Motifs généralisées et orientations symplectiques

Nanjun Yang

Communauté UNIVERSITÉ Grenoble Alpes

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Présentée par

Nanjun YANG

Thèse dirigée par **Jean FASEL, professeur des universités, Université Grenoble Alpes**

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Motifs Généralisés et Orientations Symplectiques

Generalized Motives and Symplectic Orientations

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devant le jury composé de :

M. Grégory BERHUY
Professeur des Universités, Université Grenoble Alpes, Examinateur
M. Baptiste CALMÈS
Maître de Conférences, Université d'Artois, Examinateur
M. Frédéric DÉGLISE
Directeur de Recherche, Université de Bourgogne, Examinateur
M. Jean FASEL
Professeur des Universités, Université Grenoble Alpes, Directeur de Thèse
M. Grigory GARKUSHA
Associate Professor, Swansea University, Rapporteur
M. Marc HOYOIS
Professeur Assistant, University of Southern California, Rapporteur

Abstract

In this thesis, we present a general framework to construct categories of motives and build part of the six operations formalism for these categories. In the case of MW-motivic cohomology, we prove the quaternionic projective bundle theorem and construct a Gysin triangle, which enable us to define Pontryagin classes on Chow-Witt rings for symplectic bundles. Applying these tools together, we compute the group of morphisms between smooth proper schemes in the category of (effective) MW-motives.

Key Words: Correspondences, Generalized motives, Symplectic orientations.

Résumé

Dans cet article, nous présentons une approche générale pour construire des catégories de motifs et établissons une partie du formalisme des six foncteurs pour ces catégories. Dans le cas de la cohomologie MW-motivique, nous prouvons le théorème des fibrés quaternioniques et construisons un triangle de Gysin. Ceci nous permet de définir des classes de Pontryagin sur les anneaux de Chow-Witt pour des fibrés symplectiques. Appliquant ces outils, nous calculons le groupe des morphismes entre schémas lisses et propres dans la catégorie des MW-motifs (effectifs).

Mots clés : Correspondances, Motifs généralisés, Orientations symplectiques.

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Thank you!

Mathematics guides us to discover the truth independently.

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Chapter 1

Introduction

1.1 Background

Algebraic geometry is a profound and beautiful branch of mathematics which mainly studies properties of (smooth) schemes. One of the main approach to this study is to develop suitable cohomology theories, and algebraic geometers have spent lots of time working on this. The first approach was the Chow ring ($CH^n(X)$), defined by W. L. Chow around 1956. The elements of that ring are just algebraic cycles, considered up to rational equivalence. Much later, it was realized that these groups were in fact homology groups of the so-called Rost-Schmid complex ([Ro96]) with coefficients in Milnor K-theory. The essential operations in the Chow ring are in particular products, pull-backs and push-forwards, which are in fact all defined at the level of this complex. Moreover, for any smooth scheme X and any vector bundle V on X, we have a Thom isomorphism

$$CH^n(X) \longrightarrow CH_X^{n+rk(V)}(V) \tag{1.1}$$

defined by the push-forward via the zero section of V. Using these isomorphisms, it's easy to calculate the Chow ring of a projective bundle $\mathbb{P}(V)$ in terms of the Chow ring of X, obtaining the so-called projective bundle theorem and its consequences, such as the splitting principle ([Ful98]) and the existence of Chern classes of V with coefficients in the Chow ring.

Based on the Chow ring, V. Voevodsky defined, in 2000, motivic cohomology of (smooth) schemes ([MVW06]), relating to many fields such as K-theory, Milnor K-theory and étale cohomology. This had many important applications, such as for example the Milnor conjecture. The construction is based on the notion of finite correspondences, which are special cycles and form the morphisms in the category Cor_k whose objects are smooth schemes over k. This enables in turn, given a topology t (Nisnevich or étale) on the category of smooth schemes, to consider t-sheaves on Cor_k, the so-called sheaves with transfers. The category of effective motives $DM^{eff}(k)$ is just the localization of the derived category of sheaves with transfers under the homotopy invariance conditions (making $X \times \mathbb{A}^1$ and X equivalent) and the motivic cohomology group $H^{p,q}(X, \mathbb{Z})$ is just the p^{th} hypercohomology of the motivic complex $\mathbb{Z}(q)$ constructed via the Tate twist. An important fact is that

$$H^{2p,p}(X, \mathbb{Z}) \simeq CH^p(X), \tag{1.2}$$

recovering the original Chow groups (functorially in X) as motivic cohomology groups. More generally, the general term $H^{p,q}(X, \mathbb{Z})$ corresponds to the higher Chow group $CH^q(X, 2q-p)$ defined by S. Bloch ([V02]).

There are plenty of further developments of motivic cohomology beyond the basic facts described above. First, there is the so-called Poincaré duality ([FV00]) for motives of proper schemes. This requires to stabilize $DM^{eff}(k)$, namely to formally invert the Tate twist in DM^{eff}, which is realized by the use of symmetric spectra. This, in particular, implies that the category of pure Chow motives, defined by Grothendieck, can be contravariantly embedded into $DM^{eff}(k)$. This is the so-called embedding theorem. Second, one can also construct a category Cor_S over any (smooth) scheme S, by considering finite correspondences over S ([D07]). The same techniques as above yields the category of effective (resp. stabilized) motives over S, denoted by $DM^{eff}(S)$ (resp. $DM(S)$). Then one can consider a huge and powerful mechanism called six operations formalism on the category of effective (resp. stabilized) motives, following an axiomatic approach described in [CD09] and [CD13]. The first complete version of this formalism appeared in the stable homotopy theory of schemes ([Ayo07]). It's very similar and closely related to the formalism in [CD09] and [CD13]. The former preserves more information but the latter has the important property of being oriented ([MVW06], [CD13]) for any vector bundle, which makes us possible to prove a projective bundle theorem in motivic cohomology as in the Chow ring and giving a Gysin triangle which is a motivic analogue of (1.1).

Recently, some refinements of the original ideas of Voevodsky appeared. One of them is based on the Chow-Witt groups $\widetilde{CH}^n(X, \mathscr{L})$, as defined by J. Barge and F. Morel in 2000 and completed by J. Fasel. The original goal of these groups was to determine whether a projective module has a rank one free module as a direct summand ([Fas08]), a question out of range for ordinary Chow groups. Their definition parallels the fact that Chow groups can be seen as some cohomology groups of the complex in Milnor K-theory ([Ro96]), they are cohomology groups of the Rost-Schmid complex in Milnor-Witt K-theory. A significant difference with the Chow rings is that they depend not only on a smooth scheme X, but also on a line bundle $\mathscr{L}$ on that scheme, called the *twist*. This phenomenon in the Chow-Witt rings is inherited from the Witt ring and it prevents the Chow-Witt rings from being oriented, that is, there is no projective bundle theorem, hence no Chern classes on the Chow-Witt ring ([Fas08]). It's nevertheless an interesting question to know whether it's oriented only for *symplectic bundles*, i.e. if the quaternionic projective bundle theorem as in [PW10] holds. If it's the case, we can define *Pontryagin classes* with coefficients in the Chow-Witt rings for symplectic bundles.

Mimicking the definition of ordinary motivic cohomology, one can obtain a category of motives based on the Chow-Witt rings. This is the category of MW-motives as defined by B. Calmès, F. Déglise and J. Fasel ([CF14], [DF17]). It is a better approximation of the stable homotopy theory, compared with Voevodsky's and the equation (1.2) also has an analogue there. The basic constructions in MW-motivic cohomology are very similar to motivic cohomology, where the correspondences are replaced by MW-correspondences, but there is a quite subtle difficulty at each step, that is the *calculation* of twists in the operations on Chow-Witt rings, such as product, pull-backs and push-forwards which are necessarily more complicated than in Chow ring. The serious approach to that is to regard those twists as elements in the category of virtual vector bundles ([Del87], [CF18]) and it's a delicate job to implement all calculations under the formal rules of virtual objects. This inspires us to *axiomatize* the idea of correspondences and get a general method to construct motives, even for non-oriented cohomology theories. Furthermore, to prove the *quaternionic projective bundle theorem* in Chow-Witt theory, one way is to prove its counterpart in MW-motivic cohomology first. As a consequence, it gives a computation of the Thom space of symplectic bundles in MW-motivic cohomology and gives the corresponding *Gysin triangle*. Finally, we use all the tools we developed to compute the group of morphisms in the category of (effective) motives between smooth proper schemes.

1.2 Main Results

1.2.1 Virtual Objects and Their Calculation

We provide the main tool for the calculation of virtual objects in Chapter 3, which makes a serious approach to twists possible. Let's denote by $V(Vect(X))$ the category of virtual vector bundles ([Del87]) over X.

Theorem 1.1. *(Theorem 3.1)*

1. *Suppose we have a commutative diagram of vector bundles over X with exact rows and columns*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}
$$

Then we have a commutative diagram in $V(Vect(X))$

$$
\begin{tikzcd}
V_2 \arrow[r] \arrow[d] & K + W_2 \arrow[d] \\
V_1 + C \arrow[r] & K + W_1 + C.
\end{tikzcd}
$$

2. *Suppose we have a commutative diagram of vector bundles over X with exact rows and columns*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & D \arrow[r, equal] \arrow[d] & D \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}
$$

Then we have a commutative diagram in $V(Vect(X))$

$$
\begin{tikzcd}
W_2 \arrow[r] \arrow[d] & V_2 + D \arrow[r] & V_1 + C + D \arrow[ddll, "c(C,D)"] \\
W_1 + C \arrow[d] & & \\
V_1 + D + C & &
\end{tikzcd}
$$

where $c(C, D)$ is the commutation rule between C and D in the category of virtual vector bundles.

3. *Suppose we have a commutative diagram of vector bundles over X with exact rows*

and columns

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & T \arrow[r] \arrow[d, equal] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T \arrow[r] & W_1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0.
\end{tikzcd}
$$

Then we have a commutative diagram in $V(Vect(X))$

$$
\begin{tikzcd}
V_1 \arrow[rr] \arrow[d] & & T+V_2 \arrow[r] & T+K+W_2 \arrow[dlll, "c(T,K)"] \\
K+W_1 \arrow[d] & & & \\
K+T+W_2 & & &
\end{tikzcd}
$$

where $c(T,K)$ is the commutation rule between T and K in the category of virtual vector bundles.

1.2.2 Correspondences and Generalized Motives

We propose an axiomatic definition of correspondences in Chapter 4. Then given a correspondence theory E, we establish in Chapter 5 the theory of sheaves with E-transfers. In Chapter 6, we define the category $\widetilde{DM}^{eff,-}(S)$ (resp. $\widetilde{DM}^{-}(S)$) of effective (resp. stabilized) motives over a smooth base S by using bounded above complexes (contrary to [CD09], [CD13]) and build part of its six operations formalism ($\otimes$, f^*, $f_\#$) in the general setting.

In Chapter 8, we partially show that the MW-correspondences defined in [CF14] is indeed a correspondence theory as we defined, by adopting a new perspective on the push-forward in the Chow-Witt ring.

1.2.3 Symplectic Orientations and Applications

For any $X \in Sm/S$, denote by $\widetilde{\mathbb{Z}}_S(X)$ the motive of X in $\widetilde{DM}^{eff,-}(S)$. In Chapter 7, we prove the quaternionic projective bundle theorem for MW-motivic cohomology:

Theorem 1.2. *(Theorem 7.4) Let $X \in Sm/S$ and let $(\mathscr{E}, m)$ be a symplectic vector bundle of rank $2n+2$ on X. Let $\pi : HGr_X(\mathscr{E}) \to X$ be the projection. Then, the map*

$$\widetilde{\mathbb{Z}}_S(HGr_X(\mathscr{E})) \xrightarrow{\pi \boxtimes p_1(\mathscr{U}^\vee)^i} \oplus_{i=0}^{n} \widetilde{\mathbb{Z}}_S(X)(2i)[4i]$$

is an isomorphism in $\widetilde{DM}^{eff,-}(S)$, functorial for X in Sm/S. Here, $\mathscr{U}^\vee$ is the dual tautological bundle endowed with its canonical orientation.

Hence we get the corresponding result in the Chow-Witt ring:

Proposition 1.1. *(Proposition 7.11) Let $X \in Sm/k$, $\mathscr{E}$ be a symplectic bundle of rank $2n+2$ over X and $k = min\{\lfloor \frac{j}{2} \rfloor, n\}$. Then the map*

$$\theta_j : \oplus_{i=0}^{k} \widetilde{CH}^{j-2i}(X) \xrightarrow{p^* \cdot p_1(\mathscr{U}^\vee)^i} \widetilde{CH}^{j}(HGr_X(\mathscr{E}))$$

is an isomorphism, where $j \geq 0$, $p : HGr_X(\mathscr{E}) \longrightarrow X$ is the structure map and $\mathscr{U}^\vee$ is the dual tautological bundle endowed with its canonical orientation.

As an application, we can define the Pontryagin classes (in the Chow-Witt ring) for symplectic bundles, as follows:

Definition 1.1. *(Definition 7.11) In the above proposition, set* $\zeta := p_1(\mathscr{U}^\vee)$ *and* $\theta_{2n+2}^{-1}(\zeta^{n+1}) := (\zeta_i) \in \oplus_{i=1}^{n+1}\widetilde{CH}^{2i}(X)$*. Define* $p_0(\mathscr{E}) = 1 \in \widetilde{CH}^0(X)$*, and* $p_a(\mathscr{E}) = (-1)^{a-1}\zeta_i$ *for* $1 \leq a \leq n+1$*. The class* $p_a(\mathscr{E})$ *is called the* a^{th} *Pontryagin classes of* $\mathscr{E}$*. These elements are uniquely characterized by the Pontryagin polynomial*

$$\zeta^{n+1} - p^*(p_1(\mathscr{E}))\zeta^n + \ldots + (-1)^{n+1}p^*(p_{n+1}(\mathscr{E})) = 0.$$

As a consequence, we obtain a Gysin triangle for certain closed embeddings:

Theorem 1.3. *(Theorem 7.6) Let* $X \in Sm/S$ *and let* $Y \subseteq X$ *be a smooth closed subscheme with a symplectic normal bundle with* $codim(Y) = 2n$*. Then we have a distinguished triangle*

$$\widetilde{\mathbb{Z}}_S(X \setminus Y) \longrightarrow \widetilde{\mathbb{Z}}_S(X) \longrightarrow \widetilde{\mathbb{Z}}_S(Y)(2n)[4n] \longrightarrow \widetilde{\mathbb{Z}}_S(X \setminus Y)[1]$$

in $\widetilde{DM}^{eff,-}(S)$*.*

Finally, using the theorem above, the six operations formalism of Chapter 6 and duality in the stable $\mathbb{A}^1$-derived categories ([CD13]), we can prove the following theorem.

Theorem 1.4. *(Theorem 7.7) Let* $X, Y \in Sm/k$ *with* Y *proper, then we have*

$$Hom_{\widetilde{DM}^{eff,-}(k)}(\widetilde{\mathbb{Z}}_{pt}(X), \widetilde{\mathbb{Z}}_{pt}(Y)) \cong \widetilde{CH}^{d_Y}(X \times Y, \omega_{X\times Y/X}).$$

Throughout in this article, we denote by Sm/k the category of smooth separated schemes over k ([Har77, Chapter 10]), where k is an infinite perfect field with $char(k) \neq 2$. For any $X \in Sm/k$, we denote $dimX$ by d_X and for any $f : X \longrightarrow Y$ in Sm/k, we set $d_f = d_X - d_Y$.

Chapitre 2

Introduction

2.1 Contexte

La géométrie algébrique est une branche profonde et belle des mathématiques qui étudie principalement les propriétés des schémas (lisses). Les géomètres algébristes se sont depuis longtemps attelés à définir des théorie cohomologiques permettant d'étudier ces schémas. Une des premières approches a été l'anneau de Chow ($CH^n(X)$), défini par W. L. Chow aux environs de 1956. Les éléments de cet anneau sont par définition des classes de cycles algébriques sur X à équivalence rationnelle près. Ces groupes sont apparus beaucoup plus tard comme étant la cohomologie du complexe Rost-Schmid ([Ro96]) associé à la K-théorie de Milnor. Les opérations essentielles de l'anneau de Chow sont en particulier le produit, le push-forward et le pull-back. Pour tout schéma lisse X et tout fibré vectoriel V sur X, nous avons également un isomorphisme de Thom

$$CH^n(X) \longrightarrow CH_X^{n+rk(V)}(V) \tag{2.1}$$

défini par le push-forward le long de la section nulle de V. De plus, il est facile de calculer l'anneau de Chow du fibré projectif $\mathbb{P}(V)$ en termes de l'anneau de Chow de X pour obtenir le fameux théorème de fibré projectif et le principe de scindage associé. Ce théorème permet également de définir les classes de Chern de V sur l'anneau de Chow.

Sur la base de l'anneau de Chow, V. Voevodsky a défini en 2000 la cohomologie motivique ([MVW06]), obtenant une nouvelle et magnifique théorie cohomologique associée aux schémas lisses, permettant de relier de nombreux domaines tels que la K-théorie, la K-théorie de Milnor et la cohomologie étale. De nombreuses applications importantes ont découlé de son approche, comme par exemple la preuve de la conjecture de Milnor (prix Fields). La cohomologie motivique est basée sur la théorie de l'intersection ([Sha94]), plus précisḿent sur la théorie associée à certains types de cycles algébriques, appelés correspondances finies. Ceci permet d'obtenir une catégorie Cor_k dont les objets sont les schémas lisses sur k et les morphismes des correspondances finies. La prochaine étape est de considérer les faisceaux sur cette catégorie, pour une topologie fixé t (Nisnevich ou étale), appelés les faisceaux avec transferts. La catégorie des motifs effectifs $DM^{eff}(k)$ est simplement la localisation de la catégorie dérivée de faisceaux avec transferts sous la condition d'invariance par l'homotopie (i.e. forçant $X \times \mathbb{A}^1$ à être homotope à X). Dans ce contexte, le groupe de cohomologie motivique $H^{p,q}(X, \mathbb{Z})$ n'est autre que le p-ième groupe d"hypercohomologie du complexe motivique $\mathbb{Z}(q)$, construit en considérant des produits du twist de Tate. Un théorème important spécifie que

$$H^{2p,p}(X, \mathbb{Z}) = CH^p(X), \tag{2.2}$$

récupèrant ainsi les groupes de Chow d'origine. Cette relation est compatible avec les opérations de CH citées ci-dessus. De plus, le terme général $H^{p,q}(X, \mathbb{Z})$ correspond au groupe de Chow supérieur $CH^q(X, 2q - p)$ défini par S. Bloch ([V02]).

Nous pourrions encore citer beaucoup d'autres développements de la théorie des motifs esquissée ci-dessus. Une des plus marquantes est une sorte de dualité de Poincaré ([FV00]) pour les motifs des schémas propres sur la base, mais cela nécessite de stabiliser la catégorie $DM^{eff}(k)$, à savoir d'inverser formellement le twist de Tate dans $DM^{eff}(k)$. Ceci est réalisé à l'aide de spectres symétriques. Une des conséquences de la dualitté est le fait que la catégorie des motifs effectifs de Chow, définie par Grothendieck, peut être vue comme une sous-catégorie pleine de $DM^{eff}(k)$. Plus généralement, il est possible de construire sur tout schéma lisse S une catégorie Cor_S qui considère les correspondances finies sur S ([D07]) et une catégorie des motifs effectifs (resp. stables) sur S, notée $DM^{eff}(S)$ (resp. $DM(S)$). On peut lier ces différentes catégories (effectives ou stables) à l'aide d'un ingrédient puissant, appelé formalisme des six opérations, suivant une approche axiomatique expliquée par exemple dans [CD09] et [CD13]. La première version complète de ce formalisme est apparue dans la théorie de l'homotopie stable des schémas ([Ayo07]). Le formalisme de [CD09] et [CD13] est très proche de celui d'Ayoub, mais des résultats plus forts sont disponibles du fait que les catégories considérées ont plus dee structures. En particulier, le théorème du fibré projectif est vérifié par la cohomologie motivique sur une base, ce qui permet d'obtenir le triangle de Gysin qui est un analogue motivique de (2.1).

Récemment, une théorie cohomologie plus raffinée est apparue, appelée anneau de Chow-Witt ($\widetilde{CH}^n(X, \mathscr{L})$). Elle a été définie par J. Barge et F. Morel vers 2000 et complétée par J. Fasel quelques annés plus tard. Son objectif initial était de déterminer si un module projectif avait un facteur libre de rang un ([Fas08]) en utilisant les classes d'Euler. Ce problème ne peut pas être attaqué en général en utilisant l'anneau de Chow. La définition des groupes de Chow-Witt imite le développement de [Ro96], à savoir que ces groupes sont des groupes de cohomologie du complexe de Rost-Schmid associé à la K-théorie de Milnor-Witt. Une différence significative par rapport à l'anneau de Chow est que les groupes de Chow-Witt ne dépendent pas seulement d'un schéma lisse X, mais également d'un fibré en droites L sur ce schéma, appelé *le twist*. Ce phénomène de l'anneau de Chow-Witt est hérité de l'anneau de Witt et empêche l'orientation de l'anneau de Chow-Witt, c'est-à-dire qu'il n'y a pas de théorème du fibré projectif, et pas de classe de Chern sur l'anneau de Chow-Witt ([Fas08]). Néanmoins, il était assez clair que l'anneau de Chow-Witt devait satisfaire une propriété d'orientation plus faible, i.e. qu'il était orientée uniquement pour les fibré symplectiques. En d'autres termes, les spécialistes suspectaient que le théorème des fibré projectifs quaternioniques ([PW10]) était vérifié, impliquant l'existence de *classes de Pontryagin*, associées aux fibrés symplectiques, à valeurs dans l'anneau de Chow-Witt.

Récemment, des catégories motiviques instables et stables basées sur les groupes de Chow-Witt ont éte définies par B. Calmès, F. Déglise et J. Fasel ([CF14], [DF17]) obtenant en particulier une nouvelle théorie cohomologique appelée cohomologie MW-motivique. Ces catégories de motifs sont une meilleure approximation de la théorie de l'homotopie stable en comparaison avec celle de Voevodsky et l'équation (2.2) a également un analogue ici. Les constructions de base de ces motifs ressemblent beaucoup à celles de Voevodsky : les correspondances sont remplacées par des MW-correspondances, introduisant ainsi une difficulté assez subtile à chaque étape, à savoir *le calcul* des twists impliqués dans les opérations de base de l'anneau Chow-Witt, telles que le produit, le pull-back et le push-forward. L'approche sérieuse consiste à considérer ces torsions comme des éléments de la catégorie des fibrés vectoriels virtuels ([Del87], [CF18]) et c'est un travail délicat d'implémenter tous les calculs selon les règles formelles des objets virtuels. Cela nous incite à *axiomatiser* l'idée de correspondances et à obtenir une méthode générale permettant de construire des catégories de motifs, même en partant de théories cohomologiques non orientées. Pour prouver *le théorème des fibré projectifs quaternioniques* dans l'anneau de de Chow-Witt, nous devons d'abord prouver la contrepartie en cohomologie MW-motivique. Comme conséquence, nous calculons également l'espace de Thom associé à un fibré symplectique dans nos catégories de motifs et obtenons *le triangle de Gysin* correspondant. Finalement, nous calculons le groupe des morphismes dans nos catégories entre deux schémas lisses et propres sur le corps de base.

2.2 Principaux Résultats

2.2.1 Objets Virtuels et Opérations Associées

Dans le chapitre 3, nous fournissons les outils principaux qui nous permettent de calculer les twists associés aux opérations importantes dans l'anneau de Chow-Witt. Notons $V(Vect(X))$ la catégorie des fibrés vectoriels virtuels ([Del87]) sur X.

Théorèm 2.1. *(Theorem 3.1)*

1. *Supposons que nous ayons un diagramme commutatif de fibrés vectoriels sur X, avec des lignes et des colonnes exactes*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Alors, nous avons un diagramme commutatif dans $V(Vect(X))$

$$\begin{tikzcd}
V_2 \arrow[r] \arrow[d] & K+W_2 \arrow[d] \\
V_1+C \arrow[r] & K+W_1+C.
\end{tikzcd}$$

2. *Supposons que nous ayons un diagramme commutatif de fibrés vectoriels sur X avec des lignes et des colonnes exactes*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & D \arrow[r, equal] \arrow[d] & D \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Alors, nous avons un diagramme commutatif dans $V(Vect(X))$

$$\begin{tikzcd}
W_2 \arrow[r] \arrow[d] & V_2+D \arrow[r] & V_1+C+D \arrow[lldd, "c(C,D)"] \\
W_1+C \arrow[d] & & \\
V_1+D+C. & &
\end{tikzcd}$$

3. Supposons que nous ayons un diagramme commutatif de fibrés vectoriels sur X avec des lignes et des colonnes exactes

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & T \arrow[r] \arrow[d, equal] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T \arrow[r] & W_1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Alors, nous avons un diagramme commutatif dans $V(Vect(X))$

$$\begin{tikzcd}
V_1 \arrow[r] \arrow[d] & T + V_2 \arrow[r] & T + K + W_2 \, . \arrow[ddll, "c(T,K)"] \\
K + W_1 \arrow[d] & & \\
K + T + W_2. & &
\end{tikzcd}$$

2.2.2 Correspondances et Motifs Généralisés

Nous proposons un traitement axiomatique des correspondances dans le Chapitre 4. Étant donné une théorie de correspondance E, nous établissons dans le Chapitre 5 les résultats de base de la théorie des faisceaux avec E-transferts. En particulier, nous définissons dans le chapitre 6 la catégorie $\widetilde{DM}^{eff,-}(S)$ (resp. $\widetilde{DM}^{-}(S)$) des motifs effectifs (resp. stabilisés) sur une base lisse S en utilisant des complexes de tels (différents de [CD09], [CD13]) et construisons une partie de son formalisme des six opérations ($\otimes$, f^*, $f_\#$).

Dans le chapitre 8, nous montrons partiellement que les MW-correspondance définie dans [CF14] tombent bien dans le formalisme défini ci-dessus, en adoptant une nouvelle perspective du push-forward dans l'anneau de Chow-Witt.

2.2.3 Orientations Symplectiques et Applications

Pour tout $X \in Sm/S$, notons $\widetilde{\mathbb{Z}}_S(X)$ le motif de X dans $\widetilde{DM}^{eff,-}(S)$. Dans le chapitre 7, nous prouvons le théorème des fibré projectifs quaternioniques pour la cohomologie MW-motivique :

Théorèm 2.2. *(Theorem 7.4) Soient $X \in Sm/S$ et $(\mathscr{E}, m)$ un fibré vectoriel symplectique de rang $2n+2$ sur X. Soit $\pi : HGr_X(\mathscr{E}) \to X$ la projection. Alors, le morphisme*

$$\widetilde{\mathbb{Z}}_S(HGr_X(\mathscr{E})) \xrightarrow{\pi \boxtimes p_1(\mathscr{U}^\vee)^i} \oplus_{i=0}^{n} \widetilde{\mathbb{Z}}_S(X)(2i)[4i]$$

est un isomorphisme dans $\widetilde{DM}^{eff,-}(S)$, fonctoriel pour X dans Sm/S. Ici, $\mathscr{U}^\vee$ est le fibré tautologique dual doté de son orientation canonique.

On obtient donc le résultat correspondant dans l'anneau de Chow-Witt :

Proposition 2.1. *(Proposition 7.11)Soient $X \in Sm/k$, $\mathscr{E}$ un fibré symplectique de rang $2n+2$ sur X et $k = min\{\lfloor \frac{j}{2} \rfloor, n\}$. Alors le morphisme*

$$\theta_j : \oplus_{i=0}^{k} \widetilde{CH}^{j-2i}(X) \xrightarrow{p^* \cdot p_1(\mathscr{U}^\vee)^i} \widetilde{CH}^{j}(HGr_X(\mathscr{E}))$$

est un isomorphisme, où $j \geq 0$, $p : HGr_X(\mathscr{E}) \longrightarrow X$ est le morphisme structurel et $\mathscr{U}^\vee$ est le fibré tautologique dual doté de son orientation canonique.

Comme application, nous obtenons des classes de Pontryagin à valeurs dans l'anneau de Chow-WItt associées à un fibré symplectique.

Définition 2.1. *(Definition 7.11) Dans la proposition ci-dessus, supposons que* $\zeta := p_1(\mathscr{U}^\vee)$ *et* $\theta_{2n+2}^{-1}(\zeta^{n+1}) := (\zeta_i) \in \oplus_{i=1}^{n+1}\widetilde{CH}^{2i}(X)$*. Définissons* $p_0(\mathscr{E}) = 1 \in \widetilde{CH}^0(X)$ *et* $p_a(\mathscr{E}) = (-1)^{a-1}\zeta_i$ *pour* $1 \leq a \leq n+1$*. La classe* $p_a(\mathscr{E})$ *est appelée a-ième classe de Pontryagin de* $\mathscr{E}$*. Ces classes sont caractérisées uniquement par le polynôme de Pontryagin*

$$\zeta^{n+1} - p^*(p_1(\mathscr{E}))\zeta^n + \ldots + (-1)^{n+1}p^*(p_{n+1}(\mathscr{E})) = 0.$$

Nous obtenons également un triangle de Gysin pour certains immersions fermées :

Théorèm 2.3. *(Theorem 7.6) Soient* $X \in Sm/S$ *et* $Y \subseteq X$ *un sous-schéma fermé lisse de codimension* $codim(Y) = 2n$ *avec un fibré normal symplectique. Alors, nous avons un triangle distingué*

$$\widetilde{\mathbb{Z}}_S(X \setminus Y) \longrightarrow \widetilde{\mathbb{Z}}_S(X) \longrightarrow \widetilde{\mathbb{Z}}_S(Y)(2n)[4n] \longrightarrow \widetilde{\mathbb{Z}}_S(X \setminus Y)[1]$$

dans $\widetilde{DM}^{eff,-}(S)$*.*

Enfin, en utilisant le théorème ci-dessus, le formalisme des six opérations du Chapitre 6 et la dualité dans les catégories $\mathbb{A}^1$-stables dérivées ([CD13]), nous obtenons un calcul de groupe de morphismes dans nos catégories motiviques comme ci-dessous :

Théorèm 2.4. *(Theorem 7.7) Soient* $X, Y \in Sm/k$ *avec* Y *propre sur* k*. Alors, nous avons*

$$Hom_{\widetilde{DM}^{eff,-}(k)}(\widetilde{\mathbb{Z}}_{pt}(X), \widetilde{\mathbb{Z}}_{pt}(Y)) \cong \widetilde{CH}^{d_Y}(X \times Y, \omega_{X\times Y/X}).$$

Tout au long de cet article, nous désignons par Sm/k la catégorie des schémas séparés lisses sur k ([Har77, Chapter 10]), où k est un corps parfait infini avec $char(k) \neq 2$. Pour tout $X \in Sm/k$, nous désignons $dimX$ par d_X et pour tout $f : X \longrightarrow Y$ dans Sm/k, nous posons $d_f = d_X - d_Y$.

Chapter 3

Virtual Objects and Their Calculation

In this section we will introduce the category of virtual vector bundles and explain basic techniques of calculation. The definitions all come from [Del87, Section 4], but we recall them here for clarity.

Definition 3.1. *([Del87, 4.1]) A category $\mathscr{C}$ is called a commutative Picard category if*

1. *All morphisms are isomorphisms.*
2. *There is a bifunctorial pairing*

$$+ : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

   satisfying

   (a) *For every $x, y, z \in \mathscr{C}$, an associativity isomorphism*

$$a(x, y, z) : (x + y) + z \longrightarrow x + (y + z);$$

   (b) *For every $x, y \in \mathscr{C}$, a commutativity isomorphism*

$$c(x, y) : x + y \longrightarrow y + x.$$

   Furthermore, they satisfy associativity and commutativity constraints ([Mac63]).

3. *For every $P \in \mathscr{C}$, the functors $X \longmapsto P + X$ and $X \longmapsto X + P$ are equivalences of categories. Thus there is a unit element 0 such that $0 + X \cong X$ for every $X \in \mathscr{C}$, and there is an object $-X \in \mathscr{C}$ such that $X + (-X) \cong 0$.*

Definition 3.2. *Let X be a scheme. Define $Vect(X)$ to be the category of vector bundles over X. Denote by $(Vect(X), iso)$ the subcategory of $Vect(X)$ with the same objects but picking only isomorphisms as morphisms.*

Definition 3.3. *([Del87, 4.3]) Let $\mathscr{C}$ be a commutative Picard category and let X be a scheme. A bracket functor on X (with coefficients in $\mathscr{C}$) is a covariant functor*

$$[-] : (Vect(X), iso) \longrightarrow \mathscr{C}$$

such that:

1. *For any exact sequence of vector bundles*

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0,$$

there is an isomorphism $\Sigma : [E_2] \longrightarrow [E_1] + [E_3]$ *being natural with respect to ismorphisms between exact sequences.*

2. *There is an isomophism* $z : [0] \longrightarrow 0$ *such that for every* $E \in Vect(X)$*, the composite*

$$[E] \xrightarrow{\Sigma} [0] + [E] \xrightarrow{z} 0 + [E] \longrightarrow [E]$$

is $id_{[E]}$*.*

3. *(Remark 3.1) For every consecutive subbundle inclusions* $E_1 \subseteq E_2 \subseteq E_3$*, there is a commutative diagram*

$$\begin{tikzcd} {[E_3]} \arrow[rr, "\Sigma"] \arrow[d, "\Sigma"'] & & {[E_1] + [E_3/E_1]} \arrow[d, "\Sigma"'] \\ {[E_2] + [E_3/E_2]} \arrow[rr, "\Sigma"] & & {[E_1] + [E_2/E_1] + [E_3/E_2].} \end{tikzcd}$$

4. *For every* E_1, E_2*, there is a commutative diagram*

$$\begin{tikzcd} {[E_1 \oplus E_2]} \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] & {[E_1] + [E_2]} \arrow[dl, "c(E_1,E_2)"] \\ {[E_2] + [E_1].} & \end{tikzcd}$$

The following comes from [Del87, 4.3]:

Proposition 3.1. *Let* X *be a scheme. There is a commutative Picard category* $V(Vect(X))$ *with a bracket functor on* X *such that for every commutative Picard category* $\mathscr{C}$ *with a bracket functor on* X*, there is a unique additive functor* $F : V(Vect(X)) \longrightarrow \mathscr{C}$ *making the following diagram commute*

$$\begin{tikzcd} (Vect(X), iso) \arrow[r, "{[-]}"] \arrow[d, "{[-]}"'] & V(Vect(X)) \arrow[dl, "F"'] \\ \mathscr{C}. & \end{tikzcd}$$

The category $V(Vect(X))$ *is called the category of virtual vector bundles over* X*.*

For convenience, we will still denote $[E]$ by E in the sequel. The following proposition strengthens Definition 3.3, (4) a little bit.

Proposition 3.2. *Suppose we have a commutative diagram of vector bundles over* X *with exact row and column*

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & D \arrow[d, "d"'] \arrow[dr, "v", "\cong"'] & & \\ 0 \arrow[r] & A \arrow[r, "a"] \arrow[dr, "u", "\cong"'] & B \arrow[r, "b"'] \arrow[d, "c"] & C \arrow[r] & 0 \\ & & E \arrow[d] & & \\ & & 0. & & \end{tikzcd}$$

Then the following diagram commutes in $V(Vect(X))$

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & A+C \arrow[dl, "c(E,D)\circ(u+v^{-1})"] \\
D+E. &
\end{tikzcd}$$

Proof. Since $v^{-1}\circ b$ splits d, it's a standard argument that there exists a unique $\xi : E \longrightarrow B$ such that

$$\xi \circ c = id_B - d \circ v^{-1} \circ b, c \circ \xi = id_E.$$

So we have commutative diagrams with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r, "i_1"] \arrow[d, equal] & D \oplus E \arrow[r, "p_2"] \arrow[d, "d+\xi"'] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D \arrow[r, "d"] & B \arrow[r, "c"] & E \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "i_2"] \arrow[d, "u^{-1}"'] & D \oplus E \arrow[r, "p_1"] \arrow[d, "d+\xi"'] & D \arrow[r] \arrow[d, "v"'] & 0 \\
0 \arrow[r] & A \arrow[r, "a"] & B \arrow[r, "b"] & C \arrow[r] & 0.
\end{tikzcd}$$

Hence the statement follows from the commutative diagram of Definition 3.3, (4):

$$\begin{tikzcd}
 & & A+C \\
B \arrow[rru] \arrow[r] \arrow[dr, "(d+\xi)^{-1}"'] & D+E \arrow[r, "c(D,E)"] & E+D \arrow[u, "u^{-1}+v"'] \\
 & D \oplus E. \arrow[u] \arrow[ur] &
\end{tikzcd}$$

∎

The next theorem is a fundamental tool for calculations in virtual vector bundles.

Theorem 3.1. *1. Suppose we have a commutative diagram of vector bundles over X with exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Then we have a commutative diagram in $V(Vect(X))$

$$\begin{tikzcd}
V_2 \arrow[r] \arrow[d] & K+W_2 \arrow[d] \\
V_1+C \arrow[r] & K+W_1+C.
\end{tikzcd}$$

2. *Suppose we have a commutative diagram of vector bundles over X with exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W_1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & D \arrow[r, equal] \arrow[d] & D \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Then we have a commutative diagram in $V(Vect(X))$

$$\begin{tikzcd}
W_2 \arrow[r] \arrow[d] & V_2 + D \arrow[r] & V_1 + C + D \arrow[ddll, "c(C,D)"] \\
W_1 + C \arrow[d] & & \\
V_1 + D + C. & &
\end{tikzcd}$$

3. *Suppose we have a commutative diagram of vector bundles over X with exact rows and columns*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & T \arrow[r] \arrow[d, equal] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T \arrow[r] & W_1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Then we have a commutative diagram in $V(Vect(X))$

$$\begin{tikzcd}
V_1 \arrow[r] \arrow[d] & T + V_2 \arrow[r] & T + K + W_2 \arrow[ddll, "c(T,K)"] \\
K + W_1 \arrow[d] & & \\
K + T + W_2. & &
\end{tikzcd}$$

4. *Suppose we have a commutative diagram of vector bundles over X with exact rows*

and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & W_1 \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & 0 \\
 & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Then we have a commutative diagram in $V(Vect(X))$

$$\begin{tikzcd}
W_1 \arrow[r] \arrow[d] & K + W_2 \arrow[d] \\
V_1 + C \arrow[r] & K + V_2 + C.
\end{tikzcd}$$

Proof. 1. We have injections $K \longrightarrow V_1 \longrightarrow V_2$, which gives the diagram by Definition 3.3, (3).

2. We have injections $V_1 \longrightarrow V_2 \longrightarrow W_2$ and $V_1 \longrightarrow W_1 \longrightarrow W_2$. These give two commutative diagrams by Definition 3.3, (3)

$$\begin{tikzcd}
W_2 \arrow[r] \arrow[d] & V_1 + W_2/V_1 \arrow[d, "\Sigma_1"'] & W_2 \arrow[r] \arrow[d] & V_1 + W_2/V_1 \arrow[d, "\Sigma_2"'] . \\
V_2 + D \arrow[r] & V_1 + C + D & W_1 + C \arrow[r] & V_1 + D + C
\end{tikzcd}$$

Moreover, we have a commutative diagram with exact row and column

$$\begin{tikzcd}
\Sigma_2 : & & 0 \arrow[d] & & . \\
 & & D \arrow[d] \arrow[dr, equal] & & \\
\Sigma_1 : 0 \arrow[r] & C \arrow[r] \arrow[dr, equal] & W_2/V_1 \arrow[r] \arrow[d] & D \arrow[r] & 0 \\
 & & C \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Thus we have a commutative diagram

$$\begin{tikzcd}
W_2/V_1 \arrow[r, "\Sigma_1"] \arrow[dr, "\Sigma_2"'] & C + D \arrow[d, "c(C,D)"] \\
 & D + C
\end{tikzcd}$$

by Proposition 3.2. So combining the diagrams above gives the result.

3. We denote the morphism $V_1 \longrightarrow V_2 \longrightarrow W_2$ by α. There are morphisms $ker(\alpha) \longrightarrow K$ and $ker(\alpha) \longrightarrow T$ satisfying the following commutative diagrams

$$\begin{tikzcd} ker(\alpha) \arrow[r] \arrow[d] & K \arrow[d] & ker(\alpha) \arrow[r] \arrow[d] & V_1 \arrow[d] \\ V_1 \arrow[r] & V_2 & T \arrow[r] & W_1 \end{tikzcd}$$

by the universal property of K and T as kernels. Then there is a commutative diagram with exact row and column

$$\begin{tikzcd} & \Sigma_2 : & 0 \arrow[d] & & . \\ & & K \arrow[d] \arrow[dr, equal] & & \\ \Sigma_1 : 0 \arrow[r] & T \arrow[r] \arrow[dr, equal] & ker(\alpha) \arrow[r] \arrow[d] & K \arrow[r] & 0 \\ & & T \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

Hence we have a commutative diagram

$$\begin{tikzcd} ker(\alpha) \arrow[r, "\Sigma_1"] \arrow[d, "\Sigma_2"'] & T+K \arrow[dl, "c(T,K)"] \\ K+T & \end{tikzcd}$$

by Proposition 3.2.

We have injections $T \longrightarrow ker(\alpha) \longrightarrow V_1$, $K \longrightarrow ker(\alpha) \longrightarrow V_1$, which induce the following commutative diagrams by Definition 3.3, (3):

$$\begin{tikzcd} V_1 \arrow[r] \arrow[d] & T+V_2 \arrow[d] & V_1 \arrow[r] \arrow[d] & K+W_1 \arrow[d] \\ ker(\alpha)+W_2 \arrow[r, "\Sigma_1"] & T+K+W_2 & ker(\alpha)+W_2 \arrow[r, "\Sigma_2"] & K+T+W_2 \end{tikzcd} .$$

So combining the diagrams above gives the result.

4. The diagram is a rotation and reflection of the diagram in (1).

$\square$

Remark 3.1. *We remark that (1) in the above theorem is actually the meaning of Definition 3.3, (3).*

Remark 3.2. *We would like to point out that the calculations with virtual objects are* not *trivial, especially when judging commutativity of diagrams. We will see this point in the sections below.*

Chapter 4

Correspondences from an Axiomatic Viewpoint

In this section, we are going to axiomatize the notion of correspondences, using the language of virtual vector bundles defined in the previous section. They are designed basicly to comply with properties of Chow rings or Chow-Witt rings.

Definition 4.1. *Let X be a noetherian scheme and $i \in \mathbb{N}$. We denote by $Z^i(X)$ the set of closed subsets in X whose components are all of codimension i.*

Definition 4.2. *Let $X \in Sm/k$, $C \in Z^i(X)$ and $D \in Z^j(X)$. We say that C and D intersect properly if $C \cap D \in Z^{i+j}(X)$.*

We now start our list of axioms.

Axiom 1. *(Twists) For every $X \in Sm/k$, we have a commutative Picard category (Definition 3.1) $\mathscr{P}_X$ with an additive functor $p_X : V(Vect(X)) \longrightarrow \mathscr{P}_X$ and a rank morphism $rk_X : \mathscr{P}_X \longrightarrow \mathbb{F}(\mathbb{F} = 0$ or $\mathbb{Z}/2\mathbb{Z})$ such that:*

1. *The following diagram commutes*

$$\begin{tikzcd} V(Vect(X)) \arrow[r, "rk_X"] \arrow[d, "p_Y"'] & \mathbb{Z} \arrow[d] \\ \mathscr{P}_X \arrow[r, "rk_X"] & \mathbb{F}, \end{tikzcd}$$

where the upper horizontal arrow is defined by $rk_X([E]) = rk(E)$.

2. *For every $f : X \longrightarrow Y$ in Sm/k, there is a pull-back morphism $f^* : \mathscr{P}_Y \longrightarrow \mathscr{P}_X$ such that the following diagrams commute*

$$\begin{tikzcd} \mathscr{P}_Y \arrow[r, "f^*"] \arrow[d, "rk_Y"'] & \mathscr{P}_X \arrow[dl, "rk_X"] \\ \mathbb{F} & \end{tikzcd} \qquad \begin{tikzcd} V(Vect(Y)) \arrow[r, "f^*"] \arrow[d, "p_Y"'] & V(Vect(X)) \arrow[d, "p_X"] \\ \mathscr{P}_Y \arrow[r, "f^*"] & \mathscr{P}_X, \end{tikzcd}$$

where $f^* : V(Vect(Y)) \longrightarrow V(Vect(X))$ is defined by $f^*([E]) = [f^*E]$. We have $f^*g^* = (g \circ f)^*$ for any morphisms f, g in Sm/k and $f^*(-v) = -f^*(v)$.

Remark 4.1. *In practice, the categories $\mathscr{P}_X$ should be chosen as "small" as possible. Since this will allow more isomophisms, such as orientations, as we will see in Definition 7.19.*

Axiom 2. *(Correspondences) For every $X \in Sm/k$, $i \in \mathbb{N}$, $C \in Z^i(X)$ and $v \in \mathscr{P}_X$, there exists an abelian group $E^i_C(X, v)$ which is called the group of correspondences supported on C with twist v. These groups are functorial with respect to v. Moreover, if $C = \emptyset$, then $E^i_C(X, v) = 0$.*

We are now going to describe further properties that these groups should satisfy.

Axiom 3. *(Extension of Supports) For every $X \in Sm/k$, $C_1 \subseteq C_2 \in Z^i(X)$, $i \in \mathbb{N}$, $v \in \mathscr{P}_X$, we have an injective morphism*

$$e(C_1, C_2) : E^i_{C_1}(X, v) \longrightarrow E^i_{C_2}(X, v)$$

which is called the extension of support. This map is functorial with respect to v.

For any disjoint $C_1, C_2 \in Z^i(X)$, we have

$$E^i_{C_1 \cup C_2}(X, v) \cong E^i_{C_1}(X, v) \oplus E^i_{C_2}(X, v)$$

via extension of supports. Moreover, for any $C_1 \subseteq C_2 \subseteq C_3$ we have

$$e(C_2, C_3) \circ e(C_1, C_2) = e(C_1, C_3).$$

Axiom 4. *(Products) Suppose $X \in Sm/k$, $v_1, v_2 \in \mathscr{P}_X$, $C_1, C_2 \in Z^i(X)$ and $i, j \in \mathbb{N}$. Suppose C_1 and C_2 intersect properly, then we have a product*

$$E^i_{C_1}(X, v_1) \times E^j_{C_2}(X, v_2) \xrightarrow{\cdot} E^{i+j}_{C_1 \cap C_2}(X, v_1 + v_2)\ ,$$

This product is functorial with respect to twists and extension of supports.

Axiom 5. *(Associativity) For any $X \in Sm/k$, $v_a \in \mathscr{P}_X$ and $C_a \in Z^{i_a}(X)$, $a = 1, 2, 3$, with pairwise proper intersections the following diagram commutes*

$$\begin{tikzcd}
E^{i_1}_{C_1}(X, v_1) \times E^{i_2}_{C_2}(X, v_2) \times E^{i_3}_{C_3}(X, v_3) \arrow[r, "id\times\cdot"] \arrow[d, "\cdot\times id"'] & E^{i_1}_{C_1}(X, v_1) \times E^{i_2+i_3}_{C_2 \cap C_3}(X, v_2 + v_3) \arrow[d, "\cdot"] \\
E^{i_1+i_2}_{C_1 \cap C_2}(X, v_1 + v_2) \times E^{i_3}_{C_3}(X, v_3) \arrow[d, "\cdot"'] & E^{i_1+i_2+i_3}_{C_1 \cap C_2 \cap C_3}(X, v_1 + (v_2 + v_3)) \arrow[dl, "a(v_1,v_2,v_3)^{-1}"'] \\
E^{i_1+i_2+i_3}_{C_1 \cap C_2 \cap C_3}(X, (v_1 + v_2) + v_3). &
\end{tikzcd}$$

Axiom 6. *(Conditional Commutativity) Let $X \in Sm/k$, $C_a \in Z^{i_a}(X)$, $i_a \in \mathbb{N}$, $v_a \in \mathscr{P}_X$ where $a = 1, 2$. If $(i_1 + rk_X(v_1))(i_2 + rk_X(v_2)) = 0 \in \mathbb{F}$ and C_1 and C_2 intersect properly, the following diagram commutes:*

$$\begin{tikzcd}
E^{i_1}_{C_1}(X, v_1) \times E^{i_2}_{C_2}(X, v_2) \arrow[r, "\cdot"] \arrow[d] & E^{i_1+i_2}_{C_1 \cap C_2}(X, v_1 + v_2) \arrow[d, "c(v_1,v_2)"'] \\
E^{i_2}_{C_2}(X, v_2) \times E^{i_1}_{C_1}(X, v_1) \arrow[r, "\cdot"] & E^{i_1+i_2}_{C_1 \cap C_2}(X, v_2 + v_1).
\end{tikzcd}$$

Axiom 7. *(Identity) For any $X \in Sm/k$, there is an element e in $E^0_X(X, 0)$ such that for any $v \in \mathscr{P}_X$, $i \in \mathbb{N}$ and $C \in Z^i(X)$, the following diagrams commute*

$$\begin{tikzcd}
E^i_C(X, v) \arrow[r, "e\cdot"] \arrow[d, equal] & E^i_C(X, 0 + v) \arrow[dl, "u"'] \\
E^i_C(X, v) &
\end{tikzcd}
\qquad
\begin{tikzcd}
E^i_C(X, v) \arrow[r, "\cdot e"] \arrow[d, equal] & E^i_C(X, v + 0) \arrow[dl, "u"'] \\
E^i_C(X, v), &
\end{tikzcd}$$

where u are the unit constraints in $\mathscr{P}_X$. We call e the identity and denote it by 1.

Axiom 8. *(Pull-Backs) Suppose $f : X \longrightarrow Y$ is morphism in Sm/k, $i \in \mathbb{N}$, $C \in Z^i(Y)$, $f^{-1}(C) \in Z^i(X)$ and $v \in \mathscr{P}_Y$. Then we have a pull-back morphism*

$$E^i_C(Y, v) \longrightarrow E^i_{f^{-1}(C)}(X, f^*v).$$

This morphism is functorial with respect to v and extension of supports.

Axiom 9. *(Functoriality of Pull-Backs) Let $X \xrightarrow{g} Y \xrightarrow{f} Z$ be morphisms in Sm/k, $i \in \mathbb{N}$, $C \in Z^i(Z)$, $f^{-1}(C) \in Z^i(Y)$, $g^{-1}f^{-1}(C) \in Z^i(X)$ and $v \in \mathscr{P}_Z$. We have*

$$(f \circ g)^* = g^* \circ f^*.$$

The pull-back of the identity morphism is just the identity morphism.

Axiom 10. *(Compability of Pull-Backs) Suppose that $f : X \longrightarrow Y$ is a morphism in Sm/k, and that $C_1 \in Z^i(Y)$ and $C_2 \in Z^j(Y)$ intersect properly for some $i, j \in \mathbb{N}$ (the same for their preimages). For any $v_1, v_2 \in \mathscr{P}_Y$, we have a commutative diagram*

$$\begin{tikzcd}
E^i_{C_1}(Y, v_1) \times E^j_{C_2}(Y, v_2) \arrow[r, "\cdot"] \arrow[d, "f^* \times f^*"] & E^{i+j}_{C_1 \cap C_2}(Y, v_1 + v_2) \arrow[d, "f^*"] \\
E^i_{f^{-1}(C_1)}(X, f^*(v_1)) \times E^j_{f^{-1}(C_2)}(X, f^*(v_2)) \arrow[r, "\cdot"] & E^{i+j}_{f^{-1}(C_1 \cap C_2)}(X, f^*(v_1 + v_2))
\end{tikzcd}.$$

We always have $f^*(1) = 1$.

Before proceeding further, we now recall some facts about tangent bundles and normal bundles.

Lemma 4.1. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in Sm/k.*

1. *If f, g are smooth, we have an exact sequence*

$$0 \longrightarrow T_{X/Y} \longrightarrow T_{X/Z} \longrightarrow f^*T_{Y/Z} \longrightarrow 0.$$

2. *If f is a closed immersion and g, $g \circ f$ are smooth, we have an exact sequence*

$$0 \longrightarrow T_{X/Z} \longrightarrow f^*T_{Y/Z} \longrightarrow N_{X/Y} \longrightarrow 0.$$

3. *If g is smooth and f, $g \circ f$ are closed immersions, we have an exact sequence*

$$0 \longrightarrow f^*T_{Y/Z} \longrightarrow N_{X/Y} \longrightarrow N_{X/Z} \longrightarrow 0.$$

4. *If f, g are closed immersions, we have an exact sequence*

$$0 \longrightarrow N_{X/Y} \longrightarrow N_{X/Z} \longrightarrow f^*N_{Y/Z} \longrightarrow 0.$$

Proof. See [Har77, Chapter II, Proposition 8.11, Proposition 8.12 and Theorem 8.17 and Chapter III, Proposition 10.4]. □

Lemma 4.2. *Suppose that we have a Cartesian square of schemes*

$$\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"] & Y.
\end{tikzcd}$$

Then, the composite $T_{X'/Y'} \longrightarrow T_{X'/Y} \longrightarrow v^*T_{X/Y}$ is an isomorphism.

Proof. See [Har77, Chapter II, Proposition 8.10]. □

Lemma 4.3. *Suppose that we have a Cartesian square in Sm/k*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

such that f is a closed immersion. If one of the following conditions holds:

1. *u is smooth,*
2. *u is a closed immersion and $dimX' - dimY' = dimX - dimY$,*

then the natural morphism γ defined by the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & v^*T_{X/k} & \longrightarrow & v^*f^*T_{Y/k} & \longrightarrow & v^*N_{X/Y} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & T_{X'/k} & \longrightarrow & g^*T_{Y'/k} & \longrightarrow & N_{X'/Y'} & \longrightarrow & 0 \end{array}$$

is an isomorphism.

Proof. If u is smooth, then α and β are surjective and have the same kernel by the previous two lemmas. So γ is an isomorphism by the snake lemma.

In the other case, the dimension condition implies $N_{X'/Y'}$ and $N_{X/Y}$ have the same rank. So we only have to prove $\gamma^\vee$ is surjective. We can assume that all schemes are affine. Suppose that $Y = Spec(A)$, $X = Spec(A/I)$, $Y' = Spec(A/J)$ and $X' = Spec(A/(I+J))$. Then $N_{X/Y}^\vee = I/I^2$ and $N_{X'/Y'}^\vee = (I+J)/(I^2+J)$ and the morphism γ is given by

$$\begin{array}{ccccc} I/I^2 & \otimes_{A/I} & A/(I+J) & \longrightarrow & (I+J)/(I^2+J) \\ (\bar{i} & , & \overline{a}) & \longmapsto & \overline{ai}. \end{array}$$

This is obviously surjective. □

Axiom 11. *(Push-Forwards for Smooth Morphisms) Suppose that $f : X \longrightarrow Y$ is a smooth morphism in Sm/k, that $n \in \mathbb{N}$, $v \in \mathscr{P}_X$ and that $C \in Z^{n+d_f}(X)$ is finite over Y. Then we have a morphism*

$$f_* : E_C^{n+d_f}(X, f^*v - T_{X/Y}) \longrightarrow E^n_{f(C)}(Y, v),$$

which is functorial with respect to v and the extension of supports. The push-forward of the identity morphism is just the identity morphism (using $T_{X/Y} = 0$).

We may also use the simplified notation

$$f^*v - T_{X/Y} \longrightarrow v$$

to denote f_*. Moreover, we can consider push-forwards of the form

$$f_* : E_C^{n+d_f}(X, f^*v_1 - T_{X/Y} + f^*v_2) \longrightarrow E^n_{f(C)}(Y, v_1 + v_2)$$

which are defined by the composite of the push-forward defined above and the commutativity isomorphism $c(-T_{X/Y}, f^*v_2)$.

Axiom 12. *(Functoriality of Push-Forwards for Smooth Morphisms) Suppose that* $X \xrightarrow{g} Y \xrightarrow{f} Z$ *are smooth morphisms in* Sm/k*, and that* $C \in Z^{i+d_X-d_Z}(X)$ *is finite over* Z *(*$i \in \mathbb{N}$*). Suppose moreover that* $v \in \mathscr{P}_Z$*. Then we have a commutative diagram*

$$\begin{tikzcd}
E_C^{i+d_X-d_Z}(X,(f\circ g)^*v - T_{X/Z}) \arrow[r, "\varphi"] \arrow[ddr, "(f\circ g)_*"'] & E_C^{i+d_X-d_Z}(X,(f\circ g)^*v - g^*T_{Y/Z} - T_{X/Y}) \arrow[d, "g_*"] \\
 & E_{g(C)}^{i+d_Y-d_Z}(Y, f^*v - T_{Y/Z}) \arrow[d, "f_*"] \\
 & E^i_{f(g(C))}(Z,v)
\end{tikzcd}$$

where φ *is obtained via the following composite*

$$\begin{aligned}
(f\circ g)^*v - T_{X/Z} &\longrightarrow (f\circ g)^*v - (T_{X/Y} + g^*T_{Y/Z}) \\
&\longrightarrow (f\circ g)^*v - g^*T_{Y/Z} - T_{X/Y}.
\end{aligned}$$

Axiom 13. *(Push-Forward for Closed Immersions) Suppose* $f : X \longrightarrow Y$ *is a closed immersion in* Sm/k*,* $v \in \mathscr{P}_Y$ *and* $C \in Z^{n+d_f}(X)$*. Then we have an* isomorphism

$$f_* : E_C^{n+d_f}(X, N_{X/Y} + f^*v) \longrightarrow E^n_{f(C)}(Y,v),$$

This morphism is also functorial in v *and under extension of supports. The push-forward of the identity is just the identity, by using* $N_{X/Y} = 0$*.*

So given a vector bundle V over X, the definition above gives an isomorphism $E^n_C(X,V) \cong E_C^{n+rk_X(V)}(V,0)$ (Chapter 1) via the push-forward of the zero section.

