Mathematical analysis and numerical approximation of flow models in porous media

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Mathematical analysis and numerical approximation of flow models in porous media

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À l'âme de ma mère.

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Introduction

Dans cette thèse, nous nous intéressons à l'étude théorique ainsi que numérique du comportement des fluides dans des milieux poreux.

La description précise de l'écoulement des fluides est donnée par les équations de Darcy Brinkmann Forchheimer, est importante à la réussite des modèles dans plusieurs domaines. En particulier, dans les applications d'ingénierie telle que l'industrie pétrolière et l'hydrologie souterraine. On peut citer :
le pot catalytique qui sert à réduire la nocivité des gaz d'échappement, les condenseurs et les turbines à gaz combustion, c'est une apareil tournante thermodynamique qui transforme l'oxygene de l'air ambiante comme comburant et lui fait subir des transformations pour produire de l'énergie mécanique ou cinétique.

L'écoulement des fluides dans des milieux poreux sont généralement décrits par la loi de Darcy, où le gradient de pression est linéairement proportionnel à la vitesse du fluide. Dans certaines situations physiques cette loi n'est plus suffisante tel que les écoulements des fluides possédants une vitesse élevée, des écoulements turbulents, des écoulements inertiels ou lorsque les médias contiennent des fractures.

Dans ce cas, des comportements non linéaires apparaissent et qui peuvent étre modéliser par le terme de Forchheimer qui est une correction du modèle de Darcy. En effet, en (1901) [18] Forchheimer avait ajouté le deuxsième ordre du terme vitesse pour représenter l'effet d'inertie microscopique.

Considérant l'effet macroscopique entre le fluide et les parois des pores, Brinkman (1947) [5] avait ajouté les dérivées secondes d'une vitesse à l'équation de Darcy ce qui donne l'équation de Brinkman et qui peut être considérer comme un modèle pour Mélanger deux matériaux.

L'objectif de ce travail est d'analyser des équations de Darcy Brinkman Forchheimer :

$$\begin{aligned} -\gamma\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^{\alpha}\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad & \text{sur } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{sur } \Omega. \end{aligned}$$

où $\boldsymbol{u}, \pi$ représentent respectivement le champ de vitesse, la pression et $\boldsymbol{f}$ représente le terme de force.
Les paramètres $\gamma > 0,\ a > 0,\ b > 0$ définissent respectivement le coefficient de Brinkmann, Darcy et Forchheimer.
Sauf mention précise, Ω est un domaine simplement connexe.
les objectifs principaux de cette thèse peuvent étre scindés en quatres grands axes :

- Le premier est consacré à l'étude de l'équation de Darcy Brinkman Forchheimer stationnaire avec des conditions de type Navier sur le bord.

$$\boldsymbol{u} \cdot \boldsymbol{n} = g, \quad \textbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}. \tag{1}$$

On suppose dans ce premier chapitre que $\alpha = 1$.
Plusieurs travaux ont déja été faites et qui traitent les équations de Navier Stokes ou celles de Darcy Brinkman Forchheimer que se soit sur le plan analyse ou approximation avec des conditions de type Dirichelet sur les parois, ce qui confirme l'adhération du fluide aux parois du domaine. (voir [21] V. Kalantarov et S. Zelik, [11] A.Celbi, K. Kalantarov et [34] P.Skrzypacz et D. Wei).

Cependant, ce type de conditions sur les parois n'est pas toujours rélaliste, notamment au niveau d'applications ou l'on se trouve face à des situations physiques où il est primordial de faire intervenir d'autres types de condition sur le bord pour bien décrire le comportement du fluide proche de la paroi. comme l'a signalé Serrin dans son travail [30].

En 1827, Navier [25] a proposé une condition dite de glissement avec friction à la paroi qui permet de prendre en compte le glissement du fluide près du bord et de mesurer l'effet de friction en considérant la composante tangentielle du tenseur des contraintes proportionnelle à la composante tangentielle, désigné par l'indice τ, du champ de vitesses :

$$\boldsymbol{u} \cdot \boldsymbol{n} = g, \quad ((\mathbb{D}\boldsymbol{u})_n + \beta \boldsymbol{u})_\tau = \boldsymbol{h}.$$

où β est un coefficient de friction, $\boldsymbol{n}$ est le vecteur normal exterieur sur la frontière et $\mathbb{D}\boldsymbol{u}$ représente le tenseur de déformation défini par :

$$\mathbb{D}(\boldsymbol{u})_{ij} = \frac{1}{2}(\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j} + \frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{x}_i}), \quad 1 \leqslant i, j \leqslant 3.$$

Dans une situation idéale, nous considérons β nulle,

$$2((\mathbb{D}\boldsymbol{u})\boldsymbol{n})_\tau = \nabla_\tau(\boldsymbol{u}.\boldsymbol{n}) + (\frac{\partial \boldsymbol{u}}{\partial n})_\tau - \sum_{j=1}^{2}(\frac{\partial \boldsymbol{n}}{\partial s_j}.\boldsymbol{u}_\tau)_{\tau_j}. \tag{2}$$

$\boldsymbol{u}_\tau$ représente la projection de $\boldsymbol{u}$ sur l'hyperlplane tangent à Γ et $\boldsymbol{u}_n$ exprime la composante de $\boldsymbol{u}$ dans la direction normale comme.

$$\mathbf{curl}\boldsymbol{u} \times \boldsymbol{n} = \nabla_\tau(\boldsymbol{u}.\boldsymbol{n}) - (\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}})_\tau - \sum_{j=1}^{2}(\frac{\partial \boldsymbol{n}}{\partial s_j}.\boldsymbol{u}_\tau)_{\tau_j}. \tag{3}$$

On obtient l'équivalence des conditions aux limites (2) avec $\mathbf{curl}\boldsymbol{u} \times \boldsymbol{n} + 2(\frac{\partial \boldsymbol{u}}{\partial n})_\tau$.
En considérant les conditions (3), en supposant avoir le bord plan et négligant la friction ($\beta = 0$), nous obtenons les conditions limites suivantes :

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \;\; \mathbf{curl}\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}.$$

Ce type des conditions aux limites est généralement utilisé pour simuler l'écoulement des fluides à proximité d'un bord perforé ou de parois visqueuses.
Nous donnons comme exemples W.Jagar et A.Mikelic [24], Y.Amirate, D.Bresh, J.Lemone et J.Simon [1], M.Bulicek, Malek, K.R.Rajagopal [10]. Nous mentionnons également que de telles conditions de glissement sont utilisées dans les simulations d'écoulements turbulents appliquées à l'aérodynamique, à la météorologie et à l'hémodynamique.

Récemment, une analyse des équations de Brinkman Forchheimer avec des conditions de glissement-friction est proposée par J.K Djoko et P.A.Razafimandimby [17], où l'existence et l'unicité de la solution, par la régularisation combiné avec la méthode de Faedo-Galerkin, ont été obtenus.

Supposons β tends vers l'infini, g et $\boldsymbol{h}$ sont nulles, alors on retrouve la condition classique de Dirichlet, autrement on dit que la paroi rigide freine le fluide.
Dans tout notre travail, nous considérons le cas d'un bord plat avec

$$\beta = 0.$$

La deuxiéme condition ci-dessus peut étre remplacée par une condition portant sur la composante tangentielle du tourbillon :

$$\boldsymbol{u} \cdot \boldsymbol{n} = g, \quad \mathbf{curl}\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}.$$

- le deuxième axe est dédié au cas ou la géométrie du domaine est totalement différente de celle supposée dans le reste de la thèse. Nous considérons Ω comme un domaine non simplement connexe. Ce qui entraine l'apparition des composantes connexes définies par la contrainte

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0.$$

Dans ce chapitre, nous nous sommes toujours aux équations stationnaires incompressibles

$$-\gamma\Delta \boldsymbol{u} + b|\boldsymbol{u}|^{\alpha}\boldsymbol{u} + \boldsymbol{u}.\nabla \boldsymbol{u} + \nabla p = f, \quad \text{et} \quad \operatorname{div} \boldsymbol{u} = 0, \quad \text{sur } \Omega.$$

Dans de nombreuses applications d'autres conditions aux limites sont introduites. On peut citer comme exemple les écoulements des fluides dans un tuyau, dans le cas d'une bifurcation de ce tuyau en deux tuyaux secondaires. Les conditions sur la pression ou sur les flux sont indispensables pour contrôler la quantité de fluides dans chaque tube. Pour cette raison, nous traitons ici les conditions aux limites :

$$\boldsymbol{u} \times \boldsymbol{n} = g \times \boldsymbol{n}, \quad \pi = \pi_0 \quad \text{sur } \Gamma. \tag{4}$$

Cette condition est connu par la pression statique dans le cas des équations de Stokes, elle sera remplacée par la pression dynamique

$$\boldsymbol{u} \times \boldsymbol{n} = g \times \boldsymbol{n}, \quad \pi = \pi_0 + \frac{1}{2}|\boldsymbol{u}|^2. \tag{5}$$

dans le cas des équations non linéaires.

Certains applications ont soulevée des problèmes où les conditions aux limites précédantes apparaissent naturellement. nous mentionons le travail de C. Conca, voir [16], où l'existence et l'unicité de la solution variationelle des équations de Stokes avec la pression et le flux de la vitesse imposés, sont prouvées dans un cadre hilbertien.
Des résultats analogues ont été établis concernant les équations de Navier Stokes stationnaires.

L'étude autour les problèmes stationnaires de Stokes ainsi que Navier Stokes avec des conditions limites non standard, plus particulièrement la pression est donnée sur une partie de la frontière, est déja fait par Bégue et al [5].

Ensuite cette étude fus complétée en 2002 par Bernard [6] , qui a prouvé des résultats de régularités dans H^2 puis $W^{m,r}(\Omega)$ avec $m \geqslant 2$ et $r \geqslant 2$. L'approche dans l'analyse des équations de Navier Stokes repose sur le Théorème du point fixe et le problème de Stokes non-standard. Bernard a prouvé que l'unique solution du problème linéaire non standard posséde un **curl** dans H^1.

J. Bernard [7] a abordé certains problèmes d'évolutions avec ce même type de conditions aux limites. L'auteur a adapté la méthode de Faedo-Galerkin, combiné avec des résultats de régularité déja obtenus dans [6].

D'un autre coté, en 2009 Kozono et Yanagisawa ont pris la trace $\boldsymbol{u} \times \boldsymbol{n}$ en considération dans leur papier [22], ou les résultats peuvent étre considérer comme une extension de la décomposition Rham-Hodge-Kodaira des formes C^∞ (des formes infiniment continues et differentiables) sur des variétés de Rimann compact à des champs de vecteurs $\boldsymbol{L}^r$ sur Ω. En utilisant **curl** et div , des differents estimations de la fonction $\boldsymbol{u}$ et ses dérivées sont établit.

Enfin cet axe peut étre vu comme une amélioration des résultats de C. Amrouche et N. Seloula [2] reposant sur des propriétés relevant de potentiels vecteurs. Certaines inégalités de Sobolev étaient prouvé en addition aux résultats d'existence de solutions faibles, très faibles ainsi que la régularité des solutions dans la théorie L^p.

- La troisème direction traite deux types d'équations non stationnaires. la première est muni du terme de convection :

$$\boldsymbol{u}_t - \gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^\alpha\boldsymbol{u} + \boldsymbol{u}.\nabla\boldsymbol{u} + \nabla p = f, \quad \text{et} \quad \text{div}\boldsymbol{u} = 0, \quad \text{sur } \Omega \quad (6)$$

Nous analysons l'erreur du schéma numérique obtenu à partir d'un modèle perturbé du system DBF avec des conditions de type Dirichlet sur le bord introduites par G.G.Stokes en 1845 [35].

$$\boldsymbol{u}|_{\partial\Omega} = 0, \ \forall t \in [0, T].$$

La seconde équation considère un problème légerement different de celui qui est mentionné en (6).

$$\boldsymbol{u}_t - \gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^\alpha\boldsymbol{u} + \nabla p = f, \quad \text{et} \quad \text{div}\boldsymbol{u} = 0, \quad \text{sur } \Omega \quad (7)$$

mais avec d'autres conditions aux limites. Particulièrement nous considérons des conditions de type Navier homogènes

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = 0. \tag{8}$$

L'un des majeurs difficultés pour l'approximation numérique du solution de l'équation Darcy Brinkmann Forchheimer est liée à la contrainte d'incompressibilité " $\operatorname{div} \boldsymbol{u} = 0$ " puisque le champs de vitesse $\boldsymbol{u}$ et la pression sont dissociés.
Parmi les processus employés pour la relaxation de cette contrainte, on peut citer les méthodes suivantes :

- La méthode de pénalisation, on peut mentionner comme références T. Hughes, W. Liu, A. Brooks [20], J. Shen [33], R. Temam [36] :

$$\operatorname{div} \boldsymbol{u}^\varepsilon - \varepsilon p^\varepsilon = 0.$$

- La compressibilité artificielle, nous rappelons les travaux de F.Brezzi et J.Pitkaranta ainsi que celle de R. Temam [12, 37] :

$$\operatorname{div} \boldsymbol{u}^\varepsilon - \varepsilon p_t^\varepsilon = 0.$$

- La pression stabilisée, dans le cas stationnaire, nous donnons comme référence M. Louaked, N. Seloula, S.Trabelsi [23], F. Brezzi, J. Douglas [8], F. Brezzi, J. Pitkaranta [9], J. Hughes, L.Frnca, M. Balestra [19], tandis que le cas d'évolution est réalisé par R.Rannacher et J.Shen [26, 32]

$$\operatorname{div} \boldsymbol{u}^\varepsilon - \Delta \varepsilon p^\varepsilon = 0.$$

- La méthode de pseudo-compressibilité est considéré comme l'un des mécanismes les plus efficaces utilisés pour analyser l'erreur de l'équation de Navier Stokes instable, nous citons le travail de AJ.Chorin [13], et celle de E. Weinan et J. Liu [38] et finalement le papier de J.Shen[31]

$$\operatorname{div} \boldsymbol{u}^\varepsilon - \Delta \varepsilon p_t^\varepsilon = 0.$$

Convaincu de l'efficacité du traitement par pseudo-compressibilité, nous adoptons cette approche pour analyser les équations de Darcy Brinkman Forchheimer. Notre résultat principal est l'établissement d'une estimation d'ordre 2.

- Le dérnier axe est l'approximation par la méthode de Galerkin Discontinue, les équations de Brinkman ainsi que les équations de Forchheimer avec les conditions aux limites données précédement en (1).

  Intialement, la méthode Galerkin discontinue est introduite par Hill et Reed [27] en 1973 pour l'approximation des problèmes hyperbolique.

  Dans la littérature, des nombreux résultats relatifs à l'estimation d'erreur optimal sont obtenus en appliquant la méthode de Galerkin Discontinue aux problèmes linéaires elliptiques. Richter [28] a proposé une extension des techniques hyperboliques aux problèmes elliptiques.
  Il avait montré que le taux de convergence atteint l'ordre $k + \frac{1}{2}$ lorsque on emploie des polynômes d'ordre $k \geqslant 1$.
  En 2002 dans [15] B. Cockburn, G. Kanschat, D.Schotzau ont donné l'estimation d'erreur pour les équations de Stokes stationnaires a été établie en utlisant la méthode de Galerkin discontinue locale.

  L'étude d'approximation en s'appuyant sur la méthode de Galerkin discontinue a été étendue aux problèmes elliptiques non linéaires qui ont suscité l'attention de nombreux chercheurs.
  En 1996, Bassi et Rebay [4] avaient élargit l'utilisation de la méthode des éléments finis de type Galerkin Discontinue aux équations elliptiques non linèaires plus telles les équations de Navier Stokes par la gestion du terme visqueux avec une fomulation faible mixte, où la méthode Galerkin Discontinue locale a été employée en introduisant le vecteur auxiliaire $\theta = \nabla \boldsymbol{u}$. En fait, la méthode de Gelerkin Discontinue couple la méthode des volumes finis et celle des éléments finis.

Dans ce qui suit, nous détaillons le lien entre les axes mentionnés et les chapitres de cette thèse.

Dans le premier chapitre, nous considérons les équations du Darcy Brinkman et les équations du Darcy Brinkman Forchheimer avec les mêmes conditions sur le bord.
Nous traitons ces équations dans deux sections indépendantes.
La première section est consacrée à l'étude de l'équation de Darcy Brinkman stationnaire dans Ω un domaine ouvert borné de $\mathbb{R}^3$,

$$\begin{aligned} &-\gamma \Delta \boldsymbol{u} + a\boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{dans } \Omega, \\ &\operatorname{div} \boldsymbol{u} = 0 \quad \text{dans } \Omega \end{aligned} \tag{9}$$

avec des conditions aux limites de type Navier

$$\boldsymbol{u} \cdot \boldsymbol{n} = g, \quad \mathbf{curl}\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}, \text{ sur } \Gamma. \tag{10}$$

Ici nous nous intéressons à prouver l'existence et l'unicité ainsi que la régularité de la solution faible et forte.

Dans la première sous section, nous présentons quelques notions et espaces fonctionnelles ainsi que certains résultats qui nous serons utiles dans l'établissement de nos résultats.

La deuxième sous section traite le problème linéaire dans differents situations. Nous étudions le cas homogène, le cas non homogène et enfin le cas la divergence non-nulle.
En effet, cette sous section se focalise sur l'existence et l'unicité de la solution de l'équation de Brinkman (équation linéaire) dans le cadre Hilbertien avec des conditions aux limites (10).

La troisième sous section est dédiée à la régularité de solution de l'equation linéaire dans les espace L^P.
Dans la quatrième sous section, nous prouvons l'existence et l'unicité du solution faible $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ avec $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega)]'$ dans les cas (homogéne, non homogéne et la divergence non nulle), où l'espace $[\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega)]'$ est définit comme suit :

$$[\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega)]' = \{\boldsymbol{\psi} + \nabla F,\ \boldsymbol{\psi} \in L^r(\Omega),\ F \in L^p(\Omega)\}. \tag{11}$$

Tout au long la deuxième section, nous abordons les équations du Darcy Brinkman Fochheimer

$$\begin{aligned} -\gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|\,\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \ \text{ dans } \Omega \\ \mathrm{div}\boldsymbol{u} = 0 \text{ dans } \Omega \end{aligned}$$

avec les conditions aux limites données ci dessus (10).
La non linéarité se justifie par certains situations physiques telle que les flux avec des vitesses élevées, les fluides turbulents, l'effet d'intertie ou lorsque un média contient des fractures.

Cett section est divisée en deux sous sections secondaires. L'objectif de la première sous section est de montrer l'existence et l'unicité de la solution

faible du système linéarisé sur les deux échelles à la fois pour le cas hilbertien ainsi que dans la thèorie L^p.
Nous montrons en plus, la régularité de la solution de ce problème linéarisé.

La deuxième section concerne les équations du Darcy Brinkman Forchheimer. En utilisant la méthode de Faedo-Galerkin, nous prouvons l'existence de la solution faible sur le plan hilbertien. Ensuite nous montrons que nous aboutissons aux résultats analogues en considérant que les données $\boldsymbol{f}$ appartenants à $(\boldsymbol{H}_0^{6,2}(\mathrm{div}\,,\Omega))'$. Pour $p > 2$ des résultats de régularités sont prouvés avec $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div}\,,\Omega))'$ or pour $\frac{3}{2} < p < 2$, nous établissons le point fixe pour montrer l'existence de la solution faible du problème Darcy Brinkman Forchheimer.

Dans le deuxième chapitre, nous nous intéressons à l'étude du problème Brinkman Forchheimer stationnaire avec des conditions au limites portant à la fois sur la pression et sur la composante tangentielle du champ de vitesse dans un domain toujours borné mais non simplement connexe.
Dans la première section, nous introduisons certains notations et nous offrons le cadre fonctionnel que nous avons besoin d'utiliser dans l'ensemble de ce chapitre.

Ensuite, la section suivante est consacrée au traitement du problème Stokes stationnaire avec toujours les mêmes conditions limites où nous alons améliorons les résultats déjà obtenus en (C. Amrouche [2])

$$(\mathcal{S})\begin{cases} -\gamma\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \;\text{ et }\; \mathrm{div}\,\boldsymbol{u} = 0 & \mathrm{dans}\,\Omega,\\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \quad \text{et} \quad \pi = \pi_0 & \mathrm{sur}\,\Gamma,\\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \le i \le I, \end{cases}$$

Particulièrement dans notre travail nous n'exigeons pas que la pression initial ait toujours une certaine régularitée bien précise. Cependant, nous cherchons à rendre les hypothèses de la régularité sur la fonction π_0 de moins en moins faible à fin d'avoir finalement la pression π dans l'espace L^p.

De manière analogue que le chapitre précédant, nous utilisons le Théorème de Lax Milgram ensuite le Théorème de Rham pour prouver que ce problème est bien posé dans le cadre hilbertien mais une fois nous passons à traiter ces équations dans la thèorie L^p, nous avons besoin de faire apparaitre la condition Inf-Sup qui joue un role primordial dans la solvabilité de la fonction $\boldsymbol{u}$.

La dernière section concerne la régularité de la solution du problème homogéne Brinkman Forchheimer muni du terme de convection dans la thèorie $\boldsymbol{L}^p$.

$$\begin{cases} -\gamma\Delta\, \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\, \boldsymbol{u} + b|\boldsymbol{u}|^{\alpha}\boldsymbol{u} + \nabla\, q = \boldsymbol{f} \quad \text{et} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{dans}\, \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{et} \quad q + \frac{1}{2}|\boldsymbol{u}|^2 = q_0 \ \text{ sur } \Gamma_0 \quad \text{et} \quad q + \frac{1}{2}|\boldsymbol{u}|^2 = q_0 + c_i & \text{sur}\, \Gamma_i, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases}$$

telles que $\boldsymbol{c} = (c_1, \dots, c_I) \in \mathbb{R}^I$ et $\alpha \in [1, 2]$.

En premier temps, nous signalons la nécessité d'utiliser la formulation faible du problème de Stokes qui satisfait les conditions de compatibilitées suivantes :

$$\forall \boldsymbol{v} \in \boldsymbol{K}^2_N(\Omega), \qquad \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_{\Omega} + \int_{\Gamma} \pi_0\, \boldsymbol{v}\cdot\boldsymbol{n} = 0. \tag{12}$$

Où la solution $(\boldsymbol{u}, \pi)$ est déja donnée par C. Amrouche, C. Bernardi, M. Dauge et V. Girault [3]. Par contre dans le cas où les conditions de compatiblité ci-dessus ne sont pas vérifiées, cela implique d'introduire des variantes au problème linéaire comme suit :

$$\begin{cases} -\gamma\Delta\, \boldsymbol{u} + \nabla\, q = \boldsymbol{f} \quad \text{et} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{dans}\, \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{et} \quad q = q_0 \ \text{ sur } \Gamma_0 \quad \text{et} \quad q = q_0 + c_i & \text{sur } \Gamma_i,\ 1 \leqslant i \leqslant I \\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases} \tag{13}$$

En s'appuyant sur le travail de C. Amrouche et N. Seloula [3], nous établissons la régularité de la solution du problème (13) qui est une extension des résultats obtenus dans la section précédente.

Concernant le problème non linéaire, le fait de contrôler la partie tangentielle de la fonction $\boldsymbol{u}$ qui s'annulle sur le bord et vue que

$$\int_{\Omega} (\boldsymbol{u}\cdot\nabla\,)\boldsymbol{u} = \frac{1}{2}\int_{\Gamma} |\boldsymbol{u}\cdot\boldsymbol{n}|^2(\boldsymbol{u}\cdot\boldsymbol{n}).$$

et avec l'abssence d'information concernant la partie normale du champs de vitesse, nous pouvons pas contrôler ce terme. Pour surmonter cette difficulté, nous considérons

$$\boldsymbol{u}\cdot\nabla\, \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{u}\times\boldsymbol{u} + \frac{1}{2}\nabla|\boldsymbol{u}|^2.$$

En posant $\pi = q + \frac{1}{2}|\boldsymbol{u}|^2$, notre problème se résume au problème suivant

$$(\mathcal{BFP}) \begin{cases} -\gamma\Delta\,\boldsymbol{u} + \mathbf{curl}\boldsymbol{u} \times \boldsymbol{u} + b|\boldsymbol{u}|^\alpha \boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{et} \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{dans}\,\Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{et} \quad \pi = \pi_0 \ \text{sur}\ \Gamma_0 \quad \text{et} \quad \pi = \pi_0 + c_i \qquad \text{sur}\,\Gamma_i, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \le i \le I. \end{cases}$$

Notre objectif dans la première sous section est de prouver l'existence de la solution faible du problème non linéaire $(\mathcal{BFP})$. D'abord, nous commençons l'étude par le cas $p = 2$ où nous utilisons la méthode de Galerkin pour montrer que ce problème non linéaire est bien posé. Puisque l'application de cette méthode nécessite d'avoir une base de l'espace hilbertien, nous construisons une base d'un espace sous jacent hilbertien qui nous permet d'établir une solution approchée ensuite nous vérifions une estimation à prioi concernant cette solution $\boldsymbol{u}_m$ et finalement nous passons à la limite en s'appuyant sur l'estimation à priori déja obtenue dans l'étape précédante, donc nous déduisons que la suite approchée $\boldsymbol{u}_m$ est bornée dans l'espace $\boldsymbol{H}^1(\Omega)$, nous sommes capables d'extraire une sous suite $(\boldsymbol{u}_m)_m$ tel que $\boldsymbol{u}_m$ converge faiblement vers $\boldsymbol{u}$ dans l'espace $\boldsymbol{H}^1(\Omega)$, par ailleurs en utilisant l'injection continue de Sobolev de l'espace $\boldsymbol{H}^1(\Omega)$ dans $\boldsymbol{L}^4(\Omega)$, nous concluons que la solution approchée $(\boldsymbol{u}_m)$ converge fortement vers la solution exacte $\boldsymbol{u}$.
Il suffit de passer à la limite dans la formulation variationelle pour aboutir au résultat désiré.
D'un autre coté, nous pouvons appliquer ici le théorème de Rham à fin de déduire l'existence et la régularité de la pression $\pi \in L^2(\Omega)$.

Ensuite, dans la sous section suivante, nous prouvons l'existence de la solution faible sous la thèorie L^p avec les données $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathbf{curl}\,,\Omega))'$, nous traitons le problème selon les différentes valeurs de p (avec $p \geqslant 2$) en s'appuyant sur les injections Sobolev.

La dérnière section concerne l'étude de la solution forte $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ avec $p \geqslant \frac{6}{5}$ et des hypothèses sur les données telles que $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ et la pression initiale $\pi_0 \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$.

Dans le troisième chapitre, nous avons considéré le système Darcy Brinkman Frochheimer non stationnaire (le cas d'évolution) dans un domaine ouvert borné de $\mathbb{R}^3$:

$$\begin{aligned} &\partial_t \boldsymbol{u} - \gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^\alpha \boldsymbol{u} + \boldsymbol{u}\nabla\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{dans } \Omega_T, \\ &\operatorname{div}\boldsymbol{u} = 0 \quad \text{dans } \Omega_T \end{aligned} \tag{14}$$

avec des conditions aux limites de type Dirichlet

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{sur } \Gamma \tag{15}$$

et des conditions initiales,

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \tag{16}$$

Nous avons perturbé les équations Darcy Brinkman Forchheimer précédantes (14)-(15)-(16), en utilisant une approche de type pseudo-compressiblité (i.e l'ajout du terme $\epsilon\Delta p_t$ à l'équation de conservation de masse).

