Topics beyond the Standard Model : axions, supersymmetry, string theory

Quentin Bonnefoy

université
PARIS-SACLAY

NNT : 2019SACLX034

Topics beyond the Standard Model: axions, supersymmetry, string theory

Thèse de doctorat de l'Université Paris-Saclay
préparée à l'École polytechnique

École doctorale n°576 Particules, hadrons, énergie, noyau, instrumentation, image, cosmos et simulation (PHENIICS)
Spécialité de doctorat : Physique théorique

Thèse présentée et soutenue à Palaiseau, le 15 Juillet 2019, par

Quentin Bonnefoy

Composition du Jury :

Christophe Grojean Professeur, DESY Hamburg et Humboldt Universität zu Berlin	Président
Ignatios Antoniadis Directeur de recherche, LPTHE, Sorbonne Université	Rapporteur
Michele Cicoli Professeur associé, Università di Bologna	Rapporteur
Carlo Angelantonj Professeur associé, Università di Torino	Examinateur
Adam Falkowski Directeur de recherche, LPT, Université Paris-Sud	Examinateur
Joerg Jaeckel Professeur, Universität Heidelberg	Examinateur
Emilian Dudas Directeur de recherche, CPHT, École polytechnique	Directeur de thèse

Thèse de doctorat

Abstract

The aim of this thesis is to study various but interconnected theories for new physics beyond the standard model of particle physics. Those are theories of a new kind of particles, axions, a new symmetry principle, supersymmetry, and a new description of fundamental degrees of freedom, string theory. Constant instrumental and theoretical progresses made over the years maintain those already old subjects as leading BSM candidates.

Axions are first reviewed and studied from a phenomenological perspective: we present models which disentangle the different scales which define the axion parameter space, and we discuss axions which arise in models of flavour physics. Motivated by swampland considerations, we insist on using gauge, and not global, symmetries as model building inputs.

The focus then shifts to supersymmetry. We study its breaking, both in explicit ultraviolet models which generate a low supersymmetry breaking scale from high-scale matter, and at the effective field theory level using non-linearly realized supersymmetry. In our study of the latter topic, we focus on the constrained superfield approach. Finally, we present exact classical solutions of a supersymmetric model with broad application scope, the Wess-Zumino model of a chiral superfield.

Last, we discuss string theory. We compute string spectra as illustrations of the structure of the theory and as starting points to compute one-loop vacuum amplitudes. Those are used to understand supersymmetry breaking in string theory, as well as brane interactions. Then, the latter enable us to test one of the swampland criteria, the weak gravity conjecture, in a string theory setup with broken supersymmetry. Finally, axions in string theory are scrutinized, in particular when they are charged under an anomalous abelian factor of the gauge group.

Résumé

Cette thèse a pour but l'étude de théories diverses, toutefois interconnectées, décrivant la nouvelle physique au-delà du modèle standard de la physique des particules. Ce sont des théories d'un nouveau type de particules, les axions, d'un nouveau principe de symétrie, la supersymétrie, et d'une nouvelle description des degrés de liberté fondamentaux, la théorie des cordes. Les progrès instrumentaux et théoriques constamment faits au fil des ans ont confirmé que ces théories sont des candidates privilégiées pour une description de la physique au-delà du modèle standard.

Les axions sont d'abord examinés et étudiés d'un point de vue phénoménologique: nous présentons des modèles qui désenchevêtrent les différentes échelles qui décrivent l'espace des paramètres des modèles d'axions, et nous discutons les axions présents dans des modèles de saveur. Inspirés par les recherches autour du swampland, nous nous imposons l'utilisation de symétries de jauge, et non globales, en tant que point de départ pour la construction de modèles.

Notre intérêt se porte ensuite sur la supersymétrie. Nous étudions sa brisure, à la fois dans des modèles explicites dans l'ultraviolet qui génèrent une échelle de brisure de supersymétrie basse à partir de matière à haute échelle, et au niveau des théories effectives à l'aide de la supersymétrie non-linéaire. En ce qui concerne ce dernier thème, nous nous restreignons à l'approche des superchamps contraints. Enfin, nous présentons des solutions classiques exactes d'un modèle supersymétrique dont la portée est grande, le modèle de Wess-Zumino d'un superchamp chiral.

Finalement, nous nous intéressons à la théorie des cordes. Nous calculons des spectres de cordes en guise d'illustration de la structure de la théorie et de point de départ pour le calcul d'amplitudes du vide à une boucle. Celles-ci nous permettent de tester l'une des conjectures du swampland, qui désigne la gravité comme la plus faible des forces, dans une configuration de théorie des cordes où la supersymétrie est brisée. Enfin, les axions en théorie des cordes sont analysés, en particulier lorsqu'ils sont chargés sous une symétrie de jauge abélienne anomale.

Remerciements

To the jury members for my defence, Carlo Angelantonj, Adam Falkowski, Christophe Grojean and Joerg Jaeckel, as well as my two referees Ignatios Antoniadis and Michele Cicoli, for agreeing to assess my work.

À Bernard Pire et Jean-René Chazottes qui, en tant que directeurs successifs du CPHT au cours de mon doctorat, ont permis sa réussite et son déroulement dans les meilleures conditions.

Aux merveilleuses secrétaires du laboratoire, Malika, Fadila et Florence, pour leur disponibilité, leur bonne humeur et leur sympathie.

To the members of the string theory group, Hervé, Andrea, Blaise, Kenta, Eric, Chiara, Bob, Maria, Joachim, together with those to be mentioned later on, as well as Alexandre, Sabine and Jan, some outsiders from the mathematical physics and QCD groups, for making the lab such a pleasant place.

À Marios qui, à l'aide de la branche italienne de son équipe de thésards, a animé la vie sociale du groupe à coups alternés de houblon et de bons crus, et à Guillaume, pour sa participation fidèle et joviale à la vie du groupe susmentionnée et pour sa porte ouverte, maintes fois franchie dans un sens avec des questions puis dans l'autre avec des réponses et encore plus de questions.

To Steve, for giving me the opportunity to work in a trusting environment and for supporting me during the postdoctoral applications, and to Stefan, for the same reasons and for being an inspiring example of what it means to be passionate and dedicated.

À Balthazar, Louis et Charles C.-H., pour leur camaraderie et de nombreuses discussions vivifiantes, to Severin, for being an interesting and motivated collaborator as well as a joyful friend, and to Debtosh, for the same reasons and for his successful interest in animating conversations, physics wise and beyond.

À Charles M., pour être un exemple inspirant de maturité scientifique et de compatibilité entre extravagance, érudition, désinvolture et subtilité, à Luca, pour m'avoir fait profiter de son génie social, son humanisme, son ouverture d'esprit et sa culture d'homme du monde, à Thibaut, pour son esprit rigoureux, son éthique solide, ses intérêts enthousiasmants et les échanges qui en résultent, ainsi que pour ses nombreux conseils à propos de mon manuscrit de thèse, à tous trois enfin pour être de fascinants polymathes qui m'ont intéressé et stimulé, et surtout de bons copains.

À mes amis, pour avoir régulièrement essayé de comprendre mon sujet de thèse et pour s'être adaptés aux changements dans la façon dont je le présente, et aux membres de ma famille et de ma belle-famille, pour les mêmes raisons, pour être venus de loin m'écouter parler de façon absconse pendant deux heures, et pour avoir déterminé, chacun à leur échelle, ce que je suis aujourd'hui.

À Emilian, pour sa disponibilité, sa bienveillance, sa pédagogie, son impressionnante vision d'ensemble de la physique théorique et sa capacité à saisir l'essence de chaque problème, mais surtout pour m'avoir stimulé et considéré avec respect et confiance, ce qui a grandement participé au plaisir que j'ai eu à mener à bien cette thèse.

Enfin, à Ingrid qui, au-delà d'être la partenaire essentielle de tous les aspects de ma vie, m'a gratifié d'un soutien, d'une générosité et d'un amour indéfectibles, souvent jusqu'à me surprendre, et a beaucoup sacrifié, concédé, et récemment conquis pour m'accompagner dans mon projet professionnel.

Synthèse

Le besoin de physique au-delà du modèle standard (MS) est indéniable car, par exemple, les règles et fondements du MS et de son homologue cosmologique n'incorporent pas de description de la masse des neutrinos ou de la matière noire. En plus de telles déviations avec les prédictions phénoménologiques du MS, il existe des problèmes théoriques, comme la quantification de la gravité ou le niveau d'ajustement fin que l'on espère trouver dans une théorie fondamentale. Au cours de cette thèse, nous suivons trois approches pour identifier de la nouvelle physique au-delà du MS. La première approche consiste à résoudre des problèmes du MS à l'aide de modifications "minimales" de la théorie, telles que l'ajout de symétries jaugées ou d'espace-temps, et l'ajout restreint de nouveaux champs. Une deuxième approche utilise les théories des champs effectives afin de connecter des théories motivées de façon formelle, en particulier la théorie des cordes et les théories supersymétriques, à la physique de basse énergie. Cela permet de sélectionner, parmi les modifications "minimales" mentionnées précédemment, celles qui sont susceptibles de décrire de la physique générée par ces théories, dont la structure est différente et qui sont généralement réalisées à haute énergie. Une dernière approche, suivie marginalement ici, étudie en détail la cohérence interne d'une théorie, c'est-à-dire sa compatibilité avec des principes bien justifiés, et essaye d'identifier des régimes où cette cohérence n'est plus vérifiée. Ces trois approches sont abordées dans les trois parties de cette thèse, qui ne sont cependant pas organisées selon ce découpage.

Procédons d'abord à un survol de ces trois parties, avant d'y revenir de façon plus détaillée. D'abord, en section 2, nous étudions soigneusement les signaux expérimentaux ou théoriques au niveau du MS lui-même, et nous résolvons certains problèmes à l'aide de symétries abéliennes jaugées. En particulier, nous nous intéressons au problème CP-fort de la chromodynamique quantique (CDQ) et sa solution axionique dans les sections 2.1 et 2.2, puis aux modèles de saveur qui expliquent les hiérarchies dans les matrices de masse fermioniques en section 2.3 et 2.4, en insistant sur l'approche de Froggatt-Nielsen (FN). Des axions émergent naturellement des modèles que nous considérons. Ensuite, en section 3, nous examinons une extension plus radicale du MS: la supersymétrie (SUSY), dont les bases sont rappelées en section 3.1, et qui a la triple qualité d'être motivée d'un point de vue phénoménologique, théorique et mathématique. Cependant, puisque la nature n'est pas supersymétrique, nous nous concentrons sur des modèles qui brisent la supersymétrie en section 3.2, dont certains sont liés à des modèles étudiés dans la partie précédente. Nous présentons en section 3.3 la supersymétrie non-linéaire, qui est un formalisme général décrivant la supersymétrie brisée, et nous nous permettons en section 3.4 un écart vers un modèle qui est parfaitement supersymétrique et duquel nous calculons des solutions exactes. Enfin, la section 4 traite de théorie des cordes, et de ses théories effectives, qui permettent de lier cette discussion à celles menées dans les deux parties précédentes. Après avoir présenté le spectre des théories des cordes cohérentes en section 4.1, nous nous penchons sur les moyens de briser la supersymétrie en théorie des cordes, et sur les théories effectives qui émergent alors. Parmi celles-ci, nous analysons les théories d'axions, en particulier celles associées à un mécanisme de Green-Schwarz (GS). Finalement, nous nous arrêtons sur le programme dit du swampland en section 4.4, dont l'objectif est d'énoncer des critères permettant de discriminer les théories qui peuvent possiblement procéder d'une théorie de gravité quantique. En particulier, nous utilisons la théorie des cordes avec brisure de supersymétrie pour tester la conjecture de gravité faible de façon non-triviale. Une suite d'appendices apportent pour terminer quelques précisions techniques.

La première partie est donc dédiée à l'étude des théories des champs (quadri-dimensionnelles) qui étendent le contenu en matière du MS, avec un intérêt particulier pour les modèles d'axions et de saveur. La logique implicite est d'utiliser des symétries de jauge (abéliennes) en tant que seule restriction portant sur la dynamique des particules présentes. En particulier, de petits paramètres comme l'angle θ de la CDQ ou les hiérarchies entre couplages de Yukawa n'ont pas d'explication au sein du MS, et toute nouvelle symétrie qui le permettrait doit être une conséquence de l'invariance de jauge de la théorie. Ceci peut être justifié par l'incompatibilité apparente entre les symétries globales et la gravité quantique. Les modèles d'axions de CDQ expliquent de façon dynamique les limites expérimentales portant sur l'angle θ, et nous incorporons de telles particules dans une configuration de type clockwork. Il ressort de cette étude que le modèle de clockwork jaugé de la section 2.2.2 limite très efficacement les possibles brisures de la symétrie de décalage de l'axion, de sorte que ce dernier est un bon candidat d'axion de CDQ et de matière noire légère. En revanche, les propriétés de localisation le long des modes UV, typiques du clockwork, ne sont utilisables que pour les couplages non anormaux que toute particule-de-type-axion peut avoir. Les couplages anormaux, par exemple aux gluons, ne peuvent pas être ajustés par la localisation de type clockwork, et la constante de couplage de l'axion ne peut être bien plus petite que l'échelle véritable de nouvelle physique que s'il y a un grand nombre de particules supplémentaires, afin de satisfaire les contraintes relatives à l'annulation des anomalies. Enfin, nous utilisons le mécanisme de Froggatt-Nielsen (FN) afin d'expliquer les hiérarchies entre masses et mélanges dans le secteur des quarks du MS. Suivant une logique similaire à ce qui a été détaillé ci-dessus, nous jaugeons la symétrie de FN et nous étudions si les champs lourds qui génèrent le mécanisme sont suffisants pour annuler toutes les anomalies. Nous présentons des modèles qui y parviennent, et examinons rapidement la cohérence et la phénoménologie des axions de CDQ avec propriétés de saveur qui apparaissent dans nombre de ces modèles.

La deuxième partie traite quant à elle de supersymétrie et de modèles supersymétriques en physique des particules. Nous nous concentrons sur la brisure de supersymétrie, via l'étude de modèles explicites et l'utilisation de la supersymétrie non-linéaire, qui est le formalisme universel pour construire des théories effectives avec brisure de SUSY. Nous examinons un modèle complet de brisure spontanée de la SUSY, qui est encore un modèle clockwork jaugé et qui, pour des raisons similaires à celles qui permettaient aux axions de la section 2.2.2 d'avoir une petite masse, réduit fortement l'échelle de brisure de SUSY même quand peu de particules supplémentaires sont présentes. Dans ce modèle, on trouve également des vecteurs et des axions avec des profils de type clockwork, mais ces derniers n'induisent pas d'effets remarquables lorsque l'on souhaite générer une échelle de brisure de SUSY qui soit pertinente d'un point de vue phénoménologique. Cependant, des hiérarchies de type clockwork apparaissent entre les champs auxiliaires qui brisent la

SUSY, ce qui permet d'envisager de transmettre cette brisure de façon hiérarchique à la matière observable, menant par exemple à des modèles divisés pour les superpartenaires du modèle standard supersymétrique minimal (MSSM). Nous rappelons ensuite différentes approches de la SUSY non-linéaire, à savoir la construction des classes d'équivalence et les superchamps contraints, à la fois pour la SUSY $\mathcal{N} = 1$ et $\mathcal{N} = 2$. Nous observons succintement une ambiguïté qui réside dans la contrainte de champs auxiliaires et qui peut être pertinente pour la phénoménologie, ainsi que pour obtenir des actions de Dirac-Born-Infeld (DBI) avec brisure de SUSY complète à l'aide de superchamps contraints. Enfin, nous terminons avec la présentation d'une solution exacte des équations de Bogomol'nyi-Prasad-Sommerfield (BPS) appliquées au modèle de Wess-Zumino (WZ), qui a des applications au calcul d'amplitudes avec un grand nombre de particules et au calcul de murs de domaine pour la CDQ supersymétrique fortement couplée. Pour nous rapprocher de la phénoménologie, nous espérons pouvoir bientôt inclure la brisure de SUSY.

La troisième et dernière partie commence par un examen des règles fondamentales pour construire une théorie quantique de supercordes, afin de pouvoir identifier le spectre de ces théories. Nous détaillons ces spectres pour les théories des cordes de types II et I, en insistant sur la façon dont ils permettent de calculer les amplitudes du vide à une boucle. À partir de ces dernières, en type I, nous présentons comment extraire des informations importantes, telles que la façon dont les différentes amplitudes se combinent pour faire émerger le spectre du type I ou comment leur finitude restreint le contenu en particules de la théorie, de façon complémentaire aux discussions sur les anomalies. Nous passons ensuite à la brisure de SUSY en théorie des cordes, en nous concentrant sur la brisure de SUSY à l'aide de branes et du mécanisme de Scherk-Schwarz (SS), et en identifiant de nouveau l'impact des modifications du spectre dues à la brisure de SUSY sur les amplitudes du vide à une boucle. Motivés par le fait que les spectres de théorie des cordes contiennent de nombreux axions, et que ces derniers participent parfois à une version compactifiée du mécanisme de GS, nous étudions une théorie effective inspirée des cordes, dans laquelle un axion chargé sous un groupe de jauge $U(1)$ anormal réalise un mécanisme de GS en 4D. En particulier, nous comprenons que l'axion obtient une masse de l'ordre de l'échelle de brisure de SUSY lorsque cette dernière est due à la condensation des jauginos d'un secteur caché avec une anomalie mixte par rapport au $U(1)$ anormal, de telle sorte que l'axion ne peut pas jouer le rôle d'un axion de CDQ. Cependant, cette conclusion peut être évitée en raffinant le modèle: nous adaptons donc le modèle dit 3-2 à notre situation avec un $U(1)$ anormal afin de réussir à découpler l'échelle de brisure de SUSY et la masse de l'axion. La valeur naturelle de la constante de couplage de l'axion est de l'ordre de l'échelle de grande unification dans nos modèles, mais nous mentionnons des modifications additionnelles (des modules du modèle) qui permettent de la diminuer. Nous rappelons également que les $U(1)$ anormaux peuvent être utilisés dans les modèles de saveur de FN, auquel cas l'annulation des anomalies et les conditions d'unification prédisent sans ambiguïté les couplages de l'axion. Enfin, nous présentons les conjectures du swampland, en nous concentrant sur une conjecture mentionnée plusieurs fois dans la première partie de cette thèse, qui suggère qu'il n'existe pas de symétrie globale exacte dans une théorie de gravité quantique, ainsi que sur la conjecture de gravité faible (CGF). Nous examinons en détail l'application de cette dernière à la 2-forme de la théorie des cordes de type I, avec SUSY brisée via le mécanisme de SS, ce qui génère des interactions entre branes et des potentiels de fuite. De fait, une telle configuration permet un test non trivial de la CGF, et de sa compatibilité avec les possibles solutions cosmologiques à la conjecture de de Sitter. À l'aide de notre connaissance des amplitudes du vide, qui interviennent dans le calcul des interactions entre branes via la dualité cordes ouvertes-cordes fermées, nous montrons que les interactions à une boucle portées par des modes massifs twistés impliquent une diminution de la tension des branes alors que leur charge reste fixe, en accord avec la CGF.

En définitive, cette thèse a pour but de développer et d'étudier des théories au-delà du modèle standard de la physique des particules, ce qui est un effort justifié et fécond, comme nous essayons de le démontrer dans ce texte. Les théories qui résolvent certaines des énigmes de la physique des hautes énergies sont parfois surprenamment connectées à d'autres branches de la physique, quelquefois au-delà de la physique des particules. Toutes les théories qui ont été mentionnées ici rentrent dans cette catégorie. En effet, en sections 2 et 4.3, nous étudions les axions, imaginés à l'origine pour annuler un seul paramètre du lagrangien du MS, puis utilisés en cosmologie, en physique de la saveur ainsi que comme des vestiges de basse énergie de la théorie des cordes. De façon similaire, nous avons examiné en section 3 la supersymétrie, qui émerge de principes de symétrie théoriques, qui résout d'un coup de nombreux problèmes phénoménologiques et qui est intimement liée aux spectres de la théorie des cordes. Cette dernière domine d'ailleurs toutes ces considérations, puisqu'elle est un candidat au statut de théorie du tout du point de vue de la physique des particules, comme nous l'avons mentionné en section 4. Sa portée est au demeurant encore plus grande, grâce par exemple à l'holographie. Cet état de fait suggère d'être au courant de toutes les (ou du moins de la majorité des) questions ouvertes en physique des hautes énergies, car elles peuvent toutes être interconnectées et interdépendantes. La solution BPS de la section 3.4.2, qui illustre la généralité et la portée des théories des champs (effectives) en produisant à la fois des amplitudes à grand nombre de particules et des solutions de murs de domaine en CDQS, en est un exemple. Notre jaugeage systématique en section 2 des symétries globales abéliennes rencontrées en phénoménologie, justifié par des considérations de gravité quantique discutées en section 4.4, en est un autre. À cette occasion, nous nous interrogeons sur les conséquences de la nécessité de jauger, en explorant en section 2.2 l'espace de paramètres typique des modèles d'axions clockworks jaugés, et en étudiant en section 2.4 les champs lourds, et les axions légers, qui accompagnent nécessairement un mécanisme de Froggatt-Nielsen jaugé. Notre dernier exemple concerne la brisure de SUSY: si la nature est supersymétrique à une certaine échelle, le besoin de briser la SUSY et la façon dont elle l'est peuvent interférer avec d'autres aspects de physique des particules, comme le problème CP-fort, que nous avons étudié dans un scénario inspiré de la théorie des cordes en section 4.3.5, ou la détermination de l'étendue de la nouvelle physique au-dessus de l'échelle électro-faible, comme illustré par la séparation entre l'échelle de brisure de SUSY et l'échelle du secteur qui la brise dans le modèle clockwork de la section 3.2.3, ou enfin comme l'établissement de caractères partagés par toutes les théories cohérentes qui proviennent de la gravité quantique, ce qui est attesté par l'interaction complexe entre la brisure de SUSY et la conjecture de gravité faible en section 4.4.2. Les connections entre les théories ou aspects de la physique (des particules) permettent donc de tester, de renforcer mais aussi de remettre en cause et d'élargir le formalisme et les attentes de notre époque. De cette façon, nous serons peut-être capables de mieux comprendre ce dont la physique moderne devrait être faite.

Contents

1 Prelude: particle physics at a critical time

Preparing a Ph.D. in theoretical particle physics in 2019 is a thrilling endeavor. While standing upon decades, even centuries, of mathematizing, modeling and understanding our world down to amazing experimental precision and conceptual awareness, the young physicist faces a time of crisis.

In trying to define this crisis, one quickly realizes that it is twofold. There is an aspect of it which is inherent to the research activity, for each step forward opens up dozens of questions and dissolves many prejudices. It is the aspect which makes the scientific community vibrate and the wonder remain. It will not be commented on further. The second one is on the other hand more circumstantial, but lies at the center of every aspect of this thesis. However, in order to describe it in greater details, we have to start at the beginning of the story.

1.1 What modern physics is made of

Physics in the modern era, which we initialize between the 16th and the 17th centuries, is an interplay between mathematical concepts and scales. The latter include time scales, length scales, energy scales, etc, which are all linked to different physical intuitions, but which all enter the physical game in the same way: scales define the experimental setup which the physicist has to model, translating it into mathematical concepts.

Consequently, the history of physics can be rewritten as the simultaneous progression of those two aspects. The exploration of more and more exotic scales is easy to have in mind: from fluid dynamics, one descends to chemistry, then to atoms and nuclei, from balls rolling on a table, one accelerates to sound speeds, then relativistic velocities, etc. Meanwhile, the path towards new mathematical concepts is less popular, but at least equally fascinating. At the core of it is the notion of degrees of freedom: given a physical situation, what are the best mathematical objects to describe it? Then comes the notion of physical laws: given such degrees of freedom, which mathematical principles make them understandable, or even predictable? Behind those definitions, one sees lurking all the captivating (though implicit hereafter) epistemological questions about what makes us satisfied by a given physical theory.

One fascinating aspect of physics modelling is that mathematical concepts, introduced somehow intuitively to describe a given observed situation, backreact on intuition itself by offering landmarks, mind anchors from which one can go on prospecting. Key examples of this (within a much larger set) are the introduction of kinematical variables and dynamical equations at the time of Galileo, then Newton, who also accustomed us with the notion of force.

Thus, it is easy to understand why the continuous exploration of scales and concepts went together: nature is the most efficient and imaginative provider of wonder there exists, while extracting the relevant information it carries gives strong hints about where to look next. For instance, from the observation of the atomic world came out the principles of quantum mechanics, which are now invoked as guiding lines to explore the physics of black holes, dramatically different objects. Physics is, by definition, an experimental science whose goal is to reproduce observations, but it is most importantly the art of identifying the genuine, fundamental aspects behind things which happen.

1.2 From Maxwell to the LHC: particle physics in its glory

Let us now see how this quite general discussion is illustrated by the legacy of the late 19th and the 20th centuries to particle physics: the model of elementary particles and interactions known as the standard model of particle physics (SM). Here, the progression in scales is almost chronological: bigger machines, more available energy, smaller distances scrutinized. The establishment of mathematical concepts, for its part, has reached an incomparable level of refinement and deepness. Technical names illustrating this are gauge principle, (chiral) group representations, spontaneous symmetry breaking, renormalization. Behind each of those words lies an almost unbelievable balance between formal notions and experimental facts, between 19 free parameters (and a few constants of nature) and almost every physical phenomenon taking place on Earth. However, we do not intend to cover any of this and we only focus on the SM completeness thereafter, so that we circle back to the aforementioned crisis.

In order not to inexactly tell or miss parts of the story, let us start by its end: the Large Hadron Collider (LHC) at CERN in Geneva observed in 2012 the Brout-Englert-Higgs boson [1,2] (Higgs boson in short), which was up to then the last unobserved piece of the SM. This discovery put an end to a fantastic theoretical and experimental journey over the last century, which managed to bring together originally different views on phenomena such as light, radioactivity or the mass of atoms.

Quoting Stefan Pokorski at the 61th anniversary of the Centre de Physique Théorique de l'École polytechnique, one should "be proud of the SM". Indeed, it is a remarkable success story: experiments developed were followed by model building, which was validated by further experiments, and so on. The structure of the theory which emerged is so strong and its consistency so demanding that, in the later days of its development, experiments were partly aiming at verifying what every theoretician already took for granted. Striking examples of this are the discovery of the charm quark in 1974 [3, 4], predicted by the Glashow-Iliopoulos-Maiani (GIM) mechanism [5] in 1970, or the discovery of the Higgs boson itself, necessary for the consistency of theories with massive vector bosons [6–8].

1.3 Back to the crisis: the need for BSM physics

This matter of fact brings us back to the crisis we mentioned: if the SM is complete, why should we try to extend it? And, most importantly for researchers, how should we proceed?

The why question is actually easy to answer, since several phenomena offer discrepancies with our current understanding of nature.

There are first "pure" particle-related issues. The neutrinos, which are massless particles in the SM, are understood to be massive since they oscillate [9]. There are as well mismatches (statistically significant, some long-standing) between observed and predicted quantities: dominant examples are the magnetic moment of the muon [10] or decays of B-mesons (see e.g. [11] for latest results). Theoretical troubles or specific expectations are also associated with the SM. The major and most emphasized ones are the hierarchy issues, also called naturalness problems, among which the hierarchy problem of the Higgs boson mass, the pattern of masses and mixings of fermion generations and the strong CP problem. The last two of those problems are examined in section 2. Regarding expectations about the details of a more complete theory which would supplement the SM, those are somehow linked to naturalness/coincidence problems, and concern mostly the unification of gauge coupling constants and the presence of supersymmetry, which is discussed in section 3.

There are also cosmological observations which do not have any explanation within the SM. On the one hand, the association of particle physics (under the form of the SM) and gravity (meaning here general relativity) leads to predictions in extremely precise agreement with some of the foundational cosmological observations, such as the cosmic microwave background (CMB) or the cosmological abundance of atomic elements, explained in the framework of big bang nucleosynthesis (BBN). On the other hand, approximately four fifths of the "matter-like" cosmological constituents are made out of an unknown component, dubbed dark matter (DM) [12], which moreover only explains roughly a quarter of the full energy budget of the universe, the rest being named dark energy (DE) [13, 14]. In total, matter we understand represents only 5% of the universe's content. Similarly, there is a need to describe the kind of energy which drove a phase of accelerated expansion 14 billion years ago, called inflation. On top of that, even at the level of the matter we know, the universe we observe requires that there were amounts of matter and antimatter which were extremely close to each other during the early stages of the cosmological history, so that they almost entirely co-annihilated, leaving only a very small amount of matter behind. This may be reminiscent of coincidence problems in particle physics.

Finally, there are troubles which (mostly) originate from theoretical expectations, such as the need to quantize gravity. Indeed, gravity remains the only known force which has not been unified into a quantum picture such as the SM, since quantization procedures which proved useful fail in the presence of gravity [15, 16]. On the other hand, there are reasons to consider and try to develop quantum gravity. Indeed, situations where curvature is strong and where quantum effects, if they exist, should be important are known: astrophysical black holes have been observed [17, 18] and the cosmological evolution hints towards the big bang, a singular point in spacetime. In addition, black hole thermodynamics [19, 20] suggests that black holes, which emit a blackbody radiation, are sensitive to Planck's constant $\hbar$ and are intrinsically quantum objects. Eventually, quantum gravity, once developed, may come with many features which we would call beyond-the-standard-model (BSM) physics.

Despite all this, the second question we asked, the how question, is nevertheless at the core of the crisis: contrary to the breakdown at high energies of Fermi theory of weak interactions or of the scattering of massive vectors, there is nothing which promises that new physics will undoubtedly show up in realistic, identified and upcoming experiments. There are many ways to look for new physics, many clever questions whose answers might give the right direction. We do not attempt to even mention all of them here, but we follow some.

1.4 Outline

In this thesis, we adopt three different strategies to find new physics. The first one aims at solving SM issues with "minimal" modifications of the theory, such as additional gauge or spacetime symmetries and few additional fields. A second approach is to use effective field theories (EFTs) in order to match formally motivated theories, in particular string theory and supersymmetric theories, to low-energy physics. This can suggest which of the "minimal" modifications mentioned before are susceptible to capture physics generated by those differently structured theories, which usually lie at a high energy scale. A last approach, marginally touched upon in what follows, is to thoroughly study the internal consistency of a theory, i.e. its compatibility with strongly motivated principles, and try to identify regimes where it breaks down. Those three approaches are spread over the three parts of this thesis, which are however not organized according to this splitting. Indeed, the organization of the text is as follows.

First, in section 2, we carefully consider experimental or theoretical signals at the level of the SM itself, and we handle specific issues with the help of gauged abelian symmetries. In particular, we are interested in the strong CP problem of quantum chromodynamics (QCD) and its axion solution in sections 2.1 and 2.2. We present a model, inspired by the so-called clockwork models, which addresses some of the fragile aspects of the axion solution, that have to do with the range of the axion decay constant and with the quality of the axion shift symmetry. This model is very efficient at ensuring a high quality or a very small mass to the axion, and it displays interesting properties, although of limited phenomenological interest, concerning the decay constant. Then, we look in sections 2.3 and 2.4 into flavour models which deal with the fermion mass hierarchies. In order to do so, we follow the Froggatt-Nielsen (FN) approach, with the exception that the FN symmetry is gauged. We are in particular interested in models where the fermion content is minimal, meaning that all fermions involved in the FN mechanism, and only them, are enough to cancel the gauge anomalies. In such setups, an accidental QCD axion arises and is studied.

Secondly, in section 3, we scrutinize a more radical extension of the SM: supersymmetry (SUSY), reviewed in section 3.1. As we argue, it has the triple status of being phenomenologically, theoretically and mathematically motivated. But since it is not an exact symmetry of nature, we investigate models where supersymmetry is spontaneously broken in section 3.2. In section 3.2.3, we study a model which connects with section 2.2, since it is the immediate supersymmetrization of the one presented there. It turns out to break supersymmetry in an interesting way: the breaking scale is naturally very small with respect to the scales where (most of) the BSM dynamics lives, and there is a possibility of engineering naturally split spectra for the superpartners of the SM particles. Subsequently, we present in section 3.3 the general framework for effective models with broken supersymmetry, namely non-linearly realized supersymmetry. We study constrained superfields in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry. Before closing the section, we make a step aside in section 3.4 where, in a model which maintains supersymmetry, we manage to compute exact classical solutions which are relevant for domain wall profiles in SUSY QCD and for studies of multi-particle tree-level amplitudes. The latter can exhibit unitarity violations in scalar theories while the former represent a toy model of the actual vacuum structure of QCD.

Finally, section 4 deals with string theory, and string theoretic EFTs. After reviewing what are the spectra of a set of consistent string theories in section 4.1, we discuss in section 4.2 stringy mechanisms for breaking supersymmetry, and the associated EFTs which arise, connecting with other sections. We then study in section 4.3 axions in string theory, in particular when they are linked with a Green-Schwarz mechanism. We work out under which conditions can they be identified with a QCD axion. Last, we discuss in section 4.4 the swampland program, i.e. the establishment of criteria which characterize the theories which possibly proceed from quantum gravity. In particular, we work out what string theory, in a specific context where supersymmetry is broken, has to say about one of those criteria called the weak gravity conjecture (WGC). Our analysis concludes that it is satisfied in a non-trivial way.

Appendix A presents our conventions for QFT lagrangians, in general and for the (MS)SM, as well as the list of abbreviations used in the text. Appendix B displays a calculation of axion couplings to gauge bosons in the KSVZ model of the QCD axion. Appendix C deals with supersymmetric QCD theories: a discussion of Seiberg duality, the vacua of SQCD and the non-perturbative effects, which are relevant for sections 3.4.3 and 4.3.5, can be found there in particular. Finally, appendix D presents some one-loop computations in string theory, used in section 4.

Sections where some original material is discussed are indicated with a star in the table of contents, and publications linked to the Ph.D. work, reviewed in the text, are attached at the end of it.

2 Axions, flavour, and abelian gauge symmetries

Our exploration starts closely to the standard model, since we first look at "minimal" extensions of it, meaning that we follow the rules along which the SM was built and which characterize it in our modern view. Those include the use of relativistic quantum field theory, i.e. the fact that the dynamical objects are fields of which particles are the quanta and whose dynamics is Lorentz invariant, local, causal and unitary. Building blocks are the gauge symmetries and the matter content (scalar, spinor and vector fields arranged in representations of those groups, the latter being associated to the gauge invariance) such that the dynamics is the most general allowed (up to undetermined coefficients). The only feature of the SM which we relax is its renormalizability: the theories we build are understood as effective field theories and the contributions of irrelevant operators are taken into account when necessary.

Since we do not question here (most of) the conceptual structure of the SM, we thus use as guidelines for progress some phenomenological troubles. Those are the strong CP problem, presented in section 2.1, together with its axion solution, as well as the mass and couplings hierarchies within the fermion families, introduced in section 2.3. We tackle those problems by enlarging the symmetries and the field content of the theory, such that the unexplainably small parameters arise in a natural way. A disclaimer of this approach is that we introduce many scalars for which we do not discuss in details possible hierarchy problems (even though supersymmetry remains as a viable option in the models we discuss).

We do not examine some very important classes of theories which are considered as serious candidates for physics beyond the standard model. Such theories include grand unified, composite, and extra dimensional models. A subset of the latter will be implicitly encountered in section 4, when we study string theory, but we will not go into details of their phenomenology. We nonetheless try to do them justice by inserting their names into these introductory words. An other iconic BSM candidate, supersymmetry, is discussed in section 3.

Our notations regarding the SM Lagrangian, as well as the coupling of fields to curved space, is detailed in appendix A.1.

2.1 The QCD axion and axion-like particles

Since discussions about axions pop up repeatedly along this thesis, we begin by reviewing axion models. We motivate their study by discussing the strong CP problem, axion cosmology, the string landscape and the experimental/observational achievements and perspectives, respectively in sections 2.1.1, 2.1.6, 2.1.5 and 2.1.7. We also give some details about reference models of the

QCD axion (section 2.1.2), axion EFTs (sections 2.1.3 and 2.1.4), and we focus in section 2.1.8 on two aspects of model building, namely adjusting the axion decay constant and ensuring that the axion shift symmetry is of good enough quality.

2.1.1 The strong CP problem

The strong CP (Charge-Parity) problem[1] is a statement that concerns the following P- and CP-breaking operator which can be added to the QCD Lagrangian:

$$\mathcal{L}_{\mathrm{QCD}} \supset -\frac{\theta}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} \text{ where } \tilde{G}^{a,\mu\nu} = \frac{\epsilon^{\mu\nu\rho\sigma}}{2} G^a_{\rho\sigma} \ . \tag{2.1.1}$$

This θ-term is normalized such that $\theta \in [0, 2\pi]$. This operator contributes to the neutron electric dipole moment (EDM) [24,25] via diagrams such as the one drawn in Figure 2.1, and it is implied by measurements of this neutron EDM [26] to be extremely small: $|\theta| < 10^{-10}$ (see section 2.3.2 for more details on EDMs and CP breaking in the SM).

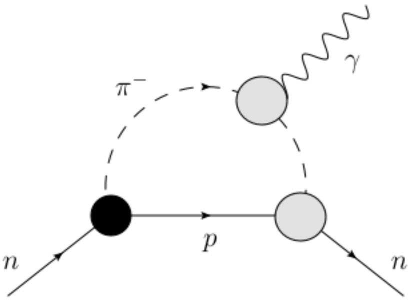


Figure 2.1: One of the diagrams contributing to the neutron EDM
Grey blobs represent CP-preserving pion-nucleon-nucleon or pion-pion-photon couplings, whereas the black one represents CP-violating ones involving θ

However, the smallness of θ remains unexplained in the SM: P and CP are respectively already significantly broken by the weak interactions and by the Cabibbo-Kobayashi-Maskawa (CKM) matrix, and a bare value of θ should be such that it cancels (up to the aforementioned experimental precision) the contribution coming from the quark fields redefinition which makes their Yukawa matrices real and diagonal (see section 2.3.1). Indeed, when $u_{L/R} \to V^u_{L/R} u_{L/R}, d_{L/R} \to V^d_{L/R} d_{L/R}$,

$$\mathcal{L}_{\mathrm{QCD}} \to \mathcal{L}_{\mathrm{QCD}} - \frac{\epsilon^{\mu\nu\rho\sigma}}{64\pi^2} \arg\det(M_u M_d) G^a_{\mu\nu} G^a_{\rho\sigma} \ , \tag{2.1.2}$$

where $M_{u,d}$ are the quark mass matrices (see appendix B for numerical details). Furthermore, anthropic reasoning does not seem to have much to say whenever $\theta \lesssim \mathcal{O}(1)$ [27,28], even though there are attempts to link the strong CP problem to the cosmological constant problem, which does have anthropic solutions [29,30].

Explanations of the strong CP problem lie presently in three main categories. A first class of solutions rely on the observation that θ is unphysical if one of the quark masses vanishes (see [31]

[1]Examples of reviews about the strong CP problem and axions, which sometimes heavily inspired the content of this section, are [21–23].

and references therein): indeed, it is then possible to redefine left- and right-handed parts of this quark such that θ is shifted away, whereas this clashes for a massive particle with the requirement of having a real mass. More precisely, the contribution to the neutron EDM is proportional to θ but also to $m_{u,d}$ so one only needs to ensure that a quark mass is small enough to wash out the effect of a non-zero θ. The vanishing of a quark mass may then be enforced by UV symmetries (such as string theory discrete symmetries), and effects mimicking a non-vanishing quark mass could be radiatively generated out of the other quark non-vanishing masses. However, recent lattice results (see e.g. [32]) seem to rule out this solution.

A second class of solutions uses the fact that, if θ is vanishing at some scale, its RG-evolution in the standard model will keep it very small [33]. Indeed, due to the flavour structure of the standard model, there is only one CP-violating phase in the CKM matrix, and it can only feed into θ in a way which respects the (spurious) $SU(3)_Q \times SU(3)_u \times SU(3)_d$ flavour symmetries of the SM. It turns out that such a consistent expression for the θ β-function can only arise from the seven loops order in perturbation theory, and if $\theta = 0$ is set at some high RG scale, the induced RG-running of θ will bring it to a value compatible with the neutron EDM measurements. Then, $\theta = 0$ would be imposed in the UV by demanding an exact P [34, 35] or CP [36, 37] symmetry, which would be spontaneously broken to explain its violation in the SM. CP-based solutions, known as Nelson-Barr models, need to radiatively generate a non-zero CKM phase while maintaining θ small.

The third class is made of Peccei-Quinn-like solutions, on which we now focus.

2.1.2 The Peccei-Quinn solution and invisible axion models

The Peccei-Quinn (PQ) solution [38] introduces a chiral $U(1)_{\mathrm{PQ}}$ symmetry with a $SU(3)^2 \times U(1)_{\mathrm{PQ}}$ anomaly, such that rotating the different fields of the model towards a "physical" basis redefines θ away. More precisely, Peccei and Quinn showed in [38] that a theory of Lagrangian

$$\mathcal{L} = -\,\mathrm{Tr}\left(\frac{F_{\mu\nu}^2}{2g^2} + \frac{\theta F_{\mu\nu}\tilde{F}^{\mu\nu}}{16\pi^2}\right) - \overline{\psi}\gamma^\mu(\partial_\mu - iA_\mu^a T^a)\psi - |\partial_\mu\sigma|^2 - V(|\sigma|^2) - (y\sigma\overline{\psi_L}\psi_R + h.c.)\ , \quad (2.1.3)$$

which is classically invariant under the $U(1)_{\mathrm{PQ}}$ symmetry $\psi \to e^{\frac{i\gamma_5\alpha}{2}}\psi, \sigma \to e^{i\alpha}\sigma$, has the property that $\theta = 0$ when the fermion ψ is defined such that its mass term $y\langle\sigma\rangle$ is real, hence inevitably solving a possible CP problem associated to the A_μ gauge field.

A direct application of this mechanism to the SM strong CP problem [39] can be made using a second Higgs doublet[2]:

$$\mathcal{L} \supset -V(|H_u|^2, |H_d|^2) - \left(\overline{u_R}H_u(Y^u)^T Q_L + \overline{d_R}H_d(Y^d)^T Q_L + \overline{e_R}H_d(Y^e)^T L_L + h.c.\right)\ . \quad (2.1.4)$$

The $U(1)_{\mathrm{PQ}}$ chiral symmetry $Q_L \to e^{\frac{i\alpha}{2}}Q_L, u_R \to e^{-\frac{i\alpha}{2}}u_R, d_R \to e^{\frac{-i\alpha}{2}}d_R, L_L \to e^{\frac{i\alpha}{2}}L_L, e_R \to e^{-\frac{i\alpha}{2}}e_R$, and $H_{u,d} \to e^{-i\alpha}H_{1,2}$ has a $SU(3)^2 \times U(1)_{\mathrm{PQ}}$ anomaly and the same conclusion about the effective value of θ in the vacuum follows.

[2]Our conventions for the Yukawa matrices in this thesis are such that they fit the MSSM extension to (2.4.1).

Soon after the PQ proposal, it was understood by Weinberg and Wilczek [40, 41] that the PQ solution predicts a light pseudoscalar, dubbed the axion, which is the Pseudo-Goldstone mode of the anomalous $U(1)_{\rm PQ}$. Its (self-)couplings are dictated by (2.1.4) and the gauge anomalies of $U(1)_{\rm PQ}$, as is discussed in the next sections. However, it was shown [40–43] that experiments at that time were in strong tension with estimated properties of the axion (it has been completely ruled out since). It was later understood that this tension can be alleviated if the spontaneous breaking of $U(1)_{\rm PQ}$ happens at a much higher energy than the weak scale $\Lambda_{\rm weak}$. The axions associated with such a high breaking scale are called invisible axions, and benchmark models of this kind are the DFSZ [44, 45] and the KSVZ [46, 47] models.

The KSVZ model is the simplest to understand since it draws from the fact that (2.1.3) does not need to assume that ψ is a SM fermion, nor σ an identified scalar, to solve the strong CP problem. Indeed, the authors of [46, 47] assumed that there is a heavy pair of Weyl fermions $\psi_{L,R}$ in the fundamental representation of $SU(3)_C$, as well as a SM singlet σ which are precisely coupled as in (2.1.3). Then, if $\langle\sigma\rangle$ is taken to be much bigger than the weak scale, any physics associated to the PQ sector is very hard to detect, and in particular the KSVZ axion is much more weakly coupled than the original PQ axion and can evade any experimental bound. This will however soon become an inexact statement since current detection techniques are getting close to the standard KSVZ/DFSZ parameter space, see section 2.1.7.

On the other hand, the DFSZ model is an extension of the SM PQ model (2.1.4) where a higher scale is introduced via a singlet scalar field σ charged under $U(1)_{\rm PQ}$:

$$\mathcal{L} \supset -V(|H_u|^2, |H_d|^2, |\sigma|^2) - \left(\overline{u_R} H_u (Y^u)^T Q_L + \overline{d_R} H_d (Y^d)^T Q_L + \overline{e_R} H_d (Y^e)^T L_L + c H_u H_d \sigma^2 + h.c.\right) \,. \tag{2.1.5}$$

$U(1)_{\rm PQ}$ now transforms SM fields as in the original PQ model if it acts on σ as $\sigma \to e^{-2i\alpha}\sigma$. Then, assuming again $\langle\sigma\rangle \gg \Lambda_{\rm weak}$, the radial part of σ is very heavy and the axion is weakly coupled to ordinary matter, even though it still solves the strong CP problem.

The couplings of the KSVZ and DFSZ axions are described by the same EFT but have very different origins, thus very different magnitude. For instance, in the DFSZ model the axion couples at tree level to SM fields since it mixes with the phases of the Higgs bosons, whereas the KSVZ axion couples to gluons via ψ loops, and to SM particles via higher order diagrams. Nevertheless, all axion models[3] share the property of describing a pseudo-Goldstone boson (PGB) with a typical anomalous coupling to gluons, thus making θ dynamical. The dynamics of QCD [48] then selects the vacuum as the one where $\theta = 0$ effectively. As we will see in section 2.1.4, this is enough knowledge to derive the axion mass and self-couplings.

[3]The word axion nowadays describes light pseudoscalars in more general contexts than the PQ solution to the strong CP problem, so we use the words QCD axion when we specifically encounter a PQ axion in what follows.

2.1.3 General axion effective field theory

The EFT of the QCD axion coupled to SM particles [49] is composed of a defining part and a model-dependent part[4]

$$\mathcal{L} = \underbrace{-\frac{1}{2}(\partial_\mu a)^2 - \frac{a}{32\pi^2 f_a} G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}}_{\mathcal{L}_{\text{def.}}} \underbrace{-\frac{\partial_\mu a}{f_a} j^\mu - \frac{\mathcal{C}_i a}{32\pi^2 f_a} F^{a_i}_{i,\mu\nu}\tilde{F}^{a_i,\mu\nu}_i}_{\mathcal{L}_{\text{dep.}}} , \tag{2.1.6}$$

where G is the field strength of the gluon field (we shifted the axion field such that there is no constant θ term), F_i stands for the field strengths of all the other SM gauge bosons and j^μ is the symmetry current associated to the PQ symmetry, which contains terms of the form $iX_\psi \overline{\psi}\gamma^\mu\gamma_5\psi + iX_\phi(\phi^*\partial^\mu\phi - h.c.)$. At tree level, only $U(1)_{\text{PQ}}$-charged particles can couple to the axion via this current term, but loops of gauge or Higgs bosons generate radiatively such couplings to the remaining uncharged particles. The defining part is made up of those terms which are necessary to call a a QCD axion, and the rest depends on the details of the UV physics associated with the axion. Notice that the dimensionful couplings are defined in terms of a scale f_a, called the axion decay constant. We defined it in (2.1.6) as the dimensionful coupling in the effective θ-term, but it is usually linked to the scale f entering the axion periodicity $a \equiv a + 2\pi f$ via a number called the domain wall number:

$$f_a = \frac{f}{N_{\text{DW}}} \tag{2.1.7}$$

for reasons which will become clear in sections 2.1.4 and 2.1.6. The scale f is itself linked to the vacuum expectation value (vev) of the scalar field whose phase is the axion in UV models such as the ones in section 2.1.2.

Axion couplings are usually rewritten in terms of low-energy particles such as photons, electrons, mesons and nucleons to match the axion theory to precise measurements (such as neutron EDM ones) via current algebra techniques or in chiral perturbation theory. In particular, the coupling to photons is of particular interest and is usually written as follows:

$$\mathcal{L} \supset -\frac{N_{\text{DW}}}{32\pi^2 f}\left(\frac{E}{N_{\text{DW}}} - \frac{2(4+m_u/m_d)}{3(1+m_u/m_d)}\right) aF_{\mu\nu}\tilde{F}^{\mu\nu} \approx -\frac{N_{\text{DW}}}{32\pi^2 f}\left(\frac{E}{N_{\text{DW}}} - 1.92\right) aF_{\mu\nu}\tilde{F}^{\mu\nu} , \tag{2.1.8}$$

where E is the electromagnetic anomaly determined in the UV, and the numerical factor which appears in combination with the pure anomaly coefficients is due to the mixing with the pions. As an illustration, there is no field charged under both $U(1)_{\text{PQ}}$ and $U(1)_{\text{em}}$ in the KSVZ model so $E = 0$, whereas in the DFSZ model $E/N_{\text{DW}} \approx 2.67$.

2.1.4 Axion mass and potential from chiral perturbation theory

Let us now evaluate the (zero temperature) mass of the QCD axion using chiral perturbation theory (χPT) [21, 50][5]. Above the scale of confinement and restricting ourselves to two flavours

[4] Due to the suppressed couplings of the axion, we only write here the first linear terms in the axion field in an expansion in inverse powers of its decay constant.

[5] Approaches to axion couplings using the dilute instanton gas approximation, certainly not reliable at zero temperature, have also been shown not to give satisfying results below $T \sim 10^6$ GeV [50]. χPT is in the contrary defined to describe QCD dynamics below the confinement scale.

of quarks, the QCD lagrangian augmented by the axion contribution is given by:

$$\mathcal{L} \supset -\frac{1}{4g^2}G^{a,2}_{\mu\nu} - \frac{1}{2}(\partial_\mu a)^2 - \overline{u}(\gamma^\mu\partial_\mu + m_u)u - \overline{d}(\gamma^\mu\partial_\mu + m_d)d - \frac{\partial_\mu a}{f_a}j^\mu - \frac{a}{32\pi^2 f_a}G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} \quad (2.1.9)$$

where we redefined the quark fields such that their masses $m_{u,d}$ are real and absorbed the shift in θ via a corresponding shift of the axion field.

When the quark masses are taken to zero (or when the mass matrix is given a spurious transformation), the lagrangian (2.1.9) has a classical $U(2)_L \times U(2)_R$ symmetry acting on the left- and right-handed projections of the quark fields. At energy scales where chiral symmetry breaking occurs, this symmetry is broken by quark condensates $\langle\overline{u}u\rangle = \langle\overline{d}d\rangle \neq 0$ down to $U(2)_{\text{V(ectorial)}} = SU(2)_W \times U(1)_B$, and the relevant degrees of freedom are the pions $\pi^0, \pi^\pm$ and the η, which are understood as the pseudo-Goldstone bosons of the $U(2)_{\text{A(xial)}}$ symmetry. They can be embedded in a 2×2 matrix:

$$\Sigma = \exp\left(\frac{i}{f_\pi}(\vec{\pi}.\vec{\sigma} + \eta\mathbb{1})\right), \quad (2.1.10)$$

where $\vec{\sigma}$ is the set of Pauli matrices. $U(2)_L \times U(2)_R$ then acts as $\Sigma \to U_L\Sigma U_R^\dagger$. The EFT which describes their interactions respects this symmetry up to spurious transformations of the mass matrix, which are such that

$$(\overline{u_R}, \overline{d_R})M\begin{pmatrix}u_L\\ d_L\end{pmatrix} \to (\overline{u_R}, \overline{d_R})U_R^\dagger M' U_L\begin{pmatrix}u_L\\ d_L\end{pmatrix} \quad (2.1.11)$$

is invariant, implying $M' = U_R M U_L^\dagger$. The EFT is thus as follows at first order:

$$\mathcal{L}_{\chi PT} = -\frac{f_\pi^2}{4}\,\mathrm{Tr}\big(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\big) + \frac{f_\pi^2 m_{\pi^0}^2}{2(m_u+m_d)}\,\mathrm{Tr}(M\Sigma + h.c.) - \frac{\partial_\mu a}{f_a}j^\mu + \mathcal{L}_{\text{anomaly}} + \text{ higher orders}, \quad (2.1.12)$$

where $M = \mathrm{diag}(m_u, m_d)$ and $\mathcal{L}_{\text{anomaly}}$ contains all the symmetry breaking terms which are generated by anomalies. For our case of study, $U(1)_{\text{A}}$ is anomalous with respect to QCD [51] and[6]

$$\mathcal{L}_{\text{anomaly}} \supset -\frac{m_\eta^2}{2}\left(\eta + \frac{f_\pi}{2f_a}a\right)^2 + \ldots, \quad (2.1.13)$$

with $m_\eta \gg m_{\pi^0}$. Restricting ourselves to neutral particles and quadratic terms, we thus get

$$\begin{aligned}\mathcal{L} &\supset -\frac{m_{\pi^0}^2}{2(m_u+m_d)}\left[m_u(\pi^0+\eta)^2 + m_d(-\pi^0+\eta)^2\right] - \frac{m_\eta^2}{2}\left(\eta + \frac{f_\pi}{2f_a}a\right)^2 \\ &\xrightarrow{\text{integrating } \pi^0,\eta} -\frac{m_a^2}{2}a^2, \text{ where } m_a \approx \frac{\sqrt{m_u m_d}\,m_{\pi^0} f_\pi}{(m_u+m_d)f_a} \approx 5.7\mu\text{eV}\left(\frac{10^{12}\text{ GeV}}{f_a}\right).\end{aligned} \quad (2.1.14)$$

Along with the axion mass, such χPT techniques can give us the full axion potential. It is easier to see it after we rotate away the effective θ-term in (2.1.9) by performing a chiral rotation of the quarks, $u/d \to e^{-i\frac{a}{4f_a}\gamma_5}u/d$, such that:

$$\mathcal{L} \to -\frac{1}{4g^2}G^{a,2}_{\mu\nu} - \frac{1}{2}(\partial_\mu a)^2 - \overline{u}\left(\gamma^\mu\partial_\mu + e^{-i\frac{a}{2f_a}\gamma_5}m_u\right)u - \overline{d}\left(\gamma^\mu\partial_\mu + e^{-i\frac{a}{2f_a}\gamma_5}m_d\right)d - \frac{\partial_\mu a}{f_a}j'^\mu, \quad (2.1.15)$$

[6] The relative factor between a and η is due to their relative coupling to gluons, which one can identify by comparing the anomalous shift induced by $U(1)_A$: $u/d \to e^{i\alpha\gamma_5}u/d$, i.e. $\eta \to \eta + 2\alpha f_\pi \implies \delta\mathcal{L} = -\frac{\epsilon^{\mu\nu\rho\sigma}}{16\pi^2}G^a_{\mu\nu}G^a_{\rho\sigma}$ and the one associated to a shift of a in (2.1.9). Said differently, the chiral transformation $U(1)_A$ is no more anomalous once it is extended by a suitable transformation of a, so the induced η mass term should be invariant under such shifts.

where the current has been shifted with respect to the original one: $j'^\mu = j^\mu - \frac{i}{4}(\overline{u}\gamma^\mu\gamma_5 u + \overline{d}\gamma^\mu\gamma_5 d)$. In this parametrisation, the χPT EFT is still as in (2.1.12), except that η has been shifted by the chiral redefinition and that $M = \mathrm{diag}(e^{-i\frac{a}{2f_a}} m_u, e^{-i\frac{a}{2f_a}} m_d)$ now. We thus get

$$\begin{aligned}
\mathcal{L} &\supset \frac{f_\pi^2 m_{\pi^0}^2}{m_u+m_d}\left[m_u \cos\left(\frac{\pi^0+\eta}{f_\pi} - \frac{a}{2f_a}\right) + m_d \cos\left(\frac{-\pi^0+\eta}{f_\pi} - \frac{a}{2f_a}\right)\right] - \frac{m_\eta^2}{2}\eta^2 \\
&= |_{\text{neglecting the heavy }\eta} \frac{f_\pi^2 m_{\pi^0}^2}{m_u+m_d}\left[m_u \cos\left(\frac{\pi^0}{f_\pi} - \frac{a}{2f_a}\right) + m_d \cos\left(\frac{\pi^0}{f_\pi} + \frac{a}{2f_a}\right)\right] \\
&= m_{\pi^0}^2 f_\pi^2 \sqrt{1 - \frac{4m_u m_d}{(m_u+m_d)^2}\sin^2\left(\frac{a}{2f_a}\right)} \cos\left(\frac{\pi^0}{f_\pi} - \phi_a\right) \;,\; \text{with } \tan(\phi_a) = \frac{m_u-m_d}{m_u+m_d}\tan\left(\frac{a}{2f_a}\right) \\
&= m_{\pi^0}^2 f_\pi^2 \sqrt{1 - \frac{4m_u m_d}{(m_u+m_d)^2}\sin^2\left(\frac{a}{2f_a}\right)} + \ldots
\end{aligned} \tag{2.1.16}$$

Higher precision expressions require more terms in the χPT EFT, and are for instance presented in [50] at NLO and taking into account lattice results. Notice that (2.1.16) has an explicit minimum at $a = 0$ as expected, and that it has a periodicity $a \to a + 2\pi f_a$, to be compared with its original periodicity discussed around (2.1.7): within the domain of a, there are N_{DW} minima of its potential. This allows for domain-walls configurations of the axion field which interpolate between different vacua. Those can have dramatic effects on cosmology, see section 2.1.6 for more details.

2.1.5 Axion-like particles and string theory axions

As we saw in the preceding sections, there are well defined generic predictions concerning the QCD axion in its minimal realization, even though lots of parametric freedom is still allowed by UV completions[7]. However, many other light pseudoscalars are often used in the litterature with a vanishing gluon coupling or other dominant sources for their potential. Those are dubbed axion-like particles (ALPs) and have an EFT close to the QCD axion one of section 2.1.3, but the relation (2.1.14) between the mass and the decay constant is relaxed, enabling model builders to scan a larger parameter space. Their mass may come for example from non-perturbative effects of additional confining gauge groups or from explicit breaking of the shift symmetry.

There are two major reasons to consider ALPs for phenomenology. The first one is pragmatically due to the fact that many experiments, which are already collecting data or close to do so (see section 2.1.7), keep on improving their sensitivity and extending their reach to broader and broader parts of the parameter space of ALPs (and in particular of the QCD axion), such that there might soon be a need to model a detected signal. The second one is theoretical, since some string theory compactifications predict loads of light ALPs [55–57]. We will have more to say about string theory and string theoretic axions in section 4.3.

[7]Actually, even the tight relation (2.1.14) can be circumvented with additional model building efforts, such as linking the θ parameter to the one of an other QCD-like group with a $\mathbb{Z}_2$ symmetry [52] or a GUT embedding [53], or by introducing copies of QCD with a specific $\mathbb{Z}_N$ symmetry [54].

2.1.6 Axion cosmology

If one has not been fully convinced by the previous arguments in favor of a thorough study of the QCD axion or ALPs, the fact that such particles may have a huge cosmological impact should provide the final impulse. Indeed, the fact that their potentials are quantum mechanically stable due to the axion shift-symmetry enables one to naturally tune their dynamics. Consequently, axions are excellent dark matter candidates [58–60], they can drive inflation [61,62] or make dark energy dynamical [63–66]. There are also mechanisms to produce axions during the cosmological history such that they can both behave as non-relativistic matter or as radiation, modifying for instance the effective number of neutrinos [67].

We do not present details about all those possibilities and we focus on why axions can play the role of dark matter. We mostly draw from [22], as in next section. The idea behind the analysis is that axions, being bosons, can have large occupation numbers of their quantum states, such that their dynamics can be understood by studying the behaviour of the classical axion field. The latter is described by a scalar field action:

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}(\partial_\mu a)^2 - V(a)\right) \ , \tag{2.1.17}$$

whose associated equation of motion

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu a) - \sqrt{-g}\frac{\partial V}{\partial a} = 0 \tag{2.1.18}$$

and energy momentum tensor

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu a\partial_\nu a - \frac{g_{\mu\nu}}{2}(\partial a\partial a + 2V) \tag{2.1.19}$$

reduce to, for a spatially homogenous field $a(t)$ on a Friedmann-Lemaître-Robertson-Walker (FLRW) background $g_{\mu\nu} = \mathrm{diag}(-1, s^2, s^2, s^2)$ with $s(t)$ the scale factor,

$$\ddot{a} + 3H\dot{a} + \frac{\partial V}{\partial a} = 0 \ , \quad \rho = \frac{\dot{a}^2}{2} + V \ , \quad p = \frac{\dot{a}^2}{2} - V \ , \tag{2.1.20}$$

where $H = \dot{s}/s$ is the Hubble rate, ρ is the energy density and p the pressure. Considering now the minimal potential $V = \frac{1}{2}m_a^2 a^2$ and late cosmological times when (assuming such times exist) $H \ll m_a$, we use the following ansatz to describe the solution of (2.1.20):

$$a = \mathcal{A}(t)\cos(m_a t + \delta) \tag{2.1.21}$$

with $\dot{\mathcal{A}}/\mathcal{A} \sim H \sim \epsilon m_a$. At first order in ϵ, we find $\mathcal{A} = \mathcal{A}_0 s^{-3/2}$, $\langle\rho\rangle = \frac{m_a^2\mathcal{A}_0^2}{2}s^{-3}$ and $\langle p\rangle = 0$ (where $\langle X\rangle$ denotes the time average of X over a period $\frac{1}{2\pi m_a}$), which defines the pressureless non-relativistic fluid usually called cold dark matter (CDM). Interestingly, the axion field undergoes oscillations at a frequency given by its mass, which is at the core of recent searches of axion DM [68,69].

Much like WIMP CDM, which only behaves as a non-interacting non-relativistic fluid after its decoupling from the thermal bath, the axion energy density only scales as s^{-3} at late times.

At early times when $H \gg m_a$, the axion is essentially frozen by the Hubble friction at a fixed value, which gives the initial condition for the axion relic density. The CDM behaviour starts when $H \sim m_a$. Cosmological observations then put a lower bound on the mass of the axion, which cannot be smaller than the Hubble scale at matter-radiation equality for instance, if it is to explain all the dark matter budget of the universe:

$$m_{\text{a, axion DM}} > 10^{-28} \text{ eV} . \tag{2.1.22}$$

The relic density can then be calculated precisely given an initial condition $a_0 \equiv |a|$ at early times if the mass is time-independent:

$$\Omega_{\text{ALP DM}} \approx 2 \times 10^2 \left(\frac{m_a}{10^{-22} \text{ eV}}\right)^{\frac{1}{2}} \left(\frac{a_0}{M_P}\right)^2 , \tag{2.1.23}$$

where $M_P = 2.4 \times 10^{18}$ GeV is the reduced Planck mass. This result can be understood (at the order of magnitude level) by assuming that the axion stays frozen at a_0 until $3H \approx m_a$ (which establishes the balance between the friction and the driving force term in (2.1.20)), after which it is given by (2.1.21).

The assumption that the mass is time-independent is actually incorrect for the QCD axion. Indeed, its mass is given by non-perturbative QCD effects which are quite sensitive to temperatures typical of the observable cosmological history, since they kick in around $T \sim \Lambda_{\text{QCD}} \sim 200$ MeV. We have on the one hand

$$H = \pi \sqrt{\frac{g_*}{90}} \frac{T^2}{M_P} \tag{2.1.24}$$

during radiation domination, where g_* is the effective number of relativistic degrees of freedom, and on the other hand (here for high temperatures $T \gtrsim 1$ GeV)

$$m_a(T) = \alpha \sqrt{m_u \Lambda_{\text{QCD}}} \frac{\Lambda_{\text{QCD}}}{f_a} \left(\frac{T}{\Lambda_{\text{QCD}}}\right)^{-n} , \tag{2.1.25}$$

where α and n are numbers derived from lattice or instanton calculations. There is no clear consensus on the precise values of those numbers, but n seem to range between 1 and 4 and $\alpha \sim 10^{-7}$. Choosing for instance $n = 4$, the onset of the axion oscillations happens at a temperature T_{osc} such that

$$3H(T_{\text{osc}}) = m(T_{\text{osc}}) . \tag{2.1.26}$$

If $f_a \gtrsim 10^{17}$ GeV, the axion oscillations happen at temperatures low enough such that the axion mass is stabilized at its zero temperature value (2.1.14) when its starts behaving as dark matter, and (2.1.23) can be safely used. For the QCD axion whose mass is related to f_a, (2.1.23) can be written

$$\Omega_{\text{QCD axion DM, high scale}} \approx 5 \times 10^3 \left(\frac{a_0^{4/3}}{10^{16} \text{ GeV} \times f_a^{1/3}}\right)^{\frac{3}{2}} . \tag{2.1.27}$$

On the contrary, if $f_a \lesssim 10^{15}$ GeV, $T_{\text{osc}} \gtrsim 1$ GeV and the mass is significantly modified along the universe cooling. This makes the potential term in (2.1.20) time-dependent and modify the solution (2.1.21). Eventually, we get

$$\Omega_{\text{QCD axion DM, low scale}} \approx 2 \times 10^4 \left(\frac{a_0^{12/7}}{10^{16} \text{ GeV} \times f_a^{5/7}}\right)^{\frac{7}{6}} . \tag{2.1.28}$$

Those formulas are modified by anharmonic coefficients once one uses the full axion potential instead of the mass term in (2.1.20), but they still give a clear picture of what is happening.

If (2.1.22) is not verified, we are considering now a cosmological fluid which cannot be interpreted as the dark matter we observe, since it would have had a constant energy density during some time after matter-radiation equality. However, such a constant energy density is welcome if we want to model dark energy and it lies at the center of axion quintessence scenarii. In such models, one wants instead to impose that the CDM-like behaviour has not started yet:

$$m_{\text{axion DE}} < H_0 \sim 10^{-33} \text{ eV} , \tag{2.1.29}$$

where H_0 is today's Hubble rate. Then, the relic density is

$$\Omega_{\text{axion DE}} \approx 8 \times 10^{-2} \left(\frac{m_a}{10^{-33} \text{ eV}}\right)^2 \left(\frac{a_0}{M_P}\right)^2 . \tag{2.1.30}$$

Note that $\Omega_a \sim 1$ today requires $a_0 > M_P$. We will comment more on this later.

Now, one may wonder what explains the origin of a_0 in (2.1.23), (2.1.30), etc. The mechanism invoked is called misalignment mechanism, and it takes place whenever a spontaneously broken global symmetry is coupled to cosmology. The schematic idea goes as follows: at high temperatures, the thermal fluctuations modify the zero temperature potential and the global symmetry is not broken. When the temperature drops below the breaking scale (which is close to the axion decay constant f_a in minimal models), the minimum of the potential shifts away from the symmetry conserving point and a Goldstone mode (our axion) appears in the spectrum. Its initial value a_0 is random, since there is no preferred one due to the global symmetry. When the temperature drops below the scale of explicit symmetry-breaking effects (as the QCD non-perturbative ones), a potential for a_0 develops and it sources an energy density.

The interplay between this picture, inflation and the notion of causal horizon is important. Indeed, when the spontaneous breaking happens, every causally connected patch picks up a symmetry-breaking vev, i.e. develops a constant background value for the axion field. If this happens before inflation, each patch grows exponentially and our observable universe is essentially made out of a single patch (which explains the temperature correlations at large angles in the CMB). Thus, one is free to consider whatever a_0 value in e.g. (2.1.23), and the only constraints come from one's taste for tuning and from the fact that the axion is periodic, with a periodicity given by the breaking scale: $a_0 \lesssim \pi f$. In this picture however, one must take into account the possible thermal quantum fluctuations experienced by any field in de Sitter space (dS), which could shift a_0 during inflation and prevent from arbitrarily tuning. The scale of those effects is given by the dS temperature $\frac{H_{\text{inflation}}}{2\pi}$ [70]. On the other hand, if the spontaneous breaking happens after inflation, no such effect is relevant but the field has to locally choose a vev, so that different values for a_0 are statistically distributed over the observable universe. Consequently, the average value of the energy density is fixed, given the average value of $\langle a_0 \rangle = \frac{\pi f_a}{\sqrt{3}}$ (calculated for a uniform distribution). This scenario thus has a strong predictive power, since it is only determined by the scale f_a. For instance, demanding that $\Omega_{\text{QCD axion}} < 0.25$, we derive[8]

$$f_a \lesssim 10^{11}\ \text{GeV}\ . \tag{2.1.31}$$

However, the existence in this picture of many different patches induces the presence of topological defects, which we will discuss soon.

We worked out the consequences of the misalignment mechanism, since it is inevitable. There are other ways of producing axions in cosmology though, and we briefly comment on them. First, the axions may be coupled to matter or additional fields. So they could be produced from the decay of heavy particles, in which case they add up to the density of relativistic species. This production mode is for instance relevant for discussions of axions in supersymmetric models, since they could be produced from the decay of their partners, sometimes necessary to avoid the cosmological moduli problem [84]. Axions could also be produced thermally from the particle bath. For weakly coupled light axions, this means that they are produced relativistically again. QCD axions, which have a defining coupling to matter, are produced thermally, but in a very negligible amount once we look at observationally consistent values for their decay constant. Finally, there can be production of axions from topological defects. Topological defects are of two kinds here: there are axion strings, which are formed when the axion value smoothly scans its periodicity range along a closed curve. Then, inside the closed curve passes a string-like region where the symmetry is unbroken, such that there is an increased energy density at the string core. There are also domain walls, which exist when the axion potential admits several minima within its periodicity range, reminding us again of the discussion around (2.1.7). Indeed, there are then classical configurations of finite energy density where the axion spatially goes from one minimum to a second, inducing again a wall-like local increased energy density. Domain walls, when they form, are arranged in a network of walls which meet on strings. Such topological defects carry an energy density and can easily upset the energy budget of the universe [85] either by the scaling of their energy density with s or due to their continuous production of axions. Interestingly, those axions have a rich spectrum and some of them are produced with low momenta, adding up to the CDM budget. The study of axion emission from strings is still very active, see e.g. [86]. If one wants to get rid of the string-wall network, there are two major options. First, one can ensure that $N_{\text{DW}} = 1$ by suitably choosing the UV completion of the axion, as in the KSVZ model. Second, one can introduce in the theory an additional controlled source of an axion potential such that the degeneracy between the vacua is lifted and the walls become unstable [85, 87, 88]. We will come back to this solution, in an albeit different context, later.

2.1.7 A short word on experimental and observational bounds

As was mentioned before, the theoretical interest in axions is supported by a remarkably active and diverse experimental program. The latter intends to scan very broad ranges of masses and

[8]This bound can also be understood as necessary for an absence of tuning in the pre-inflationary scenarii. However, there has been some significant activity aiming at relaxing this bound [71–83], which is thus to be taken with a pinch of salt.

test all the different couplings displayed in (2.1.6). There are experiments and/or observations which are aiming at axion dark matter, others which test the general fact that there exist in nature a light scalar field with couplings of the kind (2.1.6), and some which are even testing whether there exists a light scalar field. We briefly review the different approaches, not trying to list references beyond the original papers on the subject.

In the first category, one finds haloscopes, an example of which is ADMX [89], an experiment sensitive to the operator "$aF\tilde{F}$" which couples axions to photons. In a microwave cavity, a magnetic field waits for a dark matter axion to deposit its energy into the cavity. It is sensitive to the axion-photon coupling, $\frac{\mathcal{C}_\gamma}{f_a}$ in the language of (2.1.6), and to axion masses comparable with the inverse of the size of the cavity ($\sim 10^{-6}$ eV here). Updates of ADMX (see e.g. [90]) or alternative experiments testing the same operator (e.g. [68, 91]) are planned or operative. There are also experiments aiming at detecting the "$\partial a\psi\psi F$" (or "$\partial a\psi\psi$", whose associated coupling is denoted $g_{a\psi}$ in what follows) kind of operator by measuring the oscillating EDM (respectively the oscillating spin-dependent forces) which axion dark matter induces on electron or neutrons. Examples are [69, 92, 93], working in the low mass region ($\lesssim 10^{-6}$ eV).

In the second category, there are helioscopes, i.e. searches for axions emitted by the Sun via direct conversion into X-ray photons on Earth, such as CAST [94] or the future IAXO [95], sensitive again to the "$aF\tilde{F}$" operator and to sub-eV axions. Indirect astrophysical information can be obtained on the "$aF\tilde{F}$" and "$\partial a\psi\psi$" operators by looking at star cooling [96, 97] or energy loss in supernovae [98], for a large range of axion masses only bounded by the star temperature (respectively $\lesssim 100$ keV or $\lesssim 100$ MeV). At the earth-only level, there is a cute kind of approach called "light-shining-through-walls" [99, 100], which converts a photon into an axion and back again after the axion went through a wall that light cannot pass, and experiments which study the evolution of polarized light in the vacuum, such as PVLAS [101]. There are also limits imposed on spin-dependent axion-mediated forces [102], which test the "$\partial a\psi\psi$" operator for both nucleons and electrons. Finally, there are collider-based searches for rare meson decays into axions.

The last kind of observations, whose generality and reach is quite impressive, is due to the fact that spinning black holes can enhance field fluctuations in their ergosphere, in a field theory analog of the Penrose process. This fact, dubbed black hole superradiance, tends to spin black holes down, while it does not require anything more than the fact that there exists a field in the theory susceptible to be enhanced as soon as it quantum mechanically fluctuates. Thus, observations of old black holes with large spins put stringent bounds on small masses (10^{-22} eV $\lesssim m \lesssim 10^{-10}$ eV) spinless bosons [103–105].

Outputs of all those searches are usually expressed as exclusion regions in a plane (mass,coupling). An example is given in Figure 2.2 for the axion-photon coupling.

As a wrap up, strong bounds which are useful to have in mind are the following:

$$\frac{\mathcal{C}_\gamma}{f_a} \lesssim 10^{-11}\ \text{GeV}^{-1} \text{ and } g_{ae} \lesssim 10^{-13}\ \text{GeV}^{-1} \text{ if } m_a \lesssim 100\ \text{keV} \tag{2.1.32}$$

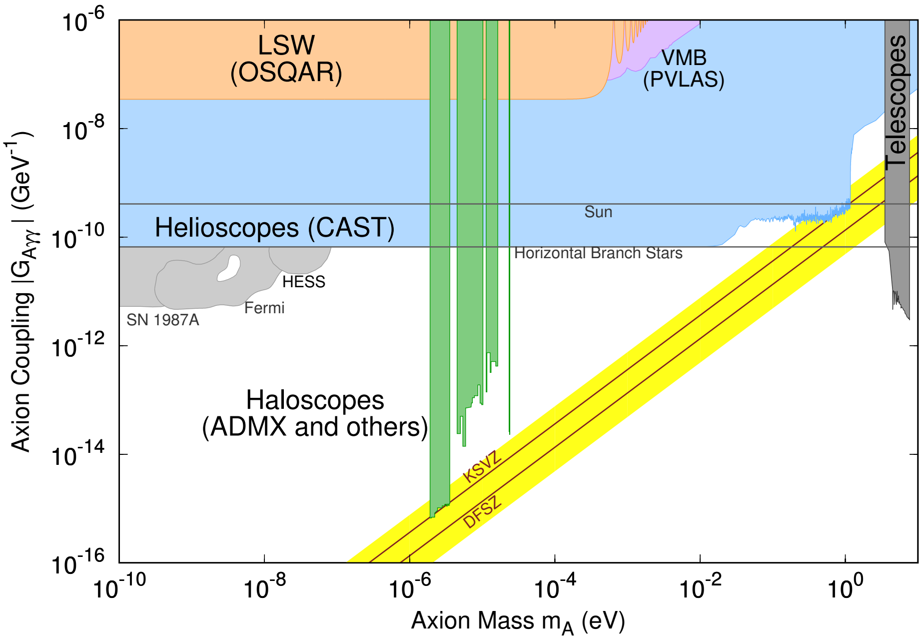


Figure 2.2: Exclusion regions for the axion-photon coupling, copied with permission from [106] The yellow band indicates where natural values are found for the benchmark models discussed in section 2.1.2

from stellar (white dwarfs and globular clusters) cooling, and

$$g_{aN} \lesssim 10^{-8} \text{ GeV}^{-1} \text{ if } m_a \lesssim 50 \text{ MeV} \tag{2.1.33}$$

from energy losses in the supernova SN1987a. None of these bounds depends on the fact that the axion makes up most of dark matter. To those bounds, one may add that axion dark matter should lead to correct structure formation up to non-linear observables, which imposes

$$m_{a,\text{DM}} > 10^{-22} \text{ eV} . \tag{2.1.34}$$

2.1.8 Challenges for axion model building: decay constants and $U(1)_{\mathbf{PQ}}$ quality

We introduce here two of the challenges sometimes faced along axion model building, which will be important in several of the discussions to come in this thesis.

The first one concerns the decay constant f_a. As we saw, it defines the mass of the QCD axion and, in minimal models, the strength of its interactions. For ALPs, one can analogously define a decay constant either using the ALP periodicity or the strength of its interactions. It turns out however that it is sometimes desirable to introduce a hierarchy between two alternative definitions of the decay constant, e.g. between an ALP periodicity and the strength of its interactions. For instance, we saw that the most stringent bounds on f_a for a QCD axion come from constraints on its coupling to photons and fermions. Hence, one could be tempted to design models where the coupling to photons or fermions is strongly suppressed (see e.g. [107–109]), enabling the axion mass to get much bigger (giving up the interpretation that the axion makes up all dark matter).

Such setups are relevant for collider-based axion searches [110–115]. Alternatively, one may want to decorrelate the scale which describes the axion dynamics from the scales defining the rest of some hidden sector. An example of when this happens has already been shown around (2.1.30), since we saw there that a big enough relic density for axion dark energy demands trans-Planckian initial field values. Similar things happen for axion inflation [61] or relaxion models [116]. This may feel worrying since scales above the Planck scale are expected to be associated to quantum gravity effects and to the breakdown of a usual EFT analysis. Quantitative views on this issue will be discussed again in the context of swampland conjectures, see section 4.4. Consequently, it would be reassuring if the axion scale was effective, smartly built from sub-Planckian fundamental scales (see e.g. [62]). In the same spirit, one could try to explain the hierarchy between usual particle physics scales, such as the GUT/see-saw scale or the Planck/string scale, and f_a, which may for instance be forced to be intermediate (remember (2.1.31)).

The second challenge has to do with the quality of the axion shift symmetry (which we call $U(1)_{\text{PQ}}$ in what follows, even if we talk about a generic ALP). By a high quality symmetry, we refer to a symmetry which is "hard" to explicitly break. There are two standard meanings for this: either the operators which break the symmetry explicitly are of high dimensions and hardly affect low energy physics, or their associated coefficients are (naively unnaturally) suppressed, if for example they are not generated perturbatively and enjoy the suppression associated to non-perturbative effects (e.g. multiplication by $e^{-8\pi^2/g^2}$ for instanton contributions).

Why are we concerned about the possible quality of $U(1)_{\text{PQ}}$? Applications of axion physics usually demand very light particles. Indeed, the QCD axion with intermediate/high f_a (due to astrophysical observations, as we said earlier) is necessarily a light particle, and it is frequent to be interested in very light ALP dark matter [117–122], for instance to solve the core-cusp problem [123]. In addition, we already emphasized that axion quintessence is a very light particle. On the other hand, uncontrolled explicit symmetry breaking corrections to the lagrangian of the theory spoil the pseudo-Goldstone nature of axions and give them an uncontrolled mass. This can be illustrated at the level of the KSVZ model of a QCD axion: if we added to the lagrangian (2.1.3) the $U(1)_{\text{PQ}}$ breaking operator

$$\mathcal{L}_{\cancel{U(1)_{\text{PQ}}}} = \frac{y}{\Lambda^{n-4}}\sigma^n + h.c. \tag{2.1.35}$$

(where Λ is a high scale of new physics), it would induce the following term in the axion potential, in addition to the usual QCD contribution:

$$\mathcal{L} \supset -\frac{m_u m_d m_{\pi^0}^2 f_\pi^2}{2(m_u+m_d)^2}\left(\frac{a}{f_a}+\theta_{\text{QCD}}\right)^2 + |y|\Lambda^4\left(\frac{f_a}{\sqrt{2}\Lambda}\right)^n \cos\left(\frac{na}{f_a}+\arg(y)\right), \tag{2.1.36}$$

where we defined $\sigma = \frac{f_a}{\sqrt{2}}e^{i\frac{a}{f_a}}$ and used that $\langle a\rangle \approx -\theta_{\text{QCD}} f_a$. Upon minimization, we find an effective θ-term

$$\frac{\langle a\rangle}{f_a} + \theta_{\text{QCD}} \approx \frac{n|y|(m_u+m_d)^2\Lambda^4\left(\frac{f_a}{\sqrt{2}\Lambda}\right)^n \sin(n\theta_{\text{QCD}}-\arg(y))}{m_u m_d m_{\pi^0}^2 f_\pi^2}\,. \tag{2.1.37}$$

Assuming that all the input parameters are unrelated and $\mathcal{O}(1)$ (which implies $n\theta_{\text{QCD}}-\arg(y) = \mathcal{O}(1)$) and noting that $\frac{m_u m_d m_{\pi^0}^2 f_\pi^2}{(m_u+m_d)^2\Lambda^4}$ is small, we understand that we have to ensure that the quantity

$|y|(f_a/\sqrt{2}\Lambda)^n$ is even smaller (such that eventually, $\frac{\langle a\rangle}{f_a}+\theta_{\rm QCD}<10^{-10}$). Since, for reasons such as the naturalness of scalar fields, we do not have any reason to expect $f_a \ll \Lambda$, we are left with the two options $|y| \ll 1$ or $n \gg 1$, illustrating our characterization of a high quality symmetry. For instance, for $\Lambda = M_P$ and $f_a = 10^{12}$ GeV, we need either $n \geq 14$ or $|y| \lesssim 10^{-55}$. Note on the other hand that the addition of (2.1.35) lifts the degeneracy between the vacua of the QCD-induced axion potential. This is the explicit breaking solution to the presence of cosmological domain walls that we mentioned earlier. Implementing it then demands a subtle balance between keeping the breaking small enough not to spoil the PQ solution, but sufficient for the walls to have a lifetime which does not exceed cosmological timescales.

At this stage, one may object: why should this explicit breaking happen in the first place? The answer to this comes from expectations about quantum gravity: quantum gravity tends to violate global symmetries [124–126], meaning that there are quantum gravitational processes which generate additional, symmetry breaking, operators in the quantum effective action even though they are not generated for symmetry reasons in field theory calculations. We will comment on this in more details when we discuss swampland conjectures, but we deal with consequences for axions here. Typically, one expects quantum gravity to generate operators such as (2.1.35), with $\Lambda \sim M_P$ and no restriction on n. To avoid spoiling the Peccei-Quinn solution to the strong CP problem, one way to proceed is to make $U(1)_{\rm PQ}$ accidental, arising as a byproduct of a choice of some gauge group and matter content, since in contrast gravity goes along with gauge symmetries. We then talk about a protected global symmetry. Such line of model building has been followed by many authors, including [88, 127–141]. Typically, strong enough protection requires either large gauge charges for scalar fields (see e.g. [88]) or many gauge groups as in quiver models (see e.g. [133, 134]). The latter can be understood as latticized versions of extra-dimensional models where the PGBs are interpreted as fifth components of vector fields, which appear as scalars in 4D, with a shift symmetry inherited from the higher-dimensional gauge invariance. More generally, protected axions can be obtained as zero modes of gauge fields, which could be vectors but also higher degree forms, in compactifications of higher-dimensional theories. We will have more to say about axions from extra dimensions and their protection when we discuss axions in string theory in section 4.3.

2.2 A clockwork model for a high protection of an axion shift symmetry

We present now a specific implementation of the ideas discussed just above in section 2.1.8, and study in section 2.2.2 a quiver model for an axion protection, which exhibits at the same time a generation of hierarchies between input and effective scales. Many features of this model are inspired by the so-called clockwork mechanism, so we first review it in section 2.2.1.

2.2.1 Clockwork mechanism

The clockwork mechanism was originally designed [142–144] to generate exponential hierarchies out of order one parameters. It has been applied to models of relaxion [142], QCD axions or ALPs [143–147], additional gauge interactions [144, 148], flavour physics [149, 150], neutrino masses [144, 151–153], dark matter [154–157], inflation [158, 159], it has been extended [160], interpreted in terms of extra-dimensional theories [144, 161–169] and it has been used within discussions of swampland conjectures [170–172]. It can be explained with a four- or five-dimensional vocabulary, which we will make explicit now.

Discrete clockwork mechanism

Discrete clockwork models are four-dimensional models with a symmetry-breaking pattern such that the properties of surviving light modes are characterized by parameters exponentially different from the input ones, for which one may choose natural values. Clockwork models have used any kind of symmetries one may try to break to generate this clockwork effect [144]: shift symmetries, chiral symmetries, gauge symmetries, diffeomorphisms... However, since it is illustrative enough and for conciseness, we focus on scalar shift symmetries in our review of clockwork models, closely following the treatment in [144].

The so-called scalar clockwork starts with N real scalars whose shift symmetry is softly broken down to a diagonal one by the following lagrangian:

$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^{N}(\partial_\mu \pi_i)^2 - \frac{m^2}{2}\sum_{i=1}^{N-1}(\pi_i - q\pi_{i+1})^2 \ , \tag{2.2.1}$$

where q is a number and the exact diagonal symmetry is seen to be[9] $\pi_i \to \pi_i + q^{N-i}\alpha f$. Consequently, one can identify a massless Goldstone scalar $\pi^{(0)} \propto \sum_{i=1}^{N} \frac{\pi_i}{q^{i-1}}$, which indeed is a zero eigenvector of the mass matrix (typical of clockwork models):

$$M^2 = m^2 \begin{pmatrix} 1 & -q & 0 & ... & 0 & 0 \\ -q & 1+q^2 & -q & ... & 0 & 0 \\ 0 & -q & 1+q^2 & ... & 0 & 0 \\ ... & .. & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1+q^2 & -q \\ 0 & 0 & 0 & ... & -q & q^2 \end{pmatrix} \tag{2.2.2}$$

The zero mode $\pi^{(0)}$ has an exponential profile along the original scalars π_i, so if matter was coupled at the i-th "site" of the clockwork, it would feel an exponentially suppressed coupling to the massless eigenstate. For instance, if π_i was coupled to the topological density of a gauge group[10]:

$$\mathcal{L} \supset -\frac{\pi_i}{32\pi^2 f} F^a_{\mu\nu}\tilde{F}^{a,\mu\nu} \implies \mathcal{L} \supset -\frac{\pi^{(0)}}{32\pi^2 f q^{i-1}} F^a_{\mu\nu}\tilde{F}^{a,\mu\nu} \ . \tag{2.2.3}$$

[9] We introduced a scale f to connect with the UV discussion below.

[10] We will sometimes implicitly assume $q \gg 1$ to avoid clumsy $\sqrt{1+q^{-2}+...+q^{-2N}}$ normalization factors.

Then, matter living at the N-th site could feel an effective axion decay constant $q^{N-1}f$, orders of magnitude bigger than the original one f. We refer to the site-dependence of any effective coupling, scale, etc, in clockwork models as clockwork localization in what follows. This property has for instance been used to design models of invisible QCD axions with additional heavy physics at the TeV scale, possibly within the reach of the LHC.

Indeed, two kinds of heavy matter come with the massless boson $\pi^{(0)}$. The first one is composed of the heavier pseudo-Goldstone modes obtained out of the π_is. For the mass matrix (2.2.2), their profiles are given by

$$\pi_i^{\text{mass}} = \sum_{j=1}^{N} O_{ij}\pi_j \ , \quad \text{with } O_{ij} \propto q \sin\left(\frac{i(j-1)\pi}{N}\right) - \sin\left(\frac{ij\pi}{N}\right) \ , \tag{2.2.4}$$

and their masses are $m_i^2 = m^2\left(q^2 + 1 - 2q\cos\left(\frac{i\pi}{N}\right)\right)$, with $i = 1, ..., N-1$. The spectrum thus features a band-like structure and, contrary to the massless mode, those heavy excitations are not exponentially localized towards one clockwork site. In particular, any matter would couple to the heavy modes with order $1/N$ suppression, e.g. with a decay constant Nf for the axion example discussed above. In such discrete models, besides being exponentially less coupled to matter leaving at the N-th site than to matter leaving at the first one, the massless boson couples exponentially less than heavy modes to matter leaving at the N-th site, while their couplings are comparable at the first site.

The second kind of heavy matter consists of partners of the π_is in a model where the shift symmetries are linearly realized and softly broken. For instance, the π_is can be understood as the phase degrees of freedom of complex fields U_i which obtain a vev:

$$U_i = \frac{f + h_i}{\sqrt{2}} e^{i\frac{\pi_i}{f}} \ , \tag{2.2.5}$$

and which are arranged in a lagrangian which softly breaks their $U(1)$ phase shifts down to the diagonal one $U_i \to e^{iq^{N-i}\alpha}U_i$:

$$\mathcal{L} = -\sum_{i=1}^{N} |\partial_\mu U_i|^2 - V(\{|U_i|^2, i = 1, ..., N\}) + \left(\lambda \sum_{i=1}^{N-1} U_i U_{i+1}^{\dagger q} + h.c.\right) \tag{2.2.6}$$

Then, defining $m^2 = \lambda\left(\frac{f}{\sqrt{2}}\right)^{q-1}$, the (second order) π_i lagrangian which proceeds from (2.2.6) is (2.2.1). Even though the couplings of the longitudinal modes h_i of the U_is do not follow from the clockwork EFT (2.2.1) and for instance depend on the potential V in (2.2.6), assuming order one coefficients locates those modes parametrically close to the scale f. Hence, the full detail of such a model could be tested with low-energy machines even though the couplings of the light mode seem to be governed by high-scale physics.

Continuous clockwork mechanism

It can be tempting to take the large N limit of (2.2.1). There is an interesting way to do so [144] which sends (2.2.1) on a free massless scalar field theory living on a five-dimensional warped spacetime called the linear dilaton background [173][11].

Indeed, keeping $\frac{m^2}{N^2}$ and q^N fixed as $N \to \infty$ has a simple geometric interpretation. To see this, let us consider a 5D theory of a free massless field, living on the 5D linear dilaton background of metric

$$ds^2 = e^{-\frac{4ky}{3}}(dx_\mu^2 + dy^2) \ , \tag{2.2.7}$$

where y is the fifth coordinate, which takes values between 0 and L, and the usual Minkowski metric contracts 4D indices. The action is

$$\begin{aligned} S &= \int d^5x\sqrt{-g}\left(-\frac{1}{2}g^{MN}\partial_M\pi\partial_N\pi\right) \\ &= \int d^4x dy e^{-2ky}\left(-\frac{1}{2}((\partial_\mu\pi)^2 + (\partial_4\pi)^2\right) \\ &= \int d^4x dy \left(-\frac{1}{2}(\partial_\mu\pi')^2 - \frac{1}{2}e^{-2ky}(\partial_4[e^{ky}\pi'])^2\right) \ . \end{aligned} \tag{2.2.8}$$

where we defined $\pi \equiv e^{ky}\pi'$. We can latticize/deconstruct this theory by defining $y_i = i\Delta_4 \equiv i\frac{L}{N}$, and replacing

$$\begin{aligned} \int_0^L dy \ f(x,y) &\to \sum_{i=1}^{N}\Delta_4 f(x,y_i) \ , \\ \partial_y f(x,y_i) &\to \frac{f(x^\mu, y_{i+1}) - f(x,y_i)}{\Delta_4} \end{aligned} \tag{2.2.9}$$

for any function f. We thus get (defining $\pi'(x,y_i) \equiv \pi'_i(x)$):

$$\begin{aligned} S &= -\frac{\Delta_4}{2}\sum_{i=1}^{N}(\partial_\mu\pi'_i)^2 - \frac{1}{2\Delta_4}\sum_{i=1}^{N-1}e^{-\frac{2ikL}{N}}\left(e^{\frac{(i+1)kL}{N}}\pi'_{i+1} - e^{\frac{ikL}{N}}\pi'_i\right)^2 \\ &= -\frac{1}{2}\sum_{i=1}^{N}(\partial_\mu\pi_i)^2 - \frac{N^2}{2L^2}\sum_{i=1}^{N-1}\left(e^{\frac{kL}{N}}\pi_{i+1} - \pi_i\right)^2 \ , \end{aligned} \tag{2.2.10}$$

with $\pi'_i \equiv \frac{\pi_i}{\sqrt{\Delta_4}}$. Identifying $q \equiv e^{\frac{kL}{N}}$ and $m^2 \equiv \frac{N^2}{L^2}$, we indeed recover (2.2.1).

In this picture, site-localized matter becomes matter living on a brane localized along the fifth dimension. In particular, couplings of the type (2.2.3) become

$$\begin{aligned} S \supset &\int d^5x\sqrt{-g}\frac{\delta(y-y_0)}{\sqrt{g_{44}}}\left[-\frac{g^{\mu\nu}g^{\rho\sigma}}{4g^2}F^a_{\mu\nu}(x^\mu)F^a_{\rho\sigma}(x^\mu) - \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g^{(4)}}}\frac{\pi(x^\mu,y)}{64\pi^2F^{3/2}}F^a_{\mu\nu}(x^\mu)F^a_{\rho\sigma}(x^\mu)\right] \\ = &\int d^4x\left[-\frac{1}{4g^2}F^a_{\mu\nu}F^{a,\mu\nu} - \frac{\epsilon^{\mu\nu\rho\sigma}q^{\frac{y_0}{\Delta_4}}\pi_{i=\frac{y_0}{\Delta_4}}(x^\mu)}{64\pi^2F^{3/2}\sqrt{\Delta_4}}F^a_{\mu\nu}F^a_{\rho\sigma}\right] \end{aligned} \tag{2.2.11}$$

where in the first line, π is normalized as in the first line of (2.2.8) and $g^{(4)}$ denotes the four-dimensional metric. When projected on the four-dimensional zero mode $\pi^{(0)}$, the last line includes

$$-\frac{\epsilon^{\mu\nu\rho\sigma}q^{\frac{y_0}{\Delta_4}}\pi_{i=\frac{y_0}{\Delta_4}}(x^\mu)}{64\pi^2F^{3/2}\sqrt{\Delta_4}}F^a_{\mu\nu}F^a_{\rho\sigma} \supset -\frac{\epsilon^{\mu\nu\rho\sigma}\pi^{(0)}}{64\pi^2F^{3/2}\sqrt{\Delta_4}}F^a_{\mu\nu}F^a_{\rho\sigma} \tag{2.2.12}$$

[11] Said differently, (2.2.1) can be understood as the deconstruction [174, 175] of this 5D theory.

and we see that such a deconstruction does not yield hierarchical couplings of the zero mode to matter localized at different sites when no y-dependence for the scale F is introduced in 5D. However, the couplings to massive modes reads

$$-\frac{\epsilon^{\mu\nu\rho\sigma}q^{\frac{y_0}{\Delta_4}}\pi_{i=\frac{y_0}{\Delta_4}}(x^\mu)}{64\pi^2F^{3/2}\sqrt{\Delta_4}}F^a_{\mu\nu}F^a_{\rho\sigma}\supset-\frac{\epsilon^{\mu\nu\rho\sigma}q^iO_{ji}\pi_j^{mass}}{64\pi^2F^{3/2}\sqrt{\Delta_4}}F^a_{\mu\nu}F^a_{\rho\sigma}\ , \tag{2.2.13}$$

where $O_{ji}=\mathcal{O}(1)$ are defined in (2.2.4). Hence, couplings of the zero and massive modes at a given site are exponentially different, and the possibility that massive modes are within reach of low-energy detectors whereas the massless mode is very weakly coupled remains. Extensive discussions on this issue, as well as 5D formulations involving both warping in the fifth dimension and brane mass terms can be found in [162, 163, 166].

2.2.2 The model

As we have said, there have been several clockwork approaches to axion models [143–147, 161]. Some of them use clockwork global symmetries, some rely on the description of the clockwork axion as a five-dimensional field, some do both. In this section, we sum up [176], attached at the end of this thesis, and study a 4D clockwork model of a QCD axion or cosmological ALP, with its associated global shift symmetry being accidentally enforced by a gauge abelian quiver with scalar bifundamental fields.

Motivation and overview

The model under study below addresses the concerns raised in section 2.1.8. Indeed, the 4D field content is such that the most general renormalizable gauge-invariant lagrangian preserves an accidental spontaneously broken global symmetry, associated with an axion mode. Furthermore, the specific 4D gauge charge assignment ensures a strong protection of this accidental symmetry from explicit breaking terms, even when the discretization is crude (i.e. when the quiver has few sites), and it generates a hierarchy between the effective axion decay constant f_a and the scale f of spontaneous symmetry breaking: f_a is reduced by a factor which grows exponentially with the number of quiver sites, in a way opposite to the usual clockwork models. Some of the features of the latter are nevertheless recovered: the axion has a clockwork profile along the quiver sites, and this profile can generate effective coupling scales which are different from f_a (which appears in the potential of the axion and its couplings to gauge fields) and bigger than f, when one considers for instance couplings of the axion to the spins of matter particles.

In what follows, we illustrate and use those properties by identifying our axion with a QCD axion or with a dark matter ALP. In the former case, the protection is only designed such that the QCD-induced mass overcomes any explicit breaking mass, as discussed in section 2.1.8. However, we show that this blessing is also a curse, since it requires a number of UV fermions which increases with the quality of the shift symmetry. On the other hand, in the latter case of a DM ALP, one does not need to generate a non-perturbative mass for the axion and can only consider any explicit breaking contribution such as the one from gravity. Then, the protection ensures that the explicit breaking mass is very small, such that the axion proves to be a particularly economical ultra-light DM candidate.

The theory's content and the axion

The 4D setup we consider is an abelian quiver model with bifundamental scalar fields [145, 161]. The precise matter content and charge assignment is given by the quiver of Figure 2.3 (where q and N are integers),

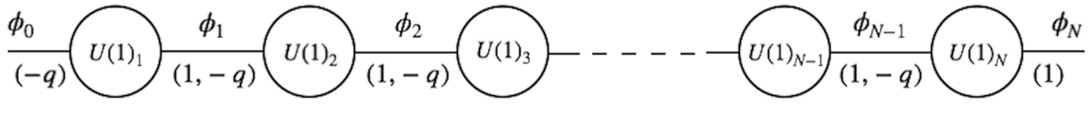


Figure 2.3: Abelian quiver of the model

with the following (most general renormalizable) lagrangian:

$$\mathcal{L} = -\sum_{i=1}^{N} \frac{1}{4g_i^2} F_{\mu\nu,i} F_i^{\mu\nu} - \sum_{k=0}^{N} (|D_\mu \phi_k|^2 + m_k^2 |\phi_k|^2) - \sum_{k,l=0}^{N} \lambda_{kl} |\phi_k|^2 |\phi_l|^2 \ , \tag{2.2.14}$$

where F_i is the field strength of the abelian vector field A_i, with coupling constant g_i, and with the covariant derivatives $D_\mu \phi_k = \big(\partial_\mu - i(1-\delta_{k,0}) A_{\mu,k} + iq(1-\delta_{k,N}) A_{\mu,k+1}\big)\phi_k$. This lagrangian has a $U(1)^{N+1}$ invariance, with a $U(1)^N$ gauged subgroup.

We choose the parameters m_k^2 and λ_{kl} of (2.2.14) so that all the scalar fields ϕ_k get vevs f_k, spontaneously breaking all the gauge symmetries. N out of the $N+1$ phases of the ϕ_k are absorbed by the gauge vectors through the Higgs effect (we write $\phi_k = \frac{f_k + r_k}{\sqrt{2}} e^{i\frac{\theta_k}{f_k}}$):

$$\mathcal{L} \supset -A_i^\mu (q f_{i-1} \partial_\mu \theta_{i-1} - f_i \partial_\mu \theta_i) \ . \tag{2.2.15}$$

The last, uneaten phase a remains in the spectrum after gauge fixing as the Goldstone boson associated to the accidental $U(1)_{\mathrm{PQ}}$ global symmetry which is the ungauged factor of the $U(1)^{N+1}$ symmetry group of (2.2.14). The profile of this boson, if the vevs are taken to be all equal to a given f, which is assumed from now on, reads:

$$a = \frac{\theta_0 + q\theta_1 + ... + q^N \theta_N}{\sqrt{1 + q^2 + ... + q^{2N}}} \ . \tag{2.2.16}$$

Eq. (2.2.16) displays the exponential localization discussed in clockwork models, and the charges of the original scalar fields under the global symmetry also match those which appear in those models. Indeed, $U(1)_{\mathrm{PQ}}$ acts here as $\phi_k \to e^{iq^k\alpha}\phi_k$.

Goldstone boson protection

The renormalizable lagrangian (2.2.14) has an accidental exact $U(1)_{\mathrm{PQ}}$ global symmetry, hence the axion a is massless. We expect however that global symmetries are broken (e.g. by gravity effects), which forces us to include all higher order operators allowed by gauge invariance in the effective theory[12]. For the quiver of Figure 2.3, these operators must be combinations of

$$|\phi_k|^2 \text{ and } \phi_0\phi_1^q...\phi_N^{q^N} \ . \tag{2.2.17}$$

Hence, operators that explicitly break the global symmetry must involve the second term and be of extremely high dimension as soon as q and N are both slightly bigger than one. We thus obtain in this setup a pseudo-Goldstone boson with a mass very well protected by the gauge symmetry, even with a reasonable number of gauge groups. More specifically, if we use (2.2.16), we find:

$$\left.\frac{\phi_0\phi_1^q...\phi_N^{q^N}}{M_c^{1+q+...q^N-4}} + h.c.\right|_{\text{axion terms}} = 2\Big(\frac{f}{\sqrt{2}M_c}\Big)^{1+q+...+q^N} M_c^4 \cos\Big(\frac{a}{f_a}\Big) \supset -\frac{1}{2}m_a^2a^2 \ , \tag{2.2.18}$$

where

$$f_a = \frac{f}{\sqrt{1+q^2+...+q^{2N}}} \tag{2.2.19}$$

and

$$m_a = \Big(\frac{f}{\sqrt{2}M_c}\Big)^{\frac{1}{2}(q+...+q^N-1)} \sqrt{1+q^2+...+q^{2N}}\, M_c \ , \tag{2.2.20}$$

and M_c is the cutoff of the theory, which we take equal to the Planck mass M_P hereafter since we consider gravity-induced breaking effects for simplicity. Note that f_a is significantly lower than f when N is large and $q > 1$ (we will come back to this soon).

The axion as a QCD axion

Let us now identify $U(1)_{\text{PQ}}$ with a Peccei-Quinn symmetry. We study first the low-energy EFT of the axion, assuming that every other massive field has been integrated out. We consider thus the following axionic coupling, effectively written here in terms of the original fields, then using (2.2.16):

$$i\log\Big(\phi_0\phi_1^q...\phi_N^{q^N}\Big)\,\text{Tr}\Big(G^{\mu\nu}\tilde{G}_{\mu\nu}\Big) + h.c. \supset -\frac{2\sqrt{1+q^2+...q^{2N}}}{f}a\,\text{Tr}\Big(G^{\mu\nu}\tilde{G}_{\mu\nu}\Big) \ , \tag{2.2.21}$$

with $G_{\mu\nu}$ the gluon field strength and we recognize the effective axion decay constant of (2.2.19). The operator in the log is, as we said before, the first gauge-invariant term capable of coupling the axion of (2.2.16) to the gluons that we could have written. This coupling has two major

[12]We already said that global symmetries are broken by Planck scale effects. The strength of the breaking is well defined in a consistent theory of quantum gravity. In what follows, we parametrize gravitational corrections as higher dimensional operators in the effective theory, suppressed by powers of the Planck scale with order one coefficients, assuming that the breaking is described correctly by the EFT approach. One may wonder whether such contributions could come from non-perturbative effects and consequently enjoy a greater suppression, as suggested by studies of axions arising from antisymmetric forms in string theory [55]. However, the kind of axions discussed here originate from charged matter fields. Even in string theory, those could in principle receive perturbative higher-order corrections to their potential, which would appear as usual higher-order terms in the EFT [177]. Furthermore, if the theory of gravity includes a heavy fermionic sector whose renormalizable couplings break the axion shift symmetry, the induced Coleman-Weinberg potential is also consistent with the effective theory point of view [133,134]. Thus, we assume that the magnitude of gravitational corrections is well described by the EFT approach, with no additional suppression.

features: it involves all the quiver sites, and it implies a *decrease* of the decay constant of the axion compared to the scale of breaking f.

This suggests that the present setup could describe intermediate scale axion decay constant obtained from high scale physics (such as string scale physics). This feature is common to most models with a global symmetry protected by gauge symmetries: the scale f_a is not identical to the original scale f of the spontaneous global symmetry breaking and one may consider the possibility of $f_a \ll f \sim M_P$. The relation between the scales f_a and f depends on the scalar fields charges, their vevs and their number, as well as on the number of gauge symmetries. Our model is then helpful in disentangling the different contributions.

When non-perturbative effects of QCD turn on, (2.2.21) induces a potential for the axion (which we shifted such that its vev includes $\theta_{\rm QCD}$):

$$\mathcal{L} \supset m_{\pi^0}^2 f_\pi^2 \sqrt{1 - \frac{4m_u m_d}{(m_u+m_d)^2}\sin^2\left(\frac{a}{2f_a}\right)} \ . \tag{2.2.22}$$

We also include every gauge-invariant term to the potential, as discussed above, and in particular generate a classical explicit breaking mass term (2.2.20) for the axion. In order to have $\left|\frac{a}{f_a}\right| < 10^{-10}$ at the minimum of the potential and solve the strong-CP problem, we must ensure [88,131, 132] that:

$$\left[m_{a,\rm QCD} \sim \frac{m_\pi f_\pi}{f_a}\right] > 10^5 \left[m_{a,\rm explicit} \sim \left(\frac{f}{\sqrt{2}M_P}\right)^{\frac{1}{2}(q+\ldots+q^N-1)} \frac{f}{f_a} M_P\right] \ . \tag{2.2.23}$$

For example, when $q=3$ and $N=2$, it implies $f \lesssim 10^{12}$ GeV. If now $q=3$, $N=3$ and $M_c = M_P$, this becomes $f \lesssim 10^{16}$ GeV. The values of the parameters q and N can be of course translated into the value of the ratio $f/f_a \sim q^N$.

Axion couplings to photons, which are the subject of most axion searches, are also part of this low-energy discussion. They can be derived when we consider the generalization of (2.2.21):

$$\mathcal{L} \supset \frac{i}{16\pi^2}\log\left(\phi_0\phi_1^q\ldots\phi_N^{q^N}\right)(\mathcal{C}G^{a,\mu\nu}\tilde{G}^a_{\mu\nu} + \mathcal{E}F^{\mu\nu}\tilde{F}_{\mu\nu}) \to -\frac{\sqrt{1+q^2+\ldots q^{2N}}}{16\pi^2 f}(\mathcal{E}-1.92\mathcal{C})aF^{\mu\nu}\tilde{F}_{\mu\nu} \tag{2.2.24}$$

(see (2.1.8)), where F is the photon field strength, $\tilde{F}$ its dual. These couplings feature the dependence on the decreased effective decay constant (2.2.19) we already encountered in (2.2.21). Couplings of the axion to fermions, such as axion-spin couplings, and their effective scales are discussed later.

Now, we study how (2.2.21) can be generated from loops of heavy fermions: (global) anomalies with respect to $SU(3)_c$ are mediated by colored fermions with some charge under the (global) symmetry, which run in triangle loops between gluons and scalars, whose phase contains part of the axion mode. The schematic procedure is (see appendix B for details and notations):

$$\begin{aligned}\mathcal{L} &= -|\partial\sigma|^2 - \overline{Q}\gamma^\mu(\partial_\mu - iG^a_\mu T^a)Q - (y\sigma\overline{Q_L}Q_R + h.c.) \ \text{ where } \sigma = \frac{f}{\sqrt{2}}e^{i\frac{a}{f}} \\ &\supset -\frac{(\partial a)^2}{2} - \overline{Q}\left(\gamma^\mu[\partial_\mu - iG^a_\mu T^a] + \frac{yf}{\sqrt{2}}\right)Q + i\frac{y}{\sqrt{2}}a\overline{Q}\gamma_5 Q \xrightarrow{Q \text{ loop}} -\frac{a}{16\pi^2 f}G\tilde{G} = \frac{i}{16\pi^2}\log(\sigma)G\tilde{G} \ ,\end{aligned} \tag{2.2.25}$$

where for compactness we defined $G\tilde{G} = \mathrm{Tr}\Big(G^{\mu\nu}\tilde{G}_{\mu\nu}\Big)$. We then see how to generate (2.2.21), starting from the following lagrangian:

$$\begin{aligned}\mathcal{L} \supset \;& -y_0\phi_0\overline{Q_{L,0}}Q_{R,0} - \phi_1\overline{Q_{L,1}^{i=1...q}}Y_{1,ij}Q_{R,1}^j + ... - \phi_N\overline{Q_{L,N}^{i=1...q^N}}Y_{N,ij}Q_{R,N}^j h.c. \\ &\xrightarrow{Q_i \text{ loops}} \frac{i}{16\pi^2}\big(\log(\phi_0) + ... + q^N\log(\phi_N)\big)G\tilde{G} = \frac{i}{16\pi^2}\log\Big(\phi_0\phi_1^q...\phi_N^{q^N}\Big)G\tilde{G}\;.\end{aligned} \tag{2.2.26}$$

This procedure, which is the straightforward one that generates an $U(1)_{\mathrm{PQ}} \times SU(3)_c^2$ anomaly without generating gauge anomalies (or said differently, that generates (2.2.21)), nonetheless displays important features shared by all other possible choices for the axion-fermions couplings. First, there is no freedom in using the axion profile to modify the effective scale of the axion-gluons coupling (here, it comes from the fact that we needed to add colored fermions at each site, in accordance with the fact that (2.2.21) involves all quiver links). Second, the number of additional fermions grows exponentially with N (this is obvious to check in (2.2.26)), or equivalently linearly with the efficiency of the protection. This observation cannot be qualitatively circumvented by modifying (2.2.26), since the new colored fermions must be such that they are unobserved at the LHC and that they do not make the QCD gauge coupling constant diverge before any scale up to which we would like a field theory incorporating QCD to be valid (e.g. the Planck scale, or the GUT scale). Figure 2.4 shows the outcome of such an analysis, performed in the full publication.

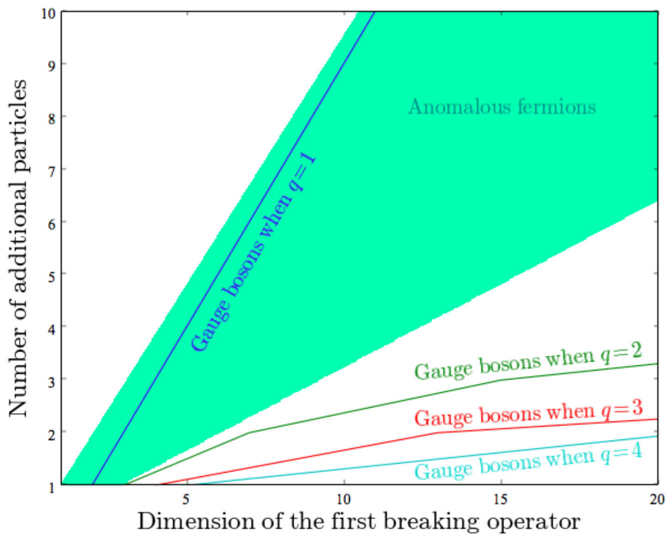


Figure 2.4: Number of additional particles, function of the first explicit breaking operator dimension

A consequence of this is that, in order to use (2.2.19) to bring a Planck scale f down to an intermediate scale $f_a = 10^{10-11}$ GeV, we need in (2.2.26) $\sim q^N \gtrsim 10^{7-8}$ additional fermions. This offers a last connection between the number of additional fermions and the ratio f/f_a.

If we do not care about large hierarchies but only about the protection of $U(1)_{\mathrm{PQ}}$, we can look at models with few additional particles and work out their phenomenology. For instance, if we consider the case $f \sim 10^{11}$ GeV, $q = 3$ and $N = 2$, which is enough to ensure the protection of $U(1)_{\mathrm{PQ}}$ as we said below (2.2.23), we get a (detectable) coupling to photons from (2.2.24), with $1 + 3 + 3^2 = 13$ additional Dirac fermions in the $\mathbf{3}$ of $SU(3)_c$:

$$\mathcal{L} \supset (8.6 \times 10^{12}\ \mathrm{GeV})^{-1} aF^{\mu\nu}\tilde{F}_{\mu\nu}\;. \tag{2.2.27}$$

The axion as a cosmological axion-like particle

Now, we study the cosmology of the axion (2.2.16) as an ALP, meaning as a pseudo-Goldstone boson not designed to solve the strong-CP problem, and whose interactions are consequently less constrained than those of the QCD axion. We focus on models where the ALP potential is entirely generated by perturbative physics in a UV theory[13], here gravitational physics ($M_c = M_P$ in (2.2.20)), which grants the ALP a small mass even for few quiver sites and which is sufficient to make it a good dark matter candidate.

Indeed, obtaining in our setup masses as low as those which appear in (2.1.34) or (2.1.29) without tuning is easy (for instance, (2.2.20) equals $\sim 10^{-33}$ eV when $f = 0.13M_P, q = 3, N = 4$). However, we saw previously that axion quintessence demands initial values which are higher than the Planck mass. This can be achieved with some tuning on a_0 or when the effective decay constant of the axion is increased compared to the mass scales of the model (as in clockwork models which, however, have no mass protection mechanism built in). Since our effective decay constant (2.2.19) is reduced, the latter is not an option while the former is not enough to reach the correct energy density (if we insist on keeping f below the Planck mass, as would be suggested by swampland conjectures, see section 4.4): indeed if we impose $m_a \lesssim 10^{-33}$ eV, we can only obtain $\Omega_a \lesssim 0.05$ and we would need at least 13 of such ALPs to reach the observed dark energy density. We thus do not go further into the analysis of axion quintessence in this setup.

In contrast, natural dark matter candidates do arise in our model. We scan the parameters f and q for some values of N which satisfy the condition that $\Omega_a = 0.3$ and (2.1.34). In (2.1.23), we write $a_0 = \epsilon_{\text{init}} f_a$ and we allow for ϵ_{init} to range from 0.1 to $\pi - 0.1$, and we include as well as a constant multiplying the potential (2.2.18) ranging from 0.033 to 30. The outcome is shown in Figure 2.5, in the (physical) (m_a, f_a) parameter space probed by our model. We also include the parameter space for the QCD axion (which, due to its temperature-dependent mass, differs for the one of other ALPs).

We see there that we obtain suitable DM candidates, and that the dependence on q and N of the mass (2.2.20) allows us to reach very low ALPs masses. These small masses, combined with the high scale f of their associated new physics, are hard to realize in a pure field theoretical framework and are usually thought of as coming from a string axiverse [55–57]. Our setup then provides an economical, in the sense of a low number of gauge groups, realization of such values. For instance, the smallest masses discussed in the literature for ultra light dark matter, $m_a \sim 10^{-21} - 10^{-22}$ eV, are obtained for $f \approx 0.2M_P$, $q = 3$ and $N = 4$. This example, as well as Figure 2.5, shows that a gravitational origin for (2.2.18) is sufficient to reproduce the cosmological relic density of dark matter.

