligned} &g_1(0) = g_1'(0) = \cdots = g_1^{(2k)}(0) = 0 \text{ et } g_1^{(2k+1)}(0) \neq 0, \\ &g_2(0) \neq 0. \end{aligned}$$

Soit $f_1 \in C^\infty(\mathbb{R}^2; \mathbb{R})$, $f_2 \in C^\infty(\mathbb{R}^2; \mathbb{R})$ *tels que*

$$\forall (u, v) \in \mathbb{R}^2, \ \ f_1(0, v) = 0, \ \ f_2(u, v) := g_1(u) g_2(v).$$

Alors, le système (8.1) *est localement contrôlable à zéro.*

La preuve du Théorème 8.3.1 s'appuie sur la méthode du retour énoncée en Théorème 6.2.1. Il reste donc à démontrer l'existence d'une trajectoire non triviale du système (8.1) qui part de $((0, 0), 0)$ et qui retourne en $((0, 0), 0)$. La construction de cette trajectoire s'appuie notamment sur la Proposition 8.2.4. On suppose que $k = 1$ pour simplifier la présentation.

Grandes lignes de la construction de la trajectoire. Premièrement, on remarque que f_2 se comporte comme une fonction cube en la variable u près de 0, au sens où

$$f_2(u, v) = \widetilde{g_1}^3(u) g_2(v), \tag{8.12}$$

avec $\widetilde{g_1}$, un C^∞ difféomorphisme local d'un voisinage de 0 dans un voisinage de 0 tel que $\widetilde{g_1}(0) = 0$.

On construit la trajectoire en deux étapes.

Première partie de la trajectoire. On se donne $\overline{u_1}$ non identiquement nulle régulière à support dans $(0, T/2) \times \omega$. On définit $\overline{v_1}$ à l'aide de $\overline{u_1}$ en résolvant le problème de Cauchy

$$\partial_t \overline{v_1} - \Delta \overline{v_1} = f_2(\overline{u_1}, \overline{v_1}), \qquad \overline{v_1}(0, .) = 0.$$

On définit alors

$$\overline{h_1} := \partial_t \overline{u_1} - \Delta \overline{u_1} - f_1(\overline{u_1}, \overline{v_1}),$$

qui est bien supporté dans $(0, T/2) \times \omega$.

Seconde partie de la tractoire : retour à $(0, 0)$**.** On sait, par la Proposition 8.2.4, qu'on peut trouver un contrôle H à support dans $(T/2, T) \times \omega$ de racine cubique régulière telle que la solution $\overline{v_2}$ associée satisfait

$$\partial_t \overline{v_2} - \Delta \overline{v_2} = H, \qquad (\overline{v_2}(T/2, .), \overline{v_2}(T, .)) = (\overline{v_1}(T/2,), 0).$$

En utilisant (8.12), on peut alors définir

$$\overline{u_2} := \widetilde{g_1}^{-1} \left(\left(\frac{H}{g_2(\overline{v_2})} \right)^{1/3} \right).$$

On peut vérifier que $\overline{u_2}$ est régulière et à support dans $(T/2, T) \times \omega$. Finalement, on pose

$$\overline{h_2} := \partial_t \overline{u_2} - \Delta \overline{u_2} - f_1(\overline{u_2}, \overline{v_2}),$$

qui est bien supporté dans $(T/2, T) \times \omega$.

Le recollement des deux trajectoires fournit alors une trajectoire partant de $((0,0),0)$ et allant à $((0,0),0)$ telle que

$$\frac{\partial f_2}{\partial u}(\overline{u_1}, \overline{v_1}) \geq \varepsilon > 0 \text{ sur } (T/8, 3T/8) \times \omega_0,$$

pour $\varepsilon > 0$ suffisamment petit et ω_0 contenu dans ω. □

8.4 Perspectives et problèmes ouverts

Une première perspective de recherche pourrait être le calcul numérique du contrôle que l'on construit dans la preuve du Théorème 8.2.2. En effet, on rappelle que la construction de ce contrôle passe par deux étapes : la construction d'un contrôle amenant la première composante du système (Power) à zéro, puis la construction d'un contrôle cubique (si $n = 3$) amenant la seconde composante à zéro. Ces deux étapes de construction peuvent s'obtenir par une méthode de dualité pénalisée : l'une dans un cadre L^2 (voir la seconde preuve du Théorème 5.2.1), l'autre dans un cadre $L^{4/3}$ (voir la preuve de la Proposition 8.2.4). Ainsi, les idées présentes dans les articles [EV09], [Boy13], [BHLR11], [BR14] pourraient s'adapter pour l'approximation numérique des contrôles amenant à zéro le système (Power).

Un problème intéressant, qui généraliserait le résultat du Théorème 8.2.2, serait le suivant.

Problème ouvert 8.4.1. Soit n un entier plus grand que 2 et $k_1, \dots, k_{n-1} \in \mathbb{N}$. On considère le problème de contrôle de type cascade non linéaire suivant

$$\begin{cases} \partial_t u_1 - \Delta u_1 = h 1_\omega & \text{dans } Q_T, \\ \partial_t u_2 - \Delta u_2 = u_1^{2k_1+1} & \text{dans } Q_T, \\ \vdots & \\ \partial_t u_n - \Delta u_n = u_{n-1}^{2k_{n-1}+1} & \text{dans } Q_T, \\ U = 0 & \text{sur } \Sigma_T, \\ U(0,.) = U_0 & \text{dans } \Omega. \end{cases} \tag{8.13}$$

Le système (8.13) est-il globalement contrôlable à zéro dans $L^\infty(\Omega)^n$?

Rappelons que le cas $n = 2$ est entièrement traité par le Théorème 8.2.2. Pour le cas $n = 3$, et $k_1 = k_2 = 1$, on peut montrer que le système (8.13) est globalement contrôlable à zéro par la méthode du retour et un argument d'homogénéité (voir [CG17]).

Chapitre 9

Contrôlabilité globale à zéro et à états positifs d'équations de la chaleur faiblement non linéaires

Le but de cette partie est de présenter les nouveaux résultats de contrôlabilité obtenus dans l'article [LB18a] pour l'équation de la chaleur semilinéaire

$$\begin{cases} \partial_t y - \Delta y + f(y) = h 1_\omega & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega, \end{cases} \tag{9.1}$$

où $f \in C^1(\mathbb{R};\mathbb{R})$ est telle que $f(0) = 0$ et f satisfait une condition de croissance à l'infini du type

$$\exists \alpha > 0, \ \frac{f(s)}{|s| \log^\alpha(1+|s|)} \to 0 \text{ quand } |s| \to +\infty. \tag{9.2}$$

Nous commençons par rappeler en Section 9.1, Section 9.2, Section 9.3, Section 9.4, Section 9.5 les principaux résultats de contrôlabilité déjà connus pour l'équation (9.1).

En Section 9.6 (voir Théorème 9.6.2), nous présentons un premier nouveau résultat issu de [LB18a]. C'est un résultat de contrôlabilité globale en temps petit à états positifs pour l'équation (9.1) sous certaines hypothèses sur la fonction f. La preuve de ce dernier résultat est reportée en Section 9.8. Elle est basée sur une estimation fine du coût de contrôle à états positifs pour l'équation de la chaleur avec potentiel énoncée en Théorème 9.8.1. Cette estimation repose sur une estimation fine de la constante d'observabilité de l'équation de la chaleur avec potentiel pour des données initiales à valeurs positives établie en Théorème 9.8.3, grâce à une nouvelle inégalité L^1 (voir Théorème 9.8.5). En comparant avec les résultats déjà connus de la littérature, on montre l'optimalité de ce résultat selon que le paramètre α de (9.2) est plus petit ou plus grand que 2.

Le Théorème 9.6.2 nous permet d'établir en Section 9.7 un nouveau résultat de contrôlabilité globale en temps long (uniforme) à zéro pour l'équation (9.1) sous certaines hypothèses sur la fonction f (voir Théorème 9.7.1). La preuve de ce dernier résultat passe par trois étapes de contrôles : amener à un état positif pour éviter l'explosion vers le bas grâce au Théorème 9.6.2, laisser dissiper en profitant de la dissipation de la fonction f sur $\mathbb{R}^+$, contrôler localement à zéro grâce au Théorème 5.3.1. En comparant à nouveau avec les résultats déjà connus de la littérature, on montre l'optimalité de ce résultat selon que le paramètre α de (9.2) est plus petit ou plus grand que 2.

9.1 Explosion sans contrôle

Dans cette Section 9.1, on considère (9.1) avec $h = 0$, et on remplace f par $-f$ pour simplifier la présentation, c'est-à-dire qu'on a

$$\begin{cases} \partial_t y - \Delta y = f(y) & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega. \end{cases} \tag{9.3}$$

Le résultat suivant montre que sans contrôle il y a explosion en temps fini pour (9.1).

Proposition 9.1.1. *On suppose que f est une fonction convexe et $f(s) > 0$ pour $s > 0$. On suppose également que la condition d'Osgood :*

$$\int_1^{+\infty} \frac{ds}{f(s)} < +\infty, \tag{9.4}$$

est vérifiée. Alors, pour tout $T > 0$, il existe une donnée initiale $y_0 \in L^\infty(\Omega)$ telle que la solution maximale de (9.3) avec $h = 0$ explose en $T^* < T$.

Démonstration. Soit φ_1 la première fonction propre, associée à la première valeur propre $\lambda_1 > 0$ de $(-\Delta, H^2 \cap H_0^1(\Omega))$, que l'on suppose de masse normalisée à 1. On pose $u(t) = \int_\Omega y(t,x)\varphi_1(x)dx$. En multipliant (9.3) par φ_1 et en intégrant en espace, on déduit par intégrations par parties et par l'inégalité de Jensen

$$u' = \int_\Omega y_t \varphi_1 dx = \int_\Omega y \Delta\varphi_1 dx + \int_\Omega f(y)\varphi_1 dx \geq -\lambda_1 u + f(u).$$

Comme f est convexe et qu'on a (9.4), on montre qu'il existe $C > 0$ tel que $f(s) \geq 2\lambda_1 s$ pour $s \geq C$. Ainsi, si la donnée initiale y_0 est choisie telle que $u(0) \geq C$, on a alors

$$u' \geq f(u) - \lambda_1 u \geq \frac{1}{2} f(u),$$

d'où

$$t/2 \leq \int_0^t \frac{u'(\tau)}{f(u(\tau))} d\tau = \int_{u(0)}^{u(t)} \frac{ds}{f(s)} \leq \int_{u(0)}^{+\infty} \frac{ds}{f(s)} < +\infty.$$

On en déduit alors une borne sur le temps maximal T^* d'existence de la solution, cette borne tendant vers 0 quand $u(0) \to +\infty$, d'où la conclusion de la preuve. □

Remarque 9.1.2. La condition de convexité de la Proposition 9.1.1 peut être remplacée par l'hypothèse $f(s) \geq \widetilde{f}(s)$ pour s grand et $\widetilde{f}$ satisfaisant les hypothèses de la proposition. Ainsi, pour des non linéarités presque linéaires, typiquement pour $f(s) = s \log^p(1 + |s|)$ avec $p > 1$, la Proposition 9.1.1 montre qu'il y a explosion.

9.2 Contrôlabilité globale à zéro en temps petit malgré l'explosion

Nous avons vu dans le Théorème 5.4.2 qu'on peut en fait lutter contre l'explosion et même faire beaucoup mieux : on peut contrôler globalement en temps petit l'équation de la chaleur semilinéaire (9.1) sous l'hypothèse que f vérifie la condition asymptotique (9.2) avec $\alpha \leq 3/2$.

9.3 Explosion quel que soit le contrôle

En fait, le résultat du Théorème 5.4.2 est très sensible au paramètre α de (9.2) puisque nous avons le résultat suivant (voir [FCZ00, Theorem 1.1]).

Théorème 9.3.1. *On pose $f(s) := \int_0^{|s|} \log^p(1+\sigma)d\sigma$ avec $p > 2$ et on suppose que $\Omega \setminus \overline{\omega} \neq \emptyset$. Alors pour tout $T > 0$, il existe une donnée initiale $y_0 \in L^\infty(\Omega)$ telle que, quel que soit le contrôle $h \in L^\infty(q_T)$, la solution maximale y de l'équation de la chaleur semilinéaire (9.1) explose en un temps $T^* < T$.*

Remarque 9.3.2. Une telle fonction f satisfait (9.2) pour tout $\alpha > p$ car $|f(s)| \sim |s| \log^p(1+|s|)$ quand $|s| \to +\infty$. Ainsi, le Théorème 9.3.1 montre que pour certaines non linéarités satisfaisant (9.2) pour $\alpha > 2$, l'équation (9.1) n'est pas globalement contrôlable à zéro pour tout temps $T > 0$.

La preuve du Théorème 9.3.1 s'appuie sur une analyse similaire à celle faite dans la preuve de la Proposition 9.1.1. Le point difficile étant que pour ne plus voir l'influence du contrôle, il va falloir multiplier non plus par une fonction propre mais par une troncature en espace convenablement choisie et localisée hors de la zone de contrôle.

Idée de la preuve du Théorème 9.3.1. On introduit une fonction de troncature en espace ρ régulière telle que

$$\rho = 0 \text{ dans } \omega, \quad \int_\Omega \rho(x)dx = 1.$$

Soit $y_0 \in L^\infty(\Omega)$, $h \in L^\infty(q_T)$ et y la solution maximale associée de (9.1). On multiplie l'équation (9.1) par ρ et on intègre sur Ω. Au regard du support de ρ, on a alors, posant $u(t) = -\int_\Omega y(t,x)\rho(x)dx$,

$$-u' = \int_\Omega \Delta y \rho - \int_\Omega f(y)\rho,$$

et donc par intégrations par parties et par parité de la fonction f,

$$u' = -\int_\Omega y\Delta\rho + \int_\Omega f(|y|)\rho. \tag{9.5}$$

Comme f est convexe, on peut introduire la fonction conjuguée de f, définie de la manière suivante :

$$\forall s \in \mathbb{R}, \;\; f^*(s) = \sup_{a\in\mathbb{R}}(as - f(a)).$$

Supposons pour le moment qu'on ait réussi à choisir la fonction ρ telle que

$$C := \int_\Omega f^* \left(\frac{2|\Delta\rho|}{\rho} \right) \rho dx < +\infty. \tag{9.6}$$

Alors, par l'inégalité de Young,

$$\begin{aligned}
\left| \int_\Omega y \Delta\rho dx \right| &\leq \int_\Omega |y| \left| \frac{\Delta\rho}{\rho} \right| \rho dx \\
&\leq \frac{1}{2} \int_\Omega f(|y|) \rho dx + \frac{1}{2} \int_\Omega f^* \left(\frac{2\Delta\rho}{\rho} \right) \rho dx \\
\left| \int_\Omega y \Delta\rho dx \right| &\leq \frac{1}{2} \int_\Omega f(|y|) \rho dx + \frac{C}{2}. \qquad (9.7)
\end{aligned}$$

Ainsi, en rassemblant (9.5) et (9.7), on en déduit que

$$u' \geq -\frac{C}{2} + \frac{1}{2} \int_\Omega f(|y|) \rho dx,$$

puis par l'inégalité de Jensen et la parité de la fonction f, on aboutit à l'inéquation différentielle

$$u' \geq -\frac{C}{2} + \frac{f(u)}{2}.$$

On conclut alors comme dans la preuve de la Proposition 9.1.1.

Il reste donc à montrer qu'on peut choisir une fonction ρ telle que (9.6) soit vérifiée. Ceci est rendu possible si $p > 2$ en exploitant le caractère asymptotique de f^* au voisinage de $+\infty$ et en recherchant des fonctions $\rho(x) = \exp(-(r - |x|)^{-m})$ où $r > 0$ est tel que $B(0, r) \subset \Omega \setminus \overline{\omega}$ (quitte à translater) et m est à choisir convenablement en fonction de p (voir les détails dans la preuve du [FCZ00, Theorem 1.1]). □

9.4 Cas dissipatif : contrôlabilité globale en temps long

Quand la non linéarité f est dissipative, c'est-à-dire quand $f(s)s \geq 0$ par exemple, alors se donnant $y_0 \in L^\infty(\Omega)$ (et $h = 0$), la solution associée de l'équation de la chaleur semilinéaire (9.1) converge dans $L^\infty(\Omega)$ vers 0. Or, par contrôlabilité locale à zéro de l'équation de la chaleur semilinéaire (voir Théorème 5.3.1), on en déduit que, si la solution s'est rapprochée suffisamment de 0, on peut trouver un contrôle qui l'y amène exactement. Ainsi, le précédent raisonnement montre qu'on peut amener toute donnée initiale à 0 en un temps suffisamment grand (qui dépend de la taille de la condition initiale a priori). On montre en fait dans le prochain résultat que pour certaines non linéarités, le temps de contrôle à zéro peut être rendu uniforme par rapport à la donnée initiale. Ce type d'argument a déjà utilisé par Jean-Michel Coron dans le contexte de l'équation de Burgers (voir [Cor07b, Theorem 8]).

Théorème 9.4.1. *On suppose que $f(s) > 0$ pour $s > 0$ et $f(s) < 0$ pour $s < 0$. En particulier, la non linéarité est dissipative. On suppose de plus que la condition d'Osgood (9.4) est vérifiée au voisinage de $\pm\infty$. Alors il existe $T > 0$ suffisamment grand tel que pour tout $y_0 \in L^\infty(\Omega)$, il existe $h \in L^\infty(q_T)$ tel que la solution y de l'équation de la chaleur semilinéaire (9.1) vérifie $y(T, .) = 0$.*

Démonstration. Soit $y_0 \in L^\infty(\Omega)$. La preuve s'effectue en deux étapes.

Etape 1 : Laisser dissiper. On pose $h_1 = 0$ sur $(0, T^*)$ avec $T^* > 0$ à fixer plus tard. On a alors par principe de comparaison que

$$\forall t \in [0, T^*],\ \text{p.p. } x \in \Omega,\ v_-(t) \leq y(t, x) \leq v_+(t), \tag{9.8}$$

où $v_\pm$ est l'unique solution de l'équation différentielle ordinaire

$$\begin{cases} \dot{v_\pm}(t) = -f(v_\pm(t)) & \text{dans } (0, +\infty), \\ v_\pm(0) = \pm\left(\|y_0\|_{L^\infty(\Omega)} + 1\right) & . \end{cases} \tag{9.9}$$

Étudions le comportement asymptotique de v_+. Un calcul simple montre que

$$\forall t \in [0, +\infty),\ v_+(t) > 0 \text{ et } F(v_+(t)) - F(v_+(0)) = t, \tag{9.10}$$

où F est définie de la manière suivante

$$\forall s > 0,\ F(s) = \int_{+\infty}^{s} \frac{-1}{f(\sigma)} d\sigma = \int_{s}^{+\infty} \frac{1}{f(\sigma)} d\sigma. \tag{9.11}$$

Notez que F est bien définie car $f(\sigma) > 0$ pour $\sigma > 0$ et $1/f \in L^1([1, +\infty))$ par hypothèse. On vérifie que F est une fonction C^1 strictement décroissante. De plus, $1/f \notin L^1((0, 1])$ car $f \in C^1(\mathbb{R}; \mathbb{R})$ et $f(0) = 0$. D'où, nous avons par (9.11)

$$\lim_{s \to 0^+} F(s) = +\infty \text{ et } \lim_{s \to +\infty} F(s) = 0. \tag{9.12}$$

Donc, on en déduit que $F : (0, +\infty) \to (0, +\infty)$ est un C^1-difféomorphisme. Notons $F^{-1} : (0, +\infty) \to (0, +\infty)$ son inverse, qui est elle aussi strictement décroissante. Alors, par (9.10), on a

$$\forall t \in [0, +\infty),\ v_+(t) = F^{-1}(t + F(v_+(0))) \leq F^{-1}(t). \tag{9.13}$$

L'estimation (9.13) est le point clef car elle nous permet de majorer v_+ par une fonction indépendante de la taille de $v_+(0)$. Or, on a

$$F^{-1}(t) \to 0 \text{ quand } t \to +\infty, \tag{9.14}$$

en utilisant (9.12).

Soit $\delta > 0$ tel que la contrôlabilité à zéro de (9.1) ait lieu dans $B_{L^\infty(\Omega)}(0, \delta)$ au temps $T = 1$. L'existence de δ est garantie par le théorème de contrôlabilité locale de l'équation de la chaleur semilinéaire : Théorème 5.3.1.

Par (9.14), on pose alors T^* suffisamment grand pour avoir

$$F^{-1}(T^*) \leq \delta. \tag{9.15}$$

Finalement, en utilisant (9.8), (9.13), (9.15) et en effectuant un raisonnement similaire pour v_-, on a

$$\text{p.p. } x \in \Omega, \ |y(T^*, x)| \leq \delta. \tag{9.16}$$

Etape 2 : Contrôlabilité locale. Par contrôlabilité locale et (9.16), on en déduit qu'il existe h_2 défini sur (T^*, T^*+1) qui amène la solution y à zéro au temps $t = T^* + 1$.

Le contrôle $h = 0$ sur $(0, T^*)$ et $h = h_2$ sur (T^*, T^*+1) amène ainsi la solution de (9.1) à zéro au temps $t = T^* + 1$. Il est à noter que $T^* + 1$ ne dépend pas de y_0. $\square$

9.5 Cas dissipatif : contrôlabilité globale en temps petit impossible

Le résultat suivant, énoncé sans preuve, montre qu'on ne peut guère espérer mieux qu'un résultat de contrôle en temps long dans le cas dissipatif (voir [AT02, Theorem 3]).

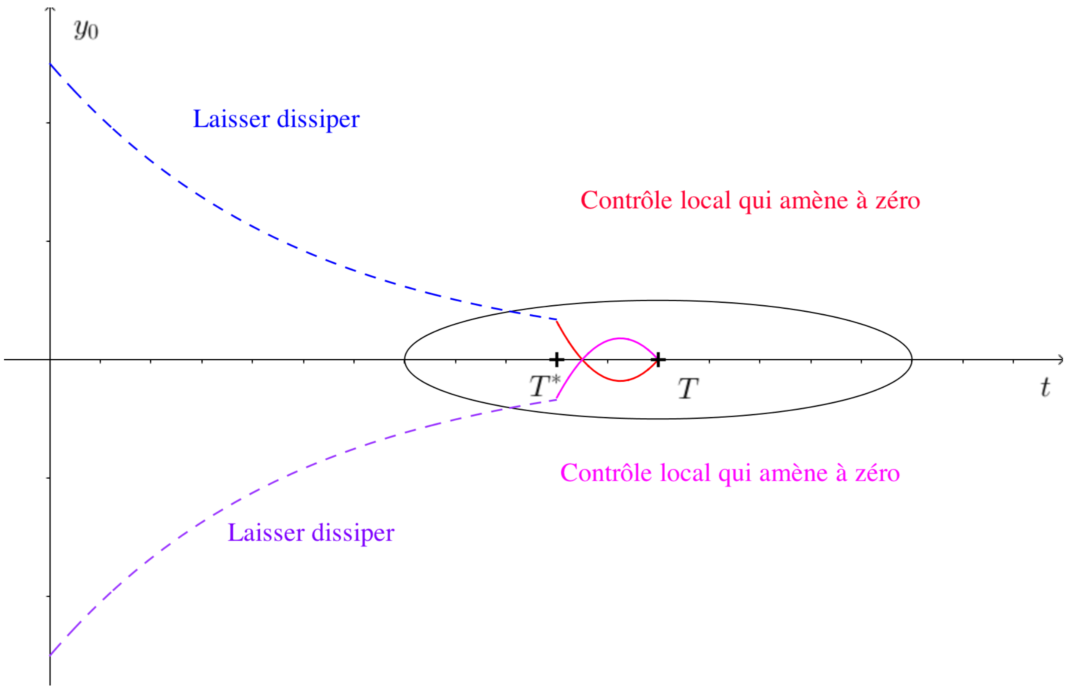

FIGURE 9.1 – Illustration de la preuve du Théorème 9.4.1

Théorème 9.5.1. *On pose* $f(s) := s \log^p(1+|s|)$ *avec* $p > 2$ *et on suppose que* $\Omega \setminus \overline{\omega} \neq \emptyset$. *Alors, il existe* $x_0 \in \Omega \setminus \overline{\omega}$, $T_0 \in (0,1)$ *tels que pour tout* $T \in (0, T_0)$, *il existe une donnée initiale* $y_0 \in L^\infty(\Omega)$ *telle que, quel que soit le contrôle* $h \in L^\infty(q_T)$, *la solution* y *de l'équation de la chaleur semilinéaire* (9.1) *satisfait* $y(T, x_0) < 0$.

Remarque 9.5.2. En particulier, pour un tel f comme dans le Théorème 9.5.1, (9.1) n'est pas globalement contrôlable en temps petit. Le Théorème 9.5.1 est dû à une majoration ponctuelle de y, solution de (9.1), qui est indépendante du contrôle h. Celle-ci est obtenue en construisant des sur-solutions appropriées.

9.6 Nouveau résultat de contrôlabilité globale à états positifs en temps petit malgré l'explosion

On introduit une nouvelle notion de contrôlabilité.

Définition 9.6.1. Soit $T > 0$. L'équation de la chaleur semilinéaire (9.1) est *globalement positivement contrôlable* (respectivement *globalement négativement contrôlable*) au temps T si pour tout $y_0 \in L^\infty(\Omega)$, il existe $h \in L^\infty(q_T)$ tel que la solution y de (9.1) satisfait :

$$y(T, .) \geq 0 \qquad \text{(respectivement } y(T, .) \leq 0). \tag{9.17}$$

Voici le premier nouveau résultat que l'on a obtenu pour (9.1) (voir [LB18a, Theorem 2.2]).

Théorème 9.6.2. *On suppose que* f *vérifie la condition asymptotique* (9.2) *pour* $\alpha \leq 2$ *et* $f(s) \geq 0$ *pour* $s \geq 0$ *(respectivement* $f(s) \leq 0$ *pour* $s \leq 0$*). Alors, pour tout* $T > 0$, *l'équation de la chaleur semilinéaire* (9.1) *est globalement positivement contrôlable (respectivement négativement contrôlable) au temps* T.

Remarque 9.6.3. Le Théorème 9.6.2 est presque optimal puisqu'il n'est pas vérifié pour $\alpha > 2$ grâce au Théorème 9.5.1. Le cas où $|f(s)| \sim |s| \log^2(1 + |s|)$ quand $|s| \to +\infty$ est ouvert.

Remarque 9.6.4. Le Théorème 9.6.2 ne traite pas le cas $f(s) = -s \log^p(1 + |s|)$ avec $p < 2$ à cause de la condition de signe.

Nous donnons les idées de la preuve du Théorème 9.6.2 dans la Section 9.8.

9.7 Nouveau résultat de contrôlabilité globale à zéro en temps long malgré l'explosion

Notre deuxième nouveau résultat est le suivant (voir [LB18a, Theorem 2.5]).

Théorème 9.7.1. *On suppose que f vérifie la condition asymptotique* (9.2) *pour* $\alpha \leq 2$, $f(s) > 0$ *pour* $s > 0$ *(respectivement* $f(s) < 0$ *pour* $s < 0$*) et la condition d'Osgood* (9.4) *est vérifiée au voisinage de* $+\infty$ *(respectivement au voisinage de* $-\infty$*). Alors il existe* $T > 0$ *suffisamment grand tel que l'équation de la chaleur semilinéaire* (9.1) *est globalement contrôlable à zéro au temps* T.

Remarque 9.7.2. Le Théorème 9.7.1 prouve que le Théorème 9.3.1 est presque optimal. En effet, prenons $f(s) = \int_0^{|s|} \log^p(1+\sigma)d\sigma$ avec $p < 2$, alors par le Théorème 9.7.1, il existe T suffisamment grand tel que (9.1) est globalement contrôlable au temps T. En particulier, on peut trouver un contrôle localisé qui empêche l'explosion. Le cas $f(s) = \int_0^{|s|} \log^2(1+\sigma)d\sigma$ est laissé ouvert.

Remarque 9.7.3. Le Théorème 9.7.1 ne traite pas le cas $f(s) = -s\log^p(1+|s|)$ avec $p < 2$ en raison de la condition de signe.

Remarque 9.7.4. La contrôlabilité à zéro en temps petit de (9.1) reste ouverte quand (9.2) a lieu pour $3/2 < \alpha \leq 2$.

Passons à la preuve du Théorème 9.7.1.

Démonstration. Soit $y_0 \in L^\infty(\Omega)$. La preuve s'effectue en trois étapes que nous allons résumer ici car elle suit de manière très proche le raisonnement de la preuve du Théorème 9.4.1.

Etape 1 : Chauffer très fort. Par le Théorème 9.6.2, on sait que l'on peut amener y_0 à un état positif au moyen d'un contrôle h_1 au temps $t = 1$ par exemple.

Etape 2 : Laisser refroidir. On pose $h_2 = 0$ sur $(1, T^*)$ où $T^* > 0$ sera fixé plus tard. Par principe de comparaison, on démontre que

$$\forall t \in [1, T^*],\ \text{p.p. } x \in \Omega,\ 0 \leq y(t,x) \leq v_+(t),$$

où v_+ est l'unique solution de

$$\begin{cases} \dot{v_+}(t) = -f(v_+(t)) & \text{dans } (1, +\infty), \\ v_+(1) = +\|y_0\|_{L^\infty(\Omega)} + 1 & . \end{cases}$$

Puis, on montre de manière similaire que

$$\forall t \in [1, +\infty),\ v_+(t) \leq G(t) \to 0 \text{ quand } t \to +\infty,$$

où G est indépendante de y_0. On en déduit que

$$\text{p.p. } x \in \Omega,\ 0 \leq y(T^*, x) \leq \delta,$$

où $\delta > 0$ est tel que la contrôlabilité locale de (9.1) ait lieu dans $B_{L^\infty(\Omega)}(0, \delta)$ au temps $t = 1$.

Etape 3 : Contrôlabilité locale. Par contrôlabilité locale, on en déduit qu'il existe h_3 défini sur $(T^*, T^* + 1)$ qui amène la solution y à zéro au temps $t = T^* + 1$.

Le contrôle $h = h_1$ sur $(0, 1)$, $h = 0$ sur $(1, T^*)$ et $h = h_3$ sur $(T^*, T^* + 1)$ amène ainsi la solution de (9.1) à zéro au temps $t = T^* + 1$. Il est à noter que $T^* + 1$ ne dépend pas de y_0. □

9.8 Estimations optimales d'observabilité pour l'équation de la chaleur avec potentiel

La preuve du théorème de contrôlabilité globale à états positifs en temps petit, i.e. le Théorème 9.6.2, est basée sur le nouveau résultat suivant, qui est un résultat de contrôle (avec estimation fine du coût) à états positifs pour l'équation de la chaleur avec potentiel $a \in L^\infty(Q_T)$:

$$\begin{cases} \partial_t y - \Delta y + a(t,x)y = h1_\omega & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega. \end{cases} \tag{9.18}$$

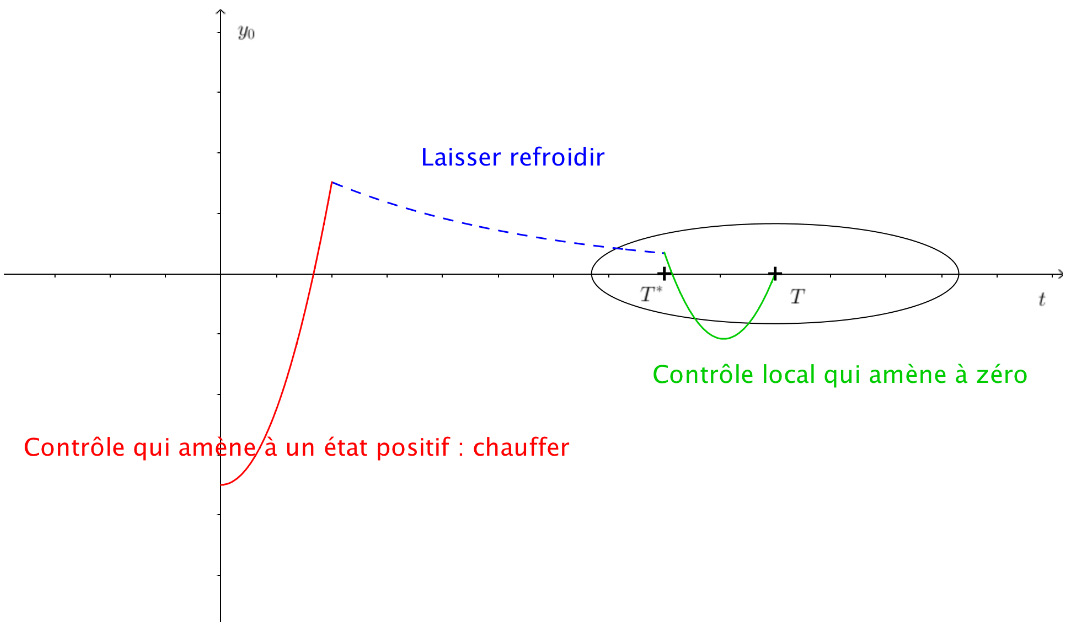

FIGURE 9.2 – Illustration de la preuve du Théorème 9.7.1

Théorème 9.8.1. *Pour tout $T > 0$, (9.18) est globalement positivement contrôlable au temps T. Plus précisément, pour tout $T > 0$, il existe $C = C(\Omega, \omega, T, a) > 0$, avec*

$$C(\Omega, \omega, T, a) = \exp\left(C(\Omega, \omega)\left(1 + \frac{1}{T} + T\|a\|_{L^\infty(Q_T)} + \|a\|_{L^\infty(Q_T)}^{1/2}\right)\right) \tag{9.19}$$

tel que pour tout $y_0 \in L^2(\Omega)$, il existe $h \in L^\infty(q_T)$ tel que

$$\|h\|_{L^\infty(q_T)} \leq C(\Omega, \omega, T, a)\|y_0\|_{L^2(\Omega)}, \tag{9.20}$$

et la solution y de (9.18) satisfait

$$y(T, .) \geq 0. \tag{9.21}$$

Pour démontrer le Théorème 9.6.2 à partir du Théorème 9.8.1, il suffit d'adapter les arguments de la preuve du théorème de contrôlabilité globale à zéro en temps petit de l'équation de la chaleur faiblement non linéaire, i.e. le Théorème 5.4.2, et d'utiliser la dissipation de f sur $\mathbb{R}^+$.

Remarque 9.8.2. On sait déjà que (9.18) est globalement positivement contrôlable au temps T car elle est globalement contrôlable à zéro au temps T (voir le Théorème 3.5.1), mais le plus intéressant est le *coût de contrôle* donné dans le Théorème 9.8.1. En particulier, l'exposant $1/2$ du terme $\|a\|_{L^\infty(Q_T)}^{1/2}$ est le point clef pour prouver le Théorème 9.6.2 pour des non linéarités vérifiant (9.2) avec $\alpha < 2 = (1/2)^{-1}$.

La preuve du Théorème 9.8.1 est une conséquence de l'inégalité d'observabilité suivante pour le système adjoint

$$\begin{cases} -\partial_t \varphi - \Delta\varphi + a(t,x)\varphi = 0 & \text{dans } Q_T, \\ \varphi = 0 & \text{sur } \Sigma_T, \\ \varphi(T, .) = \varphi_T & \text{dans } \Omega. \end{cases} \tag{9.22}$$

Théorème 9.8.3. *Pour tout $T > 0$, il existe $C = C(\Omega, \omega, T, a) > 0$ de la forme (9.19) tel que pour $\varphi_T \in L^2(\Omega; \mathbb{R}^+)$ la solution φ de (9.22) satisfait*

$$\|\varphi(0, .)\|_{L^2(\Omega)}^2 \leq C\left(\int_0^T \int_\omega \varphi^2 dx dt\right). \tag{9.23}$$

Remarque 9.8.4. Rappelons au passage que (9.23) est vérifiée avec C de la forme

$$C(\Omega, \omega, T, a) = \exp\left(C(\Omega, \omega)\left(1 + \frac{1}{T} + T\|a\|_{L^\infty(Q_T)} + \|a\|_{L^\infty(Q_T)}^{2/3}\right)\right) \tag{9.24}$$

pour tout $\varphi_T \in L^2(\Omega; \mathbb{R})$ d'après (3.24). L'exposant $2/3$ du terme $\|a\|_{L^\infty(Q_T)}^{2/3}$ est d'ailleurs le point clef pour prouver le Théorème 5.4.2. L'optimalité de l'exposant $2/3$ a été prouvée par Thomas Duyckaerts, Xu Zhang et Enrique Zuazua pour des *systèmes d'au moins* 2 *équations paraboliques* avec $N \geq 2$, N pair, N étant la dimension spatiale (voir [DZZ08, Theorem 1.1] et aussi [Zua07, Theorem 5.2] pour les principales idées de la preuve). Le Théorème 9.8.3 montre qu'on peut en fait diminuer l'exposant $2/3$ à l'exposant $1/2$ pour des données initiales (au temps $t = T$) positives.

Reprenant les notations de la Section 3.4, la preuve du Théorème 9.8.3 est basée sur l'estimation d'inspiration Carleman L^1 suivante.

Théorème 9.8.5. *Il existe deux constantes* $C := C(\Omega, \omega) > 0$ *et* $C_1 := C_1(\Omega, \omega) > 0$, *telles que*

$$\forall \lambda \geq 1, \qquad \forall s \geq s_1(\lambda) := C(\Omega, \omega) \left(T + T^2 + T^2 \|a\|^{1/2}_{L^\infty(Q_T)} \right), \tag{9.25}$$

pour tout $\varphi_T \in L^2(\Omega; \mathbb{R}^+)$, *la solution positive* φ *de* (9.22) *vérifie*

$$\lambda \int_{Q_T} e^{-s\alpha} s \xi^2 \eta^0 q + \int_{Q_T} e^{-s\alpha} \xi q \leq C_1 \lambda \int_{(0,T)\times\omega} e^{-s\alpha} s \xi^2 q dx dt. \tag{9.26}$$

Démonstration. Soit $\psi = e^{-s\alpha}\varphi$ où φ est solution de (9.22) associée à une donnée initiale $\varphi_T \in L^2(\Omega; \mathbb{R}^+)$.

On procède comme dans la preuve de l'inégalité de Carleman L^2 établie pour le Théorème 3.4.3 : on regarde l'équation satisfaite par ψ, on a :

$$M\psi = g_{s,\lambda}, \tag{9.27}$$

où on a défini

$$M\psi = -2s\lambda\xi\nabla\eta^0.\nabla\psi + \partial_t\psi + s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi + \Delta\psi + s\alpha_t\psi - a(t,x)\psi, \tag{9.28}$$

et

$$g_{s,\lambda} = g + s\lambda\Delta\eta^0\xi\psi + s\lambda^2|\nabla\eta^0|^2\xi\psi.$$

On multiplie (9.27) par η^0 et on intègre sur $(0,T) \times \Omega$

$$\begin{aligned} &\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi\eta^0 - \int_{Q_T} 2s\lambda\xi(\nabla\eta^0.\nabla\psi)\eta^0 + \int_{Q_T} (\partial_t\psi)\eta^0 + \int_{Q_T} (\Delta\psi)\eta^0 \\ &= \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi\eta^0 - \int_{Q_T} s\alpha_t\psi\eta^0 + \int_{Q_T} a(t,x)\psi\eta^0 \\ &\quad + \int_{Q_T} s\lambda\Delta\eta^0\xi\psi\eta^0. \end{aligned} \tag{9.29}$$

En utilisant la propriété du poids η^0 : (3.13), on a

$$\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi\eta^0 \geq c\int_{Q_T} s^2\lambda^2\xi^2\psi\eta^0 - c\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi\eta^0. \tag{9.30}$$

En combinant (9.29) et (9.30), on a

$$
\begin{aligned}
& c\int_{Q_T} s^2\lambda^2\xi^2\psi\eta^0 - \int_{Q_T} 2s\lambda\xi(\nabla\eta^0.\nabla\psi)\eta^0 + \int_{Q_T}(\partial_t\psi)\eta^0 + \int_{Q_T}(\Delta\psi)\eta^0 \\
& \leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi\eta^0 + \int_{Q_T} s|\alpha_t|\psi\eta^0 + \int_{Q_T}|a(t,x)|\psi\eta^0 \\
& \quad + \int_{Q_T} s\lambda|\Delta\eta^0|\xi\psi\eta^0 + c\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi\eta^0.
\end{aligned} \tag{9.31}
$$

On a de plus les intégrations par parties suivantes

$$
-\int_{Q_T} 2s\lambda\xi(\nabla\eta^0.\nabla\psi)\eta^0 = \int_{Q_T} 2s\lambda\left((\nabla\xi.\nabla\eta^0)\eta^0\psi + \xi(\Delta\eta^0)\eta^0\psi + \underbrace{\xi|\nabla\eta^0|^2\psi}_{\geq 0}\right). \tag{9.32}
$$

$$
\int_{Q_T}(\partial_t\psi)\eta^0 = \int_\Omega \eta^0(.)(\psi(T,.)-\psi(0,.)) = 0, \tag{9.33}
$$

$$
\int_{Q_T}(\Delta\psi)\eta^0 = \int_{Q_T}\psi\Delta\eta^0. \tag{9.34}
$$

De (9.31), (9.32), (9.33), (9.34) et des propriétés de η^0, on déduit

$$
\begin{aligned}
& c\int_{Q_T} s^2\lambda^2\xi^2\psi\eta^0 + 2c\int_{Q_T} s\lambda\xi\psi \\
& \leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi\eta^0 + \int_{Q_T} s|\alpha_t|\psi\eta^0 + \int_{Q_T}|a(t,x)|\psi\eta^0 \\
& \quad +3\int_{Q_T} s\lambda|\Delta\eta^0|\xi\psi\eta^0 + 2\int_{Q_T} s\lambda|\nabla\xi||\nabla\eta^0|\psi\eta^0 + \int_{Q_T}\psi|\Delta\eta^0| \\
& \quad +c\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi\eta^0 + 2c\int_{(0,T)\times\omega} s\lambda\xi\psi.
\end{aligned} \tag{9.35}
$$

En utilisant les estimations du Lemme 3.4.2, les cinq premiers termes du membre de droite (9.35) peuvent être absorbés par le premier terme de gauche si $s \geq s_1$ comme défini en (9.25). Le sixième terme du membre de droite de (9.35) peut être absorbé par le second terme de gauche si $s \geq C(\Omega,\omega)T^2$. Les deux derniers termes du membre de droite de (9.35) sont plus petits que $\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi$ si $s \geq C(\Omega,\omega)T^2$. Ce qui mène à

$$
\int_{Q_T} s^2\lambda^2\xi^2\psi\eta^0 + \int_{Q_T} s\lambda\xi\psi \leq C\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi,
$$

et donc à (9.26) en divisant par $s\lambda$. $\square$

De la même manière, on démontre dans [LB18a] une estimation fine du coût de contrôle pour les systèmes

$$
\begin{cases}
\partial_t y - \Delta y + B(t,x).\nabla y = h1_\omega & \text{dans } Q_T, \\
y = 0 & \text{sur } \Sigma_T, \\
y(0,.) = y_0 & \text{dans } \Omega,
\end{cases} \tag{9.36}
$$

où $B \in L^\infty(Q_T)^N$. En effet, nous démontrons le nouveau résultat suivant.

Théorème 9.8.6. *Pour tout $T > 0$, (9.36) est globalement positivement contrôlable au temps T avec un coût de contrôle de la forme*

$$C(\Omega, \omega, T, B) = \exp\left(C(\Omega, \omega)\left(1 + \frac{1}{T} + T\,\|B\|^2_{L^\infty(Q_T)^N} + \|B\|_{L^\infty(Q_T)^N}\right)\right). \quad (9.37)$$

Le Théorème 9.8.6 repose sur une amélioration de la quantification de la constante d'observabilité pour l'équation

$$\begin{cases} -\partial_t \varphi - \Delta\varphi - \nabla.(B(t,x)\varphi) = 0 & \text{dans } Q_T, \\ \varphi = 0 & \text{sur } \Sigma_T, \\ \varphi(T,.) = \varphi_T & \text{dans } \Omega. \end{cases} \quad (9.38)$$

qui est de l'ordre de $C(\Omega, \omega, T, B)$ comme dans (9.37) pour des données initiales $\varphi_T \in L^2(\Omega; \mathbb{R}^+)$. Il est à noter que pour $\varphi_T \in L^2(\Omega; \mathbb{R})$, la meilleure estimation de la constante d'observabilité pour (9.38) connue à ce jour (voir [DFCGBZ02]) est

$$C(\Omega, \omega, T, B) = \exp\left(C(\Omega, \omega)\left(1 + \frac{1}{T} + (1+T)\,\|B\|^2_{L^\infty(Q_T)^N}\right)\right).$$

9.9 Perspectives et problèmes ouverts

Une première piste de recherche pourrait être de voir dans quelle mesure l'approximation numérique des contrôles amenant à zéro les solutions de (9.1) en temps petit dans [FCM12] peut être adaptée pour calculer les contrôles amenant à un état positif les solutions de (9.1) en temps petit.

9.9.1 Non linéarités dépendant du gradient de l'état

On peut également se demander dans quelle mesure les résultats du Théorème 9.6.2 et du Théorème 9.7.1 peuvent être étendus à des non linéarités $F(y, \nabla y)$. Pour simplifier la discussion, on suppose que F dépend seulement du gradient de l'état, c'est-à-dire qu'on considère l'équation non linéaire suivante :

$$\begin{cases} \partial_t y - \Delta y + F(\nabla y) = h 1_\omega & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega, \end{cases} \quad (9.39)$$

où $F \in C^1(\mathbb{R}^N; \mathbb{R})$ vérifie $F(0) = 0$.

Nous disposons du résultat suivant de contrôlabilité globale à zéro (voir [DFCGBZ02, Theorem 1.1]).

Théorème 9.9.1. *On suppose que (9.2) est vérifiée pour $\alpha \leq 1/2$. Alors, pour tout $T > 0$, (9.39) est globalement contrôlable à zéro au temps T.*

On a également le résultat (négatif) suivant (voir [PZ12, Theorem 4.2]).

Théorème 9.9.2. *Il existe F satisfaisant $F(\nabla y) \sim |\nabla y| \log^p(1 + |\nabla y|)$ quand $|\nabla y| \to +\infty$ avec $p > 1$ tel que pour tout $r > 0$, il existe $y_0 \in L^\infty(\Omega)$ avec $\|y_0\|_{L^\infty(\Omega)} = r$ tel que pour tout contrôle $h \in L^\infty(q_T)$, la solution y de (9.39) satisfait $y(t,.) \neq 0$ pour tout $t < C_0 r$, où $C_0 = C_0(\Omega, \omega, F)$. En particulier, le système (9.39) n'est pas globalement contrôlable à zéro pour tout $T > 0$.*

Remarque 9.9.3. Comme déjà mentionné dans la [PZ12, Remark 4.5], il y a une marge entre le Théorème 9.9.1 et le Théorème 9.9.2. Ce qui nous amène à la question suivante : que peut-on dire des semilinéarités satisfaisant (9.2) pour $\alpha \in (1/2, 1]$? Une bonne stratégie pour répondre (partiellement) à cette question pourrait être de partir de la nouvelle estimation de coût du Théorème 9.8.6. Une des difficultés à résoudre serait alors l'adaptation de l'argument employé dans l'étape 2 de la preuve du Théorème 9.7.1. En effet, pour des non linéarités dépendant du gradient de l'état, la comparaison de la solution libre à la solution d'une équation différentielle n'est plus possible.

9.9.2 Conjecture de Landis et estimations optimales d'observabilité pour l'équation de la chaleur avec potentiel

Dans un premier temps, rappelons dans quel sens l'optimalité de l'inégalité d'observabilité de l'équation de la chaleur avec potentiel mentionnée en Remarque 9.8.4 a été prouvée par Thomas Duyckaertz, Xu Zhang et Enrique Zuazua (voir [DZZ08, Theorem 1.1]).

Théorème 9.9.4. *On suppose que la dimension spatiale $N \geq 2$ est paire et que le nombre d'équations du système parabolique est $n \geq 2$. Alors il existe $c > 0$, $\mu > 0$, une famille de potentiels (matriciels) $(A_R)_{R>0}$ tels que*

$$\|A_R\|_{L^\infty(\Omega;\mathbb{C}^{n\times n})} \to +\infty, \ \textit{quand } R \to +\infty,$$

et une famille $(\varphi_{T,R})_{R>0}$ de données initiales dans $L^2(\Omega)^n$ telle que la famille de solutions φ_R de

$$\begin{cases} -\partial_t \varphi_R - \Delta\varphi_R + A_R(x)\varphi_R = 0 & \text{dans } Q_T, \\ \varphi_R = 0 & \text{sur } \Sigma_T, \\ \varphi_R(T,.) = \varphi_{T,R} & \text{dans } \Omega, \end{cases} \tag{9.40}$$

satisfait

$$\lim_{R\to+\infty} \left\{ \inf_{T\in I_\mu} \frac{\|\varphi_R(0,.)\|^2_{L^2(\Omega)^n}}{\exp\left(c\,\|A_R\|^{2/3}_{L^\infty(\Omega)}\right) \int_0^T \int_\omega |\varphi_R(t,x)|^2 dtdx} \right\} = +\infty, \tag{9.41}$$

où $I_\mu = \left(0, \mu\,\|A_R\|^{-1/3}_{L^\infty(\Omega)}\right]$.

On redonne l'idée de la preuve du Théorème 9.9.4 (voir [Zua07, Section 5.2]).

Idée de preuve du Théorème 9.9.4. Pour simplifier la présentation, on se concentre sur le cas $N = 2$ et $n = 2$.

La preuve est basée sur le résultat suivant dû à Viktor Meshkov (voir [Mes91]).

Théorème 9.9.5. *Il existe un potentiel à valeur complexe non nul $q = q(x)$ et une solution à valeur complexe non nulle $u = u(x)$ de*

$$\Delta u = q(x)u, \text{ dans } \mathbb{R}^2,$$

et qui satisfait la propriété de décroissance

$$\forall x \in \mathbb{R}^2, \ |u(x)| \leq C \exp(-|x|^{4/3}),$$

pour une certaine constante $C > 0$.

Etape 1 : Construction sur $\mathbb{R}^N$. On se donne u et q comme dans le Théorème 9.9.5. On pose

$$u_R(x) = u(Rx), \qquad A_R(x) = R^2 q(Rx).$$

On obtient alors une famille de potentiels $(A_R)_{R>0}$ et de solutions $(u_R)_{R>0}$ vérifiant

$$\Delta u_R = A_R(x)u_R, \text{ dans } \mathbb{R}^2,$$

et

$$\forall x \in \mathbb{R}^2, \ |u(x)| \leq C \exp(-R^{4/3}|x|^{4/3}).$$

Ainsi, en posant $\psi_R(t, x) = u_R(x)$ pour $(t, x) \in (0, +\infty) \times \mathbb{R}^N$, on a

$$-\partial_t \psi_R - \Delta \psi_R + A_R \psi_R = 0 \text{ dans } (0, +\infty) \times \mathbb{R}^2,$$

et

$$\forall (t, x) \in (0, +\infty) \times \mathbb{R}^2, \ |\psi_R(t, x)| \leq C \exp(-R^{4/3}|x|^{4/3}).$$

Etape 2 : Restriction à Ω. On suppose que ω est contenu strictement dans Ω. Sans perte de généralité (par translation et changement d'échelle), on peut supposer que $B(0, 1) \subset \Omega \setminus \overline{\omega}$.

Ainsi, la famille (ψ_R) préalablement construite est solution du problème de Dirichlet non homogène suivant

$$\begin{cases} -\partial_t \psi_R - \Delta \psi_R + A_R(x)\psi_R = 0 & \text{dans } Q_T, \\ \psi_R = \varepsilon_R & \text{sur } \Sigma_T, \end{cases} \tag{9.42}$$

où $\varepsilon_R = \psi_R$ sur $\partial\Omega$.

En utilisant le fait que ω et $\partial\Omega$ sont contenus dans le complémentaire de $B(0, 1)$, on déduit que pour certaines constantes $c, C > 0$:

$$\forall (t, x) \in (0, T) \times \omega, \ |\psi_R(t, x)| \leq C \exp(-R^{4/3}), \tag{9.43}$$

$$\forall (t, x) \in (0, T) \times \partial\Omega, \ |\varepsilon_R(t, x)| \leq C \exp(-R^{4/3}), \tag{9.44}$$

$$\|\psi_R(0, .)\|^2_{L^2(\Omega)} \sim \|\psi_R(0, .)\|^2_{L^2(\mathbb{R}^N)} = \|u_R\|^2_{L^2(\mathbb{R}^N)} = \frac{1}{R^N} \|u\|^2_{L^2(\mathbb{R}^N)} = \frac{c}{R^N}, \tag{9.45}$$

$$\|A_R\|^2_{L^\infty(\Omega)} \sim \|A_R\|^2_{L^\infty(\mathbb{R}^N)} = CR^4. \tag{9.46}$$

On peut alors corriger la solution ψ_R pour qu'elle satisfasse un problème de Dirichlet homogène. Pour cela on introduit

$$\begin{cases} -\partial_t \rho_R - \Delta \rho_R + A_R(x)\rho_R = 0 & \text{dans } Q_T, \\ \rho_R = \varepsilon_R & \text{sur } \Sigma_T, \\ \rho_R(T,.) = 0, \end{cases} \tag{9.47}$$

et on pose alors

$$\varphi_R = \psi_R - \rho_R.$$

Ainsi, par (9.42) et (9.47), (φ_R) est une famille de solutions du système parabolique (9.40) avec potentiels $A_R(x) = R^2 q(Rx)$.

Le caractère exponentiellement décroissant de la donnée de Dirichlet ε_R montre que ρ_R est exponentiellement décroissant. Cela nous permet de montrer que φ_R satisfait essentiellement les mêmes propriétés que ψ_R énoncées en (9.43), (9.44), (9.45), (9.46). Ainsi, la famille (φ_R) permet de conclure la preuve du Théorème 9.9.4. □

La preuve précédente établit ainsi un lien entre l'existence de fonctions $u \in L^\infty(\mathbb{C};\mathbb{C})$ et $q \in L^\infty(\mathbb{C};\mathbb{C})$, telles que $\Delta u = q(x)u$, avec u décroissant plus qu'exponentiellement à l'infini : $\forall x \in \mathbb{C},\ |u(x)| \leq C\exp(-|x|^{4/3})$ et la quantification de l'inégalité d'observabilité pour l'équation de la chaleur avec potentiel. Il est également démontré dans [Mes91] par une inégalité de Carleman l'optimalité du Théorème 9.9.5 au sens suivant.

Théorème 9.9.6. *Soit $q \in L^\infty(\mathbb{C};\mathbb{C})$ et $u \in L^\infty(\mathbb{C};\mathbb{C})$ solution de*

$$\Delta u = q(x)u,\ \text{dans } \mathbb{C},$$

et qui satisfait la propriété de décroissance

$$\exists C > 0, \varepsilon > 0,\ \forall x \in \mathbb{C},\ |u(x)| \leq C\exp(-|x|^{4/3+\varepsilon}).$$

Alors

$$u \equiv 0.$$

Historiquement, la construction de Viktor Meshkov du Théorème 9.9.5 répondait à une question de Evgenii Landis datant de la fin des années 60 (voir [KL88]), appelée dorénavant conjecture de Landis.

Problème ouvert 9.9.7 (Conjecture de Landis)**.** Si u satisfait

$$-\Delta u + q(x)u = 0 \qquad \text{dans } \mathbb{R}^N, \tag{9.48}$$

avec $q \in L^\infty(\mathbb{R}^N)$ et u satisfait

$$\exists C > 0,\ \exists \varepsilon > 0,\ \forall x \in \mathbb{R}^N,\ |u(x)| \leq C\exp(-C|x|^{1+\varepsilon}), \tag{9.49}$$

alors

$$u \equiv 0 \qquad \text{dans } \mathbb{R}^N. \tag{9.50}$$

Cette conjecture a donc été niée par Victor Meshkov grâce à son contre-exemple du Théorème 9.9.5 dans le cas complexe. Une forme quantitative des résultats de Viktor Meshkov a par ailleurs été établie dans l'article de Jean Bourgain et Carlos Kenig (voir [BK05]) sur la résolution du problème de localisation d'Anderson. La conjecture de Landis a alors été remise au goût du jour. Est-elle vraie si u et q sont à valeurs réelles dans le Problème ouvert 9.9.7 (voir [Ken06]) ?

Le cas unidimensionnel $N = 1$ et radial a été démontré récemment par Luca Rossi (voir [Ros18]). Le cas $u \geq 0$ ou $V \geq 0$ a été démontré par Luca Rossi (voir [Ros18]) et de manière indépendante par Ari Arapostathis, Anus Biswas et Debdip Ganguly par des outils probabilistes (voir [ABG17]). Mentionnons également l'article [KSW15] dans le cas $N = 2$ avec $V \geq 0$ et les papiers qui ont suivi, qui donnent une preuve quantitative de la conjecture de Landis.

La preuve du Théorème 9.9.4 établit donc un lien entre la conjecture de Landis et la quantification de l'inégalité d'observabilité pour l'équation de la chaleur avec potentiel.

Ainsi, ayant démontré dans le Théorème 9.8.3 que la constante d'observabilité pour des données initiales à valeurs positives est de l'ordre

$$C(\Omega, \omega, T, a) = \exp\left(C(\Omega, \omega)\left(1 + \frac{1}{T} + T\,\|a\|_{L^\infty(Q_T)} + \|a\|_{L^\infty(Q_T)}^{1/2}\right)\right),$$

on peut retrouver le résultat récent : la conjecture de Landis est vérifiée pour des fonctions u à valeurs positives. En effet, il suffit de raisonner par l'absurde, i.e. de supposer l'existence d'une fonction u positive non triviale vérifiant (9.48) et (9.49). Puis, en utilisant les mêmes arguments que dans la preuve du Théorème 9.9.4, on peut alors nier la quantification préalablement établie de la constante d'observabilité pour des données initiales à valeurs positives. En fait, on peut également redémontrer de manière directe la conjecture de Landis dans le cas $u \geq 0$ en démontrant une inégalité d'inspiration Carleman elliptique L^1 similaire à celle établie dans le cas parabolique dans le Théorème 9.8.5 (voir Annexe F).

On conjecture que l'inégalité d'observabilité (9.23) pour l'équation de la chaleur avec potentiel réel (9.18) est vérifiée avec une constante C de la forme

$$C(\Omega, \omega, T, a) = \exp\left(C(\Omega, \omega)\left(1 + \frac{1}{T} + T\,\|a\|_{L^\infty(Q_T)} + \|a\|_{L^\infty(Q_T)}^{1/2}\right)\right)$$

pour tout $\varphi_T \in L^2(\Omega; \mathbb{R})$. A part dans le cas unidimensionnel où il est connu que le résultat est vrai pour des potentiels a ne dépendant que de la variable d'espace (voir [DZZ08, Section 8.5]), ce problème est largement ouvert. Ce type d'inégalité impliquerait la conjecture de Landis (dans le cas réel) en raisonnant par l'absurde et en utilisant les arguments de la preuve du Théorème 9.9.5. Elle impliquerait également la contrôlabilité globale à zéro en temps petit de l'équation (9.1) pour f satisfaisant (9.2) pour $\alpha \in (3/2, 2]$. C'est donc un problème difficile a priori. De nouvelles idées doivent voir le jour pour le résoudre.

Chapitre 10

Contrôlabilité à zéro de systèmes linéaires parabolique-transport

Le but de ce chapitre est de présenter le travail, en cours de rédaction, fait en collaboration avec Karine Beauchard et Armand Koenig. Nous nous intéressons à la contrôlabilité à zéro de systèmes linéaires parabolique-transport. Plus précisément, nous étudions

$$\begin{cases} \partial_t f - B\partial_{xx} f + A\partial_x f + Kf = u1_\omega & \text{dans } (0,T)\times\mathbb{T}, \\ f(0,.) = f_0 & \text{dans } \mathbb{T}, \end{cases} \tag{Sys}$$

où

- $T > 0$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, ω est un ensemble non vide contenu dans $\mathbb{T}$, $d \in \mathbb{N}^*$, $A, B, K \in \mathbb{R}^{d\times d}$,
- l'état du système est $f : [0,T]\times\mathbb{T} \to \mathbb{R}^d$,
- le contrôle du système est $u : [0,T]\times\mathbb{T} \to \mathbb{R}^d$.

En Section 10.1, nous formulons les hypothèses sur les matrices A, B et K du système (Sys) sous lesquelles nous allons travailler. En Section 10.2, nous énonçons notre principal résultat de contrôlabilité. Il permet de généraliser certains résultats de contrôlabilité concernant l'équation des ondes avec amortissement structurel (voir Annexe G). C'est également un des premiers résultats de contrôlabilité d'un système couplant une dynamique parabolique avec une dynamique de transport. En Section 10.3, nous énonçons sans preuve les propriétés spectrales de l'opérateur associé au système (Sys), i.e. $-B\partial_{xx} + A\partial_x + K$. Ces dernières propriétés seront d'utilité constante dans la suite. Les principaux points de la preuve du résultat négatif de contrôlabilité à zéro sont rassemblés en Section 10.5. Enfin, les principales étapes de la preuve du résultat positif de contrôlabilité à zéro sont présentées en Section 10.6.

10.1 Présentation du système et hypothèses

On suppose que

$$d = d_1 + d_2 \text{ avec } 1 \leq d_1 < d,\ 1 \leq d_2 < d, \tag{H.1}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \text{ avec } D \in \mathbb{R}^{d_2 \times d_2}, \tag{H.2}$$

$$\Re(\mathrm{Sp}(D)) \subset (0, \infty). \tag{H.3}$$

En introduisant la décomposition par blocs analogue des matrices A et K, de la fonction f et du contrôle u,

$$A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \quad f(t,x) = \begin{pmatrix} f_1(t,x) \\ f_2(t,x) \end{pmatrix}, \quad u(t,x) = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix},$$

on voit que le système (Sys) couple une équation de transport sur f_1 avec une équation parabolique sur f_2

$$\begin{cases} (\partial_t + A'\partial_x + K_{11}) f_1 + (A_{12}\partial_x + K_{12}) f_2 = u_1 1_\omega & \text{dans } (0,T) \times \mathbb{T}, \\ (\partial_t - D\partial_x^2 + A_{22}\partial_x + K_{22}) f_2 + (A_{21}\partial_x + K_{21}) f_1 = u_2 1_\omega & \text{dans } (0,T) \times \mathbb{T}, \\ (f_1, f_2)(0,.) = (f_{0,1}, f_{0,2}) & \text{dans } \mathbb{T}. \end{cases} \tag{10.1}$$

On fait de plus l'hypothèse suivante sur la matrice A'

$$A' \text{ est diagonalisable avec } \mathrm{Sp}(A') \subset \mathbb{R}. \tag{H.4}$$

On peut démontrer avec de l'analyse de Fourier vectorielle et une analyse spectrale assez fine que, pour toute donnée initiale $f_0 \in L^2(\mathbb{T}, \mathbb{C}^d)$ et pour tout contrôle $u \in L^2((0,T) \times \omega, \mathbb{C}^d)$, il existe une unique solution $f \in C^0([0,T], L^2(\mathbb{T})^d)$ de (Sys) (voir Annexe G).

10.2 Nouveau résultat de contrôlabilité

Notre principal résultat est le suivant.

Théorème 10.2.1. *On suppose* (H.1)-(H.4). *On définit*

$$\ell(\omega) := \sup\{|I|;\ I \text{ composante connexe de } \mathbb{T} \setminus \omega\},$$

$$\mu_* = \min\{|\mu|;\ \mu \in Sp(A')\},$$

et

$$T^* = \begin{cases} \frac{\ell(\omega)}{\mu_*} & si\ \mu_* > 0, \\ +\infty & si\ \mu_* = 0. \end{cases} \tag{10.2}$$

Alors

1. *le système* (Sys) *n'est pas contrôlable à zéro sur ω en tout temps $T < T^*$,*
2. *le système* (Sys) *est contrôlable à zéro sur ω en tout temps $T > T^*$.*

En particulier, quand ω est un intervalle de $\mathbb{T}$ et $\mu_* > 0$, alors le temps minimal de contrôlabilité est $T^* = \frac{2\pi - |\omega|}{\mu_*}$.

Le résultat négatif au temps $T < T^*$ est attendu en raison de la partie transport du système (Sys) (voir la composante f_1 de (10.1)), mais la preuve n'est pas évidente. En effet, en raison du couplage parabolique, il peut ne pas exister des solutions purement transportées pour le système (Sys). Notre preuve est basée sur des techniques d'analyse complexe dévelopées par Armand Koenig dans l'article [Koe17].

La preuve du résultat positif, au temps $T > T^*$ est basée sur une adaptation de la stratégie employée par Gilles Lebeau et Enrique Zuazua pour le contrôle d'un système linéaire de thermoélasticité (voir [LZ98]).

10.3 Analyse spectrale du système

Dans le but d'étudier (Sys), nous sommes ramenés à étudier l'opérateur

$$\mathcal{L} := -B\partial_{xx} + A\partial_x + K. \tag{10.3}$$

Dans tout ce chapitre, nous appelons e_n la fonction $x \mapsto \mathrm{e}^{\mathrm{i}nx}$. On remarque qu'en appliquant $\mathcal{L}$ à ϕe_n, où $\phi \in \mathbb{C}^d$, on obtient

$$\mathcal{L}(\phi e_n) = n^2 \left(B + \frac{\mathrm{i}}{n} A + \frac{1}{n^2} \right) \phi e_n. \tag{10.4}$$

Ainsi, en définissant $E(z)$ la matrice perturbée

$$\forall z \in \mathbb{C},\ E(z) = B + zA - z^2 K, \tag{10.5}$$

on déduit que $\mathcal{L}$ agit du coté Fourier comme la multiplication par $n^2 E(\mathrm{i}/n)$. Il nous faut ainsi connaître les valeurs propres et vecteurs propres de $E(z)$. Ces propriétés spectrales sont reliées à celles des matrices A et B d'après la théorie de la perturbation analytique en dimension finie (voir [Kat95, Ch. II §1 and §2]). Pour les détails des prochaines preuves, on renvoie à Annexe G.

Pour $r > 0$ et $m \in \mathbb{N}^*$, on note $\mathcal{O}_r^{m\times m}$ l'ensemble des fonctions holomorphes sur le disque $D(0, r)$ du plan complexe à valeurs dans $\mathbb{C}^{m\times m}$.

Proposition 10.3.1. *Il existe $r > 0$ et une fonction holomorphe à valeurs matricielles $P^{\mathrm{h}} \in \mathcal{O}_r^{m\times m}$ à valeurs matricielles telle que*

(i) $P^{\mathrm{h}}(0) = \begin{pmatrix} I_{d_1} & 0 \\ 0 & 0 \end{pmatrix}$,

(ii) pour tout $|z| < r$, $P^{\mathrm{h}}(z)$ est une projection qui commute avec $E(z)$,

(iii) dans la limite $z \to 0$, $E(z)P^{\mathrm{h}}(z) = O(z)$.

Pour démontrer la Proposition 10.3.1, il suffit de définir $P^{\mathrm{h}}(z)$ comme la somme des projections sur les espaces caractéristiques de $E(z)$ associées au « 0-groupe » de valeurs propres parallèlement aux autres espaces caractéristiques et d'utiliser la forme bloc de

B, voir (H.2), (H.3).

On dit que P^{h} est la « projection sur les branches hyperboliques ». On introduit $P^{\mathrm{p}}(z) = I_d - P^{\mathrm{h}}(z)$, que nous appelons la « projection sur les branches paraboliques ».

Proposition 10.3.2. *La fonction à valeurs matricielles P^{p} est dans $\mathcal{O}_r^{m\times m}$ et*

(i) $P^{\mathrm{p}}(0) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{d_2} \end{smallmatrix}\right)$,

(ii) pour tout $|z| < r$, $P^{\mathrm{p}}(z)$ est une projection qui commute avec $E(z)$,

(iii) dans la limite $z \to 0$, $E(z)P^{\mathrm{p}}(z) = B + O(z)$.

Dans la suite, nous devons être plus précis dans le développement asymptotique de la branche hyperbolique donné en Proposition 10.3.1. Nous établissons le résultat suivant.

Proposition 10.3.3. *Il existe $r > 0$ et une famille de fonctions holomorphes à valeurs matricielles $(P^{\mathrm{h}}_\mu)_{\mu\in \mathrm{Sp}(A')} \in (\mathcal{O}_r^{d\times d})^{\mathrm{Sp}(A')}$ vérifiant*

(i) pour tout $\mu \in \mathrm{Sp}(A')$ et $|z| < r$, $P^{\mathrm{h}}_\mu(z)$ est une projection non nulle qui commute avec $E(z)$,

(ii) pour tout $|z| < r$, $P^{\mathrm{h}}(z) = \sum\limits_{\mu\in\mathrm{Sp}(A')} P^{\mathrm{h}}_\mu(z)$ et pour tout $\mu \neq \mu'$, $P^{\mathrm{h}}_\mu(z)P^{\mathrm{h}}_{\mu'}(z) = 0$,

(iii) pour tout $\mu \in \mathrm{Sp}(A')$, il existe $R^{\mathrm{h}}_\mu \in \mathcal{O}_r^{d\times d}$ tel que

$$\forall |z| < r,\ E(z)P^{\mathrm{h}}_\mu(z) = \mu z P^{\mathrm{h}}_\mu(z) + z^2 R^{\mathrm{h}}_\mu(z).$$

La preuve de la Proposition 10.3.3 est basée sur le processus de réduction de Kato (voir [Kat95, Ch. II §2.3]). Ceci est rendu possible en utilisant le fait que 0 est une valeur propre semi-simple de la matrice B, c'est-à-dire que sa multiplicité algébrique coïncide avec sa multiplicité géométrique, et l'hypothèse (H.4), c'est-à-dire que la matrice A' est diagonalisable sur $\mathbb{R}$.

10.4 Observabilité du système adjoint

Par la « HUM », la contrôlabilité à zéro du système (Sys) équivaut à l'observabilité du système adjoint.

Proposition 10.4.1. *Soit $T > 0$. Le système (Sys) est contrôlable sur ω au temps T si et seulement si il existe $C > 0$ tel que pour tout $g_0 \in L^2(\mathbb{T}; \mathbb{C}^d)$, la solution g de l'équation*

$$\begin{cases} \partial_t g - B^{\mathrm{tr}}\partial_{xx} g - A^{\mathrm{tr}}\partial_x g + K^{\mathrm{tr}} g = 0 & \text{dans } (0,T)\times\mathbb{T}, \\ g(0,.) = g_0 & \text{dans } \mathbb{T}. \end{cases} \tag{10.6}$$

satisfait

$$\|g(T,.)\|^2_{L^2(\mathbb{C})} \leq C \int_0^T \int_\omega |g(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x. \tag{10.7}$$

On peut montrer que les solutions du système adjoint (10.6) sont de la forme

$$g(t,x) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{inx - tn^2 E\left(\frac{\mathrm{i}}{n}\right)^*} \varphi_n, \tag{10.8}$$

où $(\varphi_n) \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$. De plus, nous disposons d'une théorie spectrale qui est similaire aux Proposition 10.3.1, Proposition 10.3.2, Proposition 10.3.3 pour l'opérateur $E(z)^*$. Il suffit de passer à l'adjoint dans chacune des formules.

10.5 Preuve du résultat de négatif de contrôlabilité à zéro en temps petit

Le but de cette partie est de prouver le premier point du Théorème 10.2.1. Pour simplifier, on suppose que ω est un intervalle. On raisonne par l'absurde. On suppose que (Sys) est contrôlable à zéro sur ω au temps $T < T^* = \frac{2\pi - |\omega|}{\mu_*}$. Ainsi, en utilisant la Proposition 10.4.1, l'inégalité d'observabilité (10.7) est vérifiée au temps T. En adaptant la méthode d'Armand Koenig (voir [Koe17]), on démontre que l'inégalité d'observabilité (10.7) implique une estimation d'observabilité pour des polynômes à valeurs complexes que nous nions par la suite grâce au théorème de Runge.

Soit $\mu \in \mathrm{Sp}(A')$ de valeur absolue minimale. Nous démontrons dans le résultat suivante une estimation d'observabilité pour des polynômes à valeurs complexes.

Proposition 10.5.1. *Soit U un ouvert borné de $\mathbb{C}$, étoilé par rapport à 0, qui contient $\omega_T = \bigcup_{0\leq t\leq T}(\bar\omega - \mu t)$ (où $\bar\omega - \mu t$ est la rotation de $\bar\omega$ par un angle $-\mu t$, voir Figure 10.1).*

Il existe une constante $C > 0$ et un entier N tels que pour tout polynôme $p(\zeta) = \sum_{n>N} a_n \zeta^n$ où 0 est une racine de multiplicité au moins N, on a

$$|p|_{L^2(D(0,1))} \leq C|p|_{L^\infty(U)}. \tag{10.9}$$

Démonstration. Grâce à la Proposition 10.3.3, il existe $r > 0$, une fonction à valeurs projections : P_μ^{h} et une fonction à valeurs matricielles : R_μ^{h} qui sont holomorphes sur $D(0,r)$, tels que pour tout $|z| < r$,

$$P_\mu(z)E(z) = E(z)P_\mu(z) = \mu z P_\mu^{\mathrm{h}}(z) + z^2 R_\mu^{\mathrm{h}}(z). \tag{10.10}$$

Soit $\varphi_0 \neq 0$ dans l'image de $P_\mu^{\mathrm{h}}(0)^*$. Pour nier l'inégalité d'observabilité (10.7), on cherche des solutions $g(t,x)$ du système (10.6) avec des données initiales de la forme $g(0,x) = \sum a_n \mathrm{e}^{inx} P(\mathrm{i}/n)^* \varphi_0$. On suppose dans la suite que les sommes sont finies.

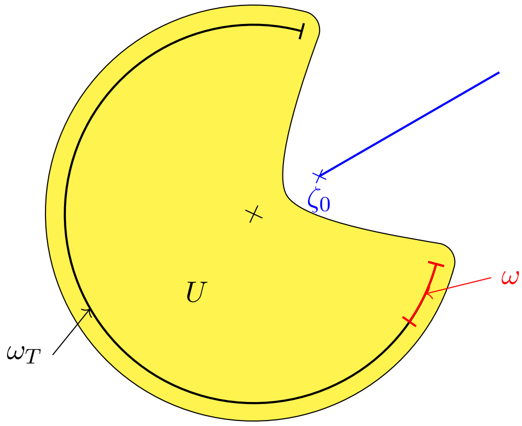

FIGURE 10.1 – Illustration d'un domaine U qui contient ω_T, contenu strictement dans $D(0,1)$.

Comme sur l'image de $P^{\mathrm{h}}_\mu(z)$, $E(z)$ agit comme $\mu z + z^2 R^{\mathrm{h}}_\mu(z)$ par (10.10), on a

$$\begin{aligned} g(t,x) &= \sum a_n \mathrm{e}^{\mathrm{i}nx} \mathrm{e}^{-tn^2 E(\frac{\mathrm{i}}{n})^*} P^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^* \varphi_0 \\ &= \sum a_n \mathrm{e}^{\mathrm{i}nx} \mathrm{e}^{-tn^2 E(\frac{\mathrm{i}}{n})^* P^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*} P^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^* \varphi_0 \\ &= \sum a_n \mathrm{e}^{\mathrm{i}nx} \mathrm{e}^{-tn^2\left(-\mu\frac{\mathrm{i}}{n} P^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^* - \frac{1}{n^2} R^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*\right)} P^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^* \varphi_0 \\ &= \sum a_n \mathrm{e}^{\mathrm{i}n(x+\mu t) + tR^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*} P^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^* \varphi_0. \end{aligned}$$

Ainsi, en définissant pour $0 \leq t \leq T$ et $n \in \mathbb{Z}$,

$$\gamma_t(n) = \mathrm{e}^{tR^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*} P^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^*, \tag{10.11}$$

on écrit $g(t,x)$ sous la forme

$$g(t,x) = \sum a_n \mathrm{e}^{\mathrm{i}n(x+\mu t)} \gamma_t(n) \varphi_0.$$

Si $\gamma_t(n) = 1$, alors $g(t,x)$ serait simplement la solution d'une équation de transport découplée, il serait alors aisé de nier (10.7). Pour traiter ce terme, nous allons utiliser le lemme suivant (pour la preuve, on renvoie à l'Annexe G).

Lemme 10.5.2. *Soit U comme dans la Proposition 10.5.1. Il existe une constante $C > 0$ et un entier N tels que pour tout polynôme $\sum_{n>N} a_n \zeta^n$ où 0 est une racine de multiplicité* *au moins N, pour tout $0 \le \tau \le T$,*

$$\Big| \sum_{n>N} a_n \zeta^n \gamma_\tau(n) \Big|_{L^\infty(\omega_T)} \le C \Big| \sum_{n>N} a_n \zeta^n \Big|_{L^\infty(U)}. \tag{10.12}$$

On suppose que $a_n = 0$ pour $n \le N$. En posant $\zeta(t,x) = \mathrm{e}^{\mathrm{i}(x+\mu t)}$, qui appartient à ω_T pour $(t,x) \in [0,T] \times \omega$, on a

$$g(t,x) = \sum_{n>N} a_n \zeta(t,x)^n \gamma_t(n) \varphi_0.$$

Soit $(t,x) \in [0,T] \times \omega$. En appliquant le Lemme 10.5.2 avec $\tau = t$, on a

$$|g(t,x)| \le C \Big| \sum_{n>N} a_n \zeta^n \Big|_{L^\infty(U)}.$$

Ainsi, le terme de droite de l'inégalité d'observabilité (10.7) satisfait

$$|g|^2_{L^2([0,T]\times\omega)} \le 2\pi T |g|^2_{L^\infty([0,T]\times\omega)} \le 2\pi T C^2 \Big| \sum_{n>N} a_n \zeta^n \Big|^2_{L^\infty(U)}. \tag{10.13}$$

On minore à présent le terme de gauche de (10.7). Grâce à l'égalité de Parseval, on a

$$|g(T,\cdot)|^2_{L^2(\mathbb{T})} = \Big| \sum_{n>N} a_n \mathrm{e}^{\mathrm{i}n(x+\mu T)} \gamma_T(n) \varphi_0 \Big|^2_{L^2(\mathbb{T})} = 2\pi \sum_{n>N} |a_n|^2 |\gamma_T(n)\varphi_0|^2. \tag{10.14}$$

Comme R est holomorphe sur $D(0,r)$, on en déduit que $z \mapsto R^{\mathrm{h}}_\mu(\overline{z})^*$ l'est aussi. En particulier, on a $C_1 := \sup_{|z|\le r/2} |R^{\mathrm{h}}_\mu(z)^*| < +\infty$. Donc, pour $n \ge 2r^{-1}$,

$$\big| \big(\mathrm{e}^{-TR^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*}\big)^{-1} \big| = \big| \mathrm{e}^{TR^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*} \big| \le \mathrm{e}^{C_1 T}. \tag{10.15}$$

De plus, φ_0 est dans l'image de $P^{\mathrm{h}}_\mu(0)^*$ et P^{h}_μ est holomorphe sur $D(0,r)$, donc il existe $r' > 0$ suffisamment petit tel que pour $|z| < r'$,

$$|P^{\mathrm{h}}_\mu(z)^* \varphi_0| \ge |\varphi_0|/2 =: c. \tag{10.16}$$

En rassemblant (10.15) et (10.16), on déduit que pour $n \ge N' := \lfloor \max(2r^{-1}, r'^{-1}) \rfloor + 1$,

$$|\gamma_T(n)\varphi_0| = \left| \mathrm{e}^{-TR^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*} P^{\mathrm{h}}_\mu\Big(\frac{\mathrm{i}}{n}\Big)^* \varphi_0 \right| \ge \mathrm{e}^{-C_1 T} c =: c'.$$

Ainsi, en supposant que $a_n = 0$ pour $n \le N'$, on a alors en utilisant la précédente estimation et l'égalité de Parseval (10.14)

$$\Big| \sum_{n>N} a_n \zeta^n \Big|^2_{L^2(D(0,1))} = \pi \sum_{n>N} \frac{|a_n|^2}{n+1} \le \frac{\pi}{c'} \sum_{n>N} \frac{|a_n|^2}{n+1} |\gamma_T(n)\varphi_0|^2 \le \frac{\pi}{c'} |g(T,\cdot)|^2_{L^2(\mathbb{T})}. \tag{10.17}$$

D'où, grâce à la minoration (10.17) et à la majoration (10.13), l'inégalité d'observabilité (10.7) implique

$$\Big|\sum_{n>N} a_n\zeta^n\Big|^2_{L^2(D(0,1))} \leq C|g(T,\cdot)|^2_{L^2(\{T\})} \leq C'|g|^2_{L^2([0,T]\times\omega)} \leq C''\Big|\sum_{n>N} a_n\zeta^n\Big|^2_{L^\infty(U)},$$

ce qui conclut la preuve de la Proposition 10.5.1. □

Vérifions à présent que l'inégalité de la Proposition 10.5.1 n'est pas vérifiée. Nous allons utiliser le théorème de Runge (voir par exemple le « Big Rudin » [Rud87, Thm. 13.11]) pour construire un contre-exemple.

Proposition 10.5.3 (Théorème de Runge)**.** *Soit U un ensemble ouvert simplement connexe de $\mathbb{C}$ et f une fonction holomorphe sur U. Alors, il existe une suite $(p_k)_{k\geq 0}$ de polynômes qui converge uniformément sur tout compact de U vers f.*

Fin de la preuve du premier point du Théorème 10.2.1. Soit $T < T^*$ et ω_T comme dans la Proposition 10.5.1. Par définition de T^*, ω_T n'est pas le cercle unité tout entier. On peut donc trouver un ouvert borné U, étoilé par rapport à 0 et qui ne contient pas $D(0,1)$ (voir Figure 10.1).

Avec un tel choix de U, il existe un nombre complexe $\zeta_0 \in D(0,1)$ qui n'est pas dans l'adhérence de U. Ainsi, par le théorème de Runge, il existe une suite de polynômes $(\tilde{p}_k)$ qui converge uniformément sur tout compact de $\mathbb{C}\setminus(\zeta_0[1,+\infty))$ vers $\zeta \mapsto (\zeta-\zeta_0)^{-1}$. On pose alors $p_k(\zeta) = \zeta^{N+1}\tilde{p}_k(\zeta)$. D'où, la suite (p_k) fournit un contre-exemple à l'inégalité sur les polynômes complexes (10.9). En effet, comme $\zeta^{N+1}(\zeta-\zeta_0)^{-1}$ est borné sur U, (p_k) est uniformément borné sur U, donc le terme de droite de l'inégalité d'observabilité (10.9) est borné. Mais comme ζ_0 appartient à $D(0,1)$, $\zeta^{N+1}(\zeta-\zeta_0)^{-1}$ a une norme L^2 infinie sur $D(0,1)$, et donc par le lemme de Fatou, $|p_k|_{L^2(D(0,1))}$ tend vers $+\infty$ quand $k \to +\infty$. □

10.6 Preuve du résultat positif de contrôlabilité à zéro en temps long

Le but de cette partie est de prouver le point 2. du Théorème 10.2.1, en adaptant la stratégie de Lebeau et Zuazua (voir [LZ98]), basée sur une décomposition spectrale. Cette analyse spectrale provient d'arguments de théorie de perturbation analytique en dimension finie (voir Section 10.3) et est donc plus générale que celle menée dans [LZ98]. A haute fréquence, le spectre se sépare en une branche parabolique et une branche hyperbolique. En projetant la dynamique sur le sous espace hyperbolique d'une part et le sous espace parabolique d'autre part, le système se décompose en deux sous systèmes faiblement couplés, le premier se comportant comme une équation de transport, le second comme une équation de la chaleur. Le système de transport est traité en utilisant les résultats récents de [ABCO17] (voir Section 10.6.3). Le système parabolique est traitée par une nouvelle méthode par blocs de type Lebeau-Robbiano [LR95] (voir). La partie basse fréquence est traitée par un argument de compacité et une propriété de continuation unique.

Dans toute la Section 10.6, le paramètre $r > 0$ est suffisamment petit pour que les Proposition 10.3.1, Proposition 10.3.2, Proposition 10.3.3 soient vérifiées.

10.6.1 Une décomposition adaptée de $L^2(\mathbb{T})^d$

Proposition 10.6.1. *Soit $n_0 \in \mathbb{N}$ tel que $\frac{1}{n_0} < r$. On a la décomposition suivante*

$$L^2(\mathbb{T})^d = F^0 \oplus F^{\mathrm{p}} \oplus F^{\mathrm{h}}, \tag{10.18}$$

où

$$F^0 := \bigoplus_{|n| \leq n_0} \mathbb{C}^d e_n, \tag{10.19}$$

$$F^{\mathrm{p}} := \bigoplus_{|n| > n_0} \mathrm{Im}\left(P^{\mathrm{p}}\left(\frac{\mathrm{i}}{n}\right)\right) e_n, \tag{10.20}$$

$$F^{\mathrm{h}} := \bigoplus_{|n| > n_0} \mathrm{Im}\left(P^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)\right) e_n. \tag{10.21}$$

De plus, les projections Π^0, Π^{p}, Π^{h} et Π définies par

$$\begin{array}{rcccccl} L^2(\mathbb{T})^d & = & F^0 & \oplus & F^{\mathrm{p}} & \oplus & F^{\mathrm{h}} \\ \Pi^0 & = & I_{F^0} & + & 0 & + & 0 \\ \Pi^{\mathrm{p}} & = & 0 & + & I_{F^{\mathrm{p}}} & + & 0 \\ \Pi^{\mathrm{h}} & = & 0 & + & 0 & + & I_{F^{\mathrm{h}}} \\ \Pi & = & 0 & + & I_{F^{\mathrm{p}}} & + & I_{F^{\mathrm{h}}} = \Pi^{\mathrm{p}} + \Pi^{\mathrm{h}} \end{array}$$

sont des opérateurs bornés sur $L^2(\mathbb{T})^d$.

La preuve de la Proposition 10.6.1 est basée essentiellement sur les Proposition 10.3.1, Proposition 10.3.2 et sur le fait que $(e_n)_{n\in\mathbb{Z}}$ est une base Hilbertienne de $L^2(\mathbb{T})$.

10.6.2 Stratégie de contrôle

Soit T^* défini comme dans (10.2) et T, T' tels que

$$T^* < T' < T. \tag{10.22}$$

Dans cette partie, on considère des contrôles u de la forme

$$u := (u_{\mathrm{h}}, u_{\mathrm{p}})^{\mathrm{tr}} \in \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}, \tag{10.23}$$

où

$$\mathrm{supp}(u_{\mathrm{h}}) \subset [0, T'] \times \overline{\omega}, \qquad \mathrm{supp}(u_{\mathrm{p}}) \subset [T', T] \times \overline{\omega}, \tag{10.24}$$

$$u_{\rm h} \in L^2((0,T') \times \mathbb{T})^{d_1}, \qquad u_{\rm p} \in L^2((T',T) \times \mathbb{T})^{d_2}.$$

Le contrôle $u_{\rm h}$ est supposé contrôler la composante hyperbolique du système tandis que le contrôle $u_{\rm p}$ est supposé agir sur la partie parabolique.

La stratégie de contrôle du système (Sys) consiste à

— d'abord prouver la contrôlabilité à zéro au temps T dans un sous espace de $L^2(\mathbb{T})^d$ de codimension finie,

— ensuite utiliser un argument de continuation unique, pour obtenir la contrôlabilité à zéro dans tout l'espace $L^2(\mathbb{T})^d$.

La première étape de la stratégie est basée sur le résultat suivant.

Proposition 10.6.2. *Il existe un sous espace $\mathcal{G}$ de $L^2(\mathbb{T})^d$ de codimension finie et un opérateur continu*

$$\begin{aligned}\mathcal{U}\colon\ \mathcal{G} &\to L^2((0,T') \times \omega)^{d_1} \times C_c^\infty((T',T) \times \omega)^{d_2}\\ f_0 &\mapsto (u_{\rm h}, u_{\rm p}),\end{aligned}$$

qui, à toute donnée initiale $f_0 \in \mathcal{G}$, associe la paire de contrôles $\mathcal{U} f_0 = (u_{\rm h}, u_{\rm p})$ telle que

$$\forall f_0 \in \mathcal{G},\ \Pi S(T; f_0, \mathcal{U} f_0) = 0. \tag{10.25}$$

Dans la Proposition 10.6.2, on entend par « opérateur continu » : pour tout $s \in \mathbb{N}$, l'application $\mathcal{U} : \mathcal{G} \to L^2((0,T') \times \omega)^{d_1} \times H^s((T',T) \times \omega)^{d_2}$ est continue : il existe $C_s > 0$ tel que

$$\forall f_0 \in \mathcal{G}\,, \quad \|u_{\rm h}\|_{L^2((0,T')\times\omega)^{d_1}} + \|u_{\rm p}\|_{H^s((T',T)\times\omega)^{d_2}} \leqslant C_s \|f_0\|_{L^2(\mathbb{T})^d}\,.$$

La méthode de preuve de la Proposition 10.6.2 consiste à scinder le problème en deux parties :

— pour toute donnée initiale f_0 et tout contrôle parabolique $u_{\rm p}$, amener les hautes fréquences hyperboliques à zéro au temps T (Proposition 10.6.3),

— pour toute donnée initiale f_0 et tout contrôle $u_{\rm h}$, amener les hautes fréquences paraboliques à zéro au temps T (Proposition 10.6.4).

Proposition 10.6.3. *Si n_0 (dans les équations (10.19)-(10.20)) est assez grand, il existe un opérateur continu*

$$\begin{aligned}\mathcal{U}^{\rm h}\colon\ L^2(\mathbb{T})^d \times L^2((T',T) \times \omega)^{d_2} &\to L^2((0,T') \times \omega)^{d_1}\\ (f_0, u_{\rm p}) &\mapsto u_{\rm h},\end{aligned}$$

tel que pour tout $(f_0, u_{\rm p}) \in L^2(\mathbb{T})^d \times L^2((T',T) \times \omega)^{d_2}$,

$$\Pi^{\rm h} S(T; f_0, (\mathcal{U}^{\rm h}(f_0, u_{\rm p}), u_{\rm p})) = 0.$$

Proposition 10.6.4. *Si n_0 (dans les équations (10.19)-(10.20)) est assez grand, il existe un opérateur continu*

$$\begin{aligned}\mathcal{U}^{\mathrm{p}}\colon L^2(\mathbb{T})^d \times L^2((0,T')\times\omega)^{d_1} &\to C_c^\infty((T',T)\times\omega)^{d_2}\\ (f_0,u_{\mathrm{h}}) &\mapsto u_{\mathrm{p}},\end{aligned}$$

tel que pour tout $(f_0,u_{\mathrm{h}}) \in L^2(\mathbb{T})^d \times L^2((0,T')\times\omega)^{d_1}$,

$$\Pi^{\mathrm{p}} S(T; f_0, (u_{\mathrm{h}}, \mathcal{U}^{\mathrm{p}}(f_0,u_{\mathrm{h}})) = 0.$$

Pour la preuve du fait que la Proposition 10.6.3 et la Proposition 10.6.4 impliquent la Proposition 10.6.2, on renvoie à l'Annexe G.

Concernant la contrôlabilité à zéro dans tout l'espace $L^2(\mathbb{T})^d$ à partir de la Proposition 10.6.2 et un argument de continuation unique, on renvoie à l'Annexe G.

Le but des deux prochaines parties est ainsi de démontrer la Proposition 10.6.3 et la Proposition 10.6.4.

10.6.3 Contrôle des hautes fréquences hyperboliques

Le but de cette sous-section est de prouver la Proposition 10.6.3. On rappelle que $T > T' > T^*$ et que le contrôle $u = (u_{\mathrm{h}}, u_{\mathrm{p}})$ vérifie (10.24).

Le but de ce paragraphe est de transformer le problème de contrôlabilité à zéro de la Proposition 10.6.3 en un problème de contrôlabilité exacte pour un système hyperbolique. Précisément, on obtient la Proposition 10.6.3 comme un corollaire du résultat suivant.

Proposition 10.6.5. *Si n_0 (dans les équations (10.19)-(10.20)) est suffisamment grand, alors, pour tout $T > T^*$, il existe un opérateur continu*

$$\begin{aligned}\underline{\mathcal{U}}^{\mathrm{h}}_T\colon F^{\mathrm{h}} &\to L^2((0,T)\times\omega)^{d_1}\\ f_T &\mapsto u_{\mathrm{h}},\end{aligned}$$

tel que pour tout $f_T \in F^{\mathrm{h}}$,

$$\Pi^{\mathrm{h}} S\left(T; 0, (\underline{\mathcal{U}}^{\mathrm{h}}_T(f_T), 0)\right) = f_T.$$

Pour la preuve de la Proposition 10.6.3 à partir de la Proposition 10.6.5, on renvoie à l'Annexe G.

Le but de ce paragraphe est de prouver la Proposition 10.6.5. Par la « HUM », la Proposition 10.6.5 est équivalente à l'inégalité d'observabilité suivante (c'est une adaptation du [Cor07a, Theorem 2.42]).

Proposition 10.6.6. *Il existe une constante $C > 0$ telle que pour tout $g_0 \in \widetilde{F^{\mathrm{h}}} := \mathrm{Im}\left((\Pi^{\mathrm{h}})^*\right)$, la solution g de (10.6) satisfait*

$$\|g_0\|^2_{L^2(\mathbb{T})^d} \leq C \int_0^{T'} \int_\omega |g_1(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x, \tag{10.26}$$

où g_1 désigne les premières d_1 composantes de g.

Démonstration. Soit $g_0 \in \widetilde{F^{\mathrm{h}}}$. En utilisant la définition de F^{h} (10.21) et la Proposition 10.3.3, g_0 se décompose de la manière suivante

$$g_0 = \sum_{\mu \in \mathrm{Sp}(A')} \sum_{|n|>n_0} P_\mu^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)^* \widehat{g}_0(n) e_n. \tag{10.27}$$

Ainsi, la solution g de (10.6) est

$$g(t) = \sum_{\mu \in \mathrm{Sp}(A')} \sum_{|n|>n_0} \mathrm{e}^{-tn^2 E\left(\frac{\mathrm{i}}{n}\right)^*} P_\mu^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)^* \widehat{g}_0(n). \tag{10.28}$$

Pour $\mu \in \mathrm{Sp}(A')$, on définit

$$g_\mu(t) = \sum_{|n|>n_0} \mathrm{e}^{-tn^2 E\left(\frac{\mathrm{i}}{n}\right)^*} P_\mu^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)^* \widehat{g}_0(n) e_n. \tag{10.29}$$

En utilisant (i) et (iii) de la Proposition 10.3.3, on a

$$\mathrm{e}^{-tn^2 E\left(\frac{\mathrm{i}}{n}\right)^*} P_\mu^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)^* = \mathrm{e}^{-tn^2\left(\mu\overline{\frac{\mathrm{i}}{n}} + \left(\frac{\mathrm{i}}{n}\right)^2 R_\mu^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)\right)^*} P_\mu^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)^* = \mathrm{e}^{-t\mu \mathrm{i} n + tR_\mu^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)^*} P_\mu^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)^*,$$

ce qui donne

$$\begin{cases} \partial_t g_\mu - \mu \partial_x g_\mu + R_\mu^{\mathrm{h}}(0)^* g_\mu = S_\mu & \text{dans } Q_{T'}, \\ g_\mu(0,.) = g_{\mu 0} & \text{dans } \mathbb{T}, \end{cases} \tag{10.30}$$

où

$$\|S_\mu\|_{L^2(Q_{T'})^d} \leq C \left(\sum_{|n|>n_0} \frac{|\widehat{g}_0(n)|^2}{n^2} \right)^{1/2}. \tag{10.31}$$

Si le terme S_μ était nul, le système (10.30) serait observable au temps $T_\mu := \frac{2\pi - |\omega|}{|\mu|}$ (voir [ABCO17, Theorem 2.2]). Ainsi, en utilisant (10.22) et (10.2), on obtient l'inégalité d'observabilité

$$\|g_{\mu 0}\|^2_{L^2(\mathbb{T})^d} \leq C \left(\int_0^{T'} \int_\omega |g_\mu(t,x)|^2 \,\mathrm{d}t \,\mathrm{d}x + \|S_\mu\|^2_{L^2(Q_{T'})^d} \right). \tag{10.32}$$

En utilisant (10.32) et (10.31), on a

$$\|g_{\mu 0}\|^2_{L^2(\mathbb{T})^d} \leq C \left(\int_0^{T'} \int_\omega |g_\mu(t,x)|^2 \,\mathrm{d}t \,\mathrm{d}x + \sum_{|n|>n_0} \frac{|\widehat{g}_0(n)|^2}{n^2} \right). \tag{10.33}$$

De plus, grâce à (i), (ii) de la Proposition 10.3.3, (10.28) et (10.29), on obtient

$$g_\mu(t) = P_\mu^{\mathrm{h}}(0)^* g(t) + Q_\mu(t),$$

avec Q_μ satisfaisant l'estimation (10.31). Ainsi, on a

$$\int_0^{T'}\int_\omega |g_\mu(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x \leq C\left(\int_0^{T'}\int_\omega |P_\mu^{\mathrm{h}}(0)^* g(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x + \sum_{|n|>n_0}\frac{|\widehat{g}_0(n)|^2}{n^2}\right). \quad (10.34)$$

D'où, par (10.33) et (10.34), on a

$$\|g_{\mu 0}\|^2_{L^2(\mathbb{T})^d} \leq C\left(\int_0^{T'}\int_\omega |P_\mu^{\mathrm{h}}(0)^* g(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x + \sum_{|n|>n_0}\frac{|\widehat{g}_0(n)|^2}{n^2}\right). \quad (10.35)$$

En sommant sur $\mu \in \mathrm{Sp}(A')$ l'estimation (10.35) et en utilisant (ii) de la Proposition 10.3.3, Proposition 10.3.1 et (10.27),on obtient l'inégalité d'observabilité faible

$$\|g_0\|^2_{L^2(\mathbb{T})^d} \leq C\left(\int_0^{T'}\int_\omega |g_1(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x + \|g_{T'}\|^2_{H^{-1}(\mathbb{T})}\right). \quad (10.36)$$

Par (10.36) et l'injection compacte $L^2(\mathbb{T}) \hookrightarrow H^{-1}(\mathbb{T})$, un argument de compacité-unicité donne l'inégalité d'observabilité (10.26) (pour les détails, on renvoie à l'Annexe G). □

10.6.4 Contrôle des hautes fréquences paraboliques

Le but de cette sous-partie est de prouver la Proposition 10.6.4. On rappelle que $0 < T' < T$ sont choisis tels que (10.22) est vérifiée et le contrôle u est tel que (10.23) et (10.24) sont vérifiées.

La stratégie est la suivante : identifier l'équation satisfaite par les d_2 composantes de l'équation parabolique (10.6) en utilisant l'asymptotique de la Proposition 10.3.3, puis construire des contrôles réguliers en adaptant la méthode de Lebeau-Robbiano aux systèmes.

Dans cette partie, pour tout vecteur $\varphi \in \mathbb{C}^d$, on désigne par φ_1 ses premières d_1 composantes et φ_2 ses dernières d_2 composantes.

Le but de ce paragraphe est de transformer le problème de contrôlabilité à zéro de la Proposition 10.6.4 en un problème de contrôlabilité à zéro pour un système parabolique. Précisément, on peut prouver que la Proposition 10.6.4 est une conséquence du résultat suivant (voir l'Annexe G).

Proposition 10.6.7. *Si n_0 est suffisamment grand, alors pour tout $T > 0$, il existe un opérateur continu*

$$\begin{aligned}\underline{\mathcal{U}}_T^{\mathrm{p}}\colon F^{\mathrm{p}} &\to C_c^\infty((0,T)\times\omega)^{d_2}\\ f_0 &\mapsto u_{\mathrm{p}},\end{aligned}$$

tel que pour tout $f_0 \in F^{\mathrm{p}}$,

$$\Pi^{\mathrm{p}} S(T; f_0, (0, \underline{\mathcal{U}}_T^{\mathrm{p}}(f_0))) = 0.$$

La Proposition 10.6.7 est démontrée grâce à une adaptation de la méthode de Lebeau-Robbiano.

On commence par prouver que si g est dans $\widetilde{F^{\mathrm{p}}} := \mathrm{Im}\left((\Pi^{\mathrm{p}})^*\right)$, alors on peut calculer ses d_1 premières composantes à partir de ses d_2 dernières. Ce qui nous permet d'écrire une équation découplée portant sur les d_2 dernières composantes de g.

Proposition 10.6.8. *Si z est suffisamment petit, alors il existe une matrice $G(z)$ telle que pour tout $\varphi \in \mathbb{C}^d$,*

$$\varphi \in \mathrm{Im}(P^{\mathrm{p}}(z)^*) \Leftrightarrow \varphi_1 = G(z)\varphi_2.$$

De plus G est holomorphe en z et $G(0) = 0$.

Démonstration. On écrit

$$P^{\mathrm{p}}(z)^* = \begin{pmatrix} p_{11}(z) & p_{12}(z) \\ p_{21}(z) & p_{22}(z) \end{pmatrix}.$$

Comme $P^{\mathrm{p}}(z)^*$ est une projection, φ est dans $\mathrm{Im}(P^{\mathrm{p}}(z)^*)$ si et seulement si

$$\begin{cases} p_{11}(z)\varphi_1 + p_{12}(z)\varphi_2 = \varphi_1 \\ p_{21}(z)\varphi_1 + p_{22}(z)\varphi_2 = \varphi_2. \end{cases}$$

En particulier, si $\varphi \in \mathrm{Im}(P^{\mathrm{p}}(z)^*)$, alors $(I_{d_1} - p_{11}(z))\varphi_1 = p_{12}(z)\varphi_2$. Et comme $P^{\mathrm{p}}(0)^* = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{d_2} \end{smallmatrix}\right)$ (voir Proposition 10.3.2), $p_{11}(0) = 0$, et donc, si z est suffisamment petit, $|p_{11}(z)| < 1$ et $I_{d_1} - p_{11}(z)$ est inversible.

Dans ce cas, $\varphi_1 = (I_{d_1} - p_{11}(z))^{-1}p_{12}(z)\varphi_2$. Cela montre que l'application

$$\varphi \in \mathrm{Im}(P^{\mathrm{p}}(z)^*) \mapsto \varphi_2 \in \mathbb{C}^{d_2}$$

est injective. Mais le rang de $P^{\mathrm{p}}(z)^*$ ne dépend pas z, et donc vaut toujours d_2. D'où l'application précédente est bijective. On note $G(z)$ les premières d_1 composantes de son inverse. Remarquons que $G(z) = (I_{d_1} - p_{11}(z))^{-1}p_{12}(z)$. D'où, si $\varphi \in \mathrm{Im}(P^{\mathrm{p}}(z)^*)$, on a

$$\varphi = (\varphi_1, \varphi_2) = (G(z)\varphi_2, \varphi_2).$$

Pour démontrer la réciproque, on remarque que l'inverse de $\varphi \in \mathrm{Im}(P^{\mathrm{p}}(z)^*) \mapsto \varphi_2$ est $\varphi_2 \in \mathbb{C}^{d_2} \mapsto (G(z)\varphi_2, \varphi_2)$. □

En augmentant n_0 si nécessaire, on peut supposer que si $|n| > n_0$, $G(\mathrm{i}/n)$ est bien défini. Ainsi, on définit l'opérateur borné G de $L^2(\mathbb{T}, \mathbb{C}^{d_2})$ dans $L^2(\mathbb{T}, \mathbb{C}^{d_1})$ par

$$G\left(\sum_{n\in\mathbb{Z}} \varphi_{n,2}e_n\right) = \sum_{|n|>n_0} G\left(\frac{\mathrm{i}}{n}\right)\varphi_{n,2}e_n. \tag{10.37}$$

Ainsi, grâce à la définition de $\widetilde{F^{\mathrm{p}}}$, on a le corollaire suivant qui nous permet de calculer les d_1 premières composantes à partir des d_2 dernières.

Corollaire 10.6.9. *Pour tout $g \in (F^0)^\perp$, on a l'équivalence $g \in \widetilde{F^{\mathrm{p}}} \Leftrightarrow g_1 = Gg_2$.*

Le Corollaire 10.6.9 nous permet d'écrire une équation satisfaite par les d_2 composantes de la solution du système adjoint (10.6) si la donnée initiale est dans $\widetilde{F^{\mathrm{p}}}$.

Proposition 10.6.10. *On définit l'opérateur $\mathfrak{D}$ par*

$$D(\mathfrak{D}) = H^2(\mathbb{T})^{d_2}, \quad \mathfrak{D} = D^{\mathrm{tr}}\partial_x^2 + A_{22}^{\mathrm{tr}}\partial_x - K_{22}^{\mathrm{tr}} + A_{12}^{\mathrm{tr}}\partial_x G - K_{12}^{\mathrm{tr}} G. \tag{10.38}$$

Soit $g_0 \in \widetilde{F^{\mathrm{p}}}$ et $g(t) = \mathrm{e}^{-t\mathcal{L}^*} g_0$. Alors, pour tout $t \geq 0$, $g_1(t) = Gg_2(t)$ et g_2 satisfait l'équation

$$\partial_t g_2(t,x) - \mathfrak{D} g_2(t,x) = 0 \qquad \text{dans } (0,T) \times \mathbb{T}. \tag{10.39}$$

Démonstration. La fonction g satisfait le système

$$(\partial_t - B^{\mathrm{tr}}\partial_x^2 - A^{\mathrm{tr}}\partial_x + K^{\mathrm{tr}})g(t,x) = 0 \qquad \text{dans } (0,T) \times \mathbb{T}.$$

Si on prend les d_2 composantes de ce système, on obtient que dans $(0,T) \times \mathbb{T}$,

$$\left(\partial_t - D^{\mathrm{tr}}\partial_x^2 - A_{22}^{\mathrm{tr}} + K_{22}^{\mathrm{tr}}\right) g_2(t,x) - \left(A_{12}^{\mathrm{tr}}\partial_x - K_{12}^{\mathrm{tr}}\right) g_1(t,x) = 0\,. \tag{10.40}$$

Comme pour tout $t \in [0,T]$, $g(t,\cdot) \in \widetilde{F^{\mathrm{p}}}$, on en déduit grâce au Corollaire 10.6.9, $g_1(t) = Gg_2(t)$. En reportant ceci dans l'équation (10.40), on obtient l'équation voulue satisfaite par g_2, i.e. (10.39). □

Ainsi, grâce à la Proposition 10.6.10, en exploitant le fait que g_2 satisfait une équation parabolique et en utilisant une méthode de Lebeau-Robbiano adaptée au cas des systèmes, on peut démontrer la Proposition 10.6.7. Pour les détails, on renvoie à l'Annexe G.

10.7 Perspectives et problèmes ouverts

Une première perspective que nous sommes en train d'aborder à l'heure actuelle avec Karine Beauchard et Armand Koenig est la suivante. Nous essayons de voir s'il est possible sous certaines hypothèses sur nos matrices (A, B, K) de retirer certains contrôles dans le système (Sys) tout en préservant la contrôlabilité à zéro en temps $T > T^*$ établis dans le Théorème 10.2.1. Nous voulons en particulier retirer le contrôle sur la composante hyperbolique (respectivement parabolique) comme Gilles Lebeau et Enrique Zuazua le font dans [LZ98] et déterminer la condition nécessaire et suffisante sur le triplet (A, B, K) pour que ceci soit possible.

Une seconde perspective pourrait être d'étudier le comportement du coût de contrôle quand ε tend vers 0 du système

$$\begin{cases} \partial_t f - B_\varepsilon \partial_{xx} f + A\partial_x f + Kf = u 1_\omega & \text{dans } (0,T) \times \mathbb{T}, \\ f(0,.) = f_0 & \text{dans } \mathbb{T}, \end{cases} \tag{Sys-ε}$$

où $B_\varepsilon = \begin{pmatrix} \varepsilon I_{d_1} & 0 \\ 0 & D \end{pmatrix}$. Pour cela, on pourrait s'inspirer des articles [CG05], [GL07] et des travaux qui ont suivi concernant la contrôlabilité d'une équation de transport en limite de viscosité évanescente.

Annexe A

Boîte à outils pour les équations paraboliques

Le but de cette section est d'énoncer des résultats classiques pour des équations paraboliques linéaires et non linéaires. Ces résultats sont énoncés dans le cas de l'équation de la chaleur avec potentiel ou pour l'équation de la chaleur semilinéaire dans le cas de conditions de Dirichlet. Ils s'adaptent au cas de conditions au bord de Neumann et aussi dans le cas des systèmes paraboliques.

Parmi les nombreuses propriétés des équations paraboliques, nous allons présenter les suivantes :

- caractère bien posé linéaire,
- caractère bien posé non linéaire,
- dissipation en temps,
- principe du maximum,
- principe de comparaison de solutions dans le cas non linéaire,
- inégalité de Harnack,
- principe du maximum fort,
- unicité rétrograde,
- analyticité,
- estimations L^p-L^q,
- régularité maximale L^p,
- régularité maximale Holdërienne.

A.1 Équations paraboliques linéaires : caractère bien posé au sens d'Hadamard

On introduit l'espace fonctionnel

$$W_T := L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \tag{A.1}$$

qui satisfait l'injection suivante (voir [Eva10, Section 5.9.2, Theorem 3])

$$W_T \hookrightarrow C([0,T];L^2(\Omega)). \tag{A.2}$$

Définition A.1.1. Soit $a \in L^\infty(Q_T)$, $F \in L^2(Q_T)$ et $y_0 \in L^2(\Omega)$. Une fonction $y \in W_T$ est solution de

$$\begin{cases} \partial_t y - \Delta y + a(t,x)y = F & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega, \end{cases} \tag{A.3}$$

si pour tout $w \in L^2(0,T;H_0^1(\Omega))$,

$$\int_0^T (\partial_t y, w)_{(H^{-1}(\Omega),H_0^1(\Omega))} + \int_{Q_T} \nabla y.\nabla w + \int_{Q_T} ayw = \int_{Q_T} Fw, \tag{A.4}$$

et

$$y(0,.) = y_0 \text{ dans } L^2(\Omega). \tag{A.5}$$

Remarque A.1.2. L'évaluation en 0 de $y \in W_T$ dans (A.5) est justifiée par l'injection (A.2).

L'équation (A.3) est bien posé au sens d'Hadamard. En effet, nous avons le résultat suivant.

Proposition A.1.3. *Soit $a \in L^\infty(Q_T)$, $F \in L^2(Q_T)$ et $y_0 \in L^2(\Omega)$. Le problème de Cauchy* (A.3) *admet une unique solution $y \in W_T$. De plus, il existe $C = C(\Omega) > 0$ tel que*

$$\|y\|_{W_T} \leq C \exp\left(CT\|a\|_{L^\infty(Q_T)}\right)\left(\|y_0\|_{L^2(\Omega)} + \|F\|_{L^2(Q_T)}\right). \tag{A.6}$$

La preuve de la Proposition A.1.3 est basée sur des *approximations de Galerkin, des estimations d'énergie et des arguments de type Gronwall* (voir [Eva10, Section 7.1.2]).

Nous avons également une estimation L^∞ pour les solutions de (A.3).

Proposition A.1.4. *Soit $a \in L^\infty(Q_T)$, $F \in L^\infty(Q_T)$ et $y_0 \in L^\infty(\Omega)$. Alors la solution y de* (A.3) *appartient à $L^\infty(Q_T)$ et il existe $C = C(\Omega) > 0$ tel que*

$$\|y\|_{L^\infty(Q_T)} \leq C \exp\left(CT\|a\|_{L^\infty(Q_T)}\right)\left(\|y_0\|_{L^\infty(\Omega)} + \|F\|_{L^\infty(Q_T)}\right). \tag{A.7}$$

La preuve de la Proposition A.1.4 est basée sur la *méthode de Stampacchia* (voir la preuve de [LSU68, Chapter 3, Paragraph 7, Theorem 7.1] ou la preuve de [WYW06, Proposition 4.2.1]).

Les solutions de (A.3) dissipent en temps au sens suivant.

Proposition A.1.5. *Soit $a \in L^\infty(Q_T)$, $y_0 \in L^2(\Omega)$ et $t_1 < t_2 \in [0,T]$. Alors il existe $C = C(\Omega) > 0$ tel que la solution $y \in W_T$ de* (A.3) *avec $F = 0$, satisfait*

$$\|y(t_2,.)\|_{L^2(\Omega)} \leq \exp\left(CT\|a\|_{L^\infty(Q_T)}\right)\|y(t_1,.)\|_{L^2(\Omega)}. \tag{A.8}$$

La preuve de la Proposition A.1.5 est basée sur l'application de la formulation variationnelle (A.4) avec $w = y$ et d'un argument de type Gronwall.

Remarque A.1.6. On dispose également d'un caractère bien posé au sens d'Hadamard pour l'équation
$$\partial_t y - \Delta y + \nabla.(B(t,x)y) + a(t,x)y = 0,$$
où $a \in L^\infty(Q_T)$ et $B \in L^\infty(Q_T)^N$. L'estimation de dissipation semblable à la Proposition A.1.5 s'écrit dans ce cas (voir [DFCGBZ02])
$$\|y(t_2,.)\|_{L^2(\Omega)} \leq \exp\left(CT\left(\|a\|_{L^\infty(Q_T)} + \|B\|^2_{L^\infty(Q_T)}\right)\right)\|y(t_1,.)\|_{L^2(\Omega)}.$$

A.2 Équations paraboliques non linéaires : caractère bien posé

Nous donnons la définition d'une solution de
$$\begin{cases} \partial_t y - \Delta y + f(y) = h1_\omega & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega, \end{cases} \tag{A.9}$$
où $f \in C^1(\mathbb{R};\mathbb{R})$.

Définition A.2.1. Soit $y_0 \in L^\infty(\Omega)$, $h \in L^\infty(Q_T)$. Une fonction $y \in W_T \cap L^\infty(Q_T)$ est solution de (A.9) si pour tout $w \in L^2(0,T;H^1_0(\Omega))$,
$$\int_0^T (\partial_t y, w)_{(H^{-1}(\Omega),H^1_0(\Omega))} + \int_{Q_T} \nabla y.\nabla w + \int_{Q_T} f(y)w = \int_{Q_T} h1_\omega w, \tag{A.10}$$
et
$$y(0,.) = y_0 \text{ dans } L^\infty(\Omega). \tag{A.11}$$

Remarque A.2.2. L'unicité de la solution de (A.9) provient du fait que f est localement Lipschitzienne car $f \in C^1(\mathbb{R};\mathbb{R})$.

A.3 Principe du maximum et de comparaison

Énonçons dans un premier temps le principe du maximum.

Proposition A.3.1. *Soit $a \in L^\infty(Q_T)$, $F \leq G \in L^2(Q_T)$ et $y_0 \leq z_0 \in L^2(\Omega)$. Soit y et z deux solutions de*
$$\begin{cases} \partial_t y - \Delta y + a(t,x)y = F, \\ y = 0, \\ y(0,.) = y_0, \end{cases} \qquad \begin{cases} \partial_t z - \Delta z + a(t,x)z = G & \text{dans } (0,T)\times\Omega, \\ y = 0 & \text{sur } (0,T)\times\partial\Omega, \\ z(0,.) = z_0 & \text{dans } \Omega. \end{cases} \tag{A.12}$$
Alors, on a le principe de comparaison suivant
$$\forall t \in [0,T],\ \ a.e.\ x \in \Omega,\ \ y(t,x) \leq z(t,x). \tag{A.13}$$

La preuve de la Proposition A.3.1 est basée sur le principe de comparaison pour des solutions régulières de (A.12) (voir [WYW06, Theorem 8.1.6]) et sur un argument de régularisation.