We may also use the simplified notation

$$N_{X/Y} + f^*v \longrightarrow v$$

to denote f_*. Moreover, we could also consider push-forwards of the form

$$f_* : E_C^{n+d_f}(X, f^*v_1 + N_{X/Y} + f^*v_2) \longrightarrow E^n_{f(C)}(Y, v_1 + v_2)$$

which are defined by the composite of the push-forward defined above and the commutativity isomorphism $c(f^*v_1, N_{X/Y})$.

Axiom 14. *(Functoriality Push-Forwards for Closed Immersions) Suppose that* $X \xrightarrow{g} Y \xrightarrow{f} Z$ *are closed immersions in* Sm/k*,* $C \in Z^{i+d_X-d_Z}(X)$ *and* $v \in \mathscr{P}_Z$*. Then we have a commutative diagram*

$$\begin{tikzcd}
E_C^{i+d_X-d_Z}(X, N_{X/Z} + (f\circ g)^*v) \arrow[r, "\varphi"] \arrow[ddr, "(f\circ g)_*"] & E_C^{i+d_X-d_Z}(X, N_{X/Y} + g^*N_{Y/Z} + (f\circ g)^*v) \arrow[d, "g_*"] \\
 & E_{g(C)}^{i+d_Y-d_Z}(Y, N_{Y/Z} + f^*v) \arrow[d, "f_*"] \\
 & E^i_{f(g(C))}(Z,v),
\end{tikzcd}$$

where φ *is induced by the isomorphism* $N_{X/Z} + (f\circ g)^*v \cong N_{X/Y} + g^*N_{Y/Z} + (f\circ g)^*v$*.*

Axiom 15. *(Base Change for Smooth Morphisms) Suppose we have a Cartesian square of smooth schemes*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y \end{tikzcd}$$

with f smooth. Let moreover $c = d_X - d_Y = d_{X'} - d_{Y'}$, $n \in \mathbb{N}$, $s \in \mathscr{P}_Y$, $C \in Z^{n+c}(X)$ finite over Y such that $v^{-1}(C) \in Z^{n+c}(X)$. Then the following diagram commutes

$$\begin{tikzcd} E_C^{n+c}(X, f^*s - T_{X/Y}) \arrow[r, "f_*"] \arrow[d, "v^*"] & E^n_{f(C)}(Y, s) \arrow[d, "u^*"] \\ E^{n+c}_{v^{-1}(C)}(X', v^*f^*s - v^*T_{X/Y}) \arrow[r, "g_*"] & E^n_{g(v^{-1}(C))}(Y', u^*s). \end{tikzcd}$$

Here we have used the canonical isomorphism $T_{X'/Y'} \longrightarrow v^*T_{X/Y}$ of Lemma 4.2.

Axiom 16. *(Base Change for Closed Immersions) Suppose that we have a Cartesian square of smooth schemes*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y \end{tikzcd}$$

with f a closed immersion. Let $c = d_X - d_Y = d_{X'} - d_{Y'}$, $s \in \mathscr{P}_Y$, $C \in Z^{n+c}(X)$ such that $v^{-1}(C) \in Z^{n+c}(X)$. Then the following diagram commutes

$$\begin{tikzcd} E_C^{n+c}(X, N_{X/Y} + f^*s) \arrow[r, "f_*"] \arrow[d, "v^*"] & E^n_{f(C)}(Y, s) \arrow[d, "u^*"] \\ E^{n+c}_{v^{-1}(C)}(X', v^*N_{X/Y} + v^*f^*s) \arrow[r, "g_*"] & E^n_{g(v^{-1}(C))}(Y', u^*s). \end{tikzcd}$$

Axiom 17. *(Projection Formula for Smooth Morphisms) Suppose that we have a smooth morphism $f : X \longrightarrow Y$ in Sm/k and that $n, m \in \mathbb{N}$. Let further $C \in Z^{n+d_f}(X)$ be finite over Y and $D \in Z^m(Y)$ be such that C and $f^{-1}(D)$ intersect properly and $v_1, v_2 \in \mathscr{P}_Y$. Then the diagrams*

$$\begin{tikzcd} E_C^{n+d_f}(X, f^*v_1 - T_{X/Y}) \times E^m_D(Y, v_2) \arrow[r, "id\times f^*"] \arrow[d, "f_*\times id"] & E_C^{n+d_f}(X, f^*v_1 - T_{X/Y}) \times E^m_{f^{-1}(D)}(X, f^*v_2) \arrow[d, "\cdot"] \\ E^n_C(Y, v_1) \times E^m_D(Y, v_2) \arrow[d, "\cdot"] & E^{n+m+d_f}_{C\cap f^{-1}(D)}(X, f^*v_1 - T_{X/Y} + f^*v_2) \arrow[dl, "f_*"] \\ E^{n+m}_{Y, f(C)\cap D}(v_1 + v_2) & \end{tikzcd}$$

and

$$\begin{tikzcd} E^m_D(Y, v_2) \times E_C^{n+d_f}(X, f^*v_1 - T_{X/Y}) \arrow[r, "f^*\times id"] \arrow[d, "id\times f_*"] & E^m_{f^{-1}(D)}(X, f^*v_2) \times E_C^{n+d_f}(X, f^*v_1 - T_{X/Y}) \arrow[d, "\cdot"] \\ E^m_D(Y, v_2) \times E^n_C(Y, v_1) \arrow[d, "\cdot"] & E^{n+m+d_f}_{C\cap f^{-1}(D)}(X, f^*v_2 + f^*v_1 - T_{X/Y}) \arrow[dl, "f_*"] \\ E^{n+m}_{f(C)\cap D}(Y, v_2 + v_1) & \end{tikzcd}$$

commute.

Axiom 18. *(Projection Formula for Closed Immersions) Suppose that we have a closed immersion $f : X \longrightarrow Y$ in Sm/k. Let $n, m \in \mathbb{N}$, $C \in Z^{n+d_f}(X)$ and $D \in Z^m(Y)$ be such that $f^{-1}(D) \in Z^m(X)$, and such that C and $f^{-1}(D)$ intersect properly. Let further $v_1, v_2 \in \mathscr{P}_Y$. Then the diagrams*

$$\begin{tikzcd}
E_C^{n+d_f}(X, N_{X/Y} + f^*v_1) \times E_D^m(Y, v_2) \arrow[r, "id \times f^*"] \arrow[d, "f_* \times id"] & E_C^{n+d_f}(X, N_{X/Y} + f^*v_1) \times E^m_{f^{-1}(D)}(X, f^*v_2) \arrow[d, "\cdot"] \\
E_C^n(Y, v_1) \times E_D^m(Y, v_2) \arrow[d, "\cdot"] & E^{n+m+d_f}_{C \cap f^{-1}(D)}(X, N_{X/Y} + f^*v_1 + f^*v_2) \arrow[dl, "f_*"] \\
E^{n+m}_{f(C) \cap D}(Y, v_1 + v_2) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
E_D^m(Y, v_2) \times E_C^{n+d_f}(X, N_{X/Y} + f^*v_1) \arrow[r, "f^* \times id"] \arrow[d, "id \times f_*"] & E^m_{f^{-1}(D)}(X, f^*v_2) \times E_C^{n+d_f}(X, N_{X/Y} + f^*v_1) \arrow[d, "\cdot"] \\
E_D^m(Y, v_2) \times E_C^n(Y, v_1) \arrow[d, "\cdot"] & E^{n+m+d_f}_{C \cap f^{-1}(D)}(X, f^*v_2 + N_{X/Y} + f^*v_1) \arrow[dl, "f_*"] \\
E^{n+m}_{f(C) \cap D}(Y, v_2 + v_1) &
\end{tikzcd}$$

commute.

We still need a compability between the two push-forwards introduced above.

Axiom 19. *(Compability between Two Push-Forwards)*

1. *Suppose that $X \xrightarrow{f} Z \xrightarrow{g} Y$ are morphisms in Sm/k, that f is a closed immersion and that g, $g \circ f$ are smooth. Let $C \in Z^{i+d_X-d_Y}(X)$ be finite over Y, $i \in \mathbb{N}$ and $v \in \mathscr{P}_Y$. Then the following diagram commutes*

$$\begin{tikzcd}
E_C^{i+d_X-d_Y}(X, N_{X/Z} + f^*g^*v - f^*T_{Z/Y}) \arrow[r, "c(N_{X/Z}, f^*g^*v)"] \arrow[d, "f_*"] & E_C^{i+d_X-d_Y}(X, f^*g^*v + N_{X/Z} - f^*T_{Z/Y}), \arrow[d, "\varphi"] \\
E_{f(C)}^{i+d_Z-d_Y}(Z, g^*v - T_{Z/Y}) \arrow[d, "g_*"] & E_C^{i+d_X-d_Y}(X, f^*g^*v - T_{X/Y}) \arrow[dl, "(g \circ f)_*"] \\
E^i_{g(f(C))}(Y, v), &
\end{tikzcd}$$

where φ is induced by Lemma 4.1, (2).

2. *Suppose that $X \xrightarrow{f} Z \xrightarrow{g} Y$ are morphisms in Sm/k with g smooth and f, $g \circ f$ closed immersions. Let $C \in Z^{i+d_X-d_Y}(X)$ be finite over Y, $i \in \mathbb{N}$ and $v \in \mathscr{P}_Y$.*

Then the following diagram commutes

$$\begin{tikzcd}
E_C^{i+d_X-d_Y}(X, N_{X/Z}+f^*g^*v-f^*T_{Z/Y}) \arrow[r, "c(N_{X/Z}+f^*g^*v,-f^*T_{Z/Y})"] \arrow[d, "f_*"'] & E_C^{i+d_X-d_Y}(X, -f^*T_{Z/Y}+N_{X/Z}+f^*g^*v) \arrow[d, "\varphi"] \\
E_{f(C)}^{i+d_Z-d_Y}(Z, g^*v-T_{Z/Y}) \arrow[d, "g_*"'] & E_C^{i+d_X-d_Y}(X, N_{X/Y}+f^*g^*v) \arrow[dl, "(g\circ f)_*"'] \\
E^i_{g(f(C))}(Y, v), &
\end{tikzcd}$$

where φ is induced by Lemma 4.1, (3).

3. *Suppose that we have a Cartesian square of smooth schemes*

$$\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"] & Y,
\end{tikzcd}$$

where u is smooth and f is a closed immersion. Let $C \in Z^{n+d_f+d_v}(X')$ be finite over Y and let $s \in \mathscr{P}_Y$. Then the following diagram commutes

$$\begin{tikzcd}
E_C^{n+d_f+d_v}(X', N_{X'/Y'}+g^*u^*s-g^*T_{Y'/Y}) \arrow[r, "g_*"] \arrow[d] & E_{g(C)}^{n+d_u}(Y', u^*s-T_{Y'/Y}) \arrow[d, "u_*"] \\
E_C^{n+d_f+d_v}(X', v^*N_{X/Y}+u^*f^*s-T_{X'/X}) \arrow[d, "v_*"'] & E^n_{u(g(C))}(Y, s) \\
E^{n+d_f}_{v(C)}(X, N_{X/Y}+f^*s). \arrow[ur, "f_*"'] &
\end{tikzcd}$$

Axiom 20. *(Étale Excision) Suppose that $f : X \longrightarrow Y$ is an étale morphism in Sm/k, that $C \in Z^i(Y)$ and that the morphism $f : f^{-1}(C) \longrightarrow C$ is an isomorphism under reduced closed subscheme structures. Then for any $i \in \mathbb{N}$ and $v \in \mathscr{P}_Y$, the pull-back morphism*

$$f^* : E^i_C(Y, v) \longrightarrow E^i_{f^{-1}(C)}(X, f^*(v))$$

is an isomorphism between abelian groups with inverse f_*.

Definition 4.3. *If the categories $\mathscr{P}_X$ and groups $E^i_C(X, v)$ satisfy all the axioms above, then they are called a correspondence theory.*

Remark 4.2. *Let R be a commutative ring. The first example of a correspondence theory is given by $E^i_C(X, v) = CH^i_C(X, v) \otimes R$, where the latter is the free R-module generated by irreducible components of C for $C \in Z^i(X)$, $\mathbb{F} = 0$ and $\mathscr{P}_X = 0$ for any X.*

We now give another example, starting with the definition of the categories $\mathscr{P}_X$.

Definition 4.4. *For a scheme X, we define a category $\mathscr{P}_X$ as follows. Its objects are sequences $\mathcal{E} := (E_1, \ldots, E_n)$, where $n \in \mathbb{N}$ and E_i are vector bundles over X for $i = 1, \ldots, n$. We attach to each object $\mathcal{E}$ a line bundle*

$$det(\mathcal{E}) = det E_1 \otimes \cdots \otimes det E_n$$

and an integer

$$rk(\mathcal{E}) = rkE_1 + \cdots + rkE_n \in \mathbb{Z}/2\mathbb{Z}.$$

The morphisms between objects $\mathcal{E} = (E_1, \ldots, E_n)$ *and* $\mathcal{F} = (F_1, \ldots, F_m)$ *are given by*

$$Hom_{\mathscr{P}_X}(\mathcal{E}, \mathcal{F}) = \begin{cases} \mathrm{Isom}_{O_X}(det(\mathcal{E}), det(\mathcal{F})) & \textit{if } rk(\mathcal{E}) = rk(\mathcal{F}). \\ \emptyset & \textit{else.} \end{cases}$$

The composition law is inherited from the category of line bundles.

Remark 4.3. *The category* $\mathscr{P}_X$ *is equivalent to the category of* $\mathbb{Z}/2\mathbb{Z}$*-graded line bundles considered in [Del87, 4.3]. However, the category* $\mathscr{P}_X$ *will be more convenient in our computations.*

To complete the definition of our correspondence theory, we set

$$\widetilde{CH}^i_C(X, v) = \widetilde{CH}^i_C(X, det(v)).$$

for every $X \in Sm/k$, $C \in Z^i(X)$, $v \in \mathscr{P}_X$. These are precisely the MW-correspondences defined in [CF14]. We will give a plan of proof of the following theorem in Chapter 8.

Theorem 4.1. *The collection of MW-correspondences form a correspondence theory with twists in* $\mathscr{P}_X$.

Chapter 5

Sheaves with E-Tranfers and Their Operations

In this section, we develop the theory of sheaves with E-transfers over a smooth base as in [D07] and [CF14], where E is a correspondence theory.

Since there will be heavy calculations involving twists, we use the abbreviation (α, v) for $\alpha \in E^i_C(X, v)$ from now on for convenience and clarity. We extend this notation to operations such as $(\alpha, v) \cdot (\beta, u)$, $f^*((\alpha, v))$. For $S \in Sm/k$, we denote the category of smooth schemes over S by Sm/S.

We will need the notion of admissible subset coming from [CF14, Definition 4.1].

Definition 5.1. *Let $X, Y \in Sm/S$. We denote by $\mathscr{A}_S(X, Y)$ the set of closed subsets T of $X \times_S Y$ whose components are all finite over X and of dimension dim(X). The elements of $\mathscr{A}_S(X, Y)$ are called admissible subsets from X to Y over S.*

Lemma 5.1. *In the definition above, T itself is also finite over X.*

Proof. For every affine open subset U of X, $T \cap U$ is affine since each of its components are affine (see [Har77, Chapter III, Exercises 3.2]). Its structure ring is a submodule of a finite $O_X(U)$-module. Hence we conclude that $T \cap U$ is finite over U. □

Definition 5.2. *Let $S \in Sm/k$, and let $X, Y \in Sm/S$. The group*

$$\widetilde{Cor}_S(X, Y) = \varinjlim_{T \in \mathscr{A}_S(X,Y)} E^{d_Y - d_S}_T(X \times_S Y, -T_{X \times_S Y/X})$$

is called the group of finite E-correspondences between X and Y over S.

We can now consider the category $\widetilde{Cor}_S(X, Y)$, whose objects are smooth schemes over S and morphisms between X and Y are just $\widetilde{Cor}_S(X, Y)$ defined above. Our aim is now to study the composition in that category.

To avoid complicated expressions, we denote for smooth schemes X, Y and Z the scheme $X \times_S Y \times_S Z$ by XYZ and the projection $X \times_S Y \times_S Z \longrightarrow Y \times_S Z$ by p^{XYZ}_{YZ}. We extend this notation to arbitrary products of schemes in an obvious way.

Given any $\alpha \in \widetilde{Cor}_S(X, Y)$ and $\beta \in \widetilde{Cor}_S(Y, Z)$, we may suppose they are defined over admissible subsets. With this in mind, the image of

$$p^{XYZ}_{XZ*}(p^{XYZ*}_{YZ}((\beta, -T_{YZ/Y})) \cdot p^{XYZ*}_{XY}((\alpha, -T_{XY/X})))$$

in $\widetilde{Cor}_S(X, Z)$ is just defined as $\beta \circ \alpha$. It is straightforward to check that this definition is compatible with extension of supports.

Proposition 5.1. *The composition law defined above is associative.*

Proof. Suppose that $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} W$ are morphisms in $\widetilde{Cor}_S$. As before, we may suppose that each correspondence is defined over an admissible subset.

Consider the Cartesian squares

$$\begin{tikzcd} XYZW \arrow[r] \arrow[d] & XZW \arrow[d] & XYZW \arrow[r] \arrow[d] & XYW \arrow[d] \\ XYZ \arrow[r] & XZ & YZW \arrow[r] & YW. \end{tikzcd}$$

Now

$$\gamma \circ (\beta \circ \alpha)$$

$$= p^{XZW}_{XW*}(p^{XZW*}_{ZW}((\gamma, -T_{ZW/Z}))p^{XZW*}_{XZ} p^{XYZ}_{XZ*}(p^{XYZ*}_{YZ}((\beta, -T_{YZ/Y}))p^{XYZ*}_{XY}((\alpha, -T_{XY/X}))))$$

by definition

$$= p^{XZW}_{XW*}(p^{XZW*}_{ZW}(\gamma)p^{XZW*}_{XZ} p^{XYZ}_{XZ*}((p^{XYZ*}_{YZ}(\beta)p^{XYZ*}_{XY}(\alpha), -T_{XYZ/XY} - T_{XYZ/XZ})))$$

by definition of the product

$$= p^{XZW}_{XW*}(p^{XZW*}_{ZW}(\gamma)p^{XYZW}_{XZW*} p^{XYZW*}_{XYZ}((p^{XYZ*}_{YZ}(\beta)p^{XYZ*}_{XY}(\alpha), -T_{XYZ/XY} - T_{XYZ/XZ})))$$

by Axiom 15 for the left square above

$$= p^{XZW}_{XW*}(p^{XZW*}_{ZW}(\gamma)p^{XYZW}_{XZW*}(p^{XYZW*}_{YZ}(\beta)p^{XYZW*}_{XY}(\alpha), -T_{XYZW/XYW} - T_{XYZW/XZW}))$$

by Axiom 9 and Axiom 10

$$= p^{XZW}_{XW*}p^{XYZW}_{XZW*}((p^{XYZW*}_{ZW}(\gamma), -T_{XYZW/XYZ})p^{XYZW*}_{YZ}(\beta)p^{XYZW*}_{XY}(\alpha))$$

by Axiom 17 for p^{XYZW}_{XZW}

$$= p^{XZW}_{XW*}p^{XYZW}_{XZW*}((\delta, -p^{XYZW*}_{XW}T_{XW/X} - p^{XYZW*}_{XZW}T_{XZW/XW} - T_{XYZW/XZW}))$$

by definition of the product where $\delta = p^{XYZW*}_{ZW}(\gamma)p^{XYZW*}_{YZ}(\beta)p^{XYZW*}_{XY}(\alpha)$

$$= p^{XYZW}_{XW*}((\delta, -p^{XYZW*}_{XW}T_{XW/X} - T_{XYZW/XW}))$$

by Axiom 12

$$= p^{XYW}_{XW*}p^{XYZW}_{XYW*}((\delta, -p^{XYZW*}_{XW}T_{XW/X} - p^{XYZW*}_{XYW}T_{XYW/XW} - T_{XYZW/XYW}))$$

by Axiom 12, note that we have used $c(-T_{XYZW/XYW}, -T_{XYZW/XZW})$

$$= p^{XYW}_{XW*}(p^{XYZW}_{XYW*}(p^{XYZW*}_{ZW}(\gamma)p^{XYZW*}_{YZ}(\beta))p^{XYW*}_{XY}(\alpha))$$

by Axiom 17 for p^{XYZW}_{XYW}

$$= p^{XYW}_{XW*}(p^{XYZW}_{XYW*}p^{XYZW*}_{YZW}(p^{YZW*}_{ZW}(\gamma)(p^{YZW*}_{YZ}(\beta), -T_{YZW/YW}))p^{XYW*}_{XY}(\alpha))$$

by Axiom 9 and Axiom 10

$$= p^{XYW}_{XW*}(p^{XYW*}_{YW}p^{YZW}_{YW*}(p^{YZW*}_{ZW}(\gamma)p^{YZW*}_{YZ}(\beta))p^{XYW*}_{XY}(\alpha))$$

by Axiom 15 for the right square above

$$= (\gamma \circ \beta) \circ \alpha$$

by definition.

□

Definition 5.3. *Consider the functor*

$$\widetilde{\gamma} : Sm/S \longrightarrow \widetilde{Cor}_S,$$

defined on objects by $\widetilde{\gamma}(X) = X$. *Given an* S*-morphism* $f : X \longrightarrow Y$, *we have the graph morphism* $\Gamma_f : X \longrightarrow X \times_S Y$ *and the natural map*

$$\Gamma_f^* T_{X \times_S Y/X} \longrightarrow N_{X/X \times_S Y}$$

of Lemma 4.1 is an isomorphism. We set $\widetilde{\gamma}(f)$ to be image of the element $1 \in E^0_X(X,0)$ of Axiom 7 under the composite

$$\begin{tikzcd} E^0_X(X,0) \arrow[r] & E^0_X(X, N_{X/X\times_S Y} - \Gamma_f^* T_{X\times_S Y/X}) \arrow[r, "\Gamma_{f*}"] & E^{d_Y-d_S}_X(X\times_S Y, -T_{X\times_S Y/X}) \arrow[d] \\ & & \widetilde{Cor}_S(X,Y). \end{tikzcd}$$

We prove in the next couple of results that $\widetilde{\gamma}$ respects the composition of both categories, starting with some easy cases.

Proposition 5.2. *Let $f : X \longrightarrow Y$ be a morphism in Sm/S and $g : Y \longrightarrow Z$ be a morphism in $\widetilde{Cor}_S$. Then we have*

$$g \circ \widetilde{\gamma}(f) = (f \times id_Z)^*(g)$$

where the right hand side is the image into the direct limit of the corresponding element.

Proof. We have a Cartesian square

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] & XY \\ XZ \arrow[u, "p_X^{XZ}"] \arrow[r, "\Gamma_f\times id_Z"] & XYZ. \arrow[u, "p_{XY}^{XYZ}"'] \end{tikzcd}$$

Denote the map $E^0_X(X,0) \longrightarrow E^0_X(X, N_{X/X\times_S Y} - \Gamma_f^* T_{X\times_S Y/X})$ by t. Suppose as usual that g is supported on an admissible subset. We have

$g \circ \widetilde{\gamma}(f)$
$=p^{XYZ}_{XZ*}(p^{XYZ*}_{YZ}((g, -T_{YZ/Y})) \cdot p^{XYZ*}_{XY}\Gamma_{f*}((t(1), N_{X/XY} - \Gamma_f^* T_{XY/X})))$
by definition
$=p^{XYZ}_{XZ*}(p^{XYZ*}_{YZ}((g, -T_{YZ/Y})) \cdot (\Gamma_f \times id_Z)_* p^{XZ*}_X(t(1)))$
by Axiom 16 for the square above
$=p^{XYZ}_{XZ*}((\Gamma_f \times id_Z)_*((\Gamma_f \times id_Z)^* p^{XYZ*}_{YZ}((g, -T_{YZ/Y})) \cdot p^{XZ*}_X(t(1))))$
by Axiom 18 for $\Gamma_f \times id_Z$
$=p^{XYZ}_{XZ*}((\Gamma_f \times id_Z)_*((f \times id_Z)^*((g, -T_{YZ/Y})) \cdot p^{XZ*}_X(t(1))))$
by Axiom 9
$=p^{XYZ}_{XZ*}(\Gamma_f \times id_Z)_*((f \times id_Z)^*(g) \cdot p^{XZ*}_X(t(1)), -T_{XZ/X} + N_{XZ/XYZ} - (\Gamma_f \times id_Z)^* T_{XYZ/XZ})$
by definition of the product and the pull-back, and Lemma 4.3
$=s(((f \times id_Z)^*(g) \cdot p^{XZ*}_X(t(1)), -T_{XZ/X} + N_{XZ/XYZ} - (\Gamma_f \times id_Z)^* T_{XYZ/XZ}))$
by Axiom 19 (here s is the isomorphism cancelling $N_{XZ/XYZ} \cong (\Gamma_f \times id_Z)^* T_{XYZ/XZ}$)
$=(f \times id_Z)^*(g) \cdot s(p^{XZ*}_X(t(1)))$
by bifunctoriality of products with respect to twists
$=(f \times id_Z)^*(g) \cdot p^{XZ*}_X(1)$
by functoriality of pull-backs with respect to twists
$=(f \times id_Z)^*(g)$
by the definition of the identity and Axiom 9.

□

Proposition 5.3. *Let $f : X \longrightarrow Y$ be a morphism in $\widetilde{Cor}_S$ and let $g : Y \longrightarrow Z$ be a smooth morphism in Sm/S. Let t be the composite*

$$\begin{aligned}
&-T_{XY/X} \\
\longrightarrow &-(id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} - T_{XY/X} \\
\longrightarrow &-(id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} - (id_X \times \Gamma_g)^* T_{XYZ/XZ} \\
\longrightarrow &-(id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} - N_{XY/XYZ} - T_{XY/XZ} \\
\longrightarrow &-(id_X \times \Gamma_g)^* T_{XYZ/XY} - T_{XY/XZ} \\
\longrightarrow &-(id_X \times g)^* T_{XZ/X} - T_{XY/XZ}.
\end{aligned}$$

Then we have

$$\widetilde{\gamma}(g) \circ f = (id_X \times g)_*(t(f)),$$

where the right side is the image into the direct limit of the corresponding element.

Proof. We have a Cartesian square

$$\begin{tikzcd}
Y \arrow[r, "\Gamma_g"] & YZ \\
XY \arrow[u, "p_Y^{XY}"] \arrow[r, "id_X \times \Gamma_g"] & XYZ, \arrow[u, "p_{YZ}^{XYZ}"']
\end{tikzcd}$$

an isomorphism $s : 0 \longrightarrow N_{Y/YZ} - \Gamma_g^* T_{YZ/Y}$ and an isomorphism

$$r : -T_{XY/X} \longrightarrow N_{XY/XYZ} - (id_X \times \Gamma_g)^* T_{XYZ/XY} - T_{XY/X}.$$

Suppose that f is supported on some admissible subset. We obtain

$$\begin{aligned}
&\widetilde{\gamma}(g) \circ f \\
=&p_{XZ*}^{XYZ}(p_{YZ}^{XYZ*}\Gamma_{g*}((s(1), N_{Y/YZ} - \Gamma_g^* T_{YZ/Y})) \cdot p_{XY}^{XYZ*}((f, -T_{XY/X})) \\
&\text{by definition} \\
=&p_{XZ*}^{XYZ}((id_X \times \Gamma_g)_* p_Y^{XY*}((s(1), N_{Y/YZ} - \Gamma_g^* T_{YZ/Y})) \cdot p_{XY}^{XYZ*}(f)) \\
&\text{by Axiom 16 for the square above} \\
=&p_{XZ*}^{XYZ}(id_X \times \Gamma_g)_*(p_Y^{XY*}((s(1), N_{Y/YZ} - \Gamma_g^* T_{YZ/Y})) \cdot (id_X \times \Gamma_g)^* p_{XY}^{XYZ*}(f)) \\
&\text{by Axiom 18 for } id_X \times \Gamma_g \\
=&p_{XZ*}^{XYZ}(id_X \times \Gamma_g)_*(r((id_X \times \Gamma_g)^* p_{XY}^{XYZ*}(f))) \\
&\text{by functoriality of pull-backs and products with respect to twists} \\
=&(id_X \times g)_*(t((id_X \times \Gamma_g)^* p_{XY}^{XYZ*}(f))) \\
&\text{by Axiom 19} \\
=&(id_X \times g)_*(t(f)) \\
&\text{by Axiom 9.}
\end{aligned}$$

□

Proposition 5.4. *Let $f : X \longrightarrow Y$ be a morphism in $\widetilde{Cor}_S$ and let $g : Y \longrightarrow Z$ be a closed immersion in Sm/S. Let t' be the composite*

$$\begin{aligned}
&-T_{XY/X} \\
\longrightarrow &-T_{XY/X} + N_{XY/XYZ} - (id_X \times \Gamma_g)^* T_{XYZ/XY} \\
\longrightarrow &-T_{XY/X} + (id_X \times \Gamma_g)^* T_{XYZ/XZ} + N_{XY/XZ} - (id_X \times \Gamma_g)^* T_{XYZ/XY} \\
\longrightarrow &-T_{XY/X} + T_{XY/X} + N_{XY/XZ} - (id_X \times \Gamma_g)^* T_{XYZ/XY} \\
\longrightarrow &N_{XY/XZ} - (id_X \times \Gamma_g)^* T_{XYZ/XY} \\
\longrightarrow &N_{XY/XZ} - (id_X \times g)^* T_{XZ/X}.
\end{aligned}$$

Then we have

$$\widetilde{\gamma}(g) \circ f = (id_X \times g)_*(t'(f)),$$

where the right side is the image into the direct limit of the corresponding element.

Proof. The same proof as in the above proposition applies. $\square$

Before proceeding further, we make the isomorphisms t and t' above more concrete in the category of virtual vector bundles.

Lemma 5.2. *Suppose that we have a commutative diagram in Sm/k*

$$\begin{tikzcd}
Y \arrow[d] \arrow[dr, equal] \arrow[dd, bend right, "j"'] & \\
X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
Z \arrow[r, "g"] & S
\end{tikzcd}$$

in which the square is Cartesian and f, g are smooth.

1. *If j is a closed immersion, then the following diagram commutes*

$$\begin{tikzcd}
T_{X/Y}|_Y + T_{Y/S} \arrow[d] & T_{X/S}|_Y \arrow[l] \arrow[r] & T_{X/Z}|_Y + T_{Z/S}|_Y \arrow[dd] \\
N_{Y/X} + T_{Y/S} \arrow[d] & & \\
T_{X/Z}|_Y + N_{Y/Z} + T_{Y/S} \arrow[rr] & & T_{X/Z}|_Y + T_{Y/S} + N_{Y/Z}.
\end{tikzcd}$$

2. *If j is smooth, then the following diagram commutes*

$$\begin{tikzcd}
T_{X/Y}|_Y + T_{Y/S} \arrow[d] & T_{X/S}|_Y \arrow[l] \arrow[r] & T_{X/Z}|_Y + T_{Z/S}|_Y \arrow[dd] \\
N_{Y/X} + T_{Y/S} \arrow[d] & & \\
N_{Y/X} + T_{Y/Z} + T_{Z/S}|_Y \arrow[rr] & & T_{Y/Z} + N_{Y/X} + T_{Z/S}|_Y.
\end{tikzcd}$$

Proof. In both cases, there is a commutative diagram with exact row and column

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & T_{Y/S} \arrow[d] \arrow[dr, equal] & & \\
0 \arrow[r] & T_{X/Y}|_Y \arrow[r] \arrow[dr, "\cong"'] & T_{X/S}|_Y \arrow[r] \arrow[d] & T_{Y/S} \arrow[r] & 0 \\
 & & N_{Y/X} \arrow[d] & & \\
 & & 0. & &
\end{tikzcd}$$

It induces a commutative diagram

$$\begin{tikzcd}
T_{X/S}|_Y \arrow[r] \arrow[d] & T_{X/Y}|_Y + T_{Y/S} \arrow[dl] \\
T_{Y/S} + N_{Y/X} &
\end{tikzcd}$$

by Theorem 3.1, (3). We now pass to the proof of the first statement. We have a commutative diagram with exact columns and rows

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & T_{Y/S} \arrow[r, equal] \arrow[d] & T_{Y/S} \arrow[d] & \\
0 \arrow[r] & T_{X/Z}|_Y \arrow[r] \arrow[d, equal] & T_{X/S}|_Y \arrow[r] \arrow[d] & T_{Z/S}|_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{X/Z}|_Y \arrow[r] & N_{Y/X} \arrow[r] \arrow[d] & N_{Y/Z} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

We deduce the following commutative diagram by Theorem 3.1, (3)

$$\begin{tikzcd}
T_{X/S}|_Y \arrow[r] \arrow[d] & T_{X/Z}|_Y + T_{Z/S}|_Y \arrow[r] & T_{X/Z}|_Y + T_{Y/S} + N_{Y/Z} \arrow[ddll] \\
T_{Y/S} + N_{Y/X} \arrow[d] & & \\
T_{Y/S} + T_{X/Z}|_Y + N_{Y/Z}. & &
\end{tikzcd}$$

Furthermore, there is an obvious commutative diagram

$$\begin{tikzcd}
N_{Y/X} + T_{Y/S} \arrow[d] & T_{Y/S} + N_{Y/X} \arrow[l] \arrow[r] & T_{Y/S} + T_{X/Z}|_Y + N_{Y/Z} \arrow[d] . \\
T_{X/Z}|_Y + N_{Y/Z} + T_{Y/S} \arrow[rr] & & T_{X/Z}|_Y + T_{Y/S} + N_{Y/Z}
\end{tikzcd}$$

So the statement follows by combining the diagrams above.

For the second statement, observe that we have a commutative diagram with exact columns and rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & T_{Y/Z} \arrow[r] \arrow[d] & T_{Y/S} \arrow[r] \arrow[d] & T_{Z/S}|_Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_{X/Z}|_Y \arrow[r] \arrow[d] & T_{X/S}|_Y \arrow[r] \arrow[d] & T_{Z/S}|_Y \arrow[r] & 0 \\
 & N_{Y/X} \arrow[r, equal] \arrow[d] & N_{Y/X} \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Then the result follows by the same method as above by applying Theorem 3.1, (2) to the diagram above.

□

Lemma 5.3. *Suppose that $X, Y, Z \in Sm/S$ and that $g : Y \longrightarrow Z$ is a morphism in Sm/S.*

1. *If g is a closed immersion, then the isomorphism t in Proposition 5.3 is equal to*
$$-T_{XY/X} \longrightarrow N_{XY/XZ} - N_{XY/XZ} - T_{XY/X} \longrightarrow N_{XY/XZ} - (id_X \times g)^* T_{XZ/X}.$$
2. *If g is smooth, then the isomorphism t' in Proposition 5.4 is equal to*
$$-T_{XY/X} \longrightarrow -(id_X \times g)^* T_{XZ/X} - T_{XY/XZ}.$$

Proof. We have a commutative diagram in Sm/k

$$\begin{tikzcd}
XY \arrow[d, "id_X\times\Gamma_g"] \arrow[dr, equal] \arrow[dd, bend right, "id_X\times g"'] & \\
XYZ \arrow[r, "p^{XYZ}_{XY}"'] \arrow[d, "p^{XYZ}_{XZ}"] & XY \arrow[d, "p^{XY}_{X}"] \\
XZ \arrow[r, "p^{XZ}_{X}"'] & X
\end{tikzcd}$$

in which the square is Cartesian. Suppose first that g is a closed immersion. In that case, we show that the composite

$$\begin{aligned}
&- T_{XY/X} \\
\longrightarrow &- T_{XY/X} + N_{XY/XYZ} - (id_X \times \Gamma_g)^* T_{XYZ/XY} \\
\longrightarrow &- T_{XY/X} + (id_X \times \Gamma_g)^* T_{XYZ/XZ} + N_{XY/XZ} - (id_X \times \Gamma_g)^* T_{XYZ/XY} \\
\longrightarrow &N_{XY/XZ} - (id_X \times \Gamma_g)^* T_{XYZ/XY} \\
\longrightarrow &N_{XY/XZ} - (id_X \times g)^* T_{XZ/X} \\
\longrightarrow &N_{XY/XZ} - N_{XY/XZ} - T_{XY/X} \\
\longrightarrow &- T_{XY/X}
\end{aligned}$$

is just $id_{-T_{XY/X}}$. Indeed, it is equal to

$$\begin{aligned}
&- T_{XY/X} \\
\longrightarrow &- T_{XY/X} + N_{XY/XYZ} - (id_X \times \Gamma_g)^* T_{XYZ/XY} \\
\longrightarrow &- T_{XY/X} - (id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} \\
\longrightarrow &- T_{XY/X} - (id_X \times g)^* T_{XZ/X} + N_{XY/XYZ} \\
\longrightarrow &- T_{XY/X} - N_{XY/XZ} - T_{XY/X} + N_{XY/XYZ} \\
\longrightarrow &- T_{XY/X} - N_{XY/XZ} - T_{XY/X} + (id_X \times \Gamma_g)^* T_{XYZ/XZ} + N_{XY/XZ} \\
\longrightarrow &- N_{XY/XZ} - T_{XY/X} + N_{XY/XZ} \\
\longrightarrow &- T_{XY/X},
\end{aligned}$$

where the sixth arrow is the cancellation map between the first and the fourth term. By Lemma 5.2, (1) and the commutative diagram above, we have a commutative diagram

$$\begin{tikzcd}
(id_X \times \Gamma_g)^* T_{XYZ/XY} + T_{XY/X} \arrow[d] & (id_X \times \Gamma_g)^* T_{XYZ/X} \arrow[l] \arrow[d] \\
N_{XY/XYZ} + T_{XY/X} \arrow[d] & (id_X \times \Gamma_g)^* T_{XYZ/XZ} + (id_X \times g)^* T_{XZ/X} \arrow[d] \\
(id_X \times \Gamma_g)^* T_{XYZ/XZ} + N_{XY/XZ} + T_{XY/X} \arrow[r] & (id_X \times \Gamma_g)^* T_{XYZ/XZ} + T_{XY/X} + N_{XY/XZ}.
\end{tikzcd}$$

Hence the composite above is equal to

$$
\begin{aligned}
& -T_{XY/X} \\
\longrightarrow & -T_{XY/X} + N_{XY/XYZ} - (id_X \times \Gamma_g)^* T_{XYZ/XY} \\
\longrightarrow & -T_{XY/X} - (id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} \\
\longrightarrow & -T_{XY/X} - N_{XY/XYZ} + N_{XY/XYZ} \\
\longrightarrow & -T_{XY/X} - N_{XY/XZ} - (id_X \times \Gamma_g)^* T_{XYZ/XZ} + N_{XY/XYZ} \\
\longrightarrow & -T_{XY/X} - N_{XY/XZ} - (id_X \times \Gamma_g)^* T_{XYZ/XZ} + (id_X \times \Gamma_g)^* T_{XYZ/XZ} + N_{XY/XZ} \\
\longrightarrow & -T_{XY/X},
\end{aligned}
$$

which gives the result.

Suppose next that g is smooth. We show that the composite

$$
\begin{aligned}
& -T_{XY/X} \\
\longrightarrow & -(id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} - T_{XY/X} \\
\longrightarrow & -(id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} - (id_X \times \Gamma_g)^* T_{XYZ/XZ} \\
\longrightarrow & -(id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} - N_{XY/XYZ} - T_{XY/XZ} \\
\longrightarrow & -(id_X \times \Gamma_g)^* T_{XYZ/XY} - T_{XY/XZ} \\
\longrightarrow & -(id_X \times g)^* T_{XZ/X} - T_{XY/XZ} \\
\longrightarrow & -T_{XY/X}
\end{aligned}
$$

is just $id_{-T_{XY/X}}$. By Lemma 5.2, (2) and the commutative diagram at the beginning of the proof, we get a commutative diagram

$$
\begin{tikzcd}
(id_X \times \Gamma_g)^* T_{XYZ/XY} + T_{XY/X} \arrow[d] & (id_X \times \Gamma_g)^* T_{XYZ/X} \arrow[l] \arrow[d] \\
N_{XY/XYZ} + T_{XY/X} \arrow[d] & (id_X \times \Gamma_g)^* T_{XYZ/XZ} + (id_X \times g)^* T_{XZ/X} \arrow[d] \\
N_{XY/XYZ} + T_{XY/XZ} + (id_X \times g)^* T_{XZ/X} \arrow[r] & T_{XY/XZ} + N_{XY/XYZ} + (id_X \times g)^* T_{XZ/X}.
\end{tikzcd}
$$

Hence the given composite is equal to

$$
\begin{aligned}
& -T_{XY/X} \\
\longrightarrow & -(id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} - T_{XY/X} \\
\longrightarrow & -(id_X \times \Gamma_g)^* T_{XYZ/XY} - T_{XY/X} + N_{XY/XYZ} \\
\longrightarrow & -N_{XY/XYZ} - T_{XY/X} + N_{XY/XYZ} \\
\longrightarrow & -N_{XY/XYZ} - (id_X \times g)^* T_{XZ/X} - T_{XY/XZ} + N_{XY/XYZ} \\
\longrightarrow & -(id_X \times g)^* T_{XZ/X} - T_{XY/XZ} \\
\longrightarrow & -T_{XY/X},
\end{aligned}
$$

where the fifth arrow is the cancellation between the first and the fourth term. The result follows. □

Proposition 5.5. *For any $X \in Sm/S$, $\widetilde{\gamma}(id_X)$ is an identity. That is, for any $X, Y \in Sm/S$, $f \in \widetilde{Cor}_S(X,Y)$, $g \in \widetilde{Cor}_S(Y,X)$, we have*

$$\widetilde{\gamma}(id_Y) \circ f = f, \quad g \circ \widetilde{\gamma}(id_X) = g.$$

Proof. The second equation follows by Proposition 5.2 and the first one follows from Lemma 5.2, (1) and Proposition 5.3. □

Combining Proposition 5.1 and Proposition 5.5, we have proved that $\widetilde{Cor}_S$ is indeed a category. We now complete the proof that $\widetilde{\gamma}$ is indeed a functor.

Proposition 5.6. *For any* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *in* Sm/S*, we have*

$$\widetilde{\gamma}(g \circ f) = \widetilde{\gamma}(g) \circ \widetilde{\gamma}(f).$$

Proof. Suppose at first that f is a closed immersion or that it is smooth. We have a Cartesian square

$$\begin{tikzcd} XZ \arrow[r, "f\times id_Z"] & YZ \\ X \arrow[u, "\Gamma_{g\circ f}"] \arrow[r, "f"] & Y \arrow[u, "\Gamma_g"'] \end{tikzcd}$$

and two isomorphisms $a : N_{Y/YZ} - \Gamma_g^* T_{YZ/Y} \longrightarrow 0$ and $b : N_{X/XZ} - \Gamma_f^* T_{XZ/X} \longrightarrow 0$. For convenience, we denote the induced morphisms at the level of correspondences still by a and b respectively. Then we have

$$\begin{aligned}
&\widetilde{\gamma}(g) \circ \widetilde{\gamma}(f)\\
=&(f \times id_Z)^*(\widetilde{\gamma}(g))\\
&\text{by Proposition 5.2}\\
=&(f \times id_Z)^*(\Gamma_{g*}(a^{-1}(1), N_{Y/YZ} - \Gamma_g^* T_{YZ/Y}))\\
&\text{by definition of } \widetilde{\gamma}\\
=&(\Gamma_{g\circ f})_* f^*(a^{-1}(1))\\
&\text{by Axiom 16 for the square above}\\
=&(\Gamma_{g\circ f})_*(b^{-1}(1))\\
&\text{by Axiom 9 and functoriality of pull-backs with respect to twists}\\
=&\widetilde{\gamma}(g \circ f)\\
&\text{by definition of } \widetilde{\gamma}.
\end{aligned}$$

Suppose now that $f = p \circ i$ in Sm/S, where p is smooth and i is a closed immersion. Then

$$\widetilde{\gamma}(g) \circ \widetilde{\gamma}(f) = \widetilde{\gamma}(g) \circ \widetilde{\gamma}(i) \circ \widetilde{\gamma}(p) = \widetilde{\gamma}(i \circ g) \circ \widetilde{\gamma}(p) = \widetilde{\gamma}(g \circ f)$$

by the statements above. □

Remark 5.1. *In [V01, Section 2] and [GP14, Section 2], the set* $Fr_n(X, Y)$ *(resp.* $\mathbb{Z}F_n(X, Y)$*) of (resp. linear) framed correspondence of level n for any* $X, Y \in Sm/k, n \in \mathbb{N}$ *is defined. Garkusha-Panin and Voevodsky define the category* $\mathbb{Z}F_*(k)$ *to be the category whose objects are those of* Sm/k *and*

$$Hom_{\mathbb{Z}F_*(k)}(X, Y) = \oplus_n \mathbb{Z}F_n(X, Y).$$

Here, any element s in $Fr_n(X, Y)$ *is given by (an equivalence class of) a commutative diagram as below*

$$\begin{tikzcd} \mathbb{A}^n_X \arrow[d, "p"'] & U \arrow[l, "a"'] \arrow[r, "g"] & \mathbb{A}^n_Y \\ X & Z \arrow[u, "i"] \arrow[r] & Y, \arrow[u, "z"'] \end{tikzcd}$$

where a is étale, $i, a\circ i$ are closed immersions, $p\circ a\circ i$ is finite, z is the zero section and the square is Cartesian. Suppose that $Z\neq\emptyset$ and denote the composite

$$U \xrightarrow{g} \mathbb{A}^n_Y \longrightarrow Y$$

by f. We have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\Gamma_f"] & U\times Y \arrow[r, "b=a\times id"] & \mathbb{A}^n_X\times Y \arrow[r, "c"] \arrow[d, "q"'] & X\times Y \arrow[d] \\ & & \mathbb{A}^n_X \arrow[r] & X \end{tikzcd}$$

in which the square is Cartesian. Then we can associate to s an element $\alpha(s)$ in $\widetilde{Cor}_k(X,Y)$ defined to be the image of 1 under the composite

$$\begin{tikzcd} E^0_Y(Y,0) \arrow[r] & E^0_Y(Y,N_z-N_z) \arrow[r, "z_*"] & E^n_Y(\mathbb{A}^n_Y,-T_{\mathbb{A}^n_Y/Y}) \arrow[dll, "g^*"'] \\ E^n_Z(U,-g^*T_{\mathbb{A}^n_Y/Y}) \arrow[r] & E^n_Z(U,N_{\Gamma_f}-N_{\Gamma_f}-g^*T_{\mathbb{A}^n_Y/Y}) \arrow[r, "b_*\Gamma_{f*}"] & E^{n+d_Y-d_S}_Z(\mathbb{A}^n_X\times Y,-T_{\mathbb{A}^n_X\times Y/\mathbb{A}^n_X}-q^*T_{\mathbb{A}^n_X/X}) \arrow[d, "c_*"] \\ & \widetilde{Cor}_k(X,Y) & E^{d_Y-d_S}_{p(Z)}(X\times Y,-T_{X\times Y/X}), \arrow[l] \end{tikzcd}$$

where we have used the isomorphism $g^*T_{\mathbb{A}^n_Y/Y}\cong a^*T_{\mathbb{A}^n_X/X}$. One checks that this induces a functor

$$\alpha:\mathbb{Z}F_*(k)\longrightarrow\widetilde{Cor}_k$$

as in [DF17, Proposition 2.1.12].

Definition 5.4. *Define $\widetilde{PSh}(S)$ to be the category of contravariant additive functors from $\widetilde{Cor}_S$ to Ab as in [DF17, Definition 1.2.1] and [MVW06, Definition 2.1]. The objects of this category are called presheaves with E-transfers over S. Further, define $\widetilde{Sh}(S)$ to be the full subcategory of objects whose restriction on Sm/S via $\widetilde{\gamma}$ are Nisnevich sheaves. We call them sheaves with E-transfers over S.*

Definition 5.5. *Let $X,Y\in Sm/S$, we define $\widetilde{c}_S(X)$ by $\widetilde{c}_S(X)(Y)=\widetilde{Cor}_S(Y,X)$. It is the presheaf with E-transfers represented by X.*

We recall the following three propositions which are the technical heart when dealing with Nisnevich sheaves:

Proposition 5.7. *Let $f:X\longrightarrow S$ be a locally of finite type morphism between locally noetherian schemes. Let I be a directed set and let $\{T_i\}$ be an inverse system of S-schemes such that for any $i_1\preceq i_2$, the morphism $T_{i_2}\longrightarrow T_{i_1}$ is affine. Then $\varprojlim_i T_i$ exists in the category of S-schemes and we have*

$$Hom_S(\varprojlim_i T_i,X)=\varinjlim_i Hom_S(T_i,X).$$

Proof. See [Pro, Lemma 2.2] and [Pro, Proposition 6.1]. $\square$

Now, let A be a noetherian ring and let $p\in \mathrm{Spec}A$. Consider the set I whose elements are pairs (B,q), where B is a connected étale A-algebra, $q\in\mathrm{Spec}B$, $q\cap A=p$ and $k(p)=k(q)$. Set $(B_1,q_1)\preceq(B_2,q_2)$ if there is an A-algebra morphism (always unique if exists) $f:B_1\longrightarrow B_2$ such that $f^{-1}(q_2)=q_1$.

Proposition 5.8. *The set I is a directed set and we have*

$$\varinjlim_{(B,q)} B \cong A_p^h,$$

where the right hand side is the Henselization of A_p.

Proof. See the remarks around [Mil80, Lemma 4.8] and see for example [Mil80, Theorem 4.2] for basic properties of Henselian rings. □

Proposition 5.9. *Let U, X, Y be locally noetherian schemes, $p : U \longrightarrow X$ be a Nisnevich covering and $f : X \longrightarrow Y$ be a finite morphism. Then, there exists for every $y \in Y$ a scheme V with an étale morphism $V \longrightarrow Y$ being Nisnevich at y such that the morphism $U \times_Y V \longrightarrow X \times_Y V$ has a section.*

Proof. Consider the following commutative diagram with Cartesian squares

$$\begin{tikzcd} U \arrow[r, "p"] & X \arrow[r, "f"] & Y \\ R_2 \arrow[u, "\gamma"] \arrow[r, "\alpha"] & R_1 \arrow[u] \arrow[r, "\beta"] & \mathrm{Spec} O_{Y,y}^h. \arrow[u] \end{tikzcd}$$

Since β is a finite morphism, R_1 is a finite direct product of Henselian rings (see [Mil80, Theorem 4.2]). Hence, α has a section s since it is Nisnevich at every maximal ideal of R_1. Pick an affine neighbourhood U_0 of y. By [Pro, Lemma 2.3] and Proposition 5.8,

$$R_1 = \left(\varprojlim_{(B,q)\succeq(O_Y(U_0),y)} \mathrm{Spec} B \right) \times_{U_0} f^{-1}(U_0) = \varprojlim_{(B,q)\succeq(O_Y(U_0),y)} (\mathrm{Spec} B \times_{U_0} f^{-1}(U_0)),$$

hence there exists a $(B_0, q) \succeq (O_Y(U_0), y)$ such that $\gamma \circ s$ factor through the projection

$$\varprojlim_{(B,q)\succeq(O_Y(U_0),y)} (\mathrm{Spec} B \times_{U_0} f^{-1}(U_0)) \longrightarrow \mathrm{Spec} B_0 \times_{U_0} f^{-1}(U_0)$$

by using Proposition 5.7 for p. Then we finally let $V = \mathrm{Spec} B_0$. □

Now we are going to prove a similar result as in [DF17, Lemma 1.2.6].

Proposition 5.10. *Let $X, U \in Sm/S$ and let $p : U \longrightarrow X$ be a Nisnevich covering. Denote the n-fold product $A \times_B A \times_B \cdots \times_B A$ by A_B^n for any schemes A and B. Then, the complex of sheaves associated to the complex*

$$\check{C}(U/X) := \cdots \longrightarrow \widetilde{c}_S(U_X^n) \xrightarrow{d_n} \cdots \longrightarrow \widetilde{c}_S(U \times_X U) \xrightarrow{d_2} \widetilde{c}_S(U) \xrightarrow{d_1} \widetilde{c}_S(X) \xrightarrow{d_0} 0\ ,$$

is exact. Here we set $p_i : U_X^n \longrightarrow U_X^{n-1}$ to be the projection omitting i-th factor and $d_n = \sum_i (-1)^{i-1} \widetilde{c}_S(p_i)$.

Proof. Given $Y \in Sm/S$, we have to prove that the complex is exact at every point $y \in Y$. Now, assume that we have an element $a \in \widetilde{Cor}_S(Y, U_X^n)$ such that $d_n(a) = 0$. We may suppose that there exists $T \in \mathscr{A}_S(Y, X)$ such that a comes from $E_{R^n}^{d_X - d_S}((Y \times_S U)_{Y\times_S X}^n, -T_{Y\times_S U_X^n/Y})$ and $d_n(a) = 0$, where R^n is defined by the following Cartesian squares ($R := R^1$)

$$\begin{tikzcd} R^n \arrow[r] \arrow[d] & Y \times_S U_X^n \arrow[r] \arrow[d] & U_X^n \arrow[d] \\ T \arrow[r] & Y \times_S X \arrow[r] & X. \end{tikzcd}$$

By Proposition 5.9, there is a Nisnevich neighbourhood V of y such that the map $p : R\times_Y V \longrightarrow T\times_Y V$ has a section s, which is both an open immersion and a closed immersion (see [Mil80, Corollary 3.12]). Let $D = (R\times_Y V)\setminus s(T\times_Y V)$. Then $d_n(a|_{V\times_S U^n_X}) = 0$. We have a commutative diagram

$$\begin{tikzcd}
V\times_S U^n_X \arrow[r] \arrow[d] & Y\times_S U^n_X \arrow[d] \\
V\times_S X \arrow[r] & Y\times_S X,
\end{tikzcd}$$

Cartesian squares

$$\begin{tikzcd}
R^n\times_Y V \arrow[r] \arrow[d] & V\times_S U^n_X \arrow[r] \arrow[d] & V \arrow[d] \\
R^n \arrow[r] & Y\times_S U^n_X \arrow[r] & Y
\end{tikzcd}, \quad
\begin{tikzcd}
T\times_Y V \arrow[r, "s"] \arrow[d, "s"'] & (V\times_S U)\setminus D \arrow[d] \\
R\times_Y V \arrow[r] & V\times_S U
\end{tikzcd},$$

equations

$$\begin{aligned}
Y\times_S U^n_X &= (Y\times_S U)^n_{Y\times_S X},\\
V\times_S U^n_X &= (V\times_S U)^n_{V\times_S X},\\
R^n &= R\times_T\cdots\times_T R = R^n_T,\\
R^n\times_Y V = (R\times_Y V)^n_{T\times_Y V} &= (T\times_{Y\times_S X}(V\times_S U))^n_{T\times_Y V},\\
(R\times_Y V)^n_{T\times_Y V} &= (R\times_Y V)^n_{V\times_S X},
\end{aligned}$$

and a diagram of Cartesian squares in which the right-hand vertical maps are étale:

$$\begin{tikzcd}
(R\times_Y V)^n_{T\times_Y V} \arrow[r] \arrow[d, "id^n\times s"] \arrow[dd, bend right=60, "id"'] & (V\times_S U)^n_{V\times_S X}\times_{(V\times_S X)}((V\times_S U)\setminus D) := W^{n+1} \arrow[d, "j_{n+1}"'] \\
(R\times_Y V)^{n+1}_{T\times_Y V} \arrow[r] \arrow[d, "p_{n+1}"] & (V\times_S U)^{n+1}_{V\times_S X} \arrow[d, "p_{n+1}"'] \\
(R\times_Y V)^n_{T\times_Y V} \arrow[r] & (V\times_S U)^n_{V\times_S X},
\end{tikzcd}$$

where p_{n+1} denotes the projection omitting the last factor. The maps

$$E^{d_X-d_S}_{R^n\times_Y V}((V\times_S U)^n_{V\times_S X}, -T_{V\times_S U^n_X/V}) \xrightarrow{(p_{n+1}\circ j_{n+1})^*} E^{d_X-d_S}_{R^n\times_Y V}(W^{n+1}, -T_{V\times_S U^{n+1}_X/V}|_{W^{n+1}})$$

and

$$E^{d_X-d_S}_{R^n\times_Y V}((V\times_S U)^{n+1}_{V\times_S X}, -T_{V\times_S U^{n+1}_X/V}) \xrightarrow{j^*_{n+1}} E^{d_X-d_S}_{R^n\times_Y V}(W^{n+1}, -T_{V\times_S U^{n+1}_X/V}|_{W^{n+1}})$$

are isomorphisms with respective inverses $(p_{n+1}\circ j_{n+1})_*$ and $(j_{n+1})_*$ by Axiom 20.