In order to conclude that such ALPs are to play a role in the cosmic evolution, we must check that their lifetime can be comparable to or bigger than the age of the universe. We refer to the

[13] There could also be instantonic contributions to the potential, associated to a confining gauge group with a $U(1)_{\text{PQ}}$ anomaly. However, since the discussion of the previous section showed us that making $U(1)_{\text{PQ}}$ anomalous demands a large number of additional fermions in the theory, especially when N grows, we restrict ourselves to those ALPs which do not have any anomalous couplings for simplicity.

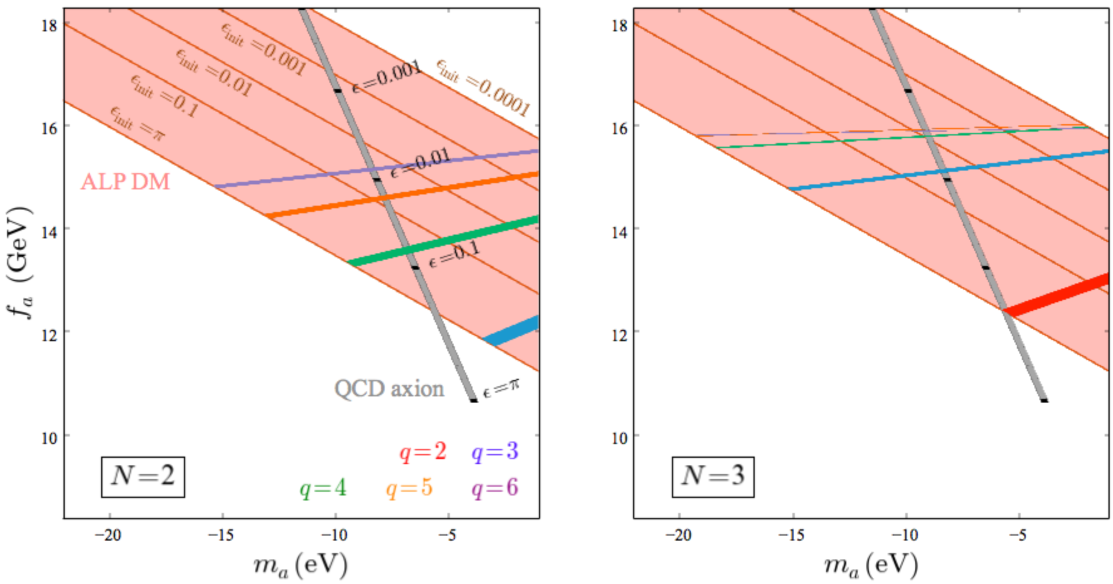


Figure 2.5: Parameter space for a DM ALP of mass $m_a \leq 10^{-2}$ eV
(The pink region indicates the parameter space where $\Omega_a = 0.3$, whereas colored bands show where DM axions are found in our model. The QCD axion parameter space is given by the grey line. Axes are log-scale)

full paper for more details on this, but the conclusion is that those axions are perfectly safe. It is due to the fact that the non-anomalous, CP-even and gauge invariant operators which could couple the axion to photons:

$$\Box a F\tilde{F}, \partial_\mu a \partial^\eta F^{\mu\nu}\tilde{F}_{\eta\nu} \text{ and } \partial_\mu a \partial^\eta \tilde{F}^{\mu\nu} F_{\eta\nu} \tag{2.2.28}$$

are very much suppressed. They are also too weak to be probed by current ALPs searches. On the other hand, there exist other operators which make the ALP detectable. Those are dimension five couplings of a to fermions, generically written [49] as follows:

$$\frac{g_{aee}}{f_a}\partial_\mu a\ \bar{e}\gamma^\mu\gamma_5 e \text{ and } \frac{g_{aNN}}{f_a}\partial_\mu a\ \overline{N}\gamma^\mu\gamma_5 N\ , \tag{2.2.29}$$

where g_{aXX}s are dimensionless coupling constants, f_a is again the axion decay constant, and N and e are respectively the nucleon and electron fields. In our setup, due to the quiver structure, it turns out that the couplings are naturally of the following magnitude:

$$\frac{-iq^i\partial_\mu a}{\sqrt{1+...+q^{2N}}f}(\overline{u}\gamma_5\gamma^\mu u + \overline{d}\gamma_5\gamma^\mu d + \overline{e}\gamma_5\gamma^\mu e)\ . \tag{2.2.30}$$

We do not discuss here details of how they are generated, but we note that, unlike anomalous couplings (2.2.21), the ALP-spin coupling of (2.2.30) is site dependent due to the clockwork profile (2.2.16). This is explained by the fact that they can be generated by matter localized at a single site of the quiver.

If the mass (2.2.20) of the ALP is such that it constitutes part of the dark matter, these couplings may soon be tested, for instance via Nuclear Magnetic Resonance (NMR) by the CASPEr-Wind experiment [92]. As an illustration, in Figure 2.6 we assume that the coupling (2.2.30) is located at site $i = 0$ of the quiver. We then see that CASPEr-Wind can detect some of the ALPs discussed in this paper (one example is for $f \lesssim 5 \times 10^{15}$ GeV, $q = 2$ and $N = 4$). Thus the present model, while invisible to experiments based on axion-photons couplings, can be probed and constrained by NMR-based ALPs searches.

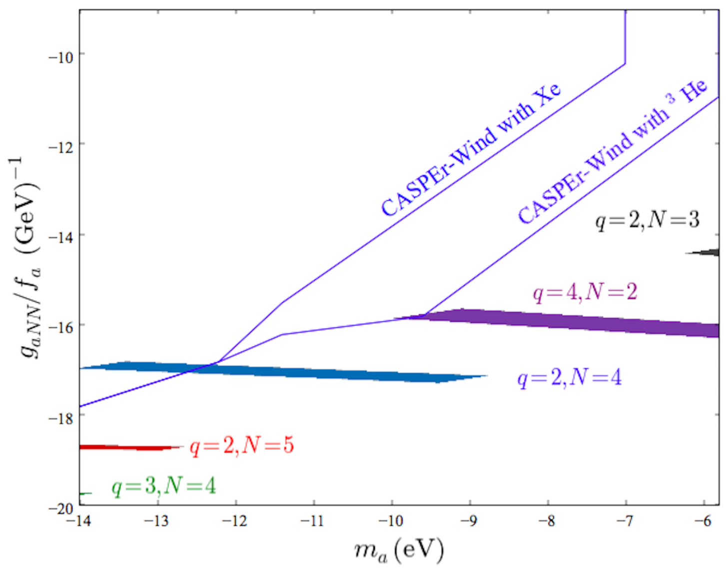


Figure 2.6: Sensitivity of CASPEr-Wind to the ALPs
(colored regions indicate axions suitable to saturate the DM relic density, blue curves set the limit of the upper left part of the plot where the sensitivity of CASPEr-Wind allows for a DM detection. Both axes are log-scale)

Summary

In this work, we were motivated by the concerns of section 2.1.8. We have thus studied a QCD axion or cosmological ALP in a model inspired by 4-dimensional clockwork models, with the global symmetry accidentally arising due to gauge symmetries in an abelian quiver with scalar bifundamental fields. We showed that with $\mathcal{O}(1)$ parameters, the mass induced by gravitational effects is exponentially damped, such that the protection is sufficiently strong.

For the QCD axion we have detailed the axion EFT and the quantitative conditions to have a protected PQ symmetry. We also remarked that the clockwork profile of the axion is useless as far as anomalous couplings are concerned. We finally mentioned the connection between the degree of protection of the axion mass, the explanation of the hierarchy $f_a \ll f$ and the number of colored fermions needed to generate anomalous couplings to gluons, all linked together by the underlying gauge symmetries.

In the DM ALPs models, assuming that their mass is solely given by gravitational corrections, we have identified the parameter space such that the scale f_a and the mass m_a combine to give the observed relic abundance. In the latter case, we have used gravitational corrections in a constructive way, without referring to any additional strongly interacting sector and its chiral anomalies. We saw that the ALP can be coupled to the standard model in a way which is sensitive to the clockwork profile of the axion, allowing our models to be for instance tested via Nuclear Magnetic Resonance experiments. Pseudo-Goldstone quintessence models of dynamical dark energy can also be obtained in such a setup, but their construction faces usual challenges, such as a trans-Planckian axion decay constant, in order to recover the observed energy density.

What we have not covered

More details about topics which were discussed above are presented in [176]. They are briefly listed below for the interested reader.

First, the model of Figure 2.3 can be understood as the deconstructed [174, 175, 178] version of a 5D abelian gauge theory in a linear dilaton background with Dirichlet boundary conditions for the 4D components of the gauge boson. The precise derivation is worked out.

Second, on top of the axion, there are massive vector and scalar modes in this model, whose profiles and masses are shown.

Third, we deepen the discussion on the relationship between the gauge protection of the axion, the hierarchy between the decay constant f_a and the scale of new physics f, and the number of necessary additional fermions to make $U(1)_{\mathrm{PQ}}$ anomalous. In particular, the number and the charges of the extra fermions are disentangled. From there, general conclusions on QCD axion models that use global symmetries that are consequences of gauge symmetries are drawn and tested on prototypical models such as [88, 133, 134]. The comparison with the latter models also helps to conclude that the model of Figure 2.3 is economical, meaning that it requires less extra particles and smaller input numbers than other models to achieve its goals.

Fourth, realizations of KSVZ- and DFSZ-like models are built, and non-gravitational explicit breaking of $U(1)_{\mathrm{PQ}}$ is discussed on this occasion. Precisely, we describe how fermionic operators explicitly breaking $U(1)_{\mathrm{PQ}}$, which may or may not be present in addition to (2.2.26) depending on the gauge charges, can be involved in the generation of (2.2.18) via a Coleman-Weinberg potential calculation.

Fifth, a complete scan over parameters which enter the discussion of ALP DM is performed. The dependence on M_c in (2.2.18) is for instance taken into account.

Last, explicit realizations of the ALP DM couplings to matter are presented, in order to show that such couplings can be generated with a small (q, N-independent) number of additional particles and that they can be site-localized along the quiver, hence being sensitive to the clockwork profile of the axion.

2.3 Flavour hierarchies and the Froggatt-Nielsen mechanism

We continue our treatment of the different motivations for BSM physics which we presented in the introduction by specifically studying the fermion mass and mixing hierarchies in the SM. This fine-tuning problem concerns the flavour sector of the SM, which has also been associated to several experimental versus theoretical discrepancies over the years, linked with e.g. the magnetic moment of the muon [10] or the decays of B-mesons (see e.g. [11] for latest results)

We make those hierarchies clear by first reviewing the flavour structure of the SM in section 2.3.1. We also present important aspects of it which have to do with CP violation in section 2.3.2 and flavour changing neutral currents in section 2.3.3. Section 2.3.4 is devoted to a slightly off-topic presentation of the notion of mass matrices textures. Then, we present the hierarchies in section 2.3.5 and a mechanism to naturally generate them, the Froggatt-Nielsen (FN) mechanism, in section 2.3.6. Some of its limitations, meant to be addressed later, are discussed in section 2.3.7.

2.3.1 Flavour structure of the SM: physical parameters

We thus start with a few words on the flavour structure of the SM. Examples of reviews are [179–181]. Our conventions for the fields of the SM are presented in appendix A.1.2. Then, with those conventions, the lagrangian of the SM is made out of kinetic terms, the Higgs potential and the Yukawa potential. We are interested in the latter, which reads:

$$\mathcal{L} \supset -(Y^u_{ji}\overline{u_{R,i}}HQ_{L,j} + Y^d_{ji}\overline{d_{R,i}}H^cQ_{L,j} + Y^e_{ji}\overline{e_{R,i}}H^cL_{L,j}) + h.c. \; . \tag{2.3.1}$$

The Ys are 3×3 matrices in flavour space, and their coefficients are free in the SM. They are not all observable though. Indeed, the phenomenological predictions are fully characterized by fermion masses $m^{u,d,e}_{i=1..3}$ and by the CKM matrix [182, 183]. Those are defined as follows.

Due to spontaneous electroweak symmetry breaking, the Higgs field H gets a vev, such that in unitary gauge, the relevant parametrization of the Higgs field is:

$$H = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \; . \tag{2.3.2}$$

Neglecting the Higgs boson h, and using a matrix notation in flavour space from now on, (2.3.1) reduces to

$$\mathcal{L} \supset -(\overline{u_L}M^u u_R + \overline{d_L}M^d d_R + \overline{e_L}M^e e_R) + h.c. \; , \tag{2.3.3}$$

with $M^{X=u,d,e} = \frac{v}{\sqrt{2}}(Y^X)^*$, are the mass matrices. The physical eigenstates are obtained by diagonalizing those mass terms. To do this, let us recall that any $n\times n$ complex matrix M can made diagonal with the help of a biunitary transformation (see e.g. chapter 4 of [184]):

$$\forall M \in \mathbb{M}(n,n), \exists V_L, V_R \text{ and } D, \text{ which verify} \begin{cases} V^{-1}_{L/R} = V^\dagger_{L/R} \\ D = \mathrm{diag}(m_i, i=1,...,n) \end{cases} \text{ such that } M = V_L D V_R^\dagger \; . \tag{2.3.4}$$

The m_is can all be taken real and positive. Using this, we write

$$M^X = V^X_L D^X V^{X\dagger}_R \; . \tag{2.3.5}$$

The physical fields are then defined as follows

$$X_L \longrightarrow V^X_L X_L, \; , \quad X_R \longrightarrow V^X_R X_R \; , \tag{2.3.6}$$

such that

$$(2.3.3) \longrightarrow -(\overline{u_L}D^u u_R + \overline{d_L}D^d d_R + \overline{e_L}D^e e_R) + h.c. \; . \tag{2.3.7}$$

The entries of the D^Xs, which are diagonal, define the $m^X_{i=1,..,3}$. On the other hand, the following happens for the couplings to the charged electroweak gauge bosons (contracting color indices for quarks, and generation indices for all fields):

$$\begin{aligned}
&\overline{Q_L}\gamma^\mu\left(\partial_\mu - i\left(W^1_\mu\frac{\sigma^1}{2}+W^2_\mu\frac{\sigma^2}{2}\right)\right)Q_L + \overline{L_L}\gamma^\mu\left(\partial_\mu - i\left(W^1_\mu\frac{\sigma^1}{2}+W^2_\mu\frac{\sigma^2}{2}\right)\right)L_L \\
&\quad= \overline{u_L}\gamma^\mu\partial_\mu u_L - \frac{i}{\sqrt{2}}\overline{u_L}\gamma^\mu W^+_\mu d_L + \overline{d_L}\gamma^\mu\partial_\mu d_L - \frac{i}{\sqrt{2}}\overline{d_L}\gamma^\mu W^-_\mu u_L \\
&\qquad+ \overline{\nu_L}\gamma^\mu\partial_\mu \nu_L - \frac{i}{\sqrt{2}}\overline{\nu_L}\gamma^\mu W^+_\mu e_L + \overline{e_L}\gamma^\mu\partial_\mu e_L - \frac{i}{\sqrt{2}}\overline{e_L}\gamma^\mu W^-_\mu \nu_L \\
&\quad\xrightarrow[(2.3.6)]{} \overline{u_L}\gamma^\mu\partial_\mu u_L - \frac{i}{\sqrt{2}}\overline{u_L}\gamma^\mu W^+_\mu V^{u\dagger}_L V^d_L d_L + \overline{d_L}\gamma^\mu\partial_\mu d_L - \frac{i}{\sqrt{2}}\overline{d_L}\gamma^\mu W^-_\mu V^{d\dagger}_L V^u_L u_L \\
&\qquad+ \overline{\nu_L}\gamma^\mu\partial_\mu \nu_L - \frac{i}{\sqrt{2}}\overline{\nu_L}\gamma^\mu W^+_\mu e_L + \overline{e_L}\gamma^\mu\partial_\mu e_L - \frac{i}{\sqrt{2}}\overline{e_L}\gamma^\mu W^-_\mu \nu_L\ ,
\end{aligned} \tag{2.3.8}$$

where σ^i are the Pauli matrices, and we made the transformation $\nu_L \longrightarrow V^e_L\nu_L$ on top of (2.3.6), using the fact that the neutrinos are massless in the SM and can be rotated without spoiling a diagonal mass term. On the other hand, we see that in the quark sector, the couplings to the gauge bosons are off-diagonal in flavour space. The matrix which appears there, defines the CKM matrix:

$$V_{\text{CKM}} \equiv V^{u\dagger}_L V^d_L = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} . \tag{2.3.9}$$

The CKM matrix is unitary, and some of the phases of its components can be absorbed into the left (and right) handed quark fields. Eventually, the irreducible information is contained in three mixing angles $\theta_{ij}(i,j=1,...,3, i<j)$ and one CP-violating phase δ, as shown in the following parametrisation of V_{CKM}:

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} , \tag{2.3.10}$$

where $c_{ij} \equiv \cos(\theta_{ij})$, $s_{ij} \equiv \sin(\theta_{ij})$.

The entries of the CKM matrix have been measured, and the fact that they form a unitary matrix can thus be tested. Those tests are usually displayed using a geometrical interpretation of the condition $V_{\text{CKM}}V^\dagger_{\text{CKM}} = \mathbb{1}$, which implies e.g.:

$$\frac{V_{ud}V^*_{ub}}{V_{cd}V^*_{cb}} + \frac{V_{td}V^*_{tb}}{V_{cd}V^*_{cb}} + 1 = 0\ , \tag{2.3.11}$$

and which can be understood as the fact that the points of the complex plane $\Omega = 0, A = 1, B = \overline{\rho} + i\overline{\eta} \equiv -\frac{V_{ud}V^*_{ub}}{V_{cd}V^*_{cb}}$ form a triangle with $\overrightarrow{AB} = \frac{V_{td}V^*_{tb}}{V_{cd}V^*_{cb}}$, as pictured in Figure 2.7. The tests of these relations are in good agreement with the theoretical expectations, as can be seen on Figure 2.8.

Of course, once the neutrinos are given masses to reproduce their oscillation pattern, there is also a need to define a "leptonic CKM matrix", called the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [185, 186]. We do not discuss it in this thesis, even though it could be of very nice interest in the models which will soon be presented. Anyway, it is important to realize that it represents the first set of BSM parameters ever measured.

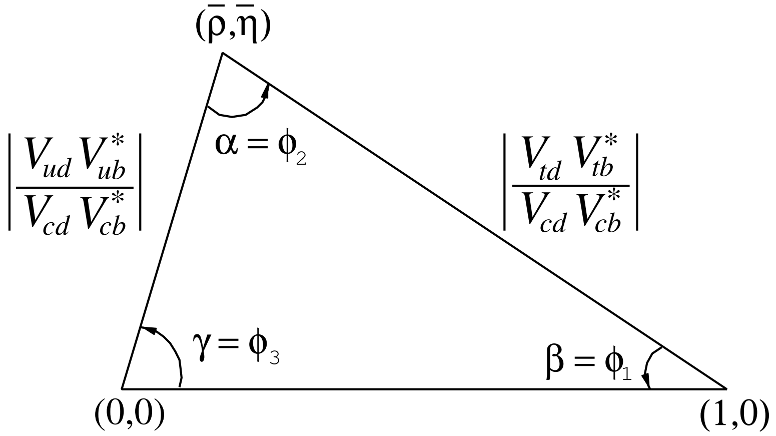


Figure 2.7: Geometrical representation of (2.3.11) in the $(\overline{\rho}, \overline{\eta})$ plane, copied with permission from [106]

2.3.2 CP violation

In the (perturbative) SM, the phase δ in (2.3.10) is the only CP-violating parameter. It leads to correlated CP violation in rare meson processes, which have been firmly observed, e.g. in [187] (see as well [106] for a complete list of tests), and remain up to date consistent with the SM prediction. It is interesting to notice that a third family of quarks was necessary to explain the violation of CP observed in neutral kaon decays [187], which led Kobayashi and Maskawa [183] to extend the CP-preserving two generations model of Cabibbo [182].

CP violation is important in cosmology, since it is one of the three Sakharov conditions [188] for a possible baryogenesis (i.e. a cosmological generation of the matter-antimatter asymmetry in the early universe). It is also an ingredient of many BSM scenarii. For instance, the PMNS matrix possibly contains a CP-violating phase (even three if the neutrino are Majorana particles) and the MSSM lagrangian incorporates fourty CP-breaking phases. Thus, tests of CP violation provide efficient probes of BSM physics.

An important example of experimental data which are very sensitive to possible BSM physics are the electric dipole moments of particles, already mentioned in section 2.1.1. Those are defined as the coefficients d_ψ of the following operators (see e.g. [189]):

$$-\frac{d_\psi}{2}\overline{\psi}\gamma^{\mu\nu}\gamma_5\psi F_{\mu\nu} \; , \tag{2.3.12}$$

with ψ a fermion and F the electromagnetic field strength. CP act as $d_\psi \to -d_\psi$ on EDMs. We already saw in section 2.1.1 that the θ-term induces an EDM for the neutron (which gives a strong bound on θ, at the source of the strong CP problem). It turns out that the CP violation in the flavour sector also generates such an EDM, but in a way which is restricted by the flavour structure of the SM: any CP-violating process induced by the flavour sector must depend on all the mixing angles and vanish if one of them does (since, as we said, one needs a mixing between at least three generations of quarks to get a physical CP-violating phase), and it must depend on δ, the CP-breaking phase. Consequently, one understands that the neutron EDM has to be

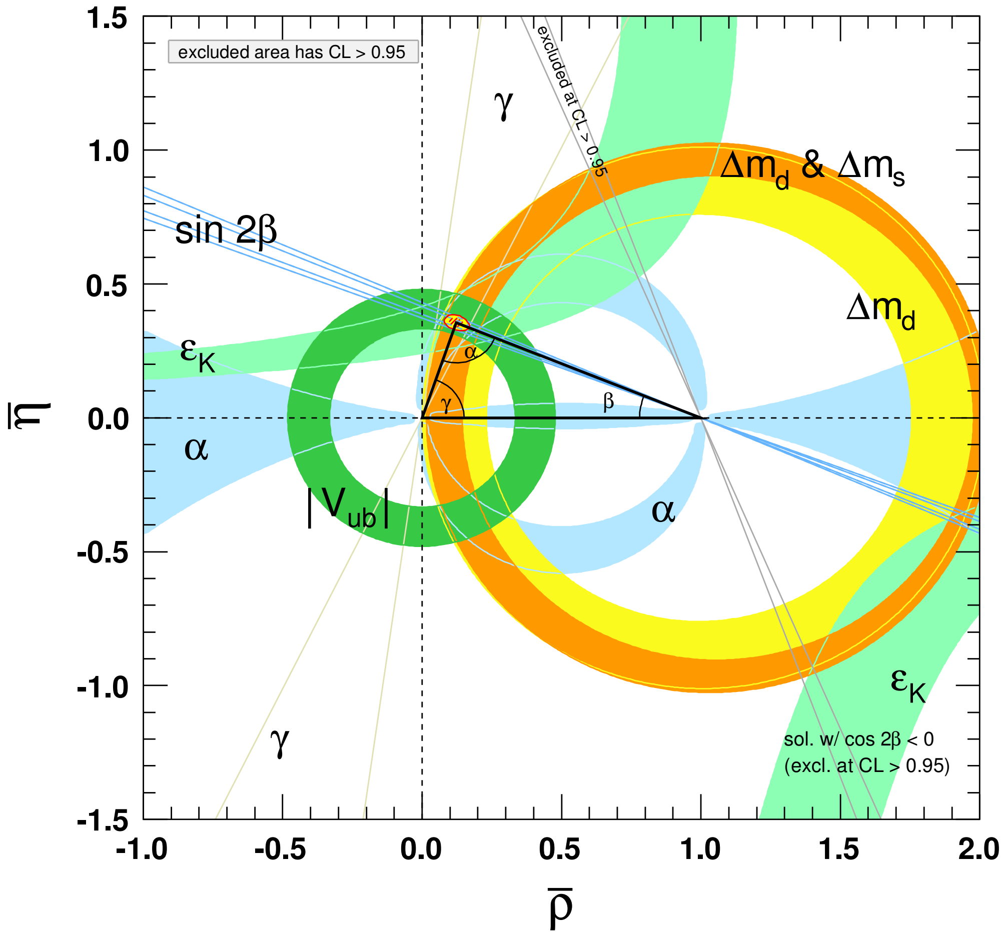


Figure 2.8: Global fit result from various experimental tests of (2.3.11) in the $(\bar{\rho}, \bar{\eta})$ plane, copied with permission from [106]

proportional to the Jarlskog invariant J [190], defined by

$$\mathrm{Im}(V_{ij}V_{kl}V_{il}^*V_{kj}^*) \equiv J(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}) \tag{2.3.13}$$

which is the first CP-breaking physical (i.e. independent on quark field rephasings) combination of elements of the CKM matrix which can be formed. It can be expressed in terms of the parameters in (2.3.10):

$$J = c_{12}c_{23}c_{13}^2 s_{12}s_{23}s_{13}\sin(\delta) \ , \tag{2.3.14}$$

which has been measured to be $J \sim 10^{-5}$. It is interesting to notice that, unlike the case of θ, it is small because of the arrangement of the cosines and sines of mixing angles (i.e. because of the flavour structure), while $\delta = \mathcal{O}(1)$. Of course, since this invariant arises via weak interaction diagrams in the calculation of the neutron EDM, there are also loop factors suppressing the result. Eventually, one gets a prediction for d_n from the flavour sector:

$$d_n \sim 10^{-32} e \cdot \text{ cm} \ , \tag{2.3.15}$$

which is still beyond the reach of current experiments (see e.g. [191, 192]).

2.3.3 The flavour changing neutral currents

The flavour structure of the SM enables other precision tests, which have to do with flavour changing neutral currents (FCNCs), or flavour changing neutral interactions, meaning the possibility of changing the flavour of a particle without modifying its electric charge. In the SM, the interactions with the uncharged electroweak gauge bosons, the gluons or the Higgs field are not sensitive to the redefinitions (2.3.6). Indeed, restricting us to to the quark fields and ignoring the gluons:

$$\begin{aligned}
&\overline{Q_L}\gamma^\mu\left(\partial_\mu - i\left(W^3_\mu\frac{\sigma^3}{2}+\frac{1}{3}B_\mu\right)\right)Q_L + \overline{u_R}\gamma^\mu\left(\partial_\mu - \frac{4i}{3}B_\mu\right)u_R + \overline{d_R}\gamma^\mu\left(\partial_\mu + \frac{2i}{3}B_\mu\right)d_R \\
&\quad + \overline{u_R}HY^uQ_L + \overline{d_R}H^cY^dQ_L \\
&= \overline{u_L}\gamma^\mu\left(\partial_\mu - i\left(\frac{W^3_\mu}{2}+\frac{1}{3}B_\mu\right)\right)u_L + \overline{d_L}\gamma^\mu\left(\partial_\mu - i\left(-\frac{W^3_\mu}{2}+\frac{1}{3}B_\mu\right)\right)d_L \\
&\quad + \overline{u_R}\gamma^\mu\left(\partial_\mu - \frac{4i}{3}B_\mu\right)u_R + \overline{d_R}\gamma^\mu\left(\partial_\mu + \frac{2i}{3}B_\mu\right)d_R + \left(1+\frac{h}{v}\right)\overline{u_R}M^{u\dagger}u_L + \left(1+\frac{h}{v}\right)\overline{d_R}M^{d\dagger}d_L \\
&\underset{(2.3.6)}{\longrightarrow} \text{ itself, except for } M^X \to D^X\ .
\end{aligned} \tag{2.3.16}$$

We understand from this that there is no tree-level FCNC in the standard model since every uncharged interaction is flavour-diagonal, and that the only flavour changing tree-level processes involve the emission of a charged $W^\pm_\mu$, as seen from (2.3.8).

On the other hand, loop diagrams such as the one of Figure 2.9 induce FCNCs. However,

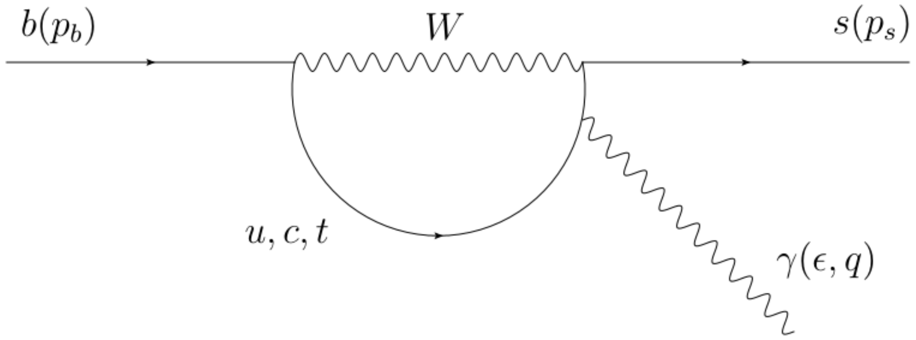


Figure 2.9: A one loop diagram inducing a FCNC

since such diagrams are very much suppressed in the SM due to the GIM mechanism [5], FCNCs set stringent bounds on many BSM scenarii.

2.3.4 Textures of the Yukawa matrices

As we said earlier, the entries of the Yukawa matrices are free parameters of the SM. There are many BSM models which predict or correlate those free parameters, in order to reduce the arbitrariness in the SM lagrangian, or at least which introduce model building tools to explain some observed but unexplained features of the SM parameters. We review some of those approaches now.

Before we really tackle flavour hierarchies, let us discuss briefly a slightly off-topic but interesting approach, which aims at increasing the predictivity of the SM. The idea is to impose constraints on the Yukawa matrices Y^X so that there are less free parameters than the usual ones. Thus, relations are set between the different angles, phases and masses, which can be tested.

The prototypical examples are due to Fritzsch [193, 194] (see e.g. [195] for a more recent treatment), and their physics can be understood at the level of a two-family case: if the quark mass matrices were such that (with $b^{X*} = b^X$)

$$M^u = \begin{pmatrix} 0 & A^u \\ A^{u*} & b^u \end{pmatrix} \ , \quad M^d = \begin{pmatrix} 0 & A^d \\ A^{d*} & b^d \end{pmatrix} \ , \tag{2.3.17}$$

we get, dropping the superscripts, assuming $b > 0$ and writing $A = ae^{i\alpha}$,

$$\begin{aligned} M &= \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a \\ a & b \end{pmatrix} \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+x_-^2}} & \frac{1}{\sqrt{1+x_+^2}} \\ -\frac{1}{\sqrt{1+x_+^2}} & \frac{1}{\sqrt{1+x_-^2}} \end{pmatrix}}_{V_L} \begin{pmatrix} m_- & 0 \\ 0 & m_+ \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+x_-^2}} & -\frac{1}{\sqrt{1+x_+^2}} \\ \frac{1}{\sqrt{1+x_+^2}} & \frac{1}{\sqrt{1+x_-^2}} \end{pmatrix} \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & 1 \end{pmatrix}}_{V_R^\dagger} \ , \end{aligned} \tag{2.3.18}$$

where $x_\pm \equiv \frac{m_\pm}{a} \equiv \frac{\sqrt{b^2+4a^2} \pm b}{2a}$, which verify $x_+ x_- = 1$, hence $x_\pm = \sqrt{\frac{m_\pm}{m_\mp}}$. Thus, reinstating the superscripts, we derive the C(KM) matrix:

$$V_{\text{CKM}} = V_L^{u\dagger} V_L^d = \begin{pmatrix} \cdots & \frac{e^{i(\alpha^d - \alpha^u)}}{\sqrt{1+x_-^{u2}}\sqrt{1+x_+^{d2}}} - \frac{1}{\sqrt{1+x_+^{u2}}\sqrt{1+x_-^{d2}}} \\ \cdots & \cdots \end{pmatrix} \equiv |_{\text{up to phases}} \begin{pmatrix} \cos(\theta_C) & -\sin(\theta_C) \\ \sin(\theta_C) & \cos(\theta_C) \end{pmatrix} \ , \tag{2.3.19}$$

and, using the hierarchies between the quark masses, one gets a prediction for the Cabibbo angle θ_C:

$$|\sin \theta_C| \approx \left| e^{i(\alpha^d - \alpha^u)} \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} \right| \ , \tag{2.3.20}$$

which is phenomenologically satisfying. (2.3.17) can for instance be justified by imposing a left-right symmetric gauge group $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, parity invariance as well as a global flavour abelian symmetry. Then, a correct phenomenology demands additional Higgs doublets and right-handed neutrinos in the model, so it comes with precise signatures.

One can extend this toy model to the (more useful) case of three generations, with an identical philosophy (but with more involved model building). There are also relations between the Yukawa matrices of the different particles once they are arranged in GUT multiplets (see e.g. [196]). We do not go further in this direction and we now discuss the flavour hierarchies.

2.3.5 The masses and mixings hierarchies

The fermion masses and the entries of the CKM matrix have been measured, and the relative differences between their magnitudes is what is usually referred to as the flavour hierarchies.

Indeed, the CKM matrix reads, in the Wolfenstein parametrisation [197]:

$$V_{\rm CKM} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4) \ , \tag{2.3.21}$$

where λ is linked to the Cabibbo angle: $\lambda = \sin(\theta_C) \approx 0.22$, and $A, \rho, \eta = \mathcal{O}(1)$. One sees clearly from this parametrisation that the entries of this matrix are not all $\mathcal{O}(1)$, not even of the same order of magnitude, although they are free parameters in the SM. For instance, it is easy to understand from this parametrisation why the Jarlskog invariant is small in the SM:

$$J \approx \lambda^6 A^2 \eta \ . \tag{2.3.22}$$

Even more strikingly, orders of magnitude for the quark and lepton masses can also be expressed in terms of λ:

$$\frac{m_u}{m_t} \sim \lambda^8 \ , \quad \frac{m_d}{m_t} \sim \lambda^7 \ , \quad \frac{m_s}{m_t} \sim \lambda^5 \ , \quad \frac{m_c}{m_t} \sim \lambda^4 \ , \quad \frac{m_b}{m_t} \sim \lambda^3 \ , \quad \frac{m_e}{m_\tau} \sim \lambda^5 \ , \quad \frac{m_\mu}{m_\tau} \sim \lambda^2 \ , \quad \frac{m_\tau}{m_t} \sim \lambda^3 \ . \tag{2.3.23}$$

The strong hierarchies between the particle masses, as well as the milder ones appearing in the CKM matrix, are unexplained input parameters in the SM. They can be traced back to hierarchies which must be present in the Yukawa matrices $Y^{u,d,e}$ of (2.3.1), and inversely, suitably chosen hierarchies in the Yukawa matrices induce such hierarchical masses and mixings.

In order to explain those features, many BSM flavour models for the mass hierarchies involve additional symmetries, whose nature and origin are diverse: they can be discrete [198], global abelian [195, 199–206], non-abelian [207] or both [208], local abelian [209–211], non-abelian [212, 213] or both [214–216]. [214] is an example of radiative generation [217], i.e. the possibility that low-masses fermions do not have tree level masses (for symmetry reasons), which are then only generated quantum mechanically from the masses of heavier fermions. Thus, they are suppressed by loop factors, which can explain the hierarchies. A nice aspect of this approach, somewhat reminiscent of our discussion on textures, is that some masses are calculable given a set of fundamental ones, leading to testable predictions. We do not study this further, and we turn now to the exploration of Froggatt-Nielsen (FN) models [218–220], which are leading candidates to account for the flavour hierarchies.

2.3.6 The Froggatt-Nielsen mechanism

Froggatt-Nielsen models address the origin of flavour hierarchies by means of a symmetry explanation: the masses and mixings arise after spontaneous breaking of a chiral symmetry, which forbids their existence when it is exact in the UV. For instance, one can postulate a global symmetry $U(1)_{\rm FN}$ acting on the different SM fields and on a complex scalar singlet ϕ. Then, $U(1)_{\rm FN}$ invariance of the Yukawa sector of the SM requires a dressing of the Yukawa matrices by powers of ϕ[14], for instance as follows:

$$\mathcal{L} \supset -Y^u_{ji}\left(\frac{\phi}{M}\right)^{n^u_{ij}} \overline{u_{R,i}} H Q_{L,j} - Y^d_{ji}\left(\frac{\phi}{M}\right)^{n^d_{ij}} \overline{d_{R,i}} H^c Q_{L,j} - Y^e_{ji}\left(\frac{\phi}{M}\right)^{n^e_{ij}} \overline{e_{R,i}} H^c L_{L,j} + h.c. \ , \tag{2.3.24}$$

[14] Powers of ϕ^* could also in principle appear, unless one talks about supersymmetric models.

where n_{ij}^X is the $U(1)_{\rm FN}$ charge of $\overline{X_{L,j}}X_{R,i}$ in units of the charge of ϕ, and M is a high scale of new physics, for instance the mass scale of heavy fermions which mix with the standard model ones, see later and in section 2.4 for explicit examples. Once $U(1)_{\rm FN}$ is spontaneously broken by a vacuum expectation value (vev) of ϕ, the hierarchies in the fermion mass matrices are naturally explained in terms of a small parameter $\epsilon = \left|\frac{\langle\phi\rangle}{M}\right|$, assumed to be $\sim \lambda$. For instance, it is possible to choose the n_{ij}^X such that, up to order one coefficients:

$$|Y^u| = \begin{pmatrix} \epsilon^8 & \epsilon^5 & \epsilon^3 \\ \epsilon^7 & \epsilon^4 & \epsilon^2 \\ \epsilon^5 & \epsilon^2 & 1 \end{pmatrix}, \quad \left|Y^d\right| = \begin{pmatrix} \epsilon^7 & \epsilon^6 & \epsilon^6 \\ \epsilon^6 & \epsilon^5 & \epsilon^5 \\ \epsilon^4 & \epsilon^3 & \epsilon^3 \end{pmatrix}, \quad |Y^e| = \begin{pmatrix} \epsilon^7 & \epsilon^6 & \epsilon^6 \\ \epsilon^6 & \epsilon^5 & \epsilon^5 \\ \epsilon^4 & \epsilon^3 & \epsilon^3 \end{pmatrix}, \tag{2.3.25}$$

where by $\left|Y^X\right|$, we mean the matrix obtained from Y^X by taking the absolute values of its elements. Then, such structures for the Yukawa matrices lead to phenomenologically acceptable masses and CKM matrix, with model building inputs of natural size. Indeed, to generate (2.3.25), one only needs $U(1)_{\rm FN}$ charges of order unity.

A question one can ask is: how to devise a UV model such that its low energy EFT is (2.3.24)? One could agnostically refer to a UV complete theory such as string theory (or explicitly compute in such a framework), in which case the correct EFT involves non-renormalizable terms as in (2.3.24), where the scale M is the scale at which the new features of the theory (such as the string nature of the particle excitations) kick in. One could also build renormalizable field theory models which effectively generate (2.3.24) at low energies. An example of the latter case goes as follows: postulate the existence of heavy fermions which mix with the SM fermions, via Yukawa-like couplings involving the field ϕ. Then, such mixings are restricted by $U(1)_{\rm FN}$ and naturally generate (2.3.24) once the heavy fermions are integrated out. Let us show in a simple case how this works (we leave the complete SM treatment for section 2.4): we want to generate a one-quark mass term

$$\mathcal{L} \supset -y\lambda^n \overline{Q_L} H Q_R + h.c. \,, \tag{2.3.26}$$

where we included an undetermined $\mathcal{O}(1)$ coefficient y and indicated the level of suppression of the mass term by λ^n, where n is an integer. The Higgs field H is here to mimic the SM case but we neglect any gauge quantum number. To explain the smallness of the mass term, FN would postulate that $\overline{Q_L}HQ_R$ has $U(1)_{\rm FN}$ charge $-n$ in units of the charge of a scalar field ϕ, for instance by assigning charges ($\phi : -1, Q_L : 0, Q_R : n, H : 0$), and understand (2.3.26) as generated by the spontaneous breaking of $U(1)_{\rm FN}$ due to the vev of ϕ:

$$(2.3.26) = -y\left(\frac{\langle\phi\rangle}{M}\right)^n \overline{Q_L} H Q_R + h.c. \,, \tag{2.3.27}$$

where we chose the potential of ϕ such that $\langle\phi\rangle \sim \lambda M$. To generate this from a renormalizable UV complete field theory, we can now introduce n pairs $(\psi_L^{i=1,\dots,n}, \psi_R^i)$ of fermions, which couple to Q and H as follows:

$$\mathcal{L} \supset -y_1\overline{Q_L}H\psi_R^1 - M_1\overline{\psi_R^1}\psi_L^1 - y_2\overline{\psi_L^1}\psi_R^2\phi - \dots - M_n\overline{\psi_R^n}\psi_L^n - y_{n+1}\overline{\psi_L^n}Q_R\phi + h.c. \,. \tag{2.3.28}$$

Such a specific lagrangian can be enforced by giving adequate $U(1)_{\rm FN}$ charges to the ψs, for instance ($\psi_L^i : i-1, \psi_R^i : i-1$). We assume that the M_is are high scales so that the fermions

ψ dynamics can be neglected when we look at SM processes. Thus, we integrate them out and eventually obtain

$$\mathcal{L} \supset -y_1 y_2 ... y_{n+1} \left(\frac{\phi}{M}\right)^n \overline{Q_L} H Q_R + h.c. \ , \tag{2.3.29}$$

where we assumed that all $M_i = M$ for simplicity, and which nicely reproduces the magnitude of (2.3.26). In this approach, we see that it is actually required that all the Yukawa couplings y_i are $\mathcal{O}(1)$ not to spoil the mechanism. On the other hand, their magnitude is not fixed and they can slightly modify the magnitude of the end result, thus its naive scaling with λ. Consequently, the FN symmetry $U(1)_{\text{FN}}$ is never completely fixed by the hierarchies in the Yukawa matrices (such as (2.3.25)), since one of the y_i could be of order λ.

A problematic feature of the FN mechanism is that there is no constraint on the scale M where the field ϕ and the fermion ψ approximately live, which could be a very high scale (e.g. close to the Planck scale), thus being associated to physics very hard to test. Luckily, precision tests in flavour physics are already able to constraint high scales.

The study of FN models has recently been revived by the focus on flavourful axions which arise in FN-like setups [199,204,205] and whose EFT is very much constrained by flavour physics [221–223]. Such flavourful axions can also be linked with dark matter studies [224]. We will come back to this point in section 2.4.2.

2.3.7 Challenges for the FN mechanism: gauging the FN symmetry

The nature of the FN symmetry is debatable, and the question of whether it can be gauged is raised, in particular in order to evade quantum gravity corrections which explicitly break global symmetries, as discussed in sections 2.1.8 and 4.4. Such gravitational breaking could in principle generate uncontrolled $U(1)_{\text{FN}}$-breaking Yukawa terms and spoil the symmetry-based hierarchies, such as (2.3.25), so one could be tempted to gauge $U(1)_{\text{FN}}$ to protect it against such effects. It was however shown that the standard model spectrum sometimes induces gauge anomalies when charged under a FN symmetry [225–228]. Indeed, we can consider

$$\left|v^6 \det(Y_u Y_d)\right| = m_u m_c m_t m_d m_s m_b \sim v^6 \lambda^{27} \ , \tag{2.3.30}$$

where v is the Higgs vev. On the other hand, if we consider for instance (2.3.24), we can compute

$$\begin{aligned}\det(Y_u Y_d) = &\left(Y^u_{11} Y^u_{22} Y^u_{33} \left(\frac{\phi}{M}\right)^{n^u_{11}+n^u_{22}+n^u_{33}} - Y^u_{11} Y^u_{23} Y^u_{32} \left(\frac{\phi}{M}\right)^{n^u_{11}+n^u_{23}+n^u_{32}} + ...\right) \\ &\times \left(Y^d_{11} Y^d_{22} Y^d_{33} \left(\frac{\phi}{M}\right)^{n^d_{11}+n^d_{22}+n^d_{33}} + ...\right) \sim \left(\frac{\phi}{M}\right)^{n^u_{11}+n^u_{22}+n^u_{33}+n^d_{11}+n^d_{22}+n^d_{33}} \ ,\end{aligned} \tag{2.3.31}$$

since we have e.g. $n^u_{ij} = \frac{q^{\text{FN}}_{U_{R,i}} - q^{\text{FN}}_{Q_{L,j}}}{q^{\text{FN}}_\phi}$, so $n^u_{23} + n^u_{32} = n^u_{22} + n^u_{33}$, etc. We recognize there the $SU(3)^2 \times U(1)_{\text{FN}}$ anomaly coefficient

$$A_3 \propto \sum_{i=1..3} (2 q^{\text{FN}}_{Q_{L,i}} - q^{\text{FN}}_{U_{R,i}} - q^{\text{FN}}_{D_{R,i}}) \propto n^u_{11} + n^u_{22} + n^u_{33} + n^d_{11} + n^d_{22} + n^d_{33} \ . \tag{2.3.32}$$

By comparing (2.3.30) and (2.3.31), we understand that (2.3.24) defines a $U(1)_{\rm FN}$ which has an $SU(3)^2 \times U(1)_{\rm FN}$ anomaly, hence which cannot be naively gauged. Actually, it should be stressed that this conclusion strongly depends on the "holomorphic" nature of (2.3.24), and it can be evaded if powers of $\overline{\phi}$ enter instead of ϕ in the Yukawa couplings. However, when one specializes to SUSY models (as [225–228] actually did), this option is not available and the issue with anomaly cancellation remains. Ways out would either extend the scalar sector, introduce additional heavy chiral fermions or rely on a Green-Schwarz-inspired mechanism [229]. We give examples of those solutions in the next section.

2.4 Minimal anomaly-free gauged FN models

In this section, which is based on ongoing work with E. Dudas and S. Pokorski, we address the concerns of section 2.3.7 and gauge the FN symmetry. Our final goal is to do so by assuming that the heavy fermions which participate in the FN mechanism (as in (2.3.28)) are chiral with respect to the (abelian) FN symmetry. It modifies the mixed anomalies with respect to the SM gauge group, which opens the possibility to gauge the FN symmetry without the need to introduce additional spectator fermions (nor using any other anomaly cancellation mechanism), while keeping mass matrices textures usually associated to anomalous flavour symmetries.

We start by reviewing in section 2.4.1 classical ways of getting rid of anomalies, which we dub "non-minimal" since they use two independent sectors, the (heavy fermionic) one which generate the FN mechanism, and a second one which takes care of the anomalies. Then, we turn in section 2.4.2 to the "minimal" case, by studying two precise anomaly-free models whose structure is typical of the models we investigated. We also show that there is an accidental Peccei-Quinn symmetry associated with a flavourful axion in some of those models, and we briefly discuss constraints on the model parameters arising from the axion phenomenology and the consistency of the model.

In this section, we focus on supersymmetric models (which will enforce holomorphic structures in the Yukawa sector), and more precisely on the minimal supersymmetric standard model (MSSM), whose flavour structure is now written in terms of chiral superfields in the superpotential[15]:

$$W = Y^u_{ij} Q_i H_u U_j + Y^d_{ij} Q_i H_d D_j + Y^e_{ij} L_i H_d E_j \ . \tag{2.4.1}$$

We do not discuss neutrino masses generation, even though it could easily be incorporated in our construction, for instance by adding right-handed neutrinos. However, such an addition would not modify the mixed anomalies or the Peccei-Quinn symmetry of the next sections, so we simply ignore it.

[15] We report a thorough discussion of supersymmetry to section 3. Our definitions and conventions for the MSSM superfields can be found in Table A.2.

2.4.1 Non-minimal models

In this section, we illustrate the usual ways, mentioned in section 2.3.7, of getting rid of the anomalies associated with the FN mechanism. We complete the SUSY generalisation of the latter:

$$W \supset Y_{ij}^u \left(\frac{\phi}{M}\right)^{n_{ij}^u} Q_i H_u U_j + Y_{ij}^d \left(\frac{\phi}{M}\right)^{n_{ij}^d} Q_i H_d D_j + Y_{ij}^e \left(\frac{\phi}{M}\right)^{n_{ij}^e} L_i H_d E_j \ , \tag{2.4.2}$$

in an anomaly-free[16] model, and we stick for concreteness to the following Yukawa matrices:

$$Y^u \left(\frac{\phi}{M}\right)^{n^u} = \begin{pmatrix} \epsilon^8 & \epsilon^5 & \epsilon^3 \\ \epsilon^7 & \epsilon^4 & \epsilon^2 \\ \epsilon^5 & \epsilon^2 & 1 \end{pmatrix}, \quad Y^d \left(\frac{\phi}{M}\right)^{n^d} = \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^3 \\ \epsilon^3 & \epsilon^2 & \epsilon^2 \\ \epsilon & 1 & 1 \end{pmatrix}, \quad Y^e \left(\frac{\phi}{M}\right)^{n^e} = \epsilon \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^3 \\ \epsilon^3 & \epsilon^2 & \epsilon^2 \\ \epsilon & 1 & 1 \end{pmatrix}, \tag{2.4.3}$$

which fit well the phenomenological values for masses and mixings when $\tan\beta$ is large.

First, we do so by adding some (heavy) spectator fields to the model. We choose for the MSSM fields the charges displayed in Table 2.1. It is then straightforward to see that the

	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$	$U(1)_{\text{FN}}$
ϕ	$\mathbf{1}$	$\mathbf{1}$	0	-1
H_u	$\mathbf{1}$	$\mathbf{2}$	1	-3
H_d	$\mathbf{1}$	$\mathbf{2}$	-1	-3
Q_i	$\mathbf{3}$	$\mathbf{2}$	$1/3$	$-17/6-(n_{11}^u - n_{i1}^u)$
U_j	$\overline{\mathbf{3}}$	$\mathbf{1}$	$-4/3$	$35/6 + n_{1j}^u$
D_j	$\overline{\mathbf{3}}$	$\mathbf{1}$	$2/3$	$35/6 + n_{1j}^d$
L_i	$\mathbf{1}$	$\mathbf{2}$	-1	$31/6-(n_{11}^e - n_{i1}^e)$
E_j	$\mathbf{1}$	$\mathbf{1}$	2	$-13/6 + n_{1j}^e$

Table 2.1: Gauge charges of the singlet and MSSM fields

$SU(3)_C^2 \times U(1)_{\text{FN}}$, $SU(2)_W^2 \times U(1)_{\text{FN}}$ and $U(1)_Y \times U(1)_{\text{FN}}^2$ anomalies cancel. However, there is then an irreducible $U(1)_Y^2 \times U(1)_{\text{FN}}$ anomaly coefficient, -144. We can now cancel this last anomaly by remarking that the superfields in Table 2.2 only feed into the $U(1)_Y^2 \times U(1)_{\text{FN}}$ anomaly coefficient, which they shift by $+144$, rendering the model anomaly-free. They are also unobservable since they can receive a mass via the following couplings:

$$W \supset \phi \left(\Psi_1, \Psi_2\right) Y \begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix} + \phi \left(\Psi_3, \Psi_4\right) Y' \begin{pmatrix} \tilde{\Psi}_3 \\ \tilde{\Psi}_4 \end{pmatrix} + \left(\tilde{\Psi}_1, \tilde{\Psi}_2\right) M \begin{pmatrix} \tilde{\Psi}_3 \\ \tilde{\Psi}_4 \end{pmatrix} \ , \tag{2.4.4}$$

where Y, Y' and M are two-by-two matrices. This is not the only possibility, one could for instance choose hypercharges ± 2 and add 8 other copies of those fermions to the theory.

Second, we use the 4D version of the Green-Schwarz mechanism [229] to cancel the anomalies (see also sections 4.3.2 and 4.3.4): we now choose to cancel the anomaly by including an axion degree of freedom with the following couplings:

$$\mathcal{L} \supset -\frac{1}{2}(\partial_\mu a - \delta_{\text{GS}} A_\mu^{FN})^2 + \frac{k_i a}{4} F^i \tilde{F}^i \ , \tag{2.4.5}$$

[16] We do not discuss the $U(1)_{\text{FN}}^3$ or $U(1)_{\text{FN}}\times$gravity anomalies, since those could be modified in a sector quite decoupled from the SM, e.g. if we added right-handed neutrinos with a $U(1)_{\text{FN}}$ charge.

	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$	$U(1)_{\rm FN}$
Ψ_1	$\mathbf{1}$	$\mathbf{1}$	6	1
Ψ_2	$\mathbf{1}$	$\mathbf{1}$	6	1
Ψ_3	$\mathbf{1}$	$\mathbf{1}$	-6	1
Ψ_4	$\mathbf{1}$	$\mathbf{1}$	-6	1
$\tilde{\Psi}_1$	$\mathbf{1}$	$\mathbf{1}$	-6	0
$\tilde{\Psi}_2$	$\mathbf{1}$	$\mathbf{1}$	-6	0
$\tilde{\Psi}_3$	$\mathbf{1}$	$\mathbf{1}$	6	0
$\tilde{\Psi}_4$	$\mathbf{1}$	$\mathbf{1}$	6	0

Table 2.2: Gauge charges of the spectator fields

where F^i symbolizes the field strength of any gauge boson. The axion undergoes a shift $a \to a + \delta_{\rm GS}\alpha$ under $U(1)_{\rm FN}$, which cancels an anomaly-induced variation of $\mathcal{L}$ if $k_i = \frac{{\rm Tr}\left(q_{\rm FN} T^{i2}\right)}{4\pi^2} \equiv C_i$, the $G_i^2 \times U(1)_{\rm FN}$ anomaly coefficient. If the k_i were free parameters, it would be easy to cancel anomalies. However, their known UV origins restrict their possible values. Indeed, they can arise after having integrated an anomalous fermion set such that the whole theory is anomaly-free [230], which amounts to finding anomaly-free fermion sets as we did just above. An other possible UV origin are superstring theories [211, 231] where there are constraints on the k_i. For instance, in heterotic superstring theory the gauge couplings and the equivalent of (2.4.5) come from the following couplings of the dilaton superfield, written here in supersymmetric language:

$$\mathcal{L} \supset \int d^4\theta \ln\left(S + \overline{S} - \delta_{\rm GS} V^{\rm FN}\right) + \int d^2\theta \frac{k_i S}{4} W^{i,2} + h.c. \ . \tag{2.4.6}$$

Both the gauge coupling and the anomalous couplings arise from the dilaton superfield: $g_i^2 = \frac{1}{k_i \langle S \rangle}$. If the couplings unify at the string scale, one gets $k_i = k_j$ and the anomalies must verify $C_i = C_j$[17]. Such conditions can be satisfied if we choose for instance $X_Q = \frac{4553}{1722}, X_L = \frac{4217}{1722}, h_u = h_d = -\frac{57}{7}$. Then, the $U(1)_Y \times U(1)_{\rm FN}^2$ anomaly vanishes and the three $U(1)_Y^2 \times U(1)_{\rm FN}$, $SU(2)_W^2 \times U(1)_{\rm FN}$ and $SU(3)_C^2 \times U(1)_{\rm FN}$ anomaly coefficients are equal (up to the normalization of $U(1)_Y$ discussed previously).

However, those solutions are decoupled from the generation of (2.4.2): if some (vector-like) fields are used to explain the suppression factors in (2.4.3), the additional ones which remove the anomalies are independent of them, as in Table 2.2.

There are also solutions which do not demand any additional field beyond the MSSM ones which contributes to the anomalies. For instance, it was shown in [228] that if (2.4.2) involved two suppression factors ϵ_1 and ϵ_2, associated to singlet fields entering the Yukawa matrices in ratios $\frac{\phi_1}{M}$ and $\frac{\phi_2}{M}$, the anomalies could be canceled without any anomalous additional sector.

[17] With our conventions, the hypercharge gauge coupling is not the one which unifies and the correct relation is $C_3 = C_2 = \frac{3}{20} C_Y$.

2.4.2 Minimal chiral models

A key assumption in (2.4.2), which we now relax, is that the only low-energy contribution of the heavy sector at scale M is the generation of the Yukawa terms. This is true if the heavy sector is vector-like with respect to the SM gauge group, but if it is chiral there could also be in the EFT anomalous couplings between (the longitudinal component of) the $U(1)_{\mathrm{FN}}$ gauge field and the SM gauge bosons [230]. In this section, we explore this possibility.

We focus on models with two singlet superfields ϕ_1 and ϕ_2, which respectively replace the flavon ϕ and the mass M in (2.4.2), such that the Yukawa sector is as follows:

$$W \supset Y^u_{ij}\left(\frac{\phi_1}{\phi_2}\right)^{n^u_{ij}} Q_i H_u U_j + Y^d_{ij}\left(\frac{\phi_1}{\phi_2}\right)^{n^d_{ij}} Q_i H_d D_j + Y^e_{ij}\left(\frac{\phi_1}{\phi_2}\right)^{n^e_{ij}} L_i H_d E_j \;, \tag{2.4.7}$$

and we allow in particular ϕ_2 to be charged under $U(1)_{\mathrm{FN}}$. The charges of the superfields which appear in (2.4.7) can be found in Table 2.3. We again assume $\langle\phi_1\rangle = \epsilon\langle\phi_2\rangle$, with $\epsilon \approx \lambda$, and

	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$	$U(1)_{\mathrm{FN}}$
ϕ_1	$\mathbf{1}$	$\mathbf{1}$	0	$-x_1$
ϕ_2	$\mathbf{1}$	$\mathbf{1}$	0	$-x_2$
H_u	$\mathbf{1}$	$\mathbf{2}$	1	h_u
H_d	$\mathbf{1}$	$\mathbf{2}$	-1	h_d
Q_i	$\mathbf{3}$	$\mathbf{2}$	1/3	$X_Q - (x_1 - x_2)(n^u_{11} - n^u_{i1})$
U_j	$\overline{\mathbf{3}}$	$\mathbf{1}$	$-4/3$	$-X_Q - h_u + (x_1 - x_2)n^u_{1j}$
D_j	$\overline{\mathbf{3}}$	$\mathbf{1}$	2/3	$-X_Q - h_d + (x_1 - x_2)n^d_{1j}$
L_i	$\mathbf{1}$	$\mathbf{2}$	-1	$X_L - (x_1 - x_2)(n^e_{11} - n^e_{i1})$
E_j	$\mathbf{1}$	$\mathbf{1}$	2	$-X_L - h_d + (x_1 - x_2)n^e_{1j}$

Table 2.3: Gauge charges of the singlet and MSSM fields,
X_Q is the $U(1)_{\mathrm{FN}}$ charge of Q_1, X_L the $U(1)_{\mathrm{FN}}$ charge of L_1

formulas to follow will encompass cases where ϕ_1 or ϕ_2 is uncharged and equivalent to a mass M. However, we always impose $x_1 \neq x_2$ such that $U(1)_{\mathrm{FN}}$ acts non-trivially on the MSSM fields.

The contribution of the MSSM fields to the mixed anomaly coefficients are as follows:

$$\begin{aligned}
SU(3)^2_C \times U(1)_{\mathrm{FN}}:\ A_{3,\mathrm{SM}} =& -3(h_u + h_d) + (x_1 - x_2)\sum_i (n^u_{ii} + n^d_{ii}) \\
SU(2)^2_W \times U(1)_{\mathrm{FN}}:\ A_{2,\mathrm{SM}} =&\ 3(3X_Q + X_L) + h_u + h_d \\
&- (x_1 - x_2)\Big(3(2n^u_{11} - n^u_{21} - n^u_{31}) + 2n^e_{11} - n^e_{21} - n^e_{31}\Big) \\
U(1)^2_Y \times U(1)_{\mathrm{FN}}:\ A_{1,\mathrm{SM}} =& -6(3X_Q + X_L) - 14(h_u + h_d) \\
&+ 4(x_1 - x_2)\Bigg(\frac{n^u_{21} + n^u_{31} - 2n^u_{11}}{6} + \frac{4(n^u_{11} + n^u_{12} + n^u_{13})}{3} \\
&\qquad + \frac{n^d_{11} + n^d_{12} + n^d_{13}}{3} + \frac{2n^e_{12} + 2n^e_{13} + n^e_{21} + n^e_{31}}{2})\Bigg) \\
U(1)_Y \times U(1)^2_{\mathrm{FN}}:\ A'_{1,\mathrm{SM}} =&\ ...
\end{aligned} \tag{2.4.8}$$

and they are generally non-vanishing, since the discussion of section 2.3.7 still applies.

Heavy sector and anomaly cancellation

We now design a UV theory which generates (2.4.7) in the IR, closely following the original FN picture. Thus, we introduce the heavy fermions shown in Table 2.4, vector-like under the SM gauge group but chiral with respect to $U(1)_{\mathrm{FN}}$, and which generate the masses of the first generation fermions. Those fields together with the MSSM fields form a renormalizable UV theory, with a superpotential formed of (here only for the first generation)

	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$	$U(1)_{\mathrm{FN}}$
Common to all quarks				
$\Psi^Q_{i=1,\ldots,n_{Q,1}\leq\max[n^u_{11},n^d_{11}]}$	$\overline{\mathbf{3}}$	$\mathbf{2}$	$-1/3$	$-X_Q+i(x_1-x_2)+x_2$
$\tilde{\Psi}^Q_{i=1,\ldots,n_{Q,1}\leq\max[n^u_{11},n^d_{11}]}$	$\mathbf{3}$	$\mathbf{2}$	$1/3$	$X_Q-i(x_1-x_2)$
For U's or D's				
$\Psi^u_{i=n_{Q,1}+1,\ldots,n^u_{11}}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$-4/3$	$-X_Q+(i-1)(x_1-x_2)-h_u$
$\tilde{\Psi}^u_{i=n_{Q,1}+1,\ldots,n^u_{11}}$	$\mathbf{3}$	$\mathbf{1}$	$4/3$	$X_Q-(i-1)(x_1-x_2)+x_2+h_u$
$\Psi^d_{i=n_{Q,1}+1,\ldots,n^d_{11}}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$2/3$	$-X_Q+(i-1)(x_1-x_2)-h_d$
$\tilde{\Psi}^d_{i=n_{Q,1}+1,\ldots,n^d_{11}}$	$\mathbf{3}$	$\mathbf{1}$	$-2/3$	$X_Q-(i-1)(x_1-x_2)+x_2+h_d$
For E's				
$\Psi^L_{i=1,\ldots,n_{L,1}\leq,n^e_{11}}$	$\mathbf{1}$	$\mathbf{2}$	1	$-X_L+i(x_1-x_2)+x_2$
$\tilde{\Psi}^L_{i=1,\ldots,n_{L,1}\leq,n^e_{11}}$	$\mathbf{1}$	$\mathbf{2}$	-1	$X_L-i(x_1-x_2)$
$\Psi^e_{i=n_{L,1}+1,\ldots,n^e_{11}}$	$\mathbf{1}$	$\mathbf{1}$	2	$-X_L+(i-1)(x_1-x_2)-h_d$
$\tilde{\Psi}^e_{i=n_{L,1}+1,\ldots,n^e_{11}}$	$\mathbf{1}$	$\mathbf{1}$	-2	$X_L-(i-1)(x_1-x_2)+x_2+h_d$

Table 2.4: Gauge charges of the FN heavy fermions

$$\begin{aligned} W \supset \phi_1 \tilde{\Psi}^X_i \Psi^X_{i+1} \text{ (meaning, e.g., } \phi_1 \tilde{\Psi}^Q_i \Psi^Q_{i+1\leq n_{Q,1}}), \quad \phi_2 \tilde{\Psi}^X_i \Psi^X_i \ , \\ H_u \tilde{\Psi}^Q_{n_{Q,1}} \Psi^u_{n_{Q,1}+1} \ , \quad H_d \tilde{\Psi}^Q_{n_{Q,1}} \Psi^d_{n_{Q,1}+1} \ , \quad H_d \tilde{\Psi}^L_{n_{L,1}} \Psi^e_{n_{L,1}+1} \ . \end{aligned} \tag{2.4.9}$$

where the Ψ^X and $\tilde{\Psi}^X$ in (2.4.9) can also be understood as being MSSM fields according to the following replacement rules:

$$Q_1 \leftrightarrow \tilde{\Psi}^Q_0 \ , \quad U_1 \leftrightarrow \Psi^u_{n^u_{11}+1} \ , \quad D_1 \leftrightarrow \Psi^d_{n^d_{11}+1} \ , \quad L_1 \leftrightarrow \tilde{\Psi}^L_0 \ , \quad E_1 \leftrightarrow \Psi^e_{n^e_{11}+1} \ . \tag{2.4.10}$$

Those couplings are (generically) the only ones one can write at renormalizable order and they are precisely the one needed to generate (2.4.7), via diagrams such as the one of Figure 2.10. Mixings to other generations can be similarly implemented via couplings between e.g. $Q_{i>1}$ and one of the $(\phi_1)\Psi^Q$ (again, see Figure 2.10 for an example of a diagram which results). However, in order to have mass matrices of rank 3 each, we need to supplement the FN fields of Table 2.4 by their equivalent for the second and third families (see e.g. [220, 232]), in which case the indices i in Table 2.4 range between 1 and $n^u_{22}, n^d_{22}, n^e_{22}$ for the second family, and between 1 and $n^u_{33}=0, n^d_{33}, n^e_{33}$ for the third one. The charges X_Q and X_L in Table 2.4 should also be replaced by $X_Q-(x_1-x_2)(n^u_{11}-n^u_{21})$ and $X_L-(x_1-x_2)(n^e_{11}-n^e_{21})$ for the second family, or by by $X_Q-(x_1-x_2)(n^u_{11}-n^u_{31})$ and $X_L-(x_1-x_2)(n^e_{11}-n^e_{31})$ for the third one.

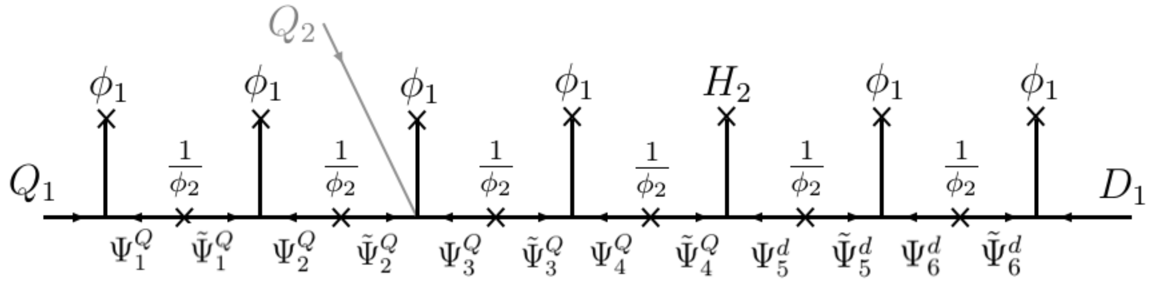


Figure 2.10: Tree diagram generating the d-quark mass, when $n^d_{11} = 6$ and $n_{Q,1} = 4$
The gray lines indicates how it should be modified to generate a mixing to Q_2 when $n^d_{21} = 4$

The contribution of the FN fields to the mixed anomaly coefficients are as follows:

$$
\begin{aligned}
A_{3,\mathrm{FN}} &= x_2\Big(2(n_{Q,1} + n_{Q,2} + n_{Q,3}) + \max(n^u_{11} - n_{Q,1}, 0) + \max(n^d_{11} - n_{Q,1}, 0) \\
&\qquad + \max(n^u_{22} - n_{Q,2}, 0) + \max(n^d_{22} - n_{Q,2}, 0) + \max(n^d_{33} - n_{Q,3}, 0)\Big) \\
A_{2,\mathrm{FN}} &= x_2\Big(3(n_{Q,1} + n_{Q,2} + n_{Q,3}) + n_{L,1} + n_{L,2} + n_{L,3}\Big) \\
A_{1,\mathrm{FN}} &= x_2\Bigg(\frac{2}{3}(n_{Q,1} + n_{Q,2} + n_{Q,3}) + 8(n_{L,1} + n_{L,2} + n_{L,3}) \\
&\qquad + \frac{16}{3}[\max(n^u_{11} - n_{Q,1}, 0) + \max(n^u_{22} - n_{Q,2}, 0)] \\
&\qquad + \frac{4}{3}[\max(n^d_{11} - n_{Q,1}, 0) + \max(n^d_{22} - n_{Q,2}, 0) + \max(n^d_{33} - n_{Q,3}, 0)] \\
&\qquad + 4(\max(n^e_{11} - n_{L,1}, 0) + \max(n^e_{22} - n_{L,2}, 0) + \max(n^e_{33} - n_{L,3}, 0))\Bigg) \\
A'_{1,\mathrm{FN}} &= ...
\end{aligned}
\tag{2.4.11}
$$

Hence, we understand that the integrating out of those FN fields generate in addition to (2.4.7) the following anomalous axionic term in the lagrangian[18]

$$
W \supset \int d^2\theta \left(\frac{A_{3,\mathrm{FN}}}{32\pi^2 x_2} \log(\phi_2)(W^a)^2 + ... \right) \ , \tag{2.4.12}
$$

where we only displayed the consequence of the QCD anomaly.

Anomaly-free models

The presence of (2.4.12) allows one to build "minimal" models where the fermions which participate in the FN mechanism, meaning those which are necessary to generate the hierarchies in masses and mixings, are sufficient to make the model anomaly-free, providing what could be called a minimal anomaly-free gauged FN model. We will not study thoroughly all possible models which achieve this, but, as proofs of principle, we restrict to two specific models.

The first one has only one singlet field ϕ_2 (and corresponds to a case where $x_1 = 0$, hence $\phi_1 = M$). It reproduces the Yukawa matrices of (2.4.3). When the FN superfields do not feature

[18] For an explicit derivation, see e.g. [230] or the appendix B. There are also (scheme-dependent) anomalous three vector couplings in the low-energy EFT, but we do not show them here for simplicity, since we will mostly focus on the axion phenomenology later on.

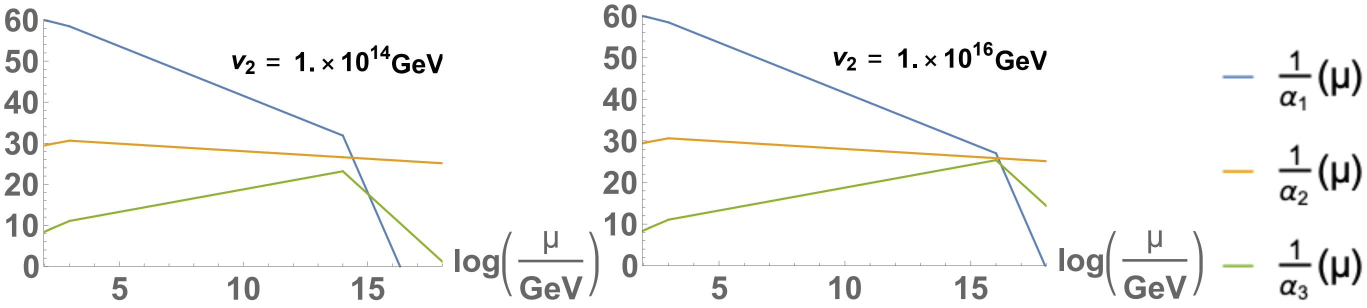


Figure 2.11: Running coupling constants of the SM, assuming $m_{\rm soft} = $ TeV

any doublet (i.e. $n_{Q,i} = n_{L,i} = 0$), choosing $h_u = h_d = 0$ and $x_2 = -\frac{3(3X_Q+X_L)}{16}$ makes all anomalies vanish (and the μ-term $\mu H_u H_d$ is allowed in the superpotential). This amounts to the usual FN model, with the exception that $\frac{\phi}{M}$ is replaced by $\frac{M}{\phi}$. If one insists on using ϕ_1 as a dynamical scalar, it is a pure singlet and there will be terms such as ϕ_1^n in the superpotential. There is no light degree of freedom in the FN sector in this scenario, which can be constrained by the running of gauge couplings. Assuming that all the superpartners kick in at a TeV and all the heavy superfields at a high scale $v_2 = \langle\phi_2\rangle$, Figure 2.11 shows the SM gauge coupling running for $v_2 = 10^{14}$ and 10^{16} GeV respectively. We see there that the hypercharge Landau pole, if it is to be below the Planck mass, imposes $v_2 \geq 10^{16}$ GeV.

On our second model, we impose the condition that the heavy FN fields should respect the qualitatively satisfying gauge coupling unification obtained in the MSSM, which can be conserved if the FN fields contribute to the (SM gauge coupling) running as $SU(5)$ multiplets (although with different $U(1)_{\rm FN}$ charges within a same "$SU(5)$ multiplet"). We thus demand that[19]

$$\begin{aligned} n_{Q,1}+n_{Q,2}+n_{Q,3} &= \max(n^u_{11}-n_{Q,1},0)+\max(n^u_{22}-n_{Q,2},0) \\ &= \max(n^e_{11}-n_{L,1},0)+\max(n^e_{22}-n_{L,2},0)+\max(n^e_{33}-n_{L,3},0)\ , \\ n_{L,1}+n_{L,2}+n_{L,3} &= \max(n^d_{11}-n_{Q,1},0)+\max(n^d_{22}-n_{Q,2},0)+\max(n^d_{33}-n_{Q,3},0)\ . \end{aligned} \tag{2.4.16}$$

One can check that we need this time two singlets ϕ_1 and ϕ_2, if we insist on not using additional

[19] This condition can be rewritten in terms of the standard model anomalies: with our conventions, $\frac{A_{A,\rm heavy}}{x_2}$ counts the heavy chiral fields which are charged under the gauge factor A (with multiplicity and charge squared for an abelian gauge factor). Thanks to the holomorphicity in our SUSY model, the same anomaly coefficients appear in the β-function for the gauge couplings:

$$\frac{1}{g_A^2(\mu)} = \frac{1}{g_A^2(\mu_0)} + \frac{b_A^{\rm SM}}{8\pi^2}\log\left(\frac{\mu}{\mu_0}\right) - \mathcal{B}\frac{A_{A,\rm heavy}}{64\pi^2}\log\left(\frac{\mu}{v_2}\right)\ , \tag{2.4.13}$$

where $\mathcal{B} = 1, 2$ respectively for a $SU(N)$ or an abelian factor of the gauge group. Respecting the gauge unification of the MSSM thus demands

$$A_{SU(3)_C,\rm heavy} = A_{SU(2)_W,\rm heavy} = \frac{3}{10}A_{U(1)_Y,\rm heavy}\ , \tag{2.4.14}$$

which, for anomaly-free models, is extended to

$$A_{SU(3)_C,\rm SM} = A_{SU(2)_W,\rm SM} = \frac{3}{10}A_{U(1)_Y,\rm SM}\ . \tag{2.4.15}$$

This relation agrees well with phenomenological mass matrices, and is the one required to implement the Green-Schwarz mechanism [225–228], consistently with the fact that the phase θ_2 of ϕ_2 does generate a GS mechanism here, once the heavy fields are integrated out.

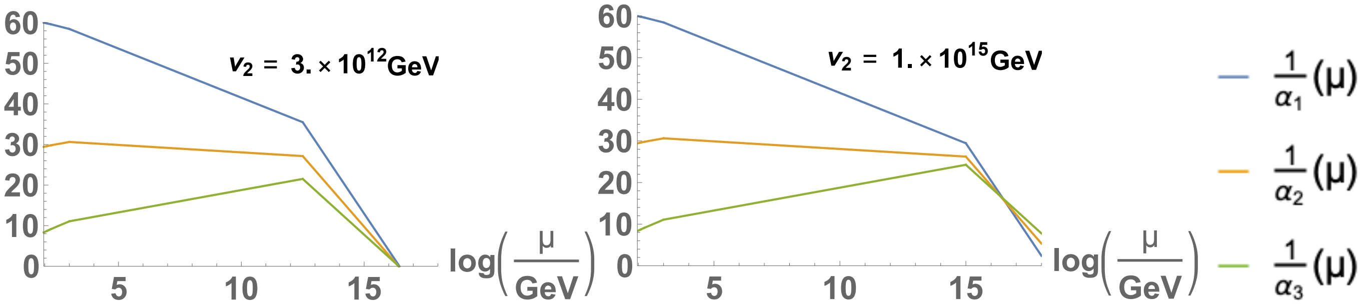


Figure 2.12: Running coupling constants of the SM, assuming $m_{\text{soft}} = \text{TeV}$

spectator fields beyond the ones which enter the FN mechanism. Choosing $x_1 = 1, x_2 = 10$, $h_u = h_d = \frac{9}{2}, X_Q = -\frac{67}{2}, X_L = -\frac{39}{2}$ and $n_{Q,1} = 4, n_{Q,2} = 2, n_{Q,3} = 0, n_{L,1} = 0, n_{L,2} = 0, n_{L,3} = 0$, all the anomalies vanish[20] and we obtain the following mass matrices (which reproduce the correct masses and mixings up to two $\mathcal{O}(\lambda)$ deviations [228])

$$Y^u \left(\frac{\phi}{M}\right)^{n^u} = \begin{pmatrix} \epsilon^8 & \epsilon^5 & \epsilon^4 \\ \epsilon^7 & \epsilon^4 & \epsilon^3 \\ \epsilon^4 & \epsilon & 1 \end{pmatrix}, \quad Y^d \left(\frac{\phi}{M}\right)^{n^d} = \begin{pmatrix} \epsilon^3 & \epsilon^3 & \epsilon^4 \\ \epsilon^2 & \epsilon^2 & \epsilon^3 \\ (\bar{\epsilon}) & (\bar{\epsilon}) & 1 \end{pmatrix}, \quad Y^e \left(\frac{\phi}{M}\right)^{n^e} = \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^3 \\ \epsilon^3 & \epsilon^2 & \epsilon^2 \\ \epsilon & 1 & 1 \end{pmatrix}, \tag{2.4.18}$$

where by the parenthesis in the last row of Y^d, we mean that those entries are forbidden by holomorphicity. However, they might be generated after field redefinitions to take care of corrections to the Kähler potential [228]. We nevertheless leave them in (2.4.18), since they indicate what we choose for the charges of the different fields. Furthermore, notice that, in order to generate the $(1,3)$ and $(2,3)$ entries of Y^d, the heavy sector in Table 2.4 should be modified such that, for instance, the index i for the d-quark-like heavy fields of the first generation is bounded by n^d_{13} instead of n^d_{11}.

In this model, the μ-term is forbidden and should be generated from the Kähler potential via the Giudice-Masiero mechanism [233], by writing

$$K \supset \frac{1}{\Lambda^2} H_u H_d \phi_2 \overline{\phi_1} \,, \tag{2.4.19}$$

assuming ϕ_1 has a non-vanishing F-term. With the same assumptions as before, Figure 2.12 shows the SM gauge coupling running for $v_2 = 3 \times 10^{12}$ and 10^{15} GeV respectively. Here, we see that the hypercharge Landau pole imposes $v_2 \gtrsim 10^{14-15}$ GeV. If we only impose that the unification happens before any Landau pole, we find that $v_2 \gtrsim 3 \times 10^2$ GeV.

An interesting aspect of this model is that it has a light mode, since out of the two phases of ϕ_1 and ϕ_2 only one is absorbed by the $U(1)_{\text{FN}}$ gauge boson, whereas the last one is left as a physical Nambu-Goldstone boson. This feature is generic of the models with two singlets, so we generally comment on it now.

[20] As a consistency check, it is straightforward to verify that

$$A_{SU(3)_C,\text{SM}} = A_{SU(2)_W,\text{SM}} = \frac{3}{10} A_{U(1)_Y,\text{SM}} = -A_{SU(3)_C,\text{heavy}} = -A_{SU(2)_W,\text{heavy}} = -\frac{3}{10} A_{U(1)_Y,\text{heavy}} = -180 \,. \tag{2.4.17}$$

An accidental flavourful Peccei-Quinn symmetry

We now turn to the systematic discussion of the physical GB which arises in models with two singlets ϕ_1 and ϕ_2. We stick to the kind of models discussed above, namely those where the heavy sector (only or mostly fields participating in the FN mechanism) gets its mass via couplings to ϕ_2.

Henceforth, we assume that the light GB is only made of the phases of ϕ_1 and ϕ_2, and that the physical pseudoscalar originating from H_u and H_d gets a large mass. This is for instance a valid assumption if the "b_μ" soft term $b_\mu H_u H_d$ is present (i.e. gauge-invariant). It turns out that, due to the large values for $\frac{\langle\phi_{1,2}\rangle}{M_W}$ imposed by the running of the gauge couplings, if a modified b_μ term, such as e.g. $b_\mu H_u H_d \phi_1$, is present, the formulas written below are still valid at leading order. For the same reason, the pseudoscalar $a_{\rm FN}$ which gives the longitudinal component of the $U(1)_{\rm FN}$ gauge boson is also given at leading order by the contribution of ϕ_1 and ϕ_2. Its expression is thus:

$$a_{\rm FN} \propto x_1 v_1 \theta_1 + x_2 v_2 \theta_2 \ , \tag{2.4.20}$$

where we wrote $\phi_{1,2} = \frac{v_{1,2}+r_{1,2}}{\sqrt{2}} e^{i\frac{\theta_{1,2}}{v_{1,2}}}$. Then, the physical leftover GB a is given by

$$a \propto x_2 v_2 \theta_1 - x_1 v_1 \theta_2 \ . \tag{2.4.21}$$

As discussed extensively in sections 2.1.8 and 2.2.2, depending on the $U(1)_{\rm FN}$ charges of the different scalar fields, the first gauge-invariant operator one could write which would violate the shift symmetry of a may be of very high dimension, thus rendering this shift symmetry accidentally protected (more on this below).

We now show that the mode a has couplings similar to the one of flavourful axions [199, 204, 205], albeit slightly different numerically, meaning that the family symmetry $U(1)_{\rm FN}$ imposes that it has anomalous couplings to gauge fields (and in particular to QCD, making it a Peccei-Quinn axion) and direct couplings to SM fermions.

In the kind of models we consider, the couplings to gauge fields is completely specified by the mass matrices. Indeed, as already mentioned in (2.4.12), the heavy sector contributes to the (axionic) anomalous couplings as

$$W \supset \int d^2\theta \left(\frac{A_{X,\rm heavy}}{16\pi^2 x_2} \log(\phi_2)\, {\rm Tr}\big(W_X^2\big) \right) \ , \tag{2.4.22}$$

where X refers to either $SU(3)_C, SU(2)_W$ or $U(1)_Y$ and we used our assumption that all mass terms come from couplings to ϕ_2. The contribution from the MSSM fields (here only focusing on QCD) is:

$$W \supset \int d^2\theta \left(\frac{1}{16\pi^2} \log\left(\left(\frac{\phi_1}{\phi_2}\right)^{\sum_i (n^u_{ii}+n^d_{ii})} (H_u H_d)^3 \right) {\rm Tr}\big(W^2_{SU(3)_C}\big) + ... \right) \ , \tag{2.4.23}$$

Neglecting $H_{u,d}$ as we assumed, $\sum_i (n^u_{ii} + n^d_{ii}) = \frac{A_{3,\rm SM}}{x_1 - x_2}$, and since anomaly cancellation imposes $A_{3,\rm heavy} = -A_{3,\rm SM}$, we end up with a total contribution

$$W \supset \int d^2\theta \left(\frac{A_{3,\rm SM}}{16\pi^2 (x_1 - x_2)} \log\left(\phi_1 \phi_2^{-\frac{x_1}{x_2}} \right) {\rm Tr}\big(W^2_{SU(3)_C}\big) + ... \right) \ , \tag{2.4.24}$$

which is obviously gauge-invariant ($a_{\rm FN}$ exactly disappears from the log), as it should.

On the other hand, (2.4.24) induces a coupling between a and the gluons, since

$$- i \log\left(\phi_1 \phi_2^{-\frac{x_1}{x_2}}\right) \supset \frac{x_2 v_2 \theta_1 - x_1 v_1 \theta_2}{x_2 v_1 v_2} = \frac{\sqrt{x_1^2 v_1^2 + x_2^2 v_2^2}}{x_2 v_1 v_2} a \ , \tag{2.4.25}$$

where we used the canonical normalization for a so that the axion decay constant[21] can be read off from (2.4.24) and (2.4.25):

$$f_a = \frac{x_2 v_1 v_2 |x_1 - x_2|}{A_{3,\rm SM}\sqrt{x_1^2 v_1^2 + x_2^2 v_2^2}} \ . \tag{2.4.26}$$

Besides the couplings to gluons, the heavy chiral fields also feed in the axion-photons coupling. A same line of reasoning gives us the latter:

$$W \supset \int d^2\theta \left(- \frac{A_{\rm em,heavy}}{16\pi^2 (x_1 - x_2)} \log\left(\phi_1 \phi_2^{-\frac{x_1}{x_2}}\right) W^2_{U(1)_{\rm em}} + ... \right) \ , \tag{2.4.27}$$

where $A_{\rm em,heavy} = \frac{A_{1,\rm heavy} + 2A_{2,\rm heavy}}{4}$ is the heavy sector electromagnetic anomaly[22], so that we understand that

$$\frac{E}{N} = \frac{A_{1,\rm heavy} + 2A_{2,\rm heavy}}{2A_{3,\rm heavy}} \ , \tag{2.4.33}$$

with the conventions of section 2.1.3. For instance, the model defined around (2.4.16) has $E/N = 8/3$, which is the same as in the DFSZ model. Thus, in this respect, this model's predictions do not deviate qualitatively from those of usual flavourful axions.

---

[21]The domain of a is given by $a = a + 2\pi f$. In the model defined around (2.4.16), $f \equiv \frac{v_1 v_2}{\sqrt{v_1^2 + 100 v_2^2}} \times \min\{|10m - n|, (m,n) \in \mathbb{Z}^2\} = \frac{v_1 v_2}{\sqrt{v_1^2 + 100 v_2^2}}$. Thus, $N_{\rm DW} = \frac{A_{3,\rm SM}}{x_2 |x_1 - x_2|} = 2$ in this model.

[22]Indeed, let us consider an arbitrarily heavy fermion $\Psi \equiv \Psi_L + \Psi_R$, which couples as follows to neutral gauge bosons

$$\begin{aligned}\mathcal{L} \supset -\overline{\Psi_1^Q} \gamma^\mu \left(\partial_\mu - i G_\mu^a T^a - \frac{i}{2} W_\mu^3 \sigma^3 - i q_Y B_\mu\right) \Psi_1^Q \supset &- \overline{\psi_{u,1}^Q} \gamma^\mu \left(\partial_\mu - i G_\mu^a T^a - i \frac{1 + q_Y}{2} A_\mu\right) \psi_{u,1}^Q \\ &- \overline{\psi_{d,1}^Q} \gamma^\mu \left(\partial_\mu - i G_\mu^a T^a - i \frac{-1 + q_Y}{2} A_\mu\right) \psi_{d,1}^Q \ ,\end{aligned} \tag{2.4.28}$$

where we decomposed $\Psi_1^Q = \begin{pmatrix} \psi_{u,1}^Q \\ \psi_{d,1}^Q \end{pmatrix}$ on its isospin components, assuming those exist. It also couples to the axion as follows

$$\mathcal{L} \supset -y \phi_2 \overline{\Psi_{R,1}^Q} \Psi_{L,1}^Q + h.c. = -y \phi_2 \left(\overline{\psi_{uR,1}^Q} \psi_{uL,1}^Q + \overline{\psi_{dR,1}^Q} \psi_{dL,1}^Q\right) + h.c. \ . \tag{2.4.29}$$

Then, using the calculation in appendix B, we find that it induces the following coupling:

$$\mathcal{L} \supset \frac{\theta_2}{16\pi^2 v_2} \left(2 \, \mathrm{Tr}\left(G\tilde{G}\right) + 6 \frac{q_Y^2 + 1}{4} F\tilde{F}\right) \tag{2.4.30}$$

at first order in ϵ (since there might be diagonalization effects in defining the heavy fermions mass eigenstates). A heavy coloured $SU(2)$ singlet, on the other hand, would only contribute

$$\mathcal{L} \supset \frac{\theta_2}{16\pi^2 v_2} \left(\mathrm{Tr}\left(G\tilde{G}\right) + 3 \frac{q_Y^2}{4} F\tilde{F}\right) \ . \tag{2.4.31}$$

Altogether, the heavy fermions contribute

$$\mathcal{L} \supset \frac{\theta_2}{16\pi^2 x_2 v_2} \left(A_{3,\rm heavy} \, \mathrm{Tr}\left(G\tilde{G}\right) + \frac{A_{1,\rm heavy} + 2A_{2,\rm heavy}}{4} F\tilde{F}\right) \tag{2.4.32}$$

where we neglected again $H_{u,d}$.

Dominant couplings between the axion and the SM fermions arise at tree-level from (2.4.7), such that the (schematic) coupling between the axion and the SM fermions is as follows:

$$\mathcal{L} \supset Y_{ij} \left(\frac{\phi_1}{\phi_2}\right)^{n_{ij}} \overline{\psi_{R,j}}\psi_{L,i}H^{(c)} \supset Y_{ij} e^{in_{ij}\left(\frac{\theta_1}{v_1}-\frac{\theta_2}{v_2}\right)}\overline{\psi_{R,j}}\psi_{L,i}H^{(c)} \supset Y_{ij} e^{i\frac{a}{f_{ij}}}\overline{\psi_{R,j}}\psi_{L,i}H^{(c)} \ , \tag{2.4.34}$$

where we neglected radial degrees of freedom in the first step, and projected the scalar phase onto the physical axion in the second. We also identified the scale of axion-fermions coupling:

$$f_{ij} = \frac{v_1 v_2 \sqrt{x_1^2 v_1^2 + x_2^2 v_2^2}}{n_{ij}(x_1 v_1^2 + x_2 v_2^2)} \ , \tag{2.4.35}$$

where we see that the axion couples more strongly to lighter generations, since those have larger charges, i.e. larger n_{ij}'s. The ratio between the axion coupling to gauge fields C_a and the coupling to fermions C_{ij} is[23]

$$\frac{C_a}{C_{ij}} \sim \frac{f_{ij}}{f_a} = \frac{A_{a,\mathrm{SM}}}{n_{ij}|x_1 - x_2|}\frac{x_1^2 v_1^2 + x_2^2 v_2^2}{x_1 v_1^2 + x_2 v_2^2} \ . \tag{2.4.37}$$

An upper bound can actually be imposed on $\langle\phi_{1,2}\rangle$ by demanding that the shift symmetry of the axion a is of high enough quality [88, 131, 132, 140, 176] to actually solve the strong CP problem once quantum gravity corrections [124–126] are taken into account. Indeed, we started with gauge symmetries considerations and did not impose any global symmetry on the model. Consequently, we expect to be able to write some gauge invariant operator which would break the shift symmetry of the physical axion. On the other hand, the presence of the $U(1)_{\mathrm{FN}}(\times G_{\mathrm{SM}})$ gauge symmetry may force such an operator to be of very high dimension such that it has no relevant impact on the axion dynamics. For instance, in the model defined around (2.4.16), the first gauge-invariant operator one could write (beyond those such as (2.4.19) which respect the axion shift symmetry) is[24] $\overline{\phi_2}\phi_1^{10}$, which can be either a contribution to the Kähler potential or a SUSY breaking term generated by gravity. In the latter case for instance, to be consistent with the measured value [26] $\theta < 10^{-10}$, we must ensure that:

$$\begin{gathered}\left[m_{a,\mathrm{QCD}} \sim \frac{m_\pi f_\pi}{f_a}\right] > 10^5 \left[m_{a,\mathrm{explicit}} \sim \epsilon^4 \left(\frac{v_2}{\sqrt{2}M_P}\right)^{\frac{9}{2}} M_P\right] \\ \text{or equivalently} \\ v_2 \lesssim \left(10^{-4}\sqrt{2}^{\frac{9}{2}}\epsilon^{-5} m_\pi f_\pi M_P^{\frac{7}{2}}\right)^{\frac{2}{11}} \sim 2\times 10^{11}\ \mathrm{GeV} \ .\end{gathered} \tag{2.4.38}$$

We immediately see that this is in tension with the perturbativity bound, even though not in contradiction since there are lots of undetermined order one numbers (e.g. the precise heavy

---

[23]As a comparison, flavourful axions models [204, 205] find

$$\frac{C_a}{C_{ij}} \sim \frac{A_{a,\mathrm{SM}}}{n_{ij}|x_1 - x_2|} \tag{2.4.36}$$

(where $x_1 - x_2$ should be understood as the $U(1)_{\mathrm{FN}}$ charge of the flavon field), which is also the order of magnitude we have here, provided $x_1 \not\gg x_2$, since $v_1 \sim \epsilon v_2 \ll v_2$. Consequently, the strong bounds on the axion decay constant f_a derived in [204, 205] by considering the bounds on $\mathrm{Br}(K^+ \to \pi^+ a)$ also apply. Those bounds, obtained from flavour physics, which state $f_a \gtrsim \kappa \times 10^{10}$ GeV (where $\kappa = \mathcal{O}(1)$ depends on the precise charge assignments and on unknown $\mathcal{O}(1)$ coefficients) then interestingly complete astrophysics bounds which require $f_a \gtrsim 10^9$ GeV, as discussed in section 2.1.7.

[24]An other dimension 11 option would be $H_u H_d \phi_1^{\dagger 9}$ but this is more suppressed since the weak scale is much below $\langle\phi_2\rangle$.

fermion mass or the coefficient in front of the operator $\overline{\phi_2}\phi_1^{10}$). Consequently, we conclude that we should have $v_2 \sim 10^{11-13}$ GeV to satisfy both bounds, also implying that explicit breaking of the Peccei-Quinn symmetry could be observable in future experiments, e.g. those aiming at better measuring the neutron (or proton) EDM [191,192]. Furthermore, this value for v_2 implies a value for f_a which is compatible with the flavourful axion to make up part or all of dark matter (see section 2.1.6).

Summary

We studied the gauging of a horizontal abelian symmetry generating the Froggatt-Nielsen mechanism, when the heavy fields in the UV completion of the mechanism are chiral with respect to this family symmetry. This for instance happens when the small parameter which explains the flavour hierarchies is composed of the vevs of two charged scalar fields which respectively mix and give masses to the heavy sector. The mixed anomalies between the Standard Model gauge group and the new symmetry are modified in this setup, such that the anomaly-free completions of the model are not the same as in the usual case when the heavy sector is vector-like.

We focused on supersymmetric models, since their holomorphicity properties usually do not leave much freedom for the anomalies to cancel. Unlike the vector-like heavy sector case, for which it has been shown that the FN symmetry is always anomalous at the level of the MSSM, in our chiral heavy sector case the mixed anomalies are enough disentangled from the mass matrices so that they sometimes vanish without adding any other spectator field than the ones which are necessary for the FN mechanism to take place. We gave specific examples where this "minimal" UV content is realized.

Finally, we emphasized the fact that such "minimal" models often come with a physical axion mode, which has couplings typical of a flavourful QCD axion. There are slight numerical differences with respect to flavourful axions originating from a global FN mechanism, recently revived as a PQ symmetry in the literature, but the qualitative phenomenology seems to be identical.

2.5 Conclusions

This first part of the thesis was devoted to the study of (4D) field theory models which extend the matter content of the standard model, with a focus on models of axions, as well as models for the flavour sector of the SM. The underlying logic behind the model building efforts here was to use (abelian) gauge symmetries as the only restriction on the dynamics for a given matter content. In particular, small parameters such as the θ-angle of QCD or the hierarchies in Yukawa couplings are undesirable, and any symmetry-based explanation of this smallness must be a consequence of the gauge invariance of the theory. This requirement can be motivated by the apparent incompatibility between global symmetries and gravity, on which we elaborate in section 4.4.1.

QCD axion models dynamically explain the bounds on the QCD θ angle, and we studied the embedding of this kind of particles in a particular gauged clockwork setup. We also relaxed the connection between the QCD θ angle and the axion in order to interpret the latter as an axion-like particle, focusing on its cosmological relevance as dark matter. Our conclusions are that the gauged clockwork model of section 2.2.2 is very efficient at controlling the breaking of the axion shift symmetry, so that the axion in the model is a good candidate both for a QCD axion or a light kind of dark matter. On the other hand, the clockwork-like localization properties of the axion along the UV modes of the model can only be used for specific type of couplings, which are the anomaly-free couplings that any ALP can have. The anomalous couplings, to gluons for instance, cannot be tuned by clockwork localization and, because of the constraints of gauge anomaly cancellation, the associated decay constant is only significantly lower than the actual new-physics scale if the model is accompanied by a large number of additional particles.

Finally, we used the Froggatt-Nielsen mechanism to explain the mass and mixings hierarchies in the quark sector of the SM. Following the logic reminded two paragraphs above, we gauged the Froggatt-Nielsen symmetry and studied if all the heavy fermions participating in the mechanism could be enough to cancel the anomalies. Our answer was positive, and we exhibited models where this happens. Some of those have a matter content such that a flavourful QCD axion arises, whose phenomenology and consistency we briefly studied.

3 Supersymmetry and supersymmetry breaking

In the spectrum of BSM field theories, the supersymmetric ones have a very special status. Indeed, supersymmetry (SUSY) stepped in the world of high energy physicists for many reasons, some of which will be mentioned several times in what follows.

Even though the fact of pairing bosons and fermions may at first seem to be an innocent and somewhat odd choice, it has dramatic consequences. First, it arises as the maximal spacetime symmetry one could imagine on top of the Lorentz group [234, 235]. It also imposes an extension of the standard model [236], called the minimal supersymmetric standard model (MSSM), where the quantum divergences are softer [237, 238], where natural candidates for dark matter are present [239] and where the unification of gauge coupling is qualitatively better than in the SM [240, 241]. In addition, SUSY is an ingredient of the consistent formulation of string theories [242, 243]. Furthermore, and quite remarkably, any attempt to gauge supersymmetry necessarily ends up in writing a theory which incorporates gravity (meaning that it has a spin-2 massless field and diffeomorphism invariance) [244, 245]. Local supersymmetry is for this reason called supergravity (SUGRA). Last (of this list, which misses many topics beyond high energy physics), supersymmetry provides a powerful tool to control and analyze theories: it helps to smooth the divergences which plague the quantization of gravity [246–249], it enables to get exact results in strongly coupled theories [250–252] or in theories on compact manifold [253], and it helps to build gravitational solutions such as supersymmetric black holes and understand their microscopics [254].

On the other hand, the use of supersymmetry in phenomenology cannot be straightforward since the observed particle spectrum is not supersymmetric. For this reason, supersymmetry, if it is an ingredient of a correct description of nature at some scale, must be spontaneously broken [255, 256]. In particular, if it is gauged, the gravitino is not a massless particle. However, the mechanism for SUSY breaking cannot be implemented in the MSSM alone and requires an extension of it, to implement the specific SUSY breaking methods known in model building. In addition, the recent results of the LHC are in tension with the naive incorporation of SUSY breaking in the MSSM under the form of soft breaking, which demands more refinement in the construction and motivation of SUSY breaking models (see e.g. [257–259]).

In what follows, we first review in section 3.1 basics about supersymmetry, supersymmetric spectra and supersymmetric dynamics, so that we can efficiently introduce SUSY breaking in section 3.2. We leave aside SUSY breaking in higher dimensions or in string theory to section 4.2. We present in section 3.2.3 an explicit model which breaks SUSY with a (very) suppressed breaking scale and which features clockwork modes, before turning in section 3.3 to the general study of spontaneously broken SUSY, using non-linear supersymmetry. Finally, we step away from SUSY breaking in section 3.4 and exhibit an exact BPS solution of the Wess-Zumino model, which has applications to the calculation of multi-particle amplitudes or as a domain wall profile for supersymmetric QCD.

We remain in this section in four dimensions, and we use the conventions (especially the two component notations for spinors) of [260].

3.1 Review of SUSY

We start our discussion by covering the basics of SUSY model building, so that we lay down the necessary mathematical language necessary to address SUSY breaking later. Even though we mention supergravity, our focus at the level of this section remains on global SUSY.

3.1.1 Algebra, representations and superfields

Following the usual textbook approach for building particle physics model, we need to specify what are the symmetries and construct their representations, using their algebra. Then, once fields are build out of the state-creating operators, one needs to write down a dynamics invariant under the chosen symmetries for those fields. We do not cover the full details of this, see e.g. [260, 261], but we describe what we need for the following of this thesis.

The supersymmetry algebra

The ($\mathcal{N} = 1$) 4D SUSY algebra involves the usual Poincaré algebra for the translations P_μ and the Lorentz generators $M_{\mu\nu}$, to which is added a Majorana spinorial charge $(Q_\alpha, \overline{Q}_{\dot\alpha})$. The (anti)commutation relations of this charge are as follows:

$$\begin{aligned} &\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \ , \quad \{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0 \ , \\ &[P_\mu, Q_\alpha] = [P_\mu, \overline{Q}_{\dot\alpha}] = 0 \ , \quad [M_{\mu\nu}, Q_\alpha] = i\sigma^{\mu\nu}{}_\alpha{}^\beta Q_\beta \ , \quad [M_{\mu\nu}, \overline{Q}^{\dot\alpha}] = i\overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} \overline{Q}^{\dot\beta} \ . \end{aligned} \tag{3.1.1}$$

It is quite remarkable that this algebra is the most general one which one can impose on the S-matrix of a relativistic quantum field theory. It is the famous graded algebra counterexample to the Coleman-Mandula theorem [234] on usual algebras, which states that, in a d(>2)-dimensional relativistic theory with a S-matrix, the only symmetries of the latter, beyond the Poincaré symmetry, are internal symmetries (i.e. symmetries which commute with every Poincaré generator). The Haag-Lopuszanski-Sohnius theorem [235] extends the Coleman-Mandula proof to the case of graded algebras, singling out supersymmetry. Their study showed that extended SUSY (see section 3.1.3) is the most general extension.

The field multiplets

The representations of this algebra (called multiplets) and the one particle states are built from (3.1.1) by boosting in a reference frame and by identifying Clifford algebras. We do not run

	Massless multiplets	Massive multiplets
Chiral multiplet	$\underbrace{\phi}_{\text{Complex scalar}}$, $\underbrace{\begin{pmatrix}\chi_\alpha \\ \overline{\chi}^{\dot\alpha}\end{pmatrix}}_{\text{Majorana fermion}}$	Same as massless chiral
Vector multiplet	$\underbrace{\begin{pmatrix}\lambda_\alpha \\ \overline{\lambda}^{\dot\alpha}\end{pmatrix}}_{\text{"Gaugino"}}$, $\underbrace{A_\mu}_{\text{Gauge boson}}$	Same as massless vector + massless chiral
Gravity multiplet	$\underbrace{\begin{pmatrix}\psi_{\mu\alpha} \\ \overline{\psi}^{\dot\alpha}_\mu\end{pmatrix}}_{\text{"Gravitino"}}$, $\underbrace{e^a_\mu}_{\text{Vierbein}}$	

Table 3.1: Fields in usual $\mathcal{N}=1$ multiplets

the argument here, and we only display in Table 3.1 the fields which capture the content of the different usual multiplets which we will encounter in what follows. From the construction of the particle states, one works out the SUSY transformation of the fields which are built out of the creation and annihilation operators for such states. For the free chiral multiplet, those read

$$\delta_\xi\phi \equiv [\xi Q + \overline{\xi Q}, \phi] = \sqrt{2}\xi\chi \ , \quad \delta_\xi\chi_\alpha = i\sqrt{2}\sigma^\mu{}_{\alpha\dot\alpha}\overline{\xi}^{\dot\alpha}\partial_\mu\phi \ , \tag{3.1.2}$$

with ξ a constant Majorana spinor which is the parameter of the SUSY transformation. In SUGRA, it is upgraded to a spacetime-dependent spinor. The free theory of a supersymmetric chiral multiplet

$$S = \int d^4x \left(-|\partial_\mu\phi|^2 - i\overline{\chi}\overline{\sigma}^\mu\partial_\mu\chi\right) \tag{3.1.3}$$

thus transforms as follows under SUSY

$$\delta_\xi S = \int d^4x \left(-\partial_\mu\phi\sqrt{2}\partial^\mu(\overline{\xi\chi}) + \overline{\chi}\overline{\sigma}^\mu\partial_\mu(\sqrt{2}\sigma^\nu\overline{\xi}\partial_\nu\phi) + h.c.\right) = \int d^4x \left(-\partial^\mu[\sqrt{2}\partial_\mu\phi\overline{\xi\chi}] + h.c.\right) \ , \tag{3.1.4}$$

which shows the invariance of the action for correct fall-offs of the fields. Two remarks are in order though. First, the transformations (3.1.2) do not close on the SUSY algebra unless the equations of motions (eoms) of (3.1.3) are imposed, in particular the fermionic one $\overline{\sigma}^\mu\partial_\mu\chi = 0$. Indeed,

$$[\delta_\xi, \delta_\eta]\chi = i(\sigma^\mu\overline{\sigma}^\nu\partial_\mu\chi)_\alpha(\eta\sigma_\nu\overline{\xi} - \xi\sigma_\nu\overline{\eta}) = \underbrace{-2i(\eta\sigma^\mu\overline{\xi} - \xi\sigma^\mu\overline{\eta})\partial_\mu\chi_\alpha}_{\text{verifies the algebra (3.1.1)}} - i(\sigma^\nu\underbrace{\overline{\sigma}^\mu\partial_\mu\chi}_{=0})_\alpha(\eta\sigma_\nu\overline{\xi} - \xi\sigma_\nu\overline{\eta}) \ . \tag{3.1.5}$$

Second, writing supersymmetric lagrangians becomes much more difficult as soon as interactions are included. To make it doable, one introduces an auxiliary field F and extends as follows the SUSY transformations:

$$\delta_\xi\phi = \sqrt{2}\xi\chi \ , \quad \delta_\xi\chi_\alpha = i\sqrt{2}\sigma^\mu{}_{\alpha\dot\alpha}\overline{\xi}^{\dot\alpha}\partial_\mu\phi + \sqrt{2}\xi_\alpha F \ , \quad \delta_\xi F = i\sqrt{2}\overline{\xi}\overline{\sigma}^\mu\partial_\mu\chi \ . \tag{3.1.6}$$

With this modification, the SUSY algebra closes off-shell, i.e. without the need of using the eoms. The supersymmetric action then becomes

$$S = \int d^4x \left(-|\partial_\mu\phi|^2 - i\overline{\chi}\overline{\sigma}^\mu\partial_\mu\chi + |F|^2\right) \ . \tag{3.1.7}$$

The equations of motion for F make it vanish, and the SUSY transformations as well as the action reduce to (3.1.2) and (3.1.3).

The superfields

The dynamics (3.1.7) can be easily generalized when written in superspace [262], which is a quite deep and interesting mathematical concept (see e.g. [263, 264]). At the level of our discussion, it is enough to understand that one can recast the content of a multiplet into a single mathematical object, called a superfield $\Psi(x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}})$, which is a function of the spacetime point and of a spinor θ (sort of "variable along the multiplet dimension"), on which the SUSY transformations (almost) act as θ shifts:

$$\delta\Psi = \left[\xi^\alpha \left(\frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu\right) + \overline{\xi}_{\dot{\alpha}} \left(\frac{\partial}{\partial\overline{\theta}_{\dot{\alpha}}} - i(\overline{\sigma}^\mu\theta)^{\dot{\alpha}}\partial_\mu\right)\right]\Psi \ . \tag{3.1.8}$$

This is consistent with the transformations (3.1.6) if one defines for the chiral multiplet:

$$\Phi(x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}) = \phi\left(x^\mu + i\theta\sigma^\mu\overline{\theta}\right) + \sqrt{2}\theta\chi\left(x^\mu + i\theta\sigma^\mu\overline{\theta}\right) + \theta^2 F\left(x^\mu + i\theta\sigma^\mu\overline{\theta}\right) \ , \tag{3.1.9}$$

where we mean in the right hand side that the fields should be Taylor-expanded around x^μ using the nilpotency properties of θ, e.g. as $\phi\left(x^\mu + i\theta\sigma^\mu\overline{\theta}\right) = \phi(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu\phi + \frac{1}{4}\theta^2\overline{\theta}^2\Box\phi$. This chiral superfield only involves the fields ϕ and F (and not their conjugates $\overline{\phi}$ and $\overline{F}$), and only the left-handed fermion χ (and not its right-handed conjugate $\overline{\chi}$). For this reason, we may sometimes use in what follows the following definition:

$$\Phi(x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}) \equiv |_{y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}} \ \Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\chi(y) + \theta^2 F(y) \ . \tag{3.1.10}$$

This is important for the holomorphicity properties of the dynamics, which we describe below. The conjugate fields appear in the anti-chiral superfield $\overline{\Phi}$.

Similar constructions hold for the vector multiplet, which is augmented by an auxiliary field D, and whose associated superfield is given (for the abelian vector boson, in a gauge called Wess-Zumino (WZ) gauge where $V^3 = 0$) by

$$V(x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}) = -\theta\sigma^\mu\overline{\theta}A_\mu + i\theta^2\overline{\theta\lambda} - i\overline{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\overline{\theta}^2 D \ . \tag{3.1.11}$$

It is also useful to know the field strength multiplet (which is gauge-independent):

$$\begin{aligned} W_\alpha(x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}) &= -\frac{1}{4}\overline{D}^2 D_\alpha V \ , \text{ with } D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu \text{ and } \overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu \\ &= -i\lambda_\alpha + \left[\delta_\alpha^\beta D - \frac{i}{2}(\sigma^\mu\overline{\sigma}^\nu)_\alpha{}^\beta F_{\mu\nu}\right]\theta_\beta + \theta^2(\sigma^\mu\partial_\mu\lambda)_\alpha \ , \end{aligned} \tag{3.1.12}$$

where the fields are to be evaluated at $x^\mu + i\theta\sigma^\mu\overline{\theta}$ like in (3.1.9) and where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of A_μ. There are non-abelian extensions of this construction, which we do not cover here. We do not cover details of the gravity multiplet either, or of the general dynamics of supergravity in what follows.

3.1.2 $\mathcal{N} = 1$ supersymmetric dynamics

The supersymmetric dynamics of the multiplets in Table 3.1 can be easily constructed using their associated superfields. The most general lagrangian with up to two spacetime derivatives acting on a set of chiral superfields $\{\Phi_i\}$ is [265, 266]

$$\mathcal{L} = K(\Phi_i, \overline{\Phi}_j)|_{\theta^2\overline{\theta}^2 \text{ term}} + W(\Phi_i)|_{\theta^2 \text{ term}} + \overline{W}(\overline{\Phi}_j)|_{\overline{\theta}^2 \text{ term}} , \tag{3.1.13}$$

where the function K, called the Kähler potential, should be real and the function W, called superpotential, should be holomorphic in terms of the superfields (i.e. it should not depend on any of the $\overline{\Phi}_j$s). In (3.1.13), the functions are understood as being again developed in Taylor series in powers of θ and $\overline{\theta}$, around x^μ for K and around y^μ for W. The name Kähler potential is due to the fact that the field expansion of (3.1.13) involves

$$\mathcal{L} = -K_{ij}\partial\phi_i\partial\overline{\phi}_j + ... \text{ where } K_{ij} \equiv \frac{\partial K}{\partial\phi_i\partial\overline{\phi}_j} , \tag{3.1.14}$$

meaning that the scalar manifold is a Kähler manifold, in the mathematical sense. The superpotential gets its name from the fact that it determines the scalar potential as follows:

$$V\left(\phi_i, \overline{\phi}_j\right) = K^{ij}\frac{\partial W}{\partial\phi_i}\frac{\partial\overline{W}}{\partial\overline{\phi}_j} , \tag{3.1.15}$$

where we defined K^{ij} such that $K^{ij}K_{kj} = \delta^i_k$ and the same for the second variable. This is obtained from (3.1.13) once the auxiliary fields $(F_i, \overline{F}_j)$ are integrated out.

One can rewrite the action (3.1.13) as an integral over the full superspace. In order to do this, one needs to define the Grassmann integration for an anticommuting variable η:

$$\text{For a function } f(\eta) = f_0 + f_1\eta , \int d\eta\, f \equiv f_1. \tag{3.1.16}$$

In particular $\int d\eta\, 1 = 0$ and $\int d\eta\, \eta = 1$. Hence, defining $d^2\theta \equiv -\frac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta}$, $d^2\overline{\theta} \equiv -\frac{1}{4}d\overline{\theta}_{\dot{\alpha}}d\overline{\theta}_{\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}$ and $d^4\theta \equiv d^2\theta d^2\overline{\theta}$, we can write (3.1.13) as

$$\mathcal{L} = \int d^4\, \theta K(\Phi_i, \overline{\Phi}_j) + \int d^2\theta\, W(\Phi_i) + \int d^2\overline{\theta}\, \overline{W}(\overline{\Phi}_j) . \tag{3.1.17}$$

The Kähler potential and the superpotential can be restricted by means of symmetries (gauge ones for instance, as we discuss next). Then, there are two kind of symmetries: those which commute with the supersymmetry generators and those which do not. Hence, the first kind acts in the same way on a full multiplet, which means that we get (here for an abelian symmetry acting on a chiral superfield) in superspace:

$$\Phi\left(x, \theta, \overline{\theta}\right) \underset{U(1)}{\longrightarrow} \Phi'\left(x, \theta, \overline{\theta}\right) = e^{iq\alpha}\Phi\left(x, \theta, \overline{\theta}\right) , \tag{3.1.18}$$

where q is the charge of the multiplet and α the transformation parameter. On the contrary, the second kind differentiates between the different components of a same multiplet. It defines what is called an R-symmetry, which we comment on again in section 3.1.3. For $\mathcal{N} = 1$ SUSY, the

R-symmetries are abelian and can be implemented in superspace via a shift of the superspace variable $\theta \to \theta' = e^{i\alpha}\theta$:

$$\Phi\left(x,\theta,\overline{\theta}\right) \xrightarrow[U(1)_R]{} \Phi'\left(x,\theta',\overline{\theta}'\right) = e^{iq\alpha}\Phi\left(x,\theta,\overline{\theta}\right) \tag{3.1.19}$$

The parameter q is the charge of the lowest component of the superfield, since (3.1.19) gives, when written in components:

$$\phi' = e^{iq\alpha}\phi\ , \quad \chi' = e^{i(q-1)\alpha}\chi\ , \quad F' = e^{i(q-2)\alpha}F\ . \tag{3.1.20}$$

From this, we understand that the superpotential needs to have R-charge 2 to define an R-symmetric theory, since the lagrangian is given by its F-component. The Kähler potential needs to be R-invariant. For a conserved non-R $U(1)$ charge, both the superpotential and the Kähler potential should be invariant.

For a set of chiral multiplets $\{\Phi_i\}$ charged under some abelian symmetries of gauge superfields $\{V^a\}$, the procedure is a bit more involved [267]. First, one needs to solve the following equations for some real functions $d^a(\Phi_i,\overline{\Phi}_j)$, given a set of charges t_i^a:

$$K_{ij}(it_j^a\overline{\Phi}_j) = i\frac{\partial d^a}{\partial\Phi_i} \text{ and } K_{ij}(-it_i^a\Phi_i) = -i\frac{\partial d^a}{\partial\overline{\Phi}_j}\ , \tag{3.1.21}$$

where the vectors $X^a(\Phi) = -it_i^a\Phi_i\frac{\partial}{\partial\Phi_i}$ and $\overline{X}^a(\overline{\Phi}) = it_j^a\overline{\Phi}_j\frac{\partial}{\partial\overline{\Phi}_j}$ are (anti-)holomorphic Killing vectors of the scalar Kähler manifold (which means that they induce the (anti-)holomorphic shifts $K(\Phi,\overline{\Phi}) \to K(\Phi,\overline{\Phi}) + F(\Phi) + \overline{F}(\overline{\Phi})$, called Kähler transformations, on K). They generate the gauge variations $\Phi' = e^{-it_i^a\Lambda^a}\Phi$, with Λ^a a chiral superfield. Then, the gauge invariant supersymmetric action is

$$\mathcal{L} = \int d^4\theta\left[K(\Phi_i,\overline{\Phi}_j) + \int_0^1 d\alpha e^{\frac{i}{2}\alpha V^a(X^a-\overline{X}^a)}V^b d^b\right] + \left[\int d^2\theta\left(W(\Phi_i) + \frac{H_{ab}(\Phi_i)}{4g_{ab}^2}W^aW^b\right) + h.c.\right]\ , \tag{3.1.22}$$

where the (holomorphic) superpotential W and gauge kinetic function H_{ab} should be gauge-invariant. Then, the scalar potential is

$$V\left(\phi_i,\overline{\phi}_j\right) = K^{ij}\frac{\partial W}{\partial\phi_i}\frac{\partial\overline{W}}{\partial\overline{\phi}_j} + \frac{g_{ab}^2}{8}H^{ab}d^ad^b\ , \tag{3.1.23}$$

where H^{ab} is the inverse of the gauge kinetic function. Once again, this potential is obtained from the full superspace action (3.1.22) once the auxiliary fields $F_i, \overline{F}_j$ and D^a are integrated out.