Prenons comme notation $\boldsymbol{u}^\epsilon, \pi^\epsilon$ respectivement le champ de la vitesse perturbée ainsi que la pression perturbée.
Le système obtenu, noté (17)-(18)-(19) est le suivant :

$$\begin{array}{l} \partial_t \boldsymbol{u}^\epsilon - \gamma\Delta \boldsymbol{u}^\epsilon + a\boldsymbol{u}^\epsilon + b|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon + B(u^\epsilon, u^\epsilon) + \nabla \pi^\epsilon = \boldsymbol{f} \quad \text{dans } \Omega_T \\ \mathrm{d}iv\boldsymbol{u}^\epsilon - \epsilon\Delta p_t^\epsilon = 0 \quad \text{dans} \quad \Omega_T \end{array} \tag{17}$$

Avec des conditions de type Dirichlet sur la vitesse et de type Neumann sur la pression

$$\boldsymbol{u}^\epsilon = \boldsymbol{0}, \quad \frac{\partial p^\epsilon}{\partial n} = 0 \quad \text{sur } \Gamma \tag{18}$$

Et données initiales :

$$\boldsymbol{u}^\epsilon(0) = \boldsymbol{u}_0, \quad p^\epsilon(0) = p_0 \quad \text{dans} \quad \Omega_T \tag{19}$$

L'objectif est de traiter l'estimation d'erreur entre le système initial DBF (14)-(15)-(16) et celui approché (17)-(18)-(19) sous les conditions

$$\boldsymbol{u}_0 \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega), \qquad \boldsymbol{f},\ \boldsymbol{f}_t \in \boldsymbol{L}^2(\Omega). \tag{20}$$

et

$$\boldsymbol{f}_t,\ \boldsymbol{f}_{tt} \in C([0,T], \boldsymbol{L}^2(\Omega)). \tag{21}$$

qui est illustré dans le théorème suivant

Theorem 0.1. *On suppose* (20) *et* (21) *alors, il existe* $C > 0$ *dépendant au donnée qui vérifie :*

$$\int_{t_0}^{t} \|\boldsymbol{u}(s) - \boldsymbol{u}^\epsilon(s)\|^2 \quad ds + \epsilon^{\frac{1}{2}} \|\boldsymbol{u}(t) - \boldsymbol{u}^\epsilon(t)\|^2 + \epsilon(\|\boldsymbol{u}(t) - \boldsymbol{u}^\epsilon(t)\|_1^2 + \|p(t) - p^\epsilon(t)\|) \leqslant C\epsilon^2. \tag{22}$$

La démonstration de ce dernier repose sur les deux lemmes

Lemma 0.2. *On suppose que* (20)-(21) *sont vérifiés. Alors, il existe un constant* $C > 0$ *dependant au données qui vérifie :*

$$\int_{t_0}^{t} \|\boldsymbol{\xi}(s)\|^2 + \epsilon^{\frac{1}{2}}\|\boldsymbol{\xi}(t)\|^2 + \epsilon(\|\boldsymbol{\xi}(t)\|_1^2 + \|\psi(t)\|^2) \leqslant C\epsilon^2, \quad \forall t \in [t_0, T_0]. \tag{23}$$

Lemma 0.3. *Si les hyothèses* (20) *et* (21) *sont valides. Alors il existe un constant* $C > 0$ *dependant des données qui satisfait :*

$$\|\boldsymbol{\eta}(t)\|^2 + \gamma \int_{t_0}^{t} \|\nabla \boldsymbol{\eta}(s)\|^2 ds + \epsilon(\|\nabla \boldsymbol{\eta}(s)\|^2 + \|\nabla \phi(t)\|^2) \leqslant C\epsilon^2, \quad \forall t \in [t_0, T_0] \tag{24}$$

Mais en premier temps, il était nécéssaire d'établir quelques résultats importants qui étaient présentés dans la première section de ce chapitre, sur les quelles nous nous appuyons dans la suite.

De manière similaire au cas continu, nous avons traité le schéma numérique discret par rapport au temps donnée par

$$\begin{aligned} \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{k} - \frac{\gamma}{2}\Delta(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n) + a\frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2} + F(\boldsymbol{u}^{n+\frac{1}{2}}) + \widetilde{B}(\boldsymbol{u}^{n+\frac{1}{2}}, \boldsymbol{u}^{n+\frac{1}{2}}) \\ +\nabla p^n = \boldsymbol{f}(t_{n+\frac{1}{2}}), \quad \text{dans} \quad \Omega. \\ \operatorname{div}\boldsymbol{u}^{n+1} - \beta k \Delta(p^{n+1} - p^n) = 0, \quad \text{dans} \quad \Omega. \\ \boldsymbol{u}^{n+1} = 0, \qquad \frac{\partial p^{n+1}}{\partial \boldsymbol{n}} = \frac{\partial p^n}{\partial \boldsymbol{n}}, \qquad \text{sur} \quad \Gamma \end{aligned} \tag{25}$$

Notons par $\boldsymbol{u}^n, \pi^n$ le champs de vitesse et la pression à l'etape t_n (avec k est le pas de discrétisation en temps et β est un constant).

Nous avons pu établir la convergence du schéma numérique discret au problème original . L'estimation d'erreur du problème discrétisé par rapport aux temps est réalisé dans le théoreme suivant

Theorem 0.4. *On suppose que les données initiales* $(\boldsymbol{u}^0, p^0)$ *vérifient les estimations :*

$$\|\boldsymbol{u}^0 - \boldsymbol{u}(t_0)\| \leqslant Ck^2, \qquad \|\nabla(\boldsymbol{u}^0 - \boldsymbol{u}(t_0))\| + \|\nabla(p^0 - p(t_0))\| \leqslant Ck. \tag{26}$$

De plus $\boldsymbol{u}^0 \in H^2(\Omega) \cap V$ *et* $\boldsymbol{f}, \boldsymbol{f}_t, \boldsymbol{f}_{tt} \in C([0,T], L^2(\Omega))$, *alors il existe* $C > 0$ *satisfait :*

$$\begin{cases} \text{Pour tout} \quad 1 \leqslant m \leqslant M = \frac{T-t_0}{k}, \quad \text{on a} \\ k\sum_{n=1}^{m} \|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|^2 + k^2\|\nabla(\boldsymbol{u}(t_m) - \boldsymbol{u}^m)\|^2 + k^2\|(p(t_m) - p^m)\|^2 \leqslant Ck^4. \end{cases} \tag{27}$$

Nous avons validé cette étude par l'implémentation sous FreeFem++ d'un test numérique classique (la cavité entrainée).

Dans la première section, nous présentons quelques concepts et résultats sur l'estimation des termes non linéaires que nous utilisons dans le reste du chapitre.

Dans la deuxième section, nous montrons l'estimation d'erreur entre la solution $(\boldsymbol{u}, \pi)$ de l'équation de Brinkman Forchheimer incompressible (14)-(15)-(16) et $(\boldsymbol{u}^\epsilon, \pi^\epsilon)$ solution du problème perturbé (17)-(18)-(19). La preuve sera divisée en deux parties, l'une correspondante au problème linéaire et l'autre pour un problème non linéaire.

La section 3 est consacrée à l'approximation en temps de l'équation de Brinkman Forchheimer, ensuite à l'anayse d'erreur que nous montrons qu'elle est d'ordre 2.

Dans la section 4, nous focalisons notre attention sur le cadre numérique oú nous présentons la cavité entraînée correspondante aux équations de Brinkman Forchheimer, de plus nous comparons nos graphes obtenus avec les résultats de Ghia.

Le chapitre 4 se rapporte à l'étude des équations incompressibles non stationnaires sans le terme de convection.

$$\boldsymbol{u}_t - \gamma \Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^\alpha \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{et} \quad \mathrm{div}\boldsymbol{u} = 0 \quad \text{dans } \Omega \times [0, T]. \tag{28}$$

avec des conditions aux limites non standards.

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \ \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{sur } \Gamma \times [0, T]. \tag{29}$$

D'abord, nous proposons une formulation variationnelle du problème (28)-(29) comme suit :

$$\begin{cases} \langle \boldsymbol{u}'_m(t), \boldsymbol{w}_i \rangle + \gamma \langle \mathbf{curl}\, \boldsymbol{u}_m(t), \mathbf{curl}\, \boldsymbol{w}_i \rangle + a\langle \boldsymbol{u}_m(t), \boldsymbol{w}_i \rangle + b\langle |\boldsymbol{u}_m(t)|^\alpha \boldsymbol{u}_m(t), \boldsymbol{w}_i \rangle \\ = \langle \boldsymbol{f}(t), \boldsymbol{w}_i \rangle_\Omega \\ \forall i = 1, \ldots, m\,,\ t \in [0, t_m], \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m} \end{cases} \tag{30}$$

L'objet de la première section est de prouver que ce problème variationelle possède au moins une solution, en utilisant la méthode de Faedo-Galerkin.

La question concernant l'unicité de la solution faible ainsi que l'existence de la solution globale dans un domaine de dimension 3 restent toujours des problèmes ouverts.

Dans la première sous section, nous introduisons une solution approchée

$$\boldsymbol{u}_m(t) = \sum_{i=1}^{m} g_{im}(t)\boldsymbol{w}_i. \tag{31}$$

et nous considérons également un problème approximatif de dimension fini

$$\begin{cases} \langle \boldsymbol{u}'_m(t), \boldsymbol{w}_i\rangle + \gamma\langle \mathbf{curl}\,\boldsymbol{u}_m(t), \mathbf{curl}\,\boldsymbol{w}_i\rangle + a\langle \boldsymbol{u}_m(t), \boldsymbol{w}_i\rangle + b\langle |\boldsymbol{u}_m(t)|^{\alpha}\boldsymbol{u}_m(t), \boldsymbol{w}_i\rangle \\ = \langle \boldsymbol{f}(t), \boldsymbol{w}_i\rangle_{\Omega} \\ \forall i = 1, \ldots, m\,,\; t \in [0, t_m], \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m} \end{cases} \tag{32}$$

Ensuite, nous montrons certains estimations à priori concernant la solution approchée $\boldsymbol{u}_m$. Ces estimations sont indépendantes de m ce qui nous conduit à conclure l'existence de la convergence faible des sous-suites et justifier par la suite le passage à la limite. Durant cette dérnière étape, nous avons besoin d'exploiter le Théorème de compacité ainsi que la transformation de Fourier pour aboutir au résultat requis.
Pour récupérer la pression, nous introduisons un opérateur fonctionel. Nous réalisons par la suite que ce dérnier est bien linéaire et continue.

Dans la sous section 2, nous montrons des estimations à priori indépendantes du paramétre ϵ.
Nous obtenons des résultats analogues même en présence du terme de convection. $\boldsymbol{u}.\nabla\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u} + \frac{1}{2}|\boldsymbol{u}|^2$ puisque il suffit de considérer $\tilde{p} = p + \frac{1}{2}|\boldsymbol{u}|^2$, en plus, le fait que $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u}\cdot\boldsymbol{u} = 0$ ce qui maintient les résultats déja obtenus durant cette section.

La section 2 de ce chapitre est dédié à l'étude du problème perturbé du DBF :

$$\begin{cases} \boldsymbol{u}^{\epsilon}_t - \gamma\Delta\boldsymbol{u}^{\epsilon} + a\boldsymbol{u}^{\epsilon} + b|\boldsymbol{u}^{\epsilon}|^{\alpha}\boldsymbol{u}^{\epsilon} + \nabla p = \boldsymbol{f}, \quad \text{dans } \Omega. \\ \mathrm{div}\boldsymbol{u}^{\epsilon} + \epsilon\Delta p^{\epsilon}_t = 0, \quad \text{dans } \Omega. \end{cases} \tag{33}$$

avec des conditions aux limites non standards données en (8).
cette section est divisée en deux sous parties. la première est consacrée à l'étude du système d'évolution du DBF perturbé en utilisant la méthode de

pseudo-compressibilité. Ensuite, on vérifie que le problème est bien posé en utilisant encore une fois la méthode de Faedo-Galerkin et le théorème de compacité.

À travers la deuxsième partie, notre objectif est de prouver la convergence de la solution des équations perturbées (33)-(8) vers celle du problème original (7). Pour cette raison, on montre d'abord certaines estimations à priori indépendantes de ϵ que nous avons besoin plus tard pour établir l'estimation d'erreur d'ordre 2 pour la solution $\boldsymbol{u}$.

Contrairement au chapitre précédant qui concerne les conditions Dirichlet ou l'inégalité de Poincaré est une clé essentielle utilisée souvent dans plusieurs passages. Nous avons besoin tout au long ce chapitre des inégalitées pour le champs de vecteur faisant intervenir les opérateurs **curl** et div .

D'abord, nous montrons en s'appuyant sur la méthode de Faedo-Galerkin l'existence de la solution faible du système perturbé. Ensuite, nous prouvons que la solution du problème perturbé converge vers la solution du système initial.

Par la suite, dans la troisème section, une analyse d'erreur similaire au chapitre précédant a été présentée où nous avons établir des estimations a priori indépendantes du ϵ qui nous servent plus tard à montrer l'estimation d'erreur d'ordre 2. Ce dernier résultat est établi en effectuant deux étapes. La première étape concerne l'estimation d'erreur corréspondant au problème linéaire tandis que la deuxième étape est liée à l'analyse d'erreur du problème non linéaire.

Dans le dernier chapitre, nous nous préocupons de l'équation non linéaire du Brinkman Forchheimer toujours stationnaire dans un domaine ouvert et borné de $\mathbb{R}^3$:

$$\begin{aligned} &-\gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^{\alpha}\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{dans } \Omega, \\ &\operatorname{div}\boldsymbol{u} = 0 \quad \text{dans } \Omega \end{aligned} \tag{34}$$

avec des conditions aux limites portant sur la composante normale du champ de vitesses et la composante tangentielle du tourbillon.

$$\boldsymbol{u}\cdot\boldsymbol{n} = g, \quad \mathbf{curl}\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}, \text{ sur } \Gamma.$$

le dérnier chapitre se focalise sur l'analyse numérique de l'équation de Darcy Brinkman Forchheimer linéairisé ainsi que l'équation de Brinkman Forchheimer. l'idée principale dans ce travail est d'analyser la méthode de Galerkin Discontinue pour le problème linéaire ainsi que pour le problème non linèaire.

La méthode de Galerkin Discontinue locale adapté pour le problème d'Oseen (le problème linéarisé correspondant aux équations stationnaires de Navier Stokes) était déja initié par B. Cockburn, voir [14]. Ensuite Cockburn et Al ont étendu l'analyse aux équations de Navier-Stokes incompressible.

Le livre de Béatrice [29] introduit la méthode Galerkin Discontinue Directe aux de Stokes stationnaires ainsi que les équations de Navier Stokes stationnaires.

Dans ce chapitre, le problème est considéré dans un domaine polyédrique Ω, nous examinons et prouvons certains résultats théoriques relatifs au problème DBF linéarisé. Nous montrons que le problème est bien posé ainsi que le schéma du modèle DBF non linéaire.
La convergence de la méthode DG est établie et des résultats numériques sont présentés.

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Contents

General Introduction

This thesis is concerned with the study of the viscous fluid flow through a porous media. Fluid flow and transport processes through porous structures is a topic of great interest in various scientific fields. In particular, in several engineering applications such as petrolum environmental and ground-water hydrology. Fluid flow in porous media are usually described by Darcy Brinkman Forchheimer equations. At first, Darcy's law (where the pressure gradient is linearly proportional to the fluid velocity) breaks down when the flow have high velocity. Forchheimer (1901) [31] added a non-linear term which presents the inertial effect. Afterwards, in 1947 [19] the second-order derivatives of velocity was added by Brinkman in order to modelize the effect between the fluid and the pore of walls. This produced the widely used Darcy Brinkman Forchheimer equations (DBF) :

$$\boldsymbol{u}_t - \gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^\alpha\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{0.0.1}$$

where $\boldsymbol{u}$ and π present respectively the velocity and the pressure. γ is Brinkman coefficient, a denotes Darcy coefficient and b defines Forchheimer coefficient. $\boldsymbol{f}$ denotes the exterior forces and $\alpha \in [1, 2]$ is a real number chosen appropriately throughout this work. If we do not say otherwise, Ω is simply connected domain in $\mathbb{R}^3$ with smooth boundary Γ. The (DBF) system must be completed by suitable boundary conditions. The most studied case is the no-slip condition:

$$\boldsymbol{u} = \boldsymbol{0}, \quad \text{on } \Gamma. \tag{0.0.2}$$

However, as quoted by Serrin, the no slip boundary condition is not always suitable since it does not yield to an accurate description of the physical boundary layer near the walls. In 1827, Navier [47] suggested another type of boundary condition (see 0.1.2), based on a balance between the fluid velocity tangent to the surface and the rate of strain at the boundary. The velocity's component normal to the surface is naturally zero, as mass cannot penetrate an impermeable solid surface.
The Navier boundary condition [1], is defined by:

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad (\text{or } = g) \quad \text{and} \qquad (\nu(\mathbb{D}\mathbf{u}\cdot\mathbf{n}) + \alpha\boldsymbol{u})_{\boldsymbol{\tau}} = \boldsymbol{0} \quad (\text{or} = \boldsymbol{h}) \quad \text{on } \Gamma, \tag{0.0.3}$$

[1] The deformation tensor $\mathbb{D}\boldsymbol{u} = (D_{ij}(\boldsymbol{u}))_{1\le i,j\le 3}$ is defined by $D_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. The outward pointing unit normal vector at Γ is denoted by $\boldsymbol{n}$, and $\forall\,\boldsymbol{v}$ defined on Γ, $\boldsymbol{v}_{\boldsymbol{\tau}}$ denotes its tangential component, $\boldsymbol{v}_{\boldsymbol{\tau}} = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$.

where the coefficient α in (0.1.2) is the friction coefficient.

This boundary condition is used to simulate flows near rough walls, such as in aerodynamics, in weather forecasts and in hemodynamics [20, 46] as well as perforated walls. Taking use of the vorticity field $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u}$ and using classical identities, one can observe that in the case of a flat boundary and when $\alpha = 0$ and $g = 0$, the conditions (0.1.2) may be replaced by:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \text{and} \qquad \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma. \tag{0.0.4}$$

We call them Navier-type boundary conditions. Moreover, in some real-life situations it is natural to prescribe the value of the pressure at least on some part of the boundary, this can arise in case of pipelines, blood vessels and different hydraulic systems involving pumps. It should be montionned that there is no physical justification for prescribing only a pressure boundary condition on the boundary and it must be completed by adding some boundary condition involving the velocity. For exemple, we can consider the tangential part of the velocity on the boundary

$$\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}, \qquad \pi = \pi_0 \qquad \text{on } \Gamma, \tag{0.0.5}$$

where π_0 is a given function. See [8, 14, 16, 28, 42]. A part of this work will be focusing in the L^p-theory for the stationary (DBF) system regarding two different types of boundary conditions for the velocity field: in the first chapter, the velocity field will have attached the Navier-type boundary conditions (0.1.2), meanwhile in the second one, we consider the boundary condition (0.0.5) involving the pressure. In the third chapter, we study the approximation of the time dependent (DBF) system with Dirichlet boundary condition (0.0.2) by the pseudo-compressibility method. This approach is introduced in [57] as a numerical approximation of the Navier-Stokes equations to overcome the difficulty that comes from the incompressibility contraint. An extension of this approach to the case of the time dependent (DBF) with Navier-type boundary condition (0.0.4) is given in the fourth Chapter. In the last Chapter, we use a (DG) for solving (DBF) equations with Navier-type boundary condition (0.0.4). This method is a finite element method which utilise discontinous basis functions in the choice of approximation spaces. (DG) method was originally introduced by Reed and Hill [51] for solving neutron transport equation. Extensive research was done on solving hyperbolic equations like Euler equations of gas dynamics. Then it was extended to solve convection diffusion problems, using the so called Local Discontinuous Galerkin (LDG) [24 26] method. (DG) method has been found to be a robust numerical algorithm to solve several varities of problems. The greatest advantage of this method is its parallelizability and the capability to handle complex geometries. It has been used to solve convection dominated problems, elliptic problems and convection-diffusion problems.

To be more precise, we describe bellow the subsequent chapters.

In **chapter 1**, the stationary (DBF) system is studied with the Navier-type boundary condition (0.1.2). The aim is to study the existence of generalized solutions in $\boldsymbol{W}^{1,p}(\Omega)$, strong solutions in $\boldsymbol{W}^{2,p}(\Omega)$. We will consider here a simply connected domain Ω and we will suppose that the Forchheimer coefficient $\alpha = 1$. This chapter has three sections. The first section is dedicated to notations to be used along the chapter and gives some useful statements playing an important role in the proof of the main result. The second section is concerned with the L^p- regularity results for the Hilbertian weak solution of the linear Darcy Brinkman system (when the Forchheimer coefficient $b = 0$):

$$\begin{cases} -\gamma\Delta \boldsymbol{u} + a\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\boldsymbol{u} = 0, \quad \text{in } \Omega. \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma. \end{cases} \tag{0.0.6}$$

The results are proved by using the regularity of the Stokes problem [6] and a suitable bootstrap argument.

The goal of the third section is to establish the existence of weak and strong solutions for the nonlinear Darcy Brinkamn Forchheimer:

$$\begin{cases} -\gamma\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\boldsymbol{u} = 0, \quad \text{in } \Omega. \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma. \end{cases} \tag{0.0.7}$$

Firstly, we prove, via the Faedo Galerkin method, in the Hilbertian case the existence of weak solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$. After, based on the Sobolev embeddings and the regularity results in the L^p theory given the second section, we prove the existence of generalized solutions in $\boldsymbol{W}^{1,p}(\Omega)$ for $p \geq 2$ and strong solutions in $\boldsymbol{W}^{2,p}(\Omega)$ for $p \geq 6/5$. Furthermore, it is possible to extend the regularity of the solution for $3/2 < p < 2$ in $\boldsymbol{W}^{1,p}(\Omega)$. The proof uses the Brower's fixed point theorem after establishing a complete L^p-theory of the linearized Darcy Brinkman Forchheimer problem.

Chapter 2 is concerned with the study of the stationary convective Darcy Brinkman Forchheimer with boundary conditions (0.0.5), where the pressure is replaced by the dynamic pressure:

$$p = \pi + \frac{1}{2}|\boldsymbol{u}|^2 \quad \text{on } \Gamma. \tag{0.0.8}$$

The non linearity $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ can be written using the following identity :

$$\boldsymbol{u}\cdot\nabla\,\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u} + \frac{1}{2}\nabla|\boldsymbol{u}|^2.$$

Then, the problem reads as:

$$\begin{cases} -\gamma\Delta\,\boldsymbol{u} + \mathbf{curl}\boldsymbol{u}\times\boldsymbol{u} + b|\boldsymbol{u}|^\alpha\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u} = 0 \qquad \text{in}\,\Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \quad \text{and} \quad p = p_0 \ \text{ on } \Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases} \tag{0.0.9}$$

We take $\alpha \in [1,\, 2]$ and we will consider here the general case where the domain is eventually multiply-connected. This leads us to supplementary difficulties in the mathematical analysis. The last condition in (2.3.1) is an extra boundary condition which is the flux through the connected component Γ_i, $1 \leq i \leq I$. The situation of bounded domains, eventually multiple-connected with boundary not connected, has been investigated by Begue-Conca-Murat-Pironneau [14, 28] for the linear and nonlinear Navier-Stokes cases (see also Ebmeyer-Frehse [30] for some mixed boundary conditions in polyhedral domains). Kozono-Yanagisawa [42] and Amrouche-Seloula [7, 55] completed this study by developing a complete L^p-theory to solve Stokes and Navier-Stokes equations by assuming that the datum verify some compatibility conditions. They also introduce a variant of the Stokes and Navier Stokes equations, where constants appear as additional unknowns on the boundary, when the compatibility conditions are not verified. In this chapter, we study analogously the following variant of the Darcy Brinkman Forchheimer: find $\boldsymbol{u}$, p and $\boldsymbol{c}$ such that:

$$\begin{cases} -\gamma\Delta\, \boldsymbol{u} + \mathbf{curl}\boldsymbol{u} \times \boldsymbol{u} + b|\boldsymbol{u}|^{\alpha}\boldsymbol{u} + \nabla\, p = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{and} \quad p = p_0 \;\text{ on } \Gamma_0 \text{ and } p = p_0 + c_i & \text{on}\, \Gamma_i,\; i = 1, ..., I \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I, & \end{cases}$$

where $\boldsymbol{c} = (c_1, \ldots, c_I) \in \mathbb{R}^I$.

Unlike the result showed in [8], two important facts are taken into account. Indeed, the results given in [8] require that the velocity and the pressure have the same regularity. First we improve the regularity results concerning the pressure for the Stokes problem with (0.0.5). Furthermore, it is possible to extend these regularity of the solution to the case of the Darcy-Brinkamn system. Secondly, a study of the $\boldsymbol{W}^{1,p}$ regularity for $p \geq 2$ (respectively the $\boldsymbol{W}^{2,p}$ regularity for $p \geq 6/5$) and not only for $3/2 < p < 3$, of the solutions for the nonlinear Darcy Brinkman Forchheimer system with (0.0.5) will be carried out. The proof is done by taking advantage of the new regularity results for the Stokes equations.

In **chapter 3**, we introduce the evolution (DBF) system

$$\begin{aligned} &\partial_t \boldsymbol{u} - \gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^{\alpha}\boldsymbol{u} + B(\boldsymbol{u}, \boldsymbol{u}) + \nabla\, p = \boldsymbol{f} \quad \text{in } \Omega_T, \\ &\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega_T, \end{aligned} \tag{0.0.10}$$

with Dirichlet boundary conditions and the initial datum as follows :

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Sigma_T \quad \text{and} \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{0.0.11}$$

where $\Omega_T = \Omega\times]0, T[$, $\Sigma_T = \partial\Omega\times]0, T[$ and the bilinear forms $B(\boldsymbol{u}, \boldsymbol{v})$ is defined as follows: $B(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}.\nabla)\boldsymbol{v}$.

Many papers are devoted to the study of (DBF) equations. The existence and uniqueness of the solution have been investigated in [40]. The continuous dependance of the solution to the coefficients has been presented in [49]. Moreover, a numerical treatment of (DBF) equations can be found in [35] where the system is solved using finite volume approach and a numerical simulations have been investigated.

It is well-known that the incompressibility constraint

$$\mathrm{div}\boldsymbol{u} = 0 \quad \text{in } \Omega \times [0, T], \tag{0.0.12}$$

introduces a real challenge to solve the problem (3.0.1)-(3.0.2) numerically because it implies to coupled the velocity and the pressure. There are several methods to overcome this difficulty (see, for instance J. Shen 1996 [57] and references therein). We can mention for example :

1. The penalty method has been introduced in 1968 by Temam [59] where it was employed to approximate Navier Stokes equations.

$$\mathrm{div}\boldsymbol{u}^\epsilon + \epsilon p^\epsilon = 0, \quad \text{in } \Omega \times [0, T],$$

   while recentley in 1995, J. Shen study again this method in [56] where optimal error estimates of the penalized problem for the non stationary Navier Stokes system are obtained.

2. The artificial compressibility method :

$$\mathrm{div}\boldsymbol{u}^\epsilon + \epsilon p_t^\epsilon = 0, \quad \text{in } \Omega \times [0, T],$$

   which is already suggested by Temam [60] and Chorin [21].

3. The projection method

$$\mathrm{div}\boldsymbol{u}^\epsilon - \epsilon \Delta p^\epsilon = 0, \quad \text{in } \Omega \times [0, T], \quad \frac{\partial p^\epsilon}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma \times [0, T]$$

   that relax the divergence-free constraint, it was first provided by Temam [59] and Chorin [21, 22] which has a high computational cost.

4. The pseudocompressibility method given by :

$$\mathrm{div}\boldsymbol{u} - \epsilon \Delta p_t^\epsilon = 0, \quad \text{in } \Omega \times [0, T], \ \frac{\partial p^\epsilon}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma \times [0, T]. \tag{0.0.13}$$

The aim of this chapter is to study the error estimate resulting after a perturbation of the Darcy Brinkman Forchheimer equations by the Pseudo-compressibility method (0.0.13).