Let's consider the element

$$b := ((j^*_{n+1})^{-1}\circ(p_{n+1}\circ j_{n+1})^*)(a|_{(V\times_S U)^n_{V\times_S X}}) \in E^{d_X-d_S}_{R^n\times_Y V}((V\times_S U)^{n+1}_{V\times_S X}, -T_{V\times_S U^{n+1}_X/V}),$$

where we have used the isomorphism

$$p^*_{n+1}T_{V\times_S U^n_X/V} \longrightarrow T_{V\times_S U^{n+1}_X/V}$$

since $U \longrightarrow X$ is étale. Then

$$d_{n+1}(b) = \sum_{i=1}^{n+1} (-1)^{i-1} \widetilde{c}_S(p_i)(b) = \sum_{i=1}^{n+1} (-1)^{i-1} p_{i*}(t_{i,n+1}(b))$$

by Proposition 5.3, where

$$t_{i,n+1} : -T_{V \times_S U_X^{n+1}/V} \longrightarrow -(id_V \times_S p_i)^* T_{(V \times_S U_X^n)/V} - T_{(V \times_S U_X^{n+1})/(V \times_S U_X^n)}$$

is the isomorphism of Proposition 5.3 applied to

$$V \xrightarrow{\quad b \quad} U_X^{n+1} \xrightarrow{\quad p_i \quad} U_X^n .$$

If $1 \leq i < n+1$, we have Cartesian squares

$$\begin{tikzcd} W^{n+1} \arrow[r, "p_{n+1} \circ j_{n+1}"'] \arrow[d, "p_i"'] & (V \times_S U)^n_{V \times_S X} \arrow[d, "p_i"] \\ W^n \arrow[r, "p_n \circ j_n"] & (V \times_S U)^{n-1}_{V \times_S X}, \end{tikzcd} \qquad \begin{tikzcd} W^{n+1} \arrow[r, "j_{n+1}"'] \arrow[d, "p_i"'] & (V \times_S U)^{n+1}_{V \times_S X} \arrow[d, "p_i"] \\ W^n \arrow[r, "j_n"] & (V \times_S U)^n_{V \times_S X}. \end{tikzcd}$$

So

$$\begin{aligned}
& p_{i*}(t_{i,n+1}(b)) \\
=& (p_{i*} \circ t_{i,n+1} \circ (j_{n+1}^*)^{-1} \circ (p_{n+1} \circ j_{n+1})^*)((a|_{(V \times_S U)^n_{V \times_S X}}, -T_{V \times_S U_X^n/V})) \\
& \text{by definition} \\
=& (p_{i*} \circ (j_{n+1}^*)^{-1} \circ j_{n+1}^*(t_{i,n+1}) \circ (p_{n+1} \circ j_{n+1})^*)(a|_{(V \times_S U)^n_{V \times_S X}}) \\
& \text{by functoriality of pullbacks with respect to twists} \\
=& ((j_n^*)^{-1} \circ p_{i*} \circ j_{n+1}^*(t_{i,n+1}) \circ (p_{n+1} \circ j_{n+1})^*)(a|_{(V \times_S U)^n_{V \times_S X}}) \\
& \text{by Axiom 15 for the right hand square above} \\
=& ((j_n^*)^{-1} \circ p_{i*} \circ (p_{n+1} \circ j_{n+1})^* \circ t_{i,n})(a|_{(V \times_S U)^n_{V \times_S X}}) \\
& \text{by functoriality of pull-backs with respect to twists} \\
=& ((j_n^*)^{-1} \circ (p_n \circ j_n)^* \circ p_{i*} \circ t_{i,n})(a|_{(V \times_S U)^n_{V \times_S X}}) \\
& \text{by Axiom 15 for the left hand square above.}
\end{aligned}$$

For $i = n+1$, we have

$$\begin{aligned}
& p_{n+1*}(t_{n+1,n+1}(b)) \\
=& (p_{n+1*} \circ t_{n+1,n+1} \circ (j_{n+1}^*)^{-1} \circ (p_{n+1} \circ j_{n+1})^*)((a|_{(V \times_S U)^n_{V \times_S X}}, -T_{V \times_S U_X^n/V})) \\
& \text{by definition} \\
=& (p_{n+1*} \circ t_{n+1,n+1} \circ j_{n+1*} \circ (p_{n+1} \circ j_{n+1})^*)(a|_{(V \times_S U)^n_{V \times_S X}}) \\
& \text{by Axiom 20} \\
=& (p_{n+1*} \circ j_{n+1*} \circ j_{n+1}^*(t_{n+1,n+1}) \circ (p_{n+1} \circ j_{n+1})^*)(a|_{(V \times_S U)^n_{V \times_S X}}) \\
& \text{by functoriality of push-forwards with respect to twists} \\
=& ((p_{n+1} \circ j_{n+1})_* \circ (p_{n+1} \circ j_{n+1})^*)(a|_{(V \times_S U)^n_{V \times_S X}}) \\
& \text{by Axiom 12 and Lemma 5.3, (2)} \\
=& a|_{(V \times_S U)^n_{V \times_S X}} \\
& \text{by Axiom 20.}
\end{aligned}$$

Hence

$$\begin{aligned}
&d_{n+1}(b)\\
=&((j_n^*)^{-1}\circ(p_n\circ j_n)^*\circ d_n)(a|_{(V\times_S U)^n_{V\times_S X}})+(-1)^n a|_{(V\times_S U)^n_{V\times_S X}}\\
=&(-1)^n a|_{(V\times_S U)^n_{V\times_S X}}.
\end{aligned}$$

So the complex is exact after Nisnevich sheafication. □

Then by the same proofs as in [DF17, 1.2.7-1.2.11], we have the following result:

Proposition 5.11. *1. The forgetful functor $\widetilde{o}: \widetilde{Sh}(S) \longrightarrow \widetilde{PSh}(S)$ has a left adjoint $\widetilde{a}$ such that the following diagram commutes:*

$$\begin{tikzcd}
PSh(S) \arrow[d, "a"'] & \widetilde{PSh}(S) \arrow[l, "\widetilde{\gamma}_*"'] \arrow[d, "\widetilde{a}"] \\
Sh(S) & \widetilde{Sh}(S), \arrow[l, "\widetilde{\gamma}_*"']
\end{tikzcd}$$

where a is the Nisnevich sheafication functor with respect to the smooth site over S (Section 7.2 for notations).

2. *The category $\widetilde{Sh}(S)$ is a Grothendieck abelian category and the functor $\widetilde{a}$ is exact.*

3. *The functor $\widetilde{\gamma}_*$ appearing in the lower line of the preceding diagram admits a left adjoint $\widetilde{\gamma}^*$ and commutes with every limits and colimits.*

Proof. The same as [DF17, Proposition 1.2.11]. □

Definition 5.6. *Given any $X \in Sm/S$, we define $\widetilde{\mathbb{Z}}_S(X) = \widetilde{a}(\widetilde{c}_S(X))$ and we denote $\widetilde{\mathbb{Z}}_S(S)$ by $\mathbb{1}_S$.*

Proposition 5.12. *Let $X \in Sm/S$ and $U_1 \cup U_2 = X$ be a Zariski covering. Then the following complex is exact as sheaves with E-transfers:*

$$0 \longrightarrow \widetilde{\mathbb{Z}}_S(U_1 \cap U_2) \longrightarrow \widetilde{\mathbb{Z}}_S(U_1) \oplus \widetilde{\mathbb{Z}}_S(U_2) \longrightarrow \widetilde{\mathbb{Z}}_S(X) \longrightarrow 0.$$

Proof. See [MVW06, Proposition 6.14] with use of Proposition 5.10. Note that this complex is left exact because for any open immersion $U \subseteq X$ in Sm/k, $\widetilde{c}_S(U)$ is a subsheaf of $\widetilde{c}_S(X)$ by Axiom 20. □

We are now going to define a tensor product on the category $\widetilde{Cor}_S$.

Definition 5.7. *Let $X_i, Y_i \in Sm/S$ for $i = 1, 2$. Let further $f_1 \in \widetilde{Cor}_S(X_1, Y_1)$ and $f_2 \in \widetilde{Cor}_S(X_2, Y_2)$. Set*

$$f_1 \times_S f_2 = p_1^* f_1 \cdot p_2^* f_2 \in \widetilde{Cor}(X_1 \times_S X_2, Y_1 \times_S Y_2)$$

where $p_i : X_1 \times_S X_2 \times_S Y_1 \times_S Y_2 \longrightarrow X_i \times_S Y_i, i = 1, 2$ are the projections. Here we have used the isomorphism $-T_{X_1X_2Y_1Y_2/X_1X_2} \longrightarrow -T_{X_1X_2Y_1Y_2/X_1X_2Y_2} - T_{X_1X_2Y_1Y_2/X_1X_2Y_1}$. We say that $f_1 \times_S f_2$ is the exterior product of f_1 and f_2.

To prove that the tensor product is well-defined, we need to verify the compatibility of the tensor products with compositions.

Lemma 5.4. *Let $X_i, Y_i, Z_i \in Sm/S$ for $i = 1, 2$. Let further $p^i_{13} : X_iY_iZ_i \longrightarrow X_iZ_i$, $a_i : X_1X_2Y_1Y_2Z_1Z_2 \longrightarrow X_iY_iZ_i$, $b_i : X_1X_2Z_1Z_2 \longrightarrow X_iZ_i$ and $p_{13} : X_1X_2Y_1Y_2Z_1Z_2 \longrightarrow X_1X_2Z_1Z_2$ be the projections. Suppose that $\alpha_i \in E^{d_{Y_i}+d_{Z_i}}_{C_i}((p^i_{13})^*v_i - T_{X_iY_iZ_i/X_iZ_i})$ where $C_i \in \mathscr{A}_S(X_i, Y_iZ_i)$ and $v_i \in \mathscr{P}_{X_iZ_i}$. Then we have*

$$b_1^*p^1_{13*}(\alpha_1) \cdot b_2^*p^2_{13*}(\alpha_2) = p_{13*}(a_1^*(\alpha_1) \cdot a_2^*(\alpha_2)),$$

where we have used the isomorphism (exchanging the middle two terms and then merging the last two terms) from

$$a_1^*(p^1_{13})^*v_1 - T_{X_1X_2Y_1Y_2Z_1Z_2/X_1X_2Y_2Z_1Z_2} + a_2^*(p^2_{13})^*v_2 - T_{X_1X_2Y_1Y_2Z_1Z_2/X_1X_2Y_1Z_1Z_2}$$

to

$$p_{13}^*(b_1^*(v_1) + b_2^*(v_2)) - T_{X_1X_2Y_1Y_2Z_1Z_2/X_1X_2Z_1Z_2}$$

in the right hand side.

Proof. We have two Cartesian squares

$$\begin{tikzcd}
X_2Y_2Z_2 \arrow[r, "p^2_{13}"] & X_2Z_2 & X_1Y_1Z_1 \arrow[r, "p^1_{13}"] & X_1Z_1 \\
X_1X_2Y_1Y_2Z_1Z_2 \arrow[u, "a_2"] \arrow[r, "q"] & X_1X_2Y_1Z_1Z_2 \arrow[u, "p_{25}"'] & X_1X_2Y_1Z_1Z_2 \arrow[u, "p"] \arrow[r, "p_{1245}"] & X_1X_2Z_1Z_2 \arrow[u, "b_1"']
\end{tikzcd}$$

and equations $p_{25} = b_2 \circ p_{1245}$, $p \circ q = a_1$ and $p_{13} = p_{1245} \circ q$. Then we have

$$\begin{aligned}
&b_1^*p^1_{13*}((\alpha_1, (p^1_{13})^*v_i - T_{X_1Y_1Z_1/X_1Z_1})) \cdot b_2^*p^2_{13*}(\alpha_2)\\
=&(p_{1245})_*p^*(\alpha_1) \cdot b_2^*p^2_{13*}(\alpha_2)\\
&\text{by Axiom 15 for the right square above}\\
=&(p_{1245})_*(p^*(\alpha_1) \cdot p_{1245}^*b_2^*p^2_{13*}(\alpha_2))\\
&\text{by Axiom 17 for } p_{1245}\\
=&(p_{1245})_*(p^*(\alpha_1) \cdot p_{25}^*p^2_{13*}(\alpha_2))\\
&\text{by Axiom 9}\\
=&(p_{1245})_*(p^*(\alpha_1) \cdot q_*a_2^*(\alpha_2))\\
&\text{by Axiom 15 for the left square above}\\
=&(p_{1245})_*q_*(q^*p^*(\alpha_1) \cdot a_2^*(\alpha_2))\\
&\text{by Axiom 17 for } q\\
=&(p_{1245})_*q_*(a_1^*(\alpha_1) \cdot a_2^*(\alpha_2))\\
&\text{by Axiom 9}\\
=&p_{13*}(a_1^*(\alpha_1) \cdot a_2^*(\alpha_2))\\
&\text{by Axiom 12.}
\end{aligned}$$

□

Proposition 5.13. *Let $X_i, Y_i, Z_i \in Sm/S$, $f_i \in \widetilde{Cor}_S(X_i, Y_i)$, $g_i \in \widetilde{Cor}_S(Y_i, Z_i)$ where $i = 1, 2$. Then*

$$(g_1 \times_S g_2) \circ (f_1 \times_S f_2) = (g_1 \circ f_1) \times_S (g_2 \circ f_2).$$

Proof. We have a commutative diagram ($i = 1, 2$)

$$\begin{tikzcd}
Y_1Y_2Z_1Z_2 \arrow[r, "q_i\times r_i"] & Y_iZ_i & \\
X_1X_2Y_1Y_2Z_1Z_2 \arrow[u, "p_{23}"] \arrow[r, "a_i"] \arrow[d, "p_{12}"] \arrow[drr, "p_{13}"] & X_iY_iZ_i \arrow[u, "p^i_{23}"'] \arrow[r, "p^i_{13}"] \arrow[d, "p^i_{12}"'] & X_iZ_i \\
X_1X_2Y_1Y_2 \arrow[r, "p_i\times q_i"] & X_iY_i & X_1X_2Z_1Z_2. \arrow[u, "b_i"']
\end{tikzcd}$$

Then

$$
\begin{aligned}
&(g_1 \times_S g_2) \circ (f_1 \times_S f_2) \\
=&p_{13*}(p_{23}^*((q_1 \times r_1)^* g_1 \cdot (q_2 \times r_2)^* g_2) \cdot p_{12}^*((q_1 \times r_1)^* f_1 \cdot (q_2 \times r_2)^* f_2) \\
&\text{by definition} \\
=&p_{13*}(a_1^*(p_{23}^1)^*(g_1) \cdot a_2^*(p_{23}^2)^*(g_2) \cdot a_1^*(p_{12}^1)^*(f_1) \cdot a_2^*(p_{12}^2)^*(f_2)) \\
&\text{by Axiom 10 and Axiom 9.} \\
=&p_{13*}(c(a_1^*(p_{23}^1)^*(g_1) \cdot a_1^*(p_{12}^1)^*(f_1) \cdot a_2^*(p_{23}^2)^*(g_2) \cdot a_2^*(p_{12}^2)^*(f_2))) \\
&\text{by Axiom 6 and Axiom 16. Here } c = c(a_1^*(p_{12}^1)^*(-T_{X_1Y_1/X_1}), a_2^*(p_{23}^2)^*(-T_{Y_2Z_2/Y_2})) \\
=&p_{13*}(c(a_1^*((p_{23}^1)^*(g_1) \cdot (p_{12}^1)^*(f_1)) \cdot a_2^*((p_{23}^2)^*(g_2) \cdot (p_{12}^2)^*(f_2)))) \\
&\text{by Axiom 10} \\
=&b_1^* p_{13*}^1((p_{23}^1)^*(g_1) \cdot (p_{12}^1)^*(f_1)) \cdot b_2^* p_{13*}^2((p_{23}^2)^*(g_2) \cdot (p_{12}^2)^*(f_2)) \\
&\text{by Lemma 5.4} \\
=&b_1^*(g_1 \circ f_1) \cdot b_2^*(g_2 \circ f_2) \\
&\text{by definition} \\
=&(g_1 \circ f_1) \times_S (g_2 \circ f_2) \\
&\text{by definition.}
\end{aligned}
$$

□

Now that we proved that the category $\widetilde{Cor}_S$ has a tensor product, we review some basic constructions that will be useful later.

For any $F \in \widetilde{PSh}(S)$ and $X \in Sm/S$, we define $F^X \in \widetilde{PSh}(S)$ by $F^X(Y) = F(X \times_S Y)$. If $F \in \widetilde{Sh}(S)$, then it's clear that $F^X \in \widetilde{Sh}(S)$ also. We define C_*F for any $F \in \widetilde{Sh}(S)$ to be the complex with $(C_*F)_n = F^{\triangle^n}$ as in [MVW06, Definition 2.14] and differentials as usual.

A pointed scheme is a pair (X, x) where $X \in Sm/S$ and $x : S \longrightarrow X$ is a S-rational point. We define $\widetilde{\mathbb{Z}}_S((X_1, x_1) \wedge \ldots \wedge (X_n, x_n))$ for pointed schemes (X_i, x_i) as the cokernel of the map

$$\theta_n : \oplus_i \widetilde{\mathbb{Z}}_S(X_1 \times \ldots \times \widehat{X_i} \times \ldots \times X_n) \xrightarrow{\sum(-1)^{i-1} id \times \ldots \times x_i \times \ldots \times id} \widetilde{\mathbb{Z}}_S(X_1 \times \ldots \times X_n)\,.$$

We denote $\widetilde{\mathbb{Z}}_S((X, x) \wedge \ldots \wedge (X, x))$ by $\widetilde{\mathbb{Z}}_S((X, x)^{\wedge n})$ and $\widetilde{\mathbb{Z}}_S(X, x)$ by $\widetilde{\mathbb{Z}}_S((X, x)^{\wedge 1})$. Then we define $\widetilde{\mathbb{Z}}_S(q) = \widetilde{\mathbb{Z}}_S((\mathbb{G}_m, 1)^{\wedge q})[-q]$ for $q \geq 0$ and we set $\widetilde{\mathbb{Z}}_S(S) = \widetilde{\mathbb{Z}}_S = \widetilde{\mathbb{Z}}_S(0) = \mathbb{1}_S$. Following the notation in [MVW06, Lemma 2.13], we let $[x_i]$ be the composite

$$X_i \longrightarrow S \xrightarrow{x_i} X_i$$

and $e_i \in \widetilde{Cor}_S(X_i, X_i)$ to be $id_{X_i} - \widetilde{\mathbb{Z}}_S([x_i])$.

Lemma 5.5. *For $n \geq 2$, the sheaf $\widetilde{\mathbb{Z}}_S((X_1, x_1) \wedge \ldots \wedge (X_n, x_n))$ is just the image of the map*

$$e_1 \times \ldots \times e_n : \widetilde{\mathbb{Z}}_S(X_1 \times \ldots \times X_n) \longrightarrow \widetilde{\mathbb{Z}}_S(X_1 \times \ldots \times X_n).$$

Moreover, the inclusion of $\widetilde{\mathbb{Z}}_S((X_1, x_1) \wedge \ldots \wedge (X_n, x_n))$ into $\widetilde{\mathbb{Z}}_S(X_1 \times \ldots \times X_n)$ as an image is a section of $e_1 \times \ldots \times e_n$.

Proof. We prove the same statements after replacing $\widetilde{\mathbb{Z}}_S$ by $\widetilde{c}_S$ and then sheafify. The first statement is tantamount to $Ker(e_1 \times \ldots \times e_n) = Im(\theta_n)$. Now, $Im(\theta_n) \subseteq Ker(e_1 \times \ldots \times e_n)$ because $e_i \circ [x_i] = 0$. On the other hand, $Ker(e_1 \times \ldots \times e_n) \subseteq Im(\theta_n)$ because

$$e_1 \times \ldots \times e_n = id_{X_1 \times \ldots \times X_n} + \sum f_{\alpha_1} \times \ldots \times f_{\alpha_n},$$

where for every n-tuple $(\alpha_1, \dots, \alpha_n)$, there exists (at least) α_i such that $f_{\alpha_i} = -[x_i]$. It follows that $f_{\alpha_1} \times \dots \times f_{\alpha_n}$ factors through $id \times \dots \times x_i \times \dots \times id$ for that i.

The second statement follows from the fact that e_i is idempotent. □

By the lemma above, we can consider the sheaf $\widetilde{\mathbb{Z}}_S((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$ as a subsheaf of $\widetilde{\mathbb{Z}}_S(X_1 \times \cdots \times X_n)$.

Lemma 5.6. *For any two pointed schemes (X_1, x_1), (X_2, x_2), we have a split exact sequence*

$$0 \longrightarrow \widetilde{\mathbb{Z}}_S((X_1, x_1) \wedge (X_2, x_2)) \longrightarrow \widetilde{\mathbb{Z}}_S(X_1 \times_S X_2, (x_1, x_2)) \longrightarrow \widetilde{\mathbb{Z}}_S(X_1, x_1) \oplus \widetilde{\mathbb{Z}}_S(X_2, x_2) \longrightarrow 0.$$

Proof. A direct computation yields the following split short exact sequence:

$$\begin{tikzcd} \widetilde{\mathbb{Z}}_S \oplus \widetilde{\mathbb{Z}}_S((X_1, x_1) \wedge (X_2, x_2)) \arrow[r, shift right, "(x_1,x_2)+id"'] & \widetilde{\mathbb{Z}}_S(X_1 \times_S X_2) \arrow[l, shift right, "(\pi, e_1 \times e_2)"'] \arrow[r, shift right, "(-e_1 \circ p_1, e_2 \circ p_2)"'] & \widetilde{\mathbb{Z}}_S(X_1, x_1) \oplus \widetilde{\mathbb{Z}}_S(X_2, x_2) \arrow[l, shift right, "-(id_{X_1}, x_2) + (x_1, id_{X_2})"'] \end{tikzcd},$$

where $\pi : X_1 \times_S X_2 \longrightarrow S$ is the structure map. The result follows from this seqeuence, after quotienting the first two terms above with $\widetilde{\mathbb{Z}}_S$ (sometimes called 'killing one point'). □

The following definitions comes from [SV00, Lemma 2.1].

Definition 5.8. *Let $n \geq 2$ and let $F_i, G \in \widetilde{PSh}(S)$ for $i = 1, \cdots, n$. A multilinear function $\varphi : F_1 \times \cdots \times F_n \longrightarrow G$ is a collection of multilinear maps of abelian groups*

$$\varphi_{(X_1, \cdots, X_n)} : F_1(X_1) \times \cdots \times F_n(X_n) \longrightarrow G(X_1 \times_S \cdots \times_S X_n)$$

for every $X_i \in Sm/S$, such that for every $f \in \widetilde{Cor}_S(X_i, X_i')$, we have a commutative diagram

$$\begin{tikzcd} \cdots \times F_i(X_i') \times \cdots \arrow[r, "\varphi(\cdots, X_i', \cdots)"] \arrow[d, "\cdots \times F(f) \times \cdots"'] & G(\cdots \times_S X_i' \times_S \cdots) \arrow[d, "G(\cdots \times f \times \cdots)"'] \\ \cdots \times F_i(X_i) \times \cdots \arrow[r, "\varphi(\cdots, X_i, \cdots)"] & G(\cdots \times_S X_i \times_S \cdots) \end{tikzcd}$$

Definition 5.9. *Let $n \geq 2$ be an integer and let $F_i, G \in \widetilde{PSh}(S)$ (resp. $\widetilde{Sh}(S)$) for $i = 1, \cdots, n$. The tensor product $F_1 \otimes_S^{pr} \cdots \otimes_S^{pr} F_n$ (resp. $F_1 \otimes_S \cdots \otimes_S F_n$) is the presheaf (resp. sheaf) with E-transfers G such that for any $H \in \widetilde{PSh}(S)$ (resp. $\widetilde{Sh}(S)$), we have*

$$Hom_S(G, H) \cong \{\textit{Multilinear functions } F_1 \times \cdots \times F_n \longrightarrow H\}$$

naturally.

For any $F, G \in \widetilde{PSh}(S)$, we can construct $F \otimes_S^{pr} G \in \widetilde{PSh}(S)$ as in the discussion before [SV00, Lemma 2.1]. Moreover, we define $\underline{Hom}_S(F, G)$ to be the presheaf with E-transfers which sends $X \in Sm/S$ to $Hom_S(F, G^X)$. If F, G are sheaves with E-transfers, we set $F \otimes_S G = \widetilde{a}(F \otimes_S^{pr} G)$. If G is a sheaf with E-transfers, it's clear that $\underline{Hom}_S(F, G)$ is also a sheaf with E-transfers. Finally, it's clear from the definition that $F \otimes_S^{pr} G \cong G \otimes_S^{pr} F$ and $F \otimes_S G \cong G \otimes_S F$.

Proposition 5.14. *For any $F, G, H \in \widetilde{PSh}(S)$, we have isomorphisms*

$$Hom_S(F \otimes_S^{pr} G, H) \cong Hom_S(F, \underline{Hom}_S(G, H)),$$

$$Hom_S(F \otimes_S^{pr} G, H) \cong Hom_S(G, \underline{Hom}_S(F, H))$$

functorial in three variables. Simililarly, for any $F, G, H \in \widetilde{Sh}(S)$, we have isomorphisms

$$Hom_S(F \otimes_S G, H) \cong Hom_S(F, \underline{Hom}_S(G, H)),$$

$$Hom_S(F \otimes_S G, H) \cong Hom_S(G, \underline{Hom}_S(F, H))$$

functorial in three variables.

Proof. This is clear from the definition of the bilinear map. □

If $F, G, H \in \widetilde{Sh}(S)$, it's easy to see using the above proposition that $(F \otimes_S G) \otimes_S H$ and $F \otimes_S (G \otimes_S H)$ are both isomorphic to $F \otimes_S G \otimes_S H$. It folllows that the tensor product defined above is associative. Finally, one checks that $\otimes_S$ (resp. $\otimes_S^{pr}$) endows $\widetilde{Sh}(S)$ (resp. $\widetilde{PSh}(S)$) with a symmetric closed monoidal structure.

Proposition 5.15. *If a morphism $f : F_1 \longrightarrow F_2$ of presheaves with E-transfers becomes an isomorphism after sheafifying, then so does the morphism $f \otimes_S^{pr} G$ for any presheaf with E-transfers G.*

Proof. The condition is equivalent to the map $Hom_S(f, H)$ being an isomorphism for any sheaf with E-transfers H. Now, we have

$$Hom_S(f \otimes_S^{pr} G, H) \cong Hom_S(f, \underline{Hom}_S(G, H))$$

by the proposition above. □

Proposition 5.16. *1. For any $X, Y \in Sm/S$, we have*

$$\widetilde{\mathbb{Z}}_S(X) \otimes_S \widetilde{\mathbb{Z}}_S(Y) \cong \widetilde{\mathbb{Z}}_S(X \times_S Y)$$

as sheaves with E-transfers.

2. For any two pointed schemes (X_1, x_1) and (X_2, x_2), we have

$$\widetilde{\mathbb{Z}}_S(X_1, x_1) \otimes_S \widetilde{\mathbb{Z}}_S(X_2, x_2) \cong \widetilde{\mathbb{Z}}_S((X_1, x_1) \wedge (X_2, x_2))$$

as sheaves with E-transfers.

Proof. We have $\widetilde{c}_S(X) \otimes_S^{pr} \widetilde{c}_S(Y) \cong \widetilde{c}_S(X \times_S Y)$ using the exterior products of correspondences. Then the statement follows by Proposition 5.15. The second statement follows by a similar method. □

Now we are going to prove some functorial properties of sheaves with E-transfers over different bases. Our approach is quite similar as [D07]. The following lemma is useful when constructing adjunctions, see [Ayo07, Definition 4.4.1] and [D07, 2.5.1].

Lemma 5.7. *Let $\varphi : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor between small categories and $\mathscr{M}$ be a category with arbitrary colimits. Then the functor*

$$\varphi_* : PreShv(\mathscr{D}, \mathscr{M}) \longrightarrow PreShv(\mathscr{C}, \mathscr{M})$$

defined by $\varphi_*(F) = F \circ \varphi$ has a left adjoint φ^*.

Proof. Suppose $G \in PreShv(\mathscr{C}, \mathscr{M})$. For every object $Y \in \mathscr{D}$, define C_Y to be the category whose objects are $Hom_{\mathscr{D}}(Y, \varphi(X))$ and morphisms from $a_1 : Y \longrightarrow \varphi(X_1)$ to $a_2 : Y \longrightarrow \varphi(X_2)$ are $b \in Hom_{\mathscr{C}}(X_1, X_2)$ such that $a_2 = \varphi(b) \circ a_1$. We have a contravariant functor

$$\theta_Y : C_Y \longrightarrow \mathscr{M}$$

defined by $\theta_Y(Y \longrightarrow \varphi(X)) = GX$. Then define $(\varphi^* G)Y = \varinjlim \theta_Y$. For any morphism $c : Y_1 \longrightarrow Y_2$ in $\mathscr{D}$, we define $(\varphi^* G)(c)$ using the following commutative diagram

$$\begin{tikzcd} \theta_{Y_2}(a) \arrow[d, "i_a"] \arrow[dr, "i_{a\circ c}"] & \\ \varinjlim \theta_{Y_2} \arrow[r, "(\varphi^* G)(c)"'] & \varinjlim \theta_{Y_1} \end{tikzcd}$$

for every $a : Y_2 \longrightarrow \varphi(X)$. One checks it is just what we want. $\square$

Definition 5.10. *Suppose that $f : S \longrightarrow T$ is a morphism in Sm/k. For any $X \in Sm/T$, set $X^S = X \times_T S \in Sm/S$. For any $X_1, X_2 \in Sm/T$, denote by p_f the projection $(X_1 \times_T X_2)^S \longrightarrow X_1 \times_T X_2$. Define*

$$\begin{array}{cccc} \varphi^f : & \widetilde{Cor}_T & \longrightarrow & \widetilde{Cor}_S \\ & X & \longmapsto & X^S \\ & g & \longmapsto & g^S \end{array},$$

where $g \longmapsto g^S : \widetilde{Cor}_T(X_1, X_2) \longrightarrow \widetilde{Cor}_S(X_1^S, X_2^S)$ is the unique map such that the following diagram commutes

$$\begin{tikzcd} E_Z^{d_{X_2}-d_T}(X_1 \times_T X_2, -T_{X_1 \times_T X_2/X_1}) \arrow[r, "p_f^*"] \arrow[d] & E_{p_f^{-1}(Z)}^{d_{X_2}-d_S}((X_1 \times_T X_2)^S, -T_{(X_1 \times_T X_2)^S/X_1^S}) \arrow[d] \\ \widetilde{Cor}_T(X_1, X_2) \arrow[r, "\varphi^f"] & \widetilde{Cor}_S(X_1^S, X_2^S) \end{tikzcd},$$

for any $Z \in \mathscr{A}_T(X_1, X_2)$.

Proposition 5.17. *Suppose that $X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} X_3$ are morphisms in $\widetilde{Cor}_T$. Then*

$$(g_2 \circ g_1)^S = g_2^S \circ g_1^S.$$

So $\varphi^f : \widetilde{Cor}_T \longrightarrow \widetilde{Cor}_S$ is indeed a functor.

Proof. We have diagrams

$$\begin{tikzcd} & X_1 \times_T X_2 \\ X_1 \times_T X_3 & X_1 \times_T X_2 \times_T X_3 \arrow[l, "p_{13}"'] \arrow[u, "p_{12}"'] \arrow[d, "p_{23}"] \\ & X_2 \times_T X_3, \end{tikzcd} \qquad \begin{tikzcd} & X_1^S \times_S X_2^S \\ X_1^S \times_S X_3^S & X_1^S \times_S X_2^S \times_S X_3^S \arrow[l, "q_{13}"'] \arrow[u, "q_{12}"'] \arrow[d, "q_{23}"] \\ & X_2^S \times_S X_3^S, \end{tikzcd}$$

and three Cartesian squares

$$\begin{tikzcd}
 & X_2 \times_S X_3 & \\
X_1 \times_T X_3 & X_1 \times_T X_2 \times_T X_3 \arrow[l, "p_{13}"'] \arrow[r, "p_{12}"] \arrow[u, "p_{23}"] & X_1 \times_S X_2 \\
X_1^S \times_S X_3^S \arrow[u, "r"] & X_1^S \times_S X_2^S \times_S X_3^S \arrow[l, "q_{13}"'] \arrow[r, "q_{12}"] \arrow[u, "t"] \arrow[d, "q_{23}"] & X_1^S \times_S X_2^S \arrow[u, "p"] \\
 & X_2^S \times_S X_3^S . \arrow[uuu, bend left, "q"] &
\end{tikzcd}$$

Suppose that g_1 and $g_2$are supported on some admissible subsets. We have

$$\begin{aligned}
& (g_2 \circ g_1)^S \\
=& r^* p_{13*}(p_{23}^*(g_2) \cdot p_{12}^*(g_1)) \\
& \text{by definition} \\
=& q_{13*} t^* (p_{23}^*(g_2) \cdot p_{12}^*(g_1)) \\
& \text{by Axiom 15 for the left square above} \\
=& q_{13*}(q_{12}^* p^*(g_2) \cdot q_{23}^* q^*(g_1)) \\
& \text{by Axiom 10 and Axiom 9} \\
=& g_2^S \circ g_1^S \\
& \text{by definition.}
\end{aligned}$$

It's then easy to verify that $\widetilde{\gamma}(id_Y)^S = \widetilde{\gamma}(id_{Y^S})$ for any $Y \in Sm/T$. So φ^f is a functor. □

It is straightforward to check that $\varphi^{f_1 \circ f_2} = \varphi^{f_2} \circ \varphi^{f_1}$.

Proposition 5.18. *Suppose* $f_i \in \widetilde{Cor}_T(X_i, Y_i)$ *where* $i = 1, 2$. *Then*

$$(f_1 \times_T f_2)^S = f_1^S \times_S f_2^S.$$

Proof. This follows from the commutative diagram

$$\begin{tikzcd}
 & (X_1Y_1X_2Y_2)^S \arrow[dl] \arrow[dr] \arrow[rrr, "p^f"] & & & X_1Y_1X_2Y_2 \arrow[dl] \arrow[dr] & \\
(X_1Y_1)^S \arrow[rrr, bend right, "p^f"'] & & (X_2Y_2)^S \arrow[rrr, bend right, "p^f"'] & X_1Y_1 & & X_2Y_2 .
\end{tikzcd}$$

□

Proposition 5.19. *In the notations above, we have an adjoint pair*

$$f^* : \widetilde{Sh}(T) \rightleftharpoons \widetilde{Sh}(S) : f_*$$

where $(f_*F)(X) = F \circ \varphi^f$ *for* $F \in \widetilde{Sh}(S)$.

Proof. Applying Lemma 5.7 to φ^f, we obtain an adjunction $\widetilde{PSh}(T) \rightleftharpoons \widetilde{PSh}(S)$.We may then apply the sheafication functor of Proposition 5.11 to get the desired result. □

Obviously, we have $(f_1 \circ f_2)^* = f_2^* \circ f_1^*$, $(f_1 \circ f_2)_* = f_{1*} \circ f_{2*}$.

Proposition 5.20. *Suppose that* $f : S \longrightarrow T$ *is a morphism in* Sm/k.

1. *For any $Y \in Sm/T$,*
$$f^*\widetilde{\mathbb{Z}}_T(Y) \cong \widetilde{\mathbb{Z}}_S(Y \times_T S)$$
as sheaves with E-transfers.

2. *For any $F \in \widetilde{Sh}(S)$ and $Y \in Sm/T$,*
$$(f_*F)^Y \cong f_*(F^{Y \times_T S})$$
as sheaves with E-transfers.

3. *For any $F \in \widetilde{Sh}(T)$ and $G \in \widetilde{Sh}(S)$,*
$$\underline{Hom}_T(F, f_*G) \cong f_*\underline{Hom}_S(f^*F, G)$$
as sheaves with E-transfers.

4. *For any $F, G \in \widetilde{Sh}(T)$, we have*
$$f^*F \otimes_S f^*G \cong f^*(F \otimes_T G)$$
as sheaves with E-transfers.

Proof. 1. We have
$$Hom_S(f^*\widetilde{\mathbb{Z}}_T(Y), -) \cong Hom_T(\widetilde{\mathbb{Z}}_T(Y), f_*-) \cong Hom_S(\widetilde{\mathbb{Z}}_S(Y \times_T S), -).$$

2. For any $Z \in Sm/T$, we get using Proposition 5.18
$$(f_*F)^Y(Z) = F((Y \times_T Z) \times_T S) \cong F((Z \times_T S) \times_S (Y \times_T S)) \cong (f_*(F^{Y \times_T S}))(Z).$$

3. For any $Y \in Sm/T$, we have
$$\begin{aligned}
\underline{Hom}_T(F, f_*G)(Y) &= Hom_T(F, (f_*G)^Y) \\
&\cong Hom_T(F, f_*(G^{Y \times_T S})) \\
&\quad \text{by (2)} \\
&\cong Hom_S(f^*F, G^{Y \times_T S}) \\
&= (f_*\underline{Hom}_S(f^*F, G))(Y).
\end{aligned}$$

4. For any $H \in \widetilde{Sh}(S)$,
$$\begin{aligned}
Hom_S(f^*F \otimes_S f^*G, H) &\cong Hom_S(f^*G, \underline{Hom}_S(f^*F, H)) \\
&\cong Hom_T(G, f_*\underline{Hom}_S(f^*F, H)) \\
&\cong Hom_T(G, \underline{Hom}_T(F, f_*H)) \\
&\quad \text{by (3)} \\
&\cong Hom_T(F \otimes_T G, f_*H) \\
&\cong Hom_S(f^*(F \otimes_T G), H).
\end{aligned}$$

$\square$

From now on in this chapter, we suppose that $f : S \longrightarrow T$ is a smooth morphism in Sm/k. Given such a morphism, we may consider any smooth S-scheme as a smooth T-scheme via f. Moreover, the fact that the diagonal map $S \to S \times_T S$ is a closed immersion implies that for any smooth S-schemes X_1 and X_2, the natural morphism

$$q_f : X_1 \times_S X_2 \to X_1 \times_T X_2$$

is a closed immersion. Indeed, this follows from the Cartesian square

$$\begin{tikzcd}
X_1 \times_S X_2 \arrow[r] \arrow[d] & X_1 \times_T X_2 \arrow[d] \\
S \arrow[r] & S \times_T S.
\end{tikzcd}$$

Definition 5.11. *For $X_1, X_2 \in Sm/S$, we define*

$$\begin{array}{rccc}
\varphi_f : & \widetilde{Cor}_S & \longrightarrow & \widetilde{Cor}_T \\
& X & \longmapsto & X \\
& g & \longmapsto & g_T
\end{array},$$

where $g \longmapsto g_T : \widetilde{Cor}_S(X_1, X_2) \longrightarrow \widetilde{Cor}_T(X_1, X_2)$ is the unique map such that the following diagram commutes

$$\begin{tikzcd}
E_Z^{d_{X_1}-d_S}(X_1 \times_S X_2 - T_{X_1 \times_S X_2/X_1}) \arrow[r, "q_{f*} \circ t_f"] \arrow[d] & E_{q_f(Z)}^{d_{X_1}-d_T}(X_1 \times_T X_2, -T_{X_1 \times_T X_2/X_1}) \arrow[d] \\
\widetilde{Cor}_T(X_1, X_2) \arrow[r, "\varphi_f"] & \widetilde{Cor}_S(X_1, X_2)
\end{tikzcd}$$

for any $Z \in \mathscr{A}_S(X_1, X_2)$. Here, t_f is the isomorphism

$$\begin{aligned}
& - T_{X_1 \times_S X_2/X_1} \\
\longrightarrow & N_{(X_1 \times_S X_2)/(X_1 \times_T X_2)} - N_{(X_1 \times_S X_2)/(X_1 \times_T X_2)} - T_{X_1 \times_S X_2/X_1} \\
\longrightarrow & N_{(X_1 \times_S X_2)/(X_1 \times_T X_2)} - q_f^* T_{X_1 \times_T X_2/X_1}.
\end{aligned}$$

For convenience of notation, we denote $T_{X/Y}$ by T_f for a smooth morphism $f : X \longrightarrow Y$ and $N_{X/Y}$ by N_f for a closed immersion f in the following few propositions.

Proposition 5.21. *Suppose that $X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} X_3$ are morphisms in $\widetilde{Cor}_S$. Then, we have*

$$(g_2 \circ g_1)_T = g_{2T} \circ g_{1T}.$$

So $\varphi_f : \widetilde{Cor}_S \longrightarrow \widetilde{Cor}_T$ is indeed a functor.

Proof. We have Cartesian squares

$$\begin{tikzcd}
X_1 \times_S X_2 \arrow[r, "i"] & X_1 \times_T X_2 \\
X_1 \times_S X_2 \times_S X_3 \arrow[r, "i'"] \arrow[u, "q'_{12}"] & X_1 \times_T (X_2 \times_S X_3), \arrow[u, "q_{12}"']
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \times_T (X_2 \times_S X_3) \arrow[r, "q"] \arrow[d, "r"'] & X_1 \times_T X_2 \times_T X_3 \arrow[d, "p_{23}"] \\
X_2 \times_S X_3 \arrow[r, "j"] & X_1 \times_T X_3,
\end{tikzcd}$$

$$\begin{tikzcd} X_1 \times_S X_2 \times_S X_3 \arrow[r] \arrow[d, "i'"'] & (X_1 \times_S X_2) \times_T X_3 \arrow[d] \\ X_1 \times_T (X_2 \times_S X_3) \arrow[r, "q"] & X_1 \times_T X_2 \times_T X_3, \end{tikzcd}$$

$$\begin{tikzcd} X_1 \times_S X_2 \times_S X_3 \arrow[r, "q'_{13}"] \arrow[d, "i'"'] & X_1 \times_S X_3 \arrow[d, "k"'] \\ X_1 \times_T (X_2 \times_S X_3) \arrow[r, "p_{13}\circ q"] & X_1 \times_T X_3, \end{tikzcd}$$

and commutative diagrams

$$\begin{tikzcd} X_1 \times_T X_2 & \\ X_1 \times_T (X_2 \times_S X_3) \arrow[u, "q_{12}"] \arrow[r, "q"] & X_1 \times_T X_2 \times_T X_3, \arrow[ul, "p_{12}"'] \end{tikzcd}$$

$$\begin{tikzcd} X_1 \times_S X_2 \times_S X_3 \arrow[r, "i'"] \arrow[dr, "q'_{23}"'] & X_1 \times_T (X_2 \times_S X_3) \arrow[d, "r"'] \\ & X_2 \times_S X_3. \end{tikzcd}$$

For g_1 and g_2 supported on admissible subsets, we have

$$\begin{aligned}
& g_{2T} \circ g_{1T} \\
=& j_* t_f(g_2) \circ i_* t_f(g_1) \\
& \text{by definition} \\
=& p_{13*}(p_{23}^* j_* t_f(g_2) \cdot p_{12}^* i_* t_f(g_1)) \\
& \text{by definition} \\
=& p_{13*}(q_* r^* t_f(g_2) \cdot p_{12}^* i_* t_f(g_1)) \\
& \text{by Axiom 16 for the second square above} \\
=& p_{13*} q_*(r^* t_f(g_2) \cdot q^* p_{12}^* i_* t_f(g_1)) \\
& \text{by Axiom 18 for } q \\
=& p_{13*} q_*(r^* t_f(g_2) \cdot q_{12}^* i_* t_f(g_1)) \\
& \text{by Axiom 9} \\
=& p_{13*} q_*(r^* t_f(g_2) \cdot i'_* q_{12}'^* t_f(g_1)) \\
& \text{by Axiom 16 for the first square above} \\
=& p_{13*} q_* i'_*(i'^* r^* t_f(g_2) \cdot q_{12}'^* t_f(g_1)) \\
& \text{by Axiom 18 for } i' \\
=& p_{13*} q_* i'_*((q_{23}'^* t_f(g_2) \cdot q_{12}'^* t_f(g_1), i'^* N_q - i'^* q^* p_{13}^* T_{X_1 \times_T X_3/X_1} + N_{i'} - i'^* q^* T_{p_{13}})) \\
& \text{by Axiom 9} \\
=& (p_{13} \circ q)_* i'_*((q_{23}'^* t_f(g_2) \cdot q_{12}'^* t_f(g_1), -i'^* q^* p_{13}^* T_{X_1 \times_T X_3/X_1} + N_{i'} - i'^* T_{p_{13}\circ q})) \\
& \text{by Axiom 19, (1) and functoriality of push-forwards with respect to twists} \\
=& k_* q'_{13*}((q_{23}'^* t_f(g_2) \cdot q_{12}'^* t_f(g_1), -q_{13}'^* k^* T_{X_1 \times_T X_3/X_1} + q_{13}'^* N_k - T_{q'_{13}})) \\
& \text{by Axiom 19, (3) for the last square above.}
\end{aligned}$$

Now, we have to treat the twists. We say that a morphism $f : A + B \longrightarrow C + D$ in a Picard category contains a switch if there are morphisms $g : A \longrightarrow D$ and $h : B \longrightarrow C$ such that $f = c(D, C) \circ (g + h)$. Conversely, we say that it doesn't contain a switch if there are morphisms $g : A \longrightarrow C$ and $h : B \longrightarrow D$ such that $f = g + h$. We have a commutative diagram in which the three squares are Cartesian

$$\begin{tikzcd}
X_1 \times_S X_2 \times_S X_3 \arrow[rr, "q'"] \arrow[dr, "u"] \arrow[dd, "i'"'] & & (X_1 \times_S X_2) \times_T X_3 \arrow[dd, "i''"] \\
& (X_1 \times_S X_3) \times_T X_2 \arrow[dr, "v"] & \\
X_1 \times_T (X_2 \times_S X_3) \arrow[rr, "q"] & & X_1 \times_T X_2 \times_T X_3.
\end{tikzcd}$$

This induces a commutative diagram (in which all arrows contain a switch)

$$\begin{tikzcd}
N_{q'} + q'^* N_{i''} \arrow[r] \arrow[d] & N_u + u^* N_v \arrow[dl] \\
N_{i'} + i'^* N_q &
\end{tikzcd}$$

since they all come from exact sequences related to $N_{i'' \circ q'} = N_{v \circ u} = N_{q \circ i'}$. Then we have a commutative diagram (no arrow contains a switch except φ)

$$\begin{tikzcd}
q'^* N_{i''} + i'^* N_q \arrow[r] \arrow[d] & N_u + N_{q'} \arrow[r] \arrow[d] & i'^* N_q + u^* N_v \arrow[d] \\
q'^* N_{i''} + N_{q'} \arrow[r] \arrow[rr, bend right, "\varphi"] & N_u + u^* N_v \arrow[r] & i'^* N_q + N_{i'}
\end{tikzcd}$$

by the diagram above. Hence the composite

$$q'^* N_{i''} + i'^* N_q \longrightarrow N_u + N_{q'} \longrightarrow i'^* N_q + u^* N_v$$

is equal to the morphism with a switch

$$q'^* N_{i''} + i'^* N_q \longrightarrow i'^* N_q + u^* N_v$$

where the morphism $q'^* N_{i''} \longrightarrow u^* N_v$ is given by the composite

$$q'^* N_{i''} \longrightarrow N_{i'} \longrightarrow u^* N_v.$$

So the composite (in which morphisms are without switch)

$$q'^*_{23} N_j + q^*_{12} N_i \longrightarrow N_{q'} + N_u \longrightarrow q'^*_{13} N_k + i'^* N_q$$

is equal to the morphism with a switch

$$q'^*_{23} N_j + q^*_{12} N_i \longrightarrow q'^*_{13} N_k + i'^* N_q$$

where the morphism $q^*_{12} N_i \longrightarrow q'^*_{13} N_k$ is given by the composite

$$q^*_{12} N_i \longrightarrow N_{i'} \longrightarrow q'^*_{13} N_k$$

and the morphism $q'^*_{23} N_j \longrightarrow i'^* N_q$ is obtained by pulling back the morphism

$$r^* N_j \longrightarrow N_q$$

along i'.

Moreover, there are commutative diagrams with Cartesian squares

$$\begin{tikzcd}
X_1 \times_S X_2 \arrow[r, "i"] & X_1 \times_T X_2 \arrow[r] & X_1 \\
X_1 \times_S X_2 \times_S X_3 \arrow[u, "q'_{12}"] \arrow[d, "i'"] \arrow[r, "u"] & (X_1 \times_S X_3) \times_T X_2 \arrow[u] \arrow[d] \arrow[r] & X_1 \times_S X_3 \arrow[u] \arrow[d] \\
X_1 \times_T (X_2 \times_S X_3) \arrow[r, "q"] & X_1 \times_T X_3 \times_T X_2 \arrow[r] & X_1 \times_T X_3
\end{tikzcd}$$

and

$$\begin{tikzcd}
X_2 \times_S X_3 \arrow[r, "j"] & X_2 \times_T X_3 \arrow[r] & X_2 \\
X_1 \times_S X_2 \times_S X_3 \arrow[u, "q'_{23}"] \arrow[d, "q'_{13}"] \arrow[r, "q'"] & (X_1 \times_S X_2) \times_T X_3 \arrow[u] \arrow[d] \arrow[r] & X_1 \times_S X_2 \arrow[u] \arrow[d] \\
X_1 \times_S X_3 \arrow[r, "k"] & X_1 \times_T X_3 \arrow[r] & X_1
\end{tikzcd}$$

which induce commutative diagrams where the right-hand vertical maps contain no switch

$$\begin{tikzcd}
i^* q'^*_{12} T_{X_1 \times_T X_2 / X_1} \arrow[r] & q'^*_{12} T_{X_1 \times_S X_2 / X_1} + q'^*_{12} N_i \;, \\
u^* T_{(X_1 \times_S X_3) \times_T X_2 / X_1 \times_S X_3} \arrow[u] \arrow[d] \arrow[r] & T_{q'_{13}} + N_u \arrow[u] \arrow[d] \\
q^* T_{p_{13}} \arrow[r] & i'^* T_{p_{13} \circ q} + i'^* N_q
\end{tikzcd}$$

$$\begin{tikzcd}
q'^*_{23} j^* T_{X_2 \times_T X_3 / X_2} \arrow[r] & q'^*_{23} T_{X_2 \times_S X_3 / X_2} + q'^*_{23} N_j \\
q'^* T_{(X_1 \times_S X_2) \times_T X_3 / X_1 \times_S X_2} \arrow[u] \arrow[d] \arrow[r] & T_{q'_{12}} + N_{q'} \arrow[u] \arrow[d] \\
k^* T_{X_1 \times_T X_3 / X_1} \arrow[r] & q'^*_{13} T_{X_1 \times_S X_3 / X_1} + q'^*_{13} N_k .
\end{tikzcd}$$

These calculations above together with the functoriality of q'_{13*} with respect to twists yield

$$\begin{aligned}
& k_* q'_{13*}((q'^*_{23} t_f(g_2) \cdot q'^*_{12} t_f(g_1), -q'^*_{13} k^* T_{X_1 \times_T X_3 / X_1} + q'^*_{13} N_k - T_{q'_{13}})) \\
= & k_* t_f(q'_{13*}(q'^*_{23}(g_2) \cdot q'^*_{12}(g_1))) \\
= & (g_2 \circ g_1)_T
\end{aligned}$$

by definition.

Finally, we have to show that $(id_X)_T = id_X$ for any $X \in Sm/S$. We have the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\triangle_S"] \arrow[dr, "\triangle_T"'] & X \times_S X \arrow[r] \arrow[d, "q_f"] & X \\
& X \times_T X \arrow[ur] &
\end{tikzcd}$$

where $\triangle$ is the diagonal map. We have to show that the following diagram commutes

$$\begin{tikzcd}
N_{\triangle_S} - N_{\triangle_S} \arrow[r] & N_{\triangle_S} - \triangle_S^* T_{X\times_S X/X} \arrow[r, "\triangle_{S*}"] \arrow[d] & -T_{X\times_S X/X} \arrow[d, "t_f"] \\
0 \arrow[u] \arrow[d] & N_{\triangle_S} + \triangle_S^*(N_{q_f} - q_f^* T_{X\times_T X/X}) \arrow[r, "\triangle_{S*}"] \arrow[d] & N_{q_f} - q_f^* T_{X\times_T X/X} \arrow[d, "q_{f*}"] \\
N_{\triangle_T} - N_{\triangle_T} \arrow[r] & N_{\triangle_T} - \triangle_T^* T_{X\times_T X/X} \arrow[r, "\triangle_{T*}"] & -T_{X\times_T X/X}.
\end{tikzcd}$$

The right-hand squares commute by functoriality of the push-forwards with respect to twists and Axiom 14. The left square comes from the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N_{\triangle_S} \arrow[r] & N_{\triangle_T} \arrow[r] & \triangle_S^* N_{q_f} \arrow[r] & 0 \\
0 \arrow[r] & \triangle_S^* T_{X\times_S X/X} \arrow[r] \arrow[u, "\cong"] & \triangle_T^* T_{X\times_T X/X} \arrow[r] \arrow[u, "\cong"] & \triangle_S^* N_{q_f} \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd}$$

□

Using Axiom 14, it is straightforward to check that $\varphi_{f_1\circ f_2} = \varphi_{f_1} \circ \varphi_{f_2}$.

Proposition 5.22. *Let $a \in \widetilde{Cor}_S(X_1, X_2)$ and let $b \in \widetilde{Cor}_T(Y_1, Y_2)$. Identifying $(X_1 \times_S X_2) \times_T Y_1 \times_T Y_2$ with $X_1 \times_S X_2 \times_S (Y_1 \times_T Y_2)^S$ and $X_1 \times_T Y_1 \times_T X_2 \times_T Y_2$ with $(X_1 \times_S Y_1^S) \times_T (X_2 \times_S Y_2^S)$, we have*

$$a_T \times_T b = (a \times_S b^S)_T.$$

Proof. We have a commutative diagram in which the square is Cartesian

$$\begin{tikzcd}
(X_1 \times_S X_2) \times_T Y_1 \times_T Y_2 \arrow[r, "r"] \arrow[d, "p_1"] \arrow[dr, "p_2"] & X_1 \times_T Y_1 \times_T X_2 \times_T Y_2 \arrow[d, "q_1"] \arrow[dr, "q_2"] & \\
X_1 \times_S X_2 \arrow[r, "t"'] & X_1 \times_T X_2 & Y_1 \times_T Y_2.
\end{tikzcd}$$

Suppose that a, b are supported on admissible subsets. Denote by θ the isomorphism

$$-q_1^* T_{X_1\times_T X_2/X_1} - q_2^* T_{Y_1\times_T Y_2/Y_1} \longrightarrow -T_{X_1\times_T X_2\times_T Y_1\times_T Y_2/X_1\times_T Y_1}$$

and by η the isomorphism

$$-p_1^* T_{X_1\times_S X_2/X_1} - p_2^* T_{Y_1\times_T Y_2/Y_1} \longrightarrow -T_{X_1\times_S X_2\times_S Y_1^S\times_S Y_2^S/X_1\times_S Y_1^S}.$$

Then

$$
\begin{aligned}
&a_T \times_T b \\
=&\theta(q_1^* t_*(t_f(a)) \cdot q_2^* b) \\
&\text{by definition} \\
=&\theta(r_* p_1^*(t_f(a)) \cdot q_2^* b) \\
&\text{by Axiom 16 for the square in the diagram} \\
=&\theta(r_*(p_1^*(t_f(a)) \cdot p_2^* b)) \\
&\text{by Proposition 18 for } r \text{ and Axiom 10} \\
=&r_* r^*(\theta)((p_1^*(t_f(a)) \cdot p_2^* b, p_1^* N_t - p_1^* t^* T_{X_1 \times_T X_2/X_1} - p_2^* T_{Y_1 \times_T Y_2/Y_1})) \\
&\text{by functoriality of push-forwards with respect to twists} \\
=&r_*(t_f(\eta(p_1^*(a) \cdot p_2^* b))) \\
=&(a \times_S b^S)_T \\
&\text{by definition.}
\end{aligned}
$$

Here the fifth equality comes from the following commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & p_2^* T_{Y_1 \times_T Y_2/Y_1} \arrow[r, equal] \arrow[d] & p_2^* T_{Y_1 \times_T Y_2/Y_1} \arrow[d] & & \\
0 \arrow[r] & r^* T_{X_1 Y_1 X_2 Y_2/X_1 Y_1} \arrow[r] \arrow[d] & T_{X_1 X_2 Y_1^S Y_2^S/X_1 Y_1^S} \arrow[r] \arrow[d] & N_r \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & r^* q_1^* T_{X_1 \times_T X_2/X_1} \arrow[r] \arrow[d] & p_1^* T_{X_1 \times_S X_2/X_1} \arrow[r] \arrow[d] & p_1^* N_t \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

and Theorem 3.1, (1). □

Applying the same proof as in Proposition 5.19 to φ_f, we get the following result.

Proposition 5.23. *There is an adjoint pair*

$$f_\# : \widetilde{Sh}(S) \rightleftharpoons \widetilde{Sh}(T) : (f_\#)',$$

where $(f_\#)'F = F \circ \varphi_f$ *for* $F \in \widetilde{Sh}(T)$.

The next lemma is important when identifying $(f_\#)'$. See also [MVW06, Exercise 1.12].

Lemma 5.8. *For any* $U \in Sm/S$, $X \in Sm/T$, *we have an adjoint pair:*

$$\widetilde{Cor}_S(U, X^S) = \widetilde{Cor}_T(U, X).$$

Proof. For any $U \in Sm/S$, $X \in Sm/T$, we have an isomorphism

$$\theta_{U,X} : U \times_S X^S \longrightarrow U \times_T X.$$

We can then define

$$\begin{array}{rccc} \lambda_{U,X}: & \widetilde{Cor}_S(U, X^S) & \longrightarrow & \widetilde{Cor}_T(U, X) \\ & W & \longmapsto & \theta_{U,X*}(W) \end{array},$$

which is obviously an isomorphism.