Let us see what it gives for a single chiral superfield Φ, a single abelian vector superfield V, a trivial Kähler potential $\Phi\overline{\Phi} = |\Phi|^2$, no superpotential $W = 0$ and a trivial gauge kinetic function $H_{ab} = \delta_{ab}$: (3.1.21) tell us that $d = c + t|\Phi|^2$, from which we get, in WZ gauge,

$$\begin{aligned}\int_0^1 d\alpha e^{\frac{i}{2}\alpha V(X-\overline{X})}Vd &= \int_0^1 d\alpha\left(1 + \frac{t}{2}\alpha V\left(\Phi\frac{\partial}{\partial\Phi} + \overline{\Phi}\frac{\partial}{\partial\overline{\Phi}}\right)\right)V\left(c + t|\Phi|^2\right) \\ &= cV + tV|\Phi|^2 + \frac{t^2}{2}V^2|\Phi|^2\ .\end{aligned} \tag{3.1.24}$$

(3.1.22) then gives us back the usual gauge-invariant theory of a chiral superfield coupled to an abelian vector multiplet:

$$\mathcal{L} = \int d^4\theta \left(\overline{\Phi} e^{tV} \Phi + cV\right) + \left(\int d^2\theta \frac{1}{4g^2} W^2 + h.c.\right) \ . \tag{3.1.25}$$

We thus see that the undetermined constants c^a which can possibly be added to the d^as in (3.1.21) are actually the Fayet-Iliopoulos (FI) terms [268] that one can choose for an abelian supersymmetric gauge theory. Those terms, which are absent for non-abelian theories, are quite useful for SUSY breaking, as we will soon see. Finally, the scalar potential reads

$$V\left(\phi, \overline{\phi}\right) = \frac{g^2}{8}\left(c + t|\phi|^2\right)^2 \ . \tag{3.1.26}$$

In SUGRA, the scalar potential is modified by gravitational contributions. Those are such that, when the Planck mass is taken to infinity, all other field values being kept fixed, the potential reduces to (3.1.23).

3.1.3 Extended supersymmetry

Around (3.1.1), we mentioned "extended" supersymmetry as the maximal (non internal) extension of the Poincaré algebra. "Extended" means that there is not only one charge $(Q, \overline{Q})$ but possibly N of them: $(Q^{i=1,\ldots,N}, \overline{Q}_j)$. We then talk about $\mathcal{N} = N$ SUSY.

The algebra is almost a straightforward extension of the $\mathcal{N} = 1$ one:

$$\{Q^i_\alpha, \overline{Q}_{\dot\alpha,j}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^i_j \ , \ [P_\mu, Q^i_\alpha] = [P_\mu, \overline{Q}_{\dot\alpha,j}] = 0 \ , \ [M_{\mu\nu}, Q^i_\alpha] = i\sigma^{\mu\nu}{}_\alpha{}^\beta Q^i_\beta \ , \ [M_{\mu\nu}, \overline{Q}^{\dot\alpha}_j] = i\overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} \overline{Q}^{\dot\beta}_j \ , \tag{3.1.27}$$

with the (fundamental) exception that one can now include central charges Z^{ij} as follows:

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij} \ , \quad \{\overline{Q}_{\dot\alpha,i}, \overline{Q}_{\dot\beta,j}\} = \epsilon_{\dot\alpha\dot\beta} \overline{Z}_{ij} \ . \tag{3.1.28}$$

The Z^{ij} are antisymmetric and thus cannot be present for $\mathcal{N} = 1$ SUSY. They are called central charges since they commute with all the other generators, so they can be diagonalized on the Hilbert space. On each state, they appear as a set of real values assembled in an antisymmetric matrix Z^{ij}. They have dramatic effects on the representations though, since they modify the standard interpretation of Q and $\overline{Q}$ as creation or annihilation operators. Massive states, of mass M, must verify

$$Z_n \leq 2M \ , \tag{3.1.29}$$

where Z_n are the positive real values obtained when the matrix Z^{ij} is diagonalized. (3.1.29) is then called the BPS bound. When it is saturated, massive multiplets have less degrees of freedom than those which do not saturate it: we talk about short, respectively long, multiplets. The short multiplets, mostly called BPS states, are also annihilated by a subset of the supersymmetry charges, meaning that they preserve some amount of supersymmetry, consistently with the fact that they have less degrees of freedom. Central charges are also usually associated to topological

	Multiplets
$\mathcal{N}=2$	Vector multiplet: $\phi, \begin{pmatrix}\lambda_{1,\alpha}\\ \overline{\lambda}_1^{\dot{\alpha}}\end{pmatrix}, \begin{pmatrix}\lambda_{2,\alpha}\\ \overline{\lambda}_2^{\dot{\alpha}}\end{pmatrix}, A_\mu$ Hypermultiplet: $\phi_1, \phi_2, \underbrace{\begin{pmatrix}\chi_{1,\alpha}\\ \overline{\chi}_2^{\dot{\alpha}}\end{pmatrix}}_{\text{Dirac spinor}}$
$\mathcal{N}=4$	Vector multiplet: $\underbrace{\varphi_{i=1,\ldots,6}}_{\text{Real scalars}}, \begin{pmatrix}\lambda_{a=1,\ldots,4,\alpha}\\ \overline{\lambda}_a^{\dot{\alpha}}\end{pmatrix}, A_\mu$
$\mathcal{N}=8$	Gravity multiplet: $\varphi_{a=1,\ldots,70}, \begin{pmatrix}\lambda_{i=1,\ldots,56,\alpha}\\ \overline{\lambda}_i^{\dot{\alpha}}\end{pmatrix}, A_{A=1,\ldots,28,\mu}, \begin{pmatrix}\psi_{I=1,\ldots,8,\mu\alpha}\\ \overline{\psi}_{I,\mu}^{\dot{\alpha}}\end{pmatrix}, e_\mu^a$

Table 3.2: Fields in usual $\mathcal{N}>1$ multiplets

defects in supersymmetric theories [269]. We will give an example where this happens in section 3.4, where we will illustrate many aspects of this discussion.

There is an other aspect of the extended algebra which is worth mentioning. The algebra without central charges is invariant under $U(N)$ relabellings of the charges, so this $U(N) = SU(N) \times U(1)$ invariance can be added to the algebra of the theory (from the Poincaré group point of view, it behaves as an internal symmetry, as required by the Coleman-Mandula theorem). It does not commute with the supersymmetry algebra and is thus the $\mathcal{N}>1$ generalization of the R-symmetry of (3.1.19). If central charges are present, they must be compatible with the R-symmetry which is chosen.

It turns out that the only Ns for which we know consistent theories in four dimensional flat space verify $N \leq 8$. Furthermore, every such theory with $N > 4$ is necessarily a theory of supergravity. Remarkable massless multiplets are those of Table 3.2. They all can be seen as assemblages of $\mathcal{N}=1$ multiplets, since every supersymmetry algebra includes a $\mathcal{N}=1$ subalgebra. The hypermultiplet can either be massless or massive BPS-saturated. The $\mathcal{N}=4$ vector multiplet is the one involved in the famous $\mathcal{N}=4$ Super-Yang-Mills (SYM) theory, dual to Type IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}_5$ via the celebrated Anti de Sitter/Conformal Field Theory (AdS/CFT) correspondance [270]. The $\mathcal{N}=8$ supergravity multiplet defines the maximally supersymmetric supergravity theory, which can be dimensionally reduced from the SUGRA with the highest-dimension known, the eleven-dimensional one [271], as well as from the ten-dimensional Type IIA SUGRA.

The indices which appear in those remarkable multiplets denote the R-symmetry properties of the theory: the two scalars of the hypermultiplet are exchanged by the $SU(2)$ R-symmetry of $\mathcal{N}=2$ SUSY, like the two fermions of the vector multiplet. The scalars and the fermions of the $\mathcal{N}=4$ multiplet belong to $SU(4)$ R-symmetry multiplets. Actually, the R-symmetry in $\mathcal{N}=4$ SYM has nice interpretations: it has to arise in this conformal theory since the R-symmetry generators are present in the superconformal algebra, and it can be understood as the $SO(6)$ rotations in internal space when $\mathcal{N}=4$ SYM in four dimensions is dimensionaly reduced from the ten-dimensional SYM on a T^6.

3.1.4 Off-shell $\mathcal{N}=2$ supersymmetry

We go now in further details about $\mathcal{N}=2$ SUSY, and present the dynamics of vector multiplets and hypermultiplets, when the latter can be written with a superfield language (see [272, 273], for pedagogic introductions).

As for $\mathcal{N}=1$ multiplets, there are cases where the $\mathcal{N}=2$ ones can be written using superspace and superfields. In what follows, we only consider a naive extension of the $\mathcal{N}=1$ superspace, and leave aside more elaborate constructions such as harmonic superspace [264]. Our extended superspace is simply the doubling of the usual one: we introduce a second set of Majorana spinors $\left(\tilde{\theta},\overline{\tilde{\theta}}\right)$ and we define superfields as functions $\Psi\left(x,\theta,\overline{\theta},\tilde{\theta},\overline{\tilde{\theta}}\right)$.

In $\mathcal{N}=1$ SUSY, an arbitrary superfield was composed of too many fields, meaning that it was a reducible representation of SUSY. Thus, we imposed constraints in order to get the irreducible representations (3.1.9) and (3.1.11). Those constraints were

$$\overline{D}_{\dot{\alpha}}\Phi=0 \text{ for the chiral superfield }, \quad V=V^\dagger \text{ for the vector superfield }, \tag{3.1.30}$$

plus a gauge fixing condition for the vector superfield, with $\overline{D}_{\dot{\alpha}}$ defined as in (3.1.12). We could also have derived the vector superfield's structure by starting with a chiral superfield W_α (i.e. which verifies $\overline{D}_{\dot{\alpha}}W_\alpha=0$) and by demanding in addition that

$$D^\alpha W_\alpha+\overline{D}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}=0\ , \tag{3.1.31}$$

which in reverse is solved if (3.1.12) is verified.

In $\mathcal{N}=2$ SUSY, the same happens. The vector multiplet is then obtained by the following construction: start with a chiral-chiral superfield $\mathcal{W}$ (i.e. which verifies $\overline{D}\mathcal{W}=\overline{\tilde{D}}\mathcal{W}=0$, with $\tilde{D}$ defined with respect to $\tilde{\theta}$ as D was with respect to θ), and impose the following constraint:

$$D\tilde{D}\mathcal{W}+\overline{D}\overline{\tilde{D}}\overline{\mathcal{W}}=0\ . \tag{3.1.32}$$

The solution of this is defined in terms of a $\mathcal{N}=1$ chiral multiplet Φ and of the field strength W_α of a $\mathcal{N}=1$ vector multiplet as follows:

$$\mathcal{W}(y,\theta,\tilde{\theta})=\Phi(y,\theta)+i\sqrt{2}\tilde{\theta}W(y,\theta)-\frac{1}{4}\tilde{\theta}^2\overline{D}^2\overline{\Phi}(y,\theta) \tag{3.1.33}$$

where $y\equiv x+i\theta\sigma\overline{\theta}+i\tilde{\theta}\sigma\overline{\tilde{\theta}}$, analogously to (3.1.10). Written like this, the first supersymmetry, defined by its action on the θ-space as in (3.1.8), is supplemented by a second one which acts on the $\tilde{\theta}$-space in the same way, which makes us recover (3.1.6):

$$\delta_{\tilde{\xi}}\Phi=i\sqrt{2}\tilde{\xi}W\ , \quad \delta_{\tilde{\xi}}W_\alpha=\sqrt{2}\left(\sigma^\mu\overline{\tilde{\xi}}\right)_\alpha\partial_\mu\Phi+\frac{i}{2\sqrt{2}}\tilde{\xi}_\alpha\overline{D}^2\overline{\Phi}\ . \tag{3.1.34}$$

Here, we see what is the rewriting of the $SU(2)$ R-symmetry in $\mathcal{N}=2$ superspace: it rotates θ and $\tilde{\theta}$ as a doublet, consistently with the fact that the two fermions in Φ and W_α are rotated into each other, or that the two supersymmetries send the scalar on each of those fermions.

The dynamics of the vector superfield is given in terms of a holomorphic function $\mathcal{F}$, called the prepotential:

$$\begin{aligned}\mathcal{L} \supset \int d^2\theta d^2\tilde{\theta}\ \mathcal{F}(\mathcal{W}) + h.c. &= \int d^2\theta \left(-\frac{1}{4}\overline{D}^2\overline{\Phi}\mathcal{F}'(\Phi) + \frac{\mathcal{F}''(\Phi)}{2}W^2\right) + h.c. \\ &= \int d^4\theta \left(\overline{\Phi}\mathcal{F}'(\Phi) + \Phi\overline{\mathcal{F}}'(\overline{\Phi})\right) + \left(\int d^2\theta\ \frac{\mathcal{F}''(\Phi)}{2}W^2 + h.c.\right) .\end{aligned} \tag{3.1.35}$$

It determines the Kähler potential and the gauge kinetic function as follows:

$$K\left(\Phi,\overline{\Phi}\right) = \overline{\Phi}\mathcal{F}'(\Phi) + \Phi\overline{\mathcal{F}}'(\overline{\Phi}) \ , \quad H(\Phi) = 2\mathcal{F}''(\Phi) \ . \tag{3.1.36}$$

In addition, one can add a FI term and a linear superpotential

$$\mathcal{L} \supset e_1 D + (e_2 F + h.c.) \ , \tag{3.1.37}$$

where F and D are the auxiliary fields in Φ and W_α respectively.

An other multiplet can be defined, which describes (the dual version of) an hypermultiplet with a shift symmetry. It is called the single-tensor multiplet, and it is built out of a $\mathcal{N}=1$ tensor multiplet [274], which is encoded in superspace by a real linear superfield L, which verifies

$$L = L^\dagger \text{ and } D^2 L = \overline{D}^2 L = 0 \ , \tag{3.1.38}$$

together with a chiral superfield Φ. One of the components of L is (the Hodge dual of) the field strength of a 2-form, which can be dualized into a scalar in 4D to form one of the two complex scalar fields which usually compose the hypermultiplet. Those two superfields are exchanged by the second SUSY transformation as follows:

$$\delta_{\tilde{\xi}}\Phi = i\sqrt{2}\overline{\tilde{\xi}D}L \ , \quad \delta_{\tilde{\xi}}L = -\frac{i}{\sqrt{2}}\left(\tilde{\xi}D\Phi + \overline{\tilde{\xi}D\Phi}\right) \ , \tag{3.1.39}$$

such that they can be arranged in a superfield $\mathcal{Z}$ in $\mathcal{N}=2$ superspace, which is chiral-anti-chiral:

$$\mathcal{Z} = \Phi + i\sqrt{2}\overline{\tilde{\theta}D}L - \frac{1}{4}\overline{\tilde{\theta}}^2\overline{D}^2\overline{\Phi} \ , \tag{3.1.40}$$

and whose dynamics is given by

$$\mathcal{L} = \int d^2\theta \left[\int d^2\tilde{\theta}\ \mathcal{G}(\mathcal{Z}) + e\Phi\right] + h.c. \ , \tag{3.1.41}$$

where $\mathcal{G}$ is again holomorphic and where we included an allowed linear superpotential for Φ. The $SU(2)$ R-symmetry is not obvious here, since there are both chiral and anti-chiral spinor coordinates, but it is manifest in the long superfield formulation of the single-tensor multiplet (see e.g. [273]), in which case it rotates as expected the scalars and leaves the fermions invariant.

3.1.5 Quantum properties

Supersymmetric theories have remarkable properties concerning the possible (perturbative) quantum corrections they undergo, called non-renormalization theorems.

$\mathcal{N} = 1$ theories can only receive quantum corrections which are written as integrals over the whole superspace, i.e. as $\int d^4\theta$ (something). In particular, the superpotential is not renormalized, and its couplings constants only run when the wave-functions renormalizations at the level of the Kähler potential are reintroduced by trivializing the kinetic terms. The gauge coupling only receives one-loop corrections (to which one also adds the wave-functions renormalizations). The non-renormalization of the superpotential can be seen in a simple example, with a nice argument [275]. Let us consider the Wess-Zumino model of a chiral superfield Φ [237], of superpotential

$$W = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \ . \tag{3.1.42}$$

Any quantum correction to W must respect its holomorphicity, both in terms of the superfield Φ and of the coupling constants m and λ. In addition, perturbative corrections respect the global symmetries of the theory. W actually does not have any symmetry, but it has spurious symmetries if the constants are granted some transformation properties. There is then a "regular" $U(1)$ symmetry, as well as a $U(1)_R$ R-symmetry, under which the different (spurious) fields transform as

$$\begin{array}{c|c|c} & U(1) & U(1)_R \\ \hline \Phi & 1 & 1 \\ m & -2 & 0 \\ \lambda & -3 & -1 \end{array} \tag{3.1.43}$$

Consequently, the only shape that the quantum-corrected holomorphic symmetry-preserving superpotential could take is the following:

$$W_{\text{eff}} = \frac{m}{2}\Phi f\left(\frac{\lambda\Phi}{m}\right) \ , \tag{3.1.44}$$

with f a holomorphic function. When $\lambda \to 0$, we should recover the tree-level superpotential of (3.1.42). Thus, we understand that

$$f\left(\frac{\lambda\Phi}{m}\right) \underset{\lambda\to 0}{\longrightarrow} 1 + \frac{2\lambda}{3m}\Phi^2 \ . \tag{3.1.45}$$

But since we could have performed the limit in a way which keeps $\frac{\lambda\Phi}{m}$ invariant, we conclude that $f\left(\frac{\lambda\Phi}{m}\right) = 1 + \frac{2\lambda}{3m}\Phi^2$ holds whatever λ, hence that the superpotential is not renormalized. We see here that holomorphy was crucial to reach this conclusion.

In $\mathcal{N} = 2$ theories, the prepotential of the vector multiplet is exact at one-loop. $\mathcal{N} = 4$ SYM, for its part, is a superconformal theory and its full lagrangian is completely determined by the gauge group, hence it does not run.

Generally, BPS states, which are short multiplets, cannot stop being BPS-saturated via quantum corrections since it would require that they get extra degrees of freedom to become a long multiplet. Thus, the saturation of the BPS bound is protected from quantum corrections.

3.2 $\mathcal{N} = 1$ explicit models for supersymmetry breaking

As we argued in the introduction to this section, there is a phenomenological need of building models of supersymmetry breaking. In this section, we deal with explicit models, meaning supersymmetric UV models whose potential displays a minimum which spontaneously breaks supersymmetry. We cover in sections 3.2.1 and 3.2.2 the basics of SUSY breaking in $\mathcal{N}=1$ SUSY, and we apply in section 3.2.3 the techniques reviewed there to a model which we already encountered in section 2.2.2, but which is now supersymmetrized. It has the nice feature of inducing a very low SUSY breaking scale, compared to the typical scales of the problem, and enables one to build split spectra, for instance for the superpartners of the SM fields.

3.2.1 Generalities

The main observation for SUSY breaking goes as follows: the algebra imposes that the hamiltonian H is given by

$$H \equiv P^0 = \frac{\overline{Q_1}Q_1 + Q_1\overline{Q}_1 + \overline{Q_2}Q_2 + Q_2\overline{Q}_2}{4} \ , \tag{3.2.1}$$

which holds as an operator equality when SUSY is not broken, implying that the charges Q are well defined on the Hilbert space. Then, we also have

$$\forall i,\ Q_i\left|0\right\rangle = \overline{Q}_i\left|0\right\rangle = 0 \tag{3.2.2}$$

on the supersymmetry-preserving vacuum $\left|0\right\rangle$. Thus we immediately conclude that $H\left|0\right\rangle = 0$, i.e. the vacuum has zero energy. Inversely, if $H\left|0\right\rangle = 0$,

$$\left\langle 0\right| H \left|0\right\rangle = \frac{\left|Q_1\left|0\right\rangle\right|^2 + \left|\overline{Q}_1\left|0\right\rangle\right|^2 + \left|Q_2\left|0\right\rangle\right|^2 + \left|\overline{Q}_2\left|0\right\rangle\right|^2}{4} = 0 \implies \forall i,\ Q_i\left|0\right\rangle = \overline{Q}_i\left|0\right\rangle = 0\ . \tag{3.2.3}$$

On the other hand, (3.1.23) shows that all the contributions to the scalar potential are positive, since both K_{ij} and H_{ab} must be definite positive since they define kinetic terms. Consequently, there is a simple criterion which is equivalent to SUSY breaking:

$$\text{SUSY is broken} \iff \left\langle 0\right| H \left|0\right\rangle > 0\ . \tag{3.2.4}$$

As for internal symmetries, when SUSY is broken there is a massless Goldstone mode in the spectrum, which is a Goldstone fermion [255] called the Goldstino. We do not discuss it deeply in this section but we will come back to it quite a lot in section 3.3.

3.2.2 F-term and D-term breaking

In (3.1.23), there are two contributions, one which comes from the superpotential (and is called the "F-term") and one which comes from the couplings to the vector multiplets (the "D-term"). They add up positively to form the potential, so we conclude from our previous discussion that

$$\text{SUSY is preserved} \iff \forall i, a, \frac{\partial W}{\partial \phi_i} = 0,\ d^a = 0\ . \tag{3.2.5}$$

Consequently, since the SUSY vacua all have (the minimal) zero energy, any scalar field configuration which verifies (3.2.5) defines a SUSY vacuum. It happens often that a continuous set of physically distinct SUSY vacua exists, parametrized by the value of one or several scalar fields. Those are called moduli, and the set is called the moduli space of vacua. It is of central interest in studies of supersymmetric non-abelian theories (see appendix C), or extended supersymmetry [276].

The conditions (3.2.5) are easier to solve than the usual minimization conditions for an arbitrary potential of the same degree as the SUSY potential (3.1.23). This announces something we will mention later, the fact that BPS conditions are easier to solve than actual equations of motion.

Instead, if one is interested in building a SUSY breaking model, it should be such that it is impossible to have (3.2.5), so that the field configuration of lowest energy still has a strictly positive energy.

It is straightforward to see that, with a single chiral superfield Φ, a trivial Kähler potential and a polynomial superpotential, it is not possible to break supersymmetry. Indeed, since the scalar ϕ in Φ is a complex field, the only superpotential which does not have any solution to $W'(\phi) = 0$ is the linear one $W(\phi) = a\phi + b$. However, it only adds a constant term $|a|^2$ to the on-shell action, which is still as much supersymmetric as if a was zero. What are then the typical models of chiral superfields which are capable of breaking SUSY? Prototypical examples of such models are O'Raifeartaigh models [277]. O'Raifeartaigh showed that, using trivial Kähler potentials, one needs at least three chiral multiplets to break SUSY. A model which works is the following:

$$W = a_0\Phi_0 + a_1\Phi_0\Phi_1^2 + a_2\Phi_1\Phi_2 \implies \begin{pmatrix} \frac{\partial W}{\partial\phi_0} = a_0 + a_1\phi_1^2 \\ \frac{\partial W}{\partial\phi_1} = 2a_1\phi_0\phi_1 + a_2\phi_2 \\ \frac{\partial W}{\partial\phi_2} = a_2\phi_1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \tag{3.2.6}$$

The Nelson-Seiberg theorem [278] gives some insight on the kind of models which break SUSY with chiral superfields. It applies to generic models, i.e. models where all the allowed operators are present, without tuning of the coefficients, once the symmetries of the theory are chosen. It then states that, in order to break SUSY with such models, there must be an R-symmetry among the initial symmetries. In addition, if a R-symmetry of the model is spontaneously broken, SUSY is also broken. The O'Raifeartaigh model (3.2.6) is consistent with the theorem since it has an R-symmetry under which the fields have charges $(\Phi_0 : 2, \Phi_1 : 0, \Phi_2 : 2)$, and it is the most general superpotential once we also impose that the fields transform as follows under a $\mathbb{Z}_2$ symmetry: $\Phi_0 \to \Phi_0, \Phi_1 \to -\Phi_1, \Phi_2 \to -\Phi_2$.

When vector multiplets are added to the theory, there are new sources of possible SUSY breaking. Indeed, there are additional conditions in (3.2.5) but no new scalar fields. On the other hand, there are less possible interaction terms in the superpotential since those are restricted by the requirement of gauge invariance. For instance, only uncharged fields can appear in the linear terms of the superpotential. One immediate outcome is that, if there are only charged fields, it is always possible to make the F-terms vanish by assigning to all the scalar fields the value zero. There, the interest of having FI terms arises. Indeed, we saw in section 3.1.2 that, for a set of charged chiral superfields Φ_i with canonical Kähler potentials and charges t_i^a under abelian gauge symmetries of vector superfields V^a, the D-terms were

$$d^a = c^a + t_i^a |\phi_i|^2 \ , \tag{3.2.7}$$

where the summation is implicit on i. Thus, if there was no FI term, it would also be possible to make the D-terms vanish by putting all scalar fields to zero. Consequently, there could not be SUSY breaking in supersymmetric abelian gauge theories without uncharged fields. With the FI terms, this is prevented from happening and SUSY breaking may occur.

Let us illustrate this using the simplest such model, which is supersymmetric quantum electrodynamics (SQED). It has two chiral superfieds Φ_+ and Φ_- of opposite charges ± 1, as indicated by the subscript, under an abelian gauge symmetry of vector superfield V. The lagrangian reads

$$\mathcal{L} = \int d^4\theta \left(\overline{\Phi}_+ e^V \Phi_+ + \overline{\Phi}_- e^{-V} \Phi_- + cV\right) + \left(\int d^2\theta \left[\frac{1}{4g^2} W^2 + m\Phi_+\Phi_-\right] + h.c.\right) \ , \tag{3.2.8}$$

and it is straightforward to see that (3.2.5) cannot be satisfied: SUSY is broken. The potential reads:

$$V = \frac{e^2}{8}\left(c + |\phi_+|^2 - |\phi_-|^2\right)^2 + |m|^2|\phi_+|^2 + |m|^2|\phi_-|^2 \ . \tag{3.2.9}$$

The minimum of the potential is here found by usual minimization techniques. Here, if we assume that $|m^2| > \frac{e^2c^2}{4}$, the minimum lies at $\phi_+ = \phi_- = 0$, and SUSY is broken with a vacuum energy $\frac{e^2c^2}{8}$. Then, the Goldstino is the gaugino, the fermion contained in the vector multiplet. Indeed, we identify it since it shifts non-linearly under a SUSY transformation, as is expected for a Goldstone mode:

$$\delta_\xi \lambda_\alpha = -\frac{ic}{2}\xi_\alpha + \text{ fields } \ . \tag{3.2.10}$$

We will have much more to say about this behaviour in section 3.3.

Slightly off-topic with respect to SUSY breaking, but enabling us to connect with previously discussed themes, it is interesting to consider the case where both the FI term c and the mass m vanish. As we argued, SUSY is not broken in the vacuum, but there is now a moduli space of vacua: all the field configurations where $|\phi_+| = |\phi_-|$ minimize the potential, and the value $f \equiv |\phi_\pm|$ is free. If it is taken different from zero, the gauge symmetry is broken, as can be seen when parametrizing $\phi_\pm = \frac{f+h_\pm}{\sqrt{2}} e^{i\frac{a_\pm}{f}}$:

$$\mathcal{L} \supset -\frac{f^2}{2}\left(\partial_\mu [a_+ - a_-] - A_\mu\right)^2 \ . \tag{3.2.11}$$

After gauge fixing, a whole chiral multiplet is absorbed by the vector multiplet to form a massive vector multiplet, consistently with Table 3.1. The remaining degrees of freedom precisely arrange in a massless chiral multiplet, as required by SUSY. This example shows nicely how supersymmetry, when it is unbroken, packages spectra according to the theory of its representations whatever the relabellings performed.

3.2.3 A clockwork model for supersymmetry breaking

In this section, we study a specific model for SUSY breaking, which is the supersymmetric version of the clockwork-inspired model of section 2.2.2. It can also be seen as a clockwork version of the model in [279]. The content of this section is based on ongoing work in collaboration with E. Dudas and S. Pokorski.

Overview

We can organize our overview of the results by comparing the model with its non-supersymmetric counterpart of section 2.2.2. First, there are again modes localized along the quiver sites. However, those modes are not only axions but can also be vectors. Moreover, their site-dependence is not the same as in the non-supersymmetric model: here, the quiver fully determines the scalar potential, the vevs of the scalars are exponentially hierarchical, which modifies the profiles and spectra of the different modes, especially the vectors. Unlike the non-supersymmetric case, there are naturally light vectors in this model.

The very efficient axion protection in section 2.2.2 has a supersymmetric counterpart: SUSY breaking, due to the specific quiver structure, can only be implemented via an operator of very high dimension. Consequently, the scale of SUSY breaking is extremely suppressed compared to the typical scales of the model. For instance, it can easily be around the TeV with a few quiver sites, even if the model lives close to the Planck scale.

Finally, the two aspects mentioned here can be put together by noticing that the SUSY breaking is not spread out homogeneously along the quiver sites: there are exponential hierarchies between the D-terms, which allows in principle to build hierarchical spectra for the MSSM superpartners. On the other hand, any attempt to generate big hierarchies using the clockwork modes of the model render SUSY breaking negligible.

The model, the vacua and the clockwork modes

One considers the quiver theory of N vector superfields $V_{i=1,\dots,N}$ and $N+1$ chiral superfields $\Phi_{j=0,\dots,N}$ displayed in Figure 3.1. The model is defined by N, by the charge q and by two free

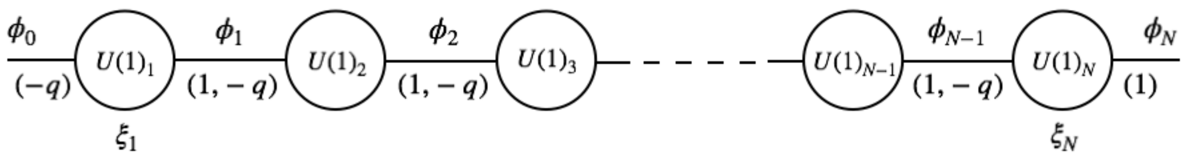


Figure 3.1: Abelian quiver of the model

Fayet-Iliopoulos parameters ξ_1, ξ_N, which determine dynamically all the fields vevs. We assume $q > 1$ and $N > 1$ in what follows. The lagrangian is

$$\mathcal{L} = \int d^4\theta \left(\overline{\Phi}_0 e^{-qV_1}\Phi_0 + \overline{\Phi}_1 e^{V_1 - qV_2}\Phi_1 + \dots + \overline{\Phi}_N e^{V_N}\Phi_N + \frac{\xi_1 V_1 + \xi_N V_N}{2} \right) + \left(\int d^2\theta \frac{1}{4g_i^2} W_i W^i + h.c \right) . \tag{3.2.12}$$

It is the most general renormalizable one allowed by the choice of gauge symmetry. This lagrangian has a $U(1)^{N+1}$ global invariance ($\Phi_i \to e^{i\alpha_i}\Phi_i$), with a $U(1)^N$ subgroup gauged by the quiver and an accidental leftover global $U(1)_a$ symmetry. If it is spontaneously broken, there is a Goldstone boson in the spectrum.

The scalar potential is:

$$V = \sum_{i=1}^{N} \frac{1}{2g_i^2} D_i^2 \quad \text{with} \quad D_i \equiv -\frac{g_i^2}{2}\left(\delta_{1,i}\xi_1 + \delta_{i,N}\xi_N + |\phi_i|^2 - q|\phi_{i-1}|^2\right) \ , \tag{3.2.13}$$

so that the model has SUSY vacua. We define $v_i \equiv |\phi_i|$. The vacua can be parametrized by a single of the vevs, v_N for instance:

$$v_{N-1}^2 = \frac{1}{q}(v_N^2+\xi_N)\ , \quad v_{1\le i<N}^2 = \frac{1}{q}v_{i+1}^2 = \frac{1}{q^{N-i}}(v_N^2+\xi_N)\ , \quad v_0^2 = \frac{1}{q}(\xi_1+v_1^2) = \frac{1}{q}[\xi_1+\frac{1}{q^{N-1}}(v_N^2+\xi_N)]\ . \tag{3.2.14}$$

Notice the increasing scalar vevs $v_1 < v_2 < \cdots v_{N-1}$, manifest from the relations

$$v_{N-1}^2 = qv_{N-2}^2 = q^2 v_{N-3}^2 = \cdots q^{N-2} v_1^2\ , \tag{3.2.15}$$

leading to an exponential hierarchy among v_1 and v_{N-1}. This (natural!) hierarchical structure is independent on the values of the FI terms ξ_i. On the other hand, the "boundary" vevs v_0 and v_N do depend on the FI terms. As it stands the model has one flat direction, which can be lifted as seen later on by adding a suitable gauge-invariant superpotential.

Gauge boson masses are generated from the Higgs mechanism:

$$L \supset \frac{1}{4}(A_{\mu,j} - qA_{\mu,j+1})^2|\phi_j|^2 \equiv \frac{1}{2}A_i^\mu \mathcal{M}_{1,ij}^2 A_{\mu,j} \tag{3.2.16}$$

with

$$\mathcal{M}_{1,ij}^2 = \frac{1}{2}\left[-qv_{i-1}^2\delta_{i-1,j} + (q^2v_{i-1}^2 + v_i^2)\delta_{i,j} - -qv_i^2\delta_{i+1,j}\right]\ . \tag{3.2.17}$$

In the SUSY vacua described above, this mass matrix takes the following form:

$$\mathcal{M}_1^2 = \frac{1}{2}\begin{pmatrix} q\xi_1 + \frac{v_{N-1}^2}{q^{N-2}}(1+q) & -\frac{v_{N-1}^2}{q^{N-3}} & 0 & \dots & 0 & 0 \\ -\frac{v_{N-1}^2}{q^{N-3}} & \frac{v_{N-1}^2}{q^{N-3}}(1+q) & -\frac{v_{N-1}^2}{q^{N-4}} & 0 & \dots & 0 \\ 0 & -\frac{v_{N-1}^2}{q^{N-4}} & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & -qv_{N-1}^2 \\ 0 & \dots & \dots & 0 & -qv_{N-1}^2 & qv_{N-1}^2(1+q) - \xi_N \end{pmatrix}\ . \tag{3.2.18}$$

Scalar masses arise from the scalar potential. We can use the symmetries to make all the vevs real. Then, writing $\phi_i = v_i + \delta\phi_i$, we find:

$$V \supset \frac{1}{2}\,\mathrm{Re}(\delta\phi_i) M_{0,ij}^2\,\mathrm{Re}(\delta\phi_j) \tag{3.2.19}$$

with

$$\mathcal{M}_{0,ij}^2 = -qv_iv_j\delta_{i-1,j} + (1+q^2)v_i^2\delta_{i,j} - qv_iv_j\delta_{i+1,j}\ . \tag{3.2.20}$$

All imaginary (axionic) parts remain massless in this case. In the (generic) case where all gauge fields are massive, N of those imaginary parts are absorbed by the gauge bosons in the Higgs mechanism. Since the vacua we discuss are supersymmetric, the leftover physical axion should pair up with a zero eigenvalue of the mass matrix above.

The spectrum depends on the parameters of the model. The different cases are easy to picture if we parametrize the flat direction of the potential in terms of v_0. All the cases of interest are summarized in Table 3.3 (where we indicated when there is a light mode of interest with respect to the clockwork properties of the model).

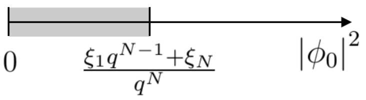
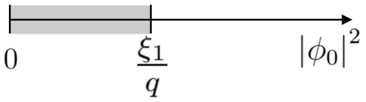
$\xi_1 < 0$ and $\xi_1 q^{N-1} + \xi_N < 0$	$v_0 = 0$: One exponentially light vector multiplet with profile $\sim \left(1, \frac{1}{q}, \frac{1}{q^2}, ...\right)$ $v_0 > 0$: One Goldstone chiral multiplet with profile $\left(\sqrt{\frac{qv_0^2-\xi_1}{qv_0^2}}, \sqrt{q}, q, ..., q^{N/2}\sqrt{\frac{q^N v_0^2 - q^{N-1}\xi_1}{q^N v_0^2 - \xi_N - q^{N-1}\xi_1}}\right)$
$\xi_1 < 0$ and $\xi_1 q^{N-1} + \xi_N = 0$	$v_0 = 0$: One massless vector multiplet with profile $\left(1, \frac{1}{q}, \frac{1}{q^2}, ...\right)$ $v_0^2 > 0$: One Goldstone chiral multiplet with profile $\left(\sqrt{\frac{qv_0^2-\xi_1}{qv_0^2}}, \sqrt{q}, q, ..., q^{N/2}\sqrt{\frac{qv_0^2-\xi_1}{qv_0^2}}\right)$
$\xi_1 < 0$ and $\xi_1 q^{N-1} + \xi_N > 0$ or $\xi_1 > 0$ and $\xi_N > 0$ Allowed values for v_0 : 	$v_0^2 > \frac{\xi_1 q^{N-1}+\xi_N}{q^N}$: One Goldstone chiral multiplet with profile $\left(\sqrt{\frac{qv_0^2-\xi_1}{qv_0^2}}, \sqrt{q}, q, ..., q^{N/2}\sqrt{\frac{q^N v_0^2 - q^{N-1}\xi_1}{q^N v_0^2 - \xi_N - q^{N-1}\xi_1}}\right)$
$\xi_1 > 0$ and $\xi_N \leq 0$ Allowed values for v_0^2 : 	$v_0^2 > \frac{\xi_1}{q}$: One Goldstone chiral multiplet with profile $\left(\sqrt{\frac{qv_0^2-\xi_1}{qv_0^2}}, \sqrt{q}, q, ..., q^{N/2}\sqrt{\frac{q^N v_0^2 - q^{N-1}\xi_1}{q^N v_0^2 - \xi_N - q^{N-1}\xi_1}}\right)$

Table 3.3: Clockwork modes of the model

The existence of the Goldstone chiral multiplets are due to the fact that the leftover $U(1)_a$ is spontaneously broken, leaving a massless Goldstone boson in the spectrum. Its profile

$$\left(\sqrt{\frac{qv_0^2 - \xi_1}{qv_0^2}}, \sqrt{q}, q, q^{\frac{3}{2}}, ..., q^{N/2}\sqrt{\frac{q^N v_0^2 - q^{N-1}\xi_1}{q^N v_0^2 - \xi_N - q^{N-1}\xi_1}}\right) \tag{3.2.21}$$

is found by demanding that it is orthogonal to the bosons absorbed by the gauge vectors. It exhibits clockwork properties, and its profile results from an interplay between the growing vevs and the quiver charges. This explains why it is not the same as in (2.2.16). Supersymmetry links it to a zero mode of the scalar mass matrix, with the same profile. The phenomenological use of this axion is the same as the one discussed in section 2.2.2: even though it is demanding to use it as a QCD axion, due to the presence of the quiver gauge symmetries, it may be possible to use it as an ALP, with possible trans-Planckian effective couplings due to the clockwork profile. Furthermore, a perturbative potential for this axion is even more suppressed than the one in section 2.2.2, since the first superpotential allowed (3.2.24) still maintains an accidental R-symmetry, which protects the axion shift symmetry. Thus, higher orders terms are needed to generate a mass for this axion.

As we said earlier, a combination of the scalar real parts completes the axion within a massless chiral multiplet. The rest of the real parts join the massive vectors within massive vector multiplets, and the profiles of the scalar mass matrix eigenvectors match the eaten Goldstone bosons profiles (which are at linear order the imaginary parts of the chiral multiplets after they acquire vevs). Masses are not arranged in a band-like way and their associated modes are not smoothly delocalized (unlike canonical models of clockwork, discussed around (2.2.4)).

The light/massless vector mode is interesting, since such modes have been used within discussions of swampland conjectures [172, 280] or can be used to naturally generate millicharges [144, 148]. When it is massless, it is due to the fact that the vevs are arranged such that two of them (v_0 and v_N) vanish and leave an abelian gauge factor unbroken[1]. It has a clockwork profile (this implies that states localized on the right of the quiver diagram have very small charges):

$$\left(1, \frac{1}{q}, \frac{1}{q^2}, ...\right) , \tag{3.2.22}$$

but it demands a tuning of the FI terms, as seen in Table 3.3. More interestingly, the same profile characterizes the lightest vector state when only v_0 vanishes (which is interesting for SUSY breaking, see below). In this case, the N gauge symmetries are broken, but not the additional $U(1)_a$ so the Φ_0 multiplet does not mix with the massive gauge bosons and stays massless. The lightness of (3.2.22) is understood from the fact that, when we switch on v_N with respect to the tuned case we mentioned just above, the massless vector almost does not feel it since Φ_N has an exponentially small charge with respect to this mode. Due to the asymmetry of the quiver charges, this conclusion would not hold if we instead turned on v_0: we would completely loose the lightness of (3.2.22). One can make this statement more quantitative by computing a bound on the smallest eigenvalue of the vector mass matrix, using the clockwork vector mode. With $|v\rangle = (1, \frac{1}{q}, ...)$:

$$\lambda_{min} \leq \frac{\langle v| \mathcal{M}_1^2 |v\rangle}{\langle v|v\rangle} = \frac{q^2}{2\langle v|v\rangle}\left[v_0^2 + \frac{v_N^2}{q^{2N}}\right] \tag{3.2.23}$$

One then sees that this bound implies an extra q^{2N} suppression in the $v_0 = 0, v_N = \mathcal{O}(1)$ case compared to the $v_0 = \mathcal{O}(1), v_N = 0$ one.

[1]The conditions $v_N = v_0 = 0$ prevent from breaking more than $N-2$ abelian symmetries, so one does not expect to identify a Goldstone boson-like particle. Nevertheless, two zero mass chiral multiplets survive the Higgs effect from the original $N+1$, and correspond to the two remaining uneaten phases and to the $(1, 0, 0...)$ and $(..., 0, 0, 1)$ zero modes of the real part mass matrix.

Finally, the case with many unbroken gauge symmetries (with $v_{0<i<N} = 0$) does not display a specifically interesting behaviour: the spectrum is composed of localized modes: one finds 2 massive vector supermultiplets located on the first and last sites (and which absorbed the Φ_0 and Φ_N chiral multiplets), when all the other scalar and vector modes remain massless. Due to the high number of massless gauge fields, this case will not be considered in phenomenological discussions.

Supersymmetry breaking: model with a superpotential

Similarly to [279] and our discussion of section 2.2.2, the model admits an extension with a superpotential that breaks supersymmetry for a region of the FI parameter space. The goal is to investigate supersymmetry breaking and to take advantage of the gauge structure of the model in order to generate a hierarchically small scale of supersymmetry breaking. The lowest order superpotential compatible with the gauge symmetries is

$$W = \frac{\lambda}{M_P^{q+q^2...+q^N-2}} \Phi_0 \Phi_1^q \Phi_2^{q^2} ... \Phi_N^{q^N} \ , \tag{3.2.24}$$

where we use M_P, since it could be generated in an EFT of quantum gravity. Of course, this scale could be an arbitrary scale of new physics. Notice the high powers involved in the superpotential, due to the peculiar clockwork-like charge assignements. The superpotential contributes to the scalar potential:

$$V = \sum_{i=1}^{N} \frac{D_i^2}{2g_i^2} + \sum_{i=0}^{N} |F_j|^2 \quad \text{with} \quad F_i = -\frac{1}{M_P^{q+q^2...+q^N-2}} q^i \left(\frac{\lambda \phi_0 \phi_1^q \phi_2^{q^2} ... \phi_N^{q^N}}{\phi_i} \right)^* \ . \tag{3.2.25}$$

These new terms do not vanish on all the previously found SUSY vacua (3.2.14), and for some choices of ξ_1 and ξ_N, there is no SUSY preserving vacuum. Let us analyse the conditions for SUSY breaking. In order to do this, let us consider (3.2.14) and try to identify which of these cancel the newly added F-terms:

- If $\xi_N \leq 0$: First, if $\xi_1 \geq 0$, $v_N^2 = -\xi_N, v_{N-1} = 0, v_0^2 = \frac{\xi_1}{q}$ sets the system on a SUSY preserving vacuum. If now $\xi_1 < 0$, one has to choose $v_{N-1}^2 > 0$ in order to satisfy $v_0^2 = \frac{\xi_1}{q} + \frac{v_{N-1}^2}{q^{N-1}} \geq 0$, and one also obtains $v_N^2 > 0$ since $v_N^2 = q v_{N-1}^2 - \xi_N$. Then, whichever choice is made, $\overline{F}_0 \equiv -\frac{\lambda}{M_P^{q+q^2...+q^N-2}} \phi_1^q \phi_2^{q^2} ... \phi_N^{q^N}$ is non zero and SUSY is broken.

- If $\xi_N \geq 0$: First, if $\xi_1 q^{N-1} + \xi_N \geq 0$, $v_N = 0, v_{N-1}^2 = \xi_N, v_0^2 = \frac{\xi_1 q^{N-1} + \xi_N}{q^N}$ defines a SUSY preserving vacuum. If now $\xi_1 q^{N-1} + \xi_N < 0$, $v_N^2 = q^N v_0^2 - (\xi_1 q^{N-1} + \xi_N) > 0$ and $v_{N-1}^2 = \frac{|\phi_N|^2 + \xi_N}{q} > 0$. There one ends up again in a situation where F_0 is non zero and SUSY is broken.

Summarizing:

$$\text{SUSY breaking} \iff \Big(\xi_N \leq 0 \ \ \& \ \ \xi_1 < 0\Big) \ \text{ or } \ \Big(\xi_N \geq 0 \ \ \& \ \ \xi_1 q^{N-1} + \xi_N < 0\Big) \ . \tag{3.2.26}$$

These constraints allow solutions with ξs of the same order of magnitude. Let us thus assume that the ξs are of "natural" size, i.e. slightly below $\mathcal{O}(M_P^2)$, and estimate the SUSY breaking scale. One can define a small parameter

$$\epsilon = \frac{v_1^q v_2^{q^2} ... v_N^{q^N-1}}{M_P^{q+q^2...+q^N-1}} \ . \tag{3.2.27}$$

Notice that due to the very high powers involved due to the clockwork charges, ϵ is extremely small even for small values of N and q. Deviations of order ϵ^0 from the solutions in (3.2.14) induce an increase in the vacuum energy of $\mathcal{O}(\epsilon^0)M_P^4$. However, there exist field configurations with a vacuum energy increase of order ϵ^2 (an explicit example is given later), so one should consider field configurations with deviations of at least order ϵ^2 from (3.2.14).

These configurations are the following (written in a way such that both sides of any identity in (3.2.28) is clearly positive when (3.2.26) is verified):

$$|\phi_0|^2 = v_0^2+a_0 \ , \quad |\phi_{1\leq i<N}|^2 = v_i^2+a_i = q^{i-1}(qv_0^2-\xi_1)+a_i \ , \quad |\phi_N|^2 = v_N^2+a_N = q^{N-1}(qv_0^2-\xi_1)-\xi_N+a_N \ , \tag{3.2.28}$$

where all a_is are of order $\mathcal{O}(\epsilon^2)$. The choice of v_0 cancels out from the calculation of the D-terms but has an impact on the F-terms which strictly increase when v_0 does. Then the minimization of V imposes the choice $v_0 = 0$, and the configuration of interest is :

$$|\phi_0|^2 = a_0 \ , \quad |\phi_i|^2 = -q^{i-1}\xi_1 + a_i \ , \quad |\phi_N|^2 = -q^{N-1}\xi_1 - \xi_N + a_N \ . \tag{3.2.29}$$

The minimization of the potential now yields (at order $\mathcal{O}(\epsilon^2)$):

$$\begin{aligned}
&[\phi_0^* \text{ eom}] \quad \phi_0 \left\{ \frac{q}{2}D_1 + \frac{\lambda^2}{M_P^{2(q+q^2...+q^N-2)}} \left| q\phi_1^{q-1}\phi_2^{q^2}...\phi_N^{q^N} \right|^2 + \frac{\lambda^2}{M_P^{2(q+q^2...+q^N-2)}} \left| q^2\phi_1^{q}\phi_2^{q^2-1}...\phi_N^{q^N} \right|^2 + ... \right\} = 0 \\
&[\phi_i^* \text{ eom}] \quad \phi_i \left\{ -\frac{1}{2}(D_i - qD_{i+1}) + \frac{\lambda^2}{M_P^{2(q+q^2...+q^N-2)}} q^i \left| \phi_1^q\phi_2^{q^2}...\phi_i^{q^i-1}...\phi_N^{q^N} \right|^2 \right\} = 0 \\
&[\phi_N^* \text{ eom}] \quad \phi_N \left\{ -\frac{1}{2}D_N + \frac{\lambda^2}{M_P^{2(q+q^2...+q^N-2)}} q^N \left| \phi_1^q\phi_2^{q^2}...\phi_N^{q^N-1} \right|^2 \right\} = 0
\end{aligned} \tag{3.2.30}$$

We considered solutions with $v_i > 0$, so from the $\phi_{i>0}^*$ eom we deduce at order $\mathcal{O}(\epsilon^2)$ the following hierarchically distributed D-terms:

$$\begin{aligned}
D_N &= 2\frac{\lambda^2}{M_P^{2(q+q^2...+q^N-2)}} q^N \left| \phi_1^q\phi_2^{q^2}...\phi_N^{q^N-1} \right|^2 = 2\lambda^2 q^N \epsilon^2 M_P^2 \ , \\
D_i &= qD_{i+1} + 2\frac{\lambda^2}{M_P^{2(q+q^2...+q^N-2)}} q^i \left| \phi_1^q\phi_2^{q^2}...\phi_i^{q^i-1}...\phi_N^{q^N} \right|^2 \\
&= 2\lambda^2 q^{2N-i}\epsilon^2 M_P^2 \left(1 + \frac{-q^{N-1}\xi_1 - \xi_N}{-q^N\xi_1} \left[1 + \frac{1}{q} + ... + \frac{1}{q^{N-i-1}} \right] \right) \ .
\end{aligned} \tag{3.2.31}$$

The ϕ_0^* eom should be treated more carefully since we don't know yet if $\phi_0 \neq 0$. If we assume so, the eom gives us:

$$\begin{aligned}
qD_1 &= -2\frac{\lambda^2}{M_P^{2(q+q^2...+q^N-2)}} \left| q\phi_1^{q-1}\phi_2^{q^2}...\phi_N^{q^N} \right|^2 - 2\frac{\lambda^2}{M_P^{2(q+q^2...+q^N-2)}} \left| q^2\phi_1^q\phi_2^{q^2-1}...\phi_N^{q^N} \right|^2 + ... \\
&= -2\lambda^2 q^{2N}\epsilon^2 M_P^2 \left(1 + \frac{-q^{N-1}\xi_1 - \xi_N}{-q^N\xi_1} \left[1 + \frac{1}{q} + ... \frac{1}{q^{N-2}} \right] \right)
\end{aligned} \tag{3.2.32}$$

which is the opposite of what was found previously. We hence need to enforce $\phi_0 = 0$. One is then left with the following vevs (recall that $D_i = \frac{1}{2}(qa_{i-1} - a_i)$):

$$|\phi_{1\leq i<N}|^2 = -q^{i-1}\xi_1 - 2(D_i + qD_{i-1} + ... + q^{i-1}D_1) , \quad |\phi_N|^2 = -q^{N-1}\xi_1 - \xi_N - 2(D_N + ... + q^{N-1}D_1) . \tag{3.2.33}$$

F-terms are zero at this order except $F_0 = -\lambda\epsilon\sqrt{-q^{N-1}\xi_1 - \xi_N}M_P$.

The scale of supersymmetry breaking is given by the auxiliary fields. Can one get this scale down to about a TeV with input scales which are not hierarchically smaller than M_P? The D-terms schematically lie between $q^N\epsilon^2 M_P^2$ and $q^{2N-1}\epsilon^2 M_P^2$, so one wants

$$q^{2N}\epsilon^2 = q^{2N+\Sigma_{i=1}^N (i-1)q^i}\left(\frac{-\xi_1}{M_P^2}\right)^{q+q^2+...q^N-1} \sim 10^{-30} , \tag{3.2.34}$$

where we neglected ξ_N compared to $q^{N-1}\xi_1$. Many values of q, ξ_1 and N can achieve this, for instance $q = 3, N = 4$ or $q = 6, N = 2$ when $\xi_1 = \frac{M_P^2}{3}$.

Notice however that the nice clockwork properties of the supersymmetric version of the model (before adding the SUSY breaking superpotential) seem to be incompatible with the requirements of supersymmetry breaking. Indeed, the Goldstone boson profile needed $v_0 \neq 0$, whereas the SUSY breaking dynamics selects $v_0 = 0$. On the other hand, a light $U(1)$ gauge boson is found in the spectrum but, for typical values which verify (3.2.34), the suppression due to profile is not significant.

A word on phenomenological perspectives concerning the MSSM

In this section, we briefly mention phenomenological questions which could be addressed using this model. First, the SUSY breaking set up allows to give masses to the MSSM sfermions by charging the MSSM superfields under different subgroups of the quiver theory, writing e.g.

$$\int d^4\theta \overline{Q} e^{V_i} Q \supset \frac{D_i}{2}|q|^2 \sim q^{2N-i}\epsilon^2 M_P^2 |q|^2 . \tag{3.2.35}$$

Since there can be a factor $q^{N-1}(\sim 10^{1-2}$ in the examples we gave) between the first and last D-terms, the spectra can thus be arranged in a way which looks like mini-split scenarii [281,282]. In addition, uncharged fields could receive masses from higher-order terms:

$$\int d^4\theta \frac{\overline{\Phi}_0\Phi_0}{M_P^2}\overline{Q}Q \supset \frac{|F_0|^2}{M_P^2}|q|^2 \sim \epsilon^2 M_P^2 |q|^2 , \tag{3.2.36}$$

in which case the hierarchies between the masses can go as high as $q^{2N-1}(\sim 10^{2-3})$.

Second, the hierarchical vevs can be used to generate effective Yukawa couplings à la Froggatt-Nielsen (see section 2.3), and the hierarchies in vevs (which scale as $q^{N/2}(\sim 10)$) could be used. Furthermore, there is more than one abelian symmetry which can possibly play the role of a FN gauge symmetry in the quiver. The different charges may then be arranged such that strong suppressions arise like in (3.2.27), with smaller charges than in the usual FN mechanism (where the ratios of charges may go as high as 10).

Summary

The supersymmetric model we put forward in this section has, for small values of the input parameters (the charges and the number of quiver sites), supersymmetric vacua with one light vector boson or one Goldstone boson with exponentially localized, clockwork-like profiles. As such, the model can accomodate milicharged particles, charged under the light gauge boson, or super-Planckian decay constants, for the Goldstone boson. It also displays naturally hierarchical vevs for the quiver bifundamental fields. It should be noted here that due to the exponential growth of those vevs, it might be necessary to tune the FI terms such that they remain within the regime of validity of the EFT (e.g. below the Planck scale).

If on the other hand we choose the parameters such that SUSY is broken, we find vacua with exponentially suppressed scale of supersymmetry breaking, of potential phenomenological interest. In particular, the input parameters are all $\mathcal{O}(1)$, and it is easy to keep the vevs below the Planck scale without too much tuning on the FI terms. However, in this case clockwork properties loose their appeal since q^N is not large enough to effectively generate large hierarchies. In the class of models we considered, there seems therefore to be a complementarity between supersymmetric vacua and clockwork properties at large N, and consistent vacua with hierarchically small supersymmetry breaking at small N. It would be interesting to find supersymmetric models in which clockwork properties coexist with a phenomenologically interesting supersymmetry breaking scale.

What we have not mentioned, or not yet studied

A few things are not covered here, even though there are of some interest. First, as in section 2.2.2, there is a five-dimensional interpretation for this model, which may be explored e.g. using the formulation of 5D SUSY theories in 4D superspace [283, 284].

Second, there are gauge anomalies in the model of Figure 3.1, which one can suppress by using a Green-Schwarz mechanism. Then, its 5D interpretation as a Chern-Simons term is worth exploring further [279]. In order to implement this Green-Schwarz mechanism, it is annoying that SUSY breaking selects the value $\phi_0 = 0$. To avoid it, it may be useful to either modify the charge assignment at the left of the quiver, or use supergravity corrections to the potential to lift the value of ϕ_0, which is made unstable by the SUGRA corrections. Then, an axion mode would be present in the SUSY breaking minimum and could be of phenomenological interest (even though, for TeV SUSY breaking, it seems to be too heavy to be a QCD axion). Definitely, more work is necessary in this direction.

Finally, the phenomenology of this model when coupled to the MSSM, barely touched upon above, is also worth scrutinizing in more details.

3.3 Non-linear supersymmetry

In this section, we present the model-independent treatment of broken SUSY known as non-linear supersymmetry. It has recently received interest within discussions of supersymmetric particle physics models [285–287], models of inflation [288, 289] or EFTs of string setups [290] with broken SUSY.

Non-linear SUSY follows from a more general construction, the coset formalism, which we review in section 3.3.1. Then, we discuss the specific case of non-linear SUSY in section 3.3.2, as well as its rewriting in terms of constrained superfields in sections 3.3.3 and 3.3.4 for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ respectively. The material in this section has been studied at the early stage of the Ph.D., and most of the work has been devoted to analyzing the impact of different constraints on phenomenological particle physics models, such as the MSSM or R-axion models, as well as realizing Born-Infeld actions with full supersymmetry breaking. However, since this work has not led to any clear-cut result, we only brush over those concerns along our review of classical aspects.

3.3.1 Spontaneous breaking and non-linear realizations

We start by presenting the general formalism of non-linear realizations, also known as the Callan-Coleman-Wess-Zumino (CCWZ) coset construction [291, 292], which gives generic rules to build EFTs with a spontaneously broken symmetry.

For simplicity, let us start by the case of a global internal symmetry. Spontaneous symmetry breaking is characterized by a symmetry group G broken down to a subgroup H. The Goldstone theorem thus states that $\dim(G) - \dim(H)$ Nambu-Goldstone bosons (GBs) are found in the spectrum. In a complete UV model, they are found among the massless physical degrees of freedom, extracted by diagonalizing the mass matrices and maybe redefining conveniently the fields. In the CCWZ formalism, they are associated to fields ξ^a, on which the broken generators A^a act non-linearly, i.e. not as matrices acting on vector indices. Typical non-linearities in those transformations are shifts, already encountered for axions in section 2 for example. The action of an element g of the group G on the ξ^as is defined as follows:

$$\xi^a \to \xi'^a \equiv g \cdot \xi^a \text{ such that } g e^{\xi^a A^a} = e^{\xi'^a A^a} h(g, \xi^a) \; , \tag{3.3.1}$$

where $h(g, \xi^a)$ belongs to H. If the generators A^a are chosen such that they are orthonormal to H (with respect to the Cartan inner product), (3.3.1) induces a linear representation of H on ξ^a:

$$h \cdot \xi^a = D^{(\xi)a}{}_b(h)\xi^b \text{ and } h(h, \xi^a) = h \; , \tag{3.3.2}$$

where $D^{(\xi)}$ is a matrix. Such a procedure is reminiscent of our parametrization of pions as Goldstone bosons in section 2.1.4.

Couplings of the GBs to matter are obtained by "dressing" appropriately interactions which are H-invariant. By this, we mean that we select a H-invariant lagrangian describing the dynamics of fields ψ lying in linear representations of H, that we upgrade their H-transformations into a full non-linear realization of G:

$$\psi \to \psi' \equiv g \cdot \psi = D(h(g, \xi^a))\psi \ , \tag{3.3.3}$$

where $D(h)$ is the matrix representation of $h \in H$ acting on ψ, and that we turn usual derivatives into G-covariant ones. In order to do so, we define the Maurer-Cartan 1-form

$$\Omega \equiv \left(e^{\xi^a A^a}\right)^{-1} d\left(e^{\xi^a A^a}\right) \equiv \omega_V^i V^i + \omega_A^a A^a \ , \tag{3.3.4}$$

where V^i are the unbroken generators of H, and we define covariant derivatives for the GBs ξ^a and the fields ψ as follows:

$$\mathcal{D}_\mu \xi^a \equiv \omega^a_{A,\mu} \ , \quad \mathcal{D}_\mu \psi \equiv (\partial_\mu + \omega^i_{V,\mu} D(V^i))\psi \ . \tag{3.3.5}$$

Then, the G-transformations of the Maurer-Cartan 1-form are such that any H- and Lorentz-invariant lagrangian built out of $\mathcal{D}_\mu \xi^a$ and $(\mathcal{D}_\mu)\psi$ is automatically G-invariant.

The transformations (3.3.1) and (3.3.3) are general: indeed, any set of fields expanded around a vacuum spontaneously breaking a symmetry can be put in the standard form described in this section. The schematic procedure is to first define the ξ^as as coordinates of the submanifold generated by the action of G on the vacuum configuration for the fields, then to pick a set of fields ψ completing the ξ^as into coordinates of the full fields manifold and linearly realizing H, which is always possible [291, 292].

Such a procedure also exists for spontaneously broken spacetime symmetries [293], but now the spacetime coordinates are understood as non-linearly realizing spacetime translations. Thus, the correct action of G (which is the symmetry group of the theory, including spacetime symmetries) on the GBs is

$$g e^{x^\mu P_\mu} e^{\xi^a(x) A^a} = e^{x'^\mu P_\mu} e^{\xi'^a(x') A^a} h(g, \xi^a(x)) \ . \tag{3.3.6}$$

An interesting aspect of the coset construction for spacetime symmetries is that not all the GBs associated to broken spacetime generators are physical [294, 295]. Then, the unphysical ones are removed by imposing non-trivial covariant constraints on the components of the Maurer-Cartan form. This is called the inverse Higgs effect.

3.3.2 Non-linear SUSY

The spontaneous breaking of SUSY can be described using the formalism of section 3.3.1. The Goldstone particle is a Goldstone fermion in this case, called the goldstino, and its SUSY transformation, the equivalent of (3.3.1), has been found by Akulov and Volkov in [255]:

$$\delta_\epsilon \lambda^\alpha = f\epsilon^\alpha - \frac{i}{f}(\lambda\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\lambda})\partial_\mu\lambda^\alpha \ , \tag{3.3.7}$$

which indeed verifies the SUSY algebra. The shift $f\epsilon^\alpha$ in $\delta_\epsilon\lambda^\alpha$, where $\sqrt{f}$ is an energy scale, is expected and reminiscent of what was found in (3.2.10). It characterizes the goldstino.

Its couplings to matter can be found by introducing matter non-linear realizations of supersymmetry [296], equivalently to (3.3.3):

$$\delta_\epsilon \psi = -\frac{i}{f}(\lambda\sigma^\mu\overline{\epsilon} - \epsilon\sigma^\mu\overline{\lambda})\partial_\mu\psi \ , \tag{3.3.8}$$

where ψ can have any Lorentz or internal index. Such realizations can always be reached from a given realization ψ' of SUSY by means of the following redefinition [297]:

$$\psi \equiv e^{\epsilon Q + \overline{\epsilon}\overline{Q}}\psi'\big|_{\epsilon=-\lambda/f} \ , \tag{3.3.9}$$

and they are such that SUSY is realized as a local diffeomorphism with parameter $\xi^\mu = \frac{i}{f}(\lambda\sigma^\mu\overline{\epsilon} - \epsilon\sigma^\mu\overline{\lambda})$. They can be coupled to matter by introducing the Akulov-Volkov vierbein

$$A_\mu{}^\nu = \delta_\mu^\nu + \frac{i}{f^2}(\lambda\sigma^\nu\partial_\mu\overline{\lambda} - \partial_\mu\lambda\sigma^\nu\overline{\lambda}) \tag{3.3.10}$$

with which we can define covariant derivatives [298, 299]

$$\mathcal{D}_\mu = (A^{-1})_\mu{}^\nu\partial_\nu \tag{3.3.11}$$

(valid for both λ and ψ) as well as a lagrangian

$$\mathcal{L} = \det(A)\left(-\frac{f^2}{2} + \mathcal{L}_{\text{matter}}(\psi, \mathcal{D}_\mu\psi, \mathcal{D}_\mu\lambda)\right) \tag{3.3.12}$$

which transforms as a total derivative under a SUSY variation. The first term in the parenthesis defines the kinetic term and self-interactions of the goldstino λ. Generalizations to gauge theories follow similar principles [298, 299].

The same procedure can (more technically) be implemented for supergravity, which yields a goldstino transformation which depends on the fields in the supergravity multiplet (here displayed up to three spinors terms) [300, 301]

$$\delta_\epsilon\lambda = \epsilon - i(\lambda\sigma^\mu\overline{\epsilon} - \epsilon\sigma^\mu\overline{\lambda})\left(\hat{\mathcal{D}}_\mu\lambda - i\frac{M}{18}\sigma_\mu\overline{\lambda}\right) + \frac{\overline{M}}{3}\epsilon(\lambda^2) + \frac{b_\mu}{3}\left(\frac{\sigma^\mu\overline{\lambda}(\lambda\epsilon)}{3} + \frac{\lambda(\epsilon\sigma^\mu\overline{\lambda})}{2} - \frac{\sigma^\mu\overline{\epsilon}(\lambda^2)}{12}\right) \ , \tag{3.3.13}$$

where we used notations of [260], units where $M_P = 1$ and each λ should be understood as λ/f. In the invariant action for the goldstino, there will be a mixing with the gravitino such that the goldstino is eaten up in unitary gauge, which defines the supersymmetric Higgs effect. Such a transformation, together with the dressing of matter/supergravity fields as in (3.3.9), is for instance relevant for EFTs of string models with broken SUSY [302].

3.3.3 Non-linear SUSY in superspace and constrained superfields

Theories which non-linearly realize SUSY can be directly expressed in superspace by building superfields out of the goldstino field, or the matter fields [300, 303]:

$$\Lambda^\alpha = e^{\theta Q + \overline{\theta}\overline{Q}}\lambda^\alpha \ , \quad \Psi = e^{\theta Q + \overline{\theta}\overline{Q}}\psi \ , \tag{3.3.14}$$

so that their general couplings (3.3.12) can be written as a superspace integral:

$$\mathcal{L} = \int d^4\theta \ \Lambda^2\overline{\Lambda}^2 \left(-\frac{1}{2f^2} + \frac{1}{f^4}\mathcal{L}_{\text{matter}}(\Psi, \partial_\mu\Psi) + ... \right) \ , \tag{3.3.15}$$

where ... means that higher order terms, or terms involving the (supersymmetric) derivatives of the different superfields, can be included.

It is interesting to note that the goldstino superfield Λ^α verifies the following constraints [303]

$$D_\beta\Lambda_\alpha = f\epsilon_{\alpha\beta} \ , \quad \overline{D}^{\dot{\beta}}\Lambda^\alpha = \frac{2i}{f}(\overline{\sigma}^\mu\Lambda)^{\dot{\beta}}\partial_\mu\Lambda^\alpha \ , \tag{3.3.16}$$

where D and $\overline{D}$ are again defined in (3.1.12), and that those constraints entirely characterize the superfield. In particular, they are enough to show that the lowest component λ^α of Λ^α transforms as the Akulov-Volkov goldstino. They can also be generalized to SUGRA in order to obtain a non-linear realization of local SUSY equivalent to (3.3.13).

This idea of imposing supersymmetric constraints on superfields to turn them into non-linear realizations has been extensively followed and applied to different kinds of superfields [304–308]. It defines the approach of constrained superfields[2]. As an illustration, a simple constraint has been proposed in [304,306,307], in which the goldstino is identified with the fermionic component of a chiral superfield which verifies

$$X^2 = 0 \ , \tag{3.3.17}$$

such that the scalar component x in X is expressed in terms of its spinor λ and auxiliary field F components:

$$x = \frac{\lambda^2}{2F} \ . \tag{3.3.18}$$

The most general action for X thus is

$$\int d^4\theta \ XX^\dagger + \left(\int d^2\theta \ fX + h.c. \right) \ , \tag{3.3.19}$$

where the scale f is the scale of SUSY breaking already introduced in (3.3.7). It gives back an action which is equivalent to the Akulov-Volkov action [309]. The interest of the constraint (3.3.17) is that it makes it easier to bridge the gap between a UV model in which SUSY is linearly realized in superspace, and the non-linear lagrangian which arises after SUSY is broken. In particular, the coupling of the goldstino superfield X to matter superfields is written as a usual superspace integral. For instance, this is relevant for couplings of the goldstino to the MSSM [285–287], for models of inflation [288, 289] or EFTs of string setups [290] with broken SUSY.

The usual coset construction applied to SUSY does not arrange matter fields in usual multiplets, as can be seen in (3.3.8). This accounts for the fact that the components within the matter superfields may be split by SUSY breaking, such that only some of them remain in the EFT below the scale of SUSY breaking. In the constrained superfield approach, this can also be done

[2]Constraints on linearly transforming fields are found beyond broken SUSY models: for instance, the pions in (2.1.10) can be obtained by considering a $SU(2)_L \times SU(2)_R$ bilinear field Σ and demanding that $\Sigma\Sigma^\dagger = 1$.

by applying constraints on the matter superfields to express some of their components in terms of others and the goldstino. A generic procedure [310] is to impose

$$XX^\dagger Y = 0 \ , \tag{3.3.20}$$

where X is the goldstino superfield which verifies (3.3.17), and Y is any superfield. This removes the lowest component of Y from the spectrum. For instance if Y is respectively equal to Φ or $D_\alpha \Phi$ for a chiral superfied Φ, (3.3.20) removes respectively the scalar and spinor component of Φ. They all have SUGRA analogs [311].

A subtlety remains in the status of auxiliary fields: using the goldstino superfield, all the auxiliary fields can be removed (by imposing for instance $XX^\dagger D^2\Phi = 0$ for a chiral superfield, or even $\frac{1}{4}D^2 X = f$ for the goldstino superfield) and replaced by combinations of other fields of the theory. For instance, the constraint proposed in [307] to remove fermions from chiral superfields, $\overline{D}_{\dot\alpha}(X\overline{\Phi}) = 0$, also removes the auxiliary field in Φ. Which of these auxiliary fields should be kept in the EFT is not clear, since different constraints can in principle be identified via the couplings between matter and goldstino, for a fixed lagrangian. This may for instance impact theories of (R-)axions (see e.g. [287]).

3.3.4 Constraints in extended SUSY and brane actions

There also exist constraints in extended supersymmetry, which are for instance introduced to describe 4D EFTs on branes preserving some amount of supersymmetry [312, 313], such as a $D3$-brane in 6D Minkowski space [314, 315]. Those describe partial breaking of supersymmetry, first argued to be impossible in 4D [316] before it was understood that such arguments rely on an ill-defined supersymmetry charge [314, 315]. Explicit realizations were later found [317–319] which introduce deformation parameters in the SUSY transformations, such that partial breaking occur. At the level of the vector superfield, it is implemented by a modification of (3.1.33) and (3.1.34):

$$\delta_{\tilde\xi}\Phi = i\sqrt{2}\tilde\xi W \ , \quad \delta_{\tilde\xi} W_\alpha = \sqrt{2}\left(\sigma^\mu \bar{\tilde\xi}\right)_\alpha \partial_\mu \Phi + i\sqrt{2}\tilde\xi_\alpha \left(\frac{\overline{D}^2\overline{\Phi}}{4} - m\right) \ , \tag{3.3.21}$$

such that a canonical prepotential $\mathcal{F} = \frac{\mathcal{W}^2}{4}$ in (3.1.35) now partially breaks $\mathcal{N} = 2$ SUSY down to $\mathcal{N} = 1$.

A constrained version of the deformed vector superfield is linked to brane actions [312] since, imposing $\mathcal{W}^2 = 0$ gives a dynamics which is the four-dimensional Born-Infeld action, i.e. the Maxwell part of the DBI action for the $D3$-brane mentioned earlier (see also (4.3.15)). Similarly, imposing $\mathcal{Z}^2 = 0$ on a deformed single-tensor multiplet yields the four-dimensional action for the fifth and sixth coordinates of the $D3$-brane [313], which appear as 4D scalars. Those constraints and actions can be recovered as infinite mass limits of linearly realized models [273].

The question of brane actions in setups where $\mathcal{N} = 2$ SUSY is fully broken down to $\mathcal{N} = 0$ is on the other hand open: the constraints mentioned earlier maintain an unbroken SUSY, and it would be useful to have constraints which yield the DBI action in setups with complete breaking. Additionally, it is not precisely known how to write a constraint which both incorporates the Maxwell and the scalar part of the DBI action for the 6D $D3$-brane, even though field redefinitions are believed to ensure the shift symmetry expected for the scalars [320]. Steps towards the identification of a constraint for the $\mathcal{N} = 2 \to \mathcal{N} = 0$ breaking have been made in [321] for both the vector and single-tensor superfields, but in a way which does not naturally yield a DBI structure.

It turns out that a well-defined procedure to build non-linearly realized actions similar to (3.3.17)-(3.3.20) exists for $\mathcal{N} = 2$, which could be used to write down a correct DBI action. Indeed, a chiral-chiral $\mathcal{N} = 2 \to \mathcal{N} = 0$ goldstino multiplet $\mathcal{X}$ was built in [322], and is such that

$$\mathcal{X}\tilde{D}_\alpha D_\beta \tilde{D}_\gamma \mathcal{X} = \mathcal{X}\tilde{D}_\alpha D_\beta D_\gamma \mathcal{X} = 0 \ , \tag{3.3.22}$$

which leaves as unconstrained fields two goldstini λ and $\tilde{\lambda}$ as its $\theta^2\tilde{\theta}$ and $\tilde{\theta}^2\theta$ components, as well as an auxiliary field F as its $\tilde{\theta}^2\theta^2$ component. Much like in (3.3.20), one can use it to impose

$$\mathcal{X}\mathcal{X}^\dagger\Phi = 0 \tag{3.3.23}$$

on any $\mathcal{N} = 2$ superfield Φ to remove its lowest component. If one does not care about a UV understanding, one can impose the following constraints on a vector superfield field strength $\mathcal{W}$:

$$\begin{aligned} \mathcal{X}\mathcal{X}^\dagger D_\alpha \mathcal{W} = \mathcal{X}\mathcal{X}^\dagger \tilde{D}_\alpha \mathcal{W} = \mathcal{X}\mathcal{X}^\dagger D_\alpha \tilde{D}_\beta \mathcal{W} = \mathcal{X}\mathcal{X}^\dagger (D^2 - \tilde{D}^2)\mathcal{W} = 0 \ , \\ \mathcal{X}\mathcal{X}^\dagger \left(\frac{1}{16} D^2 \mathcal{W} \tilde{D}^2 \mathcal{W} + \det\big(\eta_{\mu\nu} + \partial_\mu \mathcal{W} \partial_\nu \overline{\mathcal{W}}\big) \right) = 0 \ , \end{aligned} \tag{3.3.24}$$

where the first line removes from $\mathcal{W}$ everything but the complex scalar ϕ in Φ and the real part of its auxiliary field F. Then, a lagrangian

$$\mathcal{L} = eF + h.c. \tag{3.3.25}$$

yields the scalar DBI action for ϕ, plus a set of fermionic terms. This procedure, which could already be implemented at the $\mathcal{N} = 1$ level, can be generalized to explicitly encode the full (Maxwell+scalars) DBI action for the 6D $D3$-brane in the real part of F. However, this suffers from an obvious lack of UV origin, and only reintroduces in the constrained superfield language the freedom which existed in action-building at the level of the coset construction, whereas constrained superfields had the advantage of being more illuminating about the possible UV origin of the non-linear SUSY model under consideration.

3.4 An aside: a BPS classical solution for the Wess-Zumino model

Before closing this section on supersymmetry, we take a step aside and consider a model without SUSY breaking: the Wess-Zumino (WZ) model of a single chiral superfield [237]. We are interested in classical solutions of this model, and we present in section 3.4.2 such an exact classical solution, which is a Bogomol'nyi-Prasad-Sommerfield (BPS) solution (see section 3.4.1). In section 3.4.3, we then discuss two applications of our result: tree-level multi-particle amplitudes at threshold, and domain walls in supersymmetric QCD theories (a quick review of the latter topic is presented in appendix C). The content of this section follows [323], attached at the end of the thesis.

3.4.1 Generalities on the BPS condition

Let us start by reviewing the BPS condition, since we will heavily refer to it in what comes next. A field configuration is said to be BPS [324, 325] if it preserves some amount of supersymmetry. This straightforward definition has dramatic consequences, as we now see.

Let us start with the formulation of the BPS condition for a chiral superfield, which is the one we are mostly concerned with in this section. We consider scalar field configurations, with the fermions vanishing. Thus, the BPS condition amounts to requiring that fermions remain equal to zero when the preserved supersymmetry generators act. For a chiral superfield Φ such as the one of the WZ model, the variation of the fermion χ is, as we already saw above:

$$\delta_\xi \chi = i\sqrt{2}\sigma^m \overline{\xi} \partial_m \phi + \sqrt{2}\xi F \ . \tag{3.4.1}$$

In this section, we calculate multi-particle amplitudes or domain wall profiles, which are one-dimensional problems. Thus for simplicity and minimality, we restrict ourselves to configurations where $\phi(x^\mu) = \phi(z)$. Then, demanding that $\delta_\xi \chi = 0$ translates into

$$\overline{\xi}^1 \frac{d\phi}{dz} = -i\xi_1 F \text{ and } \overline{\xi}^2 \frac{d\phi}{dz} = i\xi_2 F \ . \tag{3.4.2}$$

Whenever the scalars verify $\frac{d\phi}{dz} = -e^{i2\theta}F$ for some real number θ, (3.4.2) can be satisfied. Using the on-shell value for F, for a trivial Kähler potential and a superpotential W, we find

$$\frac{d\phi}{dz} = e^{2i\theta} \frac{d\overline{W}}{d\overline{\phi}} \ . \tag{3.4.3}$$

This is the BPS condition for a chiral superfield.

The BPS condition makes it easier to solve the dynamics of a system. For instance, it is used to get non-trivial black hole solutions in SUGRA [254]. Why it helps to solve the dynamical equations can be understood as follows: (3.4.3) is a factorization of the equations of motion. Indeed, imposing the former is enough to satisfy the latter:

$$\frac{d^2\phi}{dz^2} = e^{2i\theta} \frac{d^2\overline{W}}{d\overline{\phi}^2} \frac{d\overline{\phi}}{dz} = \frac{d^2\overline{W}}{d\overline{\phi}^2} \frac{dW}{d\phi} = \frac{dV}{d\overline{\phi}} \ , \tag{3.4.4}$$

since $V = \left| \frac{dW}{d\phi}(\phi) \right|^2$ for a chiral superfield. Since the BPS condition is a first order one, it is indeed easier to solve than the equations of motion, which are second order. This is reminiscent of the fact that SUSY vacua are easier to find than usual vacua and are associated to vanishings of auxiliary fields.

The BPS condition has an other interpretation, which is the equivalent of what we saw in our $\mathcal{N}=2$ discussion of BPS states in section 3.1.3. Bogomol'nyi [324] indeed pointed out that it can be understood as the condition which minimizes the energy per unit surface of a topological defect (see [326]), such as a time independent wall [269, 327, 328]:

$$\forall\,\theta\ ,\ \mathcal{E}=\int dz\left(\left|\frac{d\phi}{dz}\right|^2+\left|\frac{dW}{d\phi}\right|^2\right)=\int dz\left|\frac{d\phi}{dz}-e^{2i\theta}\frac{d\overline{W}}{d\overline{\phi}}\right|^2+2\,\mathrm{Re}(e^{-2i\theta}\Delta W)\ , \tag{3.4.5}$$

where $\Delta W=W(z=+\infty)-W(z=-\infty)$. The fact that this condition is valid whatever θ implies the so-called BPS bound:

$$\mathcal{E}\geq 2|\Delta W|\ . \tag{3.4.6}$$

In order to saturate this bound, one must again enforce (3.4.3). We saw already something called a BPS bound in section 3.1.3, so let us connect the two discussions by computing the central charge in our setup, using the results of [269]. Writing

$$\overline{\Phi}\Phi=...+\theta^\alpha\overline{\Phi}\Phi|_{\theta\overline{\theta},\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}+\theta^2\overline{\Phi}\Phi|_{\theta^2}+\theta^2\overline{\theta}_{\dot{\alpha}}\overline{\Phi}\Phi|^{\dot{\alpha}}_{\theta^2\overline{\theta}}+\overline{\theta}^2\theta^\alpha\overline{\Phi}\Phi|_{\overline{\theta}^2\theta,\alpha}+\theta^2\overline{\theta}^2\overline{\Phi}\Phi|_{\theta^2\overline{\theta}^2} \tag{3.4.7}$$

and $W(\Phi)=W(\Phi)|_{\theta^0}+\sqrt{2}\theta^\alpha W(\Phi)|_{\theta,\alpha}+\theta^2W(\Phi)|_{\theta^2}$, the SUSY lagrangian is

$$\mathcal{L}=\overline{\Phi}\Phi\big|_{\theta^4}+\left(W(\Phi)\big|_{\theta^2}+h.c.\right) \tag{3.4.8}$$

Its variation (using (3.1.8)) is

$$\begin{aligned}\delta_\xi\mathcal{L}&=\frac{i}{2}\overline{\xi}\overline{\sigma}^m\partial_m\left(\overline{\Phi}\Phi\big|_{\overline{\theta}^2\theta}\right)+i\sqrt{2}\overline{\xi}\overline{\sigma}^m\partial_m\left(W(\Phi)\big|_\theta\right)+h.c.\\&=\xi\partial_m\Delta^m+\overline{\xi}\partial_m\overline{\Delta}^m\ \text{with}\ \Delta^m=\frac{i}{2}\sigma^m\left(\overline{\Phi}\Phi\big|_{\theta^2\overline{\theta}}\right)+i\sqrt{2}\sigma^m\overline{W}(\overline{\Phi})\big|_{\overline{\theta}}\end{aligned} \tag{3.4.9}$$

And

$$\begin{aligned}\delta_\eta\Delta^m=&\frac{i}{2}\sigma^m\left(2\overline{\eta}\overline{\Phi}\Phi\big|_{\theta^4}+i\overline{\sigma}^n\eta\partial_n\left(\overline{\Phi}\Phi\big|_{\theta^2}\right)-\frac{i}{2}\overline{\eta}_{\dot{\beta}}\overline{\sigma}^{n,\dot{\beta}\alpha}\partial_n\left(\overline{\Phi}\Phi|_{\theta\overline{\theta},\alpha\dot{\gamma}}\right)\epsilon^{\dot{\gamma}\dot{\alpha}}\right)\\&+i\sqrt{2}\sigma^m\left(i\sqrt{2}\overline{\sigma}^n\eta\partial_n\left(\overline{W}(\overline{\Phi})\big|_{\overline{\theta}^0}\right)+\sqrt{2}\overline{\eta}\overline{W}(\overline{\Phi})\big|_{\overline{\theta}^2}\right)\end{aligned} \tag{3.4.10}$$

Keeping only the ηQ contributions:

$$\begin{aligned}\delta_\alpha\Delta^m_\beta=-\frac{1}{2}\sigma^m_{\beta\dot{\beta}}\overline{\sigma}^{n,\dot{\beta}\gamma}\epsilon_{\gamma\alpha}\left(\partial_n\left(\overline{\Phi}\Phi\big|_{\theta^2}\right)+4\partial_n\left(\overline{W}(\overline{\Phi})\big|_{\overline{\theta}^0}\right)\right)&=-\frac{1}{2}\sigma^m_{\beta\dot{\beta}}\overline{\sigma}^{n,\dot{\beta}\gamma}\epsilon_{\gamma\alpha}\left(\partial_n\left(F_\phi\overline{\Phi}\right)+4\partial_n\left(\overline{W}(\overline{\phi})\right)\right)\\&=-\frac{1}{2}\sigma^m_{\beta\dot{\beta}}\overline{\sigma}^{n,\dot{\beta}\gamma}\epsilon_{\gamma\alpha}\left(\partial_n\left(-\frac{d\overline{W}}{d\overline{\phi}}\overline{\phi}\right)+4\frac{d\overline{W}}{d\overline{\phi}}\partial_n\overline{\phi}\right)\end{aligned} \tag{3.4.11}$$

From which we get

$$\{Q_\alpha,Q_\beta\}=-\int d^3x(\delta_\alpha\Delta^0_\beta+\delta_\beta\Delta^0_\alpha)=\sigma^0_{\beta\dot{\beta}}\overline{\sigma}^{n,\dot{\beta}\gamma}\epsilon_{\gamma\alpha}\int d^3x\left(-\partial_n\left(\frac{d\overline{W}}{d\overline{\phi}}\overline{\phi}\right)+4\frac{d\overline{W}}{d\overline{\phi}}\partial_n\overline{\phi}\right) \tag{3.4.12}$$

For a BPS wall of finite surface energy:

$$\begin{aligned}\{Q_\alpha,Q_\beta\}&=\sigma^0_{\beta\dot{\beta}}\overline{\sigma}^{z,\dot{\beta}\gamma}\epsilon_{\gamma\alpha}\mathrm{Volume}_{(x,y)}\int dz\left(-\frac{d}{dz}\left(\frac{d\overline{W}}{d\overline{\phi}}\overline{\phi}\right)+4\frac{d}{dz}(\overline{W}(\overline{\phi})\right)\\&=\sigma^0_{\beta\dot{\beta}}\overline{\sigma}^{z,\dot{\beta}\gamma}\epsilon_{\gamma\alpha}\mathrm{Volume}_{(x,y)}\left[-\frac{d\overline{W}}{d\overline{\phi}}\overline{\phi}+4\overline{W}(\overline{\phi})\right]_{z=-\infty}^{z=+\infty}\\&=\sigma^0_{\beta\dot{\beta}}\overline{\sigma}^{z,\dot{\beta}\gamma}\epsilon_{\gamma\alpha}\mathrm{Volume}_{(x,y)}\left[e^{-2i\theta}\frac{d\phi}{dz}\overline{\phi}+4\overline{W}(\overline{\phi})\right]_{z=-\infty}^{z=+\infty}\\&=4\sigma^0_{\beta\dot{\beta}}\overline{\sigma}^{z,\dot{\beta}\gamma}\epsilon_{\gamma\alpha}\mathrm{Volume}_{(x,y)}\Big[\overline{W}(z=+\infty)-\overline{W}(z=-\infty)\Big]\end{aligned} \tag{3.4.13}$$

since at infinity, one must have $d\phi/dz \xrightarrow[z\to\infty]{} 0$ to ensure a finite surface energy. We thus see that the variation of the superpotential ΔW, which entered the BPS bound, is as expected linked to the central charge.

In addition, the phase θ which enters the BPS equation (3.4.3) bears some geometrical meaning. Indeed:

$$\frac{d\phi}{dz} = e^{2i\theta}\frac{d\overline{W}}{d\overline{\phi}} \implies \frac{dW}{dz} = \frac{dW}{d\phi}\frac{d\phi}{dz} = e^{2i\theta}\left|\frac{dW}{d\phi}\right|^2 \implies \arg\left(\frac{dW}{dz}\right) = 2\theta \tag{3.4.14}$$

(if the trajectory does not follow a critical path in the W plane). Hence, the image $W(\phi(z))$ of the domain wall in the W-plane is a straight line making an angle 2θ with the real axis. This gives a constant of motion, which may be useful to obtain some information without knowing the full solution of (3.4.3) (see e.g. [328]).

3.4.2 The result

We now present and discuss a solution to the BPS equations for a rather general class of WZ models. Consider the following superpotential for a chiral superfield Φ:

$$W = \frac{1}{2}\Phi^2 + \frac{1}{p}\Phi^p , \tag{3.4.15}$$

where we do not place a restriction on the allowed value of the index p, and where couplings can be trivially reinstated by scaling. The associated scalar potential is

$$V = \left|\Phi + \Phi^{p-1}\right|^2 , \tag{3.4.16}$$

and if p is positive one might seek domain wall solutions between the supersymmetric minimum at $\Phi = 0$ and the $p-2$ supersymmetric minima at $\Phi = e^{i\frac{n\pi}{(p-2)}}$, $n \in \mathbb{Z}$. We look for one-dimensional BPS solutions, which have thus to verify (3.4.3), here specialized to (3.4.15):

$$\frac{d\phi}{dt} = e^{2i\theta}\left(\overline{\phi} + \overline{\phi}^{p-1}\right) , \tag{3.4.17}$$

where t is the only coordinate and θ is an arbitrary constant angle. While solving (3.4.17) with real ϕ is trivial, the conjugation on the right hand side makes a general complex solution much more difficult to find. Our central result is the following solution to (3.4.17):

$$\phi(z,\overline{z}) = \frac{z\left(1 + \frac{\overline{z}^{p-2}-z^{p-2}}{2p}\right)}{\left(\left(1 + \frac{\overline{z}^{p-2}-z^{p-2}}{2p}\right)^p + \frac{\overline{z}^{p-2}\left(\left(1-\frac{\overline{z}^{p-2}-z^{p-2}}{2p}\right)^p - \left(1+\frac{\overline{z}^{p-2}-z^{p-2}}{2p}\right)^p\right)}{\overline{z}^{p-2}-z^{p-2}}\right)^{\frac{1}{p-2}}} , \tag{3.4.18}$$

where $z = e^{t+i\theta}$.

This is a generalisation of the BPS domain-wall solution of [329] (with appropriate shifts in ϕ) which considered $p = 3$ and real ϕ. Indeed taking $\theta = \frac{\pi}{p-2}$ we find

$$\phi(t) = \left(\frac{-e^{(p-2)t}}{1 + e^{(p-2)t}}\right)^{\frac{1}{p-2}} , \tag{3.4.19}$$

which reduces for $p = 3$ to the non-singular domain wall solution

$$\phi(t) \;=\; -\frac{e^t}{1+e^t} \tag{3.4.20}$$

connecting the two minima ($\phi(-\infty) = 0$ and $\phi(\infty) = -1$) of the WZ model. It is also related to the softly broken $O(2)$ models of [330] (which took $p = 3$), examined in the context of multi-particle amplitudes on threshold, and some other works in this domain (which typically considered real ϕ). However, the solution we present here seems to have a much richer structure and is much more general than those that have been previously considered.

3.4.3 Applications

As we shall now see, (3.4.18) has applications in different areas of physics. Those concern multi-particle amplitudes, and Seiberg duality in supersymmetric QCD.

SQCD domain walls

(3.4.18) is of interest in SQCD, whose dynamics for various numbers of colours N_c and flavours N_f has been studied in great detail over the years (for some details and conventions, see appendix C). We are particularly interested in the free magnetic regime, $N_c + 1 < N_f < \frac{3}{2}N_c$, in which there exist WZ domain walls described by (3.4.18), as we shall now see.

Consider SQCD in such a phase, with a quartic superpotential

$$W^{(\mathrm{el})} \;=\; \frac{1}{\mu_X}\,\mathrm{Tr}\Big[(Q\cdot\tilde{Q})^2\Big]\;, \tag{3.4.21}$$

where the dot indicates colour contractions and the trace is over flavours. This operator could be generated by the integrating out of heavier fields of mass $\mathcal{O}(\mu_X)$, as happens in the duality cascade for example [331]. For physical consistency we therefore require that $\mu_X > \Lambda$, with Λ being the dynamical scale of the electric theory. As described in appendix C, below the scale Λ the electric SQCD theory above becomes strongly coupled, and physics is best described by a magnetic dual [332]. This theory also has N_f flavours, but $SU(N)$ gauge group, where $N = N_f - N_c$, and classical superpotential

$$W^{(\mathrm{mag})}_{\mathrm{cl}} = h\, q\Phi\tilde{q} + \frac{\mu_\Phi}{2}\,\mathrm{Tr}\big(\Phi^2\big)\;, \tag{3.4.22}$$

where Φ^i_j are the flavour mesons of the IR free theory, h is a Yukawa coupling of order unity, and q^a_i, $\tilde{q}^j_a$ are fundamental and antifundamental quarks of $SU(N)$. The Φ mass-term is $\mu_\Phi \approx \Lambda^2/\mu_X \ll \Lambda$.

This theory has supersymmetric minima at the origin, while the remaining ones are separated from it by a domain wall, where Φ develops a much larger VEV. Along this direction one is still in a pure $SU(N)$ Yang-Mills theory, but nonperturbative contributions to the superpotential become important. Including these, the complete superpotential for the mesons is of the form (3.4.15):

$$W^{(\mathrm{mag})} \;=\; \frac{\mu_\Phi}{2}\,\mathrm{Tr}\big(\Phi^2\big) + N\left(\frac{h^{N_f}\det_{N_f}\Phi}{\Lambda^{N_f-3N}}\right)^{\frac{1}{N}}\;, \tag{3.4.23}$$

where the exponent $\frac{N_f}{N} \equiv p$ is generically a rational number. In the regime of interest, $\frac{3}{2}N_c > N_f \geq N_c + 2$, we have

$$3 < p \leq \frac{N_f}{2} \; . \tag{3.4.24}$$

In principle (3.4.18) contains exact domain wall solutions for this magnetic theory, for any p.

To find them let us first locate the minima which are along $\phi_i^j = \delta_i^j \phi$ (where we use ϕ to also stand for the trace component). Setting $F_\Phi = 0$ we find nonperturbatively generated SUSY preserving minima at

$$\langle \phi_i^j \rangle = \delta_i^j \phi_0 = \delta_i^j \, \Lambda \left(-h^{\frac{N_f}{N_f - N_c}} \frac{\Lambda}{\mu_\Phi} \right)^{\frac{N_f - N_c}{N_f - 2N_c}} \; . \tag{3.4.25}$$

The exponent is negative so that $\langle \phi \rangle < \Lambda$ as required for the minima to be found in the macroscopic theory. Also note that, as there are no massless quarks, there are generically $2N_c - N_f$ solutions corresponding to the roots of -1. Now for the domain walls we define

$$\hat{\Phi} \;=\; \frac{\Phi}{|\phi_0|} \; ; \qquad \hat{W} \;=\; \frac{W}{\mu_\Phi |\phi_0|^2} \; , \tag{3.4.26}$$

giving $\hat{W} = \frac{\hat{\Phi}^2}{2} + \frac{\hat{\Phi}^p}{p}$ with $p = N_f/N$. We thus see that we recover (3.4.15) such that (3.4.19) describes the structure of the $\hat{\Phi}$ domain wall.

Multi-particle amplitudes in generalized Wess-Zumino models

Our second application concerns multi-particle amplitudes, which have been investigated for a long time [330, 333–340], and received renewed interest within discussions of the Higgsplosion mechanism [341–344]. Quantities of interest include the tree-level threshold amplitudes describing the decay of an off-shell particle to on-shell ones, all taken to be at rest. Our solution (3.4.18) can be understood in this respect as the generating function of such tree-level multi-particle amplitudes at kinematical threshold for the generalized Wess-Zumino models (3.4.15). One can indeed show that such a generating function must verify a BPS condition[3], consistently with the fact that a specific limit of (3.4.18) has been previously identified as a BPS domain wall solution [329, 346, 347]. We leave the review of standard techniques to obtain tree-level multi-particle amplitudes at kinematical threshold to [323], and we only use here the results.

We are interested in evaluating tree-level amplitudes connecting an ingoing off-shell particle to outgoing on-shell ones, all taken to be at rest[4], for the generalized Wess-Zumino model of a chiral superfield Φ with canonical Kähler potential and superpotential given in (3.4.15). We choose $p - 3 \in \mathbb{N}$ in this section. The kinematic situation is summed up in Figure 3.2. Since there are two scalar excitations, the outgoing state is labelled by two integers m and n.

[3]The fact that the generating function of multi-particle amplitudes verifies a BPS condition can be understood from [345]: smooth field configurations which solve the equations of motion and originate from a supersymmetric vacuum state must verify the BPS condition. (3.4.28) which defines the generating function (3.4.18) is thus enough to ensure that it verifies (3.4.3).

[4]Exact results are much harder to obtain at loop-level or out of threshold [337–340, 348–350].

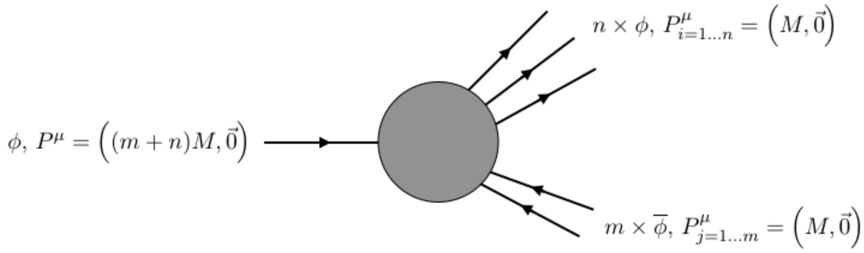


Figure 3.2: Kinematic setup (particles/anti-particles are represented using direct/reversed arrows)

We call a_{nm} the amplitude $\phi \to n \times \phi + m \times \overline{\phi}$. Defining a generating function:

$$A(z, \overline{z}) = \sum_{n,m} \frac{i a_{nm}}{n! m! ((n+m)^2 - 1)} z^n \overline{z}^m \ , \tag{3.4.27}$$

we show that it verifies

$$\begin{cases} \partial_t^2 A = (p-1) A^{p-1} \overline{A}^{p-2} + A^{p-1} + (p-1) A \overline{A}^{p-2} + A = \frac{\partial}{\partial \overline{A}} V(A, \overline{A}) \\ A(t = -\infty, \theta) = 0 \ , \quad \partial_t A(t = -\infty, \theta) = e^{i\theta} \end{cases} , \tag{3.4.28}$$

where we defined $z = e^{t + i\theta}$.

One can check that our solution (3.4.18) verifies all the conditions listed in (3.4.28) and is consequently the generating function of the diagrams of Figure 3.2. Differentiating it with respect to z and $\overline{z}$ yields the a_{nm}.

In the original papers on multi-particle amplitudes, which dealt with a model of a real scalar σ with cubic or quartic interactions, it was noticed that a_n (the $\sigma \to n \times \sigma$ amplitude) grows as $n!$, which could be a threat to unitarity at tree-level in those renormalizable theories. One question which can be asked is: what about this now? Actually, it is straightforward to see that the a_{nm}s still have this factorial growth with n and m. Indeed, the fermions of the model do not participate in the calculation of the a_{nm}s because we are at tree-level and the theory conserves fermion number. Thus, the only sign of SUSY in the calculation is the specific squared form for the potential. This proves not to change qualitatively the divergent behaviour.

3.4.4 Additional details

Some additional details can be found in [323]. First, as we said earlier, we explain there why the generating function of multi-particle amplitudes verifies (3.4.28).

Second, we study modifications of (3.4.28) and (3.4.18) when soft SUSY breaking is added to the model, which would be relevant for multi-particle production in the MSSM for instance. As an illustration, solutions are known when one adds specific soft terms to the potential (3.4.16) of the WZ model, here for $p = 3$, with reinstated couplings:

$$V = \left| \lambda \phi^2 + m \phi \right|^2 + \frac{\delta m^2}{2} \left(\frac{\phi - \overline{\phi}}{2i} \right)^2 \ , \tag{3.4.29}$$

which is associated to the following classical solution[5]:

$$\phi(z,\overline{z}) = \frac{z + \frac{\lambda}{m} i(z-\overline{z}) \frac{i(z-\bar{z}) + i\left(\sqrt{2}\frac{m_{\mathrm{Im}(\phi)}}{|m|} - 1\right)^2 (z+\bar{z})}{4\left(2\frac{m^2_{\mathrm{Im}(\phi)}}{m^2} - 1\right)}}{1 - \frac{\lambda}{m}\frac{z+\bar{z}}{2} + \left(\frac{\lambda}{m}\right)^2 \frac{(z-\bar{z})^2}{4\left(2\frac{m^2_{\mathrm{Im}(\phi)}}{m^2} - 1\right)} - \left(\frac{\lambda}{m}\right)^3 \frac{\left(\sqrt{2}\frac{m_{\mathrm{Im}(\phi)}}{|m|} - 1\right)^4 (z-\bar{z})^2 (z+\bar{z})}{8\left(2\frac{m^2_{\mathrm{Im}(\phi)}}{m^2} - 1\right)^3}} , \qquad (3.4.30)$$

where $m^2_{\mathrm{Im}(\phi)} = 2m^2 + \delta m^2$. However, the WZ model allows for the following soft terms [351] in addition to the supersymmetric potential, which are more general than (3.4.29):

$$V = \left|\lambda\phi^2 + m\phi\right|^2 + \delta m^2 |\phi|^2 + \left(\mu_3\phi^3 + \mu_2\phi^2 + h.c.\right) , \qquad (3.4.31)$$

for which no solution is known. In addition, the generalization of (3.4.30) to arbitrary p is also unknown. Intermediate results can thus be found in [323], in particular when the soft terms are as follows:

$$V = \left|\lambda\phi^2 + m\phi\right|^2 + \delta m^2 |\phi|^2 . \qquad (3.4.32)$$

Third, we comment a bit further about the supersymmetric minima of (3.4.22), in particular about their counting in connection with the Witten index [352].

Finally, we quickly describe how (3.4.18) was found, emphasizing on the different techniques used.

3.5 Conclusions

In this second part, we dealt with supersymmetry and supersymmetric model building in particle physics. Our main focus was on supersymmetry breaking, which we discussed at the level of explicit models of spontaneous SUSY breaking and using the low-energy universal framework for EFTs with broken SUSY, namely non-linear SUSY.

We investigated a full model of spontaneous breaking which is again a gauged clockwork model, and which, for similar reasons which ensured that the axions of section 2.2.2 had a small mass, strongly suppresses the SUSY breaking scale even when few additional particles are present. In this model, there are also vector or axion modes with a clockwork profile, but the latter do not induce dramatic effects if we want to generate a phenomenologically relevant SUSY breaking scale. Nevertheless, clockwork-like hierarchies appear in the SUSY breaking auxiliary fields, which opens the possibility to hierarchically mediate SUSY breaking to observable matter, in order for instance to generate split spectra for the MSSM superpartners.

We studied non-linear SUSY, which is the general framework to describe EFTs with SUSY breaking. We reminded different approaches to this, the coset construction and constrained superfields, both in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SUSY. We briefly commented on an ambiguity about

[5]This solution was originally derived in [329] in the context of softly broken $O(2)$-symmetric models. In [323], we explicitly connect our $p = 3$ solution to those of [329].

constraining auxiliary fields which may be relevant for phenomenology, as well as on how to realize DBI actions with complete SUSY breaking using constrained superfields.

Finally, we ended with a presentation of an exact BPS classical solution of the Wess-Zumino model, which is relevant for multi-particle amplitudes and as a domain wall solution in strongly coupled SQCD. We hope to be able to extend it to broken SUSY scenarii relevant for phenomenology.

4 | String theory

Many great physical questions have been given a great answer, some of which were presented earlier in this thesis. String theory doubtlessly falls into this category. Even though it is now almost fifty years old, it is still a subject of intense research and developments, due to the amazing findings which were periodically made along its study. In this last part of the thesis, we study some aspects of string theory which have to do with its phenomenology. But first, we try to do justice to some of the great ideas which are associated to its name.

A first way of being amazed by string theory is to adopt a historical perspective. Indeed, string theory was originally developped as a mathematical framework to describe strong interactions [353–355]. However, it quickly was understood to be a theory of quantum gravity [356,357], providing a fantastic tool to explore the consequences of associating gravity and quantum mechanics, as well as a theory which incorporates the usual features of particle physics. Hence, it became a serious candidate to realize a ultraviolet completion of the standard model, providing the framework to unify all forces in a single quantum description. Spectacularly, all the inconsistencies which were foreseen along the way ended up being solved in a remarkably clever way. A great example of such a phenomenon is the discovery of the Green-Schwarz (GS) mechanism for anomaly cancellation [229]. However, the current picture of string theory is still far from complete, as illustrated by the efforts towards developping an utter formulation of M-theory [358], an off-shell string field theory [359, 360], or by the consequences of the AdS-CFT correspondence [270], which remarkably grants string theory the additional status of being a holographic dual description of (strongly coupled) non-gravitationnal physics. This last feature may be a general property of gravity itself that string theory "only" precisely illustrates, much like the way it describes the microstates behind black hole entropy [361].

This historical development can be linked to a second reason of being thrilled: (many) string theory predictions seem to be necessary once one assumes the very starting point. Even though it has been a great achievement around 1905, the action describing the propagation of a point particle according to the rules of special relativity is far from telling us enough about what can happen to the particles we know, so it had to be extended to interacting theories to match observations. On the other hand, the "straightforward" string generalization of the free relativistic particle dynamics seems to already encompass all the possible string interactions, which include structures as involved as non-abelian symmetries or graviton dynamics, without much parameter freedom. Even more strikingly, the theory's consistency is enough for it to auto-reveal some of its features, such as the number of spacetime dimensions it has to live in [362] or the interpretation of D-branes as dynamical objects charged under string Ramond-Ramond forces [363]. It is thus no surprise that the structures which came out of the study of string theory managed to sometimes motivate ground-breaking research in mathematics.

Enthusiastic after such a description, one could ask: what is then the status of string theory phenomenology? There, it is tempting to say that the situation is slightly less pleasant than imagined when formally studying the theory. Indeed, the very high scales associated to string dynamics make it hard to test unambiguously. In addition, as we have said earlier, supersymmetry has not been found at the LHC, although it would have been encouraging to observe this (naive) prediction of low-energy string theory, which fitted so well the expectations of particle physicists. For this reason, we discuss again in section 4.2 supersymmetry breaking, albeit with a stringy perspective this time, since we mention mechanisms thanks to which supersymmetry is broken at the string or compactification scale. Then, we move to other aspects of string phenomenology: we first look in section 4.3 at string theory axions, which are another generic low-energy prediction of the theory, hence connecting our current focus with topics we covered earlier. Finally, we upgrade our notion of genericity to a stricter sense and discuss the swampland program [364]. The latter arose from the (discouraging at first) observation that the freedom left in defining the string setup which may describe nature, although tinier than the field theory one, is still large enough such that there is an incredibly high number of possible inequivalent physical worlds [365], which are still under scrutiny today. The swampland program then aims at trying, quite successfully as we discuss in section 4.4, to establish criteria which characterize those worlds and make them falsifiable.

As a concluding comment, it seems important to stress that, even if string theory eventually proves not to be a correct theory of our world (or even of strongly coupled physics), it has been a fantastically fruitful playground, which led to great insights about mechanisms and topics (such as extra dimensions, higher spin theories or the field content of consistent effective field theories) which may very well be incorporated in a more accurate theory. The study of string theory could in principle be justified on the basis of this remark alone.

4.1 Spectra of type IIB/I string theories

To smoothly progress towards the details of string theory we want to investigate, we first follow in this section a textbook approach and compute the spectrum of type IIB and type I string theories. It will be useful later to define the field content of string effective field theories and in order to derive one-loop vacuum amplitudes. We only present the very limited amount of string theory basics which is necessary to compute such amplitudes or extend their computations to more complicated geometrical setups, so this section should not be thought of as a thorough review. We focus on types IIB/I for conciseness and because their vacuum amplitudes will be discussed at length in sections 4.1.6, 4.1.7, 4.2 and 4.4.2. The discussion of the spectrum of type IIA string theory is almost identical, whereas the heterotic string demands some modifications of the action principle. The contents of this section heavily draws from [366–369].

4.1.1 The world-sheet theory and the constraints

String theory, as a quantum theory of fundamental interactions, is understood as the quantized version of the classical dynamics which generalizes to the string the propagation of a particle.

The classical equations describing the latter can be derived from an action principle, by demanding that the worldline element associated to the path $X^\mu(\tau)$ followed by the particle in $d-$dimensional Minkowski spacetime be extremal. For a particle of mass m, the associated action is:

$$S_{\text{particle}} = -m \int d(\text{Proper time}) = -m \int d\tau \sqrt{-\dot{X}^2} \ , \tag{4.1.1}$$

where τ is a parameter along the worldline and $\dot{X}^2 = \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}$[1]. This action is classically equivalent to the following one:

$$S'_{\text{particle}} = \frac{1}{2}\int d\tau \left(e^{-1}\dot{X}^2 - em^2\right) \ , \tag{4.1.2}$$

which has the advantage of also being valid for massless particles.

The generalization to the string leads to the Nambu-Goto action [370, 371], which measures the worldsheet area element spanned along the propagation of the string:

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{-h} \ , \tag{4.1.3}$$

with $d^2\sigma = d\tau d\sigma$, $h = \det(h_{ab})$ and $h_{ab} = \partial_a X^\mu \partial_b X_\mu$, $X^\mu(\tau,\sigma)$ being the spacetime location where the point labelled by σ along the string lies when the parametrization time is τ. The boundaries of the $\sigma \in [\sigma_b, \sigma_f]$ integral, as well as the associated boundary conditions, are defined appropriately for closed and open strings. α' is a dimensionful parameter, known as the Regge slope, and it defines the tension of the string $T = \frac{1}{2\pi\alpha'}$. The Nambu-Goto action is classically equivalent to the Polyakov action [372–374]:

$$S_{\text{P}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-g} g^{ab} h_{ab} \ , \tag{4.1.4}$$

where g_{ab} is a Minkowskian worldsheet metric. This action has symmetries: there is a global Poincaré symmetry acting on the X^μ as one would expect from special relativity, a local reparametrization invariance of the worldsheet coordinates (τ,σ) and a local Weyl rescaling of the worldsheet metric. Those two local transformations can be gauge-fixed, up to conformal transformations cancelled by a Weyl rescaling, by fixing $g_{ab} = \eta_{ab}$, such that the action becomes

$$S_{\text{P}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial^a X^\mu \partial_a X_\mu \ , \tag{4.1.5}$$

where it is now understood that worldsheet indices are contracted using η as well. This action must be supplemented by the equations of motion which arise from varying S_{P} with respect to the (non-dynamical) g before it is gauge-fixed:

$$T_{ab} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{P}}}{\delta g^{ab}} = 0 \implies \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\eta_{ab}\partial^c X^\mu \partial_c X_\mu = 0 \ , \tag{4.1.6}$$

[1] Any greek index in this section refers to a flat space index and, when contracted, is implicitely contracted using η.

where we defined the energy-momentum tensor T. (4.1.5) and (4.1.6) are the starting point for the quantization of the bosonic string dynamics.

For the type II/I theories, which are superstring theories, one actually adds to the action spinor degrees of freedom on the worldsheet[2] $\Psi^\mu(\tau,\sigma)$ to find the globally supersymmetric superstring action in the Ramond-Neveu-Schwarz (RNS) formalism [242, 243]:

$$\begin{aligned} S_{\text{RNS}} &= -\frac{1}{4\pi}\int d^2\sigma \left[\frac{1}{\alpha'}\partial^a X^\mu \partial_a X_\mu + \overline{\Psi}^\mu \gamma^a \partial_a \Psi_\mu\right] \\ &= -\frac{1}{4\pi}\int d^2\sigma \left[\frac{1}{\alpha'}\partial^a X^\mu \partial_a X_\mu - i\psi^\mu(\partial_\tau+\partial_\sigma)\psi_\mu + i\tilde{\psi}^\mu(-\partial_\tau+\partial_\sigma)\tilde{\psi}_\mu\right] \\ &= -\frac{1}{4\pi}\int d^2 z \left[\frac{2}{\alpha'}\partial X^\mu \overline{\partial} X_\mu - i\psi^\mu \overline{\partial}\psi_\mu + i\tilde{\psi}^\mu \partial \tilde{\psi}_\mu\right] , \end{aligned} \tag{4.1.7}$$

where $\gamma^{a=0,1} \equiv \left\{\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right\}$ verify the 2-dimensional Clifford algebra, $\overline{\Psi}^\mu \equiv \Psi^{\mu\dagger}(i\gamma^0)$, $\Psi^\mu \equiv \begin{pmatrix}\tilde{\psi}^\mu \\ \psi^\mu\end{pmatrix}$ verifies the Majorana condition $\Psi^{\mu *} = \Psi^\mu$ and we defined $z \equiv \sigma - \tau, \overline{z} \equiv \sigma + \tau, \partial \equiv \partial_z, \overline{\partial} \equiv \partial_{\overline{z}}$. One also adds to this action the following constraints, called Virasoro constraints:

$$\frac{2}{\alpha'}(\partial X^\mu)^2 - i\psi^\mu\partial\psi_\mu = 0 \ , \quad \psi^\mu \partial X_\mu = 0 \ , \quad \frac{2}{\alpha'}(\overline{\partial} X^\mu)^2 + i\tilde{\psi}^\mu\overline{\partial}\tilde{\psi}^\mu = 0 \ , \quad \tilde{\psi}^\mu \overline{\partial} X_\mu = 0 \ . \tag{4.1.8}$$

As with (4.1.5) and (4.1.6), (4.1.7) and (4.1.8) can be derived from a complete reparametrization invariant (and locally supersymmetric) action, see e.g. [366, 369, 375], in which case (4.1.8) could be expressed in terms of the different components of the energy-momentum tensor supermultiplet.

4.1.2 The mode expansions and the different sectors

Varying (4.1.7) with respect to the fields gives

$$\begin{aligned} \delta S_{\text{RNS}} = -\frac{1}{4\pi}\Bigg(&\int d^2 z \Big(-\frac{4}{\alpha'}\delta X^\mu \partial\overline{\partial} X_\mu - 2i\delta\psi^\mu \overline{\partial}\psi_\mu + 2i\delta\tilde{\psi}^\mu \partial \tilde{\psi}_\mu \Big) \\ &+ \left[\int d\tau \left(\frac{2}{\alpha'}\delta X^\mu \partial_\sigma X_\mu + i\delta\psi^\mu\psi_\mu - i\delta\tilde{\psi}^\mu\tilde{\psi}_\mu\right)\right]_{\sigma=\sigma_b}^{\sigma=\sigma_f}\Bigg) \ . \end{aligned} \tag{4.1.9}$$

The different cases we consider to make the boundary terms vanish depend on whether we talk about closed or open strings. We choose without loss of generality $\sigma_b = 0$ and $\sigma_f = \pi$, and the cases we consider for the boundary conditions are presented in Table 4.1[3]. For closed strings, they are chosen such that the action is uniquely defined when we go around the string once, whereas for open strings, they are chosen such that the boundary variation in (4.1.9) vanishes.

The coordinates can be Fourier-expanded along the compact σ direction. For closed strings,

[2] We call X^μ the bosonic coordinates and Ψ^μ the fermionic ones, since Ψ^μs are spinors of the worldsheet but vector with respect to the spacetime Poincaré symmetry.

[3] We first consider strings which are free to propagate in the whole d-dimensional spacetime, D-branes will be discussed later. We assumed that Lorentz invariance is respected by the boundary conditions.