Here the pseudo-compressiblity method proposed below is the pseudo-compressibility method is introduced in [57] for Navier Stokes equations, which helps us to approximate the incompressible Darcy Brinkman Forchheimer equations by the following problem :

$$\begin{array}{l} \partial_t \boldsymbol{u}^\epsilon - \gamma \Delta \boldsymbol{u}^\epsilon + a\boldsymbol{u}^\epsilon + \widetilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon) + \beta |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon + \nabla p^\epsilon = \boldsymbol{f} \quad \text{in} \quad \Omega_T \\ \mathrm{div}\boldsymbol{u}^\epsilon - \epsilon \Delta p_t^\epsilon = 0 \quad \text{in} \quad \Omega_T \end{array}, \tag{0.0.14}$$

with the following boundary conditions and initial datum :

$$\begin{aligned} &\boldsymbol{u}^\epsilon = 0 \quad \text{and} \quad \tfrac{\partial p^\epsilon}{\partial n} = 0 \quad \text{on} \quad \Sigma_T \\ &\boldsymbol{u}^\epsilon(t_0) = \boldsymbol{u}(t_0) \quad \text{and} \quad p^\epsilon(t_0) = p(t_0) \quad \text{in} \quad \Omega. \end{aligned} \tag{0.0.15}$$

The bilinear form $\widetilde{B}(\boldsymbol{u},\boldsymbol{v})$ is defined as follows: $\widetilde{B}(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}.\nabla)\boldsymbol{v} + \frac{1}{2}(\nabla\boldsymbol{u})\boldsymbol{v}$.

The outline of this chapter is the following. In the first section, we introduce some notations and results concerning the nonlinear terms that we will use in the sequel. In section 2, we consider the non perturbed problem and we prove some estimates for the velocity and the pressure. These results will play an important role in the error analysis of the perturbed problem. The section 3 is devoted to the study of the perturbed problem. By using the standard Faedo-Galerkin method and the Compactness's Theorem, we investigate some a priori estimates which ensures the well-posedness of the approximate system. Next, in order to ensure the convergence of the the perturbed system to the original problem (0.0.10)-(0.0.11), we need first to establish some ϵ independent estimates which we employ afterward to reach the optimal error estimates. In Section 4, we prove the error estimate between the incompressible Darcy Brinkman Forchheimer (0.0.10)-(0.0.11) and the perturbed problem (0.0.14)-(0.0.15). The proof will be divided into two parts: the first one corresponding to the linear problem, the second one to non linear system. Section 5 is devoted to a time discretization of Darcy Brinkman Forchheimer equations and then to give a rigorous error analysis. The scheme can be employed with any consistent space discretization associated with the pseudo-compressibility methods. Finally, in section 6, we restrict our attention to the numerical framework where we present a driven cavity to the compressible Brinkman Forchheimer equations.

In **chapter 4**, we suggest to study the evolution (DBF) system with Navier type boundary conditions:

$$\begin{cases} \boldsymbol{u}_t - \gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^\alpha\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega\times[0,T]. \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = 0, \quad \text{on } \Gamma\times[0,T]. \end{cases} \tag{0.0.16}$$

The well-posedness of this problem will be discussed by using Faedo-Galerkin method combined with the compactness property and Fourier Transform. Moreover we establish some a priori estimates useful to build the error analysis. Next, we focus on the following approximated problem via the pseudo-compressibility

$$\begin{cases} \boldsymbol{u}^\epsilon_t - \gamma\Delta\boldsymbol{u}^\epsilon + a\boldsymbol{u}^\epsilon + b|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon + \nabla p = \boldsymbol{f} & \text{in } \Omega. \\ \operatorname{div}\boldsymbol{u}^\epsilon + \epsilon\Delta p^\epsilon_t = 0 & \text{in } \Omega. \\ \boldsymbol{u}^\epsilon\cdot\boldsymbol{n} = 0, \ \mathbf{curl}\,\boldsymbol{u}^\epsilon\times\boldsymbol{n} = 0 & \text{on } \Gamma, \end{cases} \tag{0.0.17}$$

with with the following boundary conditions and initial datum :

$$\begin{aligned} &\tfrac{\partial p^\epsilon}{\partial n} = 0 \quad \text{on} \quad \Sigma_T \\ &\boldsymbol{u}^\epsilon(t_0) = \boldsymbol{u}(t_0) \quad \text{and} \quad p^\epsilon(t_0) = p(t_0) \quad \text{in} \quad \Omega. \end{aligned} \tag{0.0.18}$$

The well-posedness of the problem above will be investigated by using again Faedo-Galerkin method combined with the Theorem of compactness. Next, in order to prove the convergence of the perturbed equations (0.0.17) to the initial problem (0.0.16), we need some ϵ independent a priori estimates.

In **chapter 5**, we study an approximation of the Darcy Brinkman Forchheimer (DBF) with boundary condition (0.0.4) by means of the Discontinuous Galerkin method (DG).

Recentely, the study of (DG) approximation method was extended to cover the nonlinear elliptic problems which have been attracted the interest of many researchers. In 1996 Bassi and Rebay [13] they expanded the Discontinuous Galerkin finite element method basically considered for the hyperbolic system like Euler equations to the Navier-Stokes equations's state by the manage of the viscous term with a mixed weak formulation, where the (LDG) local discontinuous Galerkin method has been employed by introducing the auxiliary vector variable $\theta = \nabla \boldsymbol{u}$. In fact, the DG method couples two methods: finite volume and the finite element method. The paper of Brezzi, Manzini, Marini, Pietra and Russo in [18] nevertheless restrected to treat the Laplace operator in a polygonal domain but it comes to follow, analyze and give a solid background for the approach proposed by Rebay and Bassi in [13]. The analysis of the Local Discontinuous Galerkin method for Oseen problem (the linearized problem of the stationary Navier Stokes equations) was given by Cockburn in [24] while in [25] Cockburn, Bernardo, Kanschat, Guido and Schötzau, Dominik studied the (LDG) local discontinuous Galerkin method for the non linear incompressible stationary Navier-Stokes equations. The book of Béatrice [53] analyzed the Direct Discontinuous Galerkin method for both of the linear stationary Stokes equations and the nonlinear Navier stokes equations. The book of Dolejsi [29] is concerned with (DG) and its applications to the numerical simulation of the incompressible flow.

Along this chapter, the problem is treated in a polyhedral region Ω. We prove some theoritical results about the well-posedness of the linearized DBF and the nonlinear DBF model's DG scheme. The convergence of the Discontinuous Galerkin method is established. Finally, we give some numerical tests to asses the validity of the method.

0.1 State of the art

Darcy Brinkman Forchheimer (DBF) equations take attention because it modelize, with high accuracy, the behavior of the flow through porous medium. In [39], the authors prove the existence and uniqueness of solutions for all $\alpha > 1$ with Dirichlet boundary conditions and the necessary regular initial data. The arguments are based on the maximal regularity estimate for the linear problem and some modification of the non linear localization technique.

The authors in [36] prove the global in time smooth solutions for the convective Brinkman Forchheimer equations on 3D periodic domain

$$-\gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^{\alpha}\boldsymbol{u} + \nabla\pi = 0 \tag{0.1.1}$$

The theory of evolutionary systems developed by Cheskidov is used to investigate the existence of a strong global attractor.

The time behavior of solutions and the existence of global attractor to the unsteady Darcy Brinkman Forchheimer equations has been studied in [48, 63, 64] for some special values and ranges of parameters α. Moreover, the existence, decay rates and some properties of weak solutions are proved in [11].

In the paper [62], a model for two dimensional laminar steady flow and heat transfer are solved numerically by a finite difference method with second error accuracy. Furthermore, the authors cheked the fact that the inertial parameter or the viscosity parameter reduce the flow and heat transfer charachteristics when one increases Darcy or Brinkman coefficient.

We mention also the paper of J.Kou and al [41] where the numerical simulation of DBF model which describes a flow channel with high porosity are explored.
More precisely, the work suggested a semi-analytic time scheme that solves the evolution DBF equations at each time step.

0.1.1 The Navier type boundary conditions: State of the art

Generally, DBF equations are treated under Dirichlet boundary conditions where are sometimes unrealistic. We need to study this model under other boundary conditions.
In [6] a large study of the stationary Stokes equations with Navier type boundary conditions with flat boundary in a multiply-connected bounded domain Ω of $\mathbb{R}^3$ is given. Based on the inf-sup condition, the proof is established concerning the existence and uniqueness of weak and strong solution.

On the other hand, in the thesis of Rejaiba [52], the author treated the stationary Navier Stokes equations in an open bounded connexe domain Ω of $\mathbb{R}^3$ with Navier type boundary conditions defined as

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad (\text{or } = g) \quad \text{and} \qquad (\nu(\mathbb{D}\mathbf{u}\cdot\mathbf{n}) + \alpha\boldsymbol{u})_{\boldsymbol{\tau}} = \mathbf{0} \quad (\text{or} = \boldsymbol{h}) \quad \text{on } \Gamma. \tag{0.1.2}$$

In this work, the existence of a different types of solutions for Navier Stokes model under the boundary conditions given in (0.1.2) are proved.

In our study, we are concerned by the stationary Darcy Brinkman Forchheimer with Navier type boundary conditions through an open bounded simply domain Ω of $\mathbb{R}^3$. We establish the existence of the generalized solutions and strong solution for both of the Darcy Brinkman equations and Darcy Brinkman Forchheimer equations in the hilbertian case and under L^p theory.

0.1.2 The pressure boundary conditions: State of the art

The imposed pressure and the tangential component of the velocity on the boundary are given by

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, \quad \pi = \pi_0 \tag{0.1.3}$$

In 1995, Conca and al [28] consider both Stokes problem and Navier Stokes equations in a bounded domain Ω of $\mathbb{R}^3$ with the pressure and the tangential velocity given on some part of the boundary. The authors prove the equivalent between the weak formulation and the stationary Stokes system. Moreover, the authors extended the results obtained for the linear problem to the non linear equations case.

This work inspired our numerical part to validate our Discontinuous Galerkin method where the test (fluids in a pipes) served us as a validation model.

We cite also the paper [8] where the authors study the sationary Navier Stokes equations through a bounded open set, connected domain of $\mathbb{R}^3$. The proof of the existence of the solution was investigated by using the fixed point Theorem over the stationary Oseen system. The authors have given a results in the hilbertian case and L^p theory for $1 < p < \infty$.

We investigate the existence of solution for both of sationary incompressible Stokes equations

$$-\gamma\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \quad \mathrm{div}\boldsymbol{u} = 0 \quad \text{in } \Omega$$

and the stationary incompressible Brinkman Forchheimer

$$-\gamma\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + b|\boldsymbol{u}|^\alpha \boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \quad \mathrm{div}\boldsymbol{u} = 0 \quad \text{in } \Omega$$

under the same boundary conditions through a bounded open multiply-connected domain.

The improvement that we give compared to the previous works is that we prove the existence of weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ instead of $\pi \in W^{1,p}(\Omega)$.

0.1.3 The pseudo-compressibility: State of the art

It is well-known that the incompressibility constraint presents one of the serious diffuclty in the numerical scale. Because it combine both of the velocity's space and the pressure's space. In the litterature, we find many papers study the differents ways to overcome this constraint. We can cite as examples [17], [23] [38], [50], [12].
In [57], J. Shen presented a unsteady perturbed Navier Stokes equations by using the pseudo-compressibility method which is obtained by introducing a pressure stabilization term in the mass conservation's equation. The author established the first error estimate for the velocity.

In our work, we use the same type of the pseudo-compressibility method to obtain the approximate unsteady Darcy Brinkman Forchheimer equations with Dirichlet boundary condition on the velocity and we prove the first order error estimate by using a similar procedure as in the paper of J. Shen [57] however Forchheimer's non linear term need to be controled with a special care, in order to get this purpose we handle the Forchheimer term as

$$\text{for any } \boldsymbol{u} \in \boldsymbol{V}, \; F(\boldsymbol{u}) = |\boldsymbol{u}|^{\alpha}\boldsymbol{u}$$
$$\|F(\boldsymbol{u}) - F(\boldsymbol{v})\| \leqslant C\|\nabla(\boldsymbol{u} - \boldsymbol{v})\|$$

We should point out that in [58], the projection scheme to the approximate unsteady incompressible Navier Stokes equations was proposed by J. Shen. The second error estimate for the velocity is investigated.

In 2015 [44] the authors treat the unsteady Darcy Brinkman Forchheimer equations. They prove the continuous dependence of the solution on the Brinkman's and Forchheimer's coefficients as well as the initial data and external forces. They propose and study a perturbed unsteady DBF problem by using the artificial compressibility method. Afterward the authors checked a time discretization of the perturbed system by a semi-implicit Euler scheme and next a lowest-order Raviart-Thomas element is applied for spatial discretization.

Louaked and al [45] introduce the pressure stablization method for the unsteady incompressible Brinkman-Forchheimer equations. They present a time discretization scheme which can be used with any consistent finite element space approximation. Second-order error estimate is established.

In our thesis, we present a projection scheme associated with the perturbed unsteady DBF problem. We estabish the second error estimate for the velocity. We implement the driven cavity test and we observe the correctness of the method by comparing the different component of velocity for $b = 0$ with Ghia's results [32].
We extend the approach of the pseudo-compressibility method applied on the unsteady DBF problem with Navier type boundary conditions. We use some suitables functional spaces to

control the non standard boundary conditions. We obtain similar results by using different tools and techniques.

0.1.4 The Discontinuous Galerkin method: State of the art

The discontinuous Galerkin (DG) method is a class of finite element methods using completely discontinuous piecewise polynomial space. A key ingredient for the success of such method is the correct design interface numerical fluxes.
The first local discontinous method was developed by Cockburn and al [27] for the convection diffusion equations.
Later in 2002, Cockburn and al [26] interested to the LDG method for the Stokes problem in a bounded domain Ω of $\mathbb{R}^3$. Then in 2004, the authors [24] give a similar study on Oseen's system. They established an a priori estimates for L^2- norm of the errors in the velocity and pressure. The optimal order estimate are reached when the polynomials of degree k are used for each component of the velocity and polynomials of degree $k-1$ for the pressure with $k \geqslant 1$.

In 2004, a new LDG method has been applied to the incompressible steady Navier Stokes system [25]. In view of the many features of this method (the stability, the local conservating, high order accuracy), it is considered as one of the most important method.
The authors state the stability of the scheme and employ a discrete version of the classical fixed point iteration to prove the existence of solutions of the incompressible steady Navier Stokes equations by solving a sequence of Oseen problems.

While in 2008, B. Rivière in her book [53], gives a wide analyse on both of the elliptic system and the parabolic one. We mention that the author proposed a DG discretization for the non linear term (in particular the convection term) based on an upwind technique. She used Brower fixed point's theorem to check the existence of solution for Navier Stokes model.

The coupled incompressible steady Navier Stokes equations with Darcy equations was proposed by B. Rivière and al [34]. The existence of solution and some a priori estimates under some smallness conditions on the data are proved.
The authors suggested a discontinuous Galerkin scheme for discretizing the equations and presented some numerical results.

An approximation of the steady Brinkman problem defined in the nonconvex L-shaped domain $\Omega =]-1,1[\backslash]0,1[^2$ has been treated in [9] where the authors choose the values of the pyisical parameters a and γ as $a = 10$ and $\gamma = 0.1$. The approximate velocity and pressure displayed for 57898 elements and 28950 vertices.

Since the author could compute the solution analytically, the author present the Bercovier Engelman test. The researcher meshed the domain $=]0,1[\times]0,1[$ by different structured triangulation and considering p, the different component of the velocity and $\boldsymbol{w} = \mathbf{curl}\boldsymbol{u}$ are prescribed on the boundary. The velocity is taken vanish on the upper and lower boundary. The researcher conclude a good agrement between the numerical results and the theoritical results.

In our thesis, we lead with the stationary incompressible linearized Darcy Brinkman Forchheimer model

$$-\gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{d}|\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \quad \mathrm{div}\boldsymbol{u} = 0 \quad \text{in } \Omega$$

then we introduce the discretization scheme associated to the weak formulation of the linearized model and we prove the the existence of solution for the scheme obtained. The procedure is quoted from the arguments given in [25]. After, we move to check the well-posedness of the non linear steady DBF system's scheme under some smallness condition on the data. We give an approximation of DG method applied to the stationary DBF scheme in L shape geometry.
In the last part of our work, the Bercovier Engelman test was displayed. We consider the stationary DBF scheme in $\Omega =]0,1[^2$ with different component of the veocity and vorticity are given on the boundary. We choose the value of stabilization parameter as $\sigma = 50$ to get stable results.

Chapter 1

The Stationary Darcy Brinkman Forchheimer equations (DBF) with Navier-type boundary conditions

The work developed in this chapter is concerned with the existence, uniqueness and regularity of the solution for the following Darcy Brinkman Forchheimer equations:

$$\begin{cases} -\gamma\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, & \text{in } \Omega \\ \operatorname{div}\boldsymbol{u} = 0, & \text{in } \Omega, \end{cases} \tag{1.0.1}$$

attached with the Navier-type boundary conditions:

$$\boldsymbol{u}\cdot\boldsymbol{n} = g, \ \mathbf{curl}\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}, \tag{1.0.2}$$

where $\boldsymbol{u}$ is the velocity, π denotes the pressure, $\boldsymbol{f}$ are the external forces, h and g are given functions. Ω is a bounded domain of $\mathbb{R}^3$ with smooth boundary Γ. $\gamma > 0$ is the Brinkman's coefficient, $a > 0$ defines Darcy's coefficient and $b > 0$ denotes Forchheimer's coefficient.

In our work, we are focused in showing existence, uniqueness and L^p regularity of the weak and the strong solution for the problem (1.0.1)-(1.0.2). In order to prove the regularity of the weak solution in $\boldsymbol{W}^{1,p}(\Omega)$ with $p > 2$, and strong solution in $\boldsymbol{W}^{2,p}(\Omega)$ with $p \geq 6/5$ for the non linear Darcy Brinkman Forcheimer problem, we use the regularity results for the linear Darcy-Brinkman and Stokes problems combining them with a bootstrap argument. To obtain weak solution in $\boldsymbol{W}^{1,p}(\Omega)$ for $\frac{3}{2} < p < 2$, we study before the linearized problem and we can conclude the existence result for the non linear problem with fixed point theorem.

This chapter is organized as follows: the first section introduce some results and preliminaries that we need in the sequel of this chapter. The section 2 is restricted to treat the linear equations where we give all details about the well-posedness of this problem. In the last section, we deal with the study of the non linear Brinkman Forchheimer equations with respect to the boundary conditions (1.0.2). Using the results of Section 2, we prove the regularity $\boldsymbol{W}^{1,p}(\Omega)$ for $p > 2$ and in $\boldsymbol{W}^{2,p}(\Omega)$ for $p \geq 6/5$. Finally, thanks to the study of the linearized problem done in this section, we prove the regularity of the solution in $\boldsymbol{W}^{1,p}(\Omega)$ for $3/2 < p < 2$.

1.1 Notations and Preliminaries

The aim of this section is to give the functional framwork and some results already obtained which we need in this part of thesis.
For given $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$, we note the Laplace operator of vector field $\boldsymbol{u}$ as follow,

$$\Delta \boldsymbol{u} = \nabla(\mathrm{div}\boldsymbol{u}) - \mathbf{curl}\mathbf{curl}\boldsymbol{u}.$$

We shall use bold characters for the vectors or the vector spaces and the non-bold characters for the scalars.
Let us first define the spaces :

$$\boldsymbol{W}^{1,p}(\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^p(\Omega); \quad \nabla \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)\}.$$
$$\boldsymbol{W}^{1,p}_{\sigma}(\Omega) = \{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega); \quad \mathrm{div}\boldsymbol{u} = 0 \quad \text{in } \Omega\}, \quad \text{where } 1 < p < \infty.$$

We note the norm of every vector field $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} = \Big(\|\boldsymbol{u}\|_{L^p(\Omega)} + \|\nabla \boldsymbol{u}\|_{L^p(\Omega)}\Big)^{\frac{1}{p}}.$$

Let us consider $\boldsymbol{W}^{1,p'}(\Omega)$ as the dual space of $\boldsymbol{W}^{1,p}_0(\Omega)$ and $\langle . \, , \, . \rangle$ the duality between them.
We introduce the spaces :

$$\boldsymbol{H}^{p}(\mathrm{div}, \Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^p(\Omega); \mathrm{div}\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)\}.$$
$$\boldsymbol{H}^{p}(\mathbf{curl}, \Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^p(\Omega); \, \mathbf{curl}\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)\};$$

which are equipped with the following norms respectively :

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^p(\mathrm{div},\Omega)} = \Big(\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\boldsymbol{u}\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}}.$$
$$\|\boldsymbol{u}\|_{\boldsymbol{H}^p(\mathbf{curl},\Omega)} = \Big(\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)}\Big)^{\frac{1}{p}}.$$

Now, let us define the space $\boldsymbol{X}^p(\Omega)$ as $\boldsymbol{X}^p(\Omega) = \boldsymbol{H}^p(\mathrm{div}, \Omega) \cap \boldsymbol{H}^p(\mathbf{curl}, \Omega)$ with the norm,

$$\|\boldsymbol{u}\|_{X^p(\Omega)} = (\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\boldsymbol{u}\|^p_{L^p(\Omega)} + \|\mathbf{curl}\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)})^{\frac{1}{p}}.$$

We will denote $\mathcal{D}(\Omega)$ as the set of smooth functions with compact support in Ω.
Recall that $\mathcal{D}(\overline{\Omega})$ is dense both in $\boldsymbol{H}^p(\mathrm{div}, \Omega)$, $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ and $X^p(\Omega)$ see [2,8].
Any function $\boldsymbol{u}$ in $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ has a normal trace $\boldsymbol{u} \cdot \boldsymbol{n}$ in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ defined by :

$$\forall \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega), \quad \langle \boldsymbol{u} \cdot \boldsymbol{n} \, , \, \boldsymbol{v} \rangle_\Gamma = \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{v} dx + \int_\Omega (\mathrm{div}\boldsymbol{u}) \, \boldsymbol{v} \, dx; \tag{1.1.1}$$

where the duality pairing is defined by : $\langle .\,,\,. \rangle_\Gamma = \langle .\,,\,. \rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma)}$.
Moreover, any function $\boldsymbol{u} \in \boldsymbol{H}^p(\mathbf{curl},\Omega)$ has a tangential trace,

$$\forall \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega), \quad \langle \boldsymbol{u}\times\boldsymbol{n}\,,\,\boldsymbol{v}\rangle_\Gamma = \int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\boldsymbol{v}\,dx - \int_\Omega \mathbf{curl}\boldsymbol{u}\cdot\boldsymbol{v}\,dx. \tag{1.1.2}$$

We consider the subspaces :

$$\boldsymbol{H}^p_0(\mathrm{div},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma\}.$$
$$\boldsymbol{H}^p_0(\mathbf{curl},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n} = 0 \quad \text{on } \Gamma\}.$$

If $1 < p < \infty$, then p' will denote the conjugate exponent of p, i.e., $\frac{1}{p'} + \frac{1}{p} = 1$.

Let $\boldsymbol{f}$ belongs to the dual space of $\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)$, i.e $\boldsymbol{f} \in (\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega))'$ then there existe $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ and $\chi \in L^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \nabla\chi$. Moreover, $\boldsymbol{f}$ satisfies the estimate :

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\boldsymbol{f}\|_{(\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega))'}.$$

Now, let we give the subspaces $\boldsymbol{X}^p_T(\Omega)$ and $\boldsymbol{V}^p_T(\Omega)$ as follow,

$$\boldsymbol{X}^p_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma\}.$$
$$\boldsymbol{V}^p_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^p_T(\Omega);\ \mathrm{div}\boldsymbol{v} = 0 \ \text{ in } \Omega\}.$$

For s be a real number such that $s \geqslant 1$, we define the following spaces,

$$\boldsymbol{Y}^{\,s,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \mathrm{div}\boldsymbol{v} \in \boldsymbol{W}^{s-1,p}(\Omega);\ \mathbf{curl}\boldsymbol{v} \in \boldsymbol{W}^{\,s-1,p}(\Omega);\ \boldsymbol{v}\times\boldsymbol{n} \in \boldsymbol{W}^{\,s-\frac{1}{p},p}(\Gamma)\}.$$

$$\boldsymbol{N}^{\,s,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \mathrm{div}\boldsymbol{v} \in \boldsymbol{W}^{s-1,p}(\Omega);\ \mathbf{curl}\boldsymbol{v} \in \boldsymbol{W}^{\,s-1,p}(\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} \in \boldsymbol{W}^{\,s-\frac{1}{p},p}(\Gamma)\}.$$

Note that C is a positive constant which represents a different value at its several occurences. We recall the following result that we can find in [8, Theorem 3.4, Corollary 5.3].

Theorem 1.1.1. *Let Ω is of classe $\mathcal{C}^{1,1}$. The space $\boldsymbol{X}^p_T(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$ and for any function $\boldsymbol{v}$ in $\boldsymbol{X}^p_T(\Omega)$, we have the following estimate :*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant C(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\boldsymbol{v}\|_{L^p(\Omega)}). \tag{1.1.3}$$

Moreover if Ω is of classe $\mathcal{C}^{s,1}$, we have the continuous imbedding of the spaces $\boldsymbol{Y}^{s,p}(\Omega)$ and $\boldsymbol{N}^{s,p}(\Omega)$ into $\boldsymbol{W}^{s,p}(\Omega)$ with the following estimates :

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{s,p}(\Omega)} \leqslant C(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\boldsymbol{v}\|_{\boldsymbol{W}^{s-1,p}(\Omega)} + \|\mathrm{div}\boldsymbol{v}\|_{W^{\,s-1,p}(\Omega)} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{s-\frac{1}{p},p}(\Gamma)}). \tag{1.1.4}$$

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{s,p}(\Omega)} \leqslant C(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\boldsymbol{v}\|_{\boldsymbol{W}^{s-1,p}(\Omega)} + \|\mathrm{div}\boldsymbol{v}\|_{W^{\,s-1,p}(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{\boldsymbol{W}^{s-\frac{1}{p},p}(\Gamma)}). \tag{1.1.5}$$

Remark 1.1.2. If $p=2$, we have the embedding $\boldsymbol{X}^{\,2}_{T}(\Omega) \hookrightarrow \boldsymbol{H}^{\,1}(\Omega)$ which allows to obtain the estimate,

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^{\,1}(\Omega)} \leqslant C(\|\boldsymbol{u}\|_{\boldsymbol{L}^{2}(\Omega)} + \|\mathbf{curl}\boldsymbol{u}\|_{L^{\,2}(\Omega)} + \|\mathrm{div}\boldsymbol{u}\|_{L^{\,2}(\Omega)}). \tag{1.1.6}$$

Moreover, for any $\boldsymbol{u} \in \boldsymbol{H}^{\,1}(\Omega)$, with $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ, we have :

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^{2}(\Omega)} \leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}^{\,1}(\Omega)}. \tag{1.1.7}$$

We need some results concerning the solvability of the Stokes equations.

$$\begin{cases} -\gamma\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\boldsymbol{u} = 0, & \text{in } \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} = g,\ \mathbf{curl}\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma. \end{cases} \tag{1.1.8}$$

We now recall the following results concerning the weak solution and the strong solutions of Stokes problem. These results will play an important role in the study of the L^p-regularity of the solutions to the Darcy Brinkman Forchheimer problem (we refer to [6, Thoerem 4.4 and Theorem 4.8] for the proof).