Let now $U \in Sm/S$, $X_1, X_2 \in Sm/T$, $V \in \widetilde{Cor}_T(X_1, X_2)$ and $W \in \widetilde{Cor}_S(U, X_1^S)$. We want to show that

$$\lambda_{U,X_2}(V^S \circ W) = V \circ \lambda_{U,X_1}(W).$$

We have a commutative diagram

$$\begin{tikzcd}
X_1 \times_T X_2 & & U \times_T X_1 \\
X_1^S \times_S X_2^S \arrow[u, "p"'] & U \times_S X_1^S \arrow[ur, "\theta_{U,X_1}", "\cong"'] & \\
U \times_S (X_1^S \times_S X_2^S) \arrow[u, "p_{23}"'] \arrow[uu, bend left, "q_{23}"] \arrow[uurr, bend left, "q_{12}"] \arrow[ur, "p_{12}"] \arrow[r, "p_{13}"] \arrow[rr, bend right, "q_{13}"'] & U \times_S X_2^S \arrow[r, "\theta_{U,X_2}", "\cong"'] & U \times_T X_2,
\end{tikzcd}$$

where we have identified $U \times_S (X_1^S \times_S X_2^S)$ with $U \times_T X_1 \times_T X_2$ for convenience.

Suppose that V and W are supported on admissible subsets. We then have

$$\begin{aligned}
\lambda_{U,X_2}(V^S \circ W) &= \theta_{U,X_2*}p_{13*}(p_{23}^* p^* V \cdot p_{12}^* W) \\
&\quad \text{by definition} \\
&= q_{13*}(q_{23}^* V \cdot p_{12}^* W) \\
&\quad \text{by Axiom 12} \\
&= q_{13*}(q_{23}^* V \cdot q_{12}^* \theta_{U,X_1*} W) \\
&\quad \text{by Axiom 20 and Axiom 9} \\
&= V \circ \lambda_{U,X_1}(W) \\
&\quad \text{by definition.}
\end{aligned}$$

Suppose next that $U_1, U_2 \in Sm/S$, $X \in Sm/T$, $V \in \widetilde{Cor}_S(U_1, U_2)$ and $W \in \widetilde{Cor}_S(U_2, X^S)$. We want to show that

$$\lambda_{U_1,X}(W \circ V) = \lambda_{U_2,X}(W) \circ V_T.$$

We have a commutative diagram

$$\begin{tikzcd}
U_1 \times_T U_2 & U_1 \times_T (U_2 \times_S X^S) \arrow[l, "d"'] \arrow[r, "b"] \arrow[rr, bend left, "q_{23}"] \arrow[drr, "q_{13}"] & U_2 \times_S X^S \arrow[r, "\theta_{U_2,X}"] & U_2 \times_T X, \\
U_1 \times_S U_2 \arrow[u, "q_f"] & U_1 \times_S (U_2 \times_S X^S) \arrow[l, "p_{12}"'] \arrow[u, "a"] \arrow[r, "p_{13}"'] \arrow[ur, "p_{23}"] & U_1 \times_S X^S \arrow[r, "\theta_{U_1,X}"'] & U_1 \times_T X
\end{tikzcd}$$

where we have identified $U_1 \times_T (U_2 \times_S X^S)$ with $U_1 \times_T U_2 \times_T X$. If V and W are supported on admissible subsets and θ is the isomorphism

$$-T_{U_1 \times_S U_2 \times_S X^S / U_1 \times_T X} \longrightarrow N_a - a^* T_{U_1 \times_T (U_2 \times_S X^S)/U_1 \times_T X},$$

we have

$$\begin{aligned}
&\lambda_{U_1,X}(W \circ V)\\
=&\theta_{U_1,X*}p_{13*}((p_{23}^*W \cdot p_{12}^*V, -T_{U_1\times_S U_2\times_S X^S/U_2\times_S X^S} - p_{12}^*T_{U_1\times_S U_2/U_1}))\\
&\text{by definition}\\
=&q_{13*}a_*((a^*q_{23}^*\theta_{U_2,X*}W \cdot \theta(p_{12}^*V), -a^*q_{23}^*T_{U_2\times_T X/U_2} + N_a - a^*T_{U_1\times_T(U_2\times_S X^S)/U_1\times_T X}))\\
&\text{by Axiom 20 and Axiom 19, (1)}\\
=&q_{13*}((q_{23}^*\theta_{U_2,X*}W \cdot a_*\theta(p_{12}^*V), -q_{23}^*T_{U_2\times_T X/U_2} - T_{U_1\times_T(U_2\times_S X^S)/U_1\times_T X}))\\
&\text{by Axiom 18 for } a\\
=&q_{13*}((q_{23}^*\theta_{U_2,X*}W \cdot d^*q_{f*}\varphi_f(V), -q_{23}^*T_{U_2\times_T X/U_2} - d^*T_{U_1\times_T U_2/U_1}))\\
&\text{by Axiom 16 for the leftmost square in the diagram above}\\
=&\lambda_{U_2,X}(W)\circ V_T\\
&\text{by definition.}
\end{aligned}$$

□

Proposition 5.24. *Let $f : S \to T$ be a smooth morphism. Then*

$$(f_\#)' = f^*.$$

Proof. For any $Y \in Sm/S$, $\widetilde{\gamma}(id_Y) \in \widetilde{Cor}_T(Y,Y) = \widetilde{Cor}_S(Y,Y^S)$ is the initial element of C_Y in Lemma 5.7 by application of the above lemma to φ^f (see Definition 5.10). So for any $F \in \widetilde{PSh}(T)$, we have $(f^*F)(Y) = FY = ((f_\#)'F)(Y)$. This gives an isomorphism between $f^*(F)$ and $(f_\#)'(F)$ for any presheaf with E-transfers F by the lemma above. So it also gives an isomorphism after sheafication. □

Proposition 5.25. *Let $f : S \longrightarrow T$ be a smooth morphism. Then:*

1. *For any $X \in Sm/S$, we have*

   $$f_\#\widetilde{\mathbb{Z}}_S(X) \cong \widetilde{\mathbb{Z}}_T(X).$$

   as sheaves with E-transfers.

2. *For any $F \in \widetilde{Sh}(T)$ and $Y \in Sm/T$*

   $$f^*(F^Y) \cong (f^*F)^{Y\times_T S}$$

   as sheaves with E-transfers.

3. *For any $F \in \widetilde{Sh}(S)$ and $G \in \widetilde{Sh}(T)$*

   $$\underline{Hom}_T(f_\#F, H) \cong f_*\underline{Hom}_S(F, f^*H)$$

   as sheaves with E-transfers.

4. *For any $F \in \widetilde{Sh}(S)$ and $G \in \widetilde{Sh}(T)$*

   $$f_\#(F \otimes_S f^*G) \cong f_\#F \otimes_T G$$

   as sheaves with E-transfers.

Proof. 1. The result follows from the fact that any $F \in \widetilde{Sh}(T)$ we have

$$Hom_T(f_\#\widetilde{\mathbb{Z}}_S(X), F) \cong Hom_S(\widetilde{\mathbb{Z}}_S(X), f^*F) \cong (f^*F)(X) \cong F(X)$$

by the previous proposition.

2. For any $X \in Sm/S$, we get

$$(f^*(H^Y))(X) = H(Y \times_T X).$$

and

$$(f^*H)^{Y \times_T S}(X) = H((Y \times_T S) \times_S X)$$

by the above proposition. Then, we can use Proposition 5.22 to conclude.

3. For any $Y \in Sm/T$, we have

$$\begin{aligned} \underline{Hom}_T(f_\# F, H)(Y) &= Hom_T(f_\# F, H^Y) \\ &\cong Hom_S(F, f^*(H^Y)) \\ &\cong Hom_S(F, (f^*H)^{Y \times_T S}) \\ &\quad \text{by (2)} \\ &= \underline{Hom}_S(F, f^*H)(Y \times_T S) \\ &= (f_* \underline{Hom}_S(F, f^*H))(Y). \end{aligned}$$

4. For any $H \in \widetilde{Sh}(T)$, the following computation applies:

$$\begin{aligned} Hom_T(f_\#(F \otimes_S f^*G), H) &\cong Hom_S(F \otimes_S f^*G, f^*H) \\ &\cong Hom_S(f^*G, \underline{Hom}_S(F, f^*H)) \\ &\cong Hom_T(G, f_*\underline{Hom}_S(F, f^*H)) \\ &\cong Hom_T(G, \underline{Hom}_T(f_\# F, H)) \\ &\quad \text{by (3)} \\ &\cong Hom_T(f_\# F \otimes_T G, H). \end{aligned}$$

□

Chapter 6

Motivic categories

In this chapter, we construct the categories of effective (resp. stabilized) motives as a localization of the bounded above complexes ([MVW06]) of sheaves with E-transfers (resp. symmetric spectra). We then compare our construction with the constructions in [CD09], [CD13] and [DF17], where they use unbounded complexes.

6.1 Complexes of Sheaves with E-Transfers

6.1.1 Derived Categories

Denote by $D^-(S)$ (resp. $K^-(S)$) the derived (resp. homotopy) category of bounded above complexes of objects in $\widetilde{Sh}(S)$. Our first aim is to define $\otimes_S$ and $f_\#$ and f^* (Chapter 5) at the level of these categories. The method is inherited from [SV00, Corollary 2.2] and [MVW06, Lemma 8.15].

Definition 6.1. *We call a presheaf with E-transfers free if it's a direct sum of presheaves of the form $\widetilde{c}_S(X)$. We call a presheaf with E-transfers projective if it's a direct summand of a free presheaf with E-transfers. A sheaf with E-transfers is called free (resp. projective) if it's a sheafication of a free (resp. projective) presheaf with E-transfers. A bounded above complex of sheaves with E-transfers is called free (projective) if all its terms are free (projective).*

Remark 6.1. *Note that a projective presheaf with E-transfers is a projective object in the category of presheaves with E-transfers. On the other hand, this is not true anymore for projective sheaves with E-transfers.*

Definition 6.2. *A projective resolution of a bounded above complex of sheaves K is a projective complex (of sheaves) with a quasi-isomorphism $P \longrightarrow K$.*

In the definition above, if K is already projective we may take $P = K$.

Now let $S, T \in Sm/k$ and Y be a scheme with morphisms $S \xleftarrow{f} Y \xrightarrow{g} T$ where g is smooth. In this section, we consider the functors

$$\begin{array}{rccc} \varphi: & \widetilde{Cor}_S & \longrightarrow & \widetilde{Cor}_T \\ & X & \longmapsto & (X^Y)_T \cong X \times_S Y \end{array}$$

and

$$\begin{array}{rccc} \psi: & Sm_S & \longrightarrow & Sm_T \\ & X & \longmapsto & X \times_S Y \end{array}$$

determined by the triple (Y, S, T). We have a commutative diagram

$$\begin{array}{ccc} Sm/S & \xrightarrow{\psi} & Sm/T \\ {\scriptstyle\widetilde{\gamma}}\big\downarrow & & {\scriptstyle\widetilde{\gamma}}\big\downarrow \\ \widetilde{Cor}_S & \xrightarrow{\varphi} & \widetilde{Cor}_T. \end{array}$$

Recall from Lemma 5.7 the definitions of φ^* and φ_*.

Lemma 6.1. *For any $X \in Sm/S$, we have*

$$\varphi^*(\widetilde{c}_S(X)) \cong \widetilde{c}_T(\psi(X))$$

as presheaves with E-transfers.

Proof. For any $F \in \widetilde{PSh}(T)$,

$$Hom_T(\varphi^*(\widetilde{c}_S(X)), F) \cong Hom_S(\widetilde{c}_S(X), \varphi_* F) \cong F(\psi(X)).$$

□

Lemma 6.2. *The functor φ_* maps sheaves with E-transfers to sheaves with E-transfers.*

Proof. It suffices to show that for any finite Nisnevich covering $\{U_i\}$ of $X \in Sm/S$, the following sequence is exact

$$0 \longrightarrow G(X) \longrightarrow \oplus_i G(U_i) \longrightarrow \oplus_{i,j} G(U_i \times_X U_j)$$

where $G = \varphi_* F$ for some $F \in \widetilde{Sh}(T)$. This follows easily. □

The following lemma can be proved using a method similar to the one we used in the proof of Proposition 5.15.

Lemma 6.3. *Let $f : F \longrightarrow G$ be morphism in $\widetilde{PSh}(S)$ such that $\widetilde{a}(f)$ is an isomorphism, then $\widetilde{a}(\varphi^*(f))$ is also an isomorphism.*

Before stating the next result, recall that the category of presheaves with E-transfers has enough projective objects (see for instance Remark 6.1). In particular, it is possible to derive any left-exact functor (say, to the category of abelian groups).

Proposition 6.1. *For any $F \in \widetilde{PSh}(S)$,*

$$\widetilde{a}((L_i\varphi^*)\widetilde{a}(F)) \cong \widetilde{a}((L_i\varphi^*)F)$$

as sheaves with E-transfers for any $i \geq 0$, where $L_i\varphi^*$ means the i^{th} left derived functor of φ^*.

Proof. We show first that for any presheaf with E-transfers F with $\widetilde{a}(F) = 0$ we have

$$\widetilde{a}(L_i\varphi^*(F)) = 0$$

for any $i \geq 0$. Suppose that the above statement is proved. For any presheaf with E-transfers F, we can then consider the natural morphism

$$\theta : F \longrightarrow \widetilde{a}(F).$$

We have

$$\widetilde{a}(coker(\theta)) = \widetilde{a}(ker(\theta)) = 0.$$

Hence for any $i \geq 0$, we have

$$\widetilde{a}(L_i\varphi^*\widetilde{a}(F)) \cong \widetilde{a}(L_i\varphi^* Im(\theta)) \cong \widetilde{a}(L_i\varphi^* F)$$

by using long exact sequences. Hence the statement follows.

Now we prove the first claim by induction on i. The claim is true for $i = 0$ and we then suppose that it's true for $i < n$. For any $F \in \widetilde{PSh}(S)$, we have a surjection

$$\oplus_{x\in F(X)}\widetilde{c}_S(X) \longrightarrow F$$

defined by each section of F on each $X \in Sm/S$. Since $\widetilde{a}(F) = 0$, there exists for any $X \in Sm/S$ and any $x \in F(X)$ a finite Nisnevich covering $U_x \longrightarrow X$ of X such that $x|_{U_a} = 0$. Then, the composite

$$\oplus_{x\in F(X)}\widetilde{c}_S(U_x) \longrightarrow \oplus_{x\in F(X)}\widetilde{c}_S(X) \longrightarrow F$$

is trivial and we obtain a surjection

$$\oplus_{x\in F(X)} H_0(\check{C}(U_x/X)) \longrightarrow F$$

with kernel K. Proposition 5.10 implies that

$$\widetilde{a}(H_p(\check{C}(U/X))) = 0$$

for any Nisnevich covering $U \longrightarrow X$ and any $p \in \mathbb{Z}$ and consequently $\widetilde{a}(K) = 0$ as well. We have a hypercohomology spectral sequence

$$(L_p\varphi^*)H_q(\check{C}(U/X)) \Longrightarrow (\mathbb{L}_{p+q}\varphi^*)\check{C}(U/X).$$

Hence

$$\widetilde{a}((\mathbb{L}_n\varphi^*)\check{C}(U/X)) \cong \widetilde{a}((L_n\varphi^*)H_0(\check{C}(U/X)))$$

by induction hypothesis. But

$$\widetilde{a}((\mathbb{L}_n\varphi^*)\check{C}(U/X)) \cong \widetilde{a}(H_n(\varphi^*\check{C}(U/X)))$$

by definition of hypercohomology and the latter vanishes since we have

$$\varphi^*\check{C}(U/X) = \check{C}(\psi U/\psi X)$$

by the previous lemmas. So

$$\widetilde{a}((L_n\varphi^*)H_0(\check{C}(U/X))) = 0$$

and

$$\widetilde{a}(L_n\varphi^* F) \cong \widetilde{a}(L_{n-1}\varphi^* K) = 0$$

by the long exact sequence and the induction hypothesis. □

Proposition 6.2. *The functor φ^* takes acyclic projective complexes to acyclic projective complexes.*

Proof. For any projective $F \in \widetilde{Sh}(S)$, $F = \widetilde{a}(G)$ for some projective $G \in \widetilde{PSh}(S)$ by definition. So

$$\widetilde{a}((L_i\varphi^*)F) \cong \widetilde{a}((L_i\varphi^*)G) = 0$$

for any $i > 0$ by the proposition above. Let

$$0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$$

be a short exact sequence of sheaves with E-transfers with $\widetilde{a}((L_i\varphi^*)P) = 0$ for any $i > 0$. Then the sequence is still exact after applying φ^* by the long exact sequence. Then the statement follows easily. □

Proposition 6.3. *We have an exact functor*

$$L\varphi^* : D^-(S) \longrightarrow D^-(T)$$

which maps any $K \in D^-(S)$ to $\varphi^* P$, where P is a projective resolution K.

Proof. By the proposition above, the class of projective complexes is adapted (see [GM03, III.6.3]) to the functor φ^*. We may now apply [GM03, III.6.6]. □

In the sequel, we'll write φ^* in place of $L\varphi^*$ for convenience. We now apply the general results above to $\otimes_S$, $f_\#$ and f^*.

Proposition 6.4.

1. *The category $D^-(S)$ is endowed with a tensor product defined by*

$$\begin{array}{cccccc} \otimes_S : & D^-(S) & \times & D^-(S) & \longrightarrow & D^-(S) \\ & (K & , & L) & \longmapsto & P \otimes_S Q \end{array},$$

   where P, Q are projective resolutions of K, L respectively, and $P \otimes_S Q$ is the total complex of the bicomplex $\{P_i \otimes_S Q_j\}$. Moreover, for any $K \in D^-(S)$, the functor $K \otimes_S -$ is exact.

2. *Suppose that $f : S \longrightarrow T$ is a smooth morphism in Sm/k. Then, there is an exact functor*

$$f_\# : D^-(S) \to D^-(T)$$

   defined on objects by $K \mapsto f_\# P$, where P is a projective resolution of K.

3. *Suppose that $f : S \longrightarrow T$ is a morphism in Sm/k. There is an exact functor*

$$f^* : D^-(T) \to D^-(S)$$

   defined on objects by $K \mapsto f^* P$, where P is a projective resolution of K.

Proof. 1. Let $Y \in Sm/S$. In the definition of φ, we take $(Y, S, T) := (Y, S, S)$ and then $\varphi^* F \cong F \otimes_S \widetilde{\mathbb{Z}}_S(Y)$ for any $F \in \widetilde{Sh}(S)$ by Proposition 5.14.

Given an acyclic projective complex P and a projective sheaf F, the complex of sheaves $F \otimes_S P$ is also acyclic by Proposition 6.2 and by definition of projectiveness. It follows that for any projective complex K the complex $P \otimes_S K$ is also acyclic by the spectral sequence of the bicomplex $\{P_i \otimes_S K_j\}$. Then for any projective complexes P, Q, R and quasi-isomorphism $a : P \longrightarrow Q$, the morphism $a \otimes_S R$ is still a quasi-isomorphism since we have

$$Cone(a \otimes_S R) \cong Cone(a) \otimes_S R$$

and the latter one acyclic. The statement follows easily.

2. In the definition of φ, we take $(Y, S, T) := (S, S, T)$ and apply Proposition 6.3.

3. In the definition of φ, we take $(Y, S, T) := (T, S, T)$ and apply Proposition 6.3.

□

Proposition 6.5. *Let $f : S \longrightarrow T$ be a smooth morphism in Sm/k. We then have an adjoint pair*

$$f_\# : D^-(S) \rightleftharpoons D^-(T) : f^*.$$

Proof. By Proposition 5.23, it is easy to see that there is an adjunction

$$f_\# : K^-(S) \rightleftharpoons K^-(T) : f^*.$$

Since $f^* : \widetilde{Sh}(T) \longrightarrow \widetilde{Sh}(S)$ has both a left adjoint and a right adjoint, it's an exact functor and $Lf^* \cong f^*$ in this case. Suppose that $K \in D^-(S)$, $L \in D^-(T)$ and that $p : P \longrightarrow K$ is a projective resolution of K. Note then that $f_\# K = f_\# P$ by definition.

We now construct a morphism

$$\theta : Hom_{D^-(S)}(f_\# K, L) \longrightarrow Hom_{D^-(T)}(K, f^* L)$$

as follows. Suppose that $s \in Hom_{D^-(S)}(f_\# K, L)$ is written as a right roof (see [GM03, III.2.9])

$$\begin{tikzcd} & R & \\ f_\# P \arrow[ur, "a"] & & L. \arrow[ul, "b"'] \end{tikzcd}$$

By adjunction, a induces a morphism $a' : P \longrightarrow f^* R$. Then we define $\theta(s)$ to the composite of the right roof

$$\begin{tikzcd} & f^* R & \\ P \arrow[ur, "a'"] & & f^* L \arrow[ul, "f^* b"'] \end{tikzcd}$$

with p^{-1}. This morphism is well-defined since f^* is exact.

Next, we construct a morphism

$$\xi : Hom_{D^-(T)}(K, f^* L) \longrightarrow Hom_{D^-(S)}(f_\# K, L)$$

as follows. Suppose that $t \in Hom_{D^-(T)}(K, f^* L)$ and that $t \circ p$ is written as a left roof (see [GM03, III.2.8])

$$\begin{tikzcd} & R \arrow[dl, "a"'] \arrow[dr, "b"] & \\ P & & f^* L \end{tikzcd}$$

where R is also projective. By adjunction, b induces a morphism $b' : f_\# R \longrightarrow L$ and we define $\xi(t)$ to be the left roof

$$\begin{tikzcd} & f_\# R \arrow[dl, "f_\# a"'] \arrow[dr, "b'"] & \\ f_\# P & & L. \end{tikzcd}$$

This morphism is well-defined by Proposition 6.2 applied to $f_\#$. To conclude, one checks that θ and ξ are inverse to each other by direct computation. □

In [CD09, Theorem 1.7], they put a model structure $\mathfrak{M}$ on the category of unbounded complexes of sheaves with E-transfers over S. This is a cofibrantly generated model structure where the cofibrations are the I-cofibrations ([Hov07, Definition 2.1.7]) where I consists of the morphisms $S^{n+1}\widetilde{\mathbb{Z}}_S(X) \longrightarrow D^n\widetilde{\mathbb{Z}}_S(X)$ for any $X \in Sm/S$ ([CD09, 1.9] for notations) and weak equivalences are quasi-morphisms of complexes.

Proposition 6.6. *Bounded above projective complexes are cofibrant objects in $\mathfrak{M}$.*

Proof. Suppose that P is a bounded above projective complex and that we have an I-injective ([Hov07, Definition 2.1.7]) morphism $f : A \longrightarrow B$ between unbounded complexes with a morphism $g : P \longrightarrow B$. We have to show that $g = f \circ h$ for some $h : P \longrightarrow A$.

We construct h by induction. Suppose that for any $m \geq n$ we have constructed a morphism $h^m : P^m \longrightarrow A^m$ such that $g^m = f^m \circ h^m$ and $d^A \circ h^m = h^m \circ d^P$. As P is bounded above, this is certainly the case for n large enough. We now construct $h^{n-1} : P^{n-1} \longrightarrow A^{n-1}$ satisfying the same property, that is, making the following diagram commute

$$\begin{tikzcd}
A^{n-1} \arrow[rr, "d^A"] \arrow[dd, "f^{n-1}"'] & & A^n \arrow[dd, "f^n"'] & \\
& P^{n-1} \arrow[ul, "h^{n-1}"'] \arrow[rr, "d^P"] \arrow[dl, "g^{n-1}"] & & P^n \arrow[ul, "h^n"'] \arrow[dl, "g^n"] \\
B^{n-1} \arrow[rr, "d^B"] & & B^n. &
\end{tikzcd}$$

By definition, we have a split surjection $F \longrightarrow P^{n-1}$ where F is a free sheaf with E-transfers. So, we may assume that P^{n-1} is free of the form $\oplus_i\widetilde{\mathbb{Z}}_S(X_i)$ where $X_i \in Sm/S$. For every i, we have two morphisms:

$$u_i : \widetilde{\mathbb{Z}}_S(X_i) \longrightarrow P^{n-1} \longrightarrow B^{n-1} \longrightarrow B^n$$

and

$$v_i : \widetilde{\mathbb{Z}}_S(X_i) \longrightarrow P^{n-1} \longrightarrow P^n \longrightarrow A^n$$

which give a commutative square with a lifting since f is I-injective:

$$\begin{tikzcd}
S^n\widetilde{\mathbb{Z}}_S(X_i) \arrow[r, "v_i"] \arrow[d] & A \arrow[d, "f"] \\
D^{n-1}\widetilde{\mathbb{Z}}_S(X_i) \arrow[r, "u_i"] \arrow[ur, "w_i"] & B.
\end{tikzcd}$$

One checks directly that $\oplus_i w_i : P^{n-1} \longrightarrow A^{n-1}$ is the required morphism. $\square$

The model structure $\mathfrak{M}$ is stable and left proper so it induces a triangulated structure $\mathfrak{T}'$ on $D(S)$ ([Ayo07, Theoreme 4.1.49]). The classical triangulated structure of $D(S)$ or $D^-(S)$ is denoted by $\mathfrak{T}$.

Proposition 6.7. *The natural functor*

$$i : (D^-(S), \mathfrak{T}) \longrightarrow (D(S), \mathfrak{T}')$$

is fully faithful exact.

Proof. Any distinguished triangle T in $(D^-(S), \mathfrak{T})$ is isomorphic in $D^-(S)$ to a distinguished triangle in $\mathfrak{T}$ of the form

$$A \xrightarrow{f} B \longrightarrow Cone(f) \longrightarrow A[1] \,,$$

where all arrows come from explicit morphisms between chain complexes ([GM03, III.3.3 and III.3.4]). By [Hir03, Proposition 8.1.23], there exists a commutative diagram

$$\begin{tikzcd} A' \arrow[r, "g"] \arrow[d, "a"'] & B' \arrow[d, "b"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

such that (A', a) (resp. (B', b)) is a fibrant cofibrant approximation of A (resp. B) and g is a cofibration in $\mathfrak{M}$. So the triangle T is isomorphic in $D(S)$ to the distinguished triangle

$$A' \xrightarrow{g} B' \longrightarrow Cone(g) \longrightarrow A'[1]$$

in $\mathfrak{T}$. By [CD09, Lemma 1.10] and [Ayo07, Théorème 4.1.38], the shift functors $-[n]$ and $-[n]'$ in $\mathfrak{T}$ and $\mathfrak{T}'$, respectively, coincide on cofibrant objects in $\mathfrak{M}$. So we have a natural isomorphism $\eta : -[n] \longrightarrow -[n]'$ where $\eta_K = id_{K[n]}$ if K is cofibrant in $\mathfrak{M}$. It follows that the triangle above is distinguished in $\mathfrak{T}'$ by [Ayo07, Definition 4.1.45]. So the functor i is exact and it's clearly fully faithful. $\square$

Observe now that we can define $\otimes_S$, f^* and $f_\#$ on $D(S)$ by [CD09, Theorem 1.18 and Proposition 2.3].

Proposition 6.8. *1. We have a commutative diagram (up to a natural isomorphism)*

$$\begin{tikzcd} D^-(S) \times D^-(S) \arrow[r, "\otimes_S"] \arrow[d] & D^-(S) \arrow[d] \\ D(S) \times D(S) \arrow[r, "\otimes_S"] & D(S). \end{tikzcd}$$

2. *Suppose that $f : S \longrightarrow T$ is a morphism in Sm/k. We have a commutative diagram (up to a natural isomorphism)*

$$\begin{tikzcd} D^-(T) \arrow[r, "f^*"] \arrow[d] & D^-(S) \arrow[d] \\ D(T) \arrow[r, "f^*"] & D(S) \end{tikzcd}$$

3. *Suppose that $f : S \longrightarrow T$ is a smooth morphism in Sm/k. We have a commutative diagram (up to a natural isomorphism)*

$$\begin{tikzcd} D^-(S) \arrow[r, "f_\#"] \arrow[d] & D^-(T) \arrow[d] \\ D(S) \arrow[r, "f_\#"] & D(T) \end{tikzcd}$$

Proof. This follows by direct computation using Proposition 6.6. $\square$

6.1.2 Effective Motives

The following definition comes from [MVW06, Definition 9.2].

Definition 6.3. *Define $\mathscr{E}_{\mathbb{A}}$ to be the smallest thick subcategory of $D^-(S)$ such that*

1. $Cone(\widetilde{\mathbb{Z}}_S(X \times_k \mathbb{A}^1) \longrightarrow \widetilde{\mathbb{Z}}_S(X)) \in \mathscr{E}_{\mathbb{A}}$.

2. $\mathscr{E}_{\mathbb{A}}$ is closed under arbitrary direct sums if it exists in $D^-(S)$.

Set $W_{\mathbb{A}}$ to be the class of morphisms in $D^-(S)$ whose cone is in $\mathscr{E}_{\mathbb{A}}$. Define

$$\widetilde{DM}^{eff,-}(S) = D^-(S)[W_{\mathbb{A}}^{-1}]$$

to be the category of effective motives over S. The morphisms in $D^-(S)$ becoming isomorphisms after localization by $W_{\mathbb{A}}$ are called $\mathbb{A}^1$-weak equivalences.

Before proceeding further, we give an example of an $\mathbb{A}^1$-weak equivalence. Recall that a morphism $p : E \longrightarrow X$ in Sm/S is an $\mathbb{A}^n$-bundle if there is an open covering $\{U_i\}$ of X such that $p^{-1}(U_i) \cong U_i \times_k \mathbb{A}^n$ for any i.

Proposition 6.9. *Let $p : E \longrightarrow X$ in Sm/S be an $\mathbb{A}^n$-bundle. Then, $\widetilde{\mathbb{Z}}_S(p) : \widetilde{\mathbb{Z}}_S(E) \longrightarrow \widetilde{\mathbb{Z}}_S(X)$ is an $\mathbb{A}^1$-weak equivalence.*

Proof. For any $X \in Sm/S$, the projection $\widetilde{\mathbb{Z}}_S(X \times_k \mathbb{A}^n) \longrightarrow \widetilde{\mathbb{Z}}_S(X)$ is an $\mathbb{A}^1$-weak equivalence by definition. Suppose that we have two open sets U_1 and U_2 of X such that the statement is true over U_1, U_2 and $U_1 \cap U_2$ and set $E_i = p^{-1}(U_i)$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathbb{Z}}_S(E_1 \cap E_2) \arrow[r] \arrow[d] & \widetilde{\mathbb{Z}}_S(E_1) \oplus \widetilde{\mathbb{Z}}_S(E_2) \arrow[r] \arrow[d] & \widetilde{\mathbb{Z}}_S(p^{-1}(E_1 \cup E_2)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{\mathbb{Z}}_S(U_1 \cap U_2) \arrow[r] & \widetilde{\mathbb{Z}}_S(U_1) \oplus \widetilde{\mathbb{Z}}_S(U_2) \arrow[r] & \widetilde{\mathbb{Z}}_S(U_1 \cup U_2) \arrow[r] & 0
\end{tikzcd}$$

by Proposition 5.12. So the statement is also true over $U_1 \cup U_2$. To conclude, we pick a finite open covering $\{U_i\}$ of X such that $p^{-1}(U_i) \cong U_i \times_k \mathbb{A}^n$ for every i and proceed by induction on the number of open sets. $\square$

Definition 6.4. *([MVW06, Definition 9.17]) A complex $K \in D^-(S)$ is called $\mathbb{A}^1$-local if for every $\mathbb{A}^1$-equivalence $f : A \longrightarrow B$, the induced map*

$$Hom_{D^-(S)}(B, K) \longrightarrow Hom_{D^-(S)}(A, K)$$

is an isomorphism.

Before stating the next result, recall that one can associate to any complex of sheaves K its Suslin complex C_*K ([MVW06, Definition 2.14]).

Proposition 6.10. *Let $K \in D^-(S)$.*

1. The natural map $K \longrightarrow C_*K$ is an $\mathbb{A}^1$-weak equivalence.

2. If $S = pt$, the complex C_*K is $\mathbb{A}^1$-local.

3. If $S = pt$, the functor C_* induces an endofunctor of $D^-(pt)$.

Proof. 1. The proof of [MVW06, Lemma 9.15] goes through in our setting.

2. Use Remark 5.1 and mimic the proof of [DF17, Theorem 3.2.9 and Corollary 3.2.11].

3. It's easy to check that C_* induces an endofunctor of $K^-(pt)$. If $f : K \longrightarrow L$ is a quasi-isomorphism, then $Cone(f)$ is acyclic. By (1), the natural morphism $Cone(f) \longrightarrow C_*Cone(f)$ is an $\mathbb{A}^1$-equivalence. Hence it's an quasi-isomorphism by (2) and [MVW06, Lemma 9.21]. So $C_*Cone(f) = Cone(C_*f)$ is acyclic and C_*f is a quasi-isomorphism.

$\square$

We now pass to the definition of motivic cohomology.

Definition 6.5. *([MVW06, Definition 14.17]) Let $X \in Sm/k$ and let $p, q \in \mathbb{Z}, q \geq 0$. The groups*

$$H_E^{p,q}(X, \mathbb{Z}) = Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(X), \widetilde{\mathbb{Z}}_{pt}(q)[p])$$

are called E-motivic cohomology groups of X.

Proposition 6.11. *The functor φ of Proposition 6.3 induces an exact functor*

$$\varphi^* : \widetilde{DM}^{eff,-}(S) \longrightarrow \widetilde{DM}^{eff,-}(T)$$

which is determined by the following commutative diagram

$$\begin{tikzcd} D^-(S) \arrow[r, "\varphi^*"] \arrow[d] & D^-(T) \arrow[d] \\ \widetilde{DM}^{eff,-}(S) \arrow[r, "\varphi^*"] & \widetilde{DM}^{eff,-}(T). \end{tikzcd}$$

Proof. Let $\mathscr{E}$ be the full subcategory of $D^-(S)$ which consists of those complexes $K \in D^-(S)$ who satisfy $\varphi^*K \in \mathscr{E}_{\mathbb{A}}$. It's a thick subcategory of $D^-(S)$. For any $X \in Sm/S$, φ^* maps

$$\widetilde{\mathbb{Z}}_S(X \times_k \mathbb{A}^1) \longrightarrow \widetilde{\mathbb{Z}}_S(X)$$

to

$$\widetilde{\mathbb{Z}}_T((\psi X) \times_k \mathbb{A}^1) \longrightarrow \widetilde{\mathbb{Z}}_T(\psi X).$$

Therefore $\mathscr{E}_{\mathbb{A}} \subseteq \mathscr{E}$ by definition of $\mathscr{E}_{\mathbb{A}}$ and exactness of φ^*. It follows that φ^* preserves objects in $\mathscr{E}_{\mathbb{A}}$. Hence φ^* preserves $\mathbb{A}^1$-weak equivalences by exactness of φ^*. Then the statement follows from [Kra10, Proposition 4.6.2]. $\square$

Proposition 6.12. *1. There is a tensor product*

$$\otimes_S : \widetilde{DM}^{eff,-}(S) \times \widetilde{DM}^{eff,-}(S) \longrightarrow \widetilde{DM}^{eff,-}(S)$$

which is determined by the following commutative diagram

$$\begin{tikzcd} D^-(S) \times D^-(S) \arrow[r, "\otimes_S"] \arrow[d] & D^-(S) \arrow[d] \\ \widetilde{DM}^{eff,-}(S) \times \widetilde{DM}^{eff,-}(S) \arrow[r, "\otimes_S"] & \widetilde{DM}^{eff,-}(S). \end{tikzcd}$$

Furthermore, for any $K \in \widetilde{DM}^{eff,-}(S)$, the functor $K \otimes_S -$ is exact.

2. Suppose that $f : S \longrightarrow T$ is a smooth morphism in Sm/k. There is an exact functor

$$f_{\#} : \widetilde{DM}^{eff,-}(S) \longrightarrow \widetilde{DM}^{eff,-}(T)$$

which is determined by the following commutative diagram

$$\begin{tikzcd} D^-(S) \arrow[r, "f_\#"] \arrow[d] & D^-(T) \arrow[d] \\ \widetilde{DM}^{eff,-}(S) \arrow[r, "f_\#"] & \widetilde{DM}^{eff,-}(T). \end{tikzcd}$$

3. *Suppose that $f : S \longrightarrow T$ is a morphism in Sm/k. There is an exact functor*

$$f^* : \widetilde{DM}^{eff,-}(T) \longrightarrow \widetilde{DM}^{eff,-}(S)$$

which is determined by the following commutative diagram

$$\begin{tikzcd} D^-(T) \arrow[r, "f^*"] \arrow[d] & D^-(S) \arrow[d] \\ \widetilde{DM}^{eff,-}(T) \arrow[r, "f^*"] & \widetilde{DM}^{eff,-}(S). \end{tikzcd}$$

Proof. 1. Suppose that $Y \in Sm/S$. In the definition of φ, we take $(Y, S, T) := (Y, S, S)$. Then $\varphi^*F \cong F \otimes_S \widetilde{\mathbb{Z}}_S(Y)$ for any $F \in \widetilde{Sh}(S)$ by Proposition 5.14. Now, given an $\mathbb{A}^1$-weak equivalence a, $\widetilde{\mathbb{Z}}_S(Y) \otimes_S a$ is also an $\mathbb{A}^1$-weak equivalence by Proposition 6.11 (applied to φ). We may now apply the method used in the third paragraph of [MVW06, Lemma 9.5] to show that the functor $K \otimes_S - : D^-(S) \longrightarrow D^-(S)$ preserves $\mathbb{A}^1$-weak equivalences for any $K \in D^-(S)$. Finally we apply [Kra10, Proposition 4.6.2] to the functor $K \otimes_S -$.

2. In the definition of φ, we take $(Y, S, T) := (S, S, T)$ and apply Proposition 6.11.

3. In the definition of φ, we take $(Y, S, T) := (T, S, T)$ and apply Proposition 6.11.

□

Proposition 6.13. *Let $f : S \longrightarrow T$ be a smooth morphism in Sm/k. We have an adjoint pair*

$$f_\# : \widetilde{DM}^{eff,-}(S) \rightleftharpoons \widetilde{DM}^{eff,-}(T) : f^*.$$

Proof. The same method as in Proposition 6.5 applies since φ^* preserves $\mathscr{E}_{\mathbb{A}}$ by Proposition 6.11. □

Proposition 6.14. *Let $f : S \longrightarrow T$ be a morphism in Sm/k.*

1. *For any $K, L \in \widetilde{DM}^{eff,-}(T)$, we have*

$$f^*(K \otimes_S L) \cong (f^*K) \otimes_S (f^*L).$$

2. *If f is smooth, then for any $K \in \widetilde{DM}^{eff,-}(S)$ and $L \in \widetilde{DM}^{eff,-}(T)$, we have*

$$f_\#(K \otimes_S f^*L) \cong (f_\#K) \otimes_S L.$$

Proof. This follows immediately from Proposition 5.20 and Proposition 5.25. □

In [CD09, Proposition 3.5] and [DF17, Definition 3.2.1], the category $\widetilde{DM}^{eff}(S)$ is defined as the the Verdier localization of $D(S)$ with respect to the homotopy invariance conditions. Now, this localization induces a triangulated structure on $\widetilde{DM}^{eff}(S)$ ([Kra10, Lemma 4.3.1]).

Proposition 6.15. *The exact functor $D^-(S) \to D(S)$ of Proposition 6.7 induces an exact functor $\widetilde{DM}^{eff,-}(S) \longrightarrow \widetilde{DM}^{eff}(S)$ which is determined by the commutative diagram (Proposition 6.7)*

$$\begin{tikzcd} D^-(S) \arrow[r] \arrow[d] & D(S) \arrow[d] \\ \widetilde{DM}^{eff,-}(S) \arrow[r] & \widetilde{DM}^{eff}(S). \end{tikzcd}$$

This functor is fully faithful if $S = pt$.

Proof. For the first statement, we use [Kra10, Proposition 4.6.2]. For the second statement, we note that we have for any $K, L \in \widetilde{DM}^{eff,-}(pt)$ a commutative diagram

$$\begin{tikzcd} Hom_{\widetilde{DM}^{eff,-}(pt)}(K,L) \arrow[r, "u"] \arrow[d, "\alpha"'] & Hom_{\widetilde{DM}^{eff,-}(pt)}(C_*K, C_*L) \arrow[d] & Hom_{D^-(pt)}(C_*K, C_*L) \arrow[l, "\gamma"'] \arrow[d, "\cong"'] , \\ Hom_{\widetilde{DM}^{eff}(pt)}(K,L) \arrow[r, "v"] & Hom_{\widetilde{DM}^{eff}(pt)}(C_*K, C_*L)) & Hom_{D(pt)}(C_*K, C_*L) \arrow[l, "\beta"'] \end{tikzcd}$$

where u, v, γ and β are isomorphisms by Proposition 6.10. So α is an isomorphism. □

To conclude this section, we note that the versions of $\otimes_S$, f^*, $f_\#$ in both categories are compatible as in Proposition 6.8.

6.2 Symmetric Spectra

In this section, we introduce spectra in order to stabilize the category $\widetilde{DM}^{eff,-}(S)$. The main reference is [CD13, 5.3].

6.2.1 Symmetric Spectra

Let $\mathscr{A}$ be a symmetric closed monoidal abelian category with arbitrary products. We can define the category of symmetric sequences $\mathscr{A}^{\mathfrak{S}}$ as in [CD13, Definition 5.3.5]. It is also a closed symmetric monoidal abelian category by [CD13, Definition 5.3.7] and [HSS00, Lemma 2.1.6]. Here, if we have two symmetric sequences A and B, we define $A \otimes^{\mathfrak{S}} B$ by

$$(A \otimes^{\mathfrak{S}} B)_n = \oplus_p S_n \times_{S_p \times S_{n-p}} (A_p \otimes B_{n-p}).$$

Then we define $\underline{Hom}^{\mathfrak{S}}(A,B)$ by

$$\underline{Hom}^{\mathfrak{S}}(A,B)_n = \prod_p \underline{Hom}_{S_p}(A_p, B_{n+p}),$$

where $\underline{Hom}_{S_p}(A_p, B_{n+p})$ (with the obvious S_n-action) is the kernel of the map

$$\underline{Hom}(A_p, B_{n+p}) \xrightarrow{(\sigma^* - (1\times\sigma)_*)} \textstyle\prod_{\sigma \in S_p} \underline{Hom}(A_p, B_{n+p}) .$$

(see [HSS00, Definition 2.1.3] and [HSS00, Theorem 2.1.11])

Proposition 6.16. *In the context above, for any symmetric sequences A, B, C, we have*

$$Hom(A \otimes^{\mathfrak{S}} B, C) \cong Hom(A, \underline{Hom}^{\mathfrak{S}}(B,C))$$

naturally.

Proof. Giving a morphism from $A \otimes^{\mathfrak{S}} B$ to C is equivalent to giving $S_p \times S_q$-equivariant maps
$$f_{p,q} : A_p \otimes B_q \longrightarrow C_{p+q}.$$
That is equivalent to giving S_p-equivariant maps
$$g_{p,q} : A_p \longrightarrow \underline{Hom}(B_q, C_{p+q})$$
such that for any $\sigma \in S_q$,
$$\underline{Hom}(\sigma, C_{p+q}) \circ g_{p,q} = \underline{Hom}(B_q, id_{S_p} \times \sigma) \circ g_{p,q}.$$
This just says that $g_{p,q}$ factor through $\underline{Hom}_{S_q}(B_q, C_{p+q})$. $\square$

The abelian structure of $\mathscr{A}^{\mathfrak{S}}$ is just defined termwise. Moreover, we have adjunctions
$$i_0 : \mathscr{A} \rightleftharpoons \mathscr{A}^{\mathfrak{S}} : ev_0$$
and
$$-\{-i\} : \mathscr{A}^{\mathfrak{S}} \rightleftharpoons \mathscr{A}^{\mathfrak{S}} : -\{i\}(i \geq 0)$$
as in [CD13, 5.3.5.1] and [CD09, 6.4.1].

Now suppose that $R \in \mathscr{A}$. Then $Sym(R) \in \mathscr{A}^{\mathfrak{S}}$ is a commutative monoid object as in [CD13, 5.3.8]. Define $Sp_R(\mathscr{A})$ to be the category of $Sym(R)$-modules in $\mathscr{A}^{\mathfrak{S}}$. Its objects are called symmetric R-spectra. It's also a symmetric closed monoidal abelian category by [HSS00, Theorem 2.2.10] and Proposition 6.16. (The corresponding tensor product and inner-hom are just denoted by $\otimes$ and $\underline{Hom}$ for convenience)

We have an adjunction
$$Sym(R) \otimes^{\mathfrak{S}} - : \mathscr{A}^{\mathfrak{S}} \rightleftharpoons Sp_R(\mathscr{A}) : U,$$
where U is the forgetful functor. Thus we get an adjunction
$$\Sigma^\infty : \mathscr{A} \rightleftharpoons Sp_R(\mathscr{A}) : \Omega^\infty,$$
where $\Sigma^\infty = (Sym(R) \otimes^{\mathfrak{S}} -) \circ i_0$, $\Omega^\infty = ev_0 \circ U$ and Σ^∞ is monoidal.

We have a canonical identification
$$A \otimes^{\mathfrak{S}} (B\{-i\}) = (A \otimes^{\mathfrak{S}} B)\{-i\}$$
and a morphism
$$A \otimes^{\mathfrak{S}} (B\{i\}) \longrightarrow (A \otimes^{\mathfrak{S}} B)\{i\}$$
defined by the composite
$$A \otimes^{\mathfrak{S}} (B\{i\}) \longrightarrow (A \otimes^{\mathfrak{S}} (B\{i\}))\{-i\}\{i\} = (A \otimes^{\mathfrak{S}} (B\{i\}\{-i\}))\{i\} \longrightarrow (A \otimes^{\mathfrak{S}} B)\{i\}.$$
Restricting the functors $-\{-i\}$ and $-\{i\}$ on symmetric R-spectra, we get an adjunction
$$-\{-i\} : Sp_R(\mathscr{A}) \rightleftharpoons Sp_R(\mathscr{A}) : -\{i\},$$
where the module structure $Sym(R) \otimes^{\mathfrak{S}} (A\{-i\}) \longrightarrow A\{-i\}$ of $A\{-i\}$ is obtained by applying $-\{-i\}$ to the module structure of A and the module structure $Sym(R) \otimes^{\mathfrak{S}} (B\{i\}) \longrightarrow B\{i\}$ of $B\{i\}$ is obtained via the composite
$$Sym(R) \otimes^{\mathfrak{S}} (B\{i\}) \longrightarrow (Sym(R) \otimes^{\mathfrak{S}} B)\{i\} \longrightarrow B\{i\},$$
where the last arrow is just the shift of the module structure of B. Moreover, we still have an isomorphism
$$A \otimes_S (B\{-i\}) \cong (A \otimes_S B)\{-i\}$$
and a morphism
$$A \otimes_S (B\{i\}) \longrightarrow (A \otimes_S B)\{i\}$$
defined in the same way as above.

Definition 6.6. *([CD13, Definition 5.3.16]) For any $S \in Sm/k$, define*

$$\mathbb{1}_S\{1\} = Sym(coker(\widetilde{\mathbb{Z}}_S(S) \longrightarrow \widetilde{\mathbb{Z}}_S(\mathbb{G}_m)))$$

and

$$\mathbb{1}'_S\{1\} = Sym(coker(\widetilde{c}_S(S) \longrightarrow \widetilde{c}_S(\mathbb{G}_m))).$$

Then define $Sp(S)$ to be $Sp_{\mathbb{1}_S\{1\}}(\widetilde{Sh}(S))$ and $Sp'(S)$ to be $Sp_{\mathbb{1}'_S\{1\}}(\widetilde{PSh}(S))$.

We have an adjunction

$$\widetilde{a} : \widetilde{PSh}(S)^{\mathfrak{S}} \rightleftharpoons \widetilde{Sh}(S)^{\mathfrak{S}} : \widetilde{o}$$

where both functors are defined termwise (see Proposition 5.11) and $\widetilde{a}$ is monoidal by definition. Restricting the above functors on modules, we also obtain an adjunction

$$\widetilde{a} : Sp'(S) \rightleftharpoons Sp(S) : \widetilde{o},$$

where the module structure $\mathbb{1}_S\{1\} \otimes_S^{\mathfrak{S}} \widetilde{a}(A) \longrightarrow \widetilde{a}(A)$ of $\widetilde{a}(A)$ is obtained via sheafication and the module structure $\mathbb{1}'_S\{1\} \otimes_S^{\mathfrak{S}} \widetilde{o}(B) \longrightarrow \widetilde{o}(B)$ of $\widetilde{o}(B)$ is obtained via the module structure of B and the sheafication map $\mathbb{1}'_S\{1\} \otimes_S^{\mathfrak{S}} \widetilde{o}(B) \longrightarrow \mathbb{1}_S\{1\} \otimes_S^{\mathfrak{S}} B$. The functor $\widetilde{a}$ is again monoidal.

Now, let $f : S \longrightarrow T$ be a morphism in Sm/k. We have an adjunction

$$f^* : \widetilde{Sh}(T)^{\mathfrak{S}} \rightleftharpoons \widetilde{Sh}(S)^{\mathfrak{S}} : f_*$$

where both functors are defined termwise (see Proposition 5.19) and f^* is monoidal by Proposition 5.20, (4). Restricting the above functors on spectra, we also obtain an adjunction

$$f^* : Sp(T) \rightleftharpoons Sp(S) : f_*,$$

where the module structure $\mathbb{1}_S\{1\} \otimes_S^{\mathfrak{S}} f^*A \longrightarrow f^*A$ of f^*A is induced by the module structure of A via f^* and the module structure $\mathbb{1}_T\{1\} \otimes_T^{\mathfrak{S}} f_*B \longrightarrow f_*B$ of f_*B is obtained using the composite

$$\mathbb{1}_T\{1\} \otimes_T^{\mathfrak{S}} f_*B \longrightarrow f_*(\mathbb{1}_S\{1\} \otimes_S^{\mathfrak{S}} f^*f_*B) \longrightarrow f_*(\mathbb{1}_S\{1\} \otimes_S^{\mathfrak{S}} B) \longrightarrow f_*B.$$

The functor f^* is also monoidal by construction of the tensor product (see [HSS00, Lemma 2.2.2]). The same construction gives an another adjunction

$$f^* : Sp'(T) \rightleftharpoons Sp'(S) : f_*.$$

Suppose further that f is smooth. We have an adjunction

$$f_\# : \widetilde{Sh}(S)^{\mathfrak{S}} \rightleftharpoons \widetilde{Sh}(T)^{\mathfrak{S}} : f^*$$

where both functors are defined termwise (see Proposition 5.23) and

$$f_\#(A \otimes_S^{\mathfrak{S}} f^*B) \cong (f_\# A) \otimes_T^{\mathfrak{S}} B$$

also holds by Proposition 5.25, (4). Restricting the above functors on spectra, we get an adjunction

$$f_\# : Sp(S) \rightleftharpoons Sp(T) : f^*,$$

where the module structure $\mathbb{1}_T\{1\} \otimes^{\mathfrak{S}} f_\# A \longrightarrow f_\# A$ of $f_\# A$ is as follows

$$\mathbb{1}_T\{1\} \otimes_T^{\mathfrak{S}} f_\# A \cong f_\#(\mathbb{1}_S\{1\} \otimes_S^{\mathfrak{S}} A) \longrightarrow f_\# A.$$

Moreover, we also have

$$f_{\#}(A \otimes_S f^* B) \cong (f_{\#} A) \otimes_T B$$

for spectra by construction of the tensor product (see [HSS00, Lemma 2.2.2]). The same construction gives yet another adjunction

$$f_{\#} : Sp'(S) \rightleftharpoons Sp'(T) : f^*.$$

One checks that when F$=- \otimes_S A$, $f_{\#}$, f^*, $-\{-i\}$, $-\{i\}$, Σ^∞ or Ω^∞, there is a natural isomorphism $\widetilde{a} \circ F \cong F \circ \widetilde{a}$.

Let $i \geq 0$. For any $F \in \widetilde{Sh}(S)$, we have

$$(\Sigma^\infty F)\{i\} \cong \Sigma^\infty(\widetilde{\mathbb{Z}}_{tr}(\mathbb{G}_m^{\wedge 1})^{\otimes i} \otimes_S F).$$

Moreover, for any $X \in Sm/S$,

$$Hom_{Sp(S)}((\Sigma^\infty \widetilde{\mathbb{Z}}_S(X))\{-i\}, A) = A_i(X)$$

and

$$Hom_{Sp'(S)}((\Sigma^\infty \widetilde{c}_S(X))\{-i\}, B) = B_i(X).$$

So $(\Sigma^\infty \widetilde{\mathbb{Z}}_S(X))\{-i\}$ (resp. $(\Sigma^\infty \widetilde{c}_S(X))\{-i\}$) are systems of generators of $Sp(S)$ (resp. $Sp'(S)$) ([CD09, 6.7] and [CD13, 5.3.11]). This enables us to imitate the methods used in Section 6.1.

6.2.2 Derived Categories

We denote by $D^-_{Sp}(S)$ (resp. $D_{Sp}(S)$) the derived category of bounded above (resp. unbounded) complex of spectra in $Sp(S)$.

Proposition 6.17. *Let $X, U \in Sm/S$ and $p : U \longrightarrow X$ be a Nisnevich covering. For any $i \in \mathbb{N}$, the complex $(\Sigma^\infty \check{C}(U/X))\{-i\}$ (defined by termise application), is exact after sheafifying as a complex of $Sp(S)$.*

Proof. One easily see that $(\Sigma^\infty A)\{-i\} = Sym(\mathbb{Z}'_S\{1\}) \otimes_S^{\mathfrak{S}} (i_0(A)\{-i\})$ for any $A \in \widetilde{PSh}(S)$. Then the statement follows from the equality

$$\check{C}(U/X) \otimes_S^{pr} \widetilde{c}_S(Y) = \check{C}(U \times_S Y / X \times_S Y)$$

for any $Y \in Sm/S$ and Proposition 5.10. □

Definition 6.7. *We call a spectrum $A \in Sp'(S)$ free if it's a direct sum of spectra of the form $(\Sigma^\infty \widetilde{c}_S(X))\{-i\}$. We call A projective if it's a direct summand of a free spectrum. A spectrum in $Sp(S)$ is called free (resp. projective) if it's the sheafication of a free (resp. projective) spectrum in $Sp'(S)$. A bounded above complex of spectra in $Sp(S)$ is called free (projective) if all its terms are free (projective).*

Definition 6.8. *A projective resolution of a bounded above spectrum complex K is a projective complex with a quasi-isomorphism $P \longrightarrow K$.*

Now let $S, T \in Sm/k$, $j \geq 0$ and Y be a scheme with morphisms $S \xleftarrow{f} Y \xrightarrow{g} T$ where g is smooth. Consider in this section the adjunctions

$$\phi^* = \{-j\} \circ g_{\#} \circ f^* : \; Sp(S) \rightleftharpoons Sp(T) : \phi_* = f_* \circ g^* \circ \{j\}$$

$$\varphi^* = \{-j\} \circ g_{\#} \circ f^* : \; Sp'(S) \rightleftharpoons Sp'(T) : \varphi_* = f_* \circ g^* \circ \{j\}$$

and the functor

$$\psi : \begin{array}{ccc} Sm_S & \longrightarrow & Sm_T \\ X & \longmapsto & X \times_S Y \end{array}.$$

They are determined by the quadruple (Y, S, T, j).

Proposition 6.18. *For any $F \in Sp'(S)$,*

$$\widetilde{a}((L_i\varphi^*)\widetilde{a}(F)) \cong \widetilde{a}((L_i\varphi^*)F)$$

as spectra in $Sp(S)$ for any $i \geq 0$, where $L_i\varphi^*$ means the i^{th} left derived functor of φ^*.

Proof. Arguing as in the proof of Proposition 6.1, we see that it suffices to treat the case of spectra $F \in Sp'(S)$ satisfying $\widetilde{a}(F) = 0$. We prove it by induction on i. The claim is true for $i = 0$ and we suppose that it's also true for $i < n$.