Type of strings	Closed	Open
Boundary values	$X^\mu(\tau,\sigma+\pi)=X^\mu(\tau,\sigma)$ $\begin{cases}\text{R-R: } \begin{cases}\psi^\mu(\tau,\sigma)=\psi^\mu(\tau,\sigma+\pi)\\ \tilde\psi^\mu(\tau,\sigma)=\tilde\psi^\mu(\tau,\sigma+\pi)\end{cases}\\ \text{R-NS: } \begin{cases}\psi^\mu(\tau,\sigma)=\psi^\mu(\tau,\sigma+\pi)\\ \tilde\psi^\mu(\tau,\sigma)=-\tilde\psi^\mu(\tau,\sigma+\pi)\end{cases}\\ \text{... (NS-R and NS-NS)}\end{cases}$	$\partial_\sigma X^\mu(\tau,0)=\partial_\sigma X^\mu(\tau,\pi)=0$ $\psi^\mu(\tau,0)=\tilde\psi^\mu(\tau,0)$ $\begin{cases}\text{R: } \psi^\mu(\tau,\pi)=\tilde\psi^\mu(\tau,\pi)\\ \text{NS: } \psi^\mu(\tau,\pi)=-\tilde\psi^\mu(\tau,\pi)\end{cases}$

Table 4.1: Boundary conditions for the different strings and sectors, without localized D-branes NS stands for Neveu-Schwarz and R stands for Ramond

we get

$$X^\mu(\tau,\sigma)=\sum_n X_n^\mu(\tau)e^{in\sigma}\ ,\quad \begin{cases}\text{R-R: } \psi^\mu(\tau,\sigma)=\sum_n\psi_n^\mu(\tau)e^{2in\sigma}\ ,\quad \tilde\psi^\mu(\tau,\sigma)=\sum_n\tilde\psi_n^\mu(\tau)e^{2in\sigma}\\ \text{R-NS: } \psi^\mu(\tau,\sigma)=\sum_n\psi_n^\mu(\tau)e^{2in\sigma}\ ,\quad \tilde\psi^\mu(\tau,\sigma)=\sum_n\tilde\psi_n^\mu(\tau)e^{2i\left(n+\frac{1}{2}\right)\sigma}\\ \ldots\end{cases} \tag{4.1.10}$$

One the other hand, the coordinates verify free equations of motion from (4.1.9):

$$\partial\overline{\partial}X^\mu=0\ ,\quad \overline{\partial}\psi^\mu=0\ ,\quad \partial\tilde\psi^\mu=0\ , \tag{4.1.11}$$

from which we conclude that they can be expressed using functions of either z ("right movers") or $\overline{z}$ ("left movers"):

$$X^\mu(\tau,\sigma)=X_L^\mu(\tau+\sigma)+X_R^\mu(\tau-\sigma)\ \text{with}\ \begin{cases}X_R^\mu(\tau-\sigma)=\frac{1}{2}x_0^\mu+\alpha' p^\mu(\tau-\sigma)+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-2in(\tau-\sigma)}\\ X_L^\mu(\tau+\sigma)=\frac{1}{2}x_0^\mu+\alpha' p^\mu(\tau+\sigma)+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde\alpha_n^\mu}{n}e^{-2in(\tau+\sigma)}\end{cases}$$

$$\text{R-R: }\begin{cases}\psi^\mu(\tau,\sigma)=\psi^\mu(\tau-\sigma)=\sum_n\sqrt{2}\psi_n^\mu e^{-2in(\tau-\sigma)}\\ \tilde\psi^\mu(\tau,\sigma)=\tilde\psi^\mu(\tau+\sigma)=\sum_n\sqrt{2}\tilde\psi_n^\mu e^{-2in(\tau+\sigma)}\end{cases}\ ,\quad \text{R-NS: }\begin{cases}\psi^\mu(\tau,\sigma)=\sum_n\sqrt{2}\psi_n^\mu e^{-2in(\tau-\sigma)}\\ \tilde\psi^\mu(\tau,\sigma)=\sum_n\sqrt{2}\tilde\psi_n^\mu e^{-2i\left(n+\frac{1}{2}\right)(\tau+\sigma)}\end{cases}$$

$$\ldots \tag{4.1.12}$$

We refer to the αs, ψs, $\tilde\alpha$s and $\tilde\psi$s as string oscillators. The normalization of p^μ matches what is expected from the computation of the Noether charges P^μ of spacetime translations $X^\mu\to X^\mu-c^\mu$:

$$P^\mu=\frac{1}{2\pi\alpha'}\int_0^\pi\partial_\tau X^\mu=p^\mu\ , \tag{4.1.13}$$

and since X^μ is real, we have $\alpha_{-n}^\mu=\alpha_n^{\mu\dagger}$ and $\tilde\alpha_{-n}^\mu=\tilde\alpha_n^{\mu\dagger}$.

For the open string, one can complete $\partial_\sigma X^\mu$ into a smooth periodic function of period 2π thanks to the Neumann-Neumann boundary condition, while ψ^μ and $\tilde\psi^\mu$ can be completed smoothly on a 4π period. Then, upon solving the equations of motion, integrating $\partial_\sigma X^\mu$, restricting ψ^μ and

$\tilde{\psi}^\mu$ with the help of the boundary conditions, we get

$$X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}\cos(n\sigma)$$

$$\psi^\mu(\tau,\sigma) = \sum_r \psi_r^\mu e^{-ir(\tau-\sigma)} \ , \quad \tilde{\psi}^\mu(\tau,\sigma) = \sum_r \psi_r^\mu e^{-ir(\tau+\sigma)} \quad (\text{with R: } r\in\mathbb{Z}, \text{ NS: } r \in \mathbb{Z}+\frac{1}{2}) \ . \tag{4.1.14}$$

We use $r-$like latin letters for the fermionic indices from now on, keeping in mind that they can be integers or half integers.

4.1.3 Light-cone gauge quantization

To quickly get the spectrum of the RNS superstring, we follow the procedure of light-cone gauge quantization, meaning that we first gauge fix the residual gauge transformations of the Polyakov action which were not fixed by our choice $g_{ab} = \eta_{ab}$. Those are combinations of conformal transformations and Weyl rescalings of the metric. They act as conformal transformations on the coordinates and can be used to partly fix the dynamics of the X^μ. In addition, in the supersymmetric extension of the Polyakov action, these transformations are lifted to superconformal ones and can also be used to partly fix the fermionic coordinates.

Light-cone gauge fixing amounts then to define[4]:

$$X^\pm = \frac{X^0 \pm X^1}{\sqrt{2}} \ , \quad \psi^\pm = \frac{\psi^0 \pm \psi^1}{\sqrt{2}} \ , \quad \tilde{\psi}^\pm = \frac{\tilde{\psi}^0 \pm \tilde{\psi}^1}{\sqrt{2}} \ , \tag{4.1.15}$$

and to consequently completely gauge-fix the theory by imposing

$$X^+ = x^+ + 2\alpha' p^+ \tau \ , \quad \psi^+ = 0 \ , \quad \tilde{\psi}^+ = 0 \ . \tag{4.1.16}$$

This hides the Lorentz invariance of the theory, which has to be checked eventually. Why this gauge fixing is possible for closed strings X^μ can be understood as follows[5]: up to the Weyl rescaling, the leftover transformations act as conformal transformations $z \to z'(z)$, $\overline{z} \to \overline{z}'(\overline{z})$, such that $\tau' = \frac{\overline{z}'-z'}{2}$ verifies $\partial\overline{\partial}\tau' = 0$, meaning that it solves the same equation as the X^μs. In addition, it is necessarily also a periodic function of σ. Consequently, it is consistent to choose $\tau'(\tau,\sigma)$ such that

$$X'^+(\tau') \equiv X^+(\tau) = x^+ + 2\alpha' p^+ \underbrace{\left(\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left[\frac{\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}}{2\alpha' p^+ n}\right]\right)}_{\tau'} \ . \tag{4.1.17}$$

Note that this prescription fixes uniquely z' and $\overline{z}'$ as the left and right moving parts of τ', such that $\sigma'(\tau,\sigma) = \frac{z'+\overline{z}'}{2}$ is also fixed.

[4] We then have $\eta_{\mu\nu}X^\mu Y^\nu = -X^+Y^- - X^-Y^+ + \sum_{i=2\ldots d-1} X^i Y^i$. In particular, $X^2 = -2X^+X^- + \sum_{i=2\ldots d-1}(X^i)^2$.

[5] An equivalent argument for open strings or for the fermions is not hard to work out either but we do not explicitely do it here.

One can now solve the Virasoro constraints (4.1.8):

$$\begin{aligned} \partial X^- = -\frac{1}{2p^+}\left(\frac{1}{\alpha'}(\partial X^i)^2 - \frac{i}{2}\psi^i\partial\psi^i\right) , \quad \bar{\partial}X^- = \frac{1}{2p^+}\left(\frac{1}{\alpha'}(\bar{\partial}X^i)^2 + \frac{i}{2}\tilde{\psi}^i\bar{\partial}\tilde{\psi}^i\right) \\ \psi^- = -\frac{1}{\alpha' p^+}\psi^i\partial X^i , \quad \tilde{\psi}^- = \frac{1}{\alpha' p^+}\tilde{\psi}^i\partial X^i . \end{aligned} \tag{4.1.18}$$

This determines the complete mode expansions of the coordinates along the $-$ direction.

All the constraints being fixed, one can now quantize the string dynamics by following the canonical quantization procedure. From (4.1.7), we obtain the momenta associated to the coordinates:

$$\Pi^i_X = \frac{1}{2\pi\alpha'}\partial_\tau X^i , \quad \Pi^i_\psi = -\frac{i}{4\pi}\psi^i , \quad \Pi^i_{\tilde{\psi}} = -\frac{i}{4\pi}\tilde{\psi}^i , \tag{4.1.19}$$

for which we impose the canonical quantum (anti-)commutators[6]:

$$\left[\frac{\partial_\tau X^i(\tau,\sigma)}{2\pi\alpha'}, X^j(\tau,\sigma')\right] = -i\delta(\sigma-\sigma')\delta^{ij} , \ \{\psi^i(\tau,\sigma),\psi^j(\tau,\sigma')\} = \left\{\tilde{\psi}^i(\tau,\sigma),\tilde{\psi}^j(\tau,\sigma')\right\} = 2\pi\delta(\sigma-\sigma')\delta^{ij} , \tag{4.1.20}$$

all the remaining ones being zero. Commutators of the mode operators follow from such expressions. For instance, for the bosonic coordinates of the closed string, the mode expansions imply that (for $n \neq 0$)

$$\begin{aligned} \int_0^\pi d\sigma X^i(\tau,\sigma)e^{2in\sigma} &= -i\pi\sqrt{\frac{\alpha'}{2}}\frac{\alpha^i_n e^{-2in\tau} - \tilde{\alpha}^i_{-n}e^{2in\tau}}{n} \\ \int_0^\pi d\sigma\partial_\tau X^i(\tau,\sigma)e^{2in\sigma} &= \pi\sqrt{2\alpha'}(\alpha^i_n e^{-2in\tau} + \tilde{\alpha}^i_{-n}e^{2in\tau}) \end{aligned} \tag{4.1.21}$$

which leads to

$$\int\int\int_0^\pi d\tau d\sigma d\sigma' e^{2i[n(\tau+\sigma)+m(\tau+\sigma')]}\frac{\frac{mn}{\pi\alpha'}\left[X^i(\tau,\sigma),X^j(\tau,\sigma')\right] - im\left[\Pi^i_X(\tau,\sigma),X^j(\tau,\sigma')\right]}{\pi^2}\begin{cases} = [\alpha^i_n,\alpha^j_m] \\ \text{and} \\ = n\delta^{m+n,0}\delta^{ij}\end{cases} \tag{4.1.22}$$

Similar calculations can be made for all the modes, whose end results would be

$$\left[\tilde{\alpha}^i_n,\tilde{\alpha}^j_m\right] = n\delta^{m+n,0}\delta^{ij} , \quad \left\{\psi^i_r,\psi^j_s\right\} = \delta^{r+s,0}\delta^{ij} , \quad \left\{\tilde{\psi}^i_r,\tilde{\psi}^j_s\right\} = \delta^{r+s,0}\delta^{ij} , \tag{4.1.23}$$

all the remaining ones being zero again. We thus see that, with the very important exception of ψ^i_0, the different modes verify ladder operators commutation relations. Consequently, one can choose a vacuum state such that

$$\alpha^i_{n>0}\left|0\right\rangle = \tilde{\alpha}^i_{n>0}\left|0\right\rangle = \psi^i_{r>0}\left|0\right\rangle = \tilde{\psi}^i_{r>0}\left|0\right\rangle = 0 , \tag{4.1.24}$$

and the different states of the theory are reached by appliying the negative n/r operators on this vacuum. Particular care should be taken of zero modes though, since the momenta p^i and the ψ^i_0s of the R-R sector cannot be interpreted as ladder operators. The momentum, which commutes with everything (except the position x^i) is easy to treat: the vacuum is given a value for each

[6] More precisely, the fermion momenta in (4.1.19) form a (primary) second-class constraint, so the correct commutators to impose are the Dirac ones. We refer to [376] for more details.

momentum. On the other hand, the ψ_0^is form the Clifford algebra in $d-2$ dimensions, whose representations are known. The vacua are then completely labelled as follows

$$\text{NS-NS: } \left|0,k^i\right\rangle \ , \quad \text{NS-R: } \left|\tilde{s}_1,...,\tilde{s}_4,k^i\right\rangle \ , \quad \text{R-NS: } \left|s_1,...,s_4,k^i\right\rangle \ , \quad \text{R-R: } \left|s_1,...,s_4,\tilde{s}_1,...,\tilde{s}_4,k^i\right\rangle \ , \tag{4.1.25}$$

where s_i and $\tilde{s}_i = \pm\frac{1}{2}$, such that

$$p^i \left|...,k^i\right\rangle_{\text{all sectors}} = k^i \left|...,k^i\right\rangle \ , \quad \psi_0^{2a} \left|...,s_a,...\right\rangle_{\text{R-NS and R-R}} = \frac{\delta^{s_a,-\frac{1}{2}} \left|...,s_a+1,...\right\rangle + \delta^{s_a,\frac{1}{2}} \left|...,s_a-1,...\right\rangle}{\sqrt{2}} \ ,$$
$$\psi_0^{2a+1} \left|...,s_a,...\right\rangle_{\text{R-NS and R-R}} = \frac{\delta^{s_a,-\frac{1}{2}} \left|...,s_a+1,...\right\rangle - \delta^{s_a,\frac{1}{2}} \left|...,s_a-1,...\right\rangle}{\sqrt{2}i} \ , \quad ... \tag{4.1.26}$$

Note that the second line of (4.1.18) implies that $p_\mu\psi_0^\mu$ vanishes when evaluated on the R-(NS or R) vacua, meaning that they verify the massless Dirac equation.

The treatment of the open string sector is analogous, and the (independent) modes verify (4.1.23) as well. On the other hand, there exists an additional freedom one can use when defining open string states: one can add extra degrees of freedom at the ends of an open string, called Chan-Paton factors [377]:

$$|0\rangle \to |0,ij\rangle \ , \quad i,j=1,...,N \tag{4.1.27}$$

and postulate that they do not appear in the dynamical equations: they are thus constant degrees of freedom along the propagation of the endpoints of the string. They have a very rich dynamics: they introduce non-abelian symmetries in string theory interactions, and are linked with the dynamics living on D-branes. We will encounter them again in what follows.

4.1.4 Mass formula, operator ordering and the dimension of spacetime

To progress towards a complete understanding of the spectrum, we derive the mass m of the states we just identified above. Those states are built out of one of the vacua by the action of the negative n/r ladder operators. On the other hand, using the solution (4.1.18) to the Virasoro constraints, we find for the closed string

$$\alpha' m^2 \equiv \alpha'(2p^+p^- - (p^i)^2) = -\frac{2p^+}{\pi}\int_0^\pi d\sigma\partial X^- - \alpha'(p^i)^2 = 2\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i + 2\sum_r r\psi_{-r}^i\psi_r^i \ , \tag{4.1.28}$$

whereas for the open string

$$\alpha' m^2 \equiv \alpha'(2p^+p^- - (p^i)^2) = -\frac{p^+}{\pi}\int_0^{2\pi} d\sigma\partial X^- - \alpha'(p^i)^2 = \frac{1}{2}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i + \frac{1}{2}\sum_r r\psi_{-r}^i\psi_r^i \ , \tag{4.1.29}$$

In order to interpret those equations as operators equations defined at the quantum level, one should take care of ordering ambiguities in the products of classical quantities. Choosing a normal ordering on the operators, we finally write, for the open string:

$$\alpha' m^2 = \sum_{n>0}\alpha_{-n}^i\alpha_n^i + \sum_{r>0} r\psi_{-r}^i\psi_r^i + a \ , \tag{4.1.30}$$

where the constant a accounts for reordering the operators with respect to the naive classical expression. This constant, as well as the dimension of spacetime d, can be fixed by demanding that the quantum operators and the spectrum are consistent with Lorentz invariance. One can explicitly express the Lorentz generators in terms of the bosonic and fermionic modes and check their algebra: this gives a first constraint on both a and d. Then, demanding that states which transform as massless states under the Lorentz group (i.e. as representations of $SO(d-2)$) are actually massless, one can fix both constants. For instance, in the NS sector of the open string, $\psi^i_{-1/2}|0\rangle$ is such a state, and it has mass $\sqrt{\frac{1+2a}{2\alpha'}}$. Thus, we must impose $a=-1/2$ in order to match this state with a massless vector of spacetime. In the R sector, the ground states already transform as expected for a massless (spinorial) particle, so we choose $a=0$. Those choices then imply that the spacetime dimension in which our superstring theory lives is $d=10$.

For the closed string, we write

$$\alpha' m^2 = 4\left(\sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{r>0} r\psi^i_{-r}\psi^i_r + b\right) . \tag{4.1.31}$$

On the other hand, before repeating those arguments to determine b, we should note that we also could have written

$$\alpha' m^2 \equiv \alpha'(2p^+p^- - (p^i)^2) = \frac{2p^+}{\pi}\int_0^\pi d\sigma\overline{\partial}X^- - \alpha'(p^i)^2 = 2\sum_{n\neq 0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + 2\sum_r r\tilde{\psi}^i_{-r}\tilde{\psi}^i_r \ , \tag{4.1.32}$$

from which we determine the level-matching condition for physical closed string states:

$$\sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{r>0} r\psi^i_{-r}\psi^i_r + b = \sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \sum_{r>0} r\tilde{\psi}^i_{-r}\tilde{\psi}^i_r + \tilde{b} \ , \tag{4.1.33}$$

which should be understood as an operatorial equation, valid when applied on the physical states. Thus, the first states in the NS-NS sector to consider are $\psi^i_{-1/2}\tilde{\psi}^j_{-1/2}|0\rangle$, which transform as a massless symmetric tensor. Putting their mass to zero tells us that $b=\tilde{b}=-1/2$. In the NS-R sector, the state $\psi^i_{-1/2}|\tilde{s}_i\rangle$ has to be massless as well, from which we understand that $b=-1/2, \tilde{b}=0$. Similar reasoning for the other sectors enable us to conclude that the values for b and $\tilde{b}$ are derived from their open string counterparts, accordingly to the fact that closed string dynamics is a copy of two open string ones. For later purposes, we define[7]

$$L_0 \equiv \sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{r>0} r\psi^i_{-r}\psi^i_r + b \ , \quad \overline{L}_0 = \sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \sum_{r>0} r\tilde{\psi}^i_{-r}\tilde{\psi}^i_r + \tilde{b} \ . \tag{4.1.34}$$

The level matching is then $L_0 = \overline{L}_0$.

4.1.5 GSO projection, orientifold and type IIB/I spectra

At the level of our discussion, defining the consistent string theories demands a last refinement. This need can be understood by noting that the full superstring spectrum contains tachyons, as

[7] We do not respect the usual definition of L_0 and $\overline{L}_0$, which precisely equates them to the zero modes of the energy-momentum tensor, but we choose here a definition for compact formulae later on.

it was the case in the bosonic string theory. To get rid of such states, Gliozzi, Scherk and Olive (GSO) [378] imposed a projection on the spectrum to keep well behaved states. Their projection consists in removing the states in the spectrum which are odd under a fermion number. For the open string, it is defined as follows:

$$(-1)^F X^\mu = X^\mu (-1)^F \ , \quad (-1)^F \psi^\mu = -\psi^\mu (-1)^F \ , \quad \begin{cases} \text{NS: } (-1)^F |0\rangle = -|0\rangle \\ \text{R: } (-1)^F |s_a\rangle = (-1)^{\sum_a s_a} |s_a\rangle \end{cases} \ , \tag{4.1.35}$$

where we note that the open string tachyon is projected out as desired. For the closed string there are two fermion numbers $F, \tilde{F}$ respectively for the left and right movers. The closed string spectrum should then be invariant under both fermion numbers. There is a leftover freedom though, which is that the R-sector right moving vacua could be defined such that there is an extra minus sign under the action of $(-1)^{\tilde{F}}$ (when we defined the left moving one to behave as in (4.1.35) under F). Doing so or not leads respectively to type IIA and type IIB string theories. We will mention again the former in section 4.1.6.

The closed-string massless spectrum of type IIB string theory forms the particle content of one of the two ten dimensional supergravity theories with 32 supercharges, consistently known as type IIB SUGRA: a graviton, a 2-form and a dilaton, from the NS-NS sector, a self-dual 4-form, a 2-form and a 0-form from the R-R sector, and two gravitini (Majorana-Weyl spin 3/2) and dilatini (Majorana-Weyl spin 1/2) from the NS-R and R-NS sectors. The emergent spacetime supersymmetry recovered here is an output of the choice of GSO projection. In type IIB, the fermions have the same chiralities. The fact that gravitational anomalies cancel in this theory is thus not trivial, and is ensured by the contribution of the self-dual 4-form to the gravitational anomaly.

The type I string theory is obtained from type IIB by performing the so-called orientifold projection Ω [379]: this amounts to identifying left and right movers in the mode expansion of the string coordinates by gauging the theory's invariance under worldsheet parity

$$\Omega : \begin{cases} \text{For closed strings: } \sigma \to -\sigma \\ \text{For open strings: } \sigma \to \pi - \sigma \end{cases} \ , \tag{4.1.36}$$

which is uplifted to an action on the worldsheet fields, which we write here as an action on the oscillator modes:

$$\Omega : \begin{cases} \text{For closed strings: } \alpha^\mu_n \leftrightarrow \tilde{\alpha}^\mu_n \ , \quad \psi^\mu_r \leftrightarrow \tilde{\psi}^\mu_r \\ \text{For open strings: } \alpha^\mu_n \to e^{i\pi n} \alpha^\mu_n \ , \quad \psi^\mu_r \to e^{-i\pi r} \psi^\mu_r \end{cases} \ . \tag{4.1.37}$$

We note that $\Omega^2 \neq 1$ on the open-string oscillators, but the definition of the GSO projection ensures that $\Omega^2 = 1$ on the physical states. The action on the vacua is

$$\text{Closed string: } \ \Omega \cdot \begin{cases} |0\rangle_{\text{NS-NS}} \\ |0, \tilde{s}_b\rangle_{\text{NS-R}} \\ |s_a, 0\rangle_{\text{R-NS}} \\ |s_a, \tilde{s}_b\rangle_{\text{R-R}} \end{cases} = \begin{cases} |0\rangle_{\text{NS-NS}} \\ |\tilde{s}_b, 0\rangle_{\text{R-NS}} \\ |0, s_a\rangle_{\text{NS-R}} \\ -|\tilde{s}_b, s_a\rangle_{\text{R-R}} \end{cases} \qquad \text{Open string: } \ \Omega |0, i, j\rangle_{\text{NS/R}} = \omega_{\text{NS/R}} |0, j, i\rangle_{\text{NS/R}} \ , \tag{4.1.38}$$

where the minus sign for the R-R sector can be understood from the anticommutation of the two R-sector vertex operators exchanged, and we included an action on the Chan-Paton factors (4.1.27) of the open string, under the form of $\mathbb{Z}_2$ phase ω which we determine later. There is a subtlety about it in the NS-sector though, since we want $\Omega^2 = 1$ on the physical states, but $\Omega^2 = -1 = (-1)^F$ when applied on the fermionic oscillators. We must then define Ω such that $\Omega^2 = (-1)^F = -1$ on the NS-vacuum, so $\omega_{\rm NS} = -i\omega$, with ω a $\mathbb{Z}_2$ phase. $\omega_{\rm R} = \omega$ simply.

The result of gauging Ω is that the massless EFT is a ten dimensional SUGRA with only 16 supercharges, since the two gravitini of type IIB are projected down to a single Majorana-Weyl gravitino. The gravitational multiplet is then made out of the graviton, the gravitino, a 2-form, a dilaton and a dilatino. In addition, ten-dimensional $\mathcal{N} = 1$ SUSY admits matter vector multiplets [380–382], which comes from the open sector of type I theory. Indeed, as we will discuss again in section 4.1.7, anomaly cancellation demands the presence of a $SO(32)$ gauge theory in the ten-dimensional bulk, in order to implement the Green-Schwarz mechanism [229] (which we mention again in section 4.3.5). Note that at the level of the spectrum, there are $\frac{N(N+\omega)}{2}$ vectors as bosonic massless modes of the open string, as well as their gaugini superpartners:

$$\frac{\psi^i_{-\frac{1}{2}} |0, i, j\rangle + \omega\psi^i_{-\frac{1}{2}} |0, j, i\rangle}{\sqrt{2}} , \quad \frac{|s_a, i, j\rangle + \omega\, |s_a, j, i\rangle}{\sqrt{2}} . \tag{4.1.39}$$

Thus, if $\omega = 1$, we describe a $USp(N)$ gauge theory, whose adjoint has dimension $\frac{N(N+1)}{2}$. If $\omega = -1$, the gauge group is $SO(N)$. We understand then that anomalies select $\omega = -1$, as we will see in section 4.1.7.

4.1.6 The type IIB partition function

Now that we worked out the spectrum of the type IIB/I theories, we are ready to extract the one-loop vacuum amplitudes from it.

From field to string vacuum energies

We follow the field theory inspired derivation of [369]: in a theory of a massive real scalar field ϕ, of lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{M^2}{2}\phi^2 , \tag{4.1.40}$$

the vacuum energy Γ reads, in $d-$dimensional spacetime,

$$\Gamma = -\frac{V}{2}\int_\epsilon^\infty \frac{dt}{t}e^{-tM^2}\int\frac{d^dp}{(2\pi)^d}e^{-tp^2} = -\frac{V}{2(4\pi)^{\frac{d}{2}}}\int_\epsilon^\infty\frac{dt}{t^{1+\frac{d}{2}}}e^{-tM^2} , \tag{4.1.41}$$

where V is the spacetime volume and ϵ a UV regulator. In a theory with more degrees of freedom, this formula uplifts to

$$\Gamma = -\frac{V}{2(4\pi)^{\frac{d}{2}}}\int_\epsilon^\infty\frac{dt}{t^{1+\frac{d}{2}}}\mathrm{Str}\left(e^{-tM^2}\right) , \tag{4.1.42}$$

where Str counts the degrees of freedom with multiplicity and sign according to spin-statistics, for instance for a Dirac fermion of mass M:

$$\Gamma = \frac{2^{\lfloor \frac{d}{2} \rfloor} V}{2(4\pi)^{\frac{d}{2}}} \int_{\epsilon}^{\infty} \frac{dt}{t^{1+\frac{d}{2}}} e^{-tM^2} \ . \tag{4.1.43}$$

In string theory, the spectrum is determined in sections 4.1.3, 4.1.4 and 4.1.5. On the other hand, the integration region is not as in field theory, which is explained by linking the field theory calculation to worldsheet ones, which we do now.

Let us discuss the closed sector first. There, the masses are given in (4.1.31) where we remember that level matching (4.1.33) must hold on the physical Hilbert space. We can however sum over the whole Fock space and insert a δ function to impose level-matching:

$$\text{Str}\left(e^{-tM^2}\right) \to \text{Str}\left(e^{-tM^2}\delta\left(L_0 - \overline{L}_0\right)\right) = \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \ \text{Str}\left(e^{-tM^2} e^{2\pi i (L_0 - \overline{L}_0)s}\right) \tag{4.1.44}$$

Finally, the GSO projection (4.1.35) can be imposed by inserting the projector onto even fermion number states:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} ds \ \text{Str}\left(e^{-tM^2} e^{2\pi i (L_0 - \overline{L}_0)s}\right) \to \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \ \text{Str}\left(\frac{1+(-1)^F}{2}\frac{1+(-1)^{\tilde{F}}}{2} e^{-tM^2} e^{2\pi i (L_0 - \overline{L}_0)s}\right) \ , \tag{4.1.45}$$

where the presence of the Str reminds us that we have both a worldsheet and a spacetime fermion number to consider here.

The torus

Doing this actually amounts to consider the one-loop worldsheet which is present in all string theories: the torus. Once we fixed the flat metric on the torus, its geometry is characterized by a complex parameter $\tau \equiv \tau_1 + i\tau_2$, called modulus[8]

$$\text{Torus} : \{z \in \mathbb{C} \text{ such that } z \equiv z + 1 \equiv z + \tau\} \ . \tag{4.1.46}$$

Not all τs are conformally equivalent, each of them is part of an equivalence class generated by the two modular transformations T and S:

$$T : \tau \to \tau + 1 \ , \quad S : \tau \to -\frac{1}{\tau} \ , \tag{4.1.47}$$

such that the space of inequivalent τs, called fundamental domain $\mathcal{F}$, is characterized by the moduli which verify:

$$|\tau| \geq 1 \text{ and } \text{Re}(\tau) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \ . \tag{4.1.48}$$

The important part is that it does not extend to the UV region $\tau = 0$, which is the one-loop sign of the UV finiteness of string theory. The fundamental domain gives a string origin to the

[8]It should not be mistaken for the worldsheet time τ. The latter will be mentioned as "worldsheet time τ", or something close, when it is encountered in what follows.

cutoff ϵ, which is now built in and physical. This natural cutoff, together with the level matching condition, arises naturally when we write the one-loop contribution to the path integral without vertex operators, i.e. the partition function Z, on the torus. The latter can be understood by saying that a torus of modulus τ is a cylinder worldsheet of length $\pi\tau_2$, closed on itself after being skewed at its end so that the string points indexed by σ (at time $\tau = 0$) meet points of index $\sigma + \pi\tau_1$ (at time $\tau = \pi\tau_2$). This translates into the following expression :

$$Z = \mathrm{Tr}\Big(e^{2i\pi\tau_1 P - 2\pi\tau_2 H}\Big) \ , \tag{4.1.49}$$

where $P = L_0 - \tilde{L}_0$ generates translations along the space coordinate σ of the worldsheet, and the hamiltonian $H = L_0 + \tilde{L}_0$ along the time coordinate τ.

Torus amplitude in type IIB

Evaluating this trace gives precisely (4.1.43)-(4.1.45), once we define $\tau_1 \equiv s, \tau_2 \equiv \frac{t}{\alpha'\pi}$. We call this amplitude $\mathcal{T}$ (like torus) and rewrite it using the convient parametrization $q = e^{2\pi i\tau}$:

$$\mathcal{T} = -\frac{V}{2(4\pi\alpha')^{\frac{d}{2}}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^{1+\frac{d}{2}}}\mathrm{Str}\left(\frac{1+(-1)^F}{2}\frac{1+(-1)^{\tilde{F}}}{2}q^{L_0}\overline{q}^{\overline{L}_0}\right) \ . \tag{4.1.50}$$

We can now compute it, using some intermediary steps. First, let us write explicitly the Fock space states and the action of L_0 and $\overline{L}_0$ on them. In the NS-NS sector, the states are spacetime bosons and:

$$\text{For } \left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j\right\rangle = ... \left(\alpha_{-n}^i\right)^{k_n^i} ... \left(\psi_{-r}^j\right)^{l_r^j} ... \times |0\rangle \ ,$$

$$\begin{aligned} L_0\left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j\right\rangle &= \Big[\sum_i\Big(\sum_{n>0} n k_n^i + \sum_{r>0} r l_r^i\Big) - \frac{1}{2}\Big]\left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j\right\rangle \\ \overline{L}_0\left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j\right\rangle &= \Big[\sum_i\Big(\sum_{n>0} n \tilde{k}_n^i + \sum_{r>0} r \tilde{l}_r^i\Big) - \frac{1}{2}\Big]\left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j\right\rangle \ , \end{aligned} \tag{4.1.51}$$

where l_r^j and $\tilde{l}_r^j = 0, 1$, since the ψ modes are anticommuting. Consequently,

$$\begin{aligned} \mathrm{tr}\Big(q^{L_0}\overline{q}^{\overline{L}_0}\Big) &= \prod_{i,n>0,r>0}\left(\sum_{k_n^i,\tilde{k}_n^i,l_r^i,\tilde{l}_r^i}\right) q^{\sum_i\left(\sum_{n>0} n k_n^i + \sum_{r>0} r l_r^i\right) - \frac{1}{2}}\overline{q}^{\sum_i\left(\sum_{n>0} n\tilde{k}_n^i + \sum_{r>0} r\tilde{l}_r^i\right) - \frac{1}{2}} \\ &= \frac{1}{\sqrt{q\overline{q}}}\prod_{i,n>0,r>0}\underbrace{\left(\sum_{k_n^i} q^{nk_n^i}\right)}_{\frac{1}{1-q^n}}\underbrace{\left(\sum_{l_r^i} q^{rl_r^i}\right)}_{1+q^r}\left(\sum_{\tilde{k}_n^i}\overline{q}^{n\tilde{k}_n^i}\right)\left(\sum_{\tilde{l}_r^i}\overline{q}^{r\tilde{l}_r^i}\right) = \left|\frac{\prod_{r>0}(1+q^r)^8}{\sqrt{q}\prod_{n>0}(1-q^n)^8}\right|^2 , \end{aligned} \tag{4.1.52}$$

and

$$\begin{aligned} \mathrm{tr}\left(\frac{1+(-1)^F}{2}\frac{1+(-1)^{\tilde{F}}}{2}q^{L_0}\overline{q}^{\overline{L}_0}\right) &= \prod_{i,n>0,r>0}\left(\sum_{k_n^i,\tilde{k}_n^i,l_r^i,\tilde{l}_r^i}\right)\frac{1-(-1)^{\sum_r l_r^i}}{2}q^{\sum_i\left(\sum_{n>0} n k_n^i + \sum_{r>0} r l_r^i\right) - \frac{1}{2}} \times ... \\ &= \left|\frac{\prod_{r>0}(1+q^r)^8}{2\sqrt{q}\prod_{n>0}(1-q^n)^8} - \frac{\prod_{r>0}(1-q^r)^8}{2\sqrt{q}\prod_{n>0}(1-q^n)^8}\right|^2 = \left|\frac{V_8}{\eta^8}\right|^2 , \end{aligned} \tag{4.1.53}$$

where we remember that r is half-integer. The usual definitions for the Dedekind function, the $SO(8)$ characters and the Jacobi functions can be found in [369].

All the sectors put together[9] give the type IIB partition function:

$$\mathcal{T} = -\frac{V}{2(4\pi\alpha')^5}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^6}\left|\frac{V_8 - S_8}{\eta^8}\right|^2(\tau)\ , \tag{4.1.54}$$

where we specialized (4.1.50) to ten dimensions and indicated that the characters are functions of τ and $q = e^{2i\pi\tau}$. Interestingly, we can recover from (4.1.54) what we already learnt about the spectrum in section 4.1.5. Indeed, considering the large τ_2 behaviour of the characters, we find

$$\frac{V_8 - S_8}{\eta^8} \approx 8 - 8 + \mathcal{O}(q)\ . \tag{4.1.55}$$

Inserted in (4.1.54), this reminds us of two aspects of type IIB string theory. First, it is supersymmetric, so the cancellation between the fermionic and bosonic contributions to the vacuum energy is consistent. Second, remembering the interpretation of the Str in (4.1.43), we understand that there are $8^2 + 8^2$ massless bosonic degrees of freedom and $8^2 + 8^2$ fermionic ones. Those indeed respectively decompose into $128 = (35 + 28 + 1) + (35 + 28 + 1)$, i.e. the ten-dimensional massless on-shell degree of freedoms for the NS-NS sector (a graviton, an 2-form and a dilaton) and the R-R sector (a self-dual 4-form, a 2-form and a 0-form), and in $128 = 2 \times (56 + 8)$, i.e. the two gravitini (Majorana-Weyl spin 3/2) and dilatini (Majorana-Weyl spin 1/2) of the NS-R and R-NS sectors.

Other GSO projections, other theories

In section 4.1.5, we mentioned that the extension of the GSO projection (4.1.35) to the closed string is not unique. The consistency of the torus amplitude is such that, starting from the features it must have, one can find all the possible forms for the integrand, which are in turn linked to different GSO projections.

Tori paired by the modular transformations (4.1.47) are physically equivalent, and this shows up in (4.1.54): it can indeed be shown [369] that $\mathcal{T}$ is modular invariant [383, 384], i.e. invariant under the transformations (4.1.47). This is a condition for a consistent torus, and it turns out to be a strong one. In addition, it can be seen from (4.1.54) that fermions and bosons enter the amplitude with different signs, as expected from spin-statistics. The torus amplitude should also have this feature. Those conditions are demanding enough to single out three additional theories, for which $|V_8 - S_8|^2$ in (4.1.54) would be replaced by

$$\text{Type IIA: } (V_8 - C_8)(\overline{V}_8 - \overline{S}_8)\ , \quad \text{Type 0: } |O_8|^2 + |V_8|^2 + \begin{cases} S_8\overline{C}_8 + C_8\overline{S}_8 \ (\text{0A}) \\ |S_8|^2 + |C_8|^2 \ (\text{0B}) \end{cases}\ . \tag{4.1.56}$$

The type 0 theories do not have spacetime fermions, as can be seen from the sign of the different contributions in (4.1.56). They have a tachyon, as can be seen from the large τ_2 expansion of O_8, which we will encounter again in section 4.2.2, and the two types differ by their antisymmetric

[9] See appendix D.1 for the contribution of the RR sector.

forms. The type IIA theory is the other supersymmetric tachyon-free string theory, which has the type IIA SUGRA as its massless EFT: there is the same gravitational sector as type IIB, but the two gravitini (and dilatini) have different chiralities, and there are a 1-form and a 3-form in the R-R sector. The type IIA SUGRA is non-chiral, so its gravitational anomaly trivially cancels.

4.1.7 Type I amplitudes, tadpoles and $SO(32)$ gauge group

The techniques presented thus far can be applied to type I string theory. In particular, the one-loop closed string amplitude extends the torus amplitude (4.1.54) by the addition of the other non-oriented Riemann surface of Euler character[10] 0 which describes the propagation of a closed string: the Klein bottle[11].

The contribution of the Klein bottle can be written

$$\mathcal{K} = -\frac{V}{4(4\pi\alpha')^5}\int_{\mathcal{F}_\mathcal{K}} \frac{d^2\tau}{\tau_2^6}\mathrm{Str}\left(\frac{1+(-1)^F}{2}\frac{1+(-1)^{\tilde{F}}}{2}\Omega q^{L_0}\bar{q}^{\bar{L}_0}\right) , \tag{4.1.57}$$

where we denoted the integration domain for the Klein bottle by $\mathcal{K}$ and where Ω is the orientifold projection of section 4.1.5 such that, together with the halved torus amplitude, they form the vacuum amplitude for the projected closed string spectrum:

$$\frac{1}{2}\mathcal{T} + \mathcal{K} . \tag{4.1.58}$$

The integrand in the Klein bottle amplitude, which we choose to call $\mathcal{K}$ as well, as we will also do for others amplitudes, reads (see again appendix D.2)

$$\mathcal{K} = \frac{V_8 - S_8}{\eta^8}(2i\tau_2) , \tag{4.1.59}$$

where V_8 is the contribution of the NS-NS sector and S_8 the R-R sector one. The amplitude naturally depends on $2i\tau_2$, which is the modulus of the doubly covering torus of the Klein bottle. In addition, the involutive identification of some points of the torus which leads to the construction of the Klein bottle breaks the conformal group of the torus, which has the effect of leaving the full τ_2 axis as the one which parametrizes the space of physical Klein bottles:

$$\mathcal{K} = -\frac{V}{4(4\pi\alpha')^5}\int_0^\infty \frac{d\tau_2}{\tau_2^6}\frac{V_8 - S_8}{\eta_8}(2i\tau_2) . \tag{4.1.60}$$

Putting together (4.1.54) and (4.1.60) as in (4.1.58), we easily recover the massless closed-string spectrum: we find

$$\frac{1}{2}\mathcal{T} + \mathcal{K} \approx \frac{|V_8|^2 + V_8 + |S_8|^2 - S_8}{2} - \frac{V_8\overline{S}_8 + S_8\overline{V}_8}{2} = 35 + 1 + 28 - (56 + 8) + \mathcal{O}(q) , \tag{4.1.61}$$

which fits our discussion of the closed string type I massless spectrum.

[10] We recall that the Euler character is given by $2 - 2h - b - c$, where h, b and c are respectively the number of holes, boundaries and crosscaps of the surface.

[11] We refer to [369] for a discussion of the other surfaces than the torus which we encounter in what follows. In particular, we mention without defining it their doubly covering tori.

On the other hand, the self-dual 4-form is projected out of the spectrum by the orientifold projection, so only the gravitino contributes to the anomaly which does not vanish anymore. The additional contributions to the anomaly come from the bulk open string sector, and the anomaly vanishes via the GS mechanism if the gauge group is $SO(32)$ or $E_8 \times E_8$ (see e.g. [385] for a review). Those two possibilities are realized in the heterotic string [386], but type I string theory selects $SO(32)$ as the realized gauge group.

To see this, we analyze the one-loop amplitudes, using the (quite deep) fact that in string theory, anomalies are related to R-R tadpoles [387], which are themselves, like dilaton tadpoles, related to one-loop divergences [388–391]. Indeed, the integration region of the Klein bottle (4.1.60) goes all the way to $\tau_2 = 0$, which indicates a UV divergence, consistently with the fact that its presence signals a projection of the type IIB spectrum down to an anomalous one. The torus, as we discussed previously, is protected from this divergence by modular invariance. To cancel it, it is necessary that the open string sector contributes as well to the vacuum energy via two surfaces, the oriented cylinder (or annulus) $\mathcal{A}$ and the unoriented Möbius strip $\mathcal{M}$ (see appendix D.3):

$$\mathcal{A} = -\frac{N^2 V}{4(4\pi\alpha')^5}\int_0^\infty \frac{d\tau_2}{\tau_2^6}\frac{V_8 - S_8}{\eta_8}\left(i\frac{\tau_2}{2}\right) \ , \quad \mathcal{M} = -\frac{\omega N V}{4(4\pi\alpha')^5}\int_0^\infty \frac{d\tau_2}{\tau_2^6}\frac{\hat{V}_8 - \hat{S}_8}{\hat{\eta}_8}\left(i\frac{\tau_2}{2} + \frac{1}{2}\right) \ , \tag{4.1.62}$$

where once again we see the moduli of the doubly covering tori appear, and where the presence of N and ω shows that we included the contributions from (the action (4.1.38) of Ω on) the Chan-Paton factors (4.1.27). The hatted characters $\hat{V}_8, \hat{S}_8$ are defined in [369], and we do not discuss them further since we only focus on the massless modes that they describe, in which case the hats are irrelevant. To analyze (physically) the divergence, it is easier to switch to the "tree-level (gravitational) channel", by defining

$$il \equiv \begin{cases} \text{Klein bottle: } \frac{i}{2\tau_2} \\ \text{Cylinder: } \frac{2i}{\tau_2} \\ \text{Möbius strip: } \frac{i}{2\tau_2} \end{cases} \ , \tag{4.1.63}$$

which amounts to perform an S-transform, defined in (4.1.47), on the characters of the Klein bottle and the cylinder, and a $P = TST^2S$-transform on those of the Möbius strip. Up to a factor $\frac{V}{2(4\pi\alpha')^5}$, the amplitudes read

$$\tilde{\mathcal{K}} = \frac{2^5}{2}\int_0^\infty dl \frac{V_8 - S_8}{\eta^8}(il) \ , \quad \tilde{\mathcal{A}} = 2^{-10}N^2\mathcal{K} \ , \quad \tilde{\mathcal{M}} = \omega N\int_0^\infty dl \frac{\hat{V}_8 - \hat{S}_8}{\hat{\eta}^8}\left(il + \frac{1}{2}\right) \ . \tag{4.1.64}$$

We use tildes to emphasize that we talk about amplitudes written using l as the integration variable, which defines the "tree-level channel". The name comes from the fact that the one-loop open string amplitudes have a dual interpretation in terms of tree-level exchanges of closed string states, between D-branes (for the cylinder) and between D-branes and O-planes (for the Möbius amplitude). Then the Klein bottle describes a closed string exchange between O-planes only. l is understood as the length of the tube describing the tree-level propagation, and the charges, numbers and positions of the different objects can be identified from expressions such as (4.1.64).

In our case, open strings live in the ten-dimensional bulk, and the orientifold projection Ω does not leave any spacetime point fixed[12], so the amplitudes (4.1.64) describe exchanges between spacetime-filling D9-branes and/or O9-planes.

The UV-divergences of the "loop-channel" (i.e., the amplitudes expressed in terms of τ_2) are mapped to infrared divergences in the tree-level channel. There, the divergence is due to the massless contributions, which are linked to constant terms in the integrands of (4.1.64), as suggested by the (Schwinger proper time) identity

$$\frac{1}{M^2} = \int_0^\infty dl \; e^{-lM^2} \; . \tag{4.1.65}$$

In (4.1.64), the constant contributions are

$$\tilde{\mathcal{K}} + \tilde{\mathcal{A}} + \tilde{\mathcal{M}} = \int_0^\infty dl \; \frac{(32+\omega N)^2}{64} \left(8 - 8 + \mathcal{O}(q(l))\right) \tag{4.1.66}$$

and we see that demanding that this divergence vanishes selects once again $N = 32$ and $\omega = -1$, which was identified as the choice of $SO(32)$ for the gauge group in section 4.1.5.

It may seem a bit silly to fix N and ω while (the massless level of) $V_8 - S_8$ vanishes already. However, the NS-NS and the R-R contributions to the tree-level channel amplitudes have different interepretations: the NS-NS divergence has to do with the dilaton ϕ tadpole in the EFT:

$$S \sim (32 + \omega N) \int d^{10}x \sqrt{-g} e^{-\phi} \; , \tag{4.1.67}$$

whereas the R-R tadpole is linked to the overall charge neutrality of the system and to the gauge anomalies. In type I, supersymmetry relates those two tadpoles and they both vanish. However, as we will discuss in section 4.2.1, if SUSY is broken, one only needs to impose the cancellation of the R-R tadpole for the consistency of the theory.

4.2 SUSY breaking in string theory

In this section, we consider mechanisms for breaking supersymmetry which are different from those of section 3.2, since they are stringy in nature, or tied to the extra-dimensional nature of string theory. We discuss specifically two mechanisms: brane supersymmetry breaking (BSB) and Scherk-Schwarz (SS) (super)symmetry breaking. An other mechanism, which we do not cover, consists in turning on magnetic fluxes in compact directions [392]. Gaugino condensation is also often cited [393, 394], even though it is a field theory effect, as we will see in section 4.3.5.

4.2.1 Brane SUSY breaking and non-linear SUSY

First, we mention briefly a mechanism with string scale SUSY breaking, called brane supersymmetry breaking (BSB) [395–397]. It means that objects, such as (anti-)D-branes and (anti-)O-planes,

[12] This would be modified in a T-dual picture.

which do not preserve the same supersymmetries are put together in the theory. We do not try to be exhaustive about it, but we only comment the simplest ten-dimensional case where this happens [395]: let us choose 16 anti D9-branes and an exotic O9-plane, the regular ones being those which appear in the ten-dimensional supersymmetric type I theory of section 4.1.7. The resulting system still does not have a R-R tadpole (understood as the vanishing of the overall R-R charge of the two objects), but now the overall tension is not vanishing (unlike the SUSY case, where for the non-dynamical O9-plane, we defined a negative tension[13]). The end result is a theory with broken SUSY at the string scale, with an open sector which looks as follows (at the integrand level, in the loop channel):

$$\mathcal{A} = \frac{N^2}{2}\frac{V_8 - S_8}{\eta^8}\left(\frac{i\tau_2}{2}\right) \ , \quad \mathcal{M} = -\frac{N}{2}\frac{-\hat{V}_8 - \hat{S}_8}{\hat{\eta}^8}\left(\frac{i\tau_2}{2}+\frac{1}{2}\right) \ . \tag{4.2.1}$$

The fact that the R-R tadpole still vanishes in this theory is immediately seen from the fact that the R-R tree-level contribution, proportional to S_8, have not changed with respect to (4.1.64). On the other hand, the fact that SUSY is broken can be seen in several ways. First, the NS-NS contributions, given by V_8, suffer from a sign shift and do not cancel anymore, whereas there were linked to the R-R tadpole by SUSY. Second, there are $\frac{N(N+1)}{2}$ massless vectors but $\frac{N(N-1)}{2}$ fermions:

$$\mathcal{A} + \mathcal{M} \approx \frac{N(N+1)}{2}8 - \frac{N(N-1)}{2}8 + \mathcal{O}(q) \ , \tag{4.2.2}$$

so that bosons fall into the adjoint representation of $USp(N = 32)$ but the fermions are in a different representation, the antisymmetric representation. More precisely (and interestingly), the third point is that $\frac{N(N-1)}{2} = 496 = 495+1$, meaning that the fermions lie in the antisymmetric representation up to a singlet fermion: it turns out that the couplings of this fermion are precisely those of a Goldstino, with supersymmetry non-linearly realized à la Volkov-Akulov with a scale of supersymmetry breaking given by the dilaton tadpole [302]. In this kind of system, there are no supersymmetric EFT which then breaks SUSY spontaneously, SUSY is breaking at the string scale and the low-energy EFT has a built-in SUSY breaking/non-linearly realized SUSY.

4.2.2 Scherk-Schwarz mechanism and partition functions

We now turn to the SS mechanism, which is a general mechanism for spontaneously breaking symmetries from non-local effects in compactified dimensions. Unlike BSB, it is not purely stringy and can be understood at the field theory level. However, since we have not encountered higher dimensional theories before this last part of the thesis, and since we will use SS mechanism below, we choose to discuss it here. SS mechanism was first proposed at the field-theory (supergravity) level by Scherk and Schwarz [398], then applied to heterotic strings [399–401] and then to open strings [402,403]. It is the oldest and probably the most popular way of breaking supersymmetry perturbatively in string theory.

[13] D-branes being dynamical objects, we always need to define positive tensions for them.

General mechanism

The idea behind the SS mechanism goes as follows: given a theory with a symmetry group G living on a compact extra dimension, which we take to be a circle of radius R for simplicity, we can consistently impose twisted boundary conditions on the fields of the theory:

$$\phi(x^\mu, x^{d-1} + 2\pi R) \equiv g \cdot \phi(x^\mu, x^{d-1}) \cdot g^{-1} \ , \tag{4.2.3}$$

where we assume that the compact extra dimension indexed by x^{d-1} is the last spatial dimension of a complete spacetime of dimension d, and g is the element of G used to twist the boundary conditions on the field ϕ, which image under g is denoted $g \cdot \phi \cdot g^{-1}$.

An example is the theory of a set of scalar fields $\phi^{(q)}$ indexed by their charges q (assumed to quantized and integers) under an abelian global symmetry. Choosing $g = e^{i\pi Q}$, with Q the abelian generator, the twisting reads

$$\phi^{(q)}(x^\mu, x^{d-1} + 2\pi R) \equiv e^{i\pi q}\phi(x^\mu, x^{d-1}) \ . \tag{4.2.4}$$

The $(d-1)$-dimensional Kaluza-Klein (KK) modes are found by Fourier-expanding ϕ on the circle, while respecting the boundary conditions (4.2.4):

$$\phi^{(q)}(x^\mu, x^{d-1}) = \sum_{n\in\mathbb{N}} \phi_n^{(q)}(x^\mu) e^{i\left(n+\frac{q}{2}\right)\frac{x^{d-1}}{R}} \ . \tag{4.2.5}$$

The KK modes then receive mass shifts which depend on the charge q:

$$m_{\phi_n^{(q)}} = \frac{n + \frac{q}{2}}{R} \ . \tag{4.2.6}$$

Those mass contributions from the boundary conditions can be used to break SUSY [404]: use as an example of global symmetry the fermion number $\psi \to -\psi$. Then, bosons have the usual KK decompositions while the fermions have shifted masses:

$$m_{\psi_n} = \frac{n + \frac{1}{2}}{R} \ . \tag{4.2.7}$$

It is quite interesting that, even though (4.2.7) appears like an explicit breaking of SUSY, the $(d-1)$-dimensional theory has all the features of a spontaneously broken supersymmetric theory [404, 405].

(Type I) strings with Scherk-Schwarz supersymmetry breaking

In string theory, the momentum shifts such as (4.2.7) can be implemented. First, we need a compact dimension, which we choose to be $X^9 \equiv X^9 + 2\pi R$. This modifies the string spectrum by quantifying $p^9 = \frac{m}{R}$, with $m \in \mathbb{Z}$, and by introducing winding modes n (integer as well) for the closed string, which can now wrap several times the X^9-circle before closing on itself, adding an irreducible tension to its mass. Those winding modes do not exist for open strings, which are free to roll and unroll around the circle. The mass for the closed string, seen as a nine-dimensional object, is thus modified as follows:

$$m^2_{\text{closed}} = \frac{m^2}{R^2} + \frac{n^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(L_0 + \bar{L}_0) \ , \quad m^2_{\text{open}} = \frac{m^2}{R^2} + \frac{1}{\alpha'}L_0 \tag{4.2.8}$$

and vacuum amplitudes have one less $\frac{1}{\sqrt{4\pi^2\alpha'\tau_2}}$ factor since we do not integrate on a non-compact X^9 anymore, which is instead replaced by a sum over compact momenta (and windings for the closed string)

$$\Lambda_{m,n} = \sum_{m,n} q^{\frac{\alpha'}{4}\left(\frac{m}{R}+\frac{nR}{\alpha'}\right)^2} \bar{q}^{\frac{\alpha'}{4}\left(\frac{m}{R}-\frac{nR}{\alpha'}\right)^2} \ , \quad P_m = \sum_n q^{\frac{\alpha' m^2}{R^2}} \ , \tag{4.2.9}$$

for which we choose to leave implicit the sum over indices m, n in the notation $\Lambda_{m,n}$, P_m, or similar expressions, in what follows. Now, the SS mechanism is applied on the theory by demanding that the latter is invariant under a so-called freely acting orbifold g, which is the composition of a spacetime fermion number operator and a half-shift along the compact dimension:

$$g = (-1)^F \delta_{X^9 \to X^9 + \pi R} \ . \tag{4.2.10}$$

This changes the torus amplitude of type IIB (up to the $\frac{V}{2(4\pi^2\alpha'\tau_2)^{9/2}}$ factor):

$$\begin{aligned} \mathcal{T} =& \frac{1}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{11/2}} \left(\left|\frac{V_8 - S_8}{\eta^8}\right|^2 \Lambda_{m,n} + \left|\frac{V_8 + S_8}{\eta^8}\right|^2 (-1)^n \Lambda_{m,n} \right)(\tau) \\ =& \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{11/2}} \frac{(|V_8|^2 + |S_8|^2)\Lambda_{2m,n} - (V_8\bar{S}_8 + S_8\bar{V}_8)\Lambda_{2m+1,n}}{|\eta^8|^2}(\tau) \ , \end{aligned} \tag{4.2.11}$$

where it is quite clear in the second line that the bosons and the fermions underwent different momentum shifts, similarly to (4.2.7). There is a subtlety though, since (4.2.11) is not modular-invariant, as however expected for a torus amplitude. This is due to the presence of the orbifold, which brings new twisted states into the game. Those contribute to the full amplitude as follows:

$$\begin{aligned} \mathcal{T} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{11/2}} \Big\{ & (|V_8|^2 + |S_8|^2)\Lambda_{m,2n} - (V_8\bar{S}_8 + S_8\bar{V}_8)\Lambda_{m+1/2,2n} \\ & + (|O_8|^2 + |C_8|^2)\Lambda_{m,2n+1} - (O_8\bar{C}_8 + C_8\bar{O}_8)\Lambda_{m+1/2,2n+1} \Big\} \frac{1}{|\eta^8|^2}(\tau) \ , \end{aligned} \tag{4.2.12}$$

where we also rescaled the radius $R \to 2R$, so that momentum/winding states are modified as well, $|2m, n\rangle \to |m, 2n\rangle$, to keep the mass of a given state fixed as a function of the radius. Expressed in terms of this new radius, we see that the bosons have all possible momenta, whereas fermions have momenta shifted by half-integers, exactly like in (4.2.7). The action on winding modes is purely stringy and is not captured by field theory examples such as (4.2.7).

Among the new odd winding twisted states, which are states with "wrong" GSO projection, there is a tower of states starting with a scalar (coming from the character $|O_8|^2$ above) with the lightest mass given by

$$m_O^2 = -\frac{2}{\alpha'} + \frac{R^2}{\alpha'^2} \ . \tag{4.2.13}$$

For small radii $R < \sqrt{2\alpha'}$ this scalar becomes tachyonic, whereas it is very heavy in the opposite limit $R \gg \sqrt{2\alpha'}$. This scalar is a main actor in the brane-brane interactions at long distances that we discuss in section 4.4.2.

If we now turn to type I string theory, the Klein bottle amplitude is now given by

$$\mathcal{K} = \frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \frac{V_8 - S_8}{\eta^8}(2i\tau_2) \sum_m e^{-\alpha'\pi\tau_2 \frac{m^2}{R^2}} \ . \tag{4.2.14}$$

It is the same as in the superstring case, meaning that it is insensitive to the freely acting orbifold. We also see from (4.2.14) that there is no winding propagating in the Klein bottle. In particular, it reduces the number of bosonic components in the torus as in the superstring case. The one-loop open string amplitudes are given by[14]

$$\mathcal{A} = \frac{32^2}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \frac{V_8 P_m - S_8 P_{m+1/2}}{\eta^8}\left(\frac{i\tau_2}{2}\right) \ , \quad \mathcal{M} = -\frac{32}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \frac{V_8 P_m - S_8 P_{m+1/2}}{\eta^8}\left(\frac{i\tau_2}{2} + \frac{1}{2}\right) . \tag{4.2.15}$$

4.3 Axions in string theory

Within the string spectrum, there are many bosonic tensors. When the nine space dimensions in which string theory lives are compactified [406, 407] so that only three remain macroscopic, those tensors decompose into irreducible representations of the Poincaré group of the apparent four-dimensional spacetime. In particular, there are scalars in the spectrum, some of which behave like axions [408–410]. We briefly discuss in sections 4.3.1 and 4.3.2 how they arise and couple to gauge fields. The vocabulary and notations for 4D string EFTs are then presented in section 4.3.3, and specialized to anomalous setups in the heterotic string in section 4.3.4. Eventually, we turn in section 4.3.5 to the study of the compatibility between SUSY breaking via gaugino condensation and the presence of a QCD axion in a four-dimensional string-inspired EFT with a Green-Schwarz mechanism.

4.3.1 Axions in string compactifications

In string theory, we should differentiate between two kinds of axions: those which arise from the closed or the open string sectors. The latter for instance correspond to the phases of complex fields living on branes, and are close to field theory axions. Closed string axions, on the other hand, arise from the bulk antisymmetric forms and have specific features (example of reviews are [55, 411]).

Ten-dimensional bulk fields, and in particular p-forms, have a KK decomposition when spacetime is split into a direct product between the four-dimensional Minkowski spacetime $\mathcal{M}_4$ and a six-dimensional compact manifold Z_6. Schematically, the KK decomposition goes as follows:

$$\Psi(x^M) \equiv \sum_n \Psi^n_{\mathcal{M}_4}(x^\mu)\Psi^n_{Z_6}(x^m) \ , \tag{4.3.1}$$

[14] We do not turn on Wilson lines for simplicity.

where M denotes 10D spacetime indices, μ the ones of 4D Minkowski spacetime and m those of Z_6. 10D equations of motion for Ψ reduce to 4D ones for $\Psi^n_{\mathcal{M}_4}$ once specific equations are imposed on $\Psi^n_{Z_6}$. The set of all possible $\Psi^n_{Z_6}$s is then generated by a basis of the solutions to the latter equations.

In our discussions of axions, we are interested in massless 4D modes[15]. For a 10D scalar field Ψ, massless modes arise when

$$\Delta_{Z_6}\Psi^n_{Z_6} = 0 \ , \tag{4.3.2}$$

where Δ_{Z_6} is the laplacian on Z_6. This equation is only solved by a constant on the compact space Z_6 and only one massless scalar mode $\Psi^n_{\mathcal{M}_4}$ arises from our 10D scalar. For forms $C_{M_1...M_p}$, the same reasoning applies up to the fact that there is an ambiguity in (4.3.1) since no form indices appear. The correct expression is

$$C_{\mu_1..\mu_q M_{q+1}...M_p}(x^M) \equiv \sum_n C^n_{\mathcal{M}_4,\mu_1..\mu_q}(x^\mu) C^n_{Z_6,M_{q+1}...M_p}(x^m) \ , \tag{4.3.3}$$

where $C^n_{\mathcal{M}_4}$ and $C^n_{Z_6}$ are respectively a q-form on $\mathcal{M}_4$ and a $(p-q)$-form on Z_6. $C^n_{\mathcal{M}_4}$ describes a massless 4D form when

$$dC^n_{Z_6} = d^\dagger C^n_{Z_6} = 0 \tag{4.3.4}$$

where $d^\dagger$ is the adjoint of the exterior derivative d on Z_6. (4.3.4) defines what is called a harmonic $(p-q)$-form on Z^6. The number of such forms only depends on the topology of Z_6 and is called the Betti number $b_{p-q}(Z_6)$. Axions are specifically found in the KK expansion of a p-form C when the $C^n_{Z_6}$ are also p-forms, so each p-form in string theory generates $b_p(Z_6)$ axions after compactification to 4D.

In string perturbation theory, there is no potential, and in particular no mass, generated for those axions arising from forms. Their masses come entirely from non-perturbative effects, such as instantons. We only focus on gauge instantons in what follows, but instantons in string theory can also be gravitational, worldsheet or brane instantons.

4.3.2 Coupling to gauge fields: Green-Schwarz mechanism and D-branes actions

Non-perturbative gauge effects, which are welcome if one wants to use the closed string axions which we discuss here as QCD axions, need the axions to be coupled to gauge fields. We present now two ways in which forms couplings to gauge fields arise in string theory: via anomaly cancellation mechanisms and in the presence of D-branes.

First, forms in heterotic or type I string theory are involved in the Green-Schwarz mechanism and have couplings dictated by the anomaly structure of the theory. The idea behind the GS mechanism goes as follows: the gauge variation of an action due to an anomalous spectrum can be canceled by the shift of a classical term in the action. In string theory, the latter shift comes

[15]We mean by this that we do not want our axions to have masses of order the compactification scale, which is high.

from the gauge transformation of a 2-form C_2[16], which shows up from the fact that the field strength H_3 which appears in the 10D EFT

$$S \supset -\frac{1}{4\kappa_{10}^2 g_s^{2\delta}} \int H_3 \wedge *H_3 \tag{4.3.5}$$

is linked to the 2-form as follows

$$H_3 = dC_2 - \frac{\kappa_{10}^2}{g_{10}^2} \operatorname{Tr}\left(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A \right) , \tag{4.3.6}$$

where g_s is the string coupling, $\delta = 0$ or 1 respectively for type I or heterotic strings, κ_{10} is the string frame gravitational constant in 10D and g_{10} is the string frame gauge coupling associated to the gauge potential A, taken here as a matrix in the fundamental representation. Wedge products are implicit in what follows and we restrict for conciseness the discussion to pure gauge anomalies, shortly mentioning gravitational and mixed anomalies later in this section. H_3 is gauge-invariant if C_2 has the following transformation

$$\delta_\lambda C_2 = \frac{\kappa_{10}^2}{g_{10}^2} \operatorname{Tr}(\lambda dA) \tag{4.3.7}$$

when $\delta_\lambda A = d\lambda - i[A, \lambda]$. Then, a coupling of the kind

$$S \supset \int C_2 X_8(F) , \tag{4.3.8}$$

with $X_8(F)$ a 8-form built out of $F = dA - iA^2$, can cancel any anomaly whose associated anomaly polynomial is $\frac{\kappa_{10}^2}{g_{10}^2} \operatorname{Tr}(F^2) X_8(F)$. The $SO(32)$ and $E_8 \times E_8$ gauge groups are such that the anomaly polynomial indeed has this form:

$$X_8 = \frac{g_{10}^2}{384(2\pi)^5 \kappa_{10}^2} \left(\frac{\operatorname{Tr}_a(F^4)}{3} - \frac{\operatorname{Tr}_a(F^2)^2}{900} \right) , \tag{4.3.9}$$

with Tr_a indicates that the generators are taken in the adjoint representation of the gaugini. More remarkably, when gravitational anomalies are put into the game in a similar way such that C_2 also shifts under local Lorentz transformations, those two gauge groups are singled out as two of the only four gauge groups for which the full anomaly polynomial factorizes. The two other groups for which the GS mechanism is possible are $E_8 \times U(1)^{248}$ and $U(1)^{496}$ but no string theory realizing them is known.

When the 2-form C_2 is compactified, axions arise from its KK expansion as discussed in section 4.3.3. There are then two origins for the axion: the obvious way is to choose $q = 0$ in (4.3.3), "hiding" all the tensor indices of C_2 in Z_6, and the other one is linked to the fact that a four-dimensional 2-form can be dualized into an axion. Thus, the components along $\mathcal{M}_4$ of C_2 defines a 4D 2-form $C_{\mu\nu}$, which is equivalent to an axion. It can be understood as follows: one can rewrite (4.3.5) with an independent 3-form $\tilde{H}_3$ and a Lagrange multiplier a for the Bianchi identity which ensures that $\tilde{H}_3 = H_3$ as defined in (4.3.6). Indeed, if

$$S \supset -\frac{V_{Z_6}}{4\kappa_{10}^2 g_s^{2\delta}} \int \tilde{H}_3 \wedge *\tilde{H}_3 + \int a \left(d\tilde{H}_3 + \frac{\kappa_{10}^2}{g_{10}^2} \operatorname{Tr}(F^2) \right) , \tag{4.3.10}$$

[16] In type I string theory, the 2-form involved comes from the R-R sector whereas it comes from the NS sector in the heterotic string.

integrating a gives back (4.3.5). We defined V_{Z_6} as the volume of the internal compact manifold and all the integrals are four-dimensional ones. On the other hand, integrating $\tilde{H}_3$ produces

$$S \supset -\frac{\kappa_{10}^2 g_s^{2\delta}}{V_{Z_6}} \int d^4x\sqrt{-g}(\partial a)^2 + \frac{\kappa_{10}^2}{g_{10}^2} \int a \, \mathrm{Tr}\big(F^2\big) \ . \tag{4.3.11}$$

This tells us that the axion dual to $C_{\mu\nu}$ has QCD axion-like couplings to the gauge fields, hence it is a QCD axion candidate in string theory models where the SM gauge group is embedded into the unbroken gauge group of the theory[17]. In addition, its coupling are fixed: indeed, introducing the four-dimensional Planck mass $M_P^2 = \frac{V_{Z_6}}{\kappa_{10}^2 g_s^2}$ and gauge coupling $g^2 = \frac{g_{10}^2 g_s^{\delta'}}{V_{Z_6}}$, with $\delta' = 1$ or 2 for type I or heterotic strings, we find that the axion decay constant, defined as the inverse coupling of the axion to the gauge field topological density is

$$f_a = \frac{g^2 M_P}{4\sqrt{2}\pi^2 g_s^{\delta'-\delta-1}} \ , \tag{4.3.12}$$

and we see that it depends neither on the volume of Z_6 nor on the string coupling for both the type I and heterotic strings. For this reason, it is called the model independent axion, and its decay constant is fixed to be around 10^{16} GeV [55]. It is way too high to lie in the classic axion window for an absence of tuning in the initial cosmological conditions for the axion field, whose upper bound is given by (2.1.31). On the other hand, we mentioned already that the status of this bound has been the subject of (recent) debates [71–83]. Via the Green-Schwarz term, (4.3.8) is such that the model-independent axion can also be charged under an unbroken abelian gauge symmetry when the components of the gauge field parallel to Z_6 have non-trivial profiles. For instance, for a gauge group $SO(32)$, (4.3.8) implies that the 4D lagrangian contains

$$S \supset \left(\frac{g_{10}^2}{1152(2\pi)^5 \kappa_{10}^2} \int_{Z_6} \mathrm{Tr}\big(T \langle F^3 \rangle\big) \right) \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu} F_{\rho\sigma} \ , \tag{4.3.13}$$

where T is the unbroken abelian generator and $\langle F^3 \rangle$ refers to the classical profile chosen for the internal components of the gauge field. After dualization, (4.3.13) gives a minimal coupling between the abelian gauge field and the axion a. The axion which couples to the abelian gauge field becomes the longitudinal component of the latter after gauge fixing and is removed from the spectrum.

0-form KK modes of C_2 define model dependent axions, since their decay constant and their couplings depend on V_{Z_6} on top of M_P and g^2. They receive QCD axion-like couplings via (4.3.8):

$$S \supset \sum_n \left(\frac{g_{10}^2}{1152(2\pi)^5 \kappa_{10}^2} \int_{Z_6} C_{Z_6}^n \, \mathrm{Tr}\big(T^a T^b \langle F^2 \rangle\big) \right) C_{\mathcal{M}_4}^n \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^b \ , \tag{4.3.14}$$

using notations of (4.3.3). They can also be charged under an abelian generator, in which case the minimal coupling arises from the kinetic term (4.3.5).

In the full GS mechanism, there is in addition to $\mathrm{Tr}(F^2)$ in the Bianchi identity for H_3 a $\mathrm{Tr}(R^2)$ term for gravitational anomalies, and there are also gravitational corrections to X_8. Thus, the axions can couple to gravitational instantons and one must ensure that their contribution is

[17] Both $SO(32)$ and $E_8 \times E_8$ have a $SU(3) \times SU(2) \times U(1)$ subgroup.

much smaller than the QCD one if the axion is to be identified with the QCD axion. Of course, non-perturbative gauge effects beyond QCD must also be kept under control.

In type II string theories, D-branes offer a second way to couple axions to gauge fields. Indeed, the (bosonic part of the) Dirac-Born-Infeld action for N coincident Dp-branes is

$$S_{\mathrm{DBI}} = -T_p \left(\int_\Sigma d^{p+1}\xi \,\mathrm{Tr}\Big[e^{-\phi} \sqrt{-\det(g_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} \Big] + i \int_\Sigma \mathrm{Tr}\left[e^{B+2\pi\alpha' F} \wedge \sum_q C_q \right] \right) , \tag{4.3.15}$$

where T_p is the brane tension, ξ are the coordinates on the worldvolume Σ of the brane, g_{ab}, B_{ab} (and ϕ) are the NS-NS fields parallel to the brane, F_{ab} is the field strength of the gauge theory living on the brane and C_q are the R-R forms of the bulk theory. We see for instance that QCD axion-like couplings to gauge fields are generated for C_{p-3} via the Chern-Simons term. The axion decay constants are dependent on the volume in such setups, and can be brought in the classic window provided the compactification volume is large in string units [57, 412, 413].

4.3.3 Four-dimensional EFTs and moduli stabilization

Now, we present some ingredients of four-dimensional EFTs which describe string theory axions. We restrict ourselves to supersymmetric compactifications, so that SUSY breaking, when it happens, does not happen at the string scale as in section 4.2.1 for instance. On the other hand, we only consider compactifications with $\mathcal{N} = 1$ SUSY to match pseudo-realistic phenomenologies.

In this case, axions belong to the scalar part of chiral multiplets T, together with moduli of the compactification (see e.g. [414]), i.e. (mostly) dynamical fields linked to the compactification geometry. An example is the axion-dilaton superfield S. Its real part contains (a combination of fields including) the dilaton ϕ, and its imaginary part is an axion a, which corresponds to the model-independent axion from C_2 in the heterotic string or to the R-R 0-form C_0 in type II D-brane models.

The dynamics of those chiral superfields, given by a Kähler potential K and a superpotential W as discussed in section 3.1.2, is such that the (closed string) axion shift symmetries are not broken perturbatively. These symmetries, which uplift via SUSY to shifts of the superfields, such as $T \to T + i\Lambda$ with $\Lambda^\dagger = \Lambda$, restrict the possible shapes of K and W. As an illustration, typical tree-level Kähler potentials are

$$K = -c \ln\big(T + \overline{T}\big) , \tag{4.3.16}$$

where c is a constant.

The moduli in the chiral multiplets have Planck scale couplings to matter and determine the coupling constants of the EFT. Consequently, the need to be given masses so that they evade bounds coming from fifth-forces experiments. On the other hand, their masses need to be large enough so that they do not spoil the success of big bang nucleosynthesis via the so-called cosmological moduli problem [84]. The way the moduli receive masses is crucial for discussions of axions, since SUSY dynamics may link the masses of moduli and the masses of axions, see e.g. [57,414,415]. Fluxes from forms stabilize some moduli [416], there are also D-term potentials [417] as well as possible perturbative effects which give masses to moduli. In the case of D-terms, which are associated to a gauge field, one axion becomes a longitudinal component of the gauge boson and is removed from the spectrum as well. There are also non-perturbative contributions to the superpotential (examples can be found in [418]), which break the axion shift symmetry and contribute to both the moduli and the axions potential as contributions to the superpotential. Schematically, they read

$$W = W_0 + Ae^{-aT} \ , \tag{4.3.17}$$

where W_0, A and a are constants. Such contributions must be kept under control in model building with light axions (such as QCD axion models).

4.3.4 Anomalous U(1)'s in heterotic string theory

An example of different aspects of the previous discussion can be found in heterotic string theory, in the presence of an anomalous abelian factor in the gauge group, which we call $U(1)_X$ in what follows.

Anomalous means here that the 4D fermionic spectrum has a gauge anomaly in the presence of this abelian factor, even though the full theory is consistent. The consistency is ensured by a 4D SUSY version of the Green-Schwarz mechanism, made possible by the fact already mentioned in section 4.3.2 that, after compactification, 4D axions arising from the 2-form C_2 can be charged under an abelian gauge symmetry. In this case, the theory's lagrangian contains terms of the form $aF^i\tilde{F}^i$, such that the $U(1)_X$ gauge variation of the axion a cancels any mixed anomaly between $U(1)_X$ and the (semi-simple factors of the) gauge group G_i of field strength F^i.

In perturbative heterotic string theory constructions [419, 420], there is only one possible anomalous $U(1)_X$ and one superfield, the universal axion-dilaton S, whose imaginary part is the axion a aforementioned, transforming non-linearly under $U(1)_X$ gauge transformations. Those act on the different superfields involved as[18]

$$\delta V_X = \Lambda \ + \bar{\Lambda} \ , \quad \delta\phi^a \ = -2q_a\phi^a\Lambda \ , \quad \delta S = \delta_{GS}\Lambda \ , \tag{4.3.18}$$

where V_X is the vector superfield for the gauge group $U(1)_X$, ϕ^a are chiral superfields of charge q_a, and δ_{GS} is a constant, in units where $M_P = 1$ which we choose in this section.

Anomaly cancellation occurs via a 4D version of the GS mechanism, which means that the θ-term for the gauge groups G_i which have a mixed anomaly with $U(1)_X$ is given by $\mathrm{Im}(S)$ (up to a numerical coefficient). By holomorphy, we conclude that the full tree-level gauge kinetic functions are given by S:

$$f_i = k_i S \ , \tag{4.3.19}$$

where the k_is are the Kac-Moody levels of the G_i embeddings in $E_8 \times E_8$ or $SO(32)$. Anomaly

[18]We use here the same convention as in [421] to define charges of chiral superfields.

cancellation conditions relate mixed anomalies $C_i = U(1)_X \times G_i^2$, such that

$$\delta_{GS} = \frac{C_1}{k_1} = \frac{C_2}{k_2} = \cdots = \frac{C_N}{k_N} = \frac{1}{192\pi^2}\,\mathrm{Tr}(q_X)\ , \tag{4.3.20}$$

where the last expression comes from the mixed $U(1)_X$-gravitational anomaly, where $\mathrm{Tr}(q_X)$ is the sum of $U(1)_X$ charges over all the charged fermions in the spectrum.

The Kahler potential for the universal axion-dilaton is modified to account for the gauge transformations:

$$K = -\ln\big(S + \bar{S} - \delta_{GS} V_X\big) \tag{4.3.21}$$

and it encodes the Fayet-Iliopoulos term which appears in the D-term

$$D_X = q_i \phi^i \partial_i K + \frac{\delta_{GS}}{2(S+\bar{S})}\ . \tag{4.3.22}$$

We consider $\delta_{GS} > 0$ in what follows. In all known perturbative constructions there always exists in the massless spectrum a field with appropriate sign of the charge (negative in our conventions) whose vev is able to cancel perturbatively the (field-dependent) FI term and maintain supersymmetry. We consider the minimal case of one such field, called ϕ in what follows, and normalize its charge to -1, following [421].

When ϕ gets a vev, its phase becomes an Goldstone boson and combine with $\mathrm{Im}(S)$ to be absorbed by the vector in V_X. The $U(1)_X$ is thus broken and its vector modes gets a Planckian mass. The remaining combination of $\mathrm{Im}(S)$ and $\arg(\phi)$ is a physical axion, which we now study.

4.3.5 Axions and anomalous U(1)'s

In this section, we follow [211], attached at the end of this thesis, and consider again the anomalous heterotic setup of section 4.3.4 with a single charged scalar ϕ. We study the axion in S and ϕ when there is a non-perturbative superpotential arising from gaugino condensation [393, 394] in a confining hidden sector gauge group, and ask whether the physical axion can be identified with a QCD axion. Consistently with previous discussions, we see that, in the minimal gaugino condensation case, the axion mass is tied to the supersymmetry breaking scale and cannot be light enough. However, slightly refined models maintain a massless axion all the way down to the QCD scale. Both kinds of models can be extended to yield intermediate scale axion decay constants. Finally, $U(1)_X$ can be identified with the Froggatt-Nielsen symmetry in flavourful axion models. Along the way, we establish that generic anomalous $U(1)$ models coupled to charged scalars getting vacuum expectation values always have one light axion, whose mass can only come from non-perturbative effects.

Anomalous U(1) and perturbatively massless axions

We first comment on the following generic property of models with an anomalous $U(1)$[19]: at the perturbative level, and if there is at least one charged scalar field which gets a vev, such

[19] In case of additional $U(1)$ gauge symmetries, anomalous or not, the counting may be different but a similar result always applies.

models always contain a potential axion candidate, which can only get a mass by turning on non-perturbative effects.

Indeed, let us consider an abelian gauge theory in a Stueckelberg phase, coupled to charged scalars Φ_i of charges X_i, of lagrangian

$$\mathcal{L} = -|D_\mu \Phi_i|^2 - \frac{1}{4g^2} F^2_{X,\mu\nu} - \frac{1}{2}(\partial_\mu a_S + M A_{X,\mu})^2 + \cdots \ , \tag{4.3.23}$$

where $\cdots$ are other terms like axionic couplings. Since we are interested in axion-like particles, without loosing generality we only consider in what follows charged scalars having non-vanishing vevs, parametrized as

$$\Phi_i = \frac{V_i + h_i}{\sqrt{2}} e^{\frac{i\theta_i}{V_i}} \ . \tag{4.3.24}$$

From (4.3.23) one finds that the Goldstone boson which mixes in the usual way $\partial_\mu \theta_X A^\mu_X$ with the gauge field is given by (up to a normalization factor)

$$\theta_X = X_i V_i \theta_i + M a_S \ . \tag{4.3.25}$$

We have therefore $N+1$ potential axions/pseudoscalars, one of which is absorbed by the gauge field via the Higgs mechanism. The perturbative scalar potential is of the form[20]

$$V^{(\text{pert})} = \sum_\alpha \lambda_\alpha \Phi_1^{m_1^{(\alpha)}} \cdots \Phi_N^{m_N^{(\alpha)}} + \text{h.c.} \ , \tag{4.3.26}$$

and gauge invariance imposes the restriction $X_1 m_1^{(\alpha)} + \cdots + X_N m_N^{(\alpha)} = 0$. Simple matrix algebra tells us that the maximal number of independent gauge invariant operators that can be written is equal to $N-1$. On the other hand, a complete basis of such gauge invariant operators also defines the physical pseudoscalars/axions which can be expressed as a combination of the θ_i's, since their phases

$$\theta_\alpha = \frac{m_1^{(\alpha)}\theta_1}{V_1} + \cdots + \frac{m_N^{(\alpha)}\theta_N}{V_N} \tag{4.3.27}$$

are automatically orthogonal to the Goldstone boson (4.3.25). The scalar potential (4.3.26) then gives masses to at most $N-1$ pseudoscalars. Consequently, there is always (at least) one leftover massless pseudoscalar, which is a PQ axion candidate if it has the appropriate couplings. At the perturbative level, it is therefore always possible to define a PQ symmetry in models with an anomalous $U(1)_X$ gauge factor.

As one will see in the next sections, non-perturbative effects can generate gauge-invariant potential terms of the form

$$V^{(\text{non-pert})} = \sum_\beta e^{-q_\beta s_0 - i c_\beta a_S} \lambda_\beta \Phi_1^{p_1^{(\beta)}} \cdots \Phi_N^{p_N^{(\beta)}} + \text{h.c.} \ , \tag{4.3.28}$$

where s_0 is the vev of a scalar and q_β, c_β are numbers. Whenever such terms are generated, the leftover massless axion gets a mass (possibly from effects other than the usual QCD ones, as we discussed earlier).

[20]It can be checked that the argument below does not change if some of the fields in the scalar potential appear with a complex conjugation.

Gaugino condensation, anomalous $U(1)$ and axion mass

Working now in the heterotic setup of section 4.3.4, we add to the dynamics contributions arising from the gaugino condensation of a hidden sector confining gauge group. The non-perturbative contributions to the superpotential are discussed in appendix C.2, in the absence of $U(1)_X$. The non-perturbative scale $\Lambda \sim e^{-\frac{8\pi^2}{(3N_c-N_f)g^2}}$ in (C.2.1) now translates into

$$\Lambda \sim e^{-\frac{8\pi^2 k_h S}{(3N_c-N_f)}} , \tag{4.3.29}$$

where k_h is the Kac-Moody level of the hidden sector gauge group, since $k_h S$ provides the gauge coupling according to (4.3.19). Consequently, Λ in (4.3.29) has a gauge variation since S has one. However, this does not mean that gauge invariance forbids gaugino condensation to take place: [421] showed indeed that the GS cancellation of gauge anomalies restricts the non-perturbative dynamics such that the non-perturbative superpotential, as well as the scale of gaugino condensation, is precisely gauge invariant.

Let us illustrate this by taking for simplicity a SUSY-QCD model with N_c colors and $N_f < N_c$ flavours and denoting by Q $(\tilde{Q})$ the hidden sector quarks (antiquarks) of $U(1)_X$ charges q $(\tilde{q})$. The GS conditions (4.3.20) completely fix the sum of the charges to be

$$C_h = \frac{N_f(q+\tilde{q})}{4\pi^2} = \delta_{GS} k_h . \tag{4.3.30}$$

This turns out to be precisely the gauge invariance condition of the non-perturbative superpotential

$$W^{(\text{non-pert})} = (N_c - N_f) \left[\frac{e^{-8\pi^2 k_h S}}{\det\left(Q\tilde{Q}\right)} \right]^{\frac{1}{N_c-N_f}} . \tag{4.3.31}$$

We also add a perturbative coupling allowed by the gauge charges:

$$W^{(\text{pert})} = \lambda_i^{\tilde{j}} \left(\frac{\phi}{M_P} \right)^{q+\tilde{q}} Q^i \tilde{Q}_{\tilde{j}} . \tag{4.3.32}$$

Since ϕ gets a large vev of the order of the FI term via the D-term potential (4.3.22), below the scale of $U(1)_X$ gauge symmetry breaking the perturbative term (4.3.32) becomes a mass term for the hidden sector quarks and the dynamics of condensation is essentially that of supersymmetric QCD. In particular, drawing from (C.2.3), we see that the gaugino condensation scale is

$$\Lambda_L^3 = (\det\lambda)^{\frac{1}{N_c}} M_P^{3-N_f/N_c} \left(\frac{\phi}{M_P} \right)^{\frac{N_f(q+\tilde{q})}{N_c}} e^{-\frac{8\pi^2 k_h S}{N_c}} , \tag{4.3.33}$$

which is explicitly gauge invariant.

It can be explicitly shown [211] that, if the scale of condensation is much smaller than the $U(1)_X$ breaking scale, the dynamics of the mesons $Q\tilde{Q}^i_j$ is irrelevant when we discuss the axion. Consequently, we only consider S and ϕ in what follows and use as a superpotential

$$W = W_0 + N_c \Lambda_L^3 , \tag{4.3.34}$$

where the constant W_0 was added for the purpose of coupling to gravity later on. In order to identify the massless axion, it is enough to parametrize the fields by ignoring any other field than those pseudoscalars. By defining them in order to have canonical kinetic terms, we are led to the parametrization

$$S = s_0 \left(1 + i\sqrt{2}\frac{a_S}{M_P}\right) \ , \quad \phi = \frac{V}{\sqrt{2}} e^{\frac{i a_\phi}{V}} \ , \ , \tag{4.3.35}$$

where s_0 and V are vevs. One combination of those pseudoscalars

$$a_X \propto \frac{\delta_{GS}}{\sqrt{2} s_0} a_S + 2V a_\phi \tag{4.3.36}$$

is absorbed by the $U(1)_X$ gauge field, and a second one remains as a physical axion. According to our general discussion around (4.3.27), the phase of the non-perturbative term in (4.3.34) defines this physical axion, which is orthogonal to the Goldstone boson a_X precisely when the GS anomaly cancellation conditions (4.3.30) are imposed. However, it remains massless because (4.3.34) still has a global R-symmetry. The associated symmetry current gives us the expression of the physical axion a_{PQ}:

$$J_\mu \propto \frac{1}{\frac{1}{V} + \frac{8 s_0^2 V}{\delta_{GS}^2 M_P^2}} \partial_\mu \left(a_\phi - \frac{2\sqrt{2} s_0 V}{\delta_{GS} M_P} a_S \right) \equiv \ f_a \partial_\mu a_{PQ} \ , \tag{4.3.37}$$

where we identified the axion decay constant

$$\frac{1}{f_a} = \sqrt{\frac{1}{V^2} + \frac{8 s_0^2}{\delta_{GS}^2 M_P^2}} \tag{4.3.38}$$

and where we can recognize as announced the axion in the phase of (4.3.33). Natural values are of order the unification scale $f_a \sim M_{GUT}$, although smaller values are possible in orientifold models.

At the global supersymmetry level, the axion mass is protected by the R-symmetry. However, after coupling to supergravity, the constant W_0 breaks explicitly the R-symmetry and as such the axion gets a scalar potential and therefore a mass [422]. Without entering details of moduli stabilization, one expects a scalar potential of the form

$$V(a_{PQ}) \sim W_0 N_c (\det \lambda)^{\frac{1}{N_c}} M_P^{3 - N_f/N_c} \left(\frac{V}{M_P}\right)^{\frac{N_f (q+\tilde{q})}{N_c}} e^{-\frac{8\pi^2 k_h s_0}{N_c}} \cos\left(\frac{(q+\tilde{q}) N_f}{N_c} \frac{a_{PQ}}{f_a}\right) \ , \tag{4.3.39}$$

where the axion decay constant is given in (4.3.38). By using the order of magnitude value for the gravitino mass $m_{3/2} \sim W_0$, one finds that this axion can solve the strong CP problem if

$$m_{3/2} \Lambda_L^3 \ll 10^{-10} f_\pi^2 m_\pi^2 \ . \tag{4.3.40}$$

This is a very strong constraint, which favours in this minimal model low values of the gravitino mass and of the dynamical scale Λ_L. Using the fact that in the minimal model of [421, 423] supersymmetry was broken, and $m_{3/2} \sim \Lambda_L^3/(V M_P)$, one finds, without an additional source of supersymmetry breaking, the constraint $m_{3/2} \ll 10^{-14}$ eV, which is not realistic in known mediations of supersymmetry breaking. In this model therefore, an additional source of supersymmetry breaking is necessary, whereas for a gravitino mass corresponding to standard mechanisms for supersymmetry breaking, the axion is too heavy to solve the strong CP problem.

A massless axion: the 3-2 model

In the previous minimal model, the hidden sector non-perturbative dynamics was giving a mass to the axion through supergravity interactions. However, non-perturbative dynamics is as we said earlier often instrumental for stabilizing moduli, in our case the very modulus involved in the GS mechanism. The natural next step is to identify models in which the hidden sector non-perturbative dynamics leaves an exactly massless axion, even after coupling to (super)gravity. One way to achieve this goes as follows: at the perturbative level, as we said above, there is always a massless axion in models with anomalous $U(1)_X$. Suppose now that the hidden sector producing the non-perturbative dynamics has an R-symmetry itself in the limit in which the anomalous abelian gauge dynamics is turned off[21]. Then if the condensation breaks spontaneously the R-symmetry, there is another R-axion coming from the hidden sector. In total there are therefore two axions in the limit where gravity is decoupled. By turning on gravity with a constant term in the superpotential which breaks explicitly the R-symmetry, one (linear combination) of the two axions becomes massive, while the other one remains massless down to the QCD scale and behaves as an ideal candidate for a PQ QCD axion.

One explicit model of this type uses for the hidden sector the 3-2 model of supersymmetry breaking [422, 424]. The gauge group of the model is $G = G_h \times U(1)_X \times \cdots$, where $G_h = SU(3) \times SU(2)$ is the hidden sector gauge group. The nonabelian factor $SU(3)$ is confining with a dynamical scale Λ_3. The matter content in the UV contains the chiral multiplets

$$Q_i^\alpha(3,2) \ , \quad L^\alpha(1,2) \ , \quad \Big[\bar{U}^i(\bar{3},1) \ , \quad \bar{D}^i(\bar{3},1) \ \to \ \bar{Q}_\alpha^i = (\bar{D}^i, \bar{U}^i)\Big] \ , \tag{4.3.41}$$

in a self-explanatory notation (notice that the α index of $\bar{Q}$ is not gauged under $SU(2)$ and only represents a convenient repackaging). The model has two anomaly-free global symmetries, one acting like hypercharge and an R-symmetry:

$$\begin{aligned} U(1)_Y \ &: \qquad Y(Q) = \frac{1}{6} \ , \quad Y(\bar{U}) = -\frac{2}{3} \ , \quad Y(\bar{D}) = \frac{1}{3} \ , \quad Y(L) = -\frac{1}{2} \ , \\ U(1)_R \ &: \qquad R(Q) = -1 \ , \quad R(\bar{U}) = R(\bar{D}) = 0 \ , \quad R(L) = 3 \ . \end{aligned} \tag{4.3.42}$$

Below the scale of $SU(3)$ condensation, the dynamics is governed by the gauge invariant operators

$$X_1 = Q\bar{D}L \ , \quad X_2 = Q\bar{U}L \ , \quad X_3 = \det\{\bar{Q}\}_\alpha Q^\beta \tag{4.3.43}$$

and the low-energy superpotential, compatible with the symmetries and the condensation dynamics, is given by

$$W_{\text{eff}} = \lambda X_1 + \frac{2\Lambda_3^7}{X_3} \ . \tag{4.3.44}$$

The analysis of the potential, including the D-term contributions, shows that $\langle X_1 \rangle$ and $\langle X_3 \rangle$ are non-vanishing whereas $\langle X_2 \rangle$ vanishes. There are then two pseudoscalars in the hidden sector, the potential axions in the phases of X_1 and X_3. One linear combination of them gets a mass from the non-perturbative dynamics, and the second one gets a mass from couplings to (super)gravity,

[21] In pure supersymmetric QCD there is no light axion, since the only global anomaly-free symmetry in the UV is an R-symmetry which is broken explicitly by the mass term.

as in the model described in the preceding section. If we now couple this model to an anomalous $U(1)_X$, we get an additional pseudoscalar from the high-energy anomalous $U(1)_X$ sector. There is therefore one leftover axion which is massless all the way down to the QCD scale, being a good candidate for a PQ axion. To restrict the superpotential, one could use the anomalous gauge symmetry instead of imposing the hypercharge global symmetry as above. We can for instance give the following charges to the multiplets (where n is some number):

$$U(1)_X: \begin{cases} X(Q) = \frac{1}{6} + n \\ X(\bar{U}) = -\frac{1}{3} \\ X(\bar{D}) = \frac{1}{3} \\ X(L) = -\frac{1}{2} - \frac{n}{3} \end{cases} \quad \Longrightarrow \quad \begin{cases} X(X_1) = \frac{2n}{3} \\ X(X_2) = \frac{2(n-1)}{3} \\ X(X_3) = \frac{1}{3} + 2n \\ X(\Lambda_3) = \frac{1}{21} + \frac{2n}{7} \end{cases}, \tag{4.3.45}$$

where, as in the model discussed previously, the condensation scale $\Lambda_3 = e^{\frac{-8\pi^2 k_3 S}{7}}$ is not-gauge invariant anymore due to the $U(1)_X \times SU(3)^2$ anomaly:

$$U(1)_X \times SU(3)^2 : C_3 = \frac{1}{4\pi^2} \times \left(\frac{1}{3} + 2n\right) , \quad U(1)_X \times SU(2)^2 : C_2 = \frac{1}{4\pi^2} \times \frac{8n}{3} , \tag{4.3.46}$$

while the non-perturbative superpotential is:

$$W_{\text{eff}} = \lambda \left(\frac{\phi}{M_P}\right)^{\frac{2n}{3}} X_1 + \frac{2\Lambda_3^7}{X_3} . \tag{4.3.47}$$

The first term in (4.3.47) is a perturbatively generated operator if we assume that n is a multiple of $\frac{3}{2}$. If $\Lambda_3 \ll V$, analogously to the model in the previous section this axion is essentially a combination of a_S and a_ϕ. The axion decay constant will be determined as before and is therefore naturally of the order of the unification scale.

Summary

Inspired by recent studies of high-scale decay constant or flavourful QCD axions, we reviewed and clarified their existence in effective string models with anomalous $U(1)$ gauge groups.

We found that such models, when coupled to charged scalars getting vacuum expectation values, always have one light axion, whose mass can only come from non-perturbative effects (and simultaneously turning on the coupling to gravity in supersymmetric models, where an R-symmetry survives even after inclusion of non-perturbative effects). If the main non-perturbative effect is from QCD, then it becomes a Peccei-Quinn axion candidate for solving the strong CP problem. - We studied the symmetries responsible for protecting the axion and the conditions under which the axion is light enough for solving the strong CP problem in a heterotic framework with a single charged scalar and hidden sector gaugino condensation, and we concluded that realistic supersymmetry breaking is incompatible with a light enough axion. However, we also gave a refined example, the 3-2 model, where non-perturbative dynamics still preserves a massless axion all the way to the QCD scale, even after coupling to gravity.

Further comments

Besides more detailed discussions about the physical axion in the minimal gaugino condensation model (4.3.31) (taking into account the mesons for instance), a few more things can be found in [211].

First, the fact that (4.3.38) was of order the unification scale in minimal models can also be relaxed in effective models where the moduli sector is slightly more complex[22]. For example, the following model of two moduli and a charged superfield:

$$K = -\frac{3}{2}\ln\big(T_1 + \overline{T_1} - \delta_1 V_X\big) - \frac{3}{2}\ln\big(T_2 + \overline{T_2} + \delta_2 V_X\big) + \phi^\dagger e^{-2V_X}\phi \ , \tag{4.3.48}$$

coupled to two hidden strong sectors 1 and 2, with gauge kinetic functions given by:

$$f_1 = \frac{T_1}{4\pi} \ , \quad f_2 = \frac{n_2 T_1 + n_1 T_2}{4\pi} \ , \quad \text{where } n_i = \pi\delta_i \text{ are integers} \ , \tag{4.3.49}$$

such that the (stringy instanton) non-perturbative superpotential is

$$W = W_0 + A\phi^{n_1} e^{-2\pi T_1} + B e^{-2\pi(n_2 T_1 + n_1 T_2)} \tag{4.3.50}$$

now allows for a high scale stabilization of the moduli with a small or intermediate scale $V = \langle\phi\rangle$ and axion decay constant (4.3.38). A more complete discussion on this can be found in [211], as well as a generalization of such models to incorporate the 3-2 model of (4.3.44).

Then, anomalous U(1)'s appear in the phenomenological literature because Froggatt-Nielsen symmetries are often anomalous [225, 227], as we already discussed in section 2.3.7. In [211], we thus identify $U(1)_X$ with a gauged flavourful symmetry and discuss the axion couplings which arise. We show that in such a context and irrespective of the details of the model under consideration, gauge invariance fixes completely the couplings of the axion to matter when the charged scalar ϕ is used as a flavon field. The couplings to SM charged fermions are proportional to their anomalous charges and the couplings to the gauge fields to the mixed $U(1)_X \times G_a^2$ anomalies, where G_a is one of the SM gauge group factors. Gauge coupling unification conditions [231] alone then determine the ratio of the coupling to the photon to the coupling to the gluons to be $E/N = 8/3$ at the unification scale. These couplings are similar to the ones in the axi-flavon/flaxion models [204, 205], but the symmetry is now gauged.

4.4 Swampland conjectures

In the preceding sections, we studied specific aspects of string theory phenomenology: how it is tied to supersymmetry, how to break those ties, examples of particles of the string spectrum and specific features of the way the latter arise in the string EFTs.

The possibilities offered by string theory to phenomenology are much more numerous, and instead of studying in details the precise predictions of each string theory compactification, which

[22] Moduli stabilization and axions in string models with anomalous $U(1)$ were studied in various papers [423] and so was the issue of axion mass and decay constant in string theory, see e.g. [55, 57, 425, 426].

are many [365] as we said previously, one may try to establish (possibly broad) criteria which are common to all string theory compactifications. Such criteria would provide hints on how to build models with the hope that they could be completed in string theory, or ways of disproving string theory by direct observation of the violation of the criteria.

Actually, one could ask a similar question about quantum gravity in general, with the difference that most people trust the fact that a theory of quantum gravity should describe nature at some high scale, so that the disproving part of the above statement is not the one which drives research efforts. This search for criteria which are necessary for a given theory to be completable into a quantum theory of gravity is called the swampland program [364], which opposes the swampland to the landscape, which is the space of theories which can be derived from quantum theories of gravity. Beyond the usual QFT criteria to define consistent theories (absence of gauge anomalies, unitarity, etc), the additional criteria which are typical of the mutual consistency of a theory with quantum gravity are called swampland criteria, or swampland conjectures, since they all have the status of conjectures, even though there are compelling and numerous evidences for the most prominent of them.

In what follow, we first review some of those conjectures in section 4.4.1, then discuss in section 4.4.2 a string test of the weak gravity conjecture (WGC), one of the best motivated among the swampland conjectures. A recent and complete review of the swampland is [427], from which we drew quite a lot.

4.4.1 Some conjectures

Our focus is on two conjectures, and we mention additional ones to a lesser extent. The first one has to do with global symmetries, while the second one is the WGC.

No global symmetries in quantum gravity

We have already mentioned this first conjecture quite a lot in section 2, so we eventually spend some specific time discussing it. It states that, in a quantum theory of gravity, there cannot be any exact global symmetry of the spectrum and the interactions [124–126][23].

Arguments in favor of this conjecture exist in string theory, which is known to transpose any global symmetry on the worldsheet into a gauge symmetry of the interacting states [428] (a prominent example is the global super-Poincaré algebra of the worldsheet theory). In addition, there are non-perturbative effects, such as gauge, gravitational or worldsheet instantons, which break the axion shift symmetries [429, 430] as discussed in section 4.3.1.

There are also arguments which are expected to go beyond string theory, being related to properties of gravity which are expected to be universal. Those include holography, see e.g. [431, 432] where using AdS/CFT it was shown (among other things) that a global symmetry in the bulk would lead to inconsistencies in the CFT side, as well as black hole physics [126].

[23] On the other hand, gauge symmetries are fine with respect to quantum gravity.

The black hole argument for the conjecture goes as follows: imagine there is a global $U(1)$ symmetry of the theory of particles and gravity together, then consider an uncharged Schwarzschild black hole into which we send a set of particles which carries a global charge Q. The resulting black hole has mass M and a global charge Q. However, the outside of the black hole is still subject to the no hair theorem of [433] and is uniquely given by the Schwarzschild metric. In particular, the Hawking radiation is only sensitive to the physics near the black hole horizon, which is governed by the Schwarzschild metric, so it is thermal and uncharged, and the mass of the black hole decreases while its charge Q remains. Thus, it eventually reaches a given mass $M_0 < M$, with charge Q. Consequently, there are infinitely many different black hole states of mass M_0, at least one for each possible Q. However, from the outside of the black hole, an observer still measures a finite Bekenstein-Hawking entropy, which only depends on M_0, for this state of infinite degeneracy. This issue would be avoided if somehow, gravity is such that the black hole could not be associated to a constant charge Q.

The black hole evaporation can be followed to smaller charges than M_0, up to a stage where the full quantum gravity effects at the horizon deviate significantly from the Hawking calculation, such that the evaporation may end, for instance to solve the information paradox. Then, a Planck-scale object called a remnant is left behind, which has to be stable since, if it was to fully decay into particles, it should decay into a set of particles of total charge Q[24]. However, such a set might very well be heavier than M_P, as soon as the charge Q is big enough. Thus, the theory has an infinite set of stable remnants, one of each (large enough) Q. This is expected to source inconsistencies if those states run in loops as quantum states [434] or if they enter in the calculation of the maximal entropy contained in a given volume, which is bounded [435]. However, the status of those troubles regarding remnants is still unclear, and we mostly mention it here because the preceding one about black hole entropy does not apply to the WGC, unlike the remnant one.

Let us stress that this conjecture is a prototypical example of how swampland criteria are developed and strengthened: evidences for its validity come from at least two approaches, a first one which uses expected properties of gravity which can interact with known physics at any scale (such as the physics of black holes, or the holographic nature of gravity), and a second one which precisely checks the assertion in quantum theories of gravity which we understand and master, at least to some extent. These two complementary approaches have their own limitations, since arguments in the first approach may be incorrect or used beyond their validity regime (which may be the case for the interpretation of the black hole entropy in the above argument), whereas the second approach may lead to restricted, too demanding criteria (for instance, string theory may not have all the possible features of a quantum theory of gravity).

[24]It may emit some particles from an other quantum process than Hawking radiation, but at some point there is no particle left which has enough charge to be emitted, carrying away the remnant mass.

The weak gravity conjecture

The WGC is a second example of conjecture which is motivated beyond string theory. It states, in its original form [436], that for each gauge symmetry there exists a particle on which gravity is weaker than the gauge force. This is quantified by saying that, in a d-dimensional spacetime[25], there should be a particle of mass m and charge q such that

$$m \leq \sqrt{\frac{d-2}{d-3}} gq \left(M_P^{(d)}\right)^{\frac{d-2}{2}} \ , \tag{4.4.1}$$

where $M_P^{(d)}$ is the d-dimensional Planck mass and g is the gauge coupling. We call a particle which verifies (4.4.1) a WGC particle in what follows. (4.4.1) then precisely means that the repulsive gauge force dominates the gravitational attraction for a WGC particle. [436] also proposed that the WGC should also be applied to the dual gauge field and to monopoles, which led to an other version of the conjecture, called the magnetic WGC: there must be a cutoff Λ to the EFT of the gauge field and the charged states such that

$$\Lambda \lesssim g \left(M_P^{(d)}\right)^{\frac{d-2}{2}} \ . \tag{4.4.2}$$

(4.4.1) is then called the electric WGC. In the magnetic version, the cutoff was originally thought of the scale at which new physics should enter to regulate the mass of the monopole, and is mostly understood these days as the mass scale of a tower of states which become light and enter the EFT when g is decreased. It is quite remarkable to see that quantum gravity, whose effects are expected to arise near the Planck scale, can actually enforce some features on EFTs at much lower scales, here if it is coupled to a gauge field.

The arguments supporting the postulate (4.4.1)-(4.4.2) are manyfold. First, both versions of the conjectures have been extended tested and verified in string theory [437–440] or in holography [441–444].

Second, the tower interpretation of (4.4.2) is consistent with the conjecture about global symmetries in quantum gravity: indeed, there should be some quantum gravity obstruction to the fact of enforcing a global symmetry on the theory by decoupling the gauge field in the $g \to 0$ limit. (4.4.2) then makes it quite clear what this obstruction is: in this limit, the EFT breaks down since its cutoff Λ goes to zero together with g.

Third, the electric version can be connected to remnants arguments. Indeed, (4D) charged black holes must verify the extremality/BPS/no naked singularity bound $M \geq \sqrt{2} gQM_P$. Black holes which saturate this inequality are called extremal. Consequently, there are roughly[26]

$$N_{\text{BH}} \sim \frac{\Lambda}{gM_P} \tag{4.4.3}$$

different[27] eternal black hole states below a scale Λ. When g gets smaller, this number grows and might again create the problems associated to large number of remnants mentioned previously.

[25] [436] phrased the conjecture in 4D, see e.g. [437] for the d-dimensional version.

[26] Here, we assume that g is normalized such that the quantized charges q are integers.

[27] Unlike the global case, the charge of the black hole can be measured outside its horizon by interacting by the electric field it generates, such that every charge Q defines a new black hole state. This explains in particular why the entropy-based argument for the absence of global charges fails here.

Thus, one way out is to postulate that extremal black holes should be able to decay, therefore they never are remnants. This demand can be rewritten as the electric WGC (for algebraic reasons, as least one decay product of an extremal black hole must respect the electric WGC once one assumes the conservation of charge and energy).

There are also tests/derivations of the WGC which are more agnostic about quantum gravity than those which use explicitly string theory. For instance, they compute semi-classical effects to the black hole geometry and entropy [445–447] and impose consistency conditions, for instance on causality, unicity [448] or analyticity [449]. A frequent output of those calculations is that the charge to mass ratio of small black holes is modified, such that the "particle" behind the electric WGC is actually a black hole itself, such that large black holes decay in cascade to smaller black holes. In this case, the WGC "particle" is not accessible to low-energy experimentalists as a sub-Planckian state. There have also been connections between the WGC and cosmic censorship [450].

Motivated extensions of the WGC exist in several directions. First, the initial statement (4.4.1) can be refined, for instance by adding constraints about which particle must verify it: it may be the particle with the smallest charge or the smallest mass, respectively called the smallest charge or the strong WGC [436]. Further refinements include tower versions of the WGC, meaning that there is not only one WGC particle but towers of such states [437,439,451]. Second, the different lines of reasoning behind (4.4.1)-(4.4.2) can be applied to something else than a single $U(1)$ gauge theories coupled to gravity: theories of axions have been considered [436,452–454], theories with moduli-like scalars with or without gauge fields [455], dilaton theories [437], theories of multiple abelian gauge fields [456] as well as $p-$form theories [436, 437, 457]. Dilatonic $p-$form theories will be particularly interesting later (and for string theory in general), so we reproduce here the end result about the WGC: in a $p-$form theory of potential $C_{\mu_1 \dots \mu_p}$, coupled to a dilaton ϕ and gravity, with action

$$S = \int d^dx \sqrt{-g} \left(\frac{M_P^{(d)d-2}}{2} R - \frac{1}{2} (\nabla\phi)^2 - \frac{e^{-\alpha\phi}}{2(p+1)!g^2} F_{\mu_1 \dots \mu_{p+1}} F^{\mu_1 \dots \mu_{p+1}} \right) \tag{4.4.4}$$

(where $M_P^{(d)}$ is the d-dimensional Planck mass, g the gauge coupling and $F = dC$), there must exist a $(p-1)$-dimensional extended object coupled to C with charge Q and to gravity with tension T, and such that

$$8\pi G \left(\frac{\alpha}{2} + \frac{p(d-p-2)}{d-2} \right) T^2 \leq g^2 Q^2 \ . \tag{4.4.5}$$

Finally, let us mention a restriction on the predictive power of the WGC. It seems from (4.4.1) and especially (4.4.2) that upon observing a gauge force, one gets an idea of the scale at which quantum gravity should show up. However, it has been shown in [170] (see also [172] for a similar implementation in a slightly different context) that the WGC may be violated in the EFT for the light modes of a complete UV theory which respects the WGC. The idea is that specific charge assignments, iterable in a clockwork-like way, generate small effective couplings in theories where all couplings and charges satisfy the WGC. Thus, this mechanism violates the WGC in a parametric way, such that the low-energy phenomenologist cannot disentangle the true cutoff from the contributions of conspiring unknown parameters. On the other hand, specific studies in heterotic string theory were performed in [171], where it was showed that the parametric freedom in the WGC violation was limited, and that there are parameter-free estimates of where the string scale may lie.

More swampland conjectures

There are several other conjectures which arose in swampland studies. We do not go into details about those, but we mention the most prominent of them.

The completeness hypothesis [458] states that, given a gauge symmetry, all the (Dirac quantized) charges should be featured in the spectrum of the quantum gauge plus gravity theory.

The distance conjecture [459] restricts the validity regime of a theory: any (properly normalized) trans-Planckian field variation away from a reference point breaks the relevance of the EFT defined at the reference point, because the full quantum gravity theory has a tower of states which becomes exponentially massless when the field varies and which must be included in the EFT at the arrival point. It has a strong relevance for theories of inflation or dark energy. The fact that in string theory, EFT parameters are linked to the vevs of moduli can give an intuition for this conjecture.

The weak gravity conjecture has a quite unexpected consequence: super-extremal branes, which are postulated to exist by a non-supersymmetric refinement of the WGC [460], can be nucleated in non-supersymmetric AdS space, making it unstable. Thus, it was conjectured that any quantum gravity realization of non-SUSY AdS is unstable [461]. It has quite interesting consequences for particle physics [462, 463].

Finally, it was shown in [464] that there exists a bound on the scalar potential V in an EFT of quantum gravity:

$$|\nabla V| \geq \frac{c}{M_P} V \ , \tag{4.4.6}$$

where ∇ means differentiation in scalar field space and $c > 0$ and $\mathcal{O}(1)$. It has the immediate consequence to forbid de Sitter space (dS) as a local minimum of the effective potential. In particular, it justifies the interest in models of dynamical dark energy/quintessence (see e.g. [465,466]). (4.4.6) has strong implications for particle physics, where potentials have local maxima and minima, inducing the needs for specific couplings of quintessence to the SM which would generate time-dependence of couplings [467, 468]. (4.4.6) has been refined since in a way which alleviates such concerns [469].

4.4.2 The weak gravity conjecture in type I string theory with broken SUSY

In this section, we present a new string theory test of the WGC, based on [470], which is attached at the end of this thesis. We test the electric version of the WGC for the R-R 2-form of type I string theory, whose candidate WGC state are D1 branes, when supersymmetry is broken à la Scherk-Schwarz.

Motivation and overview

From a string theory viewpoint, the majority of tests of the swampland conjectures were done in the context of superstring compactifications. On the other hand, supersymmetry breaking generates precisely the ingredients needed for non-trivial tests. In particular, scalar potentials are generated and induce runaway behaviours for moduli fields, which are then potentially interpreted as dynamical dark energy, arguably the only option for an accelerated universe consistent with the de Sitter conjecture. In addition, effective brane-brane interactions arise and hint at a deviation from the BPS conditions for the branes. However, the fact that the branes are BPS in the superstring trivially verifies the WGC for the R-R forms, and the status of the latter must be reassessed when SUSY is broken.

The goal of this work is precisely to do this. We use type I string theory with SUSY breaking via compactification, the simplest and best understood way of breaking supersymmetry in string theory, which we reviewed in section 4.2.2. For a finite value of the supersymmetry breaking radius there is a runaway potential for it. While in the decompactification limit supersymmetry is restored and the weak gravity conjecture is marginally satisfied, considering the rolling field at a different value generates brane interactions and thus constraints from the point of view of the weak gravity conjecture.

We use D1 branes interactions, function of the separation in spacetime, as a test of the WGC. We take the point of view of the electric WGC and we test whether the electric repulsion between two such branes is dominating over their gravitational attraction. The output of the calculation has the double status of being a test of the WGC within a phenomenologically relevant setup, i.e. string theory with SUSY breaking at some scale, as well as a test of the compatibility between the WGC and the demands of the de Sitter conjecture, namely a rolling dynamical field, all this in a perturbative string theory setting.

We find that at short distances and at one-loop there are attractive forces which have a finite limit where the distance goes to zero, whereas at long distances those attractive forces are exponentially suppressed. Since massive (closed strings) fields do not mediate long range interactions, our interpretation is that at this order of perturbation theory the branes still have a charge to mass ratio set by the supersymmetric BPS condition. The limit of zero distance suggests that the corresponding self-energy can be interpreted as a negative quantum correction to the tension, which generates an imbalance between gauge and gravitational forces at higher loops, leading to an effective repulsion at large distances consistent with the WGC. The one-loop attractive forces, unsuppressed at small distances, induce the formation of a finite number of stable bound states of D1 branes.

Scalar potential and runaway vacua

As discussed in section 4.2.2, we implement the Scherk-Schwarz mechanism on the compactified ninth spatial dimension. The first step of our calculation is the explicit determination of the scalar potential for the radius and the Wilson lines of the D9 branes.

The scalar potential in string theory is minus the partition function, therefore

$$V(R, W_i) = -\left(\frac{1}{2}\mathcal{T} + \mathcal{K} + \mathcal{A} + \mathcal{M}\right) . \tag{4.4.7}$$

The Klein bottle is still supersymmetric and therefore it does not contribute to the scalar potential. The potential can be easily estimated in the regime where effective field theory is valid $R \gg \sqrt{2\alpha'}$. In this limit, we replace the modular functions (4.2.11) and (4.2.15) by their leading contribution and perform a Poisson resummation of the Kaluza-Klein sums to turn them into winding sums, to get

$$\mathcal{T} \simeq 128 \frac{R}{\sqrt{\alpha'}} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \sum_n [1-(-1)^n]\, e^{-\frac{\pi n^2 R^2}{\alpha' \tau_2}} \ , \quad \mathcal{A} \approx 32\, \mathcal{T} \ , \quad \mathcal{M} \approx -\mathcal{T} \ , \tag{4.4.8}$$

still without Wilson lines for the D9-branes[28]. As already stated earlier, all string amplitudes above should be multiplied by the factor $1/(4\pi^2\alpha')^{9/2}$. By including this factor and after a straightforward integration, one gets

$$\mathcal{T} = \frac{12}{\pi^{14}} \sum_n \frac{1}{(2n+1)^{10}} \frac{1}{R^9} \ , \tag{4.4.9}$$

which generate a runaway potential, typical for quintessence models: for fixed values of the Wilson lines, the 9D effective potential for the radius in the Einstein frame is of the form

$$\mathcal{L} = \frac{1}{2\kappa_9^2 R^2} (\partial R)^2 - \frac{c e^{\frac{18\phi}{7}}}{R^9} \ , \tag{4.4.10}$$

where ϕ is the dilaton field, $\frac{1}{\kappa_9^2}$ is the nine-dimensional Planck mass and $-\frac{c}{R^9}$ is obtained when summing the three contributions in (4.4.9), according to (4.4.7). After the field redefinition $R = R_0 e^\sigma$, the radion action becomes

$$\mathcal{L} = \frac{1}{2\kappa_9^2} (\partial\sigma)^2 - \frac{c e^{\frac{18\phi}{7} - 9\sigma}}{R_0^9} \ . \tag{4.4.11}$$

We recover here the well-known fact that supersymmetry breaking generates runaway scalar potentials, which generates a cosmological rolling of the corresponding field towards infinity. The example discussed in this paper is too simple to be viable as a quintessence candidate and is ruled out by time dependence of coupling constants, in particular. We note in passing that the vacuum energy is not positive unless one adds Wilson lines (see [471] for discussions of this point).