Remarque A.3.2. Si $y_0 \in L^2(\Omega)$ est négative alors la Proposition A.3.1 affirme que la solution y de l'équation de la chaleur associée reste négative (prendre $z = 0$ dans la Proposition A.3.1). C'est le principe du maximum faible.

Le résultat suivant assure l'existence d'une solution à l'équation (A.9) sans contrôle h sous réserve d'existence préalable d'une sous-solution et d'une sur-solution. De plus, la solution est alors comprise entre la sous-solution et la sur-solution.

Proposition A.3.3. *Soit $y_0 \in L^\infty(\Omega)$, $h = 0$. On suppose qu'il existe une sous-solution $\underline{y}$ et une sur-solution $\overline{y}$ in $L^\infty(Q_T)$ de (A.9), i.e., $\underline{y}$ (respectivement $\overline{y}$) satisfait (A.10), (A.11) en remplaçant l'égalité $=$ par l'inégalité $\leq$ (respectivement par l'inégalité $\geq$). De plus, on suppose que $\underline{y}$ et $\overline{y}$ sont ordonnées dans le sens suivant :*

$$\forall t \in [0, T], \ \ a.e.\ x \in \Omega, \ \ \underline{y}(t, x) \leq \overline{y}(t, x).$$

Alors, il existe une (unique) solution y de (A.9). De plus, y satisfait le principe de comparaison

$$\forall t \in [0, T], \ \ a.e.\ x \in \Omega, \ \ \underline{y}(t, x) \leq y(t, x) \leq \overline{y}(t, x). \tag{A.14}$$

Pour la preuve de la Proposition A.3.3, voir [WYW06, Corollary 12.1.1].

A.4 Inégalité de Harnack et principe du maximum fort

Nous présentons dans un premier temps l'inégalité de Harnack (voir [Eva10, Chapter 7, Section 7.1, Theorem 10] et aussi [Lie96, Chapter 2] pour le cas $a \in L^\infty(Q_T)$).

Proposition A.4.1. *Soit $a \in L^\infty(Q_T)$, $y_0 \in L^2(\Omega)$. On suppose que $y \in W_T$, solution de (A.3), appartient à $C^{1,2}(Q_T)$ et $y \geq 0$ dans Q_T. Alors, pour tout ouvert V connexe tel que $V \subset\subset \Omega$, pour tous $0 < t_1 < t_2 \leq T$, il existe une constante $C = C(V, t_1, t_2, a) > 0$ telle que*

$$\sup_V y(t_1, .) \leq C \inf_V y(t_2, .).$$

On peut déduire de la Proposition A.4.1 le principe du maximum fort (voir [Eva10, Chapter 7, Section 7.1, Theorem 11]).

Proposition A.4.2. *Soit $a \in L^\infty(Q_T)$. On suppose que $y \in W_T$, solution de (A.3), appartient à $C^{1,2}(Q_T) \cap C(\overline{Q_T})$ et on note*

$$M := \max_{\overline{Q_T}} y.$$

Si $y(t_0, x_0) = M$ pour $(t_0, x_0) \in Q_T$, alors $y \equiv M$ sur $(0, t_0) \times \Omega$.

Remarque A.4.3. D'après la Remarque A.3.2, si la donnée initiale y_0 est négative, régulière, alors la solution y de l'équation de la chaleur associée est négative. Si on suppose en plus que y_0 est strictement négative à un certain endroit de Ω, la Proposition A.4.2 affirme en plus que pour tout $(t,x) \in (0,T] \times \Omega$, $y(t,x) < 0$. Ceci exprime la *vitesse de propagation infinie* de l'équation de la chaleur.

A.5 Unicité rétrograde et analyticité pour l'équation de la chaleur

Dans cette partie, on considère l'équation de la chaleur libre :

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0,.) = y_0 & \text{dans } \Omega. \end{cases} \tag{A.15}$$

Nous disposons du résultat suivant d'unicité rétrograde.

Proposition A.5.1. *Soit $y_0 \in L^2(\Omega)$ et y l'unique solution de (A.15). Alors, on a*

$$\Big(y(T,.) = 0\Big) \Rightarrow \Big(\forall t \in [0,T],\ y(t,.) = 0\Big).$$

Pour la preuve de la Proposition A.5.1, voir [Eva10, Chapter 2, Section 2.3, Theorem 11].

Nous disposons du résultat suivant d'analyticité.

Proposition A.5.2. *Soit $y_0 \in L^2(\Omega)$ et y l'unique solution de (A.15). Alors pour tout $t \in (0,T]$, $x \in \Omega \mapsto y(t,x)$ est analytique.*

Pour la preuve de la Proposition A.5.2, voir [Mik78, Chapter 6, Section 1, Theorem 1].

A.6 Estimations de régularité

Tout d'abord, nous avons les effets régularisants L^p-L^q du semi-groupe de la chaleur (voir [CH98, Proposition 3.5.7]).

Proposition A.6.1. *Soit $1 \leq q \leq p \leq +\infty$, $y_0 \in L^2(\Omega)$ et y la solution de (A.15). Alors il existe $C = C(\Omega,p,q) > 0$ tel que pour tous $t_1 < t_2 \in (0,T)$, on a*

$$\|y(t_2,.)\|_{L^p(\Omega)} \leq C(t_2 - t_1)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|y(t_1,.)\|_{L^q(\Omega)}. \tag{A.16}$$

Pour $p \in [1,+\infty]$, introduisons l'espace fonctionnel

$$X_{T,p} = L^p(0,T;W^{2,p} \cap W_0^{1,p}(\Omega)) \cap W^{1,p}(0,T;L^p(\Omega)).$$

On dispose du résultat de régularité maximale dans L^p (voir [QS07, Theorem 48.1]).

Proposition A.6.2. *Soit* $1 < p < +\infty$, $a \in L^\infty(Q_T)$, $F \in L^p(Q_T)$, $y_0 \in C_c^\infty(\Omega)$. *L'unique solution* $y \in W_T$ *de* (A.3) *appartient à* $X_{T,p}$. *De plus, si* $y_0 = 0$, *il existe* $C = C(\Omega, T, a) > 0$ *tel que*

$$\|y\|_{X_{T,p}} \leq C \|F\|_{L^p(Q_T)} .$$

Pour gagner en régularité dans les espaces L^p à l'aide de la Proposition A.6.2 et d'un argument de bootstrap, le lemme suivant est très utile (voir [LSU68, Lemma 3.3, page 80] ou [WYW06, Chapter 1, Section 1.4.1, Theorem 1.4.1]).

Lemme A.6.3. *Soit* $p \in [1, +\infty)$. *On a*

$$X_{T,p} \hookrightarrow \begin{cases} L^{\frac{(N+2)p}{N+2-p}}(0,T;W_0^{1,\frac{(N+2)p}{N+2-p}}(\Omega)) & \text{si } p < N+2, \\ L^{2p}(0,T;W_0^{1,2p}(\Omega)) & \text{si } p = N+2, \\ C^{\alpha/2,\alpha}(Q_T),\ 0 < \alpha \leq 1 - \frac{N+2}{p} & \text{si } p > N+2. \end{cases}$$

De manière similaire à la Proposition A.6.2, on dispose du résultat suivant de régularité maximale dans les espaces de Hölder (voir [QS07, Theorem 48.2]).

Proposition A.6.4. *Soit* $\alpha \in (0,1)$ *et* Ω *un ouvert borné connexe de classe* $C^{2+\alpha}$. *Soit* $a \in C^{\alpha/2,\alpha}(\overline{Q_T})$, $F \in C^{\alpha/2,\alpha}(\overline{Q_T})$ *et* $y_0 \in C_c^\infty(\Omega)$. *L'unique solution* $y \in W_T$ *de* (A.3) *appartient à* $C^{1+\alpha/2,2+\alpha}(\overline{Q_T})$. *De plus, si* $y_0 = 0$, *il existe* $C = C(\Omega, T, a) > 0$ *tel que*

$$\|y\|_{C^{1+\alpha/2,2+\alpha}(\overline{Q_T})} \leq C \|F\|_{C^{\alpha/2,\alpha}(\overline{Q_T})} .$$

On peut déduire des Proposition A.6.2, Proposition A.6.4 et d'arguments de troncature le corollaire suivant.

Corollaire A.6.5. *On suppose que* Ω *est de classe* C^∞ *et* $\omega \subset\subset \Omega$. *Soit* $y_0 \in L^2(\Omega)$ *et* $h \in L^2(q_T)$. *Alors la solution* y *de* (A.3) *avec* $a = 0$ *et* $F = h1_\omega$ *vérifie*

$$\forall t \in (0,T],\ x \mapsto y(t,x) \in C^\infty(\Omega \setminus \overline{\omega}).$$

Annexe B

Controllability of a 4×4 quadratic reaction-diffusion system

Abstract : We consider a 4×4 nonlinear reaction-diffusion system posed on a smooth domain Ω of $\mathbb{R}^N$ ($N \geq 1$) with controls localized in some arbitrary nonempty open subset ω of the domain Ω. This system is a model for the evolution of concentrations in reversible chemical reactions. We prove the local exact controllability to stationary constant solutions of the underlying reaction-diffusion system for every $N \geq 1$ in any time $T > 0$. A specificity of this control system is the existence of some invariant quantities in the nonlinear dynamics. The proof is based on a linearization which uses return method and an adequate change of variables that creates cross diffusion which will be used as coupling terms of second order. The controllability properties of the linearized system are deduced from Carleman estimates. A Kakutani's fixed-point argument enables to go back to the nonlinear parabolic system. Then, we prove a global controllability result in large time for $1 \leq N \leq 2$ thanks to our local controllabillity result together with a known theorem on the asymptotics of the free nonlinear reaction-diffusion system.

B.1 Introduction

Let $T > 0$, $N \in \mathbb{N}^*$, Ω be a bounded, connected, open subset of $\mathbb{R}^N$ of class C^2 and let ω be a nonempty open subset of Ω. The notation $Q := (0,T) \times \Omega$ (or Q_T) will be used throughout the paper.

B.1.1 Presentation of the nonlinear reaction-diffusion system

Let $(d_1, d_2, d_3, d_4) \in (0, +\infty)^4$. We are interested in the following reaction-diffusion system

$$\begin{cases} \forall 1 \leq i \leq 4, & \\ \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) & \text{in } (0,T) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega, \end{cases} \tag{B.1}$$

where n is the outer unit normal vector to $\partial\Omega$. This system is a model for the evolution of the concentration $u_i(.,.)$ in the reversible chemical reaction

$$U_1 + U_3 \rightleftharpoons U_2 + U_4, \tag{B.2}$$

by using the law of mass action, Fick's law and the fact that no substance crosses the boundary (Neumann conditions). For this quadratic system, global existence of weak solutions holds in any dimension.

Proposition B.1.1. [Pie10, Proposition 5.12]
Let $u_0 \in L^2(\Omega)^4$, $u_0 \geq 0$. Then, there exists a global weak solution (in the sense of the definition [Pie10, Section 5, (5.12)]) to (B.1).

For dimensions $N = 1,\ 2$, it was proved that the solutions are bounded and therefore classical for bounded initial data (see [DF06], [GV10] and [HM96]). It was not known until recently whether they were bounded in higher dimension (see [Pie10, Section 7, Problem 3] and references therein for more details). But, two very recent preprints : [CGV17] and [Sou18] prove that these solutions are smooth.

B.1.2 The question

Let $(u_1^*, u_2^*, u_3^*, u_4^*) \in [0, +\infty)^4$ satisfying

$$u_1^* u_3^* = u_2^* u_4^*. \tag{B.3}$$

We will say that $(u_i^*)_{1\leq i\leq 4}$ is a stationary constant solution of (B.1).

Remark B.1.2. The nonnegative stationary solutions of (B.1) are constant (see Proposition B.6.1 in Annexe B.7). Thus, it is not restrictive to assume that $(u_1^*, u_2^*, u_3^*, u_4^*) \in [0, +\infty)^4$.

The **question** we ask is the following : **Could one reach stationary constant solutions of (B.1) with localized controls in finite time ?** From a chemical viewpoint, we wonder whether one can act on the free reaction (B.2) by a localized external force to reach in finite time T a particular steady state $(u_i^*)_{1\leq i\leq 4}$. For instance, this force can be the addition or the removal of a chemical species in a specific location of the domain Ω.

We introduce the notations :

$$j \in \{1, 2, 3\} \text{ denotes the number of internal controls,}$$

$$1_{i\leq j} := 1 \text{ if } 1 \leq i \leq j \text{ and } 0 \text{ if } i > j.$$

By symmetry of the system, we reduce our study to the case of controls entering in the first equations. Thus, we consider the following controlled system

$$\begin{cases} \forall 1 \leq i \leq 4, & \\ \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) + h_i 1_\omega 1_{i\leq j} & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{B.4}$$

Here, $(u_i)_{1\leq i\leq 4}(t,.) : \Omega \to \mathbb{R}^4$ is the *state* to be controlled and $(h_i)_{1\leq i\leq j}(t,.) : \Omega \to \mathbb{R}^j$ is the *control input* supported in ω. We are interested in the L^∞-controllability properties of (B.4) : For every $u_0 \in L^\infty(\Omega)^4$, does there exist $(h_i)_{1\leq i\leq j} \in L^\infty(Q)^j$ such that the solution u of (B.4) satisfies

$$\forall i \in \{1,2,3,4\},\ u_i(T,.) = u_i^*? \tag{B.5}$$

B.1.3 Two partial answers

Our first main outcome is a **local controllability result in $L^\infty(\Omega)$ with controls in $L^\infty(Q)$ for** (B.4), i.e. we will show that for every $1 \leq j \leq 3$, there exists $\delta > 0$ such that for every $u_0 \in X_{j,(d_i),(u_i^*)}$ (a " natural " subspace of $L^\infty(\Omega)^4$, see Annexe B.3.1), with $\|u_0 - u^*\|_{L^\infty(\Omega)^4} \leq \delta$, there exists $(h_i)_{1\leq i\leq j} \in L^\infty(Q)^j$ such that the solution u of (B.4) satisfies (B.5).

Our second main result is a **global controllability result in $L^\infty(\Omega)$ with controls in $L^\infty(Q)$ for** (B.4) **in large time and in small dimension**, i.e., we will prove that for every $1 \leq N \leq 2$, $1 \leq j \leq 3$, $u_0 \in X_{j,(d_i),(u_i^*)}$ which verifies a positivity condition (see (B.42)), there exist T^* sufficiently large and $(h_i)_{1\leq i\leq j} \in L^\infty((0,T^*) \times \Omega)^j$ such that the solution u of (B.4) (replace T with T^*) satisfies (B.5) (replace T with T^*).

The precise results are stated in Annexe B.3 (see Theorem B.3.2 and Theorem B.3.6).

B.1.4 Bibliographical comments for the null-controllability of parabolic systems with localized controls

Now, we discuss the null-controllability of parabolic coupled parabolic systems. The following results will be useful for having a proof strategy of our two main results.

Remark B.1.3. We choose to present parabolic systems with Dirichlet conditions because these results are more easy to find in the literature. However, all the following results can be adapted to the Neumann conditions.

B.1.4.1 Linear parabolic systems

The problem of null-controllability of the heat equation was solved independently by Gilles Lebeau, Luc Robbiano in 1995 (see [LR95] or the survey [LRL12]) and Andrei Fursikov, Oleg Imanuvilov in 1996 (see [FI96]) with Carleman estimates.

Theorem B.1.4. [AKBGBdT11, Corollary 2]
For every $u_0 \in L^2(\Omega)$, there exists $h \in L^2(Q)$ such that the solution u of

$$\begin{cases} \partial_t u - \Delta u = h1_\omega & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases}$$

satisfies $u(T,.) = 0$.

Then, null-controllability of linear parabolic systems was studied. A typical example is

$$\begin{cases} \partial_t u - D\Delta u = Au + Bh1_\omega & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases} \tag{B.6}$$

where $u \in C([0,T];L^2(\Omega)^k)$ is the state, $h \in L^2(Q)^l$, $1 \leq l \leq k$, is the control, $D := diag(d_1,\dots,d_k)$ with $d_i \in (0,+\infty)$ is the *diffusion matrix*, $A \in \mathcal{M}_k(\mathbb{R})$ (matrix with k lines and k columns with entries in $\mathbb{R}$) is the *coupling matrix* and $B \in \mathcal{M}_{k,l}(\mathbb{R})$ (matrix with k lines and l columns with entries in $\mathbb{R}$) represents the *distribution of controls*.

Definition B.1.5. System (B.6) is said to be null-controllable if for every $u_0 \in L^2(\Omega)^k$, there exists $h \in L^2(Q)^l$ such that the solution u of (B.6) satisfies $u(T,.) = 0$.

The triplet (D,A,B) plays an important role for null-controllabillity of (B.6) as the following theorem, proved by Farid Ammar-Khodja, Assia Benabdallah, Cédric Dupaix and Manuel Gonzalez-Burgos (which is a generalization of the well-known Kalman condition in finite dimension, see [Cor07a, Theorem 1.16]), shows us.

Theorem B.1.6. [AKBGBdT11, Theorem 5.6]
Let us denote by $(\lambda_m)_{m\geq 1}$ the sequence of positive eigenvalues of the unbounded operator $(-\Delta, H^2(\Omega)\cap H_0^1(\Omega))$ on $L^2(\Omega)$. Then, the following conditions are equivalent.

1. System (B.6) is null-controllable.
2. For every $m \geq 1$, $rank((-\lambda_m D + A)|B) = k$, where

$$((-\lambda_m D + A)|B) := \big(B, (-\lambda_m D + A)B, (-\lambda_m D + A)^2 B, \dots, (-\lambda_m D + A)^{k-1}B\big).$$

For example, let us consider the 2×2 toy-system

$$\begin{cases} \partial_t u_1 - d_1\Delta u_1 = a_{11}u_1 + a_{12}u_2 + h_1 1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t u_2 - d_2\Delta u_2 = a_{21}u_1 + a_{22}u_2 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases} \tag{B.7}$$

where $a_{i,j} \in L^\infty(Q)$ for every $1 \leq i,j \leq 2$. We easily deduce from Theorem B.1.6 the following proposition.

Proposition B.1.7. We assume $a_{ij} \in \mathbb{R}$ for every $1 \leq i,j \leq 2$. The following conditions are equivalent.

1. System (B.7) is null-controllable.
2. $a_{21} \neq 0$.

Roughly speaking, u_1 can be driven to 0 thanks to the control h_1 and u_2 can be driven to 0 thanks to the *coupling term* $a_{21}u_1$. We have the following diagram

$$h_1 \overset{controls}{\rightsquigarrow} u_1 \overset{controls}{\rightsquigarrow} u_2.$$

We also have a more general result for the toy-model (B.7).

Proposition B.1.8. [AKBGBdT11, Theorem 7.1]
We assume that for every $1 \leq i,j \leq 2$, $a_{ij} \in L^\infty(Q)$ and there exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset \omega$ and $\varepsilon > 0$ such that for almost every $(t,x) \in (t_1,t_2) \times \omega_0$, $|a_{21}(t,x)| \geq \varepsilon$. Then, system (B.7) is null-controllable.

Roughly speaking, if the coupling term a_{21} lives somewhere in the control zone, then (u_1,u_2) can be driven to $(0,0)$. The case where $supp(a_{21}) \cap \omega = \emptyset$ is more difficult even if a_{21} depends only on the spatial variable : a minimal time of control can appear (see [AKBGBdT14b], [AKBGBdT14a] and [AKBGBdT16]).

In order to reduce the number of controls entering in the equations of a linear parabolic system, a good strategy is to transform the system into a *cascade system*. This type of system has been studied by Manuel Gonzalez-Burgos and Luz de Teresa (see [GBdT10]). For example, let us consider the 3×3 toy system

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = a_{11}u_1 + a_{12}u_2 + a_{13}u_3 + h_1 1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = a_{21}u_1 + a_{22}u_2 + a_{23}u_3 & \text{in } (0,T)\times\Omega, \\ \partial_t u_3 - d_3 \Delta u_3 = \qquad\qquad a_{32}u_2 + a_{33}u_3 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases} \tag{B.8}$$

where for every $1 \leq i,j \leq 3$, $a_{ij} \in L^\infty(Q)$.

Proposition B.1.9. If there exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset \omega$ and $\varepsilon > 0$ such that for almost every $(t,x) \in (t_1,t_2) \times \omega_0$, $|a_{21}(t,x)| \geq \varepsilon$ and $|a_{32}(t,x)| \geq \varepsilon$, then system (B.8) is null-controllable.

Roughly speaking, u_1 can be driven to 0 thanks to the control h_1, u_2 can be driven to 0 thanks to the coupling term $a_{21}u_1$ (which lives somewhere in the control zone) and u_3 can be driven to 0 thanks to the coupling term $a_{32}u_2$ (which lives somewhere in the control zone). Heuristically, we have the following diagram

$$h_1 \overset{controls}{\rightsquigarrow} u_1 \overset{controls}{\rightsquigarrow} u_2 \overset{controls}{\rightsquigarrow} u_3.$$

For more general results, see the survey [AKBGBdT11, Sections 4, 5, 7] and the references therein.

We can also replace the coupling matrix A in the system (B.6) by a *differential operator of first order or second order*. In this case, there exist some similar results (see [Gao15], [BCGDT14] with a technical assumption on ω, [Dup17], [DL16], [DL18]). For example, let us consider the particular case of the 2×2 system

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = g_{11}.\nabla u_1 + g_{12}.\nabla u_2 + a_{11}u_1 + a_{12}u_2 + h_1 1_\omega & \text{in } Q_T, \\ \partial_t u_2 - d_2 \Delta u_2 = g_{21}.\nabla u_1 + g_{22}.\nabla u_2 + a_{21}u_1 + a_{22}u_2 & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases} \tag{B.9}$$

where $a_{ij} \in \mathbb{R}$, $g_{ij} \in \mathbb{R}$ for every $1 \leq i, j \leq 2$. Then, system (B.9) is null-controllable if and only if $g_{21} \neq 0$ or $a_{21} \neq 0$. This result is due to Michel Duprez and Pierre Lissy (see [DL16, Theorem 1] and [SGM18, Theorem 3.4] for a similar result). It is proved by a *fictitious control method* and *algebraic solvability*, introduced for the first time by Jean-Michel Coron in the context of stabilization of ordinary differential equations (see [Cor92]). This type of method has also been used for Navier-Stokes equations by Jean-Michel Coron and Pierre Lissy in [CL14]. However, the situation is much more complicated and is not well-understood in the case where a_{ij}, g_{ij} $(1 \leq i, j \leq 2)$ depend on the spatial variable. One can see the surprising negative result of null-controllability : [DL18, Theorem 2]. When the matrix A in (B.6) is a differential operator of second order (take $A = \widetilde{A}\Delta + C(t,x)$ with $(\widetilde{A}, C) \in \mathcal{M}_k(\mathbb{R}) \times L^\infty(Q; \mathcal{M}_k(\mathbb{R}))$ to simplify), the coupling matrix A disturbs the diagonal diffusion matrix D and creates a new "cross" diffusion matrix : $\widetilde{D} = D - \widetilde{A}$. When $\widetilde{D}$ is not diagonalizable, there are few results (see [FCGBdT15] with a technical assumption on the dimension of the Jordan Blocks of $\widetilde{D}$ and the recent preprint [LZ17, Section 3] when C does not depend on time and space).

Let us also keep in mind the following result which help to understand our analysis.

Proposition B.1.10. [Gue07, Theorem 3], [FCGBdT15, Theorem 1.5]
Let a_{11}, a_{12}, $d \in \mathbb{R}$. Let us consider the 2×2 toy system

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = a_{11} u_1 + a_{12} u_2 + h_1 1_\omega & \text{in } (0,T) \times \Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = d \Delta u_1 & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases} \tag{B.10}$$

Then, the following conditions are equivalent.
1. System (B.10) is null-controllable.
2. $d \neq 0$.

Roughly speaking, u_1 can be driven to 0 thanks to the control h_1 and u_2 can be driven to 0 thanks to the *coupling term of second order* $d\Delta u_1$.

Remark B.1.11. When it is possible, one can diagonalize the matrix $\widetilde{D} = \begin{pmatrix} d_1 & 0 \\ d & d_2 \end{pmatrix}$. Then, by a linear transformation together with Theorem B.1.6, one can prove Proposition B.1.10. However, in this paper, we choose the opposite strategy. We transform (B.4) into a system like (B.10) (with four equations). Indeed, such a system seems to be a *cascade system* with coupling terms of second order.

B.1.4.2 Nonlinear parabolic systems

Then, another challenging issue is the study of the null-controllability properties of semilinear parabolic systems. The usual strategy consists in *linearizing the system* around 0 and to deduce local controllability properties of the nonlinear system by controllability properties of the linearized system and a fixed-point argument.

For example, let us consider the 2×2 model system

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) + h_1 1_\omega & \text{in } (0,T) \times \Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases} \tag{B.11}$$

where f_1 and f_2 belong to $C^\infty(\mathbb{R}^2; \mathbb{R})$. Then, the following result is a consequence of Proposition B.1.7.

Proposition B.1.12. Let us suppose that $\frac{\partial f_2}{\partial u_1}(0,0) \neq 0$. Then, there exists $\delta > 0$ such that for every $u_0 \in L^\infty(\Omega)^2$ which satisfies $\|u_0\|_{L^\infty(\Omega)^2} \leq \delta$, there exists $h_1 \in L^\infty(Q)$ such that the solution u of (B.11) verifies $u(T,.) = 0$.

Remark B.1.13. This result is well-known but it is difficult to find it in the literature (see [AKBD06, Theorem 6] with a restriction on the dimension $1 \leq N < 6$ and other function spaces or one can adapt the arguments given in [CGR10] to get Proposition B.1.12 for any $N \in \mathbb{N}^*$). For other results in this direction, see [WZ06], [LCM+16], [GBPG06] and [CSG15].

When f_2 does not satisfy the hypothesis of Proposition B.1.12, another strategy consists in linearizing around a non trivial trajectory $(\overline{u_1}, \overline{u_2}, \overline{h_1})$ of the nonlinear system which goes from 0 to 0. This procedure is called the *return method* and was introduced by Jean-Michel Coron in [Cor92] (see [Cor07a, Chapter 6]). This method conjugated with Proposition B.1.8 gives the following result.

Proposition B.1.14. We assume that there exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset \omega$ and $\varepsilon > 0$ such that $|\frac{\partial f_2}{\partial u_1}(\overline{u_1}, \overline{u_2})| \geq \varepsilon$ on $(t_1, t_2) \times \omega_0$. Then, there exists $\delta > 0$ such that for every $u_0 \in L^\infty(\Omega)^2$ which satisfies $\|u_0\|_{L^\infty(\Omega)^2} \leq \delta$, there exists $h_1 \in L^\infty(Q)$ such that the solution u of (B.11) verifies $u(T,.) = 0$.

Proposition B.1.14 is proved in [CGR10] and used in [CGR10] with $f_2(u_1, u_2) = u_1^3 + Ru_2$, where $R \in \mathbb{R}$, [CGMR15], [CG17] and [LB18c].

Finally, Felipe Walison Chaves-Silva and Sergio Guerrero have studied the local controllability of the Keller-Segel system in which the nonlinearity involves derivative terms of order 2 (see [CSG15]). Some ideas of [CSG15] are exploited in our proof.

B.1.5 Proof strategy of the two main results

Let us return to the main question discussed in this paper (see Annexe B.1.2) and the expected results as explained in Annexe B.1.3.

The **local controllability result** is deduced from controllability properties of the linearized system around $(u_i^*)_{1 \leq i \leq 4}$ of (B.4). This strategy presents **two main difficulties**.

For the case of 3 controls (see Annexe B.4.1.1), if $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$, the linearized system is controllable and consequently the nonlinear result comes from an adaptation of Proposition B.1.12. If $(u_1^*, u_3^*, u_4^*) = (0,0,0)$, **the linearized system is not controllable**. Then, we use the *return method* to overcome this problem and the nonlinear result comes from an adaptation of Proposition B.1.14.

For the case of 2 controls and 1 control, **there exist some invariant quantities in the nonlinear system** and consequently in the linearized system, that prevent controllability from happening in the whole space $L^\infty(\Omega)^4$. Therefore, we restrict the initial data to a "natural" subspace of $L^\infty(\Omega)^4$ (see Annexe B.3.1). A modified version (for Neumann conditions) of Theorem B.1.6 cannot be applied to the linearized system of (B.4) because the rank condition is never satisfied (due to the invariant quantities). An *adequate change of variable* gets over this difficulty by creating *cross-diffusion* and by using coupling matrices of second order (see Annexe B.4.1.2 and Annexe B.4.1.3). Then, we treat the controllability properties of the linearized system by adapting Proposition B.1.9 and Proposition B.1.10.

To summarize, we must require necessary conditions on the initial data. Consequently the local controllability result depends on : the coefficients $(d_i)_{1\leq i\leq 4}$ (i.e. the diffusion matrix), the state $(u_i^*)_{1\leq i\leq 4}$ (i.e. the coupling matrix of the linearized system of (B.4)), j (i.e. the number of controls that we put in the equations).

The **global controllability result** is a corollary of our local controllability result and a result by Laurent Desvillettes, Klemens Fellner and Michel Pierre, Takashi Suzuki, Yoshio Yamada, Rong Zou concerning the asymptotics of the trajectory of (B.1) for $1 \leq N \leq 2$. Indeed, this known result claims that the solution $u(T,.)$ of (B.4) converges in $L^\infty(\Omega)^4$ to a particular positive stationary solution z of (B.1) when $T \to +\infty$ (see [DF06] or [PSZ17, Theorem 3] and [PSY18, Theorem 3]). Then, the solution of (B.4) can be exactly driven to z by our first outcome. Finally, a connectedness-compactness argument enables to steer the solution of (B.4) from z to $(u_i^*)_{1\leq i\leq 4}$.

B.2 Properties of the nonlinear controlled system

B.2.1 Definitions and usual properties

In this part, we introduce the concept of *trajectory* of (B.4). This definition requires a well-posedness result (see Proposition B.2.3).

First, we introduce some usual notations.

Let $k, l \in \mathbb{N}^*$, $\mathcal{A}$ an algebra. Then, $\mathcal{M}_k(\mathcal{A})$ (respectively $\mathcal{M}_{k,l}(\mathcal{A})$) denotes the algebra of matrices with k lines and k columns with entries in $\mathcal{A}$ (respectively the algebra of matrices with k lines and l columns with entries in $\mathcal{A}$).

For $k \in \mathbb{N}^*$ and $A \in \mathcal{M}_k(\mathbb{R})$, $Sp(M)$ denotes the set of complex eigenvalues of M,

$$Sp(M) := \{\lambda \in \mathbb{C} \ ; \ \exists X \in \mathbb{C}^k \setminus \{0\}, \ MX = \lambda X\}.$$

For $(a,b,c,d) \in \mathbb{R}^4$, we introduce

$$\forall i \in \mathbb{N}^*,\ f_i(a,b,c,d) := (-1)^i(ac-bd), \quad f(a,b,c,d) = (f_i(a,b,c,d))_{1\leq i\leq 4}. \tag{B.12}$$

Definition B.2.1. We introduce the space Y defined by

$$Y := L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))'). \tag{B.13}$$

Proposition B.2.2. From an easy adaptation of the proof of [Eva10, Section 5.9.2, Theorem 3], we have

$$Y \hookrightarrow C([0,T];L^2(\Omega)). \tag{B.14}$$

Proposition B.2.3. Let $k \in \mathbb{N}^*$, $D \in \mathcal{M}_k(\mathbb{R})$ such that D is diagonalizable and $Sp(D) \subset (0,+\infty)$, $A \in \mathcal{M}_k(L^\infty(Q))$, $u_0 \in L^2(\Omega)^k$, $g \in L^2(Q)^k$. The following Cauchy problem admits a unique weak solution $u \in Y^k$

$$\begin{cases} \partial_t u - D\Delta u = A(t,x)u + g & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases}$$

This means that u is the unique function in Y^k that satisfies the variational fomulation : $\forall w \in L^2(0,T;H^1(\Omega)^k)$,

$$\int_0^T (\partial_t u, w)_{(H^1(\Omega)^k)',H^1(\Omega)^k)} + \int_Q D\nabla u.\nabla w = \int_Q (Au+g).w, \tag{B.15}$$

and

$$u(0,.) = u_0 \text{ in } L^2(\Omega)^k. \tag{B.16}$$

Moreover, there exists $C>0$ independent of u_0 and g such that

$$\|u\|_{Y^k} \leq C\left(\|u_0\|_{L^2(\Omega)^k} + \|g\|_{L^2(Q)^k}\right). \tag{B.17}$$

Finally, if $u_0 \in L^\infty(\Omega)^k$ and $g \in L^\infty(Q)^k$, then $u \in L^\infty(Q)^k$ and there exists $C>0$ independent of u_0 and g such that

$$\|u\|_{(Y\cap L^\infty(Q))^k} \leq C\left(\|u_0\|_{L^\infty(\Omega)^k} + \|g\|_{L^\infty(Q)^k}\right). \tag{B.18}$$

Remark B.2.4. This proposition is more or less classical, but we could not find it as such in the literature and we give its proof in the Appendix (see Annexe B.7.1).

Definition B.2.5. For $u_0 \in L^\infty(\Omega)^4$, $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq j})$ is a trajectory of (B.4) if

1. $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq j}) \in (Y\cap L^\infty(Q))^4 \times L^\infty(Q)^j$,
2. $(u_i)_{1\leq i\leq 4}$ is the (unique) solution of (B.4).

Moreover, $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq j})$ is a trajectory of (B.4) reaching $(u_i^*)_{1\leq i\leq 4}$ (in time T) if

$$\forall i \in \{1,\dots,4\},\ u_i(T,.) = u_i^*.$$

Remark B.2.6. The concept of solution of (B.4) is the same as in Proposition B.2.3 (take $D = diag(d_1, d_2, d_3, d_4)$, $A = 0$ and $g = (g_i(u))^T_{1\leq i\leq 4}$ where $g_i(u) = f_i(u) + h_i 1_{i\leq j} 1_\omega$).

Remark B.2.7. The uniqueness is a consequence of the following estimate.

Let $D = diag(d_1, d_2, d_3, d_4)$, $(h_i)_{1\leq i\leq j} \in L^\infty(Q)^j$, $u = (u_i)_{1\leq i\leq 4} \in (Y \cap L^\infty(Q))^4$, $\widetilde{u} = (\widetilde{u}_i)_{1\leq i\leq 4} \in (Y \cap L^\infty(Q))^4$ be two solutions of (B.4), and $v = u - \widetilde{u}$. The function v satisfies (in the weak sense)

$$\begin{cases} \partial_t v - D\Delta v = f(u) - f(\widetilde{u}) & \text{in } (0,T)\times\Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ v(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{B.19}$$

By taking $w := v$ in the variational formulation of (B.19) (see also (B.15)) and by using the fact that the mapping $t \mapsto \|v(t)\|^2_{L^2(\Omega)^4}$ is absolutely continuous with $\frac{d}{dt}\|v(t)\|^2_{L^2(\Omega)^4} = 2(\partial_t v(t), v(t))_{(H^1(\Omega)^4)', H^1(\Omega)^4}$ for a.e. $0 \leq t \leq T$ (see [Eva10, Section 5.9.2, Theorem 3]), we find that for a.e. $0 \leq t \leq T$,

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2_{L^2(\Omega)^4}\right) + \|D\nabla v\|^2_{L^2(\Omega)^4} = (f(u) - f(\widetilde{u}), v)_{L^2(\Omega)^4, L^2(\Omega)^4}. \tag{B.20}$$

By using the facts that $(u, \widetilde{u}) \in L^\infty(Q)^4 \times L^\infty(Q)^4$, f is locally Lipschitz continuous on $\mathbb{R}^4$, we find the differential inequality

$$\frac{d}{dt}\left(\|v\|^2_{L^2(\Omega)^4}\right) \leq C\,\|v\|^2_{L^2(\Omega)^4}\,, \text{ for a.e. } 0 \leq t \leq T. \tag{B.21}$$

Gronwall's lemma and the initial condition $v(0,.) = 0$ prove that $v = 0$ in $L^2(Q)^4$. Consequently, $u = \widetilde{u}$.

B.2.2 Invariant quantities of the nonlinear dynamics

In this section, we show that in the system (B.4), some invariant quantities exist. They impose some restrictions on the initial condition for the controllability results.

B.2.2.1 Variation of the mass

Proposition B.2.8. Let $j \in \{1, 2, 3\}$, $u_0 \in L^\infty(\Omega)^4$, $((u_i)_{1\leq i\leq 4}, (h_i)_{1\leq i\leq j})$ be a trajectory of (B.4). For every $1 \leq i \leq 4$, the mapping $t \mapsto \int_\Omega u_i(t,x)dx$ is absolutely continuous with for a.e. $0 \leq t \leq T$,

$$\frac{d}{dt}\int_\Omega u_i(t,x)dx = \int_\Omega \Big\{ f_i(U(t,x)) + h_i(t,x)1_\omega(x)1_{i\leq j}\Big\} dx. \tag{B.22}$$

Proof. We fix $1 \leq i \leq 4$. By using the fact that $u_i \in Y$ and from an easy adaptation of [Eva10, Section 5.9.2, Theorem 3, (ii)], we deduce that the mapping $t \mapsto \int_\Omega u_i(t,x)dx$ is absolutely continuous and for a.e. $0 \leq t \leq T$,

$$\frac{d}{dt}\int_\Omega u_i(t,x)dx = (\partial_t u_i(t,.), 1)_{(H^1(\Omega))', H^1(\Omega)}\,.$$

Then, by using that $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq j})$ is the (unique) solution of (B.4) and by taking $w=1$ in (B.15), we find that for a.e. $0\leq t\leq T$,

$$
\begin{aligned}
&(\partial_t u_i(t,.),1)_{(H^1(\Omega))',H^1(\Omega)}\\
&= d_i(\nabla u_i(t,.),\nabla 1)_{L^2(\Omega),L^2(\Omega)} + \int_\Omega \Big\{f_i(U(t,x)) + h_i(t,x)1_\omega(x)1_{i\leq j}\Big\}dx\\
&= \int_\Omega \Big\{f_i(U(t,x)) + h_i(t,x)1_\omega(x)1_{i\leq j}\Big\}dx.
\end{aligned}
$$

□

B.2.2.2 Case of 2 controls

Proposition B.2.9. Let $j=2$, $u_0\in L^\infty(\Omega)^4$, $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq 2})$ be a trajectory of (B.4) reaching $(u_i^*)_{1\leq i\leq 4}$ in time T. Then, we have

$$\frac{1}{|\Omega|}\int_\Omega \Big(u_{3,0}(x)+u_{4,0}(x)\Big)dx = u_3^*+u_4^*, \tag{B.23}$$

$$\Big(d_3=d_4\Big)\Rightarrow\Big(u_{3,0}+u_{4,0}=u_3^*+u_4^*\Big). \tag{B.24}$$

Proof. From (B.22), we have

$$\frac{d}{dt}\left(\int_\Omega (u_3(t,x)+u_4(t,x))dx\right) = 0 \text{ for a.e. } 0\leq t\leq T.$$

Then, from Definition B.2.5, (B.23) holds.

Moreover, u_3+u_4 satisfies

$$
\begin{cases}
\partial_t(u_3+u_4) - d_4\Delta(u_3+u_4) = (d_3-d_4)\Delta u_3 & \text{in } (0,T)\times\Omega,\\
\frac{\partial(u_3+u_4)}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega.
\end{cases}
$$

If $d_3=d_4$, then the backward uniqueness for the heat equation (a corollary of Lemma B.2.11) proves that

$$\forall t\in[0,T],\ (u_3+u_4)(t,.) = (u_3+u_4)(T,.) = u_3^*+u_4^*. \tag{B.25}$$

This implies the necessary condition (B.24), stronger than (B.23), on the initial condition.

□

B.2.2.3 Case of 1 control

Proposition B.2.10. Let $j=1$, $u_0\in L^\infty(\Omega)^4$, $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq 2})$ be a trajectory of (B.4) reaching $(u_i^*)_{1\leq i\leq 4}$ in time T. Then, we have

$$
\begin{aligned}
\tfrac{1}{|\Omega|}\textstyle\int_\Omega \Big(u_{2,0}(x)+u_{3,0}(x)\Big)dx &= u_2^*+u_3^*,\\
\tfrac{1}{|\Omega|}\textstyle\int_\Omega \Big(u_{3,0}(x)+u_{4,0}(x)\Big)dx &= u_3^*+u_4^*,
\end{aligned}
\tag{B.26}
$$

$$\Big(k\neq l\in\{2,3,4\},\ d_k=d_l\Big)\Rightarrow\Big(u_{k,0}-(-1)^{k-l}u_{l,0} = u_k^*-(-1)^{k-l}u_l^*\Big). \tag{B.27}$$

Proof. From (B.22), we have for a.e. $0 \leq t \leq T$,

$$\frac{d}{dt}\left(\frac{1}{|\Omega|}\int_{\Omega}(u_2(t,x)+u_3(t,x))dx\right)=0,\ \frac{d}{dt}\left(\frac{1}{|\Omega|}\int_{\Omega}(u_3(t,x)+u_4(t,x))dx\right)=0.$$

Then, from Definition B.2.5, (B.26) holds.

Moreover, if there exists $k \neq l \in \{2,3,4\}$ such that $d_k = d_l$, by using again the backward uniqueness for the heat equation, we get

$$\begin{aligned}&\Big(k \neq l \in \{2,3,4\},\ d_k = d_l\Big)\\ &\Rightarrow \Big(\forall t \in [0,T],\ (u_k - (-1)^{k-l}u_l)(t,.) = u_k^* - (-1)^{k-l}u_l^*\Big),\end{aligned} \tag{B.28}$$

and in particular the necessary condition (B.27), stronger than (B.26), on the initial condition. ☐

B.2.3 More restrictive conditions on the initial condition when the target $(u_i^*)_{1\leq i\leq 4}$ vanishes

In the previous section, we have seen that there are invariant quantities in the dynamics of (B.4) which impose necessary conditions on the initial condition : (B.23), (B.26). Moreover, when some coefficients of diffusion d_i are equal, we have more invariant quantities in (B.4) which impose stronger necessary conditions on the initial condition : (B.24), (B.27).

B.2.3.1 The lemma of backward uniqueness

Lemma B.2.11. Backward uniqueness
Let $k \in \mathbb{N}^*$, $D = diag(d_1,\dots,d_k)$ where $d_i \in (0,+\infty)$, $C \in \mathcal{M}_k(L^\infty(Q))$, $\zeta_0 \in L^\infty(\Omega)^k$. Let $\zeta \in Y^k$ be the solution of

$$\begin{cases}\partial_t\zeta - D\Delta\zeta = C(t,x)\zeta & \text{in } (0,T)\times\Omega,\\ \frac{\partial\zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega,\\ \zeta(0,.) = \zeta_0 & \text{in } \Omega.\end{cases}$$

If $\zeta(T,.) = 0$, then for every $t \in [0,T]$, $\zeta(t,.) = 0$.

Proof. $\widetilde{\zeta}(t,x) = \exp(-t)\zeta(t,x) \in Y^k$ is the solution of the system

$$\begin{cases}\partial_t\widetilde{\zeta} - D\Delta\widetilde{\zeta} + I_k\widetilde{\zeta} = C(t,x)\widetilde{\zeta} & \text{in } (0,T)\times\Omega,\\ \frac{\partial\widetilde{\zeta}}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega,\\ \zeta(0,.) = \widetilde{\zeta}_0 & \text{in } \Omega,\end{cases}$$

which verifies $\widetilde{\zeta}(T,.) = 0$.

Let us denote $A = -D\Delta + I_k$ which is a bounded linear operator from $H^1(\Omega)^k$ to $(H^1(\Omega)^k)'$. Indeed,

$$\forall (u,v) \in (H^1(\Omega)^k)^2, (Au)(v) = \sum_{i=1}^{k} d_i(\nabla u_i,\nabla v_i)_{L^2(\Omega),L^2(\Omega)} + \sum_{i=1}^{k}(u_i,v_i)_{L^2(\Omega),L^2(\Omega)},$$

$$\|Au\|_{(H^1(\Omega)^k)'} \leq \sqrt{1+\max(d_i)}\,\|u\|_{H^1(\Omega)^k}.$$

Then, A verifies the three hypotheses : (i), (ii) and (iii) of [BT73, Proposition II.1].

(i) is satisfied because A does not depend on t.

(ii) is a consequence of

$$\forall (u,v)\in (H^1(\Omega)^k)^2, (Au)(v) = (Av)(u).$$

(iii) is satisfied because

$$(Au,u) = \sum_{i=1}^{k} d_i(\nabla u_i,\nabla u_i)_{L^2(\Omega),L^2(\Omega)} + \sum_{i=1}^{k}(u_i,u_i)_{L^2(\Omega),L^2(\Omega)} \geq \min(\min_i(d_i),1)\,\|u\|^2_{H^1(\Omega)^k}.$$

Let $B(t)$ be the family of operators in $L^2(0,T;\mathcal{L}(H^1(\Omega)^k, L^2(\Omega)^k))$ defined by

$$\forall u\in H^1(\Omega)^k,\ B(t)u(.) = C(t,.)u(.).$$

We have

$$\|B\|^2_{L^2(0,T;\mathcal{L}(H^1(\Omega)^k,L^2(\Omega)^k))} \leq \|C\|^2_{L^\infty(Q)^{k^2}}.$$

By applying [BT73, Theorem II.1], we get that for every $t\in[0,T]$, $\widetilde{\zeta}(t,.) = 0$. Then,

$$\forall t\in[0,T],\ \zeta(t,.) = 0.$$

□

B.2.3.2 Case of 2 controls

Proposition B.2.12. Let $j=2$, $u_0\in L^\infty(\Omega)^4$. If $((u_i)_{1\leq i\leq 4},(h_i)_{1\leq i\leq 2})$ is a trajectory of (B.4) reaching $(u_i^*)_{1\leq i\leq 4}$ in time T, then we have

$$\Big((u_3^*,u_4^*)=(0,0)\Big) \Rightarrow \Big((u_{3,0},u_{4,0})=(0,0)\Big). \tag{B.29}$$

Conversely, for every $u_0\in L^\infty(\Omega)^4$ such that $(u_{3,0},u_{4,0})=(0,0)$, we can find $(h_i)_{1\leq i\leq 2}\in L^\infty(Q)^2$ such that the associated solution $(u_i)_{1\leq i\leq 4}\in L^\infty(Q)^4$ of (B.4) satisfies

$$(u_1,u_2,u_3,u_4)(T,.) = (u_1^*,u_2^*,0,0).$$

Proof. If $(u_3^*,u_4^*)=(0,0)$, it results from (B.4) that

$$\begin{cases} \partial_t u_3 - d_3\Delta u_3 = -u_1u_3+u_2u_4 & \text{in } (0,T)\times\Omega,\\ \partial_t u_4 - d_4\Delta u_4 = u_1u_3-u_2u_4 & \text{in } (0,T)\times\Omega,\\ \frac{\partial u_3}{\partial n} = \frac{\partial u_4}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{B.30}$$

By using the point 1 of Definition B.2.5, we have

$$(u_1,u_2)\in L^\infty(Q)^2. \tag{B.31}$$

Then, from (B.30), (B.31), Definition B.2.5 : $(u_3, u_4)(T, .) = (0, 0)$ and Lemma B.2.11 with $k = 2$, $D = diag(d_3, d_4)$ and $C = \begin{pmatrix} -u_1 & u_2 \\ u_1 & -u_2 \end{pmatrix}$, we deduce that

$$\forall t \in [0, T],\ (u_3, u_4)(t, .) = (0, 0),$$

and in particular (B.29).

Conversely, let $u_0 \in L^\infty(\Omega)^4$ be such that $(u_{3,0}, u_{4,0}) = (0, 0)$. Then, (B.4) reduces to the following system

$$\left\{ \begin{array}{ll} \partial_t u_1 - d_1 \Delta u_1 = h_1 1_\omega & \text{in } (0, T) \times \Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = h_2 1_\omega & \text{in } (0, T) \times \Omega, \\ \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0 & \text{on } (0, T) \times \partial\Omega, \\ (u_1, u_2)(0, .) = (u_{1,0}, u_{2,0}) & \text{in } \Omega. \end{array} \right. \tag{B.32}$$

The problem reduces to the null-controllability of two decoupled heat equations in $L^\infty(\Omega)$ with two localized control in $L^\infty(Q)$ which is a solved problem (see for example [FCGBGP06b, Proposition 1]). Therefore, we can find $(h_i)_{1 \le i \le 2} \in L^\infty(Q)^2$ such that the associated solution $(u_i)_{1 \le i \le 4} \in L^\infty(Q)^4$ of (B.4) satisfies $(u_1, u_2, u_3, u_4)(T, .) = (u_1^*, u_2^*, 0, 0)$. □

Remark B.2.13. Thanks to Proposition B.2.12, we avoid the easy case $(u_3^*, u_4^*) = (0, 0)$ for 2 controls in the sequel.

B.2.3.3 Case of 1 control

Proposition B.2.14. Let $j = 1$, $u_0 \in L^\infty(\Omega)^4$. If $((u_i)_{1 \le i \le 4}, h_1)$ is a trajectory of (B.4) reaching $(u_i^*)_{1 \le i \le 4}$ in time T, then we have

$$\Big((u_3^*, u_2^*) = (0, 0)\Big) \Rightarrow \Big((u_{2,0}, u_{3,0}, u_{4,0}) = (0, 0, u_4^*)\Big), \tag{B.33}$$

$$\Big((u_3^*, u_4^*) = (0, 0)\Big) \Rightarrow \Big((u_{2,0}, u_{3,0}, u_{4,0}) = (u_2^*, 0, 0)\Big). \tag{B.34}$$

Conversely, for every $u_0 \in L^\infty(\Omega)^4$ such that $u_{3,0} = 0$, we can find $h_1 \in L^\infty(Q)$ such that the associated solution $(u_i)_{1 \le i \le 4} \in L^\infty(Q)^4$ of (B.4) satisfies
$(u_1, u_2, u_3, u_4)(T, .) = (u_1^*, u_2^*, 0, u_4^*)$.

Proof. If $u_3^* = 0$, then from (B.3), $u_2^* = 0$ or $u_4^* = 0$. We assume that $(u_3^*, u_2^*) = (0, 0)$ (the other case is similar). The backward uniqueness (i.e. Lemma B.2.11) as in Annexe B.2.3.2 leads to

$$\forall t \in [0, T],\ (u_3, u_2)(t, .) = (0, 0).$$

Then, we deduce that

$$\left\{ \begin{array}{ll} \partial_t u_4 - d_4 \Delta u_4 = 0 & \text{in } (0, T) \times \Omega, \\ \frac{\partial u_4}{\partial n} = 0 & \text{on } (0, T) \times \partial\Omega. \end{array} \right. \tag{B.35}$$

The backward uniqueness for the heat equation applied to (B.35) proves that

$$\forall t \in [0,T],\ u_4(t,.) = u_4^*,$$

and in particular (B.33) and (B.34).

Conversely, let $u_0 \in L^\infty(\Omega)^4$ such that $u_{3,0} = 0$. Then, (B.4) reduces to the following system

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = h_1 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_1}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_1(0,.) = u_{1,0} & \text{in } \Omega. \end{cases} \tag{B.36}$$

The problem reduces to the null-controllability of the heat equation in $L^\infty(\Omega)$ with a localized control in $L^\infty(Q)$ which is a solved problem (see for example [FCGBGP06b, Proposition 1]). Therefore, we can find $h_1 \in L^\infty(Q)$ such that the associated solution $(u_i)_{1\leq i\leq 4} \in L^\infty(Q)^4$ of (B.4) satisfies $(u_1,u_2,u_3,u_4)(T,.) = (u_1^*, u_2^*, 0, u_4^*)$. □

Remark B.2.15. Thanks to Proposition B.2.14, we avoid the easy case $u_3^* = 0$ for 1 control in the sequel.

B.3 Main results

In this part, we present our two main results : a local controllability result and a large-time global controllabillity result for (B.4).

B.3.1 Local controllability under constraints

In Annexe B.2.2 and Annexe B.2.3, we have highlighted necessary conditions on initial conditions when $((u_i)_{1\leq i\leq 4}, (h_i)_{1\leq i\leq j})$ is a trajectory reaching $(u_i^*)_{1\leq i\leq 4}$. They turn out to be sufficient for the existence of such trajectories at least for data close to $(u_i^*)_{1\leq i\leq 4}$. The goal of this subsection is to define subspaces of $L^\infty(\Omega)^4$ which take care of these conditions.

B.3.1.1 Case of 3 controls

We introduce

$$X_{3,(d_i),(u_i^*)} = L^\infty(\Omega)^4. \tag{B.37}$$

B.3.1.2 Case of 2 controls

The results of Annexe B.2.2.2 and Annexe B.2.3.2 are summed up in the following array.

	$(u_3^*, u_4^*) \neq (0,0)$
$d_3 = d_4$	$u_{3,0} + u_{4,0} = u_3^* + u_4^*$
$d_3 \neq d_4$	$\frac{1}{\lvert\Omega\rvert}\int_\Omega (u_{3,0} + u_{4,0}) = u_3^* + u_4^*$

(B.38)

Then, we introduce

$$X_{2,(d_i),(u_i^*)} := \{u_0 \in L^\infty(\Omega)^4 \ ; \ u_0 \text{ satisfies the condition of (B.38)}\}. \tag{B.39}$$

For example, $X_{2,(1,2,3,4),(1,1,1,1)} = \{u_0 \in L^\infty(\Omega)^4 \ ; \frac{1}{|\Omega|}\int_\Omega (u_{3,0} + u_{4,0}) = 2\}$.

B.3.1.3 Case of 1 control

The results of Annexe B.2.2.3 and Annexe B.2.3.3 are summed up in the following array.

	$u_3^* \neq 0$
$d_2 = d_3 = d_4$	$u_{2,0} + u_{3,0} = u_2^* + u_3^*, \ u_{3,0} + u_{4,0} = u_3^* + u_4^*$
$d_2 \neq d_3, \ d_3 = d_4$	$\frac{1}{\lvert\Omega\rvert}\int_\Omega (u_{2,0} + u_{3,0}) = u_2^* + u_3^*, \ u_{3,0} + u_{4,0} = u_3^* + u_4^*$
$d_2 = d_3, d_3 \neq d_4$	$u_{2,0} + u_{3,0} = u_2^* + u_3^*, \ \frac{1}{\lvert\Omega\rvert}\int_\Omega (u_{3,0} + u_{4,0}) = u_3^* + u_4^*$
$d_2 = d_4, \ d_2 \neq d_3$	$u_{2,0} - u_{4,0} = u_2^* - u_4^*, \ \frac{1}{\lvert\Omega\rvert}\int_\Omega (u_{3,0} + u_{4,0}) = u_3^* + u_4^*$
$d_2 \neq d_3, \ d_3 \neq d_4, \ d_2 \neq d_4$	$\frac{1}{\lvert\Omega\rvert}\int_\Omega (u_{2,0} + u_{3,0}) = u_2^* + u_3^*, \ \frac{1}{\lvert\Omega\rvert}\int_\Omega (u_{3,0} + u_{4,0}) = u_3^* + u_4^*$

(B.40)

Then, we introduce

$$X_{1,(d_i),(u_i^*)} := \{u_0 \in L^\infty(\Omega)^4 \ ; \ u_0 \text{ satisfies the condition of (B.40)}\}. \tag{B.41}$$

B.3.1.4 Local controllability result

Definition B.3.1. Let $j \in \{1,2,3\}$, $(u_1^*, u_2^*, u_3^*, u_4^*) \in (\mathbb{R}^+)^4$ be such that (B.3) holds. The system (B.4) is **locally controllable to the state** $(u_i^*)_{1\leq i\leq 4}$ **in** $L^\infty(\Omega)^4$ **with controls in** $L^\infty(Q)^j$ if there exists $\delta > 0$ such that for every $u_0 \in X_{j,(d_i),(u_i^*)}$ (see (B.37), (B.39) and (B.41)) verifying $\|u_0 - (u_i^*)_{1\leq i\leq 4}\|_{L^\infty(\Omega)^4} \leq \delta$, there exists $(h_i)_{1\leq i\leq j} \in L^\infty(Q)^j$ such that the solution $(u_i)_{1\leq i\leq 4} \in L^\infty(Q)^4$ to the Cauchy problem (B.4) satisfies

$$\forall i \in \{1,2,3,4\}, u_i(T,.) = u_i^*.$$

Theorem B.3.2. For every $j \in \{1,2,3\}$, for every $(u_1^*, u_2^*, u_3^*, u_4^*) \in (\mathbb{R}^+)^4$ which satisfies (B.3), the system (B.4) is **locally controllable to the state** $(u_i^*)_{1\leq i\leq 4}$ **in** $L^\infty(\Omega)^4$ **with controls in** $L^\infty(Q)^j$.

Remark B.3.3. The uniqueness of the solution $(u_i)_{1\leq i\leq 4} \in L^\infty(Q)^4$ is a consequence of Remark B.2.7. The existence of the solution $(u_i)_{1\leq i\leq 4} \in L^\infty(Q)^4$ is a consequence of a good choice of controls $(h_i)_{1\leq i\leq j} \in L^\infty(Q)^j$ and more precisely of a fixed-point argument (see Annexe B.4.5).

Remark B.3.4. As we have said in the introduction, it was not known if L^∞ blow-up occurs or not in dimension $N > 2$ for the free system (B.1) until recently (see [CGV17]). Here, our strategy of control avoids blow-up and enables the solution to reach a stationary solution of (B.1).

Remark B.3.5. In some particular cases (easy cases), this local controllability result can be improved in a global controllability result (see the case $(u_3^*, u_4^*) = (0,0)$ for 2 controls in Annexe B.2.3.2 and the case $u_3^* = 0$ for 1 control in Annexe B.2.3.3).

B.3.2 Large-time global controllability result

From Theorem B.3.2, we establish a global controllability result in large time for $N = 1, 2$.

Theorem B.3.6. We assume that $N = 1$ or 2. Let $j \in \{1, 2, 3\}$ and $(u_i^*)_{1\leq i\leq 4} \in (\mathbb{R}^+)^4$ be such that (B.3) holds. Then, for every nonnegative $u_0 \in X_{j,(d_i),(u_i^*)}$ satisfying

$$\forall (i,j) \in \Big\{(1,2),(1,4),(2,3),(3,4)\Big\},\ \frac{1}{|\Omega|}\int_\Omega (u_{i,0}+u_{j,0})(x)dx > 0. \tag{B.42}$$

there exists $T^* > 0$ (sufficiently large) and $(h_i)_{1\leq i\leq j} \in L^\infty((0,T^*)\times\Omega)^j$ such that the solution u of

$$\begin{cases} \forall 1 \leq i \leq 4, & \\ \partial_t u_i - d_i \Delta u_i = (-1)^i(u_1u_3 - u_2u_4) + h_i 1_\omega 1_{i\leq j} & \text{in } (0,T^*)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T^*)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega, \end{cases} \tag{B.43}$$

satisfies

$$u(T^*,.) = u^*. \tag{B.44}$$

Remark B.3.7. The restriction on the dimension $N \in \{1, 2\}$ is a consequence of the following property : the solution of the free system (B.1) converges in $L^\infty(\Omega)$ when $T \to +\infty$ to a particular stationary solution of (B.1) (see [DF06]). One can extend Theorem B.3.6 to $N > 2$ if the convergence in $L^\infty(\Omega)$ (of the free system) holds. For $N > 2$, one only knows that a weak solution of the free system (B.1) converges in $L^1(\Omega)$ when $T \to +\infty$ to a particular stationary solution of (B.1) (see [PSZ17, Theorem 3]). But, for example, if we assume that *the diffusion coefficients d_i are close*, the weak solution of the free system (B.1) converges in $L^\infty(\Omega)$ when $T \to +\infty$ to a particular stationary solution of (B.1) (see [CDF14, Proposition 1.3]).

Remark B.3.8. The positivity assumption (B.42) is not restrictive. One can extend the result to nonnegative initial condition $u_0 \in X_{j,(d_i),(u_i^*)}$ (see [PSZ17, Section 5]).

B.4 Proof of Theorem B.3.2 : the local controllability to constant stationary states

The aim of this section is to prove Theorem B.3.2. As usual, we study the properties of controllability of the **linearized system** around $(u_i^*)_{1\leq i\leq 4}$ of (B.4). First, we transform the problem by studying the null-controllability of a family of linear control systems (see Annexe B.4.1). The **existence of controls in $L^2(Q)$** is a consequence of a duality method : the **Hilbert Uniqueness Method** introduced by Jacques-Louis Lions (see Annexe B.4.3.1). It links the existence of controls in $L^2(Q)$ with an **observability inequality** for solution of the adjoint system. This type of inequalities is proved by **Carleman estimates** (see Annexe B.4.3.2). In order to get more regular controls (in $L^p(Q)$

sense, $p \geq 2$), we use a sophistication of Hilbert Uniqueness Method called the **penalized Hilbert Uniqueness Method** introduced by Viorel Barbu (see Annexe B.4.4.1). Indeed, this enables to have controls a bit better than $L^2(Q)$. Then, a **bootstrap method** gives controls in $L^\infty(Q)$ (see Annexe B.4.4.2). A **fixed-point argument** concludes the proof (see Annexe B.4.5).

Now, we develop a strategy in order to treat the cases of 1, 2 or 3 controls in a unified way.

We introduce the following notations

$$B_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ h^3 = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ h^2 = \begin{pmatrix} h_1 \\ h_2 \\ 0 \\ 0 \end{pmatrix},\ B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ h^1 = \begin{pmatrix} h_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{B.45}$$

Let $j \in \{1,2,3\}$, $(u_1^*, u_2^*, u_3^*, u_4^*) \in (\mathbb{R}^+)^4$ be such that (B.3) holds and $u_0 \in X_{j,(d_i),(u_i^*)}$ (see (B.37), (B.39) and (B.41)).

B.4.1 Linearization

We adopt the approach presented in Annexe B.1.4.2.

B.4.1.1 3 controls, return method when $(u_1^*, u_3^*, u_4^*) = (0,0,0)$

We linearize (B.4) around $(u_i^*)_{1 \leq i \leq 4}$ and we get the system : for every $1 \leq i \leq 4$,

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_3^* u_1 - u_4^* u_2 + u_1^* u_3 - u_2^* u_4) + h_i 1_\omega 1_{i \leq 3} & \text{in } Q_T, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } \Sigma_T, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{B.46}$$

Roughly speaking, it is easy to control u_1, u_2, u_3 thanks to h_1, h_2, h_3. The main difficulty is to control u_4. Now, we present the heuristic way of controlling u_4.

B.4.1.1.1 First case : $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$ There is a coupling term in the fourth equation of (B.46) which enables to control u_4. For example, if $u_3^* \neq 0$, then u_1 controls u_4.

Remark B.4.1. In this case, the linearized system (B.46) looks like the toy-model (B.7) and its controllability properties come from Proposition B.1.7. Consequently, the local controllability of (B.4) can be proved as in Proposition B.1.12 for system (B.11).

B.4.1.1.2 Second case : $(u_1^*, u_3^*, u_4^*) = (0,0,0)$, return method The fourth equation of (B.46) is decoupled from the other equations. In particular, if $u_4(0,.) \neq 0$, then $u_4(T,.) \neq 0$. Consequently, system (B.46) is not controllable. The idea is to linearize around a non trivial trajectory of (B.4) which comes from $(0, u_2^*, 0, 0)$ and goes to $(0, u_2^*, 0, 0)$ and which forces the appearance of a coupling term after linearization. It is the **return method**. Here, we take

$$\big((0, u_2^*, \overline{u_3}^{\sharp}, 0), (0, 0, \overline{h_3}^{\sharp})\big) := \big((0, u_2^*, g, 0), (0, 0, \partial_t g - d_3 \Delta g)\big),$$

where g satisfies the following properties

$$g \in C^{\infty}(\overline{Q}),\ \ g \geq 0,\ \ g \neq 0,\ \ supp(g) \subset (0,T) \times \omega. \tag{B.47}$$

Then, if we linearize the system (B.4) around $\big((0, u_2^*, \overline{u_3}^{\sharp}, 0), (0, 0, \overline{h_3}^{\sharp})\big)$, then the fourth equation becomes

$$\partial_t u_4 - d_4 \Delta u_4 = \overline{u_3}^{\sharp}(t,x) u_1 - u_2^* u_4 \text{ in } (0,T) \times \Omega.$$

Roughly speaking, as $\overline{u_3}^{\sharp} \neq 0$ in the control zone, then u_1 controls u_4.

Remark B.4.2. Here, the linearized system around the non trivial trajectory looks like the toy-model (B.7) and its controllability properties follow from Proposition B.1.8. Consequently, the local controllability of (B.4) can be proved as Proposition B.1.14 for (B.11).

B.4.1.1.3 Linearization in $L^{\infty}(Q)$ and null-controllability of a family of linear systems We define

$$\overline{u_3} := \begin{cases} u_3^* & \text{if } (u_1^*, u_3^*, u_4^*) \neq (0,0,0), \\ \overline{u_3}^{\sharp} & \text{if } (u_1^*, u_3^*, u_4^*) = (0,0,0), \end{cases} \quad \text{and } \overline{h_3} := \begin{cases} 0 & \text{if } (u_1^*, u_3^*, u_4^*) \neq (0,0,0), \\ \overline{h_3}^{\sharp} & \text{if } (u_1^*, u_3^*, u_4^*) = (0,0,0), \end{cases} \tag{B.48}$$

$$(\zeta, \widehat{h^3}) := (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \widehat{h_1}, \widehat{h_2}, \widehat{h_3}) := (u_1 - u_1^*, u_2 - u_2^*, u_3 - \overline{u_3}, u_4 - u_4^*, h_1, h_2, h_3 - \overline{h_3}). \tag{B.49}$$

Thus, (u, h^3) is a trajectory of (B.4) if and only if $(\zeta, \widehat{h^3})$ is a trajectory of the following system

$$\begin{cases} \forall 1 \leq i \leq 4, & \\ \partial_t \zeta_i - d_i \Delta \zeta_i & \\ \qquad = (-1)^i ((\overline{u_3} + \zeta_3)\zeta_1 - (u_4^* + \zeta_4)\zeta_2 + u_1^* \zeta_3 - u_2^* \zeta_4) + \widehat{h_i} 1_{\omega} 1_{i \leq 3} & \text{in } Q_T, \\ \frac{\partial \zeta_i}{\partial n} = 0 & \text{on } \Sigma_T, \\ \zeta_i(0,.) = u_{i,0} - u_i^* & \text{in } \Omega. \end{cases}$$

Then, $(\zeta, \widehat{h^3})$ is a trajectory of

$$\begin{cases} \partial_t \zeta - D_3 \Delta \zeta = G(\zeta)\zeta + B_3 \widehat{h^3} 1_{\omega} & \text{in } (0,T) \times \Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{B.50}$$

where

$$D_3 := \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{pmatrix}, \quad G(\zeta) := \begin{pmatrix} -\overline{u_3} - \zeta_3 & u_4^* + \zeta_4 & -u_1^* & u_2^* \\ \overline{u_3} + \zeta_3 & -u_4^* - \zeta_4 & u_1^* & -u_2^* \\ -\overline{u_3} - \zeta_3 & u_4^* + \zeta_4 & -u_1^* & u_2^* \\ \overline{u_3} + \zeta_3 & -u_4^* - \zeta_4 & u_1^* & -u_2^* \end{pmatrix}. \tag{B.51}$$

Note that $G_{41}(0,0,0,0) = \overline{u_3}$. To simplify, we suppose the following fact : if $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$, then $u_3^* \neq 0$. Otherwise, we can easily adapt our proof strategy (see Remark B.4.16). Then, from (B.47), there exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset\subset \omega$ and $M > 0$ such that

$$\forall (t,x) \in (t_1,t_2) \times \omega_0,\ G_{41}(0,0,0,0)(t,x) \geq 2/M,$$

$$\forall (k,l) \in \{1,\dots,4\}^2,\ \|G_{kl}(0,0,0,0)\|_{L^\infty(Q)} \leq M/2.$$

Consequently, we study the null-controllability of the linear systems

$$\begin{cases} \partial_t \zeta - D_3 \Delta \zeta = A\zeta + B_3 \widehat{h^3} 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{B.52}$$

where the matrix A verifies the following assumptions

$$\forall (t,x) \in (t_1,t_2) \times \omega_0,\ a_{41}(t,x) \geq 1/M, \tag{B.53}$$

$$\forall (k,l) \in \{1,\dots,4\}^2,\ \|a_{kl}\|_{L^\infty(Q)} \leq M. \tag{B.54}$$

Remark B.4.3. To simplify the notations, we now denote $\widehat{h^3}$ by h^3.

B.4.1.2 2 controls, adequate change of variables

By Annexe B.2.3.2, we can assume that $(u_3^*, u_4^*) \neq (0,0)$.

B.4.1.2.1 First case : $d_3 = d_4$

From (B.25) and (B.39), system (B.4) reduces to

$$\begin{cases} \forall 1 \leq i \leq 3, & \\ \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2(u_3^* + u_4^* - u_3)) + h_i 1_\omega 1_{i \leq 2} & \text{in } (0,T) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{B.55}$$

We do not give the complete proof of Theorem B.3.2 in this case because it is an easy adaptation of the study of the null-controllability of the linear systems (B.52) which satisfy (B.53), (B.54) (with three equations instead of four). Indeed, by linearization around $(u_i^*)_{1 \leq i \leq 4}$ of (B.55), the equation satisfied by u_3 becomes

$$\partial_t u_3 - d_3 \Delta u_3 = -u_3^* u_1 + (u_3^* + u_4^*) u_2 - (u_1^* + u_2^*) u_3 \text{ in } (0,T) \times \Omega. \tag{B.56}$$

Then, there is a coupling term in (B.56) if and only if

$$(u_3^*, u_3^* + u_4^*) \neq (0,0) \text{ i.e. } (u_3^*, u_4^*) \neq (0,0). \tag{B.57}$$

B.4.1.2.2 Second case : $d_3 \neq d_4$

We remark that

$$\boxed{(u_1, u_2, u_3, u_4)(T,.) = (u_1^*, u_2^*, u_3^*, u_4^*)} \\ \text{if and only if} \\ \boxed{(u_1, u_2, u_3, u_3 + u_4)(T,.) = (u_1^*, u_2^*, u_3^*, u_3^* + u_4^*)}. \tag{B.58}$$

Therefore, we study the system satisfied by $(v_1, v_2, v_3, v_4) := (u_1, u_2, u_3, u_3 + u_4)$,

$$\begin{cases} \forall 1 \leq i \leq 3, & \\ \partial_t v_i - d_i \Delta v_i = (-1)^i (v_1 v_3 - v_2(v_4 - v_3)) + h_i 1_\omega 1_{i \leq 2} & \text{in } (0,T) \times \Omega, \\ \partial_t v_4 - d_4 \Delta v_4 = (d_3 - d_4)\Delta v_3 & \text{in } (0,T) \times \Omega, \\ \frac{\partial v_i}{\partial n} = \frac{\partial v_4}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ (v_i, v_4)(0,.) = (u_{i,0}, u_{3,0} + u_{4,0}) & \text{in } \Omega. \end{cases} \tag{B.59}$$

Roughly speaking, v_4 can be controlled by v_3 thanks to the coupling term of second order $(d_3 - d_4)\Delta v_3$ in the second equation of (B.59) and v_3 can be controlled by v_1 or v_2 because the linearization of the first equation of (B.59) with $i = 3$ is

$$\partial_t v_3 - d_3 \Delta v_3 = -u_3^* v_1 + u_4^* v_2 - (u_1^* + u_2^*) v_3 + u_2^* v_4 \text{ in } (0,T) \times \Omega,$$

and $(u_3^*, u_4^*) \neq (0,0)$. Then, the proof of the controllability properties of the linearized-system of (B.59) follows the ideas of Proposition B.1.9 and Proposition B.1.10. The main difference is the nature of the coupling terms : one coupling term of second order $(d_3 - d_4)\Delta v_3$ and one coupling term of zero order $-u_3^* v_1$ if $u_3^* \neq 0$ or $u_4^* v_2$ if $u_4^* \neq 0$.

B.4.1.2.3 Linearization in $L^\infty(Q)$ and null-controllability of a family of linear systems when $d_3 \neq d_4$

We define

$$(\zeta, h^2) := (\zeta_1, \zeta_2, \zeta_3, \zeta_4, h_1, h_2) := (v_1 - u_1^*, v_2 - u_2^*, v_3 - u_3^*, v_4 - (u_3^* + u_4^*), h_1, h_2). \tag{B.60}$$

Then, (u, h^2) is a trajectory of (B.4) if and only if (ζ, h^2) is a trajectory of

$$\begin{cases} \partial_t \zeta - D_2 \Delta \zeta = G(\zeta)\zeta + B_2 h^2 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases}$$

where

$$\begin{aligned} D_2 &:= \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & (d_3 - d_4) & d_4 \end{pmatrix}, \\ G(\zeta) &:= \begin{pmatrix} -(u_3^* + \zeta_3) & u_4^* + \zeta_4 - \zeta_3 & -u_1^* - u_2^* & u_2^* \\ u_3^* + \zeta_3 & -(u_4^* + \zeta_4 - \zeta_3) & u_1^* + u_2^* & -u_2^* \\ -(u_3^* + \zeta_3) & u_4^* + \zeta_4 - \zeta_3 & -u_1^* - u_2^* & u_2^* \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{B.61}$$

Note that $G_{31}(0,0,0,0) = -u_3^*$ and $G_{32}(0,0,0,0) = u_4^*$. Then, $(G_{31}(0,0,0,0), G_{32}(0,0,0,0)) \neq (0,0)$. To simplify, we suppose that $G_{31}(0,0,0,0) \neq 0$. The other case is similar. There exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset\subset \omega$ and $M > 0$ such that

$$\forall (t,x) \in (t_1,t_2) \times \omega_0,\ G_{31}(0,0,0,0)(t,x) \leq -2/M,$$

$$\forall (k,l) \in \{1,\dots,3\} \times \{1,\dots,3\},\ \|G_{kl}(0,0,0,0)\|_{L^\infty(Q)} \leq M/2,$$

$$G_{14} = -G_{24} = G_{34} = u_2^*,\ G_{41} = G_{42} = G_{43} = G_{44} = 0.$$

Consequently, we study the null-controllability of the linear systems

$$\begin{cases} \partial_t \zeta - D_2 \Delta \zeta = A\zeta + B_2 h^2 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{B.62}$$

where the matrix A verifies the following assumptions

$$\forall (t,x) \in (t_1,t_2) \times \omega_0,\ a_{31}(t,x) \leq -1/M, \tag{B.63}$$

$$\forall (k,l) \in \{1,\dots,3\} \times \{1,\dots,3\},\ \|a_{kl}\|_{L^\infty(Q)} \leq M, \tag{B.64}$$

$$a_{14} = -a_{24} = a_{34} = u_2^*, \tag{B.65}$$

$$a_{41} = a_{42} = a_{43} = a_{44} = 0. \tag{B.66}$$

Remark B.4.4. Actually, we can show the null controllability of a bigger family of linear systems. Indeed, we can replace (B.65) by the more general assumption : a_{14}, a_{24}, $a_{34} \in \mathbb{R}$ because it does not change the proof of the null-controllability result of the linear systems like (B.62) (see Proposition B.4.8). But, the more general case a_{14}, a_{24}, $a_{34} \in L^\infty(Q)$ is not handled by our proof of Proposition B.4.8 (see Annexe B.4.3.5 and in particular (B.137)).

Remark B.4.5. The algebraic relation (B.66) is useful to prove the null-controllability result of the linear systems like (B.62) (see Proposition B.4.8) because it creates the cascade form of (B.62). Indeed, the fourth and the third equation of (B.62) are

$$\partial_t \zeta_4 - d_4 \Delta \zeta_4 = (d_3 - d_4)\Delta \zeta_3 \text{ in } (0,T) \times \Omega,$$

with and $d_3 - d_4 \neq 0$,

$$\partial_t \zeta_3 - d_3 \Delta \zeta_3 = a_{31}\zeta_1 + a_{32}\zeta_2 + a_{33}\zeta_3 + u_2^* \zeta_4 \text{ in } (0,T) \times \Omega,$$

with $\forall (t,x) \in (t_1,t_2) \times \omega_0,\ a_{31}(t,x) \leq -1/M$.

B.4.1.3 1 control, adequate change of variables

By Annexe B.2.3.3, we can assume that $u_3^* \neq 0$.

B.4.1.3.1 First case : $\exists k \neq l \in \{2,3,4\},\ d_k = d_l$ We treat the case $d_2 = d_3$, $d_3 \neq d_4$. The other cases are similar. From (B.28) and (B.41), system (B.4) reduces to

$$\begin{cases} \forall i \in \{1,2,4\}, & \\ \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1(u_2^* + u_3^* - u_2) - u_2 u_4) + h_i 1_\omega 1_{i \leq 1} & \text{in } Q_T, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } \Sigma_T, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{B.67}$$

We remark that

$$\begin{aligned} &(u_1, u_2, u_4)(T,.) = (u_1^*, u_2^*, u_4^*) \\ &\text{if and only if} \\ &(u_1, u_2, u_2 - u_4)(T,.) = (u_1^*, u_2^*, u_2^* - u_4^*). \end{aligned} \tag{B.68}$$

Therefore, we study the system satisfied by $(v_1, v_2, v_3) := (u_1, u_2, u_2 - u_4)$,

$$\begin{cases} \forall 1 \leq i \leq 2, & \\ \partial_t v_i - d_i \Delta v_i = (-1)^i (v_1(u_2^* + u_3^* - v_2) - v_2(v_2 - v_3)) + h_i 1_\omega 1_{i \leq 1} & \text{in } Q_T, \\ \partial_t v_3 - d_4 \Delta v_3 = (d_2 - d_4)\Delta v_2 & \text{in } Q_T, \\ \frac{\partial v_i}{\partial n} = \frac{\partial v_3}{\partial n} = 0 & \text{on } \Sigma_T, \\ (v_i(0,.), v_3(0,.)) = (u_{i,0}, u_{2,0} - u_{4,0}) & \text{in } \Omega. \end{cases} \tag{B.69}$$

We do not give the complete proof of Theorem B.3.2 in this case because it is an easy adaptation of the study of the null-controllability of the linear systems (B.62) which satisfy (B.63), (B.64), (B.65) and (B.66) (with three equations instead of four). Indeed, v_3 can be controlled by v_2 thanks to the coupling term of second order $(d_2 - d_4)\Delta v_2$ in the second equation of (B.69) and v_2 can be controlled by v_1 because the linearization of the first equation of (B.69) with $i = 2$ is

$$\partial_t v_2 - d_2 \Delta v_2 = u_3^* v_1 + (-v_1^* - 2v_2^* + v_3^*) v_2 + u_2^* v_3 \text{ in } (0,T) \times \Omega,$$

where $(v_1^*, v_2^*, v_3^*) := (u_1^*, u_2^*, u_2^* - u_4^*)$ and $u_3^* \neq 0$.

B.4.1.3.2 Second case : $d_2 \neq d_3$, $d_3 \neq d_4$, $d_2 \neq d_4$. We introduce $\alpha \neq \beta$ such that

$$\alpha(d_2 - d_4) = \beta(d_3 - d_4) = 1, \text{ i.e. } \alpha = \frac{1}{d_2 - d_4} \text{ and } \beta = \frac{1}{d_3 - d_4}. \tag{B.70}$$

Then, we define $\gamma \neq 0$ by the algebraic relation

$$\alpha - \beta + \gamma = 0, \text{ i.e. } \gamma = \beta - \alpha. \tag{B.71}$$

We remark that

$$\begin{aligned} &\boxed{(u_1, u_2, u_3, u_4)(T,.) = (u_1^*, u_2^*, u_3^*, u_4^*)} \\ &\text{if and only if} \\ &\boxed{(u_1, u_2, u_2 + u_3, \alpha u_2 + \beta u_3 + \gamma u_4)(T,.) = (u_1^*, u_2^*, u_2^* + u_3^*, \alpha u_2^* + \beta u_3^* + \gamma u_4^*)}. \end{aligned} \tag{B.72}$$

Therefore, we study the system satisfied by $(v_1, v_2, v_3, v_4) := (u_1, u_2, u_2+u_3, \alpha u_2+\beta u_3+\gamma u_4)$. We introduce the following notations

$$g_1(v_2, v_3, v_4) := \frac{\beta-\alpha}{\gamma}v_2 - \frac{\beta}{\gamma}v_3 + \frac{1}{\gamma}v_4 = u_4,\ \ g_2(v_2, v_3) := v_3 - v_2 = u_3. \tag{B.73}$$

We have

$$\begin{cases} \forall 1 \leq i \leq 2, & \\ \partial_t v_i - d_i \Delta v_i = (-1)^i \left(g_2(v_2, v_3)v_1 - g_1(v_2, v_3, v_4)v_2\right) + h_i 1_\omega 1_{i\leq 1} & \text{in } (0,T)\times\Omega, \\ \partial_t v_3 - d_3\Delta v_3 = (d_2-d_3)\Delta v_2 & \text{in } (0,T)\times\Omega, \\ \partial_t v_4 - d_4\Delta v_4 = \Delta v_3 & \text{in } (0,T)\times\Omega, \\ \frac{\partial v_i}{\partial n} = \frac{\partial v_3}{\partial n} = \frac{\partial v_4}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (v_i, v_3, v_4)(0,.) = (u_{i,0}, u_{2,0}+u_{3,0}, \alpha u_{2,0}+\beta u_{3,0}+\gamma u_{4,0}) & \text{in } \Omega. \end{cases} \tag{B.74}$$

Roughly speaking, v_4 can be controlled by v_3 thanks to the coupling term of second order Δv_3 in the third equation of (B.74) and v_3 can be controlled by v_2 thanks to the coupling term of second order $(d_2-d_3)\Delta v_2$ in the second equation of (B.74) and v_2 can be controlled by v_1 because the linearization of the first equation of (B.74) with $i=2$ is

$$\begin{aligned} \partial_t v_2 - d_2\Delta v_2 &= g_2(v_2^*, v_3^*)v_1 - g_1(v_2^*, v_3^*, v_4^*)v_2 + v_1^* g_2(v_2, v_3) - v_2^* g_1(v_2, v_3, v_4) \\ &= u_3^* v_1 - g_1(v_2^*, v_3^*, v_4^*)v_2 + v_1^* g_2(v_2, v_3) - v_2^* g_1(v_2, v_3, v_4) \end{aligned}$$

and $u_3^* \neq 0$. Then, the proof of the controllability properties of the linearized-system of (B.74) follows the ideas of Proposition B.1.9 and Proposition B.1.10. The main difference is the nature of the coupling terms : two coupling terms of second order Δv_3, $(d_2-d_3)\Delta v_2$ and one coupling term of zero order $u_3^* v_1$.

B.4.1.3.3 Linearization in $L^\infty(Q)$ and null-controllability of a family of linear systems when $d_2 \neq d_3$, $d_2 \neq d_4$, $d_3 \neq d_4$ We define

$$(\zeta, h^1) := (\zeta_1, \zeta_2, \zeta_3, \zeta_4, h_1) := (v_1 - u_1^*, v_2 - u_2^*, v_3 - (u_2^*+u_3^*), v_4 - (\alpha u_2^* + \beta u_3^* + \gamma u_4^*), h_1). \tag{B.75}$$

Then, (u, h^1) is a trajectory of (B.4) if and only if (ζ, h^1) is a trajectory of

$$\begin{cases} \partial_t \zeta - D_1\Delta\zeta = G(\zeta)\zeta + B_1 h^1 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial\zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases}$$

where

$$\begin{aligned} D_1 &:= \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & d_2-d_3 & d_3 & 0 \\ 0 & 0 & 1 & d_4 \end{pmatrix}, \\ G(\zeta) &:= \begin{pmatrix} -(u_3^* + g_2(\zeta_2,\zeta_3)) & m_1 + g_1(\zeta_2,\zeta_3,\zeta_4) & -m_2 & m_3 \\ u_3^* + g_2(\zeta_2,\zeta_3) & -(m_1 + g_1(\zeta_2,\zeta_3,\zeta_4)) & m_2 & -m_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{B.76}$$

with $m_1 := u_1^* + u_2^* + u_4^*$, $m_2 := u_1^* + \frac{\beta}{\gamma} u_2^*$ and $m_3 = \frac{1}{\gamma} u_2^*$. Note that $G_{21}(0,0,0,0) = u_3^*$. There exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset\subset \omega$ and $M > 0$ such that

$$\forall (t,x) \in (t_1,t_2) \times \omega_0,\ G_{21}(0,0,0,0)(t,x) \geq 2/M,$$

$$\forall (k,l) \in \{1,2\} \times \{1,2\},\ \|G_{kl}(0,0,0,0)\|_{L^\infty(Q)} \leq M/2,$$

$$G_{13} = -G_{23} = -m_2,\ G_{14} = -G_{24} = m_3,\ G_{kl} = 0,\ 3 \leq k \leq 4,\ 1 \leq l \leq 4.$$

Consequently, we study the null-controllability of the linear systems

$$\begin{cases} \partial_t \zeta - D_1 \Delta \zeta = A\zeta + B_1 h^1 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{B.77}$$

where the matrix A verifies the following assumptions

$$\forall (t,x) \in (t_1,t_2) \times \omega_0,\ a_{21}(t,x) \geq 1/M, \tag{B.78}$$

$$\forall (k,l) \in \{1,2\} \times \{1,2\},\ \|a_{kl}\|_{L^\infty(Q)} \leq M, \tag{B.79}$$

$$a_{13} = -a_{23} = -m_2,\ a_{14} = -a_{24} = m_3, \tag{B.80}$$

$$a_{kl} = 0,\ 3 \leq k \leq 4,\ 1 \leq l \leq 4. \tag{B.81}$$

Remark B.4.6. Actually, we can show the null controllability of a bigger family of linear systems. Indeed, we can replace (B.80) by the more general assumption : a_{13}, a_{23}, a_{14}, $a_{24} \in \mathbb{R}$ because it does not change the proof of the null-controllability result of the linear systems like (B.77) (see Proposition B.4.8). But, the more general case a_{13}, a_{23}, a_{14}, $a_{24} \in L^\infty(Q)$ is not handled by our proof of Proposition B.4.8 (see Annexe B.4.3.7 and in particular (B.158) and (B.160)).

Remark B.4.7. The algebraic relation (B.81) is useful to prove the null-controllability result of the linear systems like (B.77) (see Proposition B.4.8) because it creates the cascade form of (B.77). Indeed, the fourth, the third and the second equation of (B.77) are

$$\partial_t \zeta_4 - d_4 \Delta \zeta_4 = \Delta \zeta_3 \text{ in } (0,T) \times \Omega,$$

$$\partial_t \zeta_3 - d_3 \Delta \zeta_3 = (d_2 - d_3)\Delta \zeta_2 \text{ in } (0,T) \times \Omega,\ \text{and } (d_2 - d_3) \neq 0,$$

$$\partial_t \zeta_2 - d_2 \Delta \zeta_2 = a_{21}\zeta_1 + a_{22}\zeta_2 + m_2\zeta_3 - m_3\zeta_4 \text{ in } (0,T) \times \Omega,$$

with $\forall (t,x) \in (t_1,t_2) \times \omega_0,\ a_{21}(t,x) \geq 1/M$.

B.4.2 Null controllability in $L^2(\Omega)^4$ with controls in $L^\infty(Q)^j$ of a family of linear control systems

B.4.2.1 Main result of this subsection

We introduce the following notations,

$$\mathcal{E}_3 := \{A \in \mathcal{M}_4(L^\infty(Q)) \ ; \ A \text{ verifies the assumptions (B.53) and (B.54)}\}, \tag{B.82}$$

$$H_3 := L^2(\Omega)^4, \tag{B.83}$$

$$\mathcal{E}_2 := \{A \in \mathcal{M}_4(L^\infty(Q)) \ ; \ A \text{ verifies the assumptions (B.63), (B.64), (B.65) and (B.66)}\}, \tag{B.84}$$

$$H_2 := \left\{\zeta_0 \in L^2(\Omega)^4 \ ; \int_\Omega \zeta_{0,4} = 0\right\}, \tag{B.85}$$

$$\mathcal{E}_1 := \{A \in \mathcal{M}_4(L^\infty(Q)) \ ; \ A \text{ verifies the assumptions (B.78), (B.79), (B.80) and (B.81)}\}, \tag{B.86}$$

$$H_1 := \left\{\zeta_0 \in L^2(\Omega)^4 \ ; \int_\Omega \zeta_{0,3} = \int_\Omega \zeta_{0,4} = 0\right\}. \tag{B.87}$$

The main result of this subsection is a null-controllability result in $L^2(\Omega)^4$ with controls in $L^\infty(Q)^j$ for families of linear control systems.

Proposition B.4.8. Let $j \in \{1,2,3\}$, D_j defined by (B.51), (B.61) or (B.76). There exists $C > 0$ such that, for every $A \in \mathcal{E}_j$ and $\zeta_0 = (\zeta_{0,1}, \zeta_{0,2}, \zeta_{0,3}, \zeta_{0,4}) \in H_j$, there exists $h^j \in L^\infty(Q)^j$ satisfying

$$\left\|h^j\right\|_{L^\infty(Q)^j} \leq C \left\|\zeta_0\right\|_{L^2(\Omega)^4}, \tag{B.88}$$

such that the solution $\zeta \in Y^4$ to the Cauchy problem

$$\begin{cases} \partial_t \zeta - D_j \Delta \zeta = A\zeta + B_j h^j 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{B.89}$$

verifies

$$\zeta(T,.) = 0.$$

Remark B.4.9. For every $1 \leq j \leq 3$, the diffusion matrices D_j defined by (B.51), (B.61) or (B.76) verify the assumption of Proposition B.2.3 because they are similar to $diag(d_1, d_2, d_3, d_4)$.

B.4.2.2 Proof strategy of Proposition B.4.8 : Null controllability in $L^2(\Omega)^4$ with controls in $L^\infty(Q)^j$ of a family of linear control systems

— We let evolve the system without control in $(0, t_1)$ (take $h^j(t,.) = 0$ in $(0, t_1)$). From Proposition B.2.2 and Proposition B.2.3, we get the existence of $C > 0$ such that for every $A \in \mathcal{E}_j$, $\zeta_0 \in L^2(\Omega)^4$, the solution to the Cauchy problem satisfies

$$\|\zeta^*\|_{L^2(\Omega)^4} \leq C \|\zeta_0\|_{L^2(\Omega)^4},$$

where

$$\zeta^* = \zeta(t_1, .).$$

— Then, we find $h^j : (t_1, t_2) \times \Omega \to \mathbb{R}$ such that

$$\left\| h^j \right\|_{L^\infty((t_1,t_2)\times\Omega)^j} \leq C \left\| \zeta(t_1,.) \right\|_{L^2(\Omega)^4},$$

and the solution to the Cauchy problem

$$\begin{cases} \partial_t \zeta - D_j \Delta\zeta = A\zeta + B_j h^j 1_\omega & \text{in } (t_1,t_2)\times\Omega, \\ \frac{\partial\zeta}{\partial n} = 0 & \text{on } (t_1,t_2)\times\partial\Omega, \\ \zeta(t_1,.) = \zeta^* & \text{in } \Omega, \end{cases}$$

verifies

$$\zeta(t_2,.) = 0.$$

— Then, we set $h^j(t,.) = 0$ so that $h^j(t,.) = 0$ for $t \in (t_2, T)$.

This strategy gives

$$\zeta(T,.) = 0 \text{ and } \left\| h^j \right\|_{L^\infty((0,T)\times\omega)^j} \leq C \left\| \zeta_0 \right\|_{L^2(\Omega)^4}.$$

To simplify, we now suppose

$$(t_1, t_2) = (0, T).$$

B.4.3 First step : Controls in $L^2(Q)^j$

The goal of this section is the proof of the following result.

Proposition B.4.10. Let $j \in \{1,2,3\}$. There exists $C > 0$ such that, for every $A \in \mathcal{E}_j$ and for every $\zeta_0 \in H_j$, there exists a control $h^j \in L^2(Q)^j$ satisfying

$$\left\| h^j \right\|_{L^2(Q)^j} \leq C \left\| \zeta_0 \right\|_{L^2(\Omega)^4} \tag{B.90}$$

such that the solution $\zeta \in Y^4$ to the Cauchy problem (B.89) satisfies $\zeta(T,.) = 0$.

The proof of Proposition B.4.10 will be done in Annexe B.4.3.3 for $j = 3$, Annexe B.4.3.5 for $j = 2$, Annexe B.4.3.7 for $j = 1$. It requires technical preliminary results presented in Annexe B.4.3.1, Annexe B.4.3.2, Annexe B.4.3.4, Annexe B.4.3.6.

B.4.3.1 Hilbert Uniqueness Method

First, for $\Phi \in L^2(\Omega)$, $(\Phi)_\Omega$ denotes the mean value of Φ,

$$(\Phi)_\Omega := \frac{1}{|\Omega|} \int_\Omega \Phi,$$

and for $\Psi \in C([0,T]; L^2(\Omega))$, $t \in [0,T]$, we introduce the notation

$$(\Psi)_\Omega(t) := \frac{1}{|\Omega|} \int_\Omega \Psi(t,x)dx.$$

By the HUM (Hilbert Uniqueness Method), the null-controllability result of Proposition B.4.10 is equivalent to the following observability inequality : (B.92) (see [Cor07a, Theorem 2.44]).

Let $j \in \{1, 2, 3\}$, D_j defined by (B.51), (B.61) or (B.76). There exists $C > 0$ such that, for every $A \in \mathcal{E}_j$ and $\varphi_T \in H_j$ (see (B.82), (B.83), (B.84), (B.85), (B.86), (B.87)) the solution φ of

$$\begin{cases} -\partial_t \varphi - D_j^T \Delta \varphi = A^T \varphi & \text{in } (0,T) \times \Omega, \\ \frac{\partial \varphi}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega, \end{cases} \tag{B.91}$$

verifies

$$\int_\Omega |\varphi(0,x)|^2 dx \leq C \left(\sum_{i=1}^{j} \int\int_{(0,T)\times\omega} |\varphi_i(t,x)|^2 dxdt \right). \tag{B.92}$$

It is easy to show that it is sufficient to prove the following observability inequalities.

There exists $C > 0$ such that, for every $A \in \mathcal{E}_3$ and $\varphi_T \in L^2(\Omega)^4$, the solution φ of the adjoint system (B.91) verifies

$$\int_\Omega |\varphi(0,x)|^2 dx \leq C \left(\sum_{i=1}^{3} \int\int_{(0,T)\times\omega} |\varphi_i(t,x)|^2 dxdt \right). \tag{B.93}$$

There exists $C > 0$ such that, for every $A \in \mathcal{E}_2$ and $\varphi_T \in L^2(\Omega)^4$, the solution φ of the adjoint system (B.91) verifies

$$\sum_{i=1}^{3} \left(\|\varphi_i(0,.)\|^2_{L^2(\Omega)} \right) + \|\varphi_4(0,.) - (\varphi_4)_\Omega(0)\|^2_{L^2(\Omega)} \leq C \left(\sum_{i=1}^{2} \int\int_{(0,T)\times\omega} |\varphi_i|^2 dxdt \right). \tag{B.94}$$

There exists $C > 0$ such that, for every $A \in \mathcal{E}_1$ and $\varphi_T \in L^2(\Omega)^4$, the solution φ of the adjoint system (B.91) verifies

$$\sum_{i=1}^{2} \left(\|\varphi_i(0,.)\|^2_{L^2(\Omega)} \right) + \sum_{i=3}^{4} \left(\|\varphi_i(0,.) - (\varphi_i)_\Omega(0)\|^2_{L^2(\Omega)} \right) \leq C \left(\int\int_{(0,T)\times\omega} |\varphi_1|^2 dxdt \right). \tag{B.95}$$

B.4.3.2 Carleman estimates

We introduce several weight functions. Let $\omega'' \subset\subset \omega_0$ be a nonempty open subset and $\eta_0 \in C^2(\overline{\Omega})$ verifying

$$\forall x \in \Omega,\ \eta_0(x) > 0,\ \eta_0 = 0 \text{ on } \partial\Omega,\ \forall x \in \overline{\Omega \setminus \omega''},\ |\nabla \eta_0(x)| > 0.$$

The existence of such a function is proved in [Cor07a, Lemma 2.68]. Let $\lambda \geq 1$ a parameter. We remark that

$$1 + f(\lambda) := 1 + \exp(-\lambda \|\eta_0\|_\infty) < 2. \tag{B.96}$$

We define $\forall (t,x) \in (0,T) \times \Omega$,

$$\phi(t,x) := \frac{e^{\lambda \eta_0(x)}}{t(T-t)} > 0, \ \alpha(t,x) := \frac{e^{\lambda \eta_0(x)} - e^{2\lambda \|\eta_0\|_\infty}}{t(T-t)} < 0, \tag{B.97}$$

$$\widehat{\alpha}(t) := \min_{x \in \overline{\Omega}} \alpha(t,x) = \frac{1 - e^{2\lambda \|\eta_0\|_\infty}}{t(T-t)} < 0, \ \widehat{\phi}(t) := \min_{x \in \overline{\Omega}} \phi(t,x) = \frac{1}{t(T-t)} > 0. \tag{B.98}$$

Theorem B.4.11. Carleman inequality
Let $d \in (0, +\infty)$, ω' an open subset such that $\omega'' \subset\subset \omega' \subset\subset \omega_0$ and $\beta \in \mathbb{R}$. There exist $C = C(\Omega, \omega', \beta)$, $\lambda_0 = C(\Omega, \omega', \beta)$, $s_0 = s_0(\Omega, \omega', \beta)$ such that, for any $\lambda \geq \lambda_0$, $s \geq s_0(T + T^2)$, $\varphi_T \in L^2(\Omega)$ and $f \in L^2(Q)$, the solution φ to

$$\begin{cases} -\partial_t \varphi - d\Delta \varphi = f & \text{in } (0,T) \times \Omega, \\ \frac{\partial \varphi}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega, \end{cases}$$

satisfies

$$\begin{aligned} & I(\beta, \lambda, s, \varphi) \\ & := \int_0^T \int_\Omega e^{2s\alpha} \left(\lambda^4 (s\phi)^{\beta+3} |\varphi|^2 + \lambda^2 (s\phi)^{\beta+1} |\nabla \varphi|^2 + (s\phi)^{\beta-1} \left(|\partial_t \varphi|^2 + |\Delta \varphi|^2 \right) \right) dxdt \\ & \leq C \left(\int_0^T \int_\Omega e^{2s\alpha} (s\phi)^\beta |f|^2 dxdt + \int_0^T \int_{\omega'} \lambda^4 e^{2s\alpha} (s\phi)^{\beta+3} |\varphi|^2 dxdt \right). \end{aligned} \tag{B.99}$$

The original proof of this inequality can be found in [FI96, Lemma 1.2].

Remark B.4.12. For a general introduction to global Carleman inequalities and their applications to the controllability of parabolic systems, one can see [FCG06] (in particular, see [FCG06, Lemma 1.3]). For Neumann conditions, one can see [FCGBGP06b] and in particular [FCGBGP06b, Lemma 1].

B.4.3.2.1 A parabolic regularity result in L^2 In the following, we consider initial conditions $\varphi_T \in C_0^\infty(\Omega)^4$ in order to improve the regularity of φ, solution of (B.91), and to allow some computations.

Definition B.4.13. We define the following spaces of functions

$$H^2_{Ne}(\Omega) := \left\{ u \in H^2(\Omega) \ ; \ \frac{\partial u}{\partial n} = 0 \right\}, \qquad Y_2 := L^2(0,T; H^2_{Ne}(\Omega)) \cap H^1(0,T; L^2(\Omega)).$$

Proposition B.4.14. Let $k \in \mathbb{N}^*$, $D \in \mathcal{M}_k(\mathbb{R})$ such that $Sp(D) \subset (0, +\infty)$, $A \in \mathcal{M}_k(L^\infty(Q))$, $u_0 \in C_0^\infty(\Omega)^k$. From [DHP07, Theorem 2.1], the following Cauchy problem admits a unique solution $u \in Y_2^k$

$$\begin{cases} \partial_t u - D\Delta u = A(t,x) u & \text{in } (0,T) \times \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases}$$

B.4.3.2.2 A technical lemma for Carleman estimates By now, unless otherwise specified, we denote by C (respectively C_ε) various positive constants varying from line to line (respectively various positive constants varying from line to line and depending on the parameter ε). We insist on the fact that C and C_ε do not depend on λ and s, unless otherwise specified.

Lemma B.4.15. Let $\Phi, \Psi \in Y_2$, $a \in L^\infty(Q)$, an open subset $\widetilde{\omega} \subset \omega_0$, $\Theta \in C^\infty(\overline{\Omega}; [0, +\infty[)$ such that $supp(\Theta) \subset \widetilde{\omega}$ and $r \in \mathbb{N}$. Then, for every $\varepsilon > 0$,

$$\forall (k,l) \in \mathbb{R}^2,\ k+l = 2r,\ \forall s \geq C,$$

$$\left| \int\int_{(0,T)\times\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r a \Phi \Psi \right| \tag{B.100}$$

$$\leq \varepsilon \int\int_{(0,T)\times\Omega} e^{2s\alpha} (s\phi)^k |\Phi|^2 + C_\varepsilon \int\int_{(0,T)\times\widetilde{\omega}} e^{2s\alpha} (s\phi)^l |\Psi|^2, \tag{B.101}$$

$$\forall (k,l) \in \mathbb{R}^2,\ k+l = 2(r+2),\ \forall s \geq C,$$

$$\left| \int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r \Phi \partial_t \Psi \right| \leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha} (s\phi)^k |\Phi|^2 + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{k-4} |\partial_t \Phi|^2 \right)$$
$$+ C_\varepsilon \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^l |\Psi|^2, \tag{B.102}$$

$$\forall (k,l) \in \mathbb{R}^2,\ k+l = 2(r+2),\ \forall s \geq C,$$

$$\left| \int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r \Phi \Delta \Psi \right|$$
$$\leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha} (s\phi)^k |\Phi|^2 + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{k-2} |\nabla \Phi|^2 + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{k-4} |\Delta \Phi|^2 \right)$$
$$+ C_\varepsilon \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^l |\Psi|^2. \tag{B.103}$$

$$\forall (k,l) \in \mathbb{R}^2,\ k+l = 2r,\ \forall s \geq C,$$

$$\int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r |\nabla \Phi|^2 \leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha} (s\phi)^k |\Delta \Phi|^2 + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{k+2} |\nabla \Phi|^2 \right)$$
$$+ C_\varepsilon \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^l |\Phi|^2. \tag{B.104}$$

Proof. The inequality (B.101) is an easy consequence of Young's inequality applied to

$$\left| \int\int_{(0,T)\times\widetilde{\omega}} \Theta e^{2s\alpha} (s\phi)^r a \Phi \Psi \right| \leq C \int\int_{(0,T)\times\widetilde{\omega}} \left(\sqrt{\varepsilon} e^{s\alpha} (s\phi)^{k/2} |\Phi| \right) \left(\frac{1}{\sqrt{\varepsilon}} \Theta e^{s\alpha} (s\phi)^{l/2} |\Psi| \right).$$

For (B.102), we integrate by parts with respect to the time variable

$$-\int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha}(s\phi)^r \Phi \partial_t \Psi = \int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha}(s\phi)^r \partial_t(\Phi)\Psi + \int_0^T \int_{\widetilde{\omega}} (\Theta e^{2s\alpha}(s\phi)^r)_t \Phi\Psi.$$

Moreover, by (B.97), we have $|(\Theta e^{2s\alpha}(s\phi)^r)_t| \leq C e^{2s\alpha} s^{r+1}\phi^{r+2} \leq e^{2s\alpha} s^{r+2}\phi^{r+2}$ for $s \geq C$. Then, we get (B.102) by applying Young's inequality to

$$\begin{aligned}\left|\int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha}(s\phi)^r \Phi \partial_t \Psi\right| \leq & \int_0^T \int_{\widetilde{\omega}} \left(\sqrt{\varepsilon} e^{s\alpha}(s\phi)^{k/2-2}\partial_t \Phi\right)\left(\frac{1}{\sqrt{\varepsilon}}\Theta e^{s\alpha}(s\phi)^{l/2}\Psi\right)\\ &+ \int_0^T \int_{\widetilde{\omega}} \left(\sqrt{\varepsilon} e^{s\alpha}(s\phi)^{k/2}\Phi\right)\left(\frac{1}{\sqrt{\varepsilon}} e^{s\alpha}(s\phi)^{l/2}\Psi\right).\end{aligned}$$

For (B.103), by twice integrating by parts with respect to the spatial variable, we get

$$\int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha}(s\phi)^r \Phi \Delta \Psi = \int_0^T \int_{\widetilde{\omega}} \Delta(\Theta e^{2s\alpha}(s\phi)^r \Phi)\Psi.$$

Moreover, by (B.97), we have

$$|\Delta(\Theta e^{2s\alpha}(s\phi)^r \Phi| \leq C\left(e^{2s\alpha}(s\phi)^r|\Delta\Phi| + e^{2s\alpha}(s\phi)^{r+1}|\nabla\Phi| + e^{2s\alpha}(s\phi)^{r+2}|\Phi|\right).$$

Then, we deduce (B.103) by Young's inequality applied to

$$\begin{aligned}\left|\int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha}(s\phi)^r \Phi \Delta \Psi\right| \leq & \int_0^T \int_{\widetilde{\omega}} \left(\sqrt{\varepsilon} e^{s\alpha}(s\phi)^{k/2-2}|\Delta\Phi|\right)\left(\frac{1}{\sqrt{\varepsilon}} e^{s\alpha}(s\phi)^{l/2}\Psi\right)\\ &+ \int_0^T \int_{\widetilde{\omega}} \left(\sqrt{\varepsilon} e^{s\alpha}(s\phi)^{k/2-1}|\nabla\Phi|\right)\left(\frac{1}{\sqrt{\varepsilon}} e^{s\alpha}(s\phi)^{l/2}\Psi\right)\\ &+ \int_0^T \int_{\widetilde{\omega}} \left(\sqrt{\varepsilon} e^{s\alpha}(s\phi)^{k/2}|\Phi|\right)\left(\frac{1}{\sqrt{\varepsilon}} e^{s\alpha}(s\phi)^{l/2}\Psi\right).\end{aligned}$$

For (B.104), we integrate by parts with respect to the spatial variable,

$$\int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha}(s\phi)^r |\nabla\Phi|^2 = -\int_0^T \int_{\widetilde{\omega}} \Theta e^{2s\alpha}(s\phi)^r (\Delta\Phi)\Phi - \int_{\omega_0} \nabla(\Theta e^{2s\alpha}(s\phi)^r).(\nabla\Phi)\Phi.$$

By using $|\nabla(\Theta e^{2s\alpha}(s\phi)^r)| \leq C e^{2s\alpha}(s\phi)^{r+1}$ which is a consequence of (B.97), we get (B.104) by Young's inequality. This concludes the proof of Lemma B.4.15. □

B.4.3.3 Proof with observation on three components : (B.93)

Proof. $\boxed{j=3}$

The proof is close to the proof of [CGR10, Lemma 7].

Let $A \in \mathcal{E}_3$ (see (B.82)), $\varphi_T \in C_0^\infty(\Omega)^4$ (the general case comes from a density argument, see (B.119), Lemma B.4.21 and Lemma B.4.22), $\varphi \in Y_2^4$ be the solution of

(B.91) (see Proposition B.4.14) and ω_1 be an open subset such that $\omega'' \subset\subset \omega_1 \subset\subset \omega_0$. We have

$$\left\{\begin{array}{ll} \forall 1 \leq i \leq 4, & \\ -\partial_t \varphi_i - d_i \Delta \varphi_i = a_{1i}\varphi_1 + a_{2i}\varphi_2 + a_{3i}\varphi_3 + a_{4i}\varphi_4 & \text{in } (0,T)\times\Omega, \\ \frac{\partial \varphi_i}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega. \end{array}\right. \tag{B.105}$$

We apply (B.99) of Theorem B.4.11 to each φ_i, $1 \leq i \leq 4$, with $\omega' = \omega_1$ and $\beta = 0$. Then, we sum (by using (B.54)) : for every $\lambda \geq C$,

$$\sum_{i=1}^{4} I(0,\lambda,s,\varphi_i) \leq C \left(\sum_{i=1}^{4} \left(\int_0^T \int_\Omega e^{2s\alpha}|\varphi_i|^2 dxdt + \int_0^T \int_{\omega_1} \lambda^4 e^{2s\alpha}(s\phi)^3 |\varphi_i|^2 dxdt \right) \right). \tag{B.106}$$

We fix $\lambda \geq C$ and we take s sufficiently large, then we can absorb the first right hand side term by the left hand side term of (B.106). We get

$$\sum_{i=1}^{4} I(0,\lambda,s,\varphi_i) \leq C \sum_{i=1}^{4} \int_0^T \int_{\omega_1} e^{2s\alpha}(s\phi)^3 |\varphi_i|^2 dxdt. \tag{B.107}$$

Now, λ, s are supposed to be fixed such that (B.107) holds and the constant C may depend on λ, s.

We have to get rid of the term $\int_0^T \int_{\omega_1} e^{2s\alpha}(s\phi)^3|\varphi_4|^2 dxdt$ in order to prove the observability inequality (B.93). For this, we are going to use (B.53). So, we are going to estimate φ_4 by φ_i for every $1 \leq i \leq 3$ thanks to the first equation of (B.105) with $i = 1$.

Estimate of $\int_0^T \int_{\omega_1} e^{2s\alpha}(s\phi)^3|\varphi_4|^2 dxdt$.

Let us introduce $\chi \in C^\infty(\overline{\Omega}; [0, +\infty[)$, such that the support of χ is included in ω_0 and $\chi = 1$ in ω_1. We multiply the first equation of (B.105) with $i = 1$ by $\chi(x)e^{2s\alpha}(s\phi)^3\varphi_4$ and we integrate on $(0,T)\times\omega_0$, which leads to

$$\begin{aligned} &\int_0^T \int_{\omega_1} e^{2s\alpha}(s\phi)^3|\varphi_4|^2 dxdt \\ &\leq M \int_0^T \int_{\omega_1} e^{2s\alpha}(s\phi)^3 a_{41}|\varphi_4|^2 dxdt \text{ by (B.53)} \\ &\leq M \int_0^T \int_{\omega_0} \chi(x) e^{2s\alpha}(s\phi)^3 a_{41}|\varphi_4|^2 dxdt \\ &\leq M \int_0^T \int_{\omega_0} \chi(x) e^{2s\alpha}(s\phi)^3 \varphi_4(-\partial_t\varphi_1 - d_1\Delta\varphi_1 - a_{11}\varphi_1 - a_{21}\varphi_2 - a_{31}\varphi_3) dxdt. \end{aligned} \tag{B.108}$$

Remark B.4.16. In Annexe B.4.1.1, we suppose that if $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$, then $u_3^* \neq 0$. Consequently, we have (B.53). If, $u_1^* \neq 0$ (or respectively $u_4^* \neq 0$), we can easily adapt the preceding strategy. We can assume that

$$\forall (t,x) \in (t_1,t_2)\times\omega_0,\ a_{43}(t,x) \geq 1/M \text{ (or respectively } a_{42}(t,x) \leq -1/M),$$

and multiply the first equation of (B.105) with $i = 3$ (or respectively $i = 2$) by $\chi(x)e^{2s\alpha}(s\phi)^3\varphi_4$ (or $-\chi(x)e^{2s\alpha}(s\phi)^3\varphi_4$) and we integrate on $(0,T)\times\omega_0$.

Let $\varepsilon > 0$ which will be chosen small enough. Now, we want to estimate the right hand side term of (B.108) by $\sum_{i=1}^{3}\int_0^T\int_{\omega_1} e^{2s\alpha}(s\phi)^m|\varphi_i|^2 dxdt$ with $m\in\mathbb{N}$.