For any $F \in Sp'(S)$, we have a surjection

$$\oplus_{x \in F_t(X), t \geq 0}(\Sigma^\infty \widetilde{c}_S(X))\{-t\} \longrightarrow F$$

defined by each section of F_t on each $X \in Sm/S$. Since $\widetilde{a}(F) = 0$, there exists for any $x \in F_t(X)$ and $X \in Sm/S$ a finite Nisnevich covering $U_x \longrightarrow X$ such that $x|_{U_x} = 0$. Then, the composite

$$\oplus_{a \in F_t(X), t \geq 0}(\Sigma^\infty \widetilde{c}_S(U_a))\{-t\} \longrightarrow \oplus_{a \in F(X), t \geq 0}(\Sigma^\infty \widetilde{c}_S(X))\{-t\} \longrightarrow F$$

is trivial and we have a surjection

$$\oplus_{a \in F(X), t \geq 0} H_0((\Sigma^\infty \check{C}(U_a/X))\{-t\}) \longrightarrow F$$

with kernel K. Proposition 6.17 implies that

$$\widetilde{a}(H_p((\Sigma^\infty \check{C}(U/X))\{-t\})) = 0$$

for any Nisnevich covering $U \longrightarrow X$, $t \geq 0$ and $p \in \mathbb{Z}$ and therefore $\widetilde{a}(K) = 0$ as well. We have a hypercohomology spectral sequence

$$(L_p\varphi^*)H_q((\Sigma^\infty \check{C}(U/X))\{-t\}) \Longrightarrow (\mathbb{L}_{p+q}\varphi^*)((\Sigma^\infty \check{C}(U/X))\{-t\})$$

and consequently

$$\widetilde{a}((\mathbb{L}_n\varphi^*)((\Sigma^\infty \check{C}(U/X))\{-t\})) \cong \widetilde{a}((L_n\varphi^*)H_0((\Sigma^\infty \check{C}(U/X))\{-t\}))$$

by induction hypothesis. But

$$\widetilde{a}((\mathbb{L}_n\varphi^*)(\Sigma^\infty \check{C}(U/X))\{-t\}) \cong \widetilde{a}(H_n(\varphi^*((\Sigma^\infty \check{C}(U/X))\{-t\})))$$

by definition of hypercohomology and the latter vanishes since we have

$$\varphi^*((\Sigma^\infty \check{C}(U/X))\{-t\}) = (\Sigma^\infty \check{C}(\psi U/\psi X))\{-t-j\}.$$

So,

$$\widetilde{a}((L_n\varphi^*)H_0((\Sigma^\infty \check{C}(U/X))\{-t\})) = 0$$

and

$$\widetilde{a}(L_n\varphi^* F) \cong \widetilde{a}(L_{n-1}\varphi^* K) = 0$$

by the long exact sequence and the induction hypothesis. □

The same proofs as in Propositions 6.2 and 6.3 yield the following two propositions.

Proposition 6.19. *Let functor ϕ^* takes acyclic projective complexes to acyclic projective complexes.*

Proposition 6.20. *We have an exact functor*

$$L\phi^* : D^-_{Sp}(S) \longrightarrow D^-_{Sp}(T)$$

which maps any $K \in D^-_{Sp}(S)$ to $\phi^* P$, where P is a projective resolution K.

According to our conventions, we'll just write ϕ^* in place of $L\phi^*$. We now apply the general results above to the functors $\otimes_S$, $f_\#$ and f^*.

Proposition 6.21. *1. There is a tensor product*

$$\begin{array}{cccccc} \otimes_S : & D^-_{Sp}(S) & \times & D^-_{Sp}(S) & \longrightarrow & D^-_{Sp}(S) \\ & (K & , & L) & \longmapsto & P \otimes_S Q \end{array},$$

where P, Q are projective resolutions of K, L respectively and $P \otimes_S Q$ is the total complex of the bicomplex $\{P_i \otimes_S Q_j\}$. Moreover, for any $K \in D^-_{Sp}(S)$, the functor $K \otimes_S -$ is exact.

2. *Suppose that $f : S \longrightarrow T$ is a smooth morphism in Sm/k. Then, there is an exact functor*

$$\begin{array}{cccc} f_\# : & D^-_{Sp}(S) & \longrightarrow & D^-_{Sp}(T) \\ & K & \longmapsto & f_\# P \end{array},$$

where P is a projective resolution of K.

3. *Suppose that $f : S \longrightarrow T$ is a morphism in Sm/k. There is then an exact functor*

$$\begin{array}{cccc} f^* : & D^-_{Sp}(T) & \longrightarrow & D^-_{Sp}(S) \\ & K & \longmapsto & f^* P \end{array},$$

where P is a projective resolution of K.

4. *For $i \geq 0$ there is an exact functor*

$$\begin{array}{cccc} -\{-i\} : & D^-_{Sp}(S) & \longrightarrow & D^-_{Sp}(S) \\ & K & \longmapsto & P\{-i\} \end{array},$$

where P is a projective resolution of K.

Proof. In (1), (2) and (3), take $j = 0$ in the definition of ϕ and proceed as in the proof of Proposition 6.4. For (4), take the quadruple (S, S, S, i) and use Proposition 6.20. □

Proposition 6.22. *1. If $f : S \longrightarrow T$ is a smooth morphism in Sm/k, we have an adjoint pair*

$$f_\# : D^-_{Sp}(S) \rightleftharpoons D^-_{Sp}(T) : f^*.$$

2. *We have an adjoint pair*

$$-\{-i\} : D^-_{Sp}(S) \rightleftharpoons D^-_{Sp}(S) : -\{i\}.$$

Proof. The same as Proposition 6.5 since $-\{i\}$ is an exact functor. □

Now we are going to compare $D^-_{Sp}(S)$ with $D^-(S)$ defined in Section 6.1.

Proposition 6.23. *The functor $\Sigma^\infty : \widetilde{Sh}(S) \longrightarrow Sp(S)$ takes acyclic projective complexes of sheaves to acyclic projective complexes of spectra.*

Proof. Let P be a projective sheaf. Then

$$(\Sigma^\infty P)_n = \mathbb{1}_S\{1\}^{\otimes n} \otimes_S P_n$$

by definition and a tensor product between projective sheaves is again projective. So $\Sigma^\infty P$ is projective.

Let Q be an acyclic projective complex of sheaves. Then $\Sigma^\infty Q$ consists of complexes of the form $\mathbb{1}_S\{1\}^{\otimes n} \otimes_S Q$. They are all acyclic by Proposition 6.2. $\square$

Proposition 6.24. *There is an exact functor*

$$L\Sigma^\infty : D^-(S) \longrightarrow D^-_{Sp}(S)$$

which maps K to $\Sigma^\infty P$, where P is a projective resolution of K.

Proof. The same as Proposition 6.3. $\square$

As usual, we will write Σ^∞ instead of $L\Sigma^\infty$ for convenience.

Proposition 6.25. *There is an adjoint pair*

$$\Sigma^\infty : D^-(S) \rightleftharpoons D^-_{Sp}(S) : \Omega^\infty.$$

Proof. The same as Proposition 6.5 since Ω^∞ is an exact functor. $\square$

Proposition 6.26. *The functor $\Sigma^\infty : D^-(S) \longrightarrow D^-_{Sp}(S)$ is fully faithful.*

Proof. Let $K, L \in D^-(S)$ with respective projective resolutions P, Q. Then, there is a commutative diagram

$$\begin{array}{ccc}
Hom_{D^-(S)}(K,L) & \xrightarrow{\Sigma^\infty} & Hom_{D^-_{Sp}(S)}(\Sigma^\infty K, \Sigma^\infty L) \\
\cong\downarrow & & \cong\downarrow \\
Hom_{D^-(S)}(P,Q) & \xrightarrow{\Sigma^\infty} & Hom_{D^-_{Sp}(S)}(\Sigma^\infty P, \Sigma^\infty Q) \\
 & & \cong\downarrow \\
 & & Hom_{D^-(S)}(P, \Omega^\infty\Sigma^\infty Q).
\end{array}$$

Finally we observe that $\Omega^\infty\Sigma^\infty Q = Q$. $\square$

Proposition 6.27. 1. *We have a commutative diagram (up to a canonical isomorphism)*

$$\begin{array}{ccc}
D^-(S) \times D^-(S) & \xrightarrow{\otimes_S} & D^-(S) \\
\Sigma^\infty \times \Sigma^\infty \downarrow & & \Sigma^\infty \downarrow \\
D^-_{Sp}(S) \times D^-_{Sp}(S) & \xrightarrow{\otimes_S} & D^-_{Sp}(S)
\end{array}.$$

2. *Let $f : S \longrightarrow T$ be a morphism in Sm/k. We have a commutative diagram (up to a canonical isomorphism)*

$$\begin{array}{ccc}
D^-(T) & \xrightarrow{f^*} & D^-(S) \\
\Sigma^\infty \downarrow & & \Sigma^\infty \downarrow \\
D^-_{Sp}(T) & \xrightarrow{f^*} & D^-_{Sp}(S).
\end{array}$$

3. *Suppose that $f : S \longrightarrow T$ is a smooth morphism in Sm/k. We then have a commutative diagram (up to a canonical isomorphism)*

$$\begin{tikzcd} D^-(S) \arrow[r, "f_\#"] \arrow[d, "\Sigma^\infty"'] & D^-(T) \arrow[d, "\Sigma^\infty"'] \\ D^-_{Sp}(S) \arrow[r, "f_\#"] & D^-_{Sp}(T). \end{tikzcd}$$

Proof. This follows by direct computations. □

In [CD09, Theorem 1.7], they define a model structure $\mathfrak{M}_{Sp}$ on the category of unbounded complexes of symmetric spectra over S. This is a cofibrantly generated model structure where the cofibrations are the I-cofibrations ([Hov07, Definition 2.1.7]) where I consists of the morphisms $S^{n+1}(\Sigma^\infty\widetilde{\mathbb{Z}}_S(X)\{-i\}) \longrightarrow D^n(\Sigma^\infty\widetilde{\mathbb{Z}}_S(X)\{-i\})$ for any $X \in Sm/S$ and $i \geq 0$ and weak equivalences are quasi-morphisms between complexes. The same proof as the one of Proposition 6.6 applies to give the following result.

Proposition 6.28. *Bounded above projective complexes are cofibrant objects in $\mathfrak{M}_{Sp}$.*

Now, $\mathfrak{M}_{Sp}$ is stable and left proper so it induces a triangulated structure $\mathfrak{T}'$ on $D_{Sp}(S)$ ([Ayo07, Theoreme 4.1.49]). The classical triangulated structure of $D_{Sp}(S)$ or $D^-_{Sp}(S)$ is denoted by $\mathfrak{T}$.

Proposition 6.29. *The natural functor*

$$(D^-_{Sp}(S), \mathfrak{T}) \longrightarrow (D_{Sp}(S), \mathfrak{T}')$$

is fully faithful exact.

Proof. The same as for Proposition 6.7. □

Finally, we note that the various versions of $\otimes_S$, f^*, $f_\#$, Σ^∞, $-\{-i\}, i \geq 0$ are compatible as in Proposition 6.8.

6.2.3 Effective Motivic Spectra

Definition 6.9. *([CD13, 5.2.15]) Define $\mathscr{E}_{\mathbb{A}}$ to be the smallest thick subcategory of $D^-_{Sp}(S)$ such that*

1. $(\Sigma^\infty Cone(\widetilde{\mathbb{Z}}_S(X \times_k \mathbb{A}^1) \longrightarrow \widetilde{\mathbb{Z}}_S(X)))\{-i\} \in \mathscr{E}_{\mathbb{A}}, i \geq 0$.
2. *$\mathscr{E}_{\mathbb{A}}$ is closed under arbitrary direct sums if it exists in $D^-_{Sp}(S)$.*

Set $W_{\mathbb{A}}$ to be the class of morphisms in $D^-_{Sp}(S)$ whose cone is in $\mathscr{E}_{\mathbb{A}}$. Finally, define

$$\widetilde{DM}^{eff,-}_{Sp}(S) = D^-_{Sp}(S)[W_{\mathbb{A}}^{-1}].$$

A morphism in $D^-_{Sp}(S)$ is called a levelwise $\mathbb{A}^1$-equivalence if it becomes an isomorphism in $\widetilde{DM}^{eff,-}_{Sp}(S)$.

Definition 6.10. *([CD13, 5.3.20]) A complex $K \in D^-_{Sp}(S)$ is called levelwise $\mathbb{A}^1$-local if for every levelwise $\mathbb{A}^1$-equivalence $f : A \longrightarrow B$, the induced map*

$$Hom_{D^-_{Sp}(S)}(B, K) \longrightarrow Hom_{D^-_{Sp}(S)}(A, K)$$

is an isomorphism.

Proposition 6.30. *A complex $K = (K_n) \in D^-_{Sp}(S)$ is levelwise $\mathbb{A}^1$-local if and only if for every $n \geq 0$, the complex K_n is $\mathbb{A}^1$-local in $D^-(S)$.*

Proof. The proof of [MVW06, Lemma 9.20] applies. The complex K is levelwise $\mathbb{A}^1$-local if and only if for every $X \in Sm/S$, $n \in \mathbb{Z}$ and $i \geq 0$, the map

$$Hom_{D^-_{Sp}(S)}((\Sigma^\infty \widetilde{\mathbb{Z}}_S(X))\{-i\}[n], K) \longrightarrow Hom_{D^-_{Sp}(S)}((\Sigma^\infty \widetilde{\mathbb{Z}}_S(X \times \mathbb{A}^1))\{-i\}[n], K)$$

is an isomorphism. One uses Proposition 6.22 and Proposition 6.25 to conclude. □

For every $A = (A_n) \in Sp(S)$ and $X \in Sm/S$, we define A^X by $(A_n)^X = (A_n^X)$. The module structure $\mathbb{1}_S\{1\} \otimes^{\mathfrak{S}} A^X \longrightarrow A^X$ is given by the composite

$$\mathbb{1}_S\{1\} \otimes^{\mathfrak{S}} A^X \longrightarrow (\mathbb{1}_S\{1\} \otimes^{\mathfrak{S}} A)^X \longrightarrow A^X.$$

The functor A^X is contravariant with respect to morphisms in Sm/S. It follows that we can define the Suslin complex C_*A of A by $(C_*A)_n = C_*A_n$.

Proposition 6.31. *Let $K \in D^-_{Sp}(S)$.*

1. *The natural map $K \longrightarrow C_*K$ is a levelwise $\mathbb{A}^1$-equivalence.*
2. *If $S = pt$, the complex C_*K is levelwise $\mathbb{A}^1$-local.*
3. *If $S = pt$, the functor C_* induces an endofunctor of $D^-_{Sp}(pt)$.*

Proof. 1. We have a natural morphism $\Sigma^\infty \widetilde{\mathbb{Z}}_S(X) \otimes^{\mathfrak{S}} A^X \longrightarrow A$ defined by the composite

$$\mathbb{1}_S\{1\}^{\otimes p} \otimes_S \widetilde{\mathbb{Z}}_S(X) \otimes_S A_q^X \longrightarrow \widetilde{\mathbb{Z}}_S(X) \otimes_S (\mathbb{1}_S\{1\}^{\otimes p} \otimes_S A_q)^X \longrightarrow \widetilde{\mathbb{Z}}_S(X) \otimes_S A_{p+q}^X \longrightarrow A_{p+q}$$

for every $p, q \geq 0$. This morphism is compatible with module actions so it induces a morphism

$$\Sigma^\infty \widetilde{\mathbb{Z}}_S(X) \otimes_S A^X \longrightarrow A.$$

We then obtain a morphism

$$A^X \longrightarrow \underline{Hom}(\Sigma^\infty \widetilde{\mathbb{Z}}_S(X), A)$$

and we can use the same proof as in [MVW06, Lemma 9.15] to conclude.

2. By the proposition above and Proposition 6.10.

3. By Proposition 6.10 since quasi-isomorphisms in $D^-_{Sp}(pt)$ are defined levelwise.

□

Proposition 6.32. *A morphism $f : A \longrightarrow B$ in $D^-_{Sp}(pt)$ is a levelwise $\mathbb{A}^1$-equivalence if and only if for every $n \geq 0$, the morphism*

$$f_n = \Omega^\infty(f\{n\}) : A_n \longrightarrow B_n$$

is an $\mathbb{A}^1$-equivalence in $D^-(pt)$.

Proof. The morphism f is a levelwise $\mathbb{A}^1$-equivalence if and only if C_*f is a quasi-isomorphism by Proposition 6.31. The latter property can be checked levelwise. □

Proposition 6.33. *Let ϕ be the functor as before. We have an exact functor*

$$\phi^* : \widetilde{DM}^{eff,-}_{Sp}(S) \longrightarrow \widetilde{DM}^{eff,-}_{Sp}(T)$$

which is determined by the following commutative diagram

$$\begin{tikzcd} D^-_{Sp}(S) \arrow[r, "\phi^*"] \arrow[d] & D^-_{Sp}(T) \arrow[d] \\ \widetilde{DM}^{eff,-}_{Sp}(S) \arrow[r, "\phi^*"] & \widetilde{DM}^{eff,-}_{Sp}(T) \end{tikzcd}$$

Proof. For any $X \in Sm/S$, ϕ^* maps

$$\Sigma^\infty(\widetilde{\mathbb{Z}}_S(X \times_k \mathbb{A}^1) \longrightarrow \widetilde{\mathbb{Z}}_S(X))\{-i\}$$

to

$$\Sigma^\infty(\widetilde{\mathbb{Z}}_T((\psi X) \times_k \mathbb{A}^1) \longrightarrow \widetilde{\mathbb{Z}}_T(\psi X))\{-i-j\}.$$

So the statement follows by the same method as in Proposition 6.11. □

Proposition 6.34.

1. *Then tensor product on $D^-_{Sp}(S)$ induces a tensor product*

$$\otimes_S : \widetilde{DM}^{eff,-}_{Sp}(S) \times \widetilde{DM}^{eff,-}_{Sp}(S) \longrightarrow \widetilde{DM}^{eff,-}_{Sp}(S),$$

which is determined by the following commutative diagram

$$\begin{tikzcd} D^-_{Sp}(S) \times D^-_{Sp}(S) \arrow[r, "\otimes_S"] \arrow[d] & D^-_{Sp}(S) \arrow[d] \\ \widetilde{DM}^{eff,-}_{Sp}(S) \times \widetilde{DM}^{eff,-}_{Sp}(S) \arrow[r, "\otimes_S"] & \widetilde{DM}^{eff,-}_{Sp}(S). \end{tikzcd}$$

Furthermore, for any $K \in \widetilde{DM}^{eff,-}_{Sp}(S)$, the functor $K \otimes_S -$ is exact.

2. *Letf $: S \longrightarrow T$ be a smooth morphism in Sm/k. There is an exact functor*

$$f_\# : \widetilde{DM}^{eff,-}_{Sp}(S) \longrightarrow \widetilde{DM}^{eff,-}_{Sp}(T),$$

which is determined by the following commutative diagram

$$\begin{tikzcd} D^-_{Sp}(S) \arrow[r, "f_\#"] \arrow[d] & D^-_{Sp}(T) \arrow[d] \\ \widetilde{DM}^{eff,-}_{Sp}(S) \arrow[r, "f_\#"] & \widetilde{DM}^{eff,-}_{Sp}(T). \end{tikzcd}$$

3. *Suppose that $f : S \longrightarrow T$ is a morphism in Sm/k. There is then an exact functor*

$$f^* : \widetilde{DM}^{eff,-}_{Sp}(T) \longrightarrow \widetilde{DM}^{eff,-}_{Sp}(S),$$

which is determined by the following commutative diagram

$$\begin{tikzcd} D^-_{Sp}(T) \arrow[r, "f^*"] \arrow[d] & D^-_{Sp}(S) \arrow[d] \\ \widetilde{DM}^{eff,-}_{Sp}(T) \arrow[r, "f^*"] & \widetilde{DM}^{eff,-}_{Sp}(S). \end{tikzcd}$$

4. *For any $i \geq 0$, there is an exact functor*

$$-\{-i\} : \widetilde{DM}^{eff,-}_{Sp}(S) \longrightarrow \widetilde{DM}^{eff,-}_{Sp}(S),$$

which is determined by the following commutative diagram

$$\begin{tikzcd} D^-_{Sp}(S) \arrow[r, "-\{-i\}"] \arrow[d] & D^-_{Sp}(S) \arrow[d] \\ \widetilde{DM}^{eff,-}_{Sp}(S) \arrow[r, "-\{-i\}"] & \widetilde{DM}^{eff,-}_{Sp}(S). \end{tikzcd}$$

5. *For any $i \geq 0$, there is an exact functor*

$$-\{i\} : \widetilde{DM}^{eff,-}_{Sp}(pt) \longrightarrow \widetilde{DM}^{eff,-}_{Sp}(pt),$$

which is determined by the following commutative diagram

$$\begin{tikzcd} D^-_{Sp}(pt) \arrow[r, "-\{i\}"] \arrow[d] & D^-_{Sp}(pt) \arrow[d] \\ \widetilde{DM}^{eff,-}_{Sp}(pt) \arrow[r, "-\{i\}"] & \widetilde{DM}^{eff,-}_{Sp}(pt). \end{tikzcd}$$

Proof. For (1), (2), (3), take $j = 0$ in the definition of ϕ and proceed as in Proposition 6.12, using Proposition 6.33. For (4), take the quadruple (S, S, S, i) and use Proposition 6.33. Finally, (5) holds by Proposition 6.32. □

Proposition 6.35. 1. *Let $f : S \longrightarrow T$ be a smooth morphism in Sm/k. We have an adjoint pair*

$$f_\# : \widetilde{DM}^{eff,-}_{Sp}(S) \rightleftharpoons \widetilde{DM}^{eff,-}_{Sp}(T) : f^*.$$

2. *We have an adjoint pair*

$$-\{-i\} : \widetilde{DM}^{eff,-}_{Sp}(pt) \rightleftharpoons \widetilde{DM}^{eff,-}_{Sp}(pt) : -\{i\}.$$

Proof. The same as in Proposition 6.5. □

Proposition 6.36. 1. *There is an exact functor*

$$\Sigma^\infty : \widetilde{DM}^{eff,-}(S) \longrightarrow \widetilde{DM}^{eff,-}_{Sp}(S)$$

determined by the following commutative diagram

$$\begin{tikzcd} D^-(S) \arrow[r, "\Sigma^\infty"] \arrow[d] & D^-_{Sp}(S) \arrow[d] \\ \widetilde{DM}^{eff,-}(S) \arrow[r, "\Sigma^\infty"] & \widetilde{DM}^{eff,-}_{Sp}(S). \end{tikzcd}$$

2. *There is an exact functor*

$$\Omega^\infty : \widetilde{DM}^{eff,-}_{Sp}(pt) \longrightarrow \widetilde{DM}^{eff,-}(pt)$$

determined by the following commutative diagram

$$\begin{tikzcd} D^-_{Sp}(pt) \arrow[r, "\Omega^\infty"] \arrow[d] & D^-(pt) \arrow[d] \\ \widetilde{DM}^{eff,-}_{Sp}(pt) \arrow[r, "\Omega^\infty"] & \widetilde{DM}^{eff,-}(pt). \end{tikzcd}$$

Proof. 1. This follows from the fact that tensor products with $\widetilde{\mathbb{Z}}_S(\mathbb{G}_m^{\wedge 1})$ preserves $\mathbb{A}^1$-equivalences, together with Proposition 6.32.

2. This follows by Proposition 6.32.

□

Proposition 6.37. *There is an adjoint pair*

$$\Sigma^\infty : \widetilde{DM}^{eff,-}(pt) \rightleftharpoons \widetilde{DM}^{eff,-}_{Sp}(pt) : \Omega^\infty.$$

Proof. The same as in Proposition 6.5. □

Proposition 6.38. *The functor $\Sigma^\infty : \widetilde{DM}^{eff,-}(pt) \longrightarrow \widetilde{DM}^{eff,-}_{Sp}(pt)$ is fully faithful.*

Proof. The same as in Proposition 6.26. □

Proposition 6.39. *1. We have a commutative diagram (up to a canonical isomorphism)*

$$\begin{CD}
\widetilde{DM}^{eff,-}(S) \times \widetilde{DM}^{eff,-}(S) @>{\otimes_S}>> \widetilde{DM}^{eff,-}(S)\\
@V{\Sigma^\infty \times \Sigma^\infty}VV @V{\Sigma^\infty}VV\\
\widetilde{DM}^{eff,-}_{Sp}(S) \times \widetilde{DM}^{eff,-}_{Sp}(S) @>{\otimes_S}>> \widetilde{DM}^{eff,-}_{Sp}(S).
\end{CD}$$

2. *Suppose that $f : S \longrightarrow T$ is a morphism in Sm/k. We have a commutative diagram (up to a canonical isomorphism)*

$$\begin{CD}
\widetilde{DM}^{eff,-}(T) @>{f^*}>> \widetilde{DM}^{eff,-}(S)\\
@V{\Sigma^\infty}VV @V{\Sigma^\infty}VV\\
\widetilde{DM}^{eff,-}_{Sp}(T) @>{f^*}>> \widetilde{DM}^{eff,-}_{Sp}(S).
\end{CD}$$

3. *Suppose that $f : S \longrightarrow T$ is a smooth morphism in Sm/k. Then, we have a commutative diagram (up to a canonical isomorphism)*

$$\begin{CD}
\widetilde{DM}^{eff,-}(S) @>{f_\#}>> \widetilde{DM}^{eff,-}(T)\\
@V{\Sigma^\infty}VV @V{\Sigma^\infty}VV\\
\widetilde{DM}^{eff,-}_{Sp}(S) @>{f_\#}>> \widetilde{DM}^{eff,-}_{Sp}(T).
\end{CD}$$

Proof. This follows by direct computations. □

In [CD09, Proposition 3.5] and [CD13, Proposition 5.2.16], the category $\widetilde{DM}^{eff}_{Sp}(S)$ is defined as the the Verdier localization of $D_{Sp}(S)$ with respect to the homotopy invariance conditions. It follows that the localization induces a triangulated structure on $\widetilde{DM}^{eff}_{Sp}(S)$ ([Kra10, Lemma 4.3.1]).

Proposition 6.40. *There is an exact functor $\widetilde{DM}^{eff,-}_{Sp}(S) \longrightarrow \widetilde{DM}^{eff}_{Sp}(S)$ which is determined by the commutative diagram*

$$\begin{CD}
D^-_{Sp}(S) @>>> D_{Sp}(S)\\
@VVV @VVV\\
\widetilde{DM}^{eff,-}_{Sp}(S) @>>> \widetilde{DM}^{eff}_{Sp}(S).
\end{CD}$$

It is fully faithful when $S = pt$.

Proof. The same as Proposition 6.15 by using Proposition 6.31. □

As usual, there are compatibility results between the natural inclusion and $\otimes_S$, f^*, $f_\#$, Σ^∞, $-\{-i\}, i \geq 0$.

6.2.4 Stable Categories of Motives

Definition 6.11. *([CD13, 5.3.23]) Define $\mathscr{E}_\Omega$ to be the smallest thick subcategory of $\widetilde{DM}^{eff,-}_{Sp}(S)$ such that*

1. *$Cone((\Sigma^\infty \widetilde{\mathbb{Z}}_S(X)\{1\}\{-1\} \longrightarrow \Sigma^\infty \widetilde{\mathbb{Z}}_S(X))\{-i\}) \in \mathscr{E}_\Omega$ for every $X \in Sm/S, i \geq 0$.*
2. *$\mathscr{E}_\Omega$ is closed under arbitrary direct sums if it exists in $\widetilde{DM}^{eff,-}_{Sp}(S)$.*

Set W_Ω to be the class of morphisms in $\widetilde{DM}^{eff,-}_{Sp}(S)$ whose cone is in $\mathscr{E}_\Omega$. Define

$$\widetilde{DM}^{-}(S) = \widetilde{DM}^{eff,-}_{Sp}(S)[W_\Omega^{-1}]$$

to be the category of stable motives over S. A morphism in $\widetilde{DM}^{eff,-}_{Sp}(S)$ is called a stable $\mathbb{A}^1$-equivalence if it becomes an isomorphism in $\widetilde{DM}^{-}(S)$.

Definition 6.12. *A complex $K \in \widetilde{DM}^{eff,-}_{Sp}(S)$ is called Ω-local if for every stable $\mathbb{A}^1$-equivalence $f : A \longrightarrow B$, the induced map*

$$Hom_{\widetilde{DM}^{eff,-}_{Sp}(S)}(B, K) \longrightarrow Hom_{\widetilde{DM}^{eff,-}_{Sp}(S)}(A, K)$$

is an isomorphism.

The same method as in the proof of Proposition 6.34 yields the following proposition.

Proposition 6.41. 1. *The tensor product on $\widetilde{DM}^{eff,-}_{Sp}(S)$ induces a tensor product*

$$\otimes_S : \widetilde{DM}^{-}(S) \times \widetilde{DM}^{-}(S) \longrightarrow \widetilde{DM}^{-}(S),$$

which is determined by the following commutative diagram

$$\begin{array}{ccc} \widetilde{DM}^{eff,-}_{Sp}(S) \times \widetilde{DM}^{eff,-}_{Sp}(S) & \xrightarrow{\otimes_S} & \widetilde{DM}^{eff,-}_{Sp}(S) \\ \downarrow & & \downarrow \\ \widetilde{DM}^{-}(S) \times \widetilde{DM}^{-}(S) & \xrightarrow{\otimes_S} & \widetilde{DM}^{-}(S). \end{array}$$

Furthermore, for any $K \in \widetilde{DM}^{-}(S)$, the functor $K \otimes_S -$ is exact.

2. *Suppose that $f : S \longrightarrow T$ is a smooth morphism in Sm/k. Then, there is an exact functor*

$$f_\# : \widetilde{DM}^{-}(S) \longrightarrow \widetilde{DM}^{-}(T),$$

which is determined by the following commutative diagram

$$\begin{array}{ccc} \widetilde{DM}^{eff,-}_{Sp}(S) & \xrightarrow{f_\#} & \widetilde{DM}^{eff,-}_{Sp}(T) \\ \downarrow & & \downarrow \\ \widetilde{DM}^{-}(S) & \xrightarrow{f_\#} & \widetilde{DM}^{-}(T). \end{array}$$

3. Let $f : S \longrightarrow T$ be a morphism in Sm/k. There is an exact functor

$$f^* : \widetilde{DM}^-(T) \longrightarrow \widetilde{DM}^-(S),$$

which is determined by the following commutative diagram

$$\begin{tikzcd} \widetilde{DM}^{eff,-}_{Sp}(T) \arrow[r, "f^*"] \arrow[d] & \widetilde{DM}^{eff,-}_{Sp}(S) \arrow[d] \\ \widetilde{DM}^-(T) \arrow[r, "f^*"] & \widetilde{DM}^-(S). \end{tikzcd}$$

4. For any $i \geq 0$, there is an exact functor

$$-\{-i\} : \widetilde{DM}^-(S) \longrightarrow \widetilde{DM}^-(S),$$

which is determined by the following commutative diagram

$$\begin{tikzcd} \widetilde{DM}^{eff,-}_{Sp}(S) \arrow[r, "-\{-i\}"] \arrow[d] & \widetilde{DM}^{eff,-}_{Sp}(S) \arrow[d] \\ \widetilde{DM}^-(S) \arrow[r, "-\{-i\}"] & \widetilde{DM}^-(S). \end{tikzcd}$$

We denote by $\Sigma^{\infty,st}$ the composite

$$\widetilde{DM}^{eff,-}(S) \xrightarrow{\Sigma^\infty} \widetilde{DM}^{eff,-}_{Sp}(S) \longrightarrow \widetilde{DM}^-(S)\ .$$

Lemma 6.4. *Let $\mathscr{C}$ be a symmetric monoidal category and let $T \in \mathscr{C}$. If there exists $U \in \mathscr{C}$ such that $U \otimes T \cong \mathbb{1}$, then there are isomorphisms*

$$ev : U \otimes T \longrightarrow \mathbb{1}, coev : \mathbb{1} \longrightarrow T \otimes U$$

such that T is strongly dualizable ([CD13, 2.4.30]) with dual U under these two maps.

Proof. Let $F = - \otimes U$ and $G = - \otimes T$. Then the condition gives an endoequivalence

$$F : \mathscr{C} \rightleftharpoons \mathscr{C} : G$$

i.e. two natural isomorphisms $a : FG \longrightarrow id$ and $b : id \longrightarrow GF$. We can then construct the following two morphisms

$$\theta :\ Hom(FX, Y) \xrightarrow{G} Hom(GFX, GY) \xrightarrow{b^*} Hom(X, GY)$$

and

$$\eta :\ Hom(X, GY) \xrightarrow{F} Hom(FX, FGY) \xrightarrow{a_*} Hom(FX, Y)$$

for every $X, Y \in \mathscr{C}$. Let θ_1 be the composite

$$F \xrightarrow{id_F \times b} FGF \xrightarrow{a \times id_F} F$$

and θ_2 be

$$G \xrightarrow{b \times id_G} GFG \xrightarrow{id_G \times a} G\ .$$

Then $(\eta \circ \theta)(f) = \theta_1(X) \circ f$ and $(\theta \circ \eta)(g) = g \circ \theta_2(Y)$. So θ is an isomorphism, hence F is a left adjoint of G (vice versa). $\square$

As a straightforward consequence of the above lemma, we obtain the following result.

Proposition 6.42. *The element* $\Sigma^{\infty,st}(\widetilde{\mathbb{Z}}_S(\mathbb{G}_m^{\wedge 1}))$ *has a strong dual* $(\Sigma^{\infty,st}\mathbb{1}_S)\{-1\}$ *in* $\widetilde{DM}^-(S)$ *with the evaluation and coevaluation maps being isomorphisms.*

As a consequence, we can define $C(i)$ to be $C\otimes_S\Sigma^{\infty,st}(\mathbb{1}_S(i))$ and $C(-i)$ to be $C\{-i\}[i]$ for any $C\in\widetilde{DM}^-(S)$ and any $i\geq 0$.

Proposition 6.43. *([CD13, Proposition 5.3.25]) Suppose that* $E=\widetilde{CH}$*. Then, the functor*

$$\Sigma^{\infty,st}:\widetilde{DM}^{eff,-}(pt)\longrightarrow\widetilde{DM}^-(pt)$$

is fully faithful.

Proof. We first prove that for every projective $C\in\widetilde{DM}^{eff,-}(pt)$, $\Sigma^\infty C\in\widetilde{DM}^{eff,-}_{Sp}(pt)$ is Ω-local. Arguing as in [MVW06, Lemma 9.20], this is equivalent to the morphism

$$Hom(\Sigma^\infty\widetilde{\mathbb{Z}}_{pt}(X)\{1\}\{-1\}\{-i\},\Sigma^\infty(C[n]))\longrightarrow Hom(\Sigma^\infty\widetilde{\mathbb{Z}}_{pt}(X)\{-i\},\Sigma^\infty(C[n]))$$

being an isomorphism for any $X\in Sm/S$, any $i\geq 0$ and any $n\in\mathbb{Z}$. This follows from the following commutative diagram

$$\begin{tikzcd}
Hom(\Sigma^\infty\widetilde{\mathbb{Z}}_{pt}(X)\{1\}\{-1\}\{-i\},\Sigma^\infty(C[n])) \arrow[r] \arrow[d] & Hom(\Sigma^\infty\widetilde{\mathbb{Z}}_{pt}(X)\{-i\},\Sigma^\infty(C[n])) \arrow[d] \\
Hom(\Sigma^\infty\widetilde{\mathbb{Z}}_{pt}(X)\{1\}\{-1\},\Sigma^\infty(C[n])\{i\}) \arrow[r] \arrow[d] & Hom(\Sigma^\infty\widetilde{\mathbb{Z}}_{pt}(X),\Sigma^\infty(C[n])\{i\}) \arrow[d] \arrow[dl, "-\{1\}"'] \\
Hom(\Sigma^\infty\widetilde{\mathbb{Z}}_{pt}(X)\{1\},\Sigma^\infty(C[n])\{i+1\}) \arrow[d] & Hom(\widetilde{\mathbb{Z}}_S(X),\mathbb{1}_{pt}\{1\}^{\otimes i}\otimes(C[n])) \arrow[dl, "\mathbb{1}_{pt}\{1\}\otimes-"'] \\
Hom(\mathbb{1}_{pt}\{1\}\otimes\widetilde{\mathbb{Z}}_{pt}(X),\mathbb{1}_{pt}\{1\}^{\otimes i+1}\otimes(C[n])) &
\end{tikzcd}$$

and [FØ16, Theorem 5.0.1]. Let now $K,L\in\widetilde{DM}^{eff,-}(pt)$ be two projective resolutions of P,Q respectively. The statement follows then from the following commutative diagram

$$\begin{tikzcd}
Hom_{\widetilde{DM}^{eff,-}(pt)}(K,L) \arrow[r, "\Sigma^{\infty,st}"] \arrow[d, "\cong"'] & Hom_{\widetilde{DM}^-(pt)}(\Sigma^{\infty,st}K,\Sigma^{\infty,st}L) \arrow[d, "\cong"'] \\
Hom_{\widetilde{DM}^{eff,-}(pt)}(P,Q) \arrow[r, "\Sigma^{\infty,st}"] \arrow[dr, "\Sigma^\infty"] & Hom_{\widetilde{DM}^-(pt)}(\Sigma^{\infty,st}P,\Sigma^{\infty,st}Q) \\
 & Hom_{\widetilde{DM}^{eff,-}_{Sp}(pt)}(\Sigma^\infty P,\Sigma^\infty Q) \arrow[u, "\cong"']
\end{tikzcd}$$

and Proposition 6.38. □

Proposition 6.44. *Let* $f:S\longrightarrow T$ *be a smooth morphism in* Sm/k*. We have an adjoint pair*

$$f_\#:\widetilde{DM}^-(S)\rightleftharpoons\widetilde{DM}^-(T):f^*.$$

Proof. The same as Proposition 6.5. □

Proposition 6.45. *Suppose that $f : S \longrightarrow T$ is a morphism in Sm/k.*

1. *For any $K, L \in \widetilde{DM}^{-}(T)$, we have*
$$f^*(K \otimes_S L) \cong (f^*K) \otimes_S (f^*L).$$

2. *If f is smooth, then for any $K \in \widetilde{DM}^{-}(S)$ and $L \in \widetilde{DM}^{-}(T)$, we have*
$$f_\#(K \otimes_S f^*L) \cong (f_\# K) \otimes_S L.$$

Proof. Everything can be checked termwise by the discussion in Section 6.2.1. □

In [CD13, Proposition 5.3.23], the category $\widetilde{DM}(S)$ is defined as the the Verdier localization of $\widetilde{DM}^{eff}_{Sp}(S)$ with respect to W_Ω. As usual, the localization induces a triangulated structure on $\widetilde{DM}(S)$ ([Kra10, Lemma 4.3.1]). Here is a weak result which is enough for our purpose:

Proposition 6.46. *There is an exact functor $\widetilde{DM}^{-}(S) \longrightarrow \widetilde{DM}(S)$ which is determined by the commutative diagram*

$$\begin{tikzcd} \widetilde{DM}^{eff,-}_{Sp}(S) \arrow[r] \arrow[d] & \widetilde{DM}^{eff}_{Sp}(S) \arrow[d] \\ \widetilde{DM}^{-}(S) \arrow[r] & \widetilde{DM}(S). \end{tikzcd}$$

When $E = \widetilde{CH}$ and $S = pt$, the morphism
$$Hom_{\widetilde{DM}^{-}(S)}(X, Y) \longrightarrow Hom_{\widetilde{DM}(S)}(X, Y)$$
is an isomorphism if X and Y are of the form $(\Sigma^{\infty,st}A)\{-i\}, i \geq 0$.

Proof. The first statement follows from [Kra10, Proposition 4.6.2]. We have thus a commutative diagram (up to a natural isomorphism)

$$\begin{tikzcd} \widetilde{DM}^{eff,-}(S) \arrow[r] \arrow[d, "\Sigma^{\infty,st}"'] & \widetilde{DM}^{eff}(S) \arrow[d, "\Sigma^{\infty,st}"'] \\ \widetilde{DM}^{-}(S) \arrow[r] & \widetilde{DM}(S). \end{tikzcd}$$

Now let $E = \widetilde{CH}$ and $S = pt$. If the statement holds for X, Y, we say that $\mathscr{P}(X, Y)$ holds. If $\mathscr{P}(X, Y)$ is true, then for any $X' \cong X$ and $Y' \cong Y$, $\mathscr{P}(X', Y')$ is also true. It follows then from Proposition 6.42 and the fact that the natural inclusion is monoidal that $\mathscr{P}(X\{-1\}, Y\{-1\})$ is also true.

By Proposition 6.15, Proposition 6.43 and the diagram above, $\mathscr{P}(\Sigma^{\infty,st}A, \Sigma^{\infty,st}B)$ is true for any $A, B \in \widetilde{DM}^{eff,-}(pt)$. Hence the statement follows. □

To conclude, we note as usual that the various versions of $\otimes_S$, f^*, $f_\#$, $-\{-i\}, i \geq 0$ are compatible with the inclusion.

Chapter 7

Orientations on Symplectic Bundles and Applications

7.1 Orientations on Symplectic Bundles

In this section, we consider E-correspondences with $E = \widetilde{CH}$, i.e. MW-correspondences. We are going to prove the quaternionic projective bundle theorem and derive the existence of a Gysin triangle over any smooth base S (under some conditions). We first recall the comparison results between MW-motivic cohomology groups and Chow-Witt groups established in [DF17].

Proposition 7.1. *For any $C \in D^-(S)$ and any $i \in \mathbb{N}$, we have an isomorphism of functors $Sm_k^{op} \to \mathcal{A}b$*

$$Hom_{D^-(S)}(\widetilde{\mathbb{Z}}_S(-), C[i]) \cong \mathbb{H}^i(-, C).$$

Further, let X be a smooth scheme, $Z \subset X$ be a closed subset and $U = X \setminus Z$. Then, we have an isomorphism of functors $D^-(S) \to \mathcal{A}b$

$$Hom_{D^-(S)}(\widetilde{\mathbb{Z}}_S(X)/\widetilde{\mathbb{Z}}_S(U), -[i]) \cong \mathbb{H}^i_Z(X, -).$$

Proof. The first statement is obtained using the universal property of [GM03, page 188]. For the second statement, one first proves that

$$Hom_{\widetilde{Sh}(S)}(\widetilde{\mathbb{Z}}_S(X)/\widetilde{\mathbb{Z}}_S(U), -) \cong -_Z(X),$$

where the right hand side denotes sections with support in Z, defined by the left exact sequence

$$0 \longrightarrow F_Z(X) \longrightarrow F(X) \longrightarrow F(U).$$

Consequently, both terms have the same hypercohomology functor. Additionally, we have $\mathbb{E}xt^i(F, -) \cong Hom_{D^-(S)}(F, -[i])$ for any sheaf with MW-transfers F, yielding the second statement. $\square$

Let now $X \in Sm/S$. For any two morphisms $f_i : \widetilde{\mathbb{Z}}_S(X) \longrightarrow C_i, i = 1, 2$ in $\widetilde{DM}^{eff,-}(S)$, we denote by $f_1 \boxtimes f_2$ the composite

$$\widetilde{\mathbb{Z}}_S(X) \xrightarrow{\triangle} \widetilde{\mathbb{Z}}_S(X) \otimes_S \widetilde{\mathbb{Z}}_S(X) \xrightarrow{f_1 \otimes f_2} C_1 \otimes_S C_2 \,.$$

In case we have two morphisms $f_i : \widetilde{\mathbb{Z}}_S(X) \longrightarrow \widetilde{\mathbb{Z}}_S(n_i)[2n_i]$ in $\widetilde{DM}^{eff,-}(S)$, we denote by $f_1 f_2$ the composite

$$\widetilde{\mathbb{Z}}_S(X) \xrightarrow{f_1 \boxtimes f_2} \widetilde{\mathbb{Z}}_S(n_1)[2n_1] \otimes \widetilde{\mathbb{Z}}_S(n_2)[2n_2] \xrightarrow{\otimes} \widetilde{\mathbb{Z}}_S(n_1 + n_2)[2(n_1 + n_2)] \,.$$

Proposition 7.2. *Let $X \in Sm/k$, $Z \subseteq X$ be a closed subset and $i \geq 0$. Then*

$$\mathbb{H}^{2i}_Z(X, C_*\widetilde{\mathbb{Z}}_{pt}(i)) \cong \widetilde{CH}^i_Z(X),$$

and in particular

$$\mathbb{H}^{2i}(-, C_*\widetilde{\mathbb{Z}}_{pt}(i)) \cong \widetilde{CH}^i(-)$$

functorially in X. Moreover, the following diagram commutes for any $i, j \geq 0$ (here $\widetilde{DM}^{eff} = \widetilde{DM}^{eff,-}(pt)$)

$$\begin{tikzcd}
Hom_{\widetilde{DM}^{eff}}(\widetilde{\mathbb{Z}}_{pt}(X), \widetilde{\mathbb{Z}}_{pt}(i)[2i]) \times Hom_{\widetilde{DM}^{eff}}(\widetilde{\mathbb{Z}}_{pt}(X), \widetilde{\mathbb{Z}}_{pt}(j)[2j]) \arrow[r] \arrow[d, "\cdot"] & \widetilde{CH}^i(X) \times \widetilde{CH}^j(X) \arrow[d, "\cdot"] \\
Hom_{\widetilde{DM}^{eff}}(\widetilde{\mathbb{Z}}_{pt}(X), \widetilde{\mathbb{Z}}_{pt}(i+j)[2(i+j)]) \arrow[r] & \widetilde{CH}^{i+j}(X)
\end{tikzcd}$$

where the right-hand map is the intersection product on Chow-Witt groups. Consequently, we have isomorphisms $Hom_{\widetilde{DM}^{eff}}(\widetilde{\mathbb{Z}}_{pt}(X), \widetilde{\mathbb{Z}}_{pt}(i)[2i]) \longrightarrow \widetilde{CH}^i(X)$ which send $id_{\widetilde{\mathbb{Z}}_{pt}}$ to 1 when $i = 0$ and $X = pt$.

Proof. See [DF17, Corollary 4.2.6]. □

7.1.1 Grassmannian Bundles and Quaternionic Projective Bundles

In this section, we recall the basics on Grassmannian bundles and quaternionic projective bundles. Although these are well-known objects, we include their definitions here for the sake of notations. The reader may refer to [KL72], [Sha94] for Grassmannians, [Kle69] for Grassmannian bundles and [PW10] for quaternionic projective bundles. Let S be a k-scheme.

Definition 7.1. *Let k be a field, r be an integer and $1 \leq n \leq r$. Consider the ring*

$$A(n, r) = k[p_{i_1,\ldots,i_n} | 1 \leq i_1, \ldots, i_n \leq r]$$

and the ideal $I(n, r) \subseteq A(n, r)$ generated by

$$\begin{cases} \sum_{t=1}^{n+1} (-1)^{t-1} p_{i_1\ldots i_{n-1}j_t} p_{j_1\ldots j_{t-1},j_{t+1},\ldots j_{n+1}} & \text{with } 1 \leq i_1, \ldots, i_{n-1}, j_1, \ldots, j_{n+1} \leq r, \\ p_{i_1,\ldots,i_n} & \text{if the indices are not distinct,} \\ p_{i_1,\ldots,i_n} - sgn(\sigma) p_{\sigma(i_1),\ldots,\sigma(i_n)} & \text{for } \sigma \in S_n. \end{cases}$$

The scheme

$$Gr(n, r) = Proj(A(n, r)/I(n, r))$$

is the Grassmannian of rank n quotients of a k-vector space of rank r.

Definition 7.2. *Let X be an S-scheme, $\mathscr{E}$ be locally free of rank r on X and $1 \leq n \leq r$. Define a functor*

$$\begin{array}{rcl} F : X - Sch^{op} & \longrightarrow & Set \\ f : T \longrightarrow X & \longmapsto & \{\mathscr{F} \subseteq f^*\mathscr{E} | f^*\mathscr{E}/\mathscr{F} \text{ is locally free of rank } n\} \end{array}$$

with functorial maps defined by pull-backs.

Proposition 7.3. *The functor F is representable by an X-scheme $Gr_X(n,\mathscr{E})$, the Grassmannian bundle of rank n quotients of $\mathscr{E}$. Further, if $\mathscr{E} \cong O_X^{\oplus r}$, then $Gr_X(n,\mathscr{E}) \cong Gr(n,r)\times_k X$ over X.*

Proof. See [Kle69, Proposition 1.2]. □

Let $p : Gr_X(n,\mathscr{E}) \longrightarrow X$ be the structure map. There is a universal element $\mathscr{F} \subseteq p^*\mathscr{E}$ with quotient of rank n. The vector bundle $(p^*\mathscr{E}/\mathscr{F})^\vee$ is called the tautological bundle of $Gr_X(n,\mathscr{E})$, denoted by $\mathscr{U}$. Its dual is just called the dual tautological bundle, denoted by $\mathscr{U}^\vee$.

Definition 7.3. *Let $\mathscr{E} \neq 0$ be a locally free sheaf of rank n over a scheme X. Then $\mathscr{E}$ is called symplectic if it's equipped with a skew-symmetric ($v\cdot v = 0$) and non degenerate inner product $m : \mathscr{E}\times\mathscr{E} \longrightarrow O_X$ (hence n is always even).*

Now, let $f : X \longrightarrow Y$ be a morphism of schemes and $(\mathscr{E}, m)$ be a symplectic bundle on Y. Then $(f^*\mathscr{E}, f^*(m))$ is also a symplectic bundle, where $f^*(m)$ is the pull back of the map $\mathscr{E} \longrightarrow \mathscr{E}^\vee$ induced by m.

The following is a basic tool when dealing with non degeneracy of inner products.

Proposition 7.4. *Let $f : X \longrightarrow Y$ be a morphism between schemes and $\mathscr{E}$ be a locally free sheaf of finite rank over Y with an inner product $m : \mathscr{E}\times\mathscr{E} \longrightarrow O_X$. Then for any $x \in X$, m is non degenerate at $f(x)$ if and only if $f^*(m)$ is non degenerate at x.*

Proof. This is basically because f induces local homomorphisms between stalks. □

The following proposition can be seen from the case of vector spaces.

Proposition 7.5. *Suppose that we have an injection $i : \mathscr{E}_1 \longrightarrow \mathscr{E}_2$, where $\mathscr{E}_2$ is symplectic and $m_{\mathscr{E}_2}|_{\mathscr{E}_1}$ is non degenerate. Define $\mathscr{E}_1^\perp(U) := \mathscr{E}_1(U)^\perp$ for every U. Then $\mathscr{E}_1^\perp$ is again a symplectic bundle with inner product inherited from $\mathscr{E}_2$ and there exists a unique $p : \mathscr{E}_2 \longrightarrow \mathscr{E}_1$ with $p\circ i = id_{\mathscr{E}_1}$ and $Im(id_{\mathscr{E}_2} - i\circ p) \subseteq \mathscr{E}_1^\perp$.*

Definition 7.4. *Let X be an S-scheme and let $(\mathscr{E}, m)$ be a symplectic bundle over X. Define a functor*

$$\begin{array}{rcc} H : X-Sch^{op} & \longrightarrow & Set \\ f : T \longrightarrow X & \longmapsto & \{\mathscr{F} \subseteq f^*\mathscr{E} | f^*(m)|_{\mathscr{F}} \text{ non degenerate}, f^*\mathscr{E}/\mathscr{F} \text{ v.b. of rank } rk(\mathscr{E})-2\} \end{array}$$

with functorial maps defined by pull-backs.

Definition 7.5. *Let*

$$HP^n = D_+\left(\sum_{i=1}^{n+1} \overline{p_{i,i+n+1}}\right) \subseteq Gr(2, 2n+2),$$

where $\overline{p_{i,i+n+1}}$ means the class of $p_{i,i+n+1}$ in the quotient.

Proposition 7.6. *The functor H is representable by a scheme $HGr_X(\mathscr{E})$. Further, if $(\mathscr{E}, m) \cong \left(O_X^{\oplus 2n+2}, \begin{pmatrix} & I \\ -I & \end{pmatrix}\right)$, then $HGr_X(\mathscr{E}) \cong HP^n \times_k X$ over X.*

Proof. We have the structure map $\pi : Gr_X(2n,\mathscr{E}) \longrightarrow X$ and the tautological exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \pi^*\mathscr{E} \longrightarrow \mathscr{U}^\vee \longrightarrow 0.$$

Define

$$HGr_X(\mathscr{E}) = \{x \in Gr_X(2n,\mathscr{E}) | \pi^*(m)|_{\mathscr{F}} \text{ is non degenerate at } x\}.$$

We prove that $Hom_X(T, HGr_X(\mathscr{E})) \cong H(T)$ for any X-scheme $f : T \longrightarrow X$. By definition, $HGr_X(\mathscr{E})$ is an open subset of $Gr_X(2n, \mathscr{E})$. Then, any X-morphism $a : T \longrightarrow HGr_X(\mathscr{E})$ induces an X-morphism $b : T \longrightarrow Gr_X(2n, \mathscr{E})$ and this gives an exact sequence

$$0 \longrightarrow K \longrightarrow f^*\mathscr{E} \longrightarrow C \longrightarrow 0$$

obtained by applying b^* on the exact sequence in the beginning. By definition of $HGr_X(\mathscr{E})$, $f^*(m)|_K$ is non degenerate. Conversely, given a morphism $b : T \longrightarrow Gr_X(2n, \mathscr{E})$ such that $f^*(m)|_K$ is non degenerate as above, so $\pi^*(m)|_{\mathscr{F}}$ is non degenerate at every point in $Im(b)$ by Proposition 7.4. So $Im(b) \subseteq HGr_X(\mathscr{E})$.

For the second statement, consider an X-scheme $f : T \longrightarrow X$ and an X-morphism $b : T \longrightarrow Gr(2, 2n+2) \times_k X$. Then b factors through $HP^n \times_k X$ if and only if the composite

$$c : T \longrightarrow Gr(2, 2n+2) \times_k X \longrightarrow Gr(2, 2n+2)$$

factors through HP^n. Denote the structure map $Gr(2, 2n+2) \longrightarrow pt$ by p. Then we have the tautological exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow p^*O_{pt}^{\oplus 2n+2} \longrightarrow \mathscr{U}^\vee \longrightarrow 0$$

as in the beginning. Then one proves that c factor through HP^n if and only if the inner product $\left(p^*O_{pt}^{\oplus 2n+2}, \begin{pmatrix} & I \\ -I & \end{pmatrix}\right)$ is non degenerate after restriction to $c^*\mathscr{U}$ (taking the dual of the exact sequence above). Considering morphisms Spec $K \longrightarrow T$ where K is a field, we can assume $T =$ Spec K. Then the non vanishing of the formula $\sum_{i=1}^{n+1} p_{i,i+n+1}$ in the Definition 7.5 is just equivalent to the non degeneracy required above. $\square$

Definition 7.6. *We will call the scheme $HGr_X(\mathscr{E})$ the quaternionic projective bundle of $\mathscr{E}$.*

Let $p : HGr_X(\mathscr{E}) \longrightarrow X$ be the structure map. Then, there is a universal element $\mathscr{F} \subseteq p^*\mathscr{E}$ which is just obtained by the restriction of the universal element of the Grassmannian bundle to $HGr_X(\mathscr{E})$. The vector bundle $\mathscr{F}$ itself is called the tautological bundle of $HGr_X(\mathscr{E})$, denoted by $\mathscr{U}$. Its dual is just called the dual tautological bundle, denoted by $\mathscr{U}^\vee$. We will use the same symbol $\mathscr{U}$ for all tautological bundles defined above if there is no confusion. Note that both $\mathscr{U}$ and $\mathscr{U}^\vee$ are symplectic by Proposition 7.5.

7.1.2 Quaternionic Projective Bundle Theorem

The following proposition can also be found in [MVW06, Corollary 15.3] and [SV00, Proposition 4.3].

Proposition 7.7. *Let $S \in Sm/k$. For any correspondence theory E and $n \geq 1$, we have an isomorphism*

$$\widetilde{\mathbb{Z}}_S((\mathbb{A}^n \setminus 0) \times S) \cong \widetilde{\mathbb{Z}}_S \oplus \widetilde{\mathbb{Z}}_S(n)[2n-1]$$

in $\widetilde{DM}^{eff,-}(S)$.

Proof. We denote the point $(1, \dots, 1) \in \mathbb{A}^n$ by 1 for any n. Then it suffices to prove that

$$\widetilde{\mathbb{Z}}_S((\mathbb{A}^n \setminus 0) \times S, 1) \cong \widetilde{\mathbb{Z}}_S(n)[2n-1]$$

by induction. For $n = 1$ this is by definition.

In general, write $x_1, \dots, x_n$ for the coordinates of $\mathbb{A}^n$ and set $U_1 = D(x_1)$, $U_2 = \bigcup_{i=2}^n D(x_i)$. Note that $U_1 = (\mathbb{A}^1 \setminus 0) \times \mathbb{A}^{n-1} \times S$, $U_2 = \mathbb{A}^1 \times (\mathbb{A}^{n-1} \setminus 0) \times S$ and $U_1 \cap U_2 = (\mathbb{A}^1 \setminus 0) \times (\mathbb{A}^{n-1} \setminus 0) \times S$.

We have a commutative diagram in the category of sheaves with E-transfers:

$$\begin{tikzcd}
\widetilde{\mathbb{Z}}_S(U_1\cap U_2,1) \arrow[r] \arrow[d, equal] & \widetilde{\mathbb{Z}}_S(U_1,1)\oplus\widetilde{\mathbb{Z}}_S(U_2,1) \arrow[d] \\
\widetilde{\mathbb{Z}}_S(U_1\cap U_2,1) \arrow[r] & \widetilde{\mathbb{Z}}_S((\mathbb{A}^1\setminus 0)\times S,1)\oplus\widetilde{\mathbb{Z}}_S((\mathbb{A}^{n-1}\setminus 0)\times S,1)
\end{tikzcd}$$

where the right-hand vertical map is the sum of the respective projections. Considering the relevant sheaves as complexes concentrated in degree 0 and taking cones, we obtain a commutative diagram of triangles in $D^-(S)$

$$\begin{tikzcd}
\widetilde{\mathbb{Z}}_S(U_1\cap U_2,1) \arrow[r] \arrow[d, equal] & \widetilde{\mathbb{Z}}_S(U_1,1)\oplus\widetilde{\mathbb{Z}}_S(U_2,1) \arrow[d] \arrow[r] & C \arrow[d] \\
\widetilde{\mathbb{Z}}_S(U_1\cap U_2,1) \arrow[r] & \widetilde{\mathbb{Z}}_S((\mathbb{A}^1\setminus 0)\times S,1)\oplus\widetilde{\mathbb{Z}}_S((\mathbb{A}^{n-1}\setminus 0)\times S,1) \arrow[r] & C'
\end{tikzcd} \tag{7.1}$$

It follows from Proposition 5.10 that the map $\widetilde{\mathbb{Z}}_S(U_1,1)\oplus\widetilde{\mathbb{Z}}_S(U_2,1)\to\widetilde{\mathbb{Z}}_S((\mathbb{A}^n\setminus 0)\times S,1)$ induces a quasi-isomorphism $C\to\widetilde{\mathbb{Z}}_S((\mathbb{A}^n\setminus 0)\times S,1)$. Using now Lemma 5.6, we obtain a morphism of complexes $\widetilde{\mathbb{Z}}_S(((\mathbb{A}^1\setminus 0)\times S,1)\wedge((\mathbb{A}^{n-1}\setminus 0)\times S,1))[1]\to C'$ which is a quasi-isomorphism.