The formulae above can be generalized easily after compactification to four dimensions. We consider for simplicity a product of circles of radii R_I, $I = 1, \ldots, 6$. In the following we introduce

[28] For simplicity, all the Wilson lines for all the branes encountered in this section are put to zero. More information about what happens when they are included can be found in the full publication.

a vectorial notation for the winding numbers $\mathbf{n} = (n, n_1, \dots, n_5)$. The vacuum amplitudes, in the large radii limit, becomes

$$\mathcal{T} = \frac{3 \times 2^6 V_6}{\pi^9} \sum_{\mathbf{n}} [1 - (-1)^n] \, \frac{1}{(n^2 R^2 + n_1^2 R_1^2 + \cdots n_5 R_5^2)^5} \; , \quad \mathcal{A} = 32 \, \mathcal{T} \; , \quad \mathcal{M} = -\mathcal{T} \; , \tag{4.4.12}$$

where $V_6 = \prod_I R_I$.

Brane interactions and effective brane tensions

Now, let us turn to the computation of the brane-brane interactions. The usual calculation of [363] uses the fact that the brane-brane interactions can be captured, at large separations $r \gg \sqrt{\alpha'}$, by a field theory computation of tree-level exchange of supergravity massless fields between the branes. The setup present however some stringy features that are not fully captured by a pure field-theory analysis by keeping only the supergravity modes. Indeed, the state which mediates the first non-vanishing interactions is the SS would-be tachyon of (4.2.13), which is very heavy in the regime of interest $R \gg \sqrt{\alpha'}$ and would not be kept in a low-energy effective action. Due to this feature, we are forced to perform the computations at the string theory level, using the open-closed duality mentioned in section 4.1.7, although the results can be understood to some extent by field-theory arguments.

Let us consider two D1 branes wrapping the Scherk-Schwarz circle, charged under the RR two-form C_2 (which behave after compactification like particles coupled to a gauge field $\int_{S^1} C_2$), at a distance r in the transverse coordinates. The brane-brane potentials are contained in the cylinder amplitude:

$$\mathcal{A}_{11} = -\frac{1}{2} \, \mathrm{Str} \int \frac{dk}{2\pi} \int_0^\infty \frac{d\tau_2}{\tau_2} e^{-\pi\alpha' \tau_2 (k^2 + M^2)} \; , \tag{4.4.13}$$

with the mass operator given by

$$M^2 \;=\; \frac{1}{\alpha'} L_0 + \frac{m^2}{R^2} + \frac{r^2}{(2\pi\alpha')^2} \; , \tag{4.4.14}$$

extending the result of section 4.1.7 to branes of smaller dimensions. An explicit computation leads to the one-loop amplitude

$$\mathcal{A}_{11} = \frac{1}{2\pi\sqrt{\alpha'}} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} e^{-\frac{\tau_2 r^2}{4\pi\alpha'}} \, \left(P_m - P_{m+1/2}\right) \times \frac{\theta_2^4}{\eta^{12}} \left(\frac{i\tau_2}{2}\right) \; , \tag{4.4.15}$$

which becomes, in the (closed string) tree-level channel,

$$\tilde{\mathcal{A}}_{11} = \frac{R}{4\pi\alpha'} \int_0^\infty \frac{dl}{l^4} \; e^{-\frac{r^2}{2\pi\alpha' l}} \sum_n [1 - (-1)^n] \, e^{-\pi l \frac{n^2 R^2}{2\alpha'}} \, \frac{\theta_4^4}{\eta^{12}}(il) \; . \tag{4.4.16}$$

Notice that only massive states contribute to the D1-D1 brane interactions. In the region of interest $r, R \gg \sqrt{\alpha'}$ a standard field theory computation does not capture the string result (4.4.16). Indeed, in the region $r \gg \sqrt{\alpha'}$ the main contribution to the brane-brane interaction comes from the lightest closed string states. However, since the even winding contribution which include the supergravity states vanishes due to a cancellation between the NS-NS and the R-R sectors, the main contribution to the interaction comes from odd windings containing the would-be tachyon scalar in the closed string spectrum (in character language, O_8).

It is more illuminating to write the tree-level channel exchange potential in a way which involves an integral over the noncompact momenta of the closed strings exchanged, by using the identity

$$\int_0^\infty \frac{dl}{l^4} e^{-\frac{r^2}{2\pi\alpha' l}-\frac{\pi l}{2}\alpha' m_n^2} = \frac{\alpha'^3}{8\pi}\int d^8k \frac{e^{i\mathbf{kr}}}{k^2+m_n^2} \ . \tag{4.4.17}$$

The D1-D1 brane interactions as seen from the tree-level closed-string ("gravitational") exchange are given by

$$V_{11} = -\frac{R\alpha'^2}{2\pi^2}\sum_n \int d^8k \ e^{i\mathbf{kr}} \left[(1-1)\frac{1}{k^2+\frac{4n^2R^2}{\alpha'^2}} + \frac{1}{8}\,\frac{1}{k^2+\frac{(2n+1)^2R^2}{\alpha'^2}-\frac{2}{\alpha'}}\right] . \tag{4.4.18}$$

The contribution of the zero-mode vanishes at one-loop, according to our computation, which implies that at one-loop the interaction of D1 branes is still governed by the the BPS tree-level tension and charge $T_1 = Q_1$. Indeed, since the one-loop contribution is exclusively mediated by massive states, it is short ranged and therefore cannot be interpreted as coming from an imbalance between the tension and charge of the branes. Actually, since the would-be tachyonic scalar for large radius $R \gg \sqrt{\alpha'}$ is much heavier than the supergravity modes and also heavier than string states, one should only keep the terms with $n = 0$ and $n = -1$ in the formula above for consistency.

An important output of the computation above is the D1 brane self-energy, obtained by considering a single D1 brane and setting the spacetime distance $\mathbf{r} = 0$. The result is completely finite and is a contribution localized on the D1 brane worldvolume, it can safely interpreted as a self-energy quantum correction to the brane tension, that we compute here. One gets the approximate result

$$\tilde{\mathcal{A}}_{11} = \frac{8R}{\pi\alpha'}\int_0^\infty dl \ \sum_n e^{-\pi l\frac{(2n+1)^2R^2}{2\alpha'}} = \frac{16}{\pi^2 R}\sum_n \frac{1}{(2n+1)^2} \ . \tag{4.4.19}$$

By extracting the brane-brane self-energy, one obtains a one-loop correction to the brane tension, which can be written either as a corrected D1 brane tension or as the mass M_0 of the wrapped brane on the circle

$$T_{1,\text{eff}} = T_1 - \frac{2}{\pi^3R^2}\sum_n \frac{1}{(2n+1)^2} = T_1 - \frac{1}{2\pi R^2} \quad , \quad M_0 = 2\pi R T_{1,\text{eff}} \ , \tag{4.4.20}$$

where $T_1 = \frac{\sqrt{\pi}}{\sqrt{2}\kappa_{10}}(4\pi^2\alpha')$ is the standard type I D1 brane tension. Notice that this one-loop corrected tension is lower than the tree-level one, due to supersymmetry breaking. Indeed, since $T_1 \sim \mathcal{O}(g_s^{-1})$, the correction is of order $\mathcal{O}(g_s)$ with respect to the original value.

Notice that in a realistic compactification only four spacetime dimensions are noncompact. In this case, the brane-brane potential for $r \gg \sqrt{\alpha'}$ becomes

$$V_{11} = -\frac{R\alpha'^2}{8\pi^2 V_5}\sum_{\mathbf{p}} \int d^3k \ e^{i\mathbf{kr}} \ \frac{1}{k^2+m_{\mathbf{p}}^2+\frac{R^2}{\alpha'^2}-\frac{2}{\alpha'}} \ , \tag{4.4.21}$$

where $\sum_{\mathbf{p}}$ is the sum over all Kaluza-Klein masses in the five additional internal dimensions. The result is particularly simple if the five additional dimensions are very small, i.e. $R_I \ll R, r$, in which case one can neglect the corresponding massive modes contributions. In this limit (and using $R \gg \sqrt{\alpha'}$), the total potential energy is well approximated at large distances $r \gg \sqrt{\alpha'}$ by

$$V_{11} \sim \; - \frac{R\alpha'^2}{4V_5} \frac{e^{-r\sqrt{\frac{R^2}{\alpha'^2}-\frac{2}{\alpha'}}}}{r} \; . \tag{4.4.22}$$

Finally, until now we considered D1 branes wrapping the supersymmetry breaking circle. If on the other hand the D1 branes are perpendicular to the direction of the radius R used for supersymmetry breaking, they do not experience supersymmetry breaking. They retain therefore the BPS nature at the one-loop level and their interactions are supersymmetric.

Interactions beyond one-loop and the weak gravity conjecture

We saw that at short distances the interaction between D1 branes is attractive, such that they tend to accumulate and form bound states. There is no reason to believe that in a perturbative string setup this result would be upset to higher-orders in the perturbative expansion. At large distances however, the one-loop attraction is exponentially damped since the main contribution comes from massive closed-string states. At large distances therefore, potential higher-loop contributions generating massless gravitational (closed string) exchanges would induce infinite-range interactions, which change considerably (and dominate over) the one-loop contribution. This effect can be understood in terms of modifications of the tension and charge of D1 branes, as well as the generation of a dilaton mass, that we now try to include in the interaction potential. All of these modifications are generated by supersymmetry breaking.

Let us write the D1-D1 brane interactions in a slightly more general way as a contribution from the zero modes $V_{11}^{(0)}$ and contributions from massive states $V_{11}^{(n)}$. The contribution of the zero-mode $V_{11}^{(0)}$ vanishes at one-loop, according to our computation (4.4.18). However, since the one-loop contribution comes exclusively from massive states, it is short ranged and therefore any higher-order/loop correction leading to a zero-mode exchange changes dramatically the interaction at large distances. We consequently parametrize the zero-mode higher-loop contributions by introducing three parameters: $T_{1,\text{eff}}$ and $Q_{1,\text{eff}}$ are the quantum corrected brane tension and charge, whereas m_0 denotes the mass of the dilaton generated by quantum corrections. With these changes in mind, at large distances $r \gg \sqrt{\alpha'}$ where the main contribution comes from the lightest closed string states exchanged between the branes, we arrive at the following expression for the D1-D1 brane interaction

$$\begin{aligned} V_{11} = V_{11}^{(0)} + V_{11}^{(n)} \, , \quad \text{where} \quad V_{11}^{(0)} &= \frac{R\alpha'^2}{2\pi^2} \int d^8k \; e^{i\mathbf{kr}} \left[\frac{Q_{1,\text{eff}}^2/Q_1^2}{k^2} - \frac{T_{1,\text{eff}}^2/T_1^2}{4} \left(\frac{1}{k^2+m_0^2} + \frac{3}{k^2} \right) \right] \, , \\ V_{11}^{(n)} &= -\frac{R\alpha'^2}{8\pi^2} \int d^8k \; e^{i\mathbf{kr}} \; \frac{1}{k^2 + \frac{R^2}{\alpha'^2} - \frac{2}{\alpha'}} \; . \end{aligned} \tag{4.4.23}$$

The zero-mode contribution can also be written in terms of the supergravity 10d Planck mass

κ_{10} as usually done in the literature[29] [363]

$$V_{11}^{(0)} = 16\kappa_{10}^2 \pi R \int \frac{d^8k}{(2\pi)^8}\ e^{i\mathbf{kr}} \left[\frac{Q_{1,\text{eff}}^2}{k^2} - \frac{T_{1,\text{eff}}^2}{4} \left(\frac{1}{k^2+m_0^2} + \frac{3}{k^2} \right) \right] . \tag{4.4.24}$$

In (4.4.23), the corrected tension of the wrapped D1 brane $T_{1,\text{eff}}$ is defined in (4.4.20) and the relative factor of 1/4 (3/4) denotes the contribution of the dilaton (graviton). The one-loop corrected charge $Q_{1,\text{eff}}$ will be discussed below. The massive contributions $V_{11}^{(n)}$ contain the one-loop computation performed in (4.4.18). Notice that in a realistic compactification only four spacetime dimensions are noncompact. In this case, the brane-brane potential becomes

$$\begin{aligned} V_{11}^{(0)} &= \sum_{\mathbf{p}} \frac{16\kappa_{10}^2 \pi R}{(2\pi)^8 V_5} \int d^3k\ e^{i\mathbf{kr}} \left[\frac{Q_{1,\text{eff}}^2}{k^2+m_{\mathbf{p}}^2} - \frac{T_{1,\text{eff}}^2}{4} \left(\frac{1}{k^2+m_{\mathbf{p}}^2+m_0^2} + \frac{3}{k^2+m_{\mathbf{p}}^2} \right) \right] , \\ V_{11}^{(n)} &= -\frac{R\alpha'^2}{8\pi^2 V_5} \sum_{\mathbf{p}} \int d^3k\ e^{i\mathbf{kr}}\ \frac{1}{k^2+m_{\mathbf{p}}^2+\frac{R^2}{\alpha'^2}-\frac{2}{\alpha'}} , \end{aligned} \tag{4.4.25}$$

where $\sum_{\mathbf{p}}$ is the sum over all Kaluza-Klein masses in the five additional internal dimensions. The result is particularly simple if the five additional dimensions are much smaller than R and r, in which case one can neglect the contributions from the corresponding massive modes. In this limit, it is more transparent to express the total potential energy in terms of the four-dimensional Planck mass M_P, for which the graviton exchange provides the Newton potential in terms of the mass $M_0 = 2\pi R T_{1,\text{eff}}$ and the charge $Q_0 = 2\pi R Q_{1,\text{eff}}$ of the wrapped D1 brane. In this way, one gets the approximate potential

$$V_{11} \sim \frac{1}{M_P^2} \left[\frac{\frac{4}{3}Q_0^2 - M_0^2 - \frac{1}{3}M_0^2 e^{-m_0 r}}{r} - \frac{Q_0^2}{3}\ \frac{e^{-r\sqrt{\frac{R^2}{\alpha'^2}-\frac{2}{\alpha'}}}}{r} \right] . \tag{4.4.26}$$

This expression is valid for distances $r \gg \sqrt{\alpha'}$, whereas for shorter distances one expects the one-loop potential to be a good approximation, which has a constant limit when $r \to 0$.

The correction V_0 to the D1 brane tension is negative being generated by the massive contributions $V_{11}^{(n)}$ between the same brane ($\mathbf{r} = 0$). The correction to the charge would, on the other hand, come from a genus 3/2 computation, which was not yet performed to our knowledge. However, a quantum correction to the RR charge of the brane would be of the form $\int C_2 e^{\phi}$, where ϕ is the dilaton. Such a coupling would violate the gauge symmetry of the RR gauge field C_2, which seems implausible in perturbation theory. Corrections to the RR field kinetic terms are possible though, and this would generate a renormalization of the RR charge. A similar correction to the dilaton kinetic term should also contribute to the renormalization of the tension. However, such corrections would arise from one loop calculations and would be associated to $\mathcal{O}(g_s^2)$ corrections. We thus do not expect them to dominate the one-loop contribution to the tension, which is $\mathcal{O}(g_s)$, and therefore

$$T_{1,\text{eff}}^2 < Q_{1,\text{eff}}^2 \iff M_0^2 < Q_0^2 . \tag{4.4.27}$$

[29] The extra factor of 4 with respect to the usual formula is due to the fact that branes and their images contribute.

As a consequence, at short distances the potential is attractive whereas it is repulsive at large distances. If on the contrary the bound (4.4.27) was violated in the case of a massless dilaton, i.e. if $m_0 = 0$ (or if $M_0^2 > \frac{4}{3}Q_0^2$ for $m_0 > 0$), the potential would remain attractive also at large distances. This would violate the weak gravity conjecture. Our perturbative arguments dismiss such a possibility and we conclude that the weak gravity conjecture holds in our setup, and the massless modes exchange which it constrains determines the brane-brane dynamics at large distances.

Existence of stable bound states

An important consequence of the previous discussion on the weak gravity conjecture is that the negative self-energy of D1 branes, i.e. the decrease in the effective brane tension, and the attractive one-loop potential (4.4.16) also imply that it is energetically favorable to form bound states of D1 branes. Indeed, let us denote by $V_0 < 0$ the self-energy of one D1 brane. Then one can compare the energy of two configurations. The first is the energy $E_{N,1}$ of N coincident D1 branes and a single D1 brane at a large distance $r \gg \sqrt{\alpha'}$ from them, whereas the second is the energy $E_{N+1,0}$ of $N+1$ coincident D1 branes. They are given by

$$E_{N,1} = -(N+1)T_1 + (N^2+1)V_0 + O\left(e^{-\frac{rR}{\alpha'}}\right) \ , \quad E_{N+1,0} = -(N+1)T_1 + (N+1)^2 V_0 \ . \quad (4.4.28)$$

It is then clear that $E_{N+1,0} < E_{N,1}$ and therefore that the D1 branes tend to form bound states, which may be black holes. Consequently, black holes stability arguments, which are sometimes used in discussions about the WGC, are different in the small and large distance regions. To address this question, one needs to study the regime interpolating between large distances, where higher-order effects dominate and presumably verify the WGC as argued above, and small distances where the one-loop potential induces an attraction. Knowing the $r = 0$ value of the potential given in (4.4.19) and its asymptotic behaviour (4.4.26), we understand that it reaches a maximal value and has the shape depicted in figure 4.1.

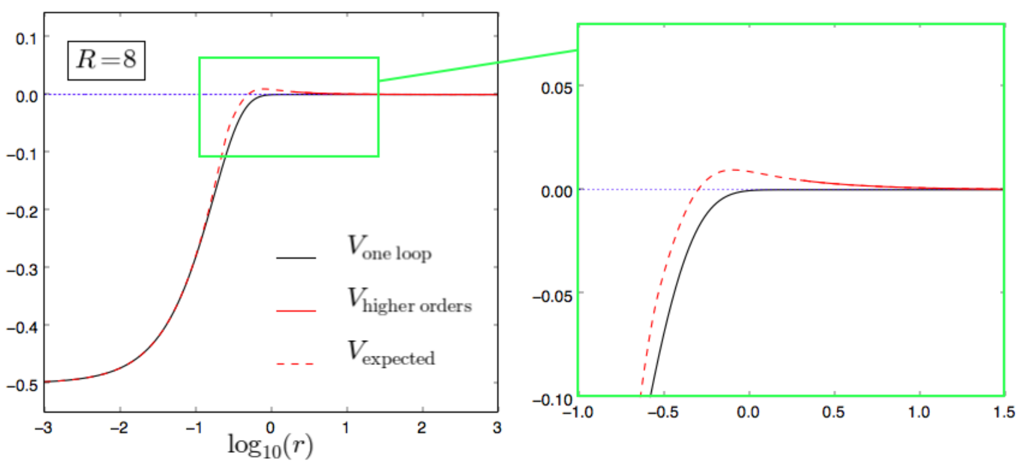


Figure 4.1: The D1-D1 potential as a function of the distance in the transverse space (the potentials and distances are expressed in units of α', we fixed $R = 8$, $g_s = 0.2$, $V_5 \sim 1.5^5$ and introduced no Wilson lines for the D1 branes)

To estimate the location r_0 of the maximum, we can use (4.4.26) if r_0 is in its validity regime. When $m_0 = 0$, we obtain

$$r_0 = -\frac{1}{\sqrt{\frac{R^2}{\alpha'^2} - \frac{2}{\alpha'}}} \left[1 + W\left(8\frac{T_{1,\text{eff}}^2 - T_1^2}{eT_1^2}\right)\right] \approx \frac{\alpha'}{R} \log\left(\frac{R^2}{g_s\alpha'}\right) , \tag{4.4.29}$$

where W is the Lambert W function.[30] This expression, obtained from (4.4.26), can be trusted if $r_0 \gg \sqrt{\alpha'}$, which can be rewritten as a constraint on the string coupling

$$g_s \ll \frac{R^3}{\alpha'^{3/2}} e^{-\frac{R}{\sqrt{\alpha'}}} . \tag{4.4.30}$$

In this case, black holes of size smaller than r_0 would be stable remnants. Such black holes could be formed from the D1 bound states about which we argued in (4.4.28) that their formation is energetically favorable. However, we expect from black hole constructions in string theory that there should only be a finite number of such remnants: from the bound state argument in (4.4.28) one can guess that if the number of D1 constituents is large and the bound state size becomes or order r_0 or larger, repulsive forces prevent more D1 branes to bind and therefore larger charge/mass remnants to form. Calculating this finite number of bound states is beyond the scope of this paper, but we could try to estimate it by comparing r_0 with the scale at which we expect the D1-branes solutions of supergravity to break down,[31] $R_S \sim \frac{N_1 g_s \alpha'^3}{V_5}$, where N_1 is the number of stacked D1-branes. Using (4.4.29), we can derive the following estimate,

$$N_{\text{crit}} \equiv N_1 \frac{r_0}{r_S} \approx \frac{1}{g_s} \frac{V_5}{\alpha'^{3/2}} \frac{\alpha'^{1/2}}{R} \log\left(\frac{R^2}{g_s\alpha'}\right) , \tag{4.4.31}$$

where all D1-branes configurations with $N_1 < N_{\text{crit}}$ correspond to situations where the attractive force is felt even in the regime where supergravity applies. In particular, this number becomes small in the decompactifcation limit $R \gg \sqrt{\alpha'}$. Furthermore, (4.4.31) also shows that the smaller g_s, the more stable bound states can exist. If $m_0 \neq 0$, r_0 becomes smaller than (4.4.29) and the appearance of such states is slightly suppressed in the limit $g_s \to 0$, but the behaviour remains qualitatively the same. Such a scaling of N_{crit} with g_s seems to be consistent with the swampland distance conjecture.

Summary and perspectives

String theory models with broken supersymmetry usually generate runaway potentials, which could lead in special cases to quintessence models of dark energy. On the other hand, the breaking of supersymmetry generates at the same time interactions between branes, which only disappear in the runaway limit. While this in itself respects the weak gravity conjecture at infinity, insisting on the rolling field cosmology could generate violations of it. With this motivation in mind, we tested the compatibility between quintessence and WGC in a type I string theory model with broken SUSY.

[30] The Lambert W function or product logarithm is defined by $W(xe^x) = x$. It has two real branches, here only the lower branch with $W \leq -1$ is relevant.

[31] This scale is the one for which the harmonic function $h(r) = 1 + \frac{R}{r}$, which defines the D1-brane solution, starts to deviate significantly from one.

At one-loop, we found a short-distance attraction between D1-branes generated by massive modes, which may naively suggest that rolling field dynamics is incompatible with the weak gravity conjecture in this perturbative and controllable string setting. However, the long-range brane-interaction carried by massless fields is vanishing at one-loop due to a cancellation between the NS-NS and the RR exchanges, so that the system feels equal tension and charge at this order of perturbation theory. The remaining one-loop attraction is exponentially damped at large distances, so we believe that higher-loop corrections are important to settle the issue about the WGC. In particular, the one-loop self-energy of a brane decreases its tension, such that the effective tension $T_{1,\text{eff}}$ and charge $Q_{1,\text{eff}}$ of D1 branes satisfy the weak gravity bound $Q_{1,\text{eff}} > T_{1,\text{eff}}$. Then, at higher-loops, a repulsive interaction generated by the exchange of massless states should appear, and should dominate over the one-loop (short range) attraction at long distances. Overall, this leads to a picture in which the weak gravity conjecture is respected at large distances, defined by the parameters (g_s, R). In the lower dimensional effective theory the D1 branes, wrapped around the Scherk-Schwarz circle, behave as particles charged under a $U(1)$-gauge symmetry with $Q_{\text{eff}} > M_{\text{eff}}$.

The stability of bound states and black holes is interesting in our setup. The one-loop short-range attraction favors the formation of D1 bound states which can potentially lead to stable black hole remnants. If the string coupling is very small, the attractive region of brane-brane potentials extends up to scales where the effective gravitational theory applies: if $g_s \lesssim \frac{R^3}{\alpha'^{3/2}} e^{-\frac{R}{\sqrt{\alpha'}}}$ (with R the radius of the supersymmetry breaking dimension), a finite number of branes well described by supergravity are sensitive to the attractive potential. This number roughly scales like $\frac{1}{g_s}$, and indicates that in the small g_s limit an increasing quantity of stable bound states is expected to arise.

There are a number of open interesting questions that are worth further exploration. It would be interesting to identify string models with broken supersymmetry where the generated moduli potentials and runaway vacua can lead to viable quintessence-like models of dark energy. There are various difficulties for progress into this direction, from generating a small acceleration of the present universe, which is highly nontrivial to achieve in string theory constructions [465, 466], to the constraints coming from time-dependence of fundamental constants and fifth force experiments. From a more theoretical string theory perspective, it would be interesting to perform higher-loop (for instance, genus 3/2) computations in order to test our result on the quantum corrected brane tension and the absence of renormalization of the brane charges at lowest order. Whereas supersymmetry breaking should generate, as usual, tadpoles which signal limitations in quantum computations at higher loops, higher-order computations of brane tensions and charges could be performed by separating two D1 branes in (our) noncompact space, in which case there should be no such problems. It would also be important to investigate stable type I models in lower dimensions with D9 Wilson lines and positive scalar potential in the class of models constructed in [471] and to investigate the D1 interaction potentials in detail. It would also be very interesting to explore quantum corrections to brane tensions and RR charges in other string models with broken supersymmetry, such as the models with brane supersymmetry breaking [395, 396]. It would also be interesting to compute if supersymmetry breaking induces corrections to the black hole extremality bound as well, to complete our test of the weak gravity conjecture. Finally, we believe it is important to test the other various recent swampland conjectures [364, 436, 459, 461, 464, 469] in explicit perturbative string theory models with broken supersymmetry. .

4.5 Conclusions

In this third part, we examined the basic rules for building a quantum theory of superstrings, working our way towards the spectrum of such theories. Then, we discussed in details the spectrum of the type IIB and type I string theories, with an emphasis on how it can be used to determine one-loop vacuum amplitudes. From the latter in type I string theories, we saw that we could extract significant information, such as how the different amplitudes combine to describe the type I spectrum or how their finiteness restrict the particle content of the theory, in a way complementary to discussions on anomalies. We then discussed supersymmetry breaking in string theory, focusing on brane supersymmetry breaking and on the Scherk-Schwarz mechanism, again relating the impact of SUSY breaking on the spectrum to its effect on one-loop vacuum amplitudes.

Motivated by the fact that string theory spectra contain lots of axions, and that such axions could play a role in the compactified version of the Green-Schwarz mechanism, we studied a string-inspired theory of an axion charged under a $U(1)$ gauge theory and realizing a 4D Green-Schwarz mechanism. In particular, we saw that when SUSY is broken by gaugino condensation in a hidden sector with a mixed anomaly with the anomalous $U(1)$, the axion gets a mass of order the supersymmetry breaking scale, preventing it to play the role of a QCD axion. However, this can be evaded by refining the model: we thus adapted the so-called 3-2 model to our anomalous $U(1)$ setup to successfully decouple the SUSY breaking scale and the axion mass. The natural value for the axion decay constant is of order the GUT scale in our models, but we mentioned further modifications (of the moduli sector) which can lower it. We also reminded that anomalous $U(1)$'s can be used in Froggatt-Nielsen models of flavour hierarchies, in which case anomaly cancellation and unification conditions unambiguously predict axion couplings.

Finally, we presented swampland conjectures, with a focus on a conjecture which was mentioned several times in the first part of this thesis, which states that no global symmetries are exact in a quantum theory of gravity, as well as on the weak gravity conjecture. We then scrutinized the latter for the 2-form of type I string theory, with broken SUSY (à la Scherk-Schwarz), which generates both brane-brane interactions and runaway potential. Thus, such a setup provides a non-trivial test of the WGC, and of its compatibility with possible cosmological ways-out from the de Sitter conjecture. Using our previous discussion on vacuum amplitudes, relevant for brane-brane interactions via the open-closed duality, we showed that one-loop interactions carried by massive twisted modes induce a decrease of the tension of the brane while the charge remains fixed, consistently with the weak gravity conjecture.

5 Outlook: a web of theories for a web of questions

This thesis was aimed at developing and studying theories beyond the standard model of particle physics, which is a well motivated and fruitful endeavor, as we have tried to demonstrate along this text. We saw that the need for BSM physics is undeniable, since, for instance, the rules and building blocks of the SM and its cosmological counterpart do not allow for neutrino masses or a dark matter particle. Besides such blatant discrepancies with the SM phenomenological predictions, there are also more theoretical puzzles, such as the quantization of gravity or the degree of fine-tuning one expects to find in a fundamental theory.

The theories which address some of those puzzles sometimes unexpectedly connect with other branches of particle physics, sometimes even beyond particle physics. All the theories we encountered here fall into this category. Indeed, in sections 2 and 4.3, we studied axions, originally designed to cancel a single parameter of the SM lagrangian, but which eventually proved to be relevant for cosmology, for flavour physics, and as low-energy probes of string theory. Likewise, we explored in section 3 supersymmetry, which emerges from theoretical symmetry principles, solves in one stroke many phenomenological problems and is intimately tied to the string theory spectrum. Finally, string theory overhangs such considerations, since it is a candidate for a theory of everything from the particle physics perspective, as we mentioned in section 4, but has a larger reach, for instance thanks to holography.

This matter of fact legitimates to be aware of all (or at least, most of) the questions which are considered in high-energy physics, since they can all be interconnected and interdependent. The BPS solution of section 3.4.2, which illustrates the generality and the reach of QFTs and EFTs by both generating multi-particle amplitudes and describing domain walls in SQCD, is an example. Our systematic gauging of phenomenologically motivated global abelian symmetries in section 2, justified by quantum gravity considerations of the kind discussed in section 4.4, is an other one. There, we asked about the consequences of this need for gauging, by exploring in section 2.2 the typical parameter space of gauged clockwork axion models, and by studying in section 2.4 the heavy fermions, and light axions, which necessarily accompany a gauged Froggatt-Nielsen mechanism for the flavour hierarchies. Our last example concerns SUSY breaking: if nature is supersymmetric at some scale, the need to break SUSY and the way it is broken may interfere with all other particle physics considerations, such as solving the strong CP problem, which we studied in a string-inspired broken SUSY scenario in section 4.3.5, or determining the spread of new physics above the weak scale, exemplified by the splitting between the SUSY breaking scale and the scale of the SUSY breaking sector in the clockwork model of section 3.2.3, or, last but not least, establishing criteria shared by all consistent theories which possibly descend from quantum gravity, as can be understood from the non-trivial interplay between SUSY breaking and the weak gravity conjecture discussed in section 4.4.2.

The connections between theories, or aspects of (particle) physics, thus enable one to test, strengthen, but also question and enlarge the formalism and expectations of the current time. This way, we may be able to know more about what modern physics ought to be made of.

Acknowledgments

I thank the Institute of Theoretical Physics at the University of Warsaw, as well as the Institute for Particle Physics Phenomenology at Durham University, for their financial support and their kind hospitality. My work in general has been partly supported by the ANR Black-dS-String and the Royal-Society/CNRS International Cost Share Award IE16059.

I would also like to indicate that the thesis of Lucien Heurtier [472] has been consulted at each critical step of the writing as an example of what a thesis should look like. Finally, Feynman graphs appearing in the thesis were drawn using JaxoDraw [473].

A Conventions and abbreviations

In this section, we collect some notations and conventions which are used throughout this thesis.

A.1 Conventions for QFT, GR and the (MS)SM

A.1.1 Generalities

In section 3, we use the exact conventions of [260]. Elsewhere in the text, and for generic QFT, our conventions are those of [474]. For the (MS)SM, we use our own conventions, defined below. Now, basic conventions are recalled.

Units are chosen such that $c = \hbar = 1$. We use the 4D metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and a curved space metric $g_{\mu\nu}$ with the same signature. Low case greek letters $\mu, \nu, ...$ denote 4D spacetime indices, whereas upper case latin ones $M, N, ...$ denote indices in any higher dimension, specified depending on the context. Gamma matrices verify the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. We define $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ in 4D, and the left-handed chirality such that $\gamma_5\psi_L = \psi_L$. Summations over indices are (almost) always implicit.

The kinetic terms for a complex scalar ϕ, a spinor ψ, a gauge field A_μ and the metric in 4D are

$$S \equiv \int d^4x\sqrt{-g}\mathcal{L} = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}R - g^{\mu\nu}D_\mu\phi(D_\nu\phi)^* - \overline{\psi}\left(\gamma^\mu D_\mu + m\right)\psi - \frac{g^{\mu\rho}g^{\nu\sigma}}{2g^2}\,\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma})\right), \tag{A.1.1}$$

where R with the Ricci scalar associated to the metric $g_{\mu\nu}$ (or the vierbein e_μ^a) with a Levi-Civita connection, $M_P = 2.43 \times 10^{18}$ GeV is the reduced Planck mass, $\gamma_\mu \equiv \gamma_a e_\mu^a$ with γ_a the flat space gamma matrices, $\overline{\psi} = \psi^\dagger(i\gamma^0)$, the covariant derivative D_μ is $\partial_\mu - iA_\mu^a T^a$ in flat space and $F_{\mu\nu} \equiv F_{\mu\nu}^a T^a \equiv (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc}A_\mu^b A_\nu^c)T^a$ is the field strengh associated to A_μ. T^a generators verify the algebra of the gauge group with structure constants f: $[T^a, T^b] = if^{abc}T^c$. As can already be seen here, in expressions involving both the gauge coupling (kinetic terms of gauge fields) and the metric of curved space, both (unrelated) quantities are denoted g. We chose not to lift this ambiguity and hope that readers will figure out which is which thanks to the context. $SU(N)$ generators are normalized such that $\mathrm{Tr}\big(T^aT^b\big) = \frac{\delta^{ab}}{2}$ in the fundamental representation. The symbol Tr denotes a trace in the fundamental representation, whereas the symbol Tr_a denotes a trace in the adjoint. Apart from the language from which the letters are taken, notations in higher dimensions are of course identical.

SM field	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$
$Q_{L,i}$	**3**	**2**	$1/3$
$u_{R,j}$	**3**	**1**	$4/3$
$d_{R,j}$	**3**	**1**	$-2/3$
$L_{L,i}$	**1**	**2**	-1
$e_{R,j}$	**1**	**1**	-2
H	**1**	**2**	1

Table A.1: Gauge charges of the SM fields
The subscripts L, R indicate the chirality of the fermion fields

A.1.2 (MS)SM conventions

The gauge couplings of the $G_{\rm SM} = SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge group of the SM are denoted g_3, g_2 and g_Y, and their respective gauge fields are called $G_\mu^{a=1,\dots,8}, W_\mu^{i=1,\dots,3}$ and B_μ. Their field strengths are denoted $G^a_{\mu\nu}, W^i_{\mu\nu}$ and $B_{\mu\nu}$.

The SM fields are taken in the representations of the gauge group displayed in Table A.1 (with our conventions, the electric charge is $Q = \frac{Y}{2} + T^3_W$). The indices $i = 1,..,3$ refer to the particle's generation. The weak doublets can be expressed in terms of the quarks and leptons we usually refer to (i.e. up and down quarks, electrons and neutrinos) as follows:

$$Q_{L,i} = \begin{pmatrix} u_{L,i} \\ d_{L,i} \end{pmatrix} \quad , \quad L_{L,i} = \begin{pmatrix} \nu_{L,i} \\ e_{L,i} \end{pmatrix} \quad , \tag{A.1.2}$$

as well as the Higgs field, for which we also define the conjugate field:

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \quad , \quad H^c \equiv i\sigma^2 H^* = \begin{pmatrix} (H^0)^* \\ -(H^+)^* \end{pmatrix} \quad , \tag{A.1.3}$$

where σ^2 is the second Pauli matrix. In unitary gauge, we have $H^+ = 0, H^0 = \frac{v+h}{\sqrt{2}}$, where $v \approx 246$ GeV is the weak scale and h is the Higgs boson.

The SM lagrangian, which encodes the dynamics of those fields, is the most general renormalizable invariant under $G_{\rm SM}$ one (up to the θ-term). Its flavour part is discussed in section 2.3.1. The physical massive vector bosons are found from it as follows:

$$\mathcal{L} \supset -|D_\mu H|^2 = -\left|\left(\partial_\mu - iW^i_\mu \frac{\sigma^i}{2} - iB_\mu\right) H\right|^2 \supset \frac{v^2}{2}\left|\begin{pmatrix} \frac{W^1_\mu - iW^2_\mu}{2} \\ -\frac{W^3_\mu}{2} + B_\mu \end{pmatrix}\right|^2 . \tag{A.1.4}$$

Reinstating the gauge couplings by performing the rescalings "$X_\mu \to g_X X_\mu$", we find the following mass terms:

$$\frac{v^2}{2}\left(g_2^2 \left(\frac{W^1_\mu + iW^2_\mu}{2}\right)\left(\frac{W^1_\mu - iW^2_\mu}{2}\right) + \left(-g_2 \frac{W^3_\mu}{2} + g_Y B_\mu\right)^2\right) \quad , \tag{A.1.5}$$

and from this, we see how to define the mass eigenstates:

$$W^+_\mu \equiv \frac{W^1_\mu - iW^2_\mu}{\sqrt{2}} \quad , \quad W^-_\mu \equiv \frac{W^1_\mu + iW^2_\mu}{\sqrt{2}} \quad , \quad Z_\mu \equiv \frac{-g_2 \frac{W^3_\mu}{2} + g_Y B_\mu}{\sqrt{\frac{g_2^2}{4} + g_Y^2}} \quad , \quad A_\mu \equiv \frac{g_Y W^3_\mu + \frac{g_2}{2} B_\mu}{\sqrt{\frac{g_2^2}{4} + g_Y^2}} \; . \tag{A.1.6}$$

MSSM superfield	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$
Q_i	$\mathbf{3}$	$\mathbf{2}$	$1/3$
$U_j \equiv u^c_{R,j}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$-4/3$
$D_j \equiv d^c_{R,j}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$2/3$
L_i	$\mathbf{1}$	$\mathbf{2}$	-1
$E_j \equiv e^c_{R,j}$	$\mathbf{1}$	$\mathbf{1}$	2
H_d	$\mathbf{1}$	$\mathbf{2}$	-1
H_u	$\mathbf{1}$	$\mathbf{2}$	1

Table A.2: Gauge charges of the MSSM fields

A_μ is the photon field, of field strength $F_{\mu\nu}$. We read the masses from (A.1.5): $m_Z^2 = \left(\frac{g_2^2}{4} + g_Y^2\right) v^2$ and $m_W^2 = \frac{g_2^2}{4} v^2$. To determine the electric charge, we write e.g.

$$\mathcal{L} \supset -2ig_Y \overline{e_R}\gamma^\mu B_\mu e_R \supset -i\frac{g_2 g_Y}{\sqrt{\frac{g_2^2}{4} + g_Y^2}} \overline{e_R}\gamma^\mu A_\mu e_R \equiv -ie\overline{e_R}\gamma^\mu A_\mu e_R \implies e = \frac{g_2 g_Y}{\sqrt{\frac{g_2^2}{4} + g_Y^2}} \,. \quad \text{(A.1.7)}$$

With such conventions, the electric charge is $q_{\text{em}} = T_W^3 + \frac{q_Y}{2}$.

The MSSM has an extra Higgs doublet, and it is easier to express all the fields as left-handed superfields (up to conjugation of the right-handed fields previously introduced), as in Table A.2. Writing $H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}$ and $H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}$, we define $\tan\beta = \frac{\langle H_u^0 \rangle}{\langle H_d^0 \rangle}$.

A.2 Abbreviations

Here, we list (hopefully) all the abbreviations which are used along the text in alphabetical order.

- AdS: anti-de Sitter space
- BBN: big bang nucleosynthesis
- BPS: Bogomol'nyi-Prasad-Sommerfield
- BSB: brane supersymmetry breaking
- BSM: beyond the standard model
- CCWZ: Callan-Coleman-Wess-Zumino
- CKM: Cabibbo-Kobayashi-Maskawa
- CMB: cosmic microwave background
- CFT: conformal field theory
- CP: charge-parity
- DBI: Dirac-Born-Infeld
- DE: dark energy
- DFSZ: Dine-Fischler-Srednicki-Zhitnitsky
- DM: dark matter
- dS: de Sitter space
- EFT: effective field theory
- eom: equation of motion
- FI: Fayet-Iliopoulos
- FN: Froggatt-Nielsen
- (P)GB: (Pseudo) Nambu-Goldstone bosons
- GIM: Glashow-Iliopoulos-Maiani
- GR: general relativity

- GS: Green-Schwarz
- IR: infrared
- KK: Kaluza-Klein
- KSVZ: Kim-Shifman-Vainshtein-Zakharov
- LHC: Large Hadron Collider
- MSSM: minimal supersymmetric standard model
- PMNS: Pontecorvo-Maki-Nakagawa-Sakata
- (S)QCD: (supersymmetric) quantum chromodynamics
- (S)QED: (supersymmetric) quantum electrodynamics
- SM: standard model (of particle physics)
- SS: Scherk-Schwarz
- SUGRA: supergravity
- SUSY: supersymmetry
- SYM: Super-Yang-Mills
- UV: ultraviolet
- vev: vacuum expectation value
- WGC: weak gravity conjecture
- WZ: Wess-Zumino

B Couplings of an axion to heavy fermions

In this appendix, we couple an axion to a set of gauge-charged heavy fermions and we derive the one-loop induced axion-gauge bosons couplings. We recover in passing the usual result for the chiral anomaly [475,476], and these results can also be used if the axion is charged under a gauge group and provides the longitudinal component of a gauge boson. More general calculations of this kind can be found in [230].

One then considers a theory with a gauge group (which we keep unspecified until the end, where we will identify it with QCD or electromagnetism) of generators T^a and vector A^a_μ (with field strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ...$), a complex scalar field σ and two chiral fermions $\psi_{L,R}$ in the fundamental representation of the gauge group, with a Yukawa coupling to the scalar:

$$\mathcal{L} = -\frac{1}{2g^2}\operatorname{Tr}\left(F^2_{\mu\nu}\right) - \overline{\psi_L}\gamma^\mu D_\mu \psi_L - \overline{\psi_R}\gamma^\mu D_\mu \psi_R - |\partial_\mu \sigma|^2 - V(|\sigma|^2) - (y\sigma\overline{\psi_L}\psi_R + h.c.) \;, \quad \text{(B.0.1)}$$

where $D_\mu = \partial_\mu - iA^a_\mu T^a$. This lagrangian has a $U(1)$ global symmetry under which $\sigma \to e^{i\alpha}\sigma$ and $\overline{\psi_L}\psi_R \to e^{-i\alpha}\overline{\psi_L}\psi_R$. The transformation of the fermion bilinear makes this global symmetry anomalous.

We choose $V(|\sigma|^2)$ so that σ gets a vev $\frac{f}{\sqrt{2}}$. We then work out the axion dynamics by parametrizing $\sigma = \frac{f}{\sqrt{2}}e^{i\frac{a}{f}}$:

$$\mathcal{L} \supset -\frac{1}{2g^2}\operatorname{Tr}\left(F^2_{\mu\nu}\right) - \overline{\psi}\left(\gamma^\mu D_\mu + \frac{yf}{\sqrt{2}}\right)\psi - \frac{1}{2}(\partial a)^2 + i\frac{y}{\sqrt{2}}a\overline{\psi}\gamma_5\psi \;, \quad \text{(B.0.2)}$$

where we only kept the linear terms in a and merged the two chiral fermions in a Dirac fermion.

One gets a coupling between a and the gauge boson A at one loop via the two following diagrams:

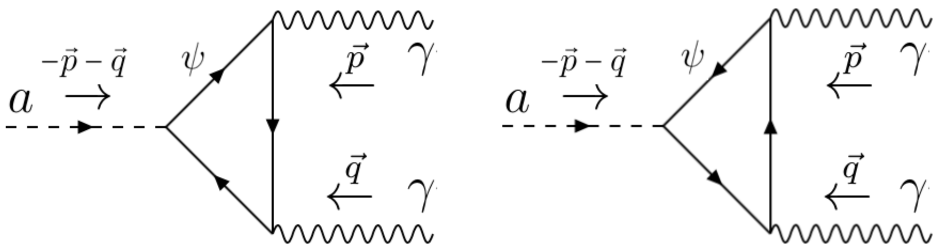


Figure B.1: Feynman diagrams leading to the axion-vector-vector couplings

The effective coupling is $\frac{c^{\mu\nu,ab}}{2} a A^a_\mu A^b_\nu$, here in momentum space with $M_\psi = \frac{yf}{\sqrt{2}}$:

$$c^{\mu\nu,ab} = \int d^4k \frac{y}{\sqrt{2}(2\pi)^4} \Bigg[\frac{\mathrm{Tr}\Big(\gamma_5(-i\not{k} + M_\psi)\gamma^\mu(-i(\not{k}+\not{p}) + M_\psi)\gamma^\nu(-i(\not{k}+\not{p}+\not{q}) + M_\psi)\Big)}{(k^2+M_\psi^2-i\epsilon)((k+p)^2+M_\psi^2-i\epsilon)((k+p+q)^2+M_\psi^2-i\epsilon)}$$
$$+ \frac{\mathrm{Tr}\Big(\gamma_5(-i\not{k} + M_\psi)\gamma^\nu(-i(\not{k}+\not{q}) + M_\psi)\gamma^\mu(-i(\not{k}+\not{p}+\not{q}) + M_\psi)\Big)}{(k^2+M_\psi^2-i\epsilon)((k+q)^2+M_\psi^2-i\epsilon)((k+p+q)^2+M_\psi^2-i\epsilon)} \Bigg]$$
$$\times \mathrm{Tr}\big(T^aT^b\big) \ . \tag{B.0.3}$$

Since the first non-zero trace including γ_5 is $\mathrm{Tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4i\epsilon_{\mu\nu\rho\sigma}$ and that traces of an odd number of gamma matrices are zero, we have:

$$\mathrm{Tr}\Big(\gamma_5(-i\not{k} + M_\psi)\gamma^\mu(-i(\not{k}+\not{p}) + M_\psi)\gamma^\nu(-i(\not{k}+\not{p}+\not{q}) + M_\psi)\Big) = 4iM_\psi\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma \ . \tag{B.0.4}$$

We also use the Feynman trick followed by a Wick rotation to calculate:

$$\int d^4k \frac{1}{(k^2+M_\psi^2-i\epsilon)((k+p)^2+M_\psi^2-i\epsilon)((k+p+q)^2+M_\psi^2-i\epsilon)}$$
$$= 2i\pi^2\frac{\Gamma(1)}{\Gamma(3)}\int_0^1 dx \int_0^{1-x} dy \frac{1}{M_\psi^2 + (x+y)(1-x-y)q^2 + x(1-x)p^2 + 2x(1-x-y)pq} \ . \tag{B.0.5}$$

We can extend this result as a series in powers of $\frac{\text{momenta}}{M_\psi}$:

$$\begin{aligned} c^{\mu\nu,ab} =& -\frac{4\pi^2\,\mathrm{Tr}\big(T^aT^b\big)}{(2\pi)^4 f}[\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma \\ &\times \int_0^1 dx\int_0^{1-x} dy\,\Big(1-(x+y)(1-x-y)\frac{q^2}{M_\psi^2} - x(1-x)\frac{p^2}{M_\psi^2} - 2x(1-x-y)\frac{pq}{M_\psi^2} + ...\Big) \\ &+ (\mu\leftrightarrow\nu, p\leftrightarrow q)] \\ =& -\frac{\mathrm{Tr}\big(T^aT^b\big)}{2\pi^2 f}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\Big(\frac{1}{2} - \frac{q^2}{12M_\psi^2} - \frac{p^2}{12M_\psi^2} - \frac{pq}{12M_\psi^2}\Big) + ... \end{aligned} \tag{B.0.6}$$

which, with the identification $pA(p) \to -i\partial A(x)$, gives finally the one-loop coupling between the axion and the vector bosons:

$$\mathcal{L} \supset -\frac{\epsilon^{\mu\nu\rho\sigma}}{64\pi^2 f} aF^a_{\mu\nu}F^a_{\rho\sigma} + \frac{g^2\epsilon^{\mu\nu\rho\sigma}}{384\pi^2M_\psi^2 f}(-\Box aF^a_{\mu\nu}F^a_{\rho\sigma} + \partial_\mu a\partial_\eta F^a_{\rho\sigma}F^{\eta,a}_\nu) + ... \ , \tag{B.0.7}$$

where the dots indicate the presence of higher order terms, and where we choosed that the gauge group is unitary and normalized the generators as follows: $\mathrm{Tr}\big(T^aT^b\big) = \frac{\delta^{ab}}{2}$ in the fundamental representation. The first term of (B.0.7) is the usual axionic coupling to gauge fields, and it gives back the usual result for the chiral anomaly:

$$\mathcal{L} \xrightarrow{\text{when } \psi\to e^{i\alpha\gamma_5}\psi,\text{ hence } a\to a+2\alpha f} \mathcal{L} - \frac{\epsilon^{\mu\nu\rho\sigma}}{32\pi^2}\alpha F^a_{\mu\nu}F^a_{\rho\sigma} \ . \tag{B.0.8}$$

If one now adds to the theory (B.0.1) an other set of fermions coupled in the following way:

$$\begin{aligned} \mathcal{L} \supset& -\overline{\psi'_L}\gamma^\mu D_\mu\psi'_L - \overline{\psi'_R}\gamma^\mu D_\mu\psi'_R - (y'\sigma^*\overline{\psi'_L}\psi'_R + h.c.) \\ &\xrightarrow{\text{terms linear in the axion}} -\overline{\psi'}\left(\gamma^\mu D_\mu - \frac{y'f}{\sqrt{2}}\right)\psi' - i\frac{y'}{\sqrt{2}}a\overline{\psi'}\gamma_5\psi' \ , \end{aligned} \tag{B.0.9}$$

there is no anomaly anymore, but there remains non-anomalous couplings to the gauge fields (where we defined $M'_\psi = y' f$):

$$\mathcal{L} \supset \frac{g^2 \epsilon^{\mu\nu\rho\sigma}}{384\pi^2 f} \left(\frac{1}{M_\psi^2} - \frac{1}{M_\psi'^2} \right) \left(-\Box a F^a_{\mu\nu} F^a_{\rho\sigma} + \partial_\mu a \partial_\eta F^a_{\rho\sigma} F_\nu{}^{\eta,a} \right) . \tag{B.0.10}$$

C Supersymmetric QCD

SQCD is defined as a supersymmetric theory of an $SU(N_c)$ gauge group, with a set of N_f (for flavours) pairs of quark chiral superfields $Q^i, \tilde{Q}_j$, respectively in the fundamental and anti-fundamental representation of $SU(N_c)$ (for reviews see [275,331]). Since it is easier to handle than non-SUSY QCD thanks to the power of non-renormalization theorems, but that it nonetheless displays features such as confinement or chiral symmetry breaking, it is sometimes understood as a laboratory for understanding actual QCD.

C.1 Phases of the theory and Seiberg duality

The dynamics of the theory is only determined by the kinetic terms. The one-loop β-function of SUSY QCD reads:

$$\beta(g) = -\frac{g^3}{16\pi^2}(3N_c - N_f) \ , \tag{C.1.1}$$

from which we can understand that, if $N_f \geq 3N_c$ the theory is free in the infrared (IR) and has a Landau pole in the ultraviolet (UV). Likewise, when $N_f \leq N_c + 1$ the theory is asymptotically free and confines in the IR. However, (C.1.1) is not enough to understand the regime when $N_c + 1 < N_f < 3N_c$. However, one output of the non-renormalization theorems discussed in section 3.1.5 is that the only renormalization of the gauge coupling beyond one-loop comes from the Kähler potential/wave-function renormalization/quark anomalous dimensions. Moreover, due to the global symmetries of the theory, all the quark have the same anomalous dimension γ_g and the exact β-function actually reads [477, 478]:

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{3N_c - N_f(1-\gamma(g^2))}{1-\frac{g^2 N_c}{8\pi}} \ , \quad \gamma(g^2) = -\frac{g^2}{8\pi^2}\frac{N_c^2-1}{N_c} + \mathcal{O}(g^4) \ . \tag{C.1.2}$$

This led Seiberg [332] to conjecture that, when $\frac{3}{2}N_c < N_f < N_c$, the theory reaches a non-trivial fixed point in the UV. It defines the so-called conformal window.

Seiberg also conjectured that, in this conformal window, there is a dual formulation of the IR physics of the theory, which has a different gauge group but the same physical content: it has a $SU(N)$ gauge group, with $N \equiv N_f - N_c$, N_f pairs of quarks in the (anti-)fundamental representation $q^i, \tilde{q}_j$, a set of uncharged fields Φ_i^j, and a superpotential

$$W = h(q\Phi\tilde{q}) \ . \tag{C.1.3}$$

The indices indicate the behaviour of the different fields under the global symmetries of the theory, which are the same for the two dual theories. Thus, Seiberg duality maps different unphysical gauge formulations of the same IR physical content. It is straighforward to check that the dual (called magnetic) theory is in the conformal window if the original (dubbed electric) theory was.

Seiberg duality thus maps the weakly coupled part to the strongly coupled part of the conformal window.

Below the conformal window, when $N_c + 1 < N_f \leq \frac{3}{2}N_c$, Seiberg duality can still be used, but the dual theory now has an infrared free limit. We then talk about the free magnetic phase. The magnetic theory's coupling constant blows up in the UV, at a scale Λ which we can identify with the scale at which, inversely, the electric theory becomes strongly coupled. Seiberg duality thus enables us to study the theory in every regime, using the most relevant description.

C.2 Moduli space of vacua and non-perturbative superpotential

The classical (gauge-invariant) vacua of SQCD are given, up to global transformations, by the values of mesons $M^i_j \equiv Q^i\tilde{Q}_j$, with color indices contracted, as well as baryons $B^{i_1,\dots,i_{N_c}} \equiv Q^{i_1}Q^{i_2}\dots Q^{i_{N_c}}$, $\tilde{B}_{j_1,\dots,j_{N_c}} \equiv \tilde{Q}_{j_1}\dots\tilde{Q}_{j_{N_c}}$ if $N_f \geq N_c$. They have no classical potential and can take arbitrary values, defining a moduli space where the gauge symmetry is spontaneously broken. At the origin only, the gauge symmetry is maintained and all the gluons are part of the low-energy description. SUSY is preserved everywhere in the moduli space.

On the other hand, SQCD can be strongly coupled in the IR (it is when $N_f \leq \frac{3}{2}N_c$, as we said above), so the quantum picture is expected to change dramatically. We only talk here about the case $N_f < N_c$. There, only the mesons exist and using the global symmetries of the theory, as well as holomorphicity, one shows that the only possible superpotential is [250, 479]

$$W^{(\text{non-pert})} = (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det(M)}\right)^{\frac{1}{N_c-N_f}} \,, \tag{C.2.1}$$

where $\frac{\Lambda}{\mu} = e^{-\frac{8\pi^2}{(3N_c-N_f)g^2}}$ defines the non-perturbative scale Λ. This non-perturbative correction, which is exact, is such that quantum mechanically, there are no vacua for the mesons whose dynamics push them towards infinity. There are three interesting aspects of (C.2.1) which we would like to emphasize. First, if $N_f = 0$, the superpotential (C.2.1) is constant and due to gaugino λ condensation, $\langle\lambda\lambda\rangle = \Lambda^3$. Second, if we include a quark mass $m^j_i Q^i\tilde{Q}_j = m^j_i M^i_j$ such that $|m| \ll \Lambda$, where $|m|$ means one eigenvalue of m, the mesons are heavy and their mass dominates over the dynamics induced by (C.2.1). Then, they can be integrated out:

$$\langle M^i_j\rangle = \left(\Lambda^{3N_c-N_f}\det(m)\right)^{\frac{1}{N_c}}\left(m^{-1}\right)^i_j \,, \tag{C.2.2}$$

such that below the scale $|m|$, there are no quarks left and the theory generates the constant superpotential linked to gaugino condensation:

$$W^{(\text{non-pert})} = \left(\Lambda^{3N_c-N_f}\det(m)\right)^{\frac{1}{N_c}} \,. \tag{C.2.3}$$

Consequently, when the mass is given by the vev of a field, this superpotential modifies the dynamics of this field. Third, with massive quarks we find N_c vacua given by (C.2.2), consistently with the Witten index calculation [352].

D More trace calculations of string amplitudes

In this appendix, we collect some more string one-loop calculations.

D.1 The R-R sector of the type IIB torus amplitude

Let us calculate a second part of the torus amplitude, which adds to (4.1.53). We have, in the R-R sector:

For $\left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j, s_a, \tilde{s}_b\right\rangle = ... \left(\alpha_{-n}^i\right)^{k_n^i} ... \left(\psi_{-r}^j\right)^{l_r^j} ... \times |s_a, \tilde{s}_b\rangle$,

$$
\begin{aligned}
L_0 \left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j, s_a, \tilde{s}_b\right\rangle &= \Big[\sum_i \Big(\sum_{n>0} n k_n^i + \sum_{r>0} r l_r^i\Big)\Big] \left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j, s_a, \tilde{s}_b\right\rangle \\
\overline{L}_0 \left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j, s_a, \tilde{s}_b\right\rangle &= \Big[\sum_i \Big(\sum_{n>0} n \tilde{k}_n^i + \sum_{r>0} r \tilde{l}_r^i\Big)\Big] \left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j, s_a, \tilde{s}_b\right\rangle \ .
\end{aligned}
\tag{D.1.1}
$$

Consequently,

$$
\begin{aligned}
\mathrm{tr}\left(\frac{1+(-1)^F}{2}\frac{1+(-1)^{\tilde{F}}}{2} q^{L_0}\overline{q}^{\overline{L}_0}\right) &= \sum_{s_a,\tilde{s}_b} \prod_{i,n>0,r>0} \left(\sum_{k_n^i,\tilde{k}_n^i,l_r^i,\tilde{l}_r^i}\right) \frac{1+(-1)^{\sum_r l_r^i + \sum_a s_a}}{2} q^{\sum_i\left(\sum_{n>0} n k_n^i + \sum_{r>0} r l_r^i\right)} \times ... \\
&= \left|16 \frac{\prod_{r>0}(1+q^r)^8}{2\prod_{n>0}(1-q^n)^8}\right|^2 = \left|\frac{S_8}{\eta^8}\right|^2(\tau) \ ,
\end{aligned}
\tag{D.1.2}
$$

where now r is integer.

D.2 The Klein bottle of type I string theory

Now, we turn to the calculation of the Klein bottle amplitude. Since the NS-R and the R-NS sector are mapped onto each other by Ω, they do not contribute to the trace, or said differently: there are no fermions which run in the Klein bottle. Furthermore, we saw that:

$$
\text{NS-NS: } \Omega\left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j\right\rangle = \left|\tilde{k}_n^i, k_n^i, \tilde{l}_r^j, l_r^j\right\rangle \ , \quad \text{R-R: } \Omega\left|k_n^i, \tilde{k}_n^i, l_r^j, \tilde{l}_r^j, s_a, \tilde{s}_b\right\rangle = -\left|\tilde{k}_n^i, k_n^i, \tilde{l}_r^j, l_r^j, \tilde{s}_b, s_a\right\rangle \ ,
\tag{D.2.1}
$$

from which we understand that all states are not sent to their linear span, such that they do not all contribute to the trace. Only the invariant states contribute:

$$\begin{aligned}\mathrm{tr}\left(\Omega q^{L_0}\bar{q}^{\bar{L}_0}\right) = &\prod_{i,n>0,r>0}\left(\sum_{k_n^i,l_r^i}\right)(q\bar{q})^{\sum_i\left(\sum_{n>0}nk_n^i+\sum_{r>0}rl_r^i\right)-\frac{1}{2}}\\ &-\sum_{s_a}\prod_{i,n>0,r'>0}\left(\sum_{k_n^i,l_{r'}^i}\right)(q\bar{q})^{\sum_i\left(\sum_{n>0}nk_n^i+\sum_{r'>0}r'l_{r'}^i\right)}\\ =&\frac{\prod_{r>0}(1+(q\bar{q})^r)^8}{\sqrt{q\bar{q}}\prod_{n>0}(1-(q\bar{q})^n)^8}-16\frac{\prod_{r'>0}(1+(q\bar{q})^{r'})^8}{\prod_{n>0}(1-(q\bar{q})^n)^8}\ ,\end{aligned} \tag{D.2.2}$$

where r is half-integer whereas r' is integer, and

$$\begin{aligned}\mathrm{tr}\left(\frac{1+(-1)^F}{2}\frac{1+(-1)^{\tilde{F}}}{2}\Omega q^{L_0}\bar{q}^{\bar{L}_0}\right) = &\prod_{i,n>0,r>0}\left(\sum_{k_n^i,l_r^i}\right)\left(\frac{1-(-1)^{\sum_r l_r^i}}{2}\right)^2(q\bar{q})^{\sum_i\left(\sum_{n>0}nk_n^i+\sum_{r>0}rl_r^i\right)-\frac{1}{2}}\\ &-\sum_{s_a}\prod_{i,n>0,r'>0}\left(\sum_{k_n^i,l_{r'}^i}\right)\left(\frac{1+(-1)^{\sum_{r'}l_{r'}^i+\sum_a s_a}}{2}\right)^2(q\bar{q})^{\sum_i\left(\sum_{n>0}nk_n^i+\sum_{r'>0}r'l_{r'}^i\right)}\\ =&\frac{1}{2}\frac{\prod_{r>0}(1+(q\bar{q})^r)^8}{\sqrt{q\bar{q}}\prod_{n>0}(1-(q\bar{q})^n)^8}-\frac{1}{2}\frac{\prod_{r>0}(1-(q\bar{q})^r)^8}{\sqrt{q\bar{q}}\prod_{n>0}(1-(q\bar{q})^n)^8}-8\frac{\prod_{r'>0}(1+(q\bar{q})^{r'})^8}{\prod_{n>0}(1-(q\bar{q})^n)^8}\\ =&\frac{V_8-S_8}{\eta^8}(2i\tau_2)\ .\end{aligned} \tag{D.2.3}$$

D.3 The annulus and the Möbius strip of type I string theory

The annulus amplitude can be calculated analogously to the torus in (4.1.50) (with a modified integration region as for the Klein bottle in (4.1.60)):

$$\mathcal{A} = -\frac{V}{4(4\pi\alpha')^{\frac{d}{2}}}\int_0^\infty\frac{d\tau_2}{\tau_2^{1+\frac{d}{2}}}\mathrm{Str}\left(\frac{1+(-1)^F}{2}q^{L_0}\right)\left(\frac{i\tau_2}{2}\right)\ , \tag{D.3.1}$$

where this time, to match with (4.1.43) given that $q=e^{2\pi i\tau}$ and that open strings have a different Regge slope than closed strings, we must choose $\tau=\frac{i\tau_2}{2}$. The open string states are

$$\text{NS sector: }\ \left|k_n^i,l_r^j,0,i,j\right\rangle\ ,\quad \text{R sector: }\ \left|k_n^i,l_r^j,s_a,i,j\right\rangle\ , \tag{D.3.2}$$

so the result is very close to the Klein bottle calculation

$$\begin{aligned}
\text{NS sector}: \operatorname{Str}\left(\frac{1+(-1)^F}{2}q^{L_0}\right) &= N^2 \prod_{i,n>0,r>0}\left(\sum_{k_n^i,l_r^i}\right)\frac{1-(-1)^{\sum_r l_r^i}}{2}q^{\sum_i\left(\sum_{n>0}nk_n^i+\sum_{r>0}rl_r^i\right)-\frac{1}{2}} \\
&= N^2\frac{V_8}{\eta^8}\ , \\
\text{R sector}: \operatorname{Str}\left(\frac{1+(-1)^F}{2}q^{L_0}\right) &= -N^2\sum_{s_a}\prod_{i,n>0,r>0}\left(\sum_{k_n^i,l_r^i}\right)\frac{1+(-1)^{\sum_r l_r^i+\sum_a s_a}}{2}q^{\sum_i\left(\sum_{n>0}nk_n^i+\sum_{r>0}rl_r^i\right)} \\
&= -N^2\frac{S_8}{\eta^8}\ .
\end{aligned} \tag{D.3.3}$$

For the Möbius strip, we get

$$\begin{aligned}
&\text{NS sector}: \operatorname{Str}\left(\frac{1+(-1)^F}{2}\Omega q^{L_0}\right) \\
&\quad = -i\omega N \prod_{i,n>0,r>0}\left(\sum_{k_n^i,l_r^i}\right)\frac{1-(-1)^{\sum_r l_r^i}}{2}(-1)^{\sum_{n>0}nk_n^i}e^{i\pi\sum_{r>0}rl_r^i}q^{\sum_i\left(\sum_{n>0}nk_n^i+\sum_{r>0}rl_r^i\right)-\frac{1}{2}} \\
&\quad = -i\omega N\left[\frac{\prod_{r>0}(1+(e^{i\pi}q)^r)^8}{2\sqrt{q}\prod_{n>0}(1-(-q)^n)^8}-\frac{\prod_{r>0}(1-(e^{i\pi}q)^r)^8}{2\sqrt{q}\prod_{n>0}(1-q^n)^8}\right] = 8\omega+\mathcal{O}(\sqrt{q})\ , \\
&\text{R sector}: \operatorname{Str}\left(\frac{1+(-1)^F}{2}\Omega q^{L_0}\right) \\
&\quad = -\omega N\sum_{s_a}\prod_{i,n>0,r>0}\left(\sum_{k_n^i,l_r^i}\right)\frac{1+(-1)^{\sum_r l_r^i+\sum_a s_a}}{2}(-q)^{\sum_i\left(\sum_{n>0}nk_n^i+\sum_{r>0}rl_r^i\right)} \\
&\quad = -16\omega N\frac{\prod_{r>0}(1+(-q)^r)^8}{2\prod_{n>0}(1-(-q)^n)^8} = -8\omega+\mathcal{O}(\sqrt{q})\ .
\end{aligned} \tag{D.3.4}$$

For the NS sector, we explicitely kept $e^{i\pi}$ to remove phase ambiguities by defining $(e^{i\pi})^{\frac{1}{2}} = e^{\frac{i\pi}{2}}$. If we redefine $-q \to q$, we see that the correct modulus becomes $\tau = \frac{1+i\tau_2}{2}$ in the Möbius strip.

Bibliography

[1] **ATLAS** Collaboration, G. Aad *et al.*, "Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC," *Phys. Lett.* **B716** (2012) 1–29, `arXiv:1207.7214 [hep-ex]`.

[2] **CMS** Collaboration, S. Chatrchyan *et al.*, "Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC," *Phys. Lett.* **B716** (2012) 30–61, `arXiv:1207.7235 [hep-ex]`.

[3] **E598** Collaboration, J. J. Aubert *et al.*, "Experimental Observation of a Heavy Particle J," *Phys. Rev. Lett.* **33** (1974) 1404–1406.

[4] **SLAC-SP-017** Collaboration, J. E. Augustin *et al.*, "Discovery of a Narrow Resonance in e^+e^- Annihilation," *Phys. Rev. Lett.* **33** (1974) 1406–1408. [Adv. Exp. Phys.5,141(1976)].

[5] S. L. Glashow, J. Iliopoulos, and L. Maiani, "Weak Interactions with Lepton-Hadron Symmetry," *Phys. Rev.* **D2** (1970) 1285–1292.

[6] F. Englert and R. Brout, "Broken Symmetry and the Mass of Gauge Vector Mesons," *Phys. Rev. Lett.* **13** (1964) 321–323. [,157(1964)].

[7] P. W. Higgs, "Broken Symmetries and the Masses of Gauge Bosons," *Phys. Rev. Lett.* **13** (1964) 508–509. [,160(1964)].

[8] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, "Global Conservation Laws and Massless Particles," *Phys. Rev. Lett.* **13** (1964) 585–587. [,162(1964)].

[9] **Super-Kamiokande** Collaboration, Y. Fukuda *et al.*, "Measurements of the solar neutrino flux from Super-Kamiokande's first 300 days," *Phys. Rev. Lett.* **81** (1998) 1158–1162, `arXiv:hep-ex/9805021 [hep-ex]`. [Erratum: Phys. Rev. Lett.81,4279(1998)].

[10] **Muon g-2** Collaboration, G. W. Bennett *et al.*, "Measurement of the negative muon anomalous magnetic moment to 0.7 ppm," *Phys. Rev. Lett.* **92** (2004) 161802, `arXiv:hep-ex/0401008 [hep-ex]`.

[11] **LHCb** Collaboration, R. Aaij *et al.*, "Search for lepton-universality violation in $B^+ \to K^+\ell^+\ell^-$ decays," `arXiv:1903.09252 [hep-ex]`.

[12] F. Zwicky, "On the Masses of Nebulae and of Clusters of Nebulae," *Astrophys. J.* **86** (1937) 217–246.

[13] **Supernova Search Team** Collaboration, A. G. Riess *et al.*, "Observational evidence from supernovae for an accelerating universe and a cosmological constant," *Astron. J.* **116** (1998) 1009–1038, `arXiv:astro-ph/9805201 [astro-ph]`.

[14] **Supernova Cosmology Project** Collaboration, S. Perlmutter *et al.*, "Measurements of Omega and Lambda from 42 high redshift supernovae," *Astrophys. J.* **517** (1999) 565–586, `arXiv:astro-ph/9812133 [astro-ph]`.

[15] G. 't Hooft and M. J. G. Veltman, "One loop divergencies in the theory of gravitation," *Ann. Inst. H. Poincare Phys. Theor.* **A20** (1974) 69–94.

[16] M. H. Goroff and A. Sagnotti, "QUANTUM GRAVITY AT TWO LOOPS," *Phys. Lett.* **160B** (1985) 81–86.

[17] **LIGO Scientific, Virgo** Collaboration, B. P. Abbott *et al.*, "Observation of Gravitational Waves from a Binary Black Hole Merger," *Phys. Rev. Lett.* **116** no. 6, (2016) 061102, `arXiv:1602.03837 [gr-qc]`.

[18] **Event Horizon Telescope** Collaboration, K. Akiyama *et al.*, "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole," *Astrophys. J.* **875** no. 1, (2019) L1.

[19] J. D. Bekenstein, "Black holes and entropy," *Phys. Rev.* **D7** (1973) 2333–2346.

[20] S. W. Hawking, "Particle Creation by Black Holes," *Commun. Math. Phys.* **43** (1975) 199–220. [,167(1975)].

[21] R. D. Peccei, "The Strong CP problem and axions," *Lect. Notes Phys.* **741** (2008) 3–17, `arXiv:hep-ph/0607268 [hep-ph]`.

[22] D. J. E. Marsh, "Axion Cosmology," *Phys. Rept.* **643** (2016) 1–79, `arXiv:1510.07633 [astro-ph.CO]`.

[23] A. Hook, "TASI Lectures on the Strong CP Problem and Axions," `arXiv:1812.02669 [hep-ph]`.

[24] V. Baluni, "CP Violating Effects in QCD," *Phys. Rev.* **D19** (1979) 2227–2230.

[25] R. J. Crewther, P. Di Vecchia, G. Veneziano, and E. Witten, "Chiral Estimate of the Electric Dipole Moment of the Neutron in Quantum Chromodynamics," *Phys. Lett.* **88B** (1979) 123. [Erratum: Phys. Lett.91B,487(1980)].

[26] C. A. Baker *et al.*, "An Improved experimental limit on the electric dipole moment of the neutron," *Phys. Rev. Lett.* **97** (2006) 131801, `arXiv:hep-ex/0602020 [hep-ex]`.

[27] L. Ubaldi, "Effects of theta on the deuteron binding energy and the triple-alpha process," *Phys. Rev.* **D81** (2010) 025011, `arXiv:0811.1599 [hep-ph]`.

[28] M. Dine, L. Stephenson Haskins, L. Ubaldi, and D. Xu, "Some Remarks on Anthropic Approaches to the Strong CP Problem," *JHEP* **05** (2018) 171, `arXiv:1801.03466 [hep-th]`.

[29] F. Takahashi, "A possible solution to the strong CP problem," *Prog. Theor. Phys.* **121** (2009) 711–725, arXiv:0804.2478 [hep-ph].

[30] N. Kaloper and J. Terning, "Landscaping the Strong CP Problem," arXiv:1710.01740 [hep-th].

[31] M. Dine, P. Draper, and G. Festuccia, "Instanton Effects in Three Flavor QCD," *Phys. Rev.* **D92** no. 5, (2015) 054004, arXiv:1410.8505 [hep-ph].

[32] S. Aoki *et al.*, "Review of lattice results concerning low-energy particle physics," *Eur. Phys. J.* **C77** no. 2, (2017) 112, arXiv:1607.00299 [hep-lat].

[33] J. R. Ellis and M. K. Gaillard, "Strong and Weak CP Violation," *Nucl. Phys.* **B150** (1979) 141–162.

[34] K. S. Babu and R. N. Mohapatra, "A Solution to the Strong CP Problem Without an Axion," *Phys. Rev.* **D41** (1990) 1286.

[35] S. M. Barr, D. Chang, and G. Senjanovic, "Strong CP problem and parity," *Phys. Rev. Lett.* **67** (1991) 2765–2768.

[36] A. E. Nelson, "Naturally Weak CP Violation," *Phys. Lett.* **136B** (1984) 387–391.

[37] S. M. Barr, "Solving the Strong CP Problem Without the Peccei-Quinn Symmetry," *Phys. Rev. Lett.* **53** (1984) 329.

[38] R. D. Peccei and H. R. Quinn, "CP Conservation in the Presence of Instantons," *Phys. Rev. Lett.* **38** (1977) 1440–1443.

[39] R. D. Peccei and H. R. Quinn, "Constraints Imposed by CP Conservation in the Presence of Instantons," *Phys. Rev.* **D16** (1977) 1791–1797.

[40] S. Weinberg, "A New Light Boson?," *Phys. Rev. Lett.* **40** (1978) 223–226.

[41] F. Wilczek, "Problem of Strong p and t Invariance in the Presence of Instantons," *Phys. Rev. Lett.* **40** (1978) 279–282.

[42] W. A. Bardeen and S. H. H. Tye, "Current Algebra Applied to Properties of the Light Higgs Boson," *Phys. Lett.* **74B** (1978) 229–232.

[43] T. W. Donnelly, S. J. Freedman, R. S. Lytel, R. D. Peccei, and M. Schwartz, "Do Axions Exist?," *Phys. Rev.* **D18** (1978) 1607.

[44] M. Dine, W. Fischler, and M. Srednicki, "A Simple Solution to the Strong CP Problem with a Harmless Axion," *Phys. Lett.* **104B** (1981) 199–202.

[45] A. R. Zhitnitsky, "On Possible Suppression of the Axion Hadron Interactions. (In Russian)," *Sov. J. Nucl. Phys.* **31** (1980) 260. [Yad. Fiz.31,497(1980)].

[46] J. E. Kim, "Weak Interaction Singlet and Strong CP Invariance," *Phys. Rev. Lett.* **43** (1979) 103.

[47] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, "Can Confinement Ensure Natural CP Invariance of Strong Interactions?," *Nucl. Phys.* **B166** (1980) 493–506.

[48] C. Vafa and E. Witten, "Parity Conservation in QCD," *Phys. Rev. Lett.* **53** (1984) 535.

[49] M. Srednicki, "Axion Couplings to Matter. 1. CP Conserving Parts," *Nucl. Phys.* **B260** (1985) 689–700.

[50] G. Grilli di Cortona, E. Hardy, J. Pardo Vega, and G. Villadoro, "The QCD axion, precisely," *JHEP* **01** (2016) 034, `arXiv:1511.02867 [hep-ph]`.

[51] G. 't Hooft, "Symmetry Breaking Through Bell-Jackiw Anomalies," *Phys. Rev. Lett.* **37** (1976) 8–11. [,226(1976)].

[52] Z. Berezhiani, L. Gianfagna, and M. Giannotti, "Strong CP problem and mirror world: The Weinberg-Wilczek axion revisited," *Phys. Lett.* **B500** (2001) 286–296, `arXiv:hep-ph/0009290 [hep-ph]`.

[53] V. A. Rubakov, "Grand unification and heavy axion," *JETP Lett.* **65** (1997) 621–624, `arXiv:hep-ph/9703409 [hep-ph]`.

[54] A. Hook, "Solving the Hierarchy Problem Discretely," *Phys. Rev. Lett.* **120** no. 26, (2018) 261802, `arXiv:1802.10093 [hep-ph]`.

[55] P. Svrcek and E. Witten, "Axions In String Theory," *JHEP* **06** (2006) 051, `arXiv:hep-th/0605206 [hep-th]`.

[56] A. Arvanitaki, S. Dimopoulos, S. Dubovsky, N. Kaloper, and J. March-Russell, "String Axiverse," *Phys. Rev.* **D81** (2010) 123530, `arXiv:0905.4720 [hep-th]`.

[57] M. Cicoli, M. Goodsell, and A. Ringwald, "The type IIB string axiverse and its low-energy phenomenology," *JHEP* **10** (2012) 146, `arXiv:1206.0819 [hep-th]`.

[58] J. Preskill, M. B. Wise, and F. Wilczek, "Cosmology of the Invisible Axion," *Phys. Lett.* **120B** (1983) 127–132.

[59] L. F. Abbott and P. Sikivie, "A Cosmological Bound on the Invisible Axion," *Phys. Lett.* **120B** (1983) 133–136.

[60] M. Dine and W. Fischler, "The Not So Harmless Axion," *Phys. Lett.* **B120** (1983) 137–141. [,URL(1982)].

[61] K. Freese, J. A. Frieman, and A. V. Olinto, "Natural inflation with pseudo - Nambu-Goldstone bosons," *Phys. Rev. Lett.* **65** (1990) 3233–3236.

[62] J. E. Kim, H. P. Nilles, and M. Peloso, "Completing natural inflation," *JCAP* **0501** (2005) 005, `arXiv:hep-ph/0409138 [hep-ph]`.

[63] B. Ratra and P. J. E. Peebles, "Cosmological Consequences of a Rolling Homogeneous Scalar Field," *Phys. Rev.* **D37** (1988) 3406.

[64] J. A. Frieman, C. T. Hill, A. Stebbins, and I. Waga, "Cosmology with ultralight pseudo nambu-goldstone bosons," *Phys. Rev. Lett.* **75** (Sep, 1995) 2077–2080. `https://link.aps.org/doi/10.1103/PhysRevLett.75.2077`.

[65] J. E. Kim and H. P. Nilles, "A Quintessential axion," *Phys. Lett.* **B553** (2003) 1–6, `arXiv:hep-ph/0210402 [hep-ph]`.

[66] J. E. Kim, "Modeling the small dark energy scale with a quintessential pseudoscalar boson," *J. Korean Phys. Soc.* **64** (2014) 795–805, `arXiv:1311.4545 [hep-ph]`.

[67] S. Weinberg, "Goldstone Bosons as Fractional Cosmic Neutrinos," *Phys. Rev. Lett.* **110** no. 24, (2013) 241301, `arXiv:1305.1971 [astro-ph.CO]`.

[68] Y. Kahn, B. R. Safdi, and J. Thaler, "Broadband and Resonant Approaches to Axion Dark Matter Detection," *Phys. Rev. Lett.* **117** no. 14, (2016) 141801, `arXiv:1602.01086 [hep-ph]`.

[69] D. Budker, P. W. Graham, M. Ledbetter, S. Rajendran, and A. Sushkov, "Proposal for a Cosmic Axion Spin Precession Experiment (CASPEr)," *Phys. Rev.* **X4** no. 2, (2014) 021030, `arXiv:1306.6089 [hep-ph]`.

[70] G. W. Gibbons and S. W. Hawking, "Cosmological Event Horizons, Thermodynamics, and Particle Creation," *Phys. Rev.* **D15** (1977) 2738–2751.

[71] P. J. Steinhardt and M. S. Turner, "Saving the Invisible Axion," *Phys. Lett.* **129B** (1983) 51.

[72] G. Lazarides, R. K. Schaefer, D. Seckel, and Q. Shafi, "Dilution of Cosmological Axions by Entropy Production," *Nucl. Phys.* **B346** (1990) 193–212.

[73] G. R. Dvali, "Removing the cosmological bound on the axion scale," `arXiv:hep-ph/9505253 [hep-ph]`.

[74] K. Choi, H. B. Kim, and J. E. Kim, "Axion cosmology with a stronger QCD in the early universe," *Nucl. Phys.* **B490** (1997) 349–364, `arXiv:hep-ph/9606372 [hep-ph]`.

[75] T. Banks and M. Dine, "The Cosmology of string theoretic axions," *Nucl. Phys.* **B505** (1997) 445–460, `arXiv:hep-th/9608197 [hep-th]`.

[76] T. Banks, M. Dine, and M. Graesser, "Supersymmetry, axions and cosmology," *Phys. Rev.* **D68** (2003) 075011, `arXiv:hep-ph/0210256 [hep-ph]`.

[77] M. Kawasaki, F. Takahashi, and M. Yamada, "Suppressing the QCD Axion Abundance by Hidden Monopoles," *Phys. Lett.* **B753** (2016) 677–681, `arXiv:1511.05030 [hep-ph]`.

[78] Y. Nomura, S. Rajendran, and F. Sanches, "Axion Isocurvature and Magnetic Monopoles," *Phys. Rev. Lett.* **116** no. 14, (2016) 141803, `arXiv:1511.06347 [hep-ph]`.

[79] P. Agrawal, G. Marques-Tavares, and W. Xue, "Opening up the QCD axion window," *JHEP* **03** (2018) 049, `arXiv:1708.05008 [hep-ph]`.

[80] M. Kawasaki, F. Takahashi, and M. Yamada, "Adiabatic suppression of the axion abundance and isocurvature due to coupling to hidden monopoles," *JHEP* **01** (2018) 053, `arXiv:1708.06047 [hep-ph]`.

[81] N. Kitajima, T. Sekiguchi, and F. Takahashi, "Cosmological abundance of the QCD axion coupled to hidden photons," *Phys. Lett.* **B781** (2018) 684–687, `arXiv:1711.06590 [hep-ph]`.

[82] P. W. Graham and A. Scherlis, "Stochastic axion scenario," *Phys. Rev.* **D98** no. 3, (2018) 035017, `arXiv:1805.07362 [hep-ph]`.

[83] F. Takahashi, W. Yin, and A. H. Guth, "QCD axion window and low-scale inflation," *Phys. Rev.* **D98** no. 1, (2018) 015042, `arXiv:1805.08763 [hep-ph]`.

[84] G. D. Coughlan, W. Fischler, E. W. Kolb, S. Raby, and G. G. Ross, "Cosmological Problems for the Polonyi Potential," *Phys. Lett.* **131B** (1983) 59–64.

[85] P. Sikivie, "Of Axions, Domain Walls and the Early Universe," *Phys. Rev. Lett.* **48** (1982) 1156–1159.

[86] M. Gorghetto, E. Hardy, and G. Villadoro, "Axions from Strings: the Attractive Solution," *JHEP* **07** (2018) 151, `arXiv:1806.04677 [hep-ph]`.

[87] B. Holdom, "DOMAIN WALLS. 1. AXION MODELS," *Phys. Rev.* **D27** (1983) 332–338.

[88] S. M. Barr and D. Seckel, "Planck-scale corrections to axion models," *Phys. Rev. D* **46** (Jul, 1992) 539–549. `https://link.aps.org/doi/10.1103/PhysRevD.46.539`.

[89] S. J. Asztalos, G. Carosi, C. Hagmann, D. Kinion, K. van Bibber, M. Hotz, L. J. Rosenberg, G. Rybka, J. Hoskins, J. Hwang, P. Sikivie, D. B. Tanner, R. Bradley, and J. Clarke, "Squid-based microwave cavity search for dark-matter axions," *Phys. Rev. Lett.* **104** (Jan, 2010) 041301. `https://link.aps.org/doi/10.1103/PhysRevLett.104.041301`.

[90] P. Sikivie, N. Sullivan, and D. B. Tanner, "Proposal for Axion Dark Matter Detection Using an LC Circuit," *Phys. Rev. Lett.* **112** no. 13, (2014) 131301, `arXiv:1310.8545 [hep-ph]`.

[91] **MADMAX Working Group** Collaboration, A. Caldwell, G. Dvali, B. Majorovits, A. Millar, G. Raffelt, J. Redondo, O. Reimann, F. Simon, and F. Steffen, "Dielectric Haloscopes: A New Way to Detect Axion Dark Matter," *Phys. Rev. Lett.* **118** no. 9, (2017) 091801, `arXiv:1611.05865 [physics.ins-det]`.

[92] P. W. Graham and S. Rajendran, "New Observables for Direct Detection of Axion Dark Matter," *Phys. Rev.* **D88** (2013) 035023, `arXiv:1306.6088 [hep-ph]`.

[93] S. P. Chang, S. Haciomeroglu, O. Kim, S. Lee, S. Park, and Y. K. Semertzidis, "Axion dark matter search using the storage ring EDM method," *PoS* **PSTP2017** (2018) 036, `arXiv:1710.05271 [hep-ex]`.

[94] **CAST** Collaboration, V. Anastassopoulos *et al.*, "New CAST Limit on the Axion-Photon Interaction," *Nature Phys.* **13** (2017) 584–590, `arXiv:1705.02290 [hep-ex]`.

[95] J. K. Vogel *et al.*, "IAXO - The International Axion Observatory," in *8th Patras Workshop on Axions, WIMPs and WISPs (AXION-WIMP 2012) Chicago, Illinois, July 18-22, 2012.* 2013. `arXiv:1302.3273 [physics.ins-det]`. `http://lss.fnal.gov/archive/2013/pub/fermilab-pub-13-699-a.pdf`.

[96] G. Raffelt and A. Weiss, "Red giant bound on the axion - electron coupling revisited," *Phys. Rev.* **D51** (1995) 1495–1498, `arXiv:hep-ph/9410205 [hep-ph]`.

[97] A. Ayala, I. Dominguez, M. Giannotti, A. Mirizzi, and O. Straniero, "Revisiting the bound on axion-photon coupling from Globular Clusters," *Phys. Rev. Lett.* **113** no. 19, (2014) 191302, `arXiv:1406.6053 [astro-ph.SR]`.

[98] M. Koshiba, "Observational neutrino astrophysics," *Phys. Rept.* **220** (1992) 229–381.

[99] K. Ehret *et al.*, "New ALPS Results on Hidden-Sector Lightweights," *Phys. Lett.* **B689** (2010) 149–155, `arXiv:1004.1313 [hep-ex]`.

[100] **OSQAR** Collaboration, R. Ballou *et al.*, "New exclusion limits on scalar and pseudoscalar axionlike particles from light shining through a wall," *Phys. Rev.* **D92** no. 9, (2015) 092002, `arXiv:1506.08082 [hep-ex]`.

[101] F. Della Valle, A. Ejlli, U. Gastaldi, G. Messineo, E. Milotti, R. Pengo, G. Ruoso, and G. Zavattini, "The PVLAS experiment: measuring vacuum magnetic birefringence and dichroism with a birefringent Fabry-Perot cavity," *Eur. Phys. J.* **C76** no. 1, (2016) 24, `arXiv:1510.08052 [physics.optics]`.

[102] A. Arvanitaki and A. A. Geraci, "Resonantly Detecting Axion-Mediated Forces with Nuclear Magnetic Resonance," *Phys. Rev. Lett.* **113** no. 16, (2014) 161801, `arXiv:1403.1290 [hep-ph]`.

[103] A. Arvanitaki and S. Dubovsky, "Exploring the String Axiverse with Precision Black Hole Physics," *Phys. Rev.* **D83** (2011) 044026, `arXiv:1004.3558 [hep-th]`.

[104] P. Pani, V. Cardoso, L. Gualtieri, E. Berti, and A. Ishibashi, "Black hole bombs and photon mass bounds," *Phys. Rev. Lett.* **109** (2012) 131102, arXiv:1209.0465 [gr-qc].

[105] M. J. Stott and D. J. E. Marsh, "Black hole spin constraints on the mass spectrum and number of axionlike fields," *Phys. Rev.* **D98** no. 8, (2018) 083006, arXiv:1805.02016 [hep-ph].

[106] M. T. et al. (Particle Data Group), "Review of particle physics," *Phys. Rev. D* **98** (2018) 030001.

[107] D. B. Kaplan, "Opening the Axion Window," *Nucl. Phys.* **B260** (1985) 215–226.

[108] L. Di Luzio, F. Mescia, and E. Nardi, "Redefining the Axion Window," *Phys. Rev. Lett.* **118** no. 3, (2017) 031801, arXiv:1610.07593 [hep-ph].

[109] N. Craig, A. Hook, and S. Kasko, "The Photophobic ALP," *JHEP* **09** (2018) 028, arXiv:1805.06538 [hep-ph].

[110] J. E. Kim and U. W. Lee, "Z^0 Decay to Photon and Variant Axion," *Phys. Lett.* **B233** (1989) 496–498.

[111] A. Djouadi, P. M. Zerwas, and J. Zunft, "Search for light pseudoscalar Higgs bosons in Z decays," *Phys. Lett.* **B259** (1991) 175–181.

[112] J. Jaeckel and M. Spannowsky, "Probing MeV to 90 GeV axion-like particles with LEP and LHC," *Phys. Lett.* **B753** (2016) 482–487, arXiv:1509.00476 [hep-ph].

[113] E. Izaguirre, T. Lin, and B. Shuve, "Searching for Axionlike Particles in Flavor-Changing Neutral Current Processes," *Phys. Rev. Lett.* **118** no. 11, (2017) 111802, arXiv:1611.09355 [hep-ph].

[114] M. Bauer, M. Neubert, and A. Thamm, "Collider Probes of Axion-Like Particles," *JHEP* **12** (2017) 044, arXiv:1708.00443 [hep-ph].

[115] M. Bauer, M. Heiles, M. Neubert, and A. Thamm, "Axion-Like Particles at Future Colliders," *Eur. Phys. J.* **C79** no. 1, (2019) 74, arXiv:1808.10323 [hep-ph].

[116] P. W. Graham, D. E. Kaplan, and S. Rajendran, "Cosmological Relaxation of the Electroweak Scale," *Phys. Rev. Lett.* **115** no. 22, (2015) 221801, arXiv:1504.07551 [hep-ph].

[117] W. H. Press, B. S. Ryden, and D. N. Spergel, "Single Mechanism for Generating Large Scale Structure and Providing Dark Missing Matter," *Phys. Rev. Lett.* **64** (1990) 1084.

[118] S.-J. Sin, "Late time cosmological phase transition and galactic halo as Bose liquid," *Phys. Rev.* **D50** (1994) 3650–3654, arXiv:hep-ph/9205208 [hep-ph].

[119] P. J. E. Peebles, "Fluid dark matter," *Astrophys. J.* **534** (2000) L127, `arXiv:astro-ph/0002495 [astro-ph]`.

[120] J. Goodman, "Repulsive dark matter," *New Astron.* **5** (2000) 103, `arXiv:astro-ph/0003018 [astro-ph]`.

[121] W. Hu, R. Barkana, and A. Gruzinov, "Cold and fuzzy dark matter," *Phys. Rev. Lett.* **85** (2000) 1158–1161, `arXiv:astro-ph/0003365 [astro-ph]`.

[122] L. Hui, J. P. Ostriker, S. Tremaine, and E. Witten, "Ultralight scalars as cosmological dark matter," *Phys. Rev.* **D95** no. 4, (2017) 043541, `arXiv:1610.08297 [astro-ph.CO]`.

[123] W. J. G. de Blok, "The Core-Cusp Problem," *Advances in Astronomy* **2010** (Jan, 2010) 789293, `arXiv:0910.3538 [astro-ph.CO]`.

[124] S. W. Hawking, "Quantum Coherence Down the Wormhole," *Phys. Lett.* **B195** (1987) 337.

[125] S. B. Giddings and A. Strominger, "Loss of Incoherence and Determination of Coupling Constants in Quantum Gravity," *Nucl. Phys.* **B307** (1988) 854–866.

[126] T. Banks and N. Seiberg, "Symmetries and Strings in Field Theory and Gravity," *Phys. Rev.* **D83** (2011) 084019, `arXiv:1011.5120 [hep-th]`.

[127] J. E. Kim, "Invisible Axion and Neutrino Oscillation in SU(11)," *Phys. Rev.* **D24** (1981) 3007.

[128] H. M. Georgi, L. J. Hall, and M. B. Wise, "Grand Unified Models With an Automatic Peccei-Quinn Symmetry," *Nucl. Phys.* **B192** (1981) 409–416.

[129] S. Dimopoulos, P. H. Frampton, H. Georgi, and M. B. Wise, "AUTOMATIC INVISIBLE AXION WITHOUT DOMAIN WALLS," *Phys. Lett.* **117B** (1982) 185.

[130] K. Kang, I.-G. Koh, and S. Ouvry, "The Strong CP Problem and Axion Invisibility," *Phys. Lett.* **119B** (1982) 361.

[131] M. Kamionkowski and J. March-Russell, "Planck scale physics and the Peccei-Quinn mechanism," *Phys. Lett.* **B282** (1992) 137–141, `arXiv:hep-th/9202003 [hep-th]`.

[132] R. Holman, S. D. H. Hsu, T. W. Kephart, E. W. Kolb, R. Watkins, and L. M. Widrow, "Solutions to the strong CP problem in a world with gravity," *Phys. Lett.* **B282** (1992) 132–136, `arXiv:hep-ph/9203206 [hep-ph]`.

[133] C. T. Hill and A. K. Leibovich, "Deconstructing 5-D QED," *Phys. Rev.* **D66** (2002) 016006, `arXiv:hep-ph/0205057 [hep-ph]`.

[134] C. T. Hill and A. K. Leibovich, "Natural theories of ultralow mass PNGB's: Axions and quintessence," *Phys. Rev.* **D66** (2002) 075010, `arXiv:hep-ph/0205237 [hep-ph]`.

[135] A. G. Dias, V. Pleitez, and M. D. Tonasse, "Naturally light invisible axion in models with large local discrete symmetries," *Phys. Rev.* **D67** (2003) 095008, arXiv:hep-ph/0211107 [hep-ph].

[136] K. Harigaya, M. Ibe, K. Schmitz, and T. T. Yanagida, "Peccei-Quinn symmetry from a gauged discrete R symmetry," *Phys. Rev.* **D88** no. 7, (2013) 075022, arXiv:1308.1227 [hep-ph].

[137] A. G. Dias, A. C. B. Machado, C. C. Nishi, A. Ringwald, and P. Vaudrevange, "The Quest for an Intermediate-Scale Accidental Axion and Further ALPs," *JHEP* **06** (2014) 037, arXiv:1403.5760 [hep-ph].

[138] A. Ringwald and K. Saikawa, "Axion dark matter in the post-inflationary Peccei-Quinn symmetry breaking scenario," *Phys. Rev.* **D93** no. 8, (2016) 085031, arXiv:1512.06436 [hep-ph]. [Addendum: Phys. Rev.D94,no.4,049908(2016)].

[139] M. Redi and R. Sato, "Composite Accidental Axions," *JHEP* **05** (2016) 104, arXiv:1602.05427 [hep-ph].

[140] H. Fukuda, M. Ibe, M. Suzuki, and T. T. Yanagida, "A "gauged" $U(1)$ Peccei-Quinn symmetry," *Phys. Lett.* **B771** (2017) 327–331, arXiv:1703.01112 [hep-ph].

[141] B. Lillard and T. M. P. Tait, "A Composite Axion from a Supersymmetric Product Group," *JHEP* **11** (2017) 005, arXiv:1707.04261 [hep-ph].

[142] K. Choi and S. H. Im, "Realizing the relaxion from multiple axions and its UV completion with high scale supersymmetry," *JHEP* **01** (2016) 149, arXiv:1511.00132 [hep-ph].

[143] D. E. Kaplan and R. Rattazzi, "Large field excursions and approximate discrete symmetries from a clockwork axion," *Phys. Rev.* **D93** no. 8, (2016) 085007, arXiv:1511.01827 [hep-ph].

[144] G. F. Giudice and M. McCullough, "A Clockwork Theory," *JHEP* **02** (2017) 036, arXiv:1610.07962 [hep-ph].

[145] R. Coy, M. Frigerio, and M. Ibe, "Dynamical Clockwork Axions," *JHEP* **10** (2017) 002, arXiv:1706.04529 [hep-ph].

[146] A. J. Long, "Cosmological Aspects of the Clockwork Axion," *JHEP* **07** (2018) 066, arXiv:1803.07086 [hep-ph].

[147] P. Agrawal, J. Fan, and M. Reece, "Clockwork Axions in Cosmology: Is Chromonatural Inflation Chrononatural?," *JHEP* **10** (2018) 193, arXiv:1806.09621 [hep-th].

[148] H. M. Lee, "Gauged $U(1)$ clockwork theory," *Phys. Lett.* **B778** (2018) 79–87, arXiv:1708.03564 [hep-ph].

[149] K. M. Patel, "Clockwork mechanism for flavor hierarchies," *Phys. Rev.* **D96** no. 11, (2017) 115013, `arXiv:1711.05393 [hep-ph]`.

[150] R. Alonso, A. Carmona, B. M. Dillon, J. F. Kamenik, J. Martin Camalich, and J. Zupan, "A clockwork solution to the flavor puzzle," *JHEP* **10** (2018) 099, `arXiv:1807.09792 [hep-ph]`.

[151] S. C. Park and C. S. Shin, "Clockwork seesaw mechanisms," *Phys. Lett.* **B776** (2018) 222–226, `arXiv:1707.07364 [hep-ph]`.

[152] A. Ibarra, A. Kushwaha, and S. K. Vempati, "Clockwork for Neutrino Masses and Lepton Flavor Violation," *Phys. Lett.* **B780** (2018) 86–92, `arXiv:1711.02070 [hep-ph]`.

[153] A. Banerjee, S. Ghosh, and T. S. Ray, "Clockworked VEVs and Neutrino Mass," *JHEP* **11** (2018) 075, `arXiv:1808.04010 [hep-ph]`.

[154] T. Hambye, D. Teresi, and M. H. G. Tytgat, "A Clockwork WIMP," *JHEP* **07** (2017) 047, `arXiv:1612.06411 [hep-ph]`.

[155] D. Teresi, "Clockwork Dark Matter," in *Proceedings, 52nd Rencontres de Moriond on Electroweak Interactions and Unified Theories: La Thuile, Italy, March 18-25, 2017*, pp. 155–161. 2017. `arXiv:1705.09698 [hep-ph]`.

[156] J. Kim and J. McDonald, "Clockwork Higgs portal model for freeze-in dark matter," *Phys. Rev.* **D98** no. 2, (2018) 023533, `arXiv:1709.04105 [hep-ph]`.

[157] A. Goudelis, K. A. Mohan, and D. Sengupta, "Clockworking FIMPs," *JHEP* **10** (2018) 014, `arXiv:1807.06642 [hep-ph]`.

[158] A. Kehagias and A. Riotto, "Clockwork Inflation," *Phys. Lett.* **B767** (2017) 73–80, `arXiv:1611.03316 [hep-ph]`.

[159] S. C. Park and C. S. Shin, "Clockwork Higgs inflation," `arXiv:1807.09952 [hep-ph]`.

[160] I. Ben-Dayan, "Generalized Clockwork Theory," `arXiv:1706.05308 [hep-ph]`.

[161] A. Ahmed and B. M. Dillon, "Clockwork Goldstone Bosons," *Phys. Rev.* **D96** no. 11, (2017) 115031, `arXiv:1612.04011 [hep-ph]`.

[162] N. Craig, I. Garcia Garcia, and D. Sutherland, "Disassembling the Clockwork Mechanism," *JHEP* **10** (2017) 018, `arXiv:1704.07831 [hep-ph]`.

[163] G. F. Giudice and M. McCullough, "Comment on "Disassembling the Clockwork Mechanism"," `arXiv:1705.10162 [hep-ph]`.

[164] A. Kehagias and A. Riotto, "The Clockwork Supergravity," *JHEP* **02** (2018) 160, `arXiv:1710.04175 [hep-th]`.

[165] I. Antoniadis, A. Delgado, C. Markou, and S. Pokorski, "The effective supergravity of Little String Theory," *Eur. Phys. J.* **C78** no. 2, (2018) 146, `arXiv:1710.05568 [hep-th]`.

[166] K. Choi, S. H. Im, and C. S. Shin, "General Continuum Clockwork," `arXiv:1711.06228 [hep-ph]`.

[167] G. F. Giudice, Y. Kats, M. McCullough, R. Torre, and A. Urbano, "Clockwork / Linear Dilaton: Structure and Phenomenology," `arXiv:1711.08437 [hep-ph]`.

[168] D. Teresi, "Clockwork without supersymmetry," `arXiv:1802.01591 [hep-ph]`.

[169] S. H. Im, H. P. Nilles, and M. Olechowski, "Heterotic M-Theory from the Clockwork Perspective," *JHEP* **01** (2019) 151, `arXiv:1811.11838 [hep-th]`.

[170] P. Saraswat, "Weak gravity conjecture and effective field theory," *Phys. Rev.* **D95** no. 2, (2017) 025013, `arXiv:1608.06951 [hep-th]`.

[171] L. E. Ibanez and M. Montero, "A Note on the WGC, Effective Field Theory and Clockwork within String Theory," *JHEP* **02** (2018) 057, `arXiv:1709.02392 [hep-th]`.

[172] N. Craig and I. Garcia Garcia, "Rescuing Massive Photons from the Swampland," *JHEP* **11** (2018) 067, `arXiv:1810.05647 [hep-th]`.

[173] I. Antoniadis, S. Dimopoulos, and A. Giveon, "Little string theory at a TeV," *JHEP* **05** (2001) 055, `arXiv:hep-th/0103033 [hep-th]`.

[174] N. Arkani-Hamed, A. G. Cohen, and H. Georgi, "(De)constructing dimensions," *Phys. Rev. Lett.* **86** (2001) 4757–4761, `arXiv:hep-th/0104005 [hep-th]`.

[175] C. T. Hill, S. Pokorski, and J. Wang, "Gauge invariant effective Lagrangian for Kaluza-Klein modes," *Phys. Rev.* **D64** (2001) 105005, `arXiv:hep-th/0104035 [hep-th]`.

[176] Q. Bonnefoy, E. Dudas, and S. Pokorski, "Axions in a highly protected gauge symmetry model," *Eur. Phys. J.* **C79** no. 1, (2019) 31, `arXiv:1804.01112 [hep-ph]`.

[177] K.-S. Choi, H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, "Accions," *Phys. Lett.* **B675** (2009) 381–386, `arXiv:0902.3070 [hep-th]`.

[178] J. de Blas, A. Falkowski, M. Perez-Victoria, and S. Pokorski, "Tools for deconstructing gauge theories in AdS(5)," *JHEP* **08** (2006) 061, `arXiv:hep-th/0605150 [hep-th]`.

[179] K. S. Babu, "TASI Lectures on Flavor Physics," in *Proceedings of Theoretical Advanced Study Institute in Elementary Particle Physics on The dawn of the LHC era (TASI 2008): Boulder, USA, June 2-27, 2008*, pp. 49–123. 2010. `arXiv:0910.2948 [hep-ph]`.

[180] B. Grinstein, "TASI-2013 Lectures on Flavor Physics," in *Theoretical Advanced Study Institute in Elementary Particle Physics: Particle Physics: The Higgs Boson and Beyond (TASI 2013) Boulder, Colorado, June 3-28, 2013*. 2015. `arXiv:1501.05283 [hep-ph]`.

[181] Y. Grossman and P. Tanedo, "Just a Taste: Lectures on Flavor Physics," in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics : Anticipating the Next Discoveries in Particle Physics (TASI 2016): Boulder, CO, USA, June 6-July 1, 2016*, pp. 109–295. 2018. `arXiv:1711.03624 [hep-ph]`.

[182] N. Cabibbo, "Unitary Symmetry and Leptonic Decays," *Phys. Rev. Lett.* **10** (1963) 531–533. [,648(1963)].

[183] M. Kobayashi and T. Maskawa, "CP Violation in the Renormalizable Theory of Weak Interaction," *Prog. Theor. Phys.* **49** (1973) 652–657.

[184] C. Giunti and C. W. Kim, *Fundamentals of Neutrino Physics and Astrophysics*. 2007.

[185] B. Pontecorvo, "Mesonium and anti-mesonium," *Sov. Phys. JETP* **6** (1957) 429. [Zh. Eksp. Teor. Fiz.33,549(1957)].

[186] Z. Maki, M. Nakagawa, and S. Sakata, "Remarks on the unified model of elementary particles," *Prog. Theor. Phys.* **28** (1962) 870–880. [,34(1962)].

[187] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, "Evidence for the 2π Decay of the K_2^0 Meson," *Phys. Rev. Lett.* **13** (1964) 138–140.

[188] A. D. Sakharov, "Violation of CP Invariance, C asymmetry, and baryon asymmetry of the universe," *Pisma Zh. Eksp. Teor. Fiz.* **5** (1967) 32–35. [Usp. Fiz. Nauk161,no.5,61(1991)].

[189] M. Pospelov and A. Ritz, "Electric dipole moments as probes of new physics," *Annals Phys.* **318** (2005) 119–169, `arXiv:hep-ph/0504231 [hep-ph]`.

[190] C. Jarlskog, "Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Violation," *Phys. Rev. Lett.* **55** (1985) 1039.

[191] K. Kirch, "EDM searches," *AIP Conf. Proc.* **1560** (2013) 90–94.

[192] V. Anastassopoulos *et al.*, "A Storage Ring Experiment to Detect a Proton Electric Dipole Moment," *Rev. Sci. Instrum.* **87** no. 11, (2016) 115116, `arXiv:1502.04317 [physics.acc-ph]`.

[193] H. Fritzsch, "Calculating the Cabibbo Angle," *Phys. Lett.* **70B** (1977) 436–440.

[194] H. Fritzsch, "Weak Interaction Mixing in the Six - Quark Theory," *Phys. Lett.* **73B** (1978) 317–322.

[195] F. Björkeroth, L. Di Luzio, F. Mescia, and E. Nardi, "$U(1)$ flavour symmetries as Peccei-Quinn symmetries," `arXiv:1811.09637 [hep-ph]`.

[196] H. Georgi and C. Jarlskog, "A New Lepton - Quark Mass Relation in a Unified Theory," *Phys. Lett.* **86B** (1979) 297–300.

[197] L. Wolfenstein, "Parametrization of the Kobayashi-Maskawa Matrix," *Phys. Rev. Lett.* **51** (1983) 1945.

[198] F. Björkeroth, E. J. Chun, and S. F. King, "Accidental Peccei-Quinn symmetry from discrete flavour symmetry and Pati-Salam," *Phys. Lett.* **B777** (2018) 428–434, arXiv:1711.05741 [hep-ph].

[199] F. Wilczek, "Axions and Family Symmetry Breaking," *Phys. Rev. Lett.* **49** (1982) 1549–1552.

[200] C. Q. Geng and J. N. Ng, "Flavor Connections and Neutrino Mass Hierarchy Invariant Invisible Axion Models Without Domain Wall Problem," *Phys. Rev.* **D39** (1989) 1449.

[201] Y. H. Ahn, "Flavored Peccei-Quinn symmetry," *Phys. Rev.* **D91** (2015) 056005, arXiv:1410.1634 [hep-ph].

[202] A. Celis, J. Fuentes-Martin, and H. Serodio, "An invisible axion model with controlled FCNCs at tree level," *Phys. Lett.* **B741** (2015) 117–123, arXiv:1410.6217 [hep-ph].

[203] T. Nomura, Y. Shimizu, and T. Yamada, "$A_4 \times U(1)_{PQ}$ model for the lepton flavor structure and the strong CP problem," *JHEP* **06** (2016) 125, arXiv:1604.07650 [hep-ph].

[204] Y. Ema, K. Hamaguchi, T. Moroi, and K. Nakayama, "Flaxion: a minimal extension to solve puzzles in the standard model," *JHEP* **01** (2017) 096, arXiv:1612.05492 [hep-ph].

[205] L. Calibbi, F. Goertz, D. Redigolo, R. Ziegler, and J. Zupan, "Minimal axion model from flavor," *Phys. Rev.* **D95** no. 9, (2017) 095009, arXiv:1612.08040 [hep-ph].

[206] Y. Ema, D. Hagihara, K. Hamaguchi, T. Moroi, and K. Nakayama, "Supersymmetric Flaxion," *JHEP* **04** (2018) 094, arXiv:1802.07739 [hep-ph].

[207] M. Linster and R. Ziegler, "A Realistic $U(2)$ Model of Flavor," *JHEP* **08** (2018) 058, arXiv:1805.07341 [hep-ph].

[208] F. Arias-Aragon and L. Merlo, "The Minimal Flavour Violating Axion," *JHEP* **10** (2017) 168, arXiv:1709.07039 [hep-ph].

[209] K. S. Babu and S. M. Barr, "Family symmetry, gravity, and the strong CP problem," *Phys. Lett.* **B300** (1993) 367–372, arXiv:hep-ph/9212219 [hep-ph].

[210] D. Suematsu, "An extension of the SM based on effective Peccei-Quinn Symmetry," *Eur. Phys. J.* **C78** no. 11, (2018) 881, arXiv:1809.06563 [hep-ph].

[211] Q. Bonnefoy and E. Dudas, "Axions and anomalous U(1)'s," *Int. J. Mod. Phys.* **A33** no. 34, (2018) 1845001, arXiv:1809.08256 [hep-ph].

[212] Z. G. Berezhiani and M. Yu. Khlopov, "Cosmology of Spontaneously Broken Gauge Family Symmetry," *Z. Phys.* **C49** (1991) 73–78.

[213] M. E. Albrecht, T. Feldmann, and T. Mannel, "Goldstone Bosons in Effective Theories with Spontaneously Broken Flavour Symmetry," *JHEP* **10** (2010) 089, `arXiv:1002.4798 [hep-ph]`.

[214] C. Cheung, "Axion Protection from Flavor," *JHEP* **06** (2010) 074, `arXiv:1003.0941 [hep-ph]`.

[215] Y. H. Ahn, "Fermion masses and flavor mixings and strong CP problem," *Nucl. Phys.* **B939** (2019) 534–565, `arXiv:1802.05044 [hep-ph]`.

[216] Y. H. Ahn, "Compact model for Quarks and Leptons via flavored-Axions," *Phys. Rev.* **D98** no. 3, (2018) 035047, `arXiv:1804.06988 [hep-ph]`.

[217] S. Weinberg, "Electromagnetic and weak masses," *Phys. Rev. Lett.* **29** (1972) 388–392.

[218] C. D. Froggatt and H. B. Nielsen, "Hierarchy of Quark Masses, Cabibbo Angles and CP Violation," *Nucl. Phys.* **B147** (1979) 277–298.

[219] S. Dimopoulos, "Natural Generation of Fermion Masses," *Phys. Lett.* **129B** (1983) 417–428.

[220] M. Leurer, Y. Nir, and N. Seiberg, "Mass matrix models," *Nucl. Phys.* **B398** (1993) 319–342, `arXiv:hep-ph/9212278 [hep-ph]`.

[221] K. Choi, S. H. Im, C. B. Park, and S. Yun, "Minimal Flavor Violation with Axion-like Particles," *JHEP* **11** (2017) 070, `arXiv:1708.00021 [hep-ph]`.

[222] F. Björkeroth, E. J. Chun, and S. F. King, "Flavourful Axion Phenomenology," *JHEP* **08** (2018) 117, `arXiv:1806.00660 [hep-ph]`.

[223] M. B. Gavela, R. Houtz, P. Quilez, R. Del Rey, and O. Sumensari, "Flavor-changing constraints on electroweak ALP couplings," `arXiv:1901.02031 [hep-ph]`.

[224] J. Jaeckel, "A Family of WISPy Dark Matter Candidates," *Phys. Lett.* **B732** (2014) 1–7, `arXiv:1311.0880 [hep-ph]`.

[225] L. E. Ibanez and G. G. Ross, "Fermion masses and mixing angles from gauge symmetries," *Phys. Lett.* **B332** (1994) 100–110, `arXiv:hep-ph/9403338 [hep-ph]`.

[226] V. Jain and R. Shrock, "Models of fermion mass matrices based on a flavor dependent and generation dependent U(1) gauge symmetry," *Phys. Lett.* **B352** (1995) 83–91, `arXiv:hep-ph/9412367 [hep-ph]`.

[227] P. Binetruy and P. Ramond, "Yukawa textures and anomalies," *Phys. Lett.* **B350** (1995) 49–57, `arXiv:hep-ph/9412385 [hep-ph]`.

[228] E. Dudas, S. Pokorski, and C. A. Savoy, "Yukawa matrices from a spontaneously broken Abelian symmetry," *Phys. Lett.* **B356** (1995) 45–55, `arXiv:hep-ph/9504292 [hep-ph]`.

[229] M. B. Green and J. H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory," *Phys. Lett.* **149B** (1984) 117–122.

[230] P. Anastasopoulos, M. Bianchi, E. Dudas, and E. Kiritsis, "Anomalies, anomalous U(1)'s and generalized Chern-Simons terms," *JHEP* **11** (2006) 057, `arXiv:hep-th/0605225 [hep-th]`.

[231] L. E. Ibanez, "Computing the weak mixing angle from anomaly cancellation," *Phys. Lett.* **B303** (1993) 55–62, `arXiv:hep-ph/9205234 [hep-ph]`.

[232] L. Calibbi, Z. Lalak, S. Pokorski, and R. Ziegler, "The Messenger Sector of SUSY Flavour Models and Radiative Breaking of Flavour Universality," *JHEP* **06** (2012) 018, `arXiv:1203.1489 [hep-ph]`.

[233] G. F. Giudice and A. Masiero, "A Natural Solution to the mu Problem in Supergravity Theories," *Phys. Lett.* **B206** (1988) 480–484.

[234] S. R. Coleman and J. Mandula, "All Possible Symmetries of the S Matrix," *Phys. Rev.* **159** (1967) 1251–1256.

[235] R. Haag, J. T. Lopuszanski, and M. Sohnius, "All Possible Generators of Supersymmetries of the s Matrix," *Nucl. Phys.* **B88** (1975) 257. [,257(1974)].

[236] P. Fayet, "Supersymmetry and Weak, Electromagnetic and Strong Interactions," *Phys. Lett.* **64B** (1976) 159.

[237] J. Wess and B. Zumino, "A Lagrangian Model Invariant Under Supergauge Transformations," *Phys. Lett.* **49B** (1974) 52.

[238] S. Dimopoulos and H. Georgi, "Softly Broken Supersymmetry and SU(5)," *Nucl. Phys.* **B193** (1981) 150–162.

[239] J. R. Ellis, J. S. Hagelin, D. V. Nanopoulos, K. A. Olive, and M. Srednicki, "Supersymmetric Relics from the Big Bang," *Nucl. Phys.* **B238** (1984) 453–476. [,223(1983)].

[240] S. Dimopoulos, S. Raby, and F. Wilczek, "Supersymmetry and the Scale of Unification," *Phys. Rev.* **D24** (1981) 1681–1683.

[241] L. E. Ibanez and G. G. Ross, "Low-Energy Predictions in Supersymmetric Grand Unified Theories," *Phys. Lett.* **105B** (1981) 439–442.

[242] A. Neveu and J. H. Schwarz, "Tachyon-free dual model with a positive-intercept trajectory," *Phys. Lett.* **34B** (1971) 517–518.

[243] P. Ramond, "Dual Theory for Free Fermions," *Phys. Rev.* **D3** (1971) 2415–2418.

[244] P. Nath and R. L. Arnowitt, "Generalized Supergauge Symmetry as a New Framework for Unified Gauge Theories," *Phys. Lett.* **56B** (1975) 177–180.