First, we treat the terms $\int_0^T\int_{\omega_0}\chi(x)e^{2s\alpha}(s\phi)^3\varphi_4 a_{j1}\varphi_j dxdt$, for every $1\le j\le 3$. By applying Lemma B.4.15 : (B.101) with $\Phi=\varphi_4$, $\Psi=\varphi_j$, $a=a_{j1}$ (recalling (B.54)), $\Theta=\chi$, $r=3$ and $(k,l)=(3,3)$, we have

$$\begin{aligned}&\left|\int\int_{(0,T)\times\omega_0}\chi(x)e^{2s\alpha}(s\phi)^3\varphi_4 a_{j1}(t,x)\varphi_j dxdt\right|\\&\le\varepsilon\int\int_{(0,T)\times\Omega}e^{2s\alpha}(s\phi)^3|\varphi_4|^2dxdt+C_\varepsilon\int\int_{(0,T)\times\omega_0}e^{2s\alpha}(s\phi)^3|\varphi_j|^2dxdt.\end{aligned}\tag{B.109}$$

Then, we treat the term $-\int_0^T\int_{\omega_0}\chi(x)e^{2s\alpha}(s\phi)^3\varphi_4\partial_t\varphi_1 dxdt$. By applying Lemma B.4.15 : (B.102) with $\Phi=\varphi_4$, $\Psi=\varphi_1$, $a=1$, $\Theta=\chi$, $r=3$ and $(k,l)=(3,7)$, we have

$$\begin{aligned}\left|\int_0^T\int_{\omega_0}\chi e^{2s\alpha}(s\phi)^3\varphi_4\partial_t\varphi_1\right|&\le\varepsilon\left(\int_0^T\int_\Omega e^{2s\alpha}\Big\{(s\phi)^3|\varphi_4|^2+(s\phi)^{-1}|\partial_t\varphi_4|^2\Big\}\right)\\&\quad+C_\varepsilon\int_0^T\int_{\omega_0}e^{2s\alpha}(s\phi)^7|\varphi_1|^2.\end{aligned}\tag{B.110}$$

Finally, the last term $-d_1\int_0^T\int_{\omega_0}\chi(x)e^{2s\alpha}(s\phi)^3\varphi_4\Delta\varphi_1 dxdt$ is estimated as follows. By applying Lemma B.4.15 : (B.103) with $\Phi=\varphi_4$, $\Psi=\varphi_1$, $a=1$, $\Theta=\chi$, $r=3$ and $(k,l)=(3,7)$, we have

$$\begin{aligned}\left|d_1\int_0^T\int_{\omega_0}\chi e^{2s\alpha}(s\phi)^3\varphi_4\Delta\varphi_1\right|&\le\varepsilon\left(\int_0^T\int_\Omega e^{2s\alpha}\Big\{(s\phi)^{-1}|\Delta\varphi_4|^2+(s\phi)|\nabla\varphi_4|^2+(s\phi)^3|\varphi_4|^2\Big\}\right)\\&\quad+C_\varepsilon\int_0^T\int_{\omega_0}e^{2s\alpha}(s\phi)^7|\varphi_1|^2.\end{aligned}\tag{B.111}$$

Gathering (B.107), (B.108), (B.109), (B.110), (B.111), we get

$$\begin{aligned}\sum_{i=1}^4 I(0,\lambda,s,\varphi_i)&\le 3\varepsilon\left(\int_0^T\int_\Omega e^{2s\alpha}\Big\{(s\phi)^3|\varphi_4|^2+(s\phi)|\nabla\varphi_4|^2+(s\phi)^{-1}\left(|\partial_t\varphi_4|^2+|\Delta\varphi_4|^2\right)\Big\}\right)\\&\quad+C_\varepsilon\left(\sum_{i=1}^3\int_0^T\int_{\omega_0}e^{2s\alpha}(s\phi)^7|\varphi_i|^2dxdt\right).\end{aligned}\tag{B.112}$$

By taking ε small enough, we get

$$\sum_{i=1}^4 I(0,\lambda,s,\varphi_i)\le C_\varepsilon\left(\sum_{i=1}^3\int_0^T\int_{\omega_0}e^{2s\alpha}(s\phi)^7|\varphi_i|^2dxdt\right).\tag{B.113}$$

In particular, we deduce from (B.113) that

$$\sum_{i=1}^{4}\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\varphi_i|^2 \leq C\left(\sum_{i=1}^{3}\int_0^T\int_{\omega_0} e^{2s\alpha}(s\phi)^7|\varphi_i|^2 dxdt\right). \tag{B.114}$$

Then, by using the facts that

$$\min_{[T/4,3T/4]\times\overline{\Omega}} e^{2s\alpha}(s\phi)^3 > 0, \tag{B.115}$$

and

$$e^{2s\alpha}(s\phi)^7 \in L^\infty((0,T)\times\Omega), \tag{B.116}$$

we get

$$\sum_{i=1}^{4}\int_{T/4}^{3T/4}\int_\Omega |\varphi_i|^2 dxdt \leq C\left(\sum_{i=1}^{3}\int_0^T\int_{\omega_0} |\varphi_i|^2 dxdt\right). \tag{B.117}$$

From the dissipation of the energy in time for (B.105) (see Lemma B.7.1 in the Appendix), we easily get

$$\|\varphi(0,.)\|^2_{L^2(\Omega)^4} \leq C\left(\sum_{i=1}^{4}\int_{T/4}^{3T/4}\int_\Omega |\varphi_i|^2 dxdt\right). \tag{B.118}$$

Then, by using (B.117) and (B.118), we obtain

$$\|\varphi(0,.)\|^2_{L^2(\Omega)^4} \leq C\left(\sum_{i=1}^{3}\int_0^T\int_{\omega_0} |\varphi_i|^2 dxdt\right). \tag{B.119}$$

This ends the proof of the observability inequality (B.93) because $\omega_0 \subset \omega$. □

Remark B.4.17. Some stronger observability inequalities
We also have the following stronger inequality than (B.119) which can be proved from (B.114), (B.115) and (B.118). It will be used to find controls in $L^2_{wght}(Q) \subset L^2(Q)$ (see Annexe B.4.4.1). We have

$$\|\varphi(0,.)\|^2_{L^2(\Omega)^4} \leq C\left(\sum_{i=1}^{3}\int_0^T\int_{\omega} e^{2s\alpha}(s\phi)^7|\varphi_i|^2 dxdt\right). \tag{B.120}$$

Moreover, we also have an even stronger inequality (see (B.114)) than (B.119) and (B.120). It will be used to find controls in $L^\infty(Q)$ (see Annexe B.4.4.2).

B.4.3.4 Density results

In this section, we show that we can assume that the data φ_T is regular i.e. $\varphi_T \in C_0^\infty(\Omega)^4$. Moreover, we also need some regularity on the coupling matrix A for the case $j = 1$. It's the purpose of Lemma B.4.18.

Lemma B.4.18. Let $a \in L^\infty(Q)$. There exists $(a_k) \in (C_0^\infty(Q))^{\mathbb{N}}$ such that

$$\|a_k\|_{L^\infty(Q)} \leq \|a\|_{L^\infty(Q)}, \tag{B.121}$$

$$a_k \underset{k\to+\infty}{\rightharpoonup^*} a \text{ in } L^\infty(Q). \tag{B.122}$$

Proof. Let $k \in \mathbb{N}^*$, $\alpha_k \in C_0^\infty((0,T);[0,1])$, $\alpha_k(t) = 1$ in $(1/k, T-1/k)$, $\beta_k \in C_0^\infty((\Omega);[0,1])$, $\beta_k(x) = 1$ in $\{x \in \Omega \ ; \ d(x,\partial\Omega) \geq 1/k\}$ and $\xi_k \in C_0^\infty(Q)$ be defined by $\xi_k(t,x) = \alpha_k(t)\beta_k(x)$. Let ρ_k be a mollifier sequence in Q such that $\int_Q \rho_k = 1$.

Then, it is easy to show that $a_k := \xi_k.(\rho_k * a)$ satisfies the conclusion of Lemma B.4.18. □

Remark B.4.19. Actually, the previous lemma shows the density of $C_0^\infty(Q)$ in $L^\infty(Q)$ for the weak-star topology.

We also recall a particular case of the Aubin-Lions' lemma which is useful for the proof of Lemma B.4.21.

Lemma B.4.20. [Sim87, Section 8, Corollary 4]
A bounded subset of Y (see Definition B.2.1) is relatively compact in $L^2(Q)$.

Lemma B.4.21. Let $j \in \{1,2,3\}$, D_j defined by (B.51), (B.61) or (B.76), $A \in \mathcal{E}_j$ (see (B.82), (B.84) and (B.86)), $\varphi_T \in L^2(\Omega)^4$. We assume that

$$\varphi_{T,k} \in C_0^\infty(\Omega)^4 \underset{k\to+\infty}{\rightarrow} \varphi_T \text{ in } L^2(\Omega)^4, \tag{B.123}$$

$$A_k \in \mathcal{M}_4(C_0^\infty(Q)) \underset{k\to+\infty}{\rightharpoonup^*} A \text{ in } L^\infty(Q)^{16}. \tag{B.124}$$

Then, the sequence of solutions $\varphi_k \in Y^4$ of

$$\begin{cases} -\partial_t\varphi_k - D_j^T\Delta\varphi_k = A_k^T\varphi_k & \text{in } (0,T)\times\Omega, \\ \frac{\partial\varphi_k}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi_k(T,.) = \varphi_{T,k} & \text{in } \Omega, \end{cases} \tag{B.125}$$

weakly converges in Y^4 and strongly converges in $L^2(Q)^4$ to φ, the solution of (B.91).

Proof. First, recalling (B.123), we remark that $(\varphi_{T,k})_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega)^4$. Secondly, recalling (B.124), we remark that (A_k) is bounded in $\mathcal{M}_4(L^\infty(Q))$. Then, from Proposition B.2.3 : (B.17), we get that $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in Y^4. Then, up to a subsequence, we can suppose that there exists $\widetilde{\varphi} \in Y^4$ such that

$$\varphi_k \underset{k\to+\infty}{\rightharpoonup} \widetilde{\varphi} \text{ in } Y^4. \tag{B.126}$$

By Proposition B.2.2, we can also suppose that

$$\varphi_k(T,.) \underset{k\to+\infty}{\rightharpoonup} \widetilde{\varphi}(T,.) \text{ in } L^2(\Omega)^4. \tag{B.127}$$

But, by (B.123), we deduce that

$$\varphi_k(T,.) = \varphi_{T,k} \underset{k\to+\infty}{\rightharpoonup} \varphi_T \text{ in } L^2(\Omega)^4. \tag{B.128}$$

Therefore, by (B.127) and (B.128), we get

$$\widetilde{\varphi}(T,.) = \varphi_T. \tag{B.129}$$

By Lemma B.4.20, up to a subsequence, we can also assume that

$$\varphi_k \underset{k\to+\infty}{\rightarrow} \widetilde{\varphi} \text{ in } L^2(Q)^4. \tag{B.130}$$

Consequently, from (B.130) and (B.124), we have

$$A_k^T\varphi_k \underset{k\to+\infty}{\rightharpoonup} A^T\widetilde{\varphi} \text{ in } L^2(Q)^4. \tag{B.131}$$

By using (B.126), (B.131), (B.129) and by letting $k \to +\infty$ in (B.125), we have

$$\begin{cases} -\partial_t\widetilde{\varphi} - D_j^T\Delta\widetilde{\varphi} = A^T\widetilde{\varphi} & \text{in } (0,T)\times\Omega, \\ \frac{\partial\widetilde{\varphi}}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{\varphi}(T,.) = \varphi_T & \text{in } \Omega. \end{cases} \tag{B.132}$$

By uniqueness in Proposition B.2.3, we have $\widetilde{\varphi} = \varphi$. Then, $(\varphi_k)_{k\in\mathbb{N}}$ only has one limit-value : φ for the weak-convergence in Y^4 and for the strong convergence in $L^2(Q)^4$. The sequence $(\varphi_k)_{k\in\mathbb{N}}$ is relatively compact in Y equipped with the weak topology and $(\varphi_k)_{k\in\mathbb{N}}$ is relatively compact in $L^2(Q)^4$ equipped with the strong topology. Therefore,

$$\varphi_k \underset{k\to+\infty}{\rightharpoonup} \varphi \text{ in } Y^4,$$

$$\varphi_k \underset{k\to+\infty}{\rightarrow} \varphi \text{ in } L^2(Q)^4.$$

This concludes the proof of Lemma B.4.21. □

Lemma B.4.22. Let us suppose that $(\varphi_k)_{k\in\mathbb{N}} \in Y^{\mathbb{N}}$ weakly converges to φ in Y and strongly converges to φ in $L^2(Q)$. Then, we have

$$\forall r \in \mathbb{N},\ \int_0^T\int_\Omega e^{2s\alpha}(s\phi)^r|\varphi_k|^2dxdt \underset{k\to+\infty}{\rightarrow} \int_0^T\int_\Omega e^{2s\alpha}(s\phi)^r|\varphi|^2dxdt,$$

$$\|\varphi(0,.)\|_{L^2(\Omega)} \leq \liminf_{k\to+\infty}\|\varphi_k(0,.)\|_{L^2(\Omega)}.$$

Proof. The result is a consequence of the fact that $e^{2s\alpha}(s\phi)^r \in L^\infty(Q)$ and Proposition B.2.2. □

B.4.3.5 Proof with observation on two components : (B.94)

B.4.3.5.1 Another parabolic regularity result For the cases $j = 2$ (2 controls) and $j = 1$ (1 control), the diffusion matrix is not diagonal (see (B.61) and (B.76)). It creates coupling terms of second order. Roughly speaking, we differentiate some equations of the adjoint system (B.91) in order to benefit from these coupling terms before applying Carleman estimates. The following lemma justifies this strategy.

Lemma B.4.23. Let $d \in (0, +\infty)$, $f \in L^2(0,T;H^2_{Ne}(\Omega))$ and $y_0 \in C_0^\infty(\Omega)$. Let $y \in Y_2$ be the solution of

$$\begin{cases} \partial_t y - d\Delta y = f & \text{in } (0,T)\times\Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega. \end{cases} \tag{B.133}$$

Then, $z := \Delta y \in Y_2$ is the solution of

$$\begin{cases} \partial_t z - d\Delta z = \Delta f & \text{in } (0,T)\times\Omega, \\ \frac{\partial z}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0,.) = \Delta y_0 & \text{in } \Omega. \end{cases} \tag{B.134}$$

Proof. Let $\widetilde{z} \in Y_2$ be the solution of

$$\begin{cases} \partial_t \widetilde{z} - d\Delta \widetilde{z} = \Delta f & \text{in } (0,T)\times\Omega, \\ \frac{\partial \widetilde{z}}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{z}(0,.) = \Delta y_0 & \text{in } \Omega. \end{cases} \tag{B.135}$$

By Proposition B.2.2, we have $\widetilde{z} \in C([0,T];L^2(\Omega))$. Moreover, a.e. $t \in [0,T]$,

$$\frac{d}{dt}\int_\Omega \widetilde{z}(t,.) = d\int_\Omega \Delta\widetilde{z}(t,.) + \int_\Omega \Delta f(t,.) = 0.$$

Then, for every $t \in [0,T]$,

$$\int_\Omega \widetilde{z}(t,.) = \int_\Omega \widetilde{z}(0,.) = \int_\Omega \Delta y_0 = 0.$$

For every $t \in [0,T]$, let $\widetilde{y}(t,.)$ be the solution of

$$\begin{cases} \Delta\widetilde{y}(t,.) = \widetilde{z}(t,.) & \text{in } \Omega, \\ \frac{\partial \widetilde{y}(t,.)}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

By elliptic regularity, $\widetilde{y} \in C([0,T];H^2_{Ne}(\Omega)) \subset L^2(0,T;H^2_{Ne}(\Omega))$, $\partial_t\widetilde{y} \in L^2(0,T;H^2_{Ne}(\Omega)) \subset L^2(0,T;L^2(\Omega))$ since $\Delta\partial_t\widetilde{y} = \partial_t\widetilde{z}$. Moreover, $\widetilde{y}$ is the solution of (B.133) (by applying the operator Δ^{-1} to (B.135) and by using $\Delta^{-1}\partial_t\widetilde{z} = \partial_t\Delta^{-1}\widetilde{z}$). Then, by uniqueness, $\widetilde{y} = y$ and $\widetilde{z} = \Delta y$ is the solution of (B.134). $\square$

B.4.3.5.2 Proof of the observability inequality : (B.94)

Proof. $\boxed{j=2}$

Let $A \in \mathcal{E}_2$ (see (B.84)), $\varphi_T \in C_0^\infty(\Omega)^4$ (the general case comes from a density argument, see (B.153), Lemma B.4.21 and Lemma B.4.22), $\varphi \in Y_2^4$ be the solution of (B.91) (see Proposition B.4.14), ω_2 and ω_1 be two open subsets such that $\omega'' \subset\subset \omega_2 \subset\subset \omega_1 \subset\subset \omega_0$. Our goal is to prove (B.94).

We have : for every $1 \leq i \leq 2$,

$$\begin{cases} -\partial_t \varphi_i - d_i \Delta \varphi_i = a_{1i}\varphi_1 + a_{2i}\varphi_2 + a_{3i}\varphi_3 & \text{in } Q_T, \\ -\partial_t \varphi_3 - d_3 \Delta \varphi_3 = a_{13}\varphi_1 + a_{23}\varphi_2 + a_{33}\varphi_3 + (d_3 - d_4)\Delta\varphi_4 & \text{in } Q_T, \\ -\partial_t \varphi_4 - d_4 \Delta \varphi_4 = u_2^*(\varphi_1 - \varphi_2 + \varphi_3) & \text{in } Q_T, \\ \frac{\partial \varphi_i}{\partial n} = \frac{\partial \varphi_3}{\partial n} = \frac{\partial \varphi_4}{\partial n} = 0 & \text{on } \Sigma_T, \\ (\varphi_i, \varphi_3, \varphi_4)(T,.) = (\varphi_{i,T}, \varphi_{3,T}, \varphi_{4,T}) & \text{in } \Omega. \end{cases} \tag{B.136}$$

From (B.136) and Lemma B.4.23, we have

$$\begin{cases} -\partial_t (\Delta\varphi_4) - d_4 \Delta(\Delta \varphi_4) = \Delta(u_2^*(\varphi_1 - \varphi_2 + \varphi_3)) & \text{in } (0,T)\times\Omega, \\ \frac{\partial \Delta\varphi_4}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \Delta\varphi_4(T,.) = \Delta\varphi_{4,T} & \text{in } \Omega. \end{cases} \tag{B.137}$$

We apply the Carleman inequality (B.99) for (B.137) with $\beta = 0$ and $\omega' = \omega_2$, for every $\lambda, s \geq C$,

$$I(0,\lambda,s,\Delta\varphi_4) \leq C\left(\int_0^T\int_\Omega e^{2s\alpha}(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2| + |\Delta\varphi_3|^2) + \int_0^T\int_{\omega_2} \lambda^4 e^{2s\alpha}(s\phi)^3 |\Delta\varphi_4|^2\right). \tag{B.138}$$

After this, we apply the Carleman inequality (B.99) for the first two equations of (B.136) with $\beta = 2$ and $\omega' = \omega_2$ to obtain (by (B.64)), for every $\lambda, s \geq C$,

$$\sum_{i=1}^3 I(2,\lambda,s,\varphi_i) \leq C\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^2(|\varphi_1|^2 + |\varphi_2|^2| + |\varphi_3|^2 + |\Delta\varphi_4|^2)\right) + C\left(\int_0^T\int_{\omega_2} \lambda^4 e^{2s\alpha}(s\phi)^5(|\varphi_1|^2 + |\varphi_2|^2| + |\varphi_3|^2)\right). \tag{B.139}$$

We sum (B.138) and (B.139), for every $\lambda, s \geq C$,

$$\sum_{i=1}^3 I(2,\lambda,s,\varphi_i) + I(0,\lambda,s,\Delta\varphi_4) \leq C\left(\int_0^T\int_\Omega e^{2s\alpha}\left((s\phi)^2(|\varphi_1|^2 + |\varphi_2|^2| + |\varphi_3|^2 + |\Delta\varphi_4|^2) + |\Delta\varphi_1|^2 + |\Delta\varphi_2|^2| + |\Delta\varphi_3|^2\right)\right) + C\left(\int_0^T\int_{\omega_2} \lambda^4 e^{2s\alpha}\left((s\phi)^5(|\varphi_1|^2 + |\varphi_2|^2| + |\varphi_3|^2) + (s\phi)^3|\Delta\varphi_4|^2\right)\right). \tag{B.140}$$

We fix $\lambda \geq C$ and we absorb the first right-hand side term of (B.140) by the left-hand side terms of (B.140), by taking s sufficiently large. Then,

$$\sum_{i=1}^{3} I(2,\lambda,s,\varphi_i) + I(0,\lambda,s,\Delta\varphi_4)$$
$$\leq C\left(\int_0^T\int_{\omega_2} e^{2s\alpha}\left((s\phi)^5(|\varphi_1|^2+|\varphi_2|^2|+|\varphi_3|^2)+(s\phi)^3|\Delta\varphi_4|^2\right)\right). \tag{B.141}$$

Now, λ, s are supposed to be fixed and the constant C may depend on λ, s.

Then, we have to get rid of $\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2dxdt$ and $\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^5|\varphi_3|^2dxdt$. For the first term, we use the coupling term of second order $(d_3-d_4)\Delta$. For the second term, we use the coupling term of zero order thanks to property (B.63).

Estimate of $\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2dxdt$.

Let us introduce $\chi_2 \in C^\infty(\overline{\Omega};[0,+\infty[)$, such that the support of χ_2 is included in ω_1 and $\chi_2 = 1$ in ω_2. We multiply the second equation of (B.136) by $sign(d_3 - d_4)\chi_2(x)e^{2s\alpha}(s\phi)^3\Delta\varphi_4$ and we integrate on $(0,T)\times\omega_1$. As $d_3 \neq d_4$, we have

$$\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2dxdt \leq \int_0^T\int_{\omega_1}\chi_2(x)e^{2s\alpha}(s\phi)^3|\Delta\varphi_4|^2dxdt$$
$$\leq C\int_0^T\int_{\omega_1}\chi_2(x)e^{2s\alpha}(s\phi)^3\Delta\varphi_4(-\partial_t\varphi_3-d_3\Delta\varphi_3-a_{13}\varphi_1-a_{23}\varphi_2-a_{33}\varphi_3)dxdt. \tag{B.142}$$

Let $\varepsilon > 0$ which will be chosen small enough. We estimate the right hand side of (B.142) in the same way as the one of (B.108) :

— for terms involving $\Delta\varphi_4 a_{i3}\varphi_i$ with $1 \leq i \leq 3$, we apply (B.101) with $\Phi = \Delta\varphi_4$, $\Psi = \varphi_i$, $a = a_{i3} \in L^\infty(Q)$, $1 \leq i \leq 3$ (recalling (B.64)), $\Theta = \chi_2$ and $r = k = l = 3$,

— for the term involving $\Delta\varphi_4\partial_t\varphi_3$, we apply (B.102) with $\Phi = \Delta\varphi_4$, $\Psi = \varphi_3$, $a = 1$, $\Theta = \chi_2$ and $r = k = 3$, $l = 7$,

— for the term involving $\Delta\varphi_4\Delta\varphi_3$, we apply (B.103) with $\Phi = \Delta\varphi_4$, $\Psi = \varphi_3$, $a = d_3$, $\Theta = \chi_2$ and $r = k = 3$, $l = 7$.

From (B.141), (B.142), we get

$$\sum_{i=1}^{3} I(2,\lambda,s,\varphi_i) + I(0,\lambda,s,\Delta\varphi_4)$$
$$\leq 3\varepsilon\left(\int_0^T\int_\Omega e^{2s\alpha}\Big\{(s\phi)^3|\Delta\varphi_4|^2+(s\phi)|\nabla\Delta\varphi_4|^2+(s\phi)^{-1}\left(|\partial_t\Delta\varphi_4|^2+|\Delta\Delta\varphi_4|^2\right)\Big\}\right)$$
$$+ C_\varepsilon\left(\sum_{i=1}^{3}\int_0^T\int_{\omega_1} e^{2s\alpha}(s\phi)^7|\varphi_i|^2\right). \tag{B.143}$$

By taking ε small enough in (B.143), we get

$$\sum_{i=1}^{3} I(2,\lambda,s,\varphi_i) + I(0,s,\Delta\varphi_4) \leq C\left(\sum_{i=1}^{3}\int_0^T\int_{\omega_1} e^{2s\alpha}(s\phi)^7|\varphi_i|^2 dxdt\right). \tag{B.144}$$

<u>Estimate of $\int_0^T\int_{\omega_1} e^{2s\alpha}(s\phi)^7|\varphi_3|^2 dxdt$.</u>

Let us introduce $\chi_1 \in C^\infty(\overline{\Omega};[0,+\infty[)$, such that the support of χ_1 is included in ω_0 and $\chi_1 = 1$ in ω_1. We multiply the first equation of the adjoint system (B.136) with $i=1$ by $-\chi_1(x)e^{2s\alpha}(s\phi)^7\varphi_3$ and we integrate on $(0,T)\times\omega_0$. By using (B.63), we have

$$\int_0^T\int_{\omega_1} e^{2s\alpha}(s\phi)^7|\varphi_3|^2 dxdt \leq \int_0^T\int_{\omega_0} \chi_1(x)e^{2s\alpha}(s\phi)^7|\varphi_3|^2 dxdt$$
$$\leq C\int_0^T\int_{\omega_0} \chi_1(x)e^{2s\alpha}(s\phi)^7\varphi_3(-\partial_t\varphi_1 - d_1\Delta\varphi_1 - a_{11}\varphi_1 - a_{21}\varphi_2)dxdt. \tag{B.145}$$

Let $\varepsilon' > 0$ which will be chosen small enough. We estimate the right hand side of (B.145) in the same way as the one of (B.108) :

- for terms involving $\varphi_3 a_{i1}\varphi_i$ with $1 \leq i \leq 2$, we apply (B.101) with $\Phi = \varphi_3$, $\Psi = \varphi_i$, $a = a_{i3} \in L^\infty(Q)$, $1 \leq i \leq 2$ (recalling (B.64)), $\Theta = \chi_1$ and $r = 7$, $k = 5$, $l = 9$,
- for the term involving $\varphi_3\partial_t\varphi_1$, we apply (B.102) with $\Phi = \varphi_3$, $\Psi = \varphi_1$, $a = 1$, $\Theta = \chi_1$ and $r = 7$, $k = 5$, $l = 13$,
- for the term involving $\varphi_3\Delta\varphi_1$, we apply (B.103) with $\Phi = \varphi_3$, $\Psi = \varphi_1$, $a = d_1$, $\Theta = \chi_1$ and $r = 7$, $k = 5$, $l = 13$.

Then, we obtain

$$\int_0^T\int_{\omega_1} e^{2s\alpha}(s\phi)^7|\varphi_3|^2$$
$$\leq 3\varepsilon'\left(\int_0^T\int_{\Omega} e^{2s\alpha}\left\{(s\phi)^5|\varphi_3|^2 + (s\phi)^3|\nabla\varphi_3|^2 + (s\phi)(|\partial_t\varphi_3|^2 + |\Delta\varphi_3|^2)\right\}\right)$$
$$+ C_{\varepsilon'}\left(\sum_{i=1}^{2}\int_0^T\int_{\omega_0} e^{2s\alpha}(s\phi)^{13}|\varphi_i|^2\right). \tag{B.146}$$

By using (B.144), (B.146) and by taking ε' sufficiently small, we get

$$\sum_{i=1}^{3} I(2,\lambda,s,\varphi_i) + I(0,\lambda,s,\Delta\varphi_4) \leq C\left(\sum_{i=1}^{2}\int\int_{(0,T)\times\omega_0} e^{2s\alpha}(s\phi)^{13}|\varphi_i|^2\right). \tag{B.147}$$

Then, we deduce from (B.147) that we have

$$\int_0^T\int_\Omega \sum_{i=1}^{3} e^{2s\alpha}(s\phi)^5|\varphi_i|^2 + e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\Delta\varphi_4|^2 \leq C\left(\sum_{i=1}^{2}\int\int_{(0,T)\times\omega_0} e^{2s\alpha}(s\phi)^{13}|\varphi_i|^2\right), \tag{B.148}$$

where $\widehat{\phi}$ and $\widehat{\alpha}$ are defined in (B.98). In particular, $\widehat{\phi}$ and $\widehat{\alpha}$ do not depend on the spatial variable x. In order to estimate φ_4 by $\Delta\varphi_4$, we use the classical lemma and the corollary that follow.

Lemma B.4.24. Poincaré-Wirtinger inequality
There exists $C = C(\Omega)$ such that

$$\forall u \in H^1(\Omega), \int_\Omega (u(x) - (u)_\Omega)^2 \, dx \leq C \int_\Omega |\nabla u(x)|^2 dx. \tag{B.149}$$

Corollary B.4.25. There exists $C = C(\Omega)$ such that

$$\forall u \in H^2_{Ne}(\Omega) := \left\{ u \in H^2(\Omega) \ ; \ \frac{\partial u}{\partial n} = 0 \right\}, \ \int_\Omega |\nabla u(x)|^2 dx \leq C \int_\Omega |\Delta u(x)|^2 dx. \tag{B.150}$$

Proof. Let $u \in H^2_{Ne}(\Omega)$ satisfying $\|\nabla u\|_{L^2(\Omega} \neq 0$. Otherwise, the inequality (B.150) is trivial. We have by an integration by parts and by using (B.149),

$$\begin{aligned}\int_\Omega |\nabla u|^2 = -\int_\Omega (\Delta u) u = -\int_\Omega (\Delta u)(u - (u)_\Omega) \leq \ & \|\Delta u\|_{L^2(\Omega)} \|u - (u)_\Omega\|_{L^2(\Omega} \\ \leq \ & C \|\Delta u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega} .\end{aligned}$$

We conclude the proof of Corollary B.4.25 by simplifying by $\|\nabla u\|_{L^2(\Omega}$. ☐

Then, by applying the Poincaré-Wirtinger inequality (B.149) and (B.150) to φ_4, we deduce from (B.148) that

$$\begin{aligned}&\int_0^T \int_\Omega \sum_{i=1}^3 e^{2s\alpha}(s\phi)^5|\varphi_i|^2 + e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_4 - (\varphi_4)_\Omega|^2 \\ &\leq C \left(\sum_{i=1}^2 \int\int_{(0,T)\times\omega_0} e^{2s\alpha}(s\phi)^{13}|\varphi_i|^2 \right).\end{aligned} \tag{B.151}$$

Now, from the dissipation in time of the energy of $(\varphi_1, \varphi_2, \varphi_3, \varphi_4 - (\varphi_4)_\Omega)$ (see Lemma B.7.1 in the Appendix), we get

$$\begin{aligned}&\sum_{i=1}^3 \left(\|\varphi_i(0,.)\|^2_{L^2(\Omega)} \right) + \|\varphi_4(0,.) - (\varphi_4)_\Omega(0)\|^2_{L^2(\Omega)} \\ &\leq C \int_{T/4}^{3T/4} \left(\sum_{i=1}^3 \left(\|\varphi_i(t,.)\|^2_{L^2(\Omega)} \right) + \|\varphi_4(t,.) - (\varphi_4)_\Omega(t)\|^2_{L^2(\Omega)} \right) dt.\end{aligned} \tag{B.152}$$

Consequently, from (B.151), (B.152) and the same arguments given between (B.114) and (B.119), we easily deduce that

$$\begin{aligned}&\sum_{i=1}^3 \left(\|\varphi_i(0,.)\|^2_{L^2(\Omega)} \right) + \|\varphi_4(0,.) - (\varphi_4)_\Omega(0)\|^2_{L^2(\Omega)} \\ &\leq C \left(\sum_{i=1}^2 \int\int_{(0,T)\times\omega} e^{2s\alpha}(s\phi)^{13}|\varphi_i|^2 dxdt \right),\end{aligned} \tag{B.153}$$

and consequently the observability inequality (B.94) because $e^{2s\alpha}(s\phi)^{13}$ is bounded.

This ends the proof of the observability inequality (B.94). $\square$

B.4.3.6 Another Carleman inequality

Theorem B.4.26. Carleman inequality
Let $d \in (0,+\infty)$, ω' an open subset such that $\omega'' \subset\subset \omega' \subset\subset \omega_0$. There exist $C = C(\Omega,\omega')$, $\lambda_0 = \lambda_0(\Omega,\omega')$ such that, for every $\lambda \geq \lambda_0$, there exists $s_0 = s_0(\Omega,\omega',\lambda)$ such that, for any $s \geq s_0(T+T^2)$, any $\varphi_T \in L^2(\Omega)$ and any $f \in L^2(0,T;H^2_{Ne}(\Omega))$, the solution φ of

$$\begin{cases} -\partial_t\varphi - d\Delta\varphi = \Delta f & \text{in } (0,T)\times\Omega, \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega, \end{cases}$$

satisfies

$$\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\varphi|^2dxdt \leq C\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4|f|^2dxdt + \int_0^T\int_{\omega'} e^{2s\alpha}(s\phi)^3|\varphi|^2dxdt\right). \tag{B.154}$$

The proof of this inequality can be found in [CSG15, Lemma A.1] (see in particular that the equality [CSG15, (A.3)] still holds for $f \in L^2(0,T;H^2_{Ne}(\Omega))$).

Remark B.4.27. The estimate (B.154) is different from (B.99) because (B.99) gives us

$$\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\varphi|^2dxdt \leq C\left(\int_0^T\int_\Omega e^{2s\alpha}|\Delta f|^2dxdt + \int_0^T\int_{\omega'} e^{2s\alpha}(s\phi)^3|\varphi|^2dxdt\right). \tag{B.155}$$

Therefore, (B.154) is useful when one wants an observation of φ in term of f (but not in term of Δf). Roughly, we remark that we have to *pay* this type of estimate with a weight $(s\phi)^4$ (see the first right hand side terms of (B.154) and (B.155)).

B.4.3.7 Proof with observation on one component : (B.95)

We have seen in Annexe B.4.3.5 that parabolic regularity allows us to apply Δ to the third equation of (B.136) (see (B.137)) in order to benefit from the coupling term of second order $(d_3 - d_4)\Delta\varphi_4$. The case $j = 1$ requires more regularity because we have to benefit from **two** terms of coupling of second order. Therefore, we need to apply $\Delta\Delta$ (see (B.158)). There are two main difficulties. First, Proposition B.4.14 only shows us that φ, the solution of (B.91) is in Y_2^4. However, we need : $\Delta\varphi \in Y_2^4$. That is why we regularize the coupling matrix $A \in \mathcal{E}_1$ (see Lemma B.4.18). Secondly, we want an observation of $\Delta\Delta\varphi_4$ in term of $\Delta\varphi_1$, $\Delta\varphi_2$ (and not in term of $\Delta\Delta\varphi_1$, $\Delta\Delta\varphi_2$ because we do not have these terms in Carleman estimates applied to φ_1 and φ_2 : see (B.162) and (B.163)). That is why we use Theorem B.4.26.

Proof. $\boxed{j=1}$

Let $A \in \mathcal{M}_4(C_0^\infty(Q)) \cap \mathcal{E}_1$ (see (B.86)), $\varphi_T \in C_0^\infty(\Omega)^4$ (the general case comes from a density argument, see (B.185), Lemma B.4.18, Lemma B.4.21 and Lemma B.4.22), $\varphi \in Y_2^4$ be the solution of (B.91) (see Proposition B.4.14), ω_3, ω_2, ω_2' and ω_1 be four open subsets such that $\omega'' \subset\subset \omega_3 \subset\subset \omega_2 \subset\subset \omega_2' \subset\subset \omega_1 \subset\subset \omega_0$. Our goal is to prove (B.95).

We have

$$\begin{cases} -\partial_t\varphi_1 - d_1\Delta\varphi_1 = a_{11}\varphi_1 + a_{21}\varphi_2 & \text{in } (0,T)\times\Omega,\\ -\partial_t\varphi_2 - d_2\Delta\varphi_2 = a_{12}\varphi_1 + a_{22}\varphi_2 + (d_2-d_3)\Delta\varphi_3 & \text{in } (0,T)\times\Omega,\\ -\partial_t\varphi_3 - d_3\Delta\varphi_3 = -m_2(\varphi_1-\varphi_2) + \Delta\varphi_4 & \text{in } (0,T)\times\Omega,\\ -\partial_t\varphi_4 - d_4\Delta\varphi_4 = m_3(\varphi_1-\varphi_2) & \text{in } (0,T)\times\Omega,\\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega,\\ \varphi(T,.) = \varphi_T & \text{in } \Omega. \end{cases} \tag{B.156}$$

First, by using the regularity : $\varphi \in Y_2^4$ and by applying consecutively Lemma B.4.23 to the fourth equation of (B.156), the third equation of (B.156), the second equation of (B.156), the first equation of (B.156), we get

$$\Delta\varphi \in L^2(0,T;H^2_{Ne}(\Omega))^4. \tag{B.157}$$

Consequently, we can apply $\Delta\Delta$ to the fourth equation of (B.156) by using (B.157) and Lemma B.4.23,

$$\begin{cases} -\partial_t(\Delta\Delta\varphi_4) - d_4\Delta(\Delta\Delta\varphi_4) = \Delta\Delta(m_3(\varphi_1-\varphi_2)) & \text{in } (0,T)\times\Omega,\\ \frac{\partial\Delta\Delta\varphi_4}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega,\\ \Delta\Delta\varphi_4(T,.) = \Delta\Delta\varphi_{4,T} & \text{in } \Omega. \end{cases} \tag{B.158}$$

Then, we use the Carleman inequality (B.154) for (B.158) with $\omega' = \omega_3$ and $f = \Delta(m_3(\varphi_1-\varphi_2)) \in L^2(0,T;H^2_{Ne}(\Omega))$, for every $\lambda, s \geq C$,

$$\begin{aligned} &\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2\\ &\leq C\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4\left(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2\right) + \int_0^T\int_{\omega_3} e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2\right). \end{aligned} \tag{B.159}$$

Remark B.4.28. Here, we have to apply the Carleman estimate (B.154) instead of (B.99) in order to get in the right hand side of (B.159) only terms of order two (and not more) in φ_1, φ_2. Otherwise, we cannot absorb the remaining terms thanks to Carleman estimates (B.99) applied to φ_1, φ_2.

Then, we apply Δ to the third equation of (B.156) thanks to (B.158) and Lemma B.4.23, for every $\lambda, s \geq C$,

$$\begin{cases} -\partial_t(\Delta\varphi_3) - d_3\Delta(\Delta\varphi_3) = \Delta(-m_2(\varphi_1-\varphi_2)) + \Delta\Delta\varphi_4 & \text{in } Q_T,\\ \frac{\partial\Delta\varphi_3}{\partial n} = 0 & \text{on } \Sigma_T,\\ \Delta\varphi_3(T,.) = \Delta\varphi_{3,T} & \text{in } \Omega. \end{cases} \tag{B.160}$$

We use the Carleman inequality (B.99) with $\omega' = \omega_3$ and $\beta = 2$, for every $\lambda, s \geq C$,

$$I(2,\lambda,s,\Delta\varphi_3) \leq C\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^2(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2|+|\Delta\Delta\varphi_4|^2)+\int_0^T\int_{\omega_2}\lambda^4 e^{2s\alpha}(s\phi)^5|\Delta\varphi_3|^2\right). \quad (B.161)$$

Then, we apply the Carleman inequality (B.99) with $\omega' = \omega_3$ and $\beta = 5$ to the second equation and the first equation of (B.156) (by (B.79)), for every $\lambda, s \geq C$,

$$\lambda I(5,\lambda,s,\varphi_2) \leq C\left(\int_0^T\int_\Omega \lambda e^{2s\alpha}(s\phi)^5(|\varphi_1|^2+|\varphi_2|^2|+|\Delta\varphi_3|^2)+\int_0^T\int_{\omega_3}\lambda^5 e^{2s\alpha}(s\phi)^8|\varphi_2|^2\right), \quad (B.162)$$

$$\lambda I(5,\lambda,s,\varphi_1) \leq C\left(\int_0^T\int_\Omega \lambda e^{2s\alpha}(s\phi)^5(|\varphi_1|^2+|\varphi_2|^2|)+\int_0^T\int_{\omega_3}\lambda^5 e^{2s\alpha}(s\phi)^8|\varphi_1|^2\right). \quad (B.163)$$

We sum (B.159), (B.161), (B.162), (B.163) and we take λ and s sufficiently large,

$$\begin{aligned}&\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2dxdt + I(2,\lambda,s,\Delta\varphi_3)+\lambda I(5,\lambda,s,\varphi_2)+\lambda I(5,\lambda,s,\varphi_1)\\ &\leq C\left(\int_0^T\int_{\omega_3} e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2dxdt+\int_0^T\int_{\omega_3}\lambda^4 e^{2s\alpha}(s\phi)^5|\Delta\varphi_3|^2dxdt\right)\\ &+C\left(\int_0^T\int_{\omega_3}\lambda^5 e^{2s\alpha}(s\phi)^8|\varphi_2|^2dxdt+\int_0^T\int_{\omega_3}\lambda^5 e^{2s\alpha}(s\phi)^8|\varphi_1|^2dxdt\right). \quad (B.164)\end{aligned}$$

Now, λ and s are supposed to be fixed. The constant C may depend on λ and s. We have

$$\begin{aligned}&\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2dxdt + I(2,\lambda,s,\Delta\varphi_3)+I(5,\lambda,s,\varphi_2)+I(5,\lambda,s,\varphi_1)\\ &\leq C\left(\int_0^T\int_{\omega_3} e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2dxdt+\int_0^T\int_{\omega_3} e^{2s\alpha}(s\phi)^5|\Delta\varphi_3|^2dxdt\right)\\ &+C\left(\int_0^T\int_{\omega_3} e^{2s\alpha}(s\phi)^8|\varphi_2|^2dxdt+\int_0^T\int_{\omega_3} e^{2s\alpha}(s\phi)^8|\varphi_1|^2dxdt\right). \quad (B.165)\end{aligned}$$

Remark B.4.29. Here, we take advantage of the two parameters λ and s in Theorem B.4.11. Indeed, if we forget λ, we would need to sum $\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2dxdt$, $I(4,s,\Delta\varphi_3)$, $I(6,s,\varphi_2)$ and $I(6,s,\varphi_1)$. Therefore, we would get in the right hand side $\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4|\Delta\Delta\varphi_4|^2dxdt$ which cannot be absorbed by the left hand side.

Then ,we have to get rid of $\int_0^T\int_{\omega_3} e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2dxdt$, $\int_0^T\int_{\omega_3} e^{2s\alpha}(s\phi)^5|\Delta\varphi_3|^2dxdt$ and
$\int_0^T\int_{\omega_3} e^{2s\alpha}(s\phi)^8|\varphi_2|^2dxdt$. For the first term, we use the coupling term of fourth order $\Delta\Delta$. For the second term, we use the coupling term of second order $(d_2-d_3)\Delta$. For the third term, we use the coupling term of zero order thanks to property (B.78).

Estimate of $\int_0^T\int_{\omega_3} e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2dxdt$.

Let us introduce $\chi_3 \in C^\infty(\overline{\Omega};[0,+\infty[)$, such that the support of χ_3 is included in ω_2 and $\chi_3 = 1$ in ω_3. We multiply the first equation (B.160) by $(\chi_3(x))^2e^{2s\alpha}(s\phi)^3\Delta\Delta\varphi_4$ and we integrate on $(0,T)\times\omega_2$. We have

$$\begin{aligned}&\int_0^T\int_{\omega_2}(\chi_3(x))^2e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2dxdt\\&\leq C\int_0^T\int_{\omega_2}(\chi_3(x))^2e^{2s\alpha}(s\phi)^3\Delta\Delta\varphi_4(-\partial_t\Delta\varphi_3-d_3\Delta\Delta\varphi_3+m_2\Delta\varphi_1-m_2\Delta\varphi_2)dxdt.\end{aligned} \tag{B.166}$$

Remark B.4.30. One can see the presence of $(\chi_3(x))^2$ instead of $\chi_3(x)$ as before (see for example (B.108)). It is purely technical (see the proofs of Lemma B.4.31 and Lemma B.4.32).

Let $\varepsilon \in (0,1)$ which will be chosen small enough. First, for every $1 \leq i \leq 2$, by applying Lemma B.4.15 : (B.101) with $\Phi = \Delta\Delta\varphi_4$, $\Psi = \Delta\varphi_i$, $a = m_2$, $\Theta = (\chi_3)^2$, $r = 3$ and $(k,l) = (3,3)$, we have

$$\begin{aligned}&\int_0^T\int_{\omega_2}\chi_3^2e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)m_2\Delta\varphi_i\\&\leq\varepsilon\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2+C_\varepsilon\int_0^T\int_{\omega_2}\chi_3^2e^{2s\alpha}(s\phi)^3|\Delta\varphi_i|^2.\end{aligned} \tag{B.167}$$

But, the other terms in the right hand side of (B.166) i.e.

$$\int_0^T\int_{\omega_2}(\chi_3(x))^2e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\partial_t\Delta\varphi_3)dxdt,$$

and

$$\int_0^T\int_{\omega_2}(\chi_3(x))^2e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\Delta\Delta\varphi_3)dxdt,$$

cannot be estimated as in Lemma B.4.15 because we have not enough derivative terms in φ_4 in the left hand side of (B.165). In order to estimate these two terms, we follow the strategy developed in the proof of [CSG15, Theorem 2.2] (see Annexe B.7.3 for the proof of the two following lemmas).

Lemma B.4.31. We have

$$\begin{aligned}&\int_0^T\int_{\omega_2}\chi_3^2e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\Delta\Delta\varphi_3)\\&\leq\varepsilon\left(\int_0^T\int_\Omega e^{2s\alpha}\Big\{(s\phi)^4(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)+(s\phi)|\Delta\Delta\varphi_3|^2+(s\phi)^3|\Delta\Delta\varphi_4|^2\Big\}\right)\\&+C_\varepsilon\left(\int_0^T\int_{\omega_2}e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2+|\nabla\Delta\varphi_3|^2)\Big\}\right).\end{aligned} \tag{B.168}$$

Lemma B.4.32. We have

$$\int_0^T \int_{\omega_2} \chi_3^2 e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\partial_t\Delta\varphi_3)$$
$$\leq \varepsilon\left(\int_0^T \int_\Omega e^{2s\alpha}\Big\{(s\phi)^4(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)+(s\phi)|\partial_t\Delta\varphi_3|^2+(s\phi)^3|\Delta\Delta\varphi_4|^2\Big\}\right)$$
$$+C_\varepsilon\left(\int_0^T \int_{\omega_2} e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2+|\nabla\Delta\varphi_3|^2)\Big\}\right). \tag{B.169}$$

Moreover, the proof of these two lemmas (see (B.311)) provides us another estimate which is useful to treat the right hand side of (B.167).

Lemma B.4.33. For every $1 \leq i \leq 2$, $\delta > 0$, we have

$$\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^3|\Delta\varphi_i|^2$$
$$\leq \delta\left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^4|\Delta\varphi_i|^2\right)$$
$$+C_\delta\left(\int_0^T \int_{\omega_2} e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}|\nabla\varphi_i|^2\Big\}\right). \tag{B.170}$$

Gathering (B.167) and (B.170) with $\delta = \varepsilon/C_\varepsilon$, we find that for $1 \leq i \leq 2$,

$$\int_0^T \int_{\omega_2} (\chi_3(x))^2 e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)m_2\Delta\varphi_i dxdt$$
$$\leq \varepsilon\left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2+\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^4|\Delta\varphi_i|^2\right)$$
$$+C_\varepsilon\left(\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22}|\nabla\varphi_i|^2\right). \tag{B.171}$$

From (B.166), (B.171), (B.168), (B.169), we get

$$\int_0^T \int_{\omega_2} (\chi_3(x))^2 e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2 dxdt$$
$$\leq \varepsilon\left(\int_0^T \int_\Omega e^{2s\alpha}\Big\{(s\phi)^4(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)+(s\phi)(|\partial_t\Delta\varphi_3|^2+|\Delta\Delta\varphi_3|^2)+(s\phi)^3|\Delta\Delta\varphi_4|^2\Big\}\right)$$
$$+C_\varepsilon\left(\int_0^T \int_{\omega_2} e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2+|\nabla\Delta\varphi_3|^2)\Big\}\right). \tag{B.172}$$

By using (B.165), (B.172) and by taking ε small enough, we have

$$\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2 dxdt + I(2,\lambda,s,\Delta\varphi_3)+I(5,\lambda,s,\varphi_2)+I(5,\lambda,s,\varphi_1)$$
$$\leq C\left(\int_0^T \int_{\omega_2} e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2+|\nabla\Delta\varphi_3|^2)\Big\}\right). \tag{B.173}$$

Estimate of $\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22}|\nabla\Delta\varphi_3|^2 dxdt$.

Let us introduce $\widetilde{\chi_2} \in C^\infty(\overline{\Omega};[0;+\infty[)$ such that $supp(\widetilde{\chi_2}) \subset \omega_2'$ and $\widetilde{\chi_2} = 1$ on ω_2. Then, by Lemma B.4.15 : (B.104) (with $\Phi = \Delta\varphi_3$, $\widetilde{\omega} = \omega_2$, $\Theta = \widetilde{\chi_2}$, $r = 22$ and $(k,l) = (1,43)$), for any $\varepsilon' > 0$, we have

$$\begin{aligned}
&\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22}|\nabla\Delta\varphi_3|^2 \\
&\leq \int_0^T \int_{\omega_2'} \widetilde{\chi_2} e^{2s\alpha}(s\phi)^{22}|\nabla\Delta\varphi_3|^2 \\
&\leq \varepsilon' \left(\int_0^T \int_\Omega e^{2s\alpha}\Big\{(s\phi)|\Delta\Delta\varphi_3|^2 + (s\phi)^3|\nabla\Delta\varphi_3|^2\Big\}\right) + C_{\varepsilon'} \int_0^T \int_{\omega_2'} e^{2s\alpha}(s\phi)^{43}|\Delta\varphi_3|^2.
\end{aligned} \tag{B.174}$$

By taking ε' small enough and by using (B.173) and (B.174), we have

$$\begin{aligned}
&\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2 dxdt + I(2,\lambda,s,\Delta\varphi_3) + I(5,\lambda,s,\varphi_2) + I(5,\lambda,s,\varphi_1) \\
&\leq C\left(\int_0^T \int_{\omega_2''} e^{2s\alpha}(s\phi)^{43}(|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) + \int_0^T \int_{\omega_2''} e^{2s\alpha}(s\phi)^{22}(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2)\right).
\end{aligned} \tag{B.175}$$

Estimate of $\int_0^T \int_{\omega_2'} e^{2s\alpha}(s\phi)^{43}|\Delta\varphi_3|^2 dxdt$.

Let us introduce $\chi_2 \in C^\infty(\overline{\Omega};[0,+\infty[)$, such that the support of χ_2 in included in ω_1 and $\chi_2 = 1$ in ω_2'. We multiply the second equation of (B.156) by $sign(d_2 - d_3)\chi_2(x)e^{2s\alpha}(s\phi)^{45}\Delta\varphi_3$ and we integrate on $(0,T)\times\omega_1$. As $d_2 \neq d_3$, we have

$$\begin{aligned}
&\int_0^T \int_{\omega_1} \chi_2(x)e^{2s\alpha}(s\phi)^{43}|\Delta\varphi_3|^2 dxdt \\
&\leq C\int_0^T \int_{\omega_1} \chi_2(x)e^{2s\alpha}(s\phi)^{43}\Delta\varphi_3(-\partial_t\varphi_2 - d_2\Delta\varphi_2 - a_{12}\varphi_1 - a_{22}\varphi_2)dxdt.
\end{aligned} \tag{B.176}$$

Let $\varepsilon'' > 0$ which will be chosen small enough. We estimate the right hand side of (B.176) in the same way as the one of (B.108) :

- for terms involving $\Delta\varphi_3 a_{i2}\varphi_i$ with $1 \leq i \leq 2$, we apply (B.101) with $\Phi = \Delta\varphi_3$, $\Psi = \varphi_i$, $a = a_{i2} \in L^\infty(Q)$, $1 \leq i \leq 2$ (recalling (B.79)), $\Theta = \chi_2$ and $r = 43$, $k = 5$, $l = 81$,
- for the term involving $\Delta\varphi_3\partial_t\varphi_2$, we apply (B.102) with $\Phi = \Delta\varphi_3$, $\Psi = \varphi_2$, $a = 1$, $\Theta = \chi_2$ and $r = 43$, $k = 5$, $l = 85$,
- for the term involving $\Delta\varphi_3\Delta\varphi_2$, we apply (B.103) with $\Phi = \Delta\varphi_3$, $\Psi = \varphi_2$, $a = d_2$, $\Theta = \chi_2$ and $r = 43$, $k = 5$, $l = 85$.

We get

$$\int_0^T \int_{\omega_1} \chi_2 e^{2s\alpha}(s\phi)^{43}|\Delta\varphi_3|^2$$
$$\leq \varepsilon'' \left(\int_0^T \int_\Omega e^{2s\alpha}\Big\{(s\phi)^5|\Delta\varphi_3|^2 + (s\phi)^3|\nabla\Delta\varphi_3|^2 + (s\phi)(|\partial_t\Delta\varphi_3|^2 + |\Delta\Delta\varphi_3|^2\Big\}\right)$$
$$+ C_{\varepsilon''}\int_0^T \int_{\omega_1} e^{2s\alpha}(s\phi)^{85}(|\varphi_1|^2 + |\varphi_2|^2). \tag{B.177}$$

By taking ε'' sufficiently small, we get from (B.175), (B.177)

$$\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2 dxdt + I(2,\lambda,s,\Delta\varphi_3) + I(5,\lambda,s,\varphi_2) + I(5,\lambda,s,\varphi_1)$$
$$\leq C\int_0^T \int_{\omega_1} e^{2s\alpha}(s\phi)^{85}(|\varphi_1|^2 + |\varphi_2|^2) + \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22}(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2). \tag{B.178}$$

<u>Estimate of $\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22}|\nabla\varphi_i|^2 dxdt$ for $1 \leq i \leq 2$.</u>

Applying Lemma B.4.15 : (B.104) (with $\Phi = \varphi_i$, $\widetilde{\omega} = \omega_1$, $\Theta = \chi_2$, $r = 22$ and $(k,l) = (4,40)$), for any $\varepsilon''' > 0$, we have

$$\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22}|\nabla\varphi_i|^2 dxdt$$
$$\leq \int_0^T \int_{\omega_1} \chi_2 e^{2s\alpha}(s\phi)^{22}|\nabla\varphi_i|^2 dxdt$$
$$\leq \varepsilon''' \left(\int_0^T \int_\Omega e^{2s\alpha}\Big\{(s\phi)^4|\Delta\varphi_i|^2 + (s\phi)^6|\nabla\varphi_i|^2\Big\} dxdt\right) + C_{\varepsilon'''}\int_0^T \int_{\omega_1} e^{2s\alpha}(s\phi)^{40}|\varphi_i|^2 dxdt. \tag{B.179}$$

By taking ε''' small enough and by using (B.178) and (B.179), we have

$$\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2 + I(2,\lambda,s,\Delta\varphi_3) + \sum_{i=1}^2 I(5,\lambda,s,\varphi_i) \leq C\int_0^T \int_{\omega_1} e^{2s\alpha}(s\phi)^{85}(|\varphi_1|^2 + |\varphi_2|^2). \tag{B.180}$$

<u>Estimate of $\int_0^T \int_{\omega_1} e^{2s\alpha}(s\phi)^{85}|\varphi_2|^2 dxdt$.</u>

Let us introduce $\chi_1 \in C^\infty(\overline{\Omega}; [0,+\infty[)$, such that the support of χ_1 in included in ω_0 and $\chi_1 = 1$ in ω_1. We multiply the first equation of (B.91) by $\chi_1(x)e^{2s\alpha}(s\phi)^{85}\varphi_2$ and we integrate on $(0,T)\times\omega_0$. Recalling (B.78), we have

$$\int_0^T \int_{\omega_0} \chi_1(x)e^{2s\alpha}(s\phi)^{85}|\varphi_2|^2 dxdt$$
$$\leq C\int_0^T \int_{\omega_0} \chi_1(x)e^{2s\alpha}(s\phi)^{85}\varphi_2(-\partial_t\varphi_1 - d_1\Delta\varphi_2 - a_{11}\varphi_1)dxdt. \tag{B.181}$$

We estimate the right hand side of (B.181) in the same way as the one of (B.108) :

— for the term involving $\varphi_2 a_{11}\varphi_1$, we apply (B.101) with $\Phi = \varphi_2$, $\Psi = \varphi_1$, $a = a_{11} \in L^\infty(Q)$ (recalling (B.79)), $\Theta = \chi_1$ and $r = 85$, $k = 8$, $l = 162$,

— for the term involving $\varphi_2 \partial_t \varphi_1$, we apply (B.102) with $\Phi = \varphi_2$, $\Psi = \varphi_1$, $a = 1$, $\Theta = \chi_1$ and $r = 85$, $k = 8$, $l = 166$,

— for the term involving $\varphi_2 \Delta\varphi_1$, we apply (B.103) with $\Phi = \varphi_2$, $\Psi = \varphi_1$, $a = d_1$, $\Theta = \chi_1$ and $r = 85$, $k = 8$, $l = 166$.

We get

$$\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2 + I(2,\lambda,s,\Delta\varphi_3) + \sum_{i=1}^{2} I(5,\lambda,s,\varphi_i) \leq C\int_0^T \int_{\omega_0} e^{2s\alpha}(s\phi)^{166}|\varphi_1|^2. \tag{B.182}$$

Then, we can deduce from (B.98) and (B.182)

$$\sum_{i=1}^{2} \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^8|\varphi_i|^2 + \int_0^T \int_\Omega e^{2s\widehat{\alpha}}\Big\{(s\widehat{\phi})^5|\Delta\varphi_3|^2 + (s\widehat{\phi})^3|\Delta\Delta\varphi_4|^2\Big\} \leq C\int_0^T \int_\omega e^{2s\alpha}(s\phi)^{166}|\varphi_1|^2. \tag{B.183}$$

Now, we use Poincaré-Wirtinger inequality as in (B.151) to get

$$\begin{aligned}&\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^8(|\varphi_1|^2 + |\varphi_2|^2) + e^{2s\widehat{\alpha}}\Big\{(s\widehat{\phi})^5|\varphi_3 - (\varphi_3)_\Omega|^2 + (s\widehat{\phi})^3|\varphi_4 - (\varphi_4)_\Omega|^2\Big\}\\ &\leq C\int_0^T \int_\omega e^{2s\alpha}(s\phi)^{166}|\varphi_1|^2.\end{aligned} \tag{B.184}$$

Now, from the dissipation of the energy of $(\varphi_1, \varphi_2, \varphi_3 - (\varphi_3)_\Omega, \varphi_4 - (\varphi_4)_\Omega)$ (see Lemma B.7.1 in Annexe B.7) and by using the same arguments as for 2 controls (see (B.152) and (B.153)), we easily get

$$\sum_{i=1}^{2} \|\varphi_i(0,.)\|^2_{L^2(\Omega)} + \sum_{i=3}^{4} \|\varphi_i(0,.) - (\varphi_i)(0,.)_\Omega\|^2_{L^2(\Omega)} \leq C\int_0^T \int_\omega e^{2s\alpha}(s\phi)^{166}|\varphi_1|^2 dxdt, \tag{B.185}$$

and consequently the observability inequality (B.95).

This ends the proof of the observability inequality (B.95). □

B.4.4 Second step : Controls in $L^\infty(Q)^j$

B.4.4.1 Penalized Hilbert Uniqueness Method

The proof in this subsection follows ideas of [Bar00] and [CGR10, Section 3.1.2]. The goal is to get more regular controls in some sense (see (B.203)) by considering a penalized problem.

Let $\varepsilon \in (0, 1)$ and

$$M_3 := 7,\ M_2 := 13,\ M_1 := 166.$$

We choose λ and s large enough such that (B.120), (B.153), (B.185) hold.

Let $j \in \{1,2,3\}$, $A \in \mathcal{E}_j$ (see (B.86), (B.84) and (B.82)), $\zeta_0 \in H_j$ (see (B.87), (B.85), (B.83)). We introduce the notation $L^2_{wght}((0,T)\times\omega)^j$ for the set of functions h^j such that for every $1 \leq i \leq j$, $(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h_i \in L^2((0,T)\times\omega)$. The set $L^2_{wght}((0,T)\times\omega_0)^j$ is an Hilbert space equipped with the inner product $(h,k) = \sum_{i=1}^{j} \int\int_{(0,T)\times\omega_0} e^{-2s\alpha}(s\phi)^{-M_j} h_i k_i dxdt$. We define

$$\forall h^j \in L^2_{wght}((0,T)\times\omega)^j,\ J(h^j) := \frac{1}{2}\int\int_{(0,T)\times\omega} e^{-2s\alpha}(s\phi)^{-M_j}|h^j|^2 dxdt + \frac{1}{2\varepsilon}\left\|\zeta(T,.)\right\|^2_{L^2(\Omega)^4},$$

where $\zeta = (\zeta_1,\zeta_2,\zeta_3,\zeta_4)$ is the solution to the Cauchy problem (B.89) associated to the control h^j.

The mapping J is a continuous, coercive, strictly convex functional on the Hilbert space $L^2_{wght}((0,T)\times\omega)^j$, then J has a unique minimum $h^{j,\varepsilon}$ with $(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^{j,\varepsilon} \in L^2(q_T)^j$. Let ζ^ε be the solution to the Cauchy problem (B.89) with control $h^{j,\varepsilon}$ and initial condition ζ_0.

The Euler-Lagrange equation gives

$$\forall h^j \in L^2_{wght}((0,T)\times\omega)^j,\ \sum_{i=1}^{j}\int\int_{(0,T)\times\omega} e^{-2s\alpha}(s\phi)^{-M_j}h_i^\varepsilon h_i + \frac{1}{\varepsilon}\int_\Omega \zeta^\varepsilon(T,.).\zeta(T,.) = 0, \tag{B.186}$$

where $\zeta = (\zeta_1,\zeta_2,\zeta_3,\zeta_4)$ is the solution to the Cauchy problem (B.89) associated to the control h^j and initial condition $\zeta_0 = 0$.

We introduce φ^ε the solution to the adjoint problem (B.91) with final condition $\varphi^\varepsilon(T,.) = -\frac{1}{\varepsilon}\zeta^\varepsilon(T,.)$. A duality argument between ζ and φ^ε gives

$$-\frac{1}{\varepsilon}\int_\Omega \zeta^\varepsilon(T,x).\zeta(T,x)dx = \sum_{i=1}^{j}\int\int_{(0,T)\times\omega_0} h_i\varphi_i^\varepsilon dxdt. \tag{B.187}$$

Then, we deduce from (B.186) and (B.187) that

$$\forall h^j \in L^2_{wght}(q_T)^j,\ \sum_{i=1}^{j}\int\int_{(0,T)\times\omega} e^{-2s\alpha}(s\phi)^{-M_j}h_i^\varepsilon h_i dxdt = \sum_{i=1}^{j}\int\int_{(0,T)\times\omega} h_i\varphi_i^\varepsilon dxdt.$$

Consequently,

$$\forall i \in \{1,\dots,j\},\ h_i^\varepsilon = e^{2s\alpha}(s\phi)^{M_j}\varphi_i^\varepsilon 1_\omega. \tag{B.188}$$

Another duality argument applied between ζ^ε and φ^ε together with (B.188) gives

$$-\frac{1}{\varepsilon}\left\|\zeta^\varepsilon(T,.)\right\|^2_{L^2(\Omega)^4} = \sum_{i=1}^{j}\int\int_{(0,T)\times\omega} e^{2s\alpha}(s\phi)^{M_j}|\varphi_i^\varepsilon|^2 dxdt + \int_\Omega \varphi^\varepsilon(0,x).\zeta_0(x)dx. \tag{B.189}$$

If $j = 2$, we have $\int_\Omega \zeta_{0,4}(x)dx = 0$. Then,

$$\int_\Omega \varphi^\varepsilon(0,x).\zeta_0(x)dx = \sum_{i=1}^{3} \int_\Omega \varphi_i^\varepsilon(0,x)\zeta_{0,i}(x)dx + \int_\Omega (\varphi_4^\varepsilon(0,x) - (\varphi_4)_\Omega(0))\zeta_{0,4}(x)dx. \tag{B.190}$$

If $j = 1$, we have $\int_\Omega \zeta_{0,3}(x)dx = 0$ and $\int_\Omega \zeta_{0,4}(x)dx = 0$. Then,

$$\int_\Omega \varphi^\varepsilon(0,x).\zeta_0(x)dx = \sum_{i=1}^{2} \int_\Omega \varphi_i^\varepsilon(0,.)\zeta_{0,i}(.) + \sum_{i=3}^{4} \int_\Omega (\varphi_i^\varepsilon(0,.) - (\varphi_i)_\Omega(0))\zeta_{0,i}(.). \tag{B.191}$$

Then, from (B.120) for $j = 3$, (B.153), (B.190) for $j = 2$, (B.185), (B.191) for $j = 1$ and (B.188), (B.189), we have

$$\frac{1}{\varepsilon}\left\|\zeta^\varepsilon(T,.)\right\|^2_{L^2(\Omega)^4} + \frac{1}{2}\left\|(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^{j,\varepsilon}\right\|^2_{L^2((0,T)\times\omega)^j} \leq C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}. \tag{B.192}$$

In particular, from (B.192),

$$\zeta^\varepsilon(T,.) \underset{\varepsilon\to 0}{\to} 0 \text{ in } L^2(\Omega)^4, \tag{B.193}$$

and

$$\left\|B_j h^{j,\varepsilon}\right\|_{L^2(Q)^j} \leq C. \tag{B.194}$$

Then, by using $A \in \mathcal{M}_4(L^\infty(Q))$ (see (B.86), (B.84) and (B.82)) and recalling (B.194), from Proposition B.2.3 applied to (B.89), we deduce that

$$\|\zeta^\varepsilon\|_{Y^4} \leq C. \tag{B.195}$$

So, from (B.195), up to a subsequence, we can suppose that there exists $\zeta \in Y^4$ such that

$$\zeta^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \zeta \text{ in } L^2(0,T;H^1(\Omega)^4), \tag{B.196}$$

$$\partial_t\zeta^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \partial_t\zeta \text{ in } L^2(0,T;(H^1(\Omega))'^4), \tag{B.197}$$

and from Proposition B.2.2,

$$\zeta^\varepsilon(0,.) \underset{\varepsilon\to 0}{\rightharpoonup} \zeta(0,.) \text{ in } L^2(\Omega)^4,\ \zeta^\varepsilon(T,.) \underset{\varepsilon\to 0}{\rightharpoonup} \zeta(T,.) \text{ in } L^2(\Omega)^4. \tag{B.198}$$

Then, as we have $\zeta^\varepsilon(0,.) = \zeta_0$ and (B.193), we deduce that

$$\zeta(0,.) = \zeta_0, \text{ and } \zeta(T,.) = 0. \tag{B.199}$$

Moreover, from (B.192), up to a subsequence, we can suppose that there exists $h^j \in L^2_{wght}((0,T)\times\omega)^j$ such that

$$(h^{j,\varepsilon}) \underset{\varepsilon\to 0}{\rightharpoonup} h^j \text{ in } L^2_{wght}((0,T)\times\omega)^j, \tag{B.200}$$

and

$$\begin{aligned}&\left\|(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^j\right\|^2_{L^2((0,T)\times\omega)^j}\\&\leq \liminf_{\varepsilon\to 0}\left\|(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^{j,\varepsilon}\right\|^2_{L^2((0,T)\times\omega)^j}\leq C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}.\end{aligned} \tag{B.201}$$

Then, from (B.196), (B.197), (B.200), we let $\varepsilon\to 0$ in the following equations

$$\begin{cases}\partial_t\zeta^\varepsilon - D\Delta\zeta^\varepsilon = A(t,x)\zeta^\varepsilon + B_j h^{j,\varepsilon}1_\omega & \text{in } (0,T)\times\Omega,\\ \frac{\partial\zeta^\varepsilon}{\partial n}=0 & \text{on } (0,T)\times\partial\Omega,\end{cases}$$

and by using (B.199), we deduce

$$\begin{cases}\partial_t\zeta - D\Delta\zeta = A(t,x)\zeta + B_j h^{j}1_\omega & \text{in } (0,T)\times\Omega,\\ \frac{\partial\zeta}{\partial n}=0 & \text{on } (0,T)\times\partial\Omega,\\ (\zeta(0,.),\zeta(T,.))=(\zeta_0,0) & \text{in } \Omega.\end{cases} \tag{B.202}$$

Therefore, we have proved the existence of a control h^j such that $(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^j\in L^2((0,T)\times\omega)^j$ that drives the solution ζ of (B.89) to 0, and we have the estimate

$$\left\|(e^{-2s\alpha}(s\phi)^{-M_j})^{1/2}h^j\right\|^2_{L^2((0,T)\times\omega)^j}\leq C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}. \tag{B.203}$$

B.4.4.2 Bootstrap method

In the previous subsection, we proved the existence of a control $h^j\in L^2_{wght}((0,T)\times\omega)^j$ i.e. a control h^j more regular than $L^2(Q)$. The key points are the link between $h^{j,\varepsilon}$ and φ^ε (i.e. (B.188)) and the weights of Carleman estimates. Now, we use an iterative process in order to find controls in $L^\infty(Q)^j$. We use the same key points together with parabolic regularity theorems. This section is inspired by [CGR10, Section 3.1.2] and [WZ06] (for the Neumann conditions). First, we are going to present the boostrap method for the case $j=3$ and after that, we explain the main differences for the case $j=2$ and $j=1$.

B.4.4.2.1 Strong observability inequalities From (B.114) for the case $j=3$, (B.151) for the case $j=2$, (B.184) for the case $j=1$, (B.188) and (B.192), we deduce these inegalities which are useful for the bootstrap method :

$$(j=3)\Rightarrow\left(\sum_{i=1}^{4}\int_0^T\int_\Omega e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_i^\varepsilon|^2dxdt\leq C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}\right), \tag{B.204}$$

$$(j=2)\Rightarrow\left(\int_0^T\int_\Omega\sum_{i=1}^{3}e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_i^\varepsilon|^2+e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_4^\varepsilon-(\varphi_4^\varepsilon)_\Omega|^2\leq C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}\right), \tag{B.205}$$

$$(j=1)\Rightarrow\left(\int_0^T\int_\Omega\sum_{i=1}^{2}e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_i^\varepsilon|^2+\sum_{i=3}^{4}e^{2s\widehat{\alpha}}(s\widehat{\phi})^3|\varphi_i^\varepsilon-(\varphi_i^\varepsilon)_\Omega|^2\leq C\left\|\zeta_0\right\|^2_{L^2(\Omega)^4}\right). \tag{B.206}$$

B.4.4.2.2 Bootstrap Let $\delta > 0$ which will be chosen sufficiently small and $(\delta_k)_{k\in\mathbb{N}} \in (\mathbb{R}^{+,*})^{\mathbb{N}}$ be a strictly increasing sequence such that $\delta_k \underset{k\to+\infty}{\to} \delta$. Let $(p_k)_{k\in\mathbb{N}}$ be the following sequence defined by induction

$$p_0 = 2,$$
$$p_{k+1} := \begin{cases} \frac{(N+2)p_k}{N+2-2p_k} & \text{if } p_k < \frac{N+2}{2}, \\ 2p_k & \text{if } p_k = \frac{N+2}{2}, \\ +\infty & \text{if } p_k > \frac{N+2}{2}. \end{cases}$$

Clearly, we have that

$$\exists l \in \mathbb{N},\ \forall k \geq l,\ p_k = +\infty. \tag{B.207}$$

Definition B.4.34. We introduce the following spaces : for every $r \in [1, +\infty]$,

$$W^{2,r}_{Ne}(\Omega) := \left\{ u \in W^{2,r}(\Omega)\ ;\ \frac{\partial u}{\partial n} = 0 \right\}, \qquad Y_r = L^r(0,T; W^{2,r}_{Ne}(\Omega)) \cap W^{1,r}(0,T;L^r(\Omega)).$$

Definition B.4.35. Let u be a function on Q. For $0 < \beta < 1$, we define

$$[u]_{\beta/2,\beta} = \sup_{(t,x),(t',x')\in Q, (t,x)\neq(t',x')} \frac{|u(t,x) - u(t',x')|}{(|t-t'| + |x-x'|^2)^{\beta/2}},$$

which is a semi-norm, and we denote by $C^{\beta/2,\beta}(\overline{Q})$ the set of all functions on Q such that $[u]_{\beta/2,\beta} < +\infty$, endowed with the norm

$$\|u\|_{\beta/2,\beta} = \left(\sup_{(t,x)\in Q} |u(t,x)| \right) + [u]_{\beta/2,\beta}.$$

Proposition B.4.36. Let $1 < p < +\infty$, $m \in \mathbb{N}^*$, $D \in \mathcal{M}_m(\mathbb{R})$ such that $Sp(D) \subset (0, +\infty)$, $A \in \mathcal{M}_m(L^\infty(Q))$, $f \in L^p(Q)^m$. From [DHP07, Theorem 2.1], the following Cauchy problem admits a unique solution $u \in Y_p^m$

$$\begin{cases} \partial_t u - D\Delta u = A(t,x)u + f & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = 0 & \text{in } \Omega. \end{cases}$$

Moreover, there exists $C > 0$ independent of f such that

$$\|u\|_{Y_p^m} \leq C \|f\|_{L^p(Q)^k}.$$

Proposition B.4.37. [WYW06, Theorem 1.4.1]
Let $r \in [1, +\infty[$, we have

$$Y_r \hookrightarrow \begin{cases} L^{\frac{(N+2)r}{N+2-2r}}(Q) & \text{if } r < \frac{N+2}{2}, \\ L^{2r}(Q) & \text{if } r = \frac{N+2}{2}, \\ C^{\beta/2,\beta}(\overline{Q}) \hookrightarrow L^\infty(Q) \text{ with } 0 < \beta \leq 2 - \frac{N+2}{r} & \text{if } r > \frac{N+2}{2}. \end{cases}$$

$\boxed{j=3}$
In the following, C denotes various positive constants varying from one line to the other and does not depend of $\|\zeta_0\|_{L^2(\Omega}$.

We define for every $k \in \mathbb{N}$,

$$\psi^{\varepsilon,k} := e^{\widehat{\alpha}(s+\delta_k)}\varphi^{\varepsilon}. \tag{B.208}$$

For $k \in \mathbb{N}^*$, by using (B.208) and the adjoint system (B.91) satisfied by φ^{ε}, we have

$$\begin{cases} -\partial_t \psi^{\varepsilon,k} - D_3 \Delta \psi^{\varepsilon,k} = A(t,x)\psi^{\varepsilon,k} + f_k & \text{in } (0,T)\times\Omega, \\ \frac{\partial \psi^{\varepsilon,k}}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \psi^{\varepsilon,k}(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{B.209}$$

with

$$f_k(t,x) = -\partial_t(e^{\widehat{\alpha}(s+\delta_k)})\varphi^{\varepsilon}.$$

By using the fact that $(\delta_k)_{k\in\mathbb{N}}$ is strictly increasing, we easily have that

$$|f_k| \leq C e^{\widehat{\alpha}(s+\delta_{k-1})}|\varphi^{\varepsilon}| = C|\psi^{\varepsilon,k-1}| \text{ in } (0,T)\times\Omega. \tag{B.210}$$

We show, by induction, that for every $0 \leq k \leq l$ (see (B.207)), we have

$$\psi^{\varepsilon,k} \in L^{p_k}(Q)^4 \text{ and } \left\|\psi^{\varepsilon,k}\right\|_{L^{p_k}(Q)^4} \leq C \left\|\zeta_0\right\|_{L^2(\Omega)^4}. \tag{B.211}$$

The case $k=0$ can be deduced from the fact that $\delta_0 > 0$ and the strong observability inequality (B.204).

Let $1 \leq k \leq l$. We suppose that

$$\psi^{\varepsilon,k-1} \in L^{p_{k-1}}(Q)^4 \text{ and } \left\|\psi^{\varepsilon,k-1}\right\|_{L^{p_{k-1}}(Q)^4} \leq C \left\|\zeta_0\right\|_{L^2(\Omega)^4}. \tag{B.212}$$

Then, from (B.209), (B.210), (B.212) and from the maximal regularity theorem : Proposition B.4.36, we get

$$\psi^{\varepsilon,k} \in X^4_{p_{k-1}} \text{ and } \left\|\psi^{\varepsilon,k}\right\|_{X^4_{p_{k-1}}} \leq C \left\|\zeta_0\right\|_{L^2(\Omega)^4}. \tag{B.213}$$

Moreover, by the Sobolev embedding Proposition B.4.37, we have

$$\psi^{\varepsilon,k} \in L^{p_k}(Q)^4 \text{ and } \left\|\psi^{\varepsilon,k}\right\|_{L^{p_k}(Q)^4} \leq C \left\|\zeta_0\right\|_{L^2(\Omega)^4}.$$

This concludes the induction.

From (B.97) and (B.98), we remark that we have the following inequality

$$\alpha \leq \frac{\widehat{\alpha}}{1+f(\lambda)}, \tag{B.214}$$

because

$$(e^{\lambda\eta_0(x)} - e^{2\lambda\|\eta_0\|_\infty})(1 + e^{-\lambda\|\eta_0\|_\infty}) = e^{\lambda\eta_0(x)} - e^{\lambda\|\eta_0\|_\infty} + 1 - e^{2\lambda\|\eta_0\|_\infty} \leq 1 - e^{2\lambda\|\eta_0\|_\infty}.$$

Moreover, from (B.96), we can pick $\delta > 0$ such that

$$2s - (1 + f(\lambda))(s + \delta) = s(2 - (1 + f(\lambda))) - \delta(1 + f(\lambda)) > 0. \tag{B.215}$$

Now, by applying consecutively (B.207), (B.188), (B.214), (B.215) and (B.211), we have for every $i \in \{1, \dots, 3\}$,

$$\begin{aligned}
\|h_i^\varepsilon\|_{L^\infty(Q)} = \|h_i^\varepsilon\|_{L^{p_l}(Q)} &= \left\|e^{2s\alpha}(s\phi)^7\varphi_i^\varepsilon\right\|_{L^{p_l}(Q)} \\
&\leq \left\|e^{\widehat{\alpha}\left(\frac{2s}{1+f(\lambda)} - (s+\delta)\right)}(s\phi)^7\right\|_{L^\infty(Q)} \left\|e^{\widehat{\alpha}(s+\delta)}\varphi_i^\varepsilon\right\|_{L^{p_l}(Q)} \\
&\leq C\left\|e^{\widehat{\alpha}(s+\delta)}\varphi_i^\varepsilon\right\|_{L^{p_l}(Q)} \\
&\leq C\left\|e^{\widehat{\alpha}(s+\delta_l)}\varphi_i^\varepsilon\right\|_{L^{p_l}(Q)} \quad (\delta_l \leq \delta \text{ and } \widehat{\alpha} < 0) \\
&\leq C\left\|\zeta_0\right\|_{L^2(\Omega)^4}.
\end{aligned} \tag{B.216}$$

Therefore, from (B.216), we get

$$\|h_i^\varepsilon\|_{L^\infty(Q)} \leq C\left\|\zeta_0\right\|_{L^2(\Omega)^4}. \tag{B.217}$$

So, $(h^{3,\varepsilon})_\varepsilon$ is bounded in $L^\infty(Q)^3$, then up to a subsequence, we can suppose that there exists $h^3 \in L^\infty(Q)^3$ such that

$$h^{3,\varepsilon} \underset{\varepsilon\to 0}{\rightharpoonup}{}^* h^3 \text{ in } L^\infty(Q)^3, \tag{B.218}$$

and

$$\left\|h^3\right\|_{L^\infty(Q)^3} \leq C\left\|\zeta_0\right\|_{L^2(\Omega)^4}.$$

From (B.196), (B.197), (B.218), (B.199), we have

$$\begin{cases}
\partial_t\zeta - D_3\Delta\zeta = A(t,x)\zeta + B_3h^3 1_\omega & \text{in } (0,T)\times\Omega, \\
\frac{\partial\zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\
(\zeta(0,.), \zeta(T,.)) = (\zeta_0, 0) & \text{in } \Omega.
\end{cases} \tag{B.219}$$

This ends the proof of Proposition B.4.8 for the case $j = 3$.

$\boxed{j = 2}$

For every $k \in \mathbb{N}$, we introduce

$$\widetilde{\varphi^\varepsilon} := (\varphi_1^\varepsilon, \varphi_2^\varepsilon, \varphi_3^\varepsilon, \varphi_4^\varepsilon - (\varphi_4^\varepsilon)_\Omega)^T, \tag{B.220}$$

$$\psi^{\varepsilon,k} := e^{\widehat{\alpha}(s+\delta_k)}\widetilde{\varphi^{\varepsilon,k}}. \tag{B.221}$$

For $k \in \mathbb{N}^*$, we have

$$\begin{cases} -\partial_t \psi^{\varepsilon,k} - D_2 \Delta \psi^{\varepsilon,k} = A(t,x)\psi^{\varepsilon,k} + f_k & \text{in } (0,T)\times\Omega, \\ \frac{\partial \psi^{\varepsilon,k}}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \psi^{\varepsilon,k}(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{B.222}$$

with

$$f_k(t,x) = -(e^{\widehat{\alpha}(s+\delta_k)})_t \widetilde{\varphi^{\varepsilon,k}} + \left(0,0,0,\left(u_2^* e^{\widehat{\alpha}(s+\delta_k)}\varphi_1^\varepsilon - u_2^* e^{\widehat{\alpha}(s+\delta_k)}\varphi_2^\varepsilon + u_2^* e^{\widehat{\alpha}(s+\delta_k)}\varphi_3^\varepsilon\right)_\Omega\right)^T,$$

because $A \in \mathcal{E}_2$ (see (B.84)). From the fact that $(\delta_k)_{k\in\mathbb{N}}$ is strictly increasing, we easily have

$$|f_k| \leq C e^{\widehat{\alpha}(s+\delta_{k-1})} |\widetilde{\varphi^\varepsilon}| = C|\psi^{\varepsilon,k-1}| \text{ in } (0,T)\times\Omega. \tag{B.223}$$

Then, the strategy of bootstrap is exactly the same. The starting point comes from the strong observability inequality (B.205).

$\boxed{j=1}$

We apply the same strategy as for the case $j=2$. For every $k \in \mathbb{N}$, we introduce

$$\widetilde{\varphi^\varepsilon} := (\varphi_1^\varepsilon, \varphi_2^\varepsilon, \varphi_3^\varepsilon - (\varphi_3^\varepsilon)_\Omega, \varphi_4^\varepsilon - (\varphi_4^\varepsilon)_\Omega)^T, \tag{B.224}$$

$$\psi^{\varepsilon,k} := e^{\widehat{\alpha}(s+\delta_k)} \widetilde{\varphi^{\varepsilon,k}}. \tag{B.225}$$

The starting point comes from the strong observability inequality (B.206).

This ends the proof of Proposition B.4.8.

B.4.5 Nonlinear problem

In order to prove Theorem B.3.2, we use Proposition B.4.8 together with a standard fixed-point argument.

B.4.5.1 Reduction to a fixed point problem

Let $j \in \{1,2,3\}$. We remark that $G : L^\infty(Q)^4 \to \mathcal{M}_4(L^\infty(Q))$ is continuous (see (B.51), (B.61) and (B.76)). Then, we get the existence of $\nu > 0$ small enough such that for every $z = (z_1, z_2, z_3, z_4) \in L^\infty(Q)^4$,

$$(\|z\|_{L^\infty(Q)^4} \leq \nu) \Rightarrow ((G(z_1,z_2,z_3,z_4)) \in \mathcal{E}_j), \tag{B.226}$$

where $\mathcal{E}_j$ are defined in (B.82), (B.84) and (B.86).

Let $\mathcal{Z}$ be the set of $z = (z_1,z_2,z_3,z_4) \in L^\infty(Q)^4$ such that $\|z\|_{L^\infty(Q)^4} \leq \nu$. From Proposition B.4.8, we have proved that there exists $C_0 > 0$ such that for all $z = (z_1,z_2,z_3,z_4) \in \mathcal{Z}$ and for all $\zeta_0 \in L^\infty(Q)^4$, there exists a control $h^j \in L^\infty(Q)^j$ satisfying

$$\left\|h^j\right\|_{L^\infty(Q)^j} \leq C_0 \left\|\zeta_0\right\|_{L^2(\Omega)^4}, \tag{B.227}$$

such that the solution $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^T \in (Y^4 \cap L^\infty(Q)^4)$ to the Cauchy problem

$$\begin{cases} \partial_t \zeta - D_j \Delta \zeta = G(z)\zeta + B_j h^j 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{B.228}$$

verifies

$$\zeta(T,.) = 0. \tag{B.229}$$

We fix $\zeta_0 \in L^\infty(Q)^4$.

We define $B : \mathcal{Z} \to L^\infty(Q)^4$ in the following way. For every $z = (z_1, z_2, z_3, z_4) \in \mathcal{Z}$, $B(z)$ is the set of $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in L^\infty(Q)^4$ solution to the Cauchy problem (B.228), associated to a control $h^j \in L^\infty(Q)^j$ satisfying (B.227), and which verifies (B.229).

Our main result (i.e. Theorem B.3.2) will be proved if we show that B has a fixed point (i.e. z is such that $z \in B(z)$).

We use the **Kakutani's fixed point theorem**.

Theorem B.4.38. Kakutani's fixed point theorem.

1. For every $z \in \mathcal{Z}$, $B(z)$ is a nonempty convex and closed subset of $L^\infty(Q)^4$.
2. There exists a convex compact set $K \subset \mathcal{Z}$ such that for every $z \in \mathcal{Z}, B(z) \subset K$.
3. B is upper semicontinuous in $L^\infty(Q)^4$, that is to say for all closed subset $\mathcal{A} \subset \mathcal{Z}$, $B^{-1}(\mathcal{A}) = \{z \in \mathcal{Z}; B(z) \cap \mathcal{A} \neq \emptyset\}$ is closed.

Then, B has a fixed point.

B.4.5.2 Hypotheses of Kakutani's fixed point theorem

B.4.5.2.1 Proof of the point 1 Let $z \in \mathcal{Z}$.

$B(z)$ is nonempty because we have proved the existence of at least one control satisfying (B.227) that drives the solution to 0.

$B(z)$ is convex because the mapping $h \in L^\infty(Q)^j \mapsto \zeta \in L^\infty(Q)^4$, where ζ is the solution to the Cauchy problem (B.228), is affine and (B.227) is clearly verified by convex combinations of controls satisfying it.

$B(z)$ is closed. Indeed, let $(\zeta_k)_{k\in\mathbb{N}}$ be a sequence of $B(z)$ such that

$$\zeta_k \underset{k\to+\infty}{\to} \zeta \text{ in } L^\infty(Q)^4. \tag{B.230}$$

We introduce $(h_k^j)_{k\in\mathbb{N}}$ the sequence of controls associated to $(\zeta_k)_{k\in\mathbb{N}}$. In particular, for every $k \in \mathbb{N}$,

$$\left\| h_k^j \right\|_{L^\infty(Q)^j} \leq C_0 \left\| \zeta_0 \right\|_{L^2(\Omega)^4}. \tag{B.231}$$

From (B.230) and (B.231), for every $k \in \mathbb{N}$,

$$\left\| G(z)\zeta_k + B_j h_k^j \right\|_{L^\infty(Q)^4} \leq C. \tag{B.232}$$

Then, from (B.232) and Proposition B.2.3 applied to ζ_k which satisfies (B.228), we deduce that for every $k \in \mathbb{N}$,

$$\|\zeta_k\|_{(Y \cap L^\infty(Q))^4} \leq C. \tag{B.233}$$

So, from (B.233), up to a subsequence, we can suppose that there exists $\zeta \in Y^4$ such that

$$\zeta_k \underset{k \to +\infty}{\rightharpoonup} \zeta \text{ in } L^2(0,T;H^1(\Omega)^4), \tag{B.234}$$

$$\partial_t \zeta_k \underset{k \to +\infty}{\rightharpoonup} \partial_t \zeta \text{ in } L^2(0,T;(H^1(\Omega))'^4), \tag{B.235}$$

and, from Proposition B.2.2,

$$\zeta_k(0,.) \underset{k \to +\infty}{\rightharpoonup} \zeta(0,.) \text{ in } L^2(\Omega)^4,\ \zeta_k(T,.) \underset{k \to +\infty}{\rightharpoonup} \zeta(T,.) \text{ in } L^2(\Omega)^4. \tag{B.236}$$

Then, as we have $\zeta_k(0,.) = \zeta_0$ and $\zeta_k(T,.) = 0$ for every $k \in \mathbb{N}$, we deduce that

$$\zeta(0,.) = \zeta_0 \text{ and } \zeta(T,.) = 0. \tag{B.237}$$

Moreover, from (B.231), up to a subsequence, we can suppose that there exists $h^j \in L^\infty(Q)^j$ such that

$$h_k^j \underset{k \to +\infty}{\overset{*}{\rightharpoonup}} h^j \text{ in } L^\infty(Q)^j, \tag{B.238}$$

and

$$\left\|h^j\right\|_{L^\infty(Q)^j} \leq \liminf_{k \to +\infty} \left\|h_k^j\right\|_{L^\infty(Q)^j} \leq C_0 \left\|\zeta_0\right\|_{L^2(\Omega)^4}. \tag{B.239}$$

Then, from (B.234), (B.235), (B.236), (B.237) and (B.238), we let $k \to +\infty$ in the following equations (i.e. passing to the limit in the variational formulation (B.15))

$$\begin{cases} \partial_t \zeta_k - D_j \Delta \zeta_k = G(z)\zeta_k + B_j h_k^j 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial \zeta_k}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ (\zeta_k(0,.), \zeta_k(T,.)) = (\zeta_0, 0) & \text{in } \Omega. \end{cases}$$

We deduce that

$$\begin{cases} \partial_t \zeta - D_j \Delta \zeta = G(z)\zeta + B_j h^j 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ (\zeta(0,.), \zeta(T,.)) = (\zeta_0, 0) & \text{in } \Omega. \end{cases} \tag{B.240}$$

Finally, from (B.240) and (B.239), we have $\zeta \in B(z)$.

B.4.5.2.2 Proof of the point 2 Let $z \in \mathcal{Z}$.

By Proposition B.2.3 and (B.227), we deduce that there exists $C_1 > 0$ such that

$$\forall z \in \mathcal{Z},\ \forall \zeta \in B(z),\ \|\zeta\|_{L^\infty(Q)^4} \leq C_1 \|\zeta_0\|_{L^\infty(\Omega)^4}.$$

Now, we suppose that $\zeta_0 \in L^\infty(\Omega)^4$ verifies

$$\|\zeta_0\|_{L^\infty(\Omega)^4} \leq \nu / C_1. \tag{B.241}$$

Then, we have

$$\forall z \in \mathcal{Z},\ B(z) \subset \mathcal{Z}. \tag{B.242}$$

Let $F \in L^\infty(Q)^4$ be the solution to the Cauchy problem

$$\begin{cases} \partial_t F - D_j \Delta F = 0 & \text{in } (0,T)\times\Omega, \\ \frac{\partial F}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ F(0,.) = \zeta_0 & \text{in } \Omega. \end{cases} \tag{B.243}$$

Let $\zeta^* = \zeta - F$, where $\zeta \in B(z)$ with $z \in \mathcal{Z}$. We also denote by h^j the control associated to ζ. Then, ζ^* is the solution to

$$\begin{cases} \partial_t \zeta^* - D_j \Delta \zeta^* = G(z)\zeta + B_j h^j 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial \zeta^*}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta^*(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{B.244}$$

From (B.226), (B.242) and (B.227), we can remark that there exists $C > 0$ such that

$$\left\| G(z)\zeta + B_j h^j 1_\omega \right\|_{L^\infty(Q)^4} \leq C. \tag{B.245}$$

From (B.245), Proposition B.4.36 with $p = N + 2$ applied to ζ^* (see (B.244)) and the Sobolev embedding theorem $Y_p \hookrightarrow C^{\beta/2,\beta}(\overline{Q})$ with $\beta > 0$ (see Proposition B.4.37), we deduce that $\zeta^* \in C^0(\overline{Q})^4$ and there exists $C_2 > 0$ such that

$$\forall (t,x) \in \overline{Q},\ \forall (t',x') \in \overline{Q},\ |\zeta^*(t,x) - \zeta^*(t',x')| \leq C_2(|t-t'|^{\beta/2} + |x-x'|^\beta). \tag{B.246}$$

Let K^* be the set of ζ^* such that (B.246) holds. Then, we have $(F+K^*)\cap\mathcal{Z}$ is a compact convex subset of $L^\infty(Q)^4$ by Ascoli's theorem and

$$\forall z \in \mathcal{Z},\ B(z) \subset (F + K^*) \cap \mathcal{Z}.$$

Then, $K := (F + K^*) \cap \mathcal{Z}$ is a convex compact subset of $\mathcal{Z}$ such that the point 2 holds.

B.4.5.2.3 Proof of the point 3 Let $\mathcal{A}$ be a closed subset of $\mathcal{Z}$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence of elements in $\mathcal{Z}$, $(\zeta_k)_{k\in\mathbb{N}}$ be a sequence of elements in $L^\infty(Q)^4$, and $z \in \mathcal{Z}$ be such that

$$z_k \underset{k\to+\infty}{\rightarrow} z \text{ in } L^\infty(Q)^4,$$
$$\forall k \in \mathbb{N},\ \zeta_k \in \mathcal{A},$$
$$\forall k \in \mathbb{N},\ \zeta_k \in B(z_k).$$

Let $(h_k^j)_{k\in\mathbb{N}}$ the sequence of controls associated to $(\zeta_k)_{k\in\mathbb{N}}$. As $\zeta_k \in B(z_k)$, we have

$$\forall k \in \mathbb{N},\ \left\| h_k^j \right\|_{L^\infty(Q)^j} \leq C_0 \left\| \zeta_0 \right\|_{L^2(\Omega)^4}.$$

By the point 2, we get that there exists a strictly increasing sequence $(k_l)_{l\in\mathbb{N}}$ of integers such that $\zeta_{k_l} \to \zeta$ in $L^\infty(Q)^4$ as $l \to +\infty$. As $\mathcal{A}$ is closed, we have $\zeta \in \mathcal{A}$, then it suffices to show that $\zeta \in B(z)$. The same arguments as in the point 1 give the result. This ends the proof of the point 3.

This concludes the proof of Theorem B.3.2.

B.5 Proof of Theorem B.3.6 : the global controllability to constant stationary states

Proof. Let $N \in \{1,2\}$, $j = 3$ (we only prove the result for this case, the other cases are similar), $u_0 \in L^\infty(\Omega)^4$ satisfying the hypothesis (B.42), $(u_i^*)_{1\leq i\leq 4} \in (\mathbb{R}^+)^4$ satisfying (B.3).

From [PSZ17, Theorem 3] and [PSY18, Theorem 3] (see also [DF06]), we deduce that the solution $u \in L^\infty((0,\infty)\times\Omega)^4$ of

$$\forall 1 \leq i \leq 4, \begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) & \text{in } (0,\infty)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega, \end{cases} \tag{B.247}$$

satisfies

$$\lim_{T\to+\infty} \|u(T,.) - z\|_{L^\infty(\Omega)^4} = 0, \tag{B.248}$$

where $z \in (\mathbb{R}^{+,*})^4$ is the unique nonnegative solution of

$$z_1 z_3 = z_2 z_4, \tag{B.249}$$
$$z_1 + z_2 = (u_{1,0})_\Omega + (u_{2,0})_\Omega,\ \ z_1 + z_4 = (u_{1,0})_\Omega + (u_{4,0})_\Omega, \tag{B.250}$$
$$z_3 + z_2 = (u_{3,0})_\Omega + (u_{2,0})_\Omega,\ \ z_3 + z_4 = (u_{3,0})_\Omega + (u_{4,0})_\Omega. \tag{B.251}$$

Case 1 : $u_3^* \neq 0$. Let us define a path γ between z and $(u_i^*)_{1\leq i\leq 4}$,

$$\gamma : \left| \begin{array}{ccl} [0,1] & \longrightarrow & \{(v_1,v_2,v_3,v_4) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{+,*} \times \mathbb{R}^+ \ ; \ v_1 v_3 = v_2 v_4\} \\ \theta & \longmapsto & \gamma(\theta), \end{array} \right. \tag{B.252}$$

where

$$\gamma(\theta) := \left(\frac{((1-\theta)z_2 + \theta u_2^*)((1-\theta)z_4 + \theta u_4^*)}{(1-\theta)z_3 + \theta u_3^*}, (1-\theta)z_2 + \theta u_2^*, (1-\theta)z_3 + \theta u_3^*, (1-\theta)z_4 + \theta u_4^* \right).$$

Let us define Φ in the following way,

$$\Phi : \left| \begin{array}{rcl} \Gamma := \{\gamma(\theta), \theta \in [0,1]\} & \longrightarrow & \mathbb{R}^{+,*} \\ (v_i) & \longmapsto & r_v, \end{array} \right. \tag{B.253}$$

where $r_v > 0$ is the radius of the ball of $L^\infty(\Omega)^4$ centered in $(v_i)_{1\leq i\leq 4}$ in which we have proved controllability to $(v_i)_{1\leq i\leq 4}$ (see Theorem B.3.2). Precisely, r_v is given by (B.241). It is straightforward but tedious to see that

$$r := \inf \Phi > 0, \tag{B.254}$$

because there exists $\varepsilon > 0$ such that for every $\theta \in [0,1]$, $v_3 = (1-\theta)z_3 + \theta u_3^* \geq \varepsilon$. For more details, one can follow the dependence of the constant $r_v = \nu/C_1$ in function of the parameters $(v_i)_{1\leq i\leq 4}$ (see (B.241), (B.226), (B.227), Proposition B.4.8 for the definition of the constant C_0, (B.51), (B.53), (B.54) and Annexe B.4.3.3 for the dependence of this constant C_0 in term of $(v_i)_{1\leq i\leq 4}$.

By (B.248), there exists $T_1 > 0$ such that $\|u(T_1,.) - z\|_{L^\infty(\Omega)^4} < r$, where u is the solution of (B.247). By (B.253) and (B.254), there exists $h^{3,1} \in L^\infty((T_1, T_1+T)\times\Omega)^3$ such that the solution u^1 of (B.4), with $(0,T) = (T_1, T_1+T)$ and $u^1(T_1,.) = u(T_1,.)$, satisfies $u^1(T_1+T,.) = z$.

The mapping γ is continuous on the compact set $[0,1]$, so γ is uniformly continuous on $[0,1]$ by Heine's theorem. Consequently, there exists $\eta > 0$ such that for every $\theta_1, \theta_2 \in [0,1]$, verifying $|\theta_1 - \theta_2| \leq \eta$, $\|\gamma(\theta_1) - \gamma(\theta_2)\|_\infty < r$. Moreover, there exists $m \in \mathbb{N}^*$ sufficiently large such that $m\eta \leq 1 < (m+1)\eta$. Therefore, let us define $\theta_k = k\eta$ for $k \in \{0,\dots,m\}$ and $\theta_{m+1} = 1$. Then, we have

$$\Gamma \subset \bigcup_{i=0}^{m+1} B(\gamma(\theta_i), r). \tag{B.255}$$

We remark that we have $\gamma(\theta_0) = z$, $\gamma(\theta_{m+1}) = u^*$ and $\|\gamma(\theta_i) - \gamma(\theta_{i+1})\|_\infty < r$ for every $i \in \{1,\dots,m\}$ by definition of η.

We have $\|z - \gamma(\theta_1)\|_\infty = \|\gamma(\theta_0) - \gamma(\theta_1)\|_\infty < r$. Then, by (B.253) and (B.254), there exists $h^{3,2} \in L^\infty((T_1+T, T_1+2T)\times\Omega)^3$ such that the solution u^2 of (B.4), with $(0,T) = (T_1+T, T_1+2T)$ and $u^1(T_1+T,.) = z$, satisfies $u^1(T_1+2T,.) = \gamma(\theta_1)$.

By repeating m times this strategy, we get the existence of a control $h^3 \in L^\infty((0, T_1 + (m+2)T)\times\Omega)$ so that $h^3(t,.) = 0$ for $t \in (0,T_1)$, $h^3(t,.) = h^{3,1}(t,.)$ for $t \in (T_1, T_1+T)$, ... , $h^3(t,.) = h^{3,m+2}(t,.)$ for $t \in (T_1+(m+1)T, T_1+(m+2)T)$, such the solution u of

$$\begin{cases} \forall 1 \leq i \leq 4, & \\ \partial_t u_i - d_i \Delta u_i = (-1)^i(u_1u_3 - u_2u_4) + h_i^3 1_\omega & \text{in } (0, T_1+(m+2)T)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0, T_1+(m+2)T)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega, \end{cases} \tag{B.256}$$

satisfies $u(T_1+(m+2)T,.) = u^*$.

<u>Case 2 : $u_3^* = 0$.</u> From (B.3), we have $u_2^* = 0$ or $u_4^* = 0$. We can assume that $u_2^* = 0$. The other case is similar. By Theorem B.3.2, we know that there exists $\widehat{r} > 0$ such that for every $\widetilde{u}^* \in B(u^*, \widehat{r})_{L^\infty(\Omega)^4}$, we can find a control $h^3 \in L^\infty((0,T)\times\Omega)^3$ that enables to go from $\widetilde{u}^*$ to u^*. Consequently, we choose β such that $0 < \beta < \widehat{r}/2$ and $\frac{\beta(u_4^*+\widehat{r}/2)}{u_1^*+\widehat{r}/2} < \widehat{r}/2$ and we set $\widetilde{u}^* := (u_1^*+\widehat{r}/2, \beta, \frac{\beta(u_4^*+\widehat{r}/2)}{u_1^*+\widehat{r}/2}, u_4^*+\widehat{r}/2) \in B(u^*, \widehat{r})$. We remark that $\widetilde{u}^*$ satisfies (B.3) and $\widetilde{u_3}^* \neq 0$. Then, from the first case of the proof, we can find a control which drives z to $\widetilde{u}^*$. Next, we can find a control which drives $\widetilde{u}^*$ to u^*. $\square$

B.6 Comments, perspectives and open problems

B.6.1 ω_i instead of ω

An interesting open problem could be the generalization of Theorem B.3.2 to the system

$$\forall 1 \leq i \leq 4, \quad \begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) + h_i 1_{\omega_i} 1_{i \leq j} & \text{in } (0,T) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega, \end{cases} \tag{B.257}$$

where for every $i \in \{1, \ldots, j\}$, ω_i are nonempty open subsets such that $\omega_i \subset \Omega$ and $\bigcap_{i=1}^{j} \omega_i = \emptyset$ (otherwise, the generalization is straightforward).

B.6.2 Stationary solutions

We only have considered nonnegative stationary **constant** solutions of (B.1). It is not restrictive because of the following proposition.

Proposition B.6.1. Let $(u_i)_{1 \leq i \leq 4} \in C^2(\overline{\Omega})^4$ be a nonnegative solution of

$$\forall 1 \leq i \leq 4, \quad \begin{cases} -d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) & \text{in } \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{B.258}$$

Then, for every $1 \leq i \leq 4$, u_i is constant.

Proof. Let $\varepsilon > 0$. For every $i \in \{1, \ldots, 4\}$, let us denote $u_i^\varepsilon = u_i + \varepsilon$ and $w_i^\varepsilon = u_i^\varepsilon (\log u_i^\varepsilon - 1) + 1$. Note that $w_i^\varepsilon \geq 0$ for every $i \in \{1, \ldots, 4\}$. We have

$$\forall 1 \leq i \leq 4, \ \nabla w_i^\varepsilon = \log(u_i^\varepsilon) \nabla u_i^\varepsilon, \ \Delta w_i^\varepsilon = \log(u_i^\varepsilon) \Delta u_i^\varepsilon + \frac{|\nabla u_i^\varepsilon|^2}{u_i^\varepsilon}. \tag{B.259}$$

Then, from (B.258) and (B.259), we have that for every $1 \leq i \leq 4$,

$$\begin{cases} -d_i \Delta w_i^\varepsilon + d_i \frac{|\nabla u_i^\varepsilon|^2}{u_i^\varepsilon} = (-1)^i \log(u_i^\varepsilon)(u_1^\varepsilon u_3^\varepsilon - u_2^\varepsilon u_4^\varepsilon - \varepsilon(u_1 + u_3 - u_2 - u_4)) & \text{in } \Omega, \\ \frac{\partial w_i^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{B.260}$$

We add the four equations of (B.260) and we integrate on Ω. We get

$$\begin{aligned} 0 &+ \int_\Omega \sum_{i=1}^4 d_i \frac{|\nabla u_i^\varepsilon|^2}{u_i^\varepsilon} \\ &= -\left(\int_\Omega (\log(u_1^\varepsilon u_3^\varepsilon) - \log(u_2^\varepsilon u_4^\varepsilon))(u_1^\varepsilon u_3^\varepsilon - u_2^\varepsilon u_4^\varepsilon) \right) \\ &\quad + \varepsilon \left(\int_\Omega (\log(u_1^\varepsilon u_3^\varepsilon) - \log(u_2^\varepsilon u_4^\varepsilon))(u_1 + u_3 - u_2 - u_4) \right) \\ &\leq \varepsilon \left(\int_\Omega (\log(u_1^\varepsilon u_3^\varepsilon) - \log(u_2^\varepsilon u_4^\varepsilon))(u_1 + u_3 - u_2 - u_4) \right). \end{aligned} \tag{B.261}$$

Moreover,

$$\forall 1 \leq i \leq 4, \ \int_\Omega d_i \frac{|\nabla u_i^\varepsilon|^2}{u_i^\varepsilon} = \int_\Omega 4d_i |\nabla \sqrt{u_i^\varepsilon}|^2. \tag{B.262}$$

Consequently, from (B.261), (B.262) and by taking ε sufficiently small, for every $1 \leq i \leq 4$,

$$\begin{aligned}\int_\Omega 4d_i |\nabla \sqrt{u_i^\varepsilon}|^2 &\leq \varepsilon \left(\int_\Omega (\log(u_1^\varepsilon u_3^\varepsilon) - \log(u_2^\varepsilon u_4^\varepsilon))(u_1 + u_3 - u_2 - u_4) \right) \\ &\leq \varepsilon \left(\int_\Omega |\log(\varepsilon^4)||u_1 + u_3 - u_2 - u_4| \right).\end{aligned}$$

Then, by letting $\varepsilon \to 0$, we get that

$$\forall 1 \leq i \leq 4, \ \int_\Omega 4d_i |\nabla \sqrt{u_i}|^2 = 0.$$

Consequently, for every $1 \leq i \leq 4$, u_i is constant. □

We can also remark that there exist non constant solutions of (B.258). For example, in the case of $(d_1, d_2, d_3, d_4) = (1, 1, 1, 1)$, $(u_1^*, u_2^*, u_3^*, u_4^*) = (\varphi_\lambda, -\varphi_\lambda, \varphi_\lambda - \lambda, -\varphi_\lambda)$, where $\lambda > 0$ and φ_λ are respectively an eigenvalue and a corresponding eigenfunction of the unbounded operator $(-\Delta, H^2_{Ne}(\Omega))$ (see Definition B.4.13), is a solution of (B.258). The result of Theorem B.3.2 is still valid for non constant stationary solutions under a natural condition of sign of $(u_1^*, u_2^*, u_3^*, u_4^*)$ on a nonempty open subset $\omega_0 \subset \omega$ (see (B.53), (B.63), (B.78) after linearization). There is only one nontrivial thing to verify. For the proof of the observability inequalities (B.94) and (B.95), the application of Δ to some equations does not create "bad" terms. A good meaning to be convinced is to look at the inequality (B.138) which becomes

$$\begin{aligned}&I(0, \lambda, s, \Delta\varphi_4) \\ &\leq C \left(\int_0^T \int_\Omega e^{2s\alpha} \left\{ \sum_{i=1}^3 |\Delta\varphi_i|^2 + |\nabla\varphi_i|^2| + |\varphi_i|^2 \right\} + \int_0^T \int_{\omega_2} e^{2s\alpha} (s\phi)^3 |\Delta\varphi_4|^2 \right).\end{aligned} \tag{B.263}$$

It is clear that $\int_0^T \int_\Omega e^{2s\alpha} \left(\sum_{i=1}^3 |\Delta\varphi_i|^2 + |\nabla\varphi_i|^2| + |\varphi_i|^2 \right)$ can be absorbed by the left hand side of (B.140) by taking s sufficiently large.

B.6.3 Nonnegative solutions and nonnegative controls

In the spirit of the works [LTZ17] and [PZ17] and in order to make the model more realistic, an interesting open problem could be : for nonnegative initial conditions $(u_{i,0})_{1 \leq i \leq 4}$, and nonnegative stationary state $(u_i^*)_{1 \leq i \leq 4}$, does there exit a control $(h_i)_{1 \leq i \leq j}$ such that the solution $(u_i)_{1 \leq i \leq 4}$ of (B.4) remains nonnegative and satisfies (B.5) ?

B.6.4 Constraints on the initial condition for the controllability of the linearized system

The goal of this section is to show that the linear transformation we do before linearization (see (B.58) and (B.72)), seems to be essential. Indeed, this adequate change of variable leads to control all possible initial conditions (see the necessary conditions on the initial conditions due to invariant quantities of the nonlinear dynamics : Annexe B.2.2). One could think about [AKBDGB09a, Theorem 5.3] which gives sufficient conditions of controllability when the rank condition of Theorem B.1.6 is not verified. But it reduces the space of initial condition once more and it becomes "artificial" in our case.

The linearized-system of (B.4) around $(u_i^*)_{1\leq i\leq 4}$ is

$$\begin{cases} \partial_t u - D\Delta u = Au + B_j h^j 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases} \tag{B.264}$$

where

$$u = (u_1,u_2,u_3,u_4)^T,\ D = diag(d_1,d_2,d_3,d_4),\ A = \begin{pmatrix} -u_3^* & u_4^* & -u_1^* & u_2^* \\ u_3^* & -u_4^* & u_1^* & -u_2^* \\ -u_3^* & u_4^* & -u_1^* & u_2^* \\ u_3^* & -u_4^* & u_1^* & -u_2^* \end{pmatrix}, \tag{B.265}$$

and B_j, h^j are defined in (B.45).

Definition B.6.2. The system (B.264) is $(u_i^*)_{1\leq i\leq 4}$-controllable if for every $u_0 \in L^2(\Omega)^4$, there exists $h^j \in L^2(Q)^j$ such that the solution u of (B.264) satisfies $u(T,.) = u^*$.

We would also use [AKBDGB09a, Theorem 1] in order to deduce the necessary and sufficient condition of controllability to $(u_i^*)_{1\leq i\leq 4}$ for (B.264). First, let us denote by $(\lambda_k)_{k\in\mathbb{N}}$ the increasing sequence of the eigenvalues of the unbounded operator $(-\Delta, H^2_{Ne}(\Omega))$ (see Definition B.4.13 for the definition of $H^2_{Ne}(\Omega)$). In particular, $\lambda_0 = 0$.

Theorem B.6.3. The system (B.264) is $(u_i^*)_{1\leq i\leq 4}$-controllable if and only if

$$\forall k \in \mathbb{N},\ rank(-\lambda_k D + A|B_j) = 4, \tag{B.266}$$

where

$$((-\lambda_k D + A)|B_j) := \big(B_j, (-\lambda_k D + A)B_j, (-\lambda_k D + A)^2 B_j, (-\lambda_k D + A)^3 B_j\big).$$

For $j = 3$, we can check that for every $k \in \mathbb{N}$, $rank(-\lambda_k D + A|B_3) = 4$ if and only if $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$. It is consistent with Section B.4.1.1.1.

For $j = 2$ and $d_3 \neq d_4$, we can check that $rank(\lambda_0 + A|B_2) < 4$, then (B.264) is not $(u_i^*)_{1\leq i\leq 4}$-controllable. It is consistent with the hypothesis we have to make for the initial condition i.e. (B.23). But, we can deduce from [AKBDGB09a, Theorem 5.3] that (B.264) is $(u_i^*)_{1\leq i\leq 4}$-controllable for initial conditions verifying

$$\forall i \in \{1,\dots,4\},\ \frac{1}{|\Omega|}\int_\Omega u_{i,0}(x) = u_i^*. \tag{B.267}$$

The condition (B.267) is a more restrictive hypothesis than (B.23). It is only a sufficient condition. Actually, we have found a necessary and sufficient condition on the initial data for $(u_i^*)_{1\leq i\leq 4}$-controllability.

Proposition B.6.4. Let $j=2$, $d_3 \neq d_4$.
For every $u_0 \in L^2(\Omega)^4$ such that $\frac{1}{|\Omega|}\int_\Omega (u_{3,0}+u_{4,0}) = u_3^*+u_4^*$, there exists $h^2 \in L^2(Q)^2$ such that the solution u of (B.264) satisfies $u(T,.) = u^*$.
If $u_0 \in L^2(\Omega)^4$ does not satisfy $\frac{1}{|\Omega|}\int_\Omega (u_{3,0}+u_{4,0}) = u_3^*+u_4^*$, for every $h^2 \in L^2(Q)^2$, the solution u of (B.264) does not satisfy $u(T,.) = u^*$.

Proof. The necessary condition of controllability is a consequence of

$$\text{a.e. } t \in [0,T],\ \frac{d}{dt}\left(\int_\Omega (u_3(t,x)+u_4(t,x))dx\right) = 0.$$

The sufficient condition of controllability is a consequence of the adequate change of variable $(v_1,v_2,v_3,v_4) := (u_1,u_2,u_3,u_3+u_4)$ and the proof of the observability inequality (B.94). □

Remark B.6.5. We chose to state our previous result in the particular case $j=2$ and $d_3 \neq d_4$ for simplicity but one can generalize this proposition to other cases.

An interesting open problem could consist in trying to find precisely the initial conditions that can be controlled for systems of the form (B.264) when (B.266) is not satisfied. This will lead to a better understanding of the controllability properties of a large class of nonlinear reaction-diffusion systems.

B.6.5 More general nonlinear reaction-diffusion systems

Let $k \in \mathbb{N}^*$, $(\alpha_1,\dots,\alpha_k) \in (\mathbb{N})^n$, $(\beta_1,\dots,\beta_k) \in (\mathbb{N})^k$ such that for every $1 \leq i \leq k$, $\alpha_i \neq \beta_i$, $(d_1,\dots,d_k) \in (0,+\infty)^k$ and $J \subset \{1,\dots,k\}$. We consider the following nonlinear controlled reaction-diffusion system :

$$\forall 1 \leq i \leq k,\ \begin{cases} \partial_t u_i - d_i \Delta u_i = & \\ \qquad (\beta_i-\alpha_i)\left(\prod_{k=1}^n u_k^{\alpha_k} - \prod_{k=1}^n u_k^{\beta_k}\right) + h_i 1_\omega 1_{i\in J} & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{B.268}$$

The article [LB18b] by the author treats the local-controllability of (B.268) around non-negative (constant) stationary states by using the same kind of change of variables as in (B.58) and (B.72). Nevertheless, the proof of observability inequalities for the linearized system cannot follow the same strategy as performed in Annexe B.4.3.7. Indeed, if we apply Carleman estimates to each equation of the adjoint system, it leads to some global terms in the right hand side of the inequality that cannot be absorbed by the left hand side. Thus, as in [FCGBdT15, Hypothesis 3], a similar technical obstruction appears. Inspired by the recent work of Pierre Lissy and Enrique Zuazua (see [LZ17, Section 3]), who obtained sharp results for the null-controllability of non-diagonalizable systems of parabolic equations, the author proves the null-controllability of the linearized system. Then, the source term method introduced by Yuning Liu, Takéo Takahashi, Marius Tucsnak (see [LTT13]) enables to go back to the nonlinear reaction-diffusion system.

B.7 Appendix

B.7.1 L^∞-estimate for parabolic systems

We give the proof of Proposition B.2.3.

Proof. By using the fact that D is diagonalizable and $Sp(D) \subset (0, +\infty)$, we only have to prove the result when $D = diag(d_1, \dots, d_k)$ with $d_i \in (0, +\infty)$.

The first point of the proof i.e. the existence and the uniqueness of the weak solution $u \in Y^k$ is based on Galerkin approximations and energy estimates. One can easily adapt the arguments given in [Eva10, Section 7.1.2] to the Neumann cases.