Applying now the exact localization functor $D^-(S)\to\widetilde{DM}^{eff,-}(S)$ to (7.1) and using Proposition 6.9, we see that the map $C\to C'$ is an isomorphism in $\widetilde{DM}^{eff,-}(S)$. Altogether, we have obtained an isomorphism in $\widetilde{DM}^{eff,-}(S)$ of the form

$$\widetilde{\mathbb{Z}}_S((\mathbb{A}^n\setminus 0)\times S,1)\to\widetilde{\mathbb{Z}}_S(((\mathbb{A}^1\setminus 0)\times S,1)\wedge((\mathbb{A}^{n-1}\setminus 0)\times S,1))[1].$$

Now, the wedge product on the right-hand side can be computed as

$$\widetilde{\mathbb{Z}}_S(((\mathbb{A}^1\setminus 0)\times S,1))\otimes_S\widetilde{\mathbb{Z}}_S(((\mathbb{A}^{n-1}\setminus 0)\times S,1))\cong\widetilde{\mathbb{Z}}_S(1)[1]\otimes_S\widetilde{\mathbb{Z}}_S(n-1)[2n-3]\cong\widetilde{\mathbb{Z}}_S(n)[2n-2]$$

in $\widetilde{DM}^{eff,-}(S)$ by Proposition 5.16 and induction hypothesis. Hence we are done. □

Recall now that for any smooth scheme X and any $v\in\mathscr{P}_X$, we have groups

$$\widetilde{CH}^i(X,v):=\widetilde{CH}^i(X,det(v))$$

for any $i\in\mathbb{N}$. We now discuss the notion of orientation of a vector bundle.

Definition 7.7. *Let $X\in Sm/k$ and let $\mathscr{E}$ be a vector bundle over X. A section $s\in det(\mathscr{E})^\vee(X)$ is called an orientation of $\mathscr{E}$ if s trivializes $det(\mathscr{E})^\vee$. A vector bundle with an orientation is called orientable.*

Definition 7.8. *Let $X\in Sm/k$ and $\mathscr{E}$ be an orientable vector bundle of rank n over X with an orientation s. Define $e(\mathscr{E})$ to be the map such that the following diagram commutes (see [Fas08, Définition 13.2.1]):*

$$\begin{tikzcd}
\widetilde{CH}^0(X) \arrow[r, "\widetilde{c}_n(\mathscr{E})"] \arrow[d, "e(\mathscr{E})"'] & \widetilde{CH}^n(X,-\mathscr{E}) \\
\widetilde{CH}^n(X) \arrow[ur, "s", "\cong"'] &
\end{tikzcd}$$

where $\widetilde{c}_n(\mathscr{E})$ is the Euler class of $\mathscr{E}$. If $n=2$, define the first Pontryagin class under the orientation s of $\mathscr{E}$ to be $-e(\mathscr{E})(1)\in\widetilde{CH}^2(X)$ (see [AF16, remark before Proposition 3.1.1]), which is denoted by $p_1(\mathscr{E})$.

Definition 7.9. *Let $\mathscr{E}$ be a vector bundle of rank n over $X \in Sm/S$ and let s be an orientation of $\mathscr{E}$. The map*

$$e(\mathscr{E}) : \widetilde{CH}^0(X) \longrightarrow \widetilde{CH}^n(X)$$

defined above gives an element

$$e(\mathscr{E})(1) \in Hom_{DM^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(X), \widetilde{\mathbb{Z}}_{pt}(n)[2n]).$$

It induces a morphism

$$\theta : \widetilde{\mathbb{Z}}_S(X) \longrightarrow \widetilde{\mathbb{Z}}_S(n)[2n]$$

by Proposition 6.13, which is called the Euler class of $\mathscr{E}$ over S under the orientation s.

If $n = 2$, then $-\theta$ is called the first Pontryagin class under the orientation s of $\mathscr{E}$ over S, which is still denoted by $p_1(\mathscr{E})$.

The following lemma is obvious.

Lemma 7.1. *Let $(\mathscr{E}, m)$ be a vector bundle of rank 2 over a scheme X with a skew-symmetric inner product. Then m is non degenerate iff the induced map $\bigwedge^2 \mathscr{E} \longrightarrow O_X$ is an isomorphism.*

Hence for any symplectic bundle of rank 2, there is a canonical orientation induced by the dual of the isomorphism in the above lemma.

Definition 7.10. *Let $\mathscr{E}_1$,$\mathscr{E}_2$ be two vector bundles over a scheme X with orientations s_1, s_2 respectively. An isomorphism $f : \mathscr{E}_1 \longrightarrow \mathscr{E}_2$ is called orientation preserving if $det(f)^\vee(s_2) = s_1$.*

Proposition 7.8. *Let $\mathscr{E}_1$,$\mathscr{E}_2$ be two orientable vector bundles of rank n over a smooth scheme X with orientations s_1, s_2, respectively. If there is an orientation preserving isomorphism $f : \mathscr{E}_1 \longrightarrow \mathscr{E}_2$, then $e(\mathscr{E}_1) = e(\mathscr{E}_2)$.*

Proof. Let E_j be the total space of $\mathscr{E}_j$, $p_j : E_j \longrightarrow X$ be the structure map and $z_j : X \longrightarrow E_j$ be the zero section for $j = 1, 2$. We have a diagram

$$\begin{tikzcd}
\widetilde{CH}^0(X) \arrow[r, "\widetilde{c}_n(\mathscr{E}_1)"] \arrow[dr, "\widetilde{c}_n(\mathscr{E}_2)"'] & \widetilde{CH}^n(X, -\mathscr{E}_1) & \widetilde{CH}^n(X) \arrow[l, "s_1"'] \arrow[dl, "s_2"] \\
& \widetilde{CH}^n(X, -\mathscr{E}_2) \arrow[u, "-f"'] &
\end{tikzcd}$$

in which the right triangle commutes since f is orientation preserving. Hence we only have to prove that the left triangle commutes. For this, use the following commutative diagrams which can be catenated:

$$\begin{tikzcd}
\widetilde{CH}^0(X) \arrow[r, "-s_1+s_1"] \arrow[ddr, "-s_2+s_2"'] & \widetilde{CH}^0(X, \mathscr{E}_1 - \mathscr{E}_1) \\
& \widetilde{CH}^0(X, \mathscr{E}_1 - \mathscr{E}_2) \arrow[u, "id_{\mathscr{E}_1}-f"'] \\
& \widetilde{CH}^0(X, \mathscr{E}_2 - \mathscr{E}_2) \arrow[u, "(f+id_{E_2})^{-1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\widetilde{CH}^0(X, \mathscr{E}_1 - \mathscr{E}_1) \arrow[r, "z_{1*}"] & \widetilde{CH}^n(E_1, -p_1^*\mathscr{E}_1) & \widetilde{CH}^n(X, -\mathscr{E}_1) \arrow[l, "p_1^*"'] \\
\widetilde{CH}^0(X, \mathscr{E}_1 - \mathscr{E}_2) \arrow[u] \arrow[r, "z_{1*}"] & \widetilde{CH}^n(E_1, -p_1^*\mathscr{E}_2) \arrow[u, "-p_1^*(f)"'] & \\
\widetilde{CH}^0(X, \mathscr{E}_2 - \mathscr{E}_2) \arrow[u] \arrow[r, "z_{2*}"] & \widetilde{CH}^n(E_2, -p_2^*\mathscr{E}_2) \arrow[u, "f^*"'] & \widetilde{CH}^n(X, -\mathscr{E}_2). \arrow[l, "p_2^*"'] \arrow[ul, "p_1^*"'] \arrow[uu, "-f"']
\end{tikzcd}$$

□

As an application, if two symplectic bundles of rank 2 are isomorphic (including their inner products) then their first Pontryagin classes under the canonical orientations are equal. Note that if they are just isomorphic as vector bundles, the statement is not true any more, since we can use automorphisms of trivial bundles.

Our next aim is to calculate the motive of HP^n. Let $x_1, \ldots, x_{2n+2}$ be the coordinates of the underlying vector space of HP^n. For any $a = 1, \ldots, n+1$, set $V_a = \sum_{i \neq a+n+1} k \cdot x_i$, $X_0^a = HP^n \setminus Gr(2, V_a)$. We have a diagram:

$$\begin{tikzcd}
& \mathrm{Spec}\ k & \\
& HP^{n-1} \arrow[u, "v"] \arrow[dr, "k"] & \\
(X_0^{n+1})^c \arrow[uur, "u"] \arrow[ur, "\pi"] \arrow[rr, "j"] & & HP^n, \arrow[uul, "w"']
\end{tikzcd} \qquad (*)$$

where u, v, w are the structure maps, $(X_0^{n+1})^c$ is the closed complement of X_0^{n+1} in HP^n,

$$k\left(\left(\begin{array}{c} x_1, \ldots, x_{2n} \\ y_1, \ldots, y_{2n} \end{array}\right)\right) = \left(\begin{array}{c} x_1, \ldots, x_n, 0, x_{n+1}, \ldots, x_{2n}, 0 \\ y_1, \ldots, y_n, 0, y_{n+1}, \ldots, y_{2n}, 0 \end{array}\right),$$

j is the inclusion and

$$\pi\left(\left(\begin{array}{c} x_1, \ldots, x_{2n+1}, 0 \\ y_1, \ldots, y_{2n+1}, 0 \end{array}\right)\right) = \left(\begin{array}{c} x_1, \ldots, x_n, x_{n+2}, \ldots, x_{2n+1} \\ y_1, \ldots, y_n, y_{n+2}, \ldots, y_{2n+1} \end{array}\right)$$

(here, $\left(\begin{array}{c} v_1 \\ v_2 \end{array}\right)$ means a two dimensional subspace written in its coordinates spanned by v_1, v_2 in a k-vector space). Note that the lower diagram doesn't commute, i.e. $k \circ \pi \neq j$.

Proposition 7.9. *The following results hold:*

1. $$\pi^*(\mathscr{U}^\vee_{HP^{n-1}}) \cong j^*(\mathscr{U}^\vee_{HP^n})$$

   as symplectic bundles.

2. *If $z : HP^n \longrightarrow \mathscr{U}^\vee$ is the zero section of $\mathscr{U}^\vee$ then there is a section s of $\mathscr{U}^\vee$ such that we have a transversal cartesian square (see [AF16, Theorem 2.4.1]):*

$$\begin{tikzcd}
(X_0^{n+1})^c \arrow[r, "j"] \arrow[d, "j"'] & HP^n \arrow[d, "z"] \\
HP^n \arrow[r, "s"'] & \mathscr{U}^\vee_{HP^n}.
\end{tikzcd}$$

Proof. See [PW10, Theorem 4.1, (d), (e)]. □

Theorem 7.1. *For any $n \geq 0$, we have*

$$\widetilde{\mathbb{Z}}_{pt}(HP^n) \cong \oplus_{i=0}^{n} \widetilde{\mathbb{Z}}_{pt}(2i)[4i]$$

in $\widetilde{DM}^{eff,-}(pt)$.

Proof. Set $U_n^a = \bigcup_{i=1}^a X_0^i \subseteq HP^n$. The normal bundle $N_{(X_0^1)^c/HP^n}$ is symplectic by replacing X_0^{n+1} byX_0^1 in Proposition 7.9. So the normal bundle $N_a := N_{(U_a \setminus X_0^1)/U_a}$ is also symplectic of rank 2 and has a canonical orientation s_a. Moreover, we have an $\mathbb{A}^2$-bundle $\pi : (X_0^1)^c \longrightarrow HP^{n-1}$ by [PW10, Theorem 3.2], and then $U_n^{a-1} \setminus X_0^1$ is also an $\mathbb{A}^2$-bundle over U_{n-1}^{a-1}.

Now we prove by induction that

$$\widetilde{\mathbb{Z}}_{pt}(U_n^a) \cong \oplus_{i=0}^{a-1} \widetilde{\mathbb{Z}}_{pt}(2i)[4i].$$

This is true for $a = 1$ by [PW10, Theorem 3.4(a)] and Proposition 6.9. We thus suppose it's true for some $a \geq 1$ and prove the result for $a + 1$. Let then

$$\theta : \widetilde{\mathbb{Z}}_{pt}(U_n^a) \longrightarrow \oplus_{i=0}^{a-1} \widetilde{\mathbb{Z}}_{pt}(2i)[4i]$$

be such an isomorphism.

We claim that the inclusion $j : \widetilde{\mathbb{Z}}_{pt}(U_n^a) \longrightarrow \widetilde{\mathbb{Z}}_{pt}(U_n^{a+1})$ splits in $\widetilde{DM}^{eff,-}(pt)$. Indeed, Proposition 7.2 yields a commutative diagram in which the vertical homomorphisms are isomorphisms

$$\begin{tikzcd}
\mathrm{Hom}_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(U_n^{a+1}), \widetilde{\mathbb{Z}}_{pt}(U_n^a)) \arrow[r, "j"] \arrow[d, "\theta"] & \mathrm{Hom}_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(U_n^a), \widetilde{\mathbb{Z}}_{pt}(U_n^a)) \arrow[d, "\theta"] \\
\mathrm{Hom}_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(U_n^{a+1}), \oplus_{i=0}^{a-1} \widetilde{\mathbb{Z}}_{pt}(2i)[4i]) \arrow[r, "j"] \arrow[d] & \mathrm{Hom}_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(U_n^a), \oplus_{i=0}^{a-1} \widetilde{\mathbb{Z}}(2i)[4i]) \arrow[d] \\
\bigoplus_{i=0}^{a-1} \widetilde{CH}^{2i}(U_n^{a+1}) \arrow[r, "j^*"] & \bigoplus_{i=0}^{a-1} \widetilde{CH}^{2i}(U_n^a).
\end{tikzcd}$$

It suffices then to prove that for any $i = 0, 2, \ldots, 2a - 2$, the pull-back

$$j^* : \widetilde{CH}^i(U_n^{a+1}) \longrightarrow \widetilde{CH}^i(U_n^a)$$

is an isomorphism since the first horizontal arrow in the above diagram will then also be an isomorphism.

We use induction on a again to prove the claim on j^*. The case $i = 0$ is easy. Hence, we may suppose that $i > 0$, which implies that $a, n > 1$. The result now follows by induction, using the following commutative diagrams (see [Fas08, Remarque 10.4.8], [Fas08, Corollaire 10.4.10] and [Fas08, Corollaire 11.3.2]) and noting that the exact rows in the first one are split by [PW10, Theorem 3.4(a)]:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{CH}^i_{U_n^{a+1} \setminus X_0^1}(U_n^{a+1}) \arrow[r] \arrow[d] & \widetilde{CH}^i(U_n^{a+1}) \arrow[r] \arrow[d] & \widetilde{CH}^i(X_0^1) \arrow[r] \arrow[d, equal] & 0, \\
0 \arrow[r] & \widetilde{CH}^i_{U_n^a \setminus X_0^1}(U_n^a) \arrow[r] & \widetilde{CH}^i(U_n^a) \arrow[r] & \widetilde{CH}^i(X_0^1) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\widetilde{CH}^{i-2}(U_n^{a+1} \setminus X_0^1) \arrow[r, "\cong", "s_{a+1}"'] \arrow[d] & \widetilde{CH}^{i-2}(U_n^{a+1} \setminus X_0^1, N_{a+1}) \arrow[r, "\cong", "Thom"'] \arrow[d] & \widetilde{CH}^i_{U_n^{a+1}\setminus X_0^1}(U_n^{a+1}) \arrow[d] \\
\widetilde{CH}^{i-2}(U_n^{a} \setminus X_0^1) \arrow[r, "\cong", "s_a"'] & \widetilde{CH}^{i-2}(U_n^{a} \setminus X_0^1, N_a) \arrow[r, "\cong", "Thom"'] & \widetilde{CH}^i_{U_n^{a}\setminus X_0^1}(U_n^{a})
\end{tikzcd},$$

$$\begin{tikzcd}
\widetilde{CH}^{i-2}(U_{n-1}^a) \arrow[r, "\cong", "\mathbb{A}^2-bundle"'] \arrow[d] & \widetilde{CH}^{i-2}(U_n^{a+1} \setminus X_0^1) \arrow[d] \\
\widetilde{CH}^{i-2}(U_{n-1}^{a-1}) \arrow[r, "\cong", "\mathbb{A}^2-bundle"'] & \widetilde{CH}^{i-2}(U_n^{a} \setminus X_0^1).
\end{tikzcd}$$

Now, we have an exact sequence of sheaves by Proposition 5.10

$$0 \longrightarrow \widetilde{\mathbb{Z}}_{pt}(U_n^a \cap X_0^{a+1}) \longrightarrow \widetilde{\mathbb{Z}}_{pt}(U_n^a) \oplus \widetilde{\mathbb{Z}}_{pt}(X_0^{a+1}) \longrightarrow \widetilde{\mathbb{Z}}_{pt}(U_n^{a+1}) \longrightarrow 0,$$

yielding an exact triangle in $\widetilde{DM}^{eff,-}(pt)$. Moreover, we have an $\mathbb{A}^1$-bundle $p : \mathbb{A}^{4n+1} \longrightarrow X_0^{a+1}$ (see [PW10, Theorem 3.4(a)]) and it follows that

$$\widetilde{\mathbb{Z}}_{pt}(U_n^a \cap X_0^{a+1}) \cong \widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^{2a} \setminus 0 \times \mathbb{A}^{4n-2a+1}) \cong \widetilde{\mathbb{Z}} \oplus \widetilde{\mathbb{Z}}_{pt}(2a)[4a-1]$$

by Proposition 7.7. Killing one point, we get a distinguished triangle in $\widetilde{DM}^{eff,-}(pt)$

$$\widetilde{\mathbb{Z}}_{pt}(2a)[4a-1] \longrightarrow \widetilde{\mathbb{Z}}_{pt}(U_n^a) \longrightarrow \widetilde{\mathbb{Z}}_{pt}(U_n^{a+1}) \longrightarrow \widetilde{\mathbb{Z}}_{pt}(2a)[4a].$$

We have proved that j splits and therefore

$$\widetilde{\mathbb{Z}}_{pt}(U_n^{a+1}) \simeq \widetilde{\mathbb{Z}}_{pt}(U_n^a) \oplus \widetilde{\mathbb{Z}}_{pt}(2a)[4a]$$

completing the induction process. □

Now we want to improve Theorem 7.1 and find an explicit isomorphism using the first Pontryagin class of the dual tautological bundle on HP^n.

The following proposition has a very similar version in [PW10, Theorem 8.1], but the twists are considered here.

Proposition 7.10. *Let $w : HP^n \to \mathrm{Spec}(k)$ be the structure map. Then the map*

$$\begin{array}{rcl} f_{n,i} : \widetilde{CH}^0(\mathrm{Spec}(k)) & \longrightarrow & \widetilde{CH}^{2i}(HP^n) \\ x & \longmapsto & w^*(x) \cdot p_1(\mathscr{U}^\vee)^i \end{array}$$

is an isomorphism of abelian groups for $i = 0, \ldots, n$. Here, $\mathscr{U}^\vee$ is endowed with its canonical orientation.

Proof. We prove the result by induction on n and use the notation of Diagram (*) above. If $n = 0$, there is nothing to prove.

We first note that $j^*(\mathscr{U}^\vee_{HP^n}) \cong N_{(X_0^{n+1})^c/HP^n}$ by Proposition 7.9. Now, we have a commutative diagram with split exact row for any $i \geq 0$ (as in Theorem 7.1)

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{CH}^{2i-2}((X_0^{n+1})^c, -j^*\mathscr{U}_{HP^n}) \arrow[r, "j_*"] & \widetilde{CH}^{2i}(HP^n) \arrow[r] & \widetilde{CH}^{2i}(X_0^{n+1}) \arrow[r] & 0, \\
& \widetilde{CH}^{2i-2}((X_0^{n+1})^c, -j^*\mathscr{U}_{HP^n} + j^*\mathscr{U}_{HP^n}) \arrow[u, "t"] \arrow[r, "j_*"] & \widetilde{CH}^{2i}(HP^n, \mathscr{U}_{HP^n}) \arrow[u, "t'"] & & \\
& \widetilde{CH}^{2i-2}((X_0^{n+1})^c) \arrow[u, "o"] & \widetilde{CH}^{2i-2}(HP^{n-1}) \arrow[l, "\pi^*"'] & \widetilde{CH}^0(\mathrm{Spec}\ k) \arrow[l, "f_{n-1,i-1}"] &
\end{tikzcd}$$

where t (resp. t') is induced (Definition 7.7) by the canonical orientation of $j^*\mathscr{U}_{HP^n}$ (resp. $\mathscr{U}_{HP^n}$) and o is the cancellation map induced by the canonical orientation. On the other hand, we have an $\mathbb{A}^1$-bundle

$$p : \mathbb{A}^{4n+1} \longrightarrow X_0^a$$

by [PW10, Theorem 3.4(a)]. It follows that the statement is true for $i = 0$. Moreover, it follows that $\widetilde{CH}^{2i}(X_0^{n+1}) = 0$ if $i > 0$. Thus j_* is an isomorphism if $i > 0$. In this case, the map $-j_* \circ t \circ o \circ \pi^* \circ f_{n-1,i-1}$ will also be an isomorphism. It suffices to show that it is equal to $f_{n,i}$ to conclude.

Pick $s \in \widetilde{CH}^0(\mathrm{Spec}\ k)$. Then

$$\begin{aligned}
&- j_*(t(o(\pi^*(f_{n-1,i-1}(s)))))\\
=&- j_*(t(o(\pi^*(v^*(s) \cdot p_1(\mathscr{U}^\vee_{HP^{n-1}})^{i-1}))))\\
&\text{by definition}\\
=&- t'(j_*(o(\pi^*(v^*(s)) \cdot j^*(p_1(\mathscr{U}^\vee_{HP^n})^{i-1}))))\\
&\text{by Axiom 10, Proposition 7.9 and the square in the diagram}\\
=&- t'(j_*(o(\pi^*(v^*(s)))) \cdot p_1(\mathscr{U}^\vee_{HP^n})^{i-1})\\
&\text{by Axiom 18 for } j\\
=&- t'(j_*(o(j^*(w^*(s)))) \cdot p_1(\mathscr{U}^\vee_{HP^n})^{i-1})\\
&\text{by Axiom 9}\\
=&- t'(j_*(j^*(w^*(s)) \cdot o(1)) \cdot p_1(\mathscr{U}^\vee_{HP^n})^{i-1})\\
&\text{by Axiom 7, Axiom 10 and functoriality of pull-back with respect to twists}\\
=&- t'(w^*(s) \cdot j_*(o(1)) \cdot p_1(\mathscr{U}^\vee_{HP^n})^{i-1})\\
&\text{by Axiom 18 for } j\\
=&- w^*(s) \cdot t'(j_*(o(1))) \cdot p_1(\mathscr{U}^\vee_{HP^n})^{i-1}\\
&\text{by functoriality of products with respect to twists.}
\end{aligned}$$

Denote the map

$$\widetilde{CH}^0(HP^n) \longrightarrow \widetilde{CH}^0(HP^n, -\mathscr{U}_{HP^n} + \mathscr{U}_{HP^n})$$

by o'. So we see that

$$t'(j_*(o(1))) = t'(j_*j^*(o'(1))) = t'(s^*(z_*(o'(1)))) = t'((p^*)^{-1}(z_*(o'(1)))) = e(\mathscr{U}^\vee_{HP^n})(1)$$

by Axiom 16, yielding the result. □

Lemma 7.2. *Let X be a smooth scheme and let $i, j \geq 0$. Then*

$$Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(X)(i)[2i], \widetilde{\mathbb{Z}}_{pt}(j)[2j]) = \begin{cases} 0 & \text{if } i > j. \\ \widetilde{CH}^{j-i}(X) & \text{if } i \leq j. \end{cases}$$

Proof. If $i \leq j$, the lemma follows from Proposition 7.2 and [FØ16, Theorem 5.0.1]. Suppose then that $i > j$. The exact sequence of sheaves with MW-transfers

$$0 \to \widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i \setminus 0) \to \widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i) \to \widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i)/\widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i \setminus 0) \to 0$$

yields an exact triangle in $\widetilde{DM}^{eff,-}(pt)$ of the form

$$\widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i \setminus 0) \to \widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i) \to \widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i)/\widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i \setminus 0) \to \widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i \setminus 0)[1]$$

As $\widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i) \simeq \widetilde{\mathbb{Z}}_{pt}(\mathrm{Spec}(k))$ by Proposition 6.9, we see that the first map is split. Consequently, we get an isomorphism

$$\widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i)/\widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i \setminus 0) \simeq \widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i \setminus 0, 1)[1]$$

and it follows from Proposition 7.7 that $\widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i)/\widetilde{\mathbb{Z}}_{pt}(\mathbb{A}^i \setminus 0) \simeq \widetilde{\mathbb{Z}}_{pt}(i)[2i]$ in $\widetilde{DM}^{eff,-}(pt)$. Therefore,

$$\widetilde{\mathbb{Z}}_{pt}(X)(i)[2i] \simeq \widetilde{\mathbb{Z}}_{pt}(X \times \mathbb{A}^i)/\widetilde{\mathbb{Z}}_{pt}(X \times (\mathbb{A}^i \setminus 0))$$

and it follows from Propositions 7.1 and 7.2 that

$$Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(X)(i)[2i], \widetilde{\mathbb{Z}}_{pt}(j)[2j]) \simeq \widetilde{CH}^j_{X\times 0}(X \times \mathbb{A}^i) = 0.$$

□

Corollary 7.1. *For any $i, j \geq 0$, we have*

$$Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(i)[2i], \widetilde{\mathbb{Z}}_{pt}(j)[2j]) = \begin{cases} 0 & \text{if } i \neq j. \\ \widetilde{CH}^0(k) & \text{if } i = j. \end{cases}$$

In other terms, the motives $\widetilde{\mathbb{Z}}_{pt}(i)[2i]$ are mutually orthogonal in the triangulated category $\widetilde{DM}^{eff,-}(pt)$.

Lemma 7.3. *Let $\mathscr{C}$ be an additive category. Let M, M_i, $i = 1, \ldots, n$ be objects in $\mathscr{C}$ such that $Hom_{\mathscr{C}}(M_i, M_j) = 0$ if $i \neq j$. Suppose that there is an isomorphism $\varphi : M \longrightarrow \oplus_i M_i$. Then, a morphism $\varphi' : M \longrightarrow \oplus_i M_i$ is an isomorphism if and only if φ'_i is a free generator of $Hom_{\mathscr{C}}(M, M_i)$ as left $End_{\mathscr{C}}(M_i)$-module for any i, where φ'_i is the composite of φ' and the i^{th} projection.*

Proof. Suppose that φ' is an isomorphism. We prove that φ'_i a free generator of $Hom_{\mathscr{C}}(M, M_i)$ as a left $End_{\mathscr{C}}(M_i)$-module. We note that the action is free since φ'_i is surjective. Now, suppose that $\psi \in Hom_{\mathscr{C}}(M, M_i)$. Since $Hom_{\mathscr{C}}(M_i, M_j) = 0$ if $i \neq j$, we see that $\psi = (\psi \circ \varphi'^{-1} \circ i_i) \circ (\varphi'_i)$ where i_i is the natural map from M_i to the direct sum. Hence ψ can be generated by φ'_i and φ'_i is indeed a free generator.

Conversely, if we have a morphism $\varphi' : M \longrightarrow \oplus_i M_i$ such that φ'_i is a free generator of $Hom_{\mathscr{C}}(M, M_i)$, then $\varphi'_i = f_i \circ \varphi_i$ for some isomorphism f_i. Hence φ' is also an isomorphism.

□

Theorem 7.2. *The map*

$$\widetilde{\mathbb{Z}}_{pt}(HP^n) \xrightarrow{\sum p_1(\mathscr{U}^\vee)^i} \oplus_{i=0}^n \widetilde{\mathbb{Z}}_{pt}(2i)[4i]$$

is an isomorphism in $\widetilde{DM}^{eff,-}(pt)$. Here, $\mathscr{U}^\vee$ is endowed with its canonical orientation.

Proof. By Theorem 7.1, Corollary 7.1 and Lemma 7.3, it remains to prove that $p_1(\mathscr{U}^\vee)^i$ is a free generator of $Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(HP^n), \widetilde{\mathbb{Z}}_{pt}(2i)[4i])$. By [FØ16, Theorem 5.0.1], the ring $End_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(2i)[4i])$ is commutative, so we only have to prove that $p := p_1(\mathscr{U}^\vee)^i$ generates $Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(HP^n), \widetilde{\mathbb{Z}}_{pt}(2i)[4i])$.

Using the notation of Diagram (*), we see that the composite

$$\begin{array}{ccc} Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}, \widetilde{\mathbb{Z}}_{pt}) & \xrightarrow{w} & Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(HP^n), \widetilde{\mathbb{Z}}_{pt}) \\ & & \downarrow{\cdot p^i} \\ & & Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(HP^n), \widetilde{\mathbb{Z}}_{pt}(2i)[4i]) \end{array}$$

is an isomorphism by Proposition 7.10. Now given a map

$$\psi \in Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(HP^n), \widetilde{\mathbb{Z}}_{pt}(2i)[4i]),$$

we can find its preimage λ under the map above. So we have a commutative diagram:

$$\begin{tikzcd}
 & & & \widetilde{\mathbb{Z}}_{pt}(2i)[4i] \\
\widetilde{\mathbb{Z}}_{pt}(HP^n) \arrow[r, "\triangle"] \arrow[urrr, bend left, "\psi"] \arrow[ddr, "p^i"'] & \widetilde{\mathbb{Z}}_{pt}(HP^n) \otimes \widetilde{\mathbb{Z}}_{pt}(HP^n) \arrow[r, "id\otimes w"] \arrow[dr, "p^i\otimes w"] & \widetilde{\mathbb{Z}}_{pt}(HP^n) \otimes \widetilde{\mathbb{Z}}_{pt} \arrow[r, "p^i\otimes\lambda"] & \widetilde{\mathbb{Z}}_{pt}(2i)[4i] \otimes \widetilde{\mathbb{Z}}_{pt} \arrow[u] \\
 & & \widetilde{\mathbb{Z}}_{pt}(2i)[4i] \otimes \widetilde{\mathbb{Z}}_{pt} \arrow[ur, "id\otimes\lambda"'] \arrow[dl] & \\
 & \widetilde{\mathbb{Z}}_{pt}(2i)[4i] & &
\end{tikzcd}$$

showing that ψ is generated by p^i. We are done. $\square$

For any $S \in Sm/k$, we have a projection $p_S : HP^n_S \longrightarrow HP^n$ and we set $\mathscr{U}_S^\vee = p_S^*\mathscr{U}^\vee$.

Theorem 7.3. *The map*

$$\widetilde{\mathbb{Z}}_S(HP^n_S) \xrightarrow{p_1(\mathscr{U}_S^\vee)^i} \oplus_{i=0}^n \widetilde{\mathbb{Z}}_S(2i)[4i]$$

is an isomorphism in $\widetilde{DM}^{eff,-}(S)$. Here, $\mathscr{U}_S^\vee$ is endowed with its canonical orientation.

Proof. We have a commutative diagram

$$\begin{tikzcd}
p^*\widetilde{\mathbb{Z}}_{pt}(HP^n) \arrow[r, "p^*(p_1(\mathscr{U}^\vee))"] & p^*\widetilde{\mathbb{Z}}_{pt}(2)[4] \\
\widetilde{\mathbb{Z}}_S(HP^n_S) \arrow[r, "p_1(\mathscr{U}_S^\vee)"] \arrow[u, "\cong"] & \widetilde{\mathbb{Z}}_S(2)[4]. \arrow[u, "\cong"]
\end{tikzcd}$$

Hence the result follows by the commutative diagram

$$\begin{tikzcd}
p^*\widetilde{\mathbb{Z}}_{pt}(HP^n) \arrow[r, "p^*(p_1(\mathscr{U}^\vee))^i"] & \oplus_{i=0}^n p^*\widetilde{\mathbb{Z}}_{pt}(2i)[4i] \\
\widetilde{\mathbb{Z}}_S(HP^n_S) \arrow[r, "p_1(\mathscr{U}_S^\vee)^i"] \arrow[u, "\cong"] & \oplus_{i=0}^n \widetilde{\mathbb{Z}}_S(2i)[4i], \arrow[u, "\cong"]
\end{tikzcd}$$

where the upper horizontal arrow is an isomorphism by the theorem above. $\square$

Theorem 7.4. *Let $X \in Sm/S$ and let $(\mathscr{E}, m)$ be a symplectic vector bundle of rank $2n+2$ on X. Let $\pi : HGr_X(\mathscr{E}) \to X$ be the projection. Then, the map*

$$\widetilde{\mathbb{Z}}_S(HGr_X(\mathscr{E})) \xrightarrow{\pi\boxtimes p_1(\mathscr{U}^\vee)^i} \oplus_{i=0}^n \widetilde{\mathbb{Z}}_S(X)(2i)[4i]$$

is an isomorphism in $\widetilde{DM}^{eff,-}(S)$ functorial for X in Sm/S. Here, $\mathscr{U}^\vee$ is endowed with its canonical orientation.

Proof. We first prove that the map

$$\widetilde{\mathbb{Z}}_S(HGr_X(\mathscr{E})) \xrightarrow{\pi\boxtimes p_1(\mathscr{U}^\vee)^i} \oplus_{i=0}^n \widetilde{\mathbb{Z}}_S(X)(2i)[4i]$$

is functorial in X. Let then $f : Y \to X$ be a morphism of S-schemes. We have a commutative diagram

$$\begin{tikzcd} HGr_Y(f^*\mathscr{E}) \arrow[r] \arrow[d, "\pi"'] & HGr_X(\mathscr{E}) \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

yielding a commutative diagram in $\widetilde{DM}^{eff,-}(S)$

$$\begin{tikzcd} \widetilde{\mathbb{Z}}_S(HGr_Y(f^*\mathscr{E})) \arrow[r] \arrow[d, "\pi"'] & \widetilde{\mathbb{Z}}_S(HGr_X(\mathscr{E})) \arrow[d, "\pi"] \\ \widetilde{\mathbb{Z}}_S(Y) \arrow[r, "f"'] & \widetilde{\mathbb{Z}}_S(X). \end{tikzcd}$$

On the other hand, we have a commutative diagram

$$\begin{tikzcd} \widetilde{\mathbb{Z}}_S(HGr_Y(f^*\mathscr{E})) \arrow[r] \arrow[d, "p_1(\mathscr{U}^\vee)^i"'] & \widetilde{\mathbb{Z}}_S(HGr_X(\mathscr{E})) \arrow[d, "p_1(\mathscr{U}^\vee)^i"] \\ \widetilde{\mathbb{Z}}_S(2i)[4i] \arrow[r, equal] & \widetilde{\mathbb{Z}}_S(2i)[4i] \end{tikzcd}$$

for any i by Proposition 7.2 and naturality of the first Pontryagin class (Proposition 7.8). Consequently, we get a commutative diagram

$$\begin{tikzcd} \widetilde{\mathbb{Z}}_S(HGr_Y(f^*\mathscr{E})) \arrow[r] \arrow[d, "\pi\boxtimes p_1(\mathscr{U}^\vee)^i"'] & \widetilde{\mathbb{Z}}_S(HGr_X(\mathscr{E})) \arrow[d, "\pi\boxtimes p_1(\mathscr{U}^\vee)^i"] \\ \oplus_i\widetilde{\mathbb{Z}}_S(Y)(2i)[4i] \arrow[r, "\oplus_i f(2i)[4i]"'] & \oplus_i\widetilde{\mathbb{Z}}_S(X)(2i)[4i] \end{tikzcd}$$

proving that the map is natural.

Let's now prove the first statement. We pick a finite open covering $\{U_\alpha\}$ of X such that

$$(\mathscr{E}, m)|_{U_\alpha} \cong \left(O_{U_\alpha}^{\oplus 2n+2}, \begin{pmatrix} & I \\ -I & \end{pmatrix}\right)$$

for every α and we work by induction on the number of the open sets. If there is just one open set, $HGr_X(\mathscr{E}) \cong HP^n \times_k X$ and we conclude tensoring the isomorphism of Theorem 7.3 with $\widetilde{\mathbb{Z}}_S(X)$.

Suppose next that $X = U_1 \cup U_2$ and the argument holds for $(\mathscr{E}, m)|_{U_1}$, $(\mathscr{E}, m)|_{U_2}$ and $(\mathscr{E}, m)|_{U_1\cap U_2}$. Set $\mathscr{E}_i$ for the restrictions of $\mathscr{E}$ to U_i and $\mathscr{E}_{12}$ for its restriction to the intersection. Using Proposition 5.10, we obtain exact triangles

$$\widetilde{\mathbb{Z}}_S(U_1 \cap U_2) \to \widetilde{\mathbb{Z}}_S(U_1) \oplus \widetilde{\mathbb{Z}}_S(U_2) \to \widetilde{\mathbb{Z}}_S(X) \to \widetilde{\mathbb{Z}}_S(U_1 \cap U_2)[1] \tag{7.2}$$

and

$$\widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}_{12})) \to \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}_1)) \oplus \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}_2)) \to \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E})) \to (\ldots)[1]. \tag{7.3}$$

Tensoring with $\widetilde{\mathbb{Z}}_S(2i)[4i]$ being exact, we obtain shifted versions of (7.2) and a diagram

$$\begin{tikzcd}
\widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}_{12})) \arrow[r] \arrow[d, "\pi\boxtimes p_1(\mathscr{U}^\vee)^i"'] & \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}_1)) \oplus \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}_2)) \arrow[r] \arrow[d, "\pi\boxtimes p_1(\mathscr{U}^\vee)^i"'] & \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E})) \arrow[r] \arrow[d, "\pi\boxtimes p_1(\mathscr{U}^\vee)^i"'] & (\ldots)[1] \arrow[d] \\
\oplus_i \widetilde{\mathbb{Z}}_S(U_1 \cap U_2)(2i)[4i] \arrow[r] & \oplus_i (\widetilde{\mathbb{Z}}_S(U_1) \oplus \widetilde{\mathbb{Z}}_S(U_2))(2i)[4i] \arrow[r] & \oplus_i \widetilde{\mathbb{Z}}_S(X)(2i)[4i] \arrow[r] & (\ldots)[1].
\end{tikzcd} \tag{7.4}$$

The two left-hand squares commute by naturality, and we now prove that the third also commutes. We have a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathbb{Z}}_S(HGr(\mathscr{E})) \arrow[r] \arrow[d, "\pi"'] & \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}_{12}))[1] \arrow[d, "\pi[1]"] \\
\widetilde{\mathbb{Z}}_S(X) \arrow[r] & \widetilde{\mathbb{Z}}_S(U_1 \cap U_2)[1].
\end{tikzcd}$$

Tensoring with the morphism corresponding to the i-th power of the first Pontryagin class $\widetilde{\mathbb{Z}}_S(HGr(\mathscr{E})) \to \widetilde{\mathbb{Z}}_S(2i)[4i]$, we obtain a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathbb{Z}}_S(HGr(\mathscr{E})) \otimes \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E})) \arrow[r] \arrow[d, "\pi\otimes p_1(\mathscr{U}^\vee)^i"'] & \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}_{12})) \otimes \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}))[1] \arrow[d, "\pi\otimes p_1(\mathscr{U}^\vee)^i[1]"] \\
\widetilde{\mathbb{Z}}_S(X) \otimes \widetilde{\mathbb{Z}}(2i)[4i] \arrow[r] & \widetilde{\mathbb{Z}}_S(U_1 \cap U_2) \otimes \widetilde{\mathbb{Z}}_S(2i)[4i][1].
\end{tikzcd} \tag{7.5}$$

On the other hand, the open cover

$$(HGr(\mathscr{E}_1) \times HGr(\mathscr{E})) \cup (HGr(\mathscr{E}_2) \times HGr(\mathscr{E})) = HGr(\mathscr{E}) \times HGr(\mathscr{E})$$

yields a Mayer-Vietoris triangle, and the commutative diagrams

$$\begin{tikzcd}
HGr(\mathscr{E}_i) \arrow[r] \arrow[d] & HGr(\mathscr{E}) \arrow[d] \\
HGr(\mathscr{E}_i) \times HGr(\mathscr{E}) \arrow[r] & HGr(\mathscr{E}) \times HGr(\mathscr{E}),
\end{tikzcd}$$

in which the first vertical arrow is the product of the identity and the inclusion and the second vertical arrow is the diagonal map, induce a morphism of Mayer-Vietoris triangles and in particular a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathbb{Z}}_S(HGr(\mathscr{E})) \arrow[r] \arrow[d, "\triangle"'] & \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}_{12}))[1] \arrow[d] \\
\widetilde{\mathbb{Z}}_S(HGr(\mathscr{E})) \otimes \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E})) \arrow[r] & \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}_{12})) \otimes \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}))[1]
\end{tikzcd} \tag{7.6}$$

where the right-hand vertical map is the tensor of the identity with the morphism $\widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}_{12})) \to \widetilde{\mathbb{Z}}_S(HGr(\mathscr{E}))$.

Concatenating Diagrams (7.5) and (7.6), we obtain that the third triangle in (7.4) also commutes. Moreover, our induction hypothesis and the five lemma imply that the third morphism in (7.4) is an isomorphism as well.

We conclude the proof of the theorem by observing that we may reduce the case of a general covering $\{U_\alpha\}$ of X to the case of a covering by two open subschemes using induction again. $\square$

Arguing as in [PW10, Theorem 8.2], we can deduce a similar version of Pontryagin classes for Chow-Witt rings.

Proposition 7.11. *Let $X \in Sm/k$, $\mathscr{E}$ be a symplectic bundle of rank $2n+2$ over X and $k = min\{\lfloor \frac{j}{2} \rfloor, n\}$. Then the map*

$$\theta_j : \oplus_{i=0}^{k} \widetilde{CH}^{j-2i}(X) \xrightarrow{p^* \cdot p_1(\mathscr{U}^\vee)^i} \widetilde{CH}^{j}(HGr_X(\mathscr{E}))$$

is an isomorphism, where $j \geq 0$, $p : HGr_X(\mathscr{E}) \longrightarrow X$ is the structure map and $\mathscr{U}^\vee$ is the dual tautological bundle endowed with its canonical orientation.

Proof. Write $\widetilde{DM}$ in place of $\widetilde{DM}^{eff,-}(pt)$ for convenience. We apply $Hom_{\widetilde{DM}}(-, \widetilde{\mathbb{Z}}_{pt}(j)[2j])$ to both sides of the isomorphism in Theorem 7.4. Note that we have an isomorphism for $i \leq \lfloor \frac{j}{2} \rfloor$

$$Hom_{\widetilde{DM}}(\widetilde{\mathbb{Z}}_{pt}(X)(2i)[4i], \widetilde{\mathbb{Z}}_{pt}(j)[2j]) \longrightarrow \widetilde{CH}^{j-2i}(X)$$

by Proposition 7.2.

Now suppose that we have an element $s \in \widetilde{CH}^{j-2i}(X)$, $i \leq k$, which corresponds to a morphism $\varphi : \widetilde{\mathbb{Z}}_{pt}(X) \longrightarrow \widetilde{\mathbb{Z}}_{pt}(j-2i)[2j-4i]$. We conclude the proof using the commutative diagrams

$$\begin{tikzcd}
\widetilde{\mathbb{Z}}_{pt}(HGr_X(\mathscr{E})) \arrow[d, "p\boxtimes p_1(\mathscr{U}^\vee)^i"] \arrow[dr, "(\varphi\circ p)\boxtimes p_1(\mathscr{U}^\vee)^i"] & & \\
\widetilde{\mathbb{Z}}_{pt}(X) \otimes \widetilde{\mathbb{Z}}_{pt}(2i)[4i] \arrow[r, "\varphi\otimes id"] & \widetilde{\mathbb{Z}}(j-2i)[2j-4i] \otimes \widetilde{\mathbb{Z}}_{pt}(2i)[4i] \arrow[r] & \widetilde{\mathbb{Z}}_{pt}(j)[2j]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\oplus_{i=0}^{k} Hom_{\widetilde{DM}}(\widetilde{\mathbb{Z}}_{pt}(X)(2i)[4i], \widetilde{\mathbb{Z}}_{pt}(j)[2j]) \arrow[r, "Hom(-,\widetilde{\mathbb{Z}}_{pt}(j)[2j])", "\cong"'] \arrow[d, "\cong"'] & Hom_{\widetilde{DM}}(\widetilde{\mathbb{Z}}_{pt}(HGr_X(\mathscr{E})), \widetilde{\mathbb{Z}}_{pt}(j)[2j]) \arrow[d, "\cong"'] \\
\oplus_{i=0}^{k} \widetilde{CH}^{j-2i}(X) \arrow[r, "p^* \cdot p_1(\mathscr{U}^\vee)^i"] & \widetilde{CH}^{j}(HGr_X(\mathscr{E})).
\end{tikzcd}$$

□

Definition 7.11. *In the above proposition, set $\zeta := p_1(\mathscr{U}^\vee)$ and*

$$\theta_{2n+2}^{-1}(\zeta^{n+1}) := (\zeta_i) \in \oplus_{i=1}^{n+1} \widetilde{CH}^{2i}(X).$$

Define $p_0(\mathscr{E}) = 1 \in \widetilde{CH}^0(X)$ and $p_a(\mathscr{E}) = (-1)^{a-1}\zeta_i$ for $1 \leq a \leq n+1$. The class $p_a(\mathscr{E})$ is called the a^{th} Pontryagin class of $\mathscr{E}$. These classes are uniquely characterized by the Pontryagin polynomial

$$\zeta^{n+1} - p^*(p_1(\mathscr{E}))\zeta^n + \ldots + (-1)^{n+1}p^*(p_{n+1}(\mathscr{E})) = 0.$$

Remark 7.1. *We show that $p_i(E) = 0$ for $i > 0$ if E is a trivial symplectic bundle. It suffices to show that $p_1(\mathscr{U}^\vee) = 0$. If $X = pt$, this is clear since $\widetilde{CH}^2(pt) = 0$. For general cases, E is the pull-back of a trivial symplectic bundle over pt, hence $p_1(\mathscr{U}^\vee)$ vanishes also.*

7.1.3 The Gysin Triangle

Definition 7.12. *Let $X \in Sm/S$ and $Y \subseteq X$ be a closed subset. For any correspondence theory E, consider the quotient sheaf with E-transfers*

$$\widetilde{M}_Y(X) := \widetilde{\mathbb{Z}}_S(X)/\widetilde{\mathbb{Z}}_S(X \setminus Y).$$

Its image in $\widetilde{DM}^{eff,-}(S)$ will be called the relative motive of X with support in Y (see [D07, Definition 2.2] and the remark before [SV00, Corollary 5.3]). By abuse of notation, we still denote it by $\widetilde{M}_Y(X)$.

Our aim in this section is to compute the relative motives in some situations. For this, we'll need the following notion.

Definition 7.13. *Suppose that $X \in Sm/S$ and that E is a vector bundle over X. For any correspondence theory, define $Th_S(E) = \widetilde{M}_X(E)$ where $X \subseteq E$ is the zero section of E. The motive $Th_S(E)$ is called the Thom space of E.*

The following result is sometimes called homotopy purity.

Proposition 7.12. *Let $X \in Sm/S$ and $Y \subseteq X$ be a smooth closed subscheme. Then for any correspondence theory, we have*

$$\widetilde{M}_Y(X) \cong Th_S(N_{Y/X})$$

in $\widetilde{DM}^{eff,-}(S)$.

Proof. Use [P09, Theorem 2.2.8] and Proposition 7.13 below. Alternatively, one may use [MV98, §3, Theorem 2.23] and the sequence of functors of [DF17, §3.2.4.a]. □

Proposition 7.13. *Let $f : X \longrightarrow Y$ be an étale morphism in Sm/S, $Z \subseteq Y$ be a closed subset of Y such that the map $f : f^{-1}(Z) \longrightarrow Z$ is an isomorphism (here, the schemes are endowed with their reduced structure). Then the map $\widetilde{M}_{f^{-1}(Z)}(X) \longrightarrow \widetilde{M}_Z(Y)$ is an isomorphism of sheaves with E-transfers for any correspondence theory E.*

Proof. By the condition given, we get a Nisnevich covering $f \amalg id : X \amalg (Y \setminus Z) \longrightarrow Y$ of Y. So we have a commutative diagram with exact (after sheafication) rows and columns by Proposition 5.10:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \widetilde{c}_S(Y \setminus Z) \arrow[r, equal] \arrow[d] & \widetilde{c}_S(Y \setminus Z) \arrow[d] & \\
\widetilde{c}_S((X \amalg (Y \setminus Z)) \times_Y (X \amalg (Y \setminus Z))) \arrow[r] \arrow[dr, "r"] & \widetilde{c}_S(X \amalg (Y \setminus Z)) \arrow[r, "f \amalg id"] \arrow[d] & \widetilde{c}_S(Y) \arrow[r] \arrow[d] & 0 \\
 & \widetilde{c}_S(X) \arrow[r, "q"] \arrow[d] & \widetilde{c}_S(Y)/\widetilde{c}(Y \setminus Z) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0. &
\end{tikzcd}$$

We want to show that $ker(q) = \widetilde{c}_S(X \setminus f^{-1}(Z))$ after sheafication, yielding the statement. We clearly have $\widetilde{c}_S(X \setminus f^{-1}(Z)) \subseteq ker(q)$ and r maps onto $ker(q)$ after sheafication.

So it suffices to show that $Im(r) \subseteq \widetilde{c}_S(X \setminus f^{-1}(Z))$. The sheaf $\widetilde{c}_S((X \amalg (Y \setminus Z)) \times_Y (X \amalg (Y \setminus Z)))$ is decomposed into four direct components

$$\widetilde{c}_S(X \times_Y X), \widetilde{c}_S(X \times_Y (Y \setminus Z)), \widetilde{c}_S((Y \setminus Z) \times_Y X), \widetilde{c}_S((Y \setminus Z) \times_Y (Y \setminus Z))$$

via disjoint unions so we just have to calculate their images under r respectively. The calculations for the last three components are easy and we only explain the computation of the first one.

We have a Cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, "\pi"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Then for any $x \in \pi^{-1}(Z)$, $p_1(x) = p_2(x)$ and the morphisms $k(p_1(x)) \longrightarrow k(x)$ induced by p_1 and p_2 are equal since $f^{-1}(Z) \cong Z$. So by [Mil80, Corollary 3.13], $p_1 = p_2$ on the connected component containing x. Hence $p_1 = p_2$ on a closed and open set U containing $\pi^{-1}(Z)$. Now, $\widetilde{c}_S(X \times_Y X) = \widetilde{c}_S(U) \oplus \widetilde{c}_S(U^c)$ and so $r|_{\widetilde{c}_S(U)} = 0$. It follows that $Im(r|_{\widetilde{c}_S(U^c)}) \subseteq \widetilde{c}_S(X \setminus f^{-1}(Z))$. So we have proved that $Im(r) \subseteq \widetilde{c}_S(X \setminus f^{-1}(Z))$. □

As a consequence, we see that the study of relative motives (of smooth schemes) reduces to the study of Thom spaces. With this in mind, suppose that X is a smooth scheme and that $(\mathscr{E}, m)$ is a symplectic vector bundle of rank $2n$ over X with total space E. We now study the Thom space of E. Recall first that, as in the discussion before [PW10, Theorem 4.1], $O_X \oplus \mathscr{E} \oplus O_X$ is also a symplectic vector bundle with inner product given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & m & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Definition 7.14. *Let X and E be as above.*

1. *Define N^- by the cartesian square*

$$\begin{tikzcd} Gr_X(2n, \mathscr{E} \oplus O_X) \arrow[r, "i"] & Gr_X(2n, O_X \oplus \mathscr{E} \oplus O_X) \\ N^- \arrow[u] \arrow[r] & HGr_X(O_X \oplus \mathscr{E} \oplus O_X), \arrow[u, "j"'] \end{tikzcd}$$

   where i is induced by the projection $p_{23} : O_X \oplus \mathscr{E} \oplus O_X \longrightarrow \mathscr{E} \oplus O_X$ and j is the inclusion (see Proposition 7.6).

2. *Set*

   $$N = \{x \in Gr_X(2n, O_X \oplus \mathscr{E} \oplus O_X) |\ \mathscr{E}' \longrightarrow p^*(O_X \oplus \mathscr{E} \oplus O_X) \longrightarrow p^*(O_X \oplus O_X) \text{ iso. at } x\},$$

   where $p : Gr_X(2n, O_X \oplus \mathscr{E} \oplus O_X) \longrightarrow X$ is the structure map and

   $$0 \longrightarrow \mathscr{E}' \longrightarrow p^*(O_X \oplus \mathscr{E} \oplus O_X) \longrightarrow \mathscr{E}'' \longrightarrow 0$$

   is the tautological exact sequence. Note that N is an open set of the Grassmannian $Gr_X(2n, O_X \oplus \mathscr{E} \oplus O_X)$.

3. Set

$$V = \{x \in Gr_X(2n, \mathscr{E} \oplus O_X) | \mathscr{F}' \longrightarrow q^*(\mathscr{E} \oplus O_X) \longrightarrow q^* O_X \text{ is an isomorphism at } x\},$$

where $q : Gr_X(2n, \mathscr{E} \oplus O_X) \longrightarrow X$ *is the structure map and*

$$0 \longrightarrow \mathscr{F}' \longrightarrow q^*(\mathscr{E} \oplus O_X) \longrightarrow \mathscr{F}'' \longrightarrow 0$$

is the tautological exact sequence. As above, note that V *is an open set of* $Gr_X(2n, \mathscr{E} \oplus O_X)$.

The notations of N^- and N come from [PW10, Theorem 4.1], but our treatment is slightly different.

Lemma 7.4. *1) Let* T *be an* X*-scheme and let* $f : T \longrightarrow Gr_X(2n, O_X \oplus \mathscr{E} \oplus O_X)$ *be an* X*-morphism. Then*

$$Im(f) \subseteq N \Longleftrightarrow f^*\mathscr{E}' \longrightarrow (p \circ f)^*(O_X \oplus \mathscr{E} \oplus O_X) \longrightarrow (p \circ f)^*(O_X \oplus O_X) \text{ is an isomorphism.}$$

Consequently, $N^- \subseteq N \cap HGr_X(O_X \oplus \mathscr{E} \oplus O_X)$.
2) Let T *be an* X*-scheme and let* $f : T \longrightarrow Gr_X(2n, \mathscr{E} \oplus O_X)$ *be an* X*-morphism. Then*

$$Im(f) \subseteq V \Longleftrightarrow f^*\mathscr{F}' \longrightarrow (q \circ f)^*(\mathscr{E} \oplus O_X) \longrightarrow (q \circ f)^* O_X \text{ is an isomorphism.}$$

Furthermore, $N^- = V$.

Proof. 1) $\Longrightarrow$ Easy. For the $\Longleftarrow$ part, set

$$C = Coker(\mathscr{E}' \longrightarrow p^*(O_X \oplus \mathscr{E} \oplus O_X) \longrightarrow p^*(O_X \oplus O_X)).$$

We see that $N = Supp(C)^c$. Since $f^{-1}(Supp(C)) = Supp(f^*C)$, $f^{-1}(Supp(C)) = \emptyset$ hence $f^{-1}(N) = T$. So $Im(f) \subseteq N$.

For the second statement, let $v : N^- \longrightarrow X$ be the structure map. The bundle N^- has a map φ towards $Gr_X(2n, O_X \oplus \mathscr{E} \oplus O_X)$ hence we have a subbundle $K \subseteq v^*(O_X \oplus \mathscr{E} \oplus O_X)$. Since φ factors through $Gr_X(2n, \mathscr{E} \oplus O_X)$, the first inclusion $v^*O_X \longrightarrow v^*(O_X \oplus \mathscr{E} \oplus O_X)$ factors through K, which makes v^*O_X a subbundle of K. Since φ also factors through $HGr_X(O_X \oplus \mathscr{E} \oplus O_X)$, the inner product is non degenerate on K. So for every $x \in N^-$, there is an affine neighborhood U of x such that $K(U)$ is a free $O_{N^-}(U)$-module with a basis $(1, 0, 0)$ and (x_1, x_2, x_3). Hence $x_3 \in O_{N^-}(U)^*$ by non degeneracy. It follows that the map $K \longrightarrow v^*(O_X \oplus \mathscr{E} \oplus O_X) \longrightarrow v^*(O_X \oplus O_X)$ is surjective on U. So we see that $N^- \subseteq N$ by the first statement.