[245] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, "Progress Toward a Theory of Supergravity," *Phys. Rev.* **D13** (1976) 3214–3218.

[246] S. Deser, J. H. Kay, and K. S. Stelle, "Renormalizability Properties of Supergravity," *Phys. Rev. Lett.* **38** (1977) 527, `arXiv:1506.03757 [hep-th]`.

[247] M. T. Grisaru, "Two Loop Renormalizability of Supergravity," *Phys. Lett.* **66B** (1977) 75–76.

[248] E. Tomboulis, "On the Two Loop Divergences of Supersymmetric Gravitation," *Phys. Lett.* **67B** (1977) 417–420.

[249] G. Chalmers, "On the finiteness of N=8 quantum supergravity," `arXiv:hep-th/0008162 [hep-th]`.

[250] I. Affleck, M. Dine, and N. Seiberg, "Dynamical Supersymmetry Breaking in Supersymmetric QCD," *Nucl. Phys.* **B241** (1984) 493–534.

[251] N. Seiberg, "Exact results on the space of vacua of four-dimensional SUSY gauge theories," *Phys. Rev.* **D49** (1994) 6857–6863, `arXiv:hep-th/9402044 [hep-th]`.

[252] K. A. Intriligator, N. Seiberg, and D. Shih, "Dynamical SUSY breaking in meta-stable vacua," *JHEP* **04** (2006) 021, `arXiv:hep-th/0602239 [hep-th]`.

[253] F. Benini, "Localization in supersymmetric field theories," *YITP Kyoto School on Strings and Fields* (2016) .

[254] A. A. Tseytlin, "Harmonic superpositions of M-branes," *Nucl. Phys.* **B475** (1996) 149–163, `arXiv:hep-th/9604035 [hep-th]`. [,286(1996)].

[255] D. V. Volkov and V. P. Akulov, "Is the Neutrino a Goldstone Particle?," *Phys. Lett.* **46B** (1973) 109–110.

[256] P. Fayet, "Spontaneously Broken Supersymmetric Theories of Weak, Electromagnetic and Strong Interactions," *Phys. Lett.* **69B** (1977) 489.

[257] J. D. Wells, "Implications of supersymmetry breaking with a little hierarchy between gauginos and scalars," in *11th International Conference on Supersymmetry and the Unification of Fundamental Interactions (SUSY 2003) Tucson, Arizona, June 5-10, 2003*. 2003. `arXiv:hep-ph/0306127 [hep-ph]`.

[258] N. Arkani-Hamed and S. Dimopoulos, "Supersymmetric unification without low energy supersymmetry and signatures for fine-tuning at the LHC," *JHEP* **06** (2005) 073, `arXiv:hep-th/0405159 [hep-th]`.

[259] G. F. Giudice and A. Romanino, "Split supersymmetry," *Nucl. Phys.* **B699** (2004) 65–89, `arXiv:hep-ph/0406088 [hep-ph]`. [Erratum: Nucl. Phys.B706,487(2005)].

[260] J. Wess and J. Bagger, *Supersymmetry and supergravity.* Princeton University Press, Princeton, NJ, USA, 1992.

[261] S. P. Martin, "A Supersymmetry primer," `arXiv:hep-ph/9709356 [hep-ph]`. [Adv. Ser. Direct. High Energy Phys.18,1(1998)].

[262] A. Salam and J. A. Strathdee, "Supergauge Transformations," *Nucl. Phys.* **B76** (1974) 477–482.

[263] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, "Superspace Or One Thousand and One Lessons in Supersymmetry," *Front. Phys.* **58** (1983) 1–548, `arXiv:hep-th/0108200 [hep-th]`.

[264] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky, and E. S. Sokatchev, *Harmonic superspace.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007. `http://www.cambridge.org/mw/academic/subjects/physics/theoretical-physics-and-mathematical-physics/harmonic-superspace?format=PB`.

[265] B. Zumino, "Supersymmetry and Kahler Manifolds," *Phys. Lett.* **87B** (1979) 203.

[266] L. Alvarez-Gaume and D. Z. Freedman, "Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma Model," *Commun. Math. Phys.* **80** (1981) 443.

[267] J. Bagger and E. Witten, "The Gauge Invariant Supersymmetric Nonlinear Sigma Model," *Phys. Lett.* **118B** (1982) 103–106.

[268] P. Fayet and J. Iliopoulos, "Spontaneously Broken Supergauge Symmetries and Goldstone Spinors," *Phys. Lett.* **51B** (1974) 461–464.

[269] J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo, and P. K. Townsend, "Topological Extensions of the Supersymmetry Algebra for Extended Objects," *Phys. Rev. Lett.* **63** (1989) 2443.

[270] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int. J. Theor. Phys.* **38** (1999) 1113–1133, `arXiv:hep-th/9711200 [hep-th]`. [Adv. Theor. Math. Phys.2,231(1998)].

[271] E. Cremmer, B. Julia, and J. Scherk, "Supergravity Theory in Eleven-Dimensions," *Phys. Lett.* **B76** (1978) 409–412. [,25(1978)].

[272] I. Antoniadis, J.-P. Derendinger, and J.-C. Jacot, "N=2 supersymmetry breaking at two different scales," *Nucl. Phys.* **B863** (2012) 471–494, `arXiv:1204.2141 [hep-th]`.

[273] I. Antoniadis, J.-P. Derendinger, and C. Markou, "Nonlinear $\mathcal{N} = 2$ global supersymmetry," *JHEP* **06** (2017) 052, arXiv:1703.08806 [hep-th].

[274] W. Siegel, "Gauge Spinor Superfield as a Scalar Multiplet," *Phys. Lett.* **85B** (1979) 333–334.

[275] K. A. Intriligator and N. Seiberg, "Lectures on supersymmetric gauge theories and electric-magnetic duality," *Nucl. Phys. Proc. Suppl.* **45BC** (1996) 1–28, arXiv:hep-th/9509066 [hep-th]. [,157(1995)].

[276] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," *Nucl. Phys.* **B426** (1994) 19–52, arXiv:hep-th/9407087 [hep-th]. [Erratum: Nucl. Phys.B430,485(1994)].

[277] L. O'Raifeartaigh, "Spontaneous Symmetry Breaking for Chiral Scalar Superfields," *Nucl. Phys.* **B96** (1975) 331–352.

[278] A. E. Nelson and N. Seiberg, "R symmetry breaking versus supersymmetry breaking," *Nucl. Phys.* **B416** (1994) 46–62, arXiv:hep-ph/9309299 [hep-ph].

[279] E. Dudas, A. Falkowski, and S. Pokorski, "Deconstructed U(1) and supersymmetry breaking," *Phys. Lett.* **B568** (2003) 281–290, arXiv:hep-th/0303155 [hep-th].

[280] M. Reece, "Photon Masses in the Landscape and the Swampland," arXiv:1808.09966 [hep-th].

[281] A. Arvanitaki, N. Craig, S. Dimopoulos, and G. Villadoro, "Mini-Split," *JHEP* **02** (2013) 126, arXiv:1210.0555 [hep-ph].

[282] N. Arkani-Hamed, A. Gupta, D. E. Kaplan, N. Weiner, and T. Zorawski, "Simply Unnatural Supersymmetry," arXiv:1212.6971 [hep-ph].

[283] E. A. Mirabelli and M. E. Peskin, "Transmission of supersymmetry breaking from a four-dimensional boundary," *Phys. Rev.* **D58** (1998) 065002, arXiv:hep-th/9712214 [hep-th].

[284] N. Arkani-Hamed, T. Gregoire, and J. G. Wacker, "Higher dimensional supersymmetry in 4-D superspace," *JHEP* **03** (2002) 055, arXiv:hep-th/0101233 [hep-th].

[285] I. Antoniadis, E. Dudas, D. M. Ghilencea, and P. Tziveloglou, "Non-linear MSSM," *Nucl. Phys.* **B841** (2010) 157–177, arXiv:1006.1662 [hep-ph].

[286] I. Antoniadis, E. Dudas, D. M. Ghilencea, and P. Tziveloglou, "Nonlinear supersymmetry and goldstino couplings to the MSSM," *Theor. Math. Phys.* **170** (2012) 26–38. [Teor. Mat. Fiz.170,34(2012)].

[287] B. Bellazzini, A. Mariotti, D. Redigolo, F. Sala, and J. Serra, "*R*-axion at colliders," *Phys. Rev. Lett.* **119** no. 14, (2017) 141804, arXiv:1702.02152 [hep-ph].

[288] I. Antoniadis, E. Dudas, S. Ferrara, and A. Sagnotti, "The Volkov-Akulov-Starobinsky supergravity," *Phys. Lett.* **B733** (2014) 32–35, arXiv:1403.3269 [hep-th].

[289] S. Ferrara, R. Kallosh, and A. Linde, "Cosmology with Nilpotent Superfields," *JHEP* **10** (2014) 143, arXiv:1408.4096 [hep-th].

[290] R. Kallosh, F. Quevedo, and A. M. Uranga, "String Theory Realizations of the Nilpotent Goldstino," *JHEP* **12** (2015) 039, arXiv:1507.07556 [hep-th].

[291] S. R. Coleman, J. Wess, and B. Zumino, "Structure of phenomenological Lagrangians. 1.," *Phys. Rev.* **177** (1969) 2239–2247.

[292] C. G. Callan, Jr., S. R. Coleman, J. Wess, and B. Zumino, "Structure of phenomenological Lagrangians. 2.," *Phys. Rev.* **177** (1969) 2247–2250.

[293] D. V. Volkov, "Phenomenological Lagrangians," *Fiz. Elem. Chast. Atom. Yadra* **4** (1973) 3–41.

[294] E. A. Ivanov and V. I. Ogievetsky, "The Inverse Higgs Phenomenon in Nonlinear Realizations," *Teor. Mat. Fiz.* **25** (1975) 164–177.

[295] I. Low and A. V. Manohar, "Spontaneously broken space-time symmetries and Goldstone's theorem," *Phys. Rev. Lett.* **88** (2002) 101602, arXiv:hep-th/0110285 [hep-th].

[296] A. I. Pashnev, "Nonlinear realisation for symmetry group with spinor parameters," *Teor. Mat. Fiz.* **20** (1974) 141–144.

[297] E. A. Ivanov and A. A. Kapustnikov, "General Relationship Between Linear and Nonlinear Realizations of Supersymmetry," *J. Phys.* **A11** (1978) 2375–2384.

[298] T. E. Clark and S. T. Love, "Goldstino couplings to matter," *Phys. Rev.* **D54** (1996) 5723–5727, arXiv:hep-ph/9608243 [hep-ph].

[299] T. E. Clark and S. T. Love, "The Akulov-Volkov Lagrangian, symmetry currents and spontaneously broken extended supersymmetry," *Phys. Rev.* **D63** (2001) 065012, arXiv:hep-th/0007225 [hep-th].

[300] A. A. Kapustnikov, "NONLINEAR REALIZATION OF EINSTEINIAN SUPERGRAVITY," *Theor. Math. Phys.* **47** (1981) 406–413. [Teor. Mat. Fiz.47,198(1981)].

[301] L. V. Delacretaz, V. Gorbenko, and L. Senatore, "The Supersymmetric Effective Field Theory of Inflation," *JHEP* **03** (2017) 063, arXiv:1610.04227 [hep-th].

[302] E. Dudas and J. Mourad, "Consistent gravitino couplings in nonsupersymmetric strings," *Phys. Lett.* **B514** (2001) 173–182, arXiv:hep-th/0012071 [hep-th].

[303] S. Samuel and J. Wess, "A Superfield Formulation of the Nonlinear Realization of Supersymmetry and Its Coupling to Supergravity," *Nucl. Phys.* **B221** (1983) 153–177.

[304] M. Rocek, "Linearizing the Volkov-Akulov Model," *Phys. Rev. Lett.* **41** (1978) 451–453.

[305] U. Lindstrom and M. Rocek, "CONSTRAINED LOCAL SUPERFIELDS," *Phys. Rev.* **D19** (1979) 2300–2303.

[306] R. Casalbuoni, S. De Curtis, D. Dominici, F. Feruglio, and R. Gatto, "Nonlinear Realization of Supersymmetry Algebra From Supersymmetric Constraint," *Phys. Lett.* **B220** (1989) 569–575.

[307] Z. Komargodski and N. Seiberg, "From Linear SUSY to Constrained Superfields," *JHEP* **09** (2009) 066, `arXiv:0907.2441 [hep-th]`.

[308] S. M. Kuzenko and S. J. Tyler, "Complex linear superfield as a model for Goldstino," *JHEP* **04** (2011) 057, `arXiv:1102.3042 [hep-th]`.

[309] S. M. Kuzenko and S. J. Tyler, "On the Goldstino actions and their symmetries," *JHEP* **05** (2011) 055, `arXiv:1102.3043 [hep-th]`.

[310] G. Dall'Agata, E. Dudas, and F. Farakos, "On the origin of constrained superfields," *JHEP* **05** (2016) 041, `arXiv:1603.03416 [hep-th]`.

[311] G. Dall'Agata and F. Farakos, "Constrained superfields in Supergravity," *JHEP* **02** (2016) 101, `arXiv:1512.02158 [hep-th]`.

[312] J. Bagger and A. Galperin, "A New Goldstone multiplet for partially broken supersymmetry," *Phys. Rev.* **D55** (1997) 1091–1098, `arXiv:hep-th/9608177 [hep-th]`.

[313] J. Bagger and A. Galperin, "The Tensor Goldstone multiplet for partially broken supersymmetry," *Phys. Lett.* **B412** (1997) 296–300, `arXiv:hep-th/9707061 [hep-th]`.

[314] J. Hughes and J. Polchinski, "Partially Broken Global Supersymmetry and the Superstring," *Nucl. Phys.* **B278** (1986) 147–169.

[315] J. Hughes, J. Liu, and J. Polchinski, "Supermembranes," *Phys. Lett.* **B180** (1986) 370–374.

[316] E. Witten, "Dynamical Breaking of Supersymmetry," *Nucl. Phys.* **B188** (1981) 513.

[317] I. Antoniadis, H. Partouche, and T. R. Taylor, "Spontaneous breaking of N=2 global supersymmetry," *Phys. Lett.* **B372** (1996) 83–87, `arXiv:hep-th/9512006 [hep-th]`.

[318] S. Ferrara, L. Girardello, and M. Porrati, "Minimal Higgs branch for the breaking of half of the supersymmetries in N=2 supergravity," *Phys. Lett.* **B366** (1996) 155–159, `arXiv:hep-th/9510074 [hep-th]`.

[319] S. Ferrara, L. Girardello, and M. Porrati, "Spontaneous breaking of N=2 to N=1 in rigid and local supersymmetric theories," *Phys. Lett.* **B376** (1996) 275–281, `arXiv:hep-th/9512180 [hep-th]`.

[320] A. A. Tseytlin, "Born-Infeld action, supersymmetry and string theory," `arXiv:hep-th/9908105 [hep-th]`.

[321] E. Dudas, S. Ferrara, and A. Sagnotti, "A superfield constraint for $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$ breaking," *JHEP* **08** (2017) 109, `arXiv:1707.03414 [hep-th]`.

[322] N. Cribiori, G. Dall'Agata, and F. Farakos, "Interactions of N Goldstini in Superspace," *Phys. Rev.* **D94** no. 6, (2016) 065019, `arXiv:1607.01277 [hep-th]`.

[323] S. Abel, Q. Bonnefoy, and D. Chowdhury, "BPS solutions for generalised Wess-Zumino models and their applications," `arXiv:1906.02201 [hep-th]`.

[324] E. B. Bogomolny, "Stability of Classical Solutions," *Sov. J. Nucl. Phys.* **24** (1976) 449. [Yad. Fiz.24,861(1976)].

[325] M. K. Prasad and C. M. Sommerfield, "An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon," *Phys. Rev. Lett.* **35** (1975) 760–762.

[326] D. Tong, "TASI lectures on solitons: Instantons, monopoles, vortices and kinks," in *Theoretical Advanced Study Institute in Elementary Particle Physics: Many Dimensions of String Theory (TASI 2005) Boulder, Colorado, June 5-July 1, 2005*. 2005. `arXiv:hep-th/0509216 [hep-th]`.

[327] P. Fendley, S. D. Mathur, C. Vafa, and N. Warner, "Integrable Deformations and Scattering Matrices for the $N = 2$ Supersymmetric Discrete Series," *Phys. Lett.* **B243** (1990) 257–264.

[328] S. Cecotti and C. Vafa, "On classification of N=2 supersymmetric theories," *Commun. Math. Phys.* **158** (1993) 569–644, `arXiv:hep-th/9211097 [hep-th]`.

[329] B. Chibisov and M. A. Shifman, "BPS saturated walls in supersymmetric theories," *Phys. Rev.* **D56** (1997) 7990–8013, `arXiv:hep-th/9706141 [hep-th]`. [Erratum: Phys. Rev.D58,109901(1998)].

[330] M. V. Libanov, V. A. Rubakov, and S. V. Troitsky, "Tree amplitudes at multiparticle threshold in a model with softly broken 0(2) symmetry," *Nucl. Phys.* **B412** (1994) 607–620, `arXiv:hep-ph/9307213 [hep-ph]`.

[331] M. J. Strassler, "The Duality cascade," in *Progress in string theory. Proceedings, Summer School, TASI 2003, Boulder, USA, June 2-27, 2003*, pp. 419–510. 2005. `arXiv:hep-th/0505153 [hep-th]`.

[332] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," *Nucl. Phys.* **B435** (1995) 129–146, `arXiv:hep-th/9411149 [hep-th]`.

[333] M. B. Voloshin, "Multiparticle amplitudes at zero energy and momentum in scalar theory," *Nucl. Phys.* **B383** (1992) 233–248.

[334] E. N. Argyres, R. H. P. Kleiss, and C. G. Papadopoulos, "Amplitude estimates for multi - Higgs production at high-energies," *Nucl. Phys.* **B391** (1993) 42–56.

[335] M. B. Voloshin, "Estimate of the onset of nonperturbative particle production at high-energy in a scalar theory," *Phys. Lett.* **B293** (1992) 389–394.

[336] L. S. Brown, "Summing tree graphs at threshold," *Phys. Rev.* **D46** (1992) R4125–R4127, arXiv:hep-ph/9209203 [hep-ph].

[337] M. V. Libanov, V. A. Rubakov, D. T. Son, and S. V. Troitsky, "Exponentiation of multiparticle amplitudes in scalar theories," *Phys. Rev.* **D50** (1994) 7553–7569, arXiv:hep-ph/9407381 [hep-ph].

[338] D. T. Son, "Semiclassical approach for multiparticle production in scalar theories," *Nucl. Phys.* **B477** (1996) 378–406, arXiv:hep-ph/9505338 [hep-ph].

[339] M. V. Libanov, V. A. Rubakov, and S. V. Troitsky, "Multiparticle processes and semiclassical analysis in bosonic field theories," *Phys. Part. Nucl.* **28** (1997) 217–240.

[340] V. V. Khoze and J. Reiness, "Review of the semiclassical formalism for multiparticle production at high energies," arXiv:1810.01722 [hep-ph].

[341] V. V. Khoze, "Multiparticle Higgs and Vector Boson Amplitudes at Threshold," *JHEP* **07** (2014) 008, arXiv:1404.4876 [hep-ph].

[342] V. V. Khoze and M. Spannowsky, "Higgsplosion: Solving the Hierarchy Problem via rapid decays of heavy states into multiple Higgs bosons," *Nucl. Phys.* **B926** (2018) 95–111, arXiv:1704.03447 [hep-ph].

[343] V. V. Khoze, "Multiparticle production in the large λn limit: realising Higgsplosion in a scalar QFT," *JHEP* **06** (2017) 148, arXiv:1705.04365 [hep-ph].

[344] V. V. Khoze, "Semiclassical computation of quantum effects in multiparticle production at large lambda n," arXiv:1806.05648 [hep-ph].

[345] D. Bazeia, J. Menezes, and M. M. Santos, "Complete factorization of equations of motion in supersymmetric field theories," *Nucl. Phys.* **B636** (2002) 132–140, arXiv:hep-th/0103041 [hep-th].

[346] G. R. Dvali and M. A. Shifman, "Dynamical compactification as a mechanism of spontaneous supersymmetry breaking," *Nucl. Phys.* **B504** (1997) 127–146, arXiv:hep-th/9611213 [hep-th].

[347] A. Kovner, M. A. Shifman, and A. V. Smilga, "Domain walls in supersymmetric Yang-Mills theories," *Phys. Rev.* **D56** (1997) 7978–7989, `arXiv:hep-th/9706089 [hep-th]`.

[348] M. B. Voloshin, "Summing one loop graphs at multiparticle threshold," *Phys. Rev.* **D47** (1993) R357–R361, `arXiv:hep-ph/9209240 [hep-ph]`.

[349] A. S. Gorsky and M. B. Voloshin, "Nonperturbative production of multiboson states and quantum bubbles," *Phys. Rev.* **D48** (1993) 3843–3851, `arXiv:hep-ph/9305219 [hep-ph]`.

[350] M. V. Libanov, D. T. Son, and S. V. Troitsky, "Exponentiation of multiparticle amplitudes in scalar theories. 2. Universality of the exponent," *Phys. Rev.* **D52** (1995) 3679–3687, `arXiv:hep-ph/9503412 [hep-ph]`.

[351] L. Girardello and M. T. Grisaru, "Soft Breaking of Supersymmetry," *Nucl. Phys.* **B194** (1982) 65.

[352] E. Witten, "Constraints on Supersymmetry Breaking," *Nucl. Phys.* **B202** (1982) 253.

[353] G. Veneziano, "Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories," *Nuovo Cim.* **A57** (1968) 190–197.

[354] M. A. Virasoro, "Alternative constructions of crossing-symmetric amplitudes with regge behavior," *Phys. Rev.* **177** (1969) 2309–2311.

[355] J. A. Shapiro, "Narrow-resonance model with regge behavior for pi pi scattering," *Phys. Rev.* **179** (1969) 1345–1353.

[356] J. Scherk and J. H. Schwarz, "Dual Models for Nonhadrons," *Nucl. Phys.* **B81** (1974) 118–144.

[357] T. Yoneya, "Connection of Dual Models to Electrodynamics and Gravidynamics," *Prog. Theor. Phys.* **51** (1974) 1907–1920.

[358] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys.* **B443** (1995) 85–126, `arXiv:hep-th/9503124 [hep-th]`. [,333(1995)].

[359] M. Kaku and K. Kikkawa, "The Field Theory of Relativistic Strings. I. Trees," *Phys. Rev.* **D10** (1974) 1110.

[360] M. B. Green and J. H. Schwarz, "Superstring Field Theory," *Nucl. Phys.* **B243** (1984) 475–536.

[361] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," *Phys. Lett.* **B379** (1996) 99–104, `arXiv:hep-th/9601029 [hep-th]`.

[362] C. Lovelace, "Pomeron form-factors and dual Regge cuts," *Phys. Lett.* **34B** (1971) 500–506.

[363] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges," *Phys. Rev. Lett.* **75** (1995) 4724–4727, `arXiv:hep-th/9510017 [hep-th]`.

[364] C. Vafa, "The String landscape and the swampland," `arXiv:hep-th/0509212 [hep-th]`.

[365] M. R. Douglas, "The Statistics of string / M theory vacua," *JHEP* **05** (2003) 046, `arXiv:hep-th/0303194 [hep-th]`.

[366] M. B. Green, J. H. Schwarz, and E. Witten, *SUPERSTRING THEORY. VOL. 1: INTRODUCTION.* Cambridge Monographs on Mathematical Physics. 1988. `http://www.cambridge.org/us/academic/subjects/physics/theoretical-physics-and-mathematical-physics/superstring-theory-volume-1`.

[367] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.

[368] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.

[369] C. Angelantonj and A. Sagnotti, "Open strings," *Phys. Rept.* **371** (2002) 1–150, `arXiv:hep-th/0204089 [hep-th]`. [Erratum: Phys. Rept.376,no.6,407(2003)].

[370] Y. Nambu, "Duality and hydrodynamics," *Lectures at the Copenhagen conference* (1970) .

[371] T. Goto, "Relativistic quantum mechanics of one-dimensional mechanical continuum and subsidiary condition of dual resonance model," *Prog. Theor. Phys.* **46** (1971) 1560–1569.

[372] S. Deser and B. Zumino, "A Complete Action for the Spinning String," *Phys. Lett.* **B65** (1976) 369–373. [,369(1976)].

[373] L. Brink, P. Di Vecchia, and P. S. Howe, "A Locally Supersymmetric and Reparametrization Invariant Action for the Spinning String," *Phys. Lett.* **65B** (1976) 471–474.

[374] A. M. Polyakov, "Quantum Geometry of Bosonic Strings," *Phys. Lett.* **B103** (1981) 207–210. [,598(1981)].

[375] E. Kiritsis, *Introduction to superstring theory*, vol. B9 of *Leuven notes in mathematical and theoretical physics.* Leuven U. Press, Leuven, 1998. `arXiv:hep-th/9709062 [hep-th]`.

[376] R. Blumenhagen, D. Lust, and S. Theisen, *Basic concepts of string theory.* Theoretical and Mathematical Physics. Springer, Heidelberg, Germany, 2013. `http://www.springer.com/physics/theoretical%2C+mathematical+%26+computational+physics/book/978-3-642-29496-9`.

[377] J. E. Paton and H.-M. Chan, "Generalized veneziano model with isospin," *Nucl. Phys.* **B10** (1969) 516–520.

[378] F. Gliozzi, J. Scherk, and D. I. Olive, "Supersymmetry, Supergravity Theories and the Dual Spinor Model," *Nucl. Phys.* **B122** (1977) 253–290.

[379] A. Sagnotti, "Open Strings and their Symmetry Groups," in *NATO Advanced Summer Institute on Nonperturbative Quantum Field Theory (Cargese Summer Institute) Cargese, France, July 16-30, 1987*, pp. 0521–528. 1987. `arXiv:hep-th/0208020 [hep-th]`.

[380] A. H. Chamseddine, "N=4 Supergravity Coupled to N=4 Matter," *Nucl. Phys.* **B185** (1981) 403.

[381] E. Bergshoeff, M. de Roo, B. de Wit, and P. van Nieuwenhuizen, "Ten-Dimensional Maxwell-Einstein Supergravity, Its Currents, and the Issue of Its Auxiliary Fields," *Nucl. Phys.* **B195** (1982) 97–136.

[382] G. F. Chapline and N. S. Manton, "Unification of Yang-Mills Theory and Supergravity in Ten-Dimensions," *Phys. Lett.* **B120** (1983) 105–109. [,105(1982)].

[383] J. A. Shapiro, "Loop graph in the dual tube model," *Phys. Rev.* **D5** (1972) 1945–1948.

[384] D. Friedan, E. J. Martinec, and S. H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," *Nucl. Phys.* **B271** (1986) 93–165.

[385] A. Bilal, "Lectures on Anomalies," `arXiv:0802.0634 [hep-th]`.

[386] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "The Heterotic String," *Phys. Rev. Lett.* **54** (1985) 502–505.

[387] J. Polchinski and Y. Cai, "Consistency of Open Superstring Theories," *Nucl. Phys.* **B296** (1988) 91–128.

[388] M. B. Green and J. H. Schwarz, "Infinity Cancellations in SO(32) Superstring Theory," *Phys. Lett.* **151B** (1985) 21–25.

[389] M. R. Douglas and B. Grinstein, "Dilaton Tadpole for the Open Bosonic String," *Phys. Lett.* **B183** (1987) 52. [Erratum: Phys. Lett.B187,442(1987)].

[390] N. Marcus and A. Sagnotti, "Group Theory from Quarks at the Ends of Strings," *Phys. Lett.* **B188** (1987) 58–64.

[391] S. Weinberg, "Cancellation of One Loop Divergences in SO(8192) String Theory," *Phys. Lett.* **B187** (1987) 278–282.

[392] C. Bachas, "A Way to break supersymmetry," `arXiv:hep-th/9503030 [hep-th]`.

[393] J. P. Derendinger, L. E. Ibanez, and H. P. Nilles, "On the Low-Energy d = 4, N=1 Supergravity Theory Extracted from the d = 10, N=1 Superstring," *Phys. Lett.* **155B** (1985) 65–70.

[394] M. Dine, R. Rohm, N. Seiberg, and E. Witten, "Gluino Condensation in Superstring Models," *Phys. Lett.* **156B** (1985) 55–60.

[395] S. Sugimoto, "Anomaly cancellations in type I D-9 - anti-D-9 system and the USp(32) string theory," *Prog. Theor. Phys.* **102** (1999) 685–699, `arXiv:hep-th/9905159 [hep-th]`.

[396] I. Antoniadis, E. Dudas, and A. Sagnotti, "Brane supersymmetry breaking," *Phys. Lett.* **B464** (1999) 38–45, `arXiv:hep-th/9908023 [hep-th]`.

[397] J. Mourad and A. Sagnotti, "An Update on Brane Supersymmetry Breaking," `arXiv:1711.11494 [hep-th]`.

[398] J. Scherk and J. H. Schwarz, "How to Get Masses from Extra Dimensions," *Nucl. Phys.* **B153** (1979) 61–88. [,79(1979)].

[399] R. Rohm, "Spontaneous Supersymmetry Breaking in Supersymmetric String Theories," *Nucl. Phys.* **B237** (1984) 553–572.

[400] C. Kounnas and M. Porrati, "Spontaneous Supersymmetry Breaking in String Theory," *Nucl. Phys.* **B310** (1988) 355–370.

[401] S. Ferrara, C. Kounnas, M. Porrati, and F. Zwirner, "Superstrings with Spontaneously Broken Supersymmetry and their Effective Theories," *Nucl. Phys.* **B318** (1989) 75–105.

[402] J. D. Blum and K. R. Dienes, "Duality without supersymmetry: The Case of the SO(16) x SO(16) string," *Phys. Lett.* **B414** (1997) 260–268, `arXiv:hep-th/9707148 [hep-th]`.

[403] J. D. Blum and K. R. Dienes, "Strong / weak coupling duality relations for nonsupersymmetric string theories," *Nucl. Phys.* **B516** (1998) 83–159, `arXiv:hep-th/9707160 [hep-th]`.

[404] J. Scherk and J. H. Schwarz, "Spontaneous Breaking of Supersymmetry Through Dimensional Reduction," *Phys. Lett.* **82B** (1979) 60–64.

[405] D. Marti and A. Pomarol, "Supersymmetric theories with compact extra dimensions in N=1 superfields," *Phys. Rev.* **D64** (2001) 105025, `arXiv:hep-th/0106256 [hep-th]`.

[406] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum Configurations for Superstrings," *Nucl. Phys.* **B258** (1985) 46–74.

[407] E. Witten, "Dimensional Reduction of Superstring Models," *Phys. Lett.* **155B** (1985) 151.

[408] E. Witten, "Some Properties of O(32) Superstrings," *Phys. Lett.* **149B** (1984) 351–356.

[409] K. Choi and J. E. Kim, "Harmful Axions in Superstring Models," *Phys. Lett.* **154B** (1985) 393. [Erratum: Phys. Lett.156B,452(1985)].

[410] K. Choi and J. E. Kim, "Compactification and Axions in E(8) x E(8)-prime Superstring Models," *Phys. Lett.* **165B** (1985) 71–75.

[411] J. E. Kim, "Light Pseudoscalars, Particle Physics and Cosmology," *Phys. Rept.* **150** (1987) 1–177.

[412] V. Balasubramanian, P. Berglund, J. P. Conlon, and F. Quevedo, "Systematics of moduli stabilisation in Calabi-Yau flux compactifications," *JHEP* **03** (2005) 007, `arXiv:hep-th/0502058 [hep-th]`.

[413] M. Cicoli, J. P. Conlon, and F. Quevedo, "General Analysis of LARGE Volume Scenarios with String Loop Moduli Stabilisation," *JHEP* **10** (2008) 105, `arXiv:0805.1029 [hep-th]`.

[414] J. P. Conlon, "The QCD axion and moduli stabilisation," *JHEP* **05** (2006) 078, `arXiv:hep-th/0602233 [hep-th]`.

[415] M. Cicoli, "Axion-like Particles from String Compactifications," in *Proceedings, 9th Patras Workshop on Axions, WIMPs and WISPs (AXION-WIMP 2013): Mainz, Germany, June 24-28, 2013*, pp. 235–242. 2013. `arXiv:1309.6988 [hep-th]`.

[416] S. B. Giddings, S. Kachru, and J. Polchinski, "Hierarchies from fluxes in string compactifications," *Phys. Rev.* **D66** (2002) 106006, `arXiv:hep-th/0105097 [hep-th]`.

[417] G. Villadoro and F. Zwirner, "D terms from D-branes, gauge invariance and moduli stabilization in flux compactifications," *JHEP* **03** (2006) 087, `arXiv:hep-th/0602120 [hep-th]`.

[418] S. Kachru, R. Kallosh, A. D. Linde, and S. P. Trivedi, "De Sitter vacua in string theory," *Phys. Rev.* **D68** (2003) 046005, `arXiv:hep-th/0301240 [hep-th]`.

[419] J. P. Derendinger, L. E. Ibanez, and H. P. Nilles, "On the Low-Energy Limit of Superstring Theories," *Nucl. Phys.* **B267** (1986) 365–414.

[420] W. Lerche, B. E. W. Nilsson, and A. N. Schellekens, "Heterotic String Loop Calculation of the Anomaly Cancelling Term," *Nucl. Phys.* **B289** (1987) 609.

[421] P. Binetruy and E. Dudas, "Gaugino condensation and the anomalous U(1)," *Phys. Lett.* **B389** (1996) 503–509, `arXiv:hep-th/9607172 [hep-th]`.

[422] J. Bagger, E. Poppitz, and L. Randall, "The R axion from dynamical supersymmetry breaking," *Nucl. Phys.* **B426** (1994) 3–18, `arXiv:hep-ph/9405345 [hep-ph]`.

[423] N. Arkani-Hamed, M. Dine, and S. P. Martin, "Dynamical supersymmetry breaking in models with a Green-Schwarz mechanism," *Phys. Lett.* **B431** (1998) 329–338, `arXiv:hep-ph/9803432 [hep-ph]`.

[424] I. Affleck, M. Dine, and N. Seiberg, "Calculable Nonperturbative Supersymmetry Breaking," *Phys. Rev. Lett.* **52** (1984) 1677.

[425] M. Cicoli, D. Klevers, S. Krippendorf, C. Mayrhofer, F. Quevedo, and R. Valandro, "Explicit de Sitter Flux Vacua for Global String Models with Chiral Matter," *JHEP* **05** (2014) 001, `arXiv:1312.0014 [hep-th]`.

[426] E. I. Buchbinder, A. Constantin, and A. Lukas, "Heterotic QCD axion," *Phys. Rev.* **D91** no. 4, (2015) 046010, `arXiv:1412.8696 [hep-th]`.

[427] E. Palti, "The Swampland: Introduction and Review," 2019. `arXiv:1903.06239 [hep-th]`.

[428] T. Banks and L. J. Dixon, "Constraints on String Vacua with Space-Time Supersymmetry," *Nucl. Phys.* **B307** (1988) 93–108.

[429] A. Hebecker, P. Mangat, S. Theisen, and L. T. Witkowski, "Can Gravitational Instantons Really Constrain Axion Inflation?," *JHEP* **02** (2017) 097, `arXiv:1607.06814 [hep-th]`.

[430] A. Hebecker, T. Mikhail, and P. Soler, "Euclidean wormholes, baby universes, and their impact on particle physics and cosmology," *Front. Astron. Space Sci.* **5** (2018) 35, `arXiv:1807.00824 [hep-th]`.

[431] D. Harlow and H. Ooguri, "Constraints on symmetry from holography," `arXiv:1810.05337 [hep-th]`.

[432] D. Harlow and H. Ooguri, "Symmetries in quantum field theory and quantum gravity," `arXiv:1810.05338 [hep-th]`.

[433] W. Israel, "Event horizons in static vacuum space-times," *Phys. Rev.* **164** (1967) 1776–1779.

[434] L. Susskind, "Trouble for remnants," `arXiv:hep-th/9501106 [hep-th]`.

[435] R. Bousso, "A Covariant entropy conjecture," *JHEP* **07** (1999) 004, `arXiv:hep-th/9905177 [hep-th]`.

[436] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, "The String landscape, black holes and gravity as the weakest force," *JHEP* **06** (2007) 060, `arXiv:hep-th/0601001 [hep-th]`.

[437] B. Heidenreich, M. Reece, and T. Rudelius, "Sharpening the Weak Gravity Conjecture with Dimensional Reduction," *JHEP* **02** (2016) 140, `arXiv:1509.06374 [hep-th]`.

[438] K. Kooner, S. Parameswaran, and I. Zavala, "Warping the Weak Gravity Conjecture," *Phys. Lett.* **B759** (2016) 402–409, `arXiv:1509.07049 [hep-th]`.

[439] B. Heidenreich, M. Reece, and T. Rudelius, "Evidence for a sublattice weak gravity conjecture," *JHEP* **08** (2017) 025, arXiv:1606.08437 [hep-th].

[440] S.-J. Lee, W. Lerche, and T. Weigand, "Tensionless Strings and the Weak Gravity Conjecture," *JHEP* **10** (2018) 164, arXiv:1808.05958 [hep-th].

[441] Y. Nakayama and Y. Nomura, "Weak gravity conjecture in the AdS/CFT correspondence," *Phys. Rev.* **D92** no. 12, (2015) 126006, arXiv:1509.01647 [hep-th].

[442] D. Harlow, "Wormholes, Emergent Gauge Fields, and the Weak Gravity Conjecture," *JHEP* **01** (2016) 122, arXiv:1510.07911 [hep-th].

[443] M. Montero, G. Shiu, and P. Soler, "The Weak Gravity Conjecture in three dimensions," *JHEP* **10** (2016) 159, arXiv:1606.08438 [hep-th].

[444] M. Montero, "A Holographic Derivation of the Weak Gravity Conjecture," *JHEP* **03** (2019) 157, arXiv:1812.03978 [hep-th].

[445] G. Shiu, P. Soler, and W. Cottrell, "Weak Gravity Conjecture and Extremal Black Hole," arXiv:1611.06270 [hep-th].

[446] Z. Fisher and C. J. Mogni, "A Semiclassical, Entropic Proof of a Weak Gravity Conjecture," arXiv:1706.08257 [hep-th].

[447] C. Cheung, J. Liu, and G. N. Remmen, "Proof of the Weak Gravity Conjecture from Black Hole Entropy," *JHEP* **10** (2018) 004, arXiv:1801.08546 [hep-th].

[448] Y. Hamada, T. Noumi, and G. Shiu, "Weak Gravity Conjecture from Unitarity and Causality," arXiv:1810.03637 [hep-th].

[449] B. Bellazzini, M. Lewandowski, and J. Serra, "Amplitudes' Positivity, Weak Gravity Conjecture, and Modified Gravity," arXiv:1902.03250 [hep-th].

[450] T. Crisford, G. T. Horowitz, and J. E. Santos, "Testing the Weak Gravity - Cosmic Censorship Connection," *Phys. Rev.* **D97** no. 6, (2018) 066005, arXiv:1709.07880 [hep-th].

[451] S. Andriolo, D. Junghans, T. Noumi, and G. Shiu, "A Tower Weak Gravity Conjecture from Infrared Consistency," *Fortsch. Phys.* **66** no. 5, (2018) 1800020, arXiv:1802.04287 [hep-th].

[452] J. Brown, W. Cottrell, G. Shiu, and P. Soler, "Fencing in the Swampland: Quantum Gravity Constraints on Large Field Inflation," *JHEP* **10** (2015) 023, arXiv:1503.04783 [hep-th].

[453] A. Hebecker, P. Henkenjohann, and L. T. Witkowski, "What is the Magnetic Weak Gravity Conjecture for Axions?," *Fortsch. Phys.* **65** no. 3-4, (2017) 1700011, arXiv:1701.06553 [hep-th].

[454] A. Hebecker and P. Soler, "The Weak Gravity Conjecture and the Axionic Black Hole Paradox," *JHEP* **09** (2017) 036, arXiv:1702.06130 [hep-th].

[455] E. Palti, "The Weak Gravity Conjecture and Scalar Fields," *JHEP* **08** (2017) 034, arXiv:1705.04328 [hep-th].

[456] C. Cheung and G. N. Remmen, "Naturalness and the Weak Gravity Conjecture," *Phys. Rev. Lett.* **113** (2014) 051601, arXiv:1402.2287 [hep-ph].

[457] M. Montero, A. M. Uranga, and I. Valenzuela, "A Chern-Simons Pandemic," *JHEP* **07** (2017) 123, arXiv:1702.06147 [hep-th].

[458] J. Polchinski, "Monopoles, duality, and string theory," *Int. J. Mod. Phys.* **A19S1** (2004) 145–156, arXiv:hep-th/0304042 [hep-th]. [,145(2003)].

[459] H. Ooguri and C. Vafa, "On the Geometry of the String Landscape and the Swampland," *Nucl. Phys.* **B766** (2007) 21–33, arXiv:hep-th/0605264 [hep-th].

[460] H. Ooguri and L. Spodyneiko, "New Kaluza-Klein instantons and the decay of AdS vacua," *Phys. Rev.* **D96** no. 2, (2017) 026016, arXiv:1703.03105 [hep-th].

[461] H. Ooguri and C. Vafa, "Non-supersymmetric AdS and the Swampland," *Adv. Theor. Math. Phys.* **21** (2017) 1787–1801, arXiv:1610.01533 [hep-th].

[462] L. E. Ibanez, V. Martin-Lozano, and I. Valenzuela, "Constraining Neutrino Masses, the Cosmological Constant and BSM Physics from the Weak Gravity Conjecture," *JHEP* **11** (2017) 066, arXiv:1706.05392 [hep-th].

[463] L. E. Ibanez, V. Martin-Lozano, and I. Valenzuela, "Constraining the EW Hierarchy from the Weak Gravity Conjecture," arXiv:1707.05811 [hep-th].

[464] G. Obied, H. Ooguri, L. Spodyneiko, and C. Vafa, "De Sitter Space and the Swampland," arXiv:1806.08362 [hep-th].

[465] P. Agrawal, G. Obied, P. J. Steinhardt, and C. Vafa, "On the Cosmological Implications of the String Swampland," *Phys. Lett.* **B784** (2018) 271–276, arXiv:1806.09718 [hep-th].

[466] M. Cicoli, S. De Alwis, A. Maharana, F. Muia, and F. Quevedo, "De Sitter vs Quintessence in String Theory," *Fortsch. Phys.* **67** no. 1-2, (2019) 1800079, arXiv:1808.08967 [hep-th].

[467] F. Denef, A. Hebecker, and T. Wrase, "de Sitter swampland conjecture and the Higgs potential," *Phys. Rev.* **D98** no. 8, (2018) 086004, arXiv:1807.06581 [hep-th].

[468] H. Murayama, M. Yamazaki, and T. T. Yanagida, "Do We Live in the Swampland?," *JHEP* **12** (2018) 032, arXiv:1809.00478 [hep-th].

[469] H. Ooguri, E. Palti, G. Shiu, and C. Vafa, "Distance and de Sitter Conjectures on the Swampland," *Phys. Lett.* **B788** (2019) 180–184, arXiv:1810.05506 [hep-th].

[470] Q. Bonnefoy, E. Dudas, and S. Lust, "On the weak gravity conjecture in string theory with broken supersymmetry," arXiv:1811.11199 [hep-th].

[471] S. Abel, E. Dudas, D. Lewis, and H. Partouche, "Stability and vacuum energy in open string models with broken supersymmetry," arXiv:1812.09714 [hep-th].

[472] L. Heurtier, *Particle physics and Cosmology beyond the Standard Model : Inflation, Dark Matter and Flavour.* PhD thesis, Ecole Polytechnique, CPHT, 2015. https://pastel.archives-ouvertes.fr/tel-01176578.

[473] D. Binosi, J. Collins, C. Kaufhold, and L. Theussl, "JaxoDraw: A Graphical user interface for drawing Feynman diagrams. Version 2.0 release notes," *Comput. Phys. Commun.* **180** (2009) 1709–1715, arXiv:0811.4113 [hep-ph].

[474] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations.* Cambridge University Press, 2005.

[475] S. L. Adler, "Axial vector vertex in spinor electrodynamics," *Phys. Rev.* **177** (1969) 2426–2438. [,241(1969)].

[476] J. S. Bell and R. Jackiw, "A PCAC puzzle: $\pi^0 \to \gamma\gamma$ in the σ model," *Nuovo Cim.* **A60** (1969) 47–61.

[477] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, "Exact Gell-Mann-Low Function of Supersymmetric Yang-Mills Theories from Instanton Calculus," *Nucl. Phys.* **B229** (1983) 381–393.

[478] M. A. Shifman and A. I. Vainshtein, "Solution of the Anomaly Puzzle in SUSY Gauge Theories and the Wilson Operator Expansion," *Nucl. Phys.* **B277** (1986) 456. [Zh. Eksp. Teor. Fiz.91,723(1986)].

[479] T. R. Taylor, G. Veneziano, and S. Yankielowicz, "Supersymmetric QCD and Its Massless Limit: An Effective Lagrangian Analysis," *Nucl. Phys.* **B218** (1983) 493–513.

Eur. Phys. J. C (2019) 79:31
https://doi.org/10.1140/epjc/s10052-018-6528-z

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Axions in a highly protected gauge symmetry model

Q. Bonnefoy[1,a], **E. Dudas**[1,b], **S. Pokorski**[2,c]

[1] Centre de Physique Théorique, École Polytechnique, CNRS, Université Paris-Saclay, Route de Saclay, 91128 Palaiseau, France
[2] Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, ul. Pasteura 5, 02-093 Warsaw, Poland

Received: 12 September 2018 / Accepted: 20 December 2018 / Published online: 16 January 2019

Abstract We study QCD axion or cosmological axion-like particles (ALPs) in a model inspired by the recent interest in 4-dimensional clockwork models, with the global symmetry being accidentally enforced by a gauge abelian quiver with scalar bifundamental fields. For the QCD axion, we analyze the connection between the degree of protection of the axion mass against gravitational corrections, the explanation of the hierarchy $f_a \ll M_P$ and the number of colored fermions needed to generate anomalous couplings to gluons, all linked together by the underlying gauge symmetries. Based on that model and on the comparison with earlier models in the literature, we derive certain general conclusions on QCD axion models that use accidental global symmetries. For the ALPs, assuming that their mass is solely given by gravitational corrections, we identify the parameter space where the decay constant and the mass are consistent with the DM abundance, and we show that this clockwork-inspired model is a particularly economical model for a very light ALP DM candidate.

1 Introduction

New pseudo-Goldstone bosons (PGB's) may play an important role in particle physics and cosmology, since they can solve the strong CP-problem (QCD axion) [1–3] and/or explain dark matter [4–6], drive inflation [7,8] or make dark energy dynamical [9–12]. The PGB playing the role of the QCD axion must have anomalous couplings to gluons whereas such couplings are not needed for the axion-like particles (ALPs) that only play the latter roles. However, in both cases one is facing several, partly similar, issues.

One is that the PGB's must be generically very light so there is a need to protect the global symmetries from a too large explicit breaking by gravitational corrections[1] [13–15]. If an axion is to solve the strong CP-problem, the non-anomalous explicit breaking must be subleading and its mass is approximately determined by the confinement scale and the axion decay constant f_a. For an ALP, the most economical possibility is that its mass is just given by the gravitational corrections, the assumption we make in this paper. The question about the proper protection of the axion and/or ALP global symmetries has been addressed by many authors [16–31]. In the field theoretical models in four dimensions, one often considers the symmetry from which the PGB's originate as an accidental consequence of gauge symmetries, i.e. as unbroken by any gauge-invariant operator up to a given dimension. Typically, strong enough protection requires either large charges of the scalars under the gauge symmetry(ies) (see e.g. [20]) or many gauge groups as in quiver models (see e.g. [23,24]). The latter can be viewed as inspired by the latticized versions of extra-dimensional models where the PGB's can be interpreted as fifth components of vector fields which appear as scalars in 4d.

Another issue is the origin of the scale f_a and of the potential hierarchy $f_a \ll M_P$. Such a hierarchy is required for the generic QCD axion window but not needed for the ALPs as dark matter (DM) candidates only, or even not acceptable for $m_{\mathrm{ALPs}} \sim O(10^{-15}–10^{-20})$ eV. One more difference between the QCD axion and the ALPs models is that the former requires a set of colored fermions to generate the anomalous couplings to gluons. Thus, the constraints are different for the two cases.

In explicit models for a protected QCD axion, one can connect the degree of protection of the axion mass, the explanation of the hierarchy $f_a \ll M_P$[2] and the number of colored fermions and their masses, all linked together by the underly-

[1] There can be other sources of such a breaking but we focus on gravitational corrections.

[2] In most models with good protection the scale f_a is not identical to the original scale f of the spontaneous global symmetry breaking and one may consider the possibility of $f_a \ll f \sim M_P$.

[a] e-mail: quentin.bonnefoy@polytechnique.edu
[b] e-mail: emilian.dudas@polytechnique.edu
[c] e-mail: stefan.pokorski@fuw.edu.pl

ing gauge symmetries. In DM ALPs models, assuming that their mass is solely given by gravitational corrections, one can identify the parameter space such that the scale f_a and the mass m_a combine to give the observed relic abundance.

In this paper we discuss those questions using as our laboratory a model inspired by the recent interest in 4-dimensional clockwork models [32–42]. Based on that model and on the comparison with earlier models in the literature, we derive certain general conclusions on the QCD axion models that use global symmetries that are consequences of gauge symmetries. Secondly, we show that the clockwork inspired model is a particularly economical model for a very light ALP DM candidate.

Our model is the 4d quiver model obtained by latticizing a 5d (abelian) gauge theory in a linear dilaton background [43], with Dirichlet boundary conditions for the 4-d components of the gauge boson [35,38]. As a result of the 5d gauge invariance, the 4d field content is such that its most general renormalizable gauge-invariant lagrangian preserves an accidental global symmetry. Furthermore, the specific 5d background, or equivalently the specific 4d gauge charge assignment, ensures a strong protection of this accidental symmetry from explicit breaking terms, even when the discretization is crude (i.e. when the quiver has few sites), and it generates a hierarchy between the effective axion decay constant f_a and the scale f of spontaneous symmetry breaking: f_a is reduced by a factor which grows exponentially with the number of quiver sites, in a way opposite to the usual clockwork models. This effective scale f_a, which appears in the potential of the axion and its couplings to gauge fields, is not the only scale parametrically different from f: the axion has a clockwork profile along the quiver sites, and this profile can generate effective coupling scales which are bigger than f when one considers for instance couplings to the spins of matter particles.

Before we proceed, let us recall that there are many arguments, from black hole ones to string theory ones, suggesting that global symmetries are broken by Planck scale effects. The strength of the breaking is well defined in a consistent theory of quantum gravity. In this paper, we will parametrize gravitational corrections as higher dimensional operators in the effective theory, suppressed by powers of the Planck scale with order one coefficients, assuming that the breaking is described correctly by the EFT approach. One may wonder whether such contributions could come from non-perturbative effects and consequently enjoy a greater suppression, as suggested by studies of axions arising from antisymmetric forms in string theory [44]. However, the kind of axions discussed in this paper originate from charged matter fields. Even in string theory, those could in principle receive perturbative higher-order corrections to their potential, which would appear as usual higher-order terms in the EFT [45]. Furthermore, if the theory of gravity includes a heavy fermionic sector whose renormalizable couplings break the axion shift symmetry, the induced Coleman-Weinberg potential is also consistent with the effective theory point of view [23,24]. Thus, we assume in this paper that the magnitude of gravitational corrections is well described by the EFT approach, with no additional suppression.

The plan of the paper is as follows: in Sect. 2 we recall the 4d model with a focus on the light scalar and examine its properties. In Sect. 3, we discuss its potential identification with a QCD axion, analyze the interplay between the three aspects mentioned earlier, compare with other models in the literature and derive some general conclusions. In Sect. 4 we consider the applications of the model to describe cosmological light particles (e.g. PGB dark matter and quintessence models) and show that it is very economical for describing an ALP as a DM candidate. We present our conclusions in the last section. Some appendices cover additional material: Appendix A contains the 5d deconstruction of an abelian vector field in a linear dilaton background whose low-energy limit matches that of our 4d picture, Appendix B discusses the massive states of the model of Sect. 2, Appendices C.1 and C.2 describe realizations of the QCD axion discussed in Sect. 3, Appendix D displays a calculation of the axion–photons couplings of Sects. 3 and 4.1 and Appendix E discusses the ranges of parameters of the model which allow the axion to be (a detectable kind of) dark matter.

2 Model

2.1 Gauge group and matter content

The (4d) setup we consider is an abelian quiver model with bifundamental scalar fields, first presented in [35] as the deconstruction [46,47] of a 5d abelian gauge theory on an orbifolded linear dilaton background with Dirichlet boundary conditions for the 4d gauge field[3] (see Appendix A), and whose low-energy theory was derived in [38] as the 4d theory obtained after chiral symmetry breaking by some confining non-abelian gauge group. The precise matter content and charge assignment is given by the quiver of Fig. 1 (where q and N are integers),

with the following (most general renormalizable) lagrangian:[4]

$$\mathcal{L} = -\frac{1}{4g_i^2}\sum_{i=1}^{N} F_{\mu\nu,i}F_i^{\mu\nu} - \sum_{k=0}^{N}(|D_\mu\phi_k|^2 + m_k^2|\phi_k|^2) - \sum_{k,l=0}^{N}\lambda_{kl}|\phi_k|^2|\phi_l|^2\ , \tag{2.1}$$

[3] See also Appendix A of [48].

[4] We use conventions of [49], in particular signature $(-+++)$.

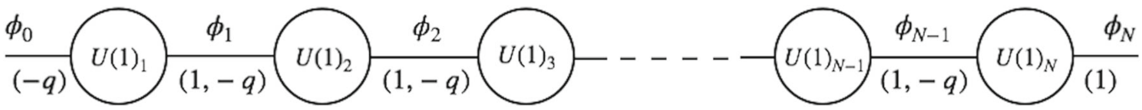


Fig. 1 Abelian quiver of the model

where F_i is the field strength of the abelian vector field A_i, with coupling constant g_i, and with the covariant derivatives $D_\mu \phi_k = \big(\partial_\mu - i(1-\delta_{k,0})A_{\mu,k} + iq(1-\delta_{k,N})A_{\mu,k+1}\big)\phi_k$. This lagrangian has a $U(1)^{N+1}$ invariance, with a $U(1)^N$ gauged subgroup.

This model is inspired by the so-called clockwork mechanism [32–42] and has been introduced in [35] as a possible realization of it, so we will comment on defining features of this mechanism if we recover them while we proceed, or discuss those which are different.

2.2 Spontaneous breaking and Goldstone mode

We are interested in obtaining Goldstone bosons, so we consider the spontaneous breaking of the full $U(1)^{N+1}$ mentioned previously by choosing the parameters m_k^2 and λ_{kl} of (2.1) so that all the scalar fields ϕ_k get vev's f_k. The spectrum then consists after gauge fixing of N massive vectors, $N+1$ massive real scalars (discussed in Appendix B) and one Goldstone boson.

Since the vev's f_k break all the gauge symmetries, N out of the $N+1$ phases of the ϕ_k are absorbed by the gauge vectors through the Higgs effect. The absorbed phase combinations depend on the charges and vev's (we write $\phi_k = \frac{f_k + r_k}{\sqrt{2}} e^{i\frac{\theta_k}{f_k}}$):

$$\mathcal{L} \supset -A_{\mu,i}(q f_{i-1}\partial_\mu \theta_{i-1} - f_i \partial_\mu \theta_i). \tag{2.2}$$

The last, uneaten phase a remains in the spectrum after gauge fixing as a Goldstone boson associated to the accidental $U(1)_a$ global symmetry which is the ungauged factor of the $U(1)^{N+1}$ symmetry group of (2.1). The profile of this boson along the original phases is orthogonal to the $q f_{i-1}\theta_{i-1} - f_i\theta_i$ gauge Goldstone bosons profiles. If we canonically normalize the field and the vev's are taken to be all equal, which will be assumed from now on[5] (we then note $f_k = f$), it reads:

$$a = \frac{\theta_0 + q\theta_1 + \cdots q^N \theta_N}{\sqrt{1+q^2+\cdots+q^{2N}}}\,. \tag{2.3}$$

Equation (2.3) displays the exponential localization discussed in clockwork models, and the charges of the original scalar fields under the global symmetry also match those which appear in those models. Indeed, $U(1)_a$ acts here as $\phi_k \to e^{iq^k\alpha}\phi_k$.

[5] In the generic case, the axion profile is, up to a normalization factor,

$$a \sim \frac{\theta_0}{q^N f_0} + \frac{\theta_1}{q^{N-1} f_1} + \cdots + \frac{\theta_{N-1}}{q f_{N-1}} + \frac{\theta_N}{f_N}\,.$$

2.3 Goldstone boson protection

The lagrangian (2.1), has an accidental exact $U(1)_a$ global symmetry at renormalizable level, hence the axion a is massless. We expect however that global symmetries are broken by gravity effects [13–15], which forces us to include all higher order operators allowed by gauge invariance in the effective theory. For the quiver of Fig. 1, these operators must be combinations of

$$|\phi_k|^2 \text{ and } \phi_0\phi_1^q \cdots \phi_N^{q^N}\,. \tag{2.4}$$

Hence, operators that explicitly break the global symmetry must involve the second term and be of extremely high dimension as soon as q and N are both slightly bigger than one. We thus obtain in this setup a pseudo-Goldstone boson with a mass very well protected by the gauge symmetry. The exponential dependence on q and N of the second operator of (2.4) can be used to make the boson mass "sufficiently" small with a reasonable number of gauge groups, as we will emphasize later on. More specifically, if we use (2.3), we find:

$$\left.\frac{\phi_0\phi_1^q\cdots\phi_N^{q^N}}{M_c^{1+q+\cdots q^N-4}} + h.c.\right|_{\text{axion terms}} = 2\left(\frac{f}{\sqrt{2}M_c}\right)^{1+q+\cdots+q^N} M_c^4 \cos\left(\frac{a}{f_a}\right) \supset -\frac{1}{2}m_a^2 a^2\,, \tag{2.5}$$

where

$$f_a = \frac{f}{\sqrt{1+q^2+\cdots q^{2N}}} \tag{2.6}$$

and

$$m_a \sim \left(\frac{f}{\sqrt{2}M_c}\right)^{\frac{1}{2}(q+\cdots+q^N-1)} \sqrt{1+q^2+\cdots q^{2N}}\, M_c\,, \tag{2.7}$$

and M_c is the cutoff of the theory, which we take close to the Planck mass M_P when we consider gravity-induced breaking effects (recall however that if a large number $\mathcal{N}$ of particles is present, the actual cutoff of the theory cannot be more than roughly $\frac{M_P}{\sqrt{\mathcal{N}}}$ [50–53]). Even though M_c may also be the scale of other breaking effects (such as the mass of heavy fermions explicitly breaking $U(1)_a$ and running in loops, see Appendices A and C.1 for discussions on this topic), we will for simplicity focus on gravitational scale breaking. Note that

f_a is significantly lower than f when N is large and $q > 1$ (we will come back to this when discussing the QCD axion).

3 QCD axion

We dedicate this section to the study of the compatibility of the Peccei–Quinn (PQ) idea [1–3] (see also [54] for a review) with the setup of Sect. 2.[6]

3.1 Accidental Peccei–Quinn symmetry in the low-energy field theory

We study in this section the low-energy effective field theory of the axion, assuming that every other massive field has been integrated out. In order to identify $U(1)_a$ with a Peccei–Quinn symmetry, we consider the following axionic coupling:[7]

$$i\log\left(\phi_0\phi_1^q\cdots\phi_N^{q^N}\right)\mathrm{Tr}(G^{\mu\nu}\tilde{G}_{\mu\nu})+h.c.\,, \tag{3.1}$$

with $G_{\mu\nu}$ the gluon field strength and $\tilde{G}_{\mu\nu}=\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$ its dual. In this section, we will not discuss the origin of this coupling, which may arise from a string theory or from a UV-complete field theory (Sect. 3.2 deals with the field theoretic case).

The operator in the log is, as we said in Sect. 2.3, the first gauge-invariant term capable of coupling the axion of (2.3) to the gluons that we could have written (using a gauge-invariant term is necessary in order not to generate any $U(1)_i\times SU(3)^2$ gauge anomaly). This coupling has two major generic features: it involves all the quiver sites, and it implies a *decrease* in the decay constant of the axion compared to the scale of breaking f. Indeed, when we plug back the axion profile (2.3) in (3.1), we obtain:

$$\begin{aligned}&i\log(\phi_0\phi_1^q\cdots\phi_N^{q^N})\mathrm{Tr}(G^{\mu\nu}\tilde{G}_{\mu\nu})+h.c.\supset\\&-\frac{2\sqrt{1+q^2+\cdots q^{2N}}}{f}a\mathrm{Tr}(G^{\mu\nu}\tilde{G}_{\mu\nu})\,,\end{aligned} \tag{3.2}$$

where we recognize the effective axion decay constant of (2.6). This suggests that the present setup could describe intermediate scale axion decay constant obtained from high scale physics (such as string scale physics).[8] This feature is common to most models with global symmetry protected by gauge symmetries: the scale f_a is not identical to the original scale f of the spontaneous global symmetry breaking and one may consider the possibility of $f_a \ll f \sim M_P$. The relation between the scales f_a and f depends on the scalar fields charges and/or the number of gauge symmetries: specific examples are (2.6) for the model under study, (3.3) and (3.4) for other models present in the literature. In the model of [20], which corresponds to $N=1$ and $q=p'/q'$, with positive coprime integers p' and q':

$$f_a=\frac{f}{\sqrt{p'^2+q'^2}}\,, \tag{3.3}$$

while in the model of [23,24], which corresponds to arbitrary N and $q=1$:

$$f_a=\frac{f}{\sqrt{N}}\,. \tag{3.4}$$

From these expressions, we can deduce that to bring f_a from the Planck scale down to the intermediate scale ($f_a\sim 10^{12}$ GeV), we require q^N (respectively $\sqrt{p'^2+q'^2}$ or $\sqrt{N}$) $\sim 10^6$, for the model under study (respectively the models introduced above). This can be achieved in our model if for instance $q=3$ and $N=13$, whereas it requires $\max(p,q)\sim 10^6$ or $N\sim 10^{12}$ in the other cases discussed. An exponential hierarchy between f and f_a in our model hence requires a much smaller number of additional gauge groups or much smaller gauge charges than in the other models analyzed.

When non-perturbative effects of QCD turn on, (3.2) induces a potential for the axion:

$$\mathcal{L}\supset m_\pi^2 f_\pi^2\cos\left(\frac{a}{f_a}-\theta_{\mathrm{QCD}}\right)\,. \tag{3.5}$$

We also include every gauge-invariant term to the potential, according to the discussion of Sect. 2.3, and in particular generate a classical explicit breaking mass term (2.7) for the axion. In order to have $|\frac{a}{f_a}-\theta_{\mathrm{QCD}}|<10^{-10}$ at the minimum of the potential and solve the strong-CP problem, we must ensure [20–22] that:

$$\begin{aligned}&\left[m_{a,\mathrm{QCD}}\sim\frac{m_\pi f_\pi}{f_a}\right]\\&\quad>10^5\left[m_{a,\mathrm{explicit}}\sim\left(\frac{f}{\sqrt{2}M_c}\right)^{\frac{1}{2}(q+\cdots+q^N-1)}\frac{f}{f_a}M_c\right]\\&\quad\text{or equivalently}\\&\quad f\lesssim\left(10^{-5}\sqrt{2}m_\pi f_\pi(\sqrt{2}M_c)^{\frac{1}{2}(q+\cdots+q^N-3)}\right)^{\frac{2}{q+\cdots+q^N+1}}\,.\end{aligned} \tag{3.6}$$

[6] Accidental PQ symmetries have been studied in many different setups, see for example [16–22,25–31].

[7] We will not pay attention to writing dimensionless quantities in the log's since it does not affect the discussion about axions which reside in the phases of the fields. Every expression in a log to appear in the rest of the paper should then be thought of as a dimensionless one [e.g. a log(scalar) means a log(scalar divided by a mass scale)].

[8] This feature, added to the fact that each site of the quiver contributes to the anomalous coupling, is qualitatively different from those of axion clockwork models, where the anomaly is generated at one site and the effective decay constant is bigger than the scale of new physics, often considered to be $\sim$ TeV.

Footnote 8 continued

For example, when $q = 3$, $N = 2$ and $M_c = M_P$, it implies $f \lesssim 10^{12}$ GeV. If now $q = 3$, $N = 3$ and $M_c = M_P$, this becomes $f \lesssim 10^{16}$ GeV.[9] The values of the parameters q and N can be of course translated into the value of the ratio $f/f_a \sim q^N$. Those numbers can be compared with those obtained in the other models we already discussed: in the model of [20]

$$m_{a,\mathrm{explicit}} \sim \left(\frac{f}{\sqrt{2}M_c}\right)^{\frac{1}{2}(p'+q'-2)} \frac{f}{f_a} M_c \,, \tag{3.7}$$

and for instance $f \sim 10^{12}$ GeV demands $p' + q' \gtrsim 13$. In the model of [23,24]:

$$m_{a,\mathrm{explicit}} \sim \left(\frac{f}{\sqrt{2}M_c}\right)^{\frac{1}{2}(N-1)} \frac{f}{f_a} M_c \,. \tag{3.8}$$

Then, $f \sim 10^{12}$ GeV demands $N \gtrsim 13$. Clearly, the larger the ratio f/f_a the better the protection, but sufficient protection is obtained already for $f/f_a = \mathcal{O}(10)$. In the present set up, this is achieved with smaller charges or smaller number of gauge groups than in the other examples described above. Unfortunately, the nice feature of the possibility of obtaining the hierarchy $f_a \ll f \sim M_P$ in QCD axion models based on global symmetries protected by gauge symmetries is overshadowed by the fermion problem discussed in Sect. 3.2.

Axion couplings to photons, which are the subject of most axion searches, are also part of this low-energy discussion. They can be derived when we consider the axionic generalizations of (3.1):

$$\begin{aligned} \mathcal{L} \supset\ & \frac{i}{32\pi^2} \log(\phi_0 \phi_1^q \cdots \phi_N^{q^N})(\mathcal{C} G^{a,\mu\nu}\tilde{G}^a_{\mu\nu} + \mathcal{E} F^{\mu\nu}\tilde{F}_{\mu\nu}) \\ \to & -\frac{\sqrt{1+q^2+\cdots q^{2N}}}{32\pi^2 f}\left(\mathcal{E} - \underbrace{\frac{2\mathcal{C}}{3}\frac{4+m_u/m_d}{1+m_u/m_d}}_{\approx 1.92\mathcal{C}}\right) a F^{\mu\nu}\tilde{F}_{\mu\nu} \end{aligned} \tag{3.9}$$

where F is the photon field strength, $\tilde{F}$ its dual, $m_{u,d}$ are quark masses, $\mathcal{C}$ and $\mathcal{E}$ anomalous constants (which we will specify when we deal with precise models in what follows), the arrow indicates that we took into account the mixing between the axion and the mesons which arises from (3.2) and (3.5) [56] and we used $m_u \approx 0.6 m_d$ under the bracket in the last line. These couplings feature the dependence on the decreased effective decay constant (2.6) we already encountered in (3.2). Couplings of the axion to fermions, such as axion–spin couplings, and their effective scales are discussed in Sect. 4.2.

[9] These values show the compatibility of our setup with astrophysical ($f_a \gtrsim 10^9$ GeV) and cosmological ($f_a \lesssim 10^{11}$ GeV) bounds on the axion decay constant (the upper bound can be relaxed, if the PQ symmetry is assumed to be broken during inflation, as soon as one allows for tuning in the cosmic initial conditions for the axion), see [55].

3.2 Axionic couplings from heavy fermion loops

We now discuss the UV origin of (3.1) in terms of loops of heavy fermions coupling to the axion.

Let us first recall how axionic couplings are generated via quarks loops. (Global) anomalies with respect to $SU(3)_c$ are mediated by colored fermions[10] with some charge under the (global) symmetry, which run in loops between gluons and scalars, whose phase contains part of the axion mode. The schematic procedure[11] is:

$$\begin{aligned} \mathcal{L} =& -|\partial\sigma|^2 - \overline{Q}\gamma^\mu(\partial_\mu - iA^a_\mu T^a)Q - (y\sigma\overline{Q_L}Q_R + h.c.) \\ & \times \text{ where } \sigma = \frac{f}{\sqrt{2}} e^{i\frac{a}{f}} \\ \supset & -\frac{(\partial a)^2}{2} - \overline{Q}\gamma^\mu\left(\partial_\mu + \frac{yf}{\sqrt{2}} - iA^a_\mu T^a\right)Q + i\frac{y}{\sqrt{2}} a\overline{Q}\gamma_5 Q \\ \times & \xrightarrow{Q \text{ triangle loop}} -\frac{a}{32\pi^2 f} G\tilde{G} = \frac{i}{32\pi^2}\log(\sigma|_a \text{terms}) G\tilde{G} \,, \end{aligned} \tag{3.10}$$

where σ is a scalar field, $Q_{L,R}$ are left and right handed colored fermions, T^a are the generators of $SU(3)_c$, A is the gluon field of field strength G and y is a Yukawa coupling.

We then see how to generate (3.1) from fermions loops, starting from the following lagrangian:[12]

$$\begin{aligned} \mathcal{L} \supset\ & -y_0\phi_0 \overline{Q_{L,0}} Q_{R,0} - \phi_1 \overline{Q^{i=1\dots q}_{L,1}} Y_{1,ij} Q^j_{R,1} \\ & -\phi_2 \overline{Q^{i=1\dots q^2}_{L,2}} Y_{2,ij} Q^j_{R,2} + \cdots + h.c. \\ \times & \xrightarrow{\text{triangle loops}} \frac{i}{32\pi^2}\big(\log(\phi_0) + \cdots + q^N \log(\phi_N)\big) G\tilde{G} \\ =\ & \frac{i}{32\pi^2}\log(\phi_0\phi_1^q \cdots \phi_N^{q^N}) G\tilde{G} \,. \end{aligned} \tag{3.11}$$

This procedure is actually the minimal one (with dimension four Yukawa couplings) that generates an $U(1)_a \times SU(3)_c^2$ anomaly without generating gauge anomalies [or said differently, that generates (3.1)]. It requires adding colored fermions at each site, in accordance with the fact that (3.1) involves all quiver links (and in particular, there is no freedom in using the axion profile to modify the effective scale of the axion–gluons coupling). In terms of couplings defined in (3.9), it has $\mathcal{C} = 1$ and $\mathcal{E} = 0$. Note that the lagrangian (3.11) [and (3.13) below] respects the global symmetry $U(1)_a$. Therefore, it cannot generate the scalar potential (2.5) by quantum corrections. However, since we had to nonetheless include (2.5) as a gravity correction, we

[10] In our case, heavy fermions must obtain their mass from Yukawa couplings since Dirac or Majorana mass terms require vector-like representations.

[11] An example of the triangle loop calculation, including the numerical coefficients, is presented in Appendix D.

[12] This procedure, as well as (3.13), is uniquely determined by the fermionic gauge charges, see Appendix C.1.

should also consider all gauge-invariant non-renormalizable fermionic operators in addition to (3.11). Those operators could classically break $U(1)_a$ and generate both the mass of the axion and its couplings to the gluons. In Appendix C.1, we present a model with such fermionic operators.

The number of additional fermions grows exponentially with N: for instance, in order to use (2.6) to bring a Planck scale f down to an intermediate scale $f_a = 10^{10-11}$ GeV, we need $\sim q^N \gtrsim 10^{7-8}$ additional fermions (which would however be close to the Planck mass and would thus not spoil gauge coupling unification, or perturbativity far below the Planck mass). Alternatively, if we start with f already at intermediate scale, the strong CP-problem is for instance solved when $f \sim 10^{11}$ GeV, $q = 3$ and $N = 2$. This is enough to ensure the gauge protection according to the discussion following (3.6), with $1+3+3^2 = 13$ additional Dirac fermions in the **3** of $SU(3)_c$. The new fermions spoil asymptotic freedom but keep perturbativity of the strong interactions below the Planck mass. In this specific example, we get a (detectable) coupling to photons from (3.9):

$$\mathcal{L} \supset (1.7 \times 10^{13}\ \text{GeV})^{-1} a F^{\mu\nu} \tilde{F}_{\mu\nu}\ . \tag{3.12}$$

If one wants to circumvent the conclusions of (3.11), one can also assign gauge charges to the fermions so that their lowest gauge-invariant mass terms are of higher dimension. One example of this type is

$$\begin{aligned}\mathcal{L} \supset\ & -y_0 \phi_0 \overline{Q_{L,0}} Q_{R,0} \\ & - \frac{1}{M_c^{q+\cdots+q^{N-1}}} \phi_1 \phi_2^q \cdots \phi_N^{q^{N-1}} \overline{Q_L^{i=1\ldots q}} Y_{ij} Q_R^j + h.c. \\ & \times \xrightarrow{\text{triangle loops}} \frac{i}{32\pi^2} \left(\log(\phi_0) + q \log\left(\phi_1 \phi_2^q \cdots \phi_N^{q^{N-1}}\right)\right) G\tilde{G} \\ & = \frac{i}{32\pi^2} \log\left(\phi_0 \phi_1^q \cdots \phi_N^{q^N}\right) G\tilde{G}\ ,\end{aligned} \tag{3.13}$$

where M_c is the cutoff of the theory. The action (3.13) couples the axion to the gluons via a number of additional fermions independent on N, but the high dimension of the second coupling in the first line of (3.13) lowers the mass of the Q_i fermions. Since these fermions are colored and unobserved at the LHC, we must impose $m_{Q_i} \gtrsim$ a few TeV, which gives, if one takes as an example $f = \frac{\sqrt{2}}{10} M_c$ and $M_c = M_P$,

$$\left(\frac{f}{\sqrt{2} M_c}\right)^{1+q+\cdots q^{N-1}} M_c \gtrsim \text{TeV} \Rightarrow \frac{q^N - 1}{q - 1} \lesssim 15\ . \tag{3.14}$$

The bound is even more stringent as soon as we decrease f in order to satisfy (3.6): $f = \frac{\sqrt{2}}{10^6} M_c$ would give $\frac{q^N-1}{q-1} \lesssim 3$. It imposes in particular that we cannot reduce the decay constant of the axion using (2.6) and (3.13) from the Planck scale down to the intermediate scale of invisible axion models. One can interpolate between (3.11) and (3.13), but then there will either be limitations on q and N due to the high dimension of the mass terms or a number of fermions that grows with N (or both). The situation is illustrated in Fig. 2: even though the use of $q \neq 1$ enables the dimension of the first $U(1)_a$-breaking operator to scale exponentially with respect to the number of quiver groups, the number of fermions necessary to make $U(1)_a$ anomalous scales linearly with respect to this operator dimension [the number of fermions is bounded above by $1 + q + \cdots + q^N$, realized in (3.11), and we bound it below by $\frac{1+q+\cdots+q^N}{3}$ according to the discussion around (3.14)].

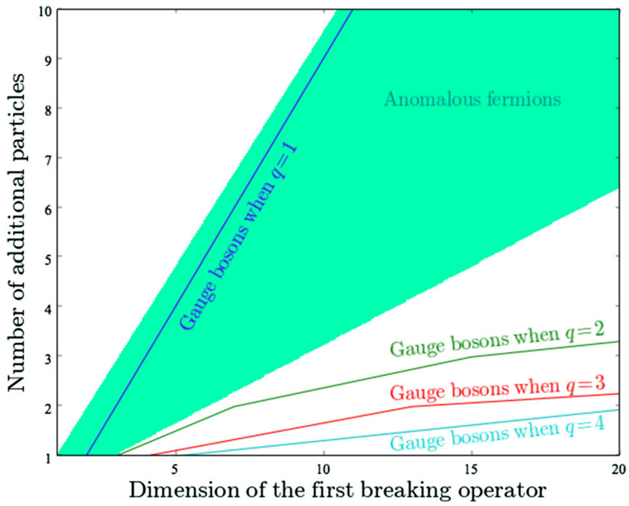


Fig. 2 Number of additional particles function of the first explicit breaking operator dimension

This discussion goes beyond the particular case of the quiver of Fig. 1 and concerns every theory with a protected PQ symmetry: the higher the quality of an accidental PQ symmetry, the higher the number of fermions required to make it anomalous with axionic couplings generated by fermion loops.[13]

Indeed, any axionic coupling term in such a theory free of gauge anomalies must be of the form:

$$i \log(\mathcal{O}) G\tilde{G} + h.c.\ , \tag{3.15}$$

where $\mathcal{O}$ is by construction gauge-invariant and not invariant under the anomalous global symmetry. If it arises from loops of heavy fermions, it is through the scheme discussed above:

$$\begin{aligned}\mathcal{L} \supset & -\sum_i (\mathcal{O}_i \overline{\psi_{i,L}} \psi_{i,R}) \\ & \times \xrightarrow{\text{triangles}} \frac{i}{32\pi^2} \log\left(\prod_i \mathcal{O}_i\right) G\tilde{G}\end{aligned} \tag{3.16}$$

[13] This applies in particular if we enlarge the scalar content of the theory depicted in Fig. 1 to additional scalar fields while keeping the quiver as the main source of protection, as in Appendix C.2.

(where we assumed that we removed from the sum every pair of vector-like fermions), and $\mathcal{O} = \prod_i \mathcal{O}_i$. However, the very notion of accidental axion symmetry means that $\mathcal{O}$ is an operator of high dimension, so the targeted quality of the axion global symmetry imposes a lower bound on $\dim(\mathcal{O}) = \sum_i \dim(\mathcal{O}_i)$, while the definition we adopt for "heavy" fermions (in our case, unobserved at the LHC) puts an upper bound on $\dim(\mathcal{O}_i)$ for each i. The two limits together imply a lower bound on the number of heavy fermions.

It is useful to analyze how such considerations show up in the other models discussed in Sect. 3.1. In the model of [20] one wants to generate $i\log(\phi_0^{q'}\phi_1^{p'})G\tilde{G} + h.c.$ and requires for this $p'+q'$ colored Dirac fermions, whereas in the model of [23,24] one wants to generate $i\log(\phi_0\phi_1\cdots\phi_N)G\tilde{G} + h.c.$ and requires for this $N+1$ colored Dirac fermions.

One can now sum up the comparison between those two models and the model under study:

- in order to protect a QCD axion for a fixed scale of spontaneous breaking f, all these models require that the first gauge-invariant $U(1)_{PQ}$-breaking operator $\mathcal{O}$ be of sufficiently high dimension $d_{\mathcal{O}}$ (e.g. $d_{\mathcal{O}} = 13$ for $f \sim 10^{12}$ GeV). This requires $N \sim d_{\mathcal{O}}$ gauge groups for the model of [23,24], $p', q' \sim d_{\mathcal{O}}$ scalar gauge charges for the one of [20] and only $N \sim \log_q(d_{\mathcal{O}})$ gauge groups for the model under study
- the effective decay constant f_a is reduced with respect to the scale f by a factor $\sim d_{\mathcal{O}}$ for our model and [20] and by a factor $\sim \sqrt{d_{\mathcal{O}}}$ for [23,24]. This can be understood by studying the $U(1)_a$ charge of $\mathcal{O}$: if the shift $a \to a + 2\pi\alpha f_a$ defines the charge normalization, $\mathcal{O}$ has charge $\sim \sqrt{d_{\mathcal{O}}}$ in [23,24] and $\sim d_{\mathcal{O}}$ in [20] and (3.1). The higher charge is however due to the high gauge charges in [20] whereas it is due to the clockwork profile of the axion as well as the expression of $\mathcal{O}$ due to the clockwork gauge charges in our model
- the number of fermions necessary to generate the axion–gluons coupling is $\sim d_{\mathcal{O}}$ in all the models.

4 Axion-like particles

In this section, we study the case of axion-like particles (ALPs), which generically refers to pseudo-Goldstone bosons not designed to solve the strong-CP problem, and whose interactions are consequently less constrained than those of the QCD axion. We will focus on models where the ALP potential is entirely generated by perturbative physics in a UV theory,[14] e.g. gravitational physics, which grants the ALP a small mass even for few quiver sites and which is sufficient to make it a good dark matter candidate. Furthermore, there exist operators which make the dark matter ALP detectable, if for example some standard model particles are charged under the quiver gauge symmetry. This only requires limited additions to the particle content of Fig. 1, even when the number of quiver sites is large. However, while the $U(1)_a$ protection by the quiver is strong enough to generate quintessence-like mass scales, the need for trans-Planckian field values of usual axion quintessence models is still present in our setup, and is exacerbated by the reduction of the axion effective decay constant.

4.1 ALPs potentials and dark sector candidates

Since most ALPs are used in cosmology, where their treatment can differ significantly from the one of the QCD axion (see [55] for a review), let us first discuss the cosmological relevance of our setup. In the non-anomalous setup that we chose to consider in this section, we think of any ALP potential as generated by some classical explicit breaking in a UV theory. The lowest-dimensional gauge-invariant potential of this type for the particle a of (2.3) is (2.5), which very weakly breaks $U(1)_a$ and entitle us to call a a pseudo-Goldstone boson, as we discussed in Sect. 2.3. It is a typical periodic potential, consistent with the ALPs origin as a periodic phase degree of freedom, and such potentials are very useful in cosmology: the smallness of the masses and the specific potential they provide make ALPs good dark matter or dynamical dark energy candidates via the misalignment mechanism. The relic density can be calculated once we are given the initial value $a_{\rm init}$ of the ALP field after inflation and its mass m_a which is taken to be constant in time (and given in (2.7) for our perturbative setup):

$$\Omega_a \approx \begin{cases} \text{Dark matter: } 2\times 10^2\left(\frac{m_a}{10^{-22}\text{eV}}\right)^{\frac{1}{2}}\left(\frac{a_{\rm init}}{M_P}\right)^2 \\ \text{Dark energy: } 8\times 10^{-2}\left(\frac{m_a}{10^{-33}\text{eV}}\right)^2\left(\frac{a_{\rm init}}{M_P}\right)^2. \end{cases} \tag{4.1}$$

$a_{\rm init}$ is given by $a_{\rm init} = \epsilon_{\rm init} f_a$, where f_a (which defines the periodicity of the ALP potential) is given in (2.6) and $\epsilon_{\rm init}$ depends on one's taste for tuning (we assume in the following that the spontaneous breaking of $U(1)_a$ happens before inflation, see the discussion of Appendix E). In order for these formulas to be valid, i.e. for the ALP to behave like CDM before radiation-matter equality or like dark energy today, we supplement (4.1) with:

[14] There could also be instantonic contributions to the potential, associated to a confining gauge group with a $U(1)_a$ anomaly. However, since

Footnote 14 continued
the discussion of Sect. 3.2 showed us that making $U(1)_a$ anomalous demands a large number of additional fermions in the theory, especially when N grows, we will for simplicity restrict ourselves to those ALPs which do not have any anomalous couplings.

$$m_a \begin{cases} \text{DM: } \gtrsim 10^{-28} \text{ eV} \\ \text{DE: } \lesssim 10^{-33} \text{ eV}, \end{cases} \tag{4.2}$$

(where the bound for DM can be pushed up to $m_a \gtrsim 10^{-22}$ eV when non-linear cosmological observables are taken into account).

In our setup, obtaining masses as low as those which appear in (4.2) without tuning is easy (for instance, (2.7) equals $\sim 10^{-33}$ eV when $f = 0.13M_P, q = 3, N = 4$). However, we can see from the comparison of (4.1) and (4.2) that axion quintessence demands initial values which are higher than the Planck mass. This can be achieved with some tuning on ϵ_{init} or when the effective decay constant of the axion is increased compared to the mass scales of the model (as in clockwork models which, however, have no mass protection mechanism built in). Since our effective decay constant (2.6) is reduced, the latter is not an option while the former is not enough to reach the correct energy density (if we insist on keeping f below the Planck mass): indeed if we impose $m_a \lesssim 10^{-33}$ eV, we can only obtain $\Omega_a \lesssim 0.05$ and would need at least 13 of such ALPs to reach the observed dark energy density.

In contrast, natural dark matter candidates do arise in our model. In Fig. 3, we scan the parameters f and M_c for some values of q and N (see Appendix E for a more complete treatment) which satisfy the condition (4.1) for $\Omega_{\text{DM}} = 0.3$ and (4.2), allowing for ϵ_{init} to range from 0.1 to $\pi - 0.1$, and allowing a constant multiplying the potential (2.5) ranging from 0.033 to 30. In Fig. 4, we focus on the case where $M_c = M_P$ and on the minimal numbers of quiver gauge groups, in order to highlight the (m_a, f_a) parameter space probed by our model. There, we allow ϵ_{init} to range from π to 0.0001, and we also include the parameter space for the QCD axion (which, due to its temperature-dependent mass, differs for the one of other ALPs).

We see in Figs. 3 and 4 that we obtain suitable DM candidates, and that the dependence on q and N of the mass (2.7) allows us to reach very low ALPs masses. These small masses, combined with the high scale f of their associated new physics, are hard to realize in a pure field theoretical framework and are usually thought of as coming from a string axiverse [44,57,58]. Our setup then provides an economical, in the sense of a low number of gauge groups, realization of such values. For instance, the smallest masses discussed in the literature for ultra light dark matter, $m_a \sim 10^{-21} - 10^{-22}$ eV, require $M_c \approx M_P$ and are obtained for $f \approx 0.2M_P$, $q = 3$ and $N = 4$ (for the choices of q and N displayed, to be compared with p', q' or $N \sim 120 - 130$ respectively for [20] and [23,24], discussed in Sects. 3.1 and 3.2). This example, as well as Fig. 4, shows that, even though we scan different values of M_c in Fig. 3, a gravitational origin ($M_c = M_P$) for (2.5) is sufficient to reproduce the cosmological relic density of dark matter.

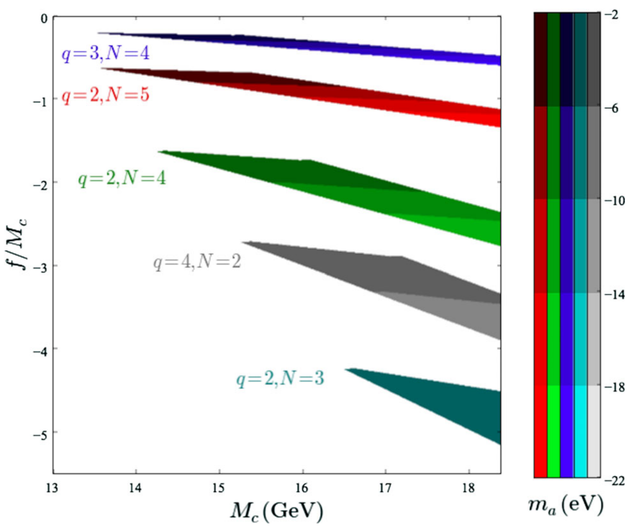


Fig. 3 Range of parameters for a DM ALP of mass $m_a \leq 10^{-2}$ eV (axions saturating the DM relic density are found in colored regions, all axes are log-scale)

In order to conclude that such ALPs are to play a role in the cosmic evolution, we must check that their lifetime can be comparable to or bigger than the age of the universe. In generic models, there is a decay channel of an ALP into two photons, usually coming from a $U(1)_{PQ} \times U(1)^2_{em}$ anomaly. Even though there is no anomaly in the models of this section, non-anomalous, CP-even and gauge invariant operators that enable this decay exist. For instance, they can arise if we couple one of the quiver sites of Fig. 1 to an anomaly-free set of electrically charged fermions displayed in Table 1, while we keep the standard model particles uncharged under the quiver gauge group.

With such charges, one can write Yukawa couplings $y_{1,2}$ to ϕ_i:

$$\mathcal{L} \supset -y_1\phi_i\overline{\psi_{R,1}}\psi_{L,1} - y_2\phi_i\overline{\psi_{L,2}}\psi_{R,2} + h.c.\,. \tag{4.3}$$

The effective operators describing the decay $a \to \gamma\gamma$ then are (see Appendix D for the computation):

$$\mathcal{L} \supset \frac{n^2 e^2 q^i}{192\pi^2\sqrt{1+q^2+\cdots q^{2N}}\,f}\left(\frac{1}{m_1^2} - \frac{1}{m_2^2}\right) \times \left(-\Box a F\tilde{F} + 2\partial_\mu a F_{\nu\eta}\partial^\eta \tilde{F}^{\mu\nu}\right), \tag{4.4}$$

where $m_1 = \frac{y_1 f}{\sqrt{2}}, m_2 = \frac{y_2 f}{\sqrt{2}}$ and F is the photon field strength. Notice that, contrary to anomalous couplings such as (3.1), non-anomalous interactions are *site localized*, and exhibit clockwork-like effects due to the profile (2.3). This feature will be present in all the operators discussed in this section. We can also see that the non-anomalous nature of the ALP-photons coupling makes this interaction of deriva-

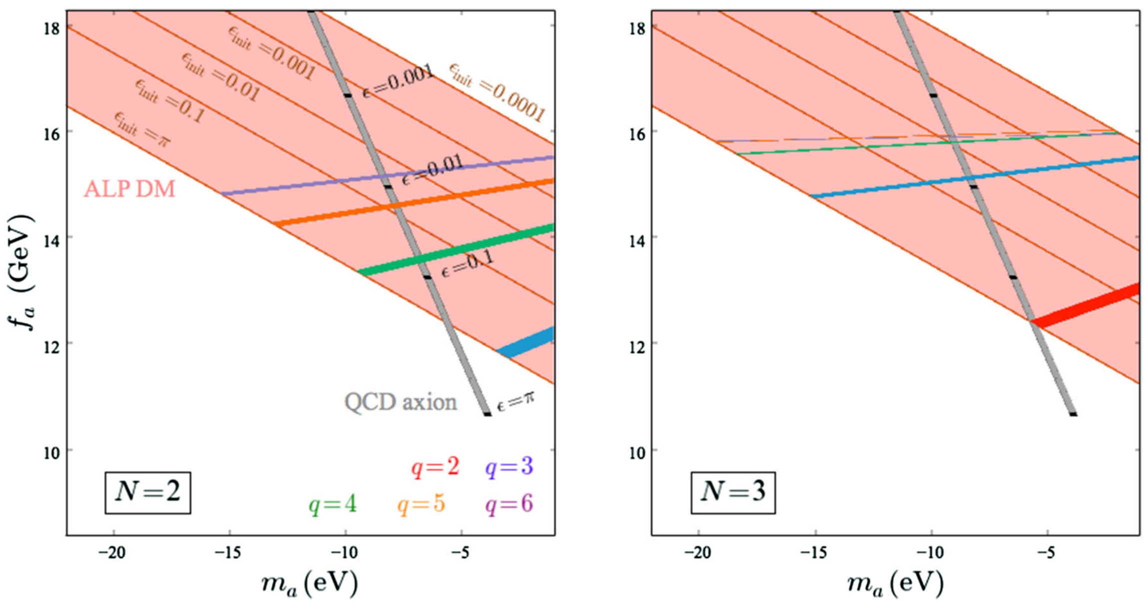


Fig. 4 Parameter space for a DM ALP of mass $m_a \leq 10^{-2}$ eV when $M_c = M_P$ (The pink region indicates the parameter space where (4.1) gives the DM relic density, whereas colored bands show where DM axions are found in our model. The QCD axion parameter space is given by the grey line. Axes are log-scale)

Table 1 Anomaly-free set of fermions coupling the ALP to the photon field (the three first columns indicate the gauge charges of the fields whereas the last one gives the PQ charges induced by (4.3), as functions of q_1 and q_2 which are arbitrary)

	$U(1)_i$	$U(1)_{i+1}$	$U(1)_{em}$	$U(1)_a$
$\psi_{L,1}$	-1	0	ne	q_1
$\psi_{R,1}$	0	$-q$	ne	q_1+q^i
$\psi_{L,2}$	1	0	$-ne$	q_2
$\psi_{R,2}$	0	q	$-ne$	q_2-q^i

tive type and of higher dimension than usual anomalous $aF\tilde{F}$ terms, so this decay does not make our ALPs unstable over the cosmic history. Indeed these couplings give a decay rate:

$$\Gamma_{a\to\gamma\gamma} = \frac{q^{2i}n^4\alpha^2 m_a^7}{1024\pi^3(1+q^2+\cdots q^{2N})f^2}\left(\frac{1}{m_1^2}-\frac{1}{m_2^2}\right)^2, \tag{4.5}$$

where α is the fine structure constant. Hence, we conclude that this decay channel is harmless with respect to the cosmic evolution of our ALPs.[15] Indeed, as guessed above, the non-anomalous nature of the ALP-photons coupling forces the m_a factor to appear in the decay rate (4.5) at a higher power than in the case of usual $aF\tilde{F}$-induced decays and ensures a long ALP lifetime. The clockwork-like dependence of (4.5) only tends to weaken the ALP couplings to photons when matter is coupled to the first quiver sites.

The example of Table 1 is a realization of the more general gauge-invariant non-anomalous operators, coupling the axion to the photon field, which we can write within the effective field theory:

$$\begin{aligned}&\frac{1}{\Lambda^4}D^\mu D_\mu\phi_i\phi_i^* F^{\mu\nu}\tilde{F}_{\mu\nu},\ \frac{1}{\Lambda^4}D_\mu\phi_i\phi_i^*\partial^\eta F^{\mu\nu}\tilde{F}_{\eta\nu}\\ &\text{and } \frac{1}{\Lambda^4}D_\mu\phi_i\phi_i^*\partial^\eta\tilde{F}^{\mu\nu}F_{\eta\nu}\\ &\times \xrightarrow{\text{terms linear in } a} \frac{iq^i f}{2\sqrt{1+\cdots q^{2N}}\Lambda^4}\big(\Box aF\tilde{F},\ \partial_\mu a\partial^\eta F^{\mu\nu}\tilde{F}_{\eta\nu}\\ &\text{and } \partial_\mu a\partial^\eta\tilde{F}^{\mu\nu}F_{\eta\nu}\big),\end{aligned} \tag{4.6}$$

where D is the covariant derivative, there is no summation over the index i and Λ the scale at which this operator is generated. For instance, in the example of Table 1 Λ is equal to the mass of the ψ fermions. Since (4.6) preserves $U(1)_a$, Λ does not have to be equal to M_c which was the scale of classical explicit breaking, even though there could also be $U(1)_a$ preserving interactions at scale M_c (for instance there could be gravitational contributions of the form (4.6) where $\Lambda = M_P$). Thus, in a minimal, agnostic approach, we should consider effective theory operators such as (4.6), supplemented by the potential (2.5) where three independent scales are used: the scale f, and f_a which follows, which are the scales of spontaneous breaking of $U(1)_a$, are given by the quiver and the renormalizable scalar potential in (2.1).

[15] For ultra-light dark matter with $m \sim 10^{-21}$ eV, if we choose $f = m_1 = 2m_2 = 0.3M_P$ and $n = 1$, the decay rate is $\sim \frac{q^{2i}}{1+q^2+\cdots q^{2N}}(10^{-300}s^{-1})$.

The scale M_c, at which $U(1)_a$ is explicitly broken, must verify $M_c > f$ for the effective lagrangian to be valid and $M_c \lesssim M_P$ since gravity anyway breaks $U(1)_a$. Finally, Λ is a scale of additional physics which generates couplings of the quiver fields to other sectors of the theory, like the SM. It must respect $\Lambda \gtrsim f$, since the new physics can lie at (almost) scale f, like in the example of Table 1, but should not be at a lower scale than the effective theory one.

From (4.6) we can calculate the decay rate of an axion into two photons:

$$\Gamma_{a\to\gamma\gamma} \sim \frac{q^{2i} m_a^7 f^2}{16\pi(1+q^2+\cdots q^{2N})\Lambda^8} . \tag{4.7}$$

This result generalizes (4.5) and of course does not spoil the conclusions made with $\Lambda \sim f$ since the dependence on m_a, responsible of the low value of $\Gamma_{a\to\gamma\gamma}$, has not changed and $\Lambda \gtrsim f$ can only weaken the decay rate.

4.2 Detection via NMR

The non-anomalous couplings to photons of (4.6), too weak to destabilize the cosmic history of our ALPs, are also too weak to be probed by current ALPs searches, which rely on a dimension 5 anomalous $aF\tilde{F}$ coupling. Non-anomalous dimension 5 generic couplings of a Goldstone pseudoscalar a to a detector's matter can be written [56] as follows:

$$\frac{g_{aee}}{f_a}\partial_\mu a\, \bar{e}\gamma^\mu\gamma_5 e \text{ and } \frac{g_{aNN}}{f_a}\partial_\mu a\, \overline{N}\gamma^\mu\gamma_5 N , \tag{4.8}$$

where g's are dimensionless coupling constants, f_a is again the axion decay constant, and N and e are respectively the nucleon and electron fields. In our setup, they can be generated in field theory if we charge the first family of the standard model under $U(1)_{i,i+1}$, according to Table 2,[16] in a way which gives them $U(1)_a$ charges.

At lowest order, the most general lagrangian is the SM lagrangian where only the first family Yukawa terms have been modified:

$$\begin{aligned}\mathcal{L} \supset &-\frac{1}{M_c}\Big(\overline{u_R}H\phi_i Y_u Q_L + \overline{d_R}(H\phi_i)^* Y_d Q_L \\ &+ \overline{e_R}(H\phi_i)^* Y_e L_L\Big) + h.c. \\ \supset &-\frac{vf}{2M_c}\Big(\overline{u}\big[e^{i\frac{q^i a}{\sqrt{1+\cdots q^{2N}}f}\gamma_5} Y_u\big]u \\ &+ \overline{d}\big[e^{-i\frac{q^i a}{\sqrt{1+\cdots q^{2N}}f}\gamma_5} Y_d\big]d + \overline{e}\big[e^{-i\frac{q^i a}{\sqrt{1+\cdots q^{2N}}f}\gamma_5} Y_e\big]e\Big),\end{aligned} \tag{4.9}$$

[16] All anomalies involving at least one standard model factor are canceled with these charges. One must however add additional fermions only charged under $U(1)_{i,i+1}$ to cancel the $U(1)_{i,i+1} \times U(1)_{i,i+1} \times U(1)_{i,i+1}$ and $U(1)_{i,i+1}$-gravity anomalies. See Appendix C.1 for explicit examples.

Table 2 SM charges that produce an ALP-spin coupling (the first columns indicate the gauge charges of the fields whereas the last one gives the PQ charges induced by (4.9), as functions of q_Q, q_H and q_L which are arbitrary)

Fields	$SU(3)$	$SU(2)$	$U(1)_Y$	$U(1)_i$	$U(1)_{i+1}$	$U(1)_a$
Q_L	**3**	**2**	$\frac{1}{6}$	0	0	q_Q
u_R	**3**	**1**	$\frac{2}{3}$	$-q$	1	$q_Q + q_H + q^i$
d_R	**3**	**1**	$-\frac{1}{3}$	q	-1	$q_Q - q_H - q^i$
L_L	**1**	**2**	$-\frac{1}{2}$	0	0	q_L
e_R	**1**	**1**	-1	q	-1	$q_L - q_H - q^i$
H	**1**	**2**	$-\frac{1}{2}$	0	0	q_H

where v is the Higgs vev and where we assumed that these higher order Yukawa couplings come from the same physics which generated (2.5), even though the fact that the precise scale M_c divides these operators is of no importance for what follows. One can make the fermion masses in (4.9) real with an appropriate chiral redefinition of the fermions, and obtain from their kinetic terms the expected couplings:

$$\mathcal{L} \supset \frac{-iq^i\partial_\mu a}{2\sqrt{1+\cdots+q^{2N}}f}(\overline{u}\gamma_5\gamma^\mu u + \overline{d}\gamma_5\gamma^\mu d + \overline{e}\gamma_5\gamma^\mu e) , \tag{4.10}$$

where no anomalous term appeared since the $U(1)_a$ symmetry is anomaly-free and where we note that, similar to what was observed previously for axion–photons couplings, the ALP-spin coupling of (4.10) is site dependent due to the clockwork profile (2.3).

If the mass (2.7) of the ALP is such that it constitutes part of the dark matter, these couplings may soon be tested via Nuclear Magnetic Resonance[17] (NMR) by the CASPEr-Wind experiment [59]. As an illustration, in Fig. 5 we assume that the coupling (4.10) is located at site $i = 0$ of the quiver and restrict ourselves to the (q, N) values displayed in Fig. 3 and to the gravitational breaking of the axionic symmetry (i.e. to the case where $M_c = M_P$, see Appendix E for a more general study). We then see that CASPEr-Wind can detect some of the ALPs discussed in this paper (one example is for $f \lesssim 5 \times 10^{15}$ GeV, $q = 2$ and $N = 4$). Thus the present model, while invisible to experiments based on axion–photons couplings, can be probed and constrained by NMR-based ALPs searches. Note however that, in order for (4.9) to be consistent with the observed values of the fermion masses, there should not be a too strong hierarchy between f and M_c. Notice also that possible FCNC effects induced by such Yukawa couplings (see for ex. [60]) are completely unobservable due to the high values of f and M_c.

[17] Bounds already exist on axion-mediated spin-dependent forces between particles, but they do not constrain models with high or intermediate scale axion decay constants.

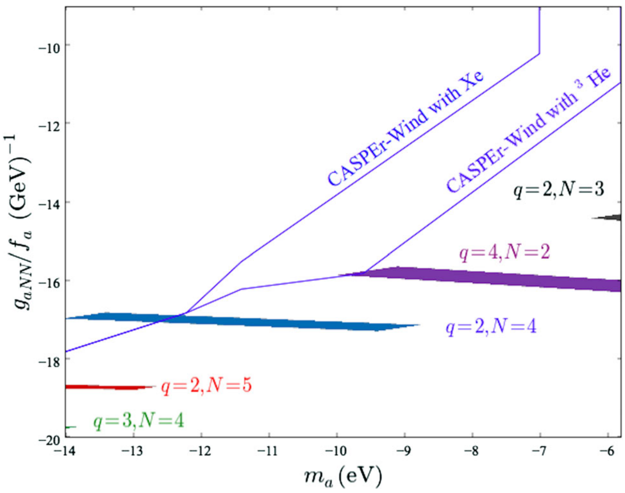


Fig. 5 Sensitivity of CASPEr-Wind to the ALPs (colored regions indicate axions suitable to saturate the DM relic density, blue curves set the limit of the upper left part of the plot where the sensitivity of CASPEr-Wind allows for a DM detection. Both axes are log-scale)

Like we did in (4.6), we can generalize such couplings in the gauge-invariant effective theory:

$$\frac{1}{\Lambda^2} D_\mu \phi_i \phi_i^* \, \bar{e} \gamma^\mu \gamma_5 e \text{ and } \frac{1}{\Lambda^2} D_\mu \phi_i \phi_i^* \, \overline{N} \gamma^\mu \gamma_5 N$$
$$\times \xrightarrow{\text{terms linear in}} a \frac{i q^i f}{2\sqrt{1 + \cdots q^{2N}} \Lambda^2} (\partial_\mu a \bar{e} \gamma^\mu \gamma_5 e \text{ and}$$
$$\times \partial_\mu a \, \overline{N} \gamma^\mu \gamma_5 N) \,, \tag{4.11}$$

Once again, the scale Λ is a priori undetermined since (4.11) does not break $U(1)_a$, for instance $\Lambda = M_P$ if (4.11) is of gravitational origin. However, taking $\Lambda = M_P$ does not allow to detect our DM candidates, contrary to the case of (4.10) where it is equal to f.

5 Conclusions

A generic problem for QCD axion models or models for ultralight PGB's as candidates for DM or quintessence is to control non-anomalous contributions to the PGB potential coming from classical explicit breaking of the global symmetry, e.g by gravitational interactions. Such contributions would shift the axion field in the minimum to unacceptably large for solving the strong CP problem values and could jeopardize the possibility of having ultralight axion-like particles as DM candidates. This problem has been often addressed in the literature in models with the global symmetry being an accidental remnant of gauge symmetries. Another issue is the origin of the scale f_a (the axion decay constant) and of the potential hierarchy $f_a \ll M_P$. Such a hierarchy is required for the generic QCD axion window but not needed for the ALPs as the dark matter candidates only, or even not acceptable for $m_{\text{ALPs}} \sim O(10^{-15} - 10^{-20})$ eV. One more difference between the QCD axion and the ALPs models is that the former requires a set of colored fermions to generate the anomalous coupling to gluons.

In this paper we have studied the QCD axion or cosmological ALPs in a model inspired by the recent interest in 4-dimensional clockwork models, with the global symmetry accidentally arising due to gauge symmetries. For the QCD axion we have analyzed the connection between the degree of protection of the axion mass against gravitational corrections, the explanation of the hierarchy $f_a \ll M_P$ and the number of colored fermions needed to generate anomalous couplings to gluons, all linked together by the underlying gauge symmetries. In the DM ALPs models, assuming that their mass is solely given by gravitational corrections, we have identified the parameter space such that the scale f_a and the mass m_a combine to give the observed relic abundance. In the latter case, we have used gravitational corrections in a constructive way, instead of invoking new anomalous gauge interactions as a source of properly adjusted explicit breaking and ignoring non-anomalous gravitational contributions.

Based on that model and on the comparison with earlier models in the literature, we have derived certain general conclusions on the QCD axion models that use global symmetries as consequences of gauge symmetries, to protect the PGB potential against large non-anomalous explicit breaking. In such models the scale f of spontaneous global symmetry breaking is not identical to the axion decay constant f_a, with $f/f_a > 1$. The larger the ratio f/f_a the better the protection but sufficient protection is obtained already for $f/f_a \sim \mathcal{O}(10)$. Furthermore, the number of colored fermions needed to generate axion anomalous couplings is approximately equal to the ratio f/f_a. Thus, the minimal sufficient protection puts the lower bound $\geq \mathcal{O}(10)$ on the number of new colored fermions.

Several results for the QCD axion are specific for our scenario based on an abelian gauge theory quiver with scalar bifundamental fields. The contributions from non-anomalous explicit breaking effects to the axion potential, and in particular to its mass are a function of the gauge charge assignment $(1, -q)$ for the scalars and the number of quiver sites. Already with $q > 1$ but $q = \mathcal{O}(1)$ and a few quiver sites ($N = \mathcal{O}(1)$, for instance $q = 3$ and $N = 2$), the mass protection against gravitational effects is sufficiently strong, with $f/f_a \sim \mathcal{O}(10)$. The number of heavy colored fermions is growing approximately exponentially with the number N of gauge groups as q^N. Whereas a large number of sites N is not needed for the mass protection of the PGB, it could be an useful option in order to decrease the axion decay constant from a large (Planck or string) value to an intermediate scale, since qualitatively $f_a \sim f/q^N$. Notice that the heavy fermions masses necessary to generate the axion–gluons coupling can be close to the Planck scale, without creating problems with the perturbativity of the low-energy theory even for large q^N. The minimal number of sites N in order to realize realistic models of this type is then way lower than for $q = 1$. Such a high N also connects the 4d model to the deconstruction of a five-dimensional abelian vector model on a linear dilaton background, with Dirichlet boundary conditions for the 4d components of the gauge field, which shares the same low-energy limit as the 4d theory. This gives some intuition to understand some features of the 4d model.

We have shown that the clockwork inspired model is a particularly economical model for a very light ALP as a DM candidate, with the observed relic abundance. Interestingly, a small number N of gauge groups is required for gravitational corrections to induce a just right ALP potential, without referring to any additional strongly interacting sector and its chiral anomalies. Such a dark matter axion-like particle can be coupled to the standard model with a small number of extra particles, if any, that does not depend on N. In particular, those couplings would be generated at a given site of the quiver and be sensitive to the clockwork profile of the axion. Such models can be tested via Nuclear Magnetic Resonance experiments, which record the matter spin precession due to the oscillation of the dark matter field. Pseudo-Goldstone quintessence models of dynamical dark energy can also be obtained in such a setup, but their construction faces usual challenges, such as a trans-Planckian axion decay constant, in order to recover the observed energy density.

Acknowledgements Q.B. and E.D. would like to thank Debtosh Chowdhury for helpful discussions. Q.B. and E.D. acknowledges partial support from the ANR Black-dS-String. The work of SP is partially supported by the National Science Centre, Poland, under research Grants DEC-2014/15/B/ST2/02157, DEC-2015/18/M/ST2/00054 and DEC-2016/23/G/ST2/04301.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: There is no additional data associated with this paper.]

A 5d deconstruction on a linear dilaton background

A.1 Abelian gauge field

We recall in this appendix the link between the 5 dimensional deconstructed theory of an abelian gauge field on a linear dilaton background and the low-energy modes of the 4d model defined in Eq. (2.1).

We start by considering a 5d manifold which is the product of the 4d Minkowski space with an interval of length L, with a 5d theory living on it:

$$\begin{aligned} \mathcal{S} &= \int d^5 z \sqrt{-g}\, \mathcal{L}(z^M) \\ &= \int d^4 x \int_0^L dy \sqrt{-g}\, \mathcal{L}(x^\mu, y)\ , \end{aligned} \tag{A.1}$$

where we split the 5d coordinates z^M into 4d Minkowski coordinates x^μ, and the position along the interval y.

We discretize the fifth dimension interval down to a regularly-spaced lattice of $N+2$ sites. Defining $\Delta_4 = \frac{L}{N+1}$, $y_i = i\Delta_4$ (where i runs from 0 to $N+1$), this amounts to replacing:

$$\begin{aligned} &\int_0^L dy\ f(x, y) \to \sum_{i=0}^N \Delta_4 f(x, y_i)\ , \\ &\partial_y f(x, y_i) \to \frac{f(x^\mu, y_{i+1}) - f(x, y_i)}{\Delta_4}\ . \end{aligned} \tag{A.2}$$

We choose to denote $f(x, y_i) = f_i(x)$ in what follows. We do not wish to study the dynamics of the background and restrict to the following static metrics:

$$\begin{aligned} ds^2 &= g_{MN} dz^M dz^N \\ &= e^{-2a(y)}(\eta_{\mu\nu} dx^\mu dx^\nu + e^{2b(y)} dy^2)\ . \end{aligned} \tag{A.3}$$

The case $a = ky$, $b = 0$ describes the so-called linear dilaton background in the conformally flat frame, whereas $a = b = ky$ is the Randall-Sundrum metric.

We will study a five-dimensional abelian theory of lagrangian

$$\begin{aligned} \sqrt{-g}\mathcal{L} &= \sqrt{-g}\left(-\frac{1}{4} g^{MP} g^{NQ} F_{MN} F_{PQ}\right) \\ &= -\frac{e^{-5a+b}}{4}\left(e^{4a} F^{\mu\nu} F_{\mu\nu} + 2e^{4a-2b} F^{\mu 4} F_{\mu 4}\right) \\ &= -\frac{e^{-a+b}}{4} F_{\mu\nu}^2 - \frac{e^{-a-b}}{2}\left(\partial_\mu A_4 - \partial_4 A_\mu\right)^2\ , \end{aligned} \tag{A.4}$$

where the 4d indices are contracted using the Minkowski metric. We impose 5d Dirichlet boundary conditions for A_μ and Neumann conditions for A_4 :

$$\begin{aligned} &A_\mu(x, y = 0, L) = 0\ , \\ &\partial_4(e^{-a-b} A_4)(x, y = 0, L) = 0\ . \end{aligned} \tag{A.5}$$

Deconstruction now yields:

$$\begin{aligned} \int_0^L dy\ \sqrt{-g}\mathcal{L} = \sum_{i=0}^N \Delta_4 \Bigg(&-\frac{e^{-a_i+b_i}}{4} F_{i,\mu\nu}^2 \\ &-\frac{e^{-a_i-b_i}}{2}\left[\partial_\mu A_{i,4} - \frac{A_{i+1,\mu} - A_{i,\mu}}{\Delta_4}\right]^2 \Bigg) \end{aligned}$$

$$= \sum_{i=0}^{N} \Bigg(-\frac{1}{4} F'^2_{i,\mu\nu} - \frac{1}{2} \left[\partial_\mu A'_{i,4} - \frac{e^{\frac{a_{i+1}-a_i-b_{i+1}-b_i}{2}} A'_{i+1,\mu} - e^{-b_i} A'_{i,\mu}}{\Delta_4} \right]^2 \Bigg), \tag{A.6}$$

where we defined $A'_{i,\mu} = e^{\frac{-a_i+b_i}{2}} \sqrt{\Delta_4} A_{i,\mu}$ (with F' its associated field strength) and $A'_{i,4} = e^{\frac{-a_i-b_i}{2}} \sqrt{\Delta_4} A_{i,4}$. Dropping the primes and using the boundary conditions, we finally obtain

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4} \sum_{i=1}^{N} F^2_{i,\mu\nu} \\ &- \frac{1}{2} \sum_{i=1}^{N-1} \left(\partial_\mu A_{i,4} - \frac{e^{\frac{a_{i+1}-a_i-b_{i+1}-b_i}{2}} A_{i+1,\mu} - e^{-b_i} A_{i,\mu}}{\Delta_4} \right)^2 \\ &- \frac{1}{2} \left(\partial_\mu A_{0,4} - \frac{e^{\frac{a_1-a_0-b_1-b_0}{2}} A_{1,\mu}}{\Delta_4} \right)^2 \\ &- \frac{1}{2} \left(\partial_\mu A_{N,4} + \frac{e^{-b_N} A_{N,\mu}}{\Delta_4} \right)^2 .\end{aligned} \tag{A.7}$$

Specializing to the linear dilaton background for which $a = ky$, $b = 0$, the lattice action now becomes

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4} \sum_{i=1}^{N} F^2_{i,\mu\nu} - \frac{1}{2} \sum_{i=1}^{N-1} \left(\partial_\mu A_{i,4} - \frac{e^{\frac{k}{2}} A_{i+1,\mu} - A_{i,\mu}}{\Delta_4} \right)^2 \\ &- \frac{1}{2} \left(\partial_\mu A_{0,4} - \frac{e^{\frac{k}{2}} A_{1,\mu}}{\Delta_4} \right)^2 - \frac{1}{2} \left(\partial_\mu A_{N,4} + \frac{A_{N,\mu}}{\Delta_4} \right)^2 ,\end{aligned} \tag{A.8}$$

where we made the replacement $k \to \frac{k}{\Delta_4}$. Defining $q = e^{\frac{k}{2}}$, $f = \frac{1}{\Delta_4}$, $\phi_i = \frac{f}{\sqrt{2}} e^{-i\frac{A_{i,4}}{f}}$, $D_\mu \phi_i = \big(\partial_\mu - i(1 - \delta_{i,0}) A_{\mu,i} + iq(1 - \delta_{i,N}) A_{\mu,i+1} \big) \phi_i$, we can rewrite (A.8) as

$$\mathcal{L} = -\frac{1}{4} \sum_{i=1}^{N} F_{\mu\nu,i} F_i^{\mu\nu} - \sum_{k=0}^{N} |D_\mu \phi_k|^2 , \tag{A.9}$$

thus establishing the link between the low-energy limit of the 4d model of Sect. 2 and the deconstruction on a linear dilaton background of an 5d abelian vector mode with boundary conditions (A.5).