The second point of the proof i.e. the L^∞ estimate is based on Stampacchia's method. We introduce

$$l(t) = (l_1(t), \dots, l_k(t))^T := l_0 \exp(tM)(1, \dots, 1)^T =: L(t)(1, \dots, 1)^T \in \mathbb{R}^k, \tag{B.269}$$

for every $t \in [0, T]$ and $l_0, M \in (0, +\infty)$ which will be chosen later. By (B.15), we have

$$\begin{aligned} &\forall w \in L^2(0,T; H^1(\Omega)^k),\\ &\int_0^T (\partial_t u, w)_{(H^1(\Omega)^k)', H^1(\Omega)^k} - \int_Q (sign(u)l').w + \int_Q D\nabla u.\nabla w\\ &= \int_Q (Au + g).w - \int_Q (sign(u)l').w, \end{aligned} \tag{B.270}$$

where $sign(u)l' = (sign(u_1)l_1', \dots, sign(u_k)l_k')^T$. We fix $t \in [0, T]$ and we apply (B.270) with w defined by $\forall (\tau, x) \in [0, T] \times \Omega$,

$$\begin{aligned} w(\tau, x) &:= sign(u)(|u|(t,x) - l(t))^+ 1_{[0,t]}(\tau)\\ &:= \big(sign(u_1)(|u_1|(t,x) - l_1(t))^+, \dots, (sign(u_k)(|u_k|(t,x) - l(t))^+\big)^T 1_{[0,t]}(\tau). \end{aligned}$$

We get

$$\int_0^t \frac{1}{2}\frac{d}{d\tau}\int_\Omega \sum_{i=0}^k \left((|u_i|(\tau,x)-l_i(\tau))^+\right)^2 dxd\tau + \int_0^t \int_\Omega \sum_{i=0}^k d_i \nabla u_i.\nabla u_i 1_{|u_i|\geq l_i}$$

$$= \int_0^t \int_\Omega \sum_{i=0}^k \left(\sum_{j=0}^k a_{ij}u_j + g_i - sign(u_i)l_i'\right) sign(u_i)(|u_i|-l_i)^+. \tag{B.271}$$

We remark that

$$-sign(u_i)l_i' sign(u_i)(|u_i|-l_i)^+ = -l_i'(|u_i|-l_i)^+.$$

Moreover, we have

$$\begin{aligned}
&\left(\sum_{j=0}^k a_{ij}u_j + g_i - sign(u_i)l_i'\right) sign(u_i)(|u_i|-l_i)^+\\
&\leq \left(\sum_{j=0}^k |a_{ij}||u_j| + |g_i| - l_i'\right)(|u_i|-l_i)^+\\
&\leq \left(\sum_{j=0}^k |a_{ij}|(|u_j|-l_j)^+ + A_i\right)(|u_i|-l_i)^+,
\end{aligned} \tag{B.272}$$

where $A_i := \sum_{j=0}^k l_j|a_{ij}| + g_i - l_i' = L\sum_{j=0}^k |a_{ij}| + g_i - ML$ (see (B.269)). We choose $l_0, M \in (0,+\infty)$ such that

$$M \geq \max_i \left\{\left\|\sum_{j=0}^k |a_{ij}|\right\|_\infty + 1\right\},\ \ l_0 = \max_i \left\{\|u_{0i}\|_\infty + \|g_i\|_\infty\right\}. \tag{B.273}$$

Then, we find

$$A_i \leq L(M-1) + l_0 - ML \leq L(M-1) + L - ML \leq 0. \tag{B.274}$$

By using $l_0 \geq \max_i \|u_{0i}\|_\infty$, $\int_0^t\int_\Omega \sum_{i=0}^k d_i\nabla u_i.\nabla u_i 1_{|u_i|\geq l_i} \geq 0$, (B.272), (B.274), together with (B.271), we have that for every $t \in [0,T]$,

$$\int_\Omega \sum_{i=0}^k \left((|u_i|(t,x)-l_i(t))^+\right)^2 dx \leq 2\int_0^t \int_\Omega \sum_{i=0}^k\sum_{j=0}^k |a_{ij}|(|u_j|-l_j)^+(|u_i|-l_i)^+ dxd\tau. \tag{B.275}$$

Cauchy-Schwartz inequality applied to the right hand side term of (B.275) gives

$$\forall t \in [0,T],\ \int_\Omega \sum_{i=0}^{k} \Big((|u_i|(t,x) - l_i(t))^+ \Big)^2 dx \leq C \int_0^t \int_\Omega \sum_{i=0}^{k} \Big((|u_i|(\tau,x) - l_i(\tau))^+ \Big)^2 dx d\tau, \tag{B.276}$$

where $C := 2k \max_{i,j} \|a_{ij}\|_\infty$. Gronwall's lemma applied to (B.276) gives

$$\forall i \in \{1,\dots,k\},\ \forall t \in [0,T],\ |u_i(t)| \leq l_i(t) = l_0 \exp(tM). \tag{B.277}$$

Therefore, from (B.277), we deduce (B.18) with our choice of l_0 (see (B.273)). □

B.7.2 Dissipation of the energy for crossed-diffusion parabolic systems

The goal of this section is to give a sketch of the proof of the dissipation of the energy (in time) for some parabolic systems.

Lemma B.7.1. Let $j \in \{1,2,3\}$, D_j defined by (B.51), (B.61), (B.76), $A \in \mathcal{E}_j$ (see (B.82), (B.84) and (B.86)), $\varphi_T \in L^2(\Omega)^4$ and φ be the solution of the following Cauchy problem

$$\begin{cases} -\varphi_t - D_j^T \Delta \varphi = A^T \varphi & \text{in } (0,T)\times\Omega, \\ \frac{\partial \varphi}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega. \end{cases}$$

Then, there exists $C > 0$ such that for every $(t_1,t_2) \in [0,T]^2, t_1 < t_2$,

$$\begin{aligned} &\sum_{i=1}^{j+1} \|\varphi_i(t_1,.)\|^2_{L^2(\Omega)} + \sum_{i=j+2}^{4} \|\varphi_i(t_1,.) - (\varphi_i)_\Omega(t_1)\|^2_{L^2(\Omega)} \\ &\leq C \left(\sum_{i=1}^{j+1} \|\varphi_i(t_2,.)\|^2_{L^2(\Omega)} + \sum_{i=j+2}^{4} \|\varphi_i(t_2,.) - (\varphi_i)_\Omega(t_2)\|^2_{L^2(\Omega)} \right). \end{aligned} \tag{B.278}$$

Proof. By using the fact that D_j is diagonalizable, we only have to prove the result when D is diagonal. First, we introduce $\psi = \Big(\varphi_1, \dots, \varphi_{j+1}, \varphi_{j+2} - (\varphi_{j+2})_\Omega(.), \dots, \varphi_4 - (\varphi_4)_\Omega(.)\Big)$. We look for the parabolic system satisfied by ψ. Then, we multiply the variational formulation (see (B.15)) by $w(t,x) = \psi(t,x) 1_{[t_1,t_2]}(t)$. By Young inequalities, we find a differential inequality as follows

$$\text{a.e. } t \in [t_1,t_2],\ \frac{d}{dt} \|\psi(t)\|^2_{L^2(\Omega)} \leq C \|\psi(t)\|^2_{L^2(\Omega)}.$$

Then, we use Gronwall's lemma to deduce (B.278). □

B.7.3 Technical estimates for the observability inequality in the case of 1 control

The goal of this section is to prove Lemma B.4.31 and Lemma B.4.32. We use the same notations as in Annexe B.4.3.7. We recall that s is supposed to be fixed and the constants C may depend on s.

First, we recall two classical facts on the heat equation for Dirichlet conditions : a well-posedness result and a regularity result.

B.7.3.1 General lemmas

Proposition B.7.2. Let $d \in (0,+\infty)$, $u_0 \in L^2(\Omega)$, $g \in L^2(Q)$. From [Eva10, Section 7.1, Theorem 3 and Theorem 4], the following Cauchy problem admits a unique weak solution $u \in Z := L^2(0,T;H_0^1(\Omega)) \cap W^{1,2}(0,T;H^{-1}(\Omega))$

$$\begin{cases} \partial_t u - d\Delta u = g & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases}$$

This means that u is the unique function in Z that satisfies the variational fomulation

$$\forall w \in L^2(0,T;H_0^1(\Omega)),\ \int_0^T (\partial_t u, w)_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_Q d\nabla u.\nabla w = \int_Q gw, \tag{B.279}$$

and

$$u(0,.) = u_0 \text{ in } L^2(\Omega). \tag{B.280}$$

Moreover, there exists $C>0$ independent of u_0 and g such that

$$\|u\|_Z \le C\left(\|u_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q)}\right).$$

Proposition B.7.3. Let $d \in (0,+\infty)$, $g \in L^2(Q)$, $u_0 \in C_0^\infty(\Omega)$. From Proposition B.7.2, the following Cauchy problem admits a unique weak solution $u \in Z$

$$\begin{cases} \partial_t u - d\Delta u = g & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases}$$

Moreover, from [Eva10, Section 7.1, Theorem 5], $u \in Z_2 := L^2(0,T,H^2(\Omega)\cap H_0^1(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega))$ and if $u_0 = 0$, then there exists $C>0$ independent of g such that

$$\|u\|_{Z_2} \le C\|g\|_{L^2(Q)}.$$

The following lemma is inspired by the proof of [CSG15, Theorem 2.2].

Lemma B.7.4. Let $d \in (0,+\infty)$, $f \in Y_2$ (see Definition B.4.13), $\Phi_T \in C_0^\infty(\Omega)$, $\widetilde{\omega}$ be an open subset such that $\widetilde{\omega} \subset\subset \omega_0$, $\chi \in C^\infty(\overline{\Omega};[0,+\infty[)$ such that $supp(\chi) \subset\subset \widetilde{\omega}$, $(r,k) \in \mathbb{R}\times[1,+\infty)$, $\Theta = \chi e^{s\alpha}(s\phi)^r$. Let $\Phi \in Z_2$ (see Proposition B.7.3) be the solution of

$$\begin{cases} -\partial_t \Phi - d\Delta\Phi = \Delta f & \text{in } (0,T)\times\Omega, \\ \Phi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \Phi(T,.) = \Phi_T & \text{in } \Omega. \end{cases} \tag{B.281}$$

We decompose

$$\Theta\Phi = \eta + \psi, \tag{B.282}$$

where $\eta \in Z_2$ and $\psi \in Z_2$ satisfy

$$\begin{cases} -\partial_t \eta - d\Delta\eta = \Theta\Delta f & \text{in } (0,T)\times\Omega, \\ \eta = 0 & \text{on } (0,T)\times\partial\Omega, \\ \eta(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{B.283}$$

$$\begin{cases} -\partial_t \psi - d\Delta\psi = -(\partial_t\Theta)\Phi - 2d\nabla\Theta.\nabla\Phi - d(\Delta\Theta)\Phi & \text{in } (0,T)\times\Omega, \\ \psi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \psi(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{B.284}$$

Then, there exist $\widetilde{\chi} \in C^\infty(\overline{\Omega};[0,+\infty[)$ such that $supp(\widetilde{\chi}) \subset\subset \widetilde{\omega}$, $\widetilde{\chi} = 1$ on $supp(\chi)$ and $C > 0$ such that

$$\|\eta\|^2_{L^2(Q)} \leq C \int_0^T \int_{\widetilde{\omega}} \widetilde{\chi}^2 e^{2s\alpha}(s\phi)^{2(r+2)}|f|^2, \tag{B.285}$$

$$\begin{aligned} &\left\| \frac{\psi}{(s\phi)^k} \right\|^2_{L^2(0,T;H_0^1(\Omega))} + \left\| \left(\frac{\psi}{(s\phi)^k} \right)_t \right\|^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq C \left(\|\eta\|^2_{L^2(Q)} + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^{2(r+2-k)}|\Phi|^2 \right). \end{aligned} \tag{B.286}$$

Proof. Let $\Gamma \in L^2(Q)$ and let $z \in Z_2$ be the solution of

$$\begin{cases} \partial_t z - d\Delta z = \Gamma & \text{in } (0,T)\times\Omega, \\ z = 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{B.287}$$

By Proposition B.7.3, we have

$$\|z\|^2_{L^2(0,T;H^2(\Omega))} \leq C \|\Gamma\|^2_{L^2(Q)}. \tag{B.288}$$

A duality argument between (B.283) and (B.287) gives

$$\int_0^T \int_\Omega \eta\Gamma dxdt = \int_0^T \int_\Omega \Theta\Delta(f) z dxdt. \tag{B.289}$$

We integrate by parts with respect to the spatial variable,

$$\int_0^T \int_\Omega \Theta \Delta(f) z dx dt = \int_0^T \int_\Omega f \Delta(\Theta z) dx dt. \tag{B.290}$$

There exists $\widetilde{\chi} \in C^\infty(\overline{\Omega}; [0, +\infty[)$ such that $supp(\widetilde{\chi}) \subset\subset \widetilde{\omega}$, $\widetilde{\chi} = 1$ on $supp(\chi)$ and

$$\forall i \in \{1, 2\},\ |D_x^i \Theta| \leq C \widetilde{\chi} (s\phi)^{r+i} e^{s\alpha} \text{ in } (0, T) \times \Omega. \tag{B.291}$$

Therefore, from (B.288) and (B.291), we can deduce that

$$\int_0^T \int_\Omega f \Delta(\Theta z) dx dt \leq \frac{1}{2} \|\Gamma\|_{L^2(Q)}^2 + C \int_0^T \int_\Omega \widetilde{\chi}^2 e^{2s\alpha} (s\phi)^{2(r+2)} |f|^2 dx dt. \tag{B.292}$$

By using (B.289), (B.290), (B.292) and by taking $\Gamma = \eta$, we deduce (B.285).

We introduce

$$\rho = (s\phi)^{-k}. \tag{B.293}$$

Then, we have

$$\begin{cases} -\partial_t(\rho\psi) - d\Delta(\rho\psi) = \rho(-(\partial_t \Theta)\Phi - 2d\nabla\Theta.\nabla\Phi - d(\Delta\Theta)\Phi) & \\ \qquad\qquad -(\partial_t \rho)\psi - 2d\nabla\rho.\nabla\psi - d(\Delta\rho)\psi & \text{in } (0, T) \times \Omega, \\ \rho\psi = 0 & \text{on } (0, T) \times \partial\Omega, \\ \rho\psi(T, .) = 0 & \text{in } \Omega. \end{cases} \tag{B.294}$$

We estimate the source term of (B.294). We have by definition of Θ, the fact that $k \geq 1$, (B.282), (B.293) and the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, the following estimates

$$\|\rho \partial_t(\Theta)\Phi\|_{L^2(Q)}^2 \leq C \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2, \tag{B.295}$$

$$\begin{aligned} \|\rho\nabla\Theta.\nabla\Phi\|_{L^2(0,T;H^{-1}(\Omega))}^2 &= \|\nabla.(\rho\Phi\nabla\Theta) - (\rho(\Delta\Theta)\Phi) - (\nabla\rho.\nabla\Theta)\Phi\|_{L^2(0,T;H^{-1}(\Omega))}^2 \\ &\leq C \left(\|\rho\Phi\nabla\Theta\|_{L^2(Q)}^2 + \|\rho(\Delta\Theta)\Phi\|_{L^2(Q)}^2 + \|(\nabla\rho.\nabla\Theta)\Phi\|_{L^2(Q)}^2 \right) \\ &\leq C \int_0^T \int_\Omega e^{2s\alpha} \left((s\phi)^{2(r+1-k)} + (s\phi)^{2(r+2-k)} + (s\phi)^{2(r+1-k)} \right) |\Phi|^2 \\ &\leq C \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2, \end{aligned} \tag{B.296}$$

$$\begin{aligned} \|(\partial_t \rho)\psi\|_{L^2(Q)}^2 &= \|(\partial_t \rho)(\Theta\Phi - \eta)\|_{L^2(Q)}^2 \\ &\leq C \left(\int_0^T \int_\Omega (s\phi)^{2(-k+1)} |\eta|^2 + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+1-k)} |\Phi|^2 \right) \\ &\leq C \left(\int_0^T \int_\Omega |\eta|^2 + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2 \right), \end{aligned} \tag{B.297}$$

$$
\begin{aligned}
\|\nabla\rho.\nabla\psi\|^2_{L^2(0,T;H^{-1}(\Omega))} &= \|\nabla.(\psi\nabla\rho)-\psi\Delta\rho\|^2_{L^2(0,T;H^{-1}(\Omega))}\\
&= \|\nabla.((\Theta\Phi-\eta)\nabla\rho)-(\Theta\Phi-\eta)\Delta\rho\|^2_{L^2(0,T;H^{-1}(\Omega))}\\
&\leq C\left(\int_0^T\int_\Omega (s\phi)^{-2k}|\eta|^2+\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^{2(r-k)}|\Phi|^2\right).\\
&\leq C\left(\int_0^T\int_\Omega |\eta|^2+\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^{2(r+2-k)}|\Phi|^2\right).
\end{aligned}\tag{B.298}
$$

By using (B.294), (B.295), (B.296), (B.297), (B.298) and Proposition B.7.2, we deduce (B.286). □

Corollary B.7.5. We take the same notations as in Lemma B.7.4 and $g \in Y_2$. Then, for every $\delta > 0$,

$$
\begin{aligned}
&\int_0^T\int_{\widetilde{\omega}} \chi e^{s\alpha}\psi\Delta g\\
&\leq \delta\left(\int_0^T\int_{\widetilde{\omega}} \widetilde{\chi}^2 e^{2s\alpha}(s\phi)^{2(r+2)}|f|^2+\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^{2(r+2-k)}|\Phi|^2\right)\\
&\quad + C_\delta\int_0^T\int_{\widetilde{\omega}} e^{2s\alpha}(s\phi)^{2(k+1)}|\nabla g|^2,
\end{aligned}\tag{B.299}
$$

$$
\begin{aligned}
&\int_0^T\int_{\widetilde{\omega}} \chi e^{s\alpha}\psi\partial_t g\\
&\leq \delta\left(\int_0^T\int_{\widetilde{\omega}} \widetilde{\chi}^2 e^{2s\alpha}(s\phi)^{2(r+2)}|f|^2+\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^{2(r+2-k)}|\Phi|^2\right) &&\text{(B.300)}\\
&\quad + C_\delta\left(\int_0^T\int_{\widetilde{\omega}} e^{2s\alpha}(s\phi)^{2(k+2)}|g|^2+\int_0^T\int_{\widetilde{\omega}} e^{2s\alpha}(s\phi)^{2k}|\nabla g|^2\right). &&\text{(B.301)}
\end{aligned}
$$

Proof. We integrate by parts with respect to the spatial variable and we use (B.286), (B.285),

$$
\begin{aligned}
&\int_0^T\int_{\widetilde{\omega}} \chi e^{s\alpha}\psi\Delta g\\
&= -\int_0^T\int_{\widetilde{\omega}} \frac{\psi}{(s\phi)^k}\nabla(\chi e^{s\alpha}(s\phi)^k).\nabla g-\int_0^T\int_{\widetilde{\omega}} \chi e^{s\alpha}(s\phi)^k\nabla\left(\frac{\psi}{(s\phi)^k}\right).\nabla g\\
&\leq \delta\left\|\frac{\psi}{(s\phi)^k}\right\|^2_{L^2(0,T;H_0^1(\Omega))}+C_\delta\int_0^T\int_{\widetilde{\omega}} e^{2s\alpha}(s\phi)^{2(k+1)}|\nabla g|^2\\
&\leq \delta\left(\|\eta\|^2_{L^2(Q)}+\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^{2(r+2-k)}|\Phi|^2\right)+C_\delta\int_0^T\int_{\widetilde{\omega}} e^{2s\alpha}(s\phi)^{2(k+1)}|\nabla g|^2\\
&\leq \delta\left(\int_0^T\int_{\widetilde{\omega}} \widetilde{\chi}^2 e^{2s\alpha}(s\phi)^{2(r+2)}|f|^2+\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^{2(r+2-k)}|\Phi|^2\right)\\
&\quad + C_\delta\int_0^T\int_{\widetilde{\omega}} e^{2s\alpha}(s\phi)^{2(k+1)}|\nabla g|^2.
\end{aligned}
$$

We integrate by parts with respect to the time variable and we use (B.286), (B.285),

$$
\begin{aligned}
&\int_0^T \int_{\widetilde{\omega}} \chi e^{s\alpha} \psi \partial_t g \\
&= - \left\langle \left(\frac{\psi}{(s\phi)^k} \right)_t, \chi e^{s\alpha} (s\phi)^k g \right\rangle_{L^2(0,T;H^{-1}(\Omega)), L^2(0,T;H_0^1(\Omega))} \\
&\quad - \int_0^T \int_{\widetilde{\omega}} \frac{\psi}{(s\phi)^k} \chi \partial_t (e^{s\alpha}(s\phi)^k) g \\
&\leq \delta \left\| \left(\frac{\psi}{(s\phi)^k} \right)_t \right\|^2_{L^2(0,T;H^{-1}(\Omega))} + C_\delta \left\| \chi e^{s\alpha} (s\phi)^k g \right\|^2_{L^2(0,T;H_0^1(\Omega))} \\
&\quad + \delta \left\| \left(\frac{\psi}{(s\phi)^k} \right) \right\|^2_{L^2(0,T;L^2(\Omega))} + C_\delta \int_0^T \int_{\widetilde{\omega}} |\partial_t (e^{s\alpha}(s\phi)^k)|^2 |g|^2 \\
&\leq \delta \left(\|\eta\|^2_{L^2(Q)} + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2 \right) \\
&\quad + C_\delta \left(\int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2(k+2)} |g|^2 + \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2k} |\nabla g|^2 \right) \\
&\leq \delta \left(\int_0^T \int_{\widetilde{\omega}} \widetilde{\chi}^2 e^{2s\alpha} (s\phi)^{2(r+2)} |f|^2 + \int_0^T \int_\Omega e^{2s\alpha} (s\phi)^{2(r+2-k)} |\Phi|^2 \right) \\
&\quad + C_\delta \left(\int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2(k+2)} |g|^2 + \int_0^T \int_{\widetilde{\omega}} e^{2s\alpha} (s\phi)^{2k} |\nabla g|^2 \right).
\end{aligned}
$$

□

B.7.3.2 Proof of technical lemmas : Lemma B.4.31 and Lemma B.4.32

Let $\varepsilon \in (0,1)$. We introduce

$$\theta = \chi_3 e^{s\alpha} (s\phi)^3. \tag{B.302}$$

The function $\theta\Delta\Delta\varphi_4$ satisfies the following parabolic system (see (B.158)),

$$
\begin{cases}
-\partial_t(\theta\Delta\Delta\varphi_4) - d_4 \Delta(\theta\Delta\Delta\varphi_4) & \\
\quad = \theta\Delta\Delta(m_3(\varphi_1 - \varphi_2)) - \partial_t\theta\Delta\Delta\varphi_4 - 2d_4 \nabla\theta.\nabla(\Delta\Delta\varphi_4) - d_4 \Delta\theta\Delta\Delta\varphi_4 & \text{in } Q_T, \\
\theta\Delta\Delta\varphi_4 = 0 & \text{on } \Sigma_T, \\
\theta\Delta\Delta\varphi_4(T,.) = 0 & \text{in } \Omega.
\end{cases}
\tag{B.303}
$$

We decompose

$$\theta\Delta\Delta\varphi_4 = \eta + \psi, \tag{B.304}$$

where η and ψ solve, respectively,

$$
\begin{cases}
-\partial_t \eta - d_4 \Delta \eta = \theta\Delta\Delta(m_3(\varphi_1 - \varphi_2)) & \text{in } (0,T) \times \Omega, \\
\eta = 0 & \text{on } (0,T) \times \partial\Omega, \\
\eta(T,.) = 0 & \text{in } \Omega,
\end{cases}
\tag{B.305}
$$

$$\begin{cases} -\partial_t\psi - d_4\Delta\psi = -\partial_t\theta\Delta\Delta\varphi_4 - 2d_4\nabla\theta.\nabla(\Delta\Delta\varphi_4) - d_4\Delta\theta\Delta\Delta\varphi_4 & \text{in } (0,T)\times\Omega, \\ \psi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \psi(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{B.306}$$

B.7.3.2.1 Proof of Lemma B.4.31 We have

$$\int_0^T\int_{\omega_2}(\chi_3(x))^2 e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\Delta\Delta\varphi_3)dxdt = \int_0^T\int_{\omega_2}\chi_3(x)e^{s\alpha}(\eta+\psi)(\Delta\Delta\varphi_3)dxdt. \tag{B.307}$$

The first term in the right-hand side of (B.307) can be estimated as follows,

$$\begin{aligned}\int_0^T\int_{\omega_2}\chi_3(x)e^{s\alpha}\eta(\Delta\Delta\varphi_3)dxdt &\leq \varepsilon\int_0^T\int_\Omega e^{2s\alpha}(s\phi)|\Delta\Delta\varphi_3|^2 + C_\varepsilon\int_0^T\int_{\omega_2}(\chi_3(x))^2(s\phi)^{-1}\eta^2 \\ &\leq \varepsilon\int_0^T\int_\Omega e^{2s\alpha}(s\phi)|\Delta\Delta\varphi_3|^2 + C_\varepsilon\int_0^T\int_\Omega \eta^2. \end{aligned} \tag{B.308}$$

Lemma B.7.6. For every $\delta > 0$,

$$\begin{aligned}&\int_0^T\int_\Omega|\eta|^2dxdt \\ &\leq \delta\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2) \\ &\quad + C_\delta\int_0^T\int_{\omega_2}e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2)\Big\}.\end{aligned} \tag{B.309}$$

Proof. The idea of the proof is to apply two times Lemma B.7.4 because the source term of (B.305) is $\theta\Delta\Delta(\dots)$.

Step 1 : We apply Lemma B.7.4 : (B.285) with $d = d_4$, $f = m_3\Delta(\varphi_1-\varphi_2)$, $\Phi_T = \Delta\Delta\varphi_{4,T}$, $\widetilde{\omega} = \omega_2$, $\chi = \chi_3$, $r = 3$, $\Theta = \theta$, $\Phi = \Delta\Delta\varphi_4$ and the decomposition (B.304). Then, there exists $\widetilde{\chi_3} \in C^\infty(\overline{\Omega};[0,+\infty[)$ such that $supp(\widetilde{\chi_3}) \subset\subset \omega_2$, $\widetilde{\chi_3} = 1$ on $supp(\chi_3)$ and

$$\|\eta\|^2_{L^2(Q)} \leq C\int_0^T\int_{\omega_2}(\widetilde{\chi_3})^2e^{2s\alpha}(s\phi)^{10}(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)dxdt. \tag{B.310}$$

Remark B.7.7. This estimate is not sufficient because we can not absorb the right hand side term of (B.310) by the left hand side term of (B.165).

Step 2 : Now, our aim is to prove that for every $i \in \{1,2\}$, $\delta > 0$, we have

$$\begin{aligned}
&\int_0^T \int_{\omega_2} (\widetilde{\chi_3})^2 e^{2s\alpha}(s\phi)^{10} |\Delta\varphi_i|^2 dxdt \\
&\leq \delta \left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)^4 |\Delta\varphi_i|^2 dxdt \right) \\
&\quad + C_\delta \left(\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) dxdt + \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22} |\nabla\varphi_i|^2 dxdt \right).
\end{aligned} \tag{B.311}$$

Remark B.7.8. This previous estimate is also useful for the proof of the observability inequality with one component (see (B.170)).

First, we remark that

$$\int_0^T \int_{\omega_2} (\widetilde{\chi_3})^2 e^{2s\alpha}(s\phi)^{10} |\Delta\varphi_i|^2 = \int_0^T \int_{\omega_2} \widetilde{\chi_3} e^{s\alpha} \widetilde{\theta} \Delta\varphi_i \Delta\varphi_i,$$

with

$$\widetilde{\theta} = \widetilde{\chi_3} e^{s\alpha}(s\phi)^{10}. \tag{B.312}$$

Moreover, $\widetilde{\theta}\Delta\varphi_i$ satisfies the following parabolic system (see (B.156) and Lemma B.4.23),

$$\begin{cases} -\partial_t(\widetilde{\theta}\Delta\varphi_i) - d_i\Delta(\widetilde{\theta}\Delta\varphi_i) & \\ \quad = \widetilde{\theta}\Delta(a_{1i}\varphi_1 + a_{2i}\varphi_2 + \delta_{i2}(d_2 - d_3)\Delta\varphi_3) & \\ \qquad -\partial_t\widetilde{\theta}\Delta\varphi_i - 2d_i\nabla\widetilde{\theta}.\nabla(\Delta\varphi_i) - d_i\Delta\widetilde{\theta}\Delta\varphi_i & \text{in } (0,T)\times\Omega, \\ \widetilde{\theta}\Delta\varphi_i = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{\theta}\Delta\varphi_i(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{B.313}$$

We decompose

$$\widetilde{\theta}\Delta\varphi_i = \widetilde{\eta_i} + \widetilde{\psi_i}, \tag{B.314}$$

where $\widetilde{\eta_i}$ and $\widetilde{\psi_i}$ solve, respectively,

$$\begin{cases} -\partial_t\widetilde{\eta_i} - d_i\Delta\widetilde{\eta_i} = \widetilde{\theta}\Delta(a_{1i}\varphi_1 + a_{2i}\varphi_2 + \delta_{i2}(d_2 - d_3)\Delta\varphi_3) & \text{in } (0,T)\times\Omega, \\ \widetilde{\eta_i} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{\eta_i}(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{B.315}$$

$$\begin{cases} -\partial_t\widetilde{\psi_i} - d_i\Delta\widetilde{\psi_i} = -\partial_t\widetilde{\theta}\Delta\varphi_i - 2d_i\nabla\widetilde{\theta}.\nabla(\Delta\varphi_i) - d_i\Delta\widetilde{\theta}\Delta\varphi_i & \text{in } (0,T)\times\Omega, \\ \widetilde{\psi_i} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{\psi_i}(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{B.316}$$

We have

$$\int_0^T \int_{\omega_2} (\widetilde{\chi_3})^2 e^{2s\alpha}(s\phi)^{10} |\Delta\varphi_i|^2 dxdt = \int_0^T \int_{\omega_2} \widetilde{\chi_3} e^{s\alpha}(\widetilde{\eta_i} + \widetilde{\psi_i})(\Delta\varphi_i) dxdt. \tag{B.317}$$

The first term in the right-hand side of (B.317) can be estimated as follows,

$$\int_0^T\int_{\omega_2}\widetilde{\chi_3}e^{s\alpha}\widetilde{\eta_i}(\Delta\varphi_i)dxdt \le \delta\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4|\Delta\varphi_i|^2dxdt+C_\delta\int_0^T\int_\Omega \widetilde{\eta_i}^2dxdt. \quad (B.318)$$

Then, we apply Lemma B.7.4 : (B.285) with $d=d_i$, $f=a_{1i}\varphi_1+a_{2i}\varphi_2+\delta_{i2}(d_2-d_3)\Delta\varphi_3\in Y_2$ (because $A\in\mathcal{M}_4(C_0^\infty(Q))$), $\Phi_T=\Delta\varphi_{i,T}$, $\widetilde{\omega}=\omega_2$, $\chi=\widetilde{\chi_3}$, $r=10$, $\Theta=\widetilde{\theta}$, $\Phi=\Delta\varphi_i$ and the decomposition (B.314). There exists $\chi_3{}^\sharp\in C^\infty(\overline{\Omega};[0,+\infty[)$ such that $supp(\chi_3{}^\sharp)\subset\subset\omega_2$ and C which depends on $\|A\|_{L^\infty(Q)}$

$$\int_0^T\int_\Omega|\widetilde{\eta_i}|^2dxdt\le C\int_0^T\int_{\omega_2}(\chi_3{}^\sharp)^2e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)dxdt. \quad (B.319)$$

Then, (B.318) and (B.319) give

$$\begin{aligned}&\int_0^T\int_{\omega_2}\widetilde{\chi_3}e^{s\alpha}\widetilde{\eta_i}(\Delta\varphi_i)dxdt\\&\le\delta\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4|\Delta\varphi_i|^2dxdt+C_\delta\int_0^T\int_{\omega_2}e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)dxdt.\end{aligned} \quad (B.320)$$

For the second term in the right-hand side of (B.317), we use Corollary B.7.5 : (B.299) with $d=d_i$, $f=a_{1i}\varphi_1+a_{2i}\varphi_2+\delta_{i2}(d_2-d_3)\Delta\varphi_3\in Y_2$, $\Phi_T=\Delta\varphi_{i,T}$, $\widetilde{\omega}=\omega_2$, $\chi=\widetilde{\chi_3}$, $(r,k)=(10,10)$, $\Theta=\widetilde{\theta}$, $\Phi=\Delta\varphi_i$ and the decomposition (B.314)). Then, we have

$$\begin{aligned}&\int_0^T\int_{\omega_2}\widetilde{\chi_3}e^{s\alpha}\widetilde{\psi_i}\Delta\varphi_i\\&\le\delta\left(\int_0^T\int_{\omega_2}(\chi_3{}^\sharp)^2e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4|\Delta\varphi_i|^2\right)\\&\quad+C_\delta\int_0^T\int_{\omega_2}e^{2s\alpha}(s\phi)^{22}|\nabla\varphi_i|^2.\end{aligned} \quad (B.321)$$

Gathering (B.317), (B.320) and (B.321), we have (B.311).

The estimates (B.310) and (B.311) give (B.309). □

End of the proof of Lemma B.4.31 : Applying Lemma B.7.6 with $\delta=\varepsilon/C_\varepsilon$, we find

$$\begin{aligned}&\int_0^T\int_\Omega|\eta|^2dxdt\\&\le\frac{\varepsilon}{C_\varepsilon}\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)dxdt\\&\quad+C'_\varepsilon\int_0^T\int_{\omega_2}e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2)\Big\}dxdt.\end{aligned} \quad (B.322)$$

Then, we put (B.322) in (B.308) to get

$$\begin{aligned}
&\int_0^T\int_{\omega_2}\chi_3(x)e^{s\alpha}\eta(\Delta\Delta\varphi_3)\\
&\leq \varepsilon\left(\int_0^T\int_\Omega e^{2s\alpha}(s\phi)|\Delta\Delta\varphi_3|^2+\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^4(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)\right)\\
&+C_\varepsilon\left(\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+\int_0^T\int_{\omega_2} e^{2s\alpha}(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2)\right).
\end{aligned} \tag{B.323}$$

Lemma B.7.9. For every $\delta>0$,

$$\begin{aligned}
&\int_0^T\int_{\omega_2}\chi_3e^{s\alpha}\psi(\Delta\Delta\varphi_3)dxdt\\
&\leq \delta\left(\int_0^T\int_{\omega_2}(\widetilde{\chi_3})^2e^{2s\alpha}(s\phi)^{10}(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)+\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2\right)\\
&\quad+C_\delta\int_0^T\int_{\omega_2}e^{2s\alpha}(s\phi)^9|\nabla\Delta\varphi_3|^2.
\end{aligned} \tag{B.324}$$

Proof. We apply Corollary B.7.5 : (B.299) with $d=d_4$, $f=m_3\Delta(\varphi_1-\varphi_2)$, $\Phi_T=\Delta\Delta\varphi_{4,T}$, $\widetilde{\omega}=\omega_2$, $\chi=\chi_3$, $(r,k)=(3,7/2)$, $\Theta=\theta$, $\Phi=\Delta\Delta\varphi_4$, the decomposition (B.304) and $g=\Delta\varphi_3$. □

Applying Lemma B.7.9 with $\delta=\varepsilon$, we find

$$\begin{aligned}
&\int_0^T\int_{\omega_2}\chi_3e^{s\alpha}\psi(\Delta\Delta\varphi_3)dxdt\\
&\leq \varepsilon\left(\int_0^T\int_{\omega_2}(\widetilde{\chi_3})^2e^{2s\alpha}(s\phi)^{10}(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)+\int_0^T\int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2\right)\\
&\quad+C_\varepsilon\int_0^T\int_{\omega_2}e^{2s\alpha}(s\phi)^9|\nabla\Delta\varphi_3|^2.
\end{aligned} \tag{B.325}$$

Then, we put (B.311) with $\delta=\varepsilon$ in (B.325) to get

$$\begin{aligned}
&\int_0^T\int_{\omega_2}\chi_3e^{s\alpha}\psi(\Delta\Delta\varphi_3)\\
&\leq \varepsilon\left(\int_0^T\int_\Omega e^{2s\alpha}\Big\{(s\phi)^4(|\Delta\varphi_1|^2+|\Delta\varphi_2|^2)+(s\phi)^3|\Delta\Delta\varphi_4|^2\Big\}\right)\\
&+C_\varepsilon\left(\int_0^T\int_{\omega_2} e^{2s\alpha}\Big\{(s\phi)^{24}(|\varphi_1|^2+|\varphi_2|^2+|\Delta\varphi_3|^2)+(s\phi)^{22}(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2+|\nabla\Delta\varphi_3|^2)\Big\}\right).
\end{aligned} \tag{B.326}$$

Therefore, recalling (B.307), (B.323), (B.326), we get (B.168) and consequently Lemma B.4.31.

B.7.3.2.2 Proof of Lemma B.4.32 We have by (B.302) and (B.304)

$$\int_0^T \int_{\omega_2} (\chi_3(x))^2 e^{2s\alpha}(s\phi)^3(\Delta\Delta\varphi_4)(\partial_t\Delta\varphi_3)dxdt = \int_0^T \int_{\omega_2} \chi_3(x)e^{s\alpha}(\eta+\psi)\partial_t(\Delta\varphi_3)dxdt. \tag{B.327}$$

We easily have by Young's inequality

$$\int_0^T \int_{\omega_2} \chi_3(x)e^{s\alpha}\eta\partial_t(\Delta\varphi_3)dxdt \leq \varepsilon \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)|\partial_t(\Delta\varphi_3)|^2 dxdt + C_\varepsilon \int_0^T \int_\Omega |\eta|^2 dxdt. \tag{B.328}$$

By using Lemma B.7.6 with $\delta = \varepsilon/C_\varepsilon$, we can deduce from (B.328) that

$$\begin{aligned}
&\int_0^T \int_{\omega_2} \chi_3(x)e^{s\alpha}\eta(\partial_t\Delta\varphi_3)dxdt \\
&\leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha}(s\phi)|\partial_t\Delta\varphi_3|^2 + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^4(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2)\right) \\
&+ C_\varepsilon \left(\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{24}(|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) + \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{22}(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2)\right).
\end{aligned} \tag{B.329}$$

Then, we estimate the other term in the right hand side of (B.327).

Lemma B.7.10. For every $\delta > 0$,

$$\begin{aligned}
&\int_0^T \int_{\omega_2} \chi_3 e^{s\alpha}\psi\partial_t\Delta\varphi_3 \\
&\leq \delta \left(\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{10}(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2) + \int_0^T \int_\Omega e^{2s\alpha}(s\phi)^3|\Delta\Delta\varphi_4|^2\right) \\
&\quad + C_\delta \left(\int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^{11}|\Delta\varphi_3|^2 + \int_0^T \int_{\omega_2} e^{2s\alpha}(s\phi)^7|\nabla\Delta\varphi_3|^2\right).
\end{aligned} \tag{B.330}$$

Proof. We apply Corollary B.7.5 : (B.301) with $d = d_4$, $f = \Delta(\varphi_1 - \varphi_2)$, $\Phi_T = \Delta\Delta\varphi_{4,T}$, $\widetilde{\omega} = \omega_2$, $\chi = \chi_3$, $(r,k) = (3, 7/2)$, $\Theta = \theta$, $\Phi = \Delta\Delta\varphi_4$, the decomposition (B.304) and $g = \Delta\varphi_3$. □

Then, we put (B.311) with $\delta = \varepsilon$ in (B.330) to get

$$\begin{aligned}
&\int_0^T \int_{\omega_2} \chi_3 e^{s\alpha}\psi(\partial_t\Delta\varphi_3) \\
&\leq \varepsilon \left(\int_0^T \int_\Omega e^{2s\alpha}\left\{(s\phi)^4(|\Delta\varphi_1|^2 + |\Delta\varphi_2|^2) + (s\phi)^3|\Delta\Delta\varphi_4|^2\right\}\right) \\
&+ C_\varepsilon \left(\int_0^T \int_{\omega_2} e^{2s\alpha}\left\{(s\phi)^{24}(|\varphi_1|^2 + |\varphi_2|^2 + |\Delta\varphi_3|^2) + (s\phi)^{22}(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2 + |\nabla\Delta\varphi_3|^2)\right\}\right).
\end{aligned} \tag{B.331}$$

Recalling (B.327), (B.329), (B.331), we get (B.169) and consequently Lemma B.4.32.

Annexe C

Local controllability of reaction-diffusion systems around nonnegative stationary states

Abstract : We consider a $n \times n$ nonlinear reaction-diffusion system posed on a smooth bounded domain Ω of $\mathbb{R}^N$. This system models reversible chemical reactions. We act on the system through m controls ($1 \leq m < n$), localized in some arbitrary nonempty open subset ω of the domain Ω. We prove the local exact controllability to nonnegative (constant) stationary states in any time $T > 0$. A specificity of this control system is the existence of some invariant quantities in the nonlinear dynamics that prevents controllability from happening in the whole space $L^\infty(\Omega)^n$. The proof relies on several ingredients. First, an adequate affine change of variables transforms the system into a cascade system with second order coupling terms. Secondly, we establish a new null-controllability result for the linearized system thanks to a spectral inequality for finite sums of eigenfunctions of the Neumann Laplacian operator, due to David Jerison, Gilles Lebeau and Luc Robbiano and precise observability inequalities for a family of finite dimensional systems. Thirdly, the source term method, introduced by Yuning Liu, Takéo Takahashi and Marius Tucsnak, is revisited in a L^∞-context. Finally, an appropriate inverse mapping theorem in suitable spaces enables to go back to the nonlinear reaction-diffusion system.

C.1 Introduction

C.1.1 Free system

Let $n \geq 2$ be an integer. We consider the following reversible chemical reaction :

$$\alpha_1 A_1 + \cdots + \alpha_n A_n \rightleftharpoons \beta_1 A_1 + \cdots + \beta_n A_n, \tag{C.1}$$

where $A_1, \ldots, A_n$ denote n chemical species and $(\alpha_1, \ldots, \alpha_n)$, $(\beta_1, \ldots, \beta_n) \in (\mathbb{N})^n$ are such that for every $1 \leq i \leq n$, $\alpha_i \neq \beta_i$. Chemically, according to the forward reaction $\rightharpoonup$

of (C.1), when α_i molecules of A_i disappear ($1 \leq i \leq n$), they are called the "reactants", then β_i molecules of A_i appear ($1 \leq i \leq n$). The backward reaction $\leftharpoondown$ of (C.1) is governed by the same law : when β_i molecules of A_i disappear ($1 \leq i \leq n$), here they are the reactants, then α_i molecules of A_i appear ($1 \leq i \leq n$).

For $1 \leq i \leq n$, let $u_i(t,.) : \Omega \to \mathbb{R}$ be the concentration of the chemical component A_i at time t. The law of mass action states that the rate of a chemical reaction is directly proportional to the product of the concentrations of the reactants. Using this law together with the Fick's law for the diffusion of the components, we obtain that u_i satisfies the following reaction rate equation (see e.g. [Per15, Section 1.2]) :

$$\begin{aligned}\partial_t u_i - \underbrace{d_i \Delta u_i}_{\text{diffusion}} + \underbrace{\alpha_i \prod_{k=1}^n u_k^{\alpha_k}}_{\text{loss of forward reacting molecules}} + \underbrace{\beta_i \prod_{k=1}^n u_k^{\beta_k}}_{\text{loss of backward reacting molecules}} \\ = \underbrace{\beta_i \prod_{k=1}^n u_k^{\alpha_k}}_{\text{gain of forward reacting molecules}} + \underbrace{\alpha_i \prod_{k=1}^n u_k^{\beta_k}}_{\text{gain of backward reacting molecules}},\end{aligned}$$

that is to say,

$$\partial_t u_i - d_i \Delta u_i = (\beta_i - \alpha_i) \left(\prod_{k=1}^n u_k^{\alpha_k} - \prod_{k=1}^n u_k^{\beta_k} \right), \tag{C.2}$$

where $d_i \in (0, +\infty)$ is the diffusion coefficient of the chemical species A_i.

For a given matrix M, we introduce the notation M^{tr} for the transpose of the matrix M.

From (C.2), by setting

$$U := (u_1, \dots, u_n)^{\mathrm{tr}},$$

we deduce that U satisfies the following reaction-diffusion system :

$$\begin{cases} \partial_t U - D\Delta U = F(U) & \text{in } (0,T) \times \Omega, \\ \frac{\partial U}{\partial \nu} = 0 & \text{on } (0,T) \times \partial\Omega, \\ U(0,.) = u_0 & \text{in } \Omega, \end{cases} \tag{C.3}$$

where

$$D := \mathrm{diag}(d_1, \dots, d_n), \tag{C.4}$$

$$F(U) := (f_i(u_1, \dots, u_n))^{\mathrm{tr}}_{1 \leq i \leq n}, \tag{C.5}$$

with

$$\forall 1 \leq i \leq n,\ f_i(u_1, \dots, u_n) := (\beta_i - \alpha_i) \left(\prod_{k=1}^n u_k^{\alpha_k} - \prod_{k=1}^n u_k^{\beta_k} \right), \tag{C.6}$$

and $T \in (0, +\infty)$, Ω is a bounded, connected, open subset of $\mathbb{R}^N$ (with $N \geq 1$) of class C^2, ν is the outer unit normal vector to $\partial\Omega$.

In general, global existence of classical solutions (in the sense of [Pie10, Definition (1.5)]) or weak solutions (in the sense of [Pie10, Definition (5.12)] replacing $\geq$ by $=$) for (C.3) with F, defined as in (C.5), (C.6), is an open problem.

- For particular semilinearities with a so-called triangular structure (see [Pie10, Section 3.3]), classical solutions exist in the time interval $[0, +\infty)$ and are unique. For example, take $n = 2$, $\alpha_1 \geq 1$, $\beta_2 = 1$, $\alpha_2 = \beta_1 = 0$ and apply [Pie10, Theorem 3.1].
- For at most quadratic nonlinearities, global existence of weak solutions holds (see [Pie10, Theorem 5.12]). For instance, take $n = 4$, $\alpha_1 = \alpha_3 = \beta_2 = \beta_4 = 1$, $\alpha_2 = \alpha_4 = \beta_1 = \beta_3 = 0$. For any spatial dimension $N \geq 1$, the recent works [CGV17] and [Sou18] (inspired by the previous work [Kan90]) prove that the solutions are bounded for bounded initial data, which ensure global existence of classical solutions.
- Without a priori L^1-bound on the nonlinearities, a challenging problem is to understand whether global solutions exist. For example, take $n = 2$, $\alpha_1 = \beta_2 = 2$, $\beta_1 = \alpha_2 = 3$ (see [Pie10, Problem 1]).

Let us also mention that global existence of renormalized solutions holds in all cases for (C.3) (see [Fis15]).

C.1.2 Control system and open question

We assume that one can act on the system through controls localized on a nonempty open subset ω of Ω. From a
chemical viewpoint, it means that one can add or remove chemical species at a specific location of the domain Ω. More precisely, let

$$J \subset \{1, \dots, n\} \text{ and } m := \#J \leq n \text{ be the number of controls.} \tag{C.7}$$

Up to a renumbering $(u_i)_{1 \leq i \leq n}$, we can assume that $J = \{1, \dots, m\}$ where J is defined in (C.7). Hence, we define

$$H^J := (h_1, \dots, h_m, 0, \dots, 0)^{\mathrm{tr}}. \tag{C.8}$$

We consider the control system :

$$\begin{cases} \partial_t U - D\Delta U = F(U) + H^J 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial U}{\partial \nu} = 0 & \text{on } (0,T) \times \partial\Omega, \\ U(0,.) = U_0 & \text{in } \Omega. \end{cases} \tag{NL-U}$$

Here, at $t \in [0, T]$, $U(t,.) : \Omega \to \mathbb{R}^n$ is the *state* to be controlled, $H^J(t,.) : \Omega \to \mathbb{R}^m$ is the *control input* supported in ω.

Let

$$U^* := (u_1^*, \dots, u_n^*)^{\mathrm{tr}}, \tag{C.9}$$

be a nonnegative stationary state of (C.3) i.e.

$$\forall 1 \leq i \leq n,\ u_i^* \in [0,+\infty) \text{ and } \prod_{k=1}^{n} u_k^{*\alpha_k} = \prod_{k=1}^{n} u_k^{*\beta_k}. \tag{C.10}$$

Note that the nonnegative stationary solutions of (C.3) do not depend on the space variable (see Proposition C.8.6 in Annexe C.8.2). Thus, it is not restrictive to assume that $U^* \in [0,+\infty)^n$.

The question we ask is the following one : For a given initial condition U_0, does there exist H^J such that the solution U of (NL-U) satisfies

$$\forall i \in \{1,\dots,n\},\ u_i(T,.) = u_i^*?$$

Under appropriate assumptions (see Hypothesis C.1.4 and Hypothesis C.1.6 below), we prove the controllability of (NL-U), in an appropriate subspace of $L^\infty(\Omega)^n$, locally around U^*, with controls in $L^\infty((0,T)\times\Omega)^m$ (see Theorem C.1.7 below).

By an adequate affine transformation, the proof relies on the study of the null-controllability of an equivalent cascade system with second order coupling terms (see Annexe C.2.1 below).

We have chosen to postpone the simple or classical proofs in Annexe C.8. Therefore, the main contributions are highlighted in the body of the article.

C.1.3 Nonlinear well-posedness result

For $\tau > 0$, we introduce

$$Q_\tau := (0,\tau)\times\Omega.$$

We define the function space

$$W_T := L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))'),$$

that satisfies the continuous embedding

$$W_T \hookrightarrow C([0,T];L^2(\Omega)). \tag{C.11}$$

We introduce the notion of solution associated to the nonlinear system (NL-U) (see Annexe C.1.2).

Definition C.1.1. Let D be defined in (C.4). For every $U_0 \in L^\infty(\Omega)^n$, $H^J \in L^\infty(Q_T)^m$, we say that $U \in \Big(W_T \cap L^\infty(Q_T)\Big)^n$ is a solution of (NL-U) if for every $V \in L^2(0,T;H^1(\Omega)^n)$,

$$\int_0^T (\partial_t U, V)_{(H^1(\Omega)^n)',H^1(\Omega)^n} + \int_{Q_T} D\nabla U.\nabla V = \int_{Q_T} \left(F(U) + H^J 1_\omega\right).V, \tag{C.12}$$

with F defined in (C.5) and

$$U(0,.) = U_0 \text{ in } L^\infty(\Omega)^n. \tag{C.13}$$

Remark C.1.2. Given $U_0 \in L^\infty(\Omega)^n$, $H^J \in L^\infty(Q_T)^m$, if a solution U of (NL-U) exists in the sense of Definition C.1.1, then it is unique because F is locally Lipschitz on $\mathbb{R}^n$ (see the proof of [LB19, Definition-Proposition 2.4]).

C.1.4 Invariant quantities of the nonlinear dynamics

In this section, we show that in the system (NL-U) (see Annexe C.1.2), when the number of controls is small, some quantities are invariant. They impose some restrictions on the initial condition, for the controllability results.

Proposition C.1.3. We assume that $m \leq n-2$. Let $U_0 \in L^\infty(\Omega)^n$, $H^J \in L^\infty(Q_T)^m$. Assume that U is a solution of (NL-U) such that $U(T,.) = U^*$ with U^* defined in (C.9). Then, we have for every $k \neq l \in \{m+1, \ldots, n\}$, $t \in [0,T]$,

$$\int_\Omega \frac{u_k(t,x) - u_k^*}{\beta_k - \alpha_k} dx = \int_\Omega \frac{u_l(t,x) - u_l^*}{\beta_l - \alpha_l} dx, \tag{C.14}$$

$$\Big(d_k = d_l\Big) \Rightarrow \left(\frac{u_k(t,.) - u_k^*}{\beta_k - \alpha_k} = \frac{u_l(t,.) - u_l^*}{\beta_l - \alpha_l}\right). \tag{C.15}$$

In particular, for every $k \neq l \in \{m+1, \ldots, n\}$,

$$\int_\Omega \frac{u_{k,0}(x) - u_k^*}{\beta_k - \alpha_k} dx = \int_\Omega \frac{u_{l,0}(x) - u_l^*}{\beta_l - \alpha_l} dx, \tag{C.16}$$

$$\Big(d_k = d_l\Big) \Rightarrow \left(\frac{u_{k,0} - u_k^*}{\beta_k - \alpha_k} = \frac{u_{l,0} - u_l^*}{\beta_l - \alpha_l}\right). \tag{C.17}$$

The proof of Proposition C.1.3 is done in Annexe C.8.3. We prove (C.14) by integrating with respect to the space variable an appropriate linear combination of equations of (NL-U) and by using the Neumann boundary conditions. We prove (C.15) by the backward uniqueness of the heat equation applied to an appropriate linear combination of equations of (NL-U).

The equation (C.15) implies that we can reduce the number of components of $(u_i)_{1 \leq i \leq n}$ of (NL-U) when some diffusion coefficients d_i are equal for $m+1 \leq i \leq n$. Thus, (NL-U) becomes more simple under this last assumption. That is why, we make the following hypothesis in order to treat the most difficult case.

Hypothesis C.1.4. For $m \leq n-2$, we suppose that for every $k \neq l \in \{m+1, \ldots, n\}$, $d_k \neq d_l$.

Remark C.1.5. It will be interesting to note that the mass condition (C.16) is obviously equivalent to

$$\forall k \geq m+2, \int_\Omega \frac{u_{k,0}(x) - u_k^*}{\beta_k - \alpha_k} dx = \int_\Omega \frac{u_{m+1,0}(x) - u_{m+1}^*}{\beta_{m+1} - \alpha_{m+1}} dx. \tag{C.18}$$

C.1.5 Main result

We will work under the following assumption that will ensure the controllability of the linearized system of (NL-U) (see Annexe C.2.2 below).

Hypothesis C.1.6. For $m \leq n-1$, we assume that

$$\partial_m f_{m+1}\Big(u_1^*, \ldots, u_n^*\Big) \neq 0, \tag{C.19}$$

where f_{m+1} is defined in (C.6).

Theorem C.1.7. Under Hypothesis C.1.4 and Hypothesis C.1.6, the system (NL-U) is locally controllable around U^*, i.e., there exists $r > 0$ such that for every $U_0 \in L^\infty(\Omega)^n$ satisfying the mass condition (C.16) and $\|U_0 - U^*\|_{L^\infty(\Omega)} \leq r$, there exists $H^J \in L^\infty(Q_T)^m$ such that the solution U of (NL-U) satisfies $U(T,.) = U^*$.

Remark C.1.8. The uniqueness of the solution $U \in L^\infty(Q_T)^n$ is a consequence of Remark C.1.2. The existence of the solution $U \in L^\infty(Q_T)^n$ is a consequence of a good choice of the control $H^J \in L^\infty(Q_T)^m$ and more precisely of an inverse mapping argument (see Annexe C.6).

Remark C.1.9. Up to renumbering the first m equations of (NL-U), we can see that Theorem C.1.7 is still valid by replacing the assumption (C.19) by

$$\exists j \in \{1, \ldots, m\},\ \partial_j f_{m+1}\Big(u_1^*, \ldots, u_n^*\Big) \neq 0. \tag{C.20}$$

Remark C.1.10. When $\alpha_m, \beta_m \geq 1$, a sufficient condition to ensure (C.19) is

$$\forall 1 \leq k \leq n,\ u_k^* \neq 0. \tag{C.21}$$

Indeed, by using (C.6), (C.10) and $\alpha_j \neq \beta_j$, if (C.21) holds true then

$$\partial_m f_{m+1}\Big(u_1^*, \ldots, u_n^*\Big) = \frac{\alpha_m - \beta_m}{u_m^*} \prod_{k=1}^{n} {u_k^*}^{\alpha_k} \neq 0.$$

Note that (C.21) is not equivalent to (C.19) as shown by the examples in Application C.1.11 (see below).

Application C.1.11. For $n = 4$, $\alpha_1 = \alpha_3 = \beta_2 = \beta_4 = 1$ and $\alpha_2 = \alpha_4 = \beta_1 = \beta_3 = 0$, we have

$$f_i(u_1, u_2, u_3, u_4) = (-1)^i (u_1 u_3 - u_2 u_4).$$

In this case, we check that (C.20) is

$$\text{for } J = \{1,2,3\}, \quad \Big(\exists j \in \{1,2,3\},\ \partial_j f_4(u_1^*, \ldots, u_4^*) \neq 0\Big) \Leftrightarrow \Big((u_1^*, u_3^*, u_4^*) \neq (0,0,0)\Big),$$

$$\text{for } J = \{1,2\}, \quad \Big(\exists j \in \{1,2\},\ \partial_j f_3(u_1^*, \ldots, u_4^*) \neq 0\Big) \Leftrightarrow \Big((u_3^*, u_4^*) \neq (0,0)\Big),$$

$$\text{for } J = \{1\}, \quad \Big(\partial_1 f_2(u_1^*, \ldots, u_4^*) \neq 0\Big) \Leftrightarrow \Big(u_3^* \neq 0\Big).$$

Thus, Theorem C.1.7 recovers the result of [LB19, Theorem 3.2] except for the case $J = \{1,2,3\}$ and $(u_1^*, u_3^*, u_4^*) = (0,0,0)$ that the proof of the present article does not treat (see Example C.7.1 for more details about the strategy of [LB19]).

We will only prove Theorem C.1.7 under the assumption $m \leq n-2$. The other cases are an easy adaptation.

C.1.6 Bibliographical comments

In this section, we recall some known results about the null-controllability of linear and semilinear parabolic systems with Neumann boundary conditions to put in perspective the statement and the proof strategy of Theorem C.1.7.

C.1.6.1 Linear results

Let $k, l \in \mathbb{N}^*$. We denote by $\mathcal{M}_k(\mathbb{R})$ (respectively $\mathcal{M}_{k,l}(\mathbb{R})$) the algebra of matrices with k lines and k columns (respectively the algebra of matrices with k lines and l columns) with entries in $\mathbb{R}$. For $M \in \mathcal{M}_k(\mathbb{R})$, $\mathrm{Sp}(M)$ is the set of complex eigenvalues of M : $\mathrm{Sp}(M) := \{\lambda \in \mathbb{C} \ ; \ \exists X \in \mathbb{C}^k \setminus \{0\}, \ MX = \lambda X\}$.

Since the pioneer works of Gilles Lebeau, Luc Robbiano in 1995 (see [LR95], [JL99] and the survey [LRL12]) and Andrei Fursikov, Oleg Imanuvilov in 1996 (see [FI96] and [FCG06]) about the null-controllability of the heat equation, the control of coupled parabolic systems has been a challenging issue in the last twenty years. For instance, in [AKBDGB09a], the authors identify necessary and sufficient conditions for the null-controllability of linear parabolic systems of the following form

$$\begin{cases} \partial_t Z - \Gamma \Delta Z = AZ + BH1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0,T) \times \partial\Omega, \\ Z(0,.) = Z_0 & \text{in } \Omega, \end{cases} \tag{C.22}$$

where Γ a diagonalizable matrix of $\mathcal{M}_k(\mathbb{R})$ with $\mathrm{Sp}(\Gamma) \subset (0, +\infty)$, $A \in \mathcal{M}_k(\mathbb{R})$, $B \in \mathcal{M}_{k,l}(\mathbb{R})$. In general, the rank of B is less that k, so that the controllability of the full system (C.22) depends strongly on the coupling present in the system.

Inspired by the works [GBdT10], [Gue07], [LZ17], a byproduct of this article is a new null-controllability result, for cascade cross-diffusion systems of arbitrary size (see Annexe C.3, Theorem C.3.1).

For a recent survey on the null-controllability of linear parabolic systems, see [AKBGBdT11] and references therein.

C.1.6.2 Semilinear results

For semilinear parabolic systems

$$\begin{cases} \partial_t Z - \Gamma \Delta Z = G(Z) + BH1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0,T) \times \partial\Omega, \\ Z(0,.) = Z_0 & \text{in } \Omega, \end{cases} \tag{C.23}$$

with $G \in C^\infty(\mathbb{R}^k; \mathbb{R}^k)$ such that $G(0) = 0$, the usual strategy consists in deducing a *local null-controllability result* for (C.23) from a (global) null-controllability result for the *linearized system* around $(Z, H) = (0, 0)$. This linear system takes the form (C.22) with the matrix A defined as follows : $a_{i,j} = \partial_j g_i(0)$ $(1 \leq i, j \leq k)$. In this paper, we use the powerful *source term method*, introduced by Yuning Liu, Takéo Takahashi and Marius

Tucsnak in [LTT13]. One of the main advantage of the method is to deduce the local null-controllability for (C.23) from the null-controllability of **only one** linear system (C.22).

In this article, we adapt the source term method in a L^∞-context in the following way.

— The source term method in L^2 enables to prove a strong observability inequality (see Corollary C.4.4). This estimate looks like a global Carleman estimate (see for example [FCG06, Lemma 1.3]), whereas the method to get it is very different.

— By using the Penalized Hilbert Uniqueness Method, introduced by Viorel Barbu in [Bar00], we construct L^∞-controls (see Theorem C.5.1).

— We use once more the source term method in L^∞ (see Proposition C.5.3).

— We conclude by an appropriate inverse mapping theorem (see Annexe C.6).

For other results using the source term method, see for instance [BM17], [FCLdM16] and [MT18].

C.2 An adequate change of variables and linearization

C.2.1 Change of variables - Cross diffusion system

The goal of this section is to transform the controlled system (NL-U) (see Annexe C.1.2) satisfied by U into another system of *cascade* type for which we better understand the controllability properties. Roughly speaking, for $1 \leq i \leq m$, the component u_i is easy to control thanks to the localized control term $h_i 1_\omega$. Thus, the challenge is to understand how the reaction term $f_i(U)$ (see (C.6)) acts on the component u_i for $m+1 \leq i \leq n$.

We multiply the $(m+1)$-th equation (respectively the $(m+2)$-th equation) of (NL-U) by

$$((\beta_{m+1} - \alpha_{m+1})(d_{m+1} - d_{m+2}))^{-1} \text{ (respectively } ((\beta_{m+2} - \alpha_{m+2})(d_{m+2} - d_{m+1}))^{-1}),$$

and we sum :

$$\partial_t v_{m+2} - d_{m+2}\Delta v_{m+2} = \frac{\Delta u_{m+1}}{\beta_{m+1} - \alpha_{m+1}},$$

where

$$v_{m+2} = \frac{u_{m+1}}{(\beta_{m+1} - \alpha_{m+1})(d_{m+1} - d_{m+2})} + \frac{u_{m+2}}{(\beta_{m+2} - \alpha_{m+2})(d_{m+2} - d_{m+1})}.$$

Roughly speaking, this linear combination enables to “kill” the reaction-term and to create a coupling term of second order.

By iterating this strategy, we construct a linear transformation $V = PU$ such that u_{m+1} acts on v_{m+2}, v_{m+2} acts on v_{m+3}, ..., v_{n-1} acts on v_n through cross diffusion terms. Moreover, we transform the problem of controllability for U to U^* into a null-controllability problem for

$$Z := P(U - U^*),$$

where P is the invertible triangular matrix defined by :

$$P := \left(\begin{array}{c|c} I_m & (0) \\ \hline (0) & * \end{array}\right), \tag{C.24}$$

with

$$\forall k, l \geq m+1,\ P_{kl} := \begin{cases} \left((\beta_l - \alpha_l) \prod_{\substack{m+1 \leq r \leq k \\ r \neq l}} (d_l - d_r) \right)^{-1} & \text{if } k \geq l, \\ 0 & \text{if } k < l, \end{cases} \tag{C.25}$$

with the convention $\prod_{\emptyset} = 1$.

We introduce the notations :

$$G(Z) := (g_1(Z), \dots, g_{m+1}(Z), 0 \dots, 0)^{\mathrm{tr}}, \tag{C.26}$$

with

$$g_i(Z) := f_i(P^{-1}Z + U^*)\ (1 \leq i \leq m),\ g_{m+1}(Z) := \frac{f_i(P^{-1}Z + U^*)}{\beta_{m+1} - \alpha_{m+1}}, \tag{C.27}$$

and

$$D_J := \left(\begin{array}{c|c} diag(d_1, \dots, d_m) & (0) \\ \hline (0) & D_\sharp \end{array}\right),\ D_\sharp := \begin{pmatrix} d_{m+1} & 0 & \dots & \dots & 0 \\ 1 & d_{m+2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & d_n \end{pmatrix}. \tag{C.28}$$

Proposition C.2.1. Let $U_0 \in L^\infty(\Omega)^n$, $H^J \in L^\infty(Q_T)^m$. Then, U is a solution of (NL-U) if and only if Z satisfies

$$\begin{cases} \partial_t Z - D_J \Delta Z = G(Z) + H^J 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0,T) \times \partial\Omega, \\ Z(0,.) = Z_0 & \text{in } \Omega. \end{cases} \tag{NL-Z}$$

The proof of Proposition C.2.1 is done in Annexe C.8.4.1.

Remark C.2.2. The concept of solution for (NL-Z) is an easy adaptation of the notion of solution for (NL-U) given in Definition C.1.1.

Let $p \in [1, +\infty]$. We introduce the following subspace of $L^p(\Omega)^n$:

$$L^p_{inv} := \left\{ Z_0 \in L^p(\Omega)^n \ ;\ \forall m+2 \leq i \leq n,\ \int_\Omega z_{i,0}(x)dx = 0 \right\}. \tag{C.29}$$

Theorem C.1.7 is equivalent to the following local null-controllability theorem for (NL-Z).

Theorem C.2.3. Under Hypothesis C.1.4 and Hypothesis C.1.6, the system (NL-Z) is locally null-controllable, i.e., there exists $r > 0$ such that for every $Z_0 \in L^{\infty}_{inv}$ verifying $\|Z_0\|_{L^{\infty}(\Omega)^n} \leq r$, there exists $H^J \in L^{\infty}(Q_T)^m$ such that the solution Z of (NL-Z) satisfies $Z(T,.) = 0$.

The equivalence between Theorem C.1.7 and Theorem C.2.3 comes from Proposition C.2.1 and the following equivalence

$$Z_0 \in L^{\infty}_{inv} \Leftrightarrow U_0 \text{ satisfies (C.16)} \Leftrightarrow U_0 \text{ satisfies (C.18) (Remark C.1.5).} \tag{C.30}$$

The proof of (C.30) is done in Annexe C.8.4.2.

From now, we will focus on the proof of Theorem C.2.3.

C.2.2 Linearization

The linearized system of (NL-Z) around $(0,0)$ is

$$\begin{cases} \partial_t Z - D_J \Delta Z = A_J Z + H^J 1_{\omega} & \text{in } (0,T) \times \Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0,T) \times \partial\Omega, \\ Z(0,.) = Z_0 & \text{in } \Omega, \end{cases} \tag{L-Z}$$

where

$$A_J = (a_{ik})_{1 \leq i,k \leq n}, \qquad a_{ik} = \begin{cases} \partial_k g_i(0,\dots,0) & \text{if } 1 \leq i \leq m+1, \\ 0 & \text{if } m+2 \leq i \leq n. \end{cases} \tag{C.31}$$

By Hypothesis C.1.6, (C.24) and (C.27), we have

$$a_{m+1,m} \neq 0. \tag{C.32}$$

Roughly speaking, we summarize the expected controllability properties in the following diagram :

$$h_1 \xrightarrow{controls} z_1,\ h_2 \xrightarrow{controls} z_2,\ \dots\ ,\ h_{m-1} \xrightarrow{controls} z_{m-1},$$
$$h_m \xrightarrow{controls} z_m \xrightarrow[a_{m+1m}z_m]{controls} z_{m+1} \xrightarrow[\Delta z_{m+1}]{controls} z_{m+2} \xrightarrow[\Delta z_{m+2}]{controls} \dots \xrightarrow[\Delta z_{n-1}]{controls} z_n.$$

C.3 Linear null-controllability under constraints in L^2

The main result of this section, stated in the following theorem, is the null-controllability in L^2_{inv} for the linear system (L-Z) (see Annexe C.2.2).

Theorem C.3.1. The system (L-Z) is null-controllable in L^2_{inv}. More precisely, there exists $C > 0$ such that for every $T > 0$ and $Z_0 \in L^2_{inv}$, there exists a control $H^J \in L^2(Q_T)^m$ verifying

$$\left\|H^J\right\|_{L^2(Q_T)^m} \leq C_T \left\|Z_0\right\|_{L^2(\Omega)^n}, \text{ where } C_T = Ce^{C/T}, \tag{C.33}$$

and such that the solution $Z \in W^n_T$ of (L-Z) satisfies $Z(T,.) = 0$.

The goal of the next two subsections is to prove Theorem C.3.1. The proof is based on the Lebeau-Robbiano's method, introduced for the first time to prove the null-controllability of the heat equation (see [LR95]). First, it consists in establishing a null-controllability result in finite dimensional subspaces of L^2_{inv} with a precise estimate of the cost of the control (see Proposition C.3.2). This first step is based on two main results : the spectral inequality for eigenfunctions of the Neumann-Laplace operator (see Lemma C.3.4) and precise observability estimates of linear finite dimensional systems associated to the adjoint system of (L-Z) (see Lemma C.3.5). Secondly, we conclude by a time-splitting procedure : the control H^J is built as a sequence of active controls and passive controls. The passive mode allows to take advantage of the natural parabolic exponential decay of the L^2 norm of the solution. This decay enables to compensate the cost of the control which steers the low frequencies to 0 (see Annexe C.3.2).

We must be careful with the dependence on the constants appearing in the estimates with respect to T (when T is small). That is why, from now and until the end of the article, we assume that

$$T \in (0,1). \tag{C.34}$$

Unless otherwise specified, we denote by C various positive constants varying from line to line.

C.3.1 A null-controllability result for the low frequencies

The unbounded operator on $L^2(\Omega)$: $(-\Delta, H^2_{Ne}(\Omega))$, where $H^2_{Ne}(\Omega)$ is defined in (C.98) (see Annexe C.8.1.2 below) is self-adjoint and has compact resolvent. Thus, we introduce the orthonormal basis $(e_k)_{k\geq 0}$ of $L^2(\Omega)$ of eigenfunctions associated to the increasing sequence of eigenvalues $(\lambda_k)_{k\geq 0}$ of the Laplacian operator, i.e., we have $-\Delta e_k = \lambda_k e_k$ and $(e_k, e_l)_{L^2(\Omega)} = \delta_{k,l}$. For $\lambda > 0$, we define the finite dimensional space $E_\lambda = \left\{ \sum\limits_{\lambda_k \leq \lambda} c_k e_k \ ; \ c_k \in \mathbb{R}^n \right\} \subset L^2(\Omega)^n$ and the orthogonal projection Π_{E_λ} onto E_λ in $L^2(\Omega)^n$.

The goal of this section is to prove the following null-controllability result in a finite dimensional subspace of L^2_{inv}.

Proposition C.3.2. There exist $C > 0$, $p_1 \in \mathbb{N}$ such that for every $\tau \in (0,T)$, $\lambda > 0$, $Z_0 \in E_\lambda \cap L^2_{inv}$, there exists a control function $H^J \in L^2(Q_\tau)$ verifying

$$\left\|H^J\right\|^2_{L^2(Q_\tau)^m} \leq \frac{C}{\tau^{p_1}} e^{C\sqrt{\lambda}} \left\|Z_0\right\|^2_{L^2(\Omega)^n}, \tag{C.35}$$

such that the solution Z of

$$\begin{cases} \partial_t Z - D_J \Delta Z = A_J Z + H^J 1_\omega & \text{in } (0,\tau)\times\Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0,\tau)\times\partial\Omega, \\ Z(0,.) = Z_0 \in E_\lambda & \text{in } \Omega, \end{cases} \tag{C.36}$$

satisfies $Z(\tau,.) = 0$.

From Proposition C.3.2, for every $\tau, \lambda > 0$ and $Z_0 \in E_\lambda \cap L^2_{inv}$, we introduce the notation :

$$H_\lambda(Z_0, 0, \tau) := H^J, \tag{C.37}$$

such that the solution Z of (C.36) satisfies $Z(\tau, .) = 0$ and H^J is the minimal-norm element of $L^2(Q_\tau)^m$ satisfying the estimate (C.35). In other words, H^J is the projection of 0 in the nonempty closed convex set of controls satisfying (C.35) and driving the solution Z of (C.36) in time τ to 0.

By the Hilbert Uniqueness Method (see [Cor07a, Theorem 2.44]), in order to prove Proposition C.3.2, we need to prove an observability inequality for the solution of the adjoint system of (C.36).

Proposition C.3.3. There exist $C > 0$, $p_1 \in \mathbb{N}$ such that for every $\tau \in (0, T)$, $\lambda > 0$ and $\varphi_\tau \in E_\lambda \cap L^2_{inv}$, the solution φ of

$$\begin{cases} -\partial_t \varphi - D_J^{\mathrm{tr}} \Delta \varphi = A_J^{\mathrm{tr}} \varphi & \text{in } (0, \tau) \times \Omega, \\ \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } (0, \tau) \times \partial\Omega, \\ \varphi(\tau, .) = \varphi_\tau & \text{in } \Omega, \end{cases} \tag{C.38}$$

satisfies

$$\|\varphi(0, .)\|^2_{L^2(\Omega)^n} \leq \frac{C}{\tau^{p_1}} e^{C\sqrt{\lambda}} \sum_{i=1}^{m} \int_0^\tau \int_\omega |\varphi_i(t, x)|^2 \, dx dt. \tag{C.39}$$

Proof. The proof is inspired by [LZ17, Section 3].

Let $\tau, \lambda > 0$ and $\varphi_\tau \in E_\lambda \cap L^2_{inv}$. We have :

$$\varphi_\tau(x) = \sum_{\lambda_k \leq \lambda} \varphi_k^\tau e_k(x),$$

with $\varphi_k^\tau \in F_k$ where $F_0 := \mathbb{R}^{m+1} \times \{0\}^{n-m-1}$ because $\varphi_\tau \in L^2_{inv}$ and $F_k := \mathbb{R}^n$ for $k \geq 1$.

Then, the solution φ of (C.38) is

$$\forall (t, x) \in (0, \tau) \times \Omega, \ \varphi(t, x) = \sum_{\lambda_k \leq \lambda} \varphi_k(t) e_k(x), \tag{C.40}$$

where φ_k is the unique solution of the ordinary differential system

$$\begin{cases} -\varphi_k' + \lambda_k D_J^{\mathrm{tr}} \varphi_k = A_J^{\mathrm{tr}} \varphi_k, & \text{in } (0, \tau), \\ \varphi_k(\tau) = \varphi_k^\tau. \end{cases} \tag{C.41}$$

We recall the spectral inequality for eigenfunctions of the Neumann-Laplace operator.

Lemma C.3.4. [JL99, Theorem 14.6]
There exists $C > 0$ such that for every sequence $(a_k)_{k \geq 0} \subset \mathbb{C}^{\mathbb{N}}$ and for every $\lambda > 0$, we have :

$$\sum_{\lambda_k \leq \lambda} |a_k|^2 = \int_\Omega \left| \sum_{\lambda_k \leq \lambda} a_k e_k(x) \right|^2 dx \leq C e^{C\sqrt{\lambda}} \int_\omega \left| \sum_{\lambda_k \leq \lambda} a_k e_k(x) \right|^2 dx. \tag{C.42}$$

By using (C.42) for $a_k = \varphi_{k,i}(t)$ with $1 \leq i \leq m$ and by summing on $1 \leq i \leq m$, we obtain that there exists $C > 0$ such that

$$\sum_{\lambda_k \leq \lambda} \sum_{i=1}^{m} |\varphi_{k,i}(t)|^2 \leq Ce^{C\sqrt{\lambda}} \sum_{i=1}^{m} \int_{\omega} \left| \sum_{\lambda_k \leq \lambda} \varphi_{k,i}(t) e_k(x) \right|^2 dx. \tag{C.43}$$

By integrating with respect to the time variable between 0 and τ the inequality (C.43), we obtain

$$\int_0^{\tau} \sum_{\lambda_k \leq \lambda} \sum_{i=1}^{m} |\varphi_{k,i}(t)|^2 dt \leq Ce^{C\sqrt{\lambda}} \sum_{i=1}^{m} \int_0^{\tau} \int_{\omega} \left| \sum_{\lambda_k \leq \lambda} \varphi_{k,i}(t) e_k(x) \right|^2 dxdt. \tag{C.44}$$

Moreover, we have the following lemma whose proof is postponed in Annexe C.8.5 (see also [Sei88]).

Lemma C.3.5. There exist $C > 0$, $(p_1, p_2) \in \mathbb{N}^2$ such that for every $\tau \in (0, 1)$, $k \in \mathbb{N}$, $\varphi_k^{\tau} \in F_k$, the solution φ_k of (C.41) satisfies

$$\|\varphi_k(0)\|^2 \leq C \left(1 + \frac{1}{\tau^{p_1}} + \lambda_k^{p_2}\right) \sum_{i=1}^{m} \int_0^{\tau} |\varphi_{k,i}(t)|^2 dt. \tag{C.45}$$

By using (C.44), (C.45), we deduce that

$$\begin{aligned} \sum_{\lambda_k \leq \lambda} \|\varphi_k(0)\|^2 &\leq \sum_{\lambda_k \leq \lambda} \frac{C}{\tau^{p_1}} (1 + \lambda_k^{p_2}) \sum_{i=1}^{m} \int_0^{\tau} |\varphi_{k,i}(t)|^2 dt \\ &\leq \frac{C}{\tau^{p_1}} e^{C\sqrt{\lambda}} \sum_{i=1}^{m} \int_0^{\tau} \int_{\omega} \left| \sum_{\lambda_k \leq \lambda} \varphi_{k,i}(t) e_k(x) \right|^2 dxdt. \end{aligned} \tag{C.46}$$

By using (C.40), we deduce (C.39) from (C.46). □

C.3.2 The Lebeau-Robbiano's method

The goal of this section is to prove Theorem C.3.1.

Proof. The proof is inspired by [LRL12, Section 6.2] (see also [LR95, Fin de la preuve du Théorème 1]). The constants C, C' will increase from line to line.

We split the interval $[0, T] = \cup_{k \in \mathbb{N}} [a_k, a_{k+1}]$ with $a_0 = 0$, $a_{k+1} = a_k + 2T_k$ and $T_k = \kappa T / 2^k$ for $k \in \mathbb{N}$ and the constant κ is chosen such that $2 \sum_{k=0}^{+\infty} T_k = T$. We also define $\mu_k = M 2^{2k}$ for $M > 0$ sufficiently large which will be defined later and for $k \in \mathbb{N}$. Then, we define the control H^J in the following way :

- if $t \in (a_k, a_k + T_k)$, $H^J = H_{\mu_k}(\Pi_{E_{\mu_k}} Z(a_k, .), a_k, T_k)$ (see the notation (C.37)) and $Z(t, .) = S(t - a_k) Z(a_k, .) + \int_{a_k}^{t} S(t - s) H^J(s, .) ds$,

— if $t \in (a_k + T_k, a_{k+1})$, $H^J = 0$ and $Z(t,.) = S(t - a_k - T_k)Z(a_k + T_k, .)$,

where $S(t)$ denotes the semigroup of the parabolic system : $S(t) = e^{t(D_J\Delta + A_J)}$. In particular, by (C.96) and (C.11), $\|S(t)\|_{\mathcal{L}(L^2(\Omega)^n)} \leq C$.

By (C.35), the choice of H^J during the interval time $[a_k, a_k + T_k]$ implies

$$\begin{aligned} \|Z(a_k + T_k, .)\|^2_{L^2(\Omega)^n} &\leq (C + C(\kappa 2^{-k}T)^{-p_1} e^{C\sqrt{M}2^k}) \|Z(a_k, .)\|^2_{L^2(\Omega)^n} \\ &\leq \frac{C}{T^{p_1}} e^{C\sqrt{M}2^k} \|Z(a_k, .)\|^2_{L^2(\Omega)^n} . \end{aligned} \tag{C.47}$$

During the passive period of the control, $t \in [a_k + T_k, a_{k+1}]$, the solution exponentially decreases :

$$\|Z(a_{k+1}, .)\|^2_{L^2(\Omega)^n} \leq C' e^{-C'M2^{2k}T_k} \|Z(a_k + T_k, .)\|^2_{L^2(\Omega)^n} . \tag{C.48}$$

Thus, by using $2^{2k}T_k = \kappa 2^k T$, (C.47) and (C.48), we have

$$\|Z(a_{k+1}, .)\|^2_{L^2(\Omega)^n} \leq \frac{C}{T^{p_1}} e^{C\sqrt{M}2^k - C'M2^kT} \|Z(a_k, .)\|^2_{L^2(\Omega)^n} ,$$

and consequently,

$$\begin{aligned} \|Z(a_{k+1}, .)\|^2_{L^2(\Omega)^n} &\leq \left(\frac{C}{T^{p_1}}\right)^{k+1} e^{\sum_{j=0}^k (C\sqrt{M}2^j - C'MT2^j)} \|Z_0\|^2_{L^2(\Omega)^n} \\ &\leq e^{C/T + (C\sqrt{M} - C'MT)2^{k+1}} \|Z_0\|^2_{L^2(\Omega)^n} . \end{aligned} \tag{C.49}$$

By taking M such that $C\sqrt{M} - C'MT < 0$, for instance $M \geq 2(C/C'T)^2$, we conclude by (C.49) that we have $\lim_{k\to+\infty} \|Z(a_k, .)\| = 0$, i.e., $Z(T, .) = 0$ because $t \mapsto Z(t, .) \in C([0,T]; L^2(\Omega)^n)$ because $H^J \in L^2(Q_T)^m$ (see Proposition C.8.2 and (C.11)) as we will show now.

We have $\|H^J\|^2_{L^2(Q_T)^m} = \sum_{k=0}^{+\infty} \|H^J\|^2_{L^2((a_k, a_k+T_k)\times\Omega)^m}$. Then, by using the estimate (C.35) of the control on each time interval $(a_k, a_k + T_k)$ and the estimate (C.49), we get :

$$\begin{aligned} &\|H^J\|^2_{L^2(Q_T)^m} \\ &\leq \left(CT_0^{-p_1} e^{C\sqrt{M}} + \sum_{k\geq 1} CT_k^{-p_1} e^{C\sqrt{M}2^k} e^{C/T + (C\sqrt{M} - C'MT)2^k}\right) \|Z_0\|^2_{L^2(\Omega)^n} \\ &\leq \left(CT^{-p_1} e^{C\sqrt{M}} + \sum_{k\geq 1} C(2^kT^{-1})^{p_1} e^{C/T} e^{(2C\sqrt{M} - C'MT)2^k}\right) \|Z_0\|^2_{L^2(\Omega)^n} . \end{aligned} \tag{C.50}$$

By taking M such that $2C\sqrt{M} - C'MT < 0$, for instance $M = 8(C/C'T)^2 \Rightarrow C\sqrt{M} - C'MT/2 = -C''/T$ with $C'' > 0$, we deduce from (C.50) that $H^J \in L^2(Q_T)^m$ and

$$\|H^J\|^2_{L^2(Q_T)^m} \leq Ce^{C/T} \int_0^{+\infty} \left(\frac{\sigma}{T}\right)^{p_1} e^{-C''\frac{\sigma}{T}} d\sigma \, \|Z_0\|^2_{L^2(\Omega)^n} \leq Ce^{C/T} \|Z_0\|^2_{L^2(\Omega)^n} ,$$

which concludes the proof of Theorem C.3.1. $\square$

C.4 The source term method in L^2

We use the source term method, introduced by Yuning Liu, Takéo Takahashi and Marius Tucsnak in [LTT13, Proposition 2.3] to deduce a local null-controllability result for a nonlinear system from the null-controllability result for only one linear system (and an estimate of the cost of the control) (see also [BM17]).

By Theorem C.3.1, we have an estimate for the control cost in L^2, then we fix $M > 0$ such that $C_T \leq Me^{M/T}$. Let $q \in (1, \sqrt{2})$ and $p > q^2/(2-q^2)$. We define the weights

$$\rho_0(t) := M^{-p} \exp\left(-\frac{Mp}{(q-1)(T-t)}\right), \tag{C.51}$$

$$\rho_{\mathcal{S}}(t) = M^{-1-p} \exp\left(-\frac{(1+p)q^2 M}{(q-1)(T-t)}\right). \tag{C.52}$$

Remark C.4.1. The assumption $p > q^2/(2-q^2) \Leftrightarrow 2p > (1+p)q^2$ implies

$$\rho_0^2/\rho_{\mathcal{S}} \in C([0,T]), \tag{C.53}$$

which will be useful for the estimate of the polynomial nonlinearity (see Annexe C.6).

Let $r \in \{2, +\infty\}$. For $S \in L^r((0,T); L^r_{inv})$, $H^J \in L^r((0,T); L^r(\Omega)^m)$, $Z_0 \in L^r_{inv}$, we introduce the following system :

$$\begin{cases} \partial_t Z - D_J \Delta Z = A_J Z + S + H^J 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0,T)\times\partial\Omega, \\ Z(0,.) = Z_0 & \text{in } \Omega. \end{cases} \tag{L+S-Z}$$

Then, we define associated spaces for the source term, the state and the control

$$\mathcal{S}_r := \left\{ S \in L^r((0,T); L^r_{inv}) \ ; \ \frac{S}{\rho_{\mathcal{S}}} \in L^r((0,T); L^r_{inv}) \right\}, \tag{C.54}$$

$$\mathcal{Z}_r := \left\{ Z \in L^r((0,T); L^r_{inv}) \ ; \ \frac{Z}{\rho_0} \in L^r((0,T); L^r_{inv}) \right\}, \tag{C.55}$$

$$\mathcal{H}_r := \left\{ H^J \in L^r((0,T); L^r(\Omega)^m) \ ; \ \frac{H^J}{\rho_0} \in L^r((0,T); L^r(\Omega)^m) \right\}. \tag{C.56}$$

Remark C.4.2. From the behaviors near $t = T$ of $\rho_{\mathcal{S}}$ and ρ_0, we deduce that each element of $\mathcal{S}_r$, $\mathcal{Z}_r$, $\mathcal{H}_r$ vanishes at $t = T$.

From the abstract result : [LTT13, Proposition 2.3], we deduce the null-controllability for (L+S-Z) in L^2_{inv}.

Proposition C.4.3. For every $S \in \mathcal{S}_2$ and $Z_0 \in L^2_{inv}$, there exists $H^J \in \mathcal{H}_2$, such that the solution Z of (L+S-Z) satisfies $Z \in \mathcal{Z}_2$. Furthermore, there exists $C > 0$, not depending on S and Z_0, such that

$$\|Z/\rho_0\|_{C([0,T];L^2(\Omega)^n)} + \left\|H^J\right\|_{\mathcal{H}_2} \leq C_T \left(\|Z_0\|_{L^2(\Omega)^n} + \|S\|_{\mathcal{S}_2}\right), \tag{C.57}$$

where $C_T = Ce^{C/T}$. In particular, since ρ_0 is a continuous function satisfying $\rho_0(T) = 0$, the above relation (C.57) yields $Z(T,.) = 0$.

For the sake of completeness, the proof of Proposition C.4.3 is in Annexe C.8.6 (see Proposition C.8.11 applied with $r = 2$).

Now, we will deduce an observability estimate for the adjoint system :

$$\begin{cases} -\partial_t \varphi - D_J^{\mathrm{tr}} \Delta \varphi = A_J^{\mathrm{tr}} \varphi & \text{in } (0,T) \times \Omega, \\ \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega. \end{cases} \tag{C.58}$$

We have the following result which is an adaptation of [LTT13, Corollary 2.6] or [IT07, Theorem 4.1] (see Annexe C.8.7 for a complete proof).

Corollary C.4.4. There exists $C > 0$ such that for every $\varphi_T \in L^2_{inv}$, the solution of (C.58) satisfies :

$$\|\varphi(0,.)\|^2_{L^2(\Omega)^n} + \int_0^T \int_\Omega |\rho_{\mathcal{S}}(t)\varphi(t,x)|^2 \leq C_T \left(\sum_{i=1}^m \int_0^T \int_\omega |\rho_0(t)\varphi_i(t,x)|^2 \right), \tag{C.59}$$

where $C_T = Ce^{C/T}$.

In the next section, we take advantage of the strong observability estimate (C.59) to get more regularity in L^p-sense for the control H^J.

C.5 Construction of L^∞-controls and the source term method in L^∞

C.5.1 The Penalized Hilbert Uniqueness Method to build L^∞-controls

The goal of this section is to prove a null-controllability result in L^∞ with an estimate of the cost of the control.

Theorem C.5.1. There exists $C > 0$ such that for every $T > 0$, $Z_0 \in L^2_{inv}$, there exists a control $H^J \in L^\infty(Q_T)^m$ verifying

$$\left\|H^J\right\|_{L^\infty(Q_T)^m} \leq C_T \left\|Z_0\right\|_{L^2(\Omega)^n}, \text{ where } C_T = Ce^{C/T}. \tag{C.60}$$

and such that the solution Z of (L-Z) (see Annexe C.2.2) satisfies $Z(T,.) = 0$.

From now and until the end of the section, we will denote by C_T various positive constants which can change from line to line and such that $C_T \leq Ce^{C/T}$.

In the next four parts, we perform the usual Penalized Hilbert Uniqueness Method, introduced for the first time by Viorel Barbu in [Bar00]. The idea is the following one : it is a well-known fact that the optimal control $H^J \in L^2((0,T) \times \Omega)^m$, i.e., the minimal-norm element in L^2, which steers the solution Z of (L-Z) to 0 in time T can be expressed as a function of a solution of the adjoint system (C.58) (see [Cor07a, Section 1.4] for more details in the context of linear finite dimensional controlled systems). By using the strong observability inequality (C.59), we will use this link by considering a penalized problem in $\mathcal{H}_2 \subset L^2((0,T) \times \Omega)^m$: the behavior at time $t = T$ of the weight ρ_0 will be the key point to produce more regular controls in L^p-sense.

C.5.1.1 The beginning of the Penalized Hilbert Uniqueness Method

Let us fix $Z_0 \in L^2_{inv}$.

We define $P_\varepsilon : \mathcal{H}_2 \to \mathbb{R}^+$, by, for every $H^J \in \mathcal{H}_2$,

$$P_\varepsilon(H^J) := \frac{1}{2} \int\int_{(0,T)\times\omega} \rho_0^{-2}(t)|H^J(t,x)|^2 dxdt + \frac{1}{2\varepsilon} \|Z(T,.)\|^2_{L^2(\Omega)^n}, \tag{C.61}$$

where Z is the solution to the Cauchy problem (L-Z) (see Annexe C.2.2) associated to the control H^J.

The functional P_ε is a C^1, coercive, strictly convex functional on the Hilbert space $\mathcal{H}_2$, then P_ε has a unique minimum $H^{J,\varepsilon} \in \mathcal{H}_2$. Let Z^ε be the solution to the Cauchy problem (L-Z) with control $H^{J,\varepsilon}$ and initial data Z_0.

The Euler-Lagrange equation gives

$$\forall H^J \in \mathcal{H}_2, \int\int_{(0,T)\times\omega} \rho_0^{-2} H^{J,\varepsilon}.H^J dxdt + \frac{1}{\varepsilon}\int_\Omega Z^\varepsilon(T,x).Z(T,x)dx = 0, \tag{C.62}$$

where Z is the solution to the Cauchy problem (L-Z) associated to the control H^J and initial data $Z_0 = 0$.

We introduce φ^ε the solution to the adjoint problem (C.58) with final condition $\varphi^\varepsilon(T,.) = -\frac{1}{\varepsilon} Z^\varepsilon(T,.)$. A duality argument between Z and φ^ε gives

$$-\frac{1}{\varepsilon}\int_\Omega Z(T,x).Z^\varepsilon(T,x)dx = \int_\Omega Z(T,x).\varphi^\varepsilon(T,x)dx = \int\int_{(0,T)\times\omega} H^J.\varphi^\varepsilon. \tag{C.63}$$

Then, we deduce from (C.62) and (C.63) that

$$\forall H^J \in \mathcal{H}_2, \int\int_{(0,T)\times\omega} \rho_0^{-2} H^{J,\varepsilon}.H^J = \int\int_{(0,T)\times\omega} \varphi^\varepsilon.H^J.$$

Consequently, we have

$$\forall i \in \{1,\dots,m\},\ h_i^\varepsilon = \rho_0^2 \varphi_i^\varepsilon 1_\omega. \tag{C.64}$$

Another duality argument applied between Z^ε and φ^ε together with (C.64) gives

$$\begin{aligned}-\frac{1}{\varepsilon}\int_\Omega |Z^\varepsilon(T,x)|^2 dx &= \int_\Omega Z^\varepsilon(T,x).\varphi^\varepsilon(T,x)dx \\ &= \int_\Omega Z_0(x).\varphi^\varepsilon(0,x)dx + \int\int_{(0,T)\times\omega} H^{J,\varepsilon}.\varphi^\varepsilon,\end{aligned}$$

which yields

$$-\frac{1}{\varepsilon}\|Z^\varepsilon(T,.)\|^2_{L^2(\Omega)^n} = \int_\Omega Z_0(x).\varphi^\varepsilon(0,x)dx + \sum_{i=1}^m \int\int_{(0,T)\times\omega} |\rho_0\varphi_i^\varepsilon|^2. \tag{C.65}$$

By Young's inequality and the observability estimate (C.59) applied to φ^ε, for $\delta > 0$, we have :

$$\begin{aligned}
&\left|\int_\Omega Z_0(x).\varphi^\varepsilon(0,x)dx\right| \\
&\leq \delta \left\|\varphi^\varepsilon(0,.)\right\|^2_{L^2(\Omega)^n} + C_\delta \left\|Z_0\right\|^2_{L^2(\Omega)^n} \\
&\leq \delta C_T \left(\sum_{i=1}^m \int\int_{(0,T)\times\omega} |\rho_0(t)\varphi_i^\varepsilon(t,x)|^2 dxdt\right) + C_\delta \left\|Z_0\right\|^2_{L^2(\Omega)^n}.
\end{aligned} \tag{C.66}$$

Then, by using (C.64), (C.65), (C.66) and by taking δ sufficiently small, we get

$$\frac{1}{\varepsilon}\left\|Z^\varepsilon(T,.)\right\|^2_{L^2(\Omega)^n} + \frac{1}{2}\left\|\rho_0^{-1}H^{J,\varepsilon}\right\|^2_{L^2((0,T)\times\omega)^n} \leq C_T \left\|Z_0\right\|^2_{L^2(\Omega)^n}. \tag{C.67}$$

Remark C.5.2. The estimate (C.67) yields Proposition C.4.3 for $S = 0$ by letting $\varepsilon \to 0$. We remark that we have only used the term $\left\|\varphi(0,.)\right\|^2_{L^2(\Omega)^n}$ in the left hand side of (C.59). The second term in the left hand side of (C.59) enables to get more regularity (in L^p-sense) for the control H^J (see Annexe C.5.1.2 below).

C.5.1.2 Bootstrap method

In the next two parts, we will use the key identity between the control $H^{J,\varepsilon}$ and the solution of the adjoint system φ^ε, i.e, (C.64) in order to deduce L^p-regularity for $H^{J,\varepsilon}$ from L^p-regularity for φ^ε. This kind of
regularity will come from the application of successive L^p-parabolic regularity theorems stated in Proposition C.8.4 to a modification of φ^ε called $\psi^{\varepsilon,r}$ (see a precise definition in (C.72) below) which is bounded from below by $\rho_0^2\varphi$. The beginning of this bootstrap argument is the strong observability inequality (C.59). Finally, we will pass to the limit $(\varepsilon \to 0)$ in $\frac{1}{\varepsilon}\left\|Z^\varepsilon(T,.)\right\|^2_{L^2(\Omega)^n} \leq C_T \left\|Z_0\right\|^2_{L^2(\Omega)^n}$ coming from (C.67) and $\left\|H^{J,\varepsilon}\right\|_{L^\infty(Q_T)} \leq C_T \left\|Z_0\right\|_{L^2(\Omega)^n}$ coming from (C.81) (see below).

By using Remark C.4.1, we introduce the positive real number

$$\gamma := 2p - (1+p)q^2 > 0. \tag{C.68}$$

Let us define a sequence of increasing positive real numbers $(\gamma_r)_{r\in\mathbb{N}}$ such that $\lim\limits_{r\to+\infty} \gamma_r = \gamma$, where γ is defined in (C.68).

We introduce for every $r \in \mathbb{N}$,

$$\rho_{\mathcal{S},r}(t) := M^{-1-p} \exp\left(-\frac{\left((1+p)q^2 + \gamma_r\right)M}{(q-1)(T-t)}\right). \tag{C.69}$$

Then, we have from (C.51), for every $r \in \mathbb{N}$,

$$\rho_0^2 \leq C_T \rho_{\mathcal{S},r}. \tag{C.70}$$

We remark that we have for every $r \in \mathbb{N}$,

$$|\rho'_{\mathcal{S},r+1}(t)| \leq C_{T,r}\rho_{\mathcal{S},r}(t). \tag{C.71}$$

We define for every $r \in \mathbb{N}$,

$$\psi^{\varepsilon,r}(t,x) := \rho_{\mathcal{S},r}(t)\varphi^{\varepsilon}(t,x). \tag{C.72}$$

From (C.58), (C.69) and (C.72), we have for every $r \in \mathbb{N}^*$,

$$\begin{cases} -\partial_t \psi^{\varepsilon,r} - D_J^{\mathrm{tr}} \Delta \psi^{\varepsilon,r} = A_J^{\mathrm{tr}} \psi^{\varepsilon,r} - \rho'_{\mathcal{S},r}(t)\varphi^{\varepsilon} & \text{in } (0,T)\times\Omega,\\ \frac{\partial \psi^{\varepsilon,r}}{\partial \nu} = 0 & \text{on } (0,T)\times\partial\Omega,\\ \psi^{\varepsilon,r}(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{C.73}$$

By using (C.71), we remark that

$$|-\rho'_{\mathcal{S},r}(t)\varphi^{\varepsilon}| \leq C_T|\psi^{\varepsilon,r-1}|. \tag{C.74}$$

Let $(p_r)_{r\in\mathbb{N}}$ be the following sequence defined by induction

$$p_0 = 2, \tag{C.75}$$

$$p_{r+1} := \begin{cases} \frac{(N+2)p_r}{N+2-2p_r} & \text{if } p_r < \frac{N+2}{2},\\ 2p_r & \text{if } p_r = \frac{N+2}{2},\\ +\infty & \text{if } p_r > \frac{N+2}{2}. \end{cases} \tag{C.76}$$

There exists $l \in \mathbb{N}^*$ such that

$$\forall r \geq l,\ p_r = +\infty. \tag{C.77}$$

We show, by induction, that for every $0 \leq r \leq l$, we have

$$\psi^{\varepsilon,r} \in L^{p_r}(Q_T)^n \text{ and } \|\psi^{\varepsilon,r}\|_{L^{p_r}(Q_T)^n} \leq C_T \|Z_0\|_{L^2(\Omega)^n}. \tag{C.78}$$

The case $r = 0$ can be deduced from the fact that $\gamma_0 > 0$ and the observability estimate (C.59) ($p_0 = 2$ by (C.75)).

Let $r \in \mathbb{N}^*$. We assume that

$$\psi^{\varepsilon,r-1} \in L^{p_{r-1}}(Q_T)^n \text{ and } \left\|\psi^{\varepsilon,r-1}\right\|_{L^{p_{r-1}}(Q_T)^n} \leq C_T \|Z_0\|_{L^2(\Omega)^n}. \tag{C.79}$$

Then, from (C.73), (C.74), (C.79) and from the maximal regularity theorem : Proposition C.8.4 applied with $p_{r-1} \in (1,+\infty)$, we get

$$\psi^{\varepsilon,r} \in X^n_{p_{r-1}} \text{ and } \|\psi^{\varepsilon,r}\|_{X^r_{p_{r-1}}} \leq C_T \|Z_0\|_{L^2(\Omega)^n}. \tag{C.80}$$

Moreover, by the Sobolev embedding : Proposition C.8.5 and (C.76), we have

$$\psi^{\varepsilon,r} \in L^{p_r}(Q_T)^n \text{ and } \|\psi^{\varepsilon,r}\|_{L^{p_r}(Q_T)^n} \leq C_T \|Z_0\|_{L^2(\Omega)^n}.$$

This concludes the induction.

C.5.1.3 The end of the Penalized Hilbert Uniqueness Method

Now, by applying consecutively (C.77) ($p_l = +\infty$), (C.64), (C.70) and (C.78), we have for every $i \in \{1, \dots, m\}$,

$$\|h_i^\varepsilon\|_{L^\infty(Q_T)} \leq C_T \|Z_0\|_{L^2(\Omega)^n} . \tag{C.81}$$

Therefore, from (C.81), $(H^{J,\varepsilon})_\varepsilon$ is uniformly bounded in $L^\infty(Q_T)^m$, then up to a subsequence, we can assume that there exists $H^J \in L^\infty(Q_T)^m$ such that

$$H^{J,\varepsilon} \underset{\varepsilon \to 0}{\rightharpoonup^*} H^J \text{ in } L^\infty(Q_T)^m, \tag{C.82}$$

$$\left\|H^J\right\|_{L^\infty(Q_T)^m} \leq C_T \|Z_0\|_{L^2(\Omega)^n} . \tag{C.83}$$

From (C.81), Proposition C.8.2 applied to (L-Z) satisfied by Z^ε, we obtain

$$\|Z^\varepsilon\|_{W_T^n} \leq C_T \|Z_0\|_{L^2(\Omega)^n} . \tag{C.84}$$

So, from (C.84), up to a subsequence, we can suppose that there exists $Z \in W_T^n$ such that

$$Z^\varepsilon \underset{\varepsilon \to 0}{\rightharpoonup} Z \text{ in } L^2(0,T;H^1(\Omega)^n),\ \partial_t Z^\varepsilon \underset{\varepsilon \to 0}{\rightharpoonup} \partial_t Z \text{ in } L^2(0,T;(H^1(\Omega))'^n), \tag{C.85}$$

and from (C.11),

$$Z^\varepsilon(0,.) \underset{\varepsilon \to 0}{\rightharpoonup} Z(0,.) \text{ in } L^2(\Omega)^n,\ Z^\varepsilon(T,.) \underset{\varepsilon \to 0}{\rightharpoonup} Z(T,.) \text{ in } L^2(\Omega)^n. \tag{C.86}$$

Then, as we have $Z^\varepsilon(0,.) = Z_0$, $Z^\varepsilon(T,.) \to 0$ from (C.67) and by uniqueness of the limit, we deduce that

$$Z(0,.) = Z_0, \text{ and } Z(T,.) = 0. \tag{C.87}$$

By letting $\varepsilon \to 0$, we have from (C.85), (C.82) and (C.87) that Z is a solution to (L-Z) satisfying $Z(T,.) = 0$ which conludes the proof of Theorem C.5.1 by using (C.83).

C.5.2 The come back to the source term method in L^∞

The goal of this section is to apply the source term method in L^∞ thanks to the null-controllability result in L^∞ : Theorem C.5.1.

To simplify the notations, we assume that the control cost in L^∞ of Theorem C.5.1 satisfies : $C_T \leq Me^{M/T}$ where M is already defined at the beginning of Annexe C.4.

From Proposition C.8.11 with $r = +\infty$ proved in Annexe C.8.6, we deduce the following null-controllability result for (L+S-Z) (see Annexe C.4) in L^∞.

Proposition C.5.3. For every $S \in \mathcal{S}_\infty$ and $Z_0 \in L^\infty_{inv}$, there exists $H^J \in \mathcal{H}_\infty$, such that the solution Z of (L+S-Z) satisfies $Z \in \mathcal{Z}_\infty$. Furthermore, there exists $C > 0$, not depending on S and Z_0, such that

$$\|Z/\rho_0\|_{L^\infty([0,T];L^\infty(\Omega)^n)} + \left\|H^J\right\|_{\mathcal{H}_\infty} \leq C \left(\|Z_0\|_{L^\infty(\Omega)^n} + \|S\|_{\mathcal{S}_\infty} \right). \tag{C.88}$$

In particular, since ρ_0 is a continuous function satisfying $\rho_0(T) = 0$, the above relation (C.88) yields $Z(T,.) = 0$.

C.6 The inverse mapping theorem in appropriate spaces

The goal of this section is to prove Theorem C.2.3. The proof is based on Proposition C.5.3 and an inverse mapping theorem in suitable spaces. It uses similar arguments to those employed, for instance, in [FI96].

Proof. Let us introduce the following space (see the definitions (C.54), (C.55) and (C.56)) :

$$E := \{(Z, H^J) \in \mathcal{Z}_\infty \times \mathcal{H}_\infty;\ \partial_t Z - D_J \Delta Z - A_J Z - H^J 1_\omega \in S_\infty\}. \tag{C.89}$$

We endow E with the following norm : for every $(Z, H^J) \in E$,

$$\left\|(Z, H^J)\right\|_E = \|Z(0,.)\|_{L^\infty} + \|Z\|_{\mathcal{Z}_\infty} + \left\|H^J\right\|_{\mathcal{H}_\infty} + \left\|\partial_t Z - D_J \Delta Z - A_J Z - H^J 1_\omega\right\|_{S_\infty}. \tag{C.90}$$

Then, $(E, \|.\|_E)$ is a Banach space.

For every $Z \in \mathcal{Z}_\infty$, we introduce the following polynomial nonlinearity of degree more than 2 :

$$Q(Z) := G(Z) - A_J Z, \tag{C.91}$$

where G is defined in (C.26). By denoting $\gamma := \max\left(\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \beta_i\right)$, we remark that for every $Z \in \mathcal{Z}_\infty$, $Q(Z) = \sum_{i=2}^{\gamma} Q_i(Z)$ where for every $2 \leq i \leq \gamma$, $Q_i(Z)$ is a polynomial term with respect to $Z = (z_1, \dots, z_n)$ of degree i. By using (C.53), we deduce that $Q(Z) \in \mathcal{S}_\infty$ and for every $2 \leq i \leq \gamma$,

$$\|Q_i(Z)\|_{\mathcal{S}_\infty} = \left\|\frac{Q_i(Z)}{\rho_{\mathcal{S}}}\right\|_{L^\infty(Q_T)^n} = \left\|\rho_0^{i-2} \frac{\rho_0^2}{\rho_{\mathcal{S}}} \frac{Q_i(Z)}{\rho_0^i}\right\|_{L^\infty(Q_T)^n} \leq C \|Z\|_{\mathcal{Z}_\infty}^i. \tag{C.92}$$

We introduce the following mapping :

$$\begin{array}{llll} \mathcal{A}: & E & \longrightarrow & F := \mathcal{S}_\infty \times L^\infty_{inv} \\ & (Z, H) & \longmapsto & (\partial_t Z - D_J \Delta Z - A_J Z - H^J 1_\omega - Q(Z), Z(0,.)). \end{array} \tag{C.93}$$

By using (C.89), the fact that for $(Z, H^J) \in E$ and $Q(Z) \in \mathcal{S}_\infty$ by (C.92), we see that $\mathcal{A}$ is well-defined. Moreover, $\mathcal{A} \in C^1(E; F)$. Indeed, all the terms in (C.93) are linear and continuous (thus C^∞) thanks to (C.90) except the term $Q(Z)$. And, for $(Z, H^J) \in E$, $Q(Z)$ is a polynomial function with respect to Z which is C^∞ thanks to (C.92).

Moreover, the differential of $\mathcal{A}$ at the point $(0, 0)$ in the direction (Z, H^J) is

$$D\mathcal{A}(0,0).(Z, H^J) = (\partial_t Z - D_J \Delta Z - A_J Z - H^J 1_\omega, Z(0,.)), \tag{C.94}$$

which is onto by using Proposition C.5.3. Then, by using the inverse mapping theorem (see [CGMR15, Theorem 2]), we deduce that there exists $r > 0$, such that for every $(S, Z_0) \in F$ satisfying $\|(S, Z_0)\|_F \leq r$, there exists $(Z, H^J) \in E$ such that $\mathcal{A}(Z, H^J) =$

(S, Z_0). By taking $S = 0$ and $Z_0 \in L^\infty_{inv}$ such that $\|Z_0\|_{L^\infty(\Omega)^n} \leq r$, we get the existence of $(Z, H^J) \in \mathcal{Z}_\infty \times \mathcal{H}_\infty$ such that

$$\begin{cases} \partial_t Z - D_J \Delta Z = \underbrace{A_J Z + Q(Z)}_{G(Z) \text{ by (C.91)}} + H^J 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0,T) \times \partial\Omega, \\ (Z(0,.), Z(T,.)) = (Z_0, 0) & \text{in } \Omega. \end{cases}$$

This concludes the proof of Theorem C.2.3. □

C.7 Comments

C.7.1 More general semilinearities

In this paper, we have only considered particular semilinearities of the form (C.6). But the main result of the article, i.e., Theorem C.1.7 holds true with more general polynomial semilinearities satisfying

$$\exists R \in \mathbb{R}[X_1, \dots, X_n],\ \forall 1 \leq i \leq n,\ \exists a_i \in \mathbb{R}^*,\ f_i = a_i R,$$

where $\mathbb{R}[X_1, \dots, X_n]$ denotes the space of multivariate polynomials with coefficients in $\mathbb{R}$. In this case, $(u_i^*)_{1 \leq i \leq n}$ is a constant nonnegative stationary state if

$$(u_i^*)_{1 \leq i \leq n} \in [0, +\infty)^n \text{ and } R(u_1^*, \dots, u_n^*) = 0.$$

C.7.2 Degenerate cases

In this part, we assume that Hypothesis C.1.6 is not satisfied. Then, the usual strategy is to perform the return method, introduced by Jean-Michel Coron in [Cor92] (see also [Cor07a, Chapter 6]). This method consists in finding a reference trajectory $(\overline{U}, \overline{H^J})$ verifying $U(0,.) = U(T,.) = U^*$ of (NL-U) (see Annexe C.1.2) such that the linearized system of (NL-U) around $(\overline{U}, \overline{H^J})$ is null-controllable (see [CGR10] for the first application of this method in the context of the null-controllability of reaction-diffusion systems).

Example C.7.1. We come back to Application C.1.11 with $J = \{1, 2, 3\}$. In this case, (C.20) is not satisfied if and only if $(u_1^*, u_3^*, u_4^*) = (0, 0, 0)$. More precisely, the linearized system around $\Big((0, u_2^*, 0, 0), (0, 0, 0)\Big)$ is not null-controllable because the fourth equation is decoupled from the others :

$$\partial_t u_4 - d_4 \Delta u_4 = -u_2^* u_4.$$

By using the return method, the author proves the local null-controllability around $(0, u_2^*, 0, 0)$ of (NL-U) (see [LB19, Section 4.1.1.2]).

C.8 Appendix

C.8.1 Toolbox for linear parabolic systems

C.8.1.1 Well-posedness results

Definition C.8.1. Let $k \in \mathbb{N}^*$, $D \in \mathcal{M}_k(\mathbb{R})$ a diagonalizable matrix such that $\mathrm{Sp}(D) \subset (0, +\infty)$, $A \in \mathcal{M}_k(\mathbb{R})$, $U_0 \in L^2(\Omega)^k$, $S \in L^2(Q_T)^k$. A function $U \in W_T^k$ is a solution to

$$\begin{cases} \partial_t U - D\Delta U = AU + S & \text{in } (0,T) \times \Omega, \\ \frac{\partial U}{\partial \nu} = 0 & \text{on } (0,T) \times \partial\Omega, \\ U(0,.) = U_0 & \text{in } \Omega, \end{cases} \tag{C.95}$$

if for every $V \in L^2(0,T; H^1(\Omega)^k)$,

$$\int_0^T (\partial_t U, V)_{(H^1(\Omega)^k)', H^1(\Omega)^k} + \int_{Q_T} D\nabla U.\nabla V = \int_{Q_T} (AU + S).V,$$

and

$$U(0,.) = U_0 \text{ in } L^2(\Omega)^k.$$

The following well-posedness result in L^2 holds for linear parabolic equations.

Proposition C.8.2. With the same notations as in Definition C.8.1, the Cauchy problem (C.95) admits a unique solution $U \in W_T^k$. Moreover, there exists $C > 0$ independent of U_0 and S such that

$$\|U\|_{W_T^k} \leq C \left(\|U_0\|_{L^2(\Omega)^k} + \|S\|_{L^2(Q_T)^k} \right). \tag{C.96}$$

We also have the following L^∞-estimate for (C.95).

Proposition C.8.3. With the same notations as in Definition C.8.1, the unique solution U of (C.95) satisfies

$$\|U\|_{L^\infty(Q_T)^k} \leq C \left(\|U_0\|_{L^\infty(\Omega)^k} + \|S\|_{L^\infty(Q_T)^k} \right). \tag{C.97}$$

with a constant $C > 0$ independent of U_0 and S.

The proofs of Proposition C.8.2 and Proposition C.8.3 can be found in [LB19, Proposition 2.3].

C.8.1.2 Maximal regularity theorems and Sobolev embeddings

In this part, we recall a maximal regularity theorem in L^p $(1 < p < +\infty)$ for parabolic systems and an embedding result for Sobolev spaces.

We introduce the following spaces : for every $r \in [1, +\infty]$,

$$W_{Ne}^{2,r}(\Omega) := \left\{ u \in W^{2,r}(\Omega) \ ; \ \frac{\partial u}{\partial \nu} = 0 \right\}, \tag{C.98}$$

$$X_r := L^r(0,T; W_{Ne}^{2,r}(\Omega)) \cap W^{1,r}(0,T; L^r(\Omega)).$$

We have the following maximal regularity theorem.

Proposition C.8.4. [DHP07, Theorem 2.1]
Let $1 < r < +\infty$, $k \in \mathbb{N}^*$, $D \in \mathcal{M}_k(\mathbb{R})$ such that $\mathrm{Sp}(D) \subset (0, +\infty)$, $A \in \mathcal{M}_k(\mathbb{R})$ and $S \in L^r(Q_T)^k$. The solution U of (C.95) satisfies

$$\|U\|_{X_r^k} \leq C \, \|S\|_{L^r(Q_T)^k} \,,$$

with C independent of S.

We have the following embedding result for Sobolev spaces.

Proposition C.8.5. [WYW06, Theorem 1.4.1]
Let $r \in [1, +\infty[$, we have

$$X_r \hookrightarrow \begin{cases} L^{\frac{(N+2)r}{N+2-2r}}(Q_T) & \text{if } r < \frac{N+2}{2}, \\ L^{2r}(Q_T) & \text{if } r = \frac{N+2}{2}, \\ L^{\infty}(Q_T) & \text{if } r > \frac{N+2}{2}. \end{cases}$$

C.8.2 Stationary states

We only have considered nonnegative stationary constant solutions of (C.3). It is not restrictive because of the following proposition.

Proposition C.8.6. Let $(u_i)_{1 \leq i \leq n} \in C^2(\overline{\Omega})^n$ be a nonnegative solution of

$$\forall 1 \leq i \leq n, \; \begin{cases} -d_i \Delta u_i = f_i(U) & \text{in } \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{C.99}$$

where $f_i(U)$ $(1 \leq i \leq n)$ is defined in (C.6). Then, for every $1 \leq i \leq n$, u_i is constant.

The proof relies on an entropy inequality : $\sum_{i=1}^{n} \log(u_i) f_i(U) \leq 0$.