2) The first statement can be proved as in 1). For the second statement, we have a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r] & v^*(O_X \oplus \mathscr{E}) \arrow[r] & \mathscr{G} \arrow[r] & 0 . \\ 0 \arrow[r] & K' \oplus O_X \arrow[r] \arrow[u, "p_1"] & v^*(O_X \oplus \mathscr{E} \oplus O_X) \arrow[r] \arrow[u, "p_{12}"] & \mathscr{G} \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

Hence there is a section in $K'(N^-)$ which maps to $(1, s, 0)$ in $v^*(O_X \oplus \mathscr{E} \oplus O_X)$. This section turns the map $K' \longrightarrow v^*(O_X \oplus \mathscr{E}) \longrightarrow v^*O_X$ into an isomorphism. So $N^- \subseteq V$. The inclusion $V \subseteq N^-$ can be proved using a similar method. $\square$

Lemma 7.5. *Let* T *be an* X*-scheme and* $f : T \longrightarrow Gr_X(2n, O_X \oplus \mathscr{E} \oplus O_X)$ *be an* X*-morphism. Let* φ *be the composite*

$$(p \circ f)^* O_X \xrightarrow{i_1} (p \circ f)^*(O_X \oplus \mathscr{E} \oplus O_X) \longrightarrow f^*\mathscr{E}'' .$$

Then

$$Im(f) \subseteq Gr_X(2n, \mathscr{E} \oplus O_X)^c \Longleftrightarrow \varphi \text{ is injective and has a locally free cokernel.}$$

Proof. We have

$$Im(f) \subseteq Gr_X(2n, \mathscr{E} \oplus O_X)^c \Longleftrightarrow \forall g : \text{Spec } K \longrightarrow T, Im(f \circ g) \subseteq Gr_X(2n, \mathscr{E} \oplus O_X)^c,$$

where K is a field. So let's assume $T = \text{Spec } K$. In this case,

$$Im(f) \subseteq Gr_X(2n, \mathscr{E} \oplus O_X)^c \Longleftrightarrow f \text{ does not factor through } Gr_X(2n, \mathscr{E} \oplus O_X),$$

and the latter condition is equivalent to $\varphi \neq 0$. Hence

$$Im(f) \subseteq Gr_X(2n, \mathscr{E} \oplus O_X)^c \Longleftrightarrow \forall g : \text{Spec } K \longrightarrow T, g^*(\varphi) \neq 0.$$

Now we may assume that T is affine and use the residue fields of T. Locally, the map φ is of the form $(a_i) : A \longrightarrow A^{\oplus 2n}$ and the condition just says that the ideal (a_i) is the unit ideal. This is equivalent to (a_i) being injective and $Coker((a_i))$ being projective. This just says that φ is injective and has a locally free cokernel. □

Consider next the following square

$$\begin{tikzcd} N^- \arrow[r, "l"] \arrow[d, "v"'] & N \arrow[d, "u"] \\ X \arrow[r, "z"'] & E \end{tikzcd}$$

where l is given by $N^- \subseteq N$ and v is just the structure map (of N^-). Let $r : N \longrightarrow X$ be the structure map of N. We have the tautological exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & r^*(O_X \oplus O_X) & \longrightarrow & r^*(O_X \oplus \mathscr{E} \oplus O_X) & \longrightarrow & r^*\mathscr{E} & \longrightarrow & 0 \\ & & (1,0) & \longmapsto & (1, s_1, 0) & & & & (**) \\ & & (0,1) & \longmapsto & (0, s_2, 1) & & & & \end{array}$$

and u is induced by s_1. Finally, z is the zero section of E.

Proposition 7.14. *The above square is a Cartesian square.*

Proof. The map l induces an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & v^*(O_X \oplus O_X) & \longrightarrow & v^*(O_X \oplus \mathscr{E} \oplus O_X) & \longrightarrow & v^*\mathscr{E} & \longrightarrow & 0. \\ & & (1,0) & \longmapsto & (1, s, 0) & & & & \end{array}$$

But $(1, 0, 0)$ belongs to the kernel, so $s = 0$. Hence the square commutes and is Cartesian. □

Now, we use the square

$$\begin{tikzcd} N \arrow[r, "w"] \arrow[d, "u"'] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"'] & X, \end{tikzcd}$$

where w is induced by s_2 in (**). We see right away that it's a Cartesian square and it follows that u is an $\mathbb{A}^{2n}$-bundle.

The third step in our calculation of $Th_S(E)$ is the following theorem. It has a similar version in [PW10, Proposition 4.3], but we are not considering the same embedding as there.

Proposition 7.15. *For any correspondence theory,*

$$Th_S(E) \cong \widetilde{M}_{N^-}(N) \cong \widetilde{M}_{N^-}(HGr_X(O_X \oplus \mathscr{E} \oplus O_X))$$

in $\widetilde{DM}^{eff,-}(S)$.

Proof. The first isomorphism comes from Proposition 7.14 and the fact that $u : N \longrightarrow E$ is an $\mathbb{A}^{2n}$-bundle. The second isomorphism follows from $N^- \subseteq N \cap HGr_X(O_X \oplus \mathscr{E} \oplus O_X)$ by Lemma 7.4 and Proposition 7.13. □

By Lemma 7.5, the natural embedding $HGr_X(\mathscr{E}) \longrightarrow HGr_X(O_X \oplus \mathscr{E} \oplus O_X)$ factors through $(N^-)^c$ and thus we have a map $i : HGr_X(\mathscr{E}) \longrightarrow (N^-)^c$.

Proposition 7.16. *For any correspondence theory,*

$$\widetilde{\mathbb{Z}}_S(i) : \widetilde{\mathbb{Z}}_S(HGr_X(\mathscr{E})) \longrightarrow \widetilde{\mathbb{Z}}_S((N^-)^c)$$

is an isomorphism in $\widetilde{DM}^{eff,-}(S)$.

Proof. Follows from the proof of [PW10, Theorem 5.2]. □

Finally, the following theorem completes the calculation. Its proof is similar to the proof of [D07, Lemma 2.12].

Theorem 7.5. *Let X be a smooth S-scheme and let E be a symplectic bundle of rank $2n$ over X. Then*

$$Th_S(E) \cong \widetilde{\mathbb{Z}}_S(X)(2n)[4n]$$

in $\widetilde{DM}^{eff,-}(S)$.

Proof. By Proposition 7.16, $\widetilde{M}_{N^-}(HGr_X(O_X \oplus \mathscr{E} \oplus O_X))$ is just the cone of the embedding $i : HGr_X(\mathscr{E}) \longrightarrow HGr_X(O_X \oplus \mathscr{E} \oplus O_X)$. By Theorem 7.4, we have a commutative diagram where the vertical arrows are isomorphisms

$$\begin{array}{ccc} \widetilde{\mathbb{Z}}_S(HGr_X(\mathscr{E})) & \longrightarrow & \widetilde{\mathbb{Z}}_S(HGr_X(O_X \oplus \mathscr{E} \oplus O_X)) \\ \downarrow & & \downarrow \\ \oplus_{i=0}^{n-1} \widetilde{\mathbb{Z}}_S(X)(2i)[4i] & \longrightarrow & \oplus_{i=0}^{n} \widetilde{\mathbb{Z}}_S(X)(2i)[4i]. \end{array}$$

Now, i pulls back the tautological bundle to the tautological bundle, giving the result. □

Putting everything together, we obtain the following result. The triangle appearing in the statement is called the Gysin triangle.

Theorem 7.6. *Let $X \in Sm/S$ be a smooth scheme and let $Y \subseteq X$ be a smooth closed subscheme of codimension $2n$ with a symplectic normal bundle. Then we have a distinguished triangle*

$$\widetilde{\mathbb{Z}}_S(X \setminus Y) \longrightarrow \widetilde{\mathbb{Z}}_S(X) \longrightarrow \widetilde{\mathbb{Z}}_S(Y)(2n)[4n] \longrightarrow \widetilde{\mathbb{Z}}_S(X \setminus Y)[1]$$

in $\widetilde{DM}^{eff,-}(S)$.

Proof. Follows from Theorem 7.5 and Proposition 7.12. □

7.2 Duality for Proper Schemes and Applications

In this section, we are going to prove that $\widetilde{\mathbb{Z}}_{pt}(X)$ is strongly dualizable in $\widetilde{DM}^{-}(pt)$ for proper $X \in Sm/k$. Then, we explicitly calculate its dual by using orientations on symplectic bundles. Finally we use our results to compute the group of morphisms in $\widetilde{DM}^{eff,-}(pt)$ between smooth proper schemes over k. For this we need to involve the stable $\mathbb{A}^1$-derived category $D_{\mathbb{A}^1}(S)$ over S introduced in [CD13, Example 5.3.31] and use the duality result on that category. For clarity, we describe our procedure using the following picture:

$$\text{Duality in } D_{\mathbb{A}^1}(S) \Longrightarrow \text{Duality in } \widetilde{DM}(S) \Longrightarrow \text{Duality in } \widetilde{DM}^{-}(pt) \Longrightarrow \text{Theorem 7.7.}$$

Let's briefly review the construction of $D_{\mathbb{A}^1}(S)$, the reader may also refer to [CD13, Section 5] and [DF17a, Section 1].

Define $Sh(S)$ to be category of Nisnevich sheaves of abelian groups on Sm/S. The Yoneda representative of the functor $F \longmapsto F(X)$ for any $X \in Sm/S$ is denoted by $\mathbb{Z}_S(X)$. The functor $\widetilde{\gamma} : Sm/S \longrightarrow \widetilde{Cor}_S$ in Proposition 5.6 and Lemma 5.7 gives us an adjunction

$$\widetilde{\gamma}^* : Sh(S) \rightleftharpoons \widetilde{Sh}(S) : \widetilde{\gamma}_*.$$

The category $Sh(S)$ is a symmetric monoidal category with $\mathbb{Z}_S(X) \otimes_S \mathbb{Z}_S(Y) \cong \mathbb{Z}_S(X \times_S Y)$ and $\widetilde{\gamma}^*$ is a monoidal functor. For any $f : S \longrightarrow T$ in Sm/k, the same method as the one used in Proposition 5.19 yields an adjunction

$$f^* : Sh(T) \rightleftharpoons Sh(S) : f_*.$$

Further, $f^*\widetilde{\gamma}^* \cong \widetilde{\gamma}^* f^*$ since there is a similar equality for their right adjoints. If f is smooth, there is an adjunction

$$f_\# : Sh(S) \rightleftharpoons Sh(T) : f^*$$

as in Proposition 5.23 and $f_\#\widetilde{\gamma}^* \cong \widetilde{\gamma}^* f_\#$ by the same argument as above.

As in Section 6.2.1, we define $SSp(S)$ to be the category of symmetric $\mathbb{1}_S\{1\}$-spectra of $Sh(S)$, where

$$\mathbb{1}_S\{1\} = Coker(\mathbb{Z}_S(S) \longrightarrow \mathbb{Z}_S(\mathbb{G}_m)).$$

There are adjunctions

$$\Sigma^\infty : Sh(S) \rightleftharpoons SSp(S) : \Omega^\infty$$

and

$$\widetilde{\gamma}^* : SSp(S) \rightleftharpoons Sp(S) : \widetilde{\gamma}_*$$

and we can also define $\otimes_S$, f^*, f_*, $f_\#$, $-\{-i\}$ and $-\{i\}$ ($i \geq 0$) on $SSp(S)$. Moreover, $\widetilde{\gamma}^*$ commutes with f^* and $f_\#$ and is monoidal as above.

In [CD09, Theorem 1.7], they put a model structure $\mathfrak{M}_S$ on the category of unbounded complexes of objects in $Sh(S)$. This is a cofibrantly generated model structure where the cofibrations are the I-cofibrations where I consists of the morphisms $S^{n+1}(\mathbb{Z}_S(X)) \longrightarrow D^n(\mathbb{Z}_S(X))$ for any $X \in Sm/S$ and weak equivalences are quasi-morphisms between complexes. The homotopy category of $\mathfrak{M}_S$ is denoted by $D_S(S)$. Moreover, $\mathfrak{M}_S$ is stable and left proper so it induces a triangulated structure on $D_S(S)$.

Localizing $D_S(S)$ with respect to the morphisms

$$\mathbb{Z}_S(X \times_k \mathbb{A}^1) \longrightarrow \mathbb{Z}_S(X)$$

as in Section 6.1, we get a category $D^{eff}_{\mathbb{A}^1}(S)$ with the induced triangulated structure.

In [CD09, Theorem 1.7], they also define a model structure $\mathfrak{M}_{SSp}$ on the category of unbounded complexes of symmetric spectra in $Sh(S)$. This is again a cofibrantly generated model structure where the cofibrations are the I-cofibrations where I consists of the morphisms $S^{n+1}(\Sigma^\infty \mathbb{Z}_S(X)\{-i\}) \longrightarrow D^n(\Sigma^\infty \widetilde{\mathbb{Z}}_S(X)\{-i\})$ for any $X \in Sm/S$ and $i \geq 0$ and weak equivalences are quasi-morphisms between complexes. The homotopy category of $\mathfrak{M}_{SSp}$ is denoted by $D_{SSp}(S)$. Moreover, $\mathfrak{M}_{SSp}$ is stable and left proper so it induces a triangulated structure on $D_{SSp}(S)$.

Localizing $D_{SSp}(S)$ with respect to the morphisms

$$(\Sigma^\infty \mathbb{Z}_S(X \times_k \mathbb{A}^1) \longrightarrow \Sigma^\infty \mathbb{Z}_S(X))\{-i\}, i \geq 0$$

as in Section 6.2.3, we get a category with the induced triangulated structure. Localizing further that category with respect to

$$(\Sigma^\infty \mathbb{Z}_S(X)\{1\}\{-1\} \longrightarrow \Sigma^\infty \mathbb{Z}_S(X))\{-i\}$$

as in Section 6.2.3, we obtain the category $D_{\mathbb{A}^1}(S)$, with the induced triangulated structure. Moreover, we have an exact functor

$$\Sigma^{\infty,st} : D^{eff}_{\mathbb{A}^1}(S) \longrightarrow D_{\mathbb{A}^1}(S).$$

The stage being set, we now calculate the inverse of the Thom space for any vector bundle, using the methods of Section 7.1.

Proposition 7.17. *For any correspondence theory, we have:*

1. *Suppose that $f : S \longrightarrow T$ is a morphism in Sm/k, that $X \in Sm/T$ and that E is a vector bundle over X. Then we have*

$$f^* Th_T(E) \cong Th_S(f^*E)$$

in $\widetilde{DM}^{eff,-}(S)$, where f^*E is the vector bundle over X^S induced by E.

2. *Suppose that $f : S \longrightarrow T$ is a smooth morphism in Sm/k, that $X \in Sm/S$ and that E is a vector bundle over X. Then we have*

$$f_\# Th_S(E) \cong Th_T(E)$$

in $\widetilde{DM}^{eff,-}(S)$.

3. *([CD13, Remark 2.4.15]) Suppose E_1 and E_2 are vector bundles over $X \in Sm/k$. Then*

$$Th_X(E_1) \otimes_X Th_X(E_2) \cong Th_X(E_1 \oplus E_2)$$

in $\widetilde{DM}^{eff,-}(X)$.

Proof. The proofs of (1) and (2) being easy, we only prove (3). The total space of $E_1 \oplus E_2$ is just $E_1 \times_X E_2$. By definition, for any vector bundle E over X, $Th_X(E)$ is quasi-isomorphic the complex

$$\widetilde{\mathbb{Z}}_S(E \setminus X) \longrightarrow \widetilde{\mathbb{Z}}_S(E).$$

Hence the left hand side is the total complex

$$\widetilde{\mathbb{Z}}_S((E_1 \setminus X) \times_X (E_2 \setminus X)) \longrightarrow \widetilde{\mathbb{Z}}_S((E_1 \setminus X) \times_X E_2) \oplus \widetilde{\mathbb{Z}}_S(E_1 \times_X (E_2 \setminus X)) \longrightarrow \widetilde{\mathbb{Z}}_S(E_1 \times_X E_2).$$

By Proposition 5.10, the complex

$$\widetilde{\mathbb{Z}}_S((E_1 \setminus X) \times_X (E_2 \setminus X)) \longrightarrow \widetilde{\mathbb{Z}}_S((E_1 \setminus X) \times_X E_2) \oplus \widetilde{\mathbb{Z}}_S(E_1 \times_X (E_2 \setminus X))$$

is quasi-isomorphic to

$$0 \longrightarrow \widetilde{\mathbb{Z}}_S((E_1 \times_X E_2) \setminus X)$$

since

$$(E_1 \times_X E_2) \setminus X = (E_1 \setminus X) \times_X E_2 \cup E_1 \times_X (E_2 \setminus X).$$

Hence we have a quasi-isomorphism

$$\begin{array}{ccccc} \widetilde{\mathbb{Z}}_S((E_1 \setminus X) \times (E_2 \setminus X)) & \longrightarrow & \widetilde{\mathbb{Z}}_S((E_1 \setminus X) \times E_2) \oplus \widetilde{\mathbb{Z}}_S(E_1 \times (E_2 \setminus X)) & \longrightarrow & \widetilde{\mathbb{Z}}_S(E_1 \times E_2) \\ \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & \widetilde{\mathbb{Z}}_S((E_1 \times E_2) \setminus X) & \longrightarrow & \widetilde{\mathbb{Z}}_S(E_1 \times E_2). \end{array}$$

□

Proposition 7.18. *Let E be a vector bundle of rank n over $X \in Sm/k$. Then we have*

$$(\Sigma^{\infty,st} Th_X(E))^{-1} \cong (\Sigma^{\infty,st} Th_X(E^\vee))(-2n)[-4n]$$

in $\widetilde{DM}^-(X)$.

Proof. By Proposition 7.17 and Theorem 7.5, we have

$$Th_X(E) \otimes_X Th_X(E^\vee) \cong Th_X(E \oplus E^\vee) \cong \mathbb{1}_X(2n)[4n]$$

in $\widetilde{DM}^{eff,-}(X)$. Now the statement follows from Proposition 6.42 and the fact that $\Sigma^{\infty,st}$ is monoidal. □

Since we have a monoidal exact functor $\widetilde{DM}^{eff,-}(X) \longrightarrow \widetilde{DM}^{eff}(X)$, the same proof as above yields the following result.

Proposition 7.19. *Let E be a vector bundle of rank n over $X \in Sm/k$. Then we have*

$$(\Sigma^{\infty,st} Th_X(E))^{-1} \cong (\Sigma^{\infty,st} Th_X(E^\vee))(-2n)[-4n]$$

in $\widetilde{DM}(X)$.

We'll need the following properties of the stable $\mathbb{A}^1$-derived category, which can be for instance found in [DF17a, 1.1.7 and Theorem 1.1.10].

Proposition 7.20. *1. For any $f : S \longrightarrow T$ in Sm/k, we have an adjoint pair of exact functors*

$$f^* : D_{\mathbb{A}^1}(T) \rightleftharpoons D_{\mathbb{A}^1}(S) : f_*.$$

2. For any smooth $f : S \longrightarrow T$ in Sm/k, we have an adjoint pair of exact functors

$$f_\# : D_{\mathbb{A}^1}(S) \rightleftharpoons D_{\mathbb{A}^1}(T) : f^*$$

and for any $A \in D_{\mathbb{A}^1}(S)$ and $B \in D_{\mathbb{A}^1}(T)$, we have

$$(f_\# A) \otimes B \cong f_\#(A \otimes f^* B).$$

3. For any $f : S \longrightarrow T$ in Sm/k, we have a functor

$$f_! : D_{\mathbb{A}^1}(S) \longrightarrow D_{\mathbb{A}^1}(T).$$

If f is proper, we have

$$f_! \cong f_*.$$

If f is smooth, we have

$$f_! \cong f_\#(- \otimes (\Sigma^{\infty,st} Th_S(T_{S/T}))^{-1}).$$

Proposition 7.21. *Let $S \in Sm/k$ and $f : X \longrightarrow S$ be a smooth proper morphism. Then $\Sigma^{\infty,st}\mathbb{Z}_S(X) \in D_{\mathbb{A}^1}(S)$ is strongly dualizable with dual $f_\#(\Sigma^{\infty,st}Th_X(T_{X/S})^{-1})$.*

Proof. For any $A, B \in D_{\mathbb{A}^1}(S)$, we have

$$\begin{aligned}
&Hom_{D_{\mathbb{A}^1}(S)}(\Sigma^{\infty,st}\mathbb{Z}_S(X) \otimes_S A, B)\\
\cong &Hom_{D_{\mathbb{A}^1}(S)}(f_\# f^* A, B)\\
&\text{by Proposition 7.20, (2)}\\
\cong &Hom_{D_{\mathbb{A}^1}(S)}(A, f_* f^* B)\\
&\text{by Proposition 7.20, (1) and (2)}\\
\cong &Hom_{D_{\mathbb{A}^1}(S)}(A, f_! f^* B)\\
&\text{by Proposition 7.20, (3)}\\
\cong &Hom_{D_{\mathbb{A}^1}(S)}(A, f_\#(f^* B \otimes_X (\Sigma^{\infty,st}Th_X(T_{X/S}))^{-1}))\\
&\text{by Proposition 7.20, (3)}\\
\cong &Hom_{D_{\mathbb{A}^1}(S)}(A, B \otimes_S f_\#(\Sigma^{\infty,st}Th_X(T_{X/S})^{-1}))\\
&\text{by Proposition 7.20, (2).}
\end{aligned}$$

. □

Proposition 7.22. *Let $S \in Sm/k$ and let $f : X \longrightarrow S$ be a smooth proper morphism. Then $\Sigma^{\infty,st}\widetilde{\mathbb{Z}}_S(X) \in \widetilde{DM}(S)$ is strongly dualizable with dual*

$$(\Sigma^{\infty,st}Th_S(\Omega_{X/S}))(-2d)[-4d],$$

where $d = d_X - d_S := dimX - dimS$.

Proof. Since we have a monoidal exact functor $\gamma^* : D_{\mathbb{A}^1}(S) \longrightarrow \widetilde{DM}(S)$ which commutes with $f_\#$ up to a natural isomorphism, $\Sigma^{\infty,st}\widetilde{\mathbb{Z}}_S(X) \in \widetilde{DM}(S)$ is strongly dualizable with dual $f_\#(\Sigma^{\infty,st}Th_X(T_{X/Y})^{-1})$ by Proposition 7.21. Now, Proposition 7.19 yields

$$(\Sigma^{\infty,st}Th_X(T_{X/S}))^{-1} \cong (\Sigma^{\infty,st}Th_X(\Omega_{X/S}))(-2d)[-4d].$$

Finally, we have

$$f_\#((\Sigma^{\infty,st}Th_X(\Omega_{X/S}))(-2d)[-4d]) \cong (\Sigma^{\infty,st}Th_S(\Omega_{X/S}))(-2d)[-4d].$$

□

Now we have a monoidal exact functor $\widetilde{DM}^-(pt) \longrightarrow \widetilde{DM}(pt)$ which commutes with $-\{-i\}, i \geq 0$ up to a natural isomorphism. Then by Proposition 6.46, we have the following result.

Proposition 7.23. *Let $X \in Sm/k$ be a proper scheme. Then $\Sigma^{\infty,st}\widetilde{\mathbb{Z}}_{pt}(X) \in \widetilde{DM}^-(pt)$ is strongly dualizable with dual*

$$(\Sigma^{\infty,st}Th_{pt}(\Omega_{X/k}))(-2d_X)[-4d_X].$$

The following theorem gives a computation of the hom-groups in the category of (effective) MW-motives in case the objects are smooth proper.

Theorem 7.7. *Let $X, Y \in Sm/k$ with Y proper. Then*

$$Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(X), \widetilde{\mathbb{Z}}_{pt}(Y)) \cong \widetilde{CH}^{d_Y}(X \times Y, -T_{X\times Y/X}).$$

Proof. Let $p : Y \longrightarrow pt$ be the structure map of Y and let $q : X \times Y \longrightarrow Y$ be the second projection. We have

$$
\begin{aligned}
& Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(X), \widetilde{\mathbb{Z}}_{pt}(Y)) \\
\cong & Hom_{\widetilde{DM}^{-}(pt)}(\Sigma^{\infty,st}\widetilde{\mathbb{Z}}_{pt}(X), \Sigma^{\infty,st}\widetilde{\mathbb{Z}}_{pt}(Y)) \\
& \text{by Proposition 6.43} \\
\cong & Hom_{\widetilde{DM}^{-}(pt)}(\Sigma^{\infty,st}\widetilde{\mathbb{Z}}_{pt}(X) \otimes (\Sigma^{\infty,st}Th_{pt}(\Omega_{Y/k}))(-2d_Y)[-4d_Y], \Sigma^{\infty,st}\mathbb{1}_{pt}) \\
& \text{by Proposition 7.23} \\
\cong & Hom_{\widetilde{DM}^{-}(pt)}(\Sigma^{\infty,st}\widetilde{\mathbb{Z}}_{pt}(X) \otimes (\Sigma^{\infty,st}Th_{pt}(\Omega_{Y/k})), \Sigma^{\infty,st}\mathbb{1}_{pt}(2d_Y)[4d_Y]) \\
& \text{by Proposition 6.42} \\
\cong & Hom_{\widetilde{DM}^{-}(pt)}(\Sigma^{\infty,st}(\widetilde{\mathbb{Z}}_{pt}(X) \otimes Th_{pt}(\Omega_{Y/k})), \Sigma^{\infty,st}\mathbb{1}_{pt}(2d_Y)[4d_Y]) \\
\cong & Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(X) \otimes Th_{pt}(\Omega_{Y/k}), \mathbb{1}_{pt}(2d_Y)[4d_Y]) \\
& \text{by Proposition 6.43} \\
\cong & Hom_{\widetilde{DM}^{eff,-}(pt)}(\widetilde{\mathbb{Z}}_{pt}(X) \otimes p_{\#}Th_Y(\Omega_{Y/k}), \mathbb{1}_{pt}(2d_Y)[4d_Y]) \\
& \text{by Proposition 7.17} \\
\cong & Hom_{\widetilde{DM}^{eff,-}(pt)}(p_{\#}(p^*\widetilde{\mathbb{Z}}_{pt}(X) \otimes Th_Y(\Omega_{Y/k})), \mathbb{1}_{pt}(2d_Y)[4d_Y]) \\
& \text{by Proposition 6.14} \\
\cong & Hom_{\widetilde{DM}^{eff,-}(pt)}(p_{\#}(\widetilde{\mathbb{Z}}_Y(X \times Y) \otimes Th_Y(\Omega_{Y/k})), \mathbb{1}_{pt}(2d_Y)[4d_Y]) \\
\cong & Hom_{\widetilde{DM}^{eff,-}(pt)}(p_{\#}(q_{\#}(\mathbb{1}_{X\times Y}) \otimes Th_Y(\Omega_{Y/k})), \mathbb{1}_{pt}(2d_Y)[4d_Y]) \\
\cong & Hom_{\widetilde{DM}^{eff,-}(pt)}(p_{\#}(q_{\#}q^*Th_Y(\Omega_{Y/k})), \mathbb{1}_{pt}(2d_Y)[4d_Y]) \\
& \text{by Proposition 6.14} \\
\cong & Hom_{\widetilde{DM}^{eff,-}(pt)}(Th_{pt}(\Omega_{X\times Y/X}), \mathbb{1}_{pt}(2d_Y)[4d_Y]) \\
& \text{by Proposition 7.17} \\
\cong & \widetilde{CH}^{d_Y}(X \times Y, -T_{X\times Y/X}) \\
& \text{by the discussion after [DF17, Remark 4.2.7].}
\end{aligned}
$$

□

Chapter 8

MW-Correspondences as a Correspondence Theory

In this section, we are going to sketch of the proof of Theorem 4.1. It's incomplete and will be completed in the future. We will always assume $E = \widetilde{CH}$ in this section.

For any scheme X and $x \in X$, set $\Omega_x = m_x/m_x^2$ and $\Lambda_x = det(m_x/m_x^2)$.

Definition 8.1. *Let G be an abelian group, and let M, N be G-sets. Define*

$$M \times_G N = M \times N/\sim, (m,n) \sim (m',n') \Longleftrightarrow (m,n) = (gm', g^{-1}n') \text{ for some } g \in G.$$

The set $M \times_G N$ is endowed with the action of G defined by $g(m,n) = (gm,n)$.

Definition 8.2. *Let G be an abelian group and let M be a G-set. We denote the group algebra of G over $\mathbb{Z}$ by $\mathbb{Z}[G]$ and the free abelian group generated by M by $\mathbb{Z}[M]$. Then $\mathbb{Z}[M]$ is a $\mathbb{Z}[G]$-module.*

The following lemma is straighforward.

Lemma 8.1. *1. Let M, N be G-sets, then*

$$\mathbb{Z}[M \times_G N] \cong \mathbb{Z}[M] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[N].$$

2. Let $G \longrightarrow H$ be a morphism of abelian groups and let M be a G-set. Then

$$\mathbb{Z}[M \times_G H] \cong \mathbb{Z}[M] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[H]$$

as $\mathbb{Z}[H]$-modules.

Definition 8.3. *Let R be a commutative ring. We set $Q(R) = R^*/(R^*)^2$ as an abelian group and for any one dimensional free R-module L we define*

$$Q(L) = L^*/\sim, x \sim y \Longleftrightarrow x = r^2 y \text{ for some } r \in R^*$$

as a $Q(R)$-sets.

The following lemma is straighforward.

Lemma 8.2. *1. Let L_1, L_2 be one dimensional free R-modules, then*

$$Q(L_1 \otimes_R L_2) \cong Q(L_1) \times_{Q(R)} Q(L_2).$$

2. Let L be a one dimensional free R-module, then

$$Q(L^\vee) \cong Hom_{Q(R)}(Q(L), Q(R)).$$

3. Let S be an R-algebra and L be a one dimensional free R-module, then

$$Q(L \otimes_R S) \cong Q(L) \times_{Q(R)} Q(S)$$

as $Q(S)$-sets.

Proposition 8.1. *The categories $\mathscr{P}_X$ (Definition 4.4) for $X \in Sm/k$ satisfy Axiom 1.*

Proof. We set $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. From the definition of $\mathscr{P}_X$, we see that for every $A = (E_1, \cdots, E_n)$, $rk(A)$ is well defined in $\mathbb{Z}/2\mathbb{Z}$, independent of isomorphisms in $\mathscr{P}_X$. Hence there is a rank morphism $rk_X : \mathscr{P}_X \longrightarrow \mathbb{Z}/2\mathbb{Z}$.

Define a bifunctor

$$\begin{array}{ccccccc} + : & \mathscr{P}_X & \times & \mathscr{P}_X & \longrightarrow & \mathscr{P}_X \\ & ((E_1, \cdots, E_n) & , & (F_1, \cdots, F_m)) & \longmapsto & (E_1, \cdots, E_n, F_1, \cdots, F_m). \end{array}$$

It is easy to see that this operation endows $\mathscr{P}_X$ with the structure of a Picard category with $-(E_1, \cdots, E_n) = (E_n^\vee, \cdots, E_1^\vee)$. For any $A, B \in \mathscr{P}_X$, we attach a commutativity isomorphism

$$c = c(A, B) : A \oplus B \longrightarrow B \oplus A$$

by

$$(-1)^{rk_X(A) rk_X(B)} id_{det(A) \otimes det(B)}.$$

This turns $\mathscr{P}_X$ into a commutative Picard category.

There is an obvious functor $i : (Vect(X), iso) \longrightarrow \mathscr{P}_X$ sending E to (E) and $f : E_1 \longrightarrow E_2$ to $det(f)$. Moreover, for every exact sequence

$$0 \longrightarrow E_1 \longrightarrow E_3 \longrightarrow E_2 \longrightarrow 0,$$

we attach the isomorphism $(E_3) \longrightarrow (E_1, E_2)$ given by the isomorphism $detE_3 \longrightarrow detE_1 \otimes detE_2$ sending $\alpha \wedge \beta$ to $\alpha \otimes \beta$ for any local base α (resp. β) of E_1 (resp. E_3). This functor satisfies all conditions given in Definition 3.3.

Finally, for any $f : X \longrightarrow Y$ in Sm/k, we define $f^* : \mathscr{P}_Y \longrightarrow \mathscr{P}_X$ by $f^*(E_1, \cdots, E_n) = (f^*E_1, \cdots, f^*E_n)$. $\square$

We set $K_n^{MW}(F, L) = K_n^{MW}(F) \otimes_{\mathbb{Z}[Q(F)]} \mathbb{Z}[Q(L)]$ ([Mor12, Remark 2.21]) for every one dimensional F-vector space L. For every $X \in Sm/k$, $x \in X$, T closed in X and $v \in \mathscr{P}_X$, define

$$K_n^{MW}(k(x), \Lambda_x^* \otimes v) = K_n^{MW}(k(x), \Lambda_x^* \otimes_{k(x)} det(v)|_{k(x)})$$

and

$$C_{RS,T}^n(X; \underline{K}_m^{MW}; v) = \bigoplus_{y \in X^{(n)} \cap T} K_{m-n}^{MW}(k(y), \Lambda_y^* \otimes v),$$

where $X^{(n)}$ is the set of points of codimension n in X ([Mor12, Chapter 4]).

Now for every $X \in Sm/k$, $i \in \mathbb{N}$, $v \in \mathscr{P}_X$ and T closed in X, we define the groups required by Axiom 2 to be of the form

$$\widetilde{CH}_T^i(X, v) = H^i(C_{RS,T}^*(X; \underline{K}_i^{MW}; v)).$$

Then, Axiom 3 just comes from the extension of supports in Chow-Witt groups.

8.1 Operations without Intersection

Lemma 8.3. *Let $f : X \longrightarrow X'$ be a smooth morphism in Sm/k, and let $x \in X$ with $[k(x) : k(f(x))] < \infty$. Then we have an isomorphism*

$$Q(\Lambda_x^*) \cong Q(\omega_{X/X'}^\vee|_{k(x)}) \times_{Q(k(x))} Q(\Lambda_{f(x)}^* \otimes k(x))$$

(the Qs will be ignored in the sequel for convenience).

Proof. If $k(x)$ is separable over $k(f(x))$, then we have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 & 0 & & \\
 & \Omega_{X/X'}|_{k(x)} \arrow[u] \arrow[r, equal] & \Omega_{X/X'}|_{k(x)} \arrow[u] & & \\
0 \arrow[r] & \Omega_x \arrow[u] \arrow[r] & \Omega_{X/k}|_{k(x)} \arrow[u] \arrow[r] & \Omega_{k(x)/k} \arrow[r] & 0 \\
0 \arrow[r] & \Omega_{f(x)} \otimes k(x) \arrow[u] \arrow[r] & \Omega_{X'/k}|_{k(f(x))} \otimes k(x) \arrow[u] \arrow[r] & \Omega_{k(f(x))/k} \otimes k(x) \arrow[u, "\cong"'] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

so we have an isomorphism

$$\Lambda_x^* \cong \omega_{X/X'}|_{k(x)}^\vee \otimes (\Lambda_{f(x)}^* \otimes k(x))$$

which induces an isomorphism

$$Q(\Lambda_x^*) \cong Q(\omega_{X/X'}|_{k(x)}^\vee) \times_{Q(k(x))} Q(\Lambda_{f(x)}^* \otimes k(x)).$$

If the field extension is not separable, we only have the horizontal exact sequences and the middle vertical arrows. But $Q(\omega_{k(x)/k}) \cong Q(\omega_{k(f(x))/k} \otimes k(x))$ still holds ([Mor12, Lemma 4.1]), so we have isomorphisms

$$\begin{aligned}
&Q(\Lambda_x^*)\\
\longrightarrow & Q(\omega_{k(x)/k}) \times_{Q(k(x))} Q(\omega_{k(x)/k}^\vee) \times_{Q(k(x))} Q(\Lambda_x^*)\\
\longrightarrow & Q(\omega_{k(x)/k}) \times_{Q(k(x))} Q(\omega_{X/k}^\vee|_{k(x)})\\
\longrightarrow & Q(\omega_{k(x)/k}) \times_{Q(k(x))} Q(\omega_{X/X'}^\vee|_{k(x)}) \times_{Q(k(x))} Q(\omega_{X'/k}^\vee|_{k(f(x))} \otimes k(x))\\
\longrightarrow & Q(\omega_{k(x)/k}) \times_{Q(k(x))} Q(\omega_{X/X'}^\vee|_{k(x)}) \times_{Q(k(x))} Q(\omega_{k(f(x))/k}^\vee \otimes k(x)) \times_{Q(k(x))} Q(\Lambda_{f(x)}^* \otimes k(x))\\
\longrightarrow & Q(\omega_{k(x)/k}) \times_{Q(k(x))} Q(\omega_{X/X'}^\vee|_{k(x)}) \times_{Q(k(x))} Q(\omega_{k(x)/k}^\vee) \times_{Q(k(x))} Q(\Lambda_{f(x)}^* \otimes k(x))\\
\longrightarrow & Q(\omega_{X/X'}^\vee|_{k(x)}) \times_{Q(k(x))} Q(\Lambda_{f(x)}^* \otimes k(x)).
\end{aligned}$$

This coincides with the isomorphism we obtained in the case of separable field extension by applying Theorem 3.1, (2) to the digram above. □

Lemma 8.4. *Let $f : X \longrightarrow X'$ be a closed immersion in Sm/k and let $x \in X$ (so that $k(x) = k(f(x))$). Then we have an isomorphism*

$$\Lambda_{f(x)}^* \cong \Lambda_x^* \otimes detN_{X/X'}^\vee|_{k(x)}.$$

Proof. This follows by the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \Omega_x \arrow[u] \arrow[r] & \Omega_{X/k}|_{k(x)} \arrow[u] \arrow[r] & \Omega_{k(x)/k} \arrow[r] & 0 \\
0 \arrow[r] & \Omega_{f(x)} \arrow[u] \arrow[r] & \Omega_{X'/k}|_{k(f(x))} \arrow[u] \arrow[r] & \Omega_{k(f(x))/k} \arrow[u, equal] \arrow[r] & 0 \\
 & N^\vee_{X/X'}|_{k(x)} \arrow[u] \arrow[r, equal] & N^\vee_{X/X'}|_{k(x)} \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

□

Lemma 8.5. *Let $f : X \longrightarrow X'$ be a smooth morphism in Sm/k, and let $x \in X$ with $codim(x) = codim(f(x))$. Then, we have an isomorphism*

$$\Omega_x \cong \Omega_{f(x)} \otimes_{k(f(x))} k(x).$$

Proof. The cotangent map

$$\Omega_{f(x)} \otimes_{k(f(x))} k(x) \longrightarrow \Omega_x$$

of f is injective and the two vector spaces have the same dimension $codim(x)$. □

Lemma 8.6. *Let $X_1, X_2 \in Sm/k$, $x_1 \in X_1$, $x_2 \in X_2$ and let y be the generic point of some component of $\overline{x_1} \times \overline{x_2}$. Then we have an isomorphism*

$$\Omega_y \cong \Omega_{x_1} \otimes_{k(x_1)} k(y) \oplus \Omega_{x_2} \otimes_{k(x_2)} k(y).$$

Proof. We have the following commutative diagram with exact rows and columns (same if we exchange X_1 and X_2)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_{x_1} \otimes k(y) \arrow[r] \arrow[d] & p_1^*\Omega_{X_1/k}|_{k(y)} \arrow[r] \arrow[d] & q_1^*\Omega_{\overline{x_1}/k}|_{k(y)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_y \arrow[r] \arrow[d] & \Omega_{X_1 \times X_2/k}|_{k(y)} \arrow[r] \arrow[d] & \Omega_{\overline{x_1} \times \overline{x_2}/k}|_{k(y)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{x_2} \otimes k(y) \arrow[r] \arrow[d] & p_2^*\Omega_{X_2/k}|_{k(y)} \arrow[r] \arrow[d] & q_2^*\Omega_{\overline{x_2}/k}|_{k(y)} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0, & 0 &
\end{tikzcd}$$

where $p_i : X_1 \times X_2 \longrightarrow X_i$ and $q_i : \overline{x_1} \times \overline{x_2} \longrightarrow \overline{x_i}$ are the projections and $\Omega_{\overline{x_1} \times \overline{x_2}/k}|_{k(y)} = \Omega_{\overline{y}/k}|_{k(y)}$. □

Definition 8.4. *(Axiom 8) Let $f : X \longrightarrow X'$ be a smooth morphism, and let $x \in X$ with $codim(x) = codim(f(x))$. For any $v \in \mathscr{P}_{X'}$, we have an obvious morphism*

$$K_n^{MW}(k(f(x)), \Lambda^*_{f(x)} \otimes v) \longrightarrow K_n^{MW}(k(x), \Lambda^*_x \otimes f^*v)$$

by Lemma 8.5. This induces a pull-back morphism ([Fas08, Corollaire 10.4.2])

$$f^* : \widetilde{CH}^n_T(X', v) \longrightarrow \widetilde{CH}^n_{f^{-1}(T)}(X, f^*(v))$$

for every $T \in Z^n(X)$. It is functorial with respect to v.

Remark 8.1. *The pull-back along closed immersions is much more difficult and we will discuss this in Section 8.2.*

The following proposition is obvious.

Proposition 8.2. *(Axiom 9) The pull-back between smooth morphisms is functorial and $f^*(1) = 1$.*

Definition 8.5. *(Axiom 11) Let $f : X \longrightarrow X'$ be a smooth morphism and let $C \in Z^{i+d_f}(X)$ be finite over X'. We define the push-forward (Proposition 8.6)*

$$f_* : \widetilde{CH}^{i+d_f}_C(X, f^*v - T_{X/X'}) \longrightarrow \widetilde{CH}^i_{f(C)}(X', v)$$

as the composite for every $x \in C \cap X^{(i+d_f)}$

$$\begin{array}{ccc} K_0^{MW}(k(x), \Lambda_x^* \otimes f^*v \otimes \omega_{X/X'}) & \longrightarrow & K_0^{MW}(k(x), \omega_{X/X'}^\vee \otimes (\Lambda^*_{f(x)} \otimes k(x)) \otimes f^*v \otimes \omega_{X/X'}) \\ & & \downarrow{\scriptstyle Tr^{k(x)}_{k(f(x))}} \\ & & K_0^{MW}(k(f(x)), \Lambda^*_{f(x)} \otimes v) \end{array}$$

where the horizontal arrow is induced by Lemma 8.3, while the vertical arrow is the trace map composed with the isomorphism of virtual vector bundles cancelling the first and last bundle. The push-forward for smooth morphisms is functorial with respect to v.

It's clear by definition that Axiom 20 is satisfied.

Definition 8.6. *(Axiom 13) Let $f : X \longrightarrow X'$ be a closed immersion and let $C \in Z^{i+d_f}(X)$. We define the push-forward (Proposition 8.7)*

$$f_* : \widetilde{CH}^{i+d_f}_C(X, N_{X/X'} + f^*v) \longrightarrow \widetilde{CH}^i_{f(C)}(X', v)$$

by the isomorphism induced by Lemma 8.4

$$K_0^{MW}(k(x), \Lambda_x^* \otimes det N_{X/X'} \otimes f^*v) \longrightarrow K_0^{MW}(k(f(x)), \Lambda^*_{f(x)} \otimes v)$$

for every $x \in C \cap X^{(i+d_f)}$. The push-forward for closed immersions is functorial with respect to v.

Remark 8.2. *Suppose that $f : X \longrightarrow X'$ is a morphism of schemes and that $C \in Z^{i+d_f}(X)$ is such that $C = \overline{x}$ for some $x \in X$. Suppose further that C is also closed in X'. Then, we have an exact sequence*

$$0 \longrightarrow T_{X/X'}|_{k(x)} \longrightarrow \Omega^*_x \longrightarrow \Omega^*_{f(x)} \longrightarrow 0;$$

if f is a closed immersion, we have an exact sequence

$$0 \longrightarrow \Omega^*_x \longrightarrow \Omega^*_{f(x)} \longrightarrow N_{X/X'}|_{k(x)} \longrightarrow 0.$$

So, we can identify Ω_x^* with $N_{\overline{x}/X}|_{k(x)}$ since the latter satisfies the same exact sequences when C is smooth. Hence in the context above, the push-forward associated to f with support C is completely determined by the composite

$$N_{C/X} + f^*v|_C - T_{X/X'}|_C \longrightarrow T_{X/X'}|_C + N_{C/Y} + f^*v|_C - T_{X/X'}|_C \longrightarrow N_{C/Y} + f^*v|_C$$

in case f is smooth and by the isomorphism

$$N_{C/X} + N_{X/X'}|_C + f^*v|_C \longrightarrow N_{C/Y} + f^*v|_C$$

if f is a closed immersion.

This inspires us to convert equations of twisted Chow-Witt groups into equations of virtual objects. Then use the method described in Chapter 3. This is the main idea we will use in this chapter.

We now explain the differentials in the Rost-Schmid complex. Suppose that $X \in Sm/k$ and that $Y = \overline{y}$ for some $y \in X$. Let further $Z = \overline{z}$ for some $z \in Y^{(1)}$ and $v \in \mathscr{P}_X$. We now define the differential

$$\partial_z^y : K_n^{MW}(k(y), \Lambda_y^* \otimes v) \longrightarrow K_{n-1}^{MW}(k(z), \Lambda_z^* \otimes v).$$

Suppose at first that Y is normal. Then the exact sequence

$$I_Y/I_Y^2 \longrightarrow \Omega_{X/k}|_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0$$

is also left exact at the stalk of z, and we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_Y/I_Y^2|_{k(z)} & \longrightarrow & \Omega_{X/k}|_{k(z)} & \longrightarrow & \Omega_{Y/k}|_{k(z)} & \longrightarrow & 0 \\ & & {\scriptstyle i}\downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & I_Z/I_Z^2|_{k(z)} & \longrightarrow & \Omega_{X/k}|_{k(z)} & \longrightarrow & \Omega_{Z/k}|_{k(z)} & \longrightarrow & 0. \end{array}$$

The map i is injective with cokernel m_z/m_z^2, where m_z is the maximal ideal of $O_{Y,z}$. Thus, we have an exact sequence

$$0 \longrightarrow (m_z/m_z^2)^\vee \longrightarrow (I_Z/I_Z^2)^\vee|_{k(z)} \longrightarrow (I_Y/I_Y^2)^\vee|_{k(z)} \longrightarrow 0.$$

Now choose a free basis a of $(I_Y/I_Y^2)_z^\vee$, e of $(m_z/m_z^2)^\vee$ and t of $det(v)_z$. Hence a is also a free basis of $\Omega_y^* = (I_Y/I_Y^2)^\vee|_{k(y)}$ and (e, a) is a free basis of $\Omega_z^* = (I_Z/I_Z^2)^\vee|_{k(z)}$ by the sequence above. We define the map ∂ by

$$\begin{array}{ccc} K_n^{MW}(k(y), \Lambda_y^* \otimes v) & \longrightarrow & K_{n-1}^{MW}(k(z), \Lambda_z^* \otimes v) \\ s \otimes a \otimes t & \longmapsto & \partial_z^e(s) \otimes (e \wedge a) \otimes t \end{array},$$

where ∂_z^e is the usual partial map for Milnor-Witt groups. This map is independent of the choice of a, e, t.

In general, let $\widetilde{Y}$ be the normalization of Y with morphism $\pi : \widetilde{Y} \longrightarrow Y$ and let $\{z_i\} = \pi^{-1}(z)$. We have an isomorphism (the same for z)

$$\Lambda_y^* \cong \omega_{k(y)/k} \otimes \omega_{X/k}^\vee|_{k(y)}.$$

Now fix z_i. We find that $\Omega_{O_{\widetilde{Y},z_i}/k}$ satisfies

$$\Omega_{O_{\widetilde{Y},z_i}/k} \otimes k(y) = \Omega_{k(y)/k}.$$

Also, we have an exact sequence

$$0 \longrightarrow m_{z_i}/m_{z_i}^2 \longrightarrow \Omega_{O_{\widetilde{Y},z_i}/k} \otimes k(z_i) \longrightarrow \Omega_{k(z_i)/k} \longrightarrow 0.$$

So, choose a free basis e_i of $(m_{z_i}/m_{z_i}^2)^\vee$, c_i of $\Omega_{O_{\widetilde{Y},z_i}/k}$, d of $(\Omega_{X/k}^\vee)_z$ and l of $det(v)_z$. We define ∂_i by the following composite

$$\begin{aligned}
&K_n^{MW}(k(y), \Lambda_y^* \otimes v)\\
\longrightarrow &K_n^{MW}(k(y), \omega_{k(y)/k} \otimes \omega_{X/k}^\vee \otimes v)\\
\longrightarrow &K_{n-1}^{MW}(k(z_i), (m_{z_i}/m_{z_i}^2)^\vee \otimes (\omega_{O_{\widetilde{Y},z_i}/k} \otimes k(z_i)) \otimes_{k(z_i)} (\omega_{X/k}^\vee|_{k(z)} \otimes_{k(z)} v|_{k(z)}))\\
\longrightarrow &K_{n-1}^{MW}(k(z_i), \omega_{k(z_i)/k} \otimes_{k(z_i)} (\omega_{X/k}^\vee|_{k(z)} \otimes_{k(z)} v|_{k(z)}))\\
\longrightarrow &K_{n-1}^{MW}(k(z_i), (\omega_{k(z)/k} \otimes k(z_i)) \otimes_{k(z_i)} (\omega_{X/k}^\vee|_{k(z)} \otimes_{k(z)} v|_{k(z)}))\\
\longrightarrow &K_{n-1}^{MW}(k(z_i), (\omega_{k(z)/k} \otimes_{k(z)} \omega_{X/k}^\vee|_{k(z)} \otimes_{k(z)} v|_{k(z)}) \otimes_{k(z)} k(z_i))\\
\longrightarrow &K_{n-1}^{MW}(k(z), \omega_{k(z)/k} \otimes_{k(z)} \omega_{X/k}^\vee|_{k(z)} \otimes_{k(z)} v|_{k(z)})\\
\longrightarrow &K_{n-1}^{MW}(k(z), \Lambda_z^* \otimes v),
\end{aligned}$$

where the second arrow is defined by

$$s \otimes c_i \otimes d \otimes l \longmapsto \partial_{z_i}^{e_i}(s) \otimes e_i \otimes (c_i \otimes 1) \otimes d \otimes l,$$

which is independent of the choice of e_i. Then we define $\partial_z^y = \sum \partial_i$. This definition coincides with the definition just given when Y is normal by applying Theorem 3.1, (4) to the following commutative diagram with exact columns and rows

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{Z/k}|_{k(z)} \arrow[r] \arrow[d, equal] & T_{Y/k}|_{k(z)} \arrow[r] \arrow[d] & (m_z/m_z^2)^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{Z/k}|_{k(z)} \arrow[r] & T_{X/k}|_{k(z)} \arrow[r] \arrow[d] & (I_Z/I_Z^2)^\vee|_{k(z)} \arrow[r] \arrow[d] & 0 \\
 & & (I_Y/I_Y^2)^\vee|_{k(z)} \arrow[r, equal] \arrow[d] & (I_Y/I_Y^2)^\vee|_{k(z)} \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Remark 8.3. *Here we would like to treat a kind of linearity of ∂_z^y. Let $s \in K_n^{MW}(k(y), \Lambda_y^* \otimes v)$.*

1. *Suppose that $f \in O_{Y,z}^*$ and that $n = 0$, we want to show that*

$$\partial_z^y([f]s) = [\overline{f}]\partial_z^y(s).$$

It suffices to show the formula for each ∂_i. We see that $\partial_i = Tr_{k(z)}^{k(z_i)} \circ \partial_{z_i}^{e_i}$ and $\partial_{z_i}^{e_i}([f]s) = \epsilon[\overline{f}]\partial_{z_i}^{e_i}(s)$. Suppose that $\partial_{z_i}^{e_i}(s) =< a > \eta$. Then

$$\begin{aligned}
&Tr_{k(z)}^{k(z_i)}(\epsilon[\overline{f}] < a > \eta)\\
=&Tr_{k(z)}^{k(z_i)}((< \overline{f} > - < 1 >) < a >)\\
=&(< \overline{f} > - < 1 >)Tr_{k(z)}^{k(z_i)}(< a >)\\
=&[\overline{f}]Tr_{k(z)}^{k(z_i)}(< a > \eta).
\end{aligned}$$

Then the claim is proved.

2. *If we have another line bundle $\mathscr{M}$ over X and m is a free basis of $\mathscr{M}_z$ (so it's also a free basis of $\mathscr{M}_y$), we have*

$$\partial_z^y(s \otimes m) = \partial_z^y(s) \otimes m.$$

We note nevertheless that this doesn't hold for a general free basis of $\mathscr{M}_y$. indeed, if we replace m by $\lambda \cdot m$, where $\lambda \in k(y)^*$, then λ plays a role in the computation of the residue maps.

Remark 8.4. *It's obvious that any morphism $v_1 \longrightarrow v_2$ in $\mathscr{P}_X$ will induce an isomorphism between the corresponding Rost-Schmid complexes.*

Definition 8.7. *([CF18]) Let $X_a \in Sm/k$ and let $x_a \in X_a$ for $a = 1, 2$. Let y be the generic point of some component of $\overline{x_1} \times \overline{x_2}$. For every $s_a \in K_n^{MW}(k(x_a), \Lambda_{x_a}^* \otimes v_a)$, we define*

$$s_1 \times s_2 = \sum_y c(p_1^*(v_1), p_2^*(\Lambda_{x_2}^*))(p_1^*(s_1) \otimes p_2^*(s_2)) \in \oplus_y K_{n+m}^{MW}(k(y), \Lambda_y^* \otimes (p_1^*(v_1) + p_2^*(v_2))),$$

where $p_i : \overline{y} \longrightarrow \overline{x_i}$ is the projection (note the use of Lemma 8.6). It is called the exterior product between s_1 and s_2. The exterior product is functorial with respect to twists and extension of supports.

We will denote $p_1^*(v_1) + p_2^*(v_2)$ by $v_1 \times v_2$ for convenience.

Now we focus of a special case of the proof that the right exterior product with an element in Chow-Witt groups (with support) is a chain complex map between Rost-Schmid complexes, while the left exterior product is not.

Proposition 8.3. *Let $X, X' \in Sm/k$, $v \in \mathscr{P}_X$, $v' \in \mathscr{P}_{X'}$ and let $Y \in Z^i(X)$, $T \in Z^j(X')$ be smooth. Suppose that $\beta \in \widetilde{CH}_T^j(X', v')$. Then the following diagram commutes*

$$\begin{tikzcd}
\oplus_{s \in (Y \times T)^{(0)}} K_n^{MW}(k(s), \Lambda_s^* \otimes (v \times v')) \arrow[r, "\partial"] & \oplus_{u \in (X \times X')^{(i+j+1)}} K_{n-1}^{MW}(k(u), \Lambda_u^* \otimes (v \times v')) . \\
\oplus_{y \in Y^{(0)}} K_n^{MW}(k(y), \Lambda_y^* \otimes v) \arrow[r, "\partial"] \arrow[u, "\times\beta"] & \oplus_{z \in Y \cap X^{(i+1)}} K_{n-1}^{MW}(k(z), \Lambda_z^* \otimes v) \arrow[u, "\times\beta"]
\end{tikzcd}$$

That is, for every $\beta \in \widetilde{CH}_T^j(X', v')$ and $\alpha \in \oplus_{y \in Y^{(0)}} K_n^{MW}(k(y), \Lambda_y^* \otimes v)$, we have

$$\partial(\alpha \times \beta) = \partial(\alpha) \times \beta.$$

Moreover, we have

$$\partial(\beta \times \alpha) =< -1 >^{j + rk_{X'}(v')} \beta \times \partial(\alpha).$$

Proof. We may assume that Y and T are irreducible. We check the commutativity after projecting to each $u \in (X \times X')^{(i+j+1)}$. It suffices to let u be a generic point of $\overline{z} \times T$, where $z \in Y \cap X^{(i+1)}$, since otherwise both terms vanish. Set $Z = \overline{z}$. We have a commutative diagram with exact columns and rows (we write $X \times Y$ by XY for short)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & N_{ZT/YT} \arrow[r, equal] \arrow[d] & N_{ZT/YT} \arrow[d] & & \\
0 \arrow[r] & N_{ZT/XT} \arrow[r] \arrow[d] & N_{ZT/XX'} \arrow[r] \arrow[d] & N_{XT/XX'}|_{ZT} \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & N_{YT/XT}|_{ZT} \arrow[r] \arrow[d] & N_{YT/XX'}|_{ZT} \arrow[r] \arrow[d] & N_{YT/YX'}|_{ZT} \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

We have projection maps $p_1 : ZT \longrightarrow Z$ and $p_2 : ZT \longrightarrow T$. By Theorem 3.1, (1), we have a commutative diagram

$$\begin{tikzcd}
p_1^*(N_{Z/Y} + N_{Y/X}|_Z + v|_Z) + p_2^*(N_{T/X'} + v'|_T) \arrow[r] \arrow[d] & p_1^*(N_{Z/Y}) + N_{YT/XX'}|_{ZT} + p_1^*(v|_Z) + p_2^*(v'|_T) \arrow[d] \\
p_1^*(N_{Z/X} + v|_Z) + p_2^*(N_{T/X'} + v'|_T) \arrow[d] & N_{ZT/YT} + N_{YT/XX'}|_{ZT} + p_1^*(v|_Z) + p_2^*(v'|_T) \arrow[d] \\
N_{ZT/XT} + p_1^*(v|_Z) + N_{YT/YX'}|_{ZT} + p_2^*(v'|_T) \arrow[r] & N_{ZT/XX'} + p_1^*(v|_Z) + p_2^*(v'|_T),
\end{tikzcd}$$

which gives the first equation. For the second one, we compute directly using the first equation using Proposition 8.4 (which still holds in this context):

$$\begin{aligned}
&\partial(\beta \times \alpha) \\
=&\partial(< -1 >^{(i+rk_X(v))(j+rk_{X'}(v'))}\ c(q_1^*v_1, q_2^*v_2)(\alpha \times \beta)) \\
=& < -1 >^{(i+rk_X(v))(j+rk_{X'}(v'))}\ c(q_1^*v_1, q_2^*v_2)(\partial(\alpha \times \beta)) \\
=& < -1 >^{(i+rk_X(v))(j+rk_{X'}(v'))}\ c(q_1^*v_1, q_2^*v_2)(\partial(\alpha) \times \beta) \\
=& < -1 >^{j+rk_{X'}(v')}\ \beta \times \partial(\alpha),
\end{aligned}$$

where q_1, q_2 are the respective projections of $X \times X'$ on the corresponding factor. $\square$

Definition 8.8. *The exterior product of Definition 8.7 induces a pairing*

$$\widetilde{CH}^{n_1}_{T_1}(X_1, v_1) \times \widetilde{CH}^{n_2}_{T_2}(X_2, v_2) \longrightarrow \widetilde{CH}^{n_1+n_2}_{T_1 \times T_2}(X_1 \times X_2, v_1 \times v_2)$$

for every $X_a \in Sm/k$, $T_a \in Z^{n_a}(X_a)$ smooth and $v_a \in \mathscr{P}_{X_a}$ for $a = 1, 2$ by Proposition 8.3. It's called the exterior product between Chow-Witt groups.