Finally, the Wilson line $e^{i\int dy\, A_4(x,y)}$ gets mapped to the $U(1)_a$-violating potential of (2.4):

$$e^{i\int dy\, \sqrt{\Delta_4} A_4(x,y)} = e^{i\sum_{i=0}^{N} q^i \Delta_4 A'_{i,4}(x)} = \frac{\phi_0 \phi_1^q \cdots \phi_N^{q^N}}{\left(\frac{f}{\sqrt{2}}\right)^{1+q+\cdots q^N}} . \tag{A.10}$$

A.2 Charged bulk fermion

Deconstructed fermions might be useful in order to get insights on how $U(1)_a$ can be made anomalous or classically broken [23,24]. However, as we will see below, this procedure is not applicable in our setup. Indeed, let us consider the action of a bulk fermion charged under the abelian symmetry of the previous section:

$$\begin{aligned}\sqrt{-g}\mathcal{L} &= \sqrt{-g} \left(-\frac{1}{2} \overline{\Psi} [\gamma^M (\partial_M - ie_4 A_M) + m] \Psi + h.c. \right) \\ &= e^{-5a+b} \Big(-\frac{1}{2} \overline{\Psi} [e^a \gamma^\mu (\partial_\mu - ie_4 A_\mu) + m] \Psi \\ &\quad - \frac{e^{a-b}}{2} \overline{\Psi} \gamma^4 [\partial_4 - ie_4 A_4] \Psi + h.c. \Big) \\ &= e^{-4a+b} \Big(-\frac{1}{2} \overline{\Psi} [\gamma^\mu (\partial_\mu - ie_4 A_\mu) + e^{-a} m] \Psi \\ &\quad - \frac{e^{-b}}{2} \overline{\Psi} \gamma^4 [\partial_4 - ie_4 A_4] \Psi + h.c. \Big),\end{aligned} \tag{A.11}$$

where we did not include the spin connection of the metric (A.3), calculable from the vielbein $e_A^M = \delta_A^M \times (e^{a - b\delta_5^M})$, since it cancels out in the action, and γ^4 can be taken equal to the 4d γ_5. Deconstructing, using the normalized bosonic fields and defining $\Psi'_i = \sqrt{\Delta_4} e^{-2a_i + \frac{b_i}{2}} \Psi_i$ we get:

$$\begin{aligned}&\int dy\, \sqrt{-g}\mathcal{L} \to \\ &\sum_{i=0}^{N} \Delta_4 e^{-4a_i+b_i} \Big(-\frac{1}{2} \overline{\Psi_i} [\gamma^\mu (\partial_\mu - ie_4 A_{i,\mu}) + e^{-a_i} m] \Psi_i \\ &\quad - \frac{e^{-b_i}}{2} \overline{\Psi_i} \gamma_5 \left[\frac{\Psi_{i+1} - \Psi_i}{\Delta_4} - ie_4 A_{i,4} \Psi_{i+1} \right] + h.c. \Big) \\ &= \sum_{i=0}^{N} \Big(-\frac{1}{2} \overline{\Psi'_i} \left[\gamma^\mu \left(\partial_\mu - i \frac{e^{\frac{a_i-b_i}{2}} e_4}{\sqrt{\Delta_4}} A'_{i,\mu} \right) + e^{-a_i} m \right] \Psi'_i \\ &\quad - \frac{1}{2} \overline{\Psi'_i} \gamma_5 \Big[\frac{e^{2a_{i+1} - 2a_i - \frac{b_{i+1}}{2} - \frac{b_i}{2}} \Psi'_{i+1} - e^{-b_i} \Psi'_i}{\Delta_4} \\ &\quad - i \frac{e_4}{\sqrt{\Delta_4}} e^{2a_{i+1} - \frac{3a_i}{2} - \frac{b_{i+1}}{2}} A'_{i,4} \Psi'_{i+1} \Big] + h.c. \Big).\end{aligned} \tag{A.12}$$

We now restrict the discussion to the linear dilaton background, with the vector boundary conditions of the previous section, to supplement with boundary conditions for the fermion. If we choose $\Psi_{0,L} = \Psi_{N+1,R} = 0$, the deconstructed lagrangian becomes (where we defined $e = \frac{e_4}{\sqrt{\Delta_4}}$, and dropped the primes):

$$\begin{aligned}\mathcal{L}_{4d} = &-\frac{1}{2} \sum_{i=1}^{N} \overline{\Psi_i} [\gamma^\mu (\partial_\mu - ie^{\frac{ki}{2}} e A_{i,\mu}) + e^{-ki} m] \Psi_i \\ &-\frac{1}{2} \overline{\Psi_{0,R}} \gamma^\mu \partial_\mu \Psi_{0,R} - \frac{1}{2} \overline{\Psi_{N+1,L}} \gamma^\mu \partial_\mu \Psi_{N+1,L}\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{N-1}\overline{\Psi_i}\gamma_5\left(\frac{e^{2k}}{\Delta_4}-ie^{k\left(\frac{i}{2}+2\right)}eA_{i,4}\right)\Psi_{i+1}\\
&-\frac{1}{2}\overline{\Psi_{0,R}}\left(\frac{e^{2k}}{\Delta_4}-ie^{k}eA_{0,4}\right)\Psi_{1,L}\\
&-\frac{1}{2}\overline{\Psi_{N,R}}\left(\frac{e^{2k}}{\Delta_4}-ie^{k\left(\frac{N}{2}+2\right)}eA_{0,4}\right)\Psi_{N+1,L}+h.c..
\end{aligned}\tag{A.13}
$$

However, we cannot UV complete this lagrangian as we did in (A.9) since its k-dependence prevents from recognizing the low-energy expansion of the ϕ_i's. Only when the background is flat ($k=0 \iff q=1$) one can follow such a procedure (when $e=1$):

$$
\begin{aligned}
\mathcal{L}_{4d\ \mathrm{UV,flat}} = &-\frac{1}{2}\sum_{i=1}^{N}\overline{\Psi_i}[\gamma^\mu(\partial_\mu+ieA_{i,\mu})+m]\Psi_i\\
&-\frac{1}{2}\overline{\Psi_{0,R}}\gamma^\mu\partial_\mu\Psi_{0,R}-\frac{1}{2}\overline{\Psi_{N+1,L}}\gamma^\mu\partial_\mu\Psi_{N+1,L}\\
&-\frac{1}{\sqrt{2}}\sum_{i=1}^{N-1}\overline{\Psi_i}\gamma_5\phi_i\Psi_{i+1}-\frac{1}{\sqrt{2}}\overline{\Psi_{0,R}}\phi_0\Psi_{1,L}\\
&-\frac{1}{\sqrt{2}}\overline{\Psi_{N,R}}\phi_N\Psi_{N+1,L}+h.c.\,.
\end{aligned}\tag{A.14}
$$

This lagrangian respects $U(1)_a$ but makes it anomalous at the loop level. If one now includes an allowed mass term $-\frac{m}{2}\overline{\Psi_{0,R}}\Psi_{N+1,L}+h.c.$, $U(1)_a$ is classically broken by non-local effects, which can then generate the potential $\phi_0\phi_1\cdots\phi_N$ from fermionic loops [23,24]. When $q\neq 1$, none of this can be implemented. This reminds us that in Sect. 3.2 we needed $\sim q^N$ fermions to make $U(1)_a$ anomalous at the loop level, while deconstruction only provides us with $\sim N$ fermions.

Nevertheless, in order to make $U(1)_a$ anomalous like in Sect. 3.2, or to classically break it like in Appendix C.1, one can consider purely four dimensional setups.

B Massive vectors of the 4d model

The model of Eq. (2.1) contains massive modes in addition to the Goldstone boson a. The vector bosons mass matrix is:

$$
\begin{aligned}
\mathcal{M}^2_{vect} &= 2\\
&\times\begin{pmatrix} q^2|\phi_0|^2+|\phi_1|^2 & -q|\phi_1|^2 & \cdots & 0\\ -q|\phi_1|^2 & q^2|\phi_1|^2+|\phi_2|^2 & \cdots & 0\\ 0 & -q|\phi_2|^2 & \cdots & -q|\phi_{N-1}|^2\\ 0 & \cdots & \cdots & q^2|\phi_{N-1}|^2+|\phi_N|^2\end{pmatrix}\\
&= f^2\begin{pmatrix} 1+q^2 & -q & \cdots & 0\\ -q & 1+q^2 & \cdots & 0\\ 0 & -q & \cdots & -q\\ 0 & \cdots & \cdots & 1+q^2\end{pmatrix}
\end{aligned}\tag{B.1}
$$

after gauge symmetry breaking, with eigenvalues and eigenvectors:

$$
\begin{aligned}
m^2_{j=1\ldots N} &= f^2\left(1+q^2-2q\cos\left(\frac{j\pi}{N+1}\right)\right)\quad\text{and}\\
A'_{j=1\ldots N} &= \left(\sin\left(\frac{jk\pi}{N+1}\right),k=1\ldots N\right)\,.
\end{aligned}\tag{B.2}
$$

All vectors are massive since all gauge symmetries are broken. We recognize in (B.2) the specific (band-like) massive spectrum of clockwork models.

The masses of the Higgs-like r_k scalar fields depend on the choices of parameters in (2.1) and do not necessarily lie in a band.

C Realizations of benchmark QCD axion models

We discuss the compatibility of usual benchmark invisible QCD axion models, namely KSVZ [61,62] and DFSZ [63,64] models, with our setup. In these models, the $U(1)_{PQ}$ anomaly with respect to QCD is respectively carried by

Table 3 Colored fermions charged under the quiver gauge group of Fig. 1, canceling $SU(3)_c^2-U(1)_i$ anomalies and leading to a QCD axion ($U(1)_a$ charges are those imposed by (C.1), functions of q_R which is arbitrary)

	$U(1)_1$	$U(1)_2$	$U(1)_3$	$\cdots$	$U(1)_N$	$SU(3)_c$	$U(1)_a$
$Q_{L,0}$	$-q$	0	0	$\cdots$	0	**3**	q_R+1
$Q_{L,1}^{i=1\ldots q}$	1	$-q$	0	$\cdots$	0	**3**	q_R+q
$Q_{L,2}^{i=1\ldots q^2}$	0	1	$-q$	$\cdots$	0	**3**	q_R+q^2
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$Q_{L,N}^{i=1\ldots q^N}$	0	0	0	$\cdots$	1	**3**	q_R+q^N
$Q_R^{i=1\cdots(1+q+\cdots q^N)}$	0	0	0	$\cdots$	0	**3**	q_R

Table 4 Colored fermions with mass terms from higher dimensional operators (q_R is arbitrary)

	$U(1)_1$	$U(1)_2$	$\cdots$	$U(1)_N$	$SU(3)_c$	$U(1)_a$
$Q_{L,0}$	$-q$	0	$\cdots$	0	**3**	q_R+1
$Q_{L,1}^{i=1\ldots q}$	1	0	$\cdots$	0	**3**	$q_R+q+q^3+\cdots+q^{q^{2N-1}}$
$Q_R^{i=1\cdots(1+q)}$	0	0	$\cdots$	0	**3**	q_R

Table 5 SM-singlet fermions charged under the quiver gauge group of Fig. 1, canceling cubic quiver anomalies of Table 3 (q_L is arbitrary)

	$U(1)_1$	$U(1)_2$	$U(1)_3$	$\cdots$	$U(1)_N$	$SU(3)_c$	$U(1)_a$
$\psi_{R,0}$	$-q$	0	0	$\cdots$	0	**1**	q_L+1
$\psi_{R,1}^{i=1\ldots q}$	1	$-q$	0	$\cdots$	0	**1**	q_L+q
$\psi_{R,2}^{i=1\ldots q^2}$	0	1	$-q$	$\cdots$	0	**1**	q_L+q^2
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\psi_{R,N}^{i=1\ldots q^N}$	0	0	0	$\cdots$	1	**1**	q_L+q^N
$\psi_{L}^{i=1\cdots(1+q+\cdots q^N)}$	0	0	0	$\cdots$	0	**1**	q_L

Table 6 Colored fermions giving the major contribution to the axion mass [$U(1)_a$ charges are those imposed by (C.3), functions of q_0 which is arbitrary)]

	$U(1)_1$	$U(1)_2$	$SU(3)_c$	$U(1)_a$
$Q_{L,0}$	0	0	**3**	q_0
$Q_{R,0}$	3	0	**3**	q_0-1
$Q_{L,1}^{i=1\ldots3}$	1	0	**3**	q_0+q+q^3
$Q_{R,1}^{i=1\ldots3}$	0	3	**3**	q_0+q^3
$Q_{L,2}^{i=1\ldots9}$	0	1	**3**	q_0+q^2
$Q_{R,2}^{i=1\ldots9}$	0	0	**3**	q_0

additional heavy colored particles or by the standard model quarks, and the PQ symmetry arises from the introduction of a SM singlet scalar field (as well as an extra Higgs doublet for the DFSZ model). The phase shift symmetry of this singlet is not gauge protected in their original realization, consequently so we replace it by the accidental symmetry of our quiver model. We will also discuss, in the case of the KSVZ model, how the additional fermions can break $U(1)_a$ and generate (2.5) as a quantum correction.

C. 1 KSVZ model: anomaly mediated by additional particles

The original KSVZ model was already (anonymously and briefly) introduced in (3.10), where σ is a SM gauge singlet, and some quiver versions of it were already described in (3.11) and (3.13). There, the needed couplings were ad hoc, in contrast with the fact that we talked about an accidental Peccei–Quinn symmetry. However, we can choose the fermions charges so that the procedure of (3.11) [respectively (3.13)] is automatically implied by the most general renormalizable gauge-invariant lagrangian (respectively the lowest-order gauge-invariant lagrangian which renders all the additional fermions massive), given the gauge charges of the different fields involved. This is for instance achieved if the fermions charges are those displayed in Table 3 (respectively Table 4).

For example, the most general renormalizable lagrangian associated with Table 3 is, with such charges:

$$\mathcal{L} \supset -\phi_0 \overline{Q_{L,0}} Y_{0,i} Q_R^i - \phi_1 \overline{Q_{L,1}^i} Y_{1,ij} Q_R^j + \cdots + h.c.\,, \tag{C.1}$$

and it defines the $U(1)_a$ charges of the fermion bilinears which make $U(1)_a$ accidentally conserved, which in turn determine the $U(1)_a \times SU(3)^2$ anomaly and justify the procedure (3.11).

Along with these colored fermions, one must also add fermions only charged under the quiver gauge group to cancel the $U(1)_i \times U(1)_j \times U(1)_k$ anomalies. A way of achieving this for (3.11) is presented in Table 5.

One can check at the level of these fermionic contents that the models are gauge-anomaly-free, and at the level of their most general renormalizable lagrangian that they preserve an anomalous $U(1)_a$ global symmetry.

Still, we only considered renormalizable lagrangian, so we could ask whether Planck-suppressed fermionic terms will be generated along with (2.5), whether such terms explicitly break $U(1)_a$ and whether they can induce quantum corrections to the axion mass. In the cases discussed above, we can supplement (C.1) by:

$$\begin{aligned}\mathcal{L} \supset &-\frac{\phi_1^{q*}\cdots\phi_N^{q^N*}}{M_P^{q+\cdots+q^N-1}}\overline{Q_{L,0}}Y'_{0,i}Q_R^i\\ &-\frac{\phi_0^*\phi_1^{(q-1)*}\phi_2^{q^2*}\cdots\phi_N^{q^N*}}{M_P^{q+\cdots+q^N-1}}\overline{Q_{L,1}^i}Y'_{1,ij}Q_R^j+\cdots+h.c.\,,\end{aligned} \tag{C.2}$$

which now explicitly breaks $U(1)_a$ and induces loop corrections to m_a^2. However, such corrections are proportional to the factor $\frac{Y'}{M_P^{q+\cdots+q^N-1}}$ since $U(1)_a$ is perturbatively preserved when those terms are equal to zero. Hence, by com-

Table 7 Matter content for the quiver DFSZ model ($U(1)_a$ charges are those imposed by (Appendix C.5), functions of q_Q, q_H, q_e, q_{EW} and q_L which are arbitrary)

Fields	$SU(3)_c$	$SU(2)_{EW}$	$U(1)_Y$	$U(1)_1$	$U(1)_2$	$U(1)_3$	$\cdots$	$U(1)_a$
Q_L	**3**	**2**	$\frac{1}{6}$	$-\frac{4q}{3}$	0	0	$\cdots$	q_Q
u_R	**3**	**1**	$\frac{2}{3}$	$-\frac{q}{3}$	0	0	$\cdots$	$q_Q + q_H$
d_R	**3**	**1**	$-\frac{1}{3}$	$-\frac{q}{3}$	0	0	$\cdots$	$q_Q - q_H - 2$
L_L	**1**	**2**	$-\frac{1}{2}$	$-\frac{4q}{3}$	0	0	$\cdots$	$q_e + q_H + 2$
e_R	**1**	**1**	-1	$-\frac{q}{3}$	0	0	$\cdots$	q_e
H_1	**1**	**2**	$\frac{1}{2}$	q	0	0	$\cdots$	q_H
H_2	**1**	**2**	$-\frac{1}{2}$	q	0	0	$\cdots$	$-q_H - 2$
$Q_{L,\mathrm{EW}}^{i=1\ldots16}$	**1**	**2**	0	q	0	0	$\cdots$	q_{EW}
$Q_{R,\mathrm{EW}}^{i=1\ldots16}$	**1**	**2**	0	0	0	0	$\cdots$	$q_{EW} + 1$
$Q_{R,0}^{i=1\ldots5}$	**3**	**1**	0	$-q$	0	0	$\cdots$	$q_L - 1$
$Q_{R,1}^{i=1\ldots q}$	**3**	**1**	0	1	$-q$	0	$\cdots$	$q_L - q$
$Q_{R,2}^{i=1\ldots q^2}$	**3**	**1**	0	0	1	$-q$	$\cdots$	$q_L - q^2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		
$Q_L^{i=1\cdots(5+q+q^2+\cdots q^N)}$	**3**	**1**	0	0	0	0	$\cdots$	q_L

paring with (2.7) where $m_a^2 \sim \frac{1}{M_P^{q+\cdots+q^N-3}}$, we conclude that (2.7) gives the leading contribution to the axion mass.[18]

However, this conclusion depends on the choice of gauge charges. For instance, if one chooses $q = 3$, $N = 2$ and the gauge charges of Table 6, one can write the following lagrangian:

$$\begin{aligned}\mathcal{L} \supset & -\overline{Q_{L,0}} M Q_{R,2} - \phi_0 \overline{Q_{L,0}} Y_{00} Q_{R,0} - \phi_1 \overline{Q_{L,1}} Y_{11} Q_{R,1} \\ & - \phi_2 \overline{Q_{L,2}} Y_{22} Q_{R,2} - \frac{\phi_2^{2*}}{M_P} \overline{Q_{L,2}} Y_{21} Q_{R,1} \\ & - \frac{\phi_1^{2*}\phi_2^{6*}}{M_P^7} \overline{Q_{L,1}} Y'_{10} Q_{R,0} + h.c.\end{aligned} \tag{C.3}$$

(where we omitted flavour indices and some gauge invariant terms which do not break $U(1)_a$ and thus have no impact on the discussion). The five first terms of (C.3) fix the $U(1)_a$ charges displayed in Table 6, whereas the last one breaks this charge assignment since it has a global charge $-1-q^2-q^4$. However, as soon as one of the M, $Y^{(')}$ is zero, $U(1)_a$ is conserved. Consequently, (C.3) induces a loop correction to m_a^2 proportionnal to $\frac{1}{M_P^8}$ whereas the square of (2.7) is proportionnal to $\frac{1}{M_P^9}$. Thus, in this case, gravitational corrections to the fermion lagrangian induce a mass for the axion which competes with the pure scalar breaking of (2.5).

[18] When one takes into account the ψ fields of Table 5, one could also write gauge-invariant Majorana mass terms for the ψ_L's, but these do not break $U(1)_a$.

C. 2 DFSZ model: anomaly mediated by standard model quarks

We focus now on the DFSZ model, since, contrary to the KSVZ model, the original model has the important feature that the anomaly is only carried by the standard model quarks. It makes uses of two Higgs doublets $H_{1,2}$, an extra singlet scalar σ and can be summarized as follows:

$$\begin{aligned}\mathcal{L} \supset & -\overline{u_R} H_1 Y_u Q_L - \overline{d_R} H_2 Y_d Q_L \\ & - \overline{e_R} H_2 Y_e L_L - \lambda H_1 H_2 \sigma^2 \\ & \xrightarrow{u,d \text{ triangles}} \frac{i}{32\pi^2} \log(H_1 H_2) G\tilde{G} - \lambda H_1 H_2 \sigma^2 .\end{aligned} \tag{C.4}$$

The first line of (C.4) is invariant under a global $U(1)$ which acts on the scalars as $\sigma \to e^{i\alpha}\sigma$, $H_{1,2} \to e^{-i\alpha} H_{1,2}$. The symmetry is spontaneously broken and, according to the second line of (C.4), anomalous with respect to QCD.

In order to adapt this construction to the case of our quiver, it is important to disentangle two features of (C.4): the $\log(H_1 H_2)$ operator originates from the Yukawa terms of the SM quarks which run into loops,[19] whereas the $H_1 H_2 \sigma^2$ term (and the rest of the tree-level lagrangian) defines which symmetry is respected.[20] Thus, if we want to apply this logic to $U(1)_a$, we must identify gauge charges of H_1 and H_2 which will preserve the accidental $U(1)_a$, and identify a gauge-invariant operator O, charged under $U(1)_a$, which will

[19] Actually, since three quark families run in the loops, the correct operator is $\log((H_1 H_2)^3)$.

[20] and thus which combination of the phases of the scalars is a genuine massless Goldstone boson. If σ is assumed to get an intermediate scale vev, this boson is mostly located on the phase of σ and evades the astrophysical constraints on an electroweak scale axion.

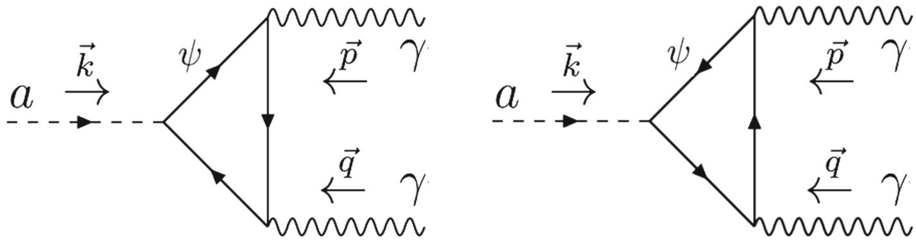


Fig. 6 Feynman diagrams leading to the axion–vector–vector couplings

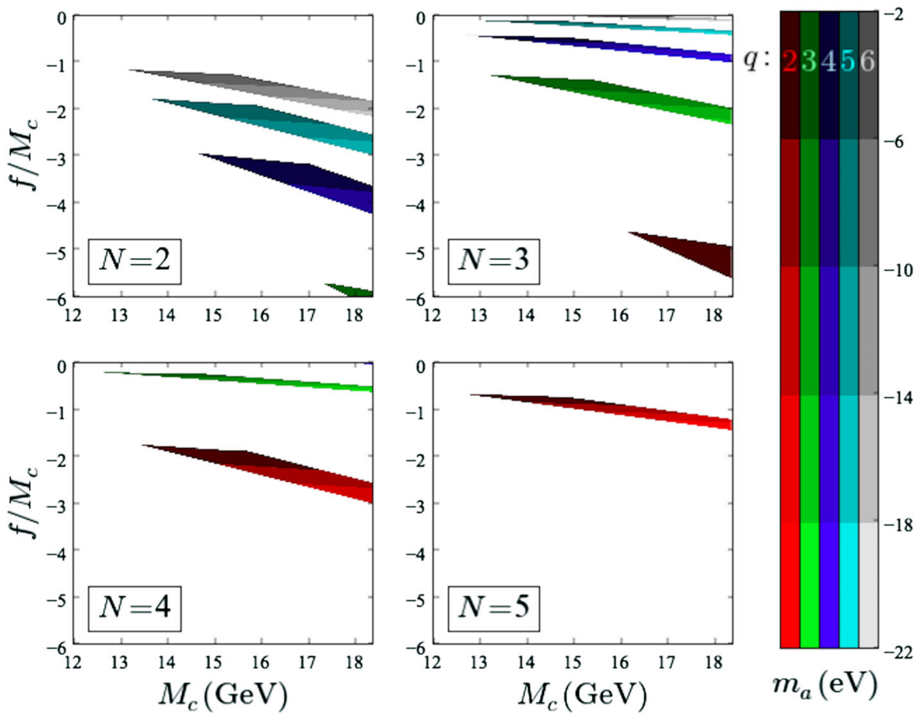


Fig. 7 Range of parameters for a DM ALP of mass $m_a \leq 10^{-2}$ eV (axions suitable to saturate the DM relic density are found in colored regions, all axes are log-scale)

induce an axionic coupling $\log(O)G\tilde{G}$ to the gluons. We can immediately understand from Sect. 3.2 that O must be of high dimension, so it must be generated by more colored particles than standard model quarks alone. It would thus be more precise to talk about a mixed DFSZ-KSVZ model, where the anomaly is mediated by both standard model quarks and additional fermions. In particular, we loose the pleasant economical quark content of the original DFSZ model, since one needs a growing number of additional particles as in the KSVZ case.

As an (unoptimized) example of this procedure, we choose the matter content and gauge charges of Table 7 in addition to that of Fig. 1.[21]

With these charges, one has the following most general renormalizable interaction terms:

$$\begin{aligned}\mathcal{L} \supset & -\overline{u_R} H_1 Y_u Q_L - \overline{d_R} H_2 Y_d Q_L - \overline{e_R} H_2 Y_e L_L \\ & - \phi_0 \overline{Q^i_{R,\mathrm{EW}}} Y_{\mathrm{EW},ij} Q^j_{L,\mathrm{EW}} - \phi_0 \overline{Q^i_{R,0}} Y_{0,ij} Q^j_L \\ & - \phi_1^* \overline{Q^i_{R,1}} Y_{1,ij} Q^j_L - \phi_2^* \overline{Q^i_{R,2}} Y_{2,ij} Q^j_L - \cdots \\ & - \lambda H_1 H_2 \phi_0^2 + h.c. \\ & \xrightarrow{\text{triangles}} -\frac{i}{32\pi^2} \log\left((H_1 H_2)^3 \phi_0^5 \phi_1^{*q} \phi_2^{*q^2} \cdots \phi_N^{*q^N}\right) G\tilde{G} \\ & - (\lambda H_1 H_2 \phi_0^2 + h.c.)\,,\end{aligned} \tag{C.5}$$

where we identify $O = (H_1 H_2)^3 \phi_0^5 \phi_1^{*q} \phi_2^{*q^2} \cdots \phi_N^{*q^N}$. $U(1)_a$ charges are assigned to $H_{1,2}$ so that $H_1 H_2 \phi_0^2$ is invariant, and $\log(O)G\tilde{G}$ makes $U(1)_a$ anomalous. The axion effective decay constant displays the same asymptotic dependence than (2.6): $f_a \sim \frac{f}{q^N}$.

It is worth noticing that a $\mu^2 H_1 H_2$ or $\mu H_1 H_2 \sigma$ term was not included in (C.4) in order to maintain a global symmetry, whereas we now cannot write something else than (C.5) that would respect gauge symmetries, which was the original goal

[21] All anomalies involving a standard model factor are canceled. The cubic, as well as the mixed abelian-gravitational anomalies of the quiver gauge group can be canceled by adding heavy SM-singlet fermions with charges identical to those of the additional fermions in Table 7, with SM representations turned into multiplicities, in the spirit of Table 5.

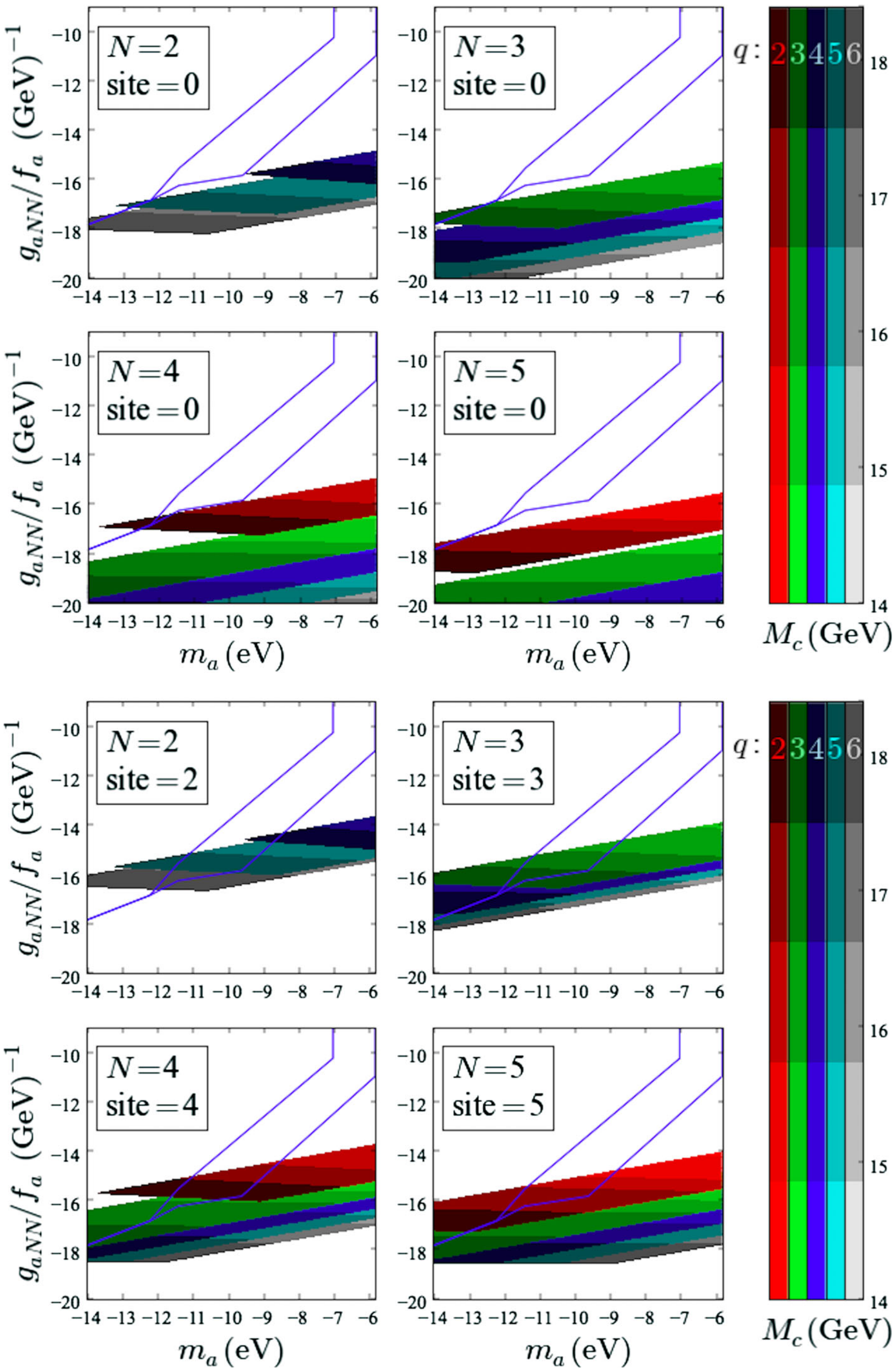


Fig. 8 Sensitivity of CASPEr-Wind to the ALPs (colored regions indicate axions suitable to saturate the DM relic density, detection happens in the upper left part of the plot, blue lines are identical to those of Fig. 5, all axes are log-scale)

when we introduced the quiver. The first allowed $U(1)_a$-violating operator is again $\phi_0\phi_1^q\cdots\phi_N^{q^N}$ and the discussion around Eq. (3.6) applies.

D Couplings of the axion to gauge vectors

We compute the axion–photon–photon coupling for the model of Fig. 1 and Table 1. However, the calculation performed here is very general and can also be seen as a derivation of (3.10).

One considers first a theory with a gauge group (which we keep unspecified until the end, where we will identify it with QCD or electromagnetism) of generators T^a, coupling constant g and vector A^a_μ (with field strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \cdots$), a complex scalar field σ and two chiral fermions $\psi_{L,R}$ in the fundamental representation of the gauge group, with a Yukawa coupling to the scalar:

$$\mathcal{L} = -\frac{1}{4g^2}F^2_{\mu\nu,a} - \overline{\psi_L}\gamma^\mu D_\mu \psi_L - \overline{\psi_R}\gamma^\mu D_\mu \psi_R - |\partial\sigma|^2 - V(|\sigma|^2) - (y\sigma\overline{\psi_L}\psi_R + h.c.)\,, \tag{D. 1}$$

where $D_\mu = \partial_\mu - iA^a_\mu T^a$. This lagrangian has a $U(1)$ global symmetry under which $\sigma \to e^{i\alpha}\sigma$ and $\overline{\psi_L}\psi_R \to e^{-i\alpha}\overline{\psi_L}\psi_R$. The transformation of the fermion bilinear makes this global symmetry anomalous.

We choose $V(|\sigma|^2)$ so that σ gets a vev f. We then work out the axion dynamics by parametrizing $\sigma = \frac{f}{\sqrt{2}}e^{i\frac{a}{f}}$:

$$\mathcal{L} \supset -\frac{1}{4g^2}F^2_{\mu\nu} -\overline{\psi}\gamma^\mu\left(D_\mu + \frac{yf}{\sqrt{2}}\right)\psi - \frac{1}{2}(\partial a)^2 + i\frac{y}{\sqrt{2}}a\overline{\psi}\gamma_5\psi\,, \tag{D. 2}$$

where we only kept the linear terms in a and merged the two chiral fermions in a Dirac fermion.

One gets a coupling between the axion a and the gauge boson A at one loop via the diagrams of Fig. 6.

The effective coupling is $c^{\mu\nu,ab}aA^a_\mu A^b_\nu$, here in momentum space with $M_\psi = \frac{yf}{\sqrt{2}}$ and at first order in $\frac{p}{M_\psi}, \frac{q}{M_\psi}$:

$$c^{\mu\nu,ab} = \frac{-1}{4\pi^2 f}\delta^{ab}\epsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma\left(\frac{1}{2} - \frac{p^2+q^2+pq}{12M^2_\psi}\right) \tag{D. 3}$$

which, with the identification $pA(p) \to -i\partial A(x)$, gives finally the one-loop coupling between the axion and the vector bosons:

$$\mathcal{L} \supset -\frac{\epsilon^{\mu\nu\rho\sigma}}{32\pi^2 f}aF^a_{\mu\nu}F^a_{\rho\sigma} + \frac{\epsilon^{\mu\nu\rho\sigma}}{192\pi^2 M^2_\psi f}(-\Box a F^a_{\mu\nu}F^a_{\rho\sigma} + 2\partial_\mu a\partial^\eta F^a_{\rho\sigma}F^a_{\nu\eta})\,. \tag{D. 4}$$

The first term of (Appendix D. 4) is the usual axionic coupling to gauge fields, while the other terms match similar calculations already performed in the literature (see for example [65]).

If one now adds to the theory (Appendix D. 1) another set of fermions coupled in the following way:

$$\mathcal{L} \supset -\overline{\psi'_L}\gamma^\mu D_\mu\psi'_L - \overline{\psi'_R}\gamma^\mu D_\mu\psi'_R - (y'\sigma^*\overline{\psi'_L}\psi'_R + h.c.) \xrightarrow{\text{axion terms}} -\overline{\psi'}\gamma^\mu\left(D_\mu + \frac{y'f}{\sqrt{2}}\right)\psi' - i\frac{y'}{\sqrt{2}}a\overline{\psi'}\gamma_5\psi'\,, \tag{D. 5}$$

there is no anomaly anymore, but there remains non-anomalous couplings to the gauge fields (where we defined $M'_\psi = y'f$):

$$\mathcal{O} \supset \frac{\epsilon^{\mu\nu\rho\sigma}}{192\pi^2 f} \times\left(\frac{1}{M^2_\psi} - \frac{1}{M'^2_\psi}\right)\left(-\Box a F^a_{\mu\nu}F^a_{\rho\sigma} + 2\partial_\mu a\partial^\eta F^a_{\rho\sigma}F^a_{\nu\eta}\right)\,. \tag{D. 6}$$

Specializing to electromagnetism, normalizing the photon field $A_\mu \to eA_\mu$ and choosing $\sigma = \phi_i = \frac{f}{\sqrt{2}}e^{i\frac{q^i a/f}{\sqrt{1+\cdots+q^{2N}}}}$, one obtains (4.4).

E Scan of the parameters which allow for (detectable) ALP DM

We extend in this appendix the analysis performed in Sect. 4 to more values of q and N, since the DM examples in Fig. 3 have been arbitrarily chosen. Figure 7 displays all DM candidates in our setup when $q \leq 6$ and $N \leq 5$, with tuning restrictions identical to those used in Fig. 3. As mentioned in Sect. 4, those results were obtained assuming that $U(1)_a$ was broken above the inflation scale. Indeed, we can see from Figs. 3, 4 and 7 that most of our ALP DM candidates require f to be high (whereas the inflation scale, given by the Hubble rate during inflation, verifies $H_{\text{inflation}} \lesssim 10^{14}$ GeV). Consequently, we only focus on the broken case (which may suffer from isocurvature fluctuations issues, which are however negligible when f is close to M_P).

We also allow in Fig. 8 (which, as Fig. 5, compares the sensitivity of the CASPEr-Wind experiment with the predictions of our model) for more values of q and N, but also for $M_c < M_P$. The upper panel of Fig. 8 couples the standard model with the first site of the quiver while the lower panel couples it to the last site of the quiver (which, as visible in the plot, increases the coupling and thus the detectability of the setup). We see from Fig. 8 that CASPEr-Wind experiments are more sensitive to high scale (e.g. gravitational) values of M_c.

References

1. R.D. Peccei, H.R. Quinn, Phys. Rev. Lett. **38**, 1440 (1977). https://doi.org/10.1103/PhysRevLett.38.1440
2. S. Weinberg, Phys. Rev. Lett. **40**, 223 (1978). https://doi.org/10.1103/PhysRevLett.40.223
3. F. Wilczek, Phys. Rev. Lett. **40**, 279 (1978). https://doi.org/10.1103/PhysRevLett.40.279
4. J. Preskill, M.B. Wise, F. Wilczek, Phys. Lett. **120B**, 127 (1983). https://doi.org/10.1016/0370-2693(83)90637-8
5. L.F. Abbott, P. Sikivie, Phys. Lett. **120B**, 133 (1983). https://doi.org/10.1016/0370-2693(83)90638-X
6. M. Dine, W. Fischler, Phys. Lett. B **120**, 137 (1983).https://doi.org/10.1016/0370-2693(83)90639-1. [URL(1982)]
7. K. Freese, J.A. Frieman, A.V. Olinto, Phys. Rev. Lett. **65**, 3233 (1990). https://doi.org/10.1103/PhysRevLett.65.3233

8. J.E. Kim, H.P. Nilles, M. Peloso, JCAP **0501**, 005 (2005). https://doi.org/10.1088/1475-7516/2005/01/005
9. B. Ratra, P.J.E. Peebles, Phys. Rev. D **37**, 3406 (1988). https://doi.org/10.1103/PhysRevD.37.3406
10. J.A. Frieman, C.T. Hill, A. Stebbins, I. Waga, Phys. Rev. Lett. **75**, 2077 (1995). https://doi.org/10.1103/PhysRevLett.75.2077
11. J.E. Kim, H.P. Nilles, Phys. Lett. B **553**, 1 (2003). https://doi.org/10.1016/S0370-2693(02)03148-9
12. J.E. Kim, J. Korean, Phys. Soc. **64**, 795 (2014). https://doi.org/10.3938/jkps.64.795
13. S.W. Hawking, Phys. Lett. B **195**, 337 (1987). https://doi.org/10.1016/0370-2693(87)90028-1
14. S.B. Giddings, A. Strominger, Nucl. Phys. B **307**, 854 (1988). https://doi.org/10.1016/0550-3213(88)90109-5
15. T. Banks, N. Seiberg, Phys. Rev. D **83**, 084019 (2011). https://doi.org/10.1103/PhysRevD.83.084019
16. J.E. Kim, Phys. Rev. D **24**, 3007 (1981). https://doi.org/10.1103/PhysRevD.24.3007
17. H.M. Georgi, L.J. Hall, M.B. Wise, Nucl. Phys. B **192**, 409 (1981). https://doi.org/10.1016/0550-3213(81)90433-8
18. S. Dimopoulos, P.H. Frampton, H. Georgi, M.B. Wise, Phys. Lett. **117B**, 185 (1982). https://doi.org/10.1016/0370-2693(82)90543-3
19. K. Kang, I.G. Koh, S. Ouvry, Phys. Lett. **119B**, 361 (1982). https://doi.org/10.1016/0370-2693(82)90689-X
20. S.M. Barr, D. Seckel, Phys. Rev. D **46**, 539 (1992). https://doi.org/10.1103/PhysRevD.46.539
21. M. Kamionkowski, J. March-Russell, Phys. Lett. B **282**, 137 (1992). https://doi.org/10.1016/0370-2693(92)90492-M
22. R. Holman, S.D.H. Hsu, T.W. Kephart, E.W. Kolb, R. Watkins, L.M. Widrow, Phys. Lett. B **282**, 132 (1992). https://doi.org/10.1016/0370-2693(92)90491-L
23. C.T. Hill, A.K. Leibovich, Phys. Rev. D **66**, 016006 (2002). https://doi.org/10.1103/PhysRevD.66.016006
24. C.T. Hill, A.K. Leibovich, Phys. Rev. D **66**, 075010 (2002). https://doi.org/10.1103/PhysRevD.66.075010
25. A.G. Dias, V. Pleitez, M.D. Tonasse, Phys. Rev. D **67**, 095008 (2003). https://doi.org/10.1103/PhysRevD.67.095008
26. K. Harigaya, M. Ibe, K. Schmitz, T.T. Yanagida, Phys. Rev. D **88**(7), 075022 (2013). https://doi.org/10.1103/PhysRevD.88.075022
27. A.G. Dias, A.C.B. Machado, C.C. Nishi, A. Ringwald, P. Vaudrevange, JHEP **06**, 037 (2014). https://doi.org/10.1007/JHEP06(2014)037
28. A. Ringwald, K. Saikawa, Phys. Rev. D **93**(8), 085031 (2016). https://doi.org/10.1103/PhysRevD.93.085031. https://doi.org/10.1103/PhysRevD.94.049908. **(Addendum: Phys. Rev. D94, no. 4, 049908(2016))**
29. M. Redi, R. Sato, JHEP **05**, 104 (2016). https://doi.org/10.1007/JHEP05(2016)104
30. H. Fukuda, M. Ibe, M. Suzuki, T.T. Yanagida, Phys. Lett. B **771**, 327 (2017). https://doi.org/10.1016/j.physletb.2017.05.071
31. B. Lillard, T.M.P. Tait, JHEP **11**, 005 (2017). https://doi.org/10.1007/JHEP11(2017)005
32. K. Choi, S.H. Im, JHEP **01**, 149 (2016). https://doi.org/10.1007/JHEP01(2016)149
33. D.E. Kaplan, R. Rattazzi, Phys. Rev. D **93**(8), 085007 (2016). https://doi.org/10.1103/PhysRevD.93.085007
34. G.F. Giudice, M. McCullough, JHEP **02**, 036 (2017). https://doi.org/10.1007/JHEP02(2017)036
35. A. Ahmed, B.M. Dillon, Phys. Rev. D **96**(11), 115031 (2017). https://doi.org/10.1103/PhysRevD.96.115031
36. T. Hambye, D. Teresi, M.H.G. Tytgat, JHEP **07**, 047 (2017). https://doi.org/10.1007/JHEP07(2017)047
37. N. Craig, I. Garcia Garcia, D. Sutherland, JHEP **10**, 018 (2017). https://doi.org/10.1007/JHEP10(2017)018
38. R. Coy, M. Frigerio, M. Ibe, JHEP **10**, 002 (2017). https://doi.org/10.1007/JHEP10(2017)002
39. G.F. Giudice, M. McCullough, A clockwork theory. JHEP **1702**, 036 (2017). https://doi.org/10.1007/JHEP02(2017)036. arXiv:1610.07962
40. K. Choi, S.H. Im, C.S. Shin, General continuum clockwork. JHEP **1807**, 113 (2018). https://doi.org/10.1007/JHEP07(2018)113. arXiv:1711.06228 [hep-ph]
41. G.F. Giudice, Y. Kats, M. McCullough, R. Torre, A. Urbano, Clockwork/linear dilaton: structure and phenomenology. JHEP **1806**, 009 (2018). https://doi.org/10.1007/JHEP06(2018)009. arXiv:1711.08437 [hep-ph]
42. D. Teresi, Clockwork without supersymmetry. Phys. Lett. B **783**, 1 (2018). https://doi.org/10.1016/j.physletb.2018.06, https://doi.org/10.1016/j.physletb.2018.06.049. arXiv:1802.01591 [hep-ph]
43. I. Antoniadis, S. Dimopoulos, A. Giveon, JHEP **05**, 055 (2001). https://doi.org/10.1088/1126-6708/2001/05/055
44. P. Svrcek, E. Witten, JHEP **06**, 051 (2006). https://doi.org/10.1088/1126-6708/2006/06/051
45. K.S. Choi, H.P. Nilles, S. Ramos-Sanchez, P.K.S. Vaudrevange, Phys. Lett. B **675**, 381 (2009). https://doi.org/10.1016/j.physletb.2009.04.028
46. N. Arkani-Hamed, A.G. Cohen, H. Georgi, Phys. Rev. Lett. **86**, 4757 (2001). https://doi.org/10.1103/PhysRevLett.86.4757
47. C.T. Hill, S. Pokorski, J. Wang, Phys. Rev. D **64**, 105005 (2001). https://doi.org/10.1103/PhysRevD.64.105005
48. J. de Blas, A. Falkowski, M. Perez-Victoria, S. Pokorski, JHEP **08**, 061 (2006). https://doi.org/10.1088/1126-6708/2006/08/061
49. S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, Cambridge (2005)
50. G. Veneziano, JHEP **06**, 051 (2002). https://doi.org/10.1088/1126-6708/2002/06/051
51. N. Arkani-Hamed, S. Dimopoulos, S. Kachru, Predictive landscapes and new physics at a TeV (2005). arXiv:hep-th/0501082
52. G. Dvali, Fortsch. Phys. **58**, 528 (2010). https://doi.org/10.1002/prop.201000009
53. G. Dvali, M. Redi, Phys. Rev. D **77**, 045027 (2008). https://doi.org/10.1103/PhysRevD.77.045027
54. R.D. Peccei, Lect. Notes Phys. **741**, 3 (2008).https://doi.org/10.1007/978-3-540-73518-2_1. [3(2006)]
55. D.J.E. Marsh, Phys. Rept. **643**, 1 (2016). https://doi.org/10.1016/j.physrep.2016.06.005
56. M. Srednicki, Nucl. Phys. B **260**, 689 (1985). https://doi.org/10.1016/0550-3213(85)90054-9
57. A. Arvanitaki, S. Dimopoulos, S. Dubovsky, N. Kaloper, J. March-Russell, Phys. Rev. D **81**, 123530 (2010). https://doi.org/10.1103/PhysRevD.81.123530
58. M. Cicoli, M. Goodsell, A. Ringwald, JHEP **10**, 146 (2012). https://doi.org/10.1007/JHEP10(2012)146
59. P.W. Graham, S. Rajendran, Phys. Rev. D **88**, 035023 (2013). https://doi.org/10.1103/PhysRevD.88.035023
60. G. Marques-Tavares, M. Teo, Light axions with large hadronic couplings. JHEP **1805**, 180 (2018). https://doi.org/10.1007/JHEP05(2018)180. arXiv:1803.07575 [hep-ph]
61. J.E. Kim, Phys. Rev. Lett. **43**, 103 (1979). https://doi.org/10.1103/PhysRevLett.43.103
62. M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B **166**, 493 (1980). https://doi.org/10.1016/0550-3213(80)90209-6
63. M. Dine, W. Fischler, M. Srednicki, Phys. Lett. **104B**, 199 (1981). https://doi.org/10.1016/0370-2693(81)90590-6
64. A.R. Zhitnitsky, Sov. J. Nucl. Phys. **31**, 260 (1980). [Yad. Fiz.31,497(1980)]
65. E. Dudas, L. Heurtier, Y. Mambrini, B. Zaldivar, JHEP **11**, 083 (2013). https://doi.org/10.1007/JHEP11(2013)083

International Journal of Modern Physics A

Axions and anomalous $U(1)$'s*

Quentin Bonnefoy†

Emilian Dudas‡

Centre de Physique Théorique, École Polytechnique, CNRS, Université Paris-Saclay
Route de Saclay, 91128 Palaiseau, France

Inspired by recent studies of high-scale decay constant or flavorful QCD axions, we review and clarify their existence in effective string models with anomalous $U(1)$ gauge groups. We find that such models, when coupled to charged scalars getting vacuum expectation values, always have one light axion, whose mass can only come from nonperturbative effects. If the main nonperturbative effect is from QCD, then it becomes a Peccei-Quinn axion candidate for solving the strong CP problem. We then study simple models with universal Green-Schwarz mechanism and only one charged scalar field: in the minimal gaugino condensation case the axion mass is tied to the supersymmetry breaking scale and cannot be light enough, but slightly refined models maintain a massless axion all the way down to the QCD scale. Both kinds of models can be extended to yield intermediate scale axion decay constants. Finally, we gauge flavorful axion models under an anomalous $U(1)$ and discuss the axion couplings which arise.

1. Introduction and Conclusions

The Peccei-Quinn (PQ) symmetry [1] and its light axion [2] (for reviews, see [3]) is probably the most elegant solution to the strong CP problem. Its implementation in string theory is natural since at tree-level in supergravity there are often continuous PQ like symmetries, usually broken to discrete subgroups by quantum corrections and nonperturbative effects. On the other hand, realistic string models often contain an "anomalous" abelian gauge symmetry, called $U(1)_X$ in what follows[a], with anomaly cancellation *à la* Green-Schwarz (GS). Such a symmetry has multiple phenomenological applications: generating hierarchical fermion masses and mixing angles via the Froggatt-Nielsen mechanism [5], relating the weak angle to anomaly coefficients [6] and breaking supersymmetry [7].

*To be published in the special issue of the International Journal of Modern Physics A "Field Theory and the Early Universe", in Commemoration of BW2018 and 15 Years of the SEENET-MTP Network.

†E-mail: quentin.bonnefoy@polytechnique.edu

‡E-mail: emilian.dudas@polytechnique.edu

[a]We consider the minimal case of one anomalous abelian symmetry, like in the original context it was studied [4], the perturbative heterotic string. Our arguments do however apply to other string models as well, in particular orientifold models, by relabeling appropriately the modulus field, as in our Section 6.

In this note we comment on one additional generic property of models with an anomalous $U(1)$: at the perturbative level, and if there is at least one charged scalar field which gets a vacuum expectation value (vev), such models always contain a potential axion candidate, which can only get a mass by turning on nonperturbative effects (and simultaneously turning on the coupling to gravity in supersymmetric models, where an R-symmetry survives even after inclusion of nonperturbative effects). We study the symmetries responsible for protecting the axion and the conditions under which the axion is light enough for solving the strong CP problem in a heterotic framework with a single charged scalar and hidden sector gaugino condensation, and we conclude that realistic supersymmetry breaking is incompatible with a light enough axion. However, we also give a refined example where nonperturbative dynamics still preserves a massless axion all the way to the QCD scale, even after coupling to gravity. Finally we show that in such a context and irrespective of the details of the model under consideration, gauge invariance fixes completely the couplings of the axion to matter when the charged scalar is used as a flavon field. The couplings to Standard Model (SM) charged fermions are proportional to their anomalous charges and the couplings to the gauge fields to the mixed $U(1)_X G_a^2$ anomalies, where $G_a = SU(3)_c, SU(2)_L, U(1)_Y$ are the SM gauge group factors. Gauge coupling unification conditions alone then determine the ratio of the coupling to the photon to the coupling to the gluons to be $E/N = 8/3$ at the unification scale. These couplings are similar to the ones in the axiflavon/flaxion models [8], but the symmetry is now gauged.

The generic value of the axion decay constant in these simple setups is of order the unification scale. Such values require a tuned or nonstandard cosmology in order to ensure a consistent relic density for the axion. We therefore discuss in the final section models of moduli stabilization which display an intermediate scale axion decay constant. However, aiming for such an intermediate scale decay constant may not be required since several (recent) studies have shown that the resulting cosmology is viable and does not necessarily involve a severe amount of tuning [9]. Moreover, new proposals for axion dark matter searches are sensitive to GUT scale values for the axion decay constant [10].

2. Anomalous $U(1)$ models

2.1. *Perturbative axion in anomalous $U(1)$ models*

In this section one will prove the following result:

Theorem. In field (string) theory models of a $U(1)_X$ gauge theory with a Stueckelberg (Green-Schwarz) mechanism and at least one charged scalar field acquiring a non-zero vacuum expectation value[b], at the perturbative level there is always a *massless pseudoscalar.*

[b]In case of additional $U(1)$ gauge symmetries, anomalous or not, the counting may be different but a similar result always applies.

Proof. Let us consider an abelian gauge theory in a Stueckelberg phase, coupled to charged scalars Φ_i of charges X_i, of lagrangian

$$\mathcal{L} = |D_\mu \Phi_i|^2 - \frac{1}{4} F_{X,\mu\nu}^2 + \frac{1}{2}(\partial_\mu a_S + M A_{X,\mu})^2 + \cdots \ , \tag{1}$$

where $\cdots$ are other terms like axionic couplings, in which case it is more appropriate to use the term Green-Schwarz rather than Stueckelberg for such a model. Since we are interested in axion-like particles, without loosing generality we only consider in what follows charged scalars having non-vanishing vev's, parametrized as

$$\Phi_i = \frac{V_i + h_i}{\sqrt{2}} e^{\frac{i\theta_i}{V_i}} \ . \tag{2}$$

Gauge transformations act as

$$\delta A_{X,\mu} = -\frac{1}{g}\partial_\mu \alpha \ , \quad \delta\theta_i = X_i V_i \alpha \ , \quad \delta a_S = \frac{M}{g}\alpha \ . \tag{3}$$

From (1) one finds that the Goldstone boson which mixes in the usual way $\partial_\mu \theta_X A_X^\mu$ with the gauge field is given by (up to a normalization factor)

$$\theta_X = g X_i V_i \theta_i + M a_S \ . \tag{4}$$

We have therefore $N+1$ potential axions/pseudoscalars, one of which is absorbed by the gauge field via the Higgs mechanism [11]. The perturbative scalar potential is of the form[c]

$$V = \sum_\alpha \lambda_\alpha \Phi_1^{m_1^{(\alpha)}} \cdots \Phi_N^{m_N^{(\alpha)}} + \text{h.c.} \ , \tag{5}$$

and gauge invariance imposes the restriction $X_1 m_1^{(\alpha)} + \cdots + X_N m_N^{(\alpha)} = 0$. Simple matrix algebra tells us that the maximal number of independent gauge invariant operators that can be written is equal to $N-1$. On the other hand, a complete basis of such gauge invariant operators also defines the physical pseudoscalars/axions which can be expressed as a combination of the θ_i's, since their phases

$$\theta_\alpha = \frac{m_1^{(\alpha)}\theta_1}{V_1} + \cdots + \frac{m_N^{(\alpha)}\theta_N}{V_N} \tag{6}$$

are automatically orthogonal to the Goldstone boson (4). It is convenient to represent the pseudoscalars above as vectors in a $N+1$ dimensional space, such as for example, up to normalization

$$\vec{\theta}_X = (gX_1V_1, \cdots, gX_NV_N, M) \ , \quad \vec{\theta}_\alpha = \left(\frac{m_1^{(\alpha)}\theta_1}{V_1}, \cdots, \frac{m_N^{(\alpha)}\theta_N}{V_N}, 0 \right) \ . \tag{7}$$

The scalar potential (5) then gives masses to at most $N-1$ pseudoscalars. Consequently, there is always (at least) one leftover massless pseudoscalar, which will

[c]It can be checked that the argument below does not change if some of the fields in the scalar potential appear with a complex conjugation.

be a PQ axion candidate if it has the appropriate couplings. At the perturbative level, it is therefore always possible to define a PQ symmetry in models with an anomalous $U(1)_X$ gauge factor. □

As one will see in the next sections, nonperturbative effects can generate gauge-invariant potential terms of the form

$$V_{np} = \sum_\beta e^{-q_\beta s_0 - i c_\beta a_S} \lambda_\beta \Phi_1^{p_1^{(\beta)}} \cdots \Phi_N^{p_N^{(\beta)}} + \text{h.c.} \ , \tag{8}$$

where s_0 is the vev of a scalar and q_β, c_β are numbers. Whenever such terms are generated, the leftover massless axion will get a mass from effects other than the usual QCD ones. Such terms can be generated by field-theory nonperturbative effects, instantonic effects in string theory or quantum gravity effects more generally.

2.2. *Anomalous $U(1)$: the heterotic case*

In perturbative heterotic string theory constructions, there is only one possible anomalous $U(1)_X$ and one field, the universal axion-dilaton S, transforming non-linearly under gauge transformations. Those act on the different superfields involved as[d]

$$\begin{aligned} &\delta V_X \ = \ \Lambda \ + \ \bar{\Lambda} \ , \quad \delta \phi^i \ = \ -2 \ q_i \ \phi^i \ \Lambda \ \equiv \ -2 X^i \Lambda \ , \\ &\delta S \ = \ \delta_{GS} \ \Lambda \ \equiv \ -2 \ X^S \Lambda \ , \end{aligned} \tag{9}$$

where X^i, X^S define the holomorphic Killing vectors. The modified Kahler potential for the universal axion-dilaton is

$$K \ = \ - \ \ln \left(S + \bar{S} - \delta_{GS} V_X \right) \tag{10}$$

and it encodes the Fayet-Iliopoulos (FI) term which appears in the D-term

$$D_X \ = \ X^I \ \partial_I G \ = \ X^I \ \partial_I K \ = \ q_i \ \phi^i \ \partial_i K \ + \ \frac{\delta_{GS}}{2(S+\bar{S})} \ , \tag{11}$$

where in (11) $G = K + \ln |W|^2$ and we used the gauge invariance of the superpotential $X^I \partial_I W = 0$. We consider $\delta_{GS} > 0$ in what follows. In all known perturbative constructions there always exists in the massless spectrum a field with appropriate sign of the charge (negative in our conventions) whose vev is able to cancel perturbatively the (field-dependent) FI term and maintain supersymmetry. We consider the minimal case of one such field, called ϕ in what follows, and normalize its charge to -1, following [7].

Anomaly cancellation conditions relate mixed anomalies $C_i = U(1)_X G_i^2$, where G_i are the various semi-simple factors of the gauge group $G = \prod_{i=1}^N G_i$, such that

$$\delta_{GS} \ = \ \frac{C_1}{k_1} \ = \ \frac{C_2}{k_2} = \cdots = \frac{C_N}{k_N} \ = \ \frac{1}{192\pi^2} Tr(q_X) \ , \tag{12}$$

[d]We use here the same convention as in [7] to define charges of chiral superfields.

where the k_i's are the Kac-Moody levels defining the tree-level gauge kinetic functions

$$f_i \ = \ k_i \ S \ . \tag{13}$$

The last term in (12) is the FI term, proportional to the mixed $U(1)_X$ - gravitational anomaly, where $Tr(q_X)$ is the sum of $U(1)_X$ charges over all the charged fermions in the spectrum. Therefore, once the FI term is generated, all mixed anomalies have to be different from zero and the theory *must* contain charged matter.

3. A light axion: gaugino condensation and anomalous $U(1)$ in heterotic theories

Gaugino condensation in heterotic theories in the presence of the (generic) anomalous $U(1)$ gauge symmetry discussed in Section 2.2 has to fulfill the consistency requirements dictated by the coexistence of two local symmetries: supersymmetry and the gauge symmetry. However, although the pure Super-Yang-Mills gaugino condensation superpotential $e^{-3S/2b_0}$, where b_0 is the beta function of the hidden sector, is not gauge invariant, gauge invariance does not forbid gaugino condensation to take place, as was discussed in the heterotic string case some time ago in [7]. It was shown there that the GS cancellation of gauge anomalies restricts the nonperturbative dynamics such that the nonperturbative superpotential is precisely gauge invariant.

Taking for simplicity a SUSY-QCD model with N_c colors and $N_f < N_c$ flavors and denoting by Q ($\tilde{Q}$) the hidden sector quarks (antiquarks) of $U(1)_X$ charges q ($\tilde{q}$), the GS conditions fix completely the sum of the charges to be

$$C_h \ = \ \frac{1}{4\pi^2} \ N_f (q+\tilde{q}) \ = \ \delta_{GS} k_h \ , \tag{14}$$

where k_h is the Kac-Moody level of the hidden sector gauge group. This turns out to be precisely the gauge invariance condition of the nonperturbative superpotential

$$W_{np} \ = \ (N_c - N_f) \left[\frac{e^{-8\pi^2 k_h S}}{\det(Q\tilde{Q})} \right]^{\frac{1}{N_c - N_f}} \ . \tag{15}$$

Notice that anomaly cancellations (12) and the structure of the D-term (11) unambiguously show that the charge of the hidden sector mesons $Q\tilde{Q}$ has the same sign as the induced FI term. Notice also that the charges allow for a perturbative coupling of the form

$$W_p = \lambda_i^{\tilde{j}} \left(\frac{\phi}{M_P} \right)^{q+\tilde{q}} Q^i \tilde{Q}_{\tilde{j}} \ . \tag{16}$$

Since ϕ gets a large vev of the order of the FI term, below the scale of $U(1)_X$ gauge symmetry breaking the perturbative term (16) becomes a mass term for the hidden sector quarks and the dynamics of condensation is essentially that of supersymmetric QCD.

3.1. *The light axion*

In supersymmetric QCD there is no light axion. The only global anomaly-free symmetry in the UV is an R-symmetry, which is broken explicitly by the mass term. In the model introduced in [7] and briefly reviewed above, the mass term is replaced by the coupling (16). Then it is easy to check that the following global R-symmetry

$$\begin{aligned} &\theta' = e^{i\alpha}\theta \ , \quad (Q,\tilde{Q})'(\theta') = e^{\frac{i(N_f-N_c)\alpha}{N_f}}(Q,\tilde{Q})(\theta) \ , \\ &\phi'(\theta') = e^{\frac{2iN_c\alpha}{N_f(q+\tilde{q})}}\phi(\theta) \ , \quad S'(\theta') = S(\theta) \ , \end{aligned} \tag{17}$$

is exact and anomaly-free with respect to $SU(N_c)$[e]. It is also spontaneously broken, therefore one expects a massless Goldstone boson. More generally, one can combine the R-symmetry above with the gauge symmetry. Indeed:

$$\begin{aligned} &\phi'(\theta') = e^{iq_\phi\alpha}\phi(\theta) \ , \quad S'(\theta') = S(\theta) - \frac{i}{2}q_S\alpha \ , \\ &Q'(\theta') = e^{iq_Q\alpha}Q(\theta) \ , \quad \tilde{Q}'(\theta') = e^{iq_{\tilde{Q}}\alpha}\tilde{Q}(\theta) \end{aligned} \tag{18}$$

is a (non-anomalous) R-symmetry of the (non-perturbative) superpotential if:

$$q_\phi = \frac{2N_c}{(q+\tilde{q})N_f} - \frac{P}{q+\tilde{q}} \ , \quad q_Q + q_{\tilde{Q}} = \frac{2(N_f-N_c)}{N_f} + P \ , \quad q_S = \frac{N_f P}{4\pi^2 k_h} \ , \tag{19}$$

where P is a number and $P=0$ corresponds to the R-symmetry (17).

The model has three pseudoscalars, on which we now concentrate. In order to identify the massless axion, it is enough to parametrize the original fields by ignoring any other field than those pseudoscalars. By defining them in order to have canonical kinetic terms, we are led to the parametrization

$$S = s_0\left(1 + i\sqrt{2}\frac{a_S}{M_P}\right) \ , \quad \phi = \frac{V}{\sqrt{2}}e^{\frac{ia_\phi}{V}} \ , \quad M = Q\tilde{Q} = M_0 I_{N_f\times N_f} e^{i\sqrt{\frac{2}{N_f M_0}}a_M} \ , \tag{20}$$

where s_0, V, and M_0 are vev's. One combination of those pseudoscalars

$$a_X \propto \frac{\delta_{GS}}{\sqrt{2}s_0}a_S + 2Va_\phi - (q+\tilde{q})\sqrt{2N_f M_0}a_M \tag{21}$$

is absorbed by the $U(1)_X$ gauge field. Another one is shifted by the symmetry (18-19), which we choose such that it leaves a_X invariant. In the limit where $M_0 \ll V, M_P$, the value of P which achieves this is

$$P = \frac{2N_c}{N_f\left(1 + \frac{\delta_{GS}^2 M_P^2}{8V^2 s_0^2}\right)} + \mathcal{O}\Big(\frac{M_0}{V, M_P}\Big) \ , \tag{22}$$

[e]The anomalies with respect to $U(1)_X$ can be canceled by fields from other sectors, for example (MS)SM fields.

and the associated symmetry current gives us the expression of the physical axion a_{PQ}:

$$J_\mu \propto \frac{1}{\frac{1}{V}+\frac{8s_0^2 V}{\delta_{GS}^2 M_P^2}} \partial_\mu \left(a_\phi - \frac{2\sqrt{2}s_0 V}{\delta_{GS} M_P} a_S \right) + \mathcal{O}\Big[\frac{M_0}{V, M_P}\Big] \equiv \; f_a \partial_\mu a_{PQ} \ , \tag{23}$$

where we identified the axion decay constant

$$\frac{1}{f_a} = \sqrt{\frac{1}{V^2}+\frac{8s_0^2}{\delta_{GS}^2 M_P^2}} \ . \tag{24}$$

Natural values are of order the unification scale $f_a \sim M_{GUT}$, although smaller values are possible in orientifold models.

3.2. *Simplified description*

If the scale of hidden sector condensation is well below the scale of $U(1)_X$ gauge symmetry breaking, which we assumed in deriving expressions (22-23), there is an approximate decoupling between the hidden sector dynamics and the $U(1)_X$ dynamics. In particular, in this limit the $SU(N_c)$ dynamics is essentially the one of supersymmetric QCD, which has no light particles, therefore no light composite axion. It should be therefore possible to describe accurately the light axion physics by integrating out the hidden sector. By doing this, one finds an effective superpotential

$$W_{eff} = W_0 + N_c (\det \lambda)^{\frac{1}{N_c}} M_P^{3-N_f/N_c} \left(\frac{\phi}{M_P}\right)^{\frac{N_f(q+\tilde{q})}{N_c}} e^{-\frac{8\pi^2 k_h S}{N_c}} \ , \tag{25}$$

where the constant W_0 was added for the purpose of coupling to gravity later on. The effect of the hidden sector condensation is therefore of generating a non-perturbative superpotential, sometimes said to be of “fractional instanton” type, as compared to “stringy instanton” effects, which would be proportional to $e^{-8\pi^2 S}$ in our conventions. According to our general discussion in Section 2, the phase of the nonperturbative term in (25) defines a physical axion, which is orthogonal to the Goldstone boson a_X precisely when the GS anomaly cancellation conditions (14) are imposed. One can write explicitly a_X and the (for now) massless axion a_{PQ} by introducing a rotation matrix

$$a_X = \cos\theta \ a_S + \sin\theta \ a_\phi \ , \quad a_{PQ} = -\sin\theta \ a_S + \cos\theta \ a_\phi \ , \tag{26}$$

with $\tan\theta = 2\sqrt{2}s_0 V/(\delta_{GS} M_P)$. Notice that a_{PQ} coincides with the leading order expression of the axionic current obtained in (23).

At the global supersymmetry level, the axion mass is protected by the R-symmetry (22). However, after coupling to supergravity, the constant W_0 breaks explicitly the R-symmetry and as such the axion will get a scalar potential and therefore a mass [12]. Without entering details of moduli stabilization, one expects a scalar potential of the form

$$V(a_{PQ}) \sim W_0 N_c (\det\lambda)^{\frac{1}{N_c}} M_P^{3-N_f/N_c} \left(\frac{V}{M_P}\right)^{\frac{N_f(q+\tilde{q})}{N_c}} e^{-\frac{8\pi^2 k_h s_0}{N_c}} \cos\left(\frac{(q+\tilde{q})N_f}{N_c}\frac{a_{PQ}}{f_a}\right) , \tag{27}$$

where the axion decay constant is given in (24). By using the order of magnitude value for the gravitino mass $m_{3/2} \sim W_0$ and the definition of the IR dynamical scale

$$\Lambda_L^3 = (\det\lambda)^{\frac{1}{N_c}} M_P^{3-N_f/N_c} \left(\frac{\phi}{M_P}\right)^{\frac{N_f(q+\tilde{q})}{N_c}} e^{-\frac{8\pi^2 k_h S}{N_c}} , \tag{28}$$

one finds that this axion can solve the strong CP problem if

$$m_{3/2}\Lambda_L^3 \ll 10^{-10} f_\pi^2 m_\pi^2 . \tag{29}$$

This is a very strong constraint, which favors in this minimal model low values of the gravitino mass and of the dynamical scale Λ_L. Using the fact that in the minimal model of [7] supersymmetry was broken, and $m_{3/2} \sim \Lambda_L^3/(VM_P)$, one finds, without an additional source of supersymmetry breaking, the constraint $m_{3/2} \ll 10^{-14}$ eV, which is not realistic in known mediations of supersymmetry breaking. In this model therefore, an additional source of supersymmetry breaking is necessary, whereas for a gravitino mass corresponding to standard mechanisms for supersymmetry breaking, the axion is too heavy to solve the strong CP problem.

3.3. *More refined analysis*

Let us perform a slightly more general analysis, by keeping the hidden sector mesons in the discussion. The hidden mesons are described by chiral (super)fields of charge $q+\tilde{q}$ and have a Kahler potential, computed along the flat directions of $SU(N_c)$, equal to

$$K = Tr(M^\dagger M)^{\frac{1}{2}} . \tag{30}$$

The hidden mesons appear in the full superpotential

$$W = W_{np} + W_p = (N_c - N_f)\left[\frac{e^{-8\pi^2 k_h S}}{\det(M)}\right]^{\frac{1}{N_c - N_f}} + \lambda_i^{\tilde{j}} \left(\frac{\phi}{M_P}\right)^{q+\tilde{q}} M_{\tilde{j}}^i \tag{31}$$

and add a pseudoscalar axionic degree of freedom a_M, that is encoded in the parametrization (20). Notice that solving for M in (31) gives back (25). Out of the original three pseudoscalars, one is the Goldstone boson absorbed by the gauge field (21) and the other two are physical, called a_1 and a_2 in what follows. They can be parametrized by the gauge invariant operators in (31) and, up to normalization, can be written as

$$a_1 \sim 8\sqrt{2}\pi^2 k_h s_0 a_S + \sqrt{\frac{2N_f}{M_0}} a_M , \quad a_2 \sim \frac{q+\tilde{q}}{V} a_\phi + \sqrt{\frac{2}{N_f M_0}} a_M . \tag{32}$$

Notice that they are both orthogonal to the Goldstone boson a_X, as enforced by gauge invariance. They are not orthogonal to each other, fact to be taken into account in what follows. The hidden sector nonperturbative dynamics is giving a mass to the linear combination

$$a_h \sim \frac{a_1}{N_c - N_f} + a_2 \ , \tag{33}$$

whereas the orthogonal combination a_l defined by $(a_h, a_l) = 0$ is the massless (at the global supersymmetric level) axion. For general vev's its expression is relatively involved. However, in the limit we considered in the previous sections $M_0 \ll V, M_P$, one can easily find that

$$a_l \sim a_1 - N_f a_2 \rightarrow a_{PQ} \tag{34}$$

is precisely the light axion (23,26), obtained by integrating-out from the start the hidden sector mesons in the simplified description.

4. A massless axion: the 3-2 model

The main problem with the previous minimal model is that the hidden sector nonperturbative dynamics was giving a mass to the axion through supergravity interactions. Nonperturbative dynamics is however often instrumental for stabilizing moduli, in our case the very modulus involved in the GS mechanism. The natural next step is to identify models in which the hidden sector nonperturbative dynamics leaves an exactly massless axion, even after coupling to (super)gravity. One way to achieve this goes as follows: at the perturbative level, as we proved in Section 2.1, there is always a massless axion in models with anomalous $U(1)_X$. Suppose now that the hidden sector producing the nonperturbative dynamics has an R-symmetry itself, in the limit in which the anomalous abelian gauge dynamics is turned off. Then if the condensation breaks spontaneously the R-symmetry, there is another R-axion coming from the hidden sector. In total there are therefore two axions in the limit where gravity is decoupled. By turning on gravity and adding a constant which breaks explicitly the R-symmetry, one (linear combination) of the two axions becomes massive. But the other one remains massless down to the QCD scale and behaves as an ideal candidate for a PQ QCD axion. Essentially the nonperturbative dynamics is not adding a potential for the axion, but is just stabilizing the GS modulus.

One explicit model of this type uses for the hidden sector the 3-2 model of supersymmetry breaking [13]. The gauge group of the model is $G = G_h \times U(1)_X \times \cdots$, where $G_h = SU(3) \times SU(2)$ is the hidden sector gauge group. The nonabelian factor $SU(3)$ is confining with a dynamical scale Λ_3. The matter content in the UV contains the chiral multiplets

$$\begin{aligned} &Q_i^\alpha(3,2) \ , \quad L^\alpha(1,2) \ , \\ &\bar{U}^i(\bar{3},1) \ , \quad \bar{D}^i(\bar{3},1) \ \rightarrow \quad \bar{Q}^i_\alpha = (\bar{D}^i, \bar{U}^i) \ , \end{aligned} \tag{35}$$

in a self-explanatory notation (notice that the α index of $\bar{Q}$ is not gauged under $SU(2)$ and only represents a convenient repackaging). The model has two anomaly-free global symmetries, one acting like hypercharge and an R-symmetry:

$$\begin{aligned} U(1)_Y \;:& \qquad Y(Q) = \frac{1}{6}\ , \quad Y(\bar{U}) = -\frac{2}{3}\ , \quad Y(\bar{D}) = \frac{1}{3}\ , \quad Y(L) = -\frac{1}{2}\ , \\ U(1)_R \;:& \qquad R(Q) = -1\ , \quad R(\bar{U}) = R(\bar{D}) = 0\ , \quad R(L) = 3\ . \end{aligned} \tag{36}$$

Below the scale of $SU(3)$ condensation, the dynamics is governed by the gauge invariant operators

$$X_1 = Q\bar{D}L\ , \quad X_2 = Q\bar{U}L\ , \quad X_3 = \det \bar{Q}_\alpha Q^\beta\ . \tag{37}$$

The low-energy superpotential, compatible with the symmetries and the condensation dynamics, is given by

$$W_{\text{eff}} = \lambda X_1 + \frac{2\Lambda_3^7}{X_3}\ . \tag{38}$$

The analysis of the potential, including the D-term contributions, shows that $\langle X_1\rangle$ and $\langle X_3\rangle$ are non-vanishing whereas $\langle X_2\rangle$ vanishes. There are then two pseudoscalars in the hidden sector, the potential axions in the phases of X_1 and X_3. One linear combination of them will get a mass from the nonperturbative dynamics, and the second one gets a mass from couplings to (super)gravity, as in the model described in the preceding section.

If we now couple this model to an anomalous $U(1)_X$, we would get an additional pseudoscalar from the high-energy anomalous $U(1)_X$ sector. There is therefore one leftover axion which is massless all the way down to the QCD scale, being a good candidate for a PQ axion. To restrict the superpotential, one could use the anomalous gauge symmetry instead of imposing the hypercharge global symmetry as above. We can for instance give the following charges to the multiplets (where n is some number):

$$U(1)_X : \begin{cases} X(Q) = \frac{1}{6} + n \\ X(\bar{U}) = -\frac{1}{3} \\ X(\bar{D}) = \frac{1}{3} \\ X(L) = -\frac{1}{2} - \frac{n}{3} \end{cases} \Longrightarrow \begin{cases} X(X_1) = \frac{2n}{3} \\ X(X_2) = \frac{2(n-1)}{3} \\ X(X_3) = \frac{1}{3} + 2n \\ X(\Lambda_3) = \frac{1}{21} + \frac{2n}{7} \end{cases} , \tag{39}$$

where, as in the model discussed previously, the condensation scale $\Lambda_3 = e^{\frac{-8\pi^2 k_3 S}{7}}$ is not-gauge invariant anymore due to the $U(1)_X SU(3)^2$ anomaly:

$$U(1)_X SU(3)^2 : C_3 = \frac{1}{4\pi^2} \times \left(\frac{1}{3} + 2n\right)\ , \quad U(1)_X SU(2)^2 : C_2 = \frac{1}{4\pi^2} \times \frac{8n}{3}\ , \tag{40}$$

while the nonperturbative superpotential is:

$$W_{\text{eff}} = \lambda \left(\frac{\phi}{M_P}\right)^{\frac{2n}{3}} X_1 + \frac{2\Lambda_3^7}{X_3}\ . \tag{41}$$

The first term in (41) is a perturbatively generated operator if we assume that n is a multiple of $\frac{3}{2}$. If $\Lambda_3 \ll V$, analogously to the model in the previous section this axion is essentially a combination of a_S and a_ϕ. The axion decay constant will be determined as before and is therefore naturally of the order of the unification scale.

5. Gauged flavor symmetry and axion couplings to matter

We now identify the $U(1)_X$ discussed in the preceding sections with a flavor symmetry [8, 14], since those are naturally anomalous due to the structure of fermion masses and lead to the GS mechanism [15]. Doing this, we will see that we generate axionic couplings for the light physical axion of the theory. Since the explicit examples discussed so far were supersymmetric, we focus on the Minimal Supersymmetric Standard Model (MSSM) in what follows.

We then charge the different MSSM superfields such that the Yukawa terms, as well as the μ-term, now explicitly involve ϕ:

$$\begin{aligned} W_{\text{MSSM}} =& \lambda_{u,ij}\Big(\frac{\phi}{M_P}\Big)^{X_{q_i}+X_{u_j}+X_{h_u}} Q_i U_j H_u + \lambda_{d,ij}\Big(\frac{\phi}{M_P}\Big)^{X_{q_i}+X_{d_j}+X_{h_d}} Q_i D_j H_d + \\ & \lambda_{e,ij}\Big(\frac{\phi}{M_P}\Big)^{X_{l_i}+X_{e_j}+X_{h_d}} L_i E_j H_d + \mu\Big(\frac{\phi}{M_P}\Big)^{X_{h_u}+X_{h_d}} H_u H_d \ . \end{aligned} \tag{42}$$

A clever choice of $U(1)_X$ charges for the MSSM fields then allows to account for, or at least soften, the mass hierarchies and the $\mu-$problem of the MSSM. We note that the $U(1)_X$ charge of ϕ makes it possible to choose most, if all, of the MSSM charges to be positive, consistently with the GS conditions (12).

Starting from this superpotential, one can work out the couplings of the physical axion to the MSSM fields. Triangle loop diagrams combined with the GS term give for instance the coupling of the axion to QCD gauge fields:

$$\mathcal{L} \supset \frac{\sum_i (2X_{q_i} + X_{u_i} + X_{d_i})}{64\pi^2} \frac{a_{PQ}}{f_a} Tr(G\tilde{G}) = \frac{C_3}{16} \frac{a_{PQ}}{f_a} Tr(G\tilde{G}) \ , \tag{43}$$

where a_{PQ} is given by the expression in (23)[f], its decay constant f_a in (24) and C_3 is the $SU(3)$ gauge anomaly coefficient which appears in (12). Note that the domain wall number $N_{\text{DW}} = \sum_i (2X_{q_i} + X_{u_i} + X_{d_i})$ can be chosen equal to 1 with a consistent choice of charges for the Higgs doublets. This expression can be understood as a modification of the QCD kinetic function (13) when the quarks are integrated out:

$$f_3 = k_3 S - \frac{C_3}{2} \ln\left(\frac{\phi}{M_P}\right) \ , \tag{44}$$

which displays clearly the two canceling contributions to the $U(1)_X SU(3)^2$ anomaly. Similar expressions hold for the other factors of the MSSM gauge groups.

[f]This assumes that the axion is mostly carried by a_ϕ and a_S, which requires that every other dynamics breaking the PQ symmetry in the hidden sector or in the MSSM happens at a much lower energy.

We can deduce from this an interesting prediction of such models if we embed them in unified theories. Indeed, in such a case the anomaly coefficients are linked at the unification scale. For instance, for $SU(5)$ unification, the MSSM gauge couplings verify $g_3^2 = g_2^3 = \frac{5}{3}g_Y^2$, while the fact that S determines all the gauge kinetic functions gives $g_Y^2 k_Y = g_2^2 k_2 = g_3^2 k_3$ and the GS conditions impose $\frac{C_3}{k_3} = \frac{C_2}{k_2} = \frac{C_Y}{k_Y}$. All this can be combined to get $C_3 = C_2 = \frac{3}{5}C_1$. Thus, the ratios of the couplings of the axion to the MSSM gauge fields are determined: for instance we get that the ratio (at the GUT scale) between the electromagnetic coupling and the gluons coupling is

$$\frac{E}{N} = \frac{8}{3} \ . \tag{45}$$

We stress that (45) is valid not only in flavor models of the type (42), but in any anomalous $U(1)$ model in which $SU(5)$ unification of gauge couplings is imposed. Indeed, (45) is enforced uniquely by unification and the kinetic function (44), determined by gauge invariance.

There are also couplings of the axion to the spin of fermions arising from (42):

$$\frac{\partial_\mu a}{f_a}(\overline{\psi_{L,I}} X_{L,I}\gamma^\mu \psi_{L,I} + \overline{\psi_{R,I}} X_{R,I}\gamma^\mu \psi_{R,I}) \ . \tag{46}$$

Their strength is given by the $U(1)_X$ charges of the MSSM fields, so the lighter generations are more coupled than the heavier ones. Besides, once expressed in terms of mass eigenstates, those couplings can be off-diagonal in flavor space, leading to possible flavor-changing currents [16]. However, if the axion dynamics lies at the string/GUT scale, all those effects are very much suppressed and evade current constraints. Still, since the couplings to the first generation of the MSSM are not specifically suppressed, recently proposed experiments [10] could have the sensitivity to probe such string scale decay constants in the near future.

6. Comments on moduli stabilization and intermediate scale decay constants

Moduli stabilization and axions in string models with anomalous $U(1)$ were studied in various papers [17] and the issue of axion mass and decay constant in string theory in various works, see e.g. [18, 19].

In the context of models of the type discussed in our note, the value of the gravitino mass is highly correlated to the stabilization of the moduli. One should distinguish the case where the coupling to supergravity lifts the axion mass, like in the model in Section 3, from the case where it does not, like in Section 4. In the first case, there is a strong correlation between the values of the gravitino mass and the axion mass such that keeping the axion light requires very small values of the gravitino mass. It was shown that in minimal models the requirement of "uplifting" the vacuum energy to zero is only compatible with large values of the gravitino mass [20]. In more sophisticated models with several charged scalars the gravitino mass can be reduced to the TeV range [21], but still far from the small values needed to keep the axion light enough. In other stabilization schemes, it is still possible to keep the axion light enough with more realistic values of the gravitino mass, see e.g. [19]. On the other hand, for models in which coupling to supergravity does not lift the axion mass, like in our Section 4, the scale of supersymmetry breaking is completely decorrelated from the axion mass, which then only gets a mass from QCD nonperturbative effects.

The moduli stabilization in Sections 3 and 4 was also enforcing a high-scale axion decay constant due to the $U(1)_X$ D-term expression (11). This can be relaxed in models where the moduli sector is slightly more complex. For example, let us consider a model of two moduli and a charged superfield:

$$K = -\frac{3}{2}\ln(T_1 + \overline{T_1} - \delta_1 V_X) - \frac{3}{2}\ln(T_2 + \overline{T_2} + \delta_2 V_X) + \phi^\dagger e^{-2V_X}\phi \ , \tag{47}$$

on which the anomalous $U(1)_X$ symmetry acts as follows:

$$\delta V_X = \Lambda + \bar{\Lambda} \ , \quad \delta\phi = 2\phi\Lambda \ , \quad \delta T_1 = \delta_1\Lambda \ , \quad \delta T_2 = -\delta_2\Lambda \ . \tag{48}$$

The $U(1)_X$ D-term potential $V_D = \frac{g_X^2}{2}(|\phi|^2 + \frac{3\delta_2}{4(T_2+\overline{T_2})} - \frac{3\delta_1}{4(T_1+\overline{T_1})})^2$ now allows for a high scale stabilization of the moduli with a small or intermediate scale ϕ. To illustrate this, we furthermore assume that there are two hidden strong sectors 1 and 2, with gauge kinetic functions given by:

$$f_1 = \frac{T_1}{4\pi} \ , \quad f_2 = \frac{n_2 T_1 + n_1 T_2}{4\pi} \ , \quad \text{where } n_i = \pi\delta_i \text{ are integers} \ , \tag{49}$$

such that the group 1 is anomalous with respect to $U(1)_X$ whereas f_2 is gauge invariant. Strong dynamics can then generate couplings of the type:[g]

$$W = W_0 + A\phi^{n_1} e^{-2\pi T_1} + B e^{-2\pi(n_2 T_1 + n_1 T_2)} \ . \tag{50}$$

In order to compute the vacuum of the theory, we assume that the uplift of the vacuum energy does not depend on the axions (e.g. *à la* KKLT [22]). Thus, as far as the axions are concerned we look at first order for the supersymmetric vacuum:[h]

$$\begin{aligned} D_\phi W &\equiv W_\phi + K_\phi W = A n_1 \phi^{n_1 - 1} e^{-2\pi T_1} + \overline{\phi} W = 0 \\ D_{T_1} W &= -2\pi A\phi^{n_1} e^{-2\pi T_1} - 2\pi n_2 B e^{-2\pi(n_2 T_1 + n_1 T_2)} - \frac{3}{2(T_1 + \overline{T_1})} W = 0 \\ D_{T_2} W &= -2\pi n_1 B e^{-2\pi(n_2 T_1 + n_1 T_2)} - \frac{3}{2(T_2 + \overline{T_2})} W = 0 \ , \end{aligned} \tag{51}$$

[g] Those nonperturbative effects have periodicity $T_i = T_i + 1$ and are called stringy instanton effects. The other option is to use fractional instanton effects, like in Section 3, which would be, with the present section notations, of the type $e^{-2\pi T_i/N}$ where $N \in \mathbb{N}$.

[h] Those three equations can be combined to check that the D-term potential vanishes.

and we solve this set of equations given the value of $m_{3/2} = \frac{|W|e^{K/2}}{M_P^2}$ and assuming that $W \approx W_0$ and $|\phi|^2 \ll T_{1,2}^{-1}$, which we eventually check to be valid:

$$\frac{T_2 + \overline{T_2}}{T_1 + \overline{T_1}} = \frac{n_2}{n_1} \ , \quad 2\pi n_2 (T_1 + \overline{T_1}) e^{-2\pi n_2 (T_1 + \overline{T_1})} = \frac{3W_0}{2B} \ , \quad |\phi| = \left| \frac{W_0 e^{\pi(T_1 + \overline{T_1})}}{n_1 A} \right|^{\frac{1}{n_1 - 2}} . \tag{52}$$

If we choose for instance $m_{3/2} = 10$ GeV, $n_1 = 3$ and $n_2 = 1$, we numerically get $T_1 + \overline{T_1} = 3(T_2 + \overline{T_2}) \approx 6M_P$ and $|\phi| \approx 10^{11}$ GeV, which implies an intermediate scale for the physical axion. However, in this setup the axion mass is tied to the supersymmetry breaking scale and cannot be light enough to provide a proper QCD axion. To cope with this, one can for instance implement the configuration (47) within the 3-2 model of Section 4. This amounts to consider the following superpotential (where all fields are those defined either above or in Section 4):

$$W = W_0 + \lambda \left(\frac{\phi}{M_P} \right)^{\frac{2n}{3}} X_1 + \frac{2\Lambda_3^7}{X_3} + B e^{-2\pi k_2 (n_2 T_1 + n_1 T_2)} \ , \ \text{with } \Lambda_3 = e^{\frac{-2\pi k_1 T_1}{7}} \ . \tag{53}$$

There is as expected a massless axion in the low-energy limit, and its associated decay constant can be of intermediate scale: for instance, choosing $n_1 = n_2 = 1, n = 6, k_1 = 17, k_2 = 4$ and $\phi \approx 10^{12}$ GeV, one finds $X_1^{1/3} \approx 10^{12}$ GeV and a gravitino mass of $\approx 10^{-4}$ eV (consistent with gauge mediation of supersymmetry breaking).

Acknowledgments

The authors acknowledge partial support from the ANR Black-dS-String, and thank Michele Cicoli and Stefan Pokorski for enlightening discussions.

References

1. R. D. Peccei and H. R. Quinn, *Phys. Rev. Lett.* **38** (1977) 1440; *Phys. Rev. D* **16** (1977) 1791.
2. F. Wilczek, *Phys. Rev. Lett.* **40** (1978) 279; S. Weinberg, *Phys. Rev. Lett.* **40** (1978) 223.
3. A. Ringwald [arXiv:1407.0546]; D. J. E. Marsh, *Phys. Rept.* **643**, 1 (2016) [arXiv:1510.07633]
4. J. P. Derendinger, L. E. Ibanez and H. P. Nilles, *Phys. Lett. B* **155** (1985) 65; M. Dine, R. Rohm, N. Seiberg and E. Witten, *Phys. Lett. B* **156** (1985) 55.
5. L. E. Ibanez and G. G. Ross, *Phys. Lett. B* **332** (1994) 100 [hep-ph/9403338]; P. Binetruy and P. Ramond, *Phys. Lett. B* **350** (1995) 49 [hep-ph/9412385].
6. L. E. Ibanez, *Phys. Lett. B* **303** (1993) 55 [hep-ph/9205234].
7. P. Binetruy and E. Dudas, *Phys. Lett. B* **389** (1996) 503 [hep-th/9607172].
8. Y. Ema, K. Hamaguchi, T. Moroi and K. Nakayama, *JHEP* **1701** (2017) 096 [arXiv:1612.05492]; L. Calibbi, F. Goertz, D. Redigolo, R. Ziegler and J. Zupan, *Phys. Rev. D* **95** (2017) no.9, 095009 [arXiv:1612.08040].
9. P. J. Steinhardt and M. S. Turner, *Phys. Lett.* **129B** (1983) 51; G. Lazarides, R. K. Schaefer, D. Seckel and Q. Shafi, *Nucl. Phys. B* **346** (1990) 193; G. R. Dvali, K. Choi, H. B. Kim and J. E. Kim, *Nucl. Phys. B* **490** (1997) 349 [hep-ph/9606372];

T. Banks and M. Dine, *Nucl. Phys. B* **505** (1997) 445 [hep-th/9608197]; T. Banks, M. Dine and M. Graesser, *Phys. Rev. D* **68** (2003) 075011 [hep-ph/0210256]; L. Visinelli and P. Gondolo, *Phys. Rev. D* **81**, 063508 (2010) [arXiv:0912.0015]; M. Kawasaki, F. Takahashi and M. Yamada, *Phys. Lett. B* **753** (2016) 677 [arXiv:1511.05030]; Y. Nomura, S. Rajendran and F. Sanches, *Phys. Rev. Lett.* **116** (2016) no.14, 141803 [arXiv:1511.06347]; L. Visinelli, *Phys. Rev. D* **96**, no. 2, 023013 (2017) [arXiv:1703.08798]; P. Agrawal, G. Marques-Tavares and W. Xue, *JHEP* **1803** (2018) 049 [arXiv:1708.05008]; M. Kawasaki, F. Takahashi and M. Yamada, *JHEP* **1801** (2018) 053 [arXiv:1708.06047]; N. Kitajima, T. Sekiguchi and F. Takahashi, *Phys. Lett. B* **781** (2018) 684 [arXiv:1711.06590]; P. W. Graham and A. Scherlis, *Phys. Rev. D* **98** (2018) no.3, 035017 [arXiv:1805.07362]; F. Takahashi, W. Yin and A. H. Guth, *Phys. Rev. D* **98** (2018) no.1, 015042 [arXiv:1805.08763].

10. D. Budker, P. W. Graham, M. Ledbetter, S. Rajendran and A. Sushkov, *Phys. Rev. X* **4** (2014) no.2, 021030 [arXiv:1306.6089]; P. Sikivie, N. Sullivan and D. B. Tanner, *Phys. Rev. Lett.* **112** (2014) no.13, 131301 [arXiv:1310.8545]; Y. Kahn, B. R. Safdi and J. Thaler, *Phys. Rev. Lett.* **117** (2016) no.14, 141801 [arXiv:1602.01086]; M. Silva-Feaver *et al.*, *IEEE Trans. Appl. Supercond.* **27** (2017) no.4, 1400204 [arXiv:1610.09344]; P. W. Graham, D. E. Kaplan, J. Mardon, S. Rajendran, W. A. Terrano, L. Trahms and T. Wilkason, *Phys. Rev. D* **97** (2018) no.5, 055006 [arXiv:1709.07852]; S. P. Chang, S. Haciomeroglu, O. Kim, S. Lee, S. Park and Y. K. Semertzidis, *PoS PSTP* **2017** (2018) 036 [arXiv:1710.05271].
11. P. Binetruy, C. Deffayet and P. Peter, *Phys. Lett. B* **441**, 52 (1998) [hep-ph/9807233].
12. J. Bagger, E. Poppitz and L. Randall, *Nucl. Phys. B* **426** (1994) 3 [hep-ph/9405345].
13. I. Affleck, M. Dine and N. Seiberg, *Nucl. Phys. B* **241** (1984) 493; *Phys. Rev. Lett.* **52** (1984) 1677.
14. F. Wilczek, *Phys. Rev. Lett.* **49** (1982) 1549; C. Q. Geng and J. N. Ng, *Phys. Rev. D* **39** (1989) 1449; Z. G. Berezhiani and M. Y. Khlopov, *Z. Phys. C* **49** (1991) 73; K. S. Babu and S. M. Barr, *Phys. Lett. B* **300** (1993) 367 [hep-ph/9212219]; M. E. Albrecht, T. Feldmann and T. Mannel, *JHEP* **1010** (2010) 089 [arXiv:1002.4798]; C. S. Fong and E. Nardi, *Phys. Rev. Lett.* **111** (2013) no.6, 061601 [arXiv:1305.1627]; A. Celis, J. Fuentes-Martn and H. Serdio, *JHEP* **1412** (2014) 167 [arXiv:1410.6218]; F. Arias-Aragon and L. Merlo, *JHEP* **1710** (2017) 168 [arXiv:1709.07039]; Y. Ema, D. Hagihara, K. Hamaguchi, T. Moroi and K. Nakayama, *JHEP* **1804** (2018) 094 [arXiv:1802.07739]; Y. H. Ahn [arXiv:1804.06988]; T. Alanne, S. Blasi and F. Goertz [arXiv:1807.10156].
15. E. Dudas, S. Pokorski and C. A. Savoy, *Phys. Lett. B* **356** (1995) 45 [hep-ph/9504292]; Y. Nir, *Phys. Lett. B* **354** (1995) 107 [hep-ph/9504312]; P. Binetruy, S. Lavignac and P. Ramond, *Nucl. Phys. B* **477** (1996) 353 [hep-ph/9601243].
16. K. Choi, S. H. Im, C. B. Park and S. Yun, *JHEP* **1711** (2017) 070 [arXiv:1708.00021]; F. Bjrkeroth, E. J. Chun and S. F. King, *JHEP* **1808** (2018) 117 [arXiv:1806.00660].
17. N. Arkani-Hamed, M. Dine and S. P. Martin, *Phys. Lett. B* **431** (1998) 329 [hep-ph/9803432].
18. P. Svrcek and E. Witten, *JHEP* **0606** (2006) 051 [hep-th/0605206]; E. I. Buchbinder, A. Constantin and A. Lukas, *Phys. Rev. D* **91** (2015) no.4, 046010 [arXiv:1412.8696].
19. M. Cicoli, D. Klevers, S. Krippendorf, C. Mayrhofer, F. Quevedo and R. Valandro, *JHEP* **1405** (2014) 001 [arXiv:1312.0014]; M. Cicoli, M. Goodsell and A. Ringwald, *JHEP* **1210**, 146 (2012) [arXiv:1206.0819]
20. E. Dudas and S. K. Vempati, *Nucl. Phys. B* **727** (2005) 139 [hep-th/0506172]; A. Achucarro, B. de Carlos, J. A. Casas and L. Doplicher [hep-th.0601190]; K. Choi and K. S. Jeong [hep-th/0605108]; E. Dudas and Y. Mambrini, *JHEP* **0610** (2006) 044

[hep-th/0607077]; K. Choi and K. S. Jeong [hep-th/0605108]; A. Achucarro, B. de Carlos, J. A. Casas and L. Doplicher [hep-th/0601190].

21. E. Dudas, Y. Mambrini, S. Pokorski and A. Romagnoni, *JHEP* **0804** (2008) 015 [arXiv:0711.4934]; E. Dudas, Y. Mambrini, S. Pokorski, A. Romagnoni and M. Trapletti, *JHEP* **0903** (2009) 011 [arXiv:0809.5064].
22. S. Kachru, R. Kallosh, A. D. Linde and S. P. Trivedi, *Phys. Rev. D* **68**, 046005 (2003) [hep-th/0301240].

CPHT-RR106.112018
IPhT-T18/142

On the weak gravity conjecture in string theory with broken supersymmetry

Quentin Bonnefoy[a1], **Emilian Dudas**[a2], **Severin Lüst**[b,a3]

[a] *CPHT, CNRS, Ecole polytechnique, IP Paris, F-91128 Palaiseau, France*

[b] *Institut de Physique Théorique, Université Paris Saclay, CEA, CNRS*
Orme des Merisiers
91191 Gif-sur-Yvette Cedex, France

Abstract

We use type I string models with supersymmetry broken by compactification (à la Scherk-Schwarz) in order to test the weak gravity conjecture in the presence of runaway potentials in a perturbative string theory setting. For a finite value of the supersymmetry breaking radius there is a runaway potential, which is the only possibility if one accepts the non-existence of de Sitter vacua. Although the weak gravity conjecture is valid in the decompactification limit, for fixed values of the radius we show that there are short-ranged attractive D1 brane-brane interactions, which would naively imply a violation of the weak gravity conjecture. We argue however that at one-loop level the effective tension of the branes decreases and becomes smaller than the effective charge such that there is a long-ranged repulsive force. Our conclusion is that the weak gravity conjecture should be respected provided that the string coupling g_s is not extremely small. For very small g_s we expect a large number of stable bound states to be present.

[1]E-mail: quentin.bonnefoy@polytechnique.edu
[2]E-mail: emilian.dudas@polytechnique.edu
[3]E-mail: severin.luest@polytechnique.edu

1 Introduction

Recently several conjectures were put forward constraining the properties of effective quantum field theories which can be consistently UV-completed by a theory of quantum gravity. These conjectures are usually based on known properties of string theory as well as black hole physics and are often dubbed swampland criteria [1]. Maybe the most prominent of them is the weak gravity conjecture (WGC) [2].[1] Closely related is the swampland distance conjecture [26] and the conjectured absence of non-supersymmetric AdS vacua [27].[2] Lately, another conjecture [34], often called the de Sitter swampland conjecture, attracted a lot of attention.[3] This conjecture constrains the scalar potential in a way that forbids the existence of (meta)stable de Sitter vacua in string theory.

In its most common formulation the weak gravity conjecture requires that in the presence of gravity for any gauge interaction there should exist at least one charged particle of mass m and charge q such that (in suitable units) $q \geq m$. This condition can be motivated by the requirement that all charged black holes in the theory should be able to decay without leaving a large number of stable remnants. Moreover, it makes it impossible to take a smooth limit towards vanishing gauge coupling and therefore ensures that gravity is always the weakest interaction. These statements allow for a natural generalization for higher-form gauge fields where the charged objects are branes. From the viewpoint of particle-particle (or brane-brane) interactions, the condition $q \geq m$ clearly implies that the electric repulsion between two such particles (or branes) is dominating over their gravitational attraction. Therefore one could reformulate the weak gravity conjecture as the requirement for the existence of at least one particle or brane for each gauge symmetry such that its effective interaction potential is repulsive. This is the point of view we want to take here.[4] It is the objective of this paper to compute such interaction potentials in explicit string theory models and to test if they obey the weak gravity conjecture.

On the other hand, runaway potentials are abundant in string theory and this was considered as a serious phenomenological problem in the past [74]. Motivated by the persistent presence of runaway potentials in string theory, it was also recently conjectured in [75] that quintessence is maybe the only realistic outcome of a theory of quantum gravity.[5] In this paper we are imposing simultaneously the weak gravity conjecture and the existence of a runaway (space) direction in which one field continues to roll. While in the decompactification limit supersymmetry is restored and the weak gravity conjecture is marginally satisfied, considering the rolling field at a different value generates brane interactions and thus constraints from the point of view of the

[1] For refinements and recent tests of the weak gravity conjecture see [3–25].

[2] For further discussions of these conjectures see also [20, 21, 24, 28–31] and [32, 33].

[3] Fundamental constraints on the consistency of de Sitter vacua have been previously pointed out in [35, 36]. After the appearance of possible counter examples [37–42] the original conjecture has been refined in [43], see also [44]. Other attempts of refinement were suggested in [45–48]. For subsequent discussions in the context of string theory see [49–73].

[4] In particular, we do not consider refined and stricter versions of the conjecture, such as for instance the strong or lattice weak gravity conjectures.

[5] This possibility was entertained earlier in various incarnations. For an earlier attempt, see e.g. [76].

weak gravity conjecture.

From a string theory viewpoint, the majority of tests of these conjectures were done in the context of superstring compactifications. On the other hand, supersymmetry breaking generates precisely the ingredients needed for non-trivial tests: runaway potentials for moduli fields, effective brane-brane interactions and the generation of scalar potentials, potentially interpreted as dark energy. The goal of the present work is to analyze the weak gravity conjecture in type I string theory with broken supersymmetry. Arguably, the simplest and best understood way of breaking supersymmetry in string theory is via compactification. This was first proposed at the field-theory (supergravity) level by Scherk and Schwarz [77], then applied to heterotic strings [78] and then to open strings [79, 80]. The usual string theory computation of brane-brane interactions [81] can be captured, at large separations $r \gg \sqrt{\alpha'}$, by a field theory computation of tree-level exchange of supergravity massless fields between the branes. The setup present however some stringy features that are not fully captured by a pure field-theory analysis by keeping only the supergravity modes. Indeed this string theory construction contains, as we review in the next section, odd-winding closed string states with a "wrong" GSO projection, which contain the would-be scalar tachyon. These states do couple to branes and do mediate brane-brane interactions. Even if in the regime of interest $R \gg \sqrt{\alpha'}$, with R the radius of the Scherk-Schwarz circle, the would-be tachyonic scalar is actually very heavy, its exchange is the main contribution to the brane-brane interactions at long distances that we compute below. Due to this feature, we are forced to perform the computations at the string theory level, although the results can be understood to some extent by field-theory arguments.

We use D1 brane interactions as a function of the separation in spacetime as a test of the WGC. We find that at short distances and at one-loop there are attractive forces which have a finite limit where the distance goes to zero, whereas at long distances those attractive forces are exponentially suppressed. Since massive (closed strings) fields do not mediate long range interactions, our interpretation is that at this order of perturbation theory the branes still have a charge to mass ratio set by the supersymmetric BPS condition. The limit of zero distance suggests that the corresponding self-energy can be interpreted as a negative quantum correction to the tension, which will generate an imbalance between gauge and gravitational forces at higher loops, leading to an effective repulsion at large distances consistent with the WGC. The one-loop attractive forces, unsuppressed at small distances, will induce the formation of a finite number of stable bound states of D1 branes. For very small string coupling, the number of such states can become very large, consistent with the swampland distance conjecture [26]. Notice, however, that those states seem to become heavier when we decrease g_s.

The structure of this paper is the following. In Section 2 we review type I string theory with supersymmetry breaking by compactification. In Section 3 we discuss in more details the resulting runaway potentials. Section 4 deals with the brane-brane interactions at one-loop and their attractive nature, which also allows us to define the quantum corrections to brane tensions. In Section 5 we notice that D1 branes not only interact among themselves, but they also experience an interaction with the D9/O9 background branes/O-planes. Section 6 contains arguments beyond one-loop, which are needed in order to clarify the fate of the weak gravity conjecture in this setup. The paper ends with some brief conclusions and future directions.

2 Type I strings with Scherk-Schwarz supersymmetry breaking

Scherk-Schwarz breaking of supersymmetry is the oldest and probably the most popular way of breaking supersymmetry perturbatively in string theory. Since we are interested in brane interactions, moduli potentials and the weak gravity conjecture, the necessary ingredients are present in the type I string and orientifolds [82]. Vacuum energy and brane-brane interactions are nicely encoded in one-loop string amplitudes: torus and Klein bottle for the propagation of closed strings, and the cylinder and the Möbius for open strings. In what follows, all string amplitudes below should be multiplied by the factor $1/(4\pi^2\alpha')^{d/2}$, where d is the number of noncompact spacetime dimensions. One will add this factor at the end of our computations, in order not to overcharge various formulae. Keeping this in mind, for 9 non-compact dimensions times a circle of radius R on which the Scherk-Schwarz mechanism is implemented, the one-loop torus amplitude is given by[6]

$$\begin{aligned}\mathcal{T} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{11/2}} \Big\{ &(|V_8|^2 + |S_8|^2)\Lambda_{m,2n} - (V_8\bar{S}_8 + S_8\bar{V}_8)\Lambda_{m+1/2,2n} \\ &+ (|O_8|^2 + |C_8|^2)\Lambda_{m,2n+1} - (O_8\bar{C}_8 + C_8\bar{O}_8)\Lambda_{m+1/2,2n+1}\Big\} \frac{1}{|\eta^8|^2}(\tau) \ .\end{aligned} \tag{1}$$

where $\mathcal{F}$ is the fundamental domain of the modular group $\mathrm{SL}(2,\mathbb{Z})$, V_8, S_8, O_8 and C_8 are $SO(8)$ characters built out of Jacobi theta functions, τ is the complex parameter of the torus and $\Lambda_{m,n} = \sum_{m,n} q^{\frac{\alpha'}{4}(\frac{m}{R}+\frac{nR}{\alpha'})^2} \bar{q}^{\frac{\alpha'}{4}(\frac{m}{R}-\frac{nR}{\alpha'})^2}$ denotes the one-dimensional lattice of states with Kaluza-Klein (KK) number m and winding number n, with $q = e^{2\pi i \tau}$. Even windings have the familiar action of spacetime fermion number: bosons have the usual KK masses, whereas fermions have a mass shifted by $1/2R$. On the other hand, odd winding states have a different, "wrong" GSO projection. In particular, this sector contains a tower of states starting with a scalar (coming from the character $|O_8|^2$ above) with the lightest mass given by

$$m_O^2 = -\frac{2}{\alpha'} + \frac{R^2}{\alpha'^2} \ . \tag{2}$$

For small radii $R < \sqrt{2\alpha'}$ this scalar becomes tachyonic, whereas it is very heavy in the opposite limit $R \gg \sqrt{2\alpha'}$. This scalar will be a main actor in the brane-brane interactions at long distances that we discuss later on. The Klein bottle amplitude provides the orientifold projection of the closed string sector and is given by

$$\mathcal{K} = \frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \frac{V_8 - S_8}{\eta^8}(2i\tau_2) \sum_m e^{-\alpha'\pi\tau_2\frac{m^2}{R^2}} \ . \tag{3}$$

[6] For notations and conventions, see [83].

Since it is the same as in the superstring case, it does not contribute to the vacuum energy and symmetrizes, as usual, the NS-NS sector which comprises the graviton g_{MN} and the dilaton Φ, whereas it antisymmetrizes the RR sector which consists of the two-from C_2. Consistency of the theory (RR tadpole conditions) requires the introduction of 16 D9 branes wrapping the circle, which can be endowed with arbitrary Wilson lines [84] $W_i = \mathrm{diag}(a_i/R, -a_i/R)$, which can be interpreted as D8 brane positions $d_i = 2\pi a R'$ on the circle after a T-duality, where $R' = \alpha'/R$ is the T-dual radius. Notice that in the T-dual interpretation the branes at positions d_i are accompanied by their images under the orientifold projection at $-d_i$. The physically distinct values of the Wilson lines can be chosen to be $0 \leq a_i \leq 1/2$, where the end-points of the interval $a_i = 0$ and $a_i = 1/2$ correspond to the location of the $O8_-$ planes. Whereas the T-dual interpretation geometrizes nicely properties of brane spectra and interactions, we should remember that the radius of the circle is large $R \gg \sqrt{\alpha'}$ in order to avoid the tachyon (and obtained dynamically by the time-evolution). Therefore the T-dual picture in the supersymmetry breaking radius is not really useful from an effective field theory description, since the T-dual radius is smaller than the string length.

The one-loop open string amplitudes are given by

$$\begin{aligned}\mathcal{A} = \sum_{i,j=1}^{16} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \Big[\frac{V_8}{\eta^8}\left(\frac{i\tau_2}{2}\right) (P_{m+a_i-a_j} + P_{m+a_i+a_j}) \\ - \frac{S_8}{\eta^8}\left(\frac{i\tau_2}{2}\right)(P_{m+1/2+a_i-a_j} + P_{m+1/2+a_i+a_j})\Big] \ , \end{aligned} \tag{4}$$

$$\mathcal{M} = -\sum_{i=1}^{16} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \left[\frac{V_8}{\eta^8}\left(\frac{i\tau_2}{2}+\frac{1}{2}\right) P_{m+2a_i} - \frac{S_8}{\eta^8}\left(\frac{i\tau_2}{2}+\frac{1}{2}\right) P_{m+1/2+2a_i)}\right] \ ,$$

where in this loop channel V_8 describes the propagation of (gauge) bosons, whereas S_8 that of charged fermions. Moreover, $P_{m+a_i} = \sum_n e^{-\pi\tau_2 \frac{\alpha'(m+a_i)^2}{R^2}}$ denotes the KK sum of open string states shifted in mass by the Wilson lines. The parameter $\tau = i\tau_2/2$ ($\tau = i\tau_2/2 + 1/2$) has the interpretation of the complex parameter of the doubly covering torus for the cylinder (Möbius) amplitude. For generic values of the Wilson lines (brane positions after T-duality), the open string gauge group is $U(1)^{16}$, whereas in their absence it is $SO(32)$. The one-loop open string amplitudes have a dual interpretation in terms of tree-level exchange of closed string states between the D-branes (for the cylinder) and between the D-branes and O-planes (for the Möbius amplitude). The corresponding string amplitudes can be obtained by appropriate modular transformations and are expressed in terms of the length l of the tube describing the tree-level propagation. Doing so, one obtains

$$\begin{aligned}\tilde{\mathcal{A}} = \frac{2^{-5}R}{\sqrt{\alpha'}} \sum_{i,j=1}^{16} \int_0^\infty dl \left[\frac{V_8 - S_8}{\eta^8}(il) \frac{1+(-1)^n}{2} + \frac{O_8 - C_8}{\eta^8}(il)\frac{1-(-1)^n}{2}\right] \\ \times \left[e^{-2\pi i n(a_i - a_j)} + e^{-2\pi i n(a_i+a_j)}\right] W_n \ , \end{aligned} \tag{5}$$

$$\tilde{\mathcal{M}} = -\frac{2R}{\sqrt{\alpha'}} \sum_{i=1}^{16} \int_0^\infty dl \; \frac{V_8 - (-1)^n S_8}{\eta^8} \left(il + \frac{1}{2} \right) e^{-4\pi i n a_i} W_{2n} \ ,$$

where $W_n = \sum_n e^{-\pi l \frac{n^2 R^2}{2\alpha'}}$ denote the (closed string) winding states couplings to the branes-O planes. In (6) V_8 (S_8) denote the couplings to the NS-NS (RR) closed string sector, whereas O_8 (C_8) denote the coupling of the odd-winding closed string states with the "wrong" GSO projection. Notice in particular the coupling of the scalar O_8 to D9 branes. The corresponding coupling to D1 brane in the next sections will play a central role in our analysis.

Supersymmetry is restored in the large radius limit $R \to \infty$. We therefore expect the dynamics to drive the radius to large values. In the region $R \gg \sqrt{\alpha'}$ the would-be tachyonic closed string scalar is very massive and should not be kept in a low-energy effective action. However, due to Jacobi function identities, $V_8 = S_8$ and the contribution of the usual NSNS-RR sectors cancel and the main contribution to D9-D9 brane interactions comes precisely from the exchange of this scalar.

3 Scalar potential and runaway vacua

The goal of this section is to write explicitly the scalar potential for the radius and the Wilson lines of the D9 branes. The scalar potential in string theory is minus the partition function, therefore

$$V(R, W_i) = -\left(\frac{1}{2}\mathcal{T} + \mathcal{K} + \mathcal{A} + \mathcal{M} \right) \equiv V_{\mathcal{T}} + V_{\mathcal{K}} + V_{\mathcal{A}} + V_{\mathcal{M}} \ . \tag{6}$$

In the Scherk-Schwarz compactification, supersymmetry is broken by global boundary conditions, which implies that the scalar potential is of field-theory origin in the open part for large radii. It is also of field-theory origin in the closed string part in the large radius limit. The Klein bottle is still supersymmetric and therefore it does not contribute to the scalar potential. Supersymmetry is restored in the decompactification limit $R \to \infty$. The potential can be easily estimated in the regime where effective field theory is valid $R \gg \sqrt{2\alpha'}$. In this limit, string oscillators in all amplitudes and winding states in the torus are very heavy and do not contribute. We can therefore replace the modular functions by their leading contribution, such that

$$\begin{aligned}
\mathcal{T} &\simeq 128 \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \sum_m \left(e^{-\alpha' \pi \tau_2 \frac{m^2}{R^2}} - e^{-\alpha' \pi \tau_2 \frac{(m+1/2)^2}{R^2}} \right) \ , \\
\mathcal{A} &\simeq 8 \sum_{i,j=1}^{16} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \left[P_{m+a_i-a_j} + P_{m+a_i+a_j} - P_{m+1/2+a_i-a_j} - P_{m+1/2+a_i+a_j} \right] \ , \\
\mathcal{M} &= -8 \sum_{i=1}^{16} \int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \left[P_{m+2a_i} - P_{m+1/2+2a_i} \right] \ .
\end{aligned} \tag{7}$$

It is convenient to perform a Poisson resummation of the Kaluza-Klein sums to turn them into winding sums, to get

$$
\begin{aligned}
\mathcal{T} &\simeq 128 \frac{R}{\sqrt{\alpha'}} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \sum_n \left[1-(-1)^n\right] e^{-\frac{\pi n^2 R^2}{\alpha' \tau_2}} \ , \\
\mathcal{A} &\simeq 8 \frac{R}{\sqrt{\alpha'}} \sum_{i,j=1}^{16} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \left[1-(-1)^n\right] \left[e^{-2\pi i (a_i - a_j) n} + e^{-2\pi i (a_i + a_j) n}\right] e^{-\frac{\pi n^2 R^2}{\alpha' \tau_2}} \ , \\
\mathcal{M} &= -8 \frac{R}{\sqrt{\alpha'}} \sum_{i=1}^{16} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \left[1-(-1)^n\right] e^{-4\pi i a_i n} e^{-\frac{\pi n^2 R^2}{\alpha' \tau_2}} \ . \qquad (8)
\end{aligned}
$$

As explained at the beginning of Section 2, all string amplitudes above should be multiplied by the factor $1/(4\pi^2\alpha')^{9/2}$. By including this factor and after a straightforward integration, one gets

$$
\begin{aligned}
\mathcal{T} &= \frac{12}{\pi^{14}} \sum_n \frac{1}{(2n+1)^{10}} \frac{1}{R^9} \ , \\
\mathcal{A} &\simeq \frac{3}{2\pi^{14}} \sum_{i,j=1}^{16} \sum_n \frac{\cos 2\pi a_i (2n+1) \cos 2\pi a_j (2n+1)}{(2n+1)^{10}} \frac{1}{R^9} \ , \\
\mathcal{M} &\simeq -\frac{3}{4\pi^{14}} \sum_{i=1}^{16} \sum_n \frac{\cos 4\pi a_i (2n+1)}{(2n+1)^{10}} \frac{1}{R^9} \ , \qquad (9)
\end{aligned}
$$

which generate a runaway potential, also typical for quintessence models. Supersymmetry breaking generates therefore runaway scalar potentials, a notoriously well-known fact. Indeed, all known ways of breaking supersymmetry generate, at some order in the perturbative expansion, a runaway potential which generates a cosmological rolling of the corresponding field towards the runaway infinity. We will not enter here into a phenomenological discussion of such potentials and their viability. The example discussed in this paper is too simple to be viable and is ruled out by time dependence of coupling constants, in particular. More important for our purposes, the vacuum energy is not positive unless one adds Wilson lines. A stability analysis including Wilson lines shows that there are no stable solutions with positive scalar potential in nine dimensions [85]. The reason is that in order to increase the value of vacuum energy some D9 branes should be displaced/separated in the T-dual version. However, as discussed in the Appendix, D8 branes (after T-duality) attract each other and such configurations are unstable. In lower dimensions, positive potential with stable brane configurations is possible [85] without changing significantly the discussion on the weak gravity conjecture below. Because of the attractive forces between the T-dual D8 branes, in the next sections we consider the case where there are no Wilson lines on D9 branes.