Proof. Let $\varepsilon > 0$ be a small parameter. For every $1 \leq i \leq n$, we introduce

$$u_{i,\varepsilon} = u_i + \varepsilon, \qquad w_{i,\varepsilon} = u_{i,\varepsilon}(\log u_{i,\varepsilon} - 1) + 1 \geq 0.$$

We have

$$\forall 1 \leq i \leq n, \; \nabla w_{i,\varepsilon} = \log(u_{i,\varepsilon}) \nabla u_{i,\varepsilon}, \qquad \Delta w_{i,\varepsilon} = \log(u_{i,\varepsilon}) \Delta u_{i,\varepsilon} + \frac{|\nabla u_{i,\varepsilon}|^2}{u_{i,\varepsilon}}. \tag{C.100}$$

Then, from (C.99) and (C.100), we have

$$\forall 1 \leq i \leq n, \; \begin{cases} -d_i \Delta w_{i,\varepsilon} + d_i \frac{|\nabla u_{i,\varepsilon}|^2}{u_{i,\varepsilon}} = \log(u_{i,\varepsilon}) f_i(U) & \text{in } \Omega, \\ \frac{\partial w_{i,\varepsilon}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{C.101}$$

We sum the n equations of (C.101), we integrate on Ω and we use the increasing of the function log :

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^n d_i \frac{|\nabla u_{i,\varepsilon}|^2}{u_{i,\varepsilon}} && \text{(C.102)}\\
&= -\left(\int_\Omega \left\{ \log\left(\prod_{i=1}^n u_{i,\varepsilon}^{\alpha_i}\right) - \log\left(\prod_{i=1}^n u_{i,\varepsilon}^{\beta_i}\right)\right\}\left\{\prod_{i=1}^n u_i^{\alpha_i} - \prod_{i=1}^n u_i^{\beta_i}\right\}\right)\\
&= -\left(\int_\Omega \left\{ \log\left(\prod_{i=1}^n u_{i,\varepsilon}^{\alpha_i}\right) - \log\left(\prod_{i=1}^n u_{i,\varepsilon}^{\beta_i}\right)\right\}\left\{\prod_{i=1}^n u_{i,\varepsilon}^{\alpha_i} - \prod_{i=1}^n u_{i,\varepsilon}^{\beta_i} + \mathcal{O}(\varepsilon)\right\}\right)\\
&\leq \int_\Omega \left| \log\left(\prod_{i=1}^n u_{i,\varepsilon}^{\alpha_i}\right) - \log\left(\prod_{i=1}^n u_{i,\varepsilon}^{\beta_i}\right)\right| \mathcal{O}(\varepsilon) \leq \left(\sum_{i=1}^n(\alpha_i+\beta_i)\right)|\log(\varepsilon)|\mathcal{O}(\varepsilon) \underset{\varepsilon\to 0}{\to} 0.
\end{aligned}$$

Moreover,

$$\forall 1 \leq i \leq n,\ \int_\Omega d_i \frac{|\nabla u_i^\varepsilon|^2}{u_i^\varepsilon} = \int_\Omega 4 d_i |\nabla \sqrt{u_i^\varepsilon}|^2. \tag{C.103}$$

Consequently, from (C.102), (C.103), we get that

$$\forall 1 \leq i \leq n,\ \int_\Omega 4 d_i |\nabla \sqrt{u_i}|^2 = 0.$$

Therefore, for every $1 \leq i \leq n$, u_i is constant. □

Our proof of Theorem C.1.7 does not treat the case of stationary states which can change of sign, contrary to the proof of [LB19, Theorem 3.2] (see [LB19, Section 6.2]).

C.8.3 Proof of the existence of invariant quantities in the system

The goal of this section is to prove Proposition C.1.3.

Proof. We introduce the notation $R := \prod_{k=1}^n u_k^{\alpha_k} - \prod_{k=1}^n u_k^{\beta_k}$.

Let $i \in \{m+1, \dots, n\}$. By using the fact that $u_i \in W_T$ and from [FC05, Lemma 3], we obtain that the mapping $t \mapsto \int_\Omega u_i(t,x)dx$ is absolutely continuous and for a.e. $0 \leq t \leq T$,

$$\frac{d}{dt}\int_\Omega u_i(t,x)dx = (\partial_t u_i(t,.), 1)_{(H^1(\Omega))', H^1(\Omega)}. \tag{C.104}$$

Then, by using that $((u_i)_{1\leq i\leq n}, (h_i)_{1\leq i\leq m})$ is a trajectory of (NL-U) and by taking $w = (0,\dots,0,\underbrace{1}_{i},0,\dots,0)^{\mathrm{tr}}$ in (C.12), we find that for a.e. $0 \leq t \leq T$,

$$\begin{aligned}
(\partial_t u_i(t,.), w)_{(H^1(\Omega))', H^1(\Omega)} &= d_i(\nabla u_i(t,.), \nabla w)_{L^2(\Omega), L^2(\Omega)} + \int_\Omega (\beta_i - \alpha_i) R && \text{(C.105)}\\
&= \int_\Omega (\beta_i - \alpha_i) R.
\end{aligned}$$

Then, by using (C.104) and (C.105), we get for a.e. $0 \leq t \leq T$,

$$\frac{d}{dt} \int_{\Omega} \frac{u_i(t,.)}{\beta_i - \alpha_i} = \int_{\Omega} R. \tag{C.106}$$

Now, let $m+1 \leq k \neq l \leq n$. By (C.106) for $i = k$ and (C.106) for $i = l$, we deduce that for a.e. $0 \leq t \leq T$,

$$\frac{d}{dt} \int_{\Omega} \left(\frac{u_k(t,.)}{\beta_k - \alpha_k} - \frac{u_l(t,.)}{\beta_l - \alpha_l} \right) = 0. \tag{C.107}$$

Therefore, from (C.107), we have for every $t \in [0,T]$,

$$\frac{1}{|\Omega|} \int_{\Omega} \left(\frac{u_k(t,x)}{\beta_k - \alpha_k} - \frac{u_l(t,x)}{\beta_l - \alpha_l} \right) dx = \frac{u_k^*}{\beta_k - \alpha_k} - \frac{u_l^*}{\beta_l - \alpha_l}.$$

If we assume that $d := d_k = d_l$, then the equation satisfied by $v := (\beta_l - \alpha_l)u_k - (\beta_k - \alpha_k)u_l$ is

$$\begin{cases} \partial_t v - d\Delta v = 0 & \text{in } (0,T) \times \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } (0,T) \times \partial\Omega, \\ v(T,.) = (\beta_l - \alpha_l)u_k^* - (\beta_k - \alpha_k)u_l^* & \text{in } \Omega. \end{cases} \tag{C.108}$$

The backward uniqueness of the heat equation (see for instance [BT73, Théorème II.1]) applied to (C.108) leads to

$$\forall t \in [0,T],\ (\beta_l - \alpha_l)u_k(t,.) - (\beta_k - \alpha_k)u_l(t,.) = (\beta_l - \alpha_l)u_k^* - (\beta_k - \alpha_k)u_l^*.$$

This yields (C.15). $\square$

C.8.4 Proofs concerning the change of variables

C.8.4.1 Proof of the equivalence of the two systems

In this section, we prove Proposition C.2.1. It is based on the following algebraic lemma.

Lemma C.8.7. Let s be an integer such that $s \geq 2$. Let $(a_1, \dots, a_s) \in \mathbb{C}^s$ be such that $a_i \neq a_j$ for $i \neq j$. Then, we have

$$\sum_{i=1}^{s} \prod_{\substack{j=1 \\ j \neq i}}^{s} \frac{1}{a_i - a_j} = 0. \tag{C.109}$$

Proof. Let $\mathbb{C}(X)$ be the field of fractional functions with coefficients in $\mathbb{C}$ and $F \in \mathbb{C}(X)$ be defined by

$$F(X) := \left(\sum_{i=1}^{s-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{s-1} \frac{1}{a_i - a_j} \right) \frac{1}{a_i - X} \right) + \prod_{j=1}^{s-1} \frac{1}{X - a_j}. \tag{C.110}$$

The partial fractional decomposition of F is the following one :

$$F(X) = \sum_{i=1}^{s-1} \frac{b_i}{X - a_i}, \qquad \text{where } b_i \in \mathbb{C}.$$

For $1 \leq i \leq s-1$, we compute each b_i by multiplying (C.110) by $(X - a_i)$ and by evaluating $X = a_i$:

$$b_i = -\prod_{\substack{j=1 \\ j \neq i}}^{s-1} \frac{1}{a_i - a_j} + \prod_{\substack{j=1 \\ j \neq i}}^{s-1} \frac{1}{a_i - a_j} = 0.$$

We deduce that $F = 0$. By remarking that

$$F(a_s) = \sum_{i=1}^{s} \prod_{\substack{j=1 \\ j \neq i}}^{s} \frac{1}{a_i - a_j} = 0,$$

we conclude the proof of (C.109) $\square$

The following result is an easy consequence of Lemma C.8.7.

Corollary C.8.8. For every $m + 2 \leq k \leq n$, we have

$$\sum_{l=m+1}^{k} P_{kl}(\beta_l - \alpha_l) = 0. \tag{C.111}$$

Proof. By (C.25), we have by taking $s = k - m$ and $a_i = d_{i+m}$ for $1 \leq i \leq k - m$ in Lemma C.8.7

$$\sum_{l=m+1}^{k} P_{kl}(\beta_l - \alpha_l) = \sum_{i=1}^{k-m} P_{k,i+m}(\beta_{i+m} - \alpha_{i+m}) = \sum_{i=1}^{k-m} \prod_{\substack{j=1 \\ j \neq i}}^{i-m} \frac{1}{d_{i+m} - d_{j+m}} = 0.$$

This ends the proof of Corollary C.8.8. $\square$

Now, we turn to the proof of Proposition C.2.1.

Proof. We introduce the following notation : $R := \prod_{k=1}^{n} u_k^{\alpha_k} - \prod_{k=1}^{n} u_k^{\beta_k}$.

We assume that (U, H^J) is a trajectory of (NL-U). The equations $1 \leq i \leq m + 1$ of

(NL-Z) are clearly satisfied. Let $m+2 \leq i \leq n$. We have :

$$\begin{aligned}
\partial_t z_i - d_i \Delta z_i &= \partial_t \left(\sum_{j=m+1}^{i} P_{ij}(u_j - u_j^*) \right) - d_i \Delta \left(\sum_{j=m+1}^{i} P_{ij}(u_j - u_j^*) \right) \\
&= \sum_{j=m+1}^{i} P_{ij}(\partial_t u_j - d_j \Delta u_j + (d_j - d_i)\Delta u_j) \\
&= \sum_{j=m+1}^{i} \left(P_{ij}((\beta_j - \alpha_j)R) + P_{ij} \underbrace{(d_j - d_i)}_{0 \text{ if } j=i} \Delta u_j \right) \qquad \text{(C.112)} \\
&= \underbrace{R \sum_{j=m+1}^{i} P_{ij}(\beta_j - \alpha_j)}_{0 \text{ by Corollary C.8.8}} + \sum_{j=m+1}^{i-1} \underbrace{P_{ij}(d_j - d_i)}_{P_{i-1,j} \text{ by (C.25)}} \Delta u_j \\
&= \Delta z_{i-1}.
\end{aligned}$$

This ends the proof of "$\Rightarrow$".

We assume that (Z, H^J) satisfies (NL-Z). Then, the equations

$$\partial_t u_i - d_i \Delta u_i = (\beta_i - \alpha_i)R, \qquad \text{(C.113)}$$

are clearly satisfied for $1 \leq i \leq m+1$. We prove (C.113) by strong induction on $i \in \{m+2, \ldots, n\}$. By using (C.112) for $i = m+2$ and (C.113) for $i = m+1$, we obtain

$$\begin{aligned}
&\sum_{j=m+1}^{m+2} P_{m+2,j}(\partial_t u_j - d_j \Delta u_j) = 0 \\
&\Leftrightarrow P_{m+2,m+2}(\partial_t u_{m+2} - d_{m+2}\Delta u_{m+2}) = -RP_{m+2,m+1}(\beta_{m+1} - \alpha_{m+1}).
\end{aligned}$$

This leads to (C.113) for $i = m+2$ by using $P_{m+2,m+1}/P_{m+2,m+2} = -(\beta_{m+2} - \alpha_{m+2})/(\beta_{m+1} - \alpha_{m+1})$ by (C.25). For $i > m+2$, by induction, we have $P_{ii}(\partial_t u_i - d_i \Delta u_i) + \sum_{j=m+1}^{i-1} P_{ij}(\beta_j - \alpha_j)R = 0$ by (C.112). Then, from Corollary C.8.8, we have $\sum_{j=m+1}^{i-1} P_{ij}(\beta_j - \alpha_j) = -P_{ii}(\beta_i - \alpha_i)$. This yields (C.113) and ends the proof of "$\Leftarrow$".

This concludes the proof of Proposition C.2.1. □

C.8.4.2 Proof of the equivalence concerning the mass condition

In this section, we prove the equivalence (C.30) which leads to the equivalence between Theorem C.2.3 and Theorem C.1.7.

Proof. Assume that $Z_0 \in L^\infty_{inv}$. Then, we have

$$\forall m+2 \leq i \leq n, \ \int_\Omega \sum_{k=m+1}^{i} P_{ik}(u_{k,0}(x) - u_k^*)dx = 0. \tag{C.114}$$

We prove (C.18) by strong induction on $k \geq m+2$. The case $k = m+2$ comes from (C.114) for $i = m+2$ and $P_{m+2,m+1}/P_{m+2,m+2} = -(\beta_{m+2} - \alpha_{m+2})/(\beta_{m+1} - \alpha_{m+1})$ by (C.25). For $i > m+2$ in (C.114), by induction, we have

$$\int_\Omega \left\{ P_{ii}(u_{i,0}(x) - u_i^*) + \sum_{k=m+1}^{i-1} P_{ik} \frac{(\beta_k - \alpha_k)(u_{m+1,0}(x) - u_{m+1}^*)}{\beta_{m+1} - \alpha_{m+1}} \right\} dx = 0.$$

Then, from Corollary C.8.8, we have $\sum\limits_{k=m+1}^{i-1} P_{ik}(\beta_k - \alpha_k) = -P_{ii}(\beta_i - \alpha_i)$. This yields (C.18) for $k = i$.

Assume (C.18) holds. From Corollary C.8.8, we have that for every $m+2 \leq i \leq n$,

$$\int_\Omega \sum_{k=m+1}^{i} P_{ik}(u_{k,0}(.) - u_k^*) = \int_\Omega \sum_{k=m+1}^{i} P_{ik} \frac{\beta_k - \alpha_k}{\beta_{m+1} - \alpha_{m+1}} (u_{m+1,0}(.) - u_{m+1}^*) = 0.$$

This ends the proof of (C.30). □

C.8.5 Proof of an observability estimate for linear finite dimensional systems

The goal of this section is to give a self-contained proof of Lemma C.3.5. By the Hilbert Uniqueness Method (see [Cor07a, Theorem 2.44]), it suffices to show the following null-controllability result for finite dimensional systems.

Proposition C.8.9. There exist $C > 0$, $p_1, p_2 \in \mathbb{N}$ such that for every $\tau \in (0,1)$, $\lambda \geq \lambda_1$ with λ_1 the first positive eigenvalue of $(-\Delta, H^2_{Ne}(\Omega))$, $y_0 \in \mathbb{R}^n$, there exists a control $h \in L^2(0,\tau;\mathbb{R}^m)$ verifying

$$\|h\|^2_{L^2(0,T;\mathbb{R}^m)} \leq C \left(1 + \frac{1}{\tau^{p_1}} + \lambda^{p_2}\right) \|y_0\|^2_{\mathbb{R}^n} \tag{C.115}$$

such that the solution $y \in L^2(0,\tau;\mathbb{R}^n)$ of

$$\begin{cases} y' = Ay + Bh, & \text{in } (0,\tau), \\ y(0) = y_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{C.116}$$

where $A = -\lambda D_J + A_J$ (see (C.28), (C.31) and (C.32)) and $B = \begin{pmatrix} I_m \\ (0) \end{pmatrix} \in \mathcal{M}_{n,m}(\mathbb{R})$, satisfies $y(\tau) = 0$.

Remark C.8.10. We do not treat the case $\lambda_0 = 0$ with initial data $y_0 \in \mathbb{R}^{m+1} \times \{0\}^{n-m-1}$ because it is a simple adaptation of the following proof.

Proof. Let $\tau \in (0,1)$, $\lambda \geq \lambda_1$, $y_0 \in \mathbb{R}^n$.

Step 1 : Construction of the control h by a Brunovsky approach. We start by defining $\overline{y}$ to be the free solution of the system (C.116) (take $h = 0$). We have $\overline{y}(t) = e^{tA}y_0 = e^{t(-\lambda D_J + A_J)}y_0$. We easily have that for any $l \geq 0$,

$$\left\| \overline{y}^{(l)} \right\|_{L^2(0,\tau;\mathbb{R}^n)} \leq C(1+\lambda^{l-1/2}) \left\| y_0 \right\|_{\mathbb{R}^n}. \tag{C.117}$$

We choose a cut-off function $\eta \in C^\infty([0,\tau];\mathbb{R})$ such that $\eta = 1$ on $[0,\tau/3]$ and $\eta = 0$ on $[2\tau/3,\tau]$ verifying :

$$\forall p \in \mathbb{N},\ \forall t \in [0,\tau],\ |\eta^{(p)}(t)| \leq \frac{C_p}{\tau^p}. \tag{C.118}$$

We start by choosing for every $i \in \{1,\dots,m-1,n\}$,

$$y_i(t) := \eta(t)\overline{y_i}(t). \tag{C.119}$$

Then, by using the cascade form of (C.116), we define by reverse induction on $i \in \{n-1, n-2, \dots, m+1\}$,

$$y_i(t) := -\frac{1}{\lambda}\left(y_{i+1}'(t) + \lambda d_{i+1} y_{i+1}(t)\right). \tag{C.120}$$

Then, y_m is defined by the equation number $(m+1)$ by

$$y_m(t) := \frac{1}{a_{m+1,m}}\left(y_{m+1}'(t) + \lambda d_{m+1} y_{m+1}(t) - \sum_{\substack{s=1\\ s\neq m}}^{n} a_{m+1,s} y_s(t) \right). \tag{C.121}$$

Finally, we set for the control

$$h := y' - Ay. \tag{C.122}$$

By (C.121) and (C.122), h is of the form $h = (h_1,\dots,h_m,0,\dots,0)$.

Step 2 : Properties of the solution y and estimate of the control h. First, we remark that,

$$\forall 1 \leq i \leq n,\ \begin{cases} y_i = \overline{y_i}, & \text{in } [0,\tau/3], \\ y_i = 0, & \text{in } [2\tau/3,\tau]. \end{cases} \tag{C.123}$$

Indeed, the property (C.123) is clear for $i \in \{1,\dots,m-1,n\}$ by definition (C.119). Then, we prove (C.123) by reverse induction on $m \leq i \leq n$ by using (C.120), (C.121) and the definition of $\overline{y}$, for instance, for $t \in [0,\tau/3]$:

$$y_{n-1}(t) = -\frac{1}{\lambda}\left(y_n'(t) + \lambda d_n y_n(t)\right) = -\frac{1}{\lambda}\left(\overline{y_n}'(t) + \lambda d_n \overline{y_n}(t)\right) = \overline{y_{n-1}}(t).$$

Now, we have by (C.119), (C.118) and (C.117) that for every $i \in \{1, \dots, m-1\}$,

$$\sum_{l=0}^{1} \left\| y_i^{(l)} \right\|_{L^2(0,\tau;\mathbb{R}^n)} \leq C \left(1 + \frac{1}{\tau^{1/2}} + \lambda^{1/2} \right) \|y_0\|_{\mathbb{R}^n} . \tag{C.124}$$

Then, we easily prove by reverse induction on $m \leq i \leq n$ by using (C.117), (C.118), (C.119), (C.120), (C.121) and (C.124)

$$\sum_{l=0}^{i+1-m} \left\| y_i^{(l)} \right\|_{L^2(0,\tau;\mathbb{R}^n)} \leq C \left(1 + \frac{1}{\tau^{n-m+1/2}} + \lambda^{n-m+1/2} \right) \|y_0\|_{L^2(0,\tau;\mathbb{R}^n)} .$$

Hence, the control h and the state y satisfy (C.115), (C.116) with $p_1 = p_2 = 2(n - m + 1/2)$ and $y(\tau) = 0$. $\square$

C.8.6 Source term method in L^r for $r \in \{2, +\infty\}$

We use the same notations as in the beginning of Annexe C.4. The goal of this section is to prove Proposition C.4.3 and Proposition C.5.3. We have the following result.

Proposition C.8.11. For every $S \in \mathcal{S}_r$ and $Z_0 \in L^r_{inv}$, there exists $H^J \in \mathcal{H}_r$, such that the solution Z of (L+S-Z) satisfies $Z \in \mathcal{Z}_r$. Furthermore, there exists $C > 0$, not depending on S and Z_0, such that

$$\|Z/\rho_0\|_{L^\infty([0,T];L^r(\Omega)^n)} + \left\|H^J\right\|_{\mathcal{H}_r} \leq C_T \left(\|Z_0\|_{L^r(\Omega)^n} + \|S\|_{\mathcal{S}_r} \right), \tag{C.125}$$

where $C_T = Ce^{C/T}$.

The proof is inspired by [BM17, Proposition 2.6] and [LTT13, Proposition 2.3].

Proof. For $k \geq 0$, we define $T_k := T(1 - q^{-k})$ where $q \in (1, \sqrt{2})$. On the one hand, let $a_0 := Z_0$ and, for $k \geq 0$, we define $a_{k+1} := Z_S(T_{k+1}^-, .)$ where Z_S is the solution to

$$\begin{cases} \partial_t Z_S - D_J \Delta Z_S = A_J Z_S + S & \text{in } (T_k, T_{k+1}) \times \Omega, \\ \frac{\partial Z_S}{\partial \nu} = 0 & \text{on } (T_k, T_{k+1}) \times \partial\Omega, \\ Z_S(T_k^+, .) = 0 & \text{in } \Omega. \end{cases}$$

From Proposition C.8.2 and Proposition C.8.3, using the estimates (C.96) and (C.11) for $r = 2$ or (C.97) and (C.11) for $r = +\infty$, we have

$$\|a_{k+1}\|_{L^r(\Omega)^n} \leq \|Z_S\|_{L^\infty([T_k,T_{k+1}];L^r(\Omega)^n)} \leq C \|S\|_{L^r((T_k,T_{k+1});L^r(\Omega)^n)} . \tag{C.126}$$

On the other hand, for $k \geq 0$, we also consider the control systems

$$\begin{cases} \partial_t Z_H - D_J \Delta Z_H = A_J Z_H + H^J 1_\omega & \text{in } (T_k, T_{k+1}) \times \Omega, \\ \frac{\partial Z_H}{\partial \nu} = 0 & \text{on } (T_k, T_{k+1}) \times \partial\Omega, \\ Z_H(T_k^+, .) = a_k & \text{in } \Omega. \end{cases}$$

Using Theorem C.3.1 for $r=2$ or Theorem C.5.1 for $r=+\infty$, we can define $H_k^J \in L^r((T_k,T_{k+1})\times\Omega)^m$ such that $Z_H(T_{k+1}^-,.)=0$ and, thanks to the cost estimate (C.33) for $r=2$ or (C.60) for $r=+\infty$ (recalling that $C_T \leq Me^{M/T}$),

$$\left\|H_k^J\right\|_{L^r((T_k,T_{k+1})\times\Omega)^m} \leq Me^{\frac{M}{T_{k+1}-T_k}} \left\|a_k\right\|_{L^2(\Omega)^n}. \tag{C.127}$$

In particular, for $k=0$, we have

$$\left\|H_0^J\right\|_{L^r((T_0,T_1)\times\Omega)^m} \leq Me^{\frac{qM}{T(q-1)}} \left\|Z_0\right\|_{L^2(\Omega)^n}.$$

And, since ρ_0 is decreasing

$$\left\|H_0^J/\rho_0\right\|_{L^r((T_0,T_1)\times\Omega)^m} \leq \rho_0^{-1}(T_1)Me^{\frac{qM}{T(q-1)}} \left\|Z_0\right\|_{L^2(\Omega)^n}. \tag{C.128}$$

For $k\geq 0$, since $\rho_{\mathcal{S}}$ is decreasing, combining (C.126) and (C.127) yields

$$\left\|H_{k+1}^J\right\|_{L^r((T_{k+1},T_{k+2})\times\Omega)^m} \leq CMe^{\frac{M}{T_{k+2}-T_{k+1}}}\rho_{\mathcal{S}}(T_k)\left\|S/\rho_{\mathcal{S}}\right\|_{L^r((T_k,T_{k+1})\times\Omega)^n}. \tag{C.129}$$

In particular, by using $Me^{\frac{M}{T_{k+2}-T_{k+1}}}\rho_{\mathcal{S}}(T_k)=\rho_0(T_{k+2})$ (see (C.51) and (C.52)), we have

$$\left\|H_{k+1}^J\right\|_{L^r((T_{k+1},T_{k+2})\times\Omega)^m} \leq C\rho_0(T_{k+2})\left\|S/\rho_{\mathcal{S}}\right\|_{L^r((T_k,T_{k+1})\times\Omega)^n}. \tag{C.130}$$

Then, from (C.130), by using the fact that ρ_0 is decreasing,

$$\left\|H_{k+1}^J/\rho_0\right\|_{L^r((T_{k+1},T_{k+2})\times\Omega)^m} \leq C\left\|S/\rho_{\mathcal{S}}\right\|_{L^r((T_k,T_{k+1})\times\Omega)^n}. \tag{C.131}$$

As in the original proof, we can paste the controls H_k^J for $k\geq 0$ together by defining

$$H^J := \sum_{k\geq 0} H_k^J 1_{(T_k,T_{k+1})}.$$

We have the estimate from (C.128) and (C.131)

$$\left\|H^J\right\|_{\mathcal{H}_r} \leq C\left\|S\right\|_{\mathcal{S}_r} + C\rho_0^{-1}(T_1)Me^{\frac{qM}{T(q-1)}}\left\|Z_0\right\|_{L^2(\Omega)^n}.$$

The state Z can also be reconstructed by concatenation of Z_S+Z_H, which are continuous at each junction T_k thanks to the construction. Then, we estimate the state. We use the energy estimate (C.96) for $r=2$ or (C.97) for $r=+\infty$ on each time interval (T_k,T_{k+1}) :

$$\left\|Z_S\right\|_{L^\infty(T_k,T_{k+1};L^r(\Omega)^n)} \leq C\left\|S\right\|_{L^r((T_k,T_{k+1})\times\Omega)^n},$$

and

$$\left\|Z_H\right\|_{L^\infty(T_k,T_{k+1};L^r(\Omega)^n)} \leq C\left(\left\|a_k\right\|_{L^r(\Omega)^n} + \left\|H_k^J\right\|_{L^r((T_k,T_{k+1})\times\Omega)^m}\right).$$

Proceeding similarly as for the estimate on the control, we obtain respectively

$$\left\|Z_S/\rho_0\right\|_{L^\infty(T_k,T_{k+1};L^r(\Omega)^n)} \leq CM^{-1}\left\|S\right\|_{\mathcal{S}_r},$$

and

$$\left\|Z_H/\rho_0\right\|_{L^\infty(T_k,T_{k+1};L^r(\Omega)^n)} \leq CM^{-1}\left\|S\right\|_{\mathcal{S}_r} + C\rho_0^{-1}(T_1)Me^{\frac{qM}{T(q-1)}}\left\|Z_0\right\|_{L^\infty(\Omega)^n}.$$

Therefore, for an appropriate choice of constant $C>0$, Z and H^J satisfy (C.125). This concludes the proof of Proposition C.8.11. $\square$

C.8.7 Proof of a strong observability inequality

We take the same notations as in the beginning of Annexe C.4. The goal of this section is to prove Corollary C.4.4.

Proof. We define $\mathcal{F}_1 : (Z_0, S) \in L^2_{inv} \times \mathcal{S}_2 \mapsto Z(T,.) \in L^2_{inv}$, where Z is the solution of (L+S-Z) with $H^J = 0$ and $\mathcal{F}_2 : H^J \in \mathcal{H}_2 \mapsto Z(T,.) \in L^2_{inv}$ is the solution of (L+S-Z) with $(Z_0, S) = (0, 0)$. It is easy to see that the null-controllability of (L+S-Z) is equivalent to $\text{Range}(\mathcal{F}_1) \subset \text{Range}(\mathcal{F}_2)$.

From [Cor07a, Lemma 2.48], we have that $\text{Range}(\mathcal{F}_1) \subset \text{Range}(\mathcal{F}_2)$ is equivalent to the observability inequality

$$\exists C_T > 0,\ \forall \varphi_T \in L^2_{inv},\ \|\mathcal{F}_1^*(\varphi_T)\|_{L^2_{inv} \times \mathcal{S}_2} \leq C_T \|\mathcal{F}_2^*(\varphi_T)\|_{\mathcal{H}_2}. \tag{C.132}$$

Consequently, by using the null-controllability result for (L+S-Z) : Proposition C.4.3, we have that (C.132) holds true. Moreover, the constant C_T in (C.132) can be chosen such that $C_T \leq Ce^{C/T}$ by using the cost estimate (C.57) (see the proof of [Cor07a, Theorem 2.44] for more details between the constant of cost estimate and the constant of observability inequality).

Duality arguments between Z, the solution of (L+S-Z), and φ, the solution of (C.58), lead to :

$$\begin{aligned}
\int_\Omega \mathcal{F}_1(Z_0, S)(x).\varphi_T(x)dx &= \int_\Omega Z_0(x).\varphi(0,x)dx + \int\int_{(0,T)\times\Omega} S.\varphi, \\
((Z_0, S), \mathcal{F}_1^*(\varphi_T))_{L^2(\Omega)^n \times \mathcal{S}_2} &= \int_\Omega Z_0(x).\varphi(0,x)dx + \int\int_{(0,T)\times\Omega} S.\varphi\rho_{\mathcal{S}}^2\rho_{\mathcal{S}}^{-2},
\end{aligned}$$

$$\begin{aligned}
\int_\Omega \mathcal{F}_2(H^J)(x).\varphi_T(x)dx &= \int\int_{(0,T)\times\omega} H^J.\varphi, \\
(H^J, \mathcal{F}_2^*(\varphi_T))_{\mathcal{H}_2} &= \sum_{i=1}^m \int\int_{(0,T)\times\Omega} h_i.\varphi_i\rho_0^2 1_\omega \rho_0^{-2}.
\end{aligned}$$

Consequently, by identification, we find

$$\mathcal{F}_1^*(\varphi_T) = (\varphi(0,.), \varphi\rho_{\mathcal{S}}^2) \in L^2(\Omega)^n \times \mathcal{S}_2, \qquad \mathcal{F}_2^*(\varphi_T) = (\varphi_i\rho_0^2 1_\omega)_{1\leq i\leq m} \in \mathcal{H}_2. \tag{C.133}$$

Finally, by putting (C.133) in (C.132), we exactly obtain (C.59) with $C_T = Ce^{C/T}$. This ends the proof of Corollary C.4.4. □

Annexe D

Null-controllability of two species reaction-diffusion system with nonlinear coupling : a new duality method

Abstract : We consider a 2×2 nonlinear reaction-diffusion system posed on a smooth bounded domain Ω of $\mathbb{R}^N$ $(N \geq 1)$. The control input is in the source term of only one equation. It is localized in some arbitrary nonempty open subset ω of the domain Ω. First, we prove a global null-controllability result in arbitrary time $T > 0$ when the coupling term in the second equation is an odd power. As the linearized system around zero is not null-controllable, the usual strategy consists in using the return method, introduced by Jean-Michel Coron, or the method of power series expansions. In this paper, we give a direct nonlinear proof, which relies on a new duality method that we call Reflexive Uniqueness Method. It is a variation in reflexive Banach spaces of the well-known Hilbert Uniqueness Method, introduced by Jacques-Louis Lions. It is based on Carleman estimates in L^p $(2 \leq p < \infty)$ obtained from the usual Carleman inequality in L^2 and parabolic regularity arguments. This strategy enables us to find a control of the heat equation, which is an odd power of a regular function. Another advantage of the method is to produce small controls for small initial data. Secondly, thanks to the return method, we also prove a local null-controllability result for more general nonlinear reaction-diffusion systems, where the coupling term in the second equation behaves as an odd power at zero.

D.1 Introduction

Let $T > 0$, $N \in \mathbb{N}^*$, Ω be a bounded, connected, open subset of $\mathbb{R}^N$ of class C^2, and let ω be a nonempty open subset of Ω.

We consider a 2×2 nonlinear reaction-diffusion system with one internal control :

$$\begin{cases} \partial_t u - \Delta u = f_1(u,v) + h 1_\omega & \text{in } (0,T) \times \Omega, \\ \partial_t v - \Delta v = f_2(u,v) & \text{in } (0,T) \times \Omega, \\ u, v = 0 & \text{on } (0,T) \times \partial\Omega, \\ (u,v)(0,.) = (u_0, v_0) & \text{in } \Omega, \end{cases} \tag{NL}$$

with $f_1, f_2 \in C^\infty(\mathbb{R}^2; \mathbb{R})$ satisfying $f_1(0,0) = f_2(0,0) = 0$. Here, $(u,v)(t,.) : \Omega \to \mathbb{R}^2$ is the *state* to be controlled, $h = h(t,.) : \Omega \to \mathbb{R}$ is the *control input* supported in ω.

We are interested in the **null-controllability** of (NL) : for any initial data (u_0, v_0), does there exist a control h such that the solution (u,v) of (NL) verifies $(u,v)(T,.) = (0,0)$?

D.1.1 Context

The problem of null-controllability of the heat equation was solved independently by Gilles Lebeau, Luc Robbiano in 1995 (see [LR95] or the survey [LRL12]) and Andrei Fursikov, Oleg Imanuvilov in 1996 (see [FI96]) with Carleman estimates.

Theorem D.1.1. [AKBGBdT11, Corollary 2]
For every $u_0 \in L^2(\Omega)$, there exists $h \in L^2((0,T) \times \Omega)$ such that the solution u of

$$\begin{cases} \partial_t u - \Delta u = h 1_\omega & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{cases}$$

satisfies $u(T,.) = 0$.

Then, null-controllability of linear and nonlinear coupled parabolic systems has been a challenging issue. For example, in [AKBDGB09a], Farid Ammar-Khodja, Assia Benabdallah, Cédric Dupaix and Manuel Gonzalez-Burgos identified sharp conditions for the control of systems of the form

$$\begin{cases} \partial_t U - D\Delta U = AU + BH 1_\omega & \text{in } (0,T) \times \Omega, \\ U = 0 & \text{on } (0,T) \times \partial\Omega, \\ U(0,.) = U_0 & \text{in } \Omega. \end{cases} \tag{D.1}$$

where $U(t,.) : \Omega \to \mathbb{R}^n$ is the state, $H = H(t,.) : \Omega \to \mathbb{R}^m$ is the control, $D := diag(d_1, \dots, d_n)$ with $d_i \in (0, +\infty)$ is the *diffusion matrix*, $A \in \mathcal{M}_n(\mathbb{R})$ (matrix with n lines and n columns with entries in $\mathbb{R}$) is the *coupling matrix* and $B \in \mathcal{M}_{n,m}(\mathbb{R})$ (matrix with n lines and m columns with entries in $\mathbb{R}$) represents the *distribution of controls*. In general, the rank of B is less than n (roughly speaking, there are less controls than equations), so that the controllability of the full system depends strongly on the (linear) coupling present in the system. We can see the survey [AKBGBdT11] for other results (and open problems) on the controllability of linear coupled parabolic problems. The introduction of the article [LB19] provides an overview of the results on the controllability of linear and nonlinear coupled parabolic problems.

Roughly speaking, the null-controllability of (NL) can be reformulated as follows : how can the component v be controlled thanks to the nonlinear coupling $f_2(u,v)$?

D.1.2 Linearization

We introduce the following notation which will be used throughout the paper,

$$\forall \tau > 0,\ Q_\tau := (0,\tau) \times \Omega.$$

The usual strategy consists in proving a **local null-controllability result** for (NL) from a **(global) null-controllability result for the linearized system of** (NL) **around** $((\overline{u},\overline{v}),\overline{h}) = ((0,0),0)$. The linearized system (L) is

$$\begin{cases} \partial_t u - \Delta u = \frac{\partial f_1}{\partial u}(0,0)u + \frac{\partial f_1}{\partial v}(0,0)v + h1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t v - \Delta v = \frac{\partial f_2}{\partial u}(0,0)u + \frac{\partial f_2}{\partial v}(0,0)v & \text{in } (0,T)\times\Omega, \\ u, v = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,.) = (u_0,v_0) & \text{in } \Omega. \end{cases} \tag{L}$$

Definition D.1.2. System (L) is said to be *null-controllable* if
for every $(u_0,v_0) \in L^2(\Omega)^2$, there exists $h \in L^2(Q_T)$ such that the solution (u,v) of (L) satisfies $(u,v)(T,.) = (0,0)$.

Proposition D.1.3. [AKBGBdT11, Theorem 7.1]
The following statements are equivalent.

1. System (L) is null-controllable.
2. $\frac{\partial f_2}{\partial u}(0,0) \neq 0$.

Indeed, if $\frac{\partial f_2}{\partial u}(0,0) = 0$, then the equation on v is decoupled from the first equation of (L). Consequently, for any initial data $(u_0,v_0) \in L^2(\Omega)^2$ such that $v_0 \neq 0$, we have $v(T,.) \neq 0$ by the backward uniqueness of the heat equation (see [BT73]). The proof of (2) $\Rightarrow$ (1) is a byproduct of Proposition D.1.7.

Roughly speaking, u can be driven to 0 thanks to the control h and v can be driven to 0 thanks to the *coupling term* $\frac{\partial f_2}{\partial u}(0,0)u$. We have the following diagram

$$h \overset{controls}{\rightsquigarrow} u \overset{controls}{\rightsquigarrow} v.$$

Definition D.1.4. [Null-controllability]

1. System (NL) is ***locally*** *null-controllable* if
   there exists $\delta > 0$ such that for every $(u_0,v_0) \in L^\infty(\Omega)^2$
   verifying $\|(u_0,v_0)\|_{L^\infty(\Omega)^2} \leq \delta$, there exists $h \in L^2(Q_T)$ such that (NL) has a (unique) solution $(u,v) \in L^\infty(Q_T)^2$ that satisfies $(u,v)(T,.) = (0,0)$.
2. System (NL) is ***globally*** *null-controllable* if
   for every $(u_0,v_0) \in L^\infty(\Omega)^2$, there exists $h \in L^2(Q_T)$ such that (NL) has a (unique) solution $(u,v) \in L^\infty(Q_T)^2$ that satisfies $(u,v)(T,.) = (0,0)$.

Now, we mention the **linear test** for (NL) which is a corollary of Proposition D.1.3.

Proposition D.1.5. [CGR10, Proof of Theorem 1]
Let us suppose that $\frac{\partial f_2}{\partial u}(0,0) \neq 0$. Then, (NL) is locally null-controllable.

Remark D.1.6. This result is well-known but it is difficult to find in the literature (see [AKBD06, Theorem 6] with a restriction on the dimension $1 \leq N < 6$ and other function spaces or one can adapt the arguments given in [CGR10] to get Proposition D.1.5 for any $N \in \mathbb{N}^*$). For other results in this direction, see [WZ06], [LCM+16], [GBPG06] and [CSG15].

The natural question is : what can we say about (NL) if the **linearized system around** $((0,0),0)$ **is not null-controllable** i.e. when $\frac{\partial f_2}{\partial u}(0,0) = 0$?

Another strategy to get local null-controllability for (NL) consists in **linearizing around a non trivial trajectory** $(\overline{u}, \overline{v}, \overline{h}) \in C^\infty(\overline{Q_T})^3$ of the nonlinear system (NL) which goes from 0 to 0. This procedure is called the **return method** and was introduced by Jean-Michel Coron in [Cor92] (see [Cor07a, Chapter 6]). The linearized system is the following one :

$$\begin{cases} \partial_t u - \Delta u = \frac{\partial f_1}{\partial u}(\overline{u}, \overline{v})u + \frac{\partial f_1}{\partial v}(\overline{u}, \overline{v})v + h 1_\omega & \text{in } (0,T) \times \Omega, \\ \partial_t v - \Delta v = \frac{\partial f_2}{\partial u}(\overline{u}, \overline{v})u + \frac{\partial f_2}{\partial v}(\overline{u}, \overline{v})v & \text{in } (0,T) \times \Omega, \\ u, v = 0 & \text{on } (0,T) \times \partial\Omega, \\ (u,v)(0,.) = (u_0, v_0) & \text{in } \Omega. \end{cases} \tag{L-Traj}$$

First, let us recall the generalization of Proposition D.1.3 when the *coupling coefficients are not constant*. Historically, the proof is due to Luz de Teresa in [dT00].

Proposition D.1.7. [AKBGBdT11, Theorem 7.1] or [GBPG06, Introduction]
We assume that there exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset \omega$ and $\varepsilon > 0$ such that $\left|\frac{\partial f_2}{\partial u}(\overline{u}(t,x), \overline{v}(t,x))\right| \geq \varepsilon$ for every $(t,x) \in (t_1,t_2) \times \omega_0$. Then, system (L-Traj) is null-controllable (in the sense of Definition D.1.2).

Then, the linear test gives the following result.

Proposition D.1.8. [CGR10, Proof of Theorem 1]
We assume that there exist $t_1 < t_2 \in (0,T)$, a nonempty open subset $\omega_0 \subset \omega$ and $\varepsilon > 0$ such that $\left|\frac{\partial f_2}{\partial u}(\overline{u}, \overline{v})\right| \geq \varepsilon$ on $(t_1,t_2) \times \omega_0$. Then, system (NL) is locally null-controllable.

Proposition D.1.8 is used in [CGR10] with $f_2(u_1,u_2) = u_1^3 + Ru_2$, where $R \in \mathbb{R}$, [CGMR15], [CG17] and [LB19].

D.1.3 The "power system"

A model-system for the question of null-controllability when the linearized system around $((0,0),0)$ is not null-controllable is the following one :

$$\begin{cases} \partial_t u - \Delta u = h 1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t v - \Delta v = u^n & \text{in } (0,T)\times\Omega, \\ u, v = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,.) = (u_0, v_0) & \text{in } \Omega, \end{cases} \tag{Power}$$

where $n \geq 2$ is an integer.

Proposition D.1.9. If n is an even integer, then (Power) is not locally null-controllable.

Indeed, by using $u^n \geq 0$ and the *maximum principle*, for any solution of (Power) associated to an initial condition (u_0, v_0) with $v_0 \geq 0$ and $v_0 \neq 0$,

$$v(T,.) \geq \widetilde{v}(T,.) \geq 0 \qquad \text{and} \qquad \widetilde{v}(T,.) \neq 0,$$

where $\widetilde{v}$ is the solution of the heat equation

$$\begin{cases} \partial_t \widetilde{v} - \Delta \widetilde{v} = 0 & \text{in } (0,T)\times\Omega, \\ \widetilde{v} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{v}(0,.) = v_0 & \text{in } \Omega. \end{cases}$$

The following result is due to Jean-Michel Coron, Sergio Guerrero and Lionel Rosier. The proof is based on the return method (see [CGR10]).

Proposition D.1.10. [CGR10, Theorem 1]
If $n = 3$, then (Power) is locally null-controllable.

Remark D.1.11. The difficult point of the proof of Proposition D.1.10 is the construction of the nontrivial trajectory (see [CGR10, Section 2]). The method can be generalized to $n = 2k+1$ for $k \in \mathbb{N}^*$ but with longer computations. The same problem appears in [Zha18, Section 4.2].

Remark D.1.12. An homogeneity argument shows that for the system (Power) the local null-controllability implies the global null-controllability (take $u_\varepsilon = \varepsilon u$, $v_\varepsilon = \varepsilon^n v$, $h_\varepsilon = \varepsilon h$). However, this strategy does not provide estimate on the control. This kind of argument is used in [CG17]. In this paper, we propose a different direct method for the global null-controllability, that provides estimates.

D.1.4 A direct approach

From now on, $k \in \mathbb{N}^*$ is fixed.

The first goal of this paper is to give *a direct proof (i.e. without return method)* of the *global null-controllability* of the system

$$\begin{cases} \partial_t u - \Delta u = h 1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t v - \Delta v = u^{2k+1} & \text{in } (0,T)\times\Omega, \\ u, v = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,.) = (u_0, v_0) & \text{in } \Omega. \end{cases} \tag{Odd}$$

Our proof is based on a *new duality method*, called **Reflexive Uniqueness Method**. The first step will consist in proving a *Carleman estimate in L^{2k+2} for the heat equation* (see Annexe D.4.1 and particularly Theorem D.4.4). The second step will consist in considering a *penalized problem in $L^{\frac{2k+2}{2k+1}}$* (see Annexe D.4.2), a generalization of the Penalized **Hilbert Uniqueness Method**, introduced by Jacques-Louis Lions (see [Lio88] and also [Zua07, Section 2] for an introduction to the Hilbert Uniqueness Method and some generalizations). This procedure enables us to find *a control of the heat equation which is an odd power of a regular function.*

The second goal of this paper is to prove a *local null-controllability* result for more general systems than (Odd) thanks to the **return method** (introduced in Annexe D.1).

D.2 Main results

D.2.1 Definitions and usual properties

We introduce the functional space

$$W_T := L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)). \tag{D.2}$$

We have the embedding (see [Eva10, Section 5.9.2, Theorem 3])

$$W_T \hookrightarrow C([0,T];L^2(\Omega)). \tag{D.3}$$

We define the notion of solution of linear parabolic systems.

Definition D.2.1. Let $l \in \mathbb{N}^*$, $y_0 \in L^2(\Omega)^l$, $g \in L^2(Q_T)^l$. We say that $y \in W_T^l$ is a solution of the Cauchy problem

$$\begin{cases} \partial_t y - \Delta y = g & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{D.4}$$

if y satisfies the variational formulation

$$\forall w \in L^2(0,T;H_0^1(\Omega)^l),\ \int_0^T (\partial_t y, w)_{(H^{-1}(\Omega)^l),H_0^1(\Omega)^l)} + \int_{Q_T} \nabla y.\nabla w = \int_{Q_T} g.w, \tag{D.5}$$

and

$$y(0,.) = y_0 \text{ in } L^2(\Omega)^l. \tag{D.6}$$

We have this well-posedness result for (D.4).

Proposition D.2.2. Let $l \in \mathbb{N}^*$, $y_0 \in L^2(\Omega)^l$, $g \in L^2(Q_T)^l$. The Cauchy problem (D.4) admits a unique solution $y \in W_T^l$. Moreover, there exists $C > 0$ independent of y_0 and g such that

$$\|y\|_{W_T^l} \leq C\left(\|y_0\|_{L^2(\Omega)^l} + \|g\|_{L^2(Q_T)^l}\right). \tag{D.7}$$

If $y_0 \in L^p(\Omega)^l$ and $g \in L^p(Q_T)^l$ with $p \in [1, +\infty]$ in addition, then $y \in L^p(Q_T)^l$ and there exists $C > 0$ independent of y_0 and g such that

$$\|y\|_{L^p(Q_T)^l} \leq C \left(\|y_0\|_{L^p(\Omega)^l} + \|g\|_{L^p(Q_T)^l} \right). \tag{D.8}$$

Proof. First, the well-posedness in W_T^l (i.e. (D.5), (D.6) and (D.7)) is based on *Galerkin approximations and energy estimates*. One can easily adapt the arguments given in [Eva10, Section 7.1.2].

Secondly, the L^p-estimate (i.e. (D.8) for $p < +\infty$) is based on the application of (D.5) with a cut-off of $w = |y|^{p-2}y$.

Finally, the L^∞-estimate (i.e. (D.8) for $p = +\infty$) is based on *Stampacchia's method* (see the proof of [LSU68, Chapter 3, Paragraph 7, Theorem 7.1]). □

We introduce the notion of solution associated to a control for (NL).

Definition D.2.3. Let $(u_0, v_0) \in L^\infty(\Omega)^2$, $h \in L^2(Q_T)$.

Let $(u, v) \in (W_T \cap L^\infty(Q_T))^2$. We say that (u, v) is a solution of (NL) if (u, v) satisfies

$$\forall (w_1, w_2) \in L^2(0, T; H_0^1(\Omega))^2,$$
$$\int_0^T (\partial_t u, w_1)_{(H^{-1}(\Omega), H_0^1(\Omega))} + \int_{Q_T} \nabla u . \nabla w_1 = \int_{Q_T} (f_1(u, v) + h 1_\omega) w_1, \tag{D.9}$$
$$\int_0^T (\partial_t v, w_2)_{(H^{-1}(\Omega), H_0^1(\Omega))} + \int_{Q_T} \nabla v . \nabla w_2 = \int_{Q_T} f_2(u, v) w_2, \tag{D.10}$$

and

$$(u, v)(0, .) = (u_0, v_0) \text{ in } L^\infty(\Omega)^2. \tag{D.11}$$

Let $(u, v) \in (W_T \cap L^\infty(Q_T))^2$ and $(\widetilde{u}, \widetilde{v}) \in (W_T \cap L^\infty(Q_T))^2$ be two solutions of (NL). Then, $(u, v) = (\widetilde{u}, \widetilde{v})$.

The following uniqueness result for the solutions of (NL) justifies the definitions of local null-controllability and global null-controllability (already introduced in Annexe D.1, see Definition D.1.4).

Proposition D.2.4. Let $(u_0, v_0) \in L^\infty(\Omega)^2$, $h \in L^2(Q_T)$. Let $(u, v) \in (W_T \cap L^\infty(Q_T))^2$ and $(\widetilde{u}, \widetilde{v}) \in (W_T \cap L^\infty(Q_T))^2$ be two solutions of (NL) in the sense of Definition D.2.3. Then, $(u, v) = (\widetilde{u}, \widetilde{v})$.

Proof. The nonlinearities f_1 and f_2 are in $C^\infty(\mathbb{R}^2, \mathbb{R})$, thus they are locally Lipschitz on $\mathbb{R}^2$. This provides the uniqueness of the solution of (NL) in $L^\infty(Q_T)$ (associated to an initial data $(u_0, v_0) \in L^\infty(\Omega)^2$ and a control $h \in L^2(Q_T)$) by a Gronwall argument (see Annexe D.7.1). □

D.2.2 Main results

Our first main result is the following one.

Theorem D.2.5. The system (Odd) is globally null-controllable (in the sense of Definition D.1.4).

More precisely, there exists $(C_p)_{p\in[2,+\infty)} \in (0,\infty)^{[2,+\infty)}$ such that for every initial data $(u_0, v_0) \in L^\infty(\Omega)^2$, there exists a control $h \in \bigcap_{p\in[2,+\infty)} L^p(Q_T)$ satisfying

$$\forall p \in [2,+\infty),\ \|h\|_{L^p(Q_T)} \leq C_p \left(\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)}^{1/(2k+1)}\right), \tag{D.12}$$

and the solution (u, v) of (Odd) verifies

$$(u, v)(T, .) = (0, 0).$$

Remark D.2.6. We give some natural extensions of this result in Annexe D.6.

Remark D.2.7. If we assume that

$$\Omega \in C^{2,\alpha}, \tag{D.13}$$

with $0 < \alpha < 1$, then Theorem D.2.5 remains true with a control $h \in L^\infty(Q_T)$ and the estimate (D.12) holds true with $p = +\infty$ (see Remark D.3.9 and Remark D.4.12).

Our second main result is a local controllability result for more general reaction-diffusion systems than (Odd).

Theorem D.2.8. Let $(g_1, g_2) \in C^\infty(\mathbb{R}; \mathbb{R})^2$ be such that

$$g_1(0) = g_1'(0) = \cdots = g_1^{(2k)}(0) = 0 \text{ and } g_1^{(2k+1)}(0) \neq 0,$$
$$g_2(0) \neq 0.$$

Let $f_1 \in C^\infty(\mathbb{R}^2; \mathbb{R})$, $f_2 \in C^\infty(\mathbb{R}^2; \mathbb{R})$ be such that

$$\forall v \in \mathbb{R},\ f_1(0, v) = 0, \tag{D.14}$$
$$\forall (u, v) \in \mathbb{R}^2,\ f_2(u, v) := g_1(u)g_2(v). \tag{D.15}$$

Then, the system (NL) is locally null-controllable (in the sense of Definition D.1.4).

Application D.2.9. By taking $f_1(u, v) = -u^{2k+1} = -f_2(u, v)$, Theorem D.2.8 shows the local null-controllability of a model for the non reversible chemical reaction (according to the law of mass action and the Fick's law)

$$(2k + 1)U \rightharpoonup V,$$

where u and v denote respectively the concentrations of the component U and V.

However, we cannot deduce from Theorem D.2.8 a local null-controllability result (which is true for $k = 1$ thanks to [CGR10, Theorem 1]) of a model for the reversible chemical reaction

$$(2k+1)U \rightleftharpoons V,$$

which corresponds to $f_1(u,v) = -u^{2k+1} + v = -f_2(u,v)$.

By taking $f_1(u,v) = (k_2 - (2k_1+1))u^{2k_1+1} + (k_5 - (2k_1+1))u^{2k_1+1}v^{k_4}$ and $f_2(u,v) = k_3 u^{2k_1+1} + (k_6 - k_4)u^{2k_1+1}v^{k_4}$ with k_1, k_2, k_3, k_4, k_5, k_6 positive integers, Theorem D.2.8 shows the local null-controllability of a model for the chemical reaction

$$\begin{cases} (2k_1+1)U & \rightharpoonup k_2 U + k_3 V, \\ (2k_1+1)U + k_4 V & \rightharpoonup k_5 U + k_6 V. \end{cases}$$

From now on, unless otherwise specified, we denote by C (respectively C_r) a positive constant (respectively a positive constant which depends on the parameter r) that may change from line to line.

D.3 Global null-controllability for the "odd power system"

The aim of this part is to prove Theorem D.2.5. We now fix $(u_0, v_0) \in L^\infty(\Omega)^2$ until the end of the section.

D.3.1 First step of the proof : steer u to 0

First, we find a control of (Odd) which steers u to 0 in time $T/2$.

Proposition D.3.1. There exists $h_1 \in L^\infty((0,T/2)\times\Omega)$ satisfying

$$\|h_1\|_{L^\infty((0,T/2)\times\Omega)} \leq C \|u_0\|_{L^\infty(\Omega)}, \tag{D.16}$$

such that the solution $(u_1, v_1) \in L^\infty((0,T/2)\times\Omega)^2$ of

$$\begin{cases} \partial_t u_1 - \Delta u_1 = h_1 1_\omega & \text{in } (0,T/2)\times\Omega, \\ \partial_t v_1 - \Delta v_1 = u_1^{2k+1} & \text{in } (0,T/2)\times\Omega, \\ u_1, v_1 = 0 & \text{on } (0,T/2)\times\partial\Omega, \\ (u_1,v_1)(0,.) = (u_0,v_0) & \text{in } \Omega, \end{cases}$$

satisfies $u_1(T/2,.) = 0$. Moreover, we have

$$\|v_1(T/2,.)\|_{L^\infty(\Omega)} \leq C\left(\|u_0\|_{L^\infty(\Omega)}^{2k+1} + \|v_0\|_{L^\infty(\Omega)}\right). \tag{D.17}$$

Proof. The proof is based on the following result (see Remark D.3.3 for some references).

Proposition D.3.2. [Null-controllability in L^∞ of the linear heat equation in any time] For every $\tau > 0$, $y_0 \in L^\infty(\Omega)$, there exists $h_\tau \in L^\infty(Q_\tau)$ satisfying

$$\|h_\tau\|_{L^\infty((0,\tau)\times\Omega)} \leq C_\tau \|y_0\|_{L^\infty(\Omega)}, \tag{D.18}$$

such that the solution $y \in L^\infty(Q_\tau)$ of

$$\begin{cases} \partial_t y - \Delta y = h_\tau 1_\omega & \text{in } (0,\tau)\times\Omega, \\ y = 0 & \text{on } (0,\tau)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{D.19}$$

satisfies $y(\tau,.) = 0$.

We use Proposition D.3.2 by taking $\tau = T/2$, $y_0 = u_0$. We get the existence of a control $h_1 \in L^\infty((0,T/2)\times\Omega)$ satisfying (D.16) which steers $u_1 \in L^\infty((0,T/2)\times\Omega)$ to 0 :

$$\begin{cases} \partial_t u_1 - \Delta u_1 = h_1 1_\omega & \text{in } (0,\tau)\times\Omega, \\ u_1 = 0 & \text{on } (0,\tau)\times\partial\Omega, \\ (u_1(0,.), u_1(T/2,.)) = (u_0, 0) & \text{in } \Omega. \end{cases} \tag{D.20}$$

Moreover, from (D.20), (D.8) with $p = +\infty$ and (D.16), we get

$$\|u_1\|_{L^\infty((0,T/2)\times\Omega)} \leq C\,\|u_0\|_{L^\infty(\Omega)}\,. \tag{D.21}$$

Then, we set $v_1 \in L^\infty((0,T/2)\times\Omega)$ (see Proposition D.2.2 with $p = +\infty$), as the solution of

$$\begin{cases} \partial_t v_1 - \Delta v_1 = u_1^{2k+1} & \text{in } (0,T/2)\times\Omega, \\ v_1 = 0 & \text{on } (0,T/2)\times\partial\Omega, \\ v_1(0,.) = v_0 & \text{in } \Omega. \end{cases} \tag{D.22}$$

From (D.22), (D.8) with $p = +\infty$ and (D.21), we have (D.17). □

Remark D.3.3. There exists at least three proofs of Proposition D.3.2. First, the common argument is the *null-controllability of the heat equation in* L^2 proved independently by Gilles Lebeau, Luc Robbiano in 1995 (see [LR95] and [LRL12]) and Andrei Fursikov, Oleg Imanuvilov in 1996 (see [FI96]). Then, the goal is to get a control in L^∞. The first method has been employed for the first time by Enrique Fernandez-Cara and Enrique Zuazua (see [FCZ00, Theorem 3.1]) and it is based on the *local regularizing effect of the heat equation* which leads to a *refined observability inequality* (see [FCZ00, Proposition 3.2]). The second method has been employed for the first time by Viorel Barbu (see [Bar00]) and it is based on a *Penalized Hilbert Uniqueness Method* (see also [CGR10, Section 3.1.2]). The more recent method is due to Olivier Bodart, Manuel Gonzalez-Burgos, Rosario Pérez-Garcia (see [BGBPGa04]) and it is sometimes called the *fictitious control method* (see [FCGBGP06b, Section 2] for the Neumann case and [DL16]).

D.3.2 Second step of the proof : steer v to 0 thanks to a control which is as an odd power

The aim of this part is to find a control of (Odd) which steers v from $v_1(T/2)$ to 0 and u from 0 to 0 at time T.

Proposition D.3.4. Let $((u_1, v_1), h_1)$ be as in Proposition D.3.1.
There exists a control $h_2 \in \bigcap_{p\in[2,+\infty)} L^p((T/2,T)\times\Omega)$ satisfying

$$\forall p \in [2,+\infty),\ \|h_2\|_{L^p((T/2,T)\times\Omega)} \leq C_p \left(\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)}^{1/(2k+1)}\right), \tag{D.23}$$

such that the solution $(u_2, v_2) \in L^\infty((T/2,T)\times\Omega)^2$ of

$$\begin{cases} \partial_t u_2 - \Delta u_2 = h_2 1_\omega & \text{in } (T/2,T)\times\Omega, \\ \partial_t v_2 - \Delta v_2 = u_2^{2k+1} & \text{in } (T/2,T)\times\Omega, \\ u_2, v_2 = 0 & \text{on } (T/2,T)\times\partial\Omega, \\ (u_2,v_2)(T/2,.) = (0, v_1(T/2,.)) & \text{in } \Omega, \end{cases}$$

satisfies $(u_2, v_2)(T,.) = (0,0)$.

Our approach consists in looking at the second equation of (Odd) like a controlled heat equation where the *state* is $v(t,.)$ and the *control input* is $u^{2k+1}(t,.)$. Here, **the question consists in proving that the heat equation is null-controllable with localized control which *is as an odd power* of a regular function.**

For the sequel, we need to introduce some usual definitions and properties.

Definition D.3.5. The mapping $x \in \mathbb{R} \mapsto x^{2k+1} \in \mathbb{R}$ is one-to-one. We note its inverse function $x \mapsto x^{\frac{1}{2k+1}}$.

Definition D.3.6. For all $\tau > 0$, $0 < \tau_1 < \tau_2$, $p \in [1,+\infty]$, we introduce the functional spaces

$$X_{\tau,p} = L^p(0,\tau; W^{2,p}\cap W_0^{1,p}(\Omega)) \cap W^{1,p}(0,\tau; L^p(\Omega)),$$

$$X_{(\tau_1,\tau_2),p} = L^p(\tau_1,\tau_2; W^{2,p}\cap W_0^{1,p}(\Omega)) \cap W^{1,p}(\tau_1,\tau_2; L^p(\Omega)),$$

The following result is *new* and it is the *key point* of this section.

Proposition D.3.7. For every $\tau > 0$, there exists $C_\tau > 0$ such that for every $y_0 \in L^\infty(\Omega)$, there exists a control $h_\tau \in L^\infty(Q_\tau)$ which verifies

$$\begin{aligned} &\left\| h_\tau^{\frac{1}{2k+1}} \right\|_{L^\infty(Q_\tau)} \leq C_\tau \|y_0\|_{L^\infty(\Omega)}^{1/(2k+1)}, \\ &h_\tau^{\frac{1}{2k+1}} \in \bigcap_{p\in[2,+\infty)} X_{\tau,p}, \\ &\forall p \in [2,+\infty),\ \exists C_{\tau,p} > 0,\ \left\| h_\tau^{\frac{1}{2k+1}} \right\|_{X_{\tau,p}} \leq C_{\tau,p} \|y_0\|_{L^\infty(\Omega)}^{1/(2k+1)}, \\ &h_\tau(0,.) = h_\tau(\tau,.) = 0, \\ &\forall t \in [0,\tau],\ supp(h_\tau(t,.)) \subset\subset \omega, \end{aligned}$$

such that the solution $y \in L^\infty(Q_\tau)$ of (D.19) satisfies $y(\tau,.) = 0$.

Remark D.3.8. Proposition D.3.7 extends Proposition D.3.2. Its proof is inspired by the *Penalized Hilbert Uniqueness Method* introduced by Barbu (see [Bar00]). It is based on the Reflexive Uniqueness Method.

Remark D.3.9. If we assume that (D.13) holds true, then we can replace $\bigcap_{p\in[2,+\infty)} X_{\tau,p}$ by $C^{1,2}(\overline{Q_T})$. It easily gives the proof of Remark D.2.7 by adapting the proof of Proposition D.3.4.

Before proving this proposition (whose proof is postponed to Annexe D.4), we apply it to our problem.

Proof. We apply Proposition D.3.7 with $(0,\tau) \leftarrow (T/2,T)$, $y_0 \leftarrow v_1(T/2,.) \in L^\infty(\Omega)$. Then, there exists a control $H \in L^\infty((T/2,T)\times\Omega)$ such that

$$H^{\frac{1}{2k+1}} \in \bigcap_{p\in[2,+\infty)} X_{(T/2,T),p}, \tag{D.24}$$

$$\forall p \in [2,+\infty),\ \left\| H^{\frac{1}{2k+1}} \right\|_{X_{(T/2,T),p}} \leq C_p \left\| v_1(T/2,.) \right\|_{L^\infty(\Omega)}^{1/(2k+1)}, \tag{D.25}$$

$$H(T/2,.) = H(T,.) = 0, \tag{D.26}$$

$$\forall t \in [T/2,T],\ supp(H(t,.)) \subset\subset \omega, \tag{D.27}$$

and the solution v_2 of

$$\begin{cases} \partial_t v_2 - \Delta v_2 = H & \text{in } (T/2,T)\times\Omega, \\ v_2 = 0 & \text{on } (T/2,T)\times\partial\Omega, \\ v_2(T/2,.) = v_1(T/2,.) & \text{in } \Omega, \end{cases} \tag{D.28}$$

satisfies

$$v_2(T,.) = 0. \tag{D.29}$$

From (D.24) and a Sobolev embedding (see for instance [WYW06, Theorem 1.4.1] or [LSU68, Lemma 3.3, page 80]), we set

$$u_2 := H^{\frac{1}{2k+1}} \in \left(\bigcap_{p\in[2,+\infty)} X_{(T/2,T),p} \right) \subset L^\infty((T/2,T)\times\Omega). \tag{D.30}$$

From (D.26) and (D.30), we have

$$u_2(T/2,.) = u_2(T,.) = 0. \tag{D.31}$$

Then, we set, from (D.30)

$$h_2 := \partial_t u_2 - \Delta u_2 \in \bigcap_{p\in[2,+\infty)} L^p((T/2,T)\times\Omega), \tag{D.32}$$

which is supported in $(T/2,T)\times\omega$ by (D.27). Moreover, from (D.25) and (D.17), we get

$$\forall p \in [2,+\infty),\ \|h_2\|_{L^p((T/2,T)\times\Omega)} \leq C_p \left(\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)}^{1/(2k+1)} \right). \tag{D.33}$$

By using (D.27), (D.28), (D.29), (D.30), (D.31), (D.32) and (D.33), we check that $((u_2,v_2),h_2)$ satisfies Proposition D.3.4. □

D.3.3 Strategy of control in the whole interval $(0, T)$

We gather Proposition D.3.1 and Proposition D.3.4 to find a control which steers (u, v) to $(0, 0)$ in time T.

Proposition D.3.10. There exists a control $h \in \bigcap_{p\in[2,+\infty)} L^p(Q_T)$ satisfying

$$\forall p \in [2, +\infty),\ \|h\|_{L^p((0,T)\times\Omega)} \leq C_p \left(\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)}^{1/(2k+1)} \right), \tag{D.34}$$

such that the solution $(u, v) \in L^\infty(Q_T)$ of (Odd) satisfies $(u, v)(T, .) = (0, 0)$.

Proof. Let $((u_1, v_1), h_1) \in L^\infty((0, T/2) \times \Omega)^3$ be as in Proposition D.3.1.

Let $((u_2, v_2), h_2) \in L^\infty((T/2, T) \times \Omega)^2 \times \bigcap_{p\in[2,+\infty)} L^p((T/2, T) \times \Omega)$ be as in Proposition D.3.4.

We define $((u, v), h) \in (W_T \cap L^\infty(Q_T))^2 \times \bigcap_{p\in[2,+\infty)} L^p(Q_T)$ by

$$\begin{array}{llll} u = u_1 & \text{in } [0, T/2] \times \Omega, & u = u_2 & \text{in } [T/2, T] \times \Omega, \\ v = v_1 & \text{in } [0, T/2] \times \Omega, & v = v_2 & \text{in } [T/2, T] \times \Omega, \\ h = h_1 & \text{in } (0, T/2) \times \Omega, & h = h_2 & \text{in } (T/2, T) \times \Omega. \end{array}$$

We deduce from Proposition D.3.1 and Proposition D.3.4 that $(u, v) \in (W_T \cap L^\infty(Q_T))^2$ is the solution of (Odd) associated to the control h and $(u, v)(T, .) = (0, 0)$. Moreover, from the bounds (D.16) and (D.23), we get the bound (D.34). $\square$

Proposition D.3.10 proves our first main result : Theorem D.2.5.

D.4 A control for the heat equation which is an odd power

The goal of this section is to prove Proposition D.3.7. We assume in the following that $\tau = T$. First, we prove a *new* Carleman estimate in L^{2k+2} for the heat equation. This type of inequality comes from the *usual Carleman inequality in* L^2 and *parabolic regularity*. Then, we get the existence of a control for the heat equation such that $h^{\frac{1}{2k+1}}$ is regular by considering a *penalized problem in* $L^{\frac{2k+2}{2k+1}}$, which is a generalization of the *usual Penalized Hilbert Uniqueness Method.*

D.4.1 A Carleman inequality in L^{2k+2}

D.4.1.1 Maximal regularity and Sobolev embeddings

We have the following
parabolic regularity result and Sobolev embedding lemma.

Proposition D.4.1. [DHP07, Theorem 2.1]
Let $1 < p < +\infty$, $g \in L^p(Q_T)$, $y_0 \in C_0^\infty(\Omega)$. The following Cauchy problem admits a

unique solution $y \in X_{T,p}$ (see Definition D.3.6)

$$\begin{cases} \partial_t y - \Delta y = g & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega. \end{cases}$$

Moreover, if $y_0 = 0$, there exists $C > 0$ independent of g such that

$$\|y\|_{X_{T,p}} \leq C \|g\|_{L^p(Q_T)}.$$

Lemma D.4.2. [LSU68, Lemma 3.3, page 80]
Let $p \in [1, +\infty)$, we have

$$X_{T,p} \hookrightarrow \begin{cases} L^{\frac{(N+2)p}{N+2-p}}(0,T; W_0^{1,\frac{(N+2)p}{N+2-p}}(\Omega)) & \text{if } p < N+2, \\ L^{2p}(0,T; W_0^{1,2p}(\Omega)) & \text{if } p = N+2, \\ L^{\infty}(0,T; W_0^{1,\infty}(\Omega)) & \text{if } p > N+2. \end{cases}$$

D.4.1.2 Carleman estimates

We define

$$\forall t \in (0,T),\ \eta(t) := \frac{1}{t(T-t)}. \tag{D.35}$$

Let ω_1 be a nonempty open subset such that

$$\omega_1 \subset\subset \omega. \tag{D.36}$$

Let us recall the usual Carleman estimate in L^2 (see [CGR10, Lemma 8] or [FCG06] for a general introduction to Carleman estimates).

Proposition D.4.3. [CGR10, Lemma 8]
There exist $C > 0$ and a function $\rho \in C^2(\overline{\Omega}; (0,+\infty))$ such that for every $\varphi_T \in C_0^\infty(\Omega)$ and for every $s \geq C$, the solution $\varphi \in X_{T,2}$ of

$$\begin{cases} -\partial_t \varphi - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega, \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega, \end{cases} \tag{D.37}$$

satisfies

$$\begin{aligned} &\int\int_{(0,T)\times\Omega} e^{-s\rho(x)\eta(t)}((s\eta)^3|\varphi|^2 + (s\eta)|\nabla\varphi|^2)dxdt \\ &\leq C\int\int_{(0,T)\times\omega_1} e^{-s\rho(x)\eta(t)}(s\eta)^3|\varphi|^2 dxdt. \end{aligned} \tag{D.38}$$

From now on, ρ is as in Proposition D.4.3. We will deduce from the above L^2-Carleman estimate the following L^{2k+2}-Carleman estimate.

Theorem D.4.4. There exist $C > 0$ and $m \in (0, +\infty)$ such that for every $\varphi_T \in C_0^\infty(\Omega)$ and for every $s \geq C$, the solution $\varphi \in X_{T,2k+2}$ of (D.37) satisfies

$$\int\int_{(0,T)\times\Omega} e^{-(k+1)s\rho(x)\eta(t)}((s\eta)^{-(k+1)m}|\varphi|^{2k+2} + (s\eta)^{-(k+1)(m+2)}|\nabla\varphi|^{2k+2})dxdt \quad \text{(D.39)}$$
$$\leq C\int\int_{(0,T)\times\omega_1} e^{-(k+1)s\rho(x)\eta(t)}(s\eta)^{3(k+1)}|\varphi|^{2k+2}dxdt.$$

Proof. Let $\varphi_T \in C_0^\infty(\Omega)$, $\varphi \in X_{T,2k+2}$ be the solution of (D.37) and $s \geq C$ where C is as in Proposition D.4.3.

By a standard parabolic regularity argument, one may deduce from the L^2-Carleman estimate (D.38) another Carleman type inequality in L^{p_0} with $p_0 = 2(N+2)/N$. If $p_0 > 2k+2$, this estimate implies (D.39). Otherwise, we iterate this strategy.

Step 1 : We introduce the sequence $(\psi_n)_{n\geq -1}$,

$$\psi_{-1} := e^{-s\rho\eta/2}(s\eta)^{3/2}\varphi, \qquad \forall n \geq 0,\ \psi_n := (s\eta)^{-2}\psi_{n-1} := e^{-s\rho\eta/2}(s\eta)^{3/2-2(n+1)}\varphi. \quad \text{(D.40)}$$

Then, we also define an increasing sequence $(p_n)_{n\geq -1}$ by

$$p_{-1} := 2, \qquad \forall n \geq 0,\ p_n := \begin{cases} \frac{(N+2)p_{n-1}}{N+2-p_{n-1}} & \text{if } p_{n-1} < N+2, \\ 2p_{n-1} & \text{if } p_{n-1} = N+2, \\ +\infty & \text{if } p_{n-1} > N+2. \end{cases} \quad \text{(D.41)}$$

Clearly, there exists a unique integer n_0 such that

$$p_{n_0} > 2k+2 \geq p_{n_0-1}. \quad \text{(D.42)}$$

We will need this technical lemma.

Lemma D.4.5. For every integer $n \in \mathbb{N}$,

$$\begin{cases} -\partial_t\psi_n - \Delta\psi_n = g_n & \text{in } (0,T)\times\Omega, \\ \psi_n = 0 & \text{on } (0,T)\times\partial\Omega, \\ \psi_n(T,.) = 0 & \text{in } \Omega, \end{cases} \quad \text{(D.43)}$$

with

$$g_n(t,x) = a_n(t,x)\psi_{n-1} + (s\eta)^{-1}\nabla\psi_{n-1}.\nabla\rho, \qquad \text{where } \|a_n\|_{L^\infty(Q_T)} \leq C_n. \quad \text{(D.44)}$$

Proof. We prove Lemma D.4.5 by induction on $n \in \mathbb{N}$.

We introduce the notation

$$\forall (t,x) \in (0,T)\times\Omega,\ \Theta(t,x) := e^{-s\rho(x)\eta(t)/2}(s\eta(t))^{3/2}. \quad \text{(D.45)}$$

Initialization : For $n = 0$, by using (D.40), (D.45) and (D.37), we have

$$\begin{aligned}
&-\partial_t\psi_0 - \Delta\psi_0 \\
&= -\partial_t((s\eta)^{-2}\psi_{-1}) - \Delta((s\eta)^{-2}\psi_{-1}) \\
&= -\partial_t((s\eta)^{-2})\psi_{-1} + (s\eta)^{-2}(-\partial_t\psi_{-1} - \Delta\psi_{-1}) \\
&= -\partial_t((s\eta)^{-2})\psi_{-1} + (s\eta)^{-2}(-(\partial_t\Theta)\varphi + \Theta(-\partial_t\varphi - \Delta\varphi) - 2\nabla\Theta.\nabla\varphi - (\Delta\Theta)\varphi),
\end{aligned}$$

$$-\partial_t\psi_0 - \Delta\psi_0 = -\partial_t((s\eta)^{-2})\psi_{-1} + (s\eta)^{-2}(-(\partial_t\Theta)\varphi - 2\nabla\Theta.\nabla\varphi - (\Delta\Theta)\varphi). \tag{D.46}$$

Straightforward computations lead to

$$\partial_t\Theta = e^{-s\rho\eta/2}\left(-\frac{1}{2}(s\eta')(s\eta)^{3/2}\rho + \frac{3}{2}(s\eta')(s\eta)^{1/2}\right), \tag{D.47}$$

$$\nabla\Theta = -\frac{1}{2}e^{-s\rho\eta/2}(s\eta)^{5/2}\nabla\rho, \quad \Delta\Theta = e^{-s\rho\eta/2}\left(\frac{(s\eta)^{7/2}}{4}|\nabla\rho|^2 - \frac{(s\eta)^{5/2}}{2}\Delta\rho\right). \tag{D.48}$$

By using (D.46), (D.47), (D.48), we get

$$\begin{aligned}
&-\partial_t\psi_0 - \Delta\psi_0 \\
&= -\partial_t((s\eta)^{-2})\psi_{-1} \\
&\quad + \left(e^{-s\rho\eta/2}\left(\frac{1}{2}(s\eta')(s\eta)^{-1/2}\rho - \frac{3}{2}(s\eta')(s\eta)^{-3/2} - \frac{(s\eta)^{3/2}}{4}|\nabla\rho|^2 + \frac{(s\eta)^{1/2}}{2}\Delta\rho\right)\right)\varphi \\
&\quad + e^{-s\rho\eta/2}(s\eta)^{1/2}\nabla\rho.\nabla\varphi.
\end{aligned} \tag{D.49}$$

Moreover, by using (D.40) and (D.48), we have

$$\psi_{-1} = e^{-s\rho\eta/2}(s\eta)^{3/2}\varphi \Leftrightarrow \varphi = e^{s\rho\eta/2}(s\eta)^{-3/2}\psi_{-1}, \tag{D.50}$$

$$\begin{aligned}
(s\eta)^{-1}\nabla\psi_{-1}.\nabla\rho &= (s\eta)^{-1}\Big((\nabla\Theta.\nabla\rho)\varphi + (\nabla\varphi.\nabla\rho)\Theta\Big) \\
&= e^{-s\rho\eta/2}\left(-\frac{(s\eta)^{3/2}}{2}|\nabla\rho|^2\varphi + (s\eta)^{1/2}\nabla\rho.\nabla\varphi\right).
\end{aligned} \tag{D.51}$$

We gather (D.49), (D.50) and (D.51) to get (D.43) and (D.44) for $n = 0$ (remark that $\eta' \leq C(\eta^2 + \eta^3)$) with

$$a_0 := -\partial_t((s\eta)^{-2}) + \eta'\left(\frac{s^{-1}\eta^{-2}}{2}\rho - \frac{3}{2}s^{-2}\eta^{-3}\right) + \frac{1}{4}|\nabla\rho|^2 + \frac{1}{2}(s\eta)^{-1}\Delta\rho \in L^\infty(Q_T), \tag{D.52}$$

Heredity : Let $n \geq 1$. We assume that (D.43) and (D.44) hold true for $n-1$. Then, by using (D.40), we have

$$\begin{aligned}
-\partial_t\psi_n - \Delta\psi_n &= -\partial_t((s\eta)^{-2}\psi_{n-1}) - \Delta((s\eta)^{-2}\psi_{n-1}) \\
&= -\partial_t((s\eta)^{-2})\psi_{n-1} + (s\eta)^{-2}(-\partial_t\psi_{n-1} - \Delta\psi_{n-1}) \\
&= -\partial_t((s\eta)^{-2})\psi_{n-1} + (s\eta)^{-2}(a_{n-1}\psi_{n-2} + (s\eta)^{-1}\nabla\psi_{n-2}.\nabla\rho) \\
&= -\partial_t((s\eta)^{-2})\psi_{n-1} + a_{n-1}\psi_{n-1} + (s\eta)^{-1}\nabla\psi_{n-1}.\nabla\rho.
\end{aligned}$$

Therefore, (D.43) and (D.44) hold true for n with

$$a_n(t,x) := -\partial_t((s\eta)^{-2}) + a_{n-1}(t,x) \in L^\infty(Q_T). \tag{D.53}$$

This ends the proof of Lemma D.4.5. $\square$

Step 2 : We show by induction that

$$\forall n \in \{0,\dots,n_0\},\ \psi_n \in X_{T,p_{n-1}},\ \|\psi_n\|_{X_{T,p_{n-1}}}^{p_{n-1}} \leq C_n \left\| e^{-s\rho\eta/2}(s\eta)^{3/2}\varphi \right\|_{L^{p_{n-1}}((0,T)\times\omega_1)}^{p_{n-1}}. \tag{D.54}$$

First, we treat the case $n=0$. By using (D.44) for $n=0$, (D.50) and (D.51), we remark that

$$\begin{aligned} g_0 &= a_0\psi_{-1} + (s\eta)^{-1}\nabla\psi_{-1}.\nabla\rho \\ &= a_0 e^{-s\rho\eta/2}(s\eta)^{3/2}\varphi + e^{-s\rho\eta/2}\left(-\frac{(s\eta)^{3/2}}{2}|\nabla\rho|^2\varphi + (s\eta)^{1/2}\nabla\rho.\nabla\varphi\right). \end{aligned} \tag{D.55}$$

Then, from (D.55), we get that $g_0 \in L^2(Q_T)$ and

$$\|g_0\|_{L^2(Q_T)}^2 \leq C \int\int_{(0,T)\times\Omega} e^{-s\rho(x)\eta(t)}((s\eta)^3|\varphi|^2 + (s\eta)|\nabla\varphi|^2)dxdt. \tag{D.56}$$

Consequently, by (D.43) (for $n=0$), (D.56) and a parabolic regularity estimate (see Proposition D.4.1 with $p=2$), we find that

$$\psi_0 \in X_{T,2} \text{ and } \|\psi_0\|_{X_{T,2}}^2 \leq C \int\int_{(0,T)\times\Omega} e^{-s\rho(x)\eta(t)}((s\eta)^3|\varphi|^2 + (s\eta)|\nabla\varphi|^2)dxdt. \tag{D.57}$$

Gathering the Carleman estimate in L^2 i.e. (D.38) and (D.57), we have

$$\|\psi_0\|_{X_{T,2}}^2 \leq C \int\int_{(0,T)\times\omega_1} e^{-s\rho(x)\eta(t)}(s\eta)^3|\varphi|^2 dxdt. \tag{D.58}$$

This concludes the proof of (D.54) for $n=0$.

Now, we assume that (D.54) holds true for an integer $n \in \{0,\dots,n_0-1\}$. By (D.41), a Sobolev embedding (see Lemma D.4.2) applied to the left hand side of (D.54), the embedding $L^{p_n}((0,T)\times\omega_1) \hookrightarrow L^{p_{n-1}}((0,T)\times\omega_1)$, applied to right hand side of (D.54), we obtain

$$\begin{aligned} &\psi_n \in L^{p_n}(0,T;W^{1,p_n}(\Omega)), \\ &\|\psi_n\|_{L^{p_n}(0,T;W^{1,p_n}(\Omega))} \leq C_n \left\| e^{-s\rho\eta/2}(s\eta)^{3/2}\varphi \right\|_{L^{p_n}((0,T)\times\omega_1)}. \end{aligned} \tag{D.59}$$

By using the parabolic equation satisfied by ψ_{n+1} i.e. (D.43), (D.44) for $(n+1)$, (D.59) and a parabolic regularity estimate (see Proposition D.4.1 with $p=p_n$), we get

$$\psi_{n+1} \in X_{T,p_n} \quad \text{and} \quad \|\psi_{n+1}\|_{X_{T,p_n}}^{p_n} \leq C_{n+1} \left\| e^{-s\rho\eta/2}(s\eta)^{3/2}\varphi \right\|_{L^{p_n}((0,T)\times\omega_1)}^{p_n}.$$

This ends the proof of (D.54).

Step 3 : We apply (D.54) with $n = n_0$ and we use a Sobolev embedding (see Lemma D.4.2) and (D.41) to get

$$\psi_{n_0} \in L^{p_{n_0}}(0,T;W^{1,p_{n_0}}(\Omega)),$$

$$\|\psi_{n_0}\|_{L^{p_{n_0}}(0,T;W^{1,p_{n_0}}(\Omega))} \leq C \left\| e^{-s\rho\eta/2}(s\eta)^{3/2}\varphi \right\|_{L^{p_{n_0-1}}((0,T)\times\omega_1)}. \tag{D.60}$$

Recalling the definition of n_0 in (D.42), and by using (D.60), together with the embedding

$$L^{p_{n_0}}(Q_T) \hookrightarrow L^{2k+2}(Q_T),$$

applied to the left hand side of (D.60) and the embedding

$$L^{2k+2}((0,T)\times\omega_1) \hookrightarrow L^{p_{n_0-1}}((0,T)\times\omega_1),$$

applied to the right hand side of (D.60), we get

$$\|\psi_{n_0}\|^{p_{n_0-1}}_{L^{2k+2}(Q_T)} + \|\nabla\psi_{n_0}\|^{p_{n_0-1}}_{L^{2k+2}(Q_T)} \leq C \left\| e^{-s\rho\eta/2}(s\eta)^{3/2}\varphi \right\|^{p_{n_0-1}}_{L^{2k+2}((0,T)\times\omega_1)}. \tag{D.61}$$

Then, from the definition (D.40) of ψ_{n_0}, we get

$$\psi_{n_0} = e^{-s\rho\eta/2}(s\eta)^{-1/2-2n_0}\varphi, \tag{D.62}$$

$$\nabla\psi_{n_0} = -\frac{1}{2}e^{-s\rho\eta/2}(s\eta)^{1/2-2n_0}\varphi\nabla\rho + e^{-s\rho\eta/2}(s\eta)^{-1/2-2n_0}\nabla\varphi. \tag{D.63}$$

Consequently, we deduce from (D.62) and (D.63) that

$$\begin{aligned} &\left\| e^{-s\rho\eta/2}(s\eta)^{-1/2-2n_0}\varphi \right\|^{p_{n_0-1}}_{L^{2k+2}(Q_T)} + \left\| e^{-s\rho\eta/2}(s\eta)^{-3/2-2n_0}\nabla\varphi \right\|^{p_{n_0-1}}_{L^{2k+2}(Q_T)} \\ &\leq C\left(\|\psi_{n_0}\|^{p_{n_0-1}}_{L^{2k+2}(Q_T)} + \left\|(s\eta)^{-1}\nabla\psi_{n_0}\right\|^{p_{n_0-1}}_{L^{2k+2}(Q_T)} \right) \\ &\leq C\left(\|\psi_{n_0}\|^{p_{n_0-1}}_{L^{2k+2}(Q_T)} + \|\nabla\psi_{n_0}\|^{p_{n_0-1}}_{L^{2k+2}(Q_T)} \right). \end{aligned} \tag{D.64}$$

By using (D.61) and (D.64), we get (D.39) with $m = 4n_0 + 1$. □

Remark D.4.6. From the L^2-Carleman estimate (see [CGR10, Lemma 8]) for the solution φ of the nonhomogeneous heat equation

$$\begin{cases} -\partial_t\varphi - \Delta\varphi = F & \text{in } (0,T)\times\Omega, \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,.) = \varphi_T & \text{in } \Omega, \end{cases} \tag{D.65}$$

and by adapting the proof of Theorem D.4.4, we can also establish a L^{2k+2}-Carleman estimate for φ. Indeed, we can show that there exist $C > 0$ and $m \in (0,+\infty)$ such that for every $\varphi_T \in C_0^\infty(\Omega)$, $F \in L^{2k+2}(Q_T)$ and for every $s \geq C$, the solution $\varphi \in X_{T,2k+2}$ of (D.65) satisfies

$$\begin{aligned}&\int\int_{(0,T)\times\Omega} e^{-(k+1)s\rho(x)\eta(t)}((s\eta)^{-(k+1)m}|\varphi|^{2k+2} + (s\eta)^{-(k+1)(m+2)}|\nabla\varphi|^{2k+2})dxdt \\ &\leq C\int\int_{(0,T)\times\Omega} e^{-(k+1)s\rho(x)\eta(t)}|F|^{2k+2}dxdt \\ &\quad + C\int\int_{(0,T)\times\omega_1} e^{-(k+1)s\rho(x)\eta(t)}(s\eta)^{3(k+1)}|\varphi|^{2k+2}dxdt.\end{aligned} \tag{D.66}$$

We consider $\chi \in C^\infty(\overline{\Omega};[0,+\infty))$ such that

$$supp(\chi) \subset\subset \omega, \qquad \chi = 1 \text{ in } \omega_1, \qquad \chi^{\frac{1}{2k+1}} \in C^\infty(\overline{\Omega};[0,+\infty)). \tag{D.67}$$

We deduce from Theorem D.4.4 the following result.

Corollary D.4.7. There exist $C > 0$ and $m \in (0,+\infty)$ such that for every $\varphi_T \in L^{2k+2}(\Omega)$ and for every $s \geq C$, the solution $\varphi \in L^{2k+2}(Q_T)$ of (D.37) satisfies

$$\begin{aligned}&\int\int_{(0,T)\times\Omega} e^{-(k+1)s\rho(x)\eta(t)}((s\eta)^{-(k+1)m}|\varphi|^{2k+2} + (s\eta)^{-(k+1)(m+2)}|\nabla\varphi|^{2k+2})dxdt \\ &\leq C\int\int_{(0,T)\times\omega} \chi^{2k+2}e^{-(k+1)s\rho(x)\eta(t)}(s\eta)^{3(k+1)}|\varphi|^{2k+2}dxdt,\end{aligned} \tag{D.68}$$

and

$$\|\varphi(0,.)\|_{L^{2k+2}(\Omega)}^{2k+2} \leq C_s\int\int_{(0,T)\times\omega} \chi^{2k+2}e^{-(k+1)s\rho(x)\eta(t)}(s\eta)^{3(k+1)}|\varphi|^{2k+2}dxdt. \tag{D.69}$$

Remark D.4.8. We insist on the fact that the constant C of the observability inequality (D.69) depends on the parameter s. It is not the case of (D.68).

Proof. **Step 1 :** We assume that $\varphi_T \in C_0^\infty(\Omega)$. We denote by $\varphi \in X_{T,2k+2}$, the solution of (D.37). Then, by Theorem D.4.4 and (D.67), for every $s \geq C$, (D.68) holds and in particular,

$$\begin{aligned}&\int\int_{(0,T)\times\Omega} e^{-(k+1)s\rho\eta}(s\eta)^{-(k+1)m}|\varphi|^{2k+2} \\ &\leq C\int\int_{(0,T)\times\omega} \chi^{2k+2}e^{-(k+1)s\rho\eta}(s\eta)^{3(k+1)}|\varphi|^{2k+2}\end{aligned} \tag{D.70}$$

We fix s sufficiently large such that (D.70) holds.

By using

$$\min_{[T/4,3T/4]\times\overline{\Omega}} e^{-(k+1)s\rho(x)\eta(t)}(s\eta)^{-(k+1)m} > 0, \tag{D.71}$$

together with (D.70), we get

$$\int_{T/4}^{3T/4}\int_{\Omega}|\varphi|^{2k+2}dxdt \leq C_s \int\int_{(0,T)\times\omega}\chi^{2k+2}e^{-(k+1)s\rho(x)\eta(t)}(s\eta)^{3(k+1)}|\varphi|^{2k+2}dxdt. \tag{D.72}$$

From the dissipation of the L^{2k+2}-norm for (D.37) (obtained by multiplying the equation (D.37) by $|\varphi|^{p-2}\varphi$ and integrating over Ω) : $\|\varphi(0,.)\|_{L^{2k+2}(\Omega)} \leq \|\varphi(t,.)\|_{L^{2k+2}(\Omega)}$ for $t \in (T/4, 3T/4)$, by integrating in time, we get

$$\|\varphi(0,.)\|_{L^{2k+2}(\Omega)}^{2k+2} \leq C\int_{T/4}^{3T/4}\int_{\Omega}|\varphi|^{2k+2}dxdt. \tag{D.73}$$

Gathering (D.72) and (D.73), we get (D.69).

Step 2 : The general case comes from a density argument by using in particular Proposition D.2.2 : (D.8) for $p = 2k+2$. The complete proof is postponed to Annexe D.7.2.

□

D.4.2 A new penalized duality method in $L^{(2k+2)/(2k+1)}$, the Reflexive Uniqueness Method

From now on, χ is a function which belongs to $C^{\infty}(\overline{\Omega};[0,+\infty))$ satisfying (D.67) and m, s are fixed by Corollary D.4.7.

We introduce the notations

$$q := \frac{2k+2}{2k+1}, \tag{D.74}$$

$$L^q_{wght}((0,T)\times\omega) := \left\{h \in L^q((0,T)\times\omega)\ ;\ e^{s\rho\eta/2}(s\eta)^{-3/2}h \in L^q((0,T)\times\omega)\right\}.$$

The goal of this section is to get a null-controllability result for the heat equation thanks to the observability inequalities of Corollary D.4.7.

Proposition D.4.9. For every $\zeta_0 \in L^q(\Omega)$, there exists a control $h \in L^q_{wght}((0,T)\times\omega)$ such that the solution ζ of

$$\begin{cases} \partial_t\zeta - \Delta\zeta = h\chi & \text{in } (0,T)\times\Omega,\\ \zeta = 0 & \text{on } (0,T)\times\partial\Omega,\\ \zeta(0,.) = \zeta_0 & \text{in } \Omega,\end{cases} \tag{D.75}$$

satisfies $\zeta(T,.) = 0$ and

$$\left\|e^{s\rho(x)\eta(t)/2}(s\eta)^{-3/2}h\right\|_{L^q((0,T)\times\omega)} \leq C\left\|\zeta_0\right\|_{L^q(\Omega)}. \tag{D.76}$$

Proof. Let $\zeta_0 \in C_0^{\infty}(\Omega)$. The general case comes from a density argument.

We first state two easy facts.

Fact D.4.10. The antiderivative of the continuous mapping $x \in \mathbb{R} \mapsto x^{\frac{1}{2k+1}}$ (see Definition D.3.5) is the strictly convex function

$$x \in \mathbb{R} \mapsto \frac{1}{q}|x|^q := \begin{cases} \frac{1}{q}\exp(q\log(|x|)) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Fact D.4.11. The real numbers $2k+2$, q belong to $(1,+\infty)$ and are conjugate :

$$\frac{1}{2k+2} + \frac{1}{q} = 1.$$

Let $\varepsilon \in (0,1)$. We consider the minimization problem

$$\inf_{h \in L^q_{wght}((0,T)\times\omega)} J(h), \tag{D.77}$$

where J is defined as follows : for every $h \in L^q_{wght}((0,T)\times\omega)$,

$$J(h) := \frac{1}{q}\int\int_{(0,T)\times\omega} e^{(q/2)s\rho(x)\eta(t)}(s\eta)^{-3q/2}|h|^q dxdt + \frac{1}{q\varepsilon}\left\|\zeta(T,.)\right\|^q_{L^q\Omega)}, \tag{D.78}$$

where $\zeta \in X_{T,q}$ is the solution of (D.75) (see Proposition D.4.1).

The mapping J is a coercive, strictly convex (see Fact D.4.10), C^1 function on the reflexive space $L^q_{wght}((0,T)\times\omega)$. Then, J has a unique minimum h^ε. We denote by $\zeta^\varepsilon \in X_{T,q}$ the solution of (D.75) associated to the control h^ε. The Euler-Lagrange equation gives

$$\begin{aligned} \forall h \in L^q_{wght}((0,T)\times\omega),\ & \int\int_{(0,T)\times\omega} e^{(q/2)s\rho(x)\eta(t)}(s\eta)^{-3q/2}(h^\varepsilon)^{1/(2k+1)}hdxdt \\ & + \frac{1}{\varepsilon}\int_\Omega (\zeta^\varepsilon(T,x))^{1/(2k+1)}\zeta(T,x)dx = 0, \end{aligned} \tag{D.79}$$

where $\zeta \in X_{T,q}$ is the solution of (D.75) (associated to the control h) with $\zeta_0 = 0$.

We introduce $\varphi^\varepsilon \in L^{2k+2}(Q_T)$ the solution of the adjoint problem

$$\begin{cases} -\partial_t\varphi^\varepsilon - \Delta\varphi^\varepsilon = 0 & \text{in } (0,T)\times\Omega, \\ \varphi^\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi^\varepsilon(T,.) = -\frac{1}{\varepsilon}(\zeta^\varepsilon(T,.))^{1/(2k+1)} =: \varphi^\varepsilon_T & \text{in } \Omega. \end{cases} \tag{D.80}$$

By a duality argument between ζ, the solution of (D.75) with $\zeta_0 = 0$ and φ^ε, we have

$$\forall h \in L^q_{wght}((0,T)\times\omega), \quad \int_\Omega \varphi^\varepsilon(T,x)\zeta(T,x)dx = \int\int_{(0,T)\times\omega} \varphi^\varepsilon h\chi dxdt. \tag{D.81}$$

Indeed, first, one can prove the result for $\varphi^\varepsilon_T \in C_0^\infty(\Omega)$ because in this case $\varphi^\varepsilon \in X_{T,2k+2}$ and $\zeta \in X_{T,q}$. This justifies the calculations for the duality argument. Then, the fact that $\overline{C_0^\infty(\Omega)}^{L^{2k+2}(\Omega)} = L^{2k+2}(\Omega)$ leads to (D.81).

From (D.80) (definition of φ_T^ε) and (D.81), we have

$$\forall h \in L^q_{wght}((0,T)\times\omega),\ -\frac{1}{\varepsilon}\int_\Omega (\zeta^\varepsilon(T,x))^{1/(2k+1)}\zeta(T,x)dx = \int\int_{(0,T)\times\omega} \varphi^\varepsilon h\chi dxdt. \tag{D.82}$$

Then, by using (D.79) and (D.82), we obtain

$$(h^\varepsilon)^{1/(2k+1)} = e^{-(q/2)s\rho(x)\eta(t)}(s\eta)^{3q/2}\varphi^\varepsilon\chi. \tag{D.83}$$

Moreover, from a duality argument between φ^ε and ζ^ε, together with (D.83), we have

$$\begin{aligned}&\int_\Omega \varphi^\varepsilon(T,x)\zeta^\varepsilon(T,x)dx \\ &= -\frac{1}{\varepsilon}\|\zeta^\varepsilon(T,.)\|^q_{L^q(\Omega)} \\ &= \int\int_{(0,T)\times\omega}\varphi^\varepsilon h^\varepsilon\chi dxdt + \int_\Omega \varphi^\varepsilon(0,x)\zeta_0(x)dx \\ &= \int\int_{(0,T)\times\omega} e^{-(k+1)s\rho\eta}(s\eta)^{3(k+1)}|\varphi^\varepsilon|^{2k+2}\chi^{2k+2} + \int_\Omega \varphi^\varepsilon(0,.)\zeta_0(.).\end{aligned} \tag{D.84}$$

By Young's inequality, we have for every $\delta > 0$,

$$\left|\int_\Omega \varphi^\varepsilon(0,x)\zeta_0(x)dx\right| \leq \delta\|\varphi^\varepsilon(0,.)\|^{2k+2}_{L^{2k+2}(\Omega)} + C_\delta\|\zeta(0,.)\|^q_{L^q(\Omega)}. \tag{D.85}$$

From (D.84), (D.85), the observability inequality (D.69) (applied to φ^ε), and by taking δ sufficiently small, we get

$$\frac{1}{\varepsilon}\|\zeta^\varepsilon(T,.)\|^q_{L^q(\Omega)} + \int\int_{(0,T)\times\omega} e^{-(k+1)s\rho\eta}(s\eta)^{3(k+1)}|\varphi^\varepsilon|^{2k+2}\chi^{2k+2} \leq C\|\zeta_0\|^q_{L^q(\Omega)}. \tag{D.86}$$

Now, plugging (D.83) into (D.86), we obtain

$$\frac{1}{\varepsilon}\|\zeta^\varepsilon(T,.)\|^q_{L^q(\Omega)} + \left\|e^{s\rho(x)\eta(t)/2}(s\eta)^{-3/2}h^\varepsilon\right\|^q_{L^q((0,T)\times\omega)} \leq C\|\zeta_0\|^q_{L^q(\Omega)}. \tag{D.87}$$

In particular, from (D.87), we have

$$\zeta^\varepsilon(T,.) \underset{\varepsilon\to 0}{\to} 0 \text{ in } L^q(\Omega), \tag{D.88}$$

and

$$\|h^\varepsilon\|_{L^q(Q_T)} \leq C. \tag{D.89}$$

We remark that

$$\zeta^\varepsilon = \zeta_1^\varepsilon + \zeta_2^\varepsilon, \tag{D.90}$$

with ζ_1^ε, ζ_2^ε satisfying

$$\left\{\begin{array}{ll} \partial_t\zeta_1^\varepsilon - \Delta\zeta_1^\varepsilon = 0 & \text{in } (0,T)\times\Omega, \\ \zeta_1^\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta_1^\varepsilon = \zeta_0 & \text{in } \Omega, \end{array}\right. \qquad \left\{\begin{array}{ll} \partial_t\zeta_2^\varepsilon - \Delta\zeta_2^\varepsilon = h^\varepsilon\chi & \text{in } (0,T)\times\Omega, \\ \zeta_2^\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta_2^\varepsilon = 0 & \text{in } \Omega. \end{array}\right. \tag{D.91}$$

Then, by using (D.90), (D.91), (D.89) and Proposition D.4.1 with $p = q$, we have

$$\|\zeta^\varepsilon\|_{X_{T,q}} \leq C. \tag{D.92}$$

So, from (D.92), up to a subsequence, we can suppose that there exists $\zeta \in X_{T,q}$ such that

$$\zeta^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \zeta \text{ in } X_{T,q}, \tag{D.93}$$

and from the embedding $X_{T,q} \hookrightarrow C([0,T];L^q(\Omega))$ (see [Eva10, Section 5.9.2, Theorem 2]),

$$\zeta^\varepsilon(0,.) \underset{\varepsilon\to 0}{\rightharpoonup} \zeta(0,.) \text{ in } L^q(\Omega),\ \zeta^\varepsilon(T,.) \underset{\varepsilon\to 0}{\rightharpoonup} \zeta(T,.) \text{ in } L^q(\Omega). \tag{D.94}$$

Then, as we have $\zeta^\varepsilon(0,.) = \zeta_0$ and (D.88), we deduce that

$$\zeta(0,.) = \zeta_0 \text{ and } \zeta(T,.) = 0. \tag{D.95}$$

Moreover, from (D.87), up to a subsequence, we can suppose that there exists $h \in L^q_{wght}((0,T)\times\omega)$ such that

$$h^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} h \text{ in } L^q_{wght}((0,T)\times\omega), \tag{D.96}$$

and

$$\begin{aligned} \left\| e^{s\rho(x)\eta(t)/2}(s\eta)^{-3/2}h \right\|^q_{L^q((0,T)\times\omega)} &\leq \liminf_{\varepsilon\to 0} \left\| e^{s\rho(x)\eta(t)/2}(s\eta)^{-3/2}h^\varepsilon \right\|^q_{L^q((0,T)\times\omega)} \\ &\leq C\,\|\zeta_0\|^q_{L^q(\Omega)}. \end{aligned} \tag{D.97}$$

Then, from (D.93), (D.96), and (D.94), we let $\varepsilon \to 0$ in the following equations

$$\left\{\begin{array}{ll} \partial_t\zeta^\varepsilon - \Delta\zeta^\varepsilon = h^\varepsilon\chi & \text{in } (0,T)\times\Omega, \\ \zeta^\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta^\varepsilon(0,.) = \zeta_0 & \text{in } \Omega, \end{array}\right. \tag{D.98}$$

and by using (D.95), we deduce

$$\left\{\begin{array}{ll} \partial_t\zeta - \Delta\zeta = h\chi & \text{in } (0,T)\times\Omega, \\ \zeta = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\zeta(0,.),\zeta(T,.)) = (\zeta_0, 0) & \text{in } \Omega. \end{array}\right. \tag{D.99}$$

Therefore, (D.99) and (D.97) conclude the proof of Proposition D.4.9. □

D.4.3 A Bootstrap argument

The goal of this section is to prove Proposition D.3.7. We keep the same notations as in the proof of Proposition D.4.9. We want to improve the regularity of the sequence $((h^\varepsilon)^{\frac{1}{2k+1}})_{\varepsilon>0}$. The *key point* is the equality (D.83). We deduce that the regularity of $(h^\varepsilon)^{\frac{1}{2k+1}}$ depends on the regularity of $e^{-(q/2)s\rho(x)\eta(t)}(s\eta)^{3q/2}\varphi^\varepsilon$. We use *parabolic regularity estimates* (see Proposition D.4.1) and a *bootstrap argument* (similar to the proof of Theorem D.4.4). The *starting point* is (D.68).

Step 1 : We introduce the sequence $(\psi_n^\varepsilon)_{n\geq -1}$,

$$\psi_{-1}^\varepsilon := e^{-s\rho\eta/2}(s\eta)^{-m/2}\varphi, \quad \forall n\geq 0,\ \psi_n := (s\eta)^{-2}\psi_{n-1}^\varepsilon = e^{-s\rho\eta/2}(s\eta)^{-m/2-2(n+1)}\varphi^\varepsilon, \tag{D.100}$$

where m is defined in Corollary D.4.7. Then, we also define an increasing sequence $(p_n)_{n\geq -1}$ by

$$p_{-1} := 2k+2, \qquad \forall n \geq 0,\ p_n := \begin{cases} \frac{(N+2)p_{n-1}}{N+2-p_{n-1}} & \text{if } p_{n-1} < N+2,\\ 2p_{n-1} & \text{if } p_{n-1} = N+2,\\ +\infty & \text{if } p_{n-1} > N+2. \end{cases} \tag{D.101}$$

We denote by l the integer such that

$$l := \min\{n \in \mathbb{N}\ ;\ p_n = +\infty\}. \tag{D.102}$$

By using (D.100) and (D.80), we show by induction that for every $n \in \mathbb{N}$,

$$\begin{cases} -\partial_t\psi_n^\varepsilon - \Delta\psi_n^\varepsilon = g_n^\varepsilon & \text{in } (0,T)\times\Omega,\\ \psi_n^\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega,\\ \psi_n^\varepsilon(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{D.103}$$

with

$$g_n(t,x) = a_n(t,x)\psi_{n-1}^\varepsilon + (s\eta)^{-1}\nabla\psi_{n-1}^\varepsilon.\nabla\rho, \qquad \text{where } \|a_n\|_{L^\infty(Q_T)} \leq C_n. \tag{D.104}$$

Indeed, straightforward computations as in the proof of Lemma D.4.5 lead to

$$a_0 := -\partial_t((s\eta)^{-2}) + \eta'\left(\frac{s^{-1}\eta^{-2}}{2}\rho + \frac{m}{2}s^{-2}\eta^{-3}\right) + \frac{1}{4}|\nabla\rho|^2 + \frac{1}{2}(s\eta)^{-1}\Delta\rho, \tag{D.105}$$

$$a_n := -\partial_t((s\eta)^{-2}) + a_{n-1}. \tag{D.106}$$

Step 2 : From (D.68), (D.86), (D.100) and (D.74), we have

$$\|\psi_{-1}\|_{L^{2k+2}(Q_T)} + \left\|(s\eta)^{-1}\nabla\psi_{-1}\right\|_{L^{2k+2}(Q_T)} \leq C\,\|\zeta_0\|_{L^q(\Omega)}^{q/(2k+2)} = C\,\|\zeta_0\|_{L^q(\Omega)}^{1/(2k+1)}. \tag{D.107}$$

Then, by using parabolic regularity estimate (see Proposition D.4.1), (D.103), (D.104), (D.107) and an induction argument (as in the proof of Theorem D.4.4), we have that

$$\forall n \in \{0,\dots,l\},\ \psi_n^\varepsilon \in X_{T,p_{n-1}} \quad\text{and}\quad \|\psi_n^\varepsilon\|_{X_{T,p_{n-1}}} \leq C_n\,\|\zeta_0\|_{L^q(\Omega)}^{1/(2k+1)}. \tag{D.108}$$

Step 3 : We apply (D.108) with $n = l$ (see (D.102)) and we use Lemma D.4.2 with $p = p_{l-1}$ to get

$$\psi_l^\varepsilon \in L^\infty(0,T;W_0^{1,\infty}(\Omega)) \qquad \text{and} \qquad \|\psi_l^\varepsilon\|_{L^\infty(0,T;W_0^{1,\infty}(\Omega))} \leq C\,\|\zeta_0\|_{L^q(\Omega)}^{1/(2k+1)}. \tag{D.109}$$

From a parabolic regularity estimate (see Proposition D.4.1) applied to the heat equation satisfied by ψ_{l+1}^ε and (D.109), we obtain

$$\psi_{l+1}^\varepsilon \in \cap_{p\in[2,+\infty)} X_{T,p} \quad \text{and} \quad \forall p \in [2,+\infty),\ \left\|\psi_{l+1}^\varepsilon\right\|_{X_{T,p}} \leq C_p\,\|\zeta_0\|_{L^q(\Omega)}^{1/(2k+1)}. \tag{D.110}$$

From (D.83), (D.100) (see in particular that $q > 1$), we have

$$\forall p \in [2,+\infty),\ \left\|(h^\varepsilon)^{1/(2k+1)}\right\|_{X_{T,p}} \leq C_p\,\|\psi_{l+1}^\varepsilon\|_{X_{T,p}}. \tag{D.111}$$

From (D.110) and (D.111), we have

$$(h^\varepsilon)^{1/(2k+1)} \in \bigcap_{p\in[2,+\infty)} X_{T,p},\ \forall p\in[2,+\infty),\ \left\|(h^\varepsilon)^{1/(2k+1)}\right\|_{X_{T,p}} \leq C_p\,\|\zeta_0\|_{L^q(\Omega)}^{1/(2k+1)}. \tag{D.112}$$

Now, by (D.87) and (D.112), we have

$$\forall p\in[2,+\infty),\ \frac{1}{\varepsilon^{1/(2k+2)}}\,\|\zeta^\varepsilon(T,.)\|_{L^q(\Omega)}^{1/(2k+1)} + \left\|(h^\varepsilon)^{1/(2k+1)}\right\|_{X_{T,p}} \leq C_p\,\|\zeta_0\|_{L^q(\Omega)}^{1/(2k+1)}. \tag{D.113}$$

Step 4 : From (D.113) and same arguments given as previously (see Annexe D.4.2), together with a diagonal extraction process, up to a subsequence, we can assume that there exist $H \in \cap_{p\in[2,+\infty)} X_{T,p}$ and $\zeta \in X_{T,p}$ such that

$$(h^\varepsilon)^{1/(2k+1)} \underset{\varepsilon\to 0}{\rightharpoonup} H \text{ in } X_{T,p}\ \forall p\in[2,+\infty), \tag{D.114}$$

$$(h^\varepsilon)^{1/(2k+1)} \underset{\varepsilon\to 0}{\rightarrow} H \text{ in } L^\infty(Q_T), \qquad \Big(\Rightarrow h^\varepsilon \underset{\varepsilon\to 0}{\rightarrow} H^{2k+1} \text{ in } L^\infty(Q_T)\Big), \tag{D.115}$$

$$H(0,.) = 0, \qquad H(T,.) = 0, \qquad \big(\text{see (D.83)}\big), \tag{D.116}$$

$$\zeta^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \zeta \text{ in } X_{T,q}, \tag{D.117}$$

$$\zeta(0,.) = \zeta_0, \qquad \zeta(T,.) = 0. \tag{D.118}$$

The strong L^∞-convergence (D.115) is a consequence of the weak $X_{T,p}$-convergence (D.114) for p sufficiently large because $X_{T,p}$ is relatively compact in $L^\infty(Q_T)$ (see [Sim87, Section 8, Corollary 4] : Aubin-Lions lemma).

By using (D.115), (D.117) and (D.118) and by letting $\varepsilon \to 0$ in the following equations

$$\begin{cases} \partial_t \zeta^\varepsilon - \Delta\zeta^\varepsilon = h^\varepsilon \chi & \text{in } (0,T)\times\Omega, \\ \zeta^\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta^\varepsilon(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{D.119}$$

we deduce

$$\begin{cases} \partial_t \zeta - \Delta \zeta = H^{2k+1}\chi & \text{in } (0,T)\times\Omega, \\ \zeta = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\zeta(0,.),\zeta(T,.)) = (\zeta_0,0) & \text{in } \Omega. \end{cases} \tag{D.120}$$

To sum up, for all $\zeta_0 \in C_0^\infty(\Omega)$, we have found a control

$$h := H^{2k+1}\chi \in L^\infty(Q_T), \tag{D.121}$$

such that

$$h^{1/(2k+1)} = H\chi^{1/(2k+1)} \in \bigcap_{[2,+\infty)} X_{T,p}, \qquad \text{(see (D.114))}, \tag{D.122}$$

$$\forall p \in [2,+\infty), \ \left\| h^{1/(2k+1)} \right\|_{X_{T,p}} \leq C_p \left\| \zeta_0 \right\|_{L^q(\Omega)}^{1/(2k+1)}, \qquad \text{(see (D.113))}, \tag{D.123}$$

$$h(0,.) = h(T,.) = 0, \qquad \text{(see (D.116))}. \tag{D.124}$$

Moreover, from (D.120), the solution $\zeta \in L^\infty(Q_T)$ of

$$\begin{cases} \partial_t \zeta - \Delta \zeta = h & \text{in } (0,T)\times\Omega, \\ \zeta = 0 & \text{on } (0,T)\times\partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega, \end{cases} \tag{D.125}$$

satisfies

$$\zeta(T,.) = 0. \tag{D.126}$$

By (D.121), (D.122), (D.123), (D.124), (D.125) and (D.126), we deduce Proposition D.3.7 for $\zeta_0 \in C_0^\infty(\Omega)$. The general case comes from $\overline{C_0^\infty(\Omega)}^{L^q(\Omega)} = L^q(\Omega)$ and the bound (D.122).

Remark D.4.12. In this paragraph, we gives the main details to get Remark D.3.9 and consequently Remark D.2.7. By (D.13), the function ρ, as in Proposition D.4.3, can be chosen such that

$$\rho \in C^{2,\alpha}(\overline{\Omega}), \tag{D.127}$$

see the proof of [Cor07a, Lemma 2.68].

Let us take β such that $1+\alpha \leq \beta < 2$. From Sobolev embedding (see [WYW06, Corollary 1.4.1]) and (D.110), we have for p sufficiently large,

$$\psi_{l+1}^\varepsilon \in X_{T,p} \hookrightarrow C^{\beta/2,\beta}(\overline{Q_T}) \text{ and } \left\| \psi_{l+1}^\varepsilon \right\|_{C^{\beta/2,\beta}(\overline{Q_T})} \leq C \left\| \zeta_0 \right\|_{L^q(\Omega)}^{1/(2k+1)}. \tag{D.128}$$

From (D.127), (D.105) and (D.106), we have $a_{l+2} \in C^{\alpha/2,\alpha}(\overline{Q_T})$. Then, we deduce from (D.103), (D.104) for $n = l+2$, (D.128), (D.13) and a parabolic regularity theorem in Hölder spaces (see [WYW06, Theorem 8.3.7 and Theorem 7.2.24]) that $\psi_{l+2}^\varepsilon \in C^{1+\alpha/2,2+\alpha}(\overline{Q_T})$. Therefore, we have

$$\frac{1}{\varepsilon^{1/(2k+2)}} \left\| \zeta^\varepsilon(T,.) \right\|_{L^q(\Omega)}^{1/(2k+1)} + \left\| (h^\varepsilon)^{1/(2k+1)} \right\|_{C^{1+\alpha/2,2+\alpha}(\overline{Q_T})} \leq C \left\| \zeta_0 \right\|_{L^q(\Omega)}^{1/(2k+1)}. \tag{D.129}$$

By (D.129), we conclude the proof of Remark D.3.9 as in the **Step 4** by using the compact embedding $C^{1+\alpha/2,2+\alpha}(\overline{Q_T}) \hookrightarrow C^{1,2}(\overline{Q_T})$ by Ascoli's theorem.

D.5 Local null-controllability of general nonlinear systems

The proof of Theorem D.2.8 relies on the return method and Proposition D.1.7. Thus, we just need to construct an appropriate reference trajectory $((\overline{u},\overline{v}),\overline{h})$. The goal of this section is to prove the existence of a nontrivial trajectory of (NL) associated to f_1 and f_2 defined in Theorem D.2.8 (see in particular (D.14) and (D.15)). More precisely, we have the following result.

Proposition D.5.1. Let f_1 and f_2 be as in Theorem D.2.8. Let ω_0 be a nonempty open subset such that $\omega_0 \subset\subset \omega$. There exist $\varepsilon > 0$, $((\overline{u},\overline{v}),\overline{h}) \in (W_T \cap L^\infty(Q_T))^2 \times \left(\bigcap_{[2,+\infty)} L^p(Q_T)\right)$ such that

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{u} = f_1(\overline{u},\overline{v}) + \overline{h}1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t \overline{v} - \Delta \overline{v} = f_2(\overline{u},\overline{v}) & \text{in } (0,T)\times\Omega, \\ \overline{u},\overline{v} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\overline{u},\overline{v})(0,.) = (0,0),\ (\overline{u},\overline{v})(T,.) = (0,0) & \text{in } \Omega, \end{cases}$$

and

$$\forall (t,x) \in (T/8, 3T/8)\times\omega_0,\ \overline{u}(t,x) \geq \varepsilon. \tag{D.130}$$

We construct the reference trajectory $((\overline{u},\overline{v}),\overline{h})$ on $(0,T/2)$ to guarantee (D.130) according to the following statement.