Theorem 1.1.3. *We suppose Ω is of classe $\mathcal{C}^{1,1}$.*

1. *Let $\boldsymbol{f} \in (\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega))'$, $g \in W^{1-\frac{1}{p},p}(\Gamma)$, $\boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$; then the Stokes equations* (1.1.8) *has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$ which satisfies the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{(\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega))'} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}). \tag{1.1.9}$$

2. *Furthermore, if Ω is of classe $\mathcal{C}^{2,1}$ and*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega),\ g \in W^{\,2-\frac{1}{p},p}(\Gamma),\ \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma);$$

then the Stokes problem (1.1.8) *has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega)\times W^{\,1,p}(\Omega)$ satisfying the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{\,1,p}(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{\,2-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}). \tag{1.1.10}$$

1.2 Darcy Brinkman equations with Navier's type boundary conditions

In this section, we are interested to the linear Brinkman problem

$$\begin{cases} -\gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} = g,\ \mathbf{curl}\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma. \end{cases} \tag{1.2.1}$$

We give some results concerning the existence and uniqueness of weak and strong solutions. These results will be used in the following to show the existence of weak solution for the nonlinear Darcy Brinkman Forchheimer problem. Since the Hilbertian case is different from the general L^p-theory, we study each case separately.

1.2.1 Hilbertian Framework

The aim of this section is to prove the existence and uniqueness of weak solution for the Brinkman problem (1.2.1) in the Hilbertian case.

Weak solution

The approach consists on dealing with Brinkman problem step by step, firstly beginning with the homogeneous case, then the non-homogeneous one and finally with the divergence not free case. The proof is fundamentally based on Lax Milgram's Theorem.

In the following, we give a result concerning the well-posedness of the incompressible Brinkman equations (1.2.1) with homogeneous boundary conditions :

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,\ \mathbf{curl}\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0}\quad \text{on } \Gamma.$$

Proposition 1.2.1. *Let* Ω *is of classe* $\mathcal{C}^{1,1}$ *and we assume that* $\boldsymbol{f}$ *belongs to* $(\boldsymbol{H}_0^2(\mathrm{div},\Omega))'$ *with* $g=0$ *and* $\boldsymbol{h}=\boldsymbol{0}$*, then the system* (1.2.1) *has a unique solution* $(\boldsymbol{u},\pi)\in\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$*. Moreover there exists* $C>0$ *depending on* Ω*,* a *and* γ *such that :*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}+\|\pi\|_{L^2(\Omega)}\leqslant C\|\boldsymbol{f}\|_{(H_0^2(\mathrm{div},\Omega))'}. \tag{1.2.2}$$

Proof. The incompressible homogeneous Brinkman problem (1.2.1) is equivalent to the following weak formulation :

$$\begin{cases} \text{Find } \boldsymbol{u}\in\boldsymbol{V}_T^2(\Omega), \text{ such that}\\ a(\boldsymbol{u},\boldsymbol{v})=l(\boldsymbol{v}),\ \text{for any } \boldsymbol{v}\in\boldsymbol{V}_T^2(\Omega), \end{cases} \tag{1.2.3}$$

where for given functions $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}_T^2(\Omega)$, the both bilinear form $a(.\,,.)$ and the linear form $l(.)$ are defined as,

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v})&=\gamma\int_\Omega \mathbf{curl}\boldsymbol{u}\cdot\mathbf{curl}\boldsymbol{v}\,dx+a\int_\Omega\boldsymbol{u}\cdot\boldsymbol{v}\,dx.\\ l(\boldsymbol{v})&=\langle\boldsymbol{f}\ ,\ \boldsymbol{v}\rangle_\Omega. \end{aligned} \tag{1.2.4}$$

The duality pairing is defined by $\langle.\,,.\rangle_\Omega=\langle.\,,.\rangle_{(\boldsymbol{H}_0^2(\mathrm{div},\Omega))'\times\boldsymbol{H}_0^2(\mathrm{div},\Omega)}$.
It is clear that the forms $a(.\,,.)$ and $l(.)$ are continuous on $\boldsymbol{X}_T^2(\Omega)$.
For any function $\boldsymbol{u}\in\boldsymbol{V}_T^2(\Omega)$, we have :

$$|a(\boldsymbol{u},\boldsymbol{u})|\geqslant\min(\gamma,a)(\|(\mathbf{curl}\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}).$$
$$|a(\boldsymbol{u},\boldsymbol{u})|\geqslant\min(\gamma,a)\|\boldsymbol{u}\|^2_{\boldsymbol{X}_T^2(\Omega)}.$$

In view of the imbedding $\boldsymbol{X}_T^2(\Omega)\hookrightarrow\boldsymbol{H}^1(\Omega)$, we derive that the bilinear form $a(.\,,.)$ is coercive and we can apply Lax Milgram Lemma to find solution $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$ to problem (1.2.3).

Based on De Rham's Theorem, one can derive the existence and uniqueness of $\pi \in L^2(\Omega)$, which completes the proof of solution's existence in the Proposition 1.2.1.

Finally, in order to achieve the proof of the Proposition 1.2.1, it suffices to prove the estimate (1.2.2). Hence by using both the coercivity of the bilinear form $a(.,.)$ and the continuity of the linear form $l(.)$, for any function $\boldsymbol{u} \in \boldsymbol{V}^2_T(\Omega)$, it follows immediately,

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\|\boldsymbol{f}\|_{[\boldsymbol{H}^2_0(\mathrm{div},\Omega)]'}. \tag{1.2.5}$$

Moreover, for given function $\pi \in L^2(\Omega)$, using the fact that,

$$\|\pi\|_{L^2(\Omega)} \leqslant C\|\nabla\pi\|_{\boldsymbol{H}^{-1}(\Omega)} \leqslant C\Big(\nu\|\Delta\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} + a\|\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)}\Big).$$

Since $\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} \leqslant C\|\boldsymbol{f}\|_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))'}$ and $\|\Delta\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)} \leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$.
The estimate (1.2.2) follows from (1.2.5).

□

Now, we move to study the incompressible Brinkman equations with non homogeneous boundary conditions. The key of the proof based on the lifting which is used and combined with the previous result in Proposition 1.2.1 about the homogeneous Brinkman equations.

Proposition 1.2.2. *Let Ω be of classe $\mathcal{C}^{1,1}$ and we assume that the datum $\boldsymbol{f}$ in $(\boldsymbol{H}^2_0(\mathrm{div},\Omega))'$, $g \in H^{\frac{1}{2}}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ with g satisfying the compatibility conditions:*

$$\int_\Gamma g\, d\sigma = 0. \tag{1.2.6}$$

Then the non homogeneous Brinkman equations (1.2.1) *has a unique solution* $(\boldsymbol{u},\pi)$ *belongs to* $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ *and satisfies the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{\boldsymbol{L}^2(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{(\boldsymbol{H}^2(\mathrm{div},\Omega))'} + \|g\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}). \tag{1.2.7}$$

Proof. Let us consider the Neumann problem,

$$\Delta\boldsymbol{\theta} = \boldsymbol{0} \text{ in } \Omega, \quad \frac{\partial\boldsymbol{\theta}}{\partial\boldsymbol{n}} = g \text{ on } \Gamma.$$

For $g \in H^{\frac{1}{2}}(\Gamma)$ satifying (1.2.6), this problem has a unique solution $\theta \in \boldsymbol{H}^2(\Omega)/\mathbb{R}$ satisfies the following estimate :

$$\|\boldsymbol{\theta}\|_{\boldsymbol{H}^2(\Omega)/\mathbb{R}} \leqslant C\|g\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}. \tag{1.2.8}$$

Next, $\boldsymbol{z} = \boldsymbol{u} - \nabla\boldsymbol{\theta}$, satisfies the system :

$$\begin{cases} -\gamma\Delta\boldsymbol{z} + a\boldsymbol{z} + \nabla\pi = \boldsymbol{f} - a\nabla\theta \quad \text{and} \quad \mathrm{div}\,\boldsymbol{z} = 0 & \text{in}\,\Omega \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}, & \text{on } \Gamma. \end{cases} \tag{1.2.9}$$

The problem obtained above is equivalent to the weak formulation (1.2.3) with the right hand side :

$$l(\boldsymbol{v}) = \langle \boldsymbol{f}\,, \boldsymbol{v}\rangle_{(\boldsymbol{H}_0^2(\mathrm{div},\Omega)'\times \boldsymbol{H}_0^2(\mathrm{div},\Omega)} + \gamma\langle \boldsymbol{h}\times\boldsymbol{n}\,,\, \boldsymbol{v}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}.$$

which is still a linear continuous form.

Thus, according to Proposition 1.2.1, we deduce the existence of a unique solution $(\boldsymbol{z},\pi)\in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ for problem (1.2.9), which satisfies the estimate :

$$\|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{\boldsymbol{L}^2(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}). \tag{1.2.10}$$

We claim that $(\boldsymbol{u},\pi) = (\boldsymbol{z}+\nabla\boldsymbol{\theta},\pi)\in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ is the unique solution for the non-homogeneous Brinkman equations (1.2.1). Finally, by combining (1.2.10) and (1.2.8), we conclude the estimate (1.2.7).

□

We can now consider the following Brinkman problem where the divergence does not vanish :

$$\begin{cases} -\gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\boldsymbol{u} = \chi & \text{in } \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} = g,\ \mathbf{curl}\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma. \end{cases} \tag{1.2.11}$$

We give the next Corollary,

Corollary 1.2.3. *We assume that Ω is of classe $\mathcal{C}^{1,1}$. Let $\boldsymbol{f},\chi,g$ and $\boldsymbol{h}$ such that :*

$$\boldsymbol{f}\in(\boldsymbol{H}_0^2(\mathrm{div},\Omega))',\ \chi\in\boldsymbol{L}^2(\Omega),\ g\in H^{\frac{1}{2}}(\Gamma),\ \boldsymbol{h}\in\boldsymbol{H}^{-\frac{1}{2}}(\Gamma),$$

with the following compatibility condition :

$$\int_\Omega \chi\,dx = \langle g\,,\,1\rangle_\Gamma. \tag{1.2.12}$$

Then the Brinkman equations (1.2.11) *has a unique solution* $(\boldsymbol{u},\pi)$ *in* $\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$. *Moreover, there exists $C>0$ such that :*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{\boldsymbol{L}^2(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^2(\mathrm{div},\Omega))'} + \|\chi\|_{\boldsymbol{L}^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}). \tag{1.2.13}$$

Proof. This proof is slight different from the previous one. we consider the unique solution $\boldsymbol{\theta}\in\boldsymbol{H}^2(\Omega)/\mathbb{R}$ of the Neumann problem :

$$\Delta\boldsymbol{\theta} = \chi \quad \text{in } \Omega, \quad \frac{\partial\boldsymbol{\theta}}{\partial\boldsymbol{n}} = g \quad \text{on } \Gamma.$$

Satisfying the estimate :

$$\|\boldsymbol{\theta}\|_{\boldsymbol{H}^2(\Omega)/\mathbb{R}} \leqslant C(\|\chi\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}). \tag{1.2.14}$$

We set $\boldsymbol{z} = \boldsymbol{u} - \nabla\theta$. Then $\boldsymbol{z}$ satisfies the system :

$$\begin{cases} -\gamma\Delta \boldsymbol{z} + a\boldsymbol{z} + \nabla\pi = \boldsymbol{F} \quad \text{and} \quad \mathrm{div}\boldsymbol{z} = 0 & \text{in } \Omega \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma, \end{cases} \tag{1.2.15}$$

where the right hand side in the first equation is defined by $\boldsymbol{F} = \boldsymbol{f} + \gamma\nabla\chi - a\nabla\theta$ which belongs also to the space $(\boldsymbol{H}_0^2(\mathrm{div},\Omega))'$. According to the Proposition 1.2.2, the problem (1.2.15) has a unique solution $(\boldsymbol{z},\pi) \in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ which satisfies the estimate (1.2.10). One conclude that $(\boldsymbol{u},\pi) = (\boldsymbol{z}+\nabla\theta,\pi) \in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ is the unique solution for Brinkman problem (1.2.1). The estimate (1.2.13) follows direclty from the relation (1.2.14) and estimate (1.2.10). □

Strong solutions:

Now, we establish the existence and uniqueness of strong solution in $\boldsymbol{H}^2(\Omega)\times H^1(\Omega)$ of the problem (1.2.1) when we impose more regular datum.

First, we consider the homogeneous case ($g = 0$, $\boldsymbol{h} = 0$) and we have only to consider more regular datum for the forces $\boldsymbol{f}$.

Proposition 1.2.4. *Let Ω be of classe $\mathcal{C}^{2,1}$. We assume that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. Then, the homogeneous Brinkman problem* (1.2.1) *has exactly one solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^2(\Omega)\times \boldsymbol{H}^1(\Omega)$. Moreover there exists a positive constant C such that :*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

Proof. Let $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ the solution given by the Proposition 1.2.1.
By setting $\boldsymbol{z} = \mathbf{curl}\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$, it follows that $\mathrm{div}\boldsymbol{z} = 0$ and $\mathbf{curl}\boldsymbol{z} \in \boldsymbol{L}^2(\Omega)$ in Ω. Moreover, we have $\boldsymbol{z}\times\boldsymbol{n} = 0$ on Γ. It follows that $\boldsymbol{z} \in \boldsymbol{Y}^{1,2}(\Omega)$.
According to the Theorem 1.1.1, we can conclude that $\boldsymbol{z}$ belongs to $\boldsymbol{H}^1(\Omega)$. As a consequence, $\boldsymbol{u}$ belongs to $\boldsymbol{N}^{2,2}(\Omega)$. Again, applying Theorem 1.1.1 we conclude that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$.
Let us consider the divergence of a first equation in Brinkman problem (1.2.1), so

$$\mathrm{div}(\nabla\pi - \boldsymbol{f}) = 0 \qquad \text{in } \Omega. \tag{1.2.16}$$

Moreover, since $\mathbf{curl}\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ on Γ implies that $\Delta\boldsymbol{u}\cdot\boldsymbol{n} = \mathbf{0}$ on Γ, we have :

$$(\nabla\pi - \boldsymbol{f})\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma. \tag{1.2.17}$$

Since $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, due to the relations (1.2.16) and (1.2.17), we derive immediately that π belongs to $H^1(\Omega)$ which achieves the proof of the Proposition 1.2.4.

□

Now, we are ready to treat the non homogeneous case.

Proposition 1.2.5. *We assume that Ω is of classe $\mathcal{C}^{2,1}$. Let $\boldsymbol{f}$, g, $\boldsymbol{h}$ such that :*

$$\boldsymbol{f} \in \boldsymbol{L}^2(\Omega),\ g \in H^{\frac{3}{2}}(\Gamma),\ \boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$$

and

$$\int_\Gamma g \, d\sigma = 0$$

Then the non homogeneous Brinkman problem (1.2.1) *has a unique solution* $(\boldsymbol{u}, \pi)$ *belongs to* $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ *which satisfies :*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}).$$

Proof. Let $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ the solution given by Proposition 1.2.2. We proceed as in the previous Proposition. We set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$. Since z belongs to $\boldsymbol{Y}^{1,2}(\Omega)$. Thanks to Theorem 1.1.1, we have $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$.

For any function $\boldsymbol{\phi}$ in $\boldsymbol{H}^2(\Omega)$, we have

$$\begin{aligned}\langle \Delta \boldsymbol{u} \cdot \boldsymbol{n} \,,\, \boldsymbol{\phi}\rangle_\Gamma &= \langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} \cdot \boldsymbol{n} \,,\, \boldsymbol{\phi}\rangle_\Gamma = -\langle \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} \,,\, \nabla\boldsymbol{\phi}\rangle_\Gamma \\ &= \langle \operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n}) \,,\, \boldsymbol{\phi}\rangle_\Gamma. \end{aligned} \tag{1.2.18}$$

Because of the relation (1.2.18), π satisfies :

$$\begin{aligned}&\operatorname{div}(\boldsymbol{f} - \nabla\pi) = 0, \quad \text{in } \Omega. \\ &(\boldsymbol{f} - \nabla\pi)\cdot\boldsymbol{n} = -\gamma \operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n}) - a\,g, \quad \text{on } \Gamma.\end{aligned}$$

Then, it infers immediately that $\pi \in \boldsymbol{H}^1(\Omega)$. □

The following result gives the regularity of the solutions of the Brinkman problem (1.2.1) when the divergence does not vanish.

Proposition 1.2.6. *Let Ω be of classe $\mathcal{C}^{2,1}$. We assume that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $\chi \in \boldsymbol{H}^1(\Omega)$, $g \in H^{\frac{3}{2}}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ satisfying the following compatibility condition:*

$$\int_\Omega \chi \, dx = \langle \boldsymbol{g} \,,\, 1\rangle_\Gamma. \tag{1.2.19}$$

Then the unique solution $(\boldsymbol{u}, \pi)$ *of the Brinkman equations* (1.2.1) *belongs to* $\boldsymbol{H}^2(\Omega) \times \boldsymbol{H}^1(\Omega)$, *furthermore there exists* $C > 0$ *such that :*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{\boldsymbol{H}^1(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\chi\|_{\boldsymbol{H}^1(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}). \tag{1.2.20}$$

Proof. Let us consider $\boldsymbol{\theta} \in \boldsymbol{H}^3(\Omega)$ to be a solution of Neumann problem :

$$\Delta\boldsymbol{\theta} = \chi \quad \text{in } \Omega, \quad \frac{\partial\boldsymbol{\theta}}{\partial\boldsymbol{n}} = g \quad \text{on } \Gamma,$$

satisfying :

$$\|\boldsymbol{\theta}\|_{\boldsymbol{H}^3(\Omega)} \leqslant C(\|\chi\|_{\boldsymbol{H}^1(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\Gamma)}). \tag{1.2.21}$$

Note that $\boldsymbol{z} = \boldsymbol{u} - \nabla\boldsymbol{\theta}$, satisfies the following problem :

$$\begin{cases} -\gamma\Delta\boldsymbol{z} + a\boldsymbol{z} + \nabla\pi = \boldsymbol{F} \quad \text{and} \quad \mathrm{div}\boldsymbol{z} = 0 \quad \text{in } \Omega. \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} \qquad\qquad \text{on } \Gamma, \end{cases} \tag{1.2.22}$$

with $\boldsymbol{F} = \boldsymbol{f} + \gamma\nabla\chi - a\nabla\theta$, since $\boldsymbol{F} \in \boldsymbol{L}^2(\Omega)$, according to the Proposition 1.2.5, we have a unique solution $(\boldsymbol{z}, \pi) \in \boldsymbol{H}^2(\Omega) \times \boldsymbol{H}^1(\Omega)$ which satisfies the estimate :

$$\|\boldsymbol{z}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Omega)}). \tag{1.2.23}$$

One can conclude that $(\boldsymbol{u}, \pi) = (\boldsymbol{z} + \nabla\theta, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ is a unique solution for the problem (1.2.1). Finally by combining the relations (1.2.23) and (1.2.21), we obtain the estimate (1.2.13). □

In order to finish the treatment of the Brinkman equations in the Hilbertian framework, we propose the following Corollary which gives an improvement of the regularity of the velocity. Indeed, we still impose the same regularities for the boundary conditions $\boldsymbol{h}$ and g but we consider the datum $\boldsymbol{f}$ in the dual space $(\boldsymbol{H}_0^2(\mathrm{div}, \Omega))'$ instead of $\boldsymbol{L}^2(\Omega)$.

Corollary 1.2.7. *We suppose Ω is of classe $\mathcal{C}^{2,1}$. Let $\boldsymbol{f} \in (\boldsymbol{H}_0^2(\mathrm{div}, \Omega))', g \in H^{\frac{3}{2}}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ satisfying* (1.2.19). *Then the non homogeneous Brinkman system* (1.2.1) *has a unique solution $(\boldsymbol{u}, \pi)$ which belongs to $\boldsymbol{H}^2(\Omega) \times L^2(\Omega)$.*

Proof. Note that $\boldsymbol{f} \in (\boldsymbol{H}_0^2(\mathrm{div}, \Omega))'$, then there exists $\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$ and $\chi \in L^2(\Omega)$ such that :

$$\boldsymbol{f} = \boldsymbol{\psi} + \nabla\chi. \tag{1.2.24}$$

By setting $\theta = \pi - \chi$, the non homgeneous Brinkman equations (1.2.1) becomes :

$$\begin{cases} -\gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + \nabla\theta = \boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\boldsymbol{u} = 0 \text{ in } \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} = g, \quad \mathbf{curl}\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} \qquad\qquad \text{on } \Gamma \end{cases} \tag{1.2.25}$$

Since $\boldsymbol{\psi} \in L^2(\Omega)$, according to Proposition 1.2.5, there exists a unique solution $(\boldsymbol{u}, \theta) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ for the problem (1.2.25).
We also derive the existence of the pressure π solution of the problem (1.2.1) defined by $\pi = \theta + \chi$ which finishes the proof of the Corollary 1.2.7.

□

1.2.2 $\boldsymbol{L}^p$-Theory

This subsection is devoted to the developement of an $\boldsymbol{L}^p$-Theory for the Brinkman problem (1.2.1), for all $1 < p < \infty$. We establish existence, uniqueness of weak and strong solutions. The proofs are based on the regularity results for the Stokes problem [8]. However, we give a complete proof in order to extend the results to the case of the non linear Brinkman-Forchheimer equations.

Proposition 1.2.8. *We suppose that* Ω *is of classe* $\mathcal{C}^{1,1}$, $p \geqslant 2$ *and let :*

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))', \;\; \chi \in L^p(\Omega), \;\; g \in W^{1-\frac{1}{p},p}(\Omega), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

with the compatibility condition (1.2.19). *Then, the Problem* (1.2.1) *has a unique solution* $(\boldsymbol{u}, \pi)$ *which belongs to* $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ *and satisfies the estimate,*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + \|\chi\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}).$$

Proof. For $p \geqslant 2$, we know that :
$(\boldsymbol{H}_0^{p'}(\mathrm{div}\,, \Omega))' \hookrightarrow (\boldsymbol{H}_0^{2}(\mathrm{div}\,, \Omega))'$, $L^p(\Omega) \hookrightarrow L^2(\Omega)$, $W^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma)$ and $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$. Then according to the hilbert case (see Corollary 1.2.3), there exists a weak solution $(\boldsymbol{u}, \pi) \in H^1(\Omega) \times L^2(\Omega)$ to problem (1.2.1), satifying the estimate (1.2.13).

Next, the regularity of the Brinkman problem will be deduced from the Stokes one. Indeed, observe that $a\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ for $2 \leqslant p \leqslant 6$.
Since, $\boldsymbol{L}^p(\Omega) \hookrightarrow (\boldsymbol{H}_0^{p'}(\mathrm{div}\,, \Omega))'$, using the argument of Stokes regularity [6, Remark 4.6] we deduce that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ for $2 \leqslant p \leqslant 6$. Now, we suppose that $p \geqslant 6$. Since $\boldsymbol{F} = \boldsymbol{f} - a\boldsymbol{u}$ belongs to $(\boldsymbol{H}_0^{p'}(\mathrm{div}\,, \Omega))'$, we have $\boldsymbol{u} \in \boldsymbol{W}^{1,6}(\Omega)$. Using the fact that $\boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$, we deduce again from the regularity of Stokes problem that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$. □

It remains to treat the case when $1 < p < 2$, which will be given in the following theorem.

Theorem 1.2.9. *We suppose that* Ω *is of class* $C^{1,1}$, *let* $1 < p < 2$, *we assume that:*

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))', \;\; \chi \in L^p(\Omega), \;\; g \in W^{1-\frac{1}{p},p}(\Omega), \boldsymbol{h} \in W^{-\frac{1}{p},p}(\Gamma), .$$

satifying the compatibility condition (1.2.19). *Then, the non homogeneous Brinkman problem* (1.2.1), *has a unique solution* $(\boldsymbol{u}, \pi)$ *which belongs to* $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

Proof. We define the space :

$$E^{p'}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p'}(\Omega); \;\; \Delta\boldsymbol{v} \in [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'\},$$

equipped with the norm :

$$\|\boldsymbol{v}\|_{E^{p'}(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\Delta \boldsymbol{v}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

First, we consider that $g = 0$ and we write the weak formulation of the non homogeneous Brinkman equations (1.2.1) as follow :

Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ where for any $(\boldsymbol{\psi}, q) \in E^{p'}(\Omega) \times L^{p'}(\Omega)$, we have :

$$\begin{aligned}\langle \boldsymbol{u}, -\gamma\Delta\boldsymbol{\psi} + a\boldsymbol{\psi} + \nabla q\rangle_{H_0^p(\mathrm{div},\Omega)\times(H_0^p(\mathrm{div},\Omega))'} - \int_\Omega \pi.\mathrm{div}\boldsymbol{\psi}\ dx = \langle \boldsymbol{f}, \boldsymbol{\psi}\rangle_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'\times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} \\ + \langle \boldsymbol{h}\times\boldsymbol{n}, \boldsymbol{\psi}\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} - \int_\Omega \chi\cdot q\, dx.\end{aligned}$$

Due to the Proposition 1.2.8, for any $(\boldsymbol{F}, \xi) \in (\boldsymbol{H}_0^p(\mathrm{div},\Omega))' \times L^{p'}(\Omega)$, the following problem :

$$\begin{cases} -\gamma\Delta\boldsymbol{\psi} + a\boldsymbol{\psi} + \nabla q = \boldsymbol{F} \quad \text{and} \quad \mathrm{div}\boldsymbol{\psi} = \xi \quad \text{in } \Omega \\ \boldsymbol{\psi}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\boldsymbol{\psi}\times\boldsymbol{n} = \mathbf{0} \qquad\qquad \text{on } \Gamma \end{cases} \tag{1.2.26}$$

has a unique solution $(\boldsymbol{\psi}, q) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}(\Omega)$ satisfying the estimate :

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|q\|_{L^{p'}(\Omega)} \leqslant C(\|\boldsymbol{F}\|_{(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'} + \|\xi\|_{L_0^{p'}(\Omega)}).$$

Furthermore, for any numbers $K, \mu \in \mathbb{R}$, we have :

$$\inf_{K,\mu\in\mathbb{R}}(\|\boldsymbol{\psi} + \mu\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|q + K\|_{L^{p'}(\Omega)}) \leqslant C(\|\boldsymbol{F}\|_{(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'} + \|\xi\|_{L_0^{p'}(\Omega)}). \tag{1.2.27}$$

Let $\boldsymbol{T}$ be a linear form defined from $(\boldsymbol{H}_0^p(\mathrm{div},\Omega))' \times L_0^{p'}(\Omega)$ onto $\mathbb{R}$ by :

$$\boldsymbol{T}\ :\ \langle \boldsymbol{F}, \xi\rangle \longmapsto \langle \boldsymbol{f}, \boldsymbol{\psi}\rangle_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \langle \boldsymbol{h}\times\boldsymbol{n}, \boldsymbol{\psi}\rangle_\Gamma - \int_\Omega \chi.q\ dx$$

Note that for any $K, \mu \in \mathbb{R}$:

$$|T(\boldsymbol{F}, \xi)| \leqslant |\langle \boldsymbol{f}, \boldsymbol{\psi} + \mu\rangle_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \langle \boldsymbol{h}\times\boldsymbol{n}, \boldsymbol{\psi} + \mu\rangle_\Gamma - \int_\Omega \chi.(q + K)\ dx|$$

Based on relation (1.2.27), we derive :

$$\begin{aligned}|T(\boldsymbol{F}, \xi)| \leqslant & C(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \|\chi\|_{L^p(\Omega)})\times \\ & (\|\boldsymbol{F}\|_{(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'} + \|\xi\|_{L_0^{p'}(\Omega)}).\end{aligned}$$

Consequently, the linear form T is continuous on $(\boldsymbol{H}_0^p(\mathrm{div},\Omega))' \times L_0^{p'}(\Omega)$.

According to Riesz representation's Theorem, we deduce a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}_0^p(\mathrm{div},\Omega)\times L^p(\Omega)$.