The formulae above can be generalized easily after compactification to four dimensions. We consider for simplicity a product of circles of radii R_I, $I = 1, \dots, 6$. In the following we introduce a vectorial notation for the winding numbers $\mathbf{n} = (n, n_1, \dots, n_5)$

and Wilson lines of the brane i, $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,6})$. The vacuum energy, in the large radii limit, becomes[7]

$$\mathcal{T} = \frac{3 \times 2^6 V_6}{\pi^9} \sum_{\mathbf{n}} [1-(-1)^n] \frac{1}{[n^2 R^2 + n_1^2 R_1^2 + \cdots n_5 R_5^2]^5} \ ,$$
$$\mathcal{A} \simeq \frac{3 \times 2^3 V_6}{\pi^9} \sum_{i,j=1}^{16} \sum_{\mathbf{n}} [1-(-1)^n] \frac{\cos(2\pi \mathbf{a}_i \mathbf{n}) \cos(2\pi \mathbf{a}_j \mathbf{n})}{[n^2 R^2 + n_1^2 R_1^2 + \cdots n_5 R_5^2]^5} \ ,$$
$$\mathcal{M} \simeq -\frac{3 \times 2^2 V_6}{4\pi^9} \sum_{i=1}^{16} \sum_{\mathbf{n}} [1-(-1)^n] \frac{\cos(4\pi \mathbf{a}_i \mathbf{n})}{[n^2 R^2 + n_1^2 R_1^2 + \cdots n_5 R_5^2]^5} \ . \tag{10}$$

where in (10) $V_6 = \prod_I R_I$. It is now possible to obtain a positive scalar potential for the radii with runaway vacua to infinity. For this, one needs to add Wilson lines and check their stability [85] .

For fixed values of the Wilson lines, the 9D effective potential for the radius in the Einstein frame is of the form

$$\mathcal{L} = \frac{1}{2\kappa_9^2 R^2} (\partial R)^2 - \frac{c e^{\frac{18\phi}{7}}}{R^9} \ , \tag{11}$$

where ϕ is the dilaton field, $\frac{1}{\kappa_9^2}$ is the nine-dimensional Planck mass and $-\frac{c}{R^9}$ is obtained when summing the three contributions in (9), according to (6). After the field redefinition $R = R_0 e^{\sigma}$, the radion action becomes

$$\mathcal{L} = \frac{1}{2\kappa_9^2} (\partial \sigma)^2 - \frac{c e^{\frac{18\phi}{7} - 9\sigma}}{R_0^9} \ . \tag{12}$$

Supersymmetry is then restored in the limit $\sigma \to \infty$. The computation above did not take into account the fact that the background spacetime is not static, due to the generated scalar potential. In particular, the Scherk-Schwarz radius is expected to run to infinity in order to restore supersymmetry. This is clearly the case if the potential is positive after compactification, which is possible after adding suitable Wilson lines. Actually, even for negative values of such a scalar potential, the large radius regime, in which supersymmetry breaking is small, is generically reached by cosmological evolution in an expanding universe, as shown in [86].

---

[7] We wrote (10) in the large radii limit. If some dimensions are small $R \sim \sqrt{2\alpha'}$, $R_I \ll \sqrt{\alpha'}$, the expressions (10) change. First of all, the winding masses along the supersymmetry breaking radius in (10) come from the "wrong" GSO closed-string sector which have a tachyonic mass contribution and we should really replace $n^2 R^2 \to n^2 R^2 - 2\alpha'$. If the five additional dimensions are small $R_I \ll \sqrt{\alpha'}$, only the windings along the supersymmetry breaking radius do contribute to the scalar potential and, whereas for large radii R_I the potential scales as $1/R^9$, for small radii it scales as $1/R^4$. Since our conclusions do not change in this case, in order not to complicate too much the discussion below we consider in most cases the limit of large radii $R, R_I \gg \sqrt{\alpha'}$.

4 Brane interactions and effective brane tensions

Type I strings contain charged D9, D5 and D1 branes. They are BPS in the superstring case with their tension equal to the RR charge $T = Q$, which guarantees no interaction between them. There is a subtlety for the D1-D9 amplitude which does not vanish, but it does so after adding the Möbius amplitude D1-O9. With supersymmetry breaking turned on, branes start to interact. Our goal is to analyze this in some detail and to understand the change in the effective tension. Consider D1 branes wrapping the Scherk-Schwarz circle, charged under the RR two-form C_2, which behave like particles after compactification, coupling to a gauge field $\int_{S^1} C_2$.

Let us consider two such D1 branes, at a distance r in the transverse coordinates. The brane-brane potentials are contained in the cylinder amplitude. Its explicit computation is very similar to a Casimir vacuum energy. The interaction is given by

$$\mathcal{A}_{11} = -\frac{1}{2}\,\mathrm{Str}\int \frac{dk}{2\pi}\int_0^\infty \frac{d\tau_2}{\tau_2} e^{-\pi\alpha'\tau_2(k^2+M^2)} \ , \tag{13}$$

with the mass operator given by

$$M^2 \ = \ \frac{1}{\alpha'}N + \frac{(m+a_i-a_j)^2}{R^2} + \frac{r^2}{(2\pi\alpha')^2} \ , \tag{14}$$

where N is the number operator for open string oscillators and $W_i = a_i/R$ are open string Wilson lines on the circle. An explicit computation, similar to the one of D9-D9 brane amplitudes leads to the one-loop amplitude

$$\mathcal{A}_{11} = \frac{1}{2\pi\sqrt{\alpha'}}\int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} e^{-\frac{\tau_2 r^2}{4\pi\alpha'}} \ \left[P_{m+a_i-a_j} + P_{m+a_i+a_j} - P_{m+1/2+a_i-a_j} - P_{m+1/2+a_i+a_j}\right] \times \frac{\theta_2^4}{2\eta^{12}}\left(\frac{i\tau_2}{2}\right) \ . \tag{15}$$

Written in the (closed string) tree-level channel, the amplitude becomes

$$\tilde{\mathcal{A}}_{11} = \frac{R}{4\pi\alpha'}\int_0^\infty \frac{dl}{l^4} e^{-\frac{r^2}{2\pi\alpha' l}} \ \left[1-(-1)^n\right]\left[e^{-2\pi i(a_i-a_j)n} + e^{-2\pi i(a_i+a_j)n}\right]\frac{\theta_4^4}{2\eta^{12}}(il)e^{-\pi l\frac{n^2R^2}{2\alpha'}} \ . \tag{16}$$

It is more illuminating to write the tree-level channel exchange potential in a way which involves an integral over the noncompact momenta of the closed strings exchanged, by using the identity

$$\int_0^\infty \frac{dl}{l^4} e^{-\frac{r^2}{2\pi\alpha' l}-\frac{\pi l}{2}\alpha' m_n^2} = \frac{\alpha'^3}{8\pi}\int d^8k \frac{e^{i\mathbf{kr}}}{k^2+m_n^2} \ . \tag{17}$$

Notice that only massive states contribute to the D1-D1 brane interactions. In the region of interest $r, R \gg \sqrt{\alpha'}$ a standard field theory computation does not capture the string result (16). Indeed, in the region $r \gg \sqrt{\alpha'}$ the main contribution to the brane-brane interaction comes from the region of a long thin tube $l \to \infty$ and therefore from the lightest closed string states. However, since the even winding contribution which include the supergravity states vanishes due to a cancellation between the NS-NS and the RR sectors, the main contribution to the interaction comes from odd windings containing the would-be tachyon scalar in the closed string spectrum (in character language, O_8). The D1-D1 brane interactions as seen from the tree-level closed-string ("gravitational") exchange are given by

$$V_{11} = -\frac{R\alpha'^2}{2\pi^2}\sum_n \int d^8k\ e^{i\mathbf{kr}} \left[(1-1)\frac{\cos[4\pi n a_i]\cos[4\pi n a_j]}{k^2+\frac{4n^2R^2}{\alpha'^2}} \right.$$
$$\left. + \frac{1}{8}\ \frac{\cos[2\pi(2n+1)a_i]\cos[2\pi(2n+1)a_j]}{k^2+\frac{(2n+1)^2R^2}{\alpha'^2}-\frac{2}{\alpha'}} \right] . \qquad (18)$$

The contribution of the zero-mode vanishes at one-loop, according to our computation, which implies that at one-loop the interaction of D1 branes is still governed by the the BPS tree-level tension and charge $T_1 = Q_1$. Indeed, since the one-loop contribution is exclusively mediated by massive states, it is short ranged and therefore cannot be interpreted as coming from an imbalance between the tension and charge of the branes. Actually, since the would-be tachyonic scalar for large radius $R \gg \sqrt{\alpha'}$ is much heavier than the supergravity modes and also heavier than string states, one should only keep the terms with $n = 0$ and $n = -1$ in the formula above for consistency.

If one fixes the values of the Wilson lines and only considers the dynamics in the dimensions perpendicular to the branes, the short-range one-loop D1-D1 brane interactions are attractive (negative potential) for coincident position of branes on the circle (zero relative Wilson line $a_i = a_j$) and are repulsive (positive potential) if the branes are separated, for example if one sits at $a_i = 0$ and the second brane sits at the other end of the interval $a_j = 1/2$. However, once the dynamics of the Wilson lines is taken into account, one sees that the potential is such that the only stable point is the attractive one $a_i = a_j$.

An important output of the computation above is the D1 brane self-energy, obtained by considering a single D1 brane of Wilson line a and setting the spacetime distance $\mathbf{r} = 0$. If the result would be divergent, more care would be needed for its interpretation. However, since the result is completely finite and is a contribution localized on the D1 brane worldvolume, it can safely interpreted as a self-energy quantum correction to the brane tension, that we compute here. The interaction is dominated in this case by the integration region $l = 0$, which is the UV region of the closed string exchange (IR region of one-loop open strings). In this case one gets the approximate result

$$\tilde{\mathcal{A}}_{11} = \frac{8R}{\pi\alpha'}\int_0^\infty dl\ \sum_n \cos^2[2\pi(2n+1))a_i]e^{-\pi l\frac{(2n+1)^2R^2}{2\alpha'}} = \frac{16}{\pi^2 R}\sum_n \frac{\cos^2[2\pi(2n+1))a_i]}{(2n+1)^2} . \qquad (19)$$

This amplitude contains brane-brane and brane-image brane interactions. By extracting the brane-brane self-energy, one obtains a correction to the brane tension. One obtains then the one-loop corrected tension of the D1 brane wrapping the circle, which can be written either as a corrected D1 brane tension or as the mass M_0 of the wrapped brane on the circle

$$T_{1,\text{eff}} = T_1 - \frac{2}{\pi^3 R^2} \sum_n \frac{1}{(2n+1)^2} \; = \; T_1 - \frac{1}{2\pi R^2} \quad , \quad M_0 = 2\pi R T_{1,\text{eff}} \; , \tag{20}$$

where $T_1 = \frac{\sqrt{\pi}}{\sqrt{2}\kappa_{10}}(4\pi^2\alpha')$ is the standard type I D1 brane tension. Notice that this one-loop corrected tension is *lower* than the tree-level one, due to supersymmetry breaking. Indeed, since $T_1 \sim \mathcal{O}(g_s^{-1})$, the correction is of order $\mathcal{O}(g_s)$ with respect to the original value. The tension becomes zero for the special value $R^2 \sim g_s\alpha'$, which is actually in the regime where type I tachyon condenses and the theory is not anymore under control.

Notice that in a realistic compactification only four spacetime dimensions are non-compact. In this case, the brane-brane potential for $r \gg \sqrt{\alpha'}$ becomes

$$V_{11} = -\frac{R\alpha'^2}{8\pi^2 V_5} \sum_{\mathbf{p}} \int d^3k \; e^{i\mathbf{kr}} \; \frac{\cos[2\pi a_i]\cos[2\pi a_j]}{k^2 + m_{\mathbf{p}}^2 + \frac{R^2}{\alpha'^2} - \frac{2}{\alpha'}} \; , \tag{21}$$

where $\sum_{\mathbf{p}}$ is the sum over all Kaluza-Klein masses in the five additional internal dimensions.

The result is particularly simple if the five additional dimensions are very small, i.e. $R_I \ll R, r$, in which case one can neglect the corresponding massive modes contributions. In this limit (and using $R \gg \sqrt{\alpha'}$), the total potential energy is well approximated at large distances $r \gg \sqrt{\alpha'}$ by

$$V_{11} \sim \; - \frac{R\alpha'^2}{4V_5} \cos[2\pi a_i]\cos[2\pi a_j] \frac{e^{-r\sqrt{\frac{R^2}{\alpha'^2} - \frac{2}{\alpha'}}}}{r} \; . \tag{22}$$

As discussed previously, despite the naive first thought that the potential (22) is negative for close values of the Wilson lines of the two branes and positive if the branes are well separated on the circle, the only minimum stable configuration is when they are coincident.

An important point for the later discussion on the weak gravity conjecture is that the negative self-energy of D1 branes and the decrease in the effective brane tension also implies that it is energetically favorable to form bound states of D1 branes. Indeed, let us denote by $V_0 < 0$ the self-energy of one D1 brane. Then one can compare the energy of two configurations. The first is the energy $E_{N,1}$ of N coincident D1 branes and a single D1 brane at a large distance $r \gg \sqrt{\alpha'}$ from them, whereas the second is the energy $E_{N+1,0}$ of $N+1$ coincident D1 branes. They are given by

$$\begin{aligned} E_{N,1} &= -(N+1)T_1 + (N^2+1)V_0 + O\left(e^{-\frac{rR}{\alpha'}}\right) \; , \\ E_{N+1,0} &= -(N+1)T_1 + (N+1)^2 V_0 \; . \end{aligned} \tag{23}$$

It is then clear that $E_{N+1,0} < E_{N,1}$ and therefore that the D1 branes tend to form bound states which eventually can lead to the formation of black holes.

Finally, until now we considered D1 branes wrapping the supersymmetry breaking circle. If on the other hand the D1 branes are perpendicular to the direction of the radius R used for supersymmetry breaking, they do not experience supersymmetry breaking. They will retain therefore the BPS nature at the one-loop level and their interactions will be supersymmetric.

5 D1 interactions with the background D9-O9

One natural question is the influence of the background D9-O9 on the potential for the Wilson lines of D1 branes. In the type I superstring there is no net interaction between D1 branes and the background D9 branes and O9 planes.[8] More precisely, the brane-brane interaction D1- D9 is cancelled by the interaction with the orientifold D1-O9, a consequence of the tadpole cancelation condition and of the BPS properties of type I branes.

In the case of supersymmetry breaking by compactification, this cancellation does not occur anymore and D1 branes feel a net interaction with the background, This generates a potential for the Wilson lines of D1 branes on the circle. In what follows, due to the discussion in Section 3 on the D9 Wilson lines and their attractive nature, we take all T-dual D8 branes to be coincident, i.e. we introduce no corresponding Wilson lines for the D9 branes. Their addition could change the minima of the D1 positions from this interaction with the background, without changing qualitatively our discussion in the next section concerning brane-brane interactions. As a consequence, as one will check here, D1 brane interactions with the background D9/O9 tend to stabilize the D1 positions a_i at the origin of the (Scherk-Schwarz) circle. The D1-D9 and D1-O9 amplitudes are then given by [87]

$$
\begin{aligned}
\mathcal{A}_{19} &= \frac{32}{2\pi\sqrt{\alpha'}} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} \left[(O_0 S_8 + V_0 C_8) P_{m+a_i} - (S_0 V_8 + C_0 O_8) P_{m+a_i+1/2} \right] \left(\frac{\eta}{\theta_4} \right)^4 , \\
\mathcal{M}_1 &= \frac{1}{4\pi\sqrt{\alpha'}} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} \left[(\hat{O}_0 \hat{V}_8 - \hat{V}_0 \hat{O}_8) P_{m+2a_i} - (\hat{S}_0 \hat{S}_8 - \hat{C}_0 \hat{C}_8) P_{m+2a_i+1/2} \right] \left(\frac{2\hat{\eta}}{\hat{\theta}_2} \right)^4 .
\end{aligned} \tag{24}
$$

In these amplitudes O_0, V_0, S_0, C_0 describe the one-loop propagation of open strings scalar, vector and spinors respectively in the two dimensional worldvolume of D1 branes in the light cone formulation, whereas O_8, V_8, S_8, C_8 describes the quantum numbers and degeneracy due to the eight Neumann-Dirichlet coordinates. The corresponding amplitudes in the tree-level / gravitational channel are given by

$$
\begin{aligned}
\tilde{\mathcal{A}}_{19} = \frac{32R}{64\pi\alpha'} \int_0^\infty dl \Big[& (V_0 O_8 - 0_0 V_8 + S_0 S_8 - C_0 C_8) e^{-4\pi i n a_i} W_{2n} \\
& + (O_0 O_8 - V_0 V_8 - S_0 C_8 + C_0 S_8) e^{-2\pi i (2n+1) a_i} W_{2n+1} \Big] \left(\frac{2\eta}{\theta_2} \right)^4 ,
\end{aligned}
$$

[8] E.D. thanks Jihad Mourad for a very helpful discussion on this issue.

$$\tilde{\mathcal{M}}_1 = \frac{R}{2\pi\alpha'} \int_0^\infty dl \left[(\hat{O}_0 \hat{V}_8 - \hat{V}_0 \hat{O}_8) - (-1)^n (\hat{S}_0 \hat{S}_8 - \hat{C}_0 \hat{C}_8) \right] e^{-4\pi i n a_i} W_{2n} \left(\frac{2\hat{\eta}}{\hat{\theta}_2} \right)^4 . \quad (25)$$

In the tree-level channel, V_0, O_0 denote propagation of NS-NS closed string fields, whereas S_0, C_0 denote propagation of RR fields. Notice that there is no net effective interaction in the RR sector exchange, due to a cancellation between the two terms. This is consistent with the fact that there is no physical RR field to be exchanged between the D1 and D9/O9 sector. The fact that the two amplitudes do not cancel anymore in the presence of supersymmetry breaking is transparent from the fact that there are odd winding states of the would-be tachyonic (for small radius $R < \sqrt{2\alpha'}$) field $O_0 O_8$ in the D1-D9 interaction, which are not present in the D1-O9 interaction. By summing the two contributions and using identities of Jacobi functions, one finds

$$\tilde{\mathcal{A}}_{19} + \tilde{\mathcal{M}}_1 = \frac{R}{2\pi\alpha'} \int_0^\infty dl \; O_0 O_8 \left(\frac{2\eta}{\theta_2} \right)^4 e^{-2\pi i (2n+1) a_i} W_{2n+1} \; . \quad (26)$$

In the limit of interest $R \gg \sqrt{\alpha'}$, one obtains the leading contribution by taking the limit $l \to 0$ in the string oscillator contributions. By doing so, one finds the final form of the potential from the 19 sector

$$V_{19} = -(\tilde{\mathcal{A}}_{19} + \tilde{\mathcal{M}}_1) = -\frac{8}{\pi^2 R} \sum_n \frac{\cos[2\pi(2n+1)a_i]}{(2n+1)^2} \; . \quad (27)$$

The minimum of the potential is at $a_i = 0$. The result (27) is valid for large five additional dimensions $R_I \gg \sqrt{\alpha'}$ and can be understood as a Casimir field-theory vacuum energy contribution on compact space dimensions. If the five additional dimensions are small $R_I \ll \sqrt{\alpha'}$, (27) changes and become parametrically of order $(\alpha')^5/V_5 R^6$. This can also be understood by T-dualizing the small dimensions, after which one gets D6 branes wrapping the supersymmetry breaking circle plus five additional large dimensions. The resulting potential energy is of order V_5'/R^6, where $V_5' \gg \alpha'^{5/2}$ is the T-dual volume. This potential is purely field theoretically and can also be understood as a Casimir energy calculation.

This interaction with the background D9 branes/O9-planes seems therefore to favor D1 branes with vanishing Wilson lines. It is unclear and rather implausible to us that this potential energy, localized on the D1 brane but Wilson line/position dependent, should be interpreted as an additional correction to the D1 brane tension. In any case, since it is of the same sign and magnitude as the self energy of the D1 brane, including it or not would not modify the qualitative features of what we discuss next.

6 Interactions beyond one-loop and the weak gravity conjecture

We consider as in Section 4 two D1 branes separated by a distance r in the three-dimensional noncompact space. Our goal is to estimate their interaction as a function of the distance r. We know that at short distances the interaction is attractive and D1 branes tend to accumulate and form bound states. There is no reason to believe that in a perturbative string setup this result would be upset to higher-orders in the perturbative expansion. At large distances however, the one-loop attraction is exponentially damped since the main contribution comes from massive closed-string states. At large distances therefore, potential higher-loop contributions generating massless gravitational (closed string) exchanges would induce infinite-range interactions, which change considerably (and dominate over) the one-loop contribution. This effect can be understood in terms of modifications of the tension and charge of D1 branes, as well as the generation of a dilaton mass, that we now try to include in the interaction potential. All of these modifications are generated by supersymmetry breaking.

Let us write the D1-D1 brane interactions in a slightly more general way as a contribution from the zero modes $V_{11}^{(0)}$ and contributions from massive states $V_{11}^{(n)}$. The contribution of the zero-mode $V_{11}^{(0)}$ vanishes at one-loop, according to our computation in Section 4. However, since the one-loop contribution comes exclusively from massive states, it is short ranged and therefore any higher-order/loop correction leading to a zero-mode exchange changes dramatically the interaction at large distances. We consequently parametrize the zero-mode higher-loop contributions by introducing three parameters: $T_{1,\text{eff}}$ and $Q_{1,\text{eff}}$ are the quantum corrected brane tension and charge, whereas m_0 denotes the mass of the dilaton generated by quantum corrections. With these changes in mind, at large distances $r \gg \sqrt{\alpha'}$ where the main contribution comes from the lightest closed string states exchanged between the branes, we arrive at the following expression for the D1-D1 brane interaction

$$\begin{aligned} V_{11} &= V_{11}^{(0)} + V_{11}^{(n)}\,, \quad \text{where} \\ V_{11}^{(0)} &= \frac{R\alpha'^2}{2\pi^2}\int d^8k\; e^{i\mathbf{kr}}\;\left[\frac{Q_{1,\text{eff}}^2/Q_1^2}{k^2} - \frac{T_{1,\text{eff}}^2/T_1^2}{4}\left(\frac{1}{k^2+m_0^2}+\frac{3}{k^2}\right)\right]\,, \\ V_{11}^{(n)} &= -\frac{R\alpha'^2}{8\pi^2}\int d^8k\; e^{i\mathbf{kr}}\;\frac{\cos[2\pi a_i]\cos[2\pi a_j]}{k^2+\frac{R^2}{\alpha'^2}-\frac{2}{\alpha'}}\,. \end{aligned} \tag{28}$$

The zero-mode contribution can also be written in terms of the supergravity 10d Planck mass κ_{10} as usually done in the literature[9] [81]

$$V_{11}^{(0)} = 16\kappa_{10}^2\pi R\int \frac{d^8k}{(2\pi)^8}\; e^{i\mathbf{kr}}\;\left[\frac{Q_{1,\text{eff}}^2}{k^2} - \frac{T_{1,\text{eff}}^2}{4}\left(\frac{1}{k^2+m_0^2}+\frac{3}{k^2}\right)\right]\,. \tag{29}$$

In (28), the corrected tension of the wrapped D1 brane $T_{1,\text{eff}}$ is defined in (20) and the relative factor of 1/4 (3/4) denotes the contribution of the dilaton (graviton). The one-loop corrected charge $Q_{1,\text{eff}}$ will be discussed below. The massive contributions $V_{11}^{(n)}$ contain the one-loop computation performed in Section 4. Notice that in a realistic compactification only four spacetime dimensions are noncompact. In this case, the

[9]The extra factor of 4 with respect to the usual formula is due to the fact that branes and their images contribute.

brane-brane potential becomes

$$V_{11}^{(0)} = \sum_{\mathbf{p}} \frac{16\kappa_{10}^2 \pi R}{(2\pi)^8 V_5} \int d^3k \; e^{i\mathbf{kr}} \left[\frac{Q_{1,\text{eff}}^2}{k^2 + m_{\mathbf{p}}^2} - \frac{T_{1,\text{eff}}^2}{4} \left(\frac{1}{k^2 + m_{\mathbf{p}}^2 + m_0^2} + \frac{3}{k^2 + m_{\mathbf{p}}^2} \right) \right] ,$$

$$V_{11}^{(n)} = -\frac{R\alpha'^2}{8\pi^2 V_5} \sum_{\mathbf{p}} \int d^3k \; e^{i\mathbf{kr}} \; \frac{\cos[2\pi a_i]\cos[2\pi a_j]}{k^2 + m_{\mathbf{p}}^2 + \frac{R^2}{\alpha'^2} - \frac{2}{\alpha'}} \; , \qquad (30)$$

where $\sum_{\mathbf{p}}$ is the sum over all Kaluza-Klein masses in the five additional internal dimensions. As we discussed in the previous sections, in the T-dual version D0 branes energetically prefer to be in the same position and coincident with the D8 branes. Therefore in what follows we can set their position to zero, i.e. we fix $a_i = 0$. Distributing D8 branes on the circle, which would change quantitatively the formulae in this section, raises stability issues and complicates the analysis, without changing qualitatively the discussion and the conclusions below.

The result is particularly simple if the five additional dimensions are much smaller than R and r, in which case one can neglect the contributions from the corresponding massive modes. In this limit, it is more transparent to express the total potential energy in terms of the four-dimensional Planck mass M_P, for which the graviton exchange provides the Newton potential in terms of the mass $M_0 = 2\pi R T_{1,\text{eff}}$ and the charge $Q_0 = 2\pi R Q_{1,\text{eff}}$ of the wrapped D1 brane. In this way, one gets the approximate potential

$$V_{11} \sim \frac{1}{M_P^2} \left[\frac{\frac{4}{3}Q_0^2 - M_0^2 - \frac{1}{3}M_0^2 e^{-m_0 r}}{r} - \frac{Q_0^2}{3} \, \frac{e^{-r\sqrt{\frac{R^2}{\alpha'^2} - \frac{2}{\alpha'}}}}{r} \right] . \qquad (31)$$

This expression is valid for distances $r \gg \sqrt{\alpha'}$, whereas for shorter distances one expects the one-loop potential to be a good approximation, which has a constant limit when $r \to 0$.

The correction V_0 to the D1 brane tension is negative being generated by the massive contributions $V_{11}^{(n)}$ between the same brane ($\mathbf{r} = 0$). The correction to the charge would, on the other hand, come from a genus 3/2 computation, which was not yet performed to our knowledge. However, a quantum correction to the RR charge of the brane would be of the form $\int C_2 e^{\phi}$, where ϕ is the dilaton. Such a coupling would violate the gauge symmetry of the RR gauge field C_2, which seems implausible in perturbation theory. Corrections to the RR field kinetic terms are possible though, and this would generate a renormalization of the RR charge.[10] A similar correction to the dilaton kinetic term should also contribute to the renormalization of the tension. However, such corrections would arise from one loop calculations and would be associated to $\mathcal{O}(g_s^2)$ corrections. We thus do not expect them to dominate the one-loop contribution to the tension, which is $\mathcal{O}(g_s)$, and therefore

$$T_{1,\text{eff}}^2 < Q_{1,\text{eff}}^2 \iff M_0^2 < Q_0^2 \, . \qquad (32)$$

[10]We thank J. Mourad for suggesting this possibility. We also thank I. Antoniadis, G. Bossard, H. Partouche, A. Sagnotti for discussions on this issue.

As a consequence, at short distances the potential is attractive whereas it is repulsive at large distances. If on the contrary the bound (32) was violated in the case of a massless dilaton, i.e. if $m_0 = 0$ (or if $M_0^2 > \frac{4}{3}Q_0^2$ for $m_0 > 0$), the potential would remain attractive also at large distances. This would violate the weak gravity conjecture. Our perturbative arguments dismiss such a possibility and we conclude that the weak gravity conjecture holds in our setup, and the massless modes exchange which it constrains determines the brane-brane dynamics at large distances.

Even if (32) holds, the one-loop potential (16) between D1 branes is attractive and unsuppressed at small distances, which entertains the possibility that stable bound states, which may be black holes, exist in this theory. Consequently, black holes stability arguments, which are sometimes used in discussions about the WGC, are different in the small and large distance regions. To address this question, one needs to study the regime interpolating between large distances, where higher-order effects dominate and presumably verify the WGC as argued above, and small distances where the one-loop potential induces an attraction. Knowing the $r = 0$ value of the potential given in (19) and its asymptotic behaviour (31), we understand that it reaches a maximal value and has the shape depicted in figure 1.

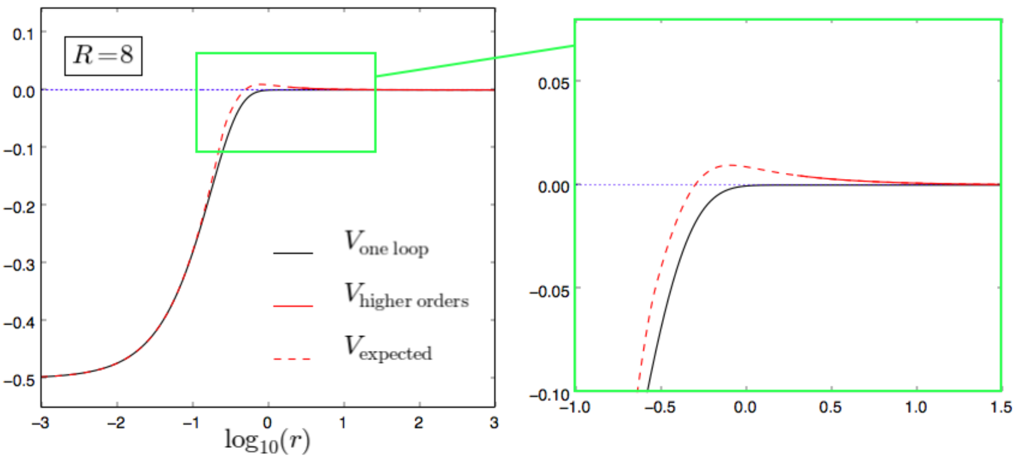


Figure 1: The D1-D1 potential as a function of the distance in the transverse space (the potentials and distances are expressed in units of α', we fixed $R = 8$, $g_s = 0.2$, $V_5 \sim 1.5^5$ and introduced no Wilson lines for the D1 branes)

To estimate the location r_0 of the maximum, we can use (31) if r_0 is in its validity regime. When $m_0 = 0$, we obtain

$$r_0 = -\frac{1}{\sqrt{\frac{R^2}{\alpha'^2} - \frac{2}{\alpha'}}} \left[1 + W\left(8\frac{T_{1,\text{eff}}^2 - T_1^2}{eT_1^2}\right)\right] \approx \frac{\alpha'}{R} \log\left(\frac{R^2}{g_s\alpha'}\right) \ , \qquad (33)$$

where W is the Lambert W function.[11] This expression, obtained from (31), can be

[11]The Lambert W function or product logarithm is defined by $W(xe^x) = x$. It has two real branches, here only the lower branch with $W \leq -1$ is relevant.

trusted if $r_0 \gg \sqrt{\alpha'}$, which can be rewritten as a constraint on the string coupling

$$g_s \ll \frac{R^3}{\alpha'^{3/2}} e^{-\frac{R}{\sqrt{\alpha'}}} \ . \tag{34}$$

In this case, black holes of size smaller than r_0 would be stable remnants. Such black holes could be formed from the D1 bound states about which we argued in (23) that their formation is energetically favorable. However, we expect from black hole constructions in string theory that there should only be a finite number of such remnants: from the bound state argument in (23) one can guess that if the number of D1 constituents is large and the bound state size becomes or order r_0 or larger, repulsive forces will prevent more D1 branes to bind and therefore larger charge/mass remnants to form. Calculating this finite number of bound states is beyond the scope of this paper, but we could try to estimate it by comparing r_0 with the scale at which we expect the D1-branes solutions of supergravity to break down,[12] $R_S \sim \frac{N_1 g_s \alpha'^3}{V_5}$, where N_1 is the number of stacked D1-branes. Using (33), we can derive the following estimate,

$$N_{\text{crit}} \equiv N_1 \frac{r_0}{r_S} \approx \frac{1}{g_s} \frac{V_5}{\alpha'^{3/2}} \frac{\alpha'^{1/2}}{R} \log\left(\frac{R^2}{g_s \alpha'}\right) \ , \tag{35}$$

where all D1-branes configurations with $N_1 < N_{\text{crit}}$ correspond to situations where the attractive force is felt even in the regime where supergravity applies. In particular, this number becomes small in the decompactifcation limit $R \gg \sqrt{\alpha'}$.

Furthermore, (35) also shows that the smaller g_s, the more stable bound states can exist. If $m_0 \neq 0$, r_0 becomes smaller than (33) and the appearance of such states is slightly suppressed in the limit $g_s \to 0$, but the behaviour remains qualitatively the same. Such a scaling of N_{crit} with g_s seems to be consistent with the swampland distance conjecture. On the other hand, when the string coupling increases, r_0 decreases and it will eventually not be consistent to use (31) and (33). Finally, when the supersymmetry breaking (Scherk-Schwarz) radius goes to infinity, as it would be if no further stabilization is added to the dynamics induced by (11) and (12), the one-loop potential vanishes since supersymmetry is recovered, no attraction nor repulsion remains, and the WGC, as well as the stability of black holes, is marginally retrieved.

Taking into account the shape of the brane-brane potential, one should clearly also consider the tunneling from large distance r to small ones when discussing the stability of brane configurations. Since we don't have a complete analytic formula, we are unable for the time being to estimate the corresponding tunneling probability. The conditions we derived are therefore necessary but apriori not sufficient to firmly establish the existence of bound states.

7 Conclusions and perspectives

String theory models with broken supersymmetry usually generate runaway potentials. Such potentials are of exponential type if one canonically normalizes the rolling field and

[12] This scale is the one for which the harmonic function $h(r) = 1 + \frac{R_S}{r}$, which defines the D1-brane solution, starts to deviate significantly from one.

could lead in special cases to quintessence models of dark energy. On the other hand, the breaking of supersymmetry generates at the same time interactions between branes, which only disappear in the runaway limit. While this in itself respects the weak gravity conjecture at infinity, insisting on the rolling field cosmology could generate violations of it at one-loop, coming from a short-distance attraction generated by massive modes. Naively one would therefore conclude that in a perturbative and controllable string setting, rolling field dynamics is incompatible with the weak gravity conjecture. Since the long-range brane-interaction at one-loop is vanishing due to a cancellation between the massless NS-NS (dilaton and graviton) and the RR exchanges, we believe however that higher-loop corrections are important to settle this issue. We gave qualitative arguments that at higher-loop a repulsive interaction generated by the exchange of massless states should appear, which at long distances should dominate over the one-loop (short range) attraction. Overall, this leads to a picture in which the weak gravity conjecture should be respected at large distances defined by the parameters (g_s, R).

The main result of this paper is that in this model, after taking one-loop corrections into account, the effective tension $T_{1,\text{eff}}$ and charge $Q_{1,\text{eff}}$ of D1 branes satisfy the weak gravity bound $Q_{1,\text{eff}} > T_{1,\text{eff}}$. This is equivalent to a repulsive interaction at long distances as here the aforementioned attractive force is exponentially suppressed. In the lower dimensional effective theory these D1 branes, wrapped around the Scherk-Schwarz circle, behave as particles charged under a $U(1)$-gauge symmetry with $Q_{\text{eff}} > M_{\text{eff}}$. To complete our test of the weak gravity conjecture, it would be interesting to compute if supersymmetry breaking induces corrections to the black hole extremality bound as well.

The stability of bound states and black holes is interesting in our setup. The one-loop short-range attraction favors the formation of D1 bound states which can potentially lead to stable black hole remnants. If the string coupling is very small, the attractive region of brane-brane potentials extends up to scales where the effective gravitational theory applies: if $g_s \lesssim \frac{R^3}{\alpha'^{3/2}} e^{-\frac{R}{\sqrt{\alpha'}}}$ (with R the radius of the supersymmetry breaking dimension), a finite number of branes well described by supergravity are sensitive to the attractive potential. This number roughly scales like $\frac{1}{g_s}$, and indicates that in the small g_s limit an increasing quantity of stable bound states is expected to arise.

There are a number of open interesting questions that are worth further exploration. It would be interesting to identify string models with broken supersymmetry where the generated moduli potentials and runaway vacua can lead to viable quintessence-like models of dark energy. There are various difficulties for progress into this direction, from generating a small acceleration of the present universe, which is highly nontrivial to achieve in string theory constructions [88], to the constraints coming from time-dependence of fundamental constants and fifth force experiments. From a more theoretical string theory perspective, it would be interesting to perform higher-loop (for instance, genus 3/2) computations in order to test our result on the quantum corrected brane tension and the absence of renormalization of the brane charges at lowest order. Whereas supersymmetry breaking should generate, as usual, tadpoles which signal limitations in quantum computations at higher loops, higher-order computations of brane tensions and charges could be performed by separating two D1 branes in (our) noncompact space, in which case there should be no such problems. It would also be important to investigate stable type I models in lower dimensions with D9 Wilson lines and positive scalar potential in the class of models constructed in [85] and to investigate the D1 interaction potentials in detail. It would also be very interesting to explore quantum corrections to brane tensions and RR charges in other string models with broken supersymmetry, such as the models with brane supersymmetry breaking [89].

Finally, we believe it is important to test the other various recent swampland conjectures [1, 2, 26, 27, 34, 43] in explicit perturbative string theory models with broken supersymmetry. Some work along these lines is in progress [90].

Acknowledgments

We are grateful to I. Antoniadis, G. Bossard and especially I. Bena, J. Mourad, H. Partouche and A. Sagnotti for enlightening discussions and correspondence. The authors acknowledge financial support from the ANR Black-dS-String. The work of S.L. is supported by the ERC Starting Grant 679278 Emergent-BH.

A D9 brane interactions

The D9 brane potentials and interactions are contained in the cylinder vacuum amplitudes. Its explicit computation is very similar to a Casimir vacuum energy and was discussed from the viewpoint of moduli potentials in Sections 2 and 3. Consider two D9 branes wrapping the circle, with Wilson lines $W_i = a_i/R$. After a T-duality, they become D8 branes localized on the circle, of positions $d_i = 2\pi a_i R'$, where R' is the T-dual radius. In the large radius limit $R \gg \sqrt{\alpha'}$, their interaction is given by

$$V_{\mathcal{A}} \simeq -\frac{3}{4\pi^{14}} \sum_n \frac{\cos 2\pi(a_i - a_j)(2n+1)}{(2n+1)^{10}} \; \frac{1}{R^9} \; , \tag{36}$$

The force experienced by the two branes can be computed from

$$F_{ij} = -\frac{\partial V_{\mathcal{A}}}{\partial_{a_{ij}}} = -\frac{3}{2\pi^{13}} \sum_n \frac{\sin 2\pi(a_i - a_j)(2n+1)}{(2n+1)^9} \; \frac{1}{R^9} \; . \tag{37}$$

This force is attractive, for any value of the radius and any separation $0 \le a_i - a_j \le 1/2$ between the brane positions / Wilson lines. D9 branes are however space-filling objects and therefore cannot be given a separation in spacetime. That is the reason we considered D1 branes to test the WGC. Indeed, D1 branes wrapped on the supersymmetry breaking circle behave as particles in four-dimensions and can be used to test the gravity as the weakest force hypothesis.

References

[1] C. Vafa, "The String landscape and the swampland," hep-th/0509212.

[2] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, "The String landscape, black holes and gravity as the weakest force," JHEP **0706** (2007) 060 [hep-th/0601001].

[3] C. Cheung and G. N. Remmen, "Naturalness and the Weak Gravity Conjecture," Phys. Rev. Lett. **113** (2014) 051601 [arXiv:1402.2287 [hep-ph]].

[4] B. Heidenreich, M. Reece and T. Rudelius, "Sharpening the Weak Gravity Conjecture with Dimensional Reduction," JHEP **1602** (2016) 140 [arXiv:1509.06374 [hep-th]].

[5] K. Kooner, S. Parameswaran and I. Zavala, "Warping the Weak Gravity Conjecture," Phys. Lett. B **759** (2016) 402 [arXiv:1509.07049 [hep-th]].

[6] J. Brown, W. Cottrell, G. Shiu and P. Soler, "Fencing in the Swampland: Quantum Gravity Constraints on Large Field Inflation," JHEP **1510** (2015) 023 [arXiv:1503.04783 [hep-th]].

[7] Y. Nakayama and Y. Nomura, "Weak gravity conjecture in the AdS/CFT correspondence," Phys. Rev. D **92** (2015) no.12, 126006 [arXiv:1509.01647 [hep-th]].

[8] D. Harlow, "Wormholes, Emergent Gauge Fields, and the Weak Gravity Conjecture," JHEP **1601** (2016) 122 [arXiv:1510.07911 [hep-th]].

[9] B. Heidenreich, M. Reece and T. Rudelius, "Evidence for a sublattice weak gravity conjecture," JHEP **1708** (2017) 025 [arXiv:1606.08437 [hep-th]].

[10] M. Montero, G. Shiu and P. Soler, "The Weak Gravity Conjecture in three dimensions," JHEP **1610** (2016) 159 [arXiv:1606.08438 [hep-th]].

[11] A. Hebecker, P. Mangat, S. Theisen and L. T. Witkowski, "Can Gravitational Instantons Really Constrain Axion Inflation?," JHEP **1702** (2017) 097 [arXiv:1607.06814 [hep-th]].

[12] P. Saraswat, "Weak gravity conjecture and effective field theory," Phys. Rev. D **95** (2017) no.2, 025013 [arXiv:1608.06951 [hep-th]].

[13] G. Shiu, P. Soler and W. Cottrell, "Weak Gravity Conjecture and Extremal Black Hole," arXiv:1611.06270 [hep-th].

[14] A. Hebecker, P. Henkenjohann and L. T. Witkowski, "What is the Magnetic Weak Gravity Conjecture for Axions?," Fortsch. Phys. **65** (2017) no.3-4, 1700011 [arXiv:1701.06553 [hep-th]].

[15] A. Hebecker and P. Soler, “The Weak Gravity Conjecture and the Axionic Black Hole Paradox,” JHEP **1709** (2017) 036 [arXiv:1702.06130 [hep-th]].

[16] M. Montero, A. M. Uranga and I. Valenzuela, “A Chern-Simons Pandemic,” JHEP **1707** (2017) 123 [arXiv:1702.06147 [hep-th]].

[17] Z. Fisher and C. J. Mogni, “A Semiclassical, Entropic Proof of a Weak Gravity Conjecture,” arXiv:1706.08257 [hep-th].

[18] B. Heidenreich, M. Reece and T. Rudelius, “The Weak Gravity Conjecture and Emergence from an Ultraviolet Cutoff,” Eur. Phys. J. C **78** (2018) no.4, 337 [arXiv:1712.01868 [hep-th]].

[19] C. Cheung, J. Liu and G. N. Remmen, “Proof of the Weak Gravity Conjecture from Black Hole Entropy,” JHEP **1810** (2018) 004 [arXiv:1801.08546 [hep-th]].

[20] S. Andriolo, D. Junghans, T. Noumi and G. Shiu, “A Tower Weak Gravity Conjecture from Infrared Consistency,” Fortsch. Phys. **66** (2018) no.5, 1800020 [arXiv:1802.04287 [hep-th]].

[21] S. J. Lee, W. Lerche and T. Weigand, “Tensionless Strings and the Weak Gravity Conjecture,” JHEP **1810** (2018) 164 [arXiv:1808.05958 [hep-th]].

[22] A. Hebecker, T. Mikhail and P. Soler, “Euclidean wormholes, baby universes, and their impact on particle physics and cosmology,” arXiv:1807.00824 [hep-th].

[23] Y. Hamada, T. Noumi and G. Shiu, “Weak Gravity Conjecture from Unitarity and Causality,” arXiv:1810.03637 [hep-th].

[24] S. J. Lee, W. Lerche and T. Weigand, “A Stringy Test of the Scalar Weak Gravity Conjecture,” arXiv:1810.05169 [hep-th].

[25] D. Klaewer, D. Lust and E. Palti, “A Spin-2 Conjecture on the Swampland,” arXiv:1811.07908 [hep-th].

[26] H. Ooguri and C. Vafa, “On the Geometry of the String Landscape and the Swampland,” Nucl. Phys. B **766** (2007) 21 [hep-th/0605264].

[27] H. Ooguri and C. Vafa, “Non-supersymmetric AdS and the Swampland,” Adv. Theor. Math. Phys. **21** (2017) 1787 [arXiv:1610.01533 [hep-th]].

[28] D. Klaewer and E. Palti, “Super-Planckian Spatial Field Variations and Quantum Gravity,” JHEP **1701** (2017) 088 [arXiv:1610.00010 [hep-th]].

[29] B. Heidenreich, M. Reece and T. Rudelius, Phys. Rev. Lett. **121** (2018) no.5, 051601 doi:10.1103/PhysRevLett.121.051601 [arXiv:1802.08698 [hep-th]].

[30] T. W. Grimm, E. Palti and I. Valenzuela, “Infinite Distances in Field Space and Massless Towers of States,” JHEP **1808** (2018) 143 [arXiv:1802.08264 [hep-th]].

[31] R. Blumenhagen, D. Klwer, L. Schlechter and F. Wolf, "The Refined Swampland Distance Conjecture in Calabi-Yau Moduli Spaces," JHEP **1806** (2018) 052 [arXiv:1803.04989 [hep-th]];

R. Blumenhagen, "Large Field Inflation/Quintessence and the Refined Swampland Distance Conjecture," PoS CORFU **2017** (2018) 175 [arXiv:1804.10504 [hep-th]].

[32] B. Freivogel and M. Kleban, "Vacua Morghulis," arXiv:1610.04564 [hep-th].

[33] U. Danielsson and G. Dibitetto, Phys. Rev. D **96** (2017) no.2, 026020 doi:10.1103/PhysRevD.96.026020 [arXiv:1611.01395 [hep-th]].

[34] G. Obied, H. Ooguri, L. Spodyneiko and C. Vafa, "De Sitter Space and the Swampland," arXiv:1806.08362 [hep-th].

[35] J. P. Conlon, "Quantum Gravity Constraints on Inflation," JCAP **1209** (2012) 019 doi:10.1088/1475-7516/2012/09/019 [arXiv:1203.5476 [hep-th]].

[36] G. Dvali, C. Gomez and S. Zell, "Quantum Break-Time of de Sitter," JCAP **1706** (2017) 028 doi:10.1088/1475-7516/2017/06/028 [arXiv:1701.08776 [hep-th]].

[37] F. Denef, A. Hebecker and T. Wrase, "de Sitter swampland conjecture and the Higgs potential," Phys. Rev. D **98** (2018) no.8, 086004 [arXiv:1807.06581 [hep-th]].

[38] J. P. Conlon, "The de Sitter swampland conjecture and supersymmetric AdS vacua," Int. J. Mod. Phys. A **33** (2018) no.29, 1850178 doi:10.1142/S0217751X18501786 [arXiv:1808.05040 [hep-th]].

[39] M. Cicoli, S. De Alwis, A. Maharana, F. Muia and F. Quevedo, "De Sitter vs Quintessence in String Theory," arXiv:1808.08967 [hep-th].

[40] H. Murayama, M. Yamazaki and T. T. Yanagida, "Do We Live in the Swampland?," arXiv:1809.00478 [hep-th].

[41] K. Choi, D. Chway and C. S. Shin, "The dS swampland conjecture with the electroweak symmetry and QCD chiral symmetry breaking," arXiv:1809.01475 [hep-th].

[42] K. Hamaguchi, M. Ibe and T. Moroi, "The swampland conjecture and the Higgs expectation value," arXiv:1810.02095 [hep-th].

[43] H. Ooguri, E. Palti, G. Shiu and C. Vafa, "Distance and de Sitter Conjectures on the Swampland," arXiv:1810.05506 [hep-th].

[44] A. Hebecker and T. Wrase, "The asymptotic dS Swampland Conjecture - a simplified derivation and a potential loophole," arXiv:1810.08182 [hep-th].

[45] D. Andriot, "On the de Sitter swampland criterion," Phys. Lett. B **785** (2018) 570 [arXiv:1806.10999 [hep-th]].

[46] G. Dvali and C. Gomez, "On Exclusion of Positive Cosmological Constant," arXiv:1806.10877 [hep-th].

[47] S. K. Garg and C. Krishnan, "Bounds on Slow Roll and the de Sitter Swampland," arXiv:1807.05193 [hep-th].

[48] D. Andriot and C. Roupec, "Further refining the de Sitter swampland conjecture," arXiv:1811.08889 [hep-th].

[49] S. Banerjee, U. Danielsson, G. Dibitetto, S. Giri and M. Schillo, "Emergent de Sitter cosmology from decaying AdS," arXiv:1807.01570 [hep-th].

[50] L. Aalsma, M. Tournoy, J. P. Van Der Schaar and B. Vercnocke, "Supersymmetric embedding of antibrane polarization," Phys. Rev. D **98** (2018) no.8, 086019 [arXiv:1807.03303 [hep-th]].

[51] C. Roupec and T. Wrase, "de Sitter extrema and the swampland," arXiv:1807.09538 [hep-th].

[52] D. Andriot, "New constraints on classical de Sitter: flirting with the swampland," arXiv:1807.09698 [hep-th].

[53] J. K. Ghosh, E. Kiritsis, F. Nitti and L. T. Witkowski, "De Sitter and Anti-de Sitter branes in self-tuning models," arXiv:1807.09794 [hep-th].

[54] C. Damian and O. Loaiza-Brito, "Two-field axion inflation and the swampland constraint in the flux-scaling scenario," arXiv:1808.03397 [hep-th].

[55] K. Dasgupta, M. Emelin, E. McDonough and R. Tatar, "Quantum Corrections and the de Sitter Swampland Conjecture," arXiv:1808.07498 [hep-th].

[56] S. Kachru and S. P. Trivedi, "A comment on effective field theories of flux vacua," arXiv:1808.08971 [hep-th].

[57] R. Kallosh, A. Linde, E. McDonough and M. Scalisi, "de Sitter Vacua with a Nilpotent Superfield," arXiv:1808.09428 [hep-th].

[58] Y. Akrami, R. Kallosh, A. Linde and V. Vardanyan, "The landscape, the swampland and the era of precision cosmology," arXiv:1808.09440 [hep-th].

[59] C. Han, S. Pi and M. Sasaki, "Quintessence Saves Higgs Instability," arXiv:1809.05507 [hep-ph].

[60] J. Moritz, A. Retolaza and A. Westphal, "On uplifts by warped anti-D3-branes," arXiv:1809.06618 [hep-th].

[61] I. Bena, E. Dudas, M. Graña and S. Lüst, "Uplifting Runaways," arXiv:1809.06861 [hep-th].

[62] R. Kallosh, A. Linde, E. McDonough and M. Scalisi, "4d models of dS uplift in KKLT," arXiv:1809.09018 [hep-th].

[63] J. Ellis, B. Nagaraj, D. V. Nanopoulos and K. A. Olive, "De Sitter Vacua in No-Scale Supergravity," arXiv:1809.10114 [hep-th].

[64] G. Buratti, E. Garca-Valdecasas and A. M. Uranga, "Supersymmetry Breaking Warped Throats and the Weak Gravity Conjecture," arXiv:1810.07673 [hep-th].

[65] F. F. Gautason, V. Van Hemelryck and T. Van Riet, "The tension between 10D supergravity and dS uplifts," arXiv:1810.08518 [hep-th].

[66] Y. Olguin-Trejo, S. L. Parameswaran, G. Tasinato and I. Zavala, "Runaway Quintessence, Out of the Swampland," arXiv:1810.08634 [hep-th].

[67] S. K. Garg, C. Krishnan and M. Zaid, "Bounds on Slow Roll at the Boundary of the Landscape," arXiv:1810.09406 [hep-th].

[68] J. Blåbäck, U. Danielsson and G. Dibitetto, "A new light on the darkest corner of the landscape," arXiv:1810.11365 [hep-th].

[69] J. J. Heckman, C. Lawrie, L. Lin and G. Zoccarato, "F-theory and Dark Energy," arXiv:1811.01959 [hep-th].

[70] J. J. Blanco-Pillado, M. A. Urkiola and J. Wachter, "Racetrack Potentials and the de Sitter Swampland Conjectures," arXiv:1811.05463 [hep-th].

[71] D. Junghans, "Weakly Coupled de Sitter Vacua with Fluxes and the Swampland," arXiv:1811.06990 [hep-th].

[72] M. Emelin and R. Tatar, "Axion Hilltops, Kahler Modulus Quintessence and the Swampland Criteria," arXiv:1811.07378 [hep-th].

[73] A. Banlaki, A. Chowdhury, C. Roupec and T. Wrase, "Scaling limits of dS vacua and the swampland," arXiv:1811.07880 [hep-th].

[74] M. Dine and N. Seiberg, "Is the Superstring Weakly Coupled?," Phys. Lett. **162B** (1985) 299.

[75] P. Agrawal, G. Obied, P. J. Steinhardt and C. Vafa, "On the Cosmological Implications of the String Swampland," Phys. Lett. B **784** (2018) 271 [arXiv:1806.09718 [hep-th]].

[76] T. Damour, F. Piazza and G. Veneziano, "Runaway dilaton and equivalence principle violations," Phys. Rev. Lett. **89** (2002) 081601 [gr-qc/0204094]; "Violations of the equivalence principle in a dilaton runaway scenario," Phys. Rev. D **66** (2002) 046007 [hep-th/0205111].

[77] J. Scherk and J. H. Schwarz, "How to Get Masses from Extra Dimensions," Nucl. Phys. B **153** (1979) 61.

[78] R. Rohm, "Spontaneous Supersymmetry Breaking in Supersymmetric String Theories," Nucl. Phys. B **237** (1984) 553; C. Kounnas and M. Porrati, "Spontaneous Supersymmetry Breaking in String Theory," Nucl. Phys. B **310** (1988) 355; S. Ferrara, C. Kounnas, M. Porrati and F. Zwirner, "Superstrings with Spontaneously Broken Supersymmetry and their Effective Theories," Nucl. Phys. B **318** (1989) 75; C. Kounnas and B. Rostand, "Coordinate Dependent Compactifications and Discrete Symmetries," Nucl. Phys. B **341** (1990) 641; I. Antoniadis and C. Kounnas, "Superstring phase transition at high temperature," Phys. Lett. B **261** (1991) 36; E. Kiritsis and C. Kounnas, "Perturbative and nonperturbative partial supersymmetry breaking: $N = 4 \to N = 2 \to N = 1$," Nucl. Phys. B **503** (1997) 117 [hep-th/9703059].

[79] J. D. Blum and K. R. Dienes, "Duality without supersymmetry: The Case of the SO(16) x SO(16) string," Phys. Lett. B **414** (1997) 260 [hep-th/9707148]; "Strong / weak coupling duality relations for nonsupersymmetric string theories," Nucl. Phys. B **516** (1998) 83 [hep-th/9707160].

[80] I. Antoniadis, E. Dudas and A. Sagnotti, "Supersymmetry breaking, open strings and M theory," Nucl. Phys. B **544** (1999) 469 [hep-th/9807011]; I. Antoniadis, G. D'Appollonio, E. Dudas and A. Sagnotti, "Partial breaking of supersymmetry, open strings and M theory," Nucl. Phys. B **553** (1999) 133 [hep-th/9812118], "Open descendants of Z(2) x Z(2) freely acting orbifolds," Nucl. Phys. B **565** (2000) 123 [hep-th/9907184].

[81] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges," Phys. Rev. Lett. **75** (1995) 4724 [hep-th/9510017]; for more detailed reviews see e.g. J. Polchinski, "Tasi lectures on D-branes," hep-th/9611050; C. P. Bachas, hep-th/9806199.

[82] A. Sagnotti, in Cargese '87, "Non-Perturbative Quantum Field Theory", eds. G. Mack et al (Pergamon Press, 1988), p. 521, arXiv:hep-th/0208020; G. Pradisi and A. Sagnotti, Phys. Lett. **B 216** (1989) 59; P. Horava, Nucl. Phys. **B 327** (1989) 461, Phys. Lett. **B 231** (1989) 251; M. Bianchi and A. Sagnotti, Phys. Lett. **B 247** (1990) 517, Nucl. Phys. **B 361** (1991) 519; M. Bianchi, G. Pradisi and A. Sagnotti, Nucl. Phys. **B 376** (1992) 365; A. Sagnotti, Phys. Lett. **B 294** (1992) 196 [arXiv:hep-th/9210127].

[83] For reviews see: C. Angelantonj and A. Sagnotti, "Open strings," Phys. Rept. **371** (2002) 1 [Erratum-ibid. **376** (2003) 339] [arXiv:hep-th/0204089]; E. Dudas, "Theory and phenomenology of type I strings and M-theory," Class. Quant. Grav. **17** (2000) R41 [arXiv:hep-ph/0006190].

[84] M. Bianchi, G. Pradisi and A. Sagnotti, "Toroidal compactification and symmetry breaking in open string theories," Nucl. Phys. B **376** (1992) 365.

[85] S. Abel, E. Dudas, D. Lewis and H. Partouche, “Stability and vacuum energy in open string models with broken supersymmetry,” arXiv:1812.09714 [hep-th].

[86] T. Coudarchet, C. Fleming and H. Partouche, “Quantum no-scale regimes in string theory,” Nucl. Phys. B **930**, 235 (2018) [arXiv:1711.09122 [hep-th]]; T. Coudarchet and H. Partouche, “Quantum no-scale regimes and moduli dynamics,” Nucl. Phys. B **933** (2018) 134 [arXiv:1804.00466 [hep-th]].

[87] E. Dudas, J. Mourad and A. Sagnotti, “Charged and uncharged D-branes in various string theories,” Nucl. Phys. B **620** (2002) 109 doi:10.1016/S0550-3213(01)00552-1 [hep-th/0107081].

[88] For reviews on the subject, see e.g. W. Fischler, A. Kashani-Poor, R. McNees and S. Paban, “The Acceleration of the universe, a challenge for string theory,” JHEP **0107** (2001) 003 [hep-th/0104181]; V. Balasubramanian, “Accelerating universes and string theory,” Class. Quant. Grav. **21** (2004) S1337 [hep-th/0404075]; M. Cicoli, S. De Alwis, A. Maharana, F. Muia and F. Quevedo, Fortsch. Phys. **2018**, 1800079 doi:10.1002/prop.201800079 [arXiv:1808.08967 [hep-th]].

[89] S. Sugimoto, “Anomaly cancellations in type I D9-D9-bar system and the USp(32) string Prog. Theor. Phys. **102** (1999) 685 [arXiv:hep-th/9905159]; I. Antoniadis, E. Dudas and A. Sagnotti, “Brane supersymmetry breaking,” Phys. Lett. **B 464** (1999) 38 [arXiv:hep-th/9908023]; C. Angelantonj, “Comments on open-string orbifolds with a non-vanishing B(ab),” Nucl. Phys. **B 566** (2000) 126 [arXiv:hep-th/9908064]; G. Aldazabal and A. M. Uranga, “Tachyon-free non-supersymmetric type IIB orientifolds via brane-antibrane JHEP **9910** (1999) 024 [arXiv:hep-th/9908072]; C. Angelantonj, I. Antoniadis, G. D’Appollonio, E. Dudas and A. Sagnotti, “Type I vacua with brane supersymmetry breaking,” Nucl. Phys. **B 572** (2000) 36 [arXiv:hep-th/9911081].

[90] I. Basile, J. Mourad and A. Sagnotti, “On Classical Stability with Broken Supersymmetry,” arXiv:1811.11448 [hep-th].

Prepared for submission to JHEP

IPPP/19/50
LPT-Orsay-19-24
CPHT-RR025.062019

BPS solutions for generalised Wess-Zumino models and their applications

Steven Abel[a,b] **, Quentin Bonnefoy**[c] **and Debtosh Chowdhury**[c,d]

[a] *Institute for Particle Physics Phenomenology, Durham University, South Road, Durham, U.K.*
[b] *Department of Mathematical Sciences, Durham University, South Road, Durham, U.K.*
[c] *Centre de Physique Théorique, CNRS, École Polytechnique, IP Paris, 91128 Palaiseau, France*
[d] *Laboratoire de Physique Théorique (UMR8627), CNRS, Univ. Paris-Sud, Université Paris-Saclay, 91405 Orsay, France*

E-mail: s.a.abel@durham.ac.uk, quentin.bonnefoy@polytechnique.edu, debtosh.chowdhury@polytechnique.edu

Abstract: We present BPS solutions to a general class of Wess-Zumino models which extend previous results in the literature. We discuss their relation to amplitudes on threshold, and their application to scalar domain walls in Supersymmetric QCD. We also find partial expressions for Wess-Zumino models with softly broken supersymmetry.

Contents

1 Introduction and summary

The purpose of this paper is to present and discuss a rather general solution to the Bogomol'nyi-Prasad-Sommerfield (BPS) equations, for a rather general class of Wess-Zumino (WZ) models. As we shall see the solution has applications in several areas, including multi-particle amplitudes on threshold, and scalar domain walls in Supersymmetric QCD (SQCD) duality.

Consider the following superpotential for a chiral superfield Φ:

$$W \;=\; \frac{1}{2}\Phi^2 + \frac{1}{p}\Phi^p \;, \tag{1.1}$$

where we do not place a restriction on the allowed value of the index p (except $p > 2$), and where couplings can be trivially reinstated by scaling. The associated scalar potential (where ϕ is the scalar component) is

$$V(\phi) \;=\; \left|\phi + \phi^{p-1}\right|^2 \;, \tag{1.2}$$

and if p is positive one might seek domain wall solutions between the supersymmetric minimum at $\phi = 0$ and the $p - 2$ supersymmetric minima at $\phi = e^{i\frac{n\pi}{(p-2)}}$, $n \in \mathbb{Z}$. Because the potential is a complete square, the equations of motion can be integrated once and factorised, yielding the familiar BPS equation (see Appendix A for a brief discussion of the latter):

$$\frac{d\phi}{dt} \;=\; e^{2i\theta}(\overline{\phi} + \overline{\phi}^{p-1}) \;, \tag{1.3}$$

where t is the coordinate across the wall and θ is an arbitrary constant angle. If we restrict ϕ to be real then solving eq. (1.3) is trivial, however the conjugation on the right hand side makes it difficult to find the general complex solution for arbitrary p. Our central result is the following solution to eq. (1.3):

$$\phi(z,\bar{z}) \;=\; \frac{z\left(1+\frac{\bar{z}^{p-2}-z^{p-2}}{2p}\right)}{\left(\left(1+\frac{\bar{z}^{p-2}-z^{p-2}}{2p}\right)^{p}+\frac{\bar{z}^{p-2}\left(\left(1-\frac{\bar{z}^{p-2}-z^{p-2}}{2p}\right)^{p}-\left(1+\frac{\bar{z}^{p-2}-z^{p-2}}{2p}\right)^{p}\right)}{\bar{z}^{p-2}-z^{p-2}}\right)^{\frac{1}{p-2}}}\,, \tag{1.4}$$

where $z=e^{t+i\theta}$ (see Appendix C for a few words on the derivation).

This is a generalisation of the BPS domain wall solution of Ref.[1] (with appropriate shifts in ϕ) which considered $p=3$ and real ϕ. Indeed taking $\theta=\frac{\pi}{p-2}$ we find

$$\phi(t) \;=\; \left(\frac{-e^{(p-2)t}}{1+e^{(p-2)t}}\right)^{\frac{1}{p-2}}\,, \tag{1.5}$$

which reduces, for $p=3$, to the non-singular domain wall solution,

$$\phi(t) \;=\; -\frac{e^{t}}{1+e^{t}}\,, \tag{1.6}$$

connecting the two minima ($\phi(-\infty)=0$ and $\phi(\infty)=-1$) of the WZ model. As an illustration, in Figure 1 we plot the generalised BPS solution as given in eq. (1.4) by setting $p=3$. There, we see that, even though $\phi(z,\bar{z})$ is singular for most values of θ, there exist

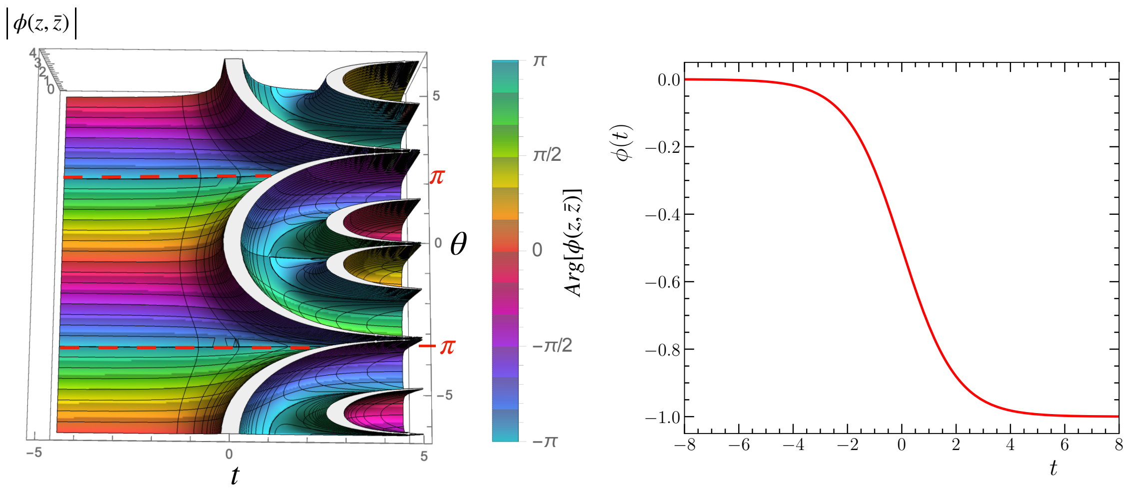


Figure 1. Plot of the solution in eq. (1.4) when $p=3$. In the left panel, the colour bar denotes the argument of the function $\phi(z,\bar{z})$. In the right panel, $\theta=\pi$ as in eq. (1.6).

smooth configurations (along the dashed lines) that correspond to domain walls connecting two minima of the function $V(\phi)$, consistent with eqs. (1.5) and (1.6). For $p=3$, when $\theta=\pm\pi$, the domain wall connects the two minima at $\phi(-\infty)=0$ and $\phi(\infty)=-1$ passing through $t=0$, as shown in the right panel of Figure 1.

The expression in eq. (1.4) is also related to the softly broken O(2) models of Ref.[2] that were examined in the context of multiparticle amplitudes on threshold (which took $p = 3$, see Appendix B), and some other work in this area (which typically considered real ϕ). However, the solution above has a richer structure and is more general than those that have been previously considered in the literature. Indeed to derive it we imposed only that ϕ scales as e^t as t goes to infinity, which is enough/required for amplitudes.

In the following section we discuss the application of our solution in the amplitude context, with particular emphasis on the recursion relations of multiparticle amplitudes on threshold, and their relationship with classical solutions of the equations of motion. After spending some time reviewing and discussing the classical ways of obtaining these amplitudes in the WZ model, we demonstrate that the general complex solution presented above translates into the ability to distinguish chiral fields and their conjugates in the possible final multiparticle states.

The arbitrariness of the exponent p also makes eq. (1.4) applicable to situations in which the second term in the superpotential of eq. (1.1) is generated non-perturbatively. In Section 3 we show that this allows one to find exact (classical) domain wall solutions for the scalar mesons in the magnetic duals of Supersymmetric QCD theories with a quartic coupling. In SU(N_c) theories with N_f flavours of quark/antiquark, this is relevant in the free-magnetic window, where $N_c + 1 < N_f < \frac{3}{2}N_c$. The exponent is given by $p = N_f/(N_f - N_c)$, so that p is generally a rational number between 3 and $N_f/2$. These non-perturbatively generated domain walls interpolate between two supersymmetric minima, going from the unbroken magnetic dual at the origin, to one of $2N_c - N_f$ pure Yang-Mills minima with meson vacuum expectation values (VEVs). This configuration is of general interest, and would appear for example in the duality cascade.

2 Multiparticle amplitudes in generalised Wess-Zumino models

Multiparticle amplitudes have been investigated for a long time [2–10], and have been the subject of renewed scrutiny recently within discussions of the so-called Higgsplosion mechanism [11–14]. The quantities of interest include the tree-level threshold amplitudes, which describe the decay of an off-shell particle to many on-shell ones, all taken to be at rest. Our solution in eq. (1.4) can be understood in this respect as the generating function of such tree-level multiparticle amplitudes at kinematic threshold for the generalised Wess-Zumino models of eq. (1.1). One can indeed show that such a generating function must satisfy a BPS condition (see Appendix A for more details), consistent with the fact that a specific limit of eq. (1.4) has been previously identified as a BPS domain wall solution [1, 15]. As we will also see, eq. (1.4) can be extended to softly broken SUSY scenarios, yielding either a complete or a partial solution depending on the choice of soft terms.

2.1 Recursion relations and classical solutions

In order to review standard techniques while simultaneously applying them to our specific problem, we will begin this section by following a diagrammatic approach to tree-level multiparticle amplitudes at kinematic threshold before linking it to classical solutions of the equations of motion. We will then show that eq. (1.4) indeed generates the amplitudes for the model of eq. (1.1), for specified numbers of emitted particles/anti-particles. In the next section we will extend the discussion to WZ models with specific sets of soft terms.

We are interested in evaluating tree-level amplitudes connecting an ingoing off-shell particle to outgoing on-shell ones, all taken to be at rest[1], for generalised Wess-Zumino models of a chiral superfield Φ. Let us take a canonical Kähler potential and for this discussion reinstate the couplings in the superpotential,

$$W = \frac{M}{2}\Phi^2 + \frac{\lambda}{p}\Phi^p , \tag{2.1}$$

where $p-3 \in \mathbb{N}$, giving rise to the following scalar potential for the complex scalar excitation ϕ:

$$V(\phi) = \left|M\phi + \lambda\phi^{p-1}\right|^2 . \tag{2.2}$$

The kinematic situation is summed up in Figure 2. Since there are two possible kinds of scalar excitation, the outgoing state is labelled by two integers m and n, denoting the number of particles and antiparticles respectively.

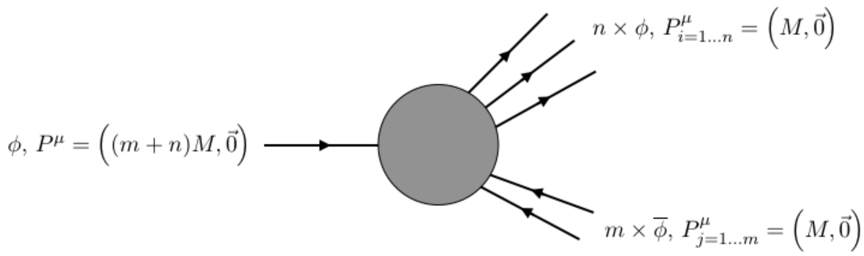


Figure 2. Kinematic setup (particles/anti-particles are represented using direct/reversed arrows).

The WZ model also includes scalar-fermion interactions. However, since we will be interested in tree-level amplitudes with initial and final states only made out of scalars, those interactions (which preserve fermion number) will not play any role.

Following earlier works on multiparticle amplitudes [3, 4], one can recursively calculate such amplitudes following the scheme of Figure 3. From this, after working out the correct combinatorics, one finds the following recursion relation:

$$\begin{aligned}\frac{a_{nm}}{m!n!} = & -\frac{(p-1)i^{2p-2}|\lambda|^2}{|M|^{2(2p-3)}} \sum_{\substack{\sum n_i = n \\ \sum m_i = m}} \frac{b_{n_1m_1}b_{n_2m_2}\ldots b_{n_{p-2}m_{p-2}}a_{n_{p-1}m_{p-1}}\ldots a_{n_{2p-3}m_{2p-3}}}{\prod_{i=1..2p-3} n_i!m_i![(n_i+m_i)^2-1]} \\ & -\frac{i^p}{|M|^{2(p-1)}} \sum_{\substack{\sum n_i = n \\ \sum m_i = m}} \frac{\lambda\overline{M}a_{n_1m_1}a_{n_2m_2}\ldots a_{n_{p-1}m_{p-1}} + (p-1)\overline{\lambda}Ma_{n_1m_1}b_{n_2m_2}\ldots b_{n_{p-1}m_{p-1}}}{\prod_{i=1\ldots p-1} n_i!m_i![(n_i+m_i)^2-1]} ,\end{aligned} \tag{2.3}$$

where a_{nm} symbolises the amplitude $\phi \longrightarrow n\times\phi + m\times\overline{\phi}$ and b_{nm} the amplitude $\overline{\phi} \longrightarrow n\times\phi + m\times\overline{\phi}$. Viewing a_{nm} as a function of λ and M, we can immediately deduce that $b_{nm}(\lambda, M) = a_{mn}(\overline{\lambda}, \overline{M})$ since $V(\phi)$ is hermitian.

[1]Exact results are much harder to obtain at loop-level or in the out of threshold regime [7–10, 16–18].

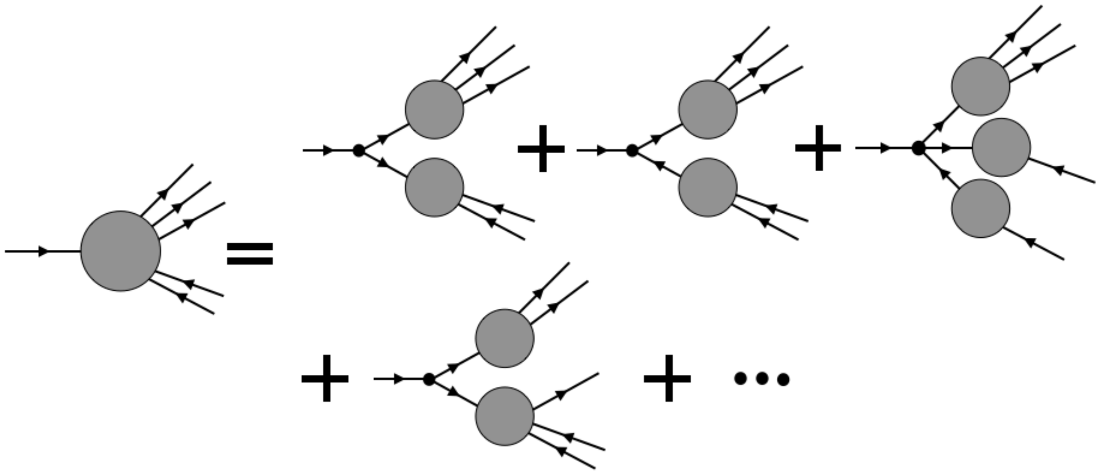


Figure 3. Recursion scheme for the amplitudes, drawn here for $p = 3$.

Inspection of the lowest amplitudes shows that the recursion is correctly initialised by imposing the following conditions:

$$\frac{a_{nm}}{[(n+m)^2-1]}\bigg|_{n=1,m=0} \;=\; -i|M|^2 \;, \qquad \frac{a_{nm}}{[(n+m)^2-1]}\bigg|_{n=0,m=1 \text{ or } n=0,m=0} \;=\; 0 \;, \tag{2.4}$$

which, combined with eq. (2.3), imply that $b_{nm} = -\overline{a_{mn}}$.

A convenient factorisation can be performed:

$$a_{nm} = -i|M|^2 A_{nm}\, n!m!\,[(n+m)^2-1]\left(\frac{\lambda}{M}\right)^{\frac{n-1}{p-2}}\left(\frac{\overline{\lambda}}{\overline{M}}\right)^{\frac{m}{p-2}} \tag{2.5}$$

with coefficients A_{nm} satisfying

$$\begin{cases} ((n+m)^2-1)A_{nm} \;=\; & (p-1)\sum A_{m_1n_1}...A_{m_{p-2}n_{p-2}}A_{n_{p-1}m_{p-1}}...A_{n_{2p-3}m_{2p-3}} \\ & +\sum\left(A_{n_1m_1}...A_{n_{p-1}m_{p-1}}+(p-1)A_{n_1m_1}A_{m_2n_2}...A_{m_{p-1}n_{p-1}}\right)\;, \\ A_{10} \;=\; 1\;, \quad A_{01} \;=\; A_{00} \;=\; 0\;, & \end{cases} \tag{2.6}$$

where the summations over indices match those in eq. (2.3). In particular, it implies that all A_{nm} are real and positive. The fact that all coupling constants disappeared from the above relation is a consequence of the (R-)symmetries of eq. (2.1) and of holomorphicity[2]. Defining a generating function

$$A(z,\overline{z}) = \sum_{n,m} A_{nm} z^n \overline{z}^m \;, \tag{2.7}$$

[2]Indeed, the effective superpotential generating tree-level diagrams can only take the form

$$W \;=\; \frac{M}{2}\Phi^2\sum_n\left(\frac{\lambda\Phi^{p-2}}{M}\right)^n \;,$$

and each amplitude has a dependence on λ, M fixed by this expression.

the recursion yields the differential equation[3]

$$\begin{cases} \left[(z\partial_z + \overline{z}\partial_{\overline{z}})^2 - 1\right] A = (p-1)A^{p-1}\overline{A}^{p-2} + A^{p-1} + (p-1)A\overline{A}^{p-2} \\ A(z=0,\overline{z}) = 0 \ , \quad \partial_z A(0,0) = 1 \ , \quad \partial_{\overline{z}} A(0,0) = 0 \end{cases} . \tag{2.8}$$

Finally, defining $z = e^{t+i\theta}$, this system becomes

$$\begin{cases} \partial_t^2 A \ = \ (p-1)A^{p-1}\overline{A}^{p-2} + A^{p-1} + (p-1)A\overline{A}^{p-2} + A \ = \ \frac{\partial}{\partial \overline{A}} V(A) \\ A(t=-\infty,\theta) = 0 \ , \quad \partial_t A(t=-\infty,\theta) = e^{i\theta} \end{cases} , \tag{2.9}$$

where the potential is as in eq. (1.2). The last equality in the first line illustrates the method of classical solutions of Ref.[6], which states that tree-level multiparticle amplitudes can be derived from the expansion of classical solutions with specific initial conditions. Besides, integrating it once and taking the square root yields the condition in eq. (1.3).

One can verify that our solution in eq. (1.4) indeed satisfies all of the conditions listed in eq. (2.9). Consequently $\phi \equiv A$ is the generating function of the diagrams of Figure 2. That is, Taylor expanding it with respect to z and $\overline{z}$ yields the amplitudes A_{nm}.

2.2 Soft terms in the Wess-Zumino model

Given the generality of the solution in eq. (1.4), it is natural to ask if one might be able to extend the analysis to non-supersymmetric cases, by deforming the theory with supersymmetry breaking operators. The renormalizable $p = 3$ WZ model allows for the following soft terms [19] in addition to the supersymmetric potential:

$$V = \left|\lambda\phi^2 + m\phi\right|^2 + \delta m^2|\phi|^2 + (\mu_3\phi^3 + \mu_2\phi^2 + h.c.) \ , \tag{2.10}$$

which can be expressed in terms of real parameters by defining $\mu_3 = c_3 + id_3$, $\mu_2 = c_2 + id_2$ and $\phi = \varphi + i\chi$ as

$$\begin{aligned} V \ = \ & \lambda^2(\varphi^2+\chi^2)^2 + (2\lambda m + 2c_3)\varphi^3 - 6d_3\varphi^2\chi + (2\lambda m - 6c_3)\varphi\chi^2 + 2d_3\chi^3 \\ & + (m^2+\delta m^2 + 2c_2)\varphi^2 - 4d_2\varphi\chi + (m^2+\delta m^2-2c_2)\chi^2 \ , \end{aligned} \tag{2.11}$$

where we take λ and m to be real by making a suitable U(1) rotation on ϕ.

Specific choices for the soft terms can be related to the softly broken O(2) model described in Appendix B. Starting with eq. (B.2) and performing rotations and shifts on φ and χ, one can only generate

$$V \ = \ \left|\lambda\phi^2 + m\phi\right|^2 + \frac{\delta m^2}{2}\left(\frac{\phi - \overline{\phi}}{2i}\right)^2 \ . \tag{2.12}$$

[3] The first condition on the second row is a slight generalisation of $A_{01} = A_{00} = 0$ since, due to the ϕ dependence of $V(\phi)$, the number of ϕ or $\overline{\phi}$ can only increase in a tree-level diagram.

Then implementing the same series of rotations and shifts on the solution obtained for the softly broken O(2) model yields a classical solution of the model in eq. (2.12):

$$\phi(z,\overline{z}) = \frac{z+\frac{\lambda}{m}i(z-\overline{z})\frac{i(z-\overline{z})+i\left(\sqrt{2}\frac{m_{\mathrm{Im}(\phi)}}{|m|}-1\right)^2(z+\overline{z})}{4\left(2\frac{m^2_{\mathrm{Im}(\phi)}}{m^2}-1\right)}}{1-\frac{\lambda}{m}\frac{z+\overline{z}}{2}+\left(\frac{\lambda}{m}\right)^2\frac{(z-\overline{z})^2}{4\left(2\frac{m^2_{\mathrm{Im}(\phi)}}{m^2}-1\right)}-\left(\frac{\lambda}{m}\right)^3\frac{\left(\sqrt{2}\frac{m_{\mathrm{Im}(\phi)}}{|m|}-1\right)^4(z-\overline{z})^2(z+\overline{z})}{8\left(2\frac{m^2_{\mathrm{Im}(\phi)}}{m^2}-1\right)^3}}\ , \tag{2.13}$$

where $m^2_{\mathrm{Im}(\phi)}=2m^2+\delta m^2$. It reduces to eqs. (1.4) and (B.1), if $\delta m^2=0$ (and $\lambda=m=1$). Its limit when $m\to 0$, which both cancels the cubic vertices and makes $\mathrm{Re}(\phi)$ massless, is the usual "φ^4" real scalar solution, where "φ" is here the imaginary part of ϕ:

$$\lim_{m\to 0}\phi(z,\overline{z}) \ = \ i\frac{\mathrm{Im}(z)}{1-\frac{\lambda}{2m^2_{\mathrm{Im}(\phi)}}\mathrm{Im}(z)^2}\ . \tag{2.14}$$

2.3 Towards a solution for symmetric soft masses

As stated in Section 2.2, there are more general soft terms than those of eq. (2.12). In particular, it is tempting to consider soft masses for the full complex scalar ϕ,

$$V \ = \ \left|\lambda\phi^2+m\phi\right|^2+\frac{\delta m^2}{2}|\phi|^2\ , \tag{2.15}$$

if we, for instance, want to leave some state light and decouple its superpartner. Thus far, we have not found a closed form solution, but we have been able to identify various limits of it. This could be used to either check or guess a more complete expression.

For simplicity, up to redefinitions in the recursion relation like the one we performed in eq. (2.5), we can restrict ourselves to the study of

$$V \ = \ \left|A^2+A\right|^2+\frac{1-\alpha}{\alpha}|A|^2\ , \tag{2.16}$$

and of the associated recursion relation/differential equation:

$$\begin{cases} ((n+m)^2-1)A_{nm} \ = \ 2\alpha^3\sum A_{m_1n_1}A_{n_2m_2}A_{n_3m_3}+\alpha^2\sum(A_{n_1m_1}A_{n_2m_2}+2A_{n_1m_1}A_{m_2n_2}) \\ \left[(z\partial_z+\overline{z}\partial_{\overline{z}})^2-1\right]A \ = \ 2\alpha^3A^2\overline{A}+\alpha^2(A^2+2A\overline{A}) \end{cases}. \tag{2.17}$$

Then, one can solve for real A, or use only the vertices $A\overline{A}^2$ or $A^2\overline{A}$ (see Appendix C for details), to determine the properties of the solution in various limits:

$$\begin{aligned} A(-\rho,-\rho) \ &= \ -\frac{\rho}{1+\alpha\rho-\alpha\frac{1-\alpha}{4}\rho^2}\ , \\ A(z,0) \ &= \ \frac{z}{\alpha(1-\frac{\alpha z}{6})^2}\ , \\ \Big(A/z\Big)(z=0,\overline{z}) \ &= \ \frac{(1+\frac{\alpha\overline{z}}{6})}{\alpha(1-\frac{\alpha\overline{z}}{6})^3}\ . \end{aligned} \tag{2.18}$$

Note that, the first of these is no longer a domain wall solution: depending on the value of α, it either diverges or it describes a regular solution oscillating once in a potential well. Indeed, if $0 < \alpha < 1$ (the positive soft mass case), the denominator vanishes for $\rho = \frac{2(\alpha \pm \sqrt{\alpha})}{\alpha(\alpha-1)}$. On the other hand, when $1 \leq \alpha$, the potential has three extrema: $A = 0$, $\frac{-3+\sqrt{\frac{9\alpha-8}{\alpha}}}{4}$, $\frac{-3-\sqrt{\frac{9\alpha-8}{\alpha}}}{4}$. The last one is the true minimum, whereas the other two being a local minimum and a local maximum, respectively. In this case the solution corresponds to the field rolling on the inverse potential, from $\phi = 0$ in the direction of the global minimum until it gets blocked by the potential barrier, then settling back at $\phi = 0$. Like a domain wall solution, it has a finite action $\int dt \left(\left| \frac{dA}{dt} \right|^2 + V \right)$ (for $\alpha = 2$ it is ≈ 0.06).

3 SQCD with quartic couplings

The solution of eq. (1.4) is also of interest in SQCD, whose dynamics for N_c colours and N_f flavours has been studied in great detail over the years (for reviews see [20–26]). We are particularly interested in the free magnetic regime, $N_c + 1 < N_f < \frac{3}{2} N_c$, in which there exist WZ domain walls described by eq. (1.4), as we shall now see.

Consider SQCD in such a phase, with a quartic superpotential

$$W^{(\mathrm{el})} = \frac{1}{\mu_X} \operatorname{Tr}\Big[(Q \cdot \tilde{Q})^2\Big] , \tag{3.1}$$

where the dot indicates colour contractions, and the trace is over flavours of quarks and antiquarks Q_i^a, $\tilde{Q}_a^j$, which are respectively in the fundamental and anti-fundamental representations of $\mathrm{SU}(N_c)$. This operator could be generated by the integrating out of heavier fields of mass $\mathcal{O}(\mu_X)$, as happens generically in the duality cascade [26]. For physical consistency we will therefore require that $\mu_X > \Lambda$, with Λ being the dynamical scale of the electric theory. Below the scale Λ, the electric SQCD theory described above becomes strongly coupled, and physics is best described by its magnetic dual. This theory also has N_f flavours, but $\mathrm{SU}(N)$ gauge group, where $N = N_f - N_c$, and a classical superpotential

$$W_{\mathrm{cl}}^{(\mathrm{mag})} = h\, q \Phi \tilde{q} + \frac{\mu_\Phi}{2} \operatorname{Tr}\big(\Phi^2\big) . \tag{3.2}$$

Here Φ_j^i are the flavour mesons of the infrared (IR) free theory, h is a Yukawa coupling of order unity, and q_i^a, $\tilde{q}_a^j$ are fundamental and anti-fundamental quarks of $\mathrm{SU}(N)$. The Φ mass term is $\mu_\Phi \approx \Lambda^2/\mu_X \ll \Lambda$.

This theory has supersymmetric minima at the origin. In order to be able to count them and compare with the original $\mathrm{SU}(N_c)$ theory, it is useful to also allow the addition of a mass term for the quarks in the electric theory, $W^{(\mathrm{el})} \supset m_Q \operatorname{Tr}\Big(Q \cdot \tilde{Q}\Big)$ which must have $m_Q < \Lambda$ (to avoid the quarks being integrated out of the electric theory before we ever reach the scale Λ). In the magnetic theory this becomes a linear meson term, $W_{\mathrm{cl}}^{(\mathrm{mag})} \supset m_Q \Lambda \operatorname{Tr}\Phi$. The conditions for supersymmetric minima then become

$$F_{\Phi_j^i} = h\, q_i . \tilde{q}^j + \mu_\Phi\, \phi_i^j + m_Q \Lambda\, \delta_i^j = 0 , \tag{3.3}$$

along with the $F_q = F_{\tilde{q}} = 0$ condition, which has solutions at $\langle q \rangle = \langle \tilde{q} \rangle = 0$ and $\langle \phi_i^j \rangle = -\delta_i^j m_Q \Lambda / \mu_\Phi$, parametrically close to the origin (whereas earlier we use ϕ_i^j to denote the scalar component of the superfield). This VEV gives a mass $|h m_Q \Lambda/\mu_\Phi|$ to all the magnetic quarks, and therefore by the usual Witten index theorem, we expect N vacua corresponding to the low energy pure SU(N) Yang-Mills theory.

The remaining supersymmetric minima are separated from the origin by domain walls, beyond which ϕ develops a much larger VEV. Along this direction one is still in a pure SU(N) Yang-Mills theory, but non-perturbative contributions to the superpotential become important. Including these (and neglecting the quark mass term), the complete superpotential for the mesons is as in eq. (1.1)

$$W^{(\mathrm{mag})} \;=\; \frac{\mu_\Phi}{2}\,\mathrm{Tr}\big(\Phi^2\big) + N\left(\frac{h^{N_f}\det_{N_f}\Phi}{\Lambda^{N_f-3N}}\right)^{\frac{1}{N}} , \tag{3.4}$$

where the effective exponent, $p \equiv \frac{N_f}{N}$, is generically a rational number. In the regime of interest, $N_c + 2 \le N_f < \frac{3}{2}N_c$, we have

$$3 \,<\, p \,\le\, \frac{N_f}{2}\,. \tag{3.5}$$

In principle using eq. (1.4) one can get the exact domain wall solutions for this magnetic theory, for any p.

To find them let us first locate the minima which are along $\phi_i^j = \delta_i^j \phi$ (where we use ϕ to also stand for the trace component of the scalar). Setting $F_\Phi = 0$, we find non-perturbatively generated SUSY preserving minima at

$$\langle \phi_i^j \rangle \;=\; \delta_i^j \phi_0 \;=\; \delta_i^j\,\Lambda\left(-h^{\frac{N_f}{N_f-N_c}}\,\frac{\Lambda}{\mu_\Phi}\right)^{\frac{N_f-N_c}{N_f-2N_c}} . \tag{3.6}$$

The exponent here is negative so that $\langle\phi\rangle < \Lambda$ as required for the minima to be found in the IR theory. Also note that, as there are no massless quarks, there are generically $2N_c - N_f$ solutions corresponding to the roots of -1. Together with the $N = N_f - N_c$ minima near the origin this gives the full complement of N_c vacua predicted by the Witten index theorem.

For the domain walls we define

$$\hat{\Phi} \;=\; \frac{\Phi}{|\phi_0|}\;; \qquad \hat{W} \;=\; \frac{W^{(\mathrm{mag})}}{\mu_\Phi|\phi_0|^2}\,, \tag{3.7}$$

giving $\hat{W} = \frac{\hat{\Phi}^2}{2} + \frac{\hat{\Phi}^p}{p}$ with $p = N_f/N$. We will henceforth drop the hats.

In order to determine the possible phases of the solution to the BPS condition in eq. (1.3), letting $\phi(t) = |\phi| e^{i(\theta+\eta)}$ we find two equations:

$$\begin{aligned} \partial_t \eta \;&=\; -\sin((p-2)\theta + p\eta)|\phi|^{p-2} - \sin(2\eta)\,, \\ \partial_t \log|\phi| \;&=\; \cos((p-2)\theta + p\eta)|\phi|^{p-2} + \cos(2\eta)\,, \end{aligned} \tag{3.8}$$

where we recall that now both W and Φ are dimensionless. It is clear that domain wall solutions with constant phase require $\eta = 0$ and $\theta = n\pi/(p-2)$ for integer n. Eq. (1.4) then has $\bar{z}^{p-2} - z^{p-2} \to 0$ along this direction, and we find

$$\phi(t) \;=\; \frac{e^{i\theta}e^{t}}{\left(1-(-1)^{n}e^{(p-2)t}\right)^{\frac{1}{p-2}}}\,, \tag{3.9}$$

which is non-singular if n is odd. Hence, there is a domain wall with constant phase between each minimum and the origin. To illustrate this, we show a solution in Figure 4 with $p = 15$. In the large p limit these solutions tend to a universal form, $\phi(t) \overset{p\to\infty}{=} 1 + \vartheta(-t)(e^{t}-1)$, where ϑ is the Heaviside theta function.

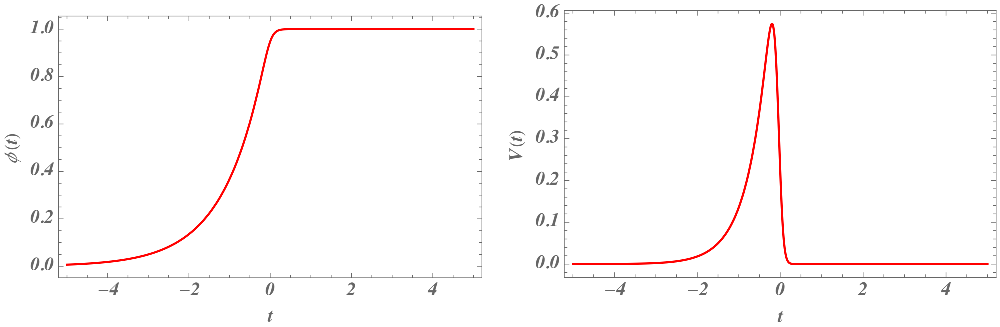


Figure 4. Domain wall solution for SQCD with $p = 15$. In the right panel we plot the potential as a function of t.

4 Conclusion

In this paper, we have presented an exact classical BPS solution of generalised Wess-Zumino models, which extends expressions previously known in the literature. We have discussed its applications as a generating function for multiparticle tree-level amplitudes on threshold and as a generalisation of known expressions for domain walls in Wess-Zumino models, which are for instance relevant for the vacuum structure of Supersymmetric QCD.

We have also pointed out natural extensions of our work, mostly in the context of models with spontaneously or softly broken supersymmetry. There, our methods yield partial expressions which would be interesting to complete, since they would be for instance of relevance for supersymmetric versions of the standard model.

Acknowledgments

We would like to thank Emilian Dudas for his collaboration at the early stages of this work, and Valya Khoze for useful discussions. This research is supported by Royal-Society/CNRS International Cost Share Award IE160590 and by the (Indo-French) CEFIPRA/IFCPAR Project No. 5404-2. Q.B. and D.C. would like to thank the Institute for Particle Physics Phenomenology at Durham University, where part of this work has been conducted, for its hospitality.

A The BPS condition

In this appendix we recap some facts about the BPS condition that underlie the discussion in the main text. A field configuration is BPS [27, 28] if it preserves some amount of supersymmetry. For scalar field configurations (transformed into fermions by supersymmetry), it amounts to requiring that fermions remain equal to zero when the preserved supersymmetry generators act. For a chiral superfield Φ such as the one in the WZ model, the fermion variation is[4]:

$$\delta_\xi \psi \;=\; i\sqrt{2}\sigma^m \bar{\xi}\partial_m \phi + \sqrt{2}\xi F \;, \tag{A.1}$$

for $\Phi = \phi + \sqrt{2}\theta\psi + \theta^2 F$. When calculating multiparticle amplitudes or domain wall profiles, we are interested in one-dimensional problems, so without loss of generality we choose $\phi(x^\mu) = \phi(x)$, x being the spatial coordinate along which the wall extends. Then, demanding that $\delta_\xi\psi = 0$ translates into

$$\bar{\xi}^2 \frac{d\phi}{dx} = i\xi_1 F \text{ and } \bar{\xi}^1 \frac{d\phi}{dx} = i\xi_2 F \;. \tag{A.2}$$

Whenever the scalars verify $\frac{d\phi}{dx} = -e^{i2\theta}F$ for some real number θ, eq. (A.2) can be satisfied. Using the on-shell value for F, for a trivial Kähler potential and a superpotential W, we find

$$\frac{d\phi}{dx} \;=\; e^{2i\theta}\frac{d\overline{W}}{d\bar{\phi}} \;. \tag{A.3}$$

For the WZ model in eq. (1.1), this reduces to eq. (1.3).

Equation (A.3) can also be understood as a factorisation of the equations of motion. Indeed, imposing the former is enough to satisfy the latter:

$$\frac{d^2\phi}{dx^2} \;=\; e^{2i\theta}\frac{d^2\overline{W}}{d\bar{\phi}^2}\frac{d\bar{\phi}}{dx} \;=\; \frac{d^2\overline{W}}{d\bar{\phi}^2}\frac{dW}{d\phi} \;=\; \frac{dV}{d\bar{\phi}} \;, \tag{A.4}$$

since $V = \left|\frac{dW}{d\phi}(\phi)\right|^2$ for a chiral superfield.

Equation (A.3) can finally be understood as the condition that minimises the energy per unit surface of a time-independent wall [30–32]:

$$\mathcal{E} \;=\; \int dx \left(\left|\frac{d\phi}{dx}\right|^2 + \left|\frac{dW}{d\phi}\right|^2\right) \;=\; \int dx \left|\frac{d\phi}{dx} - e^{2i\theta}\frac{d\overline{W}}{d\bar{\phi}}\right|^2 + 2\,\mathrm{Re}(e^{-2i\theta}\Delta W) \;, \tag{A.5}$$

where $\Delta W = W(x=+\infty) - W(x=-\infty)$. The fact that this condition is valid regardless of θ implies the so-called BPS bound:

$$\mathcal{E} \;\geq\; 2|\Delta W| \;. \tag{A.6}$$

In order to saturate this bound, one must again enforce eq. (A.3).

The fact that the generating function of multiparticle amplitudes verifies a BPS condition can be understood from [33]: smooth field configurations which solve the equations of motion and originate from a supersymmetric vacuum state must verify the BPS condition. Eq. (2.9), which defines the generating function in eq. (1.4), thus implies eq. (A.3).

[4] We use the conventions of Ref.[29].

B Link with softly broken O(2) models

Here we draw links with the special case in [2]. When $p = 3$, eq. (1.4) reduces to:

$$\begin{aligned} A(z,\bar{z}) &= \frac{z(1+\frac{\bar{z}-z}{6})}{1-\frac{z+\bar{z}}{2}+\frac{(z-\bar{z})^2}{12}-\frac{(z+\bar{z})(z-\bar{z})^2}{216}} \\ &= \Bigg|_{z=e^{t+i\theta}} \frac{e^{t+i\theta}(1-\frac{2ie^t\sin(\theta)}{6})}{1-e^t\cos(\theta)-\frac{e^{2t}\sin^2(\theta)}{3}+\frac{e^{3t}\cos(\theta)\sin^2(\theta)}{27}} \,. \end{aligned} \tag{B.1}$$

Equation (B.1) can be identified with generating functions in softly broken O(2) models [2] of two real scalar fields φ and χ, with potential

$$V(\varphi,\chi) = \mu(\varphi^2+\chi^2)^2 + \frac{m_1^2}{2}\varphi^2 + \frac{m_2^2}{2}\chi^2 \,. \tag{B.2}$$

Indeed, defining $B = A + \frac{1}{2}$, $V = \left|A^2 + A\right|^2 = \left|B^2 - \frac{1}{4}\right|^2$ matches (up to the constant term) with $V(\varphi = \mathrm{Re}(B), \chi = \mathrm{Im}(B))$ if we take $\mu = m_2^2 = -m_1^2 = 1$. Then, the "broken reflection symmetry" solution given in [2] matches eq. (B.1) once we identify $A = \varphi - \frac{1}{2} + i\chi$.

C Derivation of the solution

Here, we outline the way eq. (1.4) was found. Although one can check from the solution itself that it solves the BPS condition for the model of eq. (1.2), different methods have been used in its derivation, so we quickly list them here, following our chronological progression.

First, for the $p = 3$ case one can start by solving eq. (2.8) with $\theta = 0$ or π (i.e. z real), which makes $A(z, \bar{z} = z)$ real, giving

$$A(z,\bar{z}=z) = \frac{z}{1-z} \,. \tag{C.1}$$

In order to derive this expression, we impose that A scales as z as z goes to 0, which is enough/required for walls or amplitudes. One then seeks the multiparticle amplitudes where an incoming ϕ goes into n ϕ's (and no $\bar{\phi}$'s) in the final states. This corresponds to graphs where only ϕ propagates since, at each vertex, the number of ϕ's, or $\bar{\phi}$'s, in the out-states is always larger (or equal) than the one in the in-states. It amounts to solving the equation $\partial_t^2 A = A + A^2$, which determines $A(z, 0)$. The solution is

$$A(z,\bar{z}=0) = \frac{z}{\left(1-\frac{z}{6}\right)^2} \,. \tag{C.2}$$

Then, one can make an educated guess of the form

$$A(z,\bar{z}=0) = \frac{z}{\left(1-\frac{z}{6}\right)^2 + \bar{z}f(z,\bar{z})} \,, \tag{C.3}$$

and numerically solve the first steps of the recursion relation in eq. (2.6) to get the $(z, \bar{z})$ expansion of $f(z, \bar{z})$, from which one can surmise the following fully resummed expression:

$$\Big(A/z\Big)(z=0,\bar{z}) = \frac{\left(1+\frac{\bar{z}}{6}\right)}{\left(1-\frac{\bar{z}}{6}\right)^3} \,. \tag{C.4}$$

After some more recursive steps one can deduce the full $p = 3$ solution:

$$A(z, \bar{z}) = \frac{z(1 + \frac{\bar{z}-z}{6})}{1 - \frac{z+\bar{z}}{2} + \frac{(z-\bar{z})^2}{12} - \frac{(z+\bar{z})(z-\bar{z})^2}{216}} \; . \tag{C.5}$$

This solution turns out to be a reshuffling of the one found in [2].

Higher p solutions are derived in the following way: the Hamilton-Jacobi equation for the WZ model with $p = 4$ can be solved with a variable separation, as in [2], by defining:

$$A = \sqrt{2(\xi^2 - 1)(1 - \eta^2)} + i\sqrt{2}\xi\eta \; . \tag{C.6}$$

Ultimately this gives

$$A(z, \bar{z}) = \frac{z\left(1 + \frac{\bar{z}^2 - z^2}{8}\right)}{\sqrt{\left[1 - \frac{(z-\bar{z})^2}{4} + \frac{(z^2-\bar{z}^2)^2}{64}\right]\left[1 - \frac{(z+\bar{z})^2}{4} + \frac{(z^2-\bar{z}^2)^2}{64}\right]}} \; . \tag{C.7}$$

From this example one can guess that, for general p,

$$A(z, \bar{z}) = \frac{z\left(1 + \frac{\bar{z}^{p-2} - z^{p-2}}{2p}\right)}{P(z, \bar{z})} \; , \tag{C.8}$$

with $P(z, \bar{z})$ being a real function. This parametrisation makes it possible to solve the BPS condition of eq. (1.3), which gives a simple first order equation for $P(z, \bar{z})$ whose solution, with our boundary conditions, is eq. (1.4). The latter yields expressions in particular limits that match the results of derivations similar to the discussions for eqs. (C.1) and (C.2):

$$A(z, \bar{z} = z) = \frac{z}{(1 - z^{p-2})^{\frac{1}{p-2}}} \; , \quad A(z, \bar{z} = 0) = \frac{z}{\left(1 - \frac{z^{p-2}}{2p}\right)^{\frac{2}{p-2}}} \; . \tag{C.9}$$

References

[1] B. Chibisov and M. A. Shifman, "BPS saturated walls in supersymmetric theories," *Phys. Rev.* **D56** (1997) 7990–8013, arXiv:hep-th/9706141 [hep-th]. [Erratum: Phys. Rev.D58,109901(1998)].

[2] M. V. Libanov, V. A. Rubakov, and S. V. Troitsky, "Tree amplitudes at multiparticle threshold in a model with softly broken 0(2) symmetry," *Nucl. Phys.* **B412** (1994) 607–620, arXiv:hep-ph/9307213 [hep-ph].

[3] M. B. Voloshin, "Multiparticle amplitudes at zero energy and momentum in scalar theory," *Nucl. Phys.* **B383** (1992) 233–248.

[4] E. N. Argyres, R. H. P. Kleiss, and C. G. Papadopoulos, "Amplitude estimates for multi - Higgs production at high-energies," *Nucl. Phys.* **B391** (1993) 42–56.

[5] M. B. Voloshin, "Estimate of the onset of nonperturbative particle production at high-energy in a scalar theory," *Phys. Lett.* **B293** (1992) 389–394.

[6] L. S. Brown, "Summing tree graphs at threshold," *Phys. Rev.* **D46** (1992) R4125–R4127, arXiv:hep-ph/9209203 [hep-ph].

[7] M. V. Libanov, V. A. Rubakov, D. T. Son, and S. V. Troitsky, "Exponentiation of multiparticle amplitudes in scalar theories," *Phys. Rev.* **D50** (1994) 7553–7569, arXiv:hep-ph/9407381 [hep-ph].

[8] D. T. Son, "Semiclassical approach for multiparticle production in scalar theories," *Nucl. Phys.* **B477** (1996) 378–406, arXiv:hep-ph/9505338 [hep-ph].

[9] M. V. Libanov, V. A. Rubakov, and S. V. Troitsky, "Multiparticle processes and semiclassical analysis in bosonic field theories," *Phys. Part. Nucl.* **28** (1997) 217–240.

[10] V. V. Khoze and J. Reiness, "Review of the semiclassical formalism for multiparticle production at high energies," arXiv:1810.01722 [hep-ph].

[11] V. V. Khoze, "Multiparticle Higgs and Vector Boson Amplitudes at Threshold," *JHEP* **07** (2014) 008, arXiv:1404.4876 [hep-ph].

[12] V. V. Khoze and M. Spannowsky, "Higgsplosion: Solving the Hierarchy Problem via rapid decays of heavy states into multiple Higgs bosons," *Nucl. Phys.* **B926** (2018) 95–111, arXiv:1704.03447 [hep-ph].

[13] V. V. Khoze, "Multiparticle production in the large ?n limit: realising Higgsplosion in a scalar QFT," *JHEP* **06** (2017) 148, arXiv:1705.04365 [hep-ph].

[14] V. V. Khoze, "Semiclassical computation of quantum effects in multiparticle production at large lambda n," arXiv:1806.05648 [hep-ph].

[15] G. R. Dvali and M. A. Shifman, "Dynamical compactification as a mechanism of spontaneous supersymmetry breaking," *Nucl. Phys.* **B504** (1997) 127–146, arXiv:hep-th/9611213 [hep-th].

[16] M. B. Voloshin, "Summing one loop graphs at multiparticle threshold," *Phys. Rev.* **D47** (1993) R357–R361, arXiv:hep-ph/9209240 [hep-ph].

[17] A. S. Gorsky and M. B. Voloshin, "Nonperturbative production of multiboson states and quantum bubbles," *Phys. Rev.* **D48** (1993) 3843–3851, arXiv:hep-ph/9305219 [hep-ph].

[18] M. V. Libanov, D. T. Son, and S. V. Troitsky, "Exponentiation of multiparticle amplitudes in scalar theories. 2. Universality of the exponent," *Phys. Rev.* **D52** (1995) 3679–3687, arXiv:hep-ph/9503412 [hep-ph].

[19] L. Girardello and M. T. Grisaru, "Soft Breaking of Supersymmetry," *Nucl. Phys.* **B194** (1982) 65.

[20] J. Terning, *Modern supersymmetry: Dynamics and duality*. 2006.

[21] M. Dine, *Supersymmetry and String Theory*. Cambridge University Press, 2016. http://www.cambridge.org/academic/subjects/physics/particle-physics-and-nuclear-physics/supersymmetry-and-string-theory-beyond-standard-model-2nd-edition?format=AR&isbn=9781316405307.

[22] K. A. Intriligator and N. Seiberg, "Lectures on supersymmetric gauge theories and electric-magnetic duality," *Nucl. Phys. Proc. Suppl.* **45BC** (1996) 1–28, arXiv:hep-th/9509066 [hep-th]. [,157(1995)].

[23] M. E. Peskin, "Duality in supersymmetric Yang-Mills theory," in *Fields, strings and duality. Proceedings, Summer School, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI'96, Boulder, USA, June 2-28, 1996*, pp. 729–809. 1997.

arXiv:hep-th/9702094 [hep-th]. http://www-public.slac.stanford.edu/sciDoc/docMeta.aspx?slacPubNumber=SLAC-PUB-7393.

[24] M. A. Shifman, "Nonperturbative dynamics in supersymmetric gauge theories," *Prog. Part. Nucl. Phys.* **39** (1997) 1–116, arXiv:hep-th/9704114 [hep-th]. [,345(1995)].

[25] K. A. Intriligator, N. Seiberg, and D. Shih, "Supersymmetry breaking, R-symmetry breaking and metastable vacua," *JHEP* **07** (2007) 017, arXiv:hep-th/0703281 [hep-th].

[26] M. J. Strassler, "The Duality cascade," in *Progress in string theory. Proceedings, Summer School, TASI 2003, Boulder, USA, June 2-27, 2003*, pp. 419–510. 2005. arXiv:hep-th/0505153 [hep-th].

[27] E. B. Bogomolny, "Stability of Classical Solutions," *Sov. J. Nucl. Phys.* **24** (1976) 449. [Yad. Fiz.24,861(1976)].

[28] M. K. Prasad and C. M. Sommerfield, "An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon," *Phys. Rev. Lett.* **35** (1975) 760–762.

[29] J. Wess and J. Bagger, *Supersymmetry and supergravity.* Princeton University Press, Princeton, NJ, USA, 1992.

[30] J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo, and P. K. Townsend, "Topological Extensions of the Supersymmetry Algebra for Extended Objects," *Phys. Rev. Lett.* **63** (1989) 2443.

[31] P. Fendley, S. D. Mathur, C. Vafa, and N. Warner, "Integrable Deformations and Scattering Matrices for the $N = 2$ Supersymmetric Discrete Series," *Phys. Lett.* **B243** (1990) 257–264.

[32] S. Cecotti and C. Vafa, "On classification of N=2 supersymmetric theories," *Commun. Math. Phys.* **158** (1993) 569–644, arXiv:hep-th/9211097 [hep-th].

[33] D. Bazeia, J. Menezes, and M. M. Santos, "Complete factorization of equations of motion in supersymmetric field theories," *Nucl. Phys.* **B636** (2002) 132–140, arXiv:hep-th/0103041 [hep-th].

Titre : Quelques sujets au-delà du modèle standard: axions, supersymétrie, théorie des cordes

Mots clés : physique théorique des particules, axions, supersymétrie, théorie des cordes

Résumé : Cette thèse a pour but l'étude de théories diverses, toutefois interconnectées, décrivant la nouvelle physique au-delà du modèle standard de la physique des particules. Ce sont des théories d'un nouveau type de particules, les axions, d'un nouveau principe de symétrie, la supersymétrie, et d'une nouvelle description des degrés de liberté fondamentaux, la théorie des cordes. Les progrès instrumentaux et théoriques constamment faits au fil des ans ont confirmé que ces théories sont des candidates privilégiées pour une description de la physique au-delà du modèle standard.

Les axions sont d'abord examinés et étudiés d'un point de vue phénoménologique: nous présentons des modèles qui désenchevêtrent les différentes échelles qui décrivent l'espace des paramètres des modèles d'axions, et nous discutons les axions présents dans des modèles de saveur. Inspirés par les recherches autour du swampland, nous nous imposons l'utilisation de symétries de jauge, et non globales, en tant que point de départ pour la construction de modèles.

Notre intérêt se porte ensuite sur la supersymétrie. Nous étudions sa brisure, à la fois dans des modèles explicites dans l'ultraviolet qui génèrent une échelle de brisure de supersymétrie basse à partir de matière à haute échelle, et au niveau des théories effectives à l'aide de la supersymétrie non-linéaire. En ce qui concerne ce dernier thème, nous nous restreignons à l'approche des superchamps contraints. Enfin, nous présentons des solutions classiques exactes d'un modèle supersymétrique dont la portée est grande, le modèle de Wess-Zumino d'un superchamp chiral.

Finalement, nous nous intéressons à la théorie des cordes. Nous calculons des spectres de cordes en guise d'illustration de la structure de la théorie et de point de départ pour le calcul d'amplitudes du vide à une boucle. Celles-ci nous permettent de tester l'une des conjectures du swampland, qui désigne la gravité comme la plus faible des forces, dans une configuration de théorie des cordes où la supersymétrie est brisée. Enfin, les axions en théorie des cordes sont analysés, en particulier lorsqu'ils sont chargés sous une symétrie de jauge abélienne anomale.

Title : Topics beyond the Standard Model: axions, supersymmetry, string theory

Keywords : theoretical particle physics, axions, supersymmetry, string theory

Abstract : The aim of this thesis is to study various but interconnected theories for new physics beyond the standard model of particle physics. Those are theories of a new kind of particles, axions, a new symmetry principle, supersymmetry, and a new description of fundamental degrees of freedom, string theory. Constant instrumental and theoretical progresses made over the years maintain those already old subjects as leading BSM candidates.

Axions are first reviewed and studied from a phenomenological perspective: we present models which disentangle the different scales which define the axion parameter space, and we discuss axions which arise in models of flavour physics. Motivated by swampland considerations, we insist on using gauge, and not global, symmetries as model building inputs.

The focus then shifts to supersymmetry. We study its breaking, both in explicit ultraviolet models which generate a low supersymmetry breaking scale from high-scale matter, and at the effective field theory level using non-linearly realized supersymmetry. In our study of the latter topic, we focus on the constrained superfield approach. Finally, we present exact classical solutions of a supersymmetric model with broad application scope, the Wess-Zumino model of a chiral superfield.

Last, we discuss string theory. We compute string spectra as illustrations of the structure of the theory and as starting points to compute one-loop vacuum amplitudes. Those are used to understand supersymmetry breaking in string theory, as well as brane interactions. Then, the latter enable us to test one of the swampland criteria, the weak gravity conjecture, in a string theory setup with broken supersymmetry. Finally, axions in string theory are scrutinized, in particular when they are charged under an anomalous abelian factor of the gauge group.

Université Paris-Saclay
Espace Technologique / Immeuble Discovery
Route de l'Orme aux Merisiers RD 128 / 91190 Saint-Aubin, France