Proposition D.5.2. Let f_1 and f_2 be as in Theorem D.2.8. There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$, there exists $((\overline{u_1},\overline{v_1}),\overline{h_1}) \in L^\infty((0,T/2)\times\Omega)^2 \times L^\infty((0,T/2)\times\Omega)$ satisfying

$$\begin{cases} \partial_t \overline{u_1} - \Delta \overline{u_1} = f_1(\overline{u_1},\overline{v_1}) + \overline{h_1}1_\omega & \text{in } (0,T/2)\times\Omega, \\ \partial_t \overline{v_1} - \Delta \overline{v_1} = f_2(\overline{u_1},\overline{v_1}) & \text{in } (0,T/2)\times\Omega, \\ \overline{u_1},\overline{v_1} = 0 & \text{on } (0,T/2)\times\partial\Omega, \\ (\overline{u_1},\overline{v_1})(0,.) = (0,0) & \text{in } \Omega, \end{cases} \tag{D.131}$$

and

$$\forall (t,x) \in (T/8, 3T/8)\times\omega_0,\ \overline{u_1}(t,x) \geq \varepsilon, \tag{D.132}$$

$$\|\overline{u_1}\|_{L^\infty((0,T/2)\times\Omega)} \leq 2\varepsilon, \tag{D.133}$$

$$\|\overline{v_1}(T/2,.)\|_{L^\infty(\Omega)} \leq C\varepsilon^{2k+1}. \tag{D.134}$$

Proof. Let $\varepsilon > 0$, $\overline{u_1} \in C^\infty(\overline{(0,T/2)\times\Omega})$ such that $supp(\overline{u_1}) \subset\subset (0,T/2)\times\omega$, (D.132) and (D.133) holds. By a standard Banach fixed point argument, using (D.15) and (D.133), for $\varepsilon > 0$ small enough, there exists a unique solution $\overline{v_1} \in L^\infty((0,T/2)\times\Omega)$ of

$$\begin{cases} \partial_t \overline{v_1} - \Delta \overline{v_1} = f_2(\overline{u_1},\overline{v_1}) & \text{in } (0,T/2)\times\Omega, \\ \overline{v_1} = 0 & \text{on } (0,T/2)\times\partial\Omega, \\ \overline{v_1}(0,.) = 0 & \text{in } \Omega, \end{cases} \tag{D.135}$$

in the sense of Definition D.2.3. From (D.133), (D.15), (D.135) and Proposition D.2.2 (see (D.8)), we have (D.134). Finally, we define $\overline{h_1} \in L^\infty((0,T/2)\times\Omega)$ thanks to the property of $supp(\overline{u_1})$ and (D.14) (note that $f_1(0,.)=0$),

$$\overline{h_1} := \partial_t \overline{u_1} - \Delta \overline{u_1} - f_1(\overline{u_1}, \overline{v_1}), \tag{D.136}$$

which is supported on $(0,T/2)\times\omega$. This ends the proof of Proposition D.5.2. □

We construct the reference trajectory $((\overline{u},\overline{v}),\overline{h})$ of Proposition D.5.1 on $(T/2,T)$ to guarantee $(\overline{u},\overline{v})(T,.)=0$ according to the following statement, which relies on Proposition D.3.7 and the local invertibility of $g_1^{1/(2k+1)}$.

Proposition D.5.3. Let f_1 and f_2 be as in Theorem D.2.8. Let ε_0 be as in Proposition D.5.2.

There exists $\varepsilon_0' \in (0,\varepsilon_0)$ such that for every $\varepsilon \in (0,\varepsilon_0')$, there exists $((\overline{u_2},\overline{v_2}),\overline{h_2}) \in L^\infty((T/2,T)\times\Omega)^2 \times \cap_{[2,+\infty)} L^p((T/2,T)\times\Omega)$ satisfying

$$\begin{cases} \partial_t \overline{u_2} - \Delta \overline{u_2} = f_1(\overline{u_2},\overline{v_2}) + \overline{h_2}1_\omega & \text{in } (T/2,T)\times\Omega, \\ \partial_t \overline{v_2} - \Delta \overline{v_2} = f_2(\overline{u_2},\overline{v_2}) & \text{in } (T/2,T)\times\Omega, \\ \overline{u_2},\overline{v_2} = 0 & \text{on } (0,T/2)\times\partial\Omega, \\ (\overline{u_2},\overline{v_2})(T/2,.) = (0,\overline{v_1}(T/2,.)), \quad (\overline{u_2},\overline{v_2})(T,.) = (0,0) & \text{in } \Omega, \end{cases} \tag{D.137}$$

where $((\overline{u_1},\overline{v_1}),\overline{h_1})$ is given by Proposition D.5.2.

Proof. We recall that $f_2(u,v) = g_1(u)g_2(v)$ (see (D.15)).

Step 1 : We prove the existence of $a,\alpha,\beta>0$ and a C^∞-diffeomorphism $\widetilde{g_1}$: $(-a,a)\to(-\alpha,\beta)$ such that

$$\forall x \in (-a,a),\ g_1(x) := \widetilde{g_1}(x)^{2k+1},\ g_2(x) \neq 0. \tag{D.138}$$

From (D.15) and the Taylor formula, the map

$$\widetilde{g_1}(x) := \left(\int_0^1 \frac{(1-u)^{2k}}{(2k)!} g_1^{(2k+1)}(ux)du\right)^{1/(2k+1)} x,$$

satisfies $g_1(x) = \widetilde{g_1}(x)^{2k+1}$ for every $x\in\mathbb{R}$. Taking into account that

$$\widetilde{g_1}'(0) = \frac{1}{(2k+1)!} g_1^{(2k+1)}(0) \neq 0 \text{ and } g_2(0)\neq 0 \text{ (see (D.15))},$$

there exists $a>0$ such that $\widetilde{g_1} \in C^\infty((-a,a);\mathbb{R})$, and $\widetilde{g_1}'(x)\neq 0$, $g_2(x)\neq 0$ for every $x\in(-a,a)$.

We conclude from **Step 1** that $f_2(u,v) = \widetilde{g_1}^{2k+1}(u)g_2(v)$ locally around 0.

Step 2 : Let $\varepsilon\in(0,\varepsilon_0)$ be a small parameter which will be fixed later. Let $((\overline{u_1},\overline{v_1}),\overline{h_1})$ be as in Proposition D.5.2.

We apply Proposition D.3.7 with $(0,\tau) \leftarrow (T/2,T)$, $y_0 \leftarrow \overline{v_1}(T/2,.) \in L^\infty(\Omega)$. From (D.134), there exists a control $H \in L^\infty((T/2,T)\times\Omega)$ such that

$$\left\| H^{\frac{1}{2k+1}} \right\|_{L^\infty((T/2,T)\times\Omega)} \leq C \left\| \overline{v_1}(T/2,.)\right\|_{L^\infty(\Omega)}^{1/(2k+1)} \leq C\varepsilon, \tag{D.139}$$

$$H^{\frac{1}{2k+1}} \in \cap_{p\in[2,+\infty)} X_{(T/2,T),p}, \qquad \text{(see Definition } D.3.6\text{)}, \tag{D.140}$$

$$H(T/2,.) = H(T,.) = 0, \tag{D.141}$$

$$\forall t \in [T/2,T],\ supp(H(t,.)) \subset\subset \omega, \tag{D.142}$$

and the solution $\overline{v_2}$ of

$$\begin{cases} \partial_t \overline{v_2} - \Delta \overline{v_2} = H & \text{in } (T/2,T)\times\Omega, \\ \overline{v_2} = 0 & \text{on } (T/2,T)\times\partial\Omega, \\ \overline{v_2}(T/2,.) = \overline{v_1}(T/2,.) & \text{in } \Omega, \end{cases} \tag{D.143}$$

satisfies

$$\overline{v_2}(T,.) = 0. \tag{D.144}$$

From (D.134), (D.139), (D.143) and Proposition D.2.2, we have

$$\|\overline{v_2}\|_{L^\infty((T/2,T)\times\Omega)} \leq C\varepsilon^{2k+1}. \tag{D.145}$$

Moreover, $\overline{v_2}$ is the restriction on $(T/2,T)$ of $\overline{v}$ defined by

$$\begin{cases} \partial_t \overline{v} - \Delta \overline{v} = f_2(\overline{u_1},\overline{v_1})1_{(0,T/2)} + H1_{(T/2,T)} & \text{in } (0,T)\times\Omega, \\ \overline{v} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \overline{v}(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{D.146}$$

Then, by using Proposition D.5.2 : $(\overline{u_1},\overline{v_1}) \in L^\infty(Q_T)^2$, (D.139), (D.146), Definition D.3.6 and Proposition D.4.1, we have

$$\overline{v_2} \in \cap_{p\in[2,+\infty)} X_{(T/2,T),p}. \tag{D.147}$$

From (D.145), for ε sufficiently small, we have

$$\|\overline{v_2}\|_{L^\infty((T/2,T)\times\Omega)} < a/2, \tag{D.148}$$

where a is defined in **Step 1**. Therefore, from (D.148) and (D.138), $g_2(\overline{v_2})^{-\frac{1}{2k+1}}$ is well-defined. Moreover, from (D.139), for ε sufficiently small, we have

$$\left\| H^{\frac{1}{2k+1}} g_2(\overline{v_2})^{-\frac{1}{2k+1}} \right\|_{L^\infty((0,T/2)\times\Omega)} < \max(\alpha/2,\beta/2), \tag{D.149}$$

where α and β are defined in **Step 1**.

Then, we set

$$\overline{u_2} := \widetilde{g_1}^{-1}\left(H^{\frac{1}{2k+1}} g_2(\overline{v_2})^{-\frac{1}{2k+1}} \right) \in L^\infty((T/2,T)\times\Omega), \tag{D.150}$$

where $\widetilde{g_1}$ is defined as in **Step 1**. From the fact that $g_2^{-\frac{1}{2k+1}} \in W^{2,\infty}(-a/2, a/2)$ (see (D.138)), (D.148) and (D.147), we check that

$$g_2(\overline{v_2})^{-\frac{1}{2k+1}} \in \cap_{p\in[2,+\infty)} X_{(T/2,T),p}. \tag{D.151}$$

Taking into account that $\widetilde{g_1}^{-1} \in W^{2,\infty}(-\alpha/2, \beta/2)$, (D.140), (D.151) and (D.150), we verify that

$$\overline{u_2} \in \cap_{p\in[2,+\infty)} X_{(T/2,T),p}. \tag{D.152}$$

Finally, we define $\overline{h_2}$ thanks to (D.150) and (D.152)

$$\overline{h_2} := \partial_t \overline{u_2} - \Delta \overline{u_2} - f_1(\overline{u_2}, \overline{v_2}) \in \cap_{p\in[2,+\infty)} L^p((T/2,T)\times\Omega)., \tag{D.153}$$

which is supported on $(T/2,T)\times\omega$ by (D.142) and (D.14) (note that $f_1(0,.) = 0$). This ends the proof of Proposition D.5.3. □

D.6 Some generalizations of the global null-controllability for "odd power systems"

In this section, we generalize Theorem D.2.5 to other parabolic systems. We omit the proofs because in each case, it is a slight adaptation of the strategy of Annexe D.3.

D.6.1 Linear parabolic operators

We present a natural generalization of the global null-controllability of (Odd) to more general *linear* parabolic operators than $\partial_t - \Delta$.

Proposition D.6.1. Let $k \in \mathbb{N}^*$, $(d_1, d_2) \in (0,+\infty)^2$, $(b_1, b_2) \in (L^\infty(Q_T)^N)^2$, $(a_1, a_2) \in L^\infty(Q_T)^2$. Then,

$$\begin{cases} \partial_t u - d_1\Delta u + b_1.\nabla u + a_1 u = h1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t v - d_2\Delta v + b_2.\nabla v + a_2 v = u^{2k+1} & \text{in } (0,T)\times\Omega, \\ u, v = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,.) = (u_0, v_0) & \text{in } \Omega, \end{cases}$$

is globally null-controllable.

The proof is based on a Carleman estimate different from the one in Proposition D.4.3 which can be found in [FCG06, Lemma 2.1].

D.6.2 Global null-controllability result for particular superlinearities

We state a global null-controllability result linked with the global null-controllability of the *semilinear heat equation*.

Proposition D.6.2. Let $k \in \mathbb{N}^*$, $f \in C^\infty(\mathbb{R};\mathbb{R})$ such that

$$f(0) = 0 \text{ and } \frac{f(s)}{s\log^{3/2}(1+|s|)} \to 0 \text{ when } |s| \to +\infty.$$

Then,

$$\begin{cases} \partial_t u - \Delta u = f(u) + h1_\omega & \text{in } (0,T)\times\Omega,\\ \partial_t v - \Delta v = u^{2k+1} & \text{in } (0,T)\times\Omega,\\ u, v = 0 & \text{on } (0,T)\times\partial\Omega,\\ (u,v)(0,.) = (u_0,v_0) & \text{in } \Omega, \end{cases}$$

is globally null-controllable.

The proof is based on the global null-controllability of the semilinear heat equation with superlinearity as the function f. Particularly, we can see [FCZ00, Theorem 1.2] proved by Enrique Fernandez-Cara and Enrique Zuazua or [FCG06, Theorem 1.7].

D.6.3 Global null-controllability for all "power systems"

Let $n \in \mathbb{N}^*$. We have seen that (Power), with n an even integer, is not (globally) null-controllable by the maximum principle (see Proposition D.1.9) but (Power), with n an odd integer, is globally null-controllable (see Theorem D.2.5). In this section, we consider the following system :

$$\begin{cases} \partial_t u - \Delta u = h1_\omega & \text{in } (0,T)\times\Omega,\\ \partial_t v - \Delta v = |u|^{n-1}u & \text{in } (0,T)\times\Omega,\\ u, v = 0 & \text{on } (0,T)\times\partial\Omega,\\ (u,v)(0,.) = (u_0,v_0) & \text{in } \Omega. \end{cases} \tag{PowerO}$$

Definition D.6.3. The mapping $\Phi_n : x \in \mathbb{R} \mapsto |x|^{n-1}x \in \mathbb{R}$ is one-to-one. We note its inverse function Φ_n^{-1}.

Remark D.6.4. For n an even integer, $\Phi_n(u) = u^n$ if $u \geq 0$ and $\Phi_n(u) = -u^n$ if $u < 0$. Whereas for n an odd power, $\Phi_n(u) = u^n$ for every $u \in \mathbb{R}$.

We have a generalization of Theorem D.2.5.

Theorem D.6.5. The system (PowerO) is globally null-controllable (in the sense of Definition D.1.4).

More precisely, there exists $(C_p)_{p\in[2,+\infty)} \in (0,\infty)^{[2,+\infty)}$ such that for every initial data $(u_0,v_0) \in L^\infty(\Omega)^2$, there exists a control $h \in \bigcap_{p\in[2,+\infty)} L^p(Q_T)$ satisfying

$$\forall p \in [2,+\infty),\ \|h\|_{L^p(Q_T)} \leq C_p\left(\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)}^{1/n}\right), \tag{D.154}$$

and the solution (u,v) of (PowerO) verifies

$$(u,v)(T,.) = (0,0).$$

The strategy of proof of Theorem D.6.5 is the same as for Theorem D.2.5. It is based on the following *key* result.

Proposition D.6.6. For every $\tau > 0$, there exists $C_\tau > 0$ such that for every $y_0 \in L^\infty(\Omega)$, there exists a control $h_\tau \in L^\infty(Q_\tau)$ which verifies

$$\begin{aligned}
&\left\|\Phi_n^{-1}(h_\tau)\right\|_{L^\infty(Q_\tau)} \leq C_\tau \left\|y_0\right\|_{L^\infty(\Omega)}^{1/n},\\
&\Phi_n^{-1}(h_\tau) \in \bigcap_{p\in[2,+\infty)} X_{\tau,p},\\
&\forall p \in [2,+\infty),\ \exists C_{\tau,p} > 0,\ \left\|\Phi_n^{-1}(h_\tau)\right\|_{X_{\tau,p}} \leq C_{\tau,p} \left\|y_0\right\|_{L^\infty(\Omega)}^{1/n},\\
&h_\tau(0,.) = h_\tau(\tau,.) = 0,\\
&\forall t \in [0,\tau],\ supp(h_\tau(t,.)) \subset\subset \omega,
\end{aligned}$$

such that the solution $y \in L^\infty(Q_\tau)$ of (D.19) satisfies $y(\tau,.) = 0$.

The proof of Proposition D.6.6 is a slight adaptation of Annexe D.4. First, we get a Carleman estimate in L^{n+1} (see Theorem D.4.4). Secondly, we use a penalized duality method in $L^{(n+1)/n}$ as in Annexe D.4.2 taking into account that the antiderivative of the continuous mapping Φ_n^{-1} (see Definition D.6.3) is the strictly convex function

$$x \in \mathbb{R} \mapsto \frac{n}{n+1}\left|x\right|^{\frac{n+1}{n}} := \begin{cases} \frac{n}{n+1}\exp(\frac{n+1}{n}\log(|x|)) & \text{if } x \neq 0,\\ 0 & \text{if } x = 0.\end{cases}$$

D.6.4 Global null-controllability for "even power systems" in $\mathbb{C}$

Let $k \in \mathbb{N}^*$. We have seen in Proposition D.1.9 that global null-controllability does not hold for

$$\begin{cases} \partial_t u - \Delta u = h 1_\omega & \text{in } (0,T)\times\Omega,\\ \partial_t v - \Delta v = u^{2k} & \text{in } (0,T)\times\Omega,\\ u, v = 0 & \text{on } (0,T)\times\partial\Omega,\\ (u,v)(0,.) = (u_0,v_0) & \text{in } \Omega. \end{cases} \tag{Even}$$

A natural question, asked by Luc Robbiano, is : what happens if we consider *complex-valued* functions ? A positive answer i.e. a global null-controllability result for (Even) with $k = 1$ is given in [CGR10] (see [CGR10, Theorem 3]). Here, we want to generalize this result for every $k \in \mathbb{N}^*$. We have the following result.

Theorem D.6.7. The system (Even) is globally null-controllable (in the sense of Definition D.1.4).

More precisely, there exists $(C_p)_{p\in[2,+\infty)} \in (0,\infty)^{[2,+\infty)}$ such that for every initial data $(u_0,v_0) \in L^\infty(\Omega)^2$, there exists a control $h \in \bigcap_{p\in[2,+\infty)} L^p(Q_T;\mathbb{C})$ satisfying

$$\forall p \in [2,+\infty),\ \left\|h\right\|_{L^p(Q_T;\mathbb{C})} \leq C_p \left(\left\|u_0\right\|_{L^\infty(\Omega)} + \left\|v_0\right\|_{L^\infty(\Omega)}^{1/(2k)}\right), \tag{D.155}$$

and the solution $(u,v) \in L^\infty(Q_T,\mathbb{C})^2$ of (Even) verifies

$$(u,v)(T,.) = (0,0).$$

The strategy of proof of Theorem D.6.7 is the same as for Theorem D.2.5 (see Annexe D.3). The first step of the proof i.e. Annexe D.3.1 does not change but we have to modify some arguments given in Annexe D.3.2.

Let us fix $(u_0, v_0) \in L^\infty(\Omega)^2$ until the end of the section.

D.6.4.1 First step of the proof : steer u to 0

First, we find a control of (Even) which steers u to 0 in time $T/2$ (see the proof of Proposition D.3.1).

Proposition D.6.8. There exists $h_1 \in L^\infty((0,T/2) \times \Omega; \mathbb{R})$ satisfying

$$\|h_1\|_{L^\infty((0,T/2)\times\Omega;\mathbb{R})} \leq C \|u_0\|_{L^\infty(\Omega)}, \tag{D.156}$$

such that the solution $(u_1, v_1) \in L^\infty((0,T/2) \times \Omega; \mathbb{R})^2$ of

$$\begin{cases} \partial_t u_1 - \Delta u_1 = h_1 1_\omega & \text{in } (0,T/2) \times \Omega, \\ \partial_t v_1 - \Delta v_1 = u_1^{2k} & \text{in } (0,T/2) \times \Omega, \\ u_1, v_1 = 0 & \text{on } (0,T/2) \times \partial\Omega, \\ (u_1, v_1)(0,.) = (u_0, v_0) & \text{in } \Omega, \end{cases}$$

satisfies $u_1(T/2,.) = 0$. Moreover, we have

$$\|v_1(T/2,.)\|_{L^\infty(\Omega)} \leq C \left(\|u_0\|^{2k}_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)} \right). \tag{D.157}$$

D.6.4.2 Second step of the proof : steer v to 0

The aim of this part is to find a *complex* control of (Even) which steers v to 0 (and u from 0 to 0) in time T.

Proposition D.6.9. Let $((u_1, v_1), h_1)$ as in Proposition D.6.8.

There exists a control $h_2 \in \bigcap_{p\in[2,+\infty)} L^p((T/2,T) \times \Omega; \mathbb{C})$ satisfying

$$\forall p \in [2,+\infty),\ \|h_2\|_{L^p((T/2,T)\times\Omega);\mathbb{C})} \leq C_p \left(\|u_0\|_{L^\infty(\Omega)} + \|v_0\|^{1/(2k)}_{L^\infty(\Omega)} \right). \tag{D.158}$$

such that the solution $(u_2, v_2) \in L^\infty((T/2,T) \times \Omega; \mathbb{C})^2$ of

$$\begin{cases} \partial_t u_2 - \Delta u_2 = h_2 1_\omega & \text{in } (T/2,T) \times \Omega, \\ \partial_t v_2 - \Delta v_2 = u_2^{2k} & \text{in } (T/2,T) \times \Omega, \\ u_2, v_2 = 0 & \text{on } (T/2,T) \times \partial\Omega, \\ (u_2, v_2)(T/2,.) = (0, v_1(T/2,.)) & \text{in } \Omega, \end{cases}$$

satisfies $(u_2, v_2)(T,.) = (0,0)$.

Our approach consists in looking at the second equation of (Odd) like a controlled heat equation where the *state* is $v(t,.)$ and the *control input* is $u^{2k}(t,.)$. Here, **the question consists in proving that the heat equation is null-controllable with a localized control which *is as an even power* of a regular** *complex* function.

Now, we can prove Proposition D.6.9.

Proof. We apply Proposition D.6.6 with $n = 4k$, $(0, \tau) \leftarrow (T/2, T)$, $y_0 \leftarrow v_1(T/2, .) \in L^\infty(\Omega)$. Then, there exists a control $H \in L^\infty((T/2, T) \times \Omega)$ such that

$$\Phi_{4k}^{-1}(H) \in \bigcap_{p\in[2,+\infty)} X_{(T/2,T),p} \tag{D.159}$$

$$\forall p \in [2, +\infty), \ \left\|\Phi_{4k}^{-1}(H)\right\|_{X_{(T/2,T),p}} \leq C_p \left\|v_1(T/2, .)\right\|_{L^\infty(\Omega)}^{1/(4k)}, \tag{D.160}$$

$$H(T/2, .) = H(T, .) = 0, \tag{D.161}$$

$$\forall t \in [T/2, T], \ supp(H(t, .)) \subset\subset \omega, \tag{D.162}$$

and the solution v_2 of

$$\begin{cases} \partial_t v_2 - \Delta v_2 = H & \text{in } (T/2, T) \times \Omega, \\ v_2 = 0 & \text{on } (T/2, T) \times \partial\Omega, \\ v_2(T/2, .) = v_1(T/2, .) & \text{in } \Omega, \end{cases} \tag{D.163}$$

satisfies

$$v_2(T, .) = 0. \tag{D.164}$$

We introduce the notation

$$\alpha := e^{\frac{i\pi}{2k}} \in \mathbb{C}.$$

We take u_2, the *complex-valued function*, as

$$u_2 := \Big(((\Phi_{4k}^{-1}(H))^+ \Big)^2 + \alpha \Big(((\Phi_{4k}^{-1}(H))^- \Big)^2, \tag{D.165}$$

where the positive and negative parts of a real number x are defined as follows

$$x^+ := \max(x, 0), \qquad x^- := -\min(x, 0).$$

From Definition D.6.3 and (D.165), we verify that

$$u^{2k} = \Big(((\Phi_{4k}^{-1}(H))^+ \Big)^{4k} + \alpha^{2k} \Big(((\Phi_{4k}^{-1}(H))^- \Big)^{4k} = H^+ - H^- = H. \tag{D.166}$$

Moreover, we have

$$x \mapsto (x^+)^2 \in W_{loc}^{2,\infty}(\mathbb{R}), \qquad x \mapsto (x^-)^2 \in W_{loc}^{2,\infty}(\mathbb{R}). \tag{D.167}$$

From (D.159), (D.165) and (D.167), we have

$$u_2 \in \left(\bigcap_{p\in[2,+\infty)} X_{(T/2,T),p} \right) \cap L^\infty((T/2, T) \times \Omega; \mathbb{C}). \tag{D.168}$$

We have, from (D.161),

$$u_2(T/2, .) = u_2(T, .) = 0. \tag{D.169}$$

Then, we set, from (D.165), (D.168) and (D.162),

$$h_2 := \partial_t u_2 - \Delta u_2 \in \bigcap_{p\in[2,+\infty)} L^p((T/2,T)\times\Omega;\mathbb{C}), \tag{D.170}$$

which is supported in $(T/2,T)\times\omega$. From (D.160), (D.157) and (D.165), we have

$$\forall p \in [2,+\infty),\ \|h_2\|_{L^p((T/2,T)\times\Omega);\mathbb{C})} \leq C_p \left(\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)}^{1/(2k)}\right). \tag{D.171}$$

By using (D.162), (D.163), (D.164), (D.166), (D.168), (D.169), (D.170) and (D.171), we check
that $((u_2,v_2),h_2)$ satisfies Proposition D.6.9. □

D.7 Appendix

D.7.1 Proof of the uniqueness of the point 2. of Definition D.2.3

Proof. Let $(u_0,v_0)\in L^\infty(\Omega)^2$, $h\in L^2(Q_T)$.

Let $(u,v)\in (W_T\cap L^\infty(Q_T))^2$ and $(\widetilde{u},\widetilde{v})\in (W_T\cap L^\infty(Q_T))^2$ be two solutions of (NL). Then, the function $(\widehat{u},\widehat{v}) := (u-\widetilde{u}, v-\widetilde{v}) \in (W_T\cap L^\infty(Q_T))^2$ satisfies (in the weak sense)

$$\begin{cases} \partial_t \widehat{u} - \Delta \widehat{u} = f_1(u,v) - f_1(\widetilde{u},\widetilde{v}) & \text{in } (0,T)\times\Omega,\\ \partial_t \widehat{v} - \Delta \widehat{v} = f_2(u,v) - f_2(\widetilde{u},\widetilde{v}) & \text{in } (0,T)\times\Omega,\\ \widehat{u},\widehat{v} = 0 & \text{on } (0,T)\times\partial\Omega,\\ (\widehat{u},\widehat{v})(0,.) = (0,0) & \text{in } \Omega. \end{cases} \tag{D.172}$$

By taking $(w_1,w_2) := (\widehat{u},\widehat{v})$ in the variational formulation of (D.172) (see (D.9) and (D.10)) and by using the fact that the mapping $t\mapsto \left\|(\widehat{u}(t),\widehat{v}(t))^T\right\|^2_{L^2(\Omega)^2}$ is absolutely continuous (see [Eva10, Section 5.9.2, Theorem 3]) with for a.e. $0\leq t\leq T$,

$$\frac{d}{dt}\left\|(\widehat{u}(t),\widehat{v}(t))^T\right\|^2_{L^2(\Omega)^2} = 2\left(\left((\partial_t\widehat{u}(t),\widehat{u}(t)),(\partial_t\widehat{v}(t),\widehat{v}(t))\right)^T\right)_{(H^{-1}(\Omega)^2,H_0^1(\Omega)^2)},$$

we find that for a.e. $0\leq t\leq T$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|(\widehat{u}(t),\widehat{v}(t))^T\right\|^2_{L^2(\Omega)^2}\right) + \left\|(\nabla\widehat{u},\nabla\widehat{v})^T\right\|^2_{L^2(\Omega)^2}\\ &= \left(\left((f_1(u,v)-f_1(\widetilde{u},\widetilde{v}),\widehat{u}),(f_2(u,v)-f_2(\widetilde{u},\widetilde{v}),\widehat{v})\right)^T\right)_{(L^2(\Omega)^2,L^2(\Omega)^2)}.\end{aligned}$$

By using the facts that $(u,\widetilde{u})\in L^\infty(Q_T)^2$ and $(x,y)\mapsto f_1(x,y)$, $(x,y)\mapsto f_2(x,y)$ are locally Lipschitz on $\mathbb{R}^2$, we find the differential inequality

$$\frac{d}{dt}\left(\left\|(\widehat{u}(t),\widehat{v}(t))^T\right\|^2_{L^2(\Omega)^2}\right) \leq C\left(\left\|(\widehat{u}(t),\widehat{v}(t))^T\right\|^2_{L^2(\Omega)^2}\right) \text{ for a.e. } 0\leq t\leq T.$$

Gronwall lemma and the initial condition $(\widehat{u}(0),\widehat{v}(0)) = (0,0)$ (see (D.11)) prove
that $(\widehat{u}(t),\widehat{v}(t)) = (0,0)$. Consequently, $(u,v) = (\widetilde{u},\widetilde{v})$. □

D.7.2 Proof of the general case for Corollary D.4.7

Proof. Let $\varphi_T \in L^{2k+2}(\Omega)$ and $(\varphi_{T,n})_{n\in\mathbb{N}} \in (C_0^\infty(\Omega))^{\mathbb{N}}$ such that

$$\varphi_{T,n} \underset{n\to+\infty}{\to} \varphi_T \text{ in } L^{2k+2}(\Omega). \tag{D.173}$$

We denote by $(\varphi_n)_{n\in\mathbb{N}}$ the sequence of solutions of

$$\begin{cases} -\partial_t\varphi_n - \Delta\varphi_n = 0 & (0,T)\times\Omega,\\ \varphi_n = 0 & (0,T)\times\partial\Omega,\\ \varphi_n(T,.) = \varphi_{T,n} & \Omega. \end{cases} \tag{D.174}$$

The estimates (D.69) and (D.68) hold true for $(\varphi_n)_{n\in\mathbb{N}}$ by the **Step 1** of the proof of Corollary D.4.7. Moreover, from (D.173), (D.174), (D.37) and Proposition D.2.2 (particularly (D.8) for $p = 2k+2$), we have

$$\|\varphi_n - \varphi\|_{L^{2k+2}(Q_T)} \leq C\,\|\varphi_{T,n} - \varphi_T\|_{L^{2k+2}(\Omega)} \underset{n\to+\infty}{\to} 0, \tag{D.175}$$

where $\varphi \in L^{2k+2}(Q_T)$ is the solution of (D.37). By using

$$\chi^{2k+2}e^{-(k+1)s\rho(x)\eta(t)}(s\eta)^{3(k+1)} \in L^\infty(Q_T), \tag{D.176}$$

and (D.175), we get

$$\begin{aligned} &\int\int_{(0,T)\times\omega} \chi^{2k+2}e^{-(k+1)s\rho\eta}(s\eta)^{3(k+1)}|\varphi_n|^{2k+2}\\ &\underset{n\to+\infty}{\to} \int\int_{(0,T)\times\omega} \chi^{2k+2}e^{-(k+1)s\rho\eta}(s\eta)^{3(k+1)}|\varphi|^{2k+2}. \end{aligned} \tag{D.177}$$

From (D.69), (D.68) applied to $(\varphi_n)_{n\in\mathbb{N}}$ and (D.177), we deduce that $(\varphi_n(0,.))_{n\in\mathbb{N}}$, (respectively $(e^{-\rho s\eta/2}(s\eta)^{-m/2}\varphi_n)_{n\in\mathbb{N}}$, respectively $(e^{-\rho s\eta/2}(s\eta)^{-(m+2)/2}\nabla\varphi_n)_{n\in\mathbb{N}}$) is bounded in $L^{2k+2}(\Omega)$, (respectively $L^{2k+2}(Q_T)$, respectively $L^{2k+2}(Q_T)$) which is a Banach reflexive space. Then, up to a subsequence, we can assume that

$$\varphi_n(0,.) \underset{n\to+\infty}{\rightharpoonup} \varphi(0,.) \text{ in } L^{2k+2}(\Omega), \tag{D.178}$$

$$e^{-\rho s\eta/2}(s\eta)^{-m/2}\varphi_n \underset{n\to+\infty}{\rightharpoonup} e^{-\rho s\eta/2}(s\eta)^{-m/2}\varphi \text{ in } L^{2k+2}(Q_T), \tag{D.179}$$

$$e^{-\rho s\eta/2}(s\eta)^{-(m+2)/2}\nabla\varphi_n \underset{n\to+\infty}{\rightharpoonup} e^{-\rho s\eta/2}(s\eta)^{-(m+2)/2}\nabla\varphi \text{ in } L^{2k+2}(Q_T). \tag{D.180}$$

In particular, we have

$$\begin{aligned} \|\varphi(0,.)\|_{L^{2k+2}(\Omega)}^{2k+2} &\leq \liminf_{n\to+\infty} \|\varphi_n(0,.)\|_{L^{2k+2}(\Omega)}^{2k+2}\\ &\leq C_s \liminf_{n\to+\infty} \int\int_{(0,T)\times\omega} \chi^{2k+2}e^{-(k+1)s\rho(x)\eta(t)}(s\eta)^{3(k+1)}|\varphi_n|^{2k+2}dxdt\\ &\leq C_s \int\int_{(0,T)\times\omega} \chi^{2k+2}e^{-(k+1)s\rho(x)\eta(t)}(s\eta)^{3(k+1)}|\varphi|^{2k+2}dxdt, \end{aligned} \tag{D.181}$$

and

$$\begin{aligned}
&\int\int_{(0,T)\times\Omega} e^{-(k+1)s\rho\eta}((s\eta)^{-(k+1)m}|\varphi|^{2k+2} + (s\eta)^{-(k+1)(m+2)}|\nabla\varphi|^{2k+2}) \qquad \text{(D.182)}\\
&\leq \liminf_{n\to+\infty} \int\int_{(0,T)\times\Omega} e^{-(k+1)s\rho\eta}((s\eta)^{-(k+1)m}|\varphi_n|^{2k+2} + (s\eta)^{-(k+1)(m+2)}|\nabla\varphi_n|^{2k+2})\\
&\leq C\liminf_{n\to+\infty} \int\int_{(0,T)\times\omega} \chi^{2k+2} e^{-(k+1)s\rho\eta}(s\eta)^{3(k+1)}|\varphi_n|^{2k+2}\\
&\leq C\int\int_{(0,T)\times\omega} \chi^{2k+2} e^{-(k+1)s\rho\eta}(s\eta)^{3(k+1)}|\varphi|^{2k+2}.
\end{aligned}$$

The estimates (D.181) and (D.182) conclude the proof. □

Annexe E

Global null-controllability and nonnegative-controllability of slightly superlinear heat equations

Abstract : We consider the semilinear heat equation posed on a smooth bounded domain Ω of $\mathbb{R}^N$ with Dirichlet or Neumann boundary conditions. The control input is a source term localized in some arbitrary nonempty open subset ω of Ω. The goal of this paper is to prove the uniform large time global null-controllability for semilinearities $f(s) = \pm|s|\log^{\alpha}(2+|s|)$ where $\alpha \in [3/2, 2)$ which is the case left open by Enrique Fernandez-Cara and Enrique Zuazua in 2000. It is worth mentioning that the free solution (without control) can blow-up. First, we establish the small-time global *nonnegative-controllability* (respectively *nonpositive-controllability*) of the system, i.e., one can steer any initial data to a nonnegative (respectively nonpositive) state in arbitrary time. In particular, one can act locally thanks to the control term in order to prevent the blow-up from happening. The proof relies on precise observability estimates for the linear heat equation with a bounded potential $a(t,x)$. More precisely, we show that observability holds with a sharp constant of the order $\exp\left(C\left\|a\right\|_{\infty}^{1/2}\right)$ for *nonnegative* initial data. This inequality comes from a new L^1 Carleman estimate. A Kakutani-Leray-Schauder's fixed point argument enables to go back to the semilinear heat equation. Secondly, the uniform large time null-controllability result comes from three ingredients : the global nonnegative-controllability, a comparison principle between the free solution and the solution to the underlying ordinary differential equation which provides the convergence of the free solution toward 0 in $L^{\infty}(\Omega)$-norm, and the local null-controllability of the semilinear heat equation.

E.1 Introduction

Let $T > 0$, $N \in \mathbb{N}^*$, Ω be a bounded, connected, open subset of $\mathbb{R}^N$ of class C^2 and n be the outer unit normal vector to $\partial\Omega$. We consider the semilinear heat equation with

Neumann boundary conditions :

$$\begin{cases} \partial_t y - \Delta y + f(y) = h 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{E.1}$$

where $f \in C^1(\mathbb{R};\mathbb{R})$.

Remark E.1.1. All our results stay valid for Dirichlet boundary conditions (see Annexe E.7).

In (E.1), $y = y(t,.) : \Omega \to \mathbb{R}$ is the *state* to be controlled and $h = h(t,.) : \Omega \to \mathbb{R}$ is the *control input* supported in ω, a nonempty open subset of Ω.

We assume that f satisfies

$$f(0) = 0. \tag{E.2}$$

In this case, $y = 0$ solves (E.1) with $y_0 = 0$ and $h = 0$.

In the following, we will also assume that f satisfies the restrictive growth condition

$$\exists \alpha > 0, \ \frac{f(s)}{|s|\log^\alpha(1+|s|)} \to 0 \text{ as } |s| \to +\infty. \tag{E.3}$$

Under the hypothesis (E.3), blow-up may occur if $h = 0$ in (E.1). Take for example $f(s) = -|s|\log^\alpha(1+|s|)$ with $\alpha > 1$. The mathematical theory of blow-up for

$$\begin{cases} \partial_t y - \Delta y = |y|\log^\alpha(1+|y|) & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{E.4}$$

was established in [GV93] and [GV96]. It was shown that blow-up

— occurs *globally* in the whole domain Ω if $\alpha < 2$,

— is of *pointwise nature* if $\alpha > 2$,

— is '*regional*", i.e., it occurs in an open subset of Ω if $\alpha = 2$.

See [GV02, Section 2 and Section 5] for a survey on this problem.

The goal of this paper is to analyze the *null-controllability* properties of (E.1).

Let us define $Q_T := (0,T)\times\Omega$. We recall two classical definitions of null-controllability.

Definition E.1.2. Let $T > 0$. The system (E.1) is

— *globally null-controllable* in time T if for every $y_0 \in L^\infty(\Omega)$, there exists $h \in L^\infty(Q_T)$ such that the solution y of (E.1) satisfies $y(T,.) = 0$.

— *locally null-controllable* in time T if there exists $\delta_T > 0$ such that for every $y_0 \in L^\infty(\Omega)$ verifying $\|y_0\|_{L^\infty(\Omega)} \leq \delta_T$, there exists $h \in L^\infty(Q_T)$ such that the solution y of (E.1) satisfies $y(T,.) = 0$.

We have the following well-known local null-controllability result.

Theorem E.1.3. For every $T > 0$, (E.1) is locally null-controllable in time T.

The proof of Theorem E.1.3 is a consequence of the (global) null-controllability of the linear heat equation with a bounded potential (due to Andrei Fursikov and Oleg Imanuvilov, see [FI96] or [FCG06, Theorem 1.5]) and the *small* L^∞ *perturbations method* (see [AT02, Lemma 6] and [AKBD06], [Bar00], [LB19], [LCM+16], [WZ06] for other results in this direction).

The following global null-controllability (positive) result has been proved independently by Enrique Fernandez-Cara, Enrique Zuazua (see [FCZ00, Theorem 1.2]) and Viorel Barbu under a sign condition (see [Bar00, Theorem 2] or [Bar18, Theorem 3.6]) for Dirichlet boundary conditions. It has been extended to semilinearities which can depend on the gradient of the state and to Robin boundary conditions (then to Neumann boundary conditions) by Enrique Fernandez-Cara, Manuel Gonzalez-Burgos, Sergio Guerrero and Jean-Pierre Puel in [FCGBGP06a] (see also [DFCGBZ02] for the Dirichlet case).

Theorem E.1.4. [FCGBGP06a, Theorem 1]
We assume that (E.3) holds for $\alpha \leq 3/2$. Then, for every $T > 0$, (E.1) is globally null-controllable in time T.

Remark E.1.5. Historically, the first global null-controllability (positive) result for (E.1) with f satisfying (E.3) was proved by Enrique Fernandez-Cara in [FC97] for $\alpha \leq 1$ and for Dirichlet boundary conditions.

The following global null-controllability (negative) result has been proved by Enrique Fernandez-Cara, Enrique Zuazua (see [FCZ00]).

Theorem E.1.6. [FCZ00, Theorem 1.1]
We set $f(s) := \int_0^{|s|} \log^p(1+\sigma)d\sigma$ with $p > 2$ and we assume that $\Omega \setminus \overline{\omega} \neq \emptyset$. Then, for every $T > 0$, there exists an initial datum $y_0 \in L^\infty(\Omega)$ such that for every $h \in L^\infty(Q_T)$, the maximal solution y of (E.1) blows-up in time $T^* < T$.

Remark E.1.7. Such a function f does satisfy (E.3) for any $\alpha > p$ because $|f(s)| \sim |s|\log^p(1+|s|)$ as $|s| \to +\infty$. Then, Theorem E.1.6 shows that (E.1) can fail to be null-controllable for every $T > 0$ under the hypothesis (E.3) with $\alpha > 2$. Theorem E.1.6 comes from a localized estimate in $\Omega \setminus \overline{\omega}$ that shows that the control cannot compensate the blow-up phenomena occurring in $\Omega \setminus \overline{\omega}$ (see [FCZ00, Section 2]).

When the nonlinear term f is dissipative, i.e., $sf(s) \geq 0$ for every $s \in \mathbb{R}$, then blow-up cannot occur. Furthermore, such a nonlinearity produces energy decay for the uncontrolled equation, therefore naively one may be led to believe that it can help in steering the solution to zero in arbitrary short time. The results of Sebastian Anita and Daniel Tataru show that this is false, more precisely that for "strongly" superlinear f one needs a sufficiently large time in order to bring the solution to zero. An intuitive explanation for this is that the nonlinearity is also damping the effect of the control as it expands from the controlled region into the uncontrolled region (see [AT02]).

Theorem E.1.8. [AT02, Theorem 3]
We set $f(s) := s\log^p(1+|s|)$ with $p > 2$ and we assume that $\Omega \setminus \overline{\omega} \neq \emptyset$. Then, there exist $x_0 \in \Omega \setminus \overline{\omega}$, $T_0 \in (0,1)$ such that for every $T \in (0, T_0)$, there exists an initial datum $y_0 \in L^\infty(\Omega)$ such that for every $h \in L^\infty(Q_T)$, the solution y to (E.1) satisfies $y(T, x_0) < 0$.

Remark E.1.9. In particular, for such a f as in Theorem E.1.8, (E.1) is not globally null-controllable in small time T. Theorem E.1.8 is due to pointwise upper bounds on the solution y of (E.1) which are independent of the control h (see [AT02, Section 3]).

E.2 Main results

E.2.1 Small-time global nonnegative-controllability

We introduce a new concept of controllability.

Definition E.2.1. Let $T > 0$. The system (E.1) is *globally nonnegative-controllable* (respectively *globally nonpositive-controllable*) in time T if for every $y_0 \in L^\infty(\Omega)$, there exists $h \in L^\infty(Q_T)$ such that the solution y of (E.1) satisfies

$$y(T,.) \geq 0 \qquad \text{(respectively } y(T,.) \leq 0\text{).} \tag{E.5}$$

The first main result of this paper is a small-time global nonnegative-controllability result for (E.1).

Theorem E.2.2. We assume that (E.3) holds for $\alpha \leq 2$ and $f(s) \geq 0$ for $s \geq 0$ (respectively $f(s) \leq 0$ for $s \leq 0$). Then, for every $T > 0$, (E.1) is globally nonnegative-controllable (respectively globally nonpositive-controllable) in time T.

Remark E.2.3. Theorem E.2.2 is almost sharp because it does not hold for $\alpha > 2$ according to Theorem E.1.8. The case where $|f(s)| \sim |s|\log^2(1+|s|)$ as $|s| \to +\infty$ is open. However, by following the proof of Theorem E.2.2 (see Annexe E.5), we can prove that for a given $T > 0$, there exists a constant $\varepsilon = \varepsilon(\Omega, \omega, T)$ such that if $|f(s)| \leq \varepsilon \log^2(1+|s|)$ for $|s|$ large and $f(s) \geq 0$ for $s \geq 0$, then (E.1) is globally nonnegative-controllable in time T.

Remark E.2.4. Theorem E.2.2 does not treat the case $f(s) = -s\log^p(1+|s|)$ with $p < 2$ because of the sign condition.

E.2.2 Large time global null-controllability

The second main result of this paper is the following one.

Theorem E.2.5. We assume that (E.3) holds for $\alpha \leq 2$, $f(s) > 0$ for $s > 0$ or $f(s) < 0$ for $s < 0$ and $1/f \in L^1([1, +\infty))$. Then, there exists T sufficiently large such that (E.1) is globally null-controllable in time T.

Remark E.2.6. Theorem E.2.5 proves that Theorem E.1.6 is almost sharp. Indeed, let us take $f(s) = \int_0^{|s|} \log^p(1+\sigma)d\sigma$ with $p < 2$, then by Theorem E.2.5, there exists T sufficiently large such that (E.1) is globally null-controllable in time T. In particular, one can find a localized control which prevents the blow-up from happening. The case $f(s) = \int_0^{|s|} \log^2(1+\sigma)d\sigma$ is open.

Remark E.2.7. Theorem E.2.5 does not treat the case $f(s) = -s\log^p(1+|s|)$ with $p < 2$ because of the sign condition.

Remark E.2.8. The small-time global null-controllability of (E.1) remains open when (E.3) holds for $3/2 < \alpha \leq 2$.

E.2.3 Proof strategy of the small-time global nonnegative-controllability

We will only prove the global nonnegative-controllability result. The nonpositive-controllability result is an easy adaptation.

The proof strategy of Theorem E.2.2 will follow Enrique Fernandez-Cara and Enrique Zuazua's proof of Theorem E.1.4 (see [FCZ00]).

The starting point is to get some precise *observability estimates* for the linear heat equation with a bounded potential $a(t,x)$ for *nonnegative initial data*. More precisely, we show that observability holds with a sharp constant of the order $\exp\left(C\left\|a\right\|_\infty^{1/2}\right)$ for nonnegative initial data (see Theorem E.4.4 below). This is done thanks to a *new Carleman estimate in* L^1 (see Theorem E.4.9 below). This leads to a nonnegative-controllability result in L^∞ in the linear case with an estimate of the *control cost* of the order $\exp\left(C\left\|a\right\|_\infty^{1/2}\right)$ which is the key point of the proof (see Theorem E.4.1 below).

We end the proof of Theorem E.2.2 by a Kakutani-Leray-Schauder's fixed-point strategy. The idea of taking short control times to avoid blow-up phenomena is the same as in [FCZ00] and references therein. More precisely, the construction of the control follows two steps. The first step consists in steering the solution y of (E.1) to $y(T^*,.) \geq 0$ in time $T^* \leq T$ with an appropriate choice of the control. Then, the two conditions : $f(0) = 0$ and the dissipativity of f in $\mathbb{R}^+$ imply that the free solution y of (E.1) (with $h = 0$) defined in (T^*, T) stays nonnegative and bounded by using a comparison principle (see Annexe E.5).

E.2.4 Proof strategy of the large time global null-controllability

We will only treat the case where $f(s) > 0$ for $s > 0$. The other case, i.e., $f(s) < 0$ for $s < 0$ is an easy adaptation.

The proof strategy of Theorem E.2.5 is divided into three steps.

First, for every initial data $y_0 \in L^\infty(\Omega)$, one can steer the solution y of (E.1) in time $T_1 := 1$ (for instance) to a nonnegative state by using Theorem E.2.2.

Secondly, we let evolve the system without control and we remark that

$$\forall (t,x) \in [T_1, +\infty) \times \Omega,\ 0 \leq y(t,x) \leq G(t),$$

with G independent of $\|y(T_1,.)\|_{L^\infty(\Omega)}$ and $G(t) \to 0$ when $t \to +\infty$. This kind of argument has already been used by Jean-Michel Coron in the context of the Burgers equation (see [Cor07b, Theorem 8]).

Finally, by using the second step, for T_2 sufficiently large, $y(T_2,.)$ belongs to a small ball of $L^\infty(\Omega)$ centered at 0, where the local null-controllability holds (see Theorem E.1.3). Then, one can steer $y(T_2,.)$ to 0 with an appropriate choice of the control.

E.3 Parabolic equations : well-posedness and regularity

The goal of this section is to state well-posedness results, dissipativity in time in L^p-norm, maximum principle and L^p-L^q estimates for linear parabolic equations. We also give the definition of a solution to the semilinear heat equation (E.1). The references of these results only treat the case of Dirichlet boundary conditions but the proofs can be easily adapted to Neumann boundary conditions.

E.3.1 Well-posedness

We introduce the functional space

$$W_T := L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))'), \tag{E.6}$$

which satisfies the following embedding (see [Eva10, Section 5.9.2, Theorem 3])

$$W_T \hookrightarrow C([0,T];L^2(\Omega)). \tag{E.7}$$

E.3.1.1 Linear parabolic equations

Definition E.3.1. Let $a \in L^\infty(Q_T)$, $F \in L^2(Q_T)$ and $y_0 \in L^2(\Omega)$. A function $y \in W_T$ is a solution to

$$\left\{\begin{array}{ll} \partial_t y - \Delta y + a(t,x)y = F & \text{in } (0,T)\times\Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{array}\right. \tag{E.8}$$

if for every $w \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T (\partial_t y, w)_{((H^1(\Omega))', H^1(\Omega))} + \int_{Q_T} \nabla y.\nabla w + \int_{Q_T} ayw = \int_{Q_T} Fw, \tag{E.9}$$

and

$$y(0,.) = y_0 \text{ in } L^2(\Omega). \tag{E.10}$$

The following well-posedness result in L^2 holds for linear parabolic equations.

Proposition E.3.2. Let $a \in L^\infty(Q_T)$, $F \in L^2(Q_T)$ and $y_0 \in L^2(\Omega)$. The Cauchy problem (E.8) admits a unique weak solution $y \in W_T$. Moreover, there exists $C = C(\Omega) > 0$ such that

$$\|y\|_{W_T} \leq C \exp\left(CT\|a\|_{L^\infty(Q_T)}\right)\left(\|y_0\|_{L^2(\Omega)} + \|F\|_{L^2(Q_T)}\right). \tag{E.11}$$

The proof of Proposition E.3.2 is based on *Galerkin approximations, energy estimates and Gronwall's argument* (see [Eva10, Section 7.1.2]). Note that Proposition E.3.2, i.e., the solvability of (E.8) remains true under more general conditions on the potential a (see [LSU68, Chapter 3, Paragraph 4, Theorem 4.1]).

We also have the following classical L^∞-estimate for (E.8).

Proposition E.3.3. Let $a \in L^\infty(Q_T)$, $F \in L^\infty(Q_T)$ and $y_0 \in L^\infty(\Omega)$. Then the solution y of (E.8) belongs to $L^\infty(Q_T)$ and there exists $C = C(\Omega) > 0$ such that

$$\|y\|_{L^\infty(Q_T)} \leq C \exp\left(CT \|a\|_{L^\infty(Q_T)}\right) \left(\|y_0\|_{L^\infty(\Omega)} + \|F\|_{L^\infty(Q_T)}\right). \tag{E.12}$$

The proof of Proposition E.3.3 is based on *Stampacchia's method* (see the proof of [LSU68, Chapter 3, Paragraph 7, Theorem 7.1] or [WYW06, Proof of Proposition 4.2.1]).

Let us also mention the dissipativity in time of the L^p-norm of the heat equation with a bounded potential.

Proposition E.3.4. Let $a \in L^\infty(Q_T)$, $y_0 \in L^2(\Omega)$ and $t_1 < t_2 \in [0,T]$. Then, there exists $C = C(\Omega) > 0$ such that the solution $y \in W_T$ of (E.8) with $F = 0$, satisfies for every $p \in [1,2]$,

$$\|y(t_2,.)\|_{L^p(\Omega)} \leq C \exp\left(CT \|a\|_{L^\infty(Q_T)}\right) \|y(t_1,.)\|_{L^p(\Omega)}. \tag{E.13}$$

The proof of Proposition E.3.4 is based on the application of the variational formulation (E.9) with a cut-off of $w = |y|^{p-2}y$ and a Gronwall's argument.

E.3.1.2 Nonlinear parabolic equations

We give the definition of a solution of (E.1).

Definition E.3.5. Let $y_0 \in L^\infty(\Omega)$, $h \in L^\infty(Q_T)$. A function $y \in W_T \cap L^\infty(Q_T)$ is the solution of (E.1) if for every $w \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T (\partial_t y, w)_{((H^1(\Omega))', H^1(\Omega))} + \int_{Q_T} \nabla y . \nabla w + \int_{Q_T} f(y) w = \int_{Q_T} h 1_\omega w, \tag{E.14}$$

and

$$y(0,.) = y_0 \text{ in } L^\infty(\Omega). \tag{E.15}$$

The uniqueness of a solution to (E.1) is an easy consequence of the fact that f is locally Lipschitz because $f \in C^1(\mathbb{R};\mathbb{R})$.

E.3.2 Maximum principle

We state the maximum principle for the heat equation.

Proposition E.3.6. Let $a \in L^\infty(Q_T)$, $F \leq G \in L^2(Q_T)$ and $y_0 \leq z_0 \in L^2(\Omega)$. Let y and z be the solutions to

$$\begin{cases} \partial_t y - \Delta y + a(t,x)y = F, \\ \frac{\partial y}{\partial n} = 0, \\ y(0,.) = y_0, \end{cases} \quad \begin{cases} \partial_t z - \Delta z + a(t,x)z = G & \text{in } (0,T)\times\Omega, \\ \frac{\partial z}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0,.) = z_0 & \text{in } \Omega. \end{cases} \tag{E.16}$$

Then, we have the comparison principle

$$\forall t \in [0,T],\ \text{a.e. } x \in \Omega,\ y(t,x) \leq z(t,x). \tag{E.17}$$

The proof of Proposition E.3.6 is based on the comparison principle for smooth solutions of (E.16) (see [WYW06, Theorem 8.1.6]) and a regularization argument.

We state a comparison principle for the semilinear heat equation (E.1) without control h.

Proposition E.3.7. Let $y_0 \in L^\infty(\Omega)$, $h = 0$. We assume that there exist a subsolution $\underline{y}$ and a supersolution $\overline{y}$ in $L^\infty(Q_T)$ of (E.1), i.e., $\underline{y}$ (respectively $\overline{y}$) satisfies (E.14), (E.15) replacing the equality $=$ by the inequality $\leq$ (respectively by the inequality $\geq$). Moreover, we suppose that $\underline{y}$ and $\overline{y}$ are ordered in the following sense

$$\forall t \in [0,T],\ \text{a.e. } x \in \Omega,\ \underline{y}(t,x) \leq \overline{y}(t,x).$$

Then, there exists a (unique) solution y of (E.1). Moreover, y satisfies the comparison principle

$$\forall t \in [0,T],\ \text{a.e. } x \in \Omega,\ \underline{y}(t,x) \leq y(t,x) \leq \overline{y}(t,x). \tag{E.18}$$

For the proof of Proposition E.3.7, see [WYW06, Corollary 12.1.1].

E.3.3 L^p-L^q estimates

We have the well-known regularizing effect of the heat semigroup.

Proposition E.3.8. [CH98, Proposition 3.5.7]
Let $1 \leq q \leq p \leq +\infty$, $y_0 \in L^2(\Omega)$ and y be the solution to (E.8) with $(a, F) = (0,0)$. Then, there exists $C = C(\Omega, p, q) > 0$ such that for every $t_1 < t_2 \in (0,T)$, we have

$$\|y(t_2,.)\|_{L^p(\Omega)} \leq C(t_2 - t_1)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|y(t_1,.)\|_{L^q(\Omega)} \tag{E.19}$$

E.4 Global nonnegative-controllability of the linear heat equation with a bounded potential

E.4.1 Statement of the result

Let $a \in L^\infty(Q_T)$. We consider the heat equation with a bounded potential

$$\begin{cases} \partial_t y - \Delta y + a(t,x)y = h1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{E.20}$$

and the following adjoint equation

$$\begin{cases} -\partial_t q - \Delta q + a(t,x)q = 0 & \text{in } (0,T)\times\Omega, \\ \frac{\partial q}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ q(T,.) = q_T & \text{in } \Omega. \end{cases} \tag{E.21}$$

The goal of this section is to prove the following theorem.

Theorem E.4.1. For every $T > 0$, (E.20) is globally nonnegative-controllable in time T. More precisely, for every $T > 0$, there exists $C = C(\Omega, \omega, T, a) > 0$, with

$$C(\Omega,\omega,T,a) = \exp\left(C(\Omega,\omega)\left(1 + \frac{1}{T} + T\,\|a\|_{L^\infty(Q_T)} + \|a\|_{L^\infty(Q_T)}^{1/2}\right)\right) \tag{E.22}$$

such that for every $y_0 \in L^2(\Omega)$, there exists $h \in L^\infty(Q_T)$ such that

$$\|h\|_{L^\infty(Q_T)} \le C(\Omega,\omega,T,a)\,\|y_0\|_{L^2(\Omega)}, \tag{E.23}$$

and

$$y(T,.) \ge 0. \tag{E.24}$$

Remark E.4.2. Actually, by looking carefully at the proof of Theorem E.4.1 (see Annexe E.4.5 below), we can see that the control h in Theorem E.4.1 can be chosen constant in the time and the space variables.

Remark E.4.3. It is well-known that (E.20) is globally nonnegative-controllable in time T because it is globally null-controllable in time T (see [FCGBGP06b, Theorem 2]) but the most interesting point is the *cost of nonnegative-controllability* given in Theorem E.4.1. In particular, the exponent $1/2$ of the term $\|a\|_{L^\infty(Q_T)}^{1/2}$ will be the key point to prove Theorem E.2.2 (see Annexe E.5).

E.4.2 A precise L^2-L^1 observability inequality for the linear heat equation with bounded potential and nonnegative initial data

The proof of Theorem E.4.1 is a consequence of this kind of observability inequality.

Theorem E.4.4. For every $T > 0$, there exists $C = C(\Omega, \omega, T, a) > 0$ of the form (E.22) such that for every $q_T \in L^2(\Omega; \mathbb{R}^+)$, the solution q to (E.21) satisfies

$$\|q(0,.)\|_{L^2(\Omega)}^2 \le C\left(\int_0^T\int_\omega q\,dxdt\right)^2. \tag{E.25}$$

An immediate corollary of Theorem E.4.4 is this observability inequality L^2-L^2 that we state to discuss it below, but that will not be used in the present article.

Corollary E.4.5. For every $T > 0$, there exists $C = C(\Omega, \omega, T, a) > 0$ of the form (E.22) such that for every $q_T \in L^2(\Omega; \mathbb{R}^+)$ the solution q to (E.21) satisfies

$$\|q(0,.)\|_{L^2(\Omega)}^2 \le C\left(\int_0^T\int_\omega q^2\,dxdt\right). \tag{E.26}$$

It is well-known that null-controllability in L^2 is equivalent to an observability inequality in L^2 for every $q_T \in L^2(\Omega;\mathbb{R})$ (see [Cor07a, Theorem 2.44]). The *main idea* behind Corollary E.4.5 is the fact that nonnegative-controllability in L^2 is a consequence of an observability inequality in L^2 for every $q_T \in L^2(\Omega;\mathbb{R}^+)$ (see Annexe E.4.5).

Remark E.4.6. It is interesting to mention that (E.26) holds with C of the form

$$C(\Omega,\omega,T,a) = \exp\left(C(\Omega,\omega)\left(1+\frac{1}{T}+T\left\|a\right\|_{L^\infty(Q_T)}+\left\|a\right\|_{L^\infty(Q_T)}^{2/3}\right)\right) \tag{E.27}$$

for every $q_T \in L^2(\Omega;\mathbb{R})$ (see [FCGBGP06b, Theorem 2]). The exponent $2/3$ of the term $\|a\|_{L^\infty(Q_T)}^{2/3}$ is the key point to prove Theorem E.1.4. Note that the optimality of the exponent $2/3$ has been proved by Thomas Duyckaerts, Xu Zhang and Enrique Zuazua in the context of parabolic systems in even space dimensions $N \geq 2$ and with Dirichlet boundary conditions (see [DZZ08, Theorem 1.1] and also [Zua07, Theorem 5.2] for the main arguments of the proof). Corollary E.4.5 shows that we can actually decrease the exponent $2/3$ to the exponent $1/2$ for nonnegative initial data. In some sense, we can make the connection with the recent preprint of Camille Laurent and Matthieu Léautaud (see [LL18]). Indeed, the article [LL18] is concerned with the short-time observability constant of the heat equation. This constant is of the form $\exp(\mathcal{K}/T)$, where $\mathcal{K}$ depends only on the the geometry of Ω and ω. Luc Miller conjectured that $\mathcal{K}$ is (universally) proportional to the square of the maximal distance from ω to a point of Ω (see [Mil04]). In [LL18], the authors disprove the conjecture and show that the conjecture holds true for nonnegative initial data by using Li-Yau estimates (see [LY86]).

Remark E.4.7. In the context of the wave equation in one space dimension, the (optimal) constant of observability inequality for the linear wave equation with a bounded potential is actually $\exp\left(C\left(1+\|a\|_{L^\infty(Q_T)}^{1/2}\right)\right)$ (see [Zua93, Theorem 4]) which leads to the exact controllability of the semilinear wave equation in large time for semilinearities satisfying (E.3) with $\alpha < 2$ (see [Zua93, Theorem 1] and also [BM04, Problem 5.5] for the presentation of the related open problem in the multidimensional case). Roughly speaking, as an ordinary differential argument would indicate, this constant of observability inequality is very natural because the wave operator is of order two in the time and the space variables. Then, by analogy and by taking into account that the heat operator is of order one in the time variable and of order two in the space variable, one could rather expect a constant of obervability inequality of the order $\exp\left(C\left\|a\right\|_{L^\infty(Q_T)}\right)$ or $\exp\left(C\left\|a\right\|_{L^\infty(Q_T)}^{1/2}\right)$ which seem to be more intuitive than the term $\exp\left(C\left\|a\right\|_{L^\infty(Q_T)}^{2/3}\right)$.

E.4.3 A new L^1 Carleman estimate

The goal of this section is to establish a L^1 Carleman estimate for nonnegative initial data (see Theorem E.4.9 below). First, we introduce some classical weight functions for proving Carleman inequalities.

Lemma E.4.8. Let $\omega_0 \subset\subset \omega$ be a nonempty open subset. Then there exists $\eta^0 \in C^2(\overline{\Omega})$ such that $\eta^0 > 0$ in Ω, $\eta^0 = 0$ in $\partial\Omega$, and $|\nabla\eta^0| > 0$ in $\overline{\Omega \setminus \omega_0}$.

A proof of this lemma can be found in [Cor07a, Lemma 2.68].

Let ω_0 be a nonempty open set satisfying $\omega_0 \subset\subset \omega$ and let us set

$$\alpha(t,x) := \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{t(T-t)}, \tag{E.28}$$

$$\xi(t,x) := \frac{e^{\lambda\eta^0(x)}}{t(T-t)}, \tag{E.29}$$

for $(t,x) \in Q_T$, where η^0 is the function provided by Lemma E.4.8 for this ω_0 and $\lambda \geq 1$ is a parameter.

We have the following new L^1 Carleman estimate.

Theorem E.4.9. There exist two constants $C := C(\Omega,\omega) > 0$ and $C_1 := C_1(\Omega,\omega) > 0$, such that,

$$\forall \lambda \geq 1, \qquad \forall s \geq s_1(\lambda) := C(\Omega,\omega)e^{4\lambda\|\eta^0\|_\infty}\left(T + T^2 + T^2\,\|a\|_{L^\infty(Q_T)}^{1/2}\right), \tag{E.30}$$

for every $q_T \in L^2(\Omega;\mathbb{R}^+)$, the nonnegative solution q of (E.21) satisfies

$$\int_{Q_T} e^{-s\alpha}\xi^2 q\,dxdt \leq C_1 \int_{(0,T)\times\omega} e^{-s\alpha}\xi^2 q\,dxdt. \tag{E.31}$$

Proof. Unless otherwise specified, we denote by C various positive constants varying from line to line which may depend on Ω, ω but independent of the parameters λ and s.

We introduce other weights which are similar to α and ξ

$$\widetilde{\alpha}(t,x) := \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{-\lambda\eta^0(x)}}{t(T-t)}, \tag{E.32}$$

$$\widetilde{\xi}(t,x) := \frac{e^{-\lambda\eta^0(x)}}{t(T-t)}. \tag{E.33}$$

The following estimates

$$\begin{array}{llll} |\partial_i\alpha| = |-\partial_i\xi| & \leq C\lambda\xi, & |\partial_i\widetilde{\alpha}| = |-\partial_i\widetilde{\xi}| & \leq C\lambda\widetilde{\xi}, \\ |\partial_t\alpha| & \leq 2T\xi^2 e^{2\lambda\|\eta^0\|_\infty}, & |\partial_t\widetilde{\alpha}| & \leq 2T\widetilde{\xi}^2 e^{4\lambda\|\eta^0\|_\infty}, \\ (T/2)^2\xi & \geq 1, & (T/2)^2\widetilde{\xi} & \geq e^{-\lambda\|\eta^0\|_\infty}, \end{array} \tag{E.34}$$

will be very useful for the proof.

Let $q_T \in C_c^\infty(\Omega;\mathbb{R}^+)$. The general case comes from an easy density argument by using the fact that $C_c^\infty(\Omega;\mathbb{R}^+)$ is dense in $L^2(\Omega;\mathbb{R}^+)$ for the $L^2(\Omega;\mathbb{R})$ topology.

The solution q of (E.21) is nonnegative by applying the maximum principle given in

Proposition E.3.6 with $y = 0$ and $z(t, x) = q(t - T, x)$. Actually, if $q_T \neq 0$, the strong maximum principle gives that q is strictly positive in Q_T (see [Eva10, Chapter 7, Section 7.1, Theorem 12]).

We define

$$\psi := e^{-s\alpha} q \qquad \text{and} \qquad \widetilde{\psi} := e^{-s\widetilde{\alpha}} q.$$

The proof is divided into five steps :

- **Step 1 :** We integrate over $(0, T) \times \Omega$ an identity satisfied by ψ.
- **Step 2 :** We get an estimate which looks like (E.31) up to some boundary terms.
- **Step 3 :** We repeat the step 1 for $\widetilde{\psi}$.
- **Step 4 :** We repeat the step 2 for $\widetilde{\psi}$.
- **Step 5 :** We sum the estimates of the step 2 and the step 4 to get rid of the boundary terms.

Remark E.4.10. The "trick" of the proof to get rid of the boundary terms is inspired by the proof of the usual L^2 Carleman estimate for Neumann boundary conditions due to Andrei Fursikov and Oleg Imanuvilov (see [FI96, Chapter 1] and also [FCGBGP06b, Appendix]).

Step 1 : An identity satisfied by ψ. We readily obtain that

$$M\psi = 0, \tag{E.35}$$

where

$$\begin{aligned} M\psi = &-s\lambda^2|\nabla\eta^0|^2\xi\psi - 2s\lambda\xi\nabla\eta^0.\nabla\psi + \partial_t\psi \\ &+ s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi + \Delta\psi + s\alpha_t\psi - a(t, x)\psi \\ &- s\lambda\Delta\eta^0\xi\psi. \end{aligned} \tag{E.36}$$

Remark E.4.11. The starting point, i.e., the identity (E.35) is the same as in the classical proof developed by Andrei Fursikov and Oleg Imanuvilov in [FI96] (see also [FCG06, Proof of Lemma 1.3] or [LRL12, Section 7]). But, from now, the proof strategy of the L^1-Carleman estimate is very different from the usual one of the L^2-Carleman estimate. Indeed, we will focus on the fourth right hand side term of (E.36)

$$s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi.$$

It is nonnegative because ψ is nonnegative and it is of order two in the parameter s whereas the seventh right hand side term of (E.36)

$$a(t, x)\psi,$$

is of order 0 in the parameter s. This comparison suggests to integrate the identity (E.35) in order to obtain (E.31) for $\lambda \geq 1$ and $s \geq s_1(\lambda)$ as defined in (E.30).

We integrate (E.35) over $(0,T)\times\Omega$

$$\begin{aligned}&\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi - \int_{Q_T} 2s\lambda\xi\nabla\eta^0.\nabla\psi + \int_{Q_T}\partial_t\psi + \int_{Q_T}\Delta\psi\\ &= \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi - \int_{Q_T} s\alpha_t\psi + \int_{Q_T} a(t,x)\psi\\ &\quad + \int_{Q_T} s\lambda\Delta\eta^0\xi\psi.\end{aligned}\tag{E.37}$$

Note that all the terms in (E.37) are well-defined. Indeed, by using $q_T \in C_c^\infty(\Omega)$ and the parabolic regularity in L^2 to (E.21) (see [DHP07, Theorem 2.1]), we deduce that $q \in X_2 := L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ then $\psi \in X_2$.

Step 2 : Estimates for ψ. As a consequence of the properties of η^0 (see Lemma E.4.8), we have

$$m := \min\left\{|\nabla\eta^0(x)|^2 \ ;\ x \in \overline{\Omega\setminus\omega_0}\right\} > 0,\tag{E.38}$$

which yields

$$\begin{aligned}&\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi\\ &\geq \int_{(0,T)\times(\Omega\setminus\omega)} s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi \geq m\int_{Q_T} s^2\lambda^2\xi^2\psi - m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi.\end{aligned}\tag{E.39}$$

By combining (E.37) and (E.39), we have

$$\begin{aligned}&m\int_{Q_T} s^2\lambda^2\xi^2\psi - \int_{Q_T} 2s\lambda\xi\nabla\eta^0.\nabla\psi + \int_{Q_T}\partial_t\psi + \int_{Q_T}\Delta\psi\\ &\leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi + \int_{Q_T} s|\alpha_t|\psi + \int_{Q_T}|a(t,x)|\psi\\ &\quad + \int_{Q_T} s\lambda|\Delta\eta^0|\xi\psi + m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi.\end{aligned}\tag{E.40}$$

We have the following integration by parts

$$-\int_{Q_T} 2s\lambda\xi\nabla\eta^0.\nabla\psi = \int_{Q_T} 2s\lambda\left(\nabla\xi.\nabla\eta^0\psi + \xi\Delta\eta^0\psi\right) - \int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi d\sigma dt,\tag{E.41}$$

$$\int_{Q_T}\partial_t\psi = \int_\Omega(\psi(T,.) - \psi(0,.)) = 0,\tag{E.42}$$

$$\int_{Q_T}\Delta\psi = \int_{\Sigma_T}\frac{\partial\psi}{\partial n},\tag{E.43}$$

where $\Sigma_T := (0,T)\times\partial\Omega$.

From (E.40), (E.41), (E.42), (E.43), we have

$$\begin{aligned} & m\int_{Q_T} s^2\lambda^2\xi^2\psi - \int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi + \int_{\Sigma_T}\frac{\partial\psi}{\partial n} \\ & \leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi + \int_{Q_T} s|\alpha_t|\psi + \int_{Q_T}|a(t,x)|\psi \\ & \quad + \int_{Q_T} 3s\lambda|\Delta\eta^0|\xi\psi + \int_{Q_T} 2s\lambda|\nabla\xi||\nabla\eta^0|\psi + m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi. \end{aligned} \tag{E.44}$$

By using the first two lines of (E.34) and $\lambda \geq 1$, we have

$$\begin{aligned} & \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi + \int_{Q_T} s|\alpha_t|\psi + \int_{Q_T}|a(t,x)|\psi \\ & + \int_{Q_T} 3s\lambda|\Delta\eta^0|\xi\psi + \int_{Q_T} 2s\lambda|\nabla\xi||\nabla\eta^0|\psi \\ & \leq C\left(\int_{Q_T} s\lambda^2\xi\psi + \int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\xi^2\psi + \int_{Q_T}|a(t,x)|\psi + \int_{Q_T} s\lambda\xi\psi\right) \\ & \leq C\left(\int_{Q_T} s\lambda^2\xi\psi + \int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\xi^2\psi + \int_{Q_T}|a(t,x)|\psi\right). \end{aligned} \tag{E.45}$$

By combining (E.44) and (E.45), we get

$$\begin{aligned} & m\int_{Q_T} s^2\lambda^2\xi^2\psi - \int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi + \int_{\Sigma_T}\frac{\partial\psi}{\partial n} \\ & \leq C\left(\int_{Q_T} s\lambda^2\xi\psi + \int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\xi^2\psi + \int_{Q_T}|a(t,x)|\psi\right) \\ & \quad + m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi. \end{aligned} \tag{E.46}$$

Absorption. The goal of this intermediate step is to absorb the third right hand side terms of (E.46) by the first left hand side term of (E.46) by taking s sufficiently large. In order to do this, it is useful to keep in mind the fact that $\lambda \geq 1$ and the third line of (E.34) for the next estimates.

By taking $s \geq (T/2)^2(4C/m)$, we have $Cs\xi \leq (m/4)(s\xi)^2$ and consequently

$$C\int_{Q_T} s\lambda^2\xi\psi \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi. \tag{E.47}$$

By taking $s \geq Te^{2\lambda\|\eta^0\|_\infty}(4C/m)$, we have $Cse^{2\lambda\|\eta^0\|_\infty}T\xi^2 \leq (m/4)(\lambda s\xi)^2$ and consequently

$$C\int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\xi^2\psi \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi. \tag{E.48}$$

By taking $s \geq (T/2)^2\,\|a\|_{L^\infty(Q_T)}^{1/2}(4C/m)^{1/2}$, we have $C\,\|a\|_{L^\infty(Q_T)} \leq (m/4)(\lambda s\xi)^2$ and consequently

$$C\int_{Q_T}|a(t,x)|\psi \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi. \tag{E.49}$$

Therefore, by taking $s \geq s_1(\lambda)$ as defined in (E.30), we have from (E.47), (E.48) and (E.49) that

$$C\left(\int_{Q_T} s\lambda^2\xi\psi + \int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\xi^2\psi + \int_{Q_T} |a(t,x)|\psi\right) \leq \frac{3m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi. \tag{E.50}$$

Then, from (E.46) and (E.50), for $s \geq s_1(\lambda)$, we get

$$\frac{m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi - \int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi + \int_{\Sigma_T}\frac{\partial\psi}{\partial n} \leq m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi. \tag{E.51}$$

Step 3 : An identity satisfied by $\widetilde{\psi}$. We readily obtain that

$$\widetilde{M}\widetilde{\psi} = 0, \tag{E.52}$$

where

$$\begin{aligned}\widetilde{M}\widetilde{\psi} = &-s\lambda^2|\nabla\eta^0|^2\widetilde{\xi}\widetilde{\psi} + 2s\lambda\widetilde{\xi}\nabla\eta^0.\nabla\widetilde{\psi} + \partial_t\widetilde{\psi}\\ &+ s^2\lambda^2|\nabla\eta^0|^2\widetilde{\xi}^2\widetilde{\psi} + \Delta\widetilde{\psi} + s\widetilde{\alpha}_t\widetilde{\psi} - a(t,x)\widetilde{\psi}\\ &+ s\lambda\Delta\eta^0\widetilde{\xi}\widetilde{\psi}.\end{aligned} \tag{E.53}$$

We integrate (E.35) over $(0,T)\times\Omega$

$$\begin{aligned}&\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T} 2s\lambda\widetilde{\xi}\nabla\eta^0.\nabla\widetilde{\psi} + \int_{Q_T}\partial_t\widetilde{\psi} + \int_{Q_T}\Delta\widetilde{\psi}\\ &= \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\widetilde{\xi}\widetilde{\psi} - \int_{Q_T} s\widetilde{\alpha}_t\widetilde{\psi} + \int_{Q_T} a(t,x)\widetilde{\psi}\\ &\quad - \int_{Q_T} s\lambda\Delta\eta^0\widetilde{\xi}\widetilde{\psi}.\end{aligned} \tag{E.54}$$

Step 4 : Estimates for $\widetilde{\psi}$. By using (E.38), we have

$$\begin{aligned}&\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\widetilde{\xi}^2\widetilde{\psi}\\ &\geq \int_{(0,T)\times(\Omega\setminus\omega)} s^2\lambda^2|\nabla\eta^0|^2\widetilde{\xi}^2\widetilde{\psi} \geq m\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} - m\int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}.\end{aligned} \tag{E.55}$$

By combining (E.54) and (E.55), we have

$$\begin{aligned}&m\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T} 2s\lambda\widetilde{\xi}\nabla\eta^0.\nabla\widetilde{\psi} + \int_{Q_T}\partial_t\widetilde{\psi} + \int_{Q_T}\Delta\widetilde{\psi}\\ &\leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} s|\widetilde{\alpha}_t|\widetilde{\psi} + \int_{Q_T}|a(t,x)|\widetilde{\psi}\\ &\quad + \int_{Q_T} s\lambda|\Delta\eta^0|\widetilde{\xi}\widetilde{\psi} + m\int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}.\end{aligned} \tag{E.56}$$

We have the following integration by parts

$$\int_{Q_T} 2s\lambda\widetilde{\xi}\nabla\eta^0.\nabla\widetilde{\psi} = -\int_{Q_T} 2s\lambda\left(\nabla\widetilde{\xi}.\nabla\eta^0\widetilde{\psi} + \widetilde{\xi}\Delta\eta^0\widetilde{\psi}\right) + \int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi}, \tag{E.57}$$

$$\int_{Q_T} \partial_t\widetilde{\psi} = \int_{\Omega}(\widetilde{\psi}(T,.) - \widetilde{\psi}(0,.)) = 0, \tag{E.58}$$

$$\int_{Q_T} \Delta\widetilde{\psi} = \int_{\Sigma_T} \frac{\partial\widetilde{\psi}}{\partial n}. \tag{E.59}$$

From (E.56), (E.57), (E.58), (E.59), we have

$$\begin{aligned} &m\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} + \int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi} + \int_{\Sigma_T}\frac{\partial\widetilde{\psi}}{\partial n}\\ &\leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} s|\widetilde{\alpha}_t|\widetilde{\psi} + \int_{Q_T}|a(t,x)|\widetilde{\psi}\\ &\quad + \int_{Q_T} 3s\lambda|\Delta\eta^0|\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} 2s\lambda|\nabla\widetilde{\xi}||\nabla\eta^0|\widetilde{\psi} + m\int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \end{aligned} \tag{E.60}$$

By using the first two lines of (E.34) and the fact that $\lambda \geq 1$, we have

$$\begin{aligned} &\int_{Q_T} s\lambda^2|\nabla\eta^0|^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} s|\widetilde{\alpha}_t|\widetilde{\psi} + \int_{Q_T}|a(t,x)|\widetilde{\psi}\\ &+\int_{Q_T} 3s\lambda|\Delta\eta^0|\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} 2s\lambda|\nabla\widetilde{\xi}||\nabla\eta^0|\widetilde{\psi}\\ &\leq C\left(\int_{Q_T} s\lambda^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} se^{4\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T}|a(t,x)|\widetilde{\psi} + \int_{Q_T} s\lambda\widetilde{\xi}\widetilde{\psi}\right)\\ &\leq C\left(\int_{Q_T} s\lambda^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} se^{4\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T}|a(t,x)|\widetilde{\psi}\right) \end{aligned} \tag{E.61}$$

By combining (E.60) and (E.61), we get

$$\begin{aligned} &m\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} + \int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi} + \int_{\Sigma_T}\frac{\partial\widetilde{\psi}}{\partial n}\\ &\leq C\left(\int_{Q_T} s\lambda^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} se^{4\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T}|a(t,x)|\widetilde{\psi}\right)\\ &\quad + m\int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \end{aligned} \tag{E.62}$$

Absorption. Note that we will use the third line of (E.34) in the next four estimates. By taking $s \geq e^{\lambda\|\eta^0\|_\infty}(T/2)^2(4C/m)$, we have $Cs\widetilde{\xi} \leq (m/4)(s\widetilde{\xi})^2$ and consequently

$$C\int_{Q_T} s\lambda^2\widetilde{\xi}\widetilde{\psi} \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \tag{E.63}$$

By taking $s \geq Te^{4\lambda\|\eta^0\|_\infty}(4C/m)$, we have $Cse^{2\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2 \leq (m/4)(\lambda s\widetilde{\xi})^2$ and consequently

$$C\int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2\widetilde{\psi} \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \tag{E.64}$$

By taking $s \geq e^{\lambda\|\eta^0\|_\infty}(T/2)^2\,\|a\|^{1/2}_{L^\infty(Q_T)}\,(4C/m)^{1/2}$, we have
$C\,\|a\|_{L^\infty(Q_T)} \leq (m/4)(\lambda s\widetilde{\xi})^2$ and consequently

$$C\int_{Q_T} |a(t,x)|\widetilde{\psi} \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \tag{E.65}$$

Therefore, by taking $s \geq s_1(\lambda)$ as defined in (E.30), we have from (E.47), (E.48) and (E.65) that

$$C\left(\int_{Q_T} s\lambda^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} se^{4\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T} |a(t,x)|\widetilde{\psi}\right) \leq \frac{3m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \tag{E.66}$$

Then, from (E.62) and (E.66), for $s \geq s_1(\lambda)$, we get

$$\frac{m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} + \int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi} + \int_{\Sigma_T}\frac{\partial\widetilde{\psi}}{\partial n} \leq m\int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \tag{E.67}$$

Step 5 : Elimination of the boundary terms. From now, we take $s \geq s_1(\lambda)$. By summing (E.51) and (E.67), we get

$$\begin{aligned}
&\frac{m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi - \int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi + \int_{\Sigma_T}\frac{\partial\psi}{\partial n}\\
&+\frac{m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} + \int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi} + \int_{\Sigma_T}\frac{\partial\widetilde{\psi}}{\partial n}\\
&\leq m\left(\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi + \int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}\right).
\end{aligned} \tag{E.68}$$

Since $\eta^0 = 0$ on $\partial\Omega$, we have

$$\xi = \widetilde{\xi},\ \ \alpha = \widetilde{\alpha} \text{ and } \psi = \widetilde{\psi} \qquad \text{on } \Sigma_T, \tag{E.69}$$

which leads to

$$-\int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi + \int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi} = 0. \tag{E.70}$$

Moreover, we have

$$\partial_i\psi = e^{-s\alpha}(\partial_i q + s\lambda\partial_i\eta^0\xi q),\ \ \partial_i\widetilde{\psi} = e^{-s\widetilde{\alpha}}(\partial_i q - s\lambda\partial_i\eta^0\widetilde{\xi}q),$$

whence by using $\frac{\partial q}{\partial n} = 0$ on Σ_T, we get

$$\frac{\partial\psi}{\partial n} = s\lambda\frac{\partial\eta^0}{\partial n}\xi e^{-s\alpha}q,\ \ \frac{\partial\widetilde{\psi}}{\partial n} = -s\lambda\frac{\partial\eta^0}{\partial n}\widetilde{\xi}e^{-s\widetilde{\alpha}}q \qquad \text{on } \Sigma_T. \tag{E.71}$$

By using (E.69) and (E.71), we get

$$\int_{\Sigma_T} \frac{\partial \psi}{\partial n} + \int_{\Sigma_T} \frac{\partial \widetilde{\psi}}{\partial n} = 0. \tag{E.72}$$

We get from (E.68), (E.70) and (E.72)

$$\begin{aligned} &\frac{m}{4}\left(\int_{Q_T} s^2\lambda^2\xi^2\psi + \int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}\right) \\ &\qquad \leq C\left(\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi + \int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}\right). \end{aligned} \tag{E.73}$$

By forgetting the second term of (E.73) which is nonnegative and by using the fact that $\widetilde{\xi} \leq \xi$, $e^{-s\widetilde{\alpha}} \leq e^{-s\alpha}$ in Q_T, we get from (E.73) the Carleman estimate (E.31). This concludes the proof of Theorem E.4.9. □

E.4.4 Proof of the L^2-L^1 observability inequality : Theorem E.4.4

The goal of this subsection is to prove Theorem E.4.4, which is a consequence of Theorem E.4.9, L^p-L^q estimates and the dissipativity in time of the L^p norm of (E.21).

Proof. **Step 1 : L^1-L^1 observability inequality.** We fix $\lambda = 1$ and $s = s_1$ in Theorem E.4.9 to get

$$\int_{Q_T} t^{-2}(T-t)^{-2}e^{-s\alpha}q\,dxdt \leq C_1(\Omega,\omega)\int_{(0,T)\times\omega} t^{-2}(T-t)^{-2}e^{-s\alpha}q\,dxdt. \tag{E.74}$$

First, we observe that in $(T/4, 3T/4)\times\Omega$,

$$\begin{aligned} t^{-2}(T-t)^{-2}e^{-s\alpha} &\geq \frac{C}{T^4}\exp\left(-\frac{C(\Omega,\omega)\left(T+T^2+T^2\left\|a\right\|_{L^\infty(Q_T)}^{1/2}\right)}{T^2}\right) \\ &\geq \frac{C}{T^4}e^{-C(\Omega,\omega)\left(1+\frac{1}{T}+\|a\|_{L^\infty(Q_T)}^{1/2}\right)}. \end{aligned} \tag{E.75}$$

Secondly, from the fact that $x^2e^{-Mx} \leq C/M^2$ for every $x, M \geq 0$ used with $x = t^{-1}(T-t)^{-1}$ and $M = C(\Omega,\omega)\left(T+T^2+T^2\left\|a\right\|_{L^\infty(Q_T)}^{1/2}\right)$, we remark that in $(0,T)\times\omega$,

$$\begin{aligned} &t^{-2}(T-t)^{-2}e^{-s\alpha} \\ &\leq t^{-2}(T-t)^{-2}\exp\left(-C(\Omega,\omega)\left(T+T^2+T^2\left\|a\right\|_{L^\infty(Q_T)}^{1/2}\right)t^{-1}(T-t)^{-1}\right) \\ &\leq \frac{C}{\left(C(\Omega,\omega)\left(T+T^2+T^2\left\|a\right\|_{L^\infty(Q_T)}^{1/2}\right)\right)^2} \\ &\leq \frac{C(\Omega,\omega)}{T^4}. \end{aligned} \tag{E.76}$$

Then, we get from (E.74), (E.75) and (E.76)

$$\int_{(T/4,3T/4)\times\Omega} q dx dt \le e^{C(\Omega,\omega)\left(1+\frac{1}{T}+\|a\|_{L^\infty(Q_T)}^{1/2}\right)} \int_{(0,T)\times\omega} q dx dt. \tag{E.77}$$

On the other hand, we obtain by the dissipativity in time of the L^1-norm (see Proposition E.3.4 with $p = 1$)

$$\|q(T/4,.)\|_{L^1(\Omega)} \le \frac{2C\exp\left(CT\|a\|_{L^\infty(Q_T)}\right)}{T} \int_{T/4}^{3T/4} \|q(t,.)\|_{L^1(\Omega)} dt. \tag{E.78}$$

By using (E.77) and (E.78), we get

$$\|q(T/4,.)\|_{L^1(\Omega)} \le C(\Omega,\omega,T,a) \int_{(0,T)\times\omega} q dx dt, \tag{E.79}$$

where $C(\Omega,\omega,T,a)$ is defined in (E.22).

From now, we denote by $C(\Omega,\omega,T,a)$ various positive constants varying from line to line which are of the form (E.22).

Step 2 : Global L^2-L^1 estimate. The goal of this step is to prove that

$$\|q(0,.)\|_{L^2(\Omega)} \le C(\Omega,\omega,T,a)\|q(T/4,.)\|_{L^1(\Omega)}. \tag{E.80}$$

To simplify the notations, we set $\widehat{q}(t) := q(T-t)$ for $t \in [0,T]$. Then, (E.80) rewrites as follows

$$\left\|\widehat{q}(\widehat{T_2},.)\right\|_{L^2(\Omega)} \le C(\Omega,\omega,T,a)\left\|\widehat{q}(\widehat{T_1},.)\right\|_{L^1(\Omega)}. \tag{E.81}$$

with $\widehat{T_2} := T > \widehat{T_1} := 3T/4$.

The proof of (E.81) will be a consequence of the regularizing effect of the heat-semigroup (see Proposition E.3.8) and a bootstrap argument.

We introduce the following sequence

$$r_0 := 1, \qquad \forall k \ge 0,\ r_{k+1} := \begin{cases} \frac{Nr_k}{N-r_k} & \text{if } r_k < N, \\ 2r_k & \text{if } r_k \ge N. \end{cases} \tag{E.82}$$

We readily have from the definition (E.82) that

$$\forall k \ge 0,\ \beta_k := \frac{N}{2}\left(\frac{1}{r_k} - \frac{1}{r_{k+1}}\right) \le \frac{1}{2} < 1, \tag{E.83}$$

and

$$\exists l \ge 1,\ r_l \ge 2. \tag{E.84}$$

We also introduce a sequence of times

$$\forall k \in \{0,\dots,l\},\ \tau_k := \widehat{T_1} + \frac{k}{l}(\widehat{T_2} - \widehat{T_1}). \tag{E.85}$$

Let us remark that

$$\forall k \in \{0,\dots,l\},\ \tau_{k+1}-\tau_k = \frac{\widehat{T_2}-\widehat{T_1}}{l} = \frac{T}{4l}. \tag{E.86}$$

By induction, we will show that

$$\forall k \in \{0,\dots,l\},\ \|\widehat{q}(\tau_k,.)\|_{L^{r_k}(\Omega)} \leq C(\Omega,\omega,T,a)\, \|\widehat{q}(\tau_0,.)\|_{L^1(\Omega)}\,. \tag{E.87}$$

The case $k=0$ is obvious (take $C_0=1$). Then, by denoting by $S(t)=e^{t\Delta}$ the heat-semigroup with Neumann boundary conditions, we have for every $k \geq 0$,

$$\widehat{q}(\tau_{k+1},.) = S(\tau_{k+1}-\tau_k)\widehat{q}(\tau_k,.) + \int_{\tau_k}^{\tau_{k+1}} S(\tau_{k+1}-s)(-a(s,.)\widehat{q}(s,.))ds, \tag{E.88}$$

from the equation satisfied by $\widehat{q}$ (see (E.21)).

We assume that (E.87) holds for $k \in \{0,\dots,l\}$. From (E.88), (E.83) and the regularizing effect L^{r_k}-$L^{r_{k+1}}$ of the heat-semigroup (see Proposition E.3.8), we have

$$\begin{aligned} \|\widehat{q}(\tau_{k+1})\|_{L^{r_{k+1}}(\Omega)} &\leq C(\Omega,\omega,T,a)\Big((\tau_{k+1}-\tau_k)^{-\beta_k}\, \|\widehat{q}(\tau_k)\|_{L^{r_k}(\Omega)} \\ &\quad + \int_{\tau_k}^{\tau_{k+1}} (\tau_{k+1}-s)^{-\beta_k}\, \|a\|_{L^\infty(Q_T)}\, \|\widehat{q}(s)\|_{L^{r_k}(\Omega)}\, ds\Big) \\ &\leq C(\Omega,\omega,T,a)\left(A_{1,k}+A_{2,k}\right), \end{aligned} \tag{E.89}$$

where

$$A_{1,k} := (\tau_{k+1}-\tau_k)^{-\beta_k}\, \|\widehat{q}(\tau_k)\|_{L^{r_k}(\Omega)}\,, \tag{E.90}$$

and

$$A_{2,k} := \int_{\tau_k}^{\tau_{k+1}} (\tau_{k+1}-s)^{-\beta_k}\, \|a\|_{L^\infty(Q_T)}\, \|\widehat{q}(s)\|_{L^{r_k}(\Omega)}\, ds. \tag{E.91}$$

From (E.90), (E.86), (E.83) and (E.87), we have

$$A_{1,k} \leq CT^{-\beta_k} C(\Omega,\omega,T,a)\, \|\widehat{q}(\tau_0,.)\|_{L^1(\Omega)} \leq C(\Omega,\omega,T,a)\, \|\widehat{q}(\tau_0,.)\|_{L^1(\Omega)}\,. \tag{E.92}$$

From (E.91), the dissipativity in time of the L^{r_k}-norm (see Proposition E.3.4), the induction assumption (E.87), the integrability condition coming from (E.83) and (E.86), we have

$$\begin{aligned} A_{2,k} &\leq \|a\|_\infty \int_{\tau_k}^{\tau_{k+1}} (\tau_{k+1}-s)^{-\beta_k} C e^{CT\|a\|_\infty}\, \|\widehat{q}(\tau_k)\|_{L^{r_k}(\Omega)}\, ds \\ &\leq C\, \|a\|_\infty\, e^{CT\|a\|_\infty} C(\Omega,\omega,T,a)\, \|\widehat{q}(\tau_0,.)\|_{L^1(\Omega)}\, (\tau_{k+1}-\tau_k)^{-\beta_k+1} \\ &\leq C(\Omega,\omega,T,a)\, \|a\|_\infty\, T^{-\beta_k+1}\, \|\widehat{q}(\tau_0,.)\|_{L^1(\Omega)} \\ &\leq C(\Omega,\omega,T,a)\, \|a\|_\infty\, (T+1)\, \|\widehat{q}(\tau_0,.)\|_{L^1(\Omega)} \\ &\leq C(\Omega,\omega,T,a)\left(e^{T\|a\|_\infty} + 2e^{\|a\|_\infty^{1/2}}\right) \|\widehat{q}(\tau_0,.)\|_{L^1(\Omega)} \\ &\leq C(\Omega,\omega,T,a)\, \|\widehat{q}(\tau_0,.)\|_{L^1(\Omega)}\,. \end{aligned} \tag{E.93}$$

The estimates (E.89), (E.92) and (E.93) prove (E.87) for $(k+1)$ and concludes the induction. Thus, (E.87) holds for $k = l$, which combined with (E.84) and (E.85), yields (E.81).

Step 3 : By using (E.79) and (E.80), we prove (E.25) and consequently Theorem E.4.4. □

E.4.5 Proof of the linear global nonnegative-controllability : Theorem E.4.1

The goal of this section is to prove Theorem E.4.1. The following proof is inspired by the so-called *Hilbert Uniqueness method* due to Jacques-Louis Lions (see [Lio88] and more precisely [Zua97, Section 2.1]).

Proof. The proof is divided into two steps. First, we build a sequence of controls $h_\varepsilon \in L^\infty((0,T)\times\omega)$ with $\varepsilon > 0$ which provide the approximate nonnegative-controllability of (E.20). Secondly, we pass to the limit when ε tends to 0.

Step 1. Let us fix $T > 0$, $a \in L^\infty(Q_T)$ and $y_0 \in L^2(\Omega)$. For any $\varepsilon \in (0,1)$, we consider the following functional : for every $q_T \in L^2(\Omega;\mathbb{R}^+)$,

$$J_\varepsilon(q_T) = \frac{1}{2}\left(\int_{(0,T)\times\omega} q dxdt\right)^2 + \varepsilon \|q_T\|_{L^2(\Omega)} + \int_\Omega q(0,x)y_0(x)dx, \tag{E.94}$$

where q is the solution to (E.21).

The functional J_ε is continuous, convex and coercive on the unbounded closed convex set $L^2(\Omega;\mathbb{R}^+)$. More precisely, we will show that

$$\liminf_{\|q_T\|_{L^2(\Omega)}\to+\infty} \frac{J_\varepsilon(q_T)}{\|q_T\|_{L^2(\Omega)}} \geq \varepsilon. \tag{E.95}$$

Indeed, given a sequence $(q_{T,k})_{k\geq 0} \in L^2(\Omega)$ with $\|q_{T,k}\|_{L^2(\Omega)} \to +\infty$, we normalize it :

$$\widetilde{q}_{T,k} := \frac{q_{T,k}}{\|q_{T,k}\|_{L^2(\Omega)}},$$

and we denote by $\widetilde{q}_k$ the solution to (E.21) associated to the initial data $\widetilde{q}_{T,k}$. We have

$$\frac{J_\varepsilon(q_{T,k})}{\|q_{T,k}\|_{L^2(\Omega)}} = \frac{\|q_{T,k}\|_{L^2(\Omega)}}{2}\left(\int_{(0,T)\times\omega} \widetilde{q}_k dxdt\right)^2 + \varepsilon + \int_\Omega \widetilde{q}_k(0,x)y_0(x)dx. \tag{E.96}$$

We distinguish the following two cases.

Case 1 :

$$\liminf_{k\to+\infty} \int_{(0,T)\times\omega} \widetilde{q}_k dxdt > 0. \tag{E.97}$$

When (E.97) holds, we clearly have

$$\liminf_{k\to+\infty} \frac{J_\varepsilon(q_{T,k})}{\|q_{T,k}\|_{L^2(\Omega)}} = +\infty \geq \varepsilon$$

Case 2 :

$$\liminf_{k\to+\infty} \int_{(0,T)\times\omega} \widetilde{q}_k dxdt = 0. \tag{E.98}$$

In this case, by using the estimate (E.11) of Proposition E.3.2, the embedding (E.7) and (E.98), extracting subsequences (that we denote by the index k to simplify the notation), we deduce that there exists $\widetilde{q} \in W_T$ such that

$$\widetilde{q}_k \rightharpoonup \widetilde{q} \text{ in } W_T, \tag{E.99}$$

$$\widetilde{q}_k(0,.) \rightharpoonup \widetilde{q}(0,.) \text{ in } L^2(\Omega), \tag{E.100}$$

$$\int_{(0,T)\times\omega} \widetilde{q}_k dxdt \to 0. \tag{E.101}$$

By using Aubin Lions' lemma (see [Sim87, Section 8, Corollary 4]) and (E.99), $(\widetilde{q}_k)_{k\in\mathbb{N}}$ is relatively compact in $L^2(Q_T)$, then up to a subsequence we have

$$\widetilde{q}_k \to \widetilde{q} \text{ in } L^2(Q_T;\mathbb{R}^+). \tag{E.102}$$

In view of (E.101) and (E.102), we have

$$\widetilde{q} = 0 \text{ in } (0,T)\times\omega. \tag{E.103}$$

Then, by using (E.103) and the observability inequality (E.25), we have

$$\widetilde{q}(0,.) = 0. \tag{E.104}$$

Consequently, by combining (E.100) and (E.104), we have

$$\int_\Omega \widetilde{q}_k(0,x) y_0(x) dx \to 0,$$

which yields (E.95) thanks to (E.96).

We deduce that J_ε admits a minimum $q_{\varepsilon,T} \in L^2(\Omega;\mathbb{R}^+)$. We take

$$h_\varepsilon := \left(\int_{(0,T)\times\omega} q_\varepsilon\right) 1_\omega, \tag{E.105}$$

and we denote by $y_\varepsilon \in W_T \cap L^\infty(Q_T)$ the solution to

$$\begin{cases} \partial_t y_\varepsilon - \Delta y_\varepsilon + a(t,x) y_\varepsilon = h_\varepsilon 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y_\varepsilon}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_\varepsilon(0,.) = y_0 & \text{in } \Omega. \end{cases} \tag{E.106}$$

We use the fact that $J_\varepsilon(q_{T,\varepsilon}) \leq J_\varepsilon(0) = 0$ to get

$$\frac{1}{2}\left(\int_{(0,T)\times\omega} q_\varepsilon\right)^2 + \varepsilon \left\|q_{\varepsilon,T}\right\|_{L^2(\Omega)} \leq -\int_\Omega q_\varepsilon(0,x)y_0(x)dx. \tag{E.107}$$

By using the observability inequality (E.25), (E.105), (E.107) and Young's inequality, we obtain the following bound on the sequence of controls

$$\|h_\varepsilon\|^2_{L^\infty(Q_T)} \leq C(\Omega,\omega,T,a)\,\|y_0\|^2_{L^2(\Omega)}\,, \tag{E.108}$$

where $C(\Omega,\omega,T,a)$ is of the form (E.22).

For $\lambda > 0$ and $p_T \in L^2(\Omega;\mathbb{R}^+)$, we have

$$J_\varepsilon(q_{\varepsilon,T}) \leq J_\varepsilon(q_{\varepsilon,T} + \lambda p_T). \tag{E.109}$$

Dividing the inequality (E.109) by λ and letting $\lambda \to 0^+$, we obtain from (E.105) and the Minkowski's inequality

$$\begin{aligned} -(y_0, p(0,.))_{L^2(\Omega)} &\leq \int_{(0,T)\times\omega} h_\varepsilon p + \varepsilon \liminf_{\lambda\to 0^+} \frac{\|q_{\varepsilon,T}+\lambda p_T\|_{L^2(\Omega)} - \|q_{\varepsilon,T}\|_{L^2(\Omega)}}{\lambda} \\ &\leq \int_{(0,T)\times\omega} h_\varepsilon p + \varepsilon \left\|p_T\right\|_{L^2(\Omega)}\,, \end{aligned} \tag{E.110}$$

where p is the solution to (E.21) with initial data p_T. Since systems (E.20) and (E.21) are in duality, we have

$$\int_{(0,T)\times\omega} h_\varepsilon p = (y_\varepsilon(T,.), p_T)_{L^2(\Omega)} - (y_0, p(0,.))_{L^2(\Omega)}, \tag{E.111}$$

which, combined with (E.110), yields

$$(y_\varepsilon(T,.), p_T)_{L^2(\Omega)} \geq -\varepsilon \left\|p_T\right\|_{L^2(\Omega)}\,,\ \forall p_T \in L^2(\Omega;\mathbb{R}^+). \tag{E.112}$$

Step 2. By using (E.108), (E.106), Proposition E.3.2, Proposition E.3.3 and the embedding (E.7), up to a subsequence, we get that there exist $h \in L^\infty(Q_T)$ and $y \in W_T \cap L^\infty(Q_T)$ such that

$$h_\varepsilon \rightharpoonup^* h \text{ in } L^\infty(Q_T) \text{ as } \varepsilon \to 0, \tag{E.113}$$

$$y_\varepsilon \rightharpoonup y \text{ in } W_T \ \Rightarrow y_\varepsilon(0,.) \rightharpoonup y(0,.),\ y_\varepsilon(T,.) \rightharpoonup y(T,.) \text{ in } L^2(\Omega) \qquad \text{as } \varepsilon \to 0. \tag{E.114}$$

Then, by using (E.106), (E.113) and (E.114), we obtain that y is the solution of (E.20) associated to the control h satisfying (E.23) (by letting ε goes to 0 in (E.108)) and

$$(y(T,.), p_T)_{L^2(\Omega)} \geq 0,\ \forall p_T \in L^2(\Omega;\mathbb{R}^+). \tag{E.115}$$

Then, we deduce from (E.115) that y satisfies (E.24), which concludes the proof of Theorem E.4.1. □

E.5 A fixed-point argument to prove the small-time nonlinear global nonnegative controllability

The goal of this section is to prove Theorem E.2.2. We assume that (E.3) holds for $\alpha \leq 2$ and $f(s) \geq 0$ for $s \geq 0$.

E.5.1 A comparison principle

First, we begin with this lemma, which is a consequence of the comparison principle for subsolutions and supersolutions of (E.1) with control $h = 0$ stated in Proposition E.3.7.

Lemma E.5.1. Let $T > 0$, $y_0 \in L^\infty(\Omega)$. Assume that there exists $T^* \in (0, T]$ and a control $h^* \in L^\infty(Q_{T^*})$ such that the solution $y \in L^\infty(Q_{T^*})$ to (E.1) satisfies (E.5) (replacing $T \leftarrow T^*$). Then, if we set

$$h(t,.) := \begin{cases} h^*(t,.) & \text{for } t \in (0, T^*), \\ 0 & \text{for } t \in (T^*, T), \end{cases}$$

the solution y of (E.1) belongs to $L^\infty(Q_T)$ and satisfies (E.5). Moreover, there exists $C := C(\Omega) > 0$ such that

$$\|y\|_{L^\infty(Q_T)} \leq C \|y\|_{L^\infty(Q_{T^*})}. \tag{E.116}$$

Proof. By using the fact that $f(0) = 0$, $f(s) \geq 0$ for $s \geq 0$ and the comparison principle (see Proposition E.3.7), we have

$$\forall t \in [T^*, T],\ \text{a.e. } x \in \Omega,\ 0 \leq y(t,x) \leq \widetilde{y}(t,x), \tag{E.117}$$

where $\widetilde{y}$ is the nonnegative solution to

$$\begin{cases} \partial_t \widetilde{y} - \Delta \widetilde{y} = 0 & \text{in } (T^*, T) \times \Omega, \\ \frac{\partial \widetilde{y}}{\partial n} = 0 & \text{on } (T^*, T) \times \partial\Omega, \\ \widetilde{y}(T^*,.) = y(T^*,.) & \text{in } \Omega. \end{cases} \tag{E.118}$$

Therefore, by using Proposition E.3.3 for (E.118), we get that there exists $C := C(\Omega) > 0$ such that

$$\|\widetilde{y}\|_{L^\infty((T^*,T)\times\Omega)} \leq C \|y(T^*,.)\|_{L^\infty(\Omega)} \leq C \|y\|_{L^\infty(Q_{T^*})}. \tag{E.119}$$

By using (E.117) and (E.119), we obtain that $y \in L^\infty(Q_T)$, (E.5) and (E.116) hold. $\square$

E.5.2 The fixed-point : definition of the application

We begin with some notations. Let us set

$$g(s) = \begin{cases} \dfrac{f(s)}{s} & \text{if } s \neq 0, \\ f'(0) & \text{if } s = 0. \end{cases} \tag{E.120}$$

The function g is continuous and by using the fact that f satisfies (E.3) with $\alpha \leq 2$, we deduce that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\forall s \in \mathbb{R},\ |g(s)|^{1/2} \leq \varepsilon \log(2 + |s|) + C_\varepsilon. \tag{E.121}$$

The end of the section is devoted to the proof of Theorem E.2.2.

Proof. Let $T > 0$, $y_0 \in L^\infty(\Omega)$.

Unless otherwise specified, we denote by C various positive constants varying from line to line which may depend on Ω, ω, T.

We will perform a Kakutani-Leray-Schauder's fixed-point argument in $L^\infty(Q_T)$.

For each $z \in L^\infty(Q_T)$, we consider the linear system

$$\begin{cases} \partial_t y - \Delta y + g(z)y = h1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega. \end{cases} \tag{E.122}$$

We set

$$T_z^* := \min\left(T, \|g(z)\|_{L^\infty(Q_T)}^{-1/2}\right). \tag{E.123}$$

According to Theorem E.4.1, there exists a control $h_z \in L^\infty(Q_{T_z^*})$ satisfying

$$\begin{aligned} &\|h_z\|_{L^\infty(Q_{T_z^*})} \\ &\leq \exp\left(C\left(1 + \frac{1}{T_z^*} + T_z^* \|g(z)\|_{L^\infty(Q_T)} + \|g(z)\|_{L^\infty(Q_T)}^{1/2}\right)\right) \|y_0\|_{L^2(\Omega)} \\ &\leq \exp\left(C\left(1 + \|g(z)\|_{L^\infty(Q_T)}^{1/2}\right)\right) \|y_0\|_{L^2(\Omega)}, \end{aligned} \tag{E.124}$$

such that the solution y of (E.122) in $(0, T_z^*) \times \Omega$ with $h = h_z$ satisfies

$$y(T_z^*, .) \geq 0. \tag{E.125}$$

By extending by 0 the control h_z in (T_z^*, T), we get from (E.124)

$$\|h_z\|_{L^\infty(Q_T)} \leq \exp\left(C\left(1 + \|g(z)\|_{L^\infty(Q_T)}^{1/2}\right)\right) \|y_0\|_{L^2(\Omega)}. \tag{E.126}$$

For each $z \in L^\infty(Q_T)$, we introduce the set of controls

$$H(z) := \{h_z \in L^\infty(Q_T)\ ;\ h_z \text{ fulfills (E.126) and } h_z \equiv 0 \text{ in } (T_z^*, T) \times \Omega\}. \tag{E.127}$$

We have the following facts.

Fact E.5.2. For every $z \in L^\infty(Q_T)$, $H(z)$ is compact for the weak-star topology of $L^\infty(Q_T)$.

Fact E.5.3. Assume that $z_k \to z$ in $L^\infty(Q_T)$ and $h_k \in H(z_k) \rightharpoonup^* h$ in $L^\infty(Q_T)$ as $k \to +\infty$. Then, we have $h \in H(z)$.

We define the set-valued mapping $\Phi : L^\infty(Q_T) \to \mathcal{P}(L^\infty(Q_T))$ in the following way. For every $z \in L^\infty(Q_T)$, $\Phi(z)$ is the set of $y \in L^\infty(Q_T)$ such that for some $h_z \in H(z)$, y is the solution of (E.122) and this solution satisfies (E.125).

We recall the Kakutani-Leray-Schauder's fixed point theorem (see [Gra93, Theorem 2.2, Theorem 2.4]).

Theorem E.5.4 (Kakutani-Leray-Schauder's fixed point theorem)**.** If

1. Φ is a Kakutani map, that is to say for every $z \in L^\infty(Q_T)$, $\Phi(z)$ is a nonempty convex and closed subset of $L^\infty(Q_T)$,
2. Φ is compact, that is to say for every bounded set $B \subset L^\infty(Q_T)$, there exists a compact set $K \subset L^\infty(Q_T)$ such that for every $z \in B$, $\Phi(z) \subset K$,
3. Φ is upper semicontinuous in $L^\infty(Q_T)$, that is to say for all closed subset $\mathcal{A} \subset L^\infty(Q_T)$, $\Phi^{-1}(\mathcal{A}) = \{z \in L^\infty(Q_T)\ ;\ \Phi(z) \cap \mathcal{A} \neq \emptyset\}$ is closed,
4. $\mathcal{F} := \{y \in L^\infty(Q_T)\ ;\ \exists \lambda \in (0,1),\ y \in \lambda\Phi(y)\}$ is bounded in $L^\infty(Q_T)$,

hold.

Then Φ has a fixed point, i.e, there exists $y \in L^\infty(Q_T)$ such that $y \in \Phi(y)$.

E.5.3 Hypotheses of Kakutani-Leray-Schauder's fixed point theorem

We will check that the four hypotheses of Theorem E.5.4 hold.

The point (1) **holds.** Indeed, for every $z \in L^\infty(Q_T)$, we have seen that $\Phi(z)$ is nonempty. The convexity of $\Phi(z)$ comes from the fact that the inequality (E.125) is stable by convex combinations. Let us show that $\Phi(z)$ is closed. Let $(y_k)_{k\in\mathbb{N}}$ be a sequence of elements in $L^\infty(Q_T)$, such that for every $k \in \mathbb{N}$, $y_k \in \Phi(z)$ and $y_k \to y$ in $L^\infty(Q_T)$. Then, for every $k \in \mathbb{N}$, there exists a control $h_k \in H(z)$ such that y_k is the solution to

$$\begin{cases} \partial_t y_k - \Delta y_k + g(z) y_k = h_k 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y_k}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_k(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{E.128}$$

and this solution satisfies

$$y_k(T_z^*,.) \geq 0. \tag{E.129}$$

By using Fact E.5.2, Proposition E.3.2 and the embedding (E.7), we get that there exist a strictly increasing sequence $(k_l)_{l\in\mathbb{N}}$ of integers and $h \in H(z)$ such that

$$h_{k_l} \rightharpoonup^* h \text{ in } L^\infty(Q_T) \text{ as } l \to +\infty, \tag{E.130}$$

$$y_{k_l} \rightharpoonup y \text{ in } W_T\ \Rightarrow y_{k_l}(0,.) \rightharpoonup y_0,\ y_{k_l}(T_z^*,.) \rightharpoonup y(T_z^*,.) \text{ in } L^2(\Omega) \quad \text{as } l \to +\infty. \tag{E.131}$$

By passing to the limit as $l \to +\infty$ in (E.128), (E.129) and by using (E.130) and (E.131), we get that $y \in \Phi(z)$. This concludes the proof of the point (1).

The point (2) **holds.** Let B be a bounded set of $L^\infty(Q_T)$. By using (E.126) and Proposition E.3.3 applied to (E.122), we deduce that there exists $R > 0$ such that for every $z \in B$, for every $y \in \Phi(z)$ associated to a control $h_z \in H(z)$, we have

$$z, y, h_z \in B_R := \{\zeta \in L^\infty(Q_T)\ ;\ \|\zeta\|_{L^\infty(Q_T)} \leq R\}. \tag{E.132}$$

Let $Y \in L^\infty(Q_T)$ be the solution to the Cauchy problem

$$\begin{cases} \partial_t Y - \Delta Y = 0 & \text{in } (0,T)\times\Omega, \\ \frac{\partial Y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ Y(0,.) = y_0 & \text{in } \Omega. \end{cases} \tag{E.133}$$

Let $y^* = y - Y$, where $y \in \Phi(z)$, with $z \in B$, associated to a control $h_z \in H(z)$. Then, y^* is the solution to

$$\begin{cases} \partial_t y^* - \Delta y^* + g(z)y = h_z 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y^*}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y^*(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{E.134}$$

From (E.132), we have

$$\|-g(z)y + h_z 1_\omega\|_{L^\infty(Q_T)} \leq C_R. \tag{E.135}$$

From (E.135), a maximal parabolic regularity theorem in L^p (see [DHP07, Theorem 2.1]), with $p = N+2$, applied to y^*, solution of (E.134), we deduce that

$$y^* \in X_p := W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega)) \text{ and } \|y^*\|_{X_p} \leq C_R. \tag{E.136}$$

By the Sobolev embedding theorem $X_p \hookrightarrow C^{\beta/2,\beta}(\overline{Q_T})$ with $\beta > 0$ (see [WYW06, Theorem 1.4.1]), we deduce that $y^* \in C^0(\overline{Q_T})$ and

$$\forall (t,x) \in \overline{Q_T},\ \forall (t',x') \in \overline{Q_T},\ |y^*(t,x) - y^*(t',x')| \leq C_R(|t-t'|^{\beta/2} + |x-x'|^\beta). \tag{E.137}$$

Let K^* be the set of y^* such that (E.137) holds. Then, we have $K := (Y + K^*) \cap B_R$ is a compact convex subset of $L^\infty(Q_T)$ by Ascoli's theorem and

$$\forall z \in B,\ \Phi(z) \subset K.$$

This concludes the proof of the point (2).