By writting the first equation of the non homogeneous Brinkman problem (1.2.1) as follow :

$$-\gamma\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{G}(\boldsymbol{u}) \text{ where } \boldsymbol{G}(\boldsymbol{u}) = \boldsymbol{f} - a\boldsymbol{u}$$

It follows that $\boldsymbol{G}(\boldsymbol{u}) \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$. According to theorem 4.2.6 in [55], we conclude the existence of a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

We consider $\theta \in \boldsymbol{W}^{2,p}(\Omega)$ to be a unique solution for the following Neumann problem :

$$\Delta\theta = \chi \ \text{ in}\,\Omega \quad \frac{\partial\theta}{\partial n} = g \text{ on } \Gamma$$

Note that $\boldsymbol{z} = \boldsymbol{u} - \nabla\theta$, one can observe that $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ which finishes the proof.

□

Proposition 1.2.10. *We suppoose Ω is of classe $\mathcal{C}^{2,1}$. Let we assume that $p \geqslant 2$ and :*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega),\ \chi \in \boldsymbol{W}^{1,p}(\Omega),\ g \in W^{2-\frac{1}{p},p}(\Gamma),\ \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma).$$

Then, Brinkman equations (1.2.1) *has exactly one solution* $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

Proof. The idea of the proof is to use the regularity results for the Stokes problem.
According to Proposition 1.2.8, there exists a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ solution of the Brinkman equations (1.2.1).
We consider two cases :

- Case $2 \leqslant p \leqslant 6$. Since $a\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, the result follows immediatly using the regularity for the Stokes problem [6, Remark 4.10]
- Case $p \geqslant 6$, we know that $\boldsymbol{u} \in \boldsymbol{W}^{2,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$.
  Again using the Stokes regularity, we deduce that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

□

Now, we prove the extension of the results given in the Corollary 1.2.7 from the Hilbertian case into the $\boldsymbol{L}^p$-theory. Here we keep the datum $\boldsymbol{f}$ in the same space $(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'$ while we improve the regularity of the boundary conditions.

Corollary 1.2.11. *We suppose that Ω is of class $C^{2,1}$ and Let we assume $\boldsymbol{f}$, g, $\boldsymbol{h}$ such that :*

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))', g \in W^{2-\frac{1}{p},p}(\Gamma), \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma),$$

and satifying the compatibility condition (1.2.6). *Then, the non homogeneous Brinkman equations* (1.2.1) *has a unique solution* $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega) \times L^p(\Omega)$.

Proof. Note that $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'$, it yields to obtain the functions $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ and $\chi \in L^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \nabla\chi$.
By setting $\theta = \pi - \chi$, we reach the non homogeneous problem (1.2.1) with a datum ψ.
Due to the Proposition 1.2.10, taking into account $\boldsymbol{\psi} \in L^p(\Omega)$, we derive a unique strong solution $(\boldsymbol{u},\theta) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$, it follows that $\pi = \theta + \chi \in L^p(\Omega)$, which completes the proof.

□

1.2.3 Extension for more general datum $\boldsymbol{f} \in (\boldsymbol{H}^{r',p'}(\mathrm{div},\Omega))'$

Here we need first to introduce the following spaces, for $1 < r, p < \infty$,

$$\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega) = \{\boldsymbol{\psi} \in \boldsymbol{L}^r(\Omega),\ \mathrm{div}\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega),\ \boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}, \tag{1.2.28}$$

with $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$, which is banach space for the norm :

$$\|\boldsymbol{\psi}\|_{\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)} = \|\boldsymbol{\psi}\|_{\boldsymbol{L}^r(\Omega)} + \|\nabla\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)}.$$

The space $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)$ and its dual space can be characterized as :

$$[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]' = \{\boldsymbol{\psi} + \nabla\boldsymbol{F},\quad \boldsymbol{\psi} \in \boldsymbol{L}^r(\Omega),\ \boldsymbol{F} \in \boldsymbol{L}^p(\Omega)\}. \tag{1.2.29}$$

In the sequel, we will investigate again the existence and uniqueness of a solution for the Brinkman problem i.e we will prove that we are able to reach the same results with a datum $\boldsymbol{f}$ belongs to $(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$ which can be observed as a space with lower regularity compared with $(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'$. We start with the Hilbetian case, then we move to check the results in more general spaces $1 < r,\ p < \infty$.

Corollary 1.2.12. *Let Ω of class $C^{1,1}$ and we assume a datum $\boldsymbol{f}$ belongs to $(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'$, then the homogeneous Brinkman problem has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$.*

Proof. Due to the relation (1.2.29),

$$\exists\, \boldsymbol{\psi} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega),\quad \exists\, \boldsymbol{F} \in \boldsymbol{L}^2(\Omega) \text{ such that } \ \boldsymbol{f} = \boldsymbol{\psi} + \nabla\boldsymbol{F}.$$

Using the Green's formula, in addition to the fact that the test function $\boldsymbol{v} \in \boldsymbol{V}_T^2(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we obtain

$$\begin{aligned}\langle \boldsymbol{f}\ ,\ \boldsymbol{v}\rangle_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'\times\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)} =& \langle\boldsymbol{\psi},\boldsymbol{v}\rangle_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)\times\boldsymbol{L}^6(\Omega)} - \langle\boldsymbol{F}\ ,\ \mathrm{div}\,\boldsymbol{v}\rangle_{\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)} \\ &+ \langle\boldsymbol{F}\ ,\ \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{H^{\frac{1}{2}}(\Gamma)\times H^{-\frac{1}{2}}(\Gamma)}.\end{aligned}$$

Moreover, because of the function $\boldsymbol{v} \in \boldsymbol{V}_T^2(\Omega)$, it follows immediately that,

$$\langle\boldsymbol{f},\boldsymbol{v}\rangle_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'\times\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)} = \langle\boldsymbol{\psi},\boldsymbol{v}\rangle_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)\times\boldsymbol{L}^6(\Omega)}. \tag{1.2.30}$$

From the relation (1.2.4), one can observe that (1.2.30) is a linear continuous form on the space $(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'$.
Using exactly the same arguments in the Proposition 1.2.1, we can apply Lax Milgram Theorem in order to prove the desired result. $\square$

Now, we move to prove the extension of the results from the Hilbertian framework into the $\boldsymbol{L}^p$ theory,

Corollary 1.2.13. *We assume that Ω is of class $C^{1,1}$ and let we suppose that $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$, then the homogeneous Brinkman problem* (1.2.1) *has a unique solution* $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$

Proof. Because of $\boldsymbol{f}$ belongs to $(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$, by relation (1.2.29), there exist $\boldsymbol{f}_1 \in \boldsymbol{L}^r(\Omega)$, $\boldsymbol{f}_2 \in \boldsymbol{L}^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{f}_1 + \nabla \boldsymbol{f}_2$.

On the one hand, we consider a function $\nabla \boldsymbol{f}_2 \in (\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'$ to be a datum for the homogeneous Brinkman equations (1.2.1). According to proposition 1.2.8, we derive a unique solution $(\boldsymbol{u}_2,\pi_2) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

On the other hand, we observe $\boldsymbol{f}_1$ as a datum for the homogeneous Brinkman equations (1.2.1), by Proposition 1.2.10, we conclude the solution $(\boldsymbol{u}_1,\pi_1) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$.
Using the fact that $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow W^{1,p}(\Omega)$, where $r > \frac{6}{5}$ and $p > 2$, it infers $(\boldsymbol{u}_1,\pi_1) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$
Consequently, we deduce a unique solution $(\boldsymbol{u},\pi) = (\boldsymbol{u}_1+\boldsymbol{u}_2, \pi_1+\pi_2) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ for the homogeneous problem (1.2.1). □

Corollary 1.2.14. *Let Ω is of class $C^{1,1}$ and we assume $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$, then the homogeneous problem* (1.2.1) *has a unique solution* $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,r}(\Omega) \times L^p(\Omega)$.

Proof. Because the datum $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$. So we note that $\boldsymbol{f} = \boldsymbol{F} + \nabla \boldsymbol{\psi}$, where $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$ and $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$.
By taking $\theta = \pi - \psi$, due to the Proposition 1.2.10 the following system,

$$\begin{cases} -\Delta \boldsymbol{u}_0 + a\boldsymbol{u}_0 + \nabla\theta = \boldsymbol{F}, \quad \text{and} \quad \mathrm{div}\boldsymbol{u} = 0 \quad \text{in } \Omega. \\ \boldsymbol{u}_0 \cdot \boldsymbol{n} = 0, \ \mathbf{curl}\boldsymbol{u}_0 \times \boldsymbol{n} = 0 \qquad\qquad\qquad \text{on } \Gamma. \end{cases} \tag{1.2.31}$$

has a unique strong solution $(\boldsymbol{u}_0,\theta) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$. Then we can conclude that $\pi = \theta + \psi \in L^p(\Omega)$. □

In the next Theorem, we treat the case where the divergence is not free.

Theorem 1.2.15. *We suppose that Ω is of class $C^{1,1}$ and let we assume having :*

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))', \chi \in L^p(\Omega), g \in W^{1-\frac{1}{p},p}(\Gamma), \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$$

Then there exists a unique solution $(\boldsymbol{u},\pi) \in W^{1,p}(\Omega) \times L^p(\Omega)$ for the steady Brinkman equations (1.2.1).

Proof. We consider first $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$ to be a datum for the problem (1.2.1).
Case 1 $\boldsymbol{\chi} = 0$:
On the one hand, because the test function $\boldsymbol{v} \in V_T^p(\Omega)$, by the relation (1.2.29), we have,

$$\langle \nabla \boldsymbol{F}, \boldsymbol{v} \rangle_{(H_0^{p'}(\mathrm{div},\Omega))' \times H_0^{p'}(\mathrm{div},\Omega)} = 0.$$

According to the Proposition 1.2.8, the following problem :

$$\begin{cases} -\gamma\Delta \boldsymbol{u}_1 + a\boldsymbol{u}_1 + \nabla\pi_1 = \nabla \boldsymbol{F} \quad \text{and} \quad \text{div}\boldsymbol{u}_1 = 0 \quad \text{in } \Omega \\ \boldsymbol{u}_1\cdot\boldsymbol{n} = g, \ \text{curl}\boldsymbol{u}_1\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} \qquad\qquad \text{on } \Gamma. \end{cases}$$

has a unique solution $(\boldsymbol{u}_1, \pi_1) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$.

On the other hand, one can observe $\boldsymbol{\psi} \in \boldsymbol{L}^r(\Omega)$ as a datum for the homogeneous problem (1.2.1), due to the Proposition 1.2.10, it infers a unique strong solution $(\boldsymbol{u}_2, \pi_2)$ which belongs to $\boldsymbol{W}^{2,r}(\Omega)\times \boldsymbol{W}^{1,r}(\Omega)$, where $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and $r > \frac{6}{5}$ because of $p > 2$.
Finally, by setting $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_1+\boldsymbol{u}_2, \pi_1+\pi_2)$, we reach a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$ for a non homogeneous Brinkman problem (1.2.1).

Case 2 $\boldsymbol{\chi} \neq 0$:
We proceed similarly as the previous case. In fact, the slight difference appears when we take the function $\nabla\boldsymbol{F}$ belongs to $(\boldsymbol{H}_0^{p'}(\text{div},\Omega))'$ as a datum for the following Brinkman equations :

$$\begin{cases} -\gamma\Delta \boldsymbol{u}_1 + a\boldsymbol{u}_1 + \nabla\pi_1 = \nabla \boldsymbol{F} \quad \text{and} \quad \text{div}\boldsymbol{u}_1 = \chi \text{ in } \Omega \\ \boldsymbol{u}_1\cdot\boldsymbol{n} = g, \ \textbf{curl}\boldsymbol{u}_1\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} \qquad\qquad \text{on } \Gamma. \end{cases}$$

According to the Proposition 1.2.8, the problem above has a unique solution $(\boldsymbol{u}_1, \pi_1) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$,
Next, the function $\boldsymbol{\psi} \in \boldsymbol{L}^r(\Omega)$ can be viewed like a datum for the homogeneous problem (1.2.1), using the same arguments in the previous case, we obtain a unique solution $(\boldsymbol{u}_2, \pi_2) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$.

In conclusion, we reach the desired result by considering $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_1 + \boldsymbol{u}_2, \pi_1 + \pi_2)$ which belongs to $\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)$. □

1.3 Darcy Brinkman Forchheimer equations with Navier-type's boundary conditions

In this section, we are interested to study the Darcy-Brinkman-Forchheimer equations (1.3.59)-(1.0.2). In order to get this purpose, it requires first to have many results around the linearized problem which can be deduced based on the first section.

This section is divided into two subsections. The first subsection is preserved to examine the linearized problem. So, we check the existence of a weak solution under the both of the Hilbertian case and the Theory L^p. The subsection 2 is devoted to study the nonlinear Brinkman Forchheimer equations.

1.3.1 Linearized Darcy Brinkman Forchheimer problem

In this subsection, we study the existence of generalized weak solution for the linearized Brinkman Forchheimer equations :

$$\begin{cases} -\gamma\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{d}|\,\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega. \\ \boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{0}, \quad \mathbf{curl}\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}, \qquad\qquad \text{on } \Gamma, \end{cases} \tag{1.3.1}$$

where $\boldsymbol{d}$ is a given function.

Throughout this section, we prove first the existence of the weak solution for the linearized problem, by using Lax Milgram Theorem and Rham's Theorem when we take a datum $\boldsymbol{f}$ in $(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'$. Next, a wide study of the regularity results for the linearized Brinkman Forchheimer equations will be discussed.

Proposition 1.3.1. *We suppose Ω to be of class $C^{1,1}$ and let $\boldsymbol{d}\in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, $\boldsymbol{f}\in (\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'$, and $\boldsymbol{h}\in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$. Then the linearized problem* (1.3.1) *has a unique solution $(\boldsymbol{u},\pi)\in \boldsymbol{V}_T^2(\Omega)\times L^2(\Omega)/\mathbb{R}$ which satisfies the estimates :*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}). \tag{1.3.2}$$

$$\|\pi\|_{L^2(\Omega)/\mathbb{R}} \leqslant C(1+\|\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)})(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}). \tag{1.3.3}$$

Moreover, if $\boldsymbol{f}\in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ and $\boldsymbol{h}\in \boldsymbol{W}^{\frac{1}{6},\frac{6}{5}}(\Gamma)$, then there exists a unique pair $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{2,\frac{6}{5}}(\Omega)\times W^{1,\frac{6}{5}}(\Omega)$.

Proof. The problem : Find $\boldsymbol{v}\in \boldsymbol{V}_T^2(\Omega)$ solution of (1.3.1) is equivalent to the variational formulation

$$\begin{cases} \text{Find } \boldsymbol{u}\in \boldsymbol{V}_T^2(\Omega), \text{ such that} \\ a(\boldsymbol{u},\boldsymbol{v}) = l(\boldsymbol{v}), \text{ for any } \boldsymbol{v}\in \boldsymbol{V}_T^2(\Omega), \end{cases} \tag{1.3.4}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = \gamma\int_\Omega \mathbf{curl}\boldsymbol{u}\cdot\mathbf{curl}\boldsymbol{v}\,dx + a\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx + b\int_\Omega |\boldsymbol{d}|\boldsymbol{u}\cdot\boldsymbol{v}\,dx.$$

$$l(\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle_{(\boldsymbol{H}^{6,2}(\mathrm{div},\Omega))'\times\boldsymbol{H}^{6,2}(\mathrm{div},\Omega)} + \gamma\langle \boldsymbol{h}\times\boldsymbol{n},\boldsymbol{v}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}.$$

It is easy to verify that $l(\boldsymbol{v})$ is continuous linear form on $\boldsymbol{V}_T^2(\Omega)$. Moreover, since $\boldsymbol{d}\in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, the bilinear form $a(.,.)$ is continuous on $\boldsymbol{V}_T^2(\Omega)\times\boldsymbol{V}_T^2(\Omega)$. Next, since

$$a(\boldsymbol{u},\,\boldsymbol{u}) = \gamma\|\mathbf{curl}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + a\int_\Omega \boldsymbol{u}^2\,dx + b\int_\Omega |d||\boldsymbol{u}|^2\,dx \ge C\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)},$$

the bilinear form $a(.,.)$ is elliptic on $\boldsymbol{V}_T^2(\Omega)\times\boldsymbol{V}_T^2(\Omega)$. By applying Lax-Milgram Theorem, we derive a unique solution $\boldsymbol{u}\in\boldsymbol{V}_T^2(\Omega)$ for the problem (1.3.4). For the pressure, we note that for any $\boldsymbol{v}\in\mathcal{D}(\Omega)$ with $\operatorname{div}\boldsymbol{v}=0$, we have:

$$\langle -\gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{d}|\boldsymbol{u} - \boldsymbol{f},\boldsymbol{v}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = 0.$$

Then, by using Rham's Theorem, there exists $\pi \in L^2(\Omega)$ unique up to an additive constant such that:

$$-\gamma\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{d}|\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{in } \Omega.$$

Using the variational formulation and sobolev embedding, we obtain:(1.3.2) by

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + b\int_\Omega |d||\boldsymbol{u}|^2\, dx \leqslant C(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}),$$

where C is a constant depending on Ω, γ and a. On the other hand

$$\|\nabla\pi\|_{H^{-1}(\Omega)} \leqslant \|\boldsymbol{f}\|_{H^{-1}(\Omega)} + \gamma\|\Delta\boldsymbol{u}\|_{H^{-1}(\Omega)} + a\|\boldsymbol{u}\|_{H^{-1}(\Omega)} + b\||\boldsymbol{d}|\boldsymbol{u}\|_{H^{-1}(\Omega)}.$$

Using the fac that:

$$\|\boldsymbol{f}\|_{H^{-1}(\Omega)} \leqslant C\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'}$$

$$\|\Delta\boldsymbol{u}\|_{H^{-1}(\Omega)} + \|\boldsymbol{u}\|_{H^{-1}(\Omega)} \leqslant C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)},$$

$$\||\boldsymbol{d}|\boldsymbol{u}\|_{H^{-1}(\Omega)} \leqslant C\||\boldsymbol{d}|\boldsymbol{u}\|_{L^{\frac{6}{5}}(\Omega)} \leqslant C\|\boldsymbol{d}\|_{L^{\frac{3}{2}}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \leqslant C\|\boldsymbol{d}\|_{L^{\frac{3}{2}}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)},$$

we obtain,

$$\|\pi\|_{L^2(\Omega)} = \inf_{c\in\mathbb{R}}\|\pi + c\|_{L^2(\Omega)} \leq C(1 + \|\boldsymbol{d}\|_{L^{\frac{3}{2}}(\Omega)})(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}). \quad (1.3.5)$$

If $\boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega) \hookrightarrow (\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'$ and $\boldsymbol{h} \in \boldsymbol{W}^{\frac{1}{6},\frac{6}{5}}(\Gamma) \hookrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, then we have $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$. Thanks to the fact that $\boldsymbol{d} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, we have $\boldsymbol{f} - |\boldsymbol{d}|\boldsymbol{u} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$. Using the regularity of the Brikman problem (see Proposition 1.2.10), we deduce that $(\boldsymbol{u},\pi)$ belongs to $\boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times W^{1,\frac{6}{5}}(\Omega)$. □

Now, we want to prove some regularity results for problem (1.3.1) in L^p-theory, $1 < p < \infty$. These results will be used in the next subsection to show the existence and uniqueness of weak solution for the non linear Brinkman Forchheimer problem by using fixed point theorem. We begin with the case of strong solutions when $p \geqslant \frac{6}{5}$.

Theorem 1.3.2. *Let Ω is of class $C^{2,1}$ and we assume $p \geqslant \frac{6}{5}$,*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \;\; \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \;\; \text{and} \;\; \boldsymbol{d} \in \boldsymbol{L}^s(\Omega) \;\; \text{with}$$

$$s = \frac{3}{2} \;\; \text{if } p < \frac{3}{2}, \;\; s = \frac{3}{2} + \varepsilon \;\; \text{if } \; p = \frac{3}{2} \text{ and } \; s = p \;\; \text{if } \; p > \frac{3}{2},$$

for some arbitrary $\epsilon > 0$. Then, there problem (1.3.1) has a unique solution $(\boldsymbol{u},\pi)$ in $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ satisfying the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leqslant C(1 + \|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)})(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}). \quad (1.3.6)$$

Proof. We use the same arguments in [4]. We know by Proposition 1.3.1, that $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times W^{1,\frac{6}{5}}(\Omega)$. We consider a sequence $\boldsymbol{d}_\lambda \in \mathcal{D}(\Omega)$ such that $\boldsymbol{d}_\lambda \hookrightarrow \boldsymbol{d} \in \boldsymbol{L}^s(\Omega)$. Then, we seek for $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega)$ solution of problem

$$\begin{cases} -\gamma\Delta \boldsymbol{u}_\lambda + a\boldsymbol{u}_\lambda + b|\boldsymbol{d}_\lambda|\boldsymbol{u}_\lambda + \nabla\pi_\lambda = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u}_\lambda = 0 \quad \text{in } \Omega \\ \boldsymbol{u}_\lambda \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\boldsymbol{u}_\lambda \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \qquad \text{on } \Gamma. \end{cases} \tag{1.3.7}$$

As above, one can obtain a unique solution $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times W^{1,\frac{6}{5}}(\Omega)$ for the problem (1.3.7). Now, if $\frac{6}{5} \leqslant p \leqslant 6$, then $\boldsymbol{f} - b|\boldsymbol{d}_\lambda|\boldsymbol{u}_\lambda \in \boldsymbol{L}^6(\Omega)$. Using the regularity of the Brinkman equations (see Proposition 1.2.10), we conclude that $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Next, if $p \geqslant 6$, we still have a unique solution $\boldsymbol{u}_\lambda \in \boldsymbol{W}^{2,6}(\Omega)$ and then $|\boldsymbol{d}_\lambda|\boldsymbol{u}_\lambda$ belongs to $\boldsymbol{L}^\infty(\Omega)$. Using again the regularity for the Brinkman proplem, we conclude that $(\boldsymbol{u}_\lambda, \pi_\lambda)$ belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies:

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \leqslant C(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} + \||\boldsymbol{d}_\lambda|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}). \tag{1.3.8}$$

Next, the proof of estimate (1.3.6) is based on obtaining of strong estimates for $(\boldsymbol{u}_\lambda, \pi_\lambda)$ independently of λ.

Let $\varepsilon > 0$ with $0 < \lambda < \varepsilon/2$. We consider:

$$\boldsymbol{d}_\lambda = \boldsymbol{d}_1^\varepsilon + \boldsymbol{d}_{\lambda,2}^\varepsilon \quad \text{where} \quad \boldsymbol{d}_1^\varepsilon = \widetilde{\boldsymbol{d}} \star \rho_{\varepsilon/2} \quad \text{and} \quad \boldsymbol{d}_{\lambda,2}^\varepsilon = \boldsymbol{d}_\lambda - \boldsymbol{d}_1^\varepsilon, \tag{1.3.9}$$

being $\widetilde{\boldsymbol{d}}$ the extension by zero of $\boldsymbol{d}$ to $\mathbb{R}^3$ and $\rho_{\varepsilon/2}$ the classical mollifier. Using the decomposition (1.3.9), we want to bound the term $\||\boldsymbol{d}_\lambda|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}$. Using the Hölder inequality and the continuous imbedding

$$\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{L}^m(\Omega),$$

with $\frac{1}{m} = \frac{1}{p} - \frac{2}{3}$ if $p < \frac{3}{2}$, for any $m \geq 1$ if $p = \frac{3}{2}$ and for any $m \in [1, \infty]$ if $p > \frac{3}{2}$, we obtain

$$\||\boldsymbol{d}_{\lambda,2}^\varepsilon|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{d}_{\lambda,2}^\varepsilon\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^m(\Omega)} \leq C\|\boldsymbol{d}_{\lambda,2}^\varepsilon\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}, \tag{1.3.10}$$

where $\frac{1}{m} = \frac{1}{p} - \frac{1}{s}$, which is well defined because the definition of the real number s. Observe that

$$\|\boldsymbol{d}_{\lambda,2}^\varepsilon\|_{\boldsymbol{L}^s(\Omega)} \leq \|\boldsymbol{d}_\lambda - \boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d} - \widetilde{\boldsymbol{d}} \star \rho_{\varepsilon/2}\|_{\boldsymbol{L}^s(\Omega)} \leq \lambda + \varepsilon/2 \leq \varepsilon. \tag{1.3.11}$$

Then, we obtain

$$\||\boldsymbol{d}_{\lambda,2}^\varepsilon|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leq C\varepsilon\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}, \tag{1.3.12}$$

For $\boldsymbol{d}_1^\epsilon$, we consider two cases.

i) Case $p \leq 3/2$. Let $r \in [\frac{3}{2}, \infty]$ such that $\frac{1}{p} = \frac{1}{r} + \frac{1}{6}$ and $t \geq 1$ such that $1 + \frac{1}{r} = \frac{2}{3} + \frac{1}{t}$ satisfying:

$$\begin{aligned} \||\boldsymbol{d}_1^\varepsilon|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} &\leq \|\boldsymbol{d}_1^\varepsilon\|_{\boldsymbol{L}^r(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^6(\Omega)} \\ &\leq \|\boldsymbol{d}\|_{\boldsymbol{L}^{3/2}(\Omega)}\|\rho_{\varepsilon/2}\|_{\boldsymbol{L}^t(\mathbb{R}^3)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^6(\Omega)}. \end{aligned}$$

Using the estimate (1.3.2), we have

$$\||\boldsymbol{d}_1^{\varepsilon}|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leq C_\varepsilon \|\boldsymbol{d}\|_{\boldsymbol{L}^{3/2}(\Omega)}\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}\Big). \tag{1.3.13}$$

Thanks to the following imbeddings

$$\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{L}^{6/5}(\Omega), \qquad \boldsymbol{W}^{1-1/p,p}(\Gamma) \hookrightarrow \boldsymbol{H}^{-1/2}(\Gamma),$$

we obtain that

$$\||\boldsymbol{d}_1^{\varepsilon}|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leq C_\varepsilon \|\boldsymbol{d}\|_{\boldsymbol{L}^{s}(\Omega)}\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{p}(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\Big). \tag{1.3.14}$$

Using (1.3.12) and (1.3.14), we deduce from (1.3.8) by choosing $\epsilon > 0$ small enough:

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi_\lambda\|_{W^{1,p}(\Omega)/\mathbb{R}} \leqslant C(1+\|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)})(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}). \tag{1.3.15}$$

ii) Case $p > 3/2$. Recall the imbedding:

$$\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{2,3/2}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega).$$

As a consequence, for any ε' there exists $C'_\varepsilon > 0$ such that

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^\infty(\Omega)} \leq \varepsilon' \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + C_{\varepsilon'}\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^6(\Omega)}. \tag{1.3.16}$$

Moreover, we have

$$\||\boldsymbol{d}_1^{\epsilon}|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{d}_1^{\epsilon}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^\infty(\Omega)}. \tag{1.3.17}$$

Thanks to (1.3.16), (1.3.2), we deduce from (1.3.17)

$$\||\boldsymbol{d}_1^{\epsilon}|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leq \epsilon'\|\boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + C'_\varepsilon\|\boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)}\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\Big) \tag{1.3.18}$$

Using (1.3.18) and (1.3.12) in (1.3.8), choosing $\epsilon > 0$ small enough and $\epsilon' > 0$ such that $\epsilon'\|\boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)} < \frac{1}{2}$ give the estimate (1.3.15), where we replace $\boldsymbol{L}^p(\Omega)$ by $\boldsymbol{L}^s(\Omega)$ because in this case $s = p$.