Proposition 8.4. *(Axiom 5 and 6) In the context above, the exterior product is associative and satisfies*

$$s_1 \times s_2 =< -1 >^{(codim(x_1)+rk_{X_1}(v_1))(codim(x_2)+rk_{X_2}(v_2))}\ c(p_2^*(v_2), p_1^*(v_1))(s_2 \times s_1)$$

where $s_a \in \widetilde{CH}^{n_a}_{T_a}(X_a, v_a)$.

Proof. Associativity comes from Definition 3.3, (3) and the second statement follows from the definition of the commutativity isomorphism in Proposition 8.1. $\square$

Proposition 8.5. *(Axiom 10) Let $f_a : Y_a \longrightarrow X_a$ be smooth morphisms in Sm/k for $a = 1, 2$. Then, we have*

$$(f_1 \times f_2)^*(s_1 \times s_2) = f_1^*(s_1) \times f_2^*(s_2).$$

Proof. This follows from Lemma 8.5 and Lemma 8.6. $\square$

Now, we would like to prove a special case that the the push-forwards defined in Definition 8.5 and Definition 8.6 form a chain complex morphism between Rost-Schmid complexes, just to explain how to treat the twists.

Proposition 8.6. *Suppose that $Z \subseteq Y \subseteq X$ are schemes with X and Y smooth. Suppose that $Y = \overline{y}$ in X and that $Z = \overline{z}$ in Y for some $z \in Y^{(1)}$. Suppose moreover that $f : X \longrightarrow X'$ is a smooth morphism, that $v \in \mathscr{P}_{X'}$ and that Y is also a closed subset of X'. Then we have a commutative diagram*

$$\begin{tikzcd}
K_n^{MW}(k(y), \Lambda_y^* \otimes f^*v \otimes \omega_{X/X'}) \arrow[r, "\partial"] \arrow[d, "f_*"] & K_{n-1}^{MW}(k(z), \Lambda_z^* \otimes f^*v \otimes \omega_{X/X'}) \arrow[d, "f_*"] \\
K_n^{MW}(k(f(y)), \Lambda_{f(y)}^* \otimes v) \arrow[r, "\partial"] & K_{n-1}^{MW}(k(f(z)), \Lambda_{f(z)}^* \otimes v).
\end{tikzcd}$$

Proof. We have the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & N_{Z/Y} \arrow[r, equal] \arrow[d] & N_{Z/Y} \arrow[d] & \\
0 \arrow[r] & T_{X/X'}|_Z \arrow[r] \arrow[d, equal] & N_{Z/X} \arrow[r] \arrow[d] & N_{Z/X'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{X/X'}|_Z \arrow[r] & N_{Y/X}|_Z \arrow[r] \arrow[d] & N_{Y/X'}|_Z \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Now the statement is to prove that the following diagram commutes

$$\begin{tikzcd}
N_{Z/Y} + N_{Y/X}|_Z + f^*v|_Z - T_{X/X'}|_Z \arrow[r] \arrow[d] & N_{Z/Y} + T_{X/X'}|_Z + N_{Y/X'}|_Z + f^*v|_Z - T_{X/X'}|_Z \arrow[d] \\
N_{Z/X} + f^*v|_Z - T_{X/X'}|_Z \arrow[d] & N_{Z/Y} + N_{Y/X'}|_Z + f^*v|_Z \arrow[d] \\
T_{X/X'}|_Z + N_{Z/X'} + f^*v|_Z - T_{X/X'}|_Z \arrow[r] & N_{Z/X'} + f^*v|_Z.
\end{tikzcd}$$

We have the following commutative diagrams

$$\begin{tikzcd}
T_{X/X'}|_Z + N_{Z/X'} + f^*v|_Z - T_{X/X'}|_Z \arrow[r] & N_{Z/X'} + f^*v|_Z \\
T_{X/X'}|_Z + N_{Z/Y} + N_{Y/X'}|_Z + f^*v|_Z - T_{X/X'}|_Z \arrow[r] \arrow[u] & N_{Z/Y} + N_{Y/X'}|_Z + f^*v|_Z \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
N_{Z/Y}+N_{Y/X}|_Z+f^*v|_Z-T_{X/X'}|_Z \arrow[r] \arrow[d] & N_{Z/Y}+T_{X/X'}|_Z+N_{Y/X'}|_Z+f^*v|_Z-T_{X/X'}|_Z\ , \arrow[d] \\
N_{Z/X}+f^*v|_Z-T_{X/X'}|_Z \arrow[d] & T_{X/X'}|_Z+N_{Z/Y}+N_{Y/X'}|_Z+f^*v|_Z-T_{X/X'}|_Z \\
T_{X/X'}|_Z+N_{Z/X'}+f^*v|_Z-T_{X/X'}|_Z \arrow[ur] &
\end{tikzcd}$$

where the second one comes from Theorem 3.1, (3). Then, the result follows by combining the two diagrams above. ☐

Proposition 8.7. *Suppose that $Z \subseteq Y \subseteq X$ are schemes with X and Y smooth. Suppose that $Y = \overline{y}$ in X and that $Z = \overline{z}$ in Y for $z \in Y^{(1)}$. Suppose moreover that $f : X \longrightarrow X'$ is a closed immersion and that $v \in \mathscr{P}_{X'}$. Then we have a commutative diagram*

$$\begin{tikzcd}
K_n^{MW}(k(y), \Lambda_y^* \otimes detN_{X/X'} \otimes f^*v) \arrow[r, "\partial"] \arrow[d, "f_*"'] & K_{n-1}^{MW}(k(z), \Lambda_z^* \otimes detN_{X/X'} \otimes f^*v) \arrow[d, "f_*"'] \\
K_n^{MW}(k(f(y)), \Lambda_{f(y)}^* \otimes v) \arrow[r, "\partial"] & K_{n-1}^{MW}(k(f(z)), \Lambda_{f(z)}^* \otimes v).
\end{tikzcd}$$

Proof. The diagram commutes because of the following commutative diagram by Definition 3.3, (3)

$$\begin{tikzcd}
N_{Z/Y}+N_{Y/X}|_Z+N_{X/X'}|_Z+f^*v|_Z \arrow[r] \arrow[d] & N_{Z/X}+N_{X/X'}|_Z+f^*v|_Z \arrow[d] \\
N_{Z/Y}+N_{Y/X'}|_Z+f^*v|_Z \arrow[r] & N_{Z/X'}+f^*v|_Z.
\end{tikzcd}$$

☐

Proposition 8.8. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in Sm/k, $v \in \mathscr{P}_Z$ and let $C \in Z^{i+d_{g\circ f}}(X)$.*

1. *(Axiom 12) Suppose that f, g are smooth and that C is also a closed subset in Z. Then the following diagram commutes*

$$\begin{tikzcd}
\widetilde{CH}_C^{i+d_{g\circ f}}(X,(g\circ f)^*v-T_{X/Z}) \arrow[r] \arrow[ddr, "(g\circ f)_*"'] & \widetilde{CH}_C^{i+d_{g\circ f}}(X,(g\circ f)^*v-f^*T_{Y/Z}-T_{X/Y}) \arrow[d, "f_*"'] \\
& \widetilde{CH}_{f(C)}^{i+d_g}(Y,g^*v-T_{Y/Z}) \arrow[d, "g_*"'] \\
& \widetilde{CH}_{g(f(C))}^{i}(Z,v).
\end{tikzcd}$$

2. *(Axiom 14) Suppose that f, g are closed immersions. Then, the following diagram commutes*

$$\begin{tikzcd}
\widetilde{CH}_C^{i+d_{g\circ f}}(X,N_{X/Z}+(g\circ f)^*v) \arrow[r] \arrow[ddr, "(g\circ f)_*"'] & \widetilde{CH}_C^{i+d_{g\circ f}}(X,N_{X/Y}+f^*N_{Y/Z}+(g\circ f)^*v) \arrow[d, "f_*"'] \\
& \widetilde{CH}_{f(C)}^{i+d_g}(Y,N_{Y/Z}+g^*v) \arrow[d, "g_*"'] \\
& \widetilde{CH}_{g(f(C))}^{i}(Z,v).
\end{tikzcd}$$

3. *(Axiom 19, (1)) Suppose that f is a closed immersion, that g and $g \circ f$ are smooth and that C is also a closed subset of Z. Then the following diagram commutes*

$$\begin{tikzcd}
\widetilde{CH}^{i+d_{g\circ f}}_C(X, N_{X/Y} + f^*g^*v - f^*T_{Y/Z}) \arrow[r] \arrow[d, "f_*"'] & \widetilde{CH}^{i+d_{g\circ f}}_C(X, f^*g^*v + N_{X/Y} - f^*T_{Y/Z}) \arrow[d] \\
\widetilde{CH}^{i+d_g}_{f(C)}(Y, g^*v - T_{Y/Z}) \arrow[d, "g_*"'] & \widetilde{CH}^{i+d_{g\circ f}}_C(X, f^*g^*v - T_{X/Z}) \arrow[dl, "(g\circ f)_*"'] \\
\widetilde{CH}^{i}_{g(f(C))}(Z, v). &
\end{tikzcd}$$

4. *(Axiom 19, (2)) Suppose that g is smooth, and that f and $g\circ f$ are closed immersions. Then the following diagram commutes*

$$\begin{tikzcd}
\widetilde{CH}^{i+d_{g\circ f}}_C(X, N_{X/Y} + f^*g^*v - f^*T_{Y/Z}) \arrow[r] \arrow[d, "f_*"'] & \widetilde{CH}^{i+d_{g\circ f}}_C(X, -f^*T_{Y/Z} + N_{X/Y} + f^*g^*v) \arrow[d] \\
\widetilde{CH}^{i+d_g}_{f(C)}(Y, g^*v - T_{Y/Z}) \arrow[d, "g_*"'] & \widetilde{CH}^{i+d_{g\circ f}}_C(X, N_{X/Z} + f^*g^*v) \arrow[dl, "(g\circ f)_*"'] \\
\widetilde{CH}^{i}_{g(f(C))}(Z, v). &
\end{tikzcd}$$

Proof. 1. This follows from the following commutative diagram

$$\begin{tikzcd}
N_{C/X} + f^*g^*v|_C - T_{X/Z}|_C \arrow[r] \arrow[dddd] & N_{C/X} + f^*g^*v|_C - f^*T_{Y/Z}|_C - T_{X/Y}|_C \arrow[d] \\
& T_{X/Y}|_C + N_{C/Y} + f^*g^*v|_C - f^*T_{Y/Z}|_C - T_{X/Y}|_C \arrow[d] \\
& N_{C/Y} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[d] \\
& f^*T_{Y/Z}|_C + N_{C/Z} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[d] \\
T_{X/Z}|_C + N_{C/Z} + f^*g^*v|_C - T_{X/Z}|_C \arrow[r] & N_{C/Z} + f^*g^*v|_C
\end{tikzcd}$$

using Definition 3.3, (3).

2. Essentially the same as in (1).

3. We are going to prove that the following diagram commutes

$$\begin{tikzcd}
N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[r] \arrow[d] & N_{C/X} + f^*g^*v|_C + N_{X/Y}|_C - f^*T_{Y/Z}|_C \arrow[d] \\
N_{C/Y} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[d] & N_{C/X} + f^*g^*v|_C + N_{X/Y}|_C - N_{X/Y}|_C - T_{X/Z}|_C \arrow[d] \\
f^*T_{Y/Z}|_C + N_{C/Z} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[d] & N_{C/X} + f^*g^*v|_C - T_{X/Z}|_C \arrow[d] \\
N_{C/Z} + f^*g^*v|_C & T_{X/Z}|_C + N_{C/Z} + f^*g^*v|_C - T_{X/Z}|_C \arrow[l]
\end{tikzcd}.$$

Let $A = T_{X/Z}|_C + N_{X/Y}|_C + N_{C/Z} + f^*g^*v|_C - N_{X/Y}|_C - T_{X/Z}|_C$. We have commutative diagrams

$$\begin{tikzcd}
f^*T_{Y/Z}|_C + N_{C/Z} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[r] \arrow[d] & A \arrow[d] \\
N_{C/Z} + f^*g^*v|_C \arrow[r] & T_{X/Z}|_C + N_{C/Z} + f^*g^*v|_C - T_{X/Z}|_C
\end{tikzcd}$$

$$\begin{tikzcd}
N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - N_{X/Y}|_C - T_{X/Z}|_C \arrow[d] \arrow[dr] & \\
N_{C/X} + f^*g^*v|_C - T_{X/Z}|_C \arrow[d] & A \arrow[dl] \\
T_{X/Z}|_C + N_{C/Z} + f^*g^*v|_C - T_{X/Z}|_C. &
\end{tikzcd}$$

Furthermore, there is a commutative diagram with exact rows and columns

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & T_{X/Z}|_C \arrow[r] \arrow[d] & N_{C/X} \arrow[r] \arrow[d] & N_{C/Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & f^*T_{Y/Z}|_C \arrow[r] \arrow[d] & N_{C/Y} \arrow[r] \arrow[d] & N_{C/Z} \arrow[r] & 0 \\
& N_{X/Y}|_C \arrow[r, equal] \arrow[d] & N_{X/Y}|_C \arrow[d] & & \\
& 0 & 0 & &
\end{tikzcd}$$

and by Theorem 3.1, (2), we have a commutative diagram

$$\begin{tikzcd}
N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[r] \arrow[d] & N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[d] \\
N_{C/Y} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[d] & T_{X/Z}|_C + N_{C/Z}|_C + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[d] \\
f^*T_{Y/Z}|_C + N_{C/Z} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[r] & A.
\end{tikzcd}$$

The proof follows easily.

4. We are going to prove that the following diagram commutes

$$\begin{tikzcd}
N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[r] \arrow[d] & N_{C/X} - f^*T_{Y/Z}|_C + N_{X/Y}|_C + f^*g^*v|_C \arrow[d] \\
N_{C/Y} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[d] & N_{C/X} - f^*T_{Y/Z}|_C + f^*T_{Y/Z}|_C + N_{X/Z}|_C + f^*g^*v|_C \arrow[d] \\
f^*T_{Y/Z}|_C + N_{C/Z} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[d] & N_{C/X} + N_{X/Z}|_C + f^*g^*v|_C \arrow[dl] \\
N_{C/Z} + f^*g^*v|_C. &
\end{tikzcd}$$

We have a commutative diagram

$$\begin{tikzcd}
f^*T_{Y/Z}|_C + N_{C/Z} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[r] \arrow[d] & f^*T_{Y/Z}|_C + N_{C/X} + N_{X/Z}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[d] \\
N_{C/Z} + f^*g^*v|_C \arrow[r] & N_{C/X} + N_{X/Z}|_C + f^*g^*v|_C.
\end{tikzcd}$$

Furthermore, there is a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & f^*T_{Y/Z}|_C \arrow[r, equal] \arrow[d] & f^*T_{Y/Z}|_C \arrow[d] & \\
0 \arrow[r] & N_{C/X} \arrow[r] \arrow[d, equal] & N_{C/Y} \arrow[r] \arrow[d] & N_{X/Y}|_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_{C/X} \arrow[r] & N_{C/Z} \arrow[r] \arrow[d] & N_{X/Z}|_C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

By Theorem 3.1, (3), we have a commutative diagram

$$\begin{tikzcd}
N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[r] \arrow[d] & N_{C/X} + f^*T_{Y/Z}|_C + N_{X/Z}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[dd] \\
N_{C/Y} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[d] & \\
f^*T_{Y/Z}|_C + N_{C/Z} + f^*g^*v|_C - f^*T_{Y/Z}|_C \arrow[r] & f^*T_{Y/Z}|_C + N_{C/X} + N_{X/Z}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C
\end{tikzcd}$$

and the proof follows.

□

Proposition 8.9. *(Axiom 19, (3)) Suppose that we have a Cartesian square of smooth schemes*

$$\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"] & Y,
\end{tikzcd}$$

where u is smooth and f is a closed immersion: Let $s \in \mathscr{P}_Y$ and let $C \in Z^{n+d_f+d_v}(X')$ be also closed in Y. Then the following diagram commutes

$$\begin{tikzcd}
\widetilde{CH}^{n+d_f+d_v}_C(X', N_{X'/Y'} + g^*u^*s - g^*T_{Y'/Y}) \arrow[r, "g_*"] \arrow[d] & \widetilde{CH}^{n+d_u}_{g(C)}(X, u^*s - T_{Y'/Y}) \arrow[d, "u_*"] \\
\widetilde{CH}^{n+d_f+d_v}_C(X', v^*N_{X/Y} + u^*f^*s - T_{X'/X}) \arrow[d, "v_*"] & \widetilde{CH}^{n}_{u(g(C))}(Y, s) \\
\widetilde{CH}^{n+d_f}_{v(C)}(X, N_{X/Y} + f^*s). \arrow[ur, "f_*"] &
\end{tikzcd}$$

Proof. We are going to show that the following diagram commutes

$$\begin{tikzcd}
N_{C/X'} + N_{X'/Y'}|_C + g^*u^*s|_C - g^*T_{Y'/Y}|_C \arrow[r] \arrow[d] & N_{C/Y'} + g^*u^*s|_C - g^*T_{Y'/Y}|_C \arrow[d] \\
N_{C/X'} + v^*N_{X/Y}|_C + g^*u^*s|_C - T_{X'/X}|_C \arrow[d] & g^*T_{Y'/Y}|_C + N_{C/Y} + g^*u^*s|_C - g^*T_{Y'/Y}|_C \arrow[d] \\
T_{X'/X}|_C + N_{C/X} + v^*N_{X/Y}|_C + g^*u^*s|_C - T_{X'/X}|_C \arrow[d] & N_{C/Y} + g^*u^*s|_C \\
N_{C/X} + v^*N_{X/Y}|_C + g^*u^*s|_C. \arrow[ur] &
\end{tikzcd}$$

We have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & T_{X'/X}|_C \arrow[r, "\cong"] \arrow[d] & g^*T_{Y'/Y}|_C \arrow[d] & & \\
0 \arrow[r] & N_{C/X'} \arrow[r] \arrow[d] & N_{C/Y'} \arrow[r] \arrow[d] & N_{X'/Y'}|_C \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & N_{C/X} \arrow[r] \arrow[d] & N_{C/Y} \arrow[r] \arrow[d] & v^*N_{X/Y}|_C \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

So, we have a commutative diagram by Theorem 3.1, (1)

$$\begin{tikzcd}
N_{C/Y'} \arrow[r] \arrow[d] & N_{C/X'} + N_{X'/Y'}|_C \arrow[d] \\
g^*T_{Y'/Y}|_C + N_{C/Y} \arrow[d] & T_{X'/X}|_C + N_{C/X} + N_{X'/Y'}|_C \\
g^*T_{Y'/Y}|_C + N_{C/X} + v^*N_{X/Y}|_C. \arrow[ur] &
\end{tikzcd}$$

Then the statement follows easily from the data above. □

Proposition 8.10. *Suppose that we have a Cartesian square of smooth schemes*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y. \end{tikzcd}$$

1. *(Axiom 15) Suppose that f, u are smooth, that $s \in \mathscr{P}_Y$ and that $C \in Z^{n+d_f}(X)$ is a closed subset of Y. Then the following diagram commutes*

$$\begin{tikzcd} \widetilde{CH}^{n+d_f}_C(X, f^*s - T_{X/Y}) \arrow[r, "f_*"] \arrow[d, "v^*"] & \widetilde{CH}^n_{f(C)}(Y, s) \arrow[d, "u^*"] \\ \widetilde{CH}^{n+d_f}_{v^{-1}(C)}(X', v^*f^*s - v^*T_{X/Y}) \arrow[r, "g_*"] & \widetilde{CH}^n_{g(v^{-1}(C))}(Y', u^*s). \end{tikzcd}$$

2. *(Axiom 16) Suppose that f is a closed immersion, that $s \in \mathscr{P}_Y$ and that $C \in Z^{n+d_f}(X)$. Suppose moreover that u is smooth. Then the following diagram commutes*

$$\begin{tikzcd} \widetilde{CH}^{n+d_f}_C(X, N_{X/Y} + f^*s) \arrow[r, "f_*"] \arrow[d, "v^*"] & \widetilde{CH}^n_{f(C)}(Y, s) \arrow[d, "u^*"] \\ \widetilde{CH}^{n+d_f}_{v^{-1}(C)}(X', v^*N_{X/Y} + v^*f^*s) \arrow[r, "g_*"] & \widetilde{CH}^n_{g(v^{-1}(C))}(Y', u^*s). \end{tikzcd}$$

Proof. 1. We have a commutative diagram by functoriality of v^* with respect to twists

$$\begin{tikzcd} N_{C/X} + f^*v|_C - T_{X/Y}|_C \arrow[r] \arrow[d] & T_{X/Y}|_C + N_{C/Y} + f^*s|_C - T_{X/Y}|_C \arrow[d] \\ N_{v^{-1}(C)/X'} + v^*f^*s|_{v^{-1}(C)} - T_{X'/Y'}|_{v^{-1}(C)} \arrow[d] & N_{C/Y} + f^*s|_C \arrow[d] \\ T_{X'/Y'}|_{v^{-1}(C)} + N_{v^{-1}(C)/Y'} + v^*f^*s|_{v^{-1}(C)} - T_{X'/Y'}|_{v^{-1}(C)} \arrow[r] & N_{v^{-1}(C)/Y'} + f^*s|_{v^{-1}(C)}. \end{tikzcd}$$

2. We have a commutative diagram by functoriality of v^* with respect to twists

$$\begin{tikzcd} N_{C/X} + N_{X/Y}|_C + f^*s|_C \arrow[r] \arrow[d] & N_{C/Y} + f^*s|_C \arrow[d] \\ N_{v^{-1}(C)/X'} + N_{X'/Y'}|_{v^{-1}(C)} + v^*f^*s|_{v^{-1}(C)} \arrow[r] & N_{v^{-1}(C)/Y'} + v^*f^*s|_{v^{-1}(C)}. \end{tikzcd}$$

□

Proposition 8.11. 1. *(Axiom 17) Suppose that $f : X \longrightarrow Y$ is a smooth morphism in Sm/k, that $v \in \mathscr{P}_Y$ and that $C \in Z^{n+d_f}(X)$ is a smooth closed subset of Y. Then, for any $Z \in Sm/k$, any $v' \in \mathscr{P}_Z$ and any $D \in Z^m(Z)$, the following diagrams commute*

$$\begin{tikzcd} \widetilde{CH}^{n+d_f}_C(X, f^*v - T_{X/Y}) \times \widetilde{CH}^m_D(Z, v') \arrow[r, "\times"] \arrow[dd, "f_* \times id"'] & \widetilde{CH}^{n+d_f+m}_{C\times D}(X \times Z, (f^*v - T_{X/Y}) \times v') \arrow[d, "c"'] \\ & \widetilde{CH}^{n+d_f+m}_{C\times D}(X \times Z, (f^*v \times v') - T_{X\times Z/Y\times Z}) \arrow[d, "(f\times id)_*"'] \\ \widetilde{CH}^n_{f(C)}(Y, v) \times \widetilde{CH}^m_D(Z, v') \arrow[r, "\times"] & \widetilde{CH}^{n+m}_{f(C)\times D}(Y \times Z, v \times v') \end{tikzcd}$$

$$\begin{tikzcd}
\widetilde{CH}^m_D(Z,v') \times \widetilde{CH}^{n+d_f}_C(X, f^*v - T_{X/Y}) \arrow[r, "\times"] \arrow[d, "id\times f_*"'] & \widetilde{CH}^{n+d_f+m}_{D\times C}(Z\times X, v'\times (f^*v - T_{X/Y})) \arrow[d, "(id\times f)_*"'] \\
\widetilde{CH}^m_D(Z,v') \times \widetilde{CH}^n_{f(C)}(Y,v) \arrow[r, "\times"] & \widetilde{CH}^{n+m}_{D\times f(C)}(Z\times Y, v'\times v).
\end{tikzcd}$$

2. *(Axiom 18) Suppose that $f : X \longrightarrow Y$ is a closed immersion in Sm/k, that $v \in \mathscr{P}_Y$ and that C is a smooth closed subset of X. Then for any $Z \in Sm/k$, any $v' \in \mathscr{P}_Z$ and any $D \in Z^m(Z)$, the following diagrams commute*

$$\begin{tikzcd}
\widetilde{CH}^{n+d_f}_C(X, N_{X/Y} + f^*v) \times \widetilde{CH}^m_D(Z,v') \arrow[r, "\times"] \arrow[d, "f_*\times id"'] & \widetilde{CH}^{n+d_f+m}_{C\times D}(X\times Z, (N_{X/Y} + f^*v)\times v') \arrow[d, "(f\times id)_*"'] \\
\widetilde{CH}^n_{f(C)}(Y,v) \times \widetilde{CH}^m_D(Z,v') \arrow[r, "\times"] & \widetilde{CH}^{n+m}_{f(C)\times D}(Y\times Z, v\times v')
\end{tikzcd}$$

$$\begin{tikzcd}
\widetilde{CH}^m_D(Z,v') \times \widetilde{CH}^{n+d_f}_C(X, N_{X/Y} + f^*v) \arrow[r, "\times"] \arrow[dd, "id\times f_*"'] & \widetilde{CH}^{n+d_f+m}_{D\times C}(Z\times X, v'\times (N_{X/Y} + f^*v)) \arrow[d, "c"'] \\
 & \widetilde{CH}^{n+d_f+m}_{D\times C}(Z\times X, (v'\times f^*v) + N_{X\times Z/Y\times Z}) \arrow[d, "(id\times f)_*"'] \\
\widetilde{CH}^m_D(Z,v') \times \widetilde{CH}^n_{f(C)}(Y,v) \arrow[r, "\times"] & \widetilde{CH}^{n+m}_{D\times f(C)}(Z\times Y, v'\times v).
\end{tikzcd}$$

Proof. We have projections $p_1 : C\times D \longrightarrow C$ and $p_2 : C\times D \longrightarrow D$.

1. For the first diagram, we are going to prove that the following diagram commutes

$$\begin{tikzcd}
(N_{C/X} + f^*v|_C - T_{X/Y}|_C, N_{D/Z} + v'|_D) \arrow[r] \arrow[d, "f_*"'] & p_1^*(N_{C/X} + f^*v|_C - T_{X/Y}|_C) + p_2^*(N_{D/Z} + v'|_D) \arrow[d] \\
(N_{C/Y} + f^*v|_C, N_{D/Z} + v'|_D) \arrow[d] & N_{C\times D/X\times Z} + p_1^*(f^*v|_C) + p_2^*(v'|_D) - T_{X\times Z/Y\times Z}|_{C\times D} \arrow[d, "(f\times id)_*"'] \\
p_1^*(N_{C/Y} + f^*v|_C) + p_2^*(N_{D/Z} + v'|_D) \arrow[r] & N_{C\times D/Y\times Z} + p_1^*(f^*v|_C) + p_2^*(v'|_D).
\end{tikzcd}$$

We have a commutative diagram

$$\begin{tikzcd}
(N_{C/X} + f^*v|_C - T_{X/Y}|_C, N_{D/Z} + v'|_D) \arrow[r] \arrow[d, "f_*"'] & p_1^*(N_{C/X} + f^*v|_C - T_{X/Y}|_C) + p_2^*(N_{D/Z} + v'|_D) \arrow[ddl, "p_1^*(f_*)+p_2^*(id)"] \\
(N_{C/Y} + f^*v|_C, N_{D/Z} + v'|_D) \arrow[d] & \\
p_1^*(N_{C/Y} + f^*v|_C) + p_2^*(N_{D/Z} + v'|_D) &
\end{tikzcd}$$

and then we just have to show the following diagram commutes

$$\begin{tikzcd}
 & p_1^*(N_{C/X} + f^*v|_C - T_{X/Y}|_C) + p_2^*(N_{D/Z} + v'|_D) \arrow[d] \arrow[ddl, bend right, "p_1^*(f_*)+p_2^*(id)"'] \\
 & N_{C\times D/X\times Z} + p_1^*(f^*v|_C) + p_2^*(v'|_D) - T_{X\times Z/Y\times Z}|_{C\times D} \arrow[d, "(f\times id)_*"'] \\
p_1^*(N_{C/Y} + f^*v|_C) + p_2^*(N_{D/Z} + v'|_D) \arrow[r] & N_{C\times D/Y\times Z} + p_1^*(f^*v|_C) + p_2^*(v'|_D).
\end{tikzcd}$$

This follows from Theorem 3.1, (1) and the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & p_1^*(T_{X/Y}|_C) \arrow[r, "\cong"] \arrow[d] & T_{X\times Z/Y\times Z}|_{C\times D} \arrow[d] & & \\
0 \arrow[r] & p_1^*N_{C/X} \arrow[r] \arrow[d] & N_{C\times D/X\times Z} \arrow[r] \arrow[d] & p_2^*N_{D/Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & p_1^*N_{C/Y} \arrow[r] \arrow[d] & N_{C\times D/Y\times Z} \arrow[r] \arrow[d] & p_2^*N_{D/Z} \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

For the second diagram, we suppose that $\alpha \in \widetilde{CH}_C^{n+d_f}(X, f^*v - T_{X/Y})$ and that $\beta \in \widetilde{CH}_D^m(Z, v')$. Moreover, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & X \times Z \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "f\times id"'] & D \\
Y & Y \times Z. \arrow[l, "q_1"'] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

Then

$(id \times f)_*(\beta \times \alpha)$
$=(f \times id)_*(< -1 >^{(n+rk_Y(v))(m+rk_Z(v'))}\ c(p_1^*(f^*v - T_{X/Y}), p_2^*(v'))(\alpha \times \beta))$
by Proposition 8.4
$= < -1 >^{(n+rk_Y(v))(m+rk_Z(v'))}\ (f \times id)_*(c(p_1^*(f^*v - T_{X/Y}), p_2^*(v'))(\alpha \times \beta))$
$= < -1 >^{(n+rk_Y(v))(m+rk_Z(v'))}\ (f \times id)_*((c(p_1^*(f^*v), p_2^*(v')) \circ c(-p_1^*T_{X/Y}, p_2^*(v')))(\alpha \times \beta))$
$= < -1 >^{(n+rk_Y(v))(m+rk_Z(v'))}\ c(q_1^*(v), q_2^*(v'))((f \times id)_*(c(-p_1^*T_{X/Y}, p_2^*(v'))(\alpha \times \beta)))$
by functoriality of push-forwards with respect to twists
$= < -1 >^{(n+rk_Y(v))(m+rk_Z(v'))}\ c(q_1^*(v), q_2^*(v'))(f_*(\alpha) \times \beta))$
by the first diagram
$=\beta \times f_*(\alpha)$
by Proposition 8.4.

2. For the first diagram, we are going to prove that the following diagram commutes

$$\begin{tikzcd}
(N_{C/X} + N_{X/Y}|_C + f^*v|_C, N_{D/Z} + v') \arrow[r] \arrow[d] & p_1^*(N_{C/X} + N_{X/Y}|_C + f^*v|_C) + p_2^*(N_{D/Z} + v') \arrow[d] \\
(N_{C/Y} + f^*v|_C, N_{D/Z} + v') \arrow[d] & N_{C\times D/X\times Z} + N_{X\times Z/Y\times Z}|_{C\times D} + p_1^*(f^*v|_C) + p_2^*(v') \arrow[d] \\
p_1^*(N_{C/Y} + f^*v|_C) + p_2^*(N_{D/Z} + v') \arrow[r] & N_{C\times D/Y\times Z} + p_1^*(f^*v|_C) + p_2^*(v').
\end{tikzcd}$$

We have a commutative diagram

$$\begin{tikzcd}
(N_{C/X}+N_{X/Y}|_C+f^*v|_C, N_{D/Z}+v') \arrow[r] \arrow[d] & p_1^*(N_{C/X}+N_{X/Y}|_C+f^*v|_C)+p_2^*(N_{D/Z}+v') \arrow[ddl, "p_1^*(f_*)+p_2^*(id)"] \\
(N_{C/Y}+f^*v|_C, N_{D/Z}+v') \arrow[d] & \\
p_1^*(N_{C/Y}+f^*v|_C)+p_2^*(N_{D/Z}+v'). &
\end{tikzcd}$$

Hence we just have to show that the following diagram commutes

$$\begin{tikzcd}
 & p_1^*(N_{C/X}+N_{X/Y}|_C+f^*v|_C)+p_2^*(N_{D/Z}+v') \arrow[d] \arrow[ddl, bend right, "p_1^*(f_*)+p_2^*(id)"'] \\
 & N_{C\times D/X\times Z}+N_{X\times Z/Y\times Z}|_{C\times D}+p_1^*(f^*v|_C)+p_2^*(v') \arrow[d] \\
p_1^*(N_{C/Y}+f^*v|_C)+p_2^*(N_{D/Z}+v') \arrow[r] & N_{C\times D/Y\times Z}+p_1^*(f^*v|_C)+p_2^*(v').
\end{tikzcd}$$

This follows fromTheorem 3.1, (2) together with the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & p_1^*N_{C/X} \arrow[r] \arrow[d] & N_{C\times D/X\times Z} \arrow[r] \arrow[d] & p_2^*N_{D/Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & p_1^*N_{C/Y} \arrow[r] \arrow[d] & N_{C\times D/Y\times Z} \arrow[r] \arrow[d] & p_2^*N_{D/Z} \arrow[r] & 0 \\
 & p_1^*(N_{X/Y}|_C) \arrow[r, "\cong"] \arrow[d] & N_{X\times Z/Y\times Z}|_{C\times D} \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

The second diagram follows by the same method as in the proof of the second diagram of (1).

□

8.2 Intersection with Divisors

In this section, we discuss a special case of intersection, namely pull-backs along a divisor with smooth support. The constructions here basically come from [CF18], but the treatments of push-forwards are possibly different.

Definition 8.9. *Let $X \in Sm/k$ and let $D = \{(U_i, f_i)\}$ be a Cartier divisor on X. Suppose that $C \in Z^n(X)$, $s \in \widetilde{CH}^n_C(X, v)$ and that $dim(C \cap |D|) < dim(C)$. Let*

$$s = \sum_a s_a \otimes u_a \otimes v_a \in \oplus_{y_a \in X^{(n)}} K_0^{MW}(k(y_a), \Lambda^*_{y_a} \otimes v)$$

where $s_a \otimes u_a \otimes v_a \in K_0^{MW}(k(y_a), \Lambda_{y_a}^* \otimes v)$ and $y_a \in X^{(n)}$. For every $x \in \overline{\{y_a\}} \cap X^{(n+1)}$, suppose that $x \in U_i$ for some i (and then $y_a \in U_i$ also). Then, $f_i \in O_{X,y_a}^*$ since $y_a \notin |D|$ and consequently we have a well-defined element $\overline{f_i} \in k(y_a)$. Set

$$ord_x(D \cdot s) = \sum_{x \in \overline{y_a}} \partial_x^{y_a}(< -1 >^{codim(y_a)} [\overline{f_i}] s_a \otimes u_a \otimes f_i \otimes v_a) \in K_0^{MW}(k(x), \Lambda_x^* \otimes \mathscr{L}(-D) \otimes v).$$

Then define

$$D \cdot s = \sum_{x \in X^{(n+1)}} ord_x(D \cdot s) \in \oplus_{x \in X^{(n+1)}} K_0^{MW}(k(x), \Lambda_x^* \otimes \mathscr{L}(-D) \otimes v).$$

It's functorial with respect to v by Remark 8.4.

Lemma 8.7. *The definition of $ord_x(D \cdot s)$ above is independent of the choice of i and f_i and*

$$D \cdot s \in \widetilde{CH}^{n+1}_{C \cap |D|}(X, \mathscr{L}(-D) + v).$$

Proof. For any other j and f_j with $x \in U_j$, we have $f_j/f_i \in O_{X,x}^*$. Moreover, we have

$$\sum_{x \in \overline{y_a}} \partial_x^{y_a}(s_a \otimes u_a \otimes v_a) = 0$$

since $s \in \widetilde{CH}^n_C(X, v)$. So we have

$$\sum_{x \in \overline{y_a}} \partial_x^{y_a}(s_a \otimes u_a \otimes f_i \otimes v_a) = 0$$

and

$$\sum_{x \in \overline{y_a}} \partial_x^{y_a}(s_a \otimes u_a \otimes f_j \otimes v_a) = 0$$

by Remark 8.3, (2). Moreover,

$$\begin{aligned}
&[\overline{f_j}] s_a \otimes u_a \otimes f_j \otimes v_a \\
=&([\overline{f_j/f_i}] + < \overline{f_j/f_i} > [\overline{f_i}]) s_a \otimes u_a \otimes f_j \otimes v_a \\
=&[\overline{f_j/f_i}] s_a \otimes u_a \otimes f_j \otimes v_a + [\overline{f_i}] s_a \otimes u_a \otimes f_i \otimes v_a.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{x \in \overline{y_a}} \partial_x^{y_a}([\overline{f_j}] s_a \otimes u_a \otimes f_j \otimes v_a) \\
=&\partial_x^{y_a}(\sum_{x \in \overline{y_a}} [\overline{f_j/f_i}] s_a \otimes u_a \otimes f_j \otimes v_a) + \partial_x^{y_a}(\sum_{x \in \overline{y_a}} [\overline{f_i}] s_a \otimes u_a \otimes f_i \otimes v_a) \\
=&\partial_x^{y_a}(\sum_{x \in \overline{y_a}} [\overline{f_i}] s_a \otimes u_a \otimes f_i \otimes v_a),
\end{aligned}$$

which shows that $ord_x(D \cdot s)$ is well-defined.

If $x \notin |D|$, then $\overline{f_i} \in O^*_{\overline{y_a},x}$. So

$$\begin{aligned}
&ord_x(D \cdot s) \\
=&\sum_{x \in \overline{y_a}} \partial_x^{y_a}(< -1 >^{codim(y_a)} [\overline{f_i}] s_a \otimes u_a \otimes f_i \otimes v_a) \\
=&\sum_{x \in \overline{y_a}} [\overline{f_i}] \partial_x^{y_a}(< -1 >^{codim(y_a)} s_a \otimes u_a \otimes f_i \otimes v_a) \\
&\text{by Remark 8.3, (1)} \\
=&0.
\end{aligned}$$

Hence the support of $D \cdot s$ is contained in $C \cap |D|$.

Finally let's prove that $\partial(D \cdot s) = 0$, where for every z, we denote $\sum_{y,z\in\overline{y}} \partial_z^y$ by ∂_z and the differential map ∂ is then just (∂_z). For this, we prove that

$$\partial_u(D \cdot s) := \sum_{x\in X^{(n+1)}, u\in\overline{x}} \partial_u^x(ord_x(D \cdot s)) = 0$$

for $u \in X^{(n+2)}$. If $u \in U_i$, then

$$ord_x(D \cdot s) = \sum_{x\in\overline{y_a}} \partial_x^{y_a}(< -1 >^{codim(y_a)} [\overline{f_i}] s_a \otimes u_a \otimes f_i \otimes v_a)$$

by definition. So let $t = \sum_a < -1 >^{codim(y_a)} [\overline{f_i}] s_a \otimes u_a \otimes f_i \otimes v_a$.

$$\sum_{x\in X^{(n+1)}, u\in\overline{x}} \partial_u^x(ord_x(D \cdot s)) = \sum_{x\in X^{(n+1)}, u\in\overline{x}} \partial_u^x(\partial_x(t)) = \partial_u(\partial(t)) = 0.$$

□

Definition 8.10. *(Axiom 8) Let $X \in Sm/k$ and let D be a smooth effective Cartier divisor on X. Let $i : |D| \longrightarrow X$ be the inclusion and let $N_{D/X} \cong i^*\mathscr{L}(D)$ be its normal bundle. Suppose that $v \in \mathscr{P}_X$, that $C \in Z^n(X)$ and that $s \in \widetilde{CH}^n_C(X, v)$ and that $dim(C \cap |D|) < dim(C)$. We have a push-forward isomorphism*

$$i_* : \widetilde{CH}^n_{C\cap|D|}(|D|, i^*\mathscr{L}(D) + i^*\mathscr{L}(-D) + i^*v) \longrightarrow \widetilde{CH}^{n+1}_{C\cap|D|}(X, \mathscr{L}(-D) + v).$$

Denote by $s(\mathscr{L}(D))$ the isomorphism $i^*v \longrightarrow i^*\mathscr{L}(D) + i^*\mathscr{L}(-D) + i^*v$ and define

$$i^*(s) \in \widetilde{CH}^n_{C\cap|D|}(|D|, i^*v)$$

to be the unique element such that

$$i_*(s(\mathscr{L}(D))(i^*(s))) = D \cdot s.$$

It's functorial with respect to v.

Proposition 8.12. *Let $X_a \in Sm/k$, $v_a \in \mathscr{P}_{X_a}$ and $C_a \in Z^{n_a}(X_a)$ be smooth for $a = 1, 2$. Further, let $\alpha_a \in \widetilde{CH}^{n_a}_{C_a}(X_a, v_a)$, $p_a : X_1 \times X_2 \longrightarrow X_a$ be the projections and let D_a be smooth effective Cartier divisors on X_a. Then*

$$(D_1 \cdot \alpha_1) \times \alpha_2 = p_1^*(D_1) \cdot (\alpha_1 \times \alpha_2)$$

and

$$c(p_1^*v_1, p_2^*\mathscr{L}(-D_2))(\alpha_1 \times (D_2 \cdot \alpha_2)) = p_2^*(D_2) \cdot (\alpha_1 \times \alpha_2).$$

Proof. We prove the first assertion. Since both sides live in the group

$$\widetilde{CH}^{n_1+n_2+1}_{p_1^{-1}(|D_1|\cap C_1)\cap p_2^{-1}(C_2)}(X_1 \times X_2, p_1^*\mathscr{L}(-D_1) + (v_1 \times v_2)),$$

it suffices to check their components at any generic point u in $\overline{t_1} \times \overline{t_2}$ where $t_1 \in (|D_1| \cap C_1)^{(0)}$, $t_2 \in C_2^{(0)}$. Suppose that $D_1 = \{(U_i, f_i)\}$ and that $t_1 \in U_i$. At u, we then have

$$\begin{aligned}
&(D_1 \cdot \alpha_1) \times \alpha_2 \\
=&\partial_{t_1}(< -1 >^{n_1} [\overline{f_i}] \otimes f_i \otimes \alpha_1) \times \alpha_2 \\
=&\partial(< -1 >^{n_1} [\overline{f_i}] \otimes f_i \otimes \alpha_1) \times \alpha_2 \\
=&\partial(< -1 >^{n_1} ([\overline{f_i}] \otimes f_i \otimes \alpha_1) \times \alpha_2) \\
&\text{by Proposition 8.3.} \\
=&\partial_u(< -1 >^{n_1} ([\overline{f_i}] \otimes f_i \otimes \alpha_1) \times \alpha_2) \\
=&\partial_u(< -1 >^{n_1} ([\overline{p_1^*(f_i)}] \otimes p_1^*(f_i) \otimes (\alpha_1 \times \alpha_2)) \\
=&p_1^*(D_1) \cdot (\alpha_1 \times \alpha_2).
\end{aligned}$$

For the second assertion, we exchange the role of X_1 and X_2 as before:

$$\begin{aligned}
&c(p_1^*v_1, p_2^*\mathscr{L}(-D_2))(\alpha_1 \times (D_2 \cdot \alpha_2))\\
=&<-1>^{(n_1+rk_{X_1}(v_1))(n_2+rk_{X_2}(v_2))}\ c(p_2^*v_2, p_1^*v_1)((D_2 \cdot \alpha_2) \times \alpha_1)\\
&\text{by Proposition 8.4}\\
=&<-1>^{(n_1+rk_{X_1}(v_1))(n_2+rk_{X_2}(v_2))}\ c(p_2^*v_2, p_1^*v_1)(p_2^*(D_2) \cdot (\alpha_2 \times \alpha_1))\\
&\text{by the first equation}\\
=&<-1>^{(n_1+rk_{X_1}(v_1))(n_2+rk_{X_2}(v_2))}\ p_2^*(D_2) \cdot c(p_2^*v_2, p_1^*v_1)(\alpha_2 \times \alpha_1)\\
&\text{by the functoriality of intersections with respect to twists}\\
=&p_2^*(D_2) \cdot (\alpha_1 \times \alpha_2)\\
&\text{by Proposition 8.4.}
\end{aligned}$$

□

Proposition 8.13. *1. (Axiom 17) Let $f : X \longrightarrow Y$ be a smooth morphism in Sm/k, $C \in Z^{i+d_f}(X)$ be smooth and closed in Y, D be a Cartier divisor over Y with $dim(|D| \cap f(C)) < dim(f(C))$ and $\alpha \in \widetilde{CH}_C^{i+d_f}(X, f^*v - T_{X/Y})$. Then*

$$D \cdot f_*(\alpha) = f_*(f^*(D) \cdot \alpha).$$

2. (Axiom 18) Let $f : X \longrightarrow Y$ be a closed immersion in Sm/k, $C \in Z^{i+d_f}(X)$ be smooth, D be a Cartier divisor over Y with $dim(|D| \cap f(C)) < dim(f(C))$ and let $\alpha \in \widetilde{CH}_C^{i+d_f}(X, N_{X/Y} + f^*v)$. Then

$$D \cdot f_*(\alpha) = f_*(c(\mathscr{L}(-f^*D), N_{X/Y})(f^*(D) \cdot \alpha)).$$

Proof. 1. Both sides live in the same Chow-Witt group, so we check their components at any generic point y of $f(C) \cap |D|$. Suppose that $D = \{(U_i, f_i)\}$, $y \in U_i$. We have a commutative diagram

$$\begin{tikzcd}
(\mathscr{L}(-D)|_C, N_{C/X} + f^*v|_C - T_{X/Y}|_C) \arrow[r, "{(id,f_*)}"] \arrow[d] & (\mathscr{L}(-D)|_C, N_{C/Y} + f^*v|_C) \arrow[d] \\
\mathscr{L}(-D)|_C + N_{C/X} + f^*v|_C - T_{X/Y}|_C \arrow[r, "id+f_*"] \arrow[d] & \mathscr{L}(-D)|_C + N_{C/Y} + f^*v|_C \arrow[d] \\
N_{C/X} + \mathscr{L}(-D)|_C + f^*v|_C - T_{X/Y}|_C \arrow[r, "f_*"] & N_{C/Y} + \mathscr{L}(-D)|_C + f^*v|_C.
\end{tikzcd}$$

At y, we then have

$$\begin{aligned}
&D \cdot f_*(\alpha)\\
=&\partial_y(<-1>^i\ [\overline{f_i}] \otimes f_i \otimes f_*(\alpha))\\
=&\partial_y(<-1>^{i+d_f}\ f_*([\overline{f^*(f_i)}] \otimes f^*(f_i) \otimes \alpha))\\
&\text{by the diagram above}\\
=&f_*\partial_y(<-1>^{i+d_f}\ [\overline{f^*(f_i)}] \otimes f^*(f_i) \otimes \alpha)\\
&\text{by Proposition 8.6}\\
=&f_*(f^*(D) \cdot \alpha).
\end{aligned}$$

2. Both sides live in the same Chow-Witt group, so we check their components at any generic point y of $f(C) \cap |D|$. Suppose that $D = \{(U_i, f_i)\}$, $y \in U_i$. We then have a commutative diagram

$$\begin{array}{ccc}
(\mathscr{L}(-D)|_C, N_{C/X} + N_{X/Y}|_C + f^*v|_C) & \xrightarrow{(id,f_*)} & (\mathscr{L}(-D)|_C, N_{C/Y} + f^*v|_C) \\
\downarrow & & \downarrow \\
\mathscr{L}(-D)|_C + N_{C/X} + N_{X/Y}|_C + f^*v|_C & \xrightarrow{id+f_*} & \mathscr{L}(-D)|_C + N_{C/Y} + f^*v|_C \\
\downarrow & & \downarrow \\
N_{C/X} + N_{X/Y}|_C + \mathscr{L}(-D)|_C + f^*v|_C & \xrightarrow{f_*} & N_{C/Y} + \mathscr{L}(-D)|_C + f^*v|_C.
\end{array}$$

At y, we then have

$$\begin{aligned}
&D \cdot f_*(\alpha) \\
=&\partial_y(< -1 >^i [\overline{f_i}] \otimes f_i \otimes f_*(\alpha)) \\
=&\partial_y(< -1 >^{i+d_f} f_*([\overline{f^*(f_i)}] \otimes f^*(f_i) \otimes \alpha)) \\
&\text{by the diagram above} \\
=&f_*\partial_y(< -1 >^{i+d_f} [\overline{f^*(f_i)}] \otimes f^*(f_i) \otimes \alpha) \\
&\text{by Proposition 8.7} \\
=&f_*(c(\mathscr{L}(-f^*D), N_{X/Y})(f^*(D) \cdot \alpha)).
\end{aligned}$$

□

Now we are ready for basic formulas concerning pull-backs along divisors. We will use the notation of Definition 8.10.

Proposition 8.14. *(Axiom 10) For $a = 1, 2$, let $X_a \in Sm/k$, D_a be effective smooth divisors over X_a, $v_a \in \mathscr{P}_{X_a}$, $C_a \in Z^{n_a}(X_a)$ be smooth with $dim(C_a \cap |D_a|) < dim(C_a)$, $\alpha_a \in \widetilde{CH}^{n_a}_{C_a}(X_a, v_a)$ and $i_a : |D_a| \longrightarrow X_a$ be inclusions. Then we have*

$$i_1^*(\alpha_1) \times \alpha_2 = (i_1 \times id)^*(\alpha_1 \times \alpha_2)$$

$$\alpha_1 \times i_2^*(\alpha_2) = (id \times i_2)^*(\alpha_1 \times \alpha_2).$$

Proof. We denote the projection $X_1 \times X_2 \longrightarrow X_a$ by p_a. For the first assertion, it suffices to check the equation after application of the isomorphism $(i_1 \times id)_* \circ s(\mathscr{L}(p_1^* D_1))$ on both sides. We have

$$\begin{aligned}
&(i_1 \times id)_*(s(\mathscr{L}(p_1^*D_1))(i_1^*(\alpha_1) \times \alpha_2)) \\
=&(i_1 \times id)_*((s(\mathscr{L}(D_1))i_1^*(\alpha_1)) \times \alpha_2) \\
&\text{by bifunctoriality of exterior products with respect to twists} \\
=&i_{1*}(s(\mathscr{L}(D_1))i_1^*(\alpha_1)) \times \alpha_2 \\
&\text{by Proposition 8.11} \\
=&(D_1 \cdot \alpha_1) \times \alpha_2 \\
=&p_1^*(D_1) \cdot (\alpha_1 \times \alpha_2) \\
&\text{by Proposition 8.12} \\
=&(i_1 \times id)_*(s(\mathscr{L}(p_1^*D_1))((i_1 \times id)^*(\alpha_1 \times \alpha_2))).
\end{aligned}$$

The second equation follows by exchanging the roles of X_1 and X_2:

$$\begin{aligned}
&\alpha_1 \times i_2^*(\alpha_2)\\
&= <-1>^{(n_1+rk_{X_1}(v_1))(n_2+rk_{X_2}(v_2))}\ c(q_2^* i_2^* v_2, q_1^* v_1)(i_2^*(\alpha_2) \times \alpha_1)\\
&= <-1>^{(n_1+rk_{X_1}(v_1))(n_2+rk_{X_2}(v_2))}\ c(q_2^* i_2^* v_2, q_1^* v_1)((i_2 \times id)^*(\alpha_2 \times \alpha_1))\\
&= <-1>^{(n_1+rk_{X_1}(v_1))(n_2+rk_{X_2}(v_2))}\ (i_2 \times id)^*(c(p_2^* v_2, p_2^* v_1)(\alpha_2 \times \alpha_1))\\
&\quad \text{by functoriality of pull-backs with respect to twists}\\
&=(id \times i_2)^*(\alpha_1 \times \alpha_2).
\end{aligned}$$

□

Proposition 8.15. *Suppose that we have a Cartesian square of smooth schemes*

$$\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"] & Y,
\end{tikzcd}$$

where u is a closed immersion, $dim(X') = dim(X) - 1$ and $dim(Y') = dim(Y) - 1$.

1. *(Axiom 16) If f is a closed immersion , $s \in \mathscr{P}_Y$, $C \in Z^{n+d_f}(X)$ is smooth and $dim(u^{-1}(f(C))) < dim(f(C))$, the following diagram commutes*

$$\begin{tikzcd}
\widetilde{CH}_C^{n+d_f}(X, N_{X/Y} + f^*s) \arrow[r, "f_*"] \arrow[d, "v^*"] & \widetilde{CH}^n_{f(C)}(Y, s) \arrow[d, "u^*"] \\
\widetilde{CH}^{n+d_f}_{v^{-1}(C)}(X', v^*N_{X/Y} + v^*f^*s) \arrow[r, "g_*"] & \widetilde{CH}^n_{g(v^{-1}(C))}(Y', u^*s).
\end{tikzcd}$$

2. *(Axiom 15) If f is smooth, $s \in \mathscr{P}_Y$ and $C \in Z^{n+d_f}(X)$ is smooth and closed in Y, the following diagram commutes*

$$\begin{tikzcd}
\widetilde{CH}_C^{n+d_f}(X, f^*s - T_{X/Y}) \arrow[r, "f_*"] \arrow[d, "v^*"] & \widetilde{CH}^n_{f(C)}(Y, s) \arrow[d, "u^*"] \\
\widetilde{CH}^{n+d_f}_{v^{-1}(C)}(X', v^*f^*s - v^*T_{X/Y}) \arrow[r, "g_*"] & \widetilde{CH}^n_{g(v^{-1}(C))}(Y', u^*s).
\end{tikzcd}$$

Proof. The conditions give us a unique effective smooth divisor D (resp. D') over Y (resp. X) such that $|D| = Y'$ (resp. $|D'| = X'$). Moreover, we have $D' = f^*(D)$. It suffices to check the equation after application of $u_* \circ s(\mathscr{L}(D))$ on both sides.

1. Suppose that $\alpha \in \widetilde{CH}_C^{n+d_f}(X, N_{X/Y} + f^*s)$. We then have

$$\begin{aligned}
&u_*(s(\mathscr{L}(D))(u^* f_*(\alpha)))\\
&=D \cdot f_*(\alpha)\\
&=f_*(c(\mathscr{L}(-D'), N_{X/Y})(D' \cdot \alpha))\\
&\quad \text{by Proposition 8.13, (2)}\\
&=f_*(c(\mathscr{L}(-D'), N_{X/Y})(v_*(s(\mathscr{L}(D'))(v^*(\alpha)))))\\
&=f_*v_*((c(v^*\mathscr{L}(-D'), v^*N_{X/Y}) \circ s(\mathscr{L}(D')))(v^*(\alpha)))\\
&=u_*g_*((c(v^*\mathscr{L}(D') + v^*\mathscr{L}(-D'), v^*N_{X/Y}) \circ s(\mathscr{L}(D')))(v^*(\alpha)))\\
&\quad \text{by Proposition 8.8}\\
&=u_*(s(\mathscr{L}(D))(g_*(v^*(\alpha))))\\
&\quad \text{by functoriality of push-forwards with respect to twists.}
\end{aligned}$$

2. Suppose that $\alpha \in \widetilde{CH}_C^{n+d_f}(X, f^*s - T_{X/Y})$. We then have

$$
\begin{aligned}
& u_*(s(\mathscr{L}(D))(u^* f_*(\alpha))) \\
=& D \cdot f_*(\alpha) \\
=& f_*(D' \cdot \alpha) \\
& \text{by Proposition 8.13, (1)} \\
=& f_*(v_*((s(\mathscr{L}(D'))(v^*(\alpha)), v^*\mathscr{L}(D') - v^*\mathscr{L}(D') + v^*f^*s - v^*T_{X/Y}))) \\
=& u_*(g_*((g^*s(\mathscr{L}(D))(v^*(\alpha)), g^*u^*\mathscr{L}(D) - g^*u^*\mathscr{L}(D) + v^*f^*s - T_{X'/Y'}))) \\
& \text{by Proposition 8.9} \\
=& u_*(s(\mathscr{L}(D))(g_*v^*(\alpha))) \\
& \text{by functoriality of push-forwards with respect to twists.}
\end{aligned}
$$

□

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