The point (3) holds. Let $\mathcal{A}$ be a closed subset of $L^\infty(Q_T)$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence of elements in $L^\infty(Q_T)$, $(y_k)_{k\in\mathbb{N}}$ be a sequence of elements in $L^\infty(Q_T)$, and $z \in L^\infty(Q_T)$ be such that

$$z_k \to z \text{ in } L^\infty(Q_T) \text{ as } k \to +\infty, \tag{E.138}$$

$$\forall k \in \mathbb{N},\ y_k \in \mathcal{A}, \tag{E.139}$$

$$\forall k \in \mathbb{N},\ y_k \in \Phi(z_k). \tag{E.140}$$

By (E.140) and (E.126), for every $k \in \mathbb{N}$, there exists a control $h_k \in H(z_k)$ such that y_k is the solution to

$$\begin{cases} \partial_t y_k - \Delta y_k + g(z_k)y_k = h_k 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y_k}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_k(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{E.141}$$

and this solution satisfies

$$y_k(T^*_{z_k}, .) \geq 0. \tag{E.142}$$

By (E.138), Fact E.5.3 and the point (2) of Theorem E.5.4, we get that there exist a strictly increasing sequence $(k_l)_{l\in\mathbb{N}}$ of integers, $h \in H(z)$ and $y \in L^\infty(Q_T)$ such that

$$h_{k_l} \rightharpoonup^* h \text{ in } L^\infty(Q_T) \text{ as } l \to +\infty, \tag{E.143}$$

$$y_{k_l} \to y \text{ in } L^\infty(Q_T) \text{ as } l \to +\infty. \tag{E.144}$$

Since $\mathcal{A}$ is closed, (E.139) and (E.144) imply that $y \in \mathcal{A}$. Hence, it suffices to check that

$$y \in \Phi(z). \tag{E.145}$$

Letting $l \to +\infty$ in (E.141) and (E.142) and using (E.138), (E.143) and (E.144), we get that y satisfies (E.122) and (E.125). Hence, (E.145) holds. This concludes the proof of the point (3).

The point (4) holds. This is the difficult point. The key point is the definition of T^*_y, i.e., (E.123) and (E.126) to get the first inequality of (E.147) (see below).

Let $y \in \mathcal{F}$. Then, for some $\lambda \in (0,1)$ and $h_y \in H(y)$, we have

$$\begin{cases} \partial_t y - \Delta y + f(y) = \lambda h_y 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = \lambda y_0 & \text{in } \Omega, \end{cases}$$

and

$$y(T^*_y, .) \geq 0.$$

Therefore, by using Lemma E.5.1 and Proposition E.3.3, we have

$$\begin{aligned} \|y\|_{L^\infty(Q_T)} &\leq C\,\|y\|_{L^\infty(Q_{T^*_y})} \\ &\leq C\exp\left(CT^*_y\,\|g(y)\|_{L^\infty(Q_T)}\right)\left(\|y_0\|_{L^\infty(\Omega)} + \|h_y\|_{L^\infty(Q_T)}\right). \end{aligned} \tag{E.146}$$

Consequently, by taking into account the definition of T^*_y, i.e., (E.123) and using (E.126), (E.146), (E.121), we deduce that

$$\begin{aligned} \|y\|_{L^\infty(Q_T)} &\leq \exp\left(C\left(1 + \|g(y)\|^{1/2}_{L^\infty(Q_T)}\right)\right)\|y_0\|_{L^\infty(\Omega)} \\ &\leq \exp\left(C\left(1 + \varepsilon\log\left(2 + \|y\|_{L^\infty(Q_T)}\right) + C_\varepsilon\right)\right)\|y_0\|_{L^\infty(\Omega)} \\ &\leq \exp\left(C_\varepsilon\right)\left(2 + \|y\|_{L^\infty(Q_T)}\right)^{\varepsilon C}\|y_0\|_{L^\infty(\Omega)}. \end{aligned} \tag{E.147}$$

Therefore, by taking ε sufficiently small such that $\varepsilon C = 1/2$, we deduce from (E.147) that $\mathcal{F}$ is bounded in $L^\infty(Q_T)$. This concludes the proof of the point (4).

By Theorem E.5.4, Φ has a fixed point y. We denote by h_y the associated control. Then, by using Lemma E.5.1, y is the solution to (E.1) with control h_y such that (E.5) holds. This concludes the proof of Theorem E.2.2. □

E.6 Application of the global nonnegative-controllability to the large time global null-controllability

In this section, we prove Theorem E.2.5. We assume that (E.3) holds for $\alpha \in [3/2, 2]$, $f(s) > 0$ for $s > 0$ and $1/f \in L^1([1, +\infty))$.

Proof. Let $y_0 \in L^\infty(\Omega)$. The proof is divided into three steps.

Step 1 : Steer the solution to a nonnegative state in time $T_1 := 1$. By using Theorem E.2.2, there exists $h_1 \in L^\infty(Q_{T_1})$ such that the solution y to (E.1) replacing $T \leftarrow T_1$ satisfies

$$y_{T_1} := y(T_1, .) \geq 0.$$

Step 2 : Dissipation of f on $\mathbb{R}^+$ and comparison to an ordinary differential equation. We set

$$h_2(t, .) := 0, \text{ for } t \in [T_1, T_2],$$

with T_2 which will be determined later.

Then, by using the comparison principle given in Proposition E.3.7, we deduce that the solution y to

$$\begin{cases} \partial_t y - \Delta y = -f(y) & \text{in } (T_1, T_2) \times \Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (T_1, T_2) \times \partial\Omega, \\ y(T_1, .) = y_{T_1} & \text{in } \Omega, \end{cases}$$

satisfies

$$\forall t \in [T_1, T_2], \text{ a.e. } x \in \Omega, \ 0 \leq y(t, x) \leq v(t), \tag{E.148}$$

where v is the (global) nonnegative solution to the ordinary differential equation

$$\begin{cases} \dot{v}(t) = -f(v(t)) & \text{in } (T_1, +\infty), \\ v(T_1) = \|y_{T_1}\|_{L^\infty(\Omega)} + 1 & . \end{cases} \tag{E.149}$$

A straightforward calculation leads to

$$\forall t \in [T_1, +\infty), \ v(t) > 0 \text{ and } F(v(t)) - F(v(T_1)) = t - T_1, \tag{E.150}$$

where F is defined as follows

$$\forall s > 0, \ F(s) = \int_{+\infty}^{s} \frac{-1}{f(\sigma)} d\sigma = \int_{s}^{+\infty} \frac{1}{f(\sigma)} d\sigma. \tag{E.151}$$

Note that F is well-defined because $f(\sigma) > 0$ for every $\sigma > 0$ and $1/f \in L^1([1, +\infty))$ by hypothesis. We check that F is a C^1 strictly decreasing function. Moreover, we have $1/f \notin L^1((0, 1])$ because $f \in C^1(\mathbb{R}; \mathbb{R})$ and $f(0) = 0$. Hence, we have by (E.151)

$$\lim_{s \to 0^+} F(s) = +\infty \text{ and } \lim_{s \to +\infty} F(s) = 0. \tag{E.152}$$

Therefore, we deduce that $F : (0, +\infty) \to (0, +\infty)$ is a C^1-diffeomorphism. We denote by $F^{-1} : (0, +\infty) \to (0, +\infty)$ its inverse, which is strictly decreasing. Then, by (E.150), we have

$$\forall t \in [T_1, +\infty),\ v(t) = F^{-1}(t - T_1 + F(v(T_1)) \leq F^{-1}(t - T_1). \tag{E.153}$$

The estimate (E.153) is the key point because it states that we can upperbound v by a function independent of the size of $v(T_1)$ and we also have

$$F^{-1}(t - T_1) \to 0 \text{ as } t \to +\infty, \tag{E.154}$$

by using (E.152).

Let $\delta > 0$ be such that the null-controllability of (E.1) holds in $B_{L^\infty(\Omega)}(0, \delta)$ in time $T = 1$. The existence of δ is given by Theorem E.1.3.

By (E.154), we deduce that there exists T_2 sufficiently large such that

$$F^{-1}(T_2 - T_1) \leq \delta. \tag{E.155}$$

Consequently, by using (E.148), (E.153), (E.155), we have

$$\text{a.e. } x \in \Omega,\ 0 \leq y(T_2, x) \leq \delta. \tag{E.156}$$

Step 3 : Local null-controllability. By using Theorem E.1.3 with $T = 1$, we deduce from (E.156) that there exists a control $h_3 \in L^\infty((T_2, T_3) \times \Omega)$ with $T_3 := T_2 + 1$ such that the solution y of (E.1) replacing $(0, T) \leftarrow (T_2, T_3)$ satisfies $y(T_3, .) = 0$.

To sum up, the control

$$h(t, .) := \begin{cases} h_1(t, .) & \text{for } t \in (0, T_1), \\ h_2(t, .) & \text{for } t \in (T_1, T_2), \\ h_3(t, .) & \text{for } t \in (T_2, T_3), \end{cases}$$

steers the initial data $y_0 \in L^\infty(\Omega)$ to 0. It is worth mentioning that the final time of control T_3 does not depend on y_0. This concludes the proof of Theorem E.2.5. $\square$

E.7 Dirichlet boundary conditions

Theorem E.2.2 and Theorem E.2.5 remain valid for Dirichlet boundary conditions, as to say for

$$\begin{cases} \partial_t y - \Delta y + f(y) = h 1_\omega & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, .) = y_0 & \text{in } \Omega. \end{cases} \tag{E.157}$$

The main point is to establish a L^1-Carleman estimate similar to Theorem E.4.9 for

$$\begin{cases} -\partial_t q - \Delta q + a(t, x) q = 0 & \text{in } (0, T) \times \Omega, \\ q = 0 & \text{on } (0, T) \times \partial\Omega, \\ q(T, .) = q_T & \text{in } \Omega. \end{cases} \tag{E.158}$$

We keep the notations of Annexe E.4.3.

Theorem E.7.1. There exists two constants $C = C(\Omega, \omega) > 0$ and $C_1 := C_1(\Omega, \omega) > 0$, such that,

$$\forall \lambda \geq 1, \qquad \forall s \geq s_1(\lambda) := C(\Omega, \omega) \left(e^{2\lambda \|\eta^0\|_\infty} T + T^2 + T^2 \|a\|^{1/2}_{L^\infty(Q_T)} \right), \tag{E.159}$$

for every $q_T \in L^2(\Omega; \mathbb{R}^+)$, the nonnegative solution q of (E.158) satisfies

$$\lambda \int_{Q_T} e^{-s\alpha} s\xi^2 \eta^0 q + \int_{Q_T} e^{-s\alpha} \xi q \leq C_1 \lambda \int_{(0,T)\times\omega} e^{-s\alpha} s\xi^2 q dx dt. \tag{E.160}$$

Proof. The proof follows the one of Theorem E.4.9. This is why we omit some details. We multiply the identity (E.35) by η^0 and we integrate over $(0, T) \times \Omega$

$$\begin{aligned} &\int_{Q_T} s^2\lambda^2 |\nabla \eta^0|^2 \xi^2 \psi \eta^0 - \int_{Q_T} 2s\lambda \xi (\nabla \eta^0 . \nabla \psi) \eta^0 + \int_{Q_T} (\partial_t \psi) \eta^0 + \int_{Q_T} (\Delta \psi) \eta^0 \\ &= \int_{Q_T} s\lambda^2 |\nabla \eta^0|^2 \xi \psi \eta^0 - \int_{Q_T} s\alpha_t \psi \eta^0 + \int_{Q_T} a(t,x) \psi \eta^0 \\ &\quad + \int_{Q_T} s\lambda \Delta \eta^0 \xi \psi \eta^0. \end{aligned} \tag{E.161}$$

By the properties of η^0, we have

$$\int_{Q_T} s^2\lambda^2 |\nabla \eta^0|^2 \xi^2 \psi \eta^0 \geq m \int_{Q_T} s^2 \lambda^2 \xi^2 \psi \eta^0 - m \int_{(0,T)\times\omega} s^2\lambda^2 \xi^2 \psi \eta^0, \tag{E.162}$$

where m is defined in (E.38).

By combining (E.161) and (E.162), we have

$$\begin{aligned} &m \int_{Q_T} s^2\lambda^2 \xi^2 \psi \eta^0 - \int_{Q_T} 2s\lambda \xi (\nabla \eta^0 . \nabla \psi) \eta^0 + \int_{Q_T} (\partial_t \psi) \eta^0 + \int_{Q_T} (\Delta \psi) \eta^0 \\ &\leq \int_{Q_T} s\lambda^2 |\nabla \eta^0|^2 \xi \psi \eta^0 + \int_{Q_T} s|\alpha_t| \psi \eta^0 + \int_{Q_T} |a(t,x)| \psi \eta^0 \\ &\quad + \int_{Q_T} s\lambda |\Delta \eta^0| \xi \psi \eta^0 + m \int_{(0,T)\times\omega} s^2 \lambda^2 \xi^2 \psi \eta^0. \end{aligned} \tag{E.163}$$

We have the following integration by parts

$$-\int_{Q_T} 2s\lambda \xi (\nabla \eta^0 . \nabla \psi) \eta^0 = \int_{Q_T} 2s\lambda \left((\nabla \xi . \nabla \eta^0) \eta^0 \psi + \xi (\Delta \eta^0) \eta^0 \psi + \underbrace{\xi |\nabla \eta^0|^2 \psi}_{\geq 0} \right). \tag{E.164}$$

$$\int_{Q_T} (\partial_t \psi) \eta^0 = \int_\Omega \eta^0(.)(\psi(T,.) - \psi(0,.)) = 0, \tag{E.165}$$

$$\int_{Q_T} (\Delta \psi) \eta^0 = \int_{Q_T} \psi \Delta \eta^0. \tag{E.166}$$

From (E.163), (E.164), (E.165), (E.166) and the properties of η^0, we have

$$\begin{aligned} & m\int_{Q_T} s^2\lambda^2\xi^2\psi\eta^0 + 2m\int_{Q_T} s\lambda\xi\psi \\ & \leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi\eta^0 + \int_{Q_T} s|\alpha_t|\psi\eta^0 + \int_{Q_T} |a(t,x)|\psi\eta^0 \\ & \quad +3\int_{Q_T} s\lambda|\Delta\eta^0|\xi\psi\eta^0 + 2\int_{Q_T} s\lambda|\nabla\xi||\nabla\eta^0|\psi\eta^0 + \int_{Q_T}\psi|\Delta\eta^0| \\ & \quad +m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi\eta^0 + 2m\int_{(0,T)\times\omega} s\lambda\xi\psi. \end{aligned} \tag{E.167}$$

The first five right hand side terms of (E.167) can be absorbed by the first left hand side term provided $s \geq s_1(\lambda)$ as defined in (E.159) (see 'Step 2, Absorption" of the proof of Theorem E.4.9 for details : it is exactly the same mechanism as in the proof for the Neumann case). The sixth right hand side term of (E.167) can be absorbed by the second left hand side term provided $s \geq C(\Omega,\omega)T^2$. The two last right hand side terms of (E.167) are smaller than $\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi$ provided $s \geq C(\Omega,\omega)T^2$. This leads to

$$\int_{Q_T} s^2\lambda^2\xi^2\psi\eta^0 + \int_{Q_T} s\lambda\xi\psi \leq C\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi,$$

which yields (E.160) by dividing by $s\lambda$. □

From Theorem E.7.1, we deduce a precise L^2-L^1 observability inequality as in Theorem E.4.4 by using the second left hand side term of (E.160). It is an easy adaptation of Annexe E.4.4.

The proof of the linear global nonnegative-controllability result as Theorem E.4.1 and the fixed-point argument (see Annexe E.5) remain unchanged. This leads to the small-time global nonnegative controllability for (E.157).

The proof of the large time global null-controllability result for (E.157) follows the same lines as Annexe E.6. In particular, the comparison principle between the free solution and the solution to the ordinary differential equation, i.e., (E.148) stays valid because $v(t) > 0$ on $(T_1, T_2) \times \partial\Omega$.

E.8 Comments

E.8.1 Nonlinearities depending on the gradient of the state

We may wonder to what extent our main results, i.e., Theorem E.2.2 and Theorem E.2.5 for (E.1) can be generalized for semilinearities $F(y, \nabla y)$ as considered in [DFCGBZ02] (see also [FCGBGP06a]) under appropriate assumptions on F. We focus on the Dirichlet case and we only consider semilinearities depending on the gradient state to simplify :

$$\begin{cases} \partial_t y - \Delta y + F(\nabla y) = h1_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{E.168}$$

where $F \in C^1(\mathbb{R}^N;\mathbb{R})$ verifying $F(0) = 0$.

We have the following positive global null-controllability result (see [DFCGBZ02, Theorem 1.1]).

Theorem E.8.1. We assume that (E.3) holds for $\alpha \leq 1/2$. Then, for every $T > 0$, (E.168) is globally null-controllable in time T.

Conversely, we have the following negative global null-controllability result (see [PZ12, Theorem 4.2]).

Theorem E.8.2. There exists F satisfying $F(\nabla y) \sim |\nabla y| \log^p(1 + |\nabla y|)$ as $|\nabla y| \to +\infty$ with $p > 1$ such that for every $r > 0$, there exists $y_0 \in L^\infty(\Omega)$ with $\|y_0\|_{L^\infty(\Omega)} = r$ such that for every control $h \in L^\infty(Q_T)$, the solution y of (E.168) satisfies $y(t,.) \neq 0$ for every $t < C_0 r$, where $C_0 = C_0(\Omega, \omega, F)$. In particular, the system (E.168) fails to be null controllable at any time $T > 0$.

Remark E.8.3. As already mentioned in [PZ12, Remark 4.5], there is a gap between Theorem E.8.1 and Theorem E.8.2, it concerns semilinearities satisfying (E.3) for $\alpha \in (1/2, 1]$.

Let $B \in L^\infty(Q_T)^N$. Following the proof strategy of Theorem E.2.2, we consider the parabolic equation

$$\begin{cases} \partial_t y - \Delta y + B(t,x).\nabla y = h 1_\omega & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{E.169}$$

and the following adjoint equation

$$\begin{cases} -\partial_t q - \Delta q - \nabla.(qB(t,x)) = 0 & \text{in } (0,T) \times \Omega, \\ q = 0 & \text{on } (0,T) \times \partial\Omega, \\ q(T,.) = q_T & \text{in } \Omega. \end{cases} \tag{E.170}$$

By adaptating the proof of Theorem E.4.1 (see in particular the proof of the L^1 Carleman estimate : Theorem E.4.9), we conjecture that we can prove the following theorem.

Theorem E.8.4. For every $T > 0$, (E.169) is globally nonnegative-controllable in time T with a control cost of the following form

$$C(\Omega,\omega,T,B) = \exp\left(C(\Omega,\omega)\left(1 + \frac{1}{T} + T\|B\|^2_{L^\infty(Q_T)^N} + \|B\|_{L^\infty(Q_T)^N}\right)\right). \tag{E.171}$$

Remark E.8.5. It is interesting to mention that the null-control cost of (E.169) holds with a constant of the following form

$$C(\Omega,\omega,T,B) = \exp\left(C(\Omega,\omega)\left(1 + \frac{1}{T} + T\|B\|^2_{L^\infty(Q_T)^N} + \|B\|^2_{L^\infty(Q_T)^N}\right)\right),$$

see [DFCGBZ02, Theorem 3.1]. The exponent 2 of the term $\|B\|^2_{L^\infty(Q_T)^N}$ is the key point to prove Theorem E.8.1.

Idea of the proof of Theorem E.8.4 : By setting $\psi = e^{-s\alpha} q$ where q is the solution to (E.170) associated to a nonnegative initial data, we check that ψ satisfies

$$\begin{aligned} 0 = \quad & - s\lambda^2 |\nabla \eta^0|^2 \xi \psi - 2s\lambda\xi \nabla\eta^0 . \nabla\psi + \partial_t \psi \\ & + s^2\lambda^2 |\nabla\eta^0|^2 \xi^2 \psi + \Delta\psi + s\alpha_t \psi \\ & + \nabla.(B(t,x)\psi) - s\lambda\xi\nabla\eta^0 . B(t,x)\psi - s\lambda\Delta\eta^0 \xi\psi. \end{aligned} \tag{E.172}$$

Let us explain the key point to get Theorem E.8.4. The term $-s\lambda\xi\nabla\eta^0.B(t,x)\psi$ is of order one in the parameter s whereas the term $s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi$ is of order two. This comparison will lead to (E.171). □

We strongly believe that Theorem E.8.4 could fill the gap mentioned in Remark E.8.3 in some sense thanks to the term $\exp(C(\Omega,\omega)\, \|B\|_{L^\infty(Q_T)^N}))$ in (E.171).

E.8.2 Nonlinear reaction-diffusion systems

We may also wonder to what extent our main results, i.e., Theorem E.2.2 and Theorem E.2.5 for (E.1), can be adapted to the $m \times m$ semilinear reaction-diffusion system

$$\forall 1 \leq i \leq m, \begin{cases} \partial_t u_i - d_i \Delta u_i = f_i(u_1, \dots, u_m) + h_i 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega, \end{cases} \tag{E.173}$$

with $(d_1, \dots, d_m) \in (0,+\infty)^m$ and $(f_1,\dots,f_m) \in C^1(\mathbb{R}^m;\mathbb{R})^m$ satisfying

$$\forall i \in \{1,\dots,m\},\ f_i(0,\dots,0) = 0. \tag{E.174}$$

We assume that the nonlinearity is *strongly quasi-positive*, i.e.,

$$\forall u \in \mathbb{R}^m,\ \forall i \neq j \in \{1,\dots,m\},\ \frac{\partial f_i}{\partial u_j}(u_1,\dots,u_m) \geq 0. \tag{E.175}$$

and satisfies a "*mass-control structure*"

$$\forall u \in [0,+\infty)^m,\ \sum_{i=1}^m f_i(u) \leq C\left(1 + \sum_{i=1}^m u_i\right). \tag{E.176}$$

Lots of systems come naturally with the two properties (E.175) and (E.176) in applications (see [Pie10, Section 2]).

We have the following global-nonnegative controllability result in small time.

Theorem E.8.6. For each f_i, we assume that (E.3) holds for $\alpha \leq 2$. For every $T > 0$, the system (E.173) is *globally nonnegative-controllable* in time T.

Application E.8.7. Let $\alpha \in (0,2)$. The system

$$\begin{cases} \partial_t u - \Delta u = -u\log^\alpha(2+|u|) + h_1 1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t v - \Delta v = u\log^\alpha(2+|u|) + h_2 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u,v)(0,.) = (u_0,v_0) & \text{in } \Omega, \end{cases} \tag{E.177}$$

is globally nonnegative-controllable for every time $T > 0$.

Proof. As the proof is very similar to that of Theorem E.2.2, we limit ourselves to pointing out only the differences.

Difference 1 : A L^1-Carleman estimate for a linear parabolic system. Let $A \in L^\infty(Q_T;\mathbb{R}^{m\times m})$ be such that

$$\forall i \neq j \in \{1,\dots,m\},\ \text{a.e. } (t,x) \in Q_T,\ A_{i,j}(t,x) \geq 0. \tag{E.178}$$

Remark E.8.8. The condition (E.178) is satisfied by the linearized system of (E.173) around $(0,0)$ thanks to (E.175).

We consider the adjoint system

$$\begin{cases} -\partial_t \zeta - \Delta\zeta = A(t,x)\zeta & \text{in } (0,T)\times\Omega,\\ \frac{\partial\zeta}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega,\\ \zeta(T,.) = \zeta_T & \text{in } \Omega. \end{cases} \tag{E.179}$$

Our goal is to establish this L^1-Carleman inequality : for every $\zeta_T \in L^2(\Omega;\mathbb{R}^+)^m$, the nonnegative solution ζ of (E.179) satisfies

$$\sum_{i=1}^m \int_{Q_T} e^{-s\alpha}\xi^2\zeta_i dxdt \leq C(\Omega,\omega)\left(\sum_{i=1}^m \int_{(0,T)\times\omega} e^{-s\alpha}\xi^2\zeta_i dxdt\right), \tag{E.180}$$

for any $\lambda \geq 1$, $s \geq s_1(\lambda) := C(\Omega,\omega)e^{4\lambda\|\eta^0\|_\infty}\left(T+T^2+T^2\|A\|^{1/2}_{L^\infty(Q_T;\mathbb{R}^{m\times m})}\right)$.

In order to prove (E.180), we first remark that the nonnegativity of ζ comes from (E.178) (see [PW67, Chapter 3, Theorem 13]). Then, by applying the same proof strategy to each line of (E.179) as performed in Theorem E.4.9 and by forgetting for the moment the terms involving $A_{i,j}(t,x)\zeta_j$, we get

$$\begin{aligned}\sum_{i=1}^m \int_{Q_T} e^{-s\alpha}\lambda^2(s\xi)^2\zeta_i dxdt \leq C(\Omega,\omega)\Bigg(&\|A\|_{L^\infty(Q_T)}\int_{Q_T} e^{-s\alpha}|\zeta|dxdt\\ &+\sum_{i=1}^m\int_{(0,T)\times\omega} e^{-s\alpha}\lambda^2(s\xi)^2\zeta_i dxdt\Bigg),\end{aligned} \tag{E.181}$$

for $\lambda \geq 1$, $s \geq C(\Omega,\omega)e^{4\lambda\|\eta^0\|_\infty}\left(T+T^2\right)$. We conclude the proof of (E.180) by absorbing the first right hand side term of (E.181) provided $s \geq C(\Omega,\omega)T^2\|A\|^{1/2}_{L^\infty(Q_T)}$.

Difference 2 : Without control, the free solution associated to a nonnegative initial data of (E.173) **stays nonnegative and remains bounded.** An adaptation of Lemma E.5.1 to the system (E.173) holds true. But, the reason is different. It comes from [FMT18, Theorem 1.1] which ensures global existence of classical solutions associated to nonnegative initial data for nonlinear reaction-diffusion systems with semilinearities satisfying (E.175), (E.176) and a (super)-quadratic growth (see also [Sou18] under an additional structure assumption, the so-called dissipation of entropy).

Remark E.8.9. It is worth mentioning that if the nonlinearities of (E.173) are bounded in $L^1(Q_T)$ for all $T>0$ (which is the case of (E.177) for instance), then the solutions exist globally because the growth of the semilinearity $(f_i)_{1\leq i\leq m}$ is less than $|u|^{\frac{N+2}{N}}$ (see [Pie10, Section 1]).

This concludes the proof of Theorem E.8.6. $\square$

In the following result, we give a sufficient condition to ensure the global null-controllability of (E.173).

Theorem E.8.10. Let $\alpha \in (1,2)$. For each f_i, we assume that (E.3) holds with α and

$$\exists C>0,\ \forall r\in[0,+\infty)^m,\ \sum_{i=1}^m f_i(r) \leq -C\left(\sum_{i=1}^m r_i\right)\log^{\alpha}\left(2+\left(\sum_{i=1}^m r_i\right)\right). \tag{E.182}$$

Then, there exists T sufficiently large such that (E.173) is globally null-controllable in time T.

Application E.8.11. Let $\alpha\in(1,2)$. There exists $T>0$ such that the system

$$\begin{cases} \partial_t u - \Delta u = -u\log^{\alpha}(2+|u|+|v|) + h_1 1_{\omega} & \text{in } (0,T)\times\Omega,\\ \partial_t v - \Delta v = -v\log^{\alpha}(2+|u|+|v|) + h_2 1_{\omega} & \text{in } (0,T)\times\Omega,\\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega,\\ (u,v)(0,.) = (u_0,v_0) & \text{in } \Omega, \end{cases}$$

is globally null-controllable in time $T>0$.

Proof. As the proof is very similar to that of Theorem E.2.2, we omit the details.

The first step consists in steering the initial data to a nonnegative state in time $T_1 := 1$. This is possible thanks to Theorem E.8.6. After that, we use the following comparison principle between u, the solution to

$$\forall 1\leq i\leq m,\ \begin{cases} \partial_t u_i - d_i\Delta u_i = f_i(u_1,\dots,u_m) & \text{in } (T_1,T_2)\times\Omega,\\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (T_1,T_2)\times\partial\Omega,\\ u_i(T_1,.) = u_{i,T_1} & \text{in } \Omega, \end{cases}$$

and v, the nonnegative (global) solution to the ordinary differential system

$$\forall 1\leq i\leq m,\ \begin{cases} \dot{v}_i(t) = -f_i(v(t)) & \text{in } (T_1,+\infty),\\ v_i(T_1) = \|u_{i,T_1}\|_{L^\infty(\Omega)}+1 & , \end{cases} \tag{E.183}$$

that is to say

$$\forall i\in\{1,\dots,m\},\ \forall t\in[T_1,T_2],\ \text{a.e. } x\in\Omega,\ 0\leq u_i(t,x)\leq v_i(t). \tag{E.184}$$

This comes from the quasi-monotone nondecreasing of $(f_i)_{1\leq i\leq m}$ which is a consequence of (E.175) (see [WYW06, Theorem 12.2.1] or also [Pao92, Chapter 8, Theorem 3.1]).

Then, by using (E.182), (E.183), (E.184) and the arguments of the step 2 of the proof of Theorem E.2.5, we readily get

$$\forall i \in \{1, \ldots, m\},\ \text{a.e. } x \in \Omega,\ 0 \leq u_i(T_2, x) \leq \delta,$$

where T_2 is chosen sufficiently large and $\delta > 0$ is the radius of the ball of $L^\infty(\Omega)^m$ centered at 0 where the local null-controllability of (E.173) holds in time $T = 1$ (see for instance [FCGBdT15, Theorem 1.1] and the small L^∞ perturbations method).

Then, one can steer $u(T_2, .)$ to 0 with an appropriate choice of the control. □

Another interesting problem could be to determine if Theorem E.8.6 and Theorem E.8.10 can be generalized with fewer controls than equations in (E.173). The usual strategy of Luz de Teresa to "eliminate controls" in a linear parabolic system (see [dT00] or [AKBGBdT11, Theorem 4.1]) seems to be difficult to implement because the Carleman inequality in L^1 (see Theorem E.4.9) only provide estimates on the function (and not on its partial derivatives in time and space).

E.8.3 Numerical perspectives

It would be interesting to exploit the Carleman estimate Theorem E.4.9 in order to compute numerically "optimal" (with a sense to precise) control for (E.20) (see for instance [FCM12, Section 2] or [FCM14]). A variational characterization of the control could be probably obtained as in [FCM14, Proposition 2.1] by using Stampacchia's theorem (see [Bre11, Theorem 5.6]) instead of Lax Milgram's theorem. Then, we could implement a numerical method to find controls which steer the solution of (E.1) to a nonnegative state (see [FCM12, Section 3]).

Annexe F

The Landis conjecture for nonnegative functions : a Carleman approach

F.1 Introduction : the Landis conjecture

In the late 60's (see [KL88]), Evgeni Landis conjectured that if u satisfies

$$-\Delta u + q(x)u = 0 \qquad \text{in } \mathbb{R}^N, \tag{F.1}$$

with $q \in L^\infty(\mathbb{R}^N)$ and u satisfies the exponential decay estimate

$$\exists C > 0,\ \exists \varepsilon > 0,\ \forall x \in \mathbb{R}^N,\ |u(x)| \leq C \exp(-C|x|^{1+\varepsilon}), \tag{F.2}$$

then

$$u \equiv 0 \qquad \text{in } \mathbb{R}^N. \tag{F.3}$$

The Landis' conjecture was disproved by Viktor Meshkov in 1991 in the complex case (see [Mes91]). Indeed, he constructed such $q \in L^\infty(\mathbb{R}^2;\mathbb{C})$ and nontrivial $u \in L^\infty(\mathbb{R}^2;\mathbb{C})$ satisfying

$$\exists C > 0,\ \forall x \in \mathbb{R}^2,\ |u(x)| \leq C \exp(-C|x|^{4/3}). \tag{F.4}$$

It remains an open question whether Landis' conjecture is true for real-valued functions q and u. In this short note, we confirm Landis' conjecture for any nonnegative function by a new L^1 estimate inspired by a Carleman approach. Note that this result has recently been obtained by Luca Rossi by a different approach in the preprint [Ros18].

F.2 The L^1 estimate for nonnegative functions

Theorem F.2.1. Let $q \in L^\infty(\mathbb{R}^N;\mathbb{R})$. We assume that $u \in H^1(\mathbb{R}^N)$ satisfies

$$u \geq 0 \qquad \text{in } \mathbb{R}^N, \tag{F.5}$$

$$-\Delta u + q(x)u = 0 \qquad \text{in } \mathbb{R}^N, \tag{F.6}$$

$$\exists C > 0,\ \exists \varepsilon > 0,\ \forall x \in \mathbb{R}^N,\ |u(x)| \leq C \exp(-C|x|^{1+\varepsilon}). \tag{F.7}$$

Then, we have

$$u \equiv 0 \qquad \text{in } \mathbb{R}^N. \tag{F.8}$$

F.2.1 Weight functions

Before proving Theorem F.2.1, we introduce a sequence of functions $(\psi_n)_{\in\geq 1} \in C_c^\infty(\mathbb{R}^+;[0,1])^{\mathbb{N}}$ and $\psi \in C_c^\infty(\mathbb{R}^+;[0,1])$ satisfying the following properties :

$$\forall x \in [0,1] \cup [n+1,+\infty),\ \psi_n(x) = 0, \tag{F.9}$$

$$\forall x \in [2,n],\ \psi_n(x) = 1, \tag{F.10}$$

$$\forall x \in [3/2,2],\ \psi_n(x) \geq 1/2, \tag{F.11}$$

$$\forall x \in [0,n],\ \psi_n'(x) \geq 0, \tag{F.12}$$

$$\forall x \in [0,3/2],\ \psi_n''(x) \geq 0. \tag{F.13}$$

$$\|\psi_n\|_{L^\infty(\mathbb{R}^+)} + \left\|\psi_n'\right\|_{L^\infty(\mathbb{R}^+)} + \left\|\psi_n''\right\|_{L^\infty(\mathbb{R}^+)} \leq C, \tag{F.14}$$

$$\forall x \in \mathbb{R}^+,\ \psi_n(x) \to \psi(x),\ \psi_n'(x) \to \psi'(x),\ \psi_n''(x) \to \psi''(x) \text{ as } n \to +\infty, \tag{F.15}$$

$$\forall x \in [0,1],\ \psi(x) = 0, \tag{F.16}$$

$$\forall x \in [2,+\infty),\ \psi(x) = 1, \tag{F.17}$$

$$\forall x \in [3/2,2],\ \psi(x) \geq 1/2, \tag{F.18}$$

$$\forall x \in [0,+\infty),\ \psi'(x) \geq 0, \tag{F.19}$$

$$\forall x \in [0,3/2],\ \psi''(x) \geq 0. \tag{F.20}$$

Proof. Let $\varepsilon > 0$ be defined by (F.7) and $\alpha \in (1, 1+\varepsilon)$. Let $n \geq 1$ and $s \in (0,+\infty)$ be two parameters. We set for every $x \in \mathbb{R}^N$,

$$v_n(x) := f_n(|x|)u(x), \tag{F.21}$$

where

$$f_n(|x|) := \psi_n(|x|)e^{s|x|^\alpha}, \tag{F.22}$$

where ψ_n is defined in Annexe F.2.1.

By elliptic regularity applied to (F.6), we get that $u \in H^2(\mathbb{R}^N)$. Therefore, from (F.21), we readily check that

$$v_n \in H^2(\mathbb{R}^N). \tag{F.23}$$

Moreover, by (F.7), (F.15), (F.21) and (F.22), we get that for almost every $x \in \mathbb{R}^N$,

$$v_n(x) \to v(x) := \psi(|x|)e^{s|x|^\alpha}u(x) \text{ as } n \to +\infty \quad \text{and} \quad v_n(x) \leq v(x) \in L^1(\mathbb{R}^N). \tag{F.24}$$

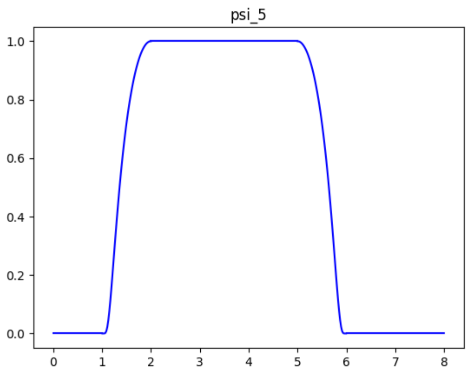

FIGURE F.1 – ψ_n for $n = 5$

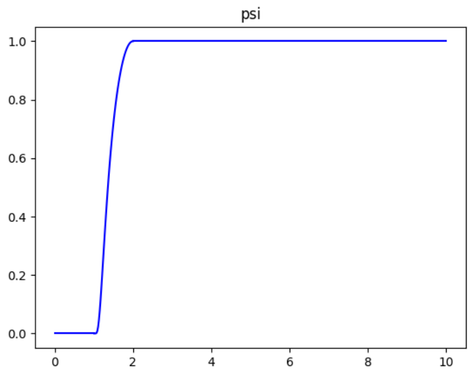

FIGURE F.2 – ψ

Step 1 : The elliptic equation satisfied by v_n. From (F.6) and (F.21), we remark that the function v_n satisfies the following elliptic equation

$$\Delta v_n - q(x)v_n = \Delta f_n(|x|)u + 2\nabla f_n(|x|).\nabla u \qquad \text{in } \mathbb{R}^N. \tag{F.25}$$

We readily have

$$\forall x \in \mathbb{R}^N,\ \nabla f_n(|x|) = f_n'(|x|)\frac{x}{|x|}, \qquad \Delta f_n(|x|) = f_n''(|x|) + \frac{N-1}{|x|}f_n'(|x|). \tag{F.26}$$

Straightforward computations lead to : for every $x \in \mathbb{R}^N$,

$$\nabla f_n(|x|) = e^{s|x|^\alpha}\left(\psi_n'(|x|)\frac{x}{|x|} + \psi_n(|x|)s\alpha|x|^{\alpha-2}x\right), \tag{F.27}$$

$$\begin{aligned}\Delta f_n(|x|) \quad &= e^{s|x|^\alpha}\Bigg(\psi_n''(|x|) + 2\psi_n'(|x|)s\alpha|x|^{\alpha-1} + \psi_n(|x|)s\alpha(\alpha-1)|x|^{\alpha-2}\\ &+\psi_n(|x|)s^2\alpha^2|x|^{2\alpha-2} + \frac{N-1}{|x|}\psi_n'(|x|) + (N-1)s\alpha|x|^{\alpha-2}\psi_n(|x|)\Bigg).\end{aligned} \tag{F.28}$$

By using $\nabla u(.) = \nabla(f_n^{-1}(|.|)v_n(.))$, we get

$$2\nabla f_n.\nabla u = -2\frac{|\nabla f_n|^2}{f^2}v_n + 2f_n^{-1}\nabla f_n.\nabla v_n. \tag{F.29}$$

Moreover, we have

$$-2\frac{|\nabla f_n|^2}{f_n^2} = -2\left(\frac{\psi_n'}{\psi_n}\right)^2 - 4\frac{\psi_n'}{\psi_n}s\alpha|x|^{\alpha-1} - 2s^2\alpha^2|x|^{2\alpha-2}, \tag{F.30}$$

$$2f_n^{-1}\nabla f_n = 2\frac{\psi_n'}{\psi_n}\frac{x}{|x|} + 2s\alpha|x|^{\alpha-2}x. \tag{F.31}$$

Therefore, by using (F.25), (F.27), (F.28), (F.29), (F.30) and (F.31), we get

$$\Delta v_n + (-A(x) - q(x))v_n - B(x).\nabla v_n = 0, \tag{F.32}$$

where

$$\begin{aligned} A(x) \;&= \frac{\psi_n''}{\psi_n} + 2\frac{\psi_n'}{\psi_n} s\alpha|x|^{\alpha-1} + s\alpha(\alpha-1)|x|^{\alpha-2} \\ &\quad + s^2\alpha^2|x|^{2\alpha-2} + \frac{N-1}{|x|}\frac{\psi_n'}{\psi_n} + (N-1)s\alpha|x|^{\alpha-2} \\ &\quad -2\left(\frac{\psi_n'}{\psi_n}\right)^2 - 4\frac{\psi_n'}{\psi_n}s\alpha|x|^{\alpha-1} - 2s^2\alpha^2|x|^{2\alpha-2}, \end{aligned} \tag{F.33}$$

$$B(x) = 2\frac{\psi_n'}{\psi_n}\frac{x}{|x|} + 2s\alpha|x|^{\alpha-2}x. \tag{F.34}$$

After simplification, we get from (F.33) that

$$\begin{aligned} A(x) \;&= \frac{\psi_n''}{\psi_n} + s\alpha(N+\alpha-2)|x|^{\alpha-2} \\ &\quad + \frac{N-1}{|x|}\frac{\psi_n'}{\psi_n} \\ &\quad -2\left(\frac{\psi_n'}{\psi_n}\right)^2 - 2\frac{\psi_n'}{\psi_n}s\alpha|x|^{\alpha-1} - s^2\alpha^2|x|^{2\alpha-2}. \end{aligned} \tag{F.35}$$

Step 2 : Integration over $\mathbb{R}^N$. We multiply (F.32) by ψ_n and we integrate over $\mathbb{R}^N$. After some integration by parts which are legitimated because v_n and $\psi_n \in H^2(\mathbb{R}^N)$ by using (F.23), we get

$$\int_{\mathbb{R}^N} \Delta v_n \psi_n = \int_{\mathbb{R}^N} v_n \Delta\psi_n = \int_{\mathbb{R}^N} v_n\psi_n'' + v_n\frac{N-1}{|x|}\psi_n', \tag{F.36}$$

$$\begin{aligned} \int_{\mathbb{R}^N} -A(x)\psi_n v_n \;&= -\int_{\mathbb{R}^N}\psi_n'' v_n - \int_{\mathbb{R}^N} s\alpha(N+\alpha-2)|x|^{\alpha-2}\psi_n v_n \\ &\quad - \int_{\mathbb{R}^N}\frac{N-1}{|x|}\psi_n' v_n + \int_{\mathbb{R}^N} 2\left(\frac{\psi_n'}{\psi_n}\right)^2\psi_n v_n \\ &\quad + \int_{\mathbb{R}^N} 2\psi_n' s\alpha|x|^{\alpha-1}v_n + \int_{\mathbb{R}^N} s^2\alpha^2|x|^{2\alpha-2}\psi_n v_n \end{aligned} \tag{F.37}$$

$$\begin{aligned} \int_{\mathbb{R}^N} -\psi_n B(x).\nabla v_n \;&= \int_{\mathbb{R}^N}\nabla.(\psi_n(|x|)B(x))v_n, \\ &= \int_{\mathbb{R}^N}\nabla.\left(2\psi_n'\frac{x}{|x|} + 2s\alpha|x|^{\alpha-2}\psi_n x\right)v_n \\ &= \int_{\mathbb{R}^N}\left(2\psi_n'' + 2(N-1)\frac{\psi_n'}{|x|}\right)v_n \\ &\quad + \int_{\mathbb{R}^N} 2s\alpha\psi_n'|x|^{\alpha-1}v_n \\ &\quad + 2s\alpha\left((\alpha-2)|x|^{\alpha-2}\psi_n + N|x|^{\alpha-2}\psi_n\right)v_n. \end{aligned} \tag{F.38}$$

By combining (F.32), (F.36), (F.37) and (F.38), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} s\alpha(N+\alpha-2)|x|^{\alpha-2}\psi_n v_n + \int_{\mathbb{R}^N} 2\frac{N-1}{|x|}\psi_n' v_n \\
&+ \int_{\mathbb{R}^N} 2\left(\frac{\psi_n'}{\psi_n}\right)^2 \psi_n v_n + \int_{\mathbb{R}^N} 4\psi_n' s\alpha|x|^{\alpha-1}v_n \\
&+ \int_{\mathbb{R}^N} s^2\alpha^2|x|^{2\alpha-2}\psi_n v_n + \int_{\mathbb{R}^N} 2\psi_n'' v_n \\
&\quad = \int_{R^N} q(x)\psi_n v_n.
\end{aligned} \tag{F.39}$$

By using (F.15), (F.14), (F.24) and the Lebesgue's dominated convergence theorem, we pass to the limit in (F.39) as $n \to +\infty$ to get

$$\begin{aligned}
&\int_{\mathbb{R}^N} s\alpha(N+\alpha-2)|x|^{\alpha-2}\psi v + \int_{\mathbb{R}^N} 2\frac{N-1}{|x|}\psi' v \\
&+ \int_{\mathbb{R}^N} 2\left(\frac{\psi'}{\psi}\right)^2 \psi v + \int_{\mathbb{R}^N} 4\psi' s\alpha|x|^{\alpha-1}v \\
&+ \int_{\mathbb{R}^N} s^2\alpha^2|x|^{2\alpha-2}\psi v + \int_{\mathbb{R}^N} 2\psi'' v \\
&\quad = \int_{R^N} q(x)\psi v.
\end{aligned} \tag{F.40}$$

By using the fact that ψ and ψ' are nonnegative functions (see (F.16), (F.17), (F.19)), we obtain from (F.40)

$$\int_{\mathbb{R}^N} s^2\alpha^2|x|^{2\alpha-2}\psi v \leq \int_{R^N} q(x)\psi v - \int_{\mathbb{R}^N} 2\psi'' v. \tag{F.41}$$

Moreover, by using (F.18), (F.20) and $v \geq 0$, we have for some constant $C > 0$,

$$-\int_{\mathbb{R}^N} 2\psi'' v = \underbrace{-\int_{|x|\leq 3/2} 2\psi'' v}_{\leq 0} - \int_{|x|>3/2} 2\psi'' v \leq C\int_{\mathbb{R}^N} \psi v. \tag{F.42}$$

Then, by using (F.41) and (F.42), we get

$$\int_{\mathbb{R}^N} s^2\alpha^2|x|^{2\alpha-2}\psi v \leq \int_{R^N} q(x)\psi v + C\int_{\mathbb{R}^N} \psi v. \tag{F.43}$$

By using (F.16), we get from (F.43)

$$\int_{|x|>1} s^2\alpha^2|x|^{2\alpha-2}\psi v \leq \int_{|x|>1} q(x)\psi v + C\int_{|x|>1} \psi v. \tag{F.44}$$

By using the fact that $2\alpha - 2 > 0$ because $\alpha > 1$ and by taking s sufficiently large in (F.44), we get

$$\int_{|x|>1} s^2\alpha^2|x|^{2\alpha-2}\psi v \leq 0. \tag{F.45}$$

We deduce from (F.17) and (F.45) that $v \equiv 0$ in $\{|x| \geq 2\}$, then $u \equiv 0$ in $\{|x| \geq 2\}$. We conclude by a well-known unique continuation theorem for (F.6) to deduce that $u \equiv 0$ in $\mathbb{R}^N$ (see for instance [LRL12, Theorem 4.2]). This concludes the proof of Theorem F.2.1. $\square$

Annexe G

Null-controllability of linear parabolic-transport systems

Abstract : Over the past two decades, the controllability of several examples of parabolic-hyperbolic systems has been investigated. The present article is the beginning of an attempt to find a unified framework that encompasses and generalizes the previous results.

We consider constant coefficients heat-transport systems with coupling of order zero and one, with a locally distributed control in the source term, posed on the one dimensional torus.

We prove the null-controllability, in optimal time (the one expected for the transport component) when there is as much controls as equations.

The whole study relies on a careful spectral analysis, based on perturbation theory. The proof of the negative result in small time uses holomorphic technics. The proof of the positive result in large time relies on a decomposition into low, asymptotically parabolic and asymptotically hyperbolic frequencies.

G.1 Introduction

G.1.1 Parabolic-transport systems

We consider the linear control system

$$\begin{cases} \partial_t f - B\partial_x^2 f + A\partial_x f + Kf = Mu1_\omega & \text{in } (0,T)\times\mathbb{T}, \\ f(0,.) = f_0 & \text{in } \mathbb{T}, \end{cases} \tag{Sys}$$

where

— $T > 0$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, ω is a nonempty open subset of $\mathbb{T}$, $d \in \mathbb{N}^*$, $m \in \{1, \ldots, d\}$, $A, B, K \in \mathbb{R}^{d\times d}$, $M \in \mathbb{R}^{d\times m}$,

— the state is $f : [0,T]\times\mathbb{T} \to \mathbb{R}^d$,

— the control is $u : [0,T]\times\mathbb{T} \to \mathbb{R}^m$.

We assume

$$d = d_1 + d_2 \text{ with } 1 \leq d_1 < d,\ 1 \leq d_2 < d, \tag{H.1}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \text{ with } D \in \mathbb{R}^{d_2 \times d_2}, \tag{H.2}$$

$$\Re(\mathrm{Sp}(D)) \subset (0, \infty). \tag{H.3}$$

Introducing the analogue block decomposition for the $d \times d$ matrices A and K, the $d \times m$ matrix M and the function f,

$$A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \quad M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \quad f(t,x) = \begin{pmatrix} f_1(t,x) \\ f_2(t,x) \end{pmatrix},$$

we see that the system (Sys) couples a transport equation on f_1 with a parabolic equation on f_2

$$\begin{cases} (\partial_t + A'\partial_x + K_{11})f_1 + (A_{12}\partial_x + K_{12})f_2 = M_1 u 1_\omega & \text{in } (0,T) \times \mathbb{T}, \\ \left(\partial_t - D\partial_x^2 + A_{22}\partial_x + K_{22}\right)f_2 + (A_{21}\partial_x + K_{21})f_1 = M_2 u 1_\omega & \text{in } (0,T) \times \mathbb{T}, \\ (f_1, f_2)(0,.) = (f_{01}, f_{02}) & \text{in } \mathbb{T}. \end{cases} \tag{G.1}$$

We make the following hypothesis on the matrix A'

$$A' \text{ is diagonalizable with } \mathrm{Sp}(A') \subset \mathbb{R}. \tag{H.4}$$

We will prove later, with vector valued Fourier series and a careful spectral analysis, that for every $f_0 \in L^2(\mathbb{T}, \mathbb{C}^d)$ and $u \in L^2((0,T) \times \mathbb{T}, \mathbb{C}^m)$, there exists a unique solution $f \in C^0([0,T], L^2(\mathbb{T})^d)$ of (Sys) (see Section G.2.3.3). In this article, we are interested in the null controllability of (Sys).

Definition G.1.1. The system (Sys) is null-controllable on ω in time T if for every $f_0 \in L^2(\mathbb{T}; \mathbb{C}^d)$, there exists a control $u \in L^2((0,T) \times \omega, \mathbb{C}^m)$ such that the solution f of (Sys) satisfies $f(T, \cdot) = 0$.

We aim at

- identifying the minimal time for null controllability,
- controlling the system with a small number of controls $m < d$,
- understanding the influence of the algebraic structure (A, B, K, M) on the above properties.

G.1.2 Statement of the results

The first result identifies the minimal time, when the control acts on each of the d equations.

Theorem G.1.2. We assume that ω is a strict open subset of $\mathbb{T}$. We also assume (H.1)–(H.4) and that the control matrix is $M = I_d$ (and so $m = d$). We define [1]

$$\ell(\omega) := \sup\{|I|;\ I \text{ connected component of } \mathbb{T} \setminus \omega\},$$

$$\mu_* = \min\{|\mu|;\ \mu \in \mathrm{Sp}(A')\},$$

and

$$T^* = \begin{cases} \frac{\ell(\omega)}{\mu_*} & \text{if } \mu_* > 0, \\ +\infty & \text{if } \mu_* = 0. \end{cases} \tag{G.2}$$

Then

1. the system (Sys) is not null-controllable on ω in time $T < T^*$,
2. the system (Sys) is null-controllable on ω in any time $T > T^*$.

In particular, when ω is an interval of $\mathbb{T}$ and $\mu_* > 0$, then the minimal time for null controllability is $T^* = \frac{2\pi - |\omega|}{\mu_*}$.

The negative result in time $T < T^*$ is expected, because of the transport component of the system, but its proof is not obvious. Indeed, because of the coupling with a parabolic component, there may not exist pure transport solutions to the system (Sys). Our proof relies on holomorphic functions technics developped by the second author [Koe17].

The proof of the positive result, in time $T > T^*$ relies on an adaptation, to systems with arbitrary size, of the strategy introduced by Lebeau and Zuazua in [LZ98] to control the system of linear thermoelasticity, that couples a scalar heat equation and a scalar wave equation. Actually, the controls are more regular than expected in Definition G.1.1 : we construct controls of the form $u = (u_1, u_2)$ where $u_1 \in L^2((0,T) \times \omega)^{d_1}$ and $u_2 \in C_c^\infty((0,T) \times \omega)^{d_2}$.

The null controllability of the system (Sys) in time $T = T^*$ is an open problem.

G.1.3 Organization of the article

This introduction ends with bibliographical comments.

Section G.2 is dedicated to preliminary results : first a careful spectral analysis of $-B\partial_x^2 + A\partial_x + K$ on $\mathbb{T}$; then the well posedness of (Sys) in $L^2(\mathbb{T})^d$ thanks to precise estimates of the Fourier components of its solution ; finally the Hilbert uniqueness method i.e. the equivalence between null controllability and observability.

In Section G.3, we prove the negative null controllability result in time $T < T^*$ of Theorem G.1.2, by constructing a counter-example to the observability inequality, thanks to holomorphic technics.

In Section G.4, we prove the positive null controllability result in time $T > T^*$ of Theorem G.1.2. The proof relies on a spectral decomposition of $L^2(\mathbb{T})^d$ into three weakly coupled parts : a high frequency hyperbolic part handled by hyperbolic methods, a high frequency parabolic part handled by the Lebeau Robbiano's method and a low frequency part handled by a compactness/uniqueness argument.

1. If $I \subset \mathbb{R}$ is measurable, we note $|I|$ its Lebesgue measure.

G.1.4 Bibliographical comments

G.1.4.1 Wave equation with structural damping

We consider the 1D wave equation with structural damping and control h

$$\partial_t^2 y - \partial_x^2 y - \partial_t \partial_x^2 y + b\partial_t y = h(t,x)\,, \tag{G.3}$$

where $b \in \mathbb{R}$. This equation can be splitted in a system of the form (Sys) by considering $z := \partial_t y - \partial_x^2 y + (b-1)y$,

$$\begin{cases} \partial_t z + z + (1-b)y = h(t,x), \\ \partial_t y - \partial_x^2 y - z + (b-1)y = 0, \end{cases} \tag{G.4}$$

i.e. (Sys) with

$$f = \begin{pmatrix} z \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 1-b \\ -1 & b-1 \end{pmatrix}. \tag{G.5}$$

Rosier and Rouchon [RR07] studied the equation (G.3) on a 1D-interval, $x \in (0,1)$, with a boundary control at $x = 1$. This is essentially equivalent to take (G.3) with $x \in (0,1)$, Dirichlet boundary conditions at $x = 0$ and $x = 1$, and a source term of the form $h(t,x) = u(t)p(x)$, where p is a fixed profile and u is a scalar control. The authors prove that this equation is not controllable.

By Theorem G.1.2, we extend this negative result to general controls h (i.e. without separate variables) for periodic boundary conditions. Here, $A' = 0$, $\mu_* = 0$, $T^* = +\infty$, the system (G.4) is not controllable even with an additional control in the second equation.

In [RR07], the authors prove that this system is not even spectrally controllable, because of an accumulation point in the spectrum. Indeed, by the moment method, a control that would steer the system from an eigenstate to another one would have a Fourier transform vanishing on a set with an accumulation point, which is not possible for an holomorphic function.

Martin, Rosier and Rouchon [MRR13], studied the null-controllability of the equation (G.3) on the 1D torus, $x \in \mathbb{T}$, with moving controls, i.e. $h(t,x) = u(t,x)1_{\omega+ct}$ with $c \in \mathbb{R}^*$. By the change of variable $x \longleftarrow (x - ct)$, this is equivalent to study the null controllability of the system

$$\begin{cases} \partial_t z - c\partial_x z + z + (1-b)y = u(t,x)1_\omega(x), \\ \partial_t y - c\partial_x y - \partial_x^2 y - z + (b-1)y = 0 \end{cases} \tag{G.6}$$

which has the form (Sys) with the same matrices f, B, K as in (G.5) and

$$A = \begin{pmatrix} -c & 0 \\ 0 & -c \end{pmatrix}.$$

In [MRR13, Theorem 1.2], for $c = 1$, the authors prove that any initial data $(y_0, y_1) \in H^{s+2} \times H^s(\mathbb{T})$ with $s > 15/2$ can be steered to 0 in time $T > 2\pi$ by mean of a control $u \in L^2((0,T) \times \omega)$.

By a future theorem (see [BKLB19]), we recover this positive null controllability result with a smaller minimal time $T > \ell(\omega)/|c|$ and a weaker regularity assumption on the initial data $(y, \partial_t y)(0) = (y_0, y_1) \in H^2 \times L^2(\mathbb{T})$ for (G.3). This corresponds to an initial data $(y, z)(0) \in L^2(\mathbb{T})^2$ for (G.6) because $z(0) = y_1 - \partial_x^2 y_0 + (b-1)y_0$. We also prove the negative result in time $T < \ell(\omega)/|c|$. Here, $\mu_* = |c|$, $A_{21} = 0$ and $K_{21} = -1$.

The limitations in [MRR13, Theorem 1.2] (regularity and time) are due to the use of controls with separate variables $u(t, x) = u_1(t)u_2(x)$. The proof relies on the moment method and the construction of a biorthogonal family. A key point in both [MRR13] and the present article is a splitting of the spectrum in one parabolic-type part, and one hyperbolic-type part.

Finally, Chaves-Silva, Rosier and Zuazua [CSRZ14] study the multi-dimensional case of equation (G.3), $x \in \Omega$, with Dirichlet boundary conditions and locally distributed moving controls $h(t, x) = u(t, x)1_{\omega(t)}(x)$. The control region $\omega(t)$ is assumed to be driven by the flow of an ODE that covers all the domain Ω within the alloted time T. Then, the authors prove the null controllability of any initial data $(y_0, y_1) \in H^2 \cap H_0^1(\Omega) \times L^2(\Omega)$ with an L^2-control.

In the particular case $\Omega = \mathbb{T}$ with a motion with constant velocity, a future theorem (see [BKLB19]) gives the same minimal time for the null controllability and also the negative result in smaller time.

The proof strategy in [CSRZ14] consists in proving Carleman estimates for the parabolic equation and the ODE in (G.4) with *the same singular weight*, adapted to the geometry of the moving support of the control.

As explained in [CSRZ14, Section 5.2], the same strategy cannot be used with periodic boundary conditions, because a weight appropriate for both the parabolic equation and the ODE does not exist. This is why, in the present article, we develop another strategy.

G.1.4.2 Wave-parabolic systems

Albano and Tataru [AT00] consider 2×2 parabolic-wave systems with boundary control, where

- the coupling term in the wave equation is given by a second order operator with respect to x,
- the coupling term in the parabolic equation is given by a first order operator with respect to (t, x).

This large class contains the linear system of thermoelasticity

$$\begin{cases} \partial_t^2 w - \Delta w + \alpha \Delta \theta = 0, & (t, x) \in (0, T) \times \Omega, \\ \partial_t \theta - \nu \Delta \theta + \beta \partial_t w = 0, & (t, x) \in (0, T) \times \Omega, \\ w(t, x) = u_1(t, x), & (t, x) \in (0, T) \times \partial\Omega, \\ \theta(t, x) = u_2(t, x), & (t, x) \in (0, T) \times \partial\Omega, \end{cases} \tag{G.7}$$

where $\alpha, \beta, \nu > 0$.

The authors of [AT00] prove the null controllability in large time of these systems. Precisely in any time $T > 2\sup\{|x|; x \in \Omega\}$ for the system (G.7). The proof relies on

Carleman estimates for the heat and the wave equation with the same singular weight. This strategy inspired Chaves-Silva, Rosier and Zuazua [CSRZ14] and does not seem adapted to periodic boundary conditions, as explained in the previous section.

Lebeau and Zuazua [LZ98] prove the null-controllability of the linear system of thermoelasticity (G.7) with a locally distributed control in the source term of the wave equation, under the geometric control condition on (Ω, ω, T). The method is based on a spectral decomposition. For high frequencies, the spectrum splits into a parabolic part and a hyperbolic part. Projecting the dynamics onto the parabolic/hyperbolic subspaces, the system is decomposed into two weakly coupled systems, the first one behaving like a wave equation, the second one like a heat equation. The wave equation is handled by using the microlocal techniques developped for the wave equation [BLR92]. The parabolic equation is treated by using the Lebeau and Robbiano's method [LR95]. The low frequency part is treated by a compactness argument relying on a unique continuation property.

The proof of the positive controllability results in the present article is an adaptation, to coupled transport-parabolic systems of any size, of this approach, introduced for a 2×2 wave-parabolic system. The transport equation is handled by using the results from Alabau-Boussouira, Coron and Olive [ABCO17].

G.1.4.3 Heat equation with memory

Gerrero and Imanuvilov [GI13] consider the heat equation with memory

$$\begin{cases} \partial_t y - \Delta y - \int_0^t \Delta y(\tau)\,\mathrm{d}\tau = u 1_\omega, & (t,x) \in (0,T)\times\Omega, \\ y(t,x) = 0, & (t,x) \in (0,T)\times\partial\Omega \end{cases} \tag{G.8}$$

In 1D, this equation can be splitted into a system of the form (Sys) by considering $v(t,x) = -\int_0^t y_x(\tau)\,\mathrm{d}\tau$:

$$\begin{cases} \partial_t v + \partial_x y = 0, \\ \partial_t y - \partial_x^2 y + v_x = h 1_\omega \end{cases} \tag{G.9}$$

i.e.

$$f = \begin{pmatrix} v \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In [GI13], the authors prove that the scalar equation (G.8) is not null controllable (whatever $T > 0$). Thus the system (G.9) is not null controllable. This example gives limitations on the possibility of controlling the whole system with a control acting only on the parabolic component. In particular the condition $K_{12} = 0$ and A_{12}^{tr} injective is not sufficient.

The present article does not provide any result for the scalar equation (G.8) because it focuses on the null controllability of all the components of the system (and not only on one of them).

G.2 Preliminary results

G.2.1 Fourier components

We want to understand the operator

$$\mathcal{L} := -B\partial_x^2 + A\partial_x + K \tag{G.10}$$

with domain

$$D(\mathcal{L}) = \left\{ f \in L^2(\mathbb{T})^d ; -B\partial_x^2 f + A\partial_x f + Kf \in L^2(\mathbb{T})^d \right\} \tag{G.11}$$

where the derivatives are considered in the distributional sense $\mathcal{D}'(\mathbb{T})$. Throughout the article, we will note e_n the function $x \mapsto \mathrm{e}^{\mathrm{i}nx}$. We remark that applying $\mathcal{L}$ to Xe_n, where $X \in \mathbb{C}^d$, we get

$$\mathcal{L}(Xe_n) = n^2 \left(B + \frac{\mathrm{i}}{n} A + \frac{1}{n^2} K \right) Xe_n. \tag{G.12}$$

Thus, if we define $E(z)$ the following perturbation of B

$$\forall z \in \mathbb{C},\ E(z) = B + zA - z^2 K, \tag{G.13}$$

then $\mathcal{L}$ acts on the Fourier side as a multiplication by $n^2 E(\mathrm{i}/n)$.

G.2.2 Perturbation theory

If we want to understand the semigroup $\mathrm{e}^{t\mathcal{L}}$, we need to know the spectrum and the eigenvectors of $E(z)$. Here, we relate the spectral properties of $E(z)$ to those of A and B, in the limit $z \to 0$. This is instrumental in all the rest of the article. Our proof are essentially self contained, but the reader unfamiliar with the analytic perturbation theory in finite dimension may read [Kat95, Ch. II §1 and §2].

For $r > 0$ and $m \in \mathbb{N}^*$, we define $\mathcal{O}_r^{m\times m}$ as the set of holomorphic functions in the complex disk $D(0,r)$ with values in $\mathbb{C}^{m\times m}$. Our first result is the following one.

Proposition G.2.1. There exist $r > 0$ and a matrix-valued holomorphic function $P^{\mathrm{h}} \in \mathcal{O}_r^{m\times m}$ such that

1. $P^{\mathrm{h}}(0) = \left(\begin{smallmatrix} I_{d_1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$,
2. for all $|z| < r$, $P^{\mathrm{h}}(z)$ is a projection that commutes with $E(z)$,
3. in the limit $z \to 0$, $E(z)P^{\mathrm{h}}(z) = O(z)$.

Proof. The spectrum of $E(z)$ is continuous in z (see [Kat95, Ch. II §1.1]). Let us consider the "0-group" of eigenvalues, i.e. the set of eigenvalues that tend to 0 as $z \to 0$. Then we note $P^{\mathrm{h}}(z)$ the sum of the projections onto the eigenspace [2] of $E(z)$ associated with eigenvalues in the 0-group along the other eigenspaces. Another way to define $P^{\mathrm{h}}(z)$

2. We stress that when we talk about "eigenspace", we mean "generalized eigenspace" (or, in the terminology of Kato, algebraic eigenspace), i.e. the space of generalized eigenvectors.

is to choose $R = \frac{1}{2}\min_{\lambda\in\mathrm{Sp}(D)}|\lambda|$ and r small enough so that for $|z| < r$, there is no eigenvalues of $E(z)$ on the circle $\partial D(0,R)$. Then, we define (see [Kat95, Ch. II §1.4, Eq. (1.16)])

$$P^{\mathrm{h}}(z) = -\frac{1}{2\mathrm{i}\pi}\int_{\partial D(0,R)} (E(z)-\zeta)^{-1}\,\mathrm{d}\zeta. \tag{G.14}$$

In the terminology of Kato, $P^{\mathrm{h}}(z)$ is the "total projection for the 0-group". Then, according to [Kat95, Ch. II §1.4], $P^{\mathrm{h}}(z)$ is the projection onto the sum of eigenspaces associated to eigenvalues of $E(z)$ lying inside $D(0,R)$ along the other eigenspaces. It is holomorphic in $|z| < r$. For $z = 0$, the formula (G.14) that defines $P^{\mathrm{h}}(0)$ becomes

$$P^{\mathrm{h}}(0) = -\frac{1}{2\mathrm{i}\pi}\int_{\partial D(0,R)} (B-\zeta)^{-1}\,\mathrm{d}\zeta.$$

Then, $P^{\mathrm{h}}(0)$ is the projection onto the eigenspace of B associated to the eigenvalue 0 along the other eigenspaces (see [Kat95, Ch. II §1.4]). So, according to the hypotheses (H.2–H.3) on the blocks of B, $P^{\mathrm{h}}(0) = \left(\begin{smallmatrix} I_{d_1} & 0 \\ 0 & 0\end{smallmatrix}\right)$. This proves *i)*.

According to the definition (G.14), $P^{\mathrm{h}}(z)$ commutes with $E(z)$. This proves *ii)*. Then we have

$$P^{\mathrm{h}}(0)E(0) = E(0)P^{\mathrm{h}}(0) = BP^{\mathrm{h}}(0) = 0,$$

which, along with the holomorphy of P^{h}, proves *iii)*. □

We say that P^{h} is the "projection on the hyperbolic branches". We note $P^{\mathrm{p}}(z) = I_d - P^{\mathrm{h}}(z)$, which we call the "projection on the parabolic branches", and satisfies properties analog to P^{h} :

Proposition G.2.2. The matrix-valued function P^{p} is in $\mathcal{O}_r^{m\times m}$ and

1. $P^{\mathrm{p}}(0) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{d_2}\end{smallmatrix}\right)$,
2. for all $|z| < r$, $P^{\mathrm{p}}(z)$ is a projection that commutes with $E(z)$,
3. in the limit $z \to 0$, $E(z)P^{\mathrm{p}}(z) = B + O(z)$.

We will need to split the hyperbolic branches further.

Proposition G.2.3. There exist $r > 0$ and a family of matrix-valued holomorphic functions $(P^{\mathrm{h}}_\mu)_{\mu\in\mathrm{Sp}(A')} \in (\mathcal{O}_r^{d\times d})^{\mathrm{Sp}(A')}$ satisfying

1. for all $\mu \in \mathrm{Sp}(A')$ and $|z| < r$, $P^{\mathrm{h}}_\mu(z)$ is a non-zero projection that commutes with $E(z)$,
2. for all $|z| < r$, $P^{\mathrm{h}}(z) = \sum_{\mu\in\mathrm{Sp}(A')} P^{\mathrm{h}}_\mu(z)$ and for all $\mu \neq \mu'$, $P^{\mathrm{h}}_\mu(z)P^{\mathrm{h}}_{\mu'}(z) = 0$,
3. for every $\mu \in \mathrm{Sp}(A')$, there exists $R^{\mathrm{h}}_\mu \in \mathcal{O}_r^{d\times d}$ such that

$$\forall |z| < r,\ E(z)P^{\mathrm{h}}_\mu(z) = \mu z P^{\mathrm{h}}_\mu(z) + z^2 R^{\mathrm{h}}_\mu(z).$$

Remark G.2.4. For $\mu \in \mathrm{Sp}(A')$, the projection P^{h}_{μ} is holomorphic and thus continuous in $D(0,r)$. Therefore, the rank of $P^{\mathrm{h}}_{\mu}(z)$, which is its trace, does not depend on $|z| < r$ (the $P^{\mathrm{h}}_{\mu}(z)$ even are similar, see [Kat95, Ch. I, §4.6, Lem. 4.10]). In the same vein, the ranks of $P^{\mathrm{h}}(z)$ and $P^{\mathrm{p}}(z)$ do not depend on z.

Proof. The proof is essentially the "reduction process" of Kato [Kat95, Ch. II §2.3]. According to Prop. G.2.1, P^{h} is holomorphic and $P^{\mathrm{h}}(z)E(z) = O(z)$. Then we define

$$E^{(1)}(z) = z^{-1}E(z)P^{\mathrm{h}}(z) = z^{-1}P^{\mathrm{h}}(z)E(z),$$

which is holomorphic in $|z| < r$. Note that we have according to Kato [Kat95, Ch. II Eq. (2.38)]

$$E^{(1)}(0) = P^{\mathrm{h}}(0)E(0)P^{\mathrm{h}}(0) = \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix}.$$

Let us assume for the moment that 0 is not an eigenvalue of A'. Then, for $\mu \in \mathrm{Sp}(A')$, we define $P^{\mathrm{h}}_{\mu}(z)$ the total projection on the μ-group of eigenvalues of $E^{(1)}(z)$. Said otherwise, and according to the definition of $E^{(1)}(z)$, $P^{\mathrm{h}}_{\mu}(z)$ is the total projection on the μz-group of eigenvalues of $E(z)$. The projection $P^{\mathrm{h}}_{\mu}(z)$ is defined and holomorphic for z small enough according to [Kat95, Ch. II, §1.4].

Since for z small enough, $P^{\mathrm{h}}_{\mu}(z)$ is the projection on some eigenspaces of $E^{(1)}(z)$ associated with non-zero eigenvalues,

$$\mathrm{Im}(P^{\mathrm{h}}_{\mu}(z)) \subset \mathrm{Im}(E^{(1)}(z)) \subset \mathrm{Im}(P^{\mathrm{h}}(z)),$$

with the last inclusion coming from the definition of $E^{(1)}(z)$. Thus $P^{\mathrm{h}}_{\mu}(z)$ is a subprojection of $P^{\mathrm{h}}(z)$. Moreover, $P^{\mathrm{h}}_{\mu}(z)$ commutes with $E^{(1)}(z)$, so it commutes with $E(z)$. This proves Item *i)* in the case $0 \notin \mathrm{Sp}(A')$.

For $\mu \neq \nu$, $P^{\mathrm{h}}_{\mu}(z)$ and $P^{\mathrm{h}}_{\nu}(z)$ are the projections on some sums of eigenspaces associated with different eigenvalues, so $P^{\mathrm{h}}_{\mu}(z)P^{\mathrm{h}}_{\nu}(z) = 0$. Let us note for convenience $Q^{\mathrm{h}}(z) = \sum_{\mu \in \mathrm{Sp}(A')} P^{\mathrm{h}}_{\mu}(z)$. Then, for z small, $Q^{\mathrm{h}}(z)$ is the projection on all the eigenspaces of $E^{(1)}(z)$ associated with non-zero eigenvalues. According to the definition of $E^{(1)}(z)$, this proves that $Q^{\mathrm{h}}(z)$ is a subprojection of $P^{\mathrm{h}}(z)$. Let us check that $Q^{\mathrm{h}}(z)$ and $P^{\mathrm{h}}(z)$ have the same rank. This will prove that for all z small enough, $Q^{\mathrm{h}}(z) = P^{\mathrm{h}}(z)$. The rank of $Q^{\mathrm{h}}(z)$, which is its trace, does not depend on z. The same is true for $P^{\mathrm{h}}(z)$. For $z = 0$, we have $E^{(1)}(0) = \left(\begin{smallmatrix} A' & 0 \\ 0 & 0 \end{smallmatrix}\right)$, so by using the fact that $0 \notin \mathrm{Sp}(A')$,

$$Q^{\mathrm{h}}(0) = \begin{pmatrix} I_{d_1} & 0 \\ 0 & 0 \end{pmatrix} = P^{\mathrm{h}}(0).$$

This proves that for all z small enough, $Q^{\mathrm{h}}(z) = P^{\mathrm{h}}(z)$, and in turn finishes the proof of Item *ii)* in the case where $0 \notin \mathrm{Sp}(A')$.

If $0 \in \mathrm{Sp}(A')$, then we add $\alpha z I$ to $E(z)$ for some $\alpha \in \mathbb{C}$. This amounts to adding $\alpha P^{\mathrm{h}}(z)$ to $E^{(1)}(z)$. This only shifts the eigenvalues of the restriction of $E^{(1)}(z)$ to

$\mathrm{Im}(P^{\mathrm{h}}(z))$ (but not of its restriction to $\mathrm{Im}(I_d - P^{\mathrm{h}}(z))$) by α, while leaving the eigenprojections unchanged. Thus, choosing α so that $0 \notin \alpha + \mathrm{Sp}(A')$, we get the Items *i)* and *ii)* in the case $0 \in \mathrm{Sp}(A')$.

We still need to prove the asymptotics of Item *iii)*. Since A' is diagonalizable, so is $E^{(1)}(0) = \left(\begin{smallmatrix} A' & 0 \\ 0 & 0 \end{smallmatrix}\right)$. So, there is no nilpotent part in the spectral decomposition of $E^{(1)}(0)$. That is to say, for all $\mu \in \mathrm{Sp}(A')$,

$$E^{(1)}(0)P^{\mathrm{h}}_{\mu}(0) = \mu P^{h}_{\mu}(0).$$

Since $z \mapsto E^{(1)}(z)P^{\mathrm{h}}_{\mu}(z)$ is holomorphic, we have

$$E^{(1)}(z)P^{\mathrm{h}}_{\mu}(z) = \mu P^{h}_{\mu}(z) + O(z).$$

Finally, we multiply by z to come back to $E(z)$, which gives us

$$E(z)P^{\mathrm{h}}_{\mu}(z) = \mu z P^{h}_{\mu}(z) + O(z^2). \qquad \square$$

G.2.3 Estimates on Fourier components and well-posedness

G.2.3.1 Dissipation of the parabolic component

The goal of this section is the proof of the following result.

Proposition G.2.5. There exist $r, K_p, c_p > 0$ such that

$$\forall |z| < r,\ \tau > 0,\ X \in \mathrm{Im}(P^{\mathrm{p}}(z)), \quad |\mathrm{e}^{-E(z)\tau} X| \leq K_p \mathrm{e}^{-c_p \tau} |X|.$$

Proof. By using Propsoition G.2.2, for $|z| \leq r$, we denote by $\mathrm{e}^{\mathrm{p}}(z)$ the restriction of $E(z)$ to the vector subspace $\mathrm{Im}[P^{\mathrm{p}}(z)]$, which is an endomorphism of $\mathrm{Im}[P^{\mathrm{p}}(z)]$.

By assumption (H.3), there exists $c > 0$ such that $\Re(\mathrm{Sp}(D)) \subset (-\infty, -c)$. There exists an open disk Ω in the complex plane such that $\mathrm{Sp}(D) \subset \Omega$ and $\max\{\Re(z);\ z \in \overline{\Omega}\} < -c$. Then, by continuity of the spectrum, for r small enough, we have, for every $|z| \leq r$, $\mathrm{Sp}(\mathrm{e}^{\mathrm{p}}(z)) \subset \Omega$.

G.2.3.1.1 Step 1 : Cauchy formula. We prove the following equality between endomorphisms of $\mathrm{Im}[P^{\mathrm{p}}(z)]$

$$\forall |z| \leq r,\ \tau \in \mathbb{R}, \quad \mathrm{e}^{-\mathrm{e}^{\mathrm{p}}(z)\tau} = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \mathrm{e}^{-\tau\xi} (\xi I - \mathrm{e}^{\mathrm{p}}(z))^{-1} \, \mathrm{d}\xi, \tag{G.15}$$

where I is the identity on $\mathrm{Im}[P^{\mathrm{p}}(z)]$. The right hand side is well defined because $\partial\Omega \cap \mathrm{Sp}(\mathrm{e}^{\mathrm{p}}(z)) = \emptyset$. Let us denote it by $\phi(\tau)$. Then

$$\begin{aligned} \phi'(\tau) &= \frac{-1}{2\pi \mathrm{i}} \int_{\partial\Omega} \mathrm{e}^{-\tau\xi} \xi (\xi I - \mathrm{e}^{\mathrm{p}}(z))^{-1} \, \mathrm{d}\xi \\ &= \frac{-1}{2\pi \mathrm{i}} \int_{\partial\Omega} \mathrm{e}^{-\tau\xi} ((\xi I - \mathrm{e}^{\mathrm{p}}(z)) + \mathrm{e}^{\mathrm{p}}(z))(\xi I - \mathrm{e}^{\mathrm{p}}(z))^{-1} \, \mathrm{d}\xi. \end{aligned}$$

By the Cauchy formula, $\int_{\partial\Omega} \mathrm{e}^{-\tau\xi}\,\mathrm{d}\xi = 0$ thus $\phi'(\tau) = -\mathrm{e}^{\mathrm{p}}(z)\phi(\tau)$. Moreover $\phi(0) = I$ because all the eigenvalues of $\mathrm{e}^{\mathrm{p}}(z)$ are inside Ω (see [Kat95, Ch. I, Problem 5.9]). Thus $\phi(\tau) = \mathrm{e}^{-\tau \mathrm{e}^{\mathrm{p}}(z)}$.

Step 2 : Estimate. We deduce from (G.15) the following equality between endomorphisms of $\mathbb{C}^d$

$$\forall |z| \leq r,\ \tau \in \mathbb{R}, \quad \mathrm{e}^{-E(z)\tau} P^{\mathrm{p}}(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial\Omega} \mathrm{e}^{-\tau\xi} (\xi I - E(z))^{-1} P^{\mathrm{p}}(z)\,\mathrm{d}\xi\,. \tag{G.16}$$

Note that, if r is small enough, then the eigenvalues of $E(z)$ are either inside Ω (parabolic branch) or close to 0 (hyperbolic branch), for instance in $\{\Re(\xi) > -c/2\}$. Thus $(\xi I - E(z))$ is invertible on $\mathbb{C}^d$ for every $\xi \in \partial\Omega$ and the above right hand side is well defined.

We deduce from (G.16) that

$$\left|\mathrm{e}^{-E(z)\tau} P^{\mathrm{p}}(z)\right| \leq \frac{1}{2\pi} \int_{\partial\Omega} \mathrm{e}^{-\tau\Re(\xi)} \left|(\xi I - E(z))^{-1} P^{\mathrm{p}}(z)\right| \mathrm{d}\xi\,.$$

The map $(\xi, z) \in \partial\Omega \times \overline{D}(0,r) \mapsto \left|(\xi I - E(z))^{-1} P^{\mathrm{p}}(z)\right|$ is continuous on a compact set thus bounded by a positive constant K. Then for every $|z| < r$ and $\tau > 0$, $\left|\mathrm{e}^{-E(z)\tau} P^{\mathrm{p}}(z)\right| \leq K\mathrm{e}^{-c\tau}$.

$\square$

G.2.3.2 Boundedness of the transport component

The goal of this section is to prove the following result.

Proposition G.2.6. There exists $r, K_h, c_h > 0$ such that

$$\forall z \in \mathrm{i}[-r,r] \setminus \{0\},\ t \in \mathbb{R},\ X \in \mathrm{Im}[P^{\mathrm{h}}(z)], \qquad \left|\exp\left(\frac{1}{z^2} E(z) t\right) X\right| \leq K_h \mathrm{e}^{c_h |t|} |X|.$$

Proof. Let $z \in \mathrm{i}[-r,r] \setminus \{0\}$, $t \in \mathbb{R}$, $\mu \in \mathrm{Sp}(A')$ and $Y \in \mathrm{Im}[P^{\mathrm{h}}_\mu(z)]$. Taking into account that $\mathrm{Im}[P^{\mathrm{h}}_\mu(z)]$ is stable by $E(z)$, we get

$$\exp\left(\frac{1}{z^2} E(z) t\right) Y = \exp\left(\frac{1}{z^2} E(z) P^{\mathrm{h}}_\mu(z) t\right) Y = \exp\left(\frac{1}{z^2}\Big(\mu z P^{\mathrm{h}}_\mu(z) + z^2 R^{\mathrm{h}}_\mu(z)\Big) t\right) Y.$$

Note that $P^{\mathrm{h}}_\mu(z)$ and $R^{\mathrm{h}}_\mu(z)$ commute because $P^{\mathrm{h}}_\mu(z)$ and $E(z)$ commute and $E(z)P^{\mathrm{h}}_\mu(z) = \mu z P^{\mathrm{h}}_\mu(z) + z^2 R^{\mathrm{h}}_\mu(z)$. Thus, by using that $\mu/z \in \mathrm{i}\mathbb{R}$, we obtain

$$\left|\exp\left(\frac{1}{z^2} E(z) t\right) Y\right| = \left|\mathrm{e}^{\mu t/z} \exp\Big(R^{\mathrm{h}}_\mu(z) t\Big) Y\right| \leq \mathrm{e}^{c_\mu |t|} |Y|$$

where $c_\mu = \max\{|R^{\mathrm{h}}_\mu(z)|; z \in \overline{D}(0,r)\}$. We conclude for $X \in \mathrm{Im}[P^{\mathrm{h}}(z)]$ that

$$\begin{aligned}\left|\exp\left(\frac{1}{z^2}E(z)t\right)X\right| &\leq \sum_{\mu\in\mathrm{Sp}(A')}\left|\exp\left(\frac{1}{z^2}E(z)t\right)P^{\mathrm{h}}_\mu(z)X\right| \\ &\leq \sum_{\mu\in\mathrm{Sp}(A')} \mathrm{e}^{c_\mu|t|}|P^{\mathrm{h}}_\mu(z)X| \leq K\mathrm{e}^{c|t|}|X|^2\end{aligned}$$

with $c = \max\{c_\mu; \mu \in \mathrm{Sp}(A')\}$ and $K = \max\left\{\sum_{\mu\in\mathrm{Sp}(A')}|P^{\mathrm{h}}_\mu(\zeta)|; \zeta \in \overline{D}(0,r)\right\}$. □

G.2.3.3 Well-posedness

By gathering the results of the previous two subsubsections, we can prove that the heat-transport system (Sys) is well-posed.

We define the Fourier coefficients by

$$\forall f \in L^2(\mathbb{T})^d, \forall n \in \mathbb{Z}, \quad \hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t \in \mathbb{C}^d.$$

We consider the operator $\mathcal{L}$ defined by (G.10) and (G.11). By Bessel-Parseval identity and (G.13),

$$D(\mathcal{L}) = \left\{f \in L^2(\mathbb{T})^d; \sum_{n\in\mathbb{Z}}\left|n^2E\left(\frac{\mathrm{i}}{n}\right)\hat{f}(n)\right|^2 < \infty\right\}, \tag{G.17}$$

The goal of this section is the proof of the following result.

Proposition G.2.7. $-\mathcal{L}$ generates a C^0 semi-group of bounded operators on $L^2(\mathbb{T}^d)$.

This result will ensure well posedness of (Sys) in the following sense.

Definition G.2.8. Let $T > 0$, $f_0 \in L^2(\mathbb{T})^d$ and $u \in L^2(Q_T)^d$. The solution of (Sys) is the function $f \in C^0([0,T]; L^2(\mathbb{T})^d)$ defined by

$$f(t) = \mathrm{e}^{-t\mathcal{L}}f_0 + \int_0^t \mathrm{e}^{-(t-\tau)\mathcal{L}}u(\tau)\,\mathrm{d}\tau \quad \text{for every } t \in [0,T].$$

Moreover, $f(t)$ satisfies the estimate

$$\forall 0 \leq t \leq T,\ \|f(t)\|_{L^2(\mathbb{T})} \leq C\big(\|f_0\|_{L^2(\mathbb{T})} + \|u\|_{L^2([0,T]\times\omega)}\big), \tag{G.18}$$

where C depends on T but not on f_0 and u. We will also note $S(t, f_0, u) := f(t)$ this solution.

Proof. We deduce from Propositions G.2.5 and G.2.6 that for every $z \in \mathrm{i}[-r,r]\setminus\{0\}$, $t > 0$ and $X \in \mathbb{C}^d$,

$$\begin{aligned}\left|\exp\left(\frac{1}{z^2}E(z)t\right)X\right| &\leq \left|\exp\left(-E(z)\frac{t}{|z|^2}\right)P^{\mathrm{p}}(z)X\right| + \left|\exp\left(\frac{1}{z^2}E(z)t\right)P^{\mathrm{h}}(z)X\right| \\ &\leq K_p\mathrm{e}^{-c_p\frac{t}{|z|^2}}|P^{\mathrm{p}}(z)X| + K_h\mathrm{e}^{c_ht}\left|P^{\mathrm{h}}(z)X\right| \\ &\leq K\mathrm{e}^{c_ht}|X|\end{aligned} \tag{G.19}$$

where $K = \max\{K_p|P^{\mathrm{p}}(z)| + K_h|P^{\mathrm{h}}(z)|; z \in \mathrm{i}[-r,r]\}$.

For $f \in L^2(\mathbb{T})^d$ and $t \in [0,\infty)$ we define

$$S(t) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{tn^2 E(\frac{\mathrm{i}}{n})} \hat{f}(n) e_n.$$

By Bessel Parseval equality and (G.19), $S(t)$ is a bounded operator on $L^2(\mathbb{T})^d$, because the number of $n \in \mathbb{Z}$ such that $\frac{1}{n} \notin [-r,r]$ is finite. The semi-group properties $S(0) = I$ and $S(t+s) = S(t)S(s)$ are clearly satisfied. For $f \in D(\mathcal{L})$, we have, by Bessel Parseval equality

$$\left\| \left(\frac{S(t)-I}{t} + \mathcal{L} \right) f \right\|^2_{L^2(\mathbb{T})^d} = \sum_{n\in\mathbb{Z}} \left| \left(\frac{\mathrm{e}^{tn^2E(\frac{\mathrm{i}}{n})} - I_d}{t} - n^2 E\left(\frac{\mathrm{i}}{n}\right) \right) \hat{f}(n) \right|^2.$$

In the right hand side, each term of the series converges to zero when $[t \to 0]$ and, thanks to (G.19), is dominated for every $t \in [0,1]$ and $n > 1/r$ by

$$\left| \left(\int_0^1 \mathrm{e}^{t\theta n^2 E(\frac{\mathrm{i}}{n})} \,\mathrm{d}\theta - I_d \right) n^2 E\left(\frac{\mathrm{i}}{n}\right) \hat{f}(n) \right|^2 \leq (K\mathrm{e}^{c_h} + 1) \left| n^2 E\left(\frac{\mathrm{i}}{n}\right) \hat{f}(n) \right|^2,$$

which can be summed over $n \in \mathbb{Z}$ because $f \in D(\mathcal{L})$, see (G.17). By the dominated convergence theorem, the sum of the series converges to zero. □

G.2.4 Adjoint system and observability

The null-controllability of a linear system is equivalent to a dual notion called "observability". We have the following general, abstract result.

Proposition G.2.9. Let H, U and V be three complex Hilbert spaces, $\mathcal{L}$ be the generator of a C^0 semigroup of bounded operators on H, $M : U \to H$ and $P : H \to V$ be bounded operators and $T > 0$. The following statements are equivalent :

1. For every $f_0 \in H$, there exists $u \in L^2([0,T]; U)$ such that the solution f of
$$\partial_t f(t) + \mathcal{L} f(t) = Mu(t), \quad f(0) = f_0 \tag{G.20}$$
satisfies $Pf(T) = 0$.
2. There exists $C > 0$ such that for every $g_0 \in V$, the solution g of
$$\partial_t g(t) = \mathcal{L}^* g(t), \quad g(0) = P^* g_0, \tag{G.21}$$
satisfies
$$\|g(T)\|^2_H \leq C \int_0^T \|M^* g(t)\|^2_U \,\mathrm{d}t. \tag{G.22}$$

If we take $V = H$ and P to be the identity map, then this proposition is just the classical duality between (null-)controllability and (final state) observability (see [Cor07a, Thm. 2.44]). The proof of Proposition G.2.9 is a straightforward adaptation of the one given by Coron in the previous reference.

In our case, this observability inequality becomes the following Proposition G.2.10.

Proposition G.2.10. Given $T > 0$, the system (Sys) is null-controllable on ω in time T if and only if there exists $C > 0$ such that for every $g_0 \in L^2(\mathbb{T};\mathbb{C}^d)$, the solution g to the equation

$$\begin{cases} \partial_t g - B^{\mathrm{tr}}\partial_x^2 g - A^{\mathrm{tr}}\partial_x g + K^{\mathrm{tr}} g = 0 & \text{in } (0,T)\times\mathbb{T}, \\ g(0,.) = g_0 & \text{in } \mathbb{T}. \end{cases} \tag{G.23}$$

satisfies

$$\|g(T,.)\|^2_{L^2(\mathbb{C})} \leq C \int_0^T \int_\omega |M^* g(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x. \tag{G.24}$$

Note that the solutions of the adjoint system (G.23) are of the form [3]

$$g(t,x) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{inx - tn^2 E(\frac{\mathrm{i}}{n})^*} \widehat{g}_0(n)\,. \tag{G.25}$$

Moreover, we have a spectral theory for the adjoint system that is similar to Prop. G.2.1–G.2.3. We just have to take the adjoint of each formulas.

G.3 Obstruction to the null-controllability in small time

The goal of this section is to prove the first point of Theorem G.1.2. It is sufficient to work with an open interval ω. Indeed, otherwise, ω is contained in an open interval $\widetilde{\omega}$ of $\mathbb{T}$ such that $\ell(\omega) = \ell(\widetilde{\omega})$ and the negative result for the large control support $\widetilde{\omega}$ implies the negative result for the small control support ω. Thus, in the whole section, we assume ω is an open interval of $\mathbb{T}$. We want to disprove the observability inequality (G.24). We do this by adapting the method used by the second author for the Grushin equation [Koe17].

G.3.1 Construction of a counterexample to the observability inequality

Let $\mu \in \mathrm{Sp}(A')$ with minimum absolute value. First, we prove the following estimate.

Proposition G.3.1. Let U be a open domain, star-shaped with respect to 0, that contains $\omega_T := \bigcup_{0\leq t\leq T}(\overline{\omega} - \mu t)$ (where $\overline{\omega} - \mu t$ is to be understood as the rotation of $\overline{\omega}$ by an angle of $-\mu t$, see figure G.1).

There exist an integer N and a constant $C > 0$ such that if the system (Sys) is null-controllable on ω in time T, then for all polynomials $p(z) = \sum_{n>N} a_n z^n$ with a zero of order at least N at 0, we have

$$|p|_{L^2(D(0,1))} \leq C |p|_{L^\infty(U)}. \tag{G.26}$$

3. When we write $E(z)^*$, it is to be understood as $(E(z))^*$. We will use the same notation for $P^{\mathrm{h}}_\mu(z)^*$ etc.

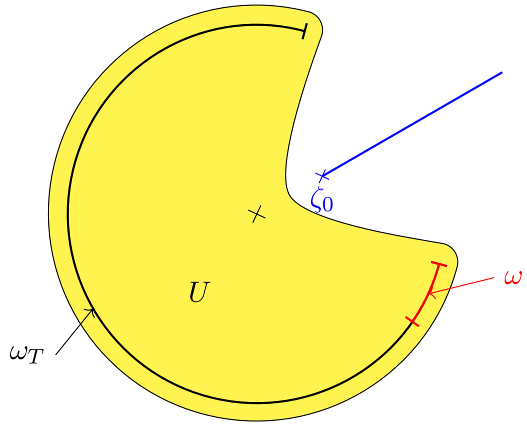

FIGURE G.1 – In yellow, an example of the domain U. The black circle arc is ω_T (once we identify $\mathbb{T}$ with the complex unit circle). The null-controllability of the parabolic-hyperbolic system implies that we can estimate the $L^2(D(0,1))$ norm of complex polynomials by their $L^\infty(U)$ norm. But if T is not too large, ω_T is not the whole unit circle, and we can choose U such that $D(0,1) \not\subset U$. Then, we can find a sequence of polynomials that converges to $\zeta \mapsto (\zeta-\zeta_0)^{-1}$ away from the blue line.

Proof. According to Proposition G.2.3, there exist $r>0$, a projection-valued function P^{h}_μ and a matrix-valued function R^{h}_μ that are holomorphic on $D(0,r)$ such that for every $|z|<r$,

$$P^{\mathrm{h}}_\mu(z)E(z) = E(z)P^{\mathrm{h}}_\mu(z) = \mu z P^{\mathrm{h}}_\mu(z) + z^2 R^{\mathrm{h}}_\mu(z). \tag{G.27}$$

Let $\varphi_0 \neq 0$ in the range of $P^{\mathrm{h}}_\mu(0)^*$. To disprove the observability inequality (G.24), we look at solutions $g(t,x)$ of the system (G.23) with initial conditions of the form $g(0,x) = \sum_{n\geq 0} a_n \mathrm{e}^{\mathrm{i}nx} P^{\mathrm{h}}_\mu(\mathrm{i}/n)^*\varphi_0$. To avoid irrelevant summability issues, we will assume that all sums are finite. Since on the range of $P^{\mathrm{h}}_\mu(z)$, $E(z)$ acts as $\mu z + z^2 R^{\mathrm{h}}_\mu(z)$ (see Eq. (G.27)), we have

$$\begin{aligned} g(t,x) &= \sum_{n\geq 0} a_n \mathrm{e}^{\mathrm{i}nx}\mathrm{e}^{-tn^2 E(\frac{\mathrm{i}}{n})^*} P^{\mathrm{h}}_\mu\Big(\frac{\mathrm{i}}{n}\Big)^*\varphi_0 \\ &= \sum_{n\geq 0} a_n \mathrm{e}^{\mathrm{i}n(x+\mu t)+tR^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*} P^{\mathrm{h}}_\mu\Big(\frac{\mathrm{i}}{n}\Big)^*\varphi_0. \end{aligned}$$

So, if we define for $0\leq t\leq T$ and $n\in\mathbb{Z}$,

$$\gamma_t(n) = \mathrm{e}^{tR^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*} P^{\mathrm{h}}_\mu\Big(\frac{\mathrm{i}}{n}\Big)^*, \tag{G.28}$$

we rewrite $g(t,x)$ as

$$g(t,x) = \sum a_n \mathrm{e}^{\mathrm{i}n(x+\mu t)}\gamma_t(n)\varphi_0.$$

If the term $\gamma_t(n)$ was equal to one, then $g(t,x)$ would just be the solution to an uncoupled transport equation, therefore it would be easy to disprove (G.24). To treat this term, we will use the following lemma, that we prove in Lemma G.3.2.

Lemma G.3.2. Let U be as in Proposition G.3.1. There exist an integer $N>0$ and a constant $C>0$ such that for every polynomials $\sum_{n>N} a_n\zeta^n$ with a zero of order N at 0, for every $0\leq\tau\leq T$,

$$\Big|\sum_{n>N} a_n\zeta^n\gamma_\tau(n)\Big|_{L^\infty(\omega_T)} \leq C\Big|\sum_{n>N} a_n\zeta^n\Big|_{L^\infty(U)} \tag{G.29}$$

From now on, we assume that $a_n = 0$ for $n \leq N$. If we note $\zeta(t,x) = \mathrm{e}^{\mathrm{i}(x+\mu t)}$, which belong to ω_T for $(t,x) \in [0,T] \times \omega$, we have

$$g(t,x) = \sum_{n>N} a_n \zeta(t,x)^n \gamma_t(n)\varphi_0.$$

Let $(t,x) \in [0,T] \times \omega$. By applying Lemma G.3.2 with $\tau = t$, we have

$$|g(t,x)| \leq C \Big| \sum_{n>N} a_n \zeta^n \Big|_{L^\infty(U)}.$$

So the right-hand side of the observability inequality (G.24) satisfies

$$|g|^2_{L^2([0,T]\times\omega)} \leq 2\pi T |g|^2_{L^\infty([0,T]\times\omega)} \leq 2\pi T C^2 \Big| \sum_{n>N} a_n \zeta^n \Big|^2_{L^\infty(U)}. \tag{G.30}$$

We now lower bound the left hand-side of the observability inequality (G.24). Thanks to Parseval's identity, we have

$$|g(T,\cdot)|^2_{L^2(\mathbb{T})} = \Big| \sum_{n>N} a_n \mathrm{e}^{\mathrm{i}n(x+\mu T)} \gamma_T(n)\varphi_0 \Big|^2_{L^2(\mathbb{T})} = 2\pi \sum_{n>N} |a_n|^2 |\gamma_T(n)\varphi_0|^2. \tag{G.31}$$

Since R is holomorphic in $D(0,r)$, so is $z \mapsto R^{\mathrm{h}}_\mu(\bar{z})^*$. In particular, we have $C_1 := \sup_{|z|\leq r/2} |R^{\mathrm{h}}_\mu(z)^*| < +\infty$. So, we have for $n \geq 2r^{-1}$,

$$\big| \big(\mathrm{e}^{-TR^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*}\big)^{-1} \big| = \big| \mathrm{e}^{TR^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*} \big| \leq \mathrm{e}^{C_1 T}. \tag{G.32}$$

Moreover, φ_0 is in the range of $P^{\mathrm{h}}_\mu(0)^*$ and P^{h}_μ is holomorphic in $D(0,r)$, so there exists $r' > 0$ sufficiently small such that for $|z| < r'$,

$$|P^{\mathrm{h}}_\mu(z)^* \varphi_0| \geq |\varphi_0|/2 =: c. \tag{G.33}$$

By gathering (G.32) and (G.33), we have for $n \geq N' := \lfloor \max(2r^{-1}, r'^{-1}) \rfloor + 1$,

$$|\gamma_T(n)\varphi_0| = \left| \mathrm{e}^{-TR^{\mathrm{h}}_\mu(\frac{\mathrm{i}}{n})^*} P^{\mathrm{h}}_\mu\Big(\frac{\mathrm{i}}{n}\Big)^* \varphi_0 \right| \geq \mathrm{e}^{-C_1 T} c =: c'.$$

So, assuming $a_n = 0$ for $n \leq N'$, we have by plugging the previous lower bound into Parseval's identity (G.31)

$$\Big| \sum_{n>N} a_n \zeta^n \Big|^2_{L^2(D(0,1))} = \pi \sum_{n>N} \frac{|a_n|^2}{n+1} \leq \frac{\pi}{c'} \sum_{n>N} \frac{|a_n|^2}{n+1} |\gamma_T(n)\varphi_0|^2 \leq \frac{1}{2c'} |g(T,\cdot)|^2_{L^2(\mathbb{T})}. \tag{G.34}$$

Thus, thanks to the lower bound (G.34) and the upper bound (G.30), the observability inequality (G.24) implies

$$\Big| \sum_{n>N} a_n \zeta^n \Big|^2_{L^2(D(0,1))} \leq C |g(T,\cdot)|^2_{L^2(\mathbb{T})} \leq C' |g|^2_{L^2([0,T]\times\omega)} \leq C'' \Big| \sum_{n>N} a_n \zeta^n \Big|^2_{L^\infty(U)},$$

which concludes the proof of Proposition G.3.1. $\square$

Let us check that the inequality of Proposition G.3.1 does not hold. We will use Runge's theorem (see for instance Rudin's textbook [Rud87, Thm. 13.9, Thm 13.11]) to construct a counterexample.

Proposition G.3.3 (Runge's theorem)**.** Let U be a simply connected open subset of $\mathbb{C}$ and f be a holomorphic function on U. Then, there exists a sequence $(p_k)_{k\geq 0}$ of polynomials that converges uniformly on every compact subset of U to f.

Proof. [Proof of Theorem G.1.2.i)] Let $T < T^*$ and ω_T be as in Proposition G.3.1. By definition of T^*, ω_T is not the whole unit circle, thus we can find an open bounded domain U that is star-shaped with respect to 0 and that does not contain $D(0,1)$ (see Fig. G.1).

With such a choice of U, there exists a complex number $\zeta_0 \in D(0,1)$ which is non-adherent to U. Then, according to Runge's theorem, there exists a sequence of polynomials $(\tilde{p}_k)$ that converges uniformly on every compact subset of $\mathbb{C} \setminus (\zeta_0[1,+\infty))$ to $\zeta \mapsto (\zeta-\zeta_0)^{-1}$. Let us define $p_k(\zeta) = \zeta^{N+1}\tilde{p}_k(\zeta)$. Then, the sequence (p_k) is a counterexample to the inequality on complex polynomials (G.26). Indeed, since $\zeta^{N+1}(\zeta-\zeta_0)^{-1}$ is bounded on U, (p_k) is uniformly bounded on U, thus, the right-hand side of the inequality (G.26) is bounded. But since ζ_0 is in $D(0,1)$, $\zeta^{N+1}(\zeta-\zeta_0)^{-1}$ has infinite L^2-norm in $D(0,1)$, and thanks to Fatou's Lemma, $|p_k|_{L^2(D(0,1))}$ tends to $+\infty$ as $k \to +\infty$. □

G.3.2 Estimate on some operators on polynomial functions

The goal of this subsection is to prove Lemma G.3.2. The main tool to prove Lemma G.3.2 is Theorem G.3.6, which is a variant of a theorem that the second author proved when studying Grushin's equation (see [Koe17, Thm. 18]).

Definition G.3.4. Let E be a Banach space. Let $R > 0$ and $\Delta_R := \{z \in \mathbb{C}, \Re(z) > R\}$. We define $\mathcal{S}_R(E)$ as the set of functions γ from Δ_R to E that are holomorphic with subexponential growth, i.e. such that for all $\varepsilon > 0$,

$$p_\varepsilon(\gamma) = \sup_{z\in\Delta_R} |\gamma(z)|e^{-\varepsilon|z|} < +\infty. \tag{G.35}$$

We endow $\mathcal{S}_R(E)$ with the topology of the seminorms p_ε for $\varepsilon > 0$.

If E is the space $\mathbb{C}^{d\times d}$ of linear maps of $\mathbb{C}^d$, we will note $\mathcal{S}_R^{d\times d} := \mathcal{S}_R(\mathcal{L}(\mathbb{C}^d))$. We will sometime call elements of $\mathcal{S}_R^{d\times d}$ *symbols.*

Remark G.3.5. If $n \leq R$, i.e. if n is not in the domain of definition of $\gamma \in \mathcal{S}_R(E)$, we will set for convenience $\gamma(n) = 0$.

Theorem G.3.6. Let $R > 0$ and $\gamma \in \mathcal{S}_R^{d\times d}$. Let H_γ be the operator on vector-valued entire functions defined by

$$H_\gamma \colon \sum_{n>R} a_n\zeta^n \longmapsto \sum_{n>R} \gamma(n)a_n\zeta^n. \tag{G.36}$$

Then, the operator H_γ is continuous on $\mathcal{O}(\mathbb{C})$. Moreover, the map $\gamma \in \mathcal{S}_R^{d\times d} \mapsto H_\gamma \in \mathcal{L}(\mathcal{O}(\mathbb{C}))$ satisfies the following continuity-like estimate : for each compact subset K of

$\mathbb{C}$ and each neighborhood V of K that is star-shaped with respect to 0, there exist a constant $C > 0$ and a seminorm p_ε of $\mathcal{S}_R^{d\times d}$ such that for every entire function f :

$$|H_\gamma(f)|_{L^\infty(K)} \leq C p_\varepsilon(\gamma)|f|_{L^\infty(V)}. \tag{G.37}$$

This theorem was proved in the case $d = 1$ by one of the authors [Koe17]. The proof follows the same lines in the general case. We provide it in Appendix G.5.1.

Now, we turn to the proof of Lemma G.3.2 which is basically an application of Theorem G.3.6.

Proof. [Proof of Lemma G.3.2] Let us define $\tilde{\gamma}_\tau(z) = \mathrm{e}^{tR_\mu^{\mathrm{h}}(\mathrm{i}/\bar{z})^*}P_\mu^{\mathrm{h}}(\mathrm{i}/\bar{z})^*$, so that $\gamma_\tau(n) = \tilde{\gamma}_\tau(n)$ (see the definition of γ_t Eq. (G.28)), and thus for every $(a_n) \in \mathbb{C}^{\mathbb{N}}$,

$$\sum_{n>N} \gamma_\tau(n) a_n \varphi_0 \zeta^n = H_{\tilde{\gamma}_\tau}\Big(\sum_{n>N} a_n \zeta^n \varphi_0\Big). \tag{G.38}$$

Let us check that $(\tilde{\gamma}_\tau)_{0\leq\tau\leq T}$ is a bounded family of $\mathcal{S}_R^{d\times d}$ for some $R > 0$. Since R_μ^{h} and P_μ^{h} are holomorphic on $D(0,r)$, $\tilde{\gamma}_\tau$ is holomorphic on $\{|z| > r^{-1}\}$, and in particular in $\{\Re(z) > r^{-1}\}$. So, for $|z| > 2r^{-1}$ and $0 \leq \tau \leq T$, we have

$$|\tilde{\gamma}_\tau(z)| \leq \mathrm{e}^{T \sup_{|z|<r/2} |R_\mu^{\mathrm{h}}(z)|} \sup_{|z|<r/2} |P_\mu^{\mathrm{h}}(z)| < +\infty.$$

Thus, with $R = 2r^{-1}$, γ_τ is in $\mathcal{S}_R^{d\times d}$, and since the previous bound is uniform in $0 \leq \tau \leq T$, the family $(\gamma_\tau)_{0\leq\tau\leq T}$ is bounded in $\mathcal{S}_R^{d\times d}$.

Let us also remind that U is star-shaped with respect to 0, and that $\omega_T \subset U$. All the conditions of Theorem G.3.6 are satisfied, so we can apply the estimate (G.37) with $K = \omega_T$ and $V = U$:

$$\big|H_{\tilde{\gamma}_\tau}\big(\sum_{n>N} a_n \varphi_0 \zeta^n\big)\big|_{L^\infty(\omega_T)} \leq C\big|\sum_{n>N} a_n \varphi_0 \zeta^n\big|_{L^\infty(U)} = C\big|\sum_{n>N} a_n \zeta^n\big|_{L^\infty(U)}, \tag{G.39}$$

where C that depends neither on the polynomial $\sum a_n \zeta^n$, neither on $0 \leq \tau \leq T$ because the family $(\tilde{\gamma}_\tau)_{0\leq\tau\leq T}$ is bounded. This, combined with (G.38), proves the inequality (G.29) and concludes the proof of Lemma G.3.2. □

G.4 Large time null-controllability

The goal of this section is to prove the point (ii) of Theorem G.1.2, by adapting the strategy of Gilles Lebeau and Enrique Zuazua [LZ98], based on a spectral decomposition. For high frequencies, the spectrum splits into a parabolic part and a hyperbolic part. Projecting the dynamics onto the parabolic/hyperbolic subspaces, the system is decomposed into 2 weakly coupled systems, the first one behaving like a transport equation, the second one like a heat equation. The transport equation is handled by using the methods developped in [ABCO17]. The parabolic equation is treated by the Lebeau-Robbiano's method [LR95], adapted to systems. The low frequency part is treated by a compactness argument and a unique continuation property.

In the whole Section G.4, the parameter $r > 0$ is assumed to be small enough so that Propositions G.2.1, G.2.2, G.2.3, G.2.5 and G.2.6 hold.

G.4.1 An adapted decomposition of $L^2(\mathbb{T})^d$

Proposition G.4.1. Let $n_0 \in \mathbb{N}$ be such that $\frac{1}{n_0} < r$. We have the following decomposition

$$L^2(\mathbb{T})^d = F^0 \oplus F^{\mathrm{p}} \oplus F^{\mathrm{h}}, \tag{G.40}$$

where

$$F^0 := \bigoplus_{|n| \leq n_0} \mathbb{C}^d e_n, \tag{G.41}$$

$$F^{\mathrm{p}} := \bigoplus_{|n| > n_0} \mathrm{Im}\left(P^{\mathrm{p}}\left(\frac{\mathrm{i}}{n}\right)\right) e_n, \tag{G.42}$$

$$F^{\mathrm{h}} := \bigoplus_{|n| > n_0} \mathrm{Im}\left(P^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)\right) e_n. \tag{G.43}$$

Moreover the projections Π^0, Π^{p}, Π^{h} and Π defined by

$$\begin{array}{rcl} L^2(\mathbb{T})^d &=& F^0 \oplus F^{\mathrm{p}} \oplus F^{\mathrm{h}} \\ \Pi^0 &=& I_{F^0} + \ 0 \ + \ 0 \\ \Pi^{\mathrm{p}} &=& \ 0 \ + I_{F^{\mathrm{p}}} + \ 0 \\ \Pi^{\mathrm{h}} &=& \ 0 \ + \ 0 \ + I_{F^{\mathrm{h}}} \\ \Pi &=& \ 0 \ + I_{F^{\mathrm{p}}} + I_{F^{\mathrm{h}}} = \Pi^{\mathrm{p}} + \Pi^{\mathrm{h}} \end{array}$$

are bounded operators on $L^2(\mathbb{T})^d$.

Proof. The function $z \in D(0,r) \mapsto P^{\mathrm{p}}(z)$ is continuous thus there exists $C > 0$ such that, for every $z \in \overline{D}(0, 1/n_0)$, $|P^{\mathrm{p}}(z)| \leq C$. Let $f \in L^2(\mathbb{T})^d$. We deduce from

$$\sum_{|n|>n_0} \left| P^{\mathrm{p}}\left(\frac{\mathrm{i}}{n}\right) \hat{f}(n) \right|^2 \leq C^2 \sum_{|n|>n_0} |\hat{f}(n)|^2 \leq C^2 \|f\|^2_{L^2(\mathbb{T})^d} \tag{G.44}$$

and Bessel-Parseval identity that the series $\sum P^{\mathrm{p}}\left(\frac{\mathrm{i}}{n}\right)\hat{f}(n)e_n$ converges in $L^2(\mathbb{T})^d$. Using $I_d = P^{\mathrm{p}}(z) + P^{\mathrm{h}}(z)$, we get the decomposition

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n)e_n = \sum_{|n|\leq n_0} \hat{f}(n)e_n + \sum_{|n|>n_0} P^{\mathrm{p}}\left(\frac{\mathrm{i}}{n}\right)\hat{f}(n)e_n + \sum_{|n|>n_0} P^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)\hat{f}(n)e_n$$

with convergent series in $L^2(\mathbb{T})^d$. This proves $L^2(\mathbb{T})^d = F^0 + F^{\mathrm{p}} + F^{\mathrm{h}}$. The sum is direct because $(e_n)_{n\in\mathbb{Z}}$ is orthogonal and $\mathrm{Im}(P^{\mathrm{p}}(z)) \cap \mathrm{Im}(P^{\mathrm{h}}(z)) = \{0\}$ when $|z| < r$. Π^0 and Π are orthogonal projections, thus bounded operators on $L^2(\mathbb{T})^d$. We deduce from Bessel-Parseval identity and (G.44) that Π^{p} is a bounded operator on $L^2(\mathbb{T})^d$ and so is $\Pi^{\mathrm{h}} = \Pi - \Pi^{\mathrm{p}}$. ☐

The operator $\mathcal{L}$ defined in (G.10) maps $D(\mathcal{L}) \cap F^0$ into F^0 thus we can define an operator $\mathcal{L}^0$ on F^0 by $D(\mathcal{L}^0) = D(\mathcal{L}) \cap F^0$ and $\mathcal{L}^0 = \mathcal{L}|_{F^0}$. Moreover, $-\mathcal{L}^0$ generates a C^0-semi-group of bounded operators on F^0 and $\mathrm{e}^{-t\mathcal{L}^0} = \mathrm{e}^{-t\mathcal{L}}|_{F^0}$. For the same reasons, we can define an operator $\mathcal{L}^{\mathrm{p}}$ on F^{p} by $D(\mathcal{L}^{\mathrm{p}}) = D(\mathcal{L}) \cap F^{\mathrm{p}}$ and $\mathcal{L}^{\mathrm{p}} = \mathcal{L}|_{F^{\mathrm{p}}}$, that generates a C^0-semi-group of bounded operators on F^{p} : $\mathrm{e}^{-t\mathcal{L}^{\mathrm{p}}} = \mathrm{e}^{-t\mathcal{L}}|_{F^{\mathrm{p}}}$. Finally, we can define an operator $\mathcal{L}^{\mathrm{h}}$ on F^{h} by $D(\mathcal{L}^{\mathrm{h}}) = D(\mathcal{L}) \cap F^{\mathrm{h}}$ and $\mathcal{L}^{\mathrm{h}} = \mathcal{L}|_{F^{\mathrm{h}}}$, that generates a C^0-semi-group of bounded operators on F^{h} : $\mathrm{e}^{-t\mathcal{L}^{\mathrm{h}}} = \mathrm{e}^{-t\mathcal{L}}|_{F^{\mathrm{h}}}$.

Proposition G.4.2. The operator $-\mathcal{L}^0$ generates a C^0 group $(\mathrm{e}^{-t\mathcal{L}^0})_{t\in\mathbb{R}}$ of bounded operators on F^0. The operator $-\mathcal{L}^{\mathrm{h}}$ generates a C^0 group $(\mathrm{e}^{-t\mathcal{L}^{\mathrm{h}}})_{t\in\mathbb{R}}$ of bounded operators on F^{h}

Proof. We just need to check that $\mathrm{e}^{-t\mathcal{L}}$ defines a bounded operator of F^0 and F^{h} when $t < 0$. It is clear for F^0 because it has finite dimension. For F^{h}, one may proceed as in the proof of Proposition G.2.7, noticing that the estimate of Proposition G.2.6 is valid for any $t \in \mathbb{R}$. $\square$

For the duality method, we will need the dual decomposition of (G.40), i.e.

$$\begin{aligned} &L^2(\mathbb{T})^d = F^0 \oplus \widetilde{F^{\mathrm{p}}} \oplus \widetilde{F^{\mathrm{h}}}, \\ &\text{where } \widetilde{F^{\mathrm{p}}} := \mathrm{Im}\left((\Pi^{\mathrm{p}})^*\right),\ \widetilde{F^{\mathrm{h}}} := \mathrm{Im}\Big((\Pi^{\mathrm{h}})^*\Big). \end{aligned} \tag{G.45}$$

By using the definitions of F^{p} and F^{h} in (G.42) and (G.43) and the fact that $(e_n)_{n\in\mathbb{Z}}$ is an Hilbert basis of $L^2(\mathbb{T})$, we get

$$\widetilde{F^{\mathrm{p}}} = \bigoplus_{|n|>n_0} \mathrm{Im}\left(P^{\mathrm{p}}\left(\frac{\mathrm{i}}{n}\right)^*\right) e_n, \tag{G.46}$$

$$\widetilde{F^{\mathrm{h}}} = \bigoplus_{|n|>n_0} \mathrm{Im}\left(P^{\mathrm{h}}\left(\frac{\mathrm{i}}{n}\right)^*\right) e_n. \tag{G.47}$$

Moreover,

$$(\mathrm{e}^{t\mathcal{L}})^* f = \mathrm{e}^{t\mathcal{L}^*} f = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-tn^2 E\left(\frac{\mathrm{i}}{n}\right)^*} \widehat{f}(n) e_n \tag{G.48}$$

and the spaces F^0, $\widetilde{F^{\mathrm{p}}}$ and $\widetilde{F^{\mathrm{h}}}$ are stable by $\mathrm{e}^{t\mathcal{L}^*}$.