The estimate (1.3.15) is uniform on λ, and therefore we can extract subsequences, that we still call $(\boldsymbol{u}_\lambda)_\lambda$ and $(\pi_\lambda)_\lambda$ such that if $\lambda \to 0$,

$$\boldsymbol{u}_\lambda \rightharpoonup \boldsymbol{u} \text{ weakly in } \boldsymbol{W}^{2,p}(\Omega).$$

and for any $k_\lambda \in \mathbb{R}$, then

$$\pi_\lambda + k_\lambda \rightharpoonup \pi \text{ weakly in } W^{1,p}(\Omega).$$

Therefore, $(\boldsymbol{u}, \pi)$ is a unique solution for the linearized problem (1.3.1) which satisfies the estimate (1.3.6). □

The following result gives the existence of generalized solution for $p \geq 2$ in the case of non homogenuous boundary condition and the case where the divergence does not vanish. Indeed, we consider the non homogeneous linearized Darcy Brinkman problem :

$$\begin{cases} -\gamma\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{d}|\,\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = \chi \quad \text{in } \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} = g, \quad \textbf{curl}\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} \qquad\qquad \text{on } \Gamma. \end{cases} \tag{1.3.19}$$

Theorem 1.3.3. *Let* Ω *is of class* $C^{1,1}$. *Suppose that* $p \geq 2$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$. *Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\operatorname{div},\Omega))'$, $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, $\chi \in W^{1,r}(\Omega)$, $g \in W^{1-\frac{1}{p},p}(\Omega)$ *where* $\boldsymbol{d} \in \boldsymbol{L}^s(\Omega)$ *such that :*

$$s = \frac{3}{2} \quad \text{if } 2 \leqslant p < 3, \quad s = \frac{3}{2} + \varepsilon \quad \text{if} \quad p = 3, \quad s = r \quad \text{if } p > 3.$$

Then, there exists a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ *for the problem* (1.3.19) *which satisfies the estimate :*

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leqslant C\Big(1 + \|\boldsymbol{d}\|_{L^s(\Omega)}\Big)^2 \Big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\operatorname{div},\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \\ + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\chi\|_{W^{1,r}(\Omega)}\Big). \end{aligned} \tag{1.3.20}$$

Proof. Taking into account that $\boldsymbol{f} \in (\boldsymbol{H}^{r',p'}(\operatorname{div},\Omega))'$ with $p \geq 2$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$, the Theorem 1.2.15 assures the existence of a unique solution $(\boldsymbol{u}_0,\pi_0) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ for the following linear Brinkman problem :

$$\begin{cases} -\gamma\Delta \boldsymbol{u}_0 + a\boldsymbol{u}_0 + \nabla\pi_0 = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u}_0 = \chi \quad \text{in } \Omega \\ \boldsymbol{u}_0\cdot\boldsymbol{n} = g, \quad \textbf{curl}\,\boldsymbol{u}_0\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} \qquad\qquad \text{on } \Gamma, \end{cases}$$

which satisfies the estimate :

$$\begin{aligned} \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_0\|_{L^p(\Omega)/\mathbb{R}} \leqslant C\Big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\operatorname{div},\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \\ + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\chi\|_{W^{1,r}(\Omega)}\Big). \end{aligned} \tag{1.3.21}$$

Moreover, we consider the following linearized Brinkman Forchheimer problem

$$\begin{cases} -\gamma\Delta \boldsymbol{z} + a\boldsymbol{z} + b|\boldsymbol{d}|\boldsymbol{z} + \nabla\pi_1 = -b|\boldsymbol{d}|\,\boldsymbol{u}_0 \quad \text{and} \quad \operatorname{div}\boldsymbol{z} = 0 \quad \text{in} \quad \Omega \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0, \quad \operatorname{curl}\boldsymbol{z}\times\boldsymbol{n} = 0 \qquad\qquad \text{on} \quad \Gamma \end{cases} \tag{1.3.22}$$

First case: $2 \leqslant p < 3$. We have $\boldsymbol{u}_0 \in \boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p*}(\Omega)$ with $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$. Moreover, $\boldsymbol{d} \in \boldsymbol{L}^{3/2}(\Omega)$. This gives that $|\boldsymbol{d}|\boldsymbol{u}_0 \in L^r(\Omega)$.

Second case: $p = 3$. We have $\boldsymbol{u}_0 \in L^{p*}(\Omega)$, for any $p* < \infty$. Since $\boldsymbol{d} \in \boldsymbol{L}^s(\Omega)$ with $s = \frac{3}{2} + \varepsilon$, we have $|\boldsymbol{d}|\boldsymbol{u}_0 \in L^r(\Omega)$.

Last case: $p > 3$. We have $\boldsymbol{u}_0 \in \boldsymbol{L}^\infty(\Omega)$ and since $\boldsymbol{d} \in \boldsymbol{L}^r(\Omega)$ we deduce that $|\boldsymbol{d}|\,\boldsymbol{u}_0 \in L^r(\Omega)$.

In view of the previous cases and due to the Proposition 1.2.10, it follows immediately that the problem (1.3.22) has an unique solution $(\boldsymbol{z}, \pi_1) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$.
We recall that $r > \frac{6}{5}$ because of $p > 2$ which infers that $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$.
We consider that $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_0 + \boldsymbol{z}, \pi_0 + \pi_1) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ to be a solution for (1.3.19), which completes the proof of the existence.

Now, in order to check the estimate (1.3.20), we proceed as follow.
Based on the previous Theorem, it yields to get the estimate for the solution $(\boldsymbol{z}, \pi_1)$ of a problem (1.3.22) as follow :

$$\begin{aligned}\|\boldsymbol{z}\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|\pi_1\|_{W^{1,r}(\Omega)} &\leqslant C(1 + \|\boldsymbol{d}\|\|_{\boldsymbol{L}^s(\Omega)})\||\boldsymbol{d}|\boldsymbol{u}_0\|_{\boldsymbol{L}^r(\Omega)}.\\ &\leqslant C(1 + \|\boldsymbol{d}\|\|_{\boldsymbol{L}^s(\Omega)})\|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^{p*}(\Omega)}.\\ &\leqslant C(1 + \|\boldsymbol{d}\|\|_{L^s(\Omega)})\|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}. \end{aligned} \tag{1.3.23}$$

Due to estimate (1.3.21), we deduce the following realtion :

$$\begin{aligned}\|\boldsymbol{z}\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|\pi_1\|_{W^{1,r}(\Omega)} \leqslant C\|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)}\Big(1 + \|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)}\Big)\Big(&\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\\ &+ \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\chi\|_{W^{1,r}(\Omega)}\Big).\end{aligned} \tag{1.3.24}$$

The estimate (1.3.20) follows by combining the above inequality and (1.3.21). □

Now, The next Theorem investigates the existence of a solution for the linearized problem (1.3.1) in the case when $\frac{3}{2} < p < 2$.

Theorem 1.3.4. *Let Ω to be of class $C^{1,1}$ and we assume $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'$, $\boldsymbol{h} \in \boldsymbol{W}^{\frac{-1}{p},p}(\Gamma)$, then the problem (1.3.1) has exactely a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^{p*}(\Omega) \times L^p(\Omega)$.*

Proof. **A**/ We introduce the space :

$$M(\Omega) = \{(\boldsymbol{w}, \theta) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}(\Omega),\ \mathrm{div}\boldsymbol{w} \in \boldsymbol{W}^{1,p*'}(\Omega),\ \boldsymbol{w}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\boldsymbol{w}\times\boldsymbol{n} = 0\}$$

Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ a solution of the system (1.3.1) is equivalent to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ with $\mathrm{div}\boldsymbol{u} = 0$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ such that :

For any $(\boldsymbol{w}, \theta) \in M(\Omega)$:

$$\begin{aligned}\langle \boldsymbol{u}, -\gamma\Delta\boldsymbol{w} + a\boldsymbol{w} - b|\boldsymbol{d}|\boldsymbol{w} + \nabla\theta\rangle_{\Omega p*,p} - \int_\Omega \pi.\mathrm{div}\boldsymbol{w}\ dx = &\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\Omega\, r',p'}\\ &+ \gamma\langle \boldsymbol{h}\times\boldsymbol{n}, \boldsymbol{w}\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}.\end{aligned} \tag{1.3.25}$$

Where the brackets $\langle.\,,\,.\rangle_{\Omega p*,p}$ represents the duality $\langle.\,,\,.\rangle_{\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega)\times(\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))'}$ and $\langle\cdot,\cdot\rangle_{\Omega r',p'}$ denotes the duality $\langle.\,,\,.\rangle_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'\times\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)}$.

Let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ be a solution of the linearized problem (1.3.1) and let consider $(\boldsymbol{w}, \theta) \in M(\Omega)$, according to Green's formula, it yields :

$$\begin{aligned}
-\gamma\langle \Delta \boldsymbol{u}, \boldsymbol{w}\rangle_{\Omega r',p'} &= \gamma \int_\Omega \mathbf{curl}\boldsymbol{u} \cdot \mathbf{curl}\boldsymbol{w}\, dx - \gamma\langle \mathbf{curl}\boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{w}\rangle_\Gamma \\
&= \gamma \int_\Omega \mathbf{curl}\boldsymbol{u} \cdot \mathbf{curl}\boldsymbol{w}\, dx - \gamma\langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{w}\rangle_\Gamma. \\
&= \gamma\langle \boldsymbol{u}, \mathbf{curl}\mathbf{curl}\boldsymbol{w}\rangle_{\Omega p*,p} - \gamma\langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{w}\rangle_\Gamma.
\end{aligned}$$

Because of $\Delta \boldsymbol{w} = \nabla(\mathrm{div}\boldsymbol{w}) - \mathbf{curl}\mathbf{curl}\boldsymbol{w}$, it follows that :

$$-\gamma\langle \Delta \boldsymbol{u}, \boldsymbol{w}\rangle_{\Omega r',p'} = \gamma\langle \boldsymbol{u}, -\Delta\boldsymbol{w} + \nabla \mathrm{div}\boldsymbol{w}\rangle_{\Omega p*,p} - \gamma\langle \boldsymbol{h} \times \boldsymbol{n}\ ,\ \boldsymbol{w}\rangle_\Gamma.$$

On the other hand, since $\mathrm{div}\boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ then :

$$\langle \boldsymbol{u}, \nabla \mathrm{div}\boldsymbol{w}\rangle_{\Omega p*,p} = -\langle \mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{w}\rangle + \langle \boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{w}\rangle_\Gamma = 0.$$

Which gives that :

$$-\gamma\langle \Delta \boldsymbol{u}, \boldsymbol{w}\rangle_{\Omega r',p'} = \gamma\langle \boldsymbol{u}, -\Delta\boldsymbol{w}\rangle_{\Omega p*,p} - \gamma\langle \boldsymbol{h} \times \boldsymbol{n}\ ,\ \boldsymbol{w}\rangle_\Gamma. \tag{1.3.26}$$

Next, one can observe that :

$$\int_\Omega \boldsymbol{u}(a\boldsymbol{w})\, dx = a \int_\Omega \boldsymbol{u}\ \boldsymbol{w}\, dx.$$

Now because of $\frac{1}{s} + \frac{1}{p*} + \frac{1}{p'*} = 1$, the following integral is well defined :

$$b\langle |\boldsymbol{d}|\boldsymbol{u}, \boldsymbol{w}\rangle_{\Omega r',p'} = b \int_\Omega |\boldsymbol{d}|\boldsymbol{u} \cdot \boldsymbol{w}\, dx = b \int_\Omega |\boldsymbol{d}|\boldsymbol{w} \cdot \boldsymbol{u}\, dx.$$

Moreover, because of the fact that $\boldsymbol{w} \cdot \boldsymbol{n} = 0$ on Γ, we obtain :

$$\langle \nabla \pi, \boldsymbol{w}\rangle_{\Omega r',p'} = -\int_\Omega \pi \mathrm{div}\boldsymbol{w}\, dx.$$

Furthermore, according to $\mathrm{div}\boldsymbol{u} = 0 \,\mathrm{in}\, \Omega$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, we have :

$$0 = -\int_\Omega \theta \mathrm{div}\boldsymbol{u}\, dx = \langle \boldsymbol{u}, \nabla\theta\rangle_{\Omega p*,p}. \tag{1.3.27}$$

We recall that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p*}(\Omega)$, which infers $\boldsymbol{u} \in \boldsymbol{H}_0^{p*,p}(\mathrm{div}, \Omega)$. In order to arrange our work, we have obtained that :

$$\begin{aligned}
\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\Omega r',p'} = \gamma\langle \boldsymbol{u}, -\Delta\boldsymbol{w}\rangle_{\Omega p*,p} &+ a \int_\Omega \boldsymbol{u}\ \boldsymbol{w}\, dx + b \int_\Omega |\boldsymbol{d}|\boldsymbol{w}\ \boldsymbol{u} \\
&- \int_\Omega \pi \mathrm{div}\boldsymbol{w}\, dx - \gamma\langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{w}\rangle_\Gamma.
\end{aligned} \tag{1.3.28}$$

Consequently the pair $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^{p*}(\Omega) \times L^p(\Omega)$ represent a solution for the problem (1.3.25).
B/ Now, we consider $\lambda > 0$, $\boldsymbol{h}_\lambda \in C^\infty(\Gamma)$ and $\boldsymbol{f}_\lambda \in \mathcal{D}(\Omega)$. by applying the density of $C^\infty(\Gamma)$ in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, we have $\boldsymbol{h}_\lambda \longrightarrow \boldsymbol{h}$ in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$.
Moreover, using the density of $\mathcal{D}(\Omega)$ in $(\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega))'$, we observe $\boldsymbol{f}_\lambda \longrightarrow \boldsymbol{f}$ in $(\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega))'$.
Let $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying :

$$\begin{cases} -\gamma\Delta \boldsymbol{u}_\lambda + a\boldsymbol{u}_\lambda + b|\boldsymbol{d}|\boldsymbol{u}_\lambda + \nabla\pi_\lambda = \boldsymbol{f}_\lambda \quad \text{and} \quad \mathrm{div}\, \boldsymbol{u}_\lambda = 0 \quad \text{in } \Omega. \\ \boldsymbol{u}_\lambda \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\, \boldsymbol{u}_\lambda \times \boldsymbol{n} = \boldsymbol{h}_\lambda \times \boldsymbol{n} \qquad \text{on } \Gamma. \end{cases} \tag{1.3.29}$$

Now, we introduce the element $\boldsymbol{w} = (\boldsymbol{u}, \pi)$ in the space $\boldsymbol{E}$ where $\boldsymbol{E} = \boldsymbol{H}_0^{p*,p}(\mathrm{div}, \Omega) \times L^p(\Omega)$.
Moreover, the norm of $\boldsymbol{w}$ can be written as :

$$\|\boldsymbol{w}\|_{\boldsymbol{E}} = \|\boldsymbol{u}\|_{\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega)} + \|\pi\|_{L^p(\Omega)}. \tag{1.3.30}$$

We define the dual space $\boldsymbol{E}' = [\boldsymbol{H}_0^{p*,p}(\mathrm{div}, \Omega)]' \times L_0^{p'}(\Omega)$ such that for any element $\boldsymbol{G} \in (\boldsymbol{F}, \phi)$, the norm is
$\|\boldsymbol{G}\|_{\boldsymbol{E}'} = \|\boldsymbol{F}\|_{[\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega)]'} + (1 + \|\boldsymbol{d}\|_{L^s(\Omega)})^2 \|\phi\|_{L_0^{p'}(\Omega)}$. By applying Hahn Banach Theorem, we obtain :

$$\begin{aligned} \|\boldsymbol{w}\|_{\boldsymbol{E}} &= \sup_{\boldsymbol{G}\in\boldsymbol{E}, \boldsymbol{G}\neq O} \frac{|\langle \boldsymbol{G}, \boldsymbol{w}\rangle|}{\|\boldsymbol{G}\|_{\boldsymbol{E}'}} \\ &= \sup_{\boldsymbol{F}\in\Omega p*,p, \boldsymbol{G}\neq O} \frac{\langle \boldsymbol{F}, \boldsymbol{u}\rangle_{\Omega p*,p} + \int_\Omega \phi\, \pi\, dx}{\|\boldsymbol{F}\|_{\Omega p*,p} + (1 + \|\boldsymbol{d}\|_{L^s(\Omega)})^2 \|\phi\|_{L^{p'}(\Omega)}}. \end{aligned} \tag{1.3.31}$$

Because of $p' > 2$, according to Theorem 1.3.3, for any element $(\boldsymbol{F}, \phi) \in [\boldsymbol{H}_0^{p*,p}(\mathrm{div}, \Omega)]' \times L^{p'}(\Omega)$, the following system :

$$\begin{cases} -\gamma\Delta \boldsymbol{w} - a\boldsymbol{w} - b|\boldsymbol{d}|\boldsymbol{w} + \nabla\theta = \boldsymbol{F}, \quad \text{and} \quad \mathrm{div}\boldsymbol{w} = -\phi \quad \text{in } \Omega. \\ \boldsymbol{w} \cdot \boldsymbol{n} = \boldsymbol{0}, \quad \mathbf{curl}\boldsymbol{w} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \qquad \text{on } \Gamma. \end{cases} \tag{1.3.32}$$

has a unique solution $(\boldsymbol{w}, \theta) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}(\Omega)$ and satisfying the estimate :

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\theta\|_{L^{p'}(\Omega)} \leqslant C(1 + \|\boldsymbol{d}\|_{L^s(\Omega)})^2 \times (\|\boldsymbol{F}\|_{\Omega p*,p} + \|\phi\|_{L^{p'}(\Omega)}) \tag{1.3.33}$$

Now, we return to relation (1.3.31), one can observe that,

$$\|\boldsymbol{u}\|_{\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega)} + \|\pi\|_{L^p(\Omega)} \tag{1.3.34}$$

$$= \sup_{\substack{\boldsymbol{F}\neq 0,\ \phi\neq 0 \\ \boldsymbol{F}\in(\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))',\ \phi\in L^{p'}(\Omega)}} \frac{\langle \boldsymbol{u}_\lambda - \gamma\Delta\boldsymbol{w} - a\boldsymbol{w} - b|\boldsymbol{d}|\boldsymbol{w} + \nabla\theta, \boldsymbol{w}\rangle_{\Omega p*,p} - \gamma\langle \boldsymbol{h}_\lambda \times \boldsymbol{n}, \boldsymbol{n}\rangle_\Gamma}{\|\boldsymbol{F}\|_{\Omega p*,p} + (1 + \|\boldsymbol{d}\|_{L^s(\Omega)})^2 \|\phi\|_{L^{p'}(\Omega)}}. \tag{1.3.35}$$

Consequently,

$$\|\boldsymbol{u}\|_{\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega)} + \|\pi\|_{L^p(\Omega)} = \sup_{\substack{\boldsymbol{F}\neq 0,\ \phi\neq 0 \\ \boldsymbol{F}\in(\boldsymbol{H}_0^{p*,p}(\mathrm{div},\Omega))',\ \phi\in L^{p'}(\Omega)}} \frac{|\langle \boldsymbol{f}_\lambda, \boldsymbol{w}\rangle_{\Omega r',p'} - \gamma\langle \boldsymbol{h}_\lambda \times \boldsymbol{n}, \boldsymbol{n}\rangle_\Gamma|}{\|\boldsymbol{F}\|_{\Omega p*,p} + (1 + \|\boldsymbol{d}\|_{L^s(\Omega)})^2 \|\phi\|_{L^{p'}(\Omega)}}. \tag{1.3.36}$$

Now, we can see that :

$$|\langle \boldsymbol{f}_\lambda, \boldsymbol{w}\rangle_{\Omega r',p'} - \gamma\langle \boldsymbol{h}_\lambda \times \boldsymbol{n}, \boldsymbol{n}\rangle_\Gamma| \leqslant \|\boldsymbol{f}_\lambda\|_{\Omega r',p'}\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\boldsymbol{h}_\lambda \times \boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\|\boldsymbol{w}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}.$$

By using the relation (1.3.33), it follows :

$$|\langle \boldsymbol{f}_\lambda, \boldsymbol{w}\rangle_{\Omega r',p'} - \gamma\langle \boldsymbol{h}_\lambda \times \boldsymbol{n}, \boldsymbol{n}\rangle_\Gamma| \leqslant C(1 + \|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)})^2(\|\boldsymbol{F}\|_{\Omega p*,p} + \|\phi\|_{L^{p'}(\Omega)})\times$$
$$(\|\boldsymbol{f}_\lambda\|_{\Omega r',p'} + \|\boldsymbol{h}_\lambda \times \boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}). \qquad (1.3.37)$$

Therefore, we obtain :

$$\|\boldsymbol{u}_\lambda\|_{\Omega p*,p} + \|\pi_\lambda\|_{L^p(\Omega)} \leqslant C(1 + \|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)})^2(\|\boldsymbol{f}_\lambda\|_{\Omega r',p'} + \|\boldsymbol{h}_\lambda \times \boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}). \qquad (1.3.38)$$

So, we conclude that $\boldsymbol{u}_\lambda$ (respectively π_λ) is bounded in $\boldsymbol{L}^{p*}(\Omega)$ (respectively in $L^p(\Omega)$). Finally, we are able to pass to the limit $(\boldsymbol{u}_\lambda, \pi_\lambda) \longrightarrow (\boldsymbol{u}, \pi) \in \boldsymbol{L}^{p*}(\Omega) \times L^p(\Omega)$ which is the unique solution for the linearized problem (1.3.1).

□

The following Theorem treats the linearized Brinkman Forcheimer problem (1.3.1) when we consider a datum $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, where $1 < p < \frac{6}{5}$.

Theorem 1.3.5. *Let Ω is of class $C^{2,1}$ and $1 < p < \frac{6}{5}$. We suppose that $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, then the problem (1.3.1) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.*

Proof. Let we have $\lambda > 0$ and f_λ be a distribution in $\mathcal{D}(\Omega)$ where $f_\lambda \hookrightarrow f$ in $\boldsymbol{L}^p(\Omega)$.
On the boundary Γ, we consider $\boldsymbol{h}_\lambda$ infinitly smooth such that :

$$\boldsymbol{h}_\lambda \hookrightarrow \boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma).$$

Now, we observe $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ as solution for :

$$\begin{cases} -\gamma\Delta \boldsymbol{u}_\lambda + a\boldsymbol{u}_\lambda + |\boldsymbol{d}|\boldsymbol{u}_\lambda + \nabla\pi_\lambda = \boldsymbol{f}_\lambda \quad \text{and} \quad \mathrm{div}\boldsymbol{u}_\lambda = 0 & \text{in } \Omega \\ \boldsymbol{u}_\lambda \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\boldsymbol{u}_\lambda \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on } \Gamma. \end{cases} \qquad (1.3.39)$$

That satisfies the estimate :

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi_\lambda\|_{W^{1,p}(\Omega)} \leqslant C(\|\boldsymbol{f}_\lambda\|_{L^p(\Omega)} + \|\boldsymbol{h}_\lambda \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} +$$
$$\||\boldsymbol{d}|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}).$$

For any $1 < p < \frac{6}{5}$ and $s = \frac{3}{2}$, it follows that $q < p*$, where $3 < p* < 6$ and we obtain :

$$\||\boldsymbol{d}|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leqslant C\|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p*}(\Omega)}.$$

Moreover, we have the compact embedding :

$$\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow L^q(\Omega).$$

Such that $q < p*$. Then there exists $\epsilon > 0$ and $C_\epsilon > 0$ satisfying the estimate :

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leqslant C(\|\boldsymbol{f}_\lambda\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} + C_\epsilon\|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p*}(\Omega)}).$$

According to the previous Theorem, it yields that :

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leqslant C(\|\boldsymbol{f}_\lambda\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} + C_\epsilon\|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)}(\|\boldsymbol{f}_\lambda\|_{[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)})).$$

Because of the embedding $\boldsymbol{L}^p(\Omega) \hookrightarrow [\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'$ and $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ which gives that :

$$\begin{aligned}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} &\leqslant C(\|\boldsymbol{f}_\lambda\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} + \\ &\quad \|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)}(\|\boldsymbol{f}_\lambda\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)})). \\ &\leqslant C(1+\|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)})(\|\boldsymbol{f}_\lambda\|_{L^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}).\end{aligned}$$

□

In the following Theorem, we will seek for investigate the existence of solution for the linearized problem (1.3.1) when $\frac{3}{2} < p < 2$.

Theorem 1.3.6. *We assume that Ω is of class $C^{1,1}$. Let $\frac{3}{2} < p < 2$ and $s = \frac{3}{2}$, we assume that :*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]', \ \boldsymbol{h} \in \boldsymbol{W}^{\frac{-1}{p},p}(\Gamma) \qquad \text{with} \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{3}.$$

Then the problem (1.3.1) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying the estimates:

1. *If $\boldsymbol{h} \neq \boldsymbol{0}$*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leqslant C(1+\|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)})^2(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}). \quad (1.3.40)$$

2. *If $\boldsymbol{h} = \boldsymbol{0}$ then $\boldsymbol{u} \in \boldsymbol{W}^{2,r}(\Omega)$ and we have*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leqslant C(1+\|\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)})\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'}. \quad (1.3.41)$$

Proof. To prove the existence, we proceed exactly as in the proof of Theorem 1.3.3.
Now, we return to prove the estimates :
1. Let $\boldsymbol{h} = 0$, because of $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'$, then there exists $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$, $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{F} + \nabla\boldsymbol{\psi}$,

$$\text{with} \quad \|\boldsymbol{F}\|_{\boldsymbol{L}^r(\Omega)} \leqslant \|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} \quad \text{and} \quad \|\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'}.$$

It follows that,

$$-\gamma\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{d}|\boldsymbol{u} + \nabla\pi = \boldsymbol{F}$$

and we can suppose that $\boldsymbol{f} \in \boldsymbol{L}^r(\Omega)$. We consider the subsequence $(\boldsymbol{f}_\lambda)_\lambda \subset \mathcal{D}(\Omega)$ where $\boldsymbol{f}_\lambda \longrightarrow \boldsymbol{f}$ in $\boldsymbol{L}^r(\Omega)$. Let $\widetilde{\boldsymbol{f}}$ be an extension by zero of $\boldsymbol{f}$ to $\mathbb{R}^3$, for any $t \in]0,1[$, we have

$$\rho_t * \widetilde{\boldsymbol{f}}_\lambda \in \mathcal{D}(\mathbb{R}^3), \ \lim_{\rho_t \longrightarrow 0} \lim_{\lambda \longrightarrow \infty} \rho_t * \widetilde{\boldsymbol{f}}_\lambda = \widetilde{\boldsymbol{f}}$$

$$\text{and} \quad \rho_t * \widetilde{\boldsymbol{f}}_\lambda|_\Omega \longrightarrow \boldsymbol{f} \quad \text{when } t \longrightarrow 0, \lambda \longrightarrow \infty.$$

Furthermore, because of $\frac{1}{q} = \frac{3}{2} - \frac{1}{p}$,

$$\|\rho_t * \widetilde{\boldsymbol{f}}_\lambda\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \leqslant \|\rho_t\|_{L^q(\mathbb{R}^3)} \|\widetilde{\boldsymbol{f}}_\lambda\|_{L^r(\mathbb{R}^3)} \leqslant \frac{4}{3}\pi t^{-\frac{3}{q'}} \|\boldsymbol{f}_\lambda\|_{L^{r'}(\mathbb{R}^3)}. \tag{1.3.42}$$

We take $t = \lambda^{-\beta}$ with $\beta > 0$.
Now by putting $\boldsymbol{F}_\lambda = \rho_t * \widetilde{\boldsymbol{f}}_\lambda|_\Omega$, we obtain $\boldsymbol{F}_\lambda \longrightarrow \boldsymbol{f}$ in $\boldsymbol{L}^r(\Omega)$ when $\lambda \longrightarrow \infty$. Next, we know that the following problem

$$\begin{cases} -\gamma\Delta \boldsymbol{u}_\lambda + a\boldsymbol{u}_\lambda + b|\boldsymbol{d}|\boldsymbol{u}_\lambda + \nabla\pi_\lambda = \boldsymbol{F}_\lambda \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u}_\lambda = 0 \quad \text{in } \Omega \\ \boldsymbol{u}_\lambda \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{u}_\lambda \times \boldsymbol{n} = 0 \qquad\qquad\qquad\qquad \text{on } \Gamma \end{cases} \tag{1.3.43}$$

has a unique solution $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \boldsymbol{W}^{2,r}(\Omega) \times L^p(\Omega)$, according to the embedding $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and using Brinkman regularity, it infers :

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\lambda\|_{L^p(\Omega)} \leqslant \|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|\pi_\lambda\|_{L^p(\Omega)} \leqslant C(\|\boldsymbol{F}_\lambda\|_{\boldsymbol{L}^r(\Omega)} + \||\boldsymbol{d}|\boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)}). \tag{1.3.44}$$

Now, we investigate the existence of a positive constant $C > 0$ not depending on $\boldsymbol{d}$, such that :

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p*}(\Omega)} \leqslant C\|\boldsymbol{F}_\lambda\|_{\boldsymbol{L}^r(\Omega)}, \quad \text{for any } \lambda \in \mathbb{N}* . \tag{1.3.45}$$

So let assume per abssurdum the validity of the previous estimate. then for $n \in \mathbb{N}*$ there exists $k_n \in \mathbb{N}, \boldsymbol{F}_{k_n} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ such that

$$\begin{cases} -\gamma\Delta \boldsymbol{u}_{k_n} + a\boldsymbol{u}_{k_n} + b|\boldsymbol{d}|\boldsymbol{u}_{k_n} + \nabla\pi_{k_n} = \boldsymbol{F}_{k_n} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u}_{k_n} = 0 \quad \text{in } \Omega \\ \boldsymbol{u}_{k_n} \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{u}_{k_n} \times \boldsymbol{n} = 0 \qquad\qquad\qquad\qquad \text{on } \Gamma. \end{cases} \tag{1.3.46}$$

Where the following inequality hold

$$\|\boldsymbol{u}_{k_n}\|_{\boldsymbol{L}^{p*}(\Omega)} > n\|\boldsymbol{F}_{k_n}\|_{\boldsymbol{L}^r(\Omega)}. \tag{1.3.47}$$

Now, we put

$$\boldsymbol{w}_n = \frac{\boldsymbol{u}_{kn}}{\|\boldsymbol{u}_{kn}\|_{L^{p*}(\Omega)}}, \quad \theta_n = \frac{\pi_{k_n}}{\|\boldsymbol{u}_{kn}\|_{L^{p*}(\Omega)}} \quad \text{and} \quad \boldsymbol{G}_n = \frac{\boldsymbol{F}_{k_n}}{\|\boldsymbol{u}_{kn}\|_{L^{p*}(\Omega)}}. \tag{1.3.48}$$

For any $n \in \mathbb{N}*$, we have

$$\begin{cases} -\gamma\Delta \boldsymbol{w}_n + a\boldsymbol{w}_n + b|\boldsymbol{d}_n|\boldsymbol{w}_n + \nabla\pi_n = \boldsymbol{G}_n \quad \text{and} \quad \mathrm{div}\boldsymbol{w}_n = 0 \quad \text{in } \Omega \\ \boldsymbol{w}_n \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\boldsymbol{w}_n \times \boldsymbol{n} = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \Gamma. \end{cases} \tag{1.3.49}$$

Because of $\boldsymbol{G}_n \in \boldsymbol{L}^{\frac{6}{5}}(\Omega) \hookrightarrow (\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'$, using the Corollary 1.2.12, it yields to obtain the estimate :

$$\|\boldsymbol{w}_n\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\|\boldsymbol{G}_n\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} = \frac{C}{\|\boldsymbol{u}_n\|_{\boldsymbol{L}^{p*}(\Omega)}}\|\boldsymbol{F}_n\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} = \frac{C}{\|\boldsymbol{u}_n\|_{\boldsymbol{L}^{p*}(\Omega)}}\|\rho_t * \widetilde{\boldsymbol{f}}_{k_n|\Omega}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} \tag{1.3.50}$$

According to the estimate (1.3.47), it follows :

$$\|\boldsymbol{w}_n\|_{\boldsymbol{H}^1(\Omega)} \leqslant \frac{C}{n\|\boldsymbol{F}_{k_n}\|_{\boldsymbol{L}^r(\Omega)}}\|\rho_t * \widetilde{\boldsymbol{f}}_{k_n|\Omega}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}. \tag{1.3.51}$$

Taking $t = n^{-\beta}$ with $0 < \beta < \frac{q'}{3}$, and according to the relation (1.3.42), it gives :

$$\|\boldsymbol{w}_n\|_{\boldsymbol{H}^1(\Omega)} \leqslant \frac{4\pi}{3n^{1-\frac{3\beta}{q'}}}\frac{C}{\|\boldsymbol{F}_{k_n}\|_{\boldsymbol{L}^r(\Omega)}}\|\widetilde{\boldsymbol{f}}_{k_n}\|_{\boldsymbol{L}^r(\Omega)}. \tag{1.3.52}$$

When $n \longrightarrow \infty$, we conclude that $\boldsymbol{w}_n \longrightarrow 0$ in $\boldsymbol{H}^1(\Omega)$. it means that $\boldsymbol{w}_n \longrightarrow 0$ in $\boldsymbol{L}^q(\Omega)$, for any $1 \leqslant q \leqslant 6$.

On the other hand, due to relation (1.3.48), one can obtain that $\|\boldsymbol{w}_n\|_{\boldsymbol{L}^{p*}(\Omega)} = 1$ where $3 < p* < 6$, which leads to a contradiction, so consequently we affirm the validity of the estimate (1.3.45).
Now according to estimate (1.3.44) and (1.3.45), we have :

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\lambda\|_{L^p(\Omega)} \leqslant C(1 + \|\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)})\|\boldsymbol{F}_\lambda\|_{\boldsymbol{L}^r(\Omega)}. \tag{1.3.53}$$

Then, we can extract a subsequence of $\boldsymbol{u}_\lambda$ and π_λ, still denoted $\boldsymbol{u}_\lambda$ and π_λ such that

$$\boldsymbol{u}_\lambda \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}^{1,p}(\Omega) \quad \pi_\lambda + c_\lambda \rightharpoonup \pi \quad \text{in} \quad L^p(\Omega),$$

and $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfies the linearized problem (1.3.1) when $\boldsymbol{h} = 0$ and establish the estimate,

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leqslant C(1 + \|\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)})\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'}. \tag{1.3.54}$$

2. Let us consider $\boldsymbol{h} \neq 0$, due to Theorem 1.2.15, the Brinkman problem

$$\begin{cases} -\gamma\Delta \boldsymbol{u}_0 + a\boldsymbol{u}_0 + \nabla\pi_0 = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\boldsymbol{u}_0 = 0 \quad \text{in } \Omega \\ \boldsymbol{u}_0 \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\boldsymbol{u}_0 \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \qquad\qquad \text{on } \Gamma. \end{cases} \tag{1.3.55}$$

has a unique solution $(\boldsymbol{u}_0, q_0) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ which satisfies the estimate

$$\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|q_0\|_{L^p(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}). \tag{1.3.56}$$

Moreover, based on the results above, then there exists a unique solution $(\boldsymbol{z}, \theta) \in \boldsymbol{W}^{2,r}(\Omega) \times \boldsymbol{W}^{1,r}(\Omega)$ of the problem (1.3.22) satifying the estimate

$$\begin{aligned}
\|\boldsymbol{z}\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|\theta\|_{\boldsymbol{W}^{1,r}(\Omega)} &\leqslant C(1 + \|\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}) \times \||\boldsymbol{d}|\boldsymbol{u}_0\|_{\boldsymbol{L}^r(\Omega)} \\
&\leqslant C(1 + \|\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}) \times \|\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^{p*}(\Omega)} \\
&\leqslant C(1 + \|\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}) \times \|\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}.
\end{aligned} \tag{1.3.57}$$

Now, because of the embedding $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$, it gives that $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_0 + \boldsymbol{z}, \pi_0 + \theta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfies the problem (1.3.58), furthermore according to the relations (1.3.56) and (1.3.57), the estimate (1.3.40) will be established. □

1.3.2 Darcy Brinkman Forchheimer problem

In this subsection, we use the results obtained in the previous sections to study the following non linear Darcy-Brinkmann-Forchheimer equations:

$$\begin{cases} -\gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\boldsymbol{u} = \chi \quad \text{in} \quad \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} = g, \quad \mathbf{curl}\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \qquad\qquad \text{on} \quad \Gamma. \end{cases} \tag{1.3.58}$$

Where b is the coefficient of Forchheimer and the inertial term $|\boldsymbol{u}|\boldsymbol{u}$ is a monotone operator. This subsection is splited into three parts. The first one is devoted to check the existence and uniqueness of the weak solution in $\boldsymbol{H}^1(\Omega)$ for Brinkman Forchheimer problem based on Faedo Galerkin's method. Afterward, we prove the existence of solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$. for $p \geq 2$. In the last part, the fixed point's Theorem is employed to achieve the study of the weak solution when $\frac{3}{2} \leqslant p < 2$.

Hilbertian case

We begin by proving the existence of a solution in $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ for the Brinkman Forchheimer equations. We use Galerkin method and some compactness results.

Let us introduce the (DBF) with the homogeneous boundary conditions :

$$\begin{cases} -\gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\boldsymbol{u} = 0 \quad \text{in} \quad \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \qquad\qquad \text{on} \quad \Gamma. \end{cases} \tag{1.3.59}$$

The variational formulation of Brinkman Forchheimer equations (1.3.59) with considering $\boldsymbol{v} \in \boldsymbol{V}$ to be the test function such that the space :
$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega),\, \mathrm{div}\boldsymbol{v} = 0 \,\mathrm{in}\, \Omega,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \,\mathrm{on}\, \Gamma\}$, can be written as follow :

$$a(\boldsymbol{u}, \boldsymbol{v}) + O(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}), \tag{1.3.60}$$

where

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= \gamma \int_\Omega \mathbf{curl}\boldsymbol{u} \cdot \mathbf{curl}\boldsymbol{v}\, dx + a \int_\Omega \boldsymbol{u}\, \boldsymbol{v}\, dx. \\ O(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) &= b \int_\Omega |\boldsymbol{u}|\boldsymbol{u}\, \boldsymbol{v}\, dx. \\ l(\boldsymbol{v}) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))' \times \boldsymbol{H}^2_0(\mathrm{div},\Omega)}. \end{aligned} \tag{1.3.61}$$

The variational problem above (1.3.60) and the system (1.3.59) are equivalents, it means that the existence of solution for the variational formulation leads naturally to the existence of solution for the homogeneous (DBF) equations (1.3.59).

Step 1 : *Construction of approximating solutions :*

We will approximate the above variational formulation using Faedo Galerkin's method.
We define $\boldsymbol{V}_m$ to be the orthogonal space to $\boldsymbol{V}$ with basis $\langle \boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_m \rangle$. For each $m \geqslant 1$ fixed integer, we consider $\boldsymbol{u}_m$ to be the approximate solution of the variational formulation (1.3.60) such that,

$$\boldsymbol{u}_m = \sum_{i=1}^{m} \xi_{i,m} \boldsymbol{w}_i, \quad \xi_{i,m} \in \mathbb{R}.$$

We can rewrite the variational formulation (1.3.60) as below :

$$\begin{aligned} \gamma \int_\Omega \mathbf{curl}\ \boldsymbol{u}_m . \mathbf{curl}\, \boldsymbol{w}_i\, dx + a \int_\Omega \boldsymbol{u}_m\, \boldsymbol{w}_i\, dx + b \int_\Omega |\boldsymbol{u}_m| \boldsymbol{u}_m\, \boldsymbol{w}_i\, dx = \\ \langle \boldsymbol{f}, \boldsymbol{w}_i \rangle, \quad i = 1, ..., m. \end{aligned} \tag{1.3.62}$$

Step 2 : *A priori estimates :*
We consider the operator : $Q_m : \boldsymbol{V}_m \hookrightarrow \boldsymbol{V}_m$ such that for any $\boldsymbol{u}$, we associate $Q_m(\boldsymbol{u})$.
Defined for any $\boldsymbol{u}$ and $\boldsymbol{v}$ belongs to $\boldsymbol{V}_m$ by :

$$\begin{aligned} \int_\Omega Q_m(\boldsymbol{u}).\boldsymbol{v}\, dx =& \gamma \int_\Omega \mathbf{curl}\boldsymbol{u}.\mathbf{curl}\boldsymbol{v}\, dx + a \int_\Omega \boldsymbol{u}\, \boldsymbol{v}\, dx + b \int_\Omega |\boldsymbol{u}|\boldsymbol{u}\, \boldsymbol{v}\, dx + \\ & - \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))' \times \boldsymbol{H}^2_0(\mathrm{div},\Omega)}. \end{aligned}$$

It is clear that $a(.\,,.)$ and $l(.)$ are continuous. So, it remains to prove the continuity for the inertial term. Indeed, using (1.1.7)

$$\begin{aligned} \int_\Omega |\boldsymbol{u}|\boldsymbol{u}.\boldsymbol{v}\, dx &\leqslant \| |\boldsymbol{u}|^2 \|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}. \\ &\leqslant \|\boldsymbol{u}\|^2_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}. \end{aligned}$$

Using the embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega)$, we obtain

$$\int_\Omega |\boldsymbol{u}|\boldsymbol{u}\,\boldsymbol{v}\,dx \leqslant C\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}.$$

Then, we deduce the continuity for the operator Q_m.
Next for $\boldsymbol{v} = \boldsymbol{u}$, we obtain :

$$\begin{aligned}\int_\Omega Q_m(\boldsymbol{u}).\boldsymbol{u}\,dx &= \gamma\|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + a\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + b\|\boldsymbol{u}\|^3_{\boldsymbol{L}^3(\Omega)} - \langle \boldsymbol{f}, \boldsymbol{u}\rangle_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))'\times \boldsymbol{H}^2_0(\mathrm{div},\Omega)}\\
&\geqslant C(\gamma,a)\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} - \|\boldsymbol{f}\|_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))'}\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.\\
&\geqslant \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}(C(\gamma,a)\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} - \|\boldsymbol{f}\|_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))'}).\end{aligned}$$

So, it follows that $\int_\Omega Q_m(\boldsymbol{u}).\boldsymbol{u}\,dx > 0$ for $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = k > 0$, with $k > \frac{1}{C(\gamma,a)}\|\boldsymbol{f}\|_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))'}$. Using a classical argument (see [61, Lemma 1.4]), we derive the existence of the solution $\boldsymbol{u}_m \in \boldsymbol{V}_m$ such that $Q_m(\boldsymbol{u}_m) = 0$, which means that the variational problem (1.3.62) has a solution $\boldsymbol{u}_m$.

We multiply the variational formulation (1.3.62) with ξ_{im} then summing up the result equation from $i = 1...m$, using Cauchy-Schwarz :

$$\begin{aligned}\gamma\|\mathbf{curl}\,\boldsymbol{u}_m\|^2_{\boldsymbol{L}^2(\Omega)} + a\|\boldsymbol{u}_m\|^2_{\boldsymbol{L}^2(\Omega)} + b\|\boldsymbol{u}_m\|^3_{\boldsymbol{L}^3(\Omega)} &\leqslant \|\boldsymbol{f}\|_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))'}\|\boldsymbol{u}_m\|_{\boldsymbol{L}^2(\Omega)}.\\
&\leqslant \frac{1}{2a}\|\boldsymbol{f}\|^2_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))'} + \frac{a}{2}\|\boldsymbol{u}_m\|^2_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Thanks to the Theorem 1.1.1, we obtain :

$$c(\gamma)\|\boldsymbol{u}_m\|^2_{\boldsymbol{H}^1(\Omega)} + \frac{a}{2}\|\boldsymbol{u}_m\|^2 + b\|\boldsymbol{u}_m\|^3_{\boldsymbol{L}^3(\Omega)} \leqslant \frac{1}{2a}\|\boldsymbol{f}\|^2_{(\boldsymbol{H}^2_0(\mathrm{div},\Omega))'}. \tag{1.3.63}$$

Step 3 : Passage to the limit :

Now let $m \longrightarrow \infty$, thanks to the last estimate, we can extract a subsequence $(\boldsymbol{u}_m)_m$ and some $\boldsymbol{u}$ in $\boldsymbol{V}$ which satisfy the following convergence :

$$\boldsymbol{u}_m \longrightarrow \boldsymbol{u} \quad \text{weak} \quad \text{in} \quad \boldsymbol{V} \tag{1.3.64}$$

$$|\boldsymbol{u}_m|\boldsymbol{u}_m \longrightarrow \boldsymbol{w} \quad \text{weak} \quad \text{in} \quad \boldsymbol{L}^{\frac{3}{2}}(\Omega). \tag{1.3.65}$$

Since the embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we can extract a subsequence $(\boldsymbol{u}_m)_m$ denoted again by $(\boldsymbol{u}_m)_m$ which converge strongly to $\boldsymbol{u}$ in $\boldsymbol{L}^2(\Omega)$. Then, il follows from (1.3.65) that $\boldsymbol{w} = |\boldsymbol{u}|\boldsymbol{u}$. Hence, We have established that as m goes to infinity, the sequence $(\boldsymbol{u}_m)_m$ converges to $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ the solution of the homogeneous Brinkman Forchheimer system (1.3.59).

Next we turn to the pressure π, taking $\boldsymbol{v} \in \mathcal{D}(\Omega)$ such that $\mathrm{div}\boldsymbol{v} = 0$ gives:

$$\langle -\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|\boldsymbol{u} - \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = 0.$$

By using De Rham's Theorem we can conclude the existence of the distribution π belongs to $\mathcal{D}'(\Omega)$ defined uniquely up to an addative constant such that $-\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|\boldsymbol{u} - \boldsymbol{f} = \nabla\pi$. Since $|\boldsymbol{u}|\boldsymbol{u} \in \boldsymbol{L}^3(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$, then $\nabla\pi \in \boldsymbol{H}^{-1}(\Omega)$. Thanks to [3] the pressure π belongs to $L^2(\Omega)$.

Proposition 1.3.7. *We assume that Ω is of class $C^{1,1}$ and the datum $\boldsymbol{f}$ belongs to $(\boldsymbol{H}_0^2(\mathrm{div},\Omega))'$, hence we obtain at least $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ as a solution of the homogeneous Brinkman Forchheimer equations* (1.3.59).

Now we will move to prove the uniqueness of the solution for Brinkman Forchheimer equations (1.3.59).

Proposition 1.3.8. *Under the assumptions of the previous proposition, the Brinkman Forchheimer equations* (1.3.59) *has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$*

Proof. Let assume having the existence of two different solutions $(\boldsymbol{u}_1,\pi_1)$ and $(\boldsymbol{u}_2,\pi_2)$ for Brinkman Forchheimer equations (1.3.59). We set $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\pi = \pi_1 - \pi_2$. Then $(\boldsymbol{u},\pi)$ satisfies the problem:

$$\begin{aligned} -\gamma\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}_1|\,\boldsymbol{u}_1 - b|\boldsymbol{u}_2|\,\boldsymbol{u}_2 + \pi = \boldsymbol{0}, \text{ in } \Omega \\ \mathrm{div}\boldsymbol{u} = 0 \text{ in } \Omega \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \ \mathbf{curl}\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma \end{aligned} \tag{1.3.66}$$

Multiplying the first equation of (1.3.66) by $\boldsymbol{u}$ and integrating by parts, we obtain

$$\gamma\|\mathbf{curl}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + a\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + b\int_\Omega (|\boldsymbol{u}_1|\boldsymbol{u}_1 - |\boldsymbol{u}_2|\boldsymbol{u}_2).\boldsymbol{u}\ dx = 0.$$

Due to the monotonocity of operator $F(\boldsymbol{u}) = |\boldsymbol{u}|\boldsymbol{u}$, we obtain:

$$\gamma\|\mathbf{curl}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + a\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leqslant 0,$$

which gives the uniqueness of the solution for the Brinkman Forchheimer equations (1.3.59).

□

Next, using exactly the same arguments in the proof of Proposition 1.3.7 and Proposition 1.3.8, we can prove the following result given with any function $\boldsymbol{f}$ in $(\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'$.

Corollary 1.3.9. *Let Ω to be of class $C^{1,1}$ and we assume $\boldsymbol{f} \in (\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega))'$, then there exists a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$*

L^p Theory

Now, we treat the general case $1 < p < \infty$. We start with the generalized solution for $p \geqslant 2$.

Theorem 1.3.10. *(Generalized solution for $p \geqslant 2$)*
We suppose that Ω is of class $C^{1,1}$ and $p \geqslant 2$. Let $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega))'$. Then, (1.3.59) *has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$. Moreover, we have the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leqslant C\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega))'}.$$

Proof. Since $(\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega))' \hookrightarrow (\boldsymbol{H}_0^{2}(\mathrm{div}\,,\Omega))'$, thanks to Proposition 1.3.7, problem (1.3.59) has solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, $\pi \in L^2(\Omega)$. As a consequence $|\boldsymbol{u}|\boldsymbol{u} \in \boldsymbol{L}^3(\Omega)$. We write $-\gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + \nabla\pi = \boldsymbol{f} - b|\boldsymbol{u}|\boldsymbol{u}$ and we distinguish the following cases :

1. $2 \leqslant p \leqslant 3$: $\boldsymbol{f} - b|\boldsymbol{u}|\boldsymbol{u} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega))'$. As a consequence, using the regularity for the Darcy Brinkman problem (see Proposition 1.2.8), we deduce that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

2. $p > 3$, We know that $\boldsymbol{u} \in \boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^t(\Omega)$, $t \geqslant 1$. Then $|\boldsymbol{u}|\boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$ for any finite q with $q \geq \frac{1}{2}$. So, we have : $\boldsymbol{f} - b|\boldsymbol{u}|\boldsymbol{u} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega))'$. We apply again Proposition 1.2.8, we deduce that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

Now, we give an extension for more general data $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div}\,,\Omega))'$. □

Corollary 1.3.11. *We assume that Ω is of class $C^{1,1}$. We suppose that $p \geqslant 2$ and $r \geqslant \frac{6}{5}$. Let $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div}\,,\Omega))'$. Then, problem* (1.3.59) *has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$. Moreover, we have the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leqslant C\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div}\,,\Omega))'}.$$

Proof. We suppose that $p \geqslant 2$ and $r \geqslant \frac{6}{5}$. Since $(\boldsymbol{H}_0^{r',p'}(\mathrm{div}\,,\Omega))' \hookrightarrow (\boldsymbol{H}_0^{6,2}(\mathrm{div}\,,\Omega))'$, according to Corollary 1.3.9, we have a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$. Then, $|\boldsymbol{u}|\boldsymbol{u}$ still belongs to $\boldsymbol{L}^3(\Omega)$. Next, we distinguish according to the values for r and p the different following cases :

1. If $\frac{6}{5} \leqslant r \leqslant 3$ and $p \geqslant 2$, observe that $\boldsymbol{f} - b|\boldsymbol{u}|\boldsymbol{u} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div}\,,\Omega))'$ as a consequence, using Corrollary 1.2.13, we deduce that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

2. If $3 < r$ and $p \geqslant 2$, we have $\boldsymbol{f} - b|\boldsymbol{u}|\boldsymbol{u}$ belongs to $(\boldsymbol{H}_0^{\frac{3}{2},p'}(\mathrm{div}\,,\Omega))'$. Thanks again to Corrollary 1.3.9, we deduce that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

□

We are interested now on strong solution for the Darcy Brinkamn Forchheimer problem (1.3.59) for $p \geq \frac{6}{5}$.

Theorem 1.3.12. *(Strong solution with $p \geqslant \frac{6}{5}$) We suppose that Ω is of class $C^{2,1}$ and $p \geqslant \frac{6}{5}$. Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$. Then* (1.3.59) *has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Moreover, we have the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leqslant C\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Proof. The proof is based on the use of the regularity of the linear Darcy-Brinkman problem. Since $\boldsymbol{L}^p(\Omega) \hookrightarrow (\boldsymbol{H}_0^{6,2}(\mathrm{div}\,,\Omega))'$. Then, by Corollary 1.3.9, There exists $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ solution of (1.3.59). we consider two cases :

1. $\frac{6}{5} \leqslant p \leqslant 3$. We have : $\boldsymbol{f} - b|\boldsymbol{u}|\boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$, $q \leqslant 3$ and the result follows directly from the regularity of Darcy-Brinkman problem.

2. $p > 3$, we know that $\boldsymbol{u} \in \boldsymbol{W}^{2,3}(\Omega)$ and $\pi \in W^{1,3}(\Omega)$. Since $\boldsymbol{W}^{2,3}(\Omega) \hookrightarrow \boldsymbol{L}^{\infty}(\Omega)$, we deduce that $|\boldsymbol{u}|\boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$ for any $q \in [1, +\infty[$.
   Thanks again to the regularity of Darcy-Brinkman problem, we have $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

□

Weak solution of Brinkman Forchheimer problem for $\frac{3}{2} < p < 2$ via Banach's fixed point Theorem

The aim of this subsection is to treat the weak solution of a steady Brinkman Forchheimer equations (1.3.58), with $\frac{3}{2} < p < 2$ and $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega))'$.