G.4.2 Control strategy

Let T^* be as in (G.2) and T, T' be such that

$$T^* < T' < T. \tag{G.49}$$

In this section, we consider controls u of the form

$$u := (u_{\mathrm{h}}, u_{\mathrm{p}})^{\mathrm{tr}} \in \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}, \tag{G.50}$$

where

$$\begin{aligned} &\operatorname{supp}(u_{\mathrm{h}}) \subset [0,T'] \times \overline{\omega}, &&\operatorname{supp}(u_{\mathrm{p}}) \subset [T',T] \times \overline{\omega}, \\ &u_{\mathrm{h}} \in L^2((0,T') \times \mathbb{T})^{d_1}, &&u_{\mathrm{p}} \in L^2((T',T) \times \mathbb{T})^{d_2}. \end{aligned} \tag{G.51}$$

the control u_{h} is intended to control the hyperbolic component of the system and the control u_{p} the parabolic component.

The control strategy for system (Sys) consists in

- first proving the null controllability in time T in a subspace of $L^2(\mathbb{T})^d$ with finite codimension,
- then using a unique continuation argument, to get the full null controllability.

The first step of this strategy is given by the following statement.

Proposition G.4.3. There exists a subspace $\mathcal{G}$ of $L^2(\mathbb{T})^d$ with finite codimension and a continuous operator

$$\begin{aligned} \mathcal{U}\colon\ \mathcal{G} &\to L^2((0,T') \times \omega)^{d_1} \times C_c^\infty((T',T) \times \omega)^{d_2} \\ f_0 &\mapsto (u_{\mathrm{h}}, u_{\mathrm{p}}), \end{aligned}$$

that associates with each $f_0 \in \mathcal{G}$ a pair of controls $\mathcal{U} f_0 = (u_{\mathrm{h}}, u_{\mathrm{p}})$ such that

$$\forall f_0 \in \mathcal{G},\ \Pi S(T; f_0, \mathcal{U} f_0) = 0. \tag{G.52}$$

By 'continuous operator', we mean that, for every $s \in \mathbb{N}$, the map $\mathcal{U} : \mathcal{G} \mapsto L^2((0,T') \times \omega)^{d_1} \times H^s((T',T) \times \omega)^{d_2}$ is continuous : there exists $C_s > 0$ such that

$$\forall f_0 \in \mathcal{G}, \quad \|u_{\mathrm{h}}\|_{L^2((0,T') \times \omega)^{d_1}} + \|u_{\mathrm{p}}\|_{H^s((T',T) \times \omega)^{d_2}} \leqslant C_s \|f_0\|_{L^2(\mathbb{T})^d}\,.$$

The proof strategy of Proposition G.4.3 consists in splitting the problem in 2 parts :

- for any initial data f_0 and parabolic control u_{p}, steer the hyperbolic high frequences to zero at time T (Proposition G.4.4),
- for any initial data f_0 and hyperbolic control u_{h}, steer the parabolic high frequences to zero at time T (Proposition G.4.5).

Proposition G.4.4. If n_0 (in Eq. (G.41–G.42)) is large enough, there exists a continuous operator

$$\begin{aligned} \mathcal{U}^{\mathrm{h}}\colon\ L^2(\mathbb{T})^d \times L^2((T',T) \times \omega)^{d_2} &\to L^2((0,T') \times \omega)^{d_1} \\ (f_0, u_{\mathrm{p}}) &\mapsto u_{\mathrm{h}}, \end{aligned}$$

such that for every $(f_0, u_{\mathrm{p}}) \in L^2(\mathbb{T})^d \times L^2((T',T) \times \omega)^{d_2}$,

$$\Pi^{\mathrm{h}} S(T; f_0, (\mathcal{U}^{\mathrm{h}}(f_0, u_{\mathrm{p}}), u_{\mathrm{p}})) = 0.$$

Proposition G.4.5. If n_0 is large enough, there exists a continuous operator

$$\begin{aligned} \mathcal{U}^{\mathrm{p}}\colon\ L^2(\mathbb{T})^d \times L^2((0,T') \times \omega)^{d_1} &\to C_c^\infty((T',T) \times \omega)^{d_2} \\ (f_0, u_{\mathrm{h}}) &\mapsto u_{\mathrm{p}}, \end{aligned}$$

such that for every $(f_0, u_{\mathrm{h}}) \in L^2(\mathbb{T})^d \times L^2((0,T') \times \omega)^{d_1}$,

$$\Pi^{\mathrm{p}} S(T; f_0, (u_{\mathrm{h}}, \mathcal{U}^{\mathrm{p}}(f_0, u_{\mathrm{h}})) = 0.$$

Admitting that Proposition G.4.4 and Proposition G.4.5 hold, we can now prove Proposition G.4.3.

Proof. We observe that the relation $\Pi S(T; f_0, (u_{\rm h}, u_{\rm p})) = 0$ holds if the two following equations are simultaneously satisfied

$$\begin{aligned} u_{\rm h} &= \mathcal{U}^{\rm h}(f_0, u_{\rm p}) = \mathcal{U}_1^{\rm h}(f_0) + \mathcal{U}_2^{\rm h}(u_{\rm p}), \\ u_{\rm p} &= \mathcal{U}^{\rm p}(f_0, u_{\rm h}) = \mathcal{U}_1^{\rm p}(f_0) + \mathcal{U}_2^{\rm p}(u_{\rm h}). \end{aligned} \tag{G.53}$$

If we set

$$C := \mathcal{U}_1^{\rm p} + \mathcal{U}_2^{\rm p}\mathcal{U}_1^{\rm h} : L^2(\mathbb{T})^d \to C_c^\infty((T', T) \times \mathbb{T})^{d_2},$$

then solving system (G.53) is equivalent to

$$\text{find } u_{\rm p} \in C_c^\infty((T', T) \times \mathbb{T})^{d_2}, \text{ such that } Cf_0 = (I - \mathcal{U}_2^{\rm p}\mathcal{U}_2^{\rm h})u_{\rm p}. \tag{G.54}$$

$\mathcal{U}_2^{\rm p}\mathcal{U}_2^{\rm h}$ is a compact operator of $L^2((T', T) \times \mathbb{T})^{d_2}$ because it takes values in $C_c^\infty((T', T) \times \mathbb{T})^{d_2}$. Thus, by Fredhlom's alternative (see [Bre11, Thm. 6.6]), there exist $N \in \mathbb{N}$ and $l_1, \ldots, l_N$ linear continuous forms on $L^2((T', T) \times \mathbb{T})^{d_2}$ such that the equation (G.54) has a solution $u_{\rm p} \in L^2((T', T) \times \mathbb{T})^{d_2}$ if and only if

$$\forall j \in \{1, \ldots, N\},\ l_j(C(f_0)) = 0. \tag{G.55}$$

Under these conditions (G.55), the equation (G.54) has a solution $u_{\rm p} = L(f_0)$ given by a continuous map $L : \mathcal{G} \to L^2((T', T) \times \mathbb{T})^{d_2}$ defined on

$$\mathcal{G} := \{f_0 \in L^2(\mathbb{T})^d\ ;\ l_j(Cf_0) = 0,\ 1 \le j \le N\}. \tag{G.56}$$

Then $L(f_0) = u_{\rm p} = \mathcal{U}_2^{\rm p}\mathcal{U}_2^{\rm h}u_{\rm p} + Cf_0$ belongs to $C_c^\infty((T', T) \times \omega)$. We get the conclusion with

$$\forall f_0 \in \mathcal{G},\ \mathcal{U}(f_0) := (\mathcal{U}^{\rm h}(f_0, L(f_0)), L(f_0)). \qquad \square$$

Proposition G.4.4 is proved in Section G.4.3. Proposition G.4.5 is proved in Section G.4.4. The unique continuation argument to control the low frequencies is presented in Annexe G.4.5.

G.4.3 Control of the hyperbolic high frequencies

The goal of this subsection is to prove Proposition G.4.4. We remind that $T > T' > T^*$ and that the control $u = (u_{\rm h}, u_{\rm p})$ satisfies (G.51).

G.4.3.1 Reduction to an exact controllability problem

The goal of this paragraph is to transform the null-controllability problem of Proposition G.4.4 into an exact controllability problem associated with an hyperbolic system. Precisely, we will get Proposition G.4.4 as a corollary of the following result.

Proposition G.4.6. If n_0 (in Eq. (G.41–G.42)) is large enough, then, for every $T > T^*$, there exists a continuous operator

$$\begin{aligned} \underline{\mathcal{U}}^{\mathrm{h}}_T \colon F^{\mathrm{h}} &\to L^2((0,T)\times\omega)^{d_1} \\ f_T &\mapsto u_{\mathrm{h}}, \end{aligned}$$

such that for every $f_T \in F^{\mathrm{h}}$,

$$\Pi^{\mathrm{h}} S\Big(T; 0, (\underline{\mathcal{U}}^{\mathrm{h}}_T(f_T), 0)\Big) = f_T.$$

Proposition G.4.6 will be proved in Section G.4.3.2. Now, we prove Proposition G.4.4 thanks to Proposition G.4.6.

Proof. [Proof of Proposition G.4.4] Let $(f_0, u_{\mathrm{p}}) \in L^2(\mathbb{T})^d \times L^2((T',T)\times\omega)^{d_2}$. We have to find $u_{\mathrm{h}} \in L^2((0,T')\times\omega)^{d_1}$ such that

$$\Pi^{\mathrm{h}} S(T; f_0, (u_{\mathrm{h}}, u_{\mathrm{p}})) = 0,$$

or, equivalently,

$$\Pi^{\mathrm{h}} S(T; 0, (u_{\mathrm{h}}, 0)) = -\Pi^{\mathrm{h}} S(T; f_0, (0, u_{\mathrm{p}})). \tag{G.57}$$

According to the well-posedness of the system (Sys) and the continuity of the projection Π^{h} (Definition G.2.8 and Proposition G.4.1), the linear map

$$(f_0, u_{\mathrm{p}}) \mapsto -\Pi^{\mathrm{h}} S(T; f_0, (0, u_{\mathrm{p}})), \tag{G.58}$$

is continuous from $L^2(\mathbb{T})^d \times L^2((T',T)\times\omega)^{d_2}$ into F^{h}. Since u_{h} is supported in $(0,T')\times\omega$ by (G.51), we have

$$\Pi^{\mathrm{h}} S(T; 0, (u_{\mathrm{h}}, 0)) = \mathrm{e}^{-(T-T')\mathcal{L}^{\mathrm{h}}} \Pi^{\mathrm{h}} S(T'; 0, (u_{\mathrm{h}}, 0)). \tag{G.59}$$

As pointed out in Proposition G.4.2, $\mathrm{e}^{t\mathcal{L}^{\mathrm{h}}}$ is well-defined for all $t \in \mathbb{R}$. Therefore, by using (G.58) and (G.59), (G.57) is equivalent to

$$\Pi^{\mathrm{h}} S(T'; 0, (u_{\mathrm{h}}, 0)) = -\mathrm{e}^{(T-T')\mathcal{L}^h} \Pi^{\mathrm{h}} S(T; f_0, (0, u_{\mathrm{p}})) \in F^{\mathrm{h}}. \tag{G.60}$$

We get the conclusion with

$$\mathcal{U}^{\mathrm{h}}(f_0, u_p) = \underline{\mathcal{U}}^{\mathrm{h}}_{T'}\Big(-\mathrm{e}^{(T-T')\mathcal{L}^h} \Pi^{\mathrm{h}} S(T; f_0, (0, u_{\mathrm{p}}))\Big). \qquad \square$$

G.4.3.2 Exact controllability of the hyperbolic part

The goal of this section is to prove Proposition G.4.6. By the Hilbert Uniqueness Method, Proposition G.4.6 is equivalent to the following observability inequality (it is an adaptation of [Cor07a, Thm. 2.42]).

Proposition G.4.7. There exists a constant $C > 0$ such that for every $g_0 \in \widetilde{F^{\mathrm{h}}}$, the solution g of (G.23) satisfies

$$\|g_0\|^2_{L^2(\mathbb{T})^d} \leq C \int_0^{T'} \int_\omega |g_1(t,x)|^2 \,\mathrm{d}t \,\mathrm{d}x, \tag{G.61}$$

where g_1 denotes the first d_1 components of g.

Proof. Let $g_0 \in \widetilde{F^{\mathrm{h}}}$. By using the definition of F^{h} (G.47) and the perturbative theory (Prop. G.2.3), g_0 decomposes as follows

$$g_0 = \sum_{\mu \in \mathrm{Sp}(A')} \sum_{|n|>n_0} P^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^* \widehat{g}_0(n) e_n. \tag{G.62}$$

Then, the solution g of (G.23) is

$$g(t) = \sum_{\mu \in \mathrm{Sp}(A')} \sum_{|n|>n_0} \mathrm{e}^{-tn^2 E\left(\frac{\mathrm{i}}{n}\right)^*} P^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^* \widehat{g}_0(n). \tag{G.63}$$

For $\mu \in \mathrm{Sp}(A')$, let us define

$$g_\mu(t) = \sum_{|n|>n_0} \mathrm{e}^{-tn^2 E\left(\frac{\mathrm{i}}{n}\right)^*} P^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^* \widehat{g}_0(n) e_n. \tag{G.64}$$

By using *i)* and *iii)* of Proposition G.2.3, we have

$$\mathrm{e}^{-tn^2 E\left(\frac{\mathrm{i}}{n}\right)^*} P^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^* = \mathrm{e}^{-tn^2\left(\mu\frac{\mathrm{i}}{n}+\left(\frac{\mathrm{i}}{n}\right)^2 R^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)\right)^*} P^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^* = \mathrm{e}^{-t\mu\mathrm{i}n + tR^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^*} P^{\mathrm{h}}_\mu\left(\frac{\mathrm{i}}{n}\right)^*,$$

which leads to

$$\begin{cases} \partial_t g_\mu - \mu \partial_x g_\mu + R^{\mathrm{h}}_\mu(0)^* g_\mu = S_\mu & \text{in } Q_{T'}, \\ g_\mu(0,.) = g_{\mu 0} & \text{in } \mathbb{T}, \end{cases} \tag{G.65}$$

where

$$\|S_\mu\|_{L^2(Q_{T'})^d} \leq C \left(\sum_{|n|>n_0} \frac{|\widehat{g}_0(n)|^2}{n^2} \right)^{1/2}. \tag{G.66}$$

If the term S_μ was $= 0$, the system (G.65) would be observable in time $T_\mu := \frac{2\pi - |\omega|}{|\mu|}$ (see for instance [ABCO17, Theorem 2.2]). Then, by using (G.49) and (G.2), we obtain the following observability estimate

$$\|g_{\mu 0}\|^2_{L^2(\mathbb{T})^d} \leq C \left(\int_0^{T'} \int_\omega |g_\mu(t,x)|^2 \,\mathrm{d}t \,\mathrm{d}x + \|S_\mu\|^2_{L^2(Q_{T'})^d} \right). \tag{G.67}$$

It can be deduced from the observability inequality for the solution of (G.65) without source term ($S_\mu = 0$), thanks to the triangle inequality, because the L^2-distance between the solutions of (G.65) with and without source term is bounded by $C\|S_\mu\|_{L^2}$. By using (G.67) and (G.66), we have

$$\|g_{\mu 0}\|^2_{L^2(\mathbb{T})^d} \leq C\left(\int_0^{T'}\int_\omega |g_\mu(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x + \sum_{|n|>n_0} \frac{|\widehat{g}_0(n)|^2}{n^2}\right). \tag{G.68}$$

Moreover, from *i)*, *ii)* of Proposition G.2.3, (G.63) and (G.64), we get

$$g_\mu(t) = P^{\mathrm{h}}_\mu(0)^* g(t) + Q_\mu(t),$$

with Q_μ satisfying the estimate (G.66). Thus, we have

$$\int_0^{T'}\int_\omega |g_\mu(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x \leq C\left(\int_0^{T'}\int_\omega |P^{\mathrm{h}}_\mu(0)^* g(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x + \sum_{|n|>n_0} \frac{|\widehat{g}_0(n)|^2}{n^2}\right). \tag{G.69}$$

Thus, from (G.68) and (G.69), we get

$$\|g_{\mu 0}\|^2_{L^2(\mathbb{T})^d} \leq C\left(\int_0^{T'}\int_\omega |P^{\mathrm{h}}_\mu(0)^* g(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x + \sum_{|n|>n_0} \frac{|\widehat{g}_0(n)|^2}{n^2}\right). \tag{G.70}$$

By summing for $\mu \in \mathrm{Sp}(A')$ the estimate (G.70) then by using *ii)* of Proposition G.2.3, Proposition G.2.1 and (G.62), we get the weak observability inequality

$$\|g_0\|^2_{L^2(\mathbb{T})^d} \leq C\left(\int_0^{T'}\int_\omega |g_1(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x + \|g_{T'}\|^2_{H^{-1}(\mathbb{T})}\right). \tag{G.71}$$

From (G.71) and the compact embedding $L^2(\mathbb{T}) \hookrightarrow H^{-1}(\mathbb{T})$, a classical compactness-uniqueness argument gives the observability inequality (G.61) (see for instance [DO18, Lemma 2.1 and Remark 2.2]).

Indeed, by Peetre's lemma (see [Pee61, Lemma 3]), we have from (G.71) that

$$N_{T'} := \{g_0 \in \widetilde{F^{\mathrm{h}}};\ g_1 = 0 \text{ in } (0,T') \times \omega\},$$

is finite-dimensional. Moreover, from [Pee61, Lemma 4], to prove (G.61), we need only to show that $N_{T'}$ is reduced to zero. First, by definition, we remark that $N_{T'}$ decreases as T' increases. By a small perturbation of T', we may therefore assume that $N_T = N_{T'}$ for $T - T'$ small thus $N_{T'}$ is stable by $\mathrm{e}^{t\mathcal{L}^{*\mathrm{h}}}$ where $\mathcal{L}^{*\mathrm{h}}$ is the restriction of $\mathcal{L}^*$ to $\widetilde{F^{\mathrm{h}}}$. Then, if $N_{T'}$ is not reduced to zero, it contains an eigenfunction of $\mathcal{L}^{*\mathrm{h}}$. But, by the construction of $N_{T'}$, the first components of that eigenfunction would wanish on ω. Therefore, $e_k(\cdot) = \mathrm{e}^{\mathrm{i}k\cdot} \equiv 0$ in ω for some k and this is a contradiction. $\square$

G.4.4 Control of the parabolic high frequencies

The goal of this subsection is to prove Proposition G.4.5. We recall that $0 < T' < T$ are chosen such that (G.49) holds and the control u is such that (G.50) and (G.51) hold.

The strategy is the following one : identify the equation satisfied by the last d_2 components of the parabolic equation (G.23) with the help of the asymptotics of Proposition G.2.3, then construct smooth controls by adapting the Lebeau-Robbiano's method to systems.

In this section, for every vector $\varphi \in \mathbb{C}^d$, we will note φ_1 its first d_1 components and φ_2 its last d_2 components.

G.4.4.1 Reduction to a null-controllability problem

The goal of this paragraph is to transform the null-controllability problem of Proposition G.4.5 into a null-controllability problem associated to a parabolic system. Precisely, we will prove that Proposition G.4.5 is a consequence of the following result.

Proposition G.4.8. If n_0 is large enough, then for every $T > 0$, there exists a continuous operator

$$\begin{aligned} \underline{\mathcal{U}}_T^{\mathrm{p}} \colon F^{\mathrm{p}} &\to C_c^\infty((0,T) \times \omega)^{d_2} \\ f_0 &\mapsto u_{\mathrm{p}}, \end{aligned}$$

such that for every $f_0 \in F^{\mathrm{p}}$,

$$\Pi^{\mathrm{p}} S(T; f_0, (0, \underline{\mathcal{U}}_T^{\mathrm{p}}(f_0))) = 0.$$

Proposition G.4.8 will be proved thanks to an adaptation of Lebeau and Robbiano's method in Section G.4.4.4, after 2 sections of necessary preliminary results. Now we prove Proposition G.4.5 thanks to Proposition G.4.8.

Proof. [Proof of Proposition G.4.5] Let $(f_0, u_{\mathrm{h}}) \in L^2(\mathbb{T})^d \times L^2((0,T') \times \omega)^{d_1}$. We have to find $u_{\mathrm{p}} \in C_c^\infty((T',T) \times \omega)^{d_2}$ such that

$$\Pi^{\mathrm{p}} S(T; f_0, (u_{\mathrm{h}}, u_{\mathrm{p}})) = 0. \tag{G.72}$$

or equivalently,

$$\Pi^{\mathrm{p}} S(T; 0, (0, u_{\mathrm{p}})) = -\Pi^{\mathrm{p}} S(T; f_0, (u_{\mathrm{h}}, 0)). \tag{G.73}$$

In view of the support of the controls in (G.51), the equality (G.73) is equivalent to

$$\Pi^{\mathrm{p}} S(T - T'; 0, (0, u_{\mathrm{p}}(\cdot - T'))) = -\mathrm{e}^{-(T-T')\mathcal{L}^{\mathrm{p}}} \Pi^{\mathrm{p}} S(T'; f_0, (u_{\mathrm{h}}, 0)), \tag{G.74}$$

or

$$\Pi^{\mathrm{p}} S\Big(T - T'; \mathrm{e}^{-(T-T')\mathcal{L}^{\mathrm{p}}} \Pi^{\mathrm{p}} S(T'; f_0, (u_{\mathrm{h}}, 0)), (0, u_{\mathrm{p}}(\cdot - T'))\Big) = 0. \tag{G.75}$$

By using Definition G.2.8 and Proposition G.4.1, we see that the mapping $(f_0, u_{\mathrm{h}}) \mapsto \Pi^{\mathrm{p}} S(T'; f_0, (u_{\mathrm{h}}, 0))$ is continuous from $L^2(\mathbb{T})^d \times L^2((0,T') \times \omega)^{d_1}$ into F^{p} . Thus we get the conclusion with

$$\forall t \in (T', T),\ \mathcal{U}^{\mathrm{p}}(f_0, u_h)(t) = \underline{\mathcal{U}}_{(T-T')}^{\mathrm{p}}\Big(\mathrm{e}^{-(T-T')\mathcal{L}^{\mathrm{p}}} \Pi^{\mathrm{p}} S(T'; f_0, (u_{\mathrm{h}}, 0))\Big)(t - T'). \qquad \square$$

G.4.4.2 Equation satisfied by the parabolic components of the free system

We begin by proving that if g is in $\widetilde{F^{\mathrm{p}}}$ then we can compute the first d_1 components of g from the last d_2. This will allow us to write an uncoupled equation for these components.

Proposition G.4.9. If z is small enough, there exists a matrix $G(z)$ such that for every $\varphi \in \mathbb{C}^d$,

$$\varphi \in \operatorname{Im}(P^{\mathrm{p}}(z)^*) \iff \varphi_1 = G(z)\varphi_2.$$

Moreover, G is holomorphic in z and $G(0) = 0$.

Proof. We write

$$P^{\mathrm{p}}(z)^* = \begin{pmatrix} p_{11}(z) & p_{12}(z) \\ p_{21}(z) & p_{22}(z) \end{pmatrix}.$$

Since $P^{\mathrm{p}}(z)^*$ is a projection, φ is in $\operatorname{Im}(P^{\mathrm{p}}(z)^*)$ if and only if

$$\begin{cases} p_{11}(z)\varphi_1 + p_{12}(z)\varphi_2 = \varphi_1 \\ p_{21}(z)\varphi_1 + p_{22}(z)\varphi_2 = \varphi_2. \end{cases}$$

In particular, if $\varphi \in \operatorname{Im}(P^{\mathrm{p}}(z)^*)$, then $(I_{d_1} - p_{11}(z))\varphi_1 = p_{12}(z)\varphi_2$. And since $P^{\mathrm{p}}(0)^* = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{d_2} \end{smallmatrix}\right)$ (see Proposition G.2.2), $p_{11}(0) = 0$, and so, if z is small enough, $|p_{11}(z)| < 1$ and $I_{d_1} - p_{11}(z)$ is invertible.

In that case, $\varphi_1 = (I_{d_1} - p_{11}(z))^{-1} p_{12}(z)\varphi_2$. This proves that the map

$$\varphi \in \operatorname{Im}(P^{\mathrm{p}}(z)^*) \mapsto \varphi_2 \in \mathbb{C}^{d_2}$$

is one to one. But the rank of $P^{\mathrm{p}}(z)^*$ does not depend on z (Remark G.2.4), and so it is always d_2. So the previous map is bijective. We note $G(z)$ the first d_1 component of its inverse. Note that we have $G(z) = (I_{d_1} - p_{11}(z))^{-1} p_{12}(z)$. Then, if $\varphi \in \operatorname{Im}(P^{\mathrm{p}}(z)^*)$, we have

$$\varphi = (\varphi_1, \varphi_2) = (G(z)\varphi_2, \varphi_2).$$

To prove the converse, note that the inverse of $\varphi \in \operatorname{Im}(P^{\mathrm{p}}(z)^*) \mapsto \varphi_2$ is $\varphi_2 \in \mathbb{C}^{d_2} \mapsto (G(z)\varphi_2, \varphi_2)$. $\square$

Increasing n_0 if necessary, we may assume that for $|n| > n_0$, $G(\mathrm{i}/n)$ is well-defined. Then, we define the (bounded) operator G from $L^2(\mathbb{T}, \mathbb{C}^{d_2})$ to $L^2(\mathbb{T}, \mathbb{C}^{d_1})$ by

$$G\left(\sum_{n\in\mathbb{Z}} \varphi_{n,2} e_n\right) = \sum_{|n|>n_0} G\left(\frac{\mathrm{i}}{n}\right)\varphi_{n,2} e_n. \tag{G.76}$$

Then, according to the definition of $\widetilde{F^{\mathrm{p}}}$, we have the following corollary that allows us to compute the first d_1 components from the last d_2.

Corollary G.4.10. For every $g \in (F^0)^\perp$ (the space of functions with no components along frequencies less than n_0), we have the equivalence $g \in \widetilde{F^{\mathrm{p}}} \Leftrightarrow g_1 = Gg_2$.

The Corollary G.4.10 makes it easy to write an equation on the last d_2 components of the adjoint system (G.23) if the initial condition is in $\widetilde{F^{\mathrm{p}}}$.

Proposition G.4.11. We define the operator $\mathfrak{D}$ by

$$D(\mathfrak{D}) = H^2(\mathbb{T})^{d_2}, \quad \mathfrak{D} = D^{\mathrm{tr}}\partial_x^2 + A_{22}^{\mathrm{tr}}\partial_x - K_{22}^{\mathrm{tr}} + A_{12}^{\mathrm{tr}}\partial_x G - K_{12}^{\mathrm{tr}} G. \tag{G.77}$$

Let $g_0 \in \widetilde{F^{\mathrm{p}}}$ and $g(t) = \mathrm{e}^{-t\mathcal{L}^*} g_0$. Then, for all $t \geq 0$, $g_1(t) = Gg_2(t)$ and g_2 satisfies the following equation

$$\partial_t g_2(t,x) - \mathfrak{D} g_2(t,x) = 0 \qquad \text{in } (0,T)\times\mathbb{T}. \tag{G.78}$$

Proof. The function g satisfies the system

$$(\partial_t - B^{\mathrm{tr}}\partial_x^2 - A^{\mathrm{tr}}\partial_x + K^{\mathrm{tr}})g(t,x) = 0 \qquad \text{in } (0,T)\times\mathbb{T}.$$

If we take the last d_2 components of this system, we get, in $(0,T)\times\mathbb{T}$,

$$\big(\partial_t - D^{\mathrm{tr}}\partial_x^2 - A_{22}^{\mathrm{tr}} + K_{22}^{\mathrm{tr}}\big)g_2(t,x) - \big(A_{12}^{\mathrm{tr}}\partial_x - K_{12}^{\mathrm{tr}}\big)g_1(t,x) = 0\,. \tag{G.79}$$

But for all $t \in [0,T]$, $g(t,\cdot) \in \widetilde{F^{\mathrm{p}}}$, so, according to Corollary G.4.10, $g_1(t) = Gg_2(t)$. Substituting this inside the equation (G.79) gives the stated equation (G.78). $\square$

G.4.4.3 Smooth control for a finite number of parabolic frequencies

For $N > n_0$ we introduce

$$F_N^{\mathrm{p}} := \bigoplus_{n_0<|n|\leq N} \mathrm{Im}\Big(P^{\mathrm{p}}\Big(\frac{\mathrm{i}}{n}\Big)\Big)e_n\,, \tag{G.80}$$

$$F_{>N}^{\mathrm{p}} := \bigoplus_{|n|>N} \mathrm{Im}\Big(P^{\mathrm{p}}\Big(\frac{\mathrm{i}}{n}\Big)\Big)e_n.$$

and the projection Π_N^{p} defined by

$$\begin{aligned} L^2(\mathbb{T})^d &= F^0 \oplus F_N^{\mathrm{p}} \oplus F_{>N}^{\mathrm{p}} \oplus F^{\mathrm{h}} \\ \Pi_N^{\mathrm{p}} &= \;\; 0 \;\; + I_{F_N^{\mathrm{p}}} + \;\; 0 \;\; + \;\; 0 \end{aligned}$$

which is a bounded operator on $L^2(\mathbb{T})^d$ (compostion of the bounded operator Π^{p} with an orthogonal projection). The goal of this section is to prove the following result.

Proposition G.4.12. There exists $\mathcal{C} > 0$ such that, for every $T \in (0,1]$ and $N > n_0$, there exists a linear map

$$\mathcal{K}_{T,N} \colon F^{\mathrm{p}} \to C_0^\infty((0,T)\times\omega)\,[4]$$

4. This space means that the function is supported on $[0,T]\times K$ where K is a compact subset of ω, and all the derivatives vanish on ω at time $t=0$ and $t=T$.

such that, for every $f_0 \in F^{\mathrm{p}}$ and $s \in \mathbb{N}$

$$\Pi^{\mathrm{p}}_N S\Big(T; f_0, (0, \mathcal{K}_{T,N}(f_0))\Big) = 0,$$

$$\|\mathcal{K}_{T,N}(f_0)\|_{H^s((0,T)\times\mathbb{T})} \leq \frac{\mathcal{C}}{T^{s+1}} N^{2s} \mathrm{e}^{\mathcal{C}N} \|f_0\|_{L^2(\mathbb{T})^d}.$$

Proof. Let $f_0 \in F^{\mathrm{p}}$. Throughout this proof, we will note $E_2(n)$ the $d_2 \times d_2$ matrices defined by

$$\forall n \in \mathbb{Z}\setminus\{0\},\ E_2(n) := D^{\mathrm{tr}} - \frac{\mathrm{i}}{n} A^{\mathrm{tr}}_{22} + \frac{1}{n^2} K^{\mathrm{tr}}_{22} - \Big(\frac{\mathrm{i}}{n} A^{\mathrm{tr}}_{12} - \frac{1}{n^2} K^{\mathrm{tr}}_{12}\Big) G\Big(\frac{\mathrm{i}}{n}\Big).$$

G.4.4.3.1 Step 1 : We prove that $u_2 \in C^\infty_0((0,T)\times\omega)$ satisfies $\Pi^{\mathrm{p}}_N S(T; f_0, (0, u_2)) = 0$ if and only if u_2 solves the following moments problem in $\mathbb{C}^{d_2}$

$$\forall n_0 < |n| \leq N,\ \int_0^T \int_\omega \mathrm{e}^{-n^2(T-t)E_2(n)^*} u_2(t,x) \mathrm{e}^{-\mathrm{i}nx}\, \mathrm{d}x\, \mathrm{d}t = F_n$$

$$\text{where } F_n = -\mathrm{e}^{-n^2 T E_2(n)^*} \Big(G\Big(\frac{\mathrm{i}}{n}\Big)^* \widehat{f}_{01}(n) + \widehat{f}_{02}(n) \Big) \tag{G.81}$$

and $E_2(n)^* = \overline{E_2(n)}^{\mathrm{tr}}$.

We first recall that, if P is a projection operator on $\mathbb{R}^d$ and $x \in \mathrm{Im}(P)$, then

$$(x = 0) \Leftrightarrow (\forall z \in \mathrm{Im}(P^*), \langle x, z\rangle = 0)$$

because $|x|^2 = \langle x, x\rangle = \langle Px, x\rangle = \langle x, P^* x\rangle$.

As a consequence, the relation $\Pi^{\mathrm{p}}_N S(T; f_0, (0, u_2)) = 0$ is equivalent to

$$\forall g_T \in \widetilde{F^{\mathrm{p}}_N},\ \langle S(T; f_0, ((0, u_p)), g_T\rangle = 0 \tag{G.82}$$

where $\langle \cdot, \cdot\rangle$ is the scalar product of $L^2(\mathbb{T}, \mathbb{C}^d)$ and

$$\widetilde{F^{\mathrm{p}}_N} := \bigoplus_{n_0 < |n| \leq N} \mathrm{Im}\Big(P^{\mathrm{p}}\Big(\frac{\mathrm{i}}{n}\Big)^*\Big) e_n\,.$$

For $g_T \in \widetilde{F^{\mathrm{p}}_N}$, we denote by $g(t) = \mathrm{e}^{-\mathcal{L}^*(T-t)} g_T$ the solution of the adjoint system. Then, by Proposition G.4.11, $g = (g_1, g_2)$, where $g_1 = G(g_2)$ and

$$\langle S(T; f_0, ((0, u_p)), g_T\rangle = \langle f_0, g(0)\rangle + \int_0^T \int_\omega \langle u_2(t,x), g_2(t,x)\rangle\, \mathrm{d}x\, \mathrm{d}t.$$

where the first 2 scalar products are in $L^2(\mathbb{T})^{d_2}$ and the last one is in $\mathbb{C}^{d_2}$. By Corollary G.4.10, the assertion (G.82) is equivalent to

$$\forall g_2^T \in L^2(\mathbb{T}, \mathbb{C}^{d_2}),\ \int_0^T \int_\omega \langle u_2(t,x), g_2(t,x)\rangle\, \mathrm{d}x\, \mathrm{d}t = -\big\langle f_0, \big(G(g_2^0), g_2^0\big)\big\rangle,$$

where $g_2(t) = \mathrm{e}^{-\mathfrak{D}(T-t)} g_2^T$ and $g_2^0 = g_2(0)$. By considering $g_2^T = Xe_n$ with $X \in \mathbb{C}^{d_2}$ and $n_0 < |n| \leq N$, we obtain

$$g_2(t) = \mathrm{e}^{-n^2(T-t)E_2(n)} Xe_n \text{ and } G(g_2^0) = G\left(\frac{\mathrm{i}}{n}\right)\mathrm{e}^{-n^2TE_2(n)} Xe_n.$$

The previous property is equivalent to

$$\begin{aligned}\forall n_0 < |n| \leq N,\ \forall X \in \mathbb{C}^{d_2},\ &\int_0^T \int_\omega \langle u_2(t,x), \mathrm{e}^{-n^2(T-t)E_2(n)} X\rangle \mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x\,\mathrm{d}t \\ &= -\langle f_{01}, G(\mathrm{i}/n)\mathrm{e}^{-n^2TE_2(n)} Xe_n\rangle - \langle f_{02}, \mathrm{e}^{-n^2TE_2(n)} Xe_n\rangle\end{aligned}$$

or, equivalently,

$$\begin{aligned}\forall n_0 < |n| \leq N,\ \forall X \in \mathbb{C}^{d_2},\ &\left\langle \int_0^T \int_\omega \mathrm{e}^{-n^2(T-t)E_2(n)^*} u_2(t,x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x\,\mathrm{d}t, X\right\rangle \\ &= -\left\langle \mathrm{e}^{-n^2TE_2(n)^*} G(\mathrm{i}/n)^* \widehat{f}_{01}(n) + \mathrm{e}^{-n^2TE_2(n)^*} \widehat{f}_{02}(n), X\right\rangle\end{aligned}$$

which proves (G.81).

G.4.4.3.2 Step 2 : Solving the moment problem. We look for a solution $u_2 \in C_0^\infty((0,T)\times\omega)$ of the moment problem (G.81) of the form

$$u_2(t,x) = \rho(t,x)v_2(t,x) \tag{G.83}$$

where $v_2 \in C^\infty((0,T)\times\mathbb{T})^{d_2}$ and $\rho \in C_0^\infty((0,T)\times\omega)$ is a scalar function with an appropriate support. More precisely, let

- $\widehat{\omega}$ be an open subset such that $\widehat{\omega} \subset\subset \omega$ and $\rho_2 \in C_c^\infty(\omega,\mathbb{R}_+)$ such that $\rho_2 = 1$ on $\widehat{\omega}$,
- $\rho_1 \in C^\infty([0,1],\mathbb{R}_+)$ such that $\rho_1(0) = \rho_1(1) = 0$ and (see Appendix G.5.2)

$$\exists C_0 > 0, \forall \gamma > 0, \quad \int_0^1 \rho_1(\tau)\mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau \geq \frac{1}{C_0}\mathrm{e}^{-C_0\sqrt{\gamma}}. \tag{G.84}$$

Then we choose $\rho(t,x) = \rho_1((T-t)/T)\rho_2(x)$. We also look for v_2 of the form

$$v_2(t,x) = \sum_{n_0<|n|\leq N} \mathrm{e}^{-k^2(T-t)E_2(k)} V_k \mathrm{e}^{\mathrm{i}kx} \text{ where } V_k \in \mathbb{C}^{d_2}. \tag{G.85}$$

The construction of v_2 will use the following algebraic result.

Lemma G.4.13. There exists $\mathcal{C} > 0$ such that, for every $N > n_0$ and $T \in (0,1]$ the matrix A in $\mathbb{C}^{(2(N-n_0)d_2)\times(2(N-n_0)d_2)}$, defined by blocks $A = (A_{n,k})_{\substack{n_0<|n|\leq N \\ n_0<|k|\leq N}}$ by

$$A_{n,k} = \int_0^T \int_\omega \mathrm{e}^{-n^2(T-t)E_2(n)^*} \mathrm{e}^{-k^2(T-t)E_2(k)} \mathrm{e}^{\mathrm{i}(k-n)x} \rho(t,x)\,\mathrm{d}x\,\mathrm{d}t \ \in \mathbb{C}^{d_2\times d_2},$$

is invertible and

$$\forall F \in \mathbb{C}^{2(N-n_0)d_2},\ |A^{-1}F| \leq \frac{\mathcal{C}}{T}\mathrm{e}^{\mathcal{C}N}|F|,$$

where $|\cdot|$ is the hermitian norm on $\mathbb{C}^{2(N-n_0)d_2}$.

Remark G.4.14. For instance, when $N = n_0 + 2$, then A is given by

$$A = \begin{pmatrix} A_{-n_0-2,-n_0-2} & A_{-n_0-2,-n_0-1} & A_{-n_0-2,n_0+1} & A_{-n_0-2,n_0+2} \\ A_{-n_0-1,-n_0-2} & A_{-n_0-1,-n_0-1} & A_{-n_0-1,n_0+1} & A_{-n_0-1,n_0+2} \\ A_{n_0+1,-n_0-2} & A_{n_0+1,-n_0-1} & A_{n_0+1,n_0+1} & A_{n_0+1,n_0+2} \\ A_{n_0+2,-n_0-2} & A_{n_0+2,-n_0-1} & A_{n_0+2,n_0+1} & A_{n_0+2,n_0+2} \end{pmatrix}.$$

For $X \in \mathbb{C}^{4d_2}$ with block decomposition

$$X = \begin{pmatrix} X_{-n_0-2} \\ X_{-n_0-1} \\ X_{n_0+1} \\ X_{n_0+2} \end{pmatrix}$$

where $X_k \in \mathbb{C}^{d_2}$ for every $n_0 < |k| \leqslant n_0 + 2$, we have

$$AX = \begin{pmatrix} \sum\limits_{n_0<|k|\leq n_0+2} A_{-n_0-2,k}X_k \\ \sum\limits_{n_0<|k|\leq n_0+2} A_{-n_0-1,k}X_k \\ \sum\limits_{n_0<|k|\leq n_0+2} A_{n_0+1,k}X_k \\ \sum\limits_{n_0<|k|\leq n_0+2} A_{n_0+2,k}X_k \end{pmatrix}.$$

Thus $\langle X, AX\rangle = \sum_{n_0<|n|,|k|\leqslant n_0+2} X_n^* A_{n,k} X_k$.

Proof. [Proof of Lemma G.4.13] The proof relies on the following spectral inequality, due to Lebeau and Robbiano (see [LR95] and also [LRL12, Thm. 5.4]) :

$$\exists C_1 > 0,\ \forall N \in \mathbb{N},\ \forall (a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}},\ \sum_{n=-N}^{+N} |a_n|^2 \leq C_1 \mathrm{e}^{C_1 N} \int_{\widehat{\omega}} \left| \sum_{n=-N}^{+N} a_n \mathrm{e}^{\mathrm{i}nx} \right|^2 \mathrm{d}x. \quad \text{(G.86)}$$

By summing the components, the same inequality holds when a_n is a vector, $a_n \in \mathbb{C}^{d_2}$, and $|\cdot|$ denotes the hermitian norm on $\mathbb{C}^{d_2}$.

Let $N > n_0$ and $X \in \mathbb{C}^{2(N-n_0)d_2}$ written by blocks $X = (X_k)_{n_0<|k|\leq N}$ with $X_k \in \mathbb{C}^{d_2}$. Then, by using the definition of A, ρ, the properties of ρ_2 and the above spectral inequality in vectorial form, we obtain

$$\begin{aligned}
\langle AX, X\rangle &= \sum_{n_0<|n|,|k|\leq N} X_n^* A_{n,k} X_k \\
&= \int_0^T \int_\omega \Big| \sum_{n_0<|k|\leq N} \mathrm{e}^{-k^2(T-t)E_2(k)} X_k \mathrm{e}^{\mathrm{i}kx} \Big|^2 \rho(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_0^T \int_{\widehat{\omega}} \Big| \sum_{n_0<|k|\leq N} \mathrm{e}^{-k^2(T-t)E_2(k)} X_k \mathrm{e}^{\mathrm{i}kx} \Big|^2 \rho_1\left(\frac{T-t}{T}\right) \mathrm{d}x\,\mathrm{d}t \\
&\geq \frac{\mathrm{e}^{-C_1 N}}{C_1} \int_0^T \sum_{n_0<|k|\leq N} \Big| \mathrm{e}^{-k^2(T-t)E_2(k)} X_k \Big|^2 \rho_1\left(\frac{T-t}{T}\right) \mathrm{d}t.
\end{aligned}$$

There exists $c > 0$ such that, for every $|k| > n_0$, $|E_2(k)| \leq c$. Then,

$$\forall |k| > n_0,\ \tau > 0,\ Y \in \mathbb{C}^{d_2},\ |\mathrm{e}^{E_2(k)\tau} Y| \leq \mathrm{e}^{c\tau}|Y|.$$

Then, by considering $\tau = k^2(T-t)$ and $Y = \exp\big(-k^2(T-t)E_2(k)\big)X_k$, we obtain

$$\forall |k| > n_0,\ t \in (0,T),\ \big|\mathrm{e}^{-k^2(T-t)E_2(k)} X_k\big| \geq \mathrm{e}^{-ck^2(T-t)}|X_k|.$$

Therefore, by using the change of variable $\tau = \frac{T-t}{T}$ and (G.84), we get

$$\begin{aligned}
\langle AX, X\rangle &\geq \frac{T\mathrm{e}^{-C_1 N}}{C_1} \sum_{n_0<|k|\leq N} |X_k|^2 \int_0^T \mathrm{e}^{-2ck^2 T\tau} \rho_1(\tau)\,\mathrm{d}\tau \\
&\geq \frac{T\mathrm{e}^{-C_1 N}}{C_1 C_0} \sum_{n_0<|k|\leq N} |X_k|^2 \mathrm{e}^{-C_0 k\sqrt{2cT}} \\
&\geq \frac{T}{C_1 C_0} \mathrm{e}^{-(C_1 + C_0\sqrt{2cT})N} |X|^2.
\end{aligned}$$

The above relation, valid for any $X \in \mathbb{C}^{2(N-n_0)d_2}$ proves that any eigenvalue of A is positive, thus A is invertible. Moreover, for any $F \in \mathbb{C}^{2(N-n_0)d_2} \setminus \{0\}$, the vector $X = A^{-1}F$ satisfies

$$\frac{T}{C_1 C_0} \mathrm{e}^{-(C_1 + C_0\sqrt{2cT})N} |X|^2 \leq \langle AX, X\rangle = \langle F, X\rangle \leq |F||X|.$$

Thus

$$|X| \leq \frac{C_1 C_0}{T} \mathrm{e}^{(C_1 + C_0\sqrt{2cT})N} |F|.$$

This gives the conclusion with $\mathcal{C} = \max\{C_1 C_0;\ C_1 + C_0\sqrt{2c}\}$. $\Diamond$

Now, let us come back to the proof of Proposition G.4.12. For such a control, (G.81) writes

$$\forall n_0 < |n| \leq N, \sum_{n_0<|k|\leq N} A_{n,k} V_k = F_n$$

or equivalently $AV = F$ with the notations of Proposition G.4.13. Thus, it is sufficient to take $V = A^{-1}F$. By the definition of F in (G.81), and Bessel-Parseval identity there exists $C_2 > 0$ independent of (T, N) such that

$$|F| = \Big(\sum_{n_0 < |n| \leq N} |F_n|^2 \Big)^{1/2} \leq C_2 \|f_0\|_{L^2(\mathbb{T})^d} .$$

Thus, by Proposition G.4.13

$$|V| = \Big(\sum_{n_0 < |k| \leq N} |V_k|^2 \Big)^{1/2} \leq \frac{C_2 \mathcal{C}}{T} \mathrm{e}^{\mathcal{C}N} \|f_0\|_{L^2(\mathbb{T})^d}. \tag{G.87}$$

G.4.4.3.3 Step 3 : Estimates on u_2. Let $s \in \mathbb{N}^*$. By (G.83), there exists $C = C(\rho, s) > 0$ such that

$$\|u_2\|_{H^s((0,T)\times\omega)} \leq \frac{C}{T^s} \|v_2\|_{H^s((0,T)\times\mathbb{T})}. \tag{G.88}$$

For any $s_1, s_2 \in \mathbb{N}$ such that $s_1 + s_2 \leq s$ we have,

$$\partial_t^{s_1} \partial_x^{s_2} v_2(t,x) = \sum_{n_0 < |k| \leq N} k^{2s_1} E_2(k)^{s_1} \mathrm{e}^{-k^2(T-t)E_2(k)} V_k (\mathrm{i}k)^{s_2} \mathrm{e}^{\mathrm{i}vkx}.$$

By Bessel-Parseval identity, we have

$$\begin{aligned}
\|\partial_t^{s_1} \partial_x^{s_2} v_2\|^2_{L^2((0,T)\times\mathbb{T})} &= \int_0^T \sum_{n_0 < |k| \leq N} \left| k^{2s_1+s_2} E_2(k)^{s_1} \exp\big[-k^2(T-t)E_2(k)\big] V_k \right|^2 \mathrm{d}t \\
&\leq C \int_0^T \sum_{n_0 < |k| \leq N} k^{4s} \left| \mathrm{e}^{-k^2(T-t)E_2(k)} V_k \right|^2 dt
\end{aligned}$$

By working as in the proof of Proposition G.2.5, we obtain, for n_0 large enough, positive constants $K_p, c_p > 0$ such that

$$\begin{aligned}
\|\partial_t^{s_1} \partial_x^{s_2} v_2\|^2_{L^2((0,T)\times\mathbb{T})} &\leq C \sum_{n_0 < |k| \leq N} k^{4s} K_p^2 \int_0^T \mathrm{e}^{-2c_p k^2 (T-t)} \,\mathrm{d}t \, |V_k|^2 \\
&\leq \frac{C K_p^2}{2c_p} \sum_{n_0 < |k| \leq N} k^{4s-2} |V_k|^2 \leq \frac{C K_p^2}{2c_p} N^{4s-2} |V|^2
\end{aligned}$$

By (G.87),

$$\|\partial_t^{s_1} \partial_x^{s_2} v_2\|_{L^2((0,T)\times\mathbb{T})} \leq \sqrt{\frac{C}{2c_p}} K_p N^{2s-1} \frac{C_2 \mathcal{C}}{T} \mathrm{e}^{\mathcal{C}N} \|f_0\|_{L^2(\mathbb{T})^d}.$$

This provides a constant $C > 0$ independant of (T, N) such that

$$\|v_2\|_{H^s((0,T)\times\mathbb{T})} \leq \frac{C}{T} N^{2s-1} \mathrm{e}^{\mathcal{C}N} \|f_0\|_{L^2(\mathbb{T})^d}$$

and (G.88) gives the expected estimate on u in H^s. $\square$

G.4.4.4 Lebeau-Robbiano's method

The goal of this section is to prove Proposition G.4.8. Let $T > 0$. We fix $\delta \in (0, T/2)$ and $\rho \in (0,1)$. For $\ell \in \mathbb{N}^*$, we set $N_\ell = 2^\ell$, $T_\ell = A2^{\rho\ell}$ where $A > 0$ is such that $2\sum_{\ell=1}^{\infty} T_\ell = T - 2\delta$. Let $f_0 \in F^{\mathrm{p}}$. We define

$$\begin{cases} f_1 = \mathrm{e}^{-\delta\mathcal{L}^{\mathrm{p}}} f_0\,, \\ g_\ell = \Pi^{\mathrm{p}} S(T_\ell; f_\ell, u_\ell) \text{ where } u_\ell = (0, K_{T_\ell,N_\ell}(f_\ell))\,, \\ f_{\ell+1} = \mathrm{e}^{-T_\ell\mathcal{L}^{\mathrm{p}}} g_\ell\,, \end{cases}$$

where K_{T_ℓ,N_ℓ} is the control operator introduced in Proposition G.4.12. By construction $\Pi^{\mathrm{p}}_{N_\ell} g_\ell = 0$ and therefore, by Proposition G.2.5

$$\begin{aligned} \|f_{\ell+1}\|^2_{L^2(\mathbb{T})^d} = \|\mathrm{e}^{-T_\ell\mathcal{L}^{\mathrm{p}}} g_\ell\|^2_{L^2(\mathbb{T})^d} &= \sum_{n>N_\ell} \left|\mathrm{e}^{-n^2 E(\mathrm{i}/n)T_\ell}\widehat{g}_\ell(n)\right|^2 \\ &\le \sum_{n>N_0} K_p^2 \mathrm{e}^{-2n^2 c_p T_\ell} |\widehat{g}_\ell(n)|^2 \le K_p^2 \mathrm{e}^{-2c_p N_\ell^2 T_\ell} \|g_\ell\|^2_{L^2(\mathbb{T})^d}\,. \end{aligned}$$

By the semi-group property proved in Proposition G.2.7, there exists positive constants K and c such that

$$\forall f \in L^2(\mathbb{T})^d, t \ge 0 \quad \|\mathrm{e}^{-t\mathcal{L}} f\|_{L^2(\mathbb{T})^d} \le K e^{ct} \|f\|_{L^2(\mathbb{T})^d}.$$

Then, according to the triangle inequality and Cauchy-Schwarz inequality,

$$\begin{aligned} \|g_\ell\|_{L^2(\mathbb{T})^d} \le \|S(T_\ell; f_\ell, u_\ell)\| &\le K\mathrm{e}^{cT_\ell}\|f_\ell\|_{L^2(\mathbb{T})^d} + \int_0^{T_\ell} K\mathrm{e}^{c(T_\ell - t)}\|u_\ell(t)\|_{L^2(\mathbb{T})}\,\mathrm{d}t \\ &\le K\mathrm{e}^{cT_\ell}\Big(\|f_\ell\|_{L^2(\mathbb{T})^d} + \sqrt{T_\ell}\|u_\ell\|_{L^2((0,T_\ell)\times\omega)}\Big), \end{aligned}$$

and by Proposition G.4.12

$$\|u_\ell\|_{L^2((0,T_\ell)\times\omega)} \le \frac{\mathcal{C}}{T_\ell}\mathrm{e}^{\mathcal{C}N_\ell}\|f_\ell\|_{L^2(\mathbb{T})^d}.$$

Thus

$$\|g_\ell\|_{L^2(\mathbb{T})^d} \le K\mathrm{e}^{cT_\ell}\left(1 + \frac{\mathcal{C}}{\sqrt{T_\ell}}\mathrm{e}^{\mathcal{C}N_\ell}\right)\|f_\ell\|_{L^2(\mathbb{T})^d}.$$

By setting

$$m_\ell = K_p \mathrm{e}^{-c_p N_\ell^2 T_\ell} K\mathrm{e}^{cT_\ell}\left(1 + \frac{\mathcal{C}}{\sqrt{T_\ell}}\mathrm{e}^{\mathcal{C}N_\ell}\right),$$

we get

$$\|f_{\ell+1}\|_{L^2(\mathbb{T})^d} \le m_\ell \|f_\ell\|_{L^2(\mathbb{T})^d}.$$

It is easy to see that there exists $C_1, C_2 > 0$ such that $m_\ell \le C_1 \mathrm{e}^{-C_2 2^{(2-\rho)\ell}}$. Thus $\|f_\ell\|_{L^2(\mathbb{T})^d} \to 0$ and more precisely there exists positive constants $C_3, C_4 > 0$ such that

$$\|f_\ell\|_{L^2(\mathbb{T})^d} \le C_3 \exp\Big(-C_4 2^{(2-\rho)\ell}\Big)\|f_0\|_{L^2(\mathbb{T})^d}.$$

Moreover

$$\sum_{\ell=1}^{\infty}\|u_\ell\|^2_{L^2((0,T_\ell)\times\omega)} \le \mathcal{C}\sum_{\ell=1}^{\infty}\frac{e^{\mathcal{C}N_\ell}}{T_\ell}C_3\exp(-C_4 2^{(2-\rho)\ell})\|f_0\|_{L^2(\mathbb{T})^d} < \infty\,. \tag{G.89}$$

We set $a_0=\delta$, $a_2=\delta+2T_1$, $\dots$, $a_\ell=a_{\ell-1}+2T_\ell$. We have $a_\ell\to(T-\delta)$ as $\ell\to\infty$. Then, for any $f_0\in F^{\mathrm{p}}$, we define the control

$$\underline{\mathcal{U}}_T^{\mathrm{p}}(f_0)(t,x)=\begin{cases} K_{T_\ell,N_\ell}(f_\ell)(t-a_{\ell-1}) & \text{for } a_{\ell-1}\le t\le a_{\ell-1}+T_\ell\,,\\ 0 & \text{for } a_{\ell-1}+T_\ell\le t\le a_{\ell-1}+2T_\ell=a_\ell,\\ 0 & \text{for } T-\delta\le t\le T\,.\end{cases}$$

Then, $\underline{\mathcal{U}}_T^{\mathrm{p}}(f_0)\in C_0^\infty((\delta,T-\delta)\times\omega)^{d_2}$ because all its derivatives vanish at times $t=a_\ell$. Thus $\underline{\mathcal{U}}_T^{\mathrm{p}}(f_0)\in C_c^\infty((0,T)\times\omega)^{d_2}$.

By (G.89), $\underline{\mathcal{U}}_T^{\mathrm{p}}(f_0)\in L^2((0,T)\times\omega)^d$ thus $S(T-\delta;f_0,\underline{\mathcal{U}}_T^{\mathrm{p}}(f_0))$ is the limit in $L^2(\mathbb{T})^d$ of the sequence $S(a_\ell;f_0,\underline{\mathcal{U}}_T^{\mathrm{p}}(f_0))$. As a consequence, $\Pi^{\mathrm{p}}S(T-\delta;f_0,\underline{\mathcal{U}}_T^{\mathrm{p}}(f_0))$ is the limit in $L^2(\mathbb{T})$ of the sequence $\Pi^{\mathrm{p}}S(a_\ell;f_0,\underline{\mathcal{U}}_T^{\mathrm{p}}(f_0))=f_{\ell+1}$. Finally,

$$\Pi^{\mathrm{p}}S(T;f_0,\underline{\mathcal{U}}_T^{\mathrm{p}}(f_0))=\Pi^{\mathrm{p}}S(T-\delta;f_0,\underline{\mathcal{U}}_T^{\mathrm{p}}(f_0))=0\,.$$

By Proposition G.4.12, for any $s\in\mathbb{N}^*$,

$$\left\|\underline{\mathcal{U}}_T^{\mathrm{p}}(f_0)\right\|_{H^s((0,T)\times\omega)}\le\sum_{\ell=1}^{\infty}\frac{\mathcal{C}}{T_\ell^{s+1}}N_\ell^{2s}e^{\mathcal{C}N_\ell}C_3\exp\left(-C_4 2^{(2-\rho)\ell}\right)\|f_0\|_{L^2(\mathbb{T})^d}<\infty\,.$$

G.4.5 Control of the low frequencies

The goal of this subsection is to prove Theorem G.1.2. Let $T>T^*$ where T^* is defined in (G.2). Then, there exists $T'>0$ such that (G.49) holds. Let $\mathcal{G}$ and $\mathcal{U}$ be as in Proposition G.4.3.

By extending $\mathcal{U}$ on $\mathcal{G}+F^0$ by 0 on a supplementary W of $\mathcal{G}$ in $\mathcal{G}+F^0$ and by replacing $\mathcal{G}$ by $\mathcal{G}+F^0$, and F^0 by F^0+W, one may assume that $F^0\subset\mathcal{G}$.

Implicitly, $\mathcal{G}$ is equipped with the topology of the $L^2(\mathbb{T})^d$-norm. The operator S is defined in Definition G.2.8.

We introduce the vector subspace of $L^2(\mathbb{T})^d$ defined by

$$\mathcal{F}_T=\left\{f_0\in L^2(\mathbb{T})^d;\ \exists u\in L^2((0,T')\times\omega)^{d_1}\times C_c^\infty((T',T)\times\omega)^{d_2}/S(T;f_0,u)=0\right\}.$$

G.4.5.0.1 Step 1 : We prove that $\mathcal{F}_T$ is a closed subspace of $L^2(\mathbb{T})^d$ with finite codimension. For $f_0\in\mathcal{G}$, the function $S(T;f_0,\mathcal{U}f_0)$ belongs to F^0, thus

$$\mathcal{K}(f_0):=-e^{T\mathcal{L}^0}S(T;f_0,\mathcal{U}f_0) \tag{G.90}$$

is well defined in F^0 by Proposition G.4.2. Then, $\mathcal{K}$ is a compact operator on $\mathcal{G}$ because it has finite rank. By the Fredholm alternative, $(I+\mathcal{K})(\mathcal{G})$ is a closed subspace of $\mathcal{G}$ and there exists a closed subspace $\mathcal{G}'$ of $\mathcal{G}$, with finite codimension in $\mathcal{G}$, such that $(I+\mathcal{K})$ is a bijection from $\mathcal{G}'$ to $(I+\mathcal{K})(\mathcal{G})$. Note that $\mathcal{G}'$ is also a closed subspace with finite codimension in $L^2(\mathbb{T})^d$.

For any $f_0 \in \mathcal{G}'$, by using that $\mathcal{K}(f_0) \in F^0$ and (G.90), we obtain

$$S(T, \mathcal{K}(f_0), 0) = \mathrm{e}^{-T\mathcal{L}}\mathcal{K}(f_0) = \mathrm{e}^{-T\mathcal{L}^0}\mathcal{K}(f_0) = -S(T, f_0, \mathcal{U}f_0)$$

thus

$$S(T, f_0 + \mathcal{K}(f_0), \mathcal{U}f_0) = S(T, f_0, \mathcal{U}f_0) + S(T, \mathcal{K}(f_0), 0) = 0\,.$$

This proves that $\mathcal{F}_T$ contains $(I+\mathcal{K})(\mathcal{G}')$, which is a closed subspace with finite codimension in $L^2(\mathbb{T})^d$. Therefore, there exists a finite dimensional subspace $F_\sharp$ of $L^2(\mathbb{T})^d$ such that $\mathcal{F}_T = (I+\mathcal{K})(\mathcal{G}') \oplus F_\sharp$. This gives the conclusion of Step 1.

G.4.5.0.2 Step 2 : We prove that, up to a possibly smaller choice of $T > T^*$, there exists $\delta > 0$ such that $\mathcal{F}_{T'} = \mathcal{F}_T$ for every $T' \in [T, T+\delta]$. When $0 < T' < T''$, by extending controls defined on $(0, T')$ by zero on (T', T''), we see that $\mathcal{F}_{T'} \subset \mathcal{F}_{T''}$. Thus, the map $T' \mapsto \mathrm{codim}(\mathcal{F}_{T'})$ is decreasing and takes integer values. As a consequence the discontinuities on $(T^*, T+1]$ are isolated. If T is not such a discontinuity point, then there exists $\delta > 0$ such that $\mathrm{codim}(\mathcal{F}_{T'}) = \mathrm{codim}(\mathcal{F}_T)$ for every $T' \in [T, T+\delta]$. In case T is such a discontinuity point, one may replace T by a smaller value, still such that $T > T^*$, for which this holds.

G.4.5.0.3 Step 3 : We prove that $\left(\mathrm{e}^{-t\mathcal{L}^*}\mathcal{F}_T^\perp\right)^\perp \subset \mathcal{F}_T$ for every $t \in (0, \delta)$. Let $t \in (0, \delta)$ and $g_0 \in L^2(\mathbb{T})^d$ be such that $\langle g_0, \mathrm{e}^{-t\mathcal{L}^*} f_0\rangle = 0$ for every $f_0 \in \mathcal{F}_T^\perp$. Then $\langle \mathrm{e}^{-t\mathcal{L}} g_0, f_0\rangle = 0$ for every $f_0 \in \mathcal{F}_T^\perp$, i.e. $\mathrm{e}^{-t\mathcal{L}} g_0 \in (\mathcal{F}_T^\perp)^\perp$. By Step 1, $\mathcal{F}_T$ is a closed subspace of $L^2(\mathbb{T})^d$ thus $(\mathcal{F}_T^\perp)^\perp = \mathcal{F}_T$. Therefore $\mathrm{e}^{-t\mathcal{L}} g_0 \in \mathcal{F}_T$. By definition of $\mathcal{F}_T$, this implies that $g_0 \in \mathcal{F}_{T+t}$. By Step 2, we get $g_0 \in \mathcal{F}_T$, which ends the proof of Step 3.

G.4.5.0.4 Step 4 : We prove that $\mathcal{F}_T^\perp$ is left invariant by $\mathrm{e}^{-t\mathcal{L}^*}$, i.e. $\mathcal{F}_T^\perp = \mathrm{e}^{-t\mathcal{L}^*}\mathcal{F}_T^\perp$ for every $t > 0$. The subspace $\mathrm{e}^{-t\mathcal{L}^*}\mathcal{F}_T^\perp$ is closed in $L^2(\mathbb{T})^d$ because it has finite dimension. Thus $\left(\left(\mathrm{e}^{-t\mathcal{L}^*}\mathcal{F}_T^\perp\right)^\perp\right)^\perp = \mathrm{e}^{-t\mathcal{L}^*}\mathcal{F}_T^\perp$ and we deduce from Step 3 that, for every $t \in (0, \delta)$, $\mathcal{F}_T^\perp \subset \mathrm{e}^{-t\mathcal{L}^*}\mathcal{F}_T^\perp$. Taking into account that $\dim(\mathrm{e}^{-t\mathcal{L}^*}\mathcal{F}_T^\perp) \leqslant \dim(\mathcal{F}_T^\perp)$, we obtain $\mathcal{F}_T^\perp = \mathrm{e}^{-t\mathcal{L}^*}\mathcal{F}_T^\perp$ for every $t \in (0, \delta)$. By the semi-group property, this equality holds for every $t > 0$.

G.4.5.0.5 Step 5 : We prove the existence of $N \in \mathbb{N}$ such that any $f_0 \in \mathcal{F}_T^\perp$ can be written

$$f_0 = \sum_{k \leq N} \varphi_k e_k \quad \text{with } \varphi_k \in \mathbb{C}^d. \tag{G.91}$$

Let $S(t)^*$ be the restriction of the semigroup $\mathrm{e}^{t\mathcal{L}^*}$ to $\mathcal{F}_T^\perp$: $S(t)^* = \mathrm{e}^{-t\mathcal{L}^*}|_{\mathcal{F}_T^\perp}$. Then $S(t)^* = \mathrm{e}^{tM}$ where M is a matrix such that $\mathcal{L}^* f_0 = M f_0$ for every $f_0 \in \mathcal{F}_T^\perp$. But then $\ker(M - \overline{\lambda})^j = \ker(\mathcal{L}^* - \overline{\lambda})^j \cap \mathcal{F}_T^\perp$. The Kernel decomposition theorem applied to M, and the structure of the generalized eigenspaces of $\mathcal{L}^*$ gives the conclusion of Step 4.

G.4.5.0.6 Step 6 : We prove that any element of $L^2(\mathbb{T})^d$ can be steered to $\mathcal{F}_T$ in an arbitrary short time , i.e. for every $\varepsilon > 0$ and $f_0 \in L^2(\mathbb{T})^d$, there exists $u \in L^2((0,T') \times \omega)^{d_1} \times C_c^\infty((T',T) \times \omega)^{d_2}$ such that $S(\varepsilon; f_0, u) \in \mathcal{F}_T$. By the Hilbert Uniqueness Method, it is sufficient to prove an observability inequality for $S(t)^*$. By using the finite-dimensionality of $\mathcal{F}_T^\perp$, it is equivalent to prove that the following unique continuation property holds : if $f(t,.) = \mathrm{e}^{tM} f_0$ with $f = 0$ in $(0,\varepsilon) \times \omega$, then $f_0 = 0$. By using the spectral inequality of Lebeau-Robbiano, i.e., (G.86) and (G.91), we readily get the result.

G.4.5.0.7 Step 7 : Conclusion. Step 5 implies the controllability of the system in any time $\tau > T$. As T is an arbitrary time such that $T > T^*$, this concludes the null-controllability in any time $T > T^*$. □

G.5 Appendix

G.5.1 Proof of an estimate on some operators on polynomial functions

In this part, we give the proof of Theorem G.3.6.

Proof. To prove Theorem G.3.6, it is enough to prove the estimate (G.37). Let K and V be as in Theorem G.3.6. Let us fix $R' > 0$ large enough so that $\bar{V} \subset D(0,R')$. Let f be any entire function that we write $f(z) = \sum f_n z^n$. According to Cauchy's integral formula, we have $f_n = \frac{1}{2i\pi} \oint_{\partial D(0,R')} \zeta^{-n-1} f(\zeta)\,\mathrm{d}\zeta$. Then,

$$\begin{aligned} H_\gamma(f)(z) &= \sum_n \gamma(n) f_n z^n \\ &= \sum_n \gamma(n) \frac{1}{2i\pi} \oint_{\partial D(0,R')} \frac{f(\zeta)}{\zeta^{n+1}} z^n \,\mathrm{d}\zeta \\ &= \oint_{\partial D(0,R')} \frac{1}{2i\pi\zeta} K_\gamma\left(\frac{z}{\zeta}\right) f(\zeta)\,\mathrm{d}\zeta, \end{aligned} \tag{G.92}$$

with

$$K_\gamma(\zeta) := \sum \gamma(n) \zeta^n. \tag{G.93}$$

According to the subexponential growth of $\gamma(n)$, the Taylor series in (G.93) is convergent for $|z| < 1$. We will prove that it can be analytically extended to $\mathbb{C} \setminus [1,+\infty)$.

Proposition G.5.1. Let $\gamma \in \mathcal{S}_R^{d\times d}$. Then, K_γ can be extended to a holomorphic function on $\mathbb{C} \setminus [1,+\infty)$. Moreover, $\gamma \in \mathcal{S}_R^{d\times d} \mapsto K_\gamma \in \mathcal{O}(\mathbb{C} \setminus [1,+\infty))$ is continuous, i.e., for every compact subset K of $\mathbb{C} \setminus [1,+\infty)$, there exist $C > 0$ and a seminorm p_ε of $\mathcal{S}_R^{d\times d}$ such that

$$|K_\gamma|_{L^\infty(K)} \leq C p_\varepsilon(\gamma). \tag{G.94}$$

Let us finish the proof of Theorem G.3.6 before proving Proposition G.5.1. Let us remind that V is a neighborhood of K that is star-shaped with respect to 0. So, we can choose V' a *smooth*, open, star-shaped with respect to 0 neighborhood of K such that $V' \subset V$. Let $c = \partial V'$ with clockwise orientation. Since V' is star-shaped with respect to 0, for $z \in V'$ and $\zeta \in c$, we never have $z/\zeta \in [1, +\infty)$, so $K(z/\zeta)$ is well-defined. This justifies the change of integration path in the expression of H_γ as a kernel operator (G.92) from $\partial D(0, R')$ to c (see for instance [How03, Theorem 6.7]). Therefore, we have

$$\begin{aligned} |H_\gamma(f)(z)| &= \frac{1}{2\pi}\left|\oint_c \frac{1}{\zeta} K_\gamma\left(\frac{z}{\zeta}\right) f(\zeta)\,\mathrm{d}\zeta\right| \\ &\leq \frac{\mathrm{length}(c)}{2\pi} \sup_{\zeta\in c}|\zeta|^{-1} \sup_{z\in K,\,\zeta\in c}\left|K_\gamma\left(\frac{z}{\zeta}\right)\right| |f|_{L^\infty(c)}. \end{aligned}$$

According to the estimation (G.94) on the kernel K_γ applying with the compact $K \leftarrow K' := \{z/\zeta\ ;\ (z,\zeta) \in K \times c\}$, there exists $C > 0$ and $\varepsilon > 0$ independent of γ such that

$$\sup_{z\in K,\,\zeta\in c}\left|K_\gamma\left(\frac{z}{\zeta}\right)\right| \leq C p_\varepsilon(\gamma).$$

So,

$$|H_\gamma(f)(z)| \leq C' p_\varepsilon(\gamma)|f|_{L^\infty(c)} \leq C' p_\varepsilon(\gamma)|f|_{L^\infty(V)}.$$

This is the inequality (G.37) we wanted to prove. □

Proposition G.5.1 was essentially already proved by Lindelöf [Lin89], and then slightly generalized by Arakelyan in [Ara84] and rediscovered by the second author in [Koe17]. We will use here Lindelöf method, based on the Residue theorem, instead of the method based on the Poisson summation formula in the other reference.

Proof. [Proof of Proposition G.5.1] We only prove it when K is a compact subset of $\mathbb{C} \setminus \mathbb{R}_+$ (instead of $\mathbb{C} \setminus [1, +\infty)$). Since the Taylor series $K_\gamma(z) = \sum \gamma(n) z^n$ is convergent for $|z| < 1$, this is enough.

First, let us choose $R' \notin \mathbb{N}$ between R and $R+1$ and let us define $\Gamma := \{R' - it\ ;\ t \in \mathbb{R}\}$. For $z \in \mathbb{C} \setminus [1, +\infty)$, we introduce

$$\widehat{K_\gamma}(z) = \int_\Gamma \frac{\gamma(\zeta) z^\zeta}{e^{2i\pi\zeta} - 1}\,\mathrm{d}\zeta = \int_\Gamma g(z,\zeta)\,\mathrm{d}\zeta, \tag{G.95}$$

with

$$g(z,\zeta) := \gamma(\zeta) z^\zeta (e^{2i\pi\zeta} - 1)^{-1}.$$

Note that when ζ is not to close to the integers, say distance$(\zeta, \mathbb{N}) \geq \delta > 0$, then there exists $C_\delta > 0$ such that

$$|(\mathrm{e}^{2i\pi\zeta} - 1)^{-1}| \leq C_\delta \mathrm{e}^{2\pi \min(\Im(\zeta), 0)}. \tag{G.96}$$

Step 1 : By using the theorem of holomorphy under the integral sign, we prove that $\widehat{K_\gamma} \in \mathcal{O}(\mathbb{C} \setminus [1, +\infty))$.

For any $\zeta \in \Gamma$, $z \mapsto g(z,\zeta) \in \mathcal{O}(\mathbb{C} \setminus [1,+\infty))$.

Let us check the domination hypothesis. Let K be a compact set of $\mathbb{C} \setminus [1,+\infty)$. For any $z \in K$, for any $\zeta \in \Gamma$, by using (G.96), we have

$$|g(z,\zeta)| \leq |\gamma(\zeta)||\mathrm{e}^{\zeta \ln(z)}||(\mathrm{e}^{2i\pi\zeta} - 1)^{-1}| \leq C|\gamma(\zeta)|\mathrm{e}^{\Re(\zeta)\ln|z| - \Im(\zeta)\arg(z) + 2\pi\min(\Im(\zeta),0)},$$

which, by denoting $C_K := \sup_{z\in K} \ln|z|$, yields for $\Im(\zeta) > 0$,

$$|g(z,\zeta)| \leq C|\gamma(\zeta)|\mathrm{e}^{C_K R' - c_{K,1}|\Im(\zeta)|}, \text{ with } c_{K,1} = \inf_{z\in K} \arg(z) > 0,$$

and for $\Im(\zeta) < 0$,

$$|g(z,\zeta)| \leq C|\gamma(\zeta)|\mathrm{e}^{C_K R' - (2\pi - c_{K,2})|\Im(\zeta)|}, \text{ with } c_{K,2} = \sup_{z\in K} \arg(z) < 2\pi.$$

Thus, in both cases, we find with $c_K := \min(c_{K,1}, 2\pi - c_{K,2}) > 0$,

$$\forall (z,\zeta) \in K \times \Gamma,\ |g(z,\zeta)| \leq C|\gamma(\zeta)|\mathrm{e}^{C_K R' - c_K|\Im(\zeta)|} \tag{G.97}$$

By using the fact that $\gamma \in \mathcal{S}_R^{d\times d}$, we have

$$\forall \varepsilon > 0,\ \forall \zeta \in \Gamma,\ |\gamma(\zeta)| \leq C p_\varepsilon(\gamma)\mathrm{e}^{\varepsilon|\zeta|} \leq C p_\varepsilon(\gamma)\mathrm{e}^{\varepsilon R'}\mathrm{e}^{\varepsilon|\Im(\zeta)|} \tag{G.98}$$

Then, by using (G.97) and (G.98) with $\varepsilon = c_K/2$, we get

$$\forall (z, \zeta = R' - it) \in K \times \Gamma,\ |g(z,\zeta)| \leq C p_\varepsilon(\gamma)\mathrm{e}^{C'_K R'}\mathrm{e}^{-(c_K/2)|t|} \in L^1_t(\mathbb{R}). \tag{G.99}$$

By the theorem of holomorphy under the integral sign, we find that $\widehat{K_\gamma} \in \mathcal{O}(\mathbb{C} \setminus [1,+\infty))$ and by using (G.99), we deduce the bound (G.94) for $\widehat{K_\gamma}$.

Step 2 : By the Residue Theorem, we prove that

$$\forall z \in D(0,1) \setminus [0,1),\ K_\gamma(z) = \frac{\widehat{K_\gamma}(z)}{2i\pi}. \tag{G.100}$$

According to the Residue Theorem (see Figure G.2), we have

$$\forall z \in D(0,1) \setminus [0,1),\ \int_{\Gamma_k} g(z,\zeta)\,\mathrm{d}\zeta = \int_{\Gamma_k} \frac{\gamma(\zeta)z^\zeta}{e^{2i\pi\zeta} - 1}\,\mathrm{d}\zeta = 2i\pi \sum_{n>R}^{R'+k} \gamma(n)z^n, \tag{G.101}$$

where $\Gamma_k = \{R' - it\ ;\ t \in [-k,k]\} \cup \Gamma'_k$ with $\Gamma'_k := \{R' + ke^{i\phi}, -\pi/2 \leq \phi \leq \pi/2\}$. Let $z \in D(0,1) \setminus [0,1)$. Arguing as before, using $\ln(|z|) < 0$ and (G.96), we show that there exists $c_z > 0$ such that for every $\zeta = R' + k\cos(\phi) + ik\sin(\phi) \in \Gamma'_k$,

$$|g(z,\zeta)| \leq C|\gamma(\zeta)|\mathrm{e}^{\Re(\zeta)\ln|z| - \Im(\zeta)\arg(z) + 2\pi\min(\Im(\zeta),0)} \leq C|\gamma(\zeta)|\mathrm{e}^{-c_z k\cos(\phi) - c_z k|\sin(\phi)|}.$$

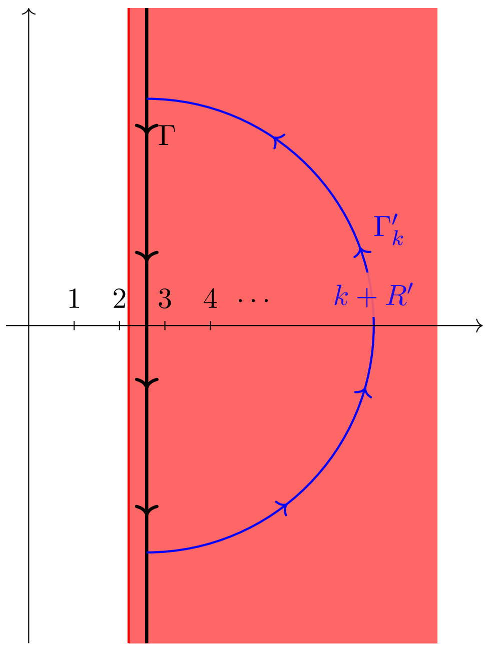

FIGURE G.2 – Path for the Residue theorem.

Consequently, by using that there exists $c_1 > 0$ such that for every $\phi \in [-\pi/2, \pi/2]$, $\cos(\phi) + |\sin(\phi)| \geq c_1$, and the fact that $\gamma \in \mathcal{S}_R^{d\times d}$, we get that there exists $c' > 0$ such that for every $\varepsilon > 0$, for every $\zeta \in \Gamma'_k$, we have

$$|g(z,\zeta)| \leq C\mathrm{e}^{-c'k+\varepsilon k}.$$

Then, by taking $\varepsilon = c'/2$ in the previous bound, we show that

$$\left|\int_{\Gamma'_k} g(z,\zeta)\,\mathrm{d}\zeta\right| \leq Ck\mathrm{e}^{-(c'/2)k} \to 0 \text{ as } k \to +\infty. \tag{G.102}$$

Then, by using (G.101), (G.102), and by passing to the limit $k \to 0$, we finally get (G.100) from the definitions of K_γ and $\widehat{K_\gamma}$ given in (G.93), (G.95).

Step 3 : Conclusion. By using the relation between K_γ and $\widehat{K_\gamma}$, i.e. (G.100), the facts that $\widehat{K_\gamma}$ is holomorphic in $\mathbb{C}\setminus[1,+\infty)$ and satisfies the continuity estimate (G.94) by the step 1 of the proof, we conclude the proof of Proposition G.5.1. □

G.5.2 Example of a particular function

The goal of this section is to construct a function ρ_1 such that the lower bound (G.84) holds.

Let $\rho_1 \in C^\infty([0,1],\mathbb{R}_+)$ be such that $\rho_1(\tau) = \rho_1(1-\tau) = \mathrm{e}^{-\frac{1}{\tau}}$ for $\tau \in (0,1/4)$. Then for every $\gamma > 0$, the change of variable $s = \sqrt{\gamma}\tau$ gives

$$\int_0^1 \rho_1(\tau)\mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau \geq \frac{1}{\sqrt{\gamma}}\int_0^{\sqrt{\gamma}/4} \mathrm{e}^{-\sqrt{\gamma}\phi(s)}\,\mathrm{d}s$$

where $\phi(s) = \frac{1}{s} + s$. The function ϕ takes its minimal value at $s_* = 1$ and $\phi''(1) = 2 > 0$ thus, by the Laplace's method (see [QZ13, Chapitre 9, Théorème VI.1]),

$$\int_0^2 \mathrm{e}^{-\sqrt{\gamma}\phi(s)}\,\mathrm{d}s \underset{\gamma\to\infty}{\sim} \frac{\sqrt{\pi}}{\sqrt[4]{\gamma}}\mathrm{e}^{-2\sqrt{\gamma}}.$$

This proves that, for γ large enough,

$$\int_0^1 \rho_1(\tau)\mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau \geq \mathrm{e}^{-3\sqrt{\gamma}}.$$

Bibliographie

[AB00] Sebastian Anita and Viorel Barbu. Null controllability of nonlinear convective heat equations. *ESAIM : COCV*, 5 :157–173, 2000.

[ABCO17] Fatiha Alabau-Boussouira, Jean-Michel Coron, and Guillaume Olive. Internal controllability of first order quasi-linear hyperbolic systems with a reduced number of controls. *SIAM J. Control Optim.*, 55(1) :300–323, 2017.

[ABG17] Ari Arapostathis, Anup Biswas, and Debdip Ganguly. Certain Liouville properties of eigenfunctions of elliptic operators. *arXiv e-prints*, page arXiv :1708.09640, Aug 2017.

[ABL11] Fatiha Alabau-Boussouira and Matthieu Léautaud. Indirect controllability of locally coupled systems under geometric conditions. *C. R. Math. Acad. Sci. Paris*, 349(7-8) :395–400, 2011.

[AE08] Giovanni Alessandrini and Luis Escauriaza. Null-controllability of one-dimensional parabolic equations. *ESAIM Control Optim. Calc. Var.*, 14(2) :284–293, 2008.

[AEWZ14] Jone Apraiz, Luis Escauriaza, Gengshang Wang, and Can Zhang. Observability inequalities and measurable sets. *J. Eur. Math. Soc. (JEMS)*, 16(11) :2433–2475, 2014.

[AKBD06] Farid Ammar-Khodja, Assia Benabdallah, and Cédric Dupaix. Null-controllability of some reaction-diffusion systems with one control force. *J. Math. Anal. Appl.*, 320(2) :928–943, 2006.

[AKBDGB09a] Farid Ammar-Khodja, Assia Benabdallah, Cédric Dupaix, and Manuel González-Burgos. A Kalman rank condition for the localized distributed controllability of a class of linear parbolic systems. *J. Evol. Equ.*, 9(2) :267–291, 2009.

[AKBDGB09b] Farid Ammar-Khodja, Assia Benabdallah, Cédrix Dupaix, and Manuel González-Burgos. A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems. *Differ. Equ. Appl.*, 1(3) :427–457, 2009.

[AKBDK03] Farid Ammar Khodja, Assia Benabdallah, Cédric Dupaix, and Ilya Kostin. Controllability to the trajectories of phase-field models by one control force. *SIAM J. Control Optim.*, 42(5) :1661–1680, 2003.

[AKBGBdT11] Farid Ammar-Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz de Teresa. Recent results on the controllability of linear coupled parabolic problems : a survey. *Math. Control Relat. Fields*, 1(3) :267–306, 2011.

[AKBGBdT14a] Farid Ammar-Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz de Teresa. Minimal time for the null controllability of parabolic systems : the effect of the condensation index of complex sequences. *J. Funct. Anal.*, 267(7) :2077–2151, 2014.

[AKBGBdT14b] Farid Ammar-Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz de Teresa. Minimal time of controllability of two parabolic equations with disjoint control and coupling domains. *C. R. Math. Acad. Sci. Paris*, 352(5) :391–396, 2014.

[AKBGBdT16] Farid Ammar-Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz de Teresa. New phenomena for the null controllability of parabolic systems : minimal time and geometrical dependence. *J. Math. Anal. Appl.*, 444(2) :1071–1113, 2016.

[Ali79] Nicholas D. Alikakos. L^p bounds of solutions of reaction-diffusion equations. *Comm. Partial Differential Equations*, 4(8) :827–868, 1979.

[Ara84] Norair U. Arakelyan. Effective analytic continuation of power series. *Mat. Sb. (N.S.)*, 124(166)(1) :24–44, 1984.

[AT00] Paolo Albano and Daniel Tataru. Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system. *Electron. J. Differential Equations*, pages No. 22, 15, 2000.

[AT02] Sebastian Anita and Daniel Tataru. Null controllability for the dissipative semilinear heat equation. *Appl. Math. Optim.*, 46(2-3) :97–105, 2002. Special issue dedicated to the memory of Jacques-Louis Lions.

[Bar00] Viorel Barbu. Exact controllability of the superlinear heat equation. *Appl. Math. Optim.*, 42(1) :73–89, 2000.

[Bar02] Viorel Barbu. Local controllability of the phase field system. *Nonlinear Anal.*, 50(3, Ser. A : Theory Methods) :363–372, 2002.

[Bar18] Viorel Barbu. *Controllability and stabilization of parabolic equations*, volume 90 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2018. Subseries in Control.

[BCGDT14] Assia Benabdallah, Michel Cristofol, Patricia Gaitan, and Luz De Teresa. Controllability to trajectories for some parabolic systems of three and two equations by one control force. *Math. Control Relat. Fields*, 4(1) :17–44, 2014.

[BDLR07] Assia Benabdallah, Yves Dermenjian, and Jérôme Le Rousseau. Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem. *J. Math. Anal. Appl.*, 336(2) :865–887, 2007.

[BGBPGa04] Olivier Bodart, Manuel González-Burgos, and Rosario Pérez-Garcí a. Existence of insensitizing controls for a semilinear heat equation with a superlinear nonlinearity. *Comm. Partial Differential Equations*, 29(7-8) :1017–1050, 2004.

[BHLR11] Franck Boyer, Florence Hubert, and Jérôme Le Rousseau. Uniform controllability properties for space/time-discretized parabolic equations. *Numerische Mathematik*, 118(4) :601–661, Aug 2011.

[BK05] Jean Bourgain and Carlos E. Kenig. On localization in the continuous Anderson-Bernoulli model in higher dimension. *Invent. Math.*, 161(2) :389–426, 2005.

[BKLB19] Karine Beauchard, Armand Koenig, and Kévin Le Balc'h. Null-controllability of linear parabolic-transport systems. *in preparation*, 2019.

[BLR92] Claude Bardos, Gilles Lebeau, and Jeffrey Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30(5) :1024–1065, 1992.

[BM04] Vincent D. Blondel and Alexandre Megretski, editors. *Unsolved problems in mathematical systems and control theory.* Princeton University Press, Princeton, NJ, 2004.

[BM17] Karine Beauchard and Frédéric Marbach. Unexpected quadratic behaviors for the small-time local null controllability of scalar-input parabolic equations. *ArXiv e-prints :1712.09790*, December 2017.

[Boy13] Franck Boyer. On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems. In *CANUM 2012, 41e Congrès National d'Analyse Numérique*, volume 41 of *ESAIM Proc.*, pages 15–58. EDP Sci., Les Ulis, 2013.

[BPS18] Karine Beauchard and Karel Pravda-Starov. Null-controllability of hypoelliptic quadratic differential equations. *J. Éc. polytech. Math.*, 5 :1–43, 2018.

[BR14] Franck Boyer and Jérôme Le Rousseau. Carleman estimates for semi-discrete parabolic operators and application to the controllability of semi-linear semi-discrete parabolic equations. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 31(5) :1035 – 1078, 2014.

[Bre11] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[BT73] Claude Bardos and Luc Tartar. Sur l'unicité rétrograde des équations paraboliques et quelques questions voisines. *Arch. Rational Mech. Anal.*, 50 :10–25, 1973.

[CDF14] José A. Canizo, Laurent Desvillettes, and Klemens Fellner. Improved duality estimates and applications to reaction-diffusion equations. *Comm. Partial Differential Equations*, 39(6) :1185–1204, 2014.

[CFCLM13] Haroldo R. Clark, Enrique Fernández-Cara, Juan Limaco, and Luis Adauto Medeiros. Theoretical and numerical local null controllability for a parabolic system with local and nonlocal nonlinearities. *Appl. Math. Comput.*, 223 :483–505, 2013.

[CG05] Jean-Michel Coron and Sergio Guerrero. Singular optimal control : a linear 1-D parabolic-hyperbolic example. *Asymptot. Anal.*, 44(3-4) :237–257, 2005.

[CG17] Jean-Michel Coron and Jean-Philippe Guilleron. Control of three heat equations coupled with two cubic nonlinearities. *SIAM J. Control Optim.*, 55(2) :989–1019, 2017.

[CGMR15] Jean-Michel Coron, Sergio Guerrero, Philippe Martin, and Lionel Rosier. Homogeneity applied to the controllability of a system of parabolic equations. In *2015 European Control Conference (ECC 2015)*, Proceedings of the 2015 European Control Conference (ECC 2015), pages 2470–2475, Linz, Austria, July 2015.

[CGR10] Jean-Michel Coron, Sergio Guerrero, and Lionel Rosier. Null controllability of a parabolic system with a cubic coupling term. *SIAM J. Control Optim.*, 48(8) :5629–5653, 2010.

[CGV17] Cristina Caputo, Thierry Goudon, and Alexis F Vasseur. Solutions of the 4-species quadratic reaction-diffusion system are bounded and C^∞, in any space dimension. *ArXiv e-prints :1709.05694*, September 2017.

[CH98] Thierry Cazenave and Alain Haraux. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors.

[CL14] Jean-Michel Coron and Pierre Lissy. Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components. *Invent. Math.*, 198(3) :833–880, 2014.

[CN17] Jean-Michel Coron and Hoai-Minh Nguyen. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. *Arch. Ration. Mech. Anal.*, 225(3) :993–1023, 2017.

[Cor92] Jean-Michel Coron. Global asymptotic stabilization for controllable systems without drift. *Math. Control Signals Systems*, 5(3) :295–312, 1992.

[Cor07a] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[Cor07b] Jean-Michel Coron. Some open problems on the control of nonlinear partial differential equations. In *Perspectives in nonlinear partial diffe-*

rential equations, volume 446 of *Contemp. Math.*, pages 215–243. Amer. Math. Soc., Providence, RI, 2007.

[CSB15] Felipe Wallison Chaves-Silva and Mostafa Bendahmane. Uniform null controllability for a degenerating reaction-diffusion system approximating a simplified cardiac model. *SIAM J. Control Optim.*, 53(6) :3483–3502, 2015.

[CSG15] Felipe Wallison Chaves-Silva and Sergio Guerrero. A uniform controllability result for the Keller-Segel system. *Asymptot. Anal.*, 92(3-4) :313–338, 2015.

[CSRZ14] Felipe W. Chaves-Silva, Lionel Rosier, and Enrique Zuazua. Null controllability of a system of viscoelasticity with a moving control. *J. Math. Pures Appl. (9)*, 101(2) :198–222, 2014.

[CSZZ17] Felipe Walison Chaves-Silva, Xu Zhang, and Enrique Zuazua. Controllability of evolution equations with memory. *SIAM J. Control Optim.*, 55(4) :2437–2459, 2017.

[CT04] Jean-Michel Coron and Emmanuel Trélat. Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM J. Control Optim.*, 43(2) :549–569, 2004.

[DE19] Jérémi Dardé and Sylvain Ervedoza. On the cost of observability in small times for the one-dimensional heat equation. *Anal. PDE*, 12(6) :1455–1488, 2019.

[DF06] Laurent Desvillettes and Klemens Fellner. Exponential decay toward equilibrium via entropy methods for reaction-diffusion equations. *J. Math. Anal. Appl.*, 319(1) :157–176, 2006.

[DFCGBZ02] Anna Doubova, Enrique Fernández-Cara, Manuel González-Burgos, and Enrique Zuazua. On the controllability of parabolic systems with a nonlinear term involving the state and the gradient. *SIAM J. Control Optim.*, 41(3) :798–819, 2002.

[DHP07] Robert Denk, Matthias Hieber, and Jan Prüss. Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data. *Math. Z.*, 257(1) :193–224, 2007.

[DL16] Michel Duprez and Pierre Lissy. Indirect controllability of some linear parabolic systems of m equations with $m-1$ controls involving coupling terms of zero or first order. *J. Math. Pures Appl. (9)*, 106(5) :905–934, 2016.

[DL18] Michel Duprez and Pierre Lissy. Positive and negative results on the internal controllability of parabolic equations coupled by zero- and first-order terms. *Journal of Evolution Equations*, 18(2) :659–680, Jun 2018.

[DO18] Michel Duprez and Guillaume Olive. Compact perturbations of controlled systems. *Math. Control Relat. Fields*, 8(2) :397–410, 2018.

[dT00] Luz de Teresa. Insensitizing controls for a semilinear heat equation. *Comm. Partial Differential Equations*, 25(1-2) :39–72, 2000.

[Dup15] Michel Duprez. *Controllability of some systems governed by parabolic equations*. Theses, Université de Franche-Comté, November 2015.

[Dup17] Michel Duprez. Controllability of a 2×2 parabolic system by one force with space-dependent coupling term of order one. *ESAIM Control Optim. Calc. Var.*, 23(4) :1473–1498, 2017.

[DZZ08] Thomas Duyckaerts, Xu Zhang, and Enrique Zuazua. On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(1) :1–41, 2008.

[Erv08] Sylvain Ervedoza. Control and stabilization properties for a singular heat equation with an inverse-square potential. *Comm. Partial Differential Equations*, 33(10-12) :1996–2019, 2008.

[EV09] Sylvain Ervedoza and Julie Valein. On the observability of abstract time-discrete linear parabolic equations. *Revista Matemática Complutense*, 23(1) :163, Nov 2009.

[Eva10] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[EZ11a] Sylvain Ervedoza and Enrique Zuazua. Observability of heat processes by transmutation without geometric restrictions. *Math. Control Relat. Fields*, 1(2) :177–187, 2011.

[EZ11b] Sylvain Ervedoza and Enrique Zuazua. Sharp observability estimates for heat equations. *Arch. Ration. Mech. Anal.*, 202(3) :975–1017, 2011.

[FC97] Enrique Fernández-Cara. Null controllability of the semilinear heat equation. *ESAIM Control Optim. Calc. Var.*, 2 :87–103, 1997.

[FC05] Enrique Fernández-Cara. A review of basic theoretical results concerning the Navier-Stokes and other similar equations. *Bol. Soc. Esp. Mat. Apl. SeMA*, (32) :45–73, 2005.

[FCG06] Enrique Fernández-Cara and Sergio Guerrero. Global Carleman inequalities for parabolic systems and applications to controllability. *SIAM J. Control Optim.*, 45(4) :1399–1446 (electronic), 2006.

[FCGBdT15] Enrique Fernández-Cara, Manuel González-Burgos, and Luz de Teresa. Controllability of linear and semilinear non-diagonalizable parabolic systems. *ESAIM Control Optim. Calc. Var.*, 21(4) :1178–1204, 2015.

[FCGBGP06a] Enrique Fernández-Cara, Manuel González-Burgos, Sergio Guerrero, and Jean-Pierre Puel. Exact controllability to the trajectories of the heat equation with Fourier boundary conditions : the semilinear case. *ESAIM Control Optim. Calc. Var.*, 12(3) :466–483, 2006.

[FCGBGP06b] Enrique Fernández-Cara, Manuel González-Burgos, Sergio Guerrero, and Jean-Pierre Puel. Null controllability of the heat equation with boundary Fourier conditions : the linear case. *ESAIM Control Optim. Calc. Var.*, 12(3) :442–465, 2006.

[FCLdM12] Enrique Fernández-Cara, Juan Limaco, and Silvano B. de Menezes. Null controllability for a parabolic equation with nonlocal nonlinearities. *Systems Control Lett.*, 61(1) :107–111, 2012.

[FCLdM16] Enrique Fernández-Cara, Juan Limaco, and Silvano Bezerra de Menezes. Controlling linear and semilinear systems formed by one elliptic and two parabolic PDEs with one scalar control. *ESAIM Control Optim. Calc. Var.*, 22(4) :1017–1039, 2016.

[FCLZ16] Enrique Fernández-Cara, Qi Lü, and Enrique Zuazua. Null controllability of linear heat and wave equations with nonlocal spatial terms. *SIAM J. Control Optim.*, 54(4) :2009–2019, 2016.

[FCM12] Enrique Fernández-Cara and Arnaud Münch. Numerical null controllability of semi-linear 1-D heat equations : fixed point, least squares and Newton methods. *Math. Control Relat. Fields*, 2(3) :217–246, 2012.

[FCM14] Enrique Fernández-Cara and Arnaud Münch. Numerical exact controllability of the 1D heat equation : duality and Carleman weights. *J. Optim. Theory Appl.*, 163(1) :253–285, 2014.

[FCZ00] Enrique Fernández-Cara and Enrique Zuazua. Null and approximate controllability for weakly blowing up semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17(5) :583–616, 2000.

[FI96] Andrei V. Fursikov and Oleg Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[Fis15] Julian Fischer. Global existence of renormalized solutions to entropy-dissipating reaction-diffusion systems. *Arch. Ration. Mech. Anal.*, 218(1) :553–587, 2015.

[FMT18] Klemens Fellner, Jeff Morgan, and Bao Quoc Tang. Global classical solutions to quadratic systems with mass conservation in arbitrary dimensions. *arXiv e-prints :1808.01315*, Aug 2018.

[FPZ95] Caroline Fabre, Jean-Pierre Puel, and Enrike Zuazua. Approximate controllability of the semilinear heat equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 125(1) :31–61, 1995.

[FR71] Hector O. Fattorini and David L. Russell. Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.*, 43 :272–292, 1971.

[Gao15] Peng Gao. Null controllability with constraints on the state for the reaction-diffusion system. *Comput. Math. Appl.*, 70(5) :776–788, 2015.

[GBdT10] Manuel González-Burgos and Luz de Teresa. Controllability results for cascade systems of m coupled parabolic PDEs by one control force. *Port. Math.*, 67(1) :91–113, 2010.

[GBPG06] Manuel González-Burgos and Rosario Pérez-García. Controllability results for some nonlinear coupled parabolic systems by one control force. *Asymptot. Anal.*, 46(2) :123–162, 2006.

[GI13] Sergio Guerrero and Oleg Yurievich Imanuvilov. Remarks on non controllability of the heat equation with memory. *ESAIM Control Optim. Calc. Var.*, 19(1) :288–300, 2013.

[GL07] Sergio Guerrero and Gilles Lebeau. Singular optimal control for a transport-diffusion equation. *Comm. Partial Differential Equations*, 32(10-12) :1813–1836, 2007.

[Gou08] Xavier Gourdon. *Les maths en tête. Analyse*. Les maths en tête. Ellipses, Seonde édition 2008.

[Gra93] Andrzej Granas. On the Leray-Schauder alternative. *Topol. Methods Nonlinear Anal.*, 2(2) :225–231, 1993.

[Gue07] Sergio Guerrero. Null controllability of some systems of two parabolic equations with one control force. *SIAM J. Control Optim.*, 46(2) :379–394, 2007.

[GV93] Victor A. Galaktionov and Juan L. Vázquez. Regional blow up in a semilinear heat equation with convergence to a Hamilton-Jacobi equation. *SIAM J. Math. Anal.*, 24(5) :1254–1276, 1993.

[GV96] Victor A. Galaktionov and Juan L. Vazquez. Blow-up for quasilinear heat equations described by means of nonlinear Hamilton-Jacobi equations. *J. Differential Equations*, 127(1) :1–40, 1996.

[GV02] Victor A. Galaktionov and Juan L. Vázquez. The problem of blow-up in nonlinear parabolic equations. *Discrete Contin. Dyn. Syst.*, 8(2) :399–433, 2002. Current developments in partial differential equations (Temuco, 1999).

[GV10] Thierry Goudon and Alexis Vasseur. Regularity analysis for systems of reaction-diffusion equations. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(1) :117–142, 2010.

[GZ16] Bao-Zhu Guo and Liang Zhang. Local exact controllability to positive trajectory for parabolic system of chemotaxis. *Math. Control Relat. Fields*, 6(1) :143–165, 2016.

[HM96] Selwin L. Hollis and Jeffrey J. Morgan. On the blow-up of solutions to some semilinear parabolic systems arising in chemical reaction modelling. In *World Congress of Nonlinear Analysts '92, Vol. I–IV (Tampa, FL, 1992)*, pages 2273–2280. de Gruyter, Berlin, 1996.

[HMT18] Nicolas Hegoburu, Pierre Magal, and Marius Tucsnak. Controllability with positivity constraints of the Lotka-McKendrick system. *SIAM J. Control Optim.*, 56(2) :723–750, 2018.

[Hor03] Dirk Horstmann. From 1970 until present : the Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein.*, 105(3) :103–165, 2003.

[How03] John M. Howie. *Complex analysis.* Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London, 2003.

[HT17] Nicolas Hegoburu and Marius Tucsnak. Null controllability of the Lotka-McKendrick system with spatial diffusion. November 2017. working paper or preprint : https ://hal.archives-ouvertes.fr/hal-01648088.

[IT07] Oleg Imanuvilov and Takéo Takahashi. Exact controllability of a fluid-rigid body system. *J. Math. Pures Appl. (9)*, 87(4) :408–437, 2007.

[JL99] David Jerison and Gilles Lebeau. Nodal sets of sums of eigenfunctions. In *Harmonic analysis and partial differential equations (Chicago, IL, 1996)*, Chicago Lectures in Math., pages 223–239. Univ. Chicago Press, Chicago, IL, 1999.

[Kan90] Jacob I. Kanel. Solvability in the large of a system of reaction-diffusion equations with the balance condition. *Differentsial nye Uravneniya*, 26(3) :448–458, 549, 1990.

[Kat95] Tosio Kato. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[Ken06] Carlos E. Kenig. Some recent quantitative unique continuation theorems. In *Séminaire : Équations aux Dérivées Partielles. 2005–2006*, Sémin. Équ. Dériv. Partielles, pages Exp. No. XX, 12. École Polytech., Palaiseau, 2006.

[KL88] Vladimir A. Kondratev and Evgeni M Landis. Qualitative properties of the solutions of a second-order nonlinear equation. *Mat. Sb. (N.S.)*, 135(177)(3) :346–360, 415, 1988.

[Koe17] Armand Koenig. Non-null-controllability of the Grushin operator in 2D. *C. R. Math. Acad. Sci. Paris*, 355(12) :1215–1235, 2017.

[KS71] Evelyn F. Keller and Lee A. Segel. Model for chemotaxis. *Journal of Theoretical Biology*, 30(2) :225 – 234, 1971.

[KSW15] Carlos Kenig, Luis Silvestre, and Jenn-Nan Wang. On Landis' conjecture in the plane. *Comm. Partial Differential Equations*, 40(4) :766–789, 2015.

[LB18a] Kévin Le Balc'h. Global null-controllability and nonnegative-controllability of slightly superlinear heat equations. *arXiv e-prints :1810.12232*, Oct 2018.

[LB18b] Kévin Le Balc'h. Local controllability of reaction-diffusion systems around nonnegative stationary states. *arXiv e-prints :1809.05303, to appear in ESAIM Control Optim. Calc. Var.*, Sep 2018.

[LB18c] Kévin Le Balc'h. Null-controllability of two species reaction-diffusion system with nonlinear coupling : a new duality method. *arXiv e-prints :1802.09187, to appear in SIAM J. Control Optim.*, Feb 2018.

[LB19] Kévin Le Balc'h. Controllability of a 4×4 quadratic reaction-diffusion system. *J. Differential Equations*, 266(6) :3100–3188, 2019.

[LCM$^+$16] Juan Límaco, Marcondes Clark, Alexandro Marinho, Silvado B. de Menezes, and Aldo T. Louredo. Null controllability of some reaction-diffusion systems with only one control force in moving domains. *Chin. Ann. Math. Ser. B*, 37(1) :29–52, 2016.

[Lie96] Gary M. Lieberman. *Second order parabolic differential equations.* World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[Lin89] Ernst Lindelöf. *Le calcul des résidus et ses applications à la théorie des fonctions.* Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Sceaux, 1989. Reprint of the 1905 original.

[Lio88] Jacques-Louis Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM Rev.*, 30(1) :1–68, 1988.

[LL18] Camille Laurent and Matthieu Léautaud. Observability of the heat equation, geometric constants in control theory, and a conjecture of Luc Miller. *ArXiv e-prints : 1806.00969*, June 2018.

[LR95] Gilles Lebeau and Luc Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2) :335–356, 1995.

[LR07] Jérôme Le Rousseau. Carleman estimates and controllability results for the one-dimensional heat equation with BV coefficients. *J. Differential Equations*, 233(2) :417–447, 2007.

[LRL12] Jérôme Le Rousseau and Gilles Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM Control Optim. Calc. Var.*, 18(3) :712–747, 2012.

[LSU68] Olga Aleksandrovna. Ladyzenskaja, Vsevolod Alekseevich Solonnikov, and Nina Nikolaevna Uralceva. *Linear and quasilinear equations of parabolic type.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[LTT13] Yuning Liu, Takéo Takahashi, and Marius Tucsnak. Single input controllability of a simplified fluid-structure interaction model. *ESAIM Control Optim. Calc. Var.*, 19(1) :20–42, 2013.

[LTZ17] Jérôme Lohéac, Emmanuel Trélat, and Enrique Zuazua. Minimal controllability time for the heat equation under unilateral state or control constraints. *Math. Models Methods Appl. Sci.*, 27(9) :1587–1644, 2017.

[LY86] Peter Li and Shing-Tung Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(3-4) :153–201, 1986.

[LZ98] Gilles Lebeau and Enrique Zuazua. Null-controllability of a system of linear thermoelasticity. *Arch. Rational Mech. Anal.*, 141(4) :297–329, 1998.

[LZ17] Pierre Lissy and Enrique Zuazua. Internal observability for coupled systems of linear partial differential equations. *hal e-prints :01480301*, March 2017. working paper or preprint.

[Mau12] Karine Mauffrey. *Controllability of systems governed by partial differential equations.* Theses, Université de Franche-Comté, October 2012.

[Mau13] Karine Mauffrey. On the null controllability of a 3×3 parabolic system with non-constant coefficients by one or two control forces. *J. Math. Pures Appl. (9)*, 99(2) :187–210, 2013.

[Mes91] Viktor Z. Meshkov. On the possible rate of decrease at infinity of the solutions of second-order partial differential equations. *Mat. Sb.*, 182(3) :364–383, 1991.

[Mik78] V. P. Mikhaĭlov. *Partial differential equations.* "Mir", Moscow ; distributed by Imported Publications, Inc., Chicago, Ill., 1978. Translated from the Russian by P. C. Sinha.

[Mil04] Luc Miller. Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time. *J. Differential Equations*, 204(1) :202–226, 2004.

[Mil06] Luc Miller. The control transmutation method and the cost of fast controls. *SIAM J. Control Optim.*, 45(2) :762–772, 2006.

[Mil10] Luc Miller. A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4) :1465–1485, 2010.

[MRR13] Philippe Martin, Lionel Rosier, and Pierre Rouchon. Null controllability of the structurally damped wave equation with moving control. *SIAM J. Control Optim.*, 51(1) :660–684, 2013.

[MT18] Sorin Micu and Takéo Takahashi. Local controllability to stationary trajectories of a Burgers equation with nonlocal viscosity. *J. Differential Equations*, 264(5) :3664–3703, 2018.

[MTZ18] Debayan Maity, Marius Tucsnak, and Enrique Zuazua. Controllability and Positivity Constraints in Population Dynamics with Age Structuring and Diffusion. *Journal de Mathématiques Pures et Appliquées*, 2018.

[Oli13] Guillaume Olive. *Contrôlabilité de systèmes paraboliques linéaires couplés.* PhD thesis, Université d'Aix Marseille, 2013.

[Pao92] C. V. Pao. *Nonlinear parabolic and elliptic equations.* Plenum Press, New York, 1992.

[Pee61] Jaak Peetre. Another approach to elliptic boundary problems. *Comm. Pure Appl. Math.*, 14 :711–731, 1961.

[Per15] Benoît Perthame. *Parabolic equations in biology.* Lecture Notes on Mathematical Modelling in the Life Sciences. Springer, Cham, 2015. Growth, reaction, movement and diffusion.

[Pie10] Michel Pierre. Global existence in reaction-diffusion systems with control of mass : a survey. *Milan J. Math.*, 78(2) :417–455, 2010.

[PSY18] Michel Pierre, Takashi Suzuki, and Yoshio Yamada. Dissipative reaction diffusion systems with quadratic growth. *Indiana University Mathematics Journal, to appear*, 2018.

[PSZ17] Michel Pierre, Takashi Suzuki, and Rong Zou. Asymptotic behavior of solutions to chemical reaction–diffusion systems. *J. Math. Anal. Appl.*, 450(1) :152–168, 2017.

[PTZ19] Camille Pouchol, Emmanuel Trélat, and Enrique Zuazua. Phase portrait control for 1D monostable and bistable reaction-diffusion equations. *Nonlinearity*, 32 :884, Mar 2019.

[PW67] Murray H. Protter and Hans F. Weinberger. *Maximum principles in differential equations.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967.

[PZ12] Alessio Porretta and Enrique Zuazua. Null controllability of viscous Hamilton-Jacobi equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(3) :301–333, 2012.

[PZ17] Dario Pighin and Enrique Zuazua. Controllability under positivity constraints of semilinear heat equations. *arXiv e-prints :1711.07678*, Nov 2017.

[QS07] Pavol Quittner and Philippe Souplet. *Superlinear parabolic problems.* Birkhäuser Advanced Texts : Basler Lehrbücher. [Birkhäuser Advanced Texts : Basel Textbooks]. Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.

[QZ13] Hervé Queffélec and Claude Zuily. *Analyse pour l'agrégation - 4e éd.* Sciences Sup. Dunod, 2013.

[Ros18] Luca Rossi. The Landis conjecture with sharp rate of decay : 1807.00341. *ArXiv e-prints*, July 2018.

[RR07] Lionel Rosier and Pierre Rouchon. On the controllability of a wave equation with structural damping. *Int. J. Tomogr. Stat.*, 5(W07) :79–84, 2007.

[Rud87] Walter Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.

[Sei88] Thomas I. Seidman. How violent are fast controls ? *Mathematics of Control, Signals and Systems*, 1(1) :89–95, Feb 1988.

[SGM18] Drew Steeves, Bahman Gharesifard, and Abdol-Reza Mansouri. Controllability of coupled parabolic systems with multiple underactuations. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 5488–5493, Dec 2018.

[Sim87] Jacques Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146 :65–96, 1987.

[Sou18] Philippe Souplet. Global existence for reaction-diffusion systems with dissipation of mass and quadratic growth. *Journal of Evolution Equations, In press, ArXiv e-prints :1804.05193*, April 2018.

[TGY12] Qiang Tao, Hang Gao, and Ying Yang. Controllability results for weakly blowing up reaction-diffusion system. *Electron. J. Qual. Theory Differ. Equ.*, pages No. 11, 19, 2012.

[Tré05] Emmanuel Trélat. *Contrôle optimal.* Mathématiques Concrètes. [Concrete Mathematics]. Vuibert, Paris, 2005. Théorie & applications. [Theory and applications].

[TT07] G. Tenenbaum and M. Tucsnak. New blow-up rates for fast controls of schrödinger and heat equations. *Journal of Differential Equations*, 243(1) :70 – 100, 2007.

[TW09] Marius Tucsnak and George Weiss. *Observation and control for operator semigroups.* Birkhäuser Advanced Texts : Basler Lehrbücher. [Birkhäuser Advanced Texts : Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.

[VZ08] Judith Vancostenoble and Enrique Zuazua. Null controllability for the heat equation with singular inverse-square potentials. *J. Funct. Anal.*, 254(7) :1864–1902, 2008.

[WYW06] Zhuoqun Wu, Jingxue Yin, and Chunpeng Wang. *Elliptic & parabolic equations.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[WZ06] Gensheng Wang and Liang Zhang. Exact local controllability of a one-control reaction-diffusion system. *J. Optim. Theory Appl.*, 131(3) :453–467, 2006.

[Zei86] Eberhard Zeidler. *Nonlinear functional analysis and its applications. I.* Springer-Verlag, New York, 1986. Fixed-point theorems, Translated from the German by Peter R. Wadsack.

[Zha18] Christophe Zhang. Internal controllability of systems of semilinear coupled one-dimensional wave equations with one control. *SIAM J. Control Optim.*, 56(4) :3092–3127, 2018.

[Zua93] Enrique Zuazua. Exact controllability for semilinear wave equations in one space dimension. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(1) :109–129, 1993.

[Zua97] Enrique Zuazua. Finite-dimensional null controllability for the semilinear heat equation. *J. Math. Pures Appl. (9)*, 76(3) :237–264, 1997.

[Zua07] Enrique Zuazua. Controllability and observability of partial differential equations : some results and open problems. In *Handbook of differential equations : evolutionary equations. Vol. III*, Handb. Differ. Equ., pages 527–621. Elsevier/North-Holland, Amsterdam, 2007.

Titre : Contrôlabilité de systèmes de réaction-diffusion non linéaires

Mots clés : théorie du contrôle, équations aux dérivées partielles, contrôle non linéaire, contrôlabilité à zéro, équation de la chaleur, systèmes paraboliques, systèmes de réaction-diffusion, systèmes de type parabolique-transport, inégalités de Carleman, méthode de dualité Hilbertienne, méthode du retour

Résumé : Cette thèse est consacrée au contrôle de quelques équations aux dérivées partielles non linéaires. On s'intéresse notamment à des systèmes paraboliques de réaction-diffusion non linéaires issus de la cinétique chimique. L'objectif principal est de démontrer des résultats de contrôlabilité locale ou globale, en temps petit, ou en temps grand.

Dans une première partie, on démontre un résultat de contrôlabilité locale à des états stationnaires positifs en temps petit, pour un système de réaction-diffusion non linéaire.

Dans une deuxième partie, on résout une question de contrôlabilité globale à zéro en temps petit pour un système 2×2 de réaction-diffusion non linéaire avec un couplage impair.

La troisième partie est consacrée au célèbre problème ouvert d'Enrique Fernández-Cara et d'Enrique Zuazua des années 2000 concernant la contrôlabilité globale à zéro de l'équation de la chaleur faiblement non linéaire. On démontre un résultat de contrôlabilité globale à états positifs en temps petit et un résultat de contrôlabilité globale à zéro en temps long.

La dernière partie, rédigée en collaboration avec Karine Beauchard et Armand Koenig, est une incursion vers l'hyperbolique. On étudie des systèmes linéaires à coefficients constants, couplant une dynamique transport avec une dynamique parabolique. On identifie leur temps minimal de contrôle et l'influence de leur structure algébrique sur leurs propriétés de contrôle.

Title : Controllability of nonlinear reaction-diffusion sytems

Keywords : control theory, partial differential equations, nonlinear control, null-controllability, heat equation, parabolic systems, reaction-diffusion systems, parabolic-transport systems, Carleman inequalities, Hilbert Uniqueness Method, return method

Abstract : This thesis is devoted to the control of nonlinear partial differential equations. We are mostly interested in nonlinear parabolic reaction-diffusion systems in reaction kinetics. Our main goal is to prove local or global controllability results in small time or in large time.

In a first part, we prove a local controllability result to nonnegative stationary states in small time, for a nonlinear reaction-diffusion system.

In a second part, we solve a question concerning the global null-controllability in small time for a 2×2 nonlinear reaction-diffusion system with an odd coupling term.

The third part focuses on the famous open problem due to Enrique Fernndez-Cara and Enrique Zuazua in 2000, concerning the global null-controllability of the weak semi-linear heat equation. We show that the equation is globally nonnegative controllable in small time and globally null-controllable in large time.

The last part, which is a joint work with Karine Beauchard and Armand Koenig, enters the hyperbolic world. We study linear parabolic-transport systems with constant coefficients. We identify their minimal time of control and the influence of their algebraic structure on the controllability properties.