Theorem 1.3.13. *Let Ω to be of class $C^{1,1}$. We suppose $\frac{3}{2} < p < 2$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$. We assume that :*

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega))',\ \chi \in L^p(\Omega),\ \boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma),\ g \in W^{1-\frac{1}{p},p}(\Gamma).$$

There exists a constant $\alpha_1 > 0$ such that, if

$$\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leqslant \alpha_1.$$

Then $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ is a solution for the problem (1.3.58) *which satisfies the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)}).$$

Proof. In order to prove the existence of a generalized solutions, we need to apply Bannach's fixed point Theorem over the linearized Brinkman Forchheimer equations.
We are searching for a fixed point for the mapping T,

$$\begin{aligned} T\ :\ \boldsymbol{W}^{1,p}(\Omega) &\longrightarrow \boldsymbol{W}^{1,p}(\Omega) \\ \boldsymbol{d} &\longrightarrow \boldsymbol{u}. \end{aligned}$$

Such that $\boldsymbol{d} \in \boldsymbol{W}^{1,p}(\Omega)$, $T\boldsymbol{d} = \boldsymbol{u}$ is the unique solution for the non linearized problem (1.3.19). In order to apply the fixed point Theorem, we have to introduce a neighborhood $\boldsymbol{B}_\lambda$ in the form :

$$\boldsymbol{B}_\lambda = \{\boldsymbol{d} \in \boldsymbol{W}^{1,p}(\Omega), \quad \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant \lambda\}.$$

Now, let $\boldsymbol{d}_1, \boldsymbol{d}_2 \in \boldsymbol{B}_\lambda$, using the contraction method, we must prove that there exists $\theta \in]0,1[$ such that :

$$\|T\boldsymbol{d}_1 - T\boldsymbol{d}_2\|_{\boldsymbol{W}^{1,p}(\Omega)} = \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant \theta \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{1.3.67}$$

In order to investigate the previous estimate, we denote that for each $k = 1, 2$, we have $(\boldsymbol{u}_k, \pi_k) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfies the system

$$\begin{cases} -\gamma \Delta \boldsymbol{u}_k + a\boldsymbol{u}_k + |\boldsymbol{d}_k|\, \boldsymbol{u}_k + \nabla \pi_k = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_k = \chi & \text{in} \quad \Omega \\ \boldsymbol{u}_k \cdot \boldsymbol{n} = g, \quad \mathbf{curl}\, \boldsymbol{u}_k \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on} \quad \Gamma, \end{cases} \tag{1.3.68}$$

with

$$\begin{aligned} \|\boldsymbol{u}_k\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leqslant C(1 + \|\boldsymbol{d}_k\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)})^2 (\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\chi\|_{L^p(\Omega)} + \\ (\|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)}). \end{aligned} \tag{1.3.69}$$

Next, we consider $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, \pi_1 - \pi_2)$ and $\boldsymbol{d} = \boldsymbol{d}_1 - \boldsymbol{d}_2$ to be a solution for the following system:

$$\begin{cases} -\gamma \Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{d}_1|\, \boldsymbol{u} + \nabla \pi = -b(|\boldsymbol{d}_1| - |\boldsymbol{d}_2|)\boldsymbol{u}_2, & \text{in} \quad \Omega \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \mathbf{curl} \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on} \quad \Gamma, \end{cases} \tag{1.3.70}$$

which satisfies the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leqslant C(1 + \|\boldsymbol{d}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}) \|(|\boldsymbol{d}_1| - |\boldsymbol{d}_2|)\boldsymbol{u}_2\|_{\boldsymbol{L}^r(\Omega)}$$

So, we have

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} &\leqslant C(1 + \|\boldsymbol{d}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}) \||\boldsymbol{d}_1| - |\boldsymbol{d}_2|\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\boldsymbol{u}_2\|_{\boldsymbol{L}^{p*}(\Omega)} \\ &\leqslant C(1 + \|\boldsymbol{d}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}) \|\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}(\Omega)}. \end{aligned}$$

Furthermore, let $C_1 > 0$, such that

$$\|\boldsymbol{d}\|_{L^{\frac{3}{2}}(\Omega)} \leqslant C_1 \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{1.3.71}$$

Now, according to relations (1.3.69) and (1.3.71), we conclude the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant \alpha C^2 C_1 (1 + C_1 \|\boldsymbol{d}_1\|_{\boldsymbol{W}^{1,p}(\Omega)}) \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,p}(\Omega)} (1 + C_1 \|\boldsymbol{d}_2\|_{\boldsymbol{W}^{1,p}(\Omega)})^2,$$

where $\alpha = \|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega))'} + \|\chi\|_{W^{-\frac{1}{p},p}(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)}$. Therefore the relation (1.3.67) will be established if we take λ where :

$$C^2C_1\alpha(1+C_1\lambda)^3 < 1.$$

For example, we choose

$$\lambda = \frac{1}{C_1}[(2C^2C_1)^{-\frac{1}{3}} - 1] \quad \text{and} \quad \alpha < \frac{1}{2C^2C_1}. \tag{1.3.72}$$

If the relations (1.3.72) are satisfied, then the fixed point $\boldsymbol{u}* \in \boldsymbol{W}^{1,p}(\Omega)$ assures

$$\|\boldsymbol{u}*\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant C\alpha(1 + C_1\|\boldsymbol{u}*\|_{\boldsymbol{W}^{1,p}(\Omega)}). \tag{1.3.73}$$

That is,

$$\|\boldsymbol{u}*\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant \boldsymbol{x}*. \tag{1.3.74}$$

Such that $\boldsymbol{x}*$ characterizes the smallest solution of the equation $C_1a\boldsymbol{x}^2 + (2a-1)\boldsymbol{x} + \dfrac{a}{C_1}.$, with $a = CC_1\alpha$, i.e $\boldsymbol{x}* = \frac{1-2a-\sqrt{1-4a}}{2C_1a}$ where $a < \frac{1}{4}$. Consequently we obtain :

$$\|\boldsymbol{u}*\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant \frac{2a}{C_1(1-2a+\sqrt{1-4a})} \leqslant \frac{4a}{C_1},$$

which achieves the proof of the theorem. □

Chapter 2

The Stationary Brinkman Forchheimer with pressure boundary conditions

Introduction

Through this chapter, we consider Ω as a bounded set of $\mathbb{R}^3$, possibily multiply-connected with boundary Γ such that $\Gamma = \overset{I}{\underset{i=0}{\cup}} \Gamma_i$ where Γ_i are the connected components of Γ.
We suppose that there exsits J connected open surfaces, called cuts, contained in Ω, such that each surface Σ_j is an open part of a smooth manifold and the boundary of each Σ_j is contained in Γ. The intersections $\overline{\Sigma_i} \cap \overline{\Sigma_j}$ are empty for $i \neq j$ and the open set $\overset{\circ}{\Omega} = \Omega \setminus \overset{J}{\underset{j=0}{\cup}} \Sigma_j$ is simply connected.

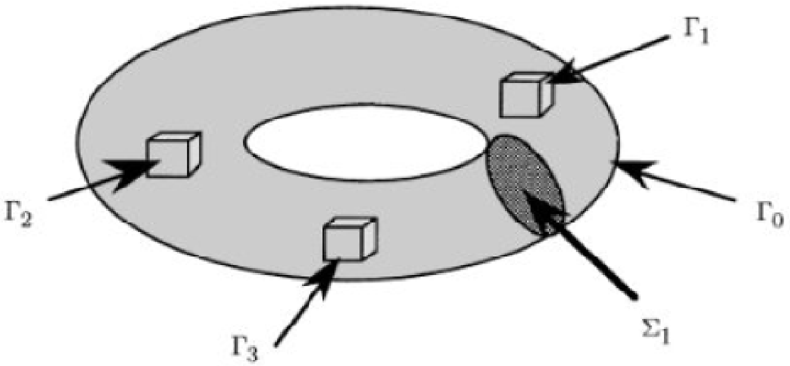


Figure 1: The domain Ω

For $J = 1$, $I = 3$, see for exemple Figure 1.

The work developed in this chapter is concerned with the existence and regularity of solution for the following stationary convective Brinkman-Forchheimer system with a pressure boundary condition:

$$\begin{cases} -\gamma\Delta\, \boldsymbol{u} + \mathbf{curl}\boldsymbol{u} \times \boldsymbol{u} + b|\boldsymbol{u}|^{\alpha}\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{and} \quad \pi = \pi_0 \ \text{ on } \Gamma_0 \quad \text{and} \quad \pi = \pi_0 + c_i & \text{on}\, \Gamma_i, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases} \tag{2.0.1}$$

where π_0, $\boldsymbol{g}$ and $\boldsymbol{f}$ are given functions. $\boldsymbol{n}$ is the unit outward normal on the boundary. $\gamma > 0$ is the Brinkman coefficient and $b > 0$ defines the Forchheimer coefficient, $\alpha \in [1, 2]$.

In [28], Conca and al precise one of the situations where we found this kind of boundary conditions which is the pipe flows. When we have a pipe bifurcates into two secondaries pipes hence the condition on the pressure and the flux are indispensables to control the amount of fluid flows into each pipe.

The boundary conditions on the pressure has interested many researchers such as Bégue and al [14] where the authors study both of the stationary linear Stokes problem and the stationary non linear Navier Stokes. This work was followed and completed by Bernard [15] which present a regularity results in $\boldsymbol{H}^2$ on the hilbertian case and regularity solutions in $\boldsymbol{W}^{m,r}$ with $m \geqslant 2$ and $r \geqslant 2$ in $\boldsymbol{L}^p$ theory. The outline of this chapter is as follows. In section 1, we introduces the functional framework and recall some usefull results that we need in the sequel of this chapter. The section 2 is devoted to treat the stationary Stokes equations with pressure boundary conditions where we improve the results already given in [8]. The section 3 is preserved to present the weak solution of BF problem (2.0.1) in the hilbertien case. The last section contains the results related to (2.0.1) where we prove the existence of both of weak solution and strong solution in L^p theory.

2.1 Notations and some useful results

Through this work, we consider $\Omega \subset \mathbb{R}^3$ a bounded domain with boundary Γ of classe $\mathcal{C}^{1,1}$. In the case that Γ is more regular, we will point it out.
Let us consider the following Stokes problem :

$$(\mathcal{S}) \begin{cases} -\gamma\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \ \text{ and } \operatorname{div} \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 & \text{on}\, \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I, \end{cases}$$

and let us introduce the following spaces :

$$\boldsymbol{X}^{\,p}_{N}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega), \quad \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \operatorname{div} \boldsymbol{u} \in \boldsymbol{L}^p(\Omega),\ \boldsymbol{v} \times \boldsymbol{n} = 0 \text{ on}\, \Gamma\}. \tag{2.1.1}$$

$$\boldsymbol{V}^{p}_{N}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^{p}_{N}(\Omega),\ \operatorname{div}\boldsymbol{v} = 0,\ \text{and } \langle \boldsymbol{v} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Gamma_i},\, 1 \leqslant i \leqslant I\}. \tag{2.1.2}$$

We state now the following well-known lemma, its proof can be found in [8]. It gives an Inf-Sup condition on the space $\boldsymbol{V}_N^p(\Omega)$ which plays an important role on the solvability of the velocity $\boldsymbol{u}$.

Lemma 2.1.1. *We have the following Inf-Sup condition:*

$$\inf_{\substack{\boldsymbol{v}\in \boldsymbol{V}_N^{p'}(\Omega)\\ \boldsymbol{v}\neq 0}} \sup_{\substack{\boldsymbol{u}\in \boldsymbol{V}_N^{p}(\Omega)\\ \boldsymbol{u}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)}} > 0. \tag{2.1.3}$$

Moreover, we consider the kernel space as

$$\boldsymbol{K}_N^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}_N^p(\Omega),\ \mathbf{curl}\,\boldsymbol{v} = 0,\ \mathrm{div}\boldsymbol{v} = 0 \text{ in } \Omega\}, \tag{2.1.4}$$

which is of finite dimension and is spanned by the functions ∇q_i^N, $1 \leqslant i \leqslant N$, where q_i^N is the unique solution in $\boldsymbol{W}^{2,p}(\Omega)$ of the problem (see [8, Corollary 4.2]).

$$\begin{cases} -\Delta q_i^N = 0 \text{ in } \Omega, \\ q_i^N|_{\Gamma_0} = 0 \quad \text{and} \quad q_i^N|\Gamma_k = C_k,\ 1 \leqslant k \leqslant I, \\ \langle \partial_n q_i^N, 1\rangle_{\Gamma_0} = -1,\ \text{and}\ \langle \partial_n q_i^N\ ,\ 1\rangle_{\Gamma_k} = \delta_{ik},\ 1 \leqslant k \leqslant I. \end{cases}$$

We introduce the space $(\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega))'$ which can be characterized as : a distribution $\boldsymbol{f}$ belongs to $(\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega))'$ if and only if there exist $\boldsymbol{F} \in \boldsymbol{L}^p(\Omega)$ and $\psi \in L^p(\Omega)$ such that :

$$\boldsymbol{f} = \boldsymbol{F} + \mathbf{curl}\,\psi. \tag{2.1.5}$$

More generally, we define the following Banach space :

$$\boldsymbol{H}_0^{r,p}(\mathbf{curl}\,,\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^r(\Omega),\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0},\ \text{on } \Gamma\},\ r \leqslant p \text{ and } \frac{1}{r} \leqslant \frac{1}{p} + \frac{1}{3},$$

whose dual space can be charaterized as follows :
$\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathbf{curl}\,,\Omega))'$ if and only if there exists, $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$ and $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ such that :

$$\boldsymbol{f} = \boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\psi}. \tag{2.1.6}$$

These characterizations are proven in [7, Lemma 2.5].
Then, the following Lemma are concerned with the existence and uniqueness of weak solutions of the Stokes problem $(\mathcal{S})$. See [7, Theorem 2.1] for details of the proof.

Proposition 2.1.2. *1. Let $\boldsymbol{f}$, $\boldsymbol{g}$, π_0 with*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)]',\ \boldsymbol{g}\times\boldsymbol{n} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma),\ \pi_0 \in W^{1-\frac{1}{p},p}(\Gamma),$$

satisfying the compatibility condition :

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega + \int_\Gamma \pi_0\, \boldsymbol{v}\cdot\boldsymbol{n} = 0, \tag{2.1.7}$$

where $\langle .\,,\,.\rangle_\Omega = \langle .\,,\,.\rangle_{(\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega))'\times \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}$. *Then the Stokes problem* $(\mathcal{S}')$ *has a unique solution* $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{1,p}(\Omega)\times W^{1,p}(\Omega)$ *satisfying the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \le C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\Big). \tag{2.1.8}$$

2. *Moreover, if* Ω *is of classe* $\mathcal{C}^{2,1}$, $\boldsymbol{f}\in \boldsymbol{L}^p(\Omega)$ *and* $\boldsymbol{g}\times\boldsymbol{n}\in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma)$, *then the solution* $(\boldsymbol{u},\pi)$ *belongs to* $\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ *and satisfies the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \le C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma)}\Big). \tag{2.1.9}$$

3. *Furthermore, if* $\operatorname{div}\boldsymbol{f}=0$ *in* Ω, $\pi_0=0$ *and* $\boldsymbol{g}\times\boldsymbol{n}=\boldsymbol{0}$, *then* $\pi=0$.

Observe that the condition (2.1.7) is a necessary compatibility condition for the existence of solution for the Stokes problem $(\mathcal{S})$. When the datum does not satisfy this compatibility condition, we introduce the following variant of the Stokes problem $(\mathcal{S}')$ (see [7]): Find functions $\boldsymbol{u},\pi$ and constants c_i, for $i=1,...,I$ such that :

$$(\mathcal{S}')\begin{cases} -\gamma\Delta\,\boldsymbol{u}+\nabla\,\pi=\boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u}=0 & \text{in}\,\Omega,\\ \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{g}\times\boldsymbol{n} \quad \text{and} \quad \pi=\pi_0 \ \ \text{on}\,\Gamma_0,\ \pi=\pi_0+c_i & \text{on}\,\Gamma_i, \quad 1\leqslant i\leqslant I\\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}=0, \quad 1\le i\le I, \end{cases}$$

We recall the following results (see [7, Theorem 2.3] and [7, Theorem 2.6]).

Proposition 2.1.3. *We suppose* Ω *to be of class* $\mathcal{C}^{1,1}$. *Let* $\boldsymbol{f},\boldsymbol{g}$ *and* π_0 *such that*

$$\boldsymbol{f}\in[\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)]',\ \ \boldsymbol{g}\times\boldsymbol{n}\in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma),\ \ \pi_0\in W^{1-\frac{1}{p},p}(\Gamma), \tag{2.1.10}$$

and constants $c_1,...,c_I$ *satisfying the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\pi\|_{W^{1,p}(\Omega)} \le C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'}+\|\pi_0\|_{W^{1-1/p,p}(\Gamma)}+\|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\Big). \tag{2.1.11}$$

and where $c_1,...,c_I$ *are given by*

$$c_i=\langle\boldsymbol{f},\,\nabla q_i^N\rangle_\Omega-\langle\pi_0\,,\,\nabla q_i^N\cdot\boldsymbol{n}\rangle_\Gamma. \tag{2.1.12}$$

Moreover, if Ω *is of classe* $\mathcal{C}^{2,1}$, $\boldsymbol{f}\in\boldsymbol{L}^p(\Omega)$ *and* $\boldsymbol{g}\times\boldsymbol{n}\in\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma)$, *then* $\boldsymbol{u}\in\boldsymbol{W}^{2,p}(\Omega)$.

Proposition 2.1.4. *We assume that* Ω *is of classe* $\mathcal{C}^{2,1}$, *let* $\boldsymbol{f}$, $\boldsymbol{g}$ *and* π_0 *such that*

$$\boldsymbol{f}\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}\,,\Omega)]',\ \ \boldsymbol{g}\times\boldsymbol{n}\in\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma),\ \ \pi_0\in W^{1-\frac{1}{r},r}(\Gamma), \tag{2.1.13}$$

with $r \leqslant p$ *and* $\dfrac{1}{r} \leqslant \dfrac{1}{p} + \dfrac{1}{3}$. *Then, the problem* $(\mathcal{S}')$ *has a unique solution* $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\pi \in W^{1,r}(\Omega)$ *and constants* $c_1, ..., c_I$ *satisfying the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,r}(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} + \|\boldsymbol{g} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\Big). \tag{2.1.14}$$

and $c_1, ..., c_I$ *are given by* (2.1.12), *where we replace the duality brackets on* Ω *by :*

$$\langle .\,,\,.\rangle_\Omega = \langle .\,,\,.\rangle_{(\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega))' \times \boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)}.$$

The following continuous embedding $\boldsymbol{H}^1 \hookrightarrow \boldsymbol{L}^q(\Omega)$ for any $q \leqslant 5$, is obvious and it verifies the estimate :

$$\forall \boldsymbol{v} \in \boldsymbol{L}^q(\Omega), \quad \|\boldsymbol{v}\|_{\boldsymbol{L}^q(\Omega)} \leqslant \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}. \tag{2.1.15}$$

2.2 Regularity results for the Stokes equations with pressure boundary conditions

We want to revise the results given in the previous section concerning Stokes problem $(\mathcal{S}')$, and to give some additional regularity results.

First we suppose that $\boldsymbol{g} = \boldsymbol{0}$ and $\boldsymbol{f}$, π_0 are given functions such that :

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega))', \;\; \pi_0 \in H^{-\frac{1}{2}}(\Gamma).$$

We define the space :

$$\boldsymbol{V}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \; \mathrm{div}\; \boldsymbol{u} = 0 \text{ in } \Omega, \; \boldsymbol{v} \times \boldsymbol{n} = 0 \text{ on } \Gamma, \; \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \; 1 \leqslant i \leqslant I\}.$$

We consider the following problem :

$$\text{Find } \boldsymbol{u} \in \boldsymbol{V}(\Omega) \text{ such that for any } \boldsymbol{v} \in \boldsymbol{V}(\Omega): \quad \gamma \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} = \langle \boldsymbol{f}\,,\,\boldsymbol{v}\rangle_\Omega - \langle \pi_0\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma. \tag{2.2.1}$$

and find constants $c = (c_1, ..., c_I)$ satisfying :

$$c_i = \langle \boldsymbol{f}\,,\,\nabla q_i^N\rangle_\Omega - \langle \pi_0\,,\,\nabla q_i^N \cdot \boldsymbol{n}\rangle_\Gamma. \tag{2.2.2}$$

where : $\langle .\,,\,.\rangle = \langle .\,,\,.\rangle_{(\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega))' \times \boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)}$ and $\langle .\,,\,.\rangle_\Gamma = \langle .\,,\,.\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}$.

Theorem 2.2.1. *The variational formulation* (2.2.1)-(2.2.2) *is equivalent to find* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ *and* $\pi \in L^2(\Omega)$ *solution of* $(\mathcal{S}')$.

Proof. **i)** We begin by proving that if $(\boldsymbol{u},\, \pi,\, \boldsymbol{c})$ is a solution of $(\mathcal{S}')$, then $\boldsymbol{u}$ is a solution of the variational problem (2.2.1) and $\boldsymbol{c}$ satisfies (2.2.2). Multiplying the first equation in $(\mathcal{S}')$ by $\boldsymbol{v}$ in $\boldsymbol{V}(\Omega)$, integrating by parts in Ω, and using the fact that $\operatorname{div} \boldsymbol{u} = 0$ in Ω, we obtain, taking into account the boundary condition that verify the pressure π on Γ, that $\boldsymbol{u}$ is a solution of (2.2.1). It remains to prove the relation (2.2.2). Let $\boldsymbol{v} \in \boldsymbol{H}^1_\sigma(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ and we set

$$\boldsymbol{v}_0 = \boldsymbol{v} - \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \nabla q_i^N. \tag{2.2.3}$$

Observe that $\boldsymbol{v}$ belongs to $\boldsymbol{V}(\Omega)$. Multiplying the first equation of $(\mathcal{S}')$ by $\boldsymbol{v}$, integrating by parts in Ω and using the relation (2.2.1) with the test function $\boldsymbol{v}_0$, we obtain

$$\sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \int_\Omega \boldsymbol{f} \cdot \nabla q_i^N \, \mathrm{d}\boldsymbol{x} = \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \Big((c_i) + \langle \pi_0,\, \nabla q_i^N \cdot \boldsymbol{n} \rangle_\Gamma \Big).$$

It suffices to take $\boldsymbol{v} = \nabla q_j^N$, we deduce the required relation (2.2.2).

ii) Reciprocally, let $\boldsymbol{u} \in \boldsymbol{V}(\Omega)$ a solution of (2.2.1) and $c_1, \ldots, c_I$ constants satisfying (2.2.2). To prove the first equation of the problem $(\mathcal{S}')$, we take $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ as a test function in (2.2.1) and we use the De Rham's Theorem. Moreover, since $\Delta \boldsymbol{u} \in \boldsymbol{H}^{-1}(\Omega)$, we deduce that $\nabla \pi \in \boldsymbol{H}^{-1}(\Omega)$. Due to [7], π belongs to $L^2(\Omega)$. Now, applying divergence operator to the first equations of problem $(\mathcal{S}')$, we obtain:

$$\Delta \pi = \operatorname{div} \boldsymbol{f} \in W^{-1,6/5}(\Omega). \tag{2.2.4}$$

Since $\pi \in L^2(\Omega)$, we can prove that the trace of π on Γ belongs to $H^{-1/2}(\Gamma)$ (see [5]). It remains to prove the boundary condition on the pressure. For this, we can use exactly the same arguments in [8, Theorem 3.2].

□

Now, we discuss the solvability of the system $(\mathcal{S}')$.

Theorem 2.2.2. *For any* $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl}\,,\Omega)]'$ *and* $\pi_0 \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, *the Stokes system* $(\mathcal{S}')$ *admits a unique solution* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, $\pi \in L^2(\Omega)$ *and constants* $c_1, ..., c_I$. *satisfying the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} \Big)$$

and where $c_1, ..., c_I$ *are given by* (2.2.2).

Proof. According to [8, Corollary 3.2], for any $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, the seminorm $\boldsymbol{v} \longmapsto \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}$ is equivalent to the norm $\|.\|_{\boldsymbol{X}^2(\Omega)}$.
Then, applying Lax Milgram Lemma, problem (2.2.1) admitis a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$.

Next, thanks to De Rham's Lemma, there exsits a unique solution $\pi \in L^2(\Omega)$.

Now, we will consider the case where the external forces $\boldsymbol{f}$ belongs to $(\boldsymbol{H}_0^{r',2}(\mathbf{curl}\,,\Omega))'$ with $\frac{6}{5} \leqslant r \leqslant 2$.
We recall that there exist function : $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$ and $\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)$ such that : $\boldsymbol{f} = \boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\psi}$. Observe that, using again the Lax-Milgram Lemma, the velocity $\boldsymbol{u}$ still belongs to $\boldsymbol{H}^1(\Omega)$. But the regularity of the pressure can be improved. Indeed , if we apply the divergence operator to the both sides of the first equation of problem $(\mathcal{S}')$, we find that the pressure is solution of the problem :

$$\Delta\pi = \operatorname{div}\boldsymbol{F} \text{ in } \Omega \quad \text{and} \quad \pi = \pi_0 \text{ on } \Gamma_0,\ \pi = \pi_0 + c_i \text{ on } \Gamma_i \tag{2.2.5}$$

Hence, by choosing a convenient boundary condition π_0, we can improve the reguralrity of the solution π. □

Theorem 2.2.3. *Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',2}(\mathbf{curl}\,,\Omega))'$ *with* $\frac{6}{5} \leqslant r \leqslant 2$.

i) If $\pi_0 \in W^{-1/r^*,r^*}(\Gamma)$, with $r^* = \frac{3r}{3-r}$, then the solution π *of* (2.2.5) belongs to $L^{r^*}(\Omega)$.

ii) If $\pi_0 \in W^{1-1/r,r}(\Gamma)$, then the solution π *of the problem* (2.2.5) *belongs to* $W^{1,r}(\Omega)$.

Proof. We write π satisfying (2.2.5) in the form : $\pi = \lambda + \mu$ where λ and μ satisfy :

$$\Delta\,\lambda = \operatorname{div}\boldsymbol{F} \quad \text{in } \Omega \qquad \text{and} \qquad \lambda = 0 \quad \text{on}\,\Gamma. \tag{2.2.6}$$

$$\Delta\,\mu = \mathbf{0} \quad \text{in}\,\Omega \qquad \text{and} \qquad \mu = \pi_0 \quad \text{on } \Gamma_0 \text{ and } \mu = \pi_0 + c_i \text{ on } \Gamma_i. \tag{2.2.7}$$

1. Since $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',2}(\mathbf{curl}\,,\Omega))'$, $\operatorname{div}\boldsymbol{f} = \operatorname{div}\boldsymbol{F}$ belongs to $\boldsymbol{W}^{-1,r}(\Omega)$. Then the solution λ of problem (2.2.6) belongs to $\boldsymbol{W}^{1,r}(\Omega) \hookrightarrow \boldsymbol{L}^{r*}(\Omega)$. Next, it is clear that if $\pi_0 \in W^{-1/r^*,r^*}(\Gamma)$ then the solution μ of problem (2.2.7) belongs to $\boldsymbol{L}^{r*}(\Omega)$. So, we deduce that the solution π of problem (2.2.5) belongs to $\boldsymbol{L}^{r*}(\Omega)$.

2. Now, if $\pi_0 \in W^{1-1/r,r}(\Gamma)$, the solution μ of (2.2.7) belongs to $\boldsymbol{W}^{1,r}(\Omega)$. Since the solution of λ of (2.2.6) still belongs to $\boldsymbol{W}^{1,r}(\Omega)$, we have immediatly $\pi \in W^{1,r}(\Omega)$.

□

Remark 2.2.4. Observe that we choose π_0. So that the solution μ of (2.2.7) belongs to a class of spaces containing spaces for $\lambda \in \boldsymbol{W}^{1,r}(\Omega)$ solution of (2.2.6).
For $\frac{6}{5} \leqslant r \leqslant 2$, the minimal regularity for λ is $\boldsymbol{W}^{1,\frac{6}{5}}(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$, while the maximal regularity is $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$.

Now, we want to investigate the case $1 < p < \infty$. So, we will try to extend the previous result with datum $\boldsymbol{f}$ only in $(\boldsymbol{H}_0^{r',2}(\mathbf{curl}\,,\Omega))'$ to the case $(\boldsymbol{H}_0^{r',p'}(\mathbf{curl}\,,\Omega))'$, with $r \leqslant p$ and $\frac{1}{r} \leqslant \frac{1}{p} + \frac{1}{3}$.

We start by showing the existence and uniqueness of weak solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

Theorem 2.2.5. *For any* $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathbf{curl}\,,\Omega))'$ *with* $r \leqslant p$, $\dfrac{1}{r} \leqslant \dfrac{1}{p} + \dfrac{1}{3}$ *and* $\pi_0 \in W^{-\frac{1}{p},p}(\Gamma)$. *Then the problem* $(\mathcal{S}')$ *has a unique solution* $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ *and constants* $c_1, ..., c_i$ *satisfying the estimate :*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{W^{-1/p,p}(\Gamma)}\Big). \tag{2.2.8}$$

and where c_i, $i = 1, ..., I$ *are given by* (2.2.2) *with the dualities :*

$$\langle\cdot\,,\,\cdot\rangle_\Omega = \langle\cdot\,,\,\cdot\rangle_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'\times[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]}, \qquad \langle\cdot\,,\,\cdot\rangle_\Gamma = \langle\cdot\,,\,\cdot\rangle_{W^{-1/p,p}(\Gamma)\times W^{1-1/p,\,p}(\Gamma)}. \tag{2.2.9}$$

Proof. It is easy to verify that $(\mathcal{S}')$ is equivalent to the problem :

$$\begin{aligned} &\text{Find } \boldsymbol{u} \in \boldsymbol{V}_N^p(\Omega) \text{ such that, for any } \boldsymbol{v} \in \boldsymbol{V}_N^{p'}(\Omega) : \\ &\gamma\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,dx = \langle \boldsymbol{f}\,,\,\boldsymbol{v}\rangle_\Omega - \langle \pi_0\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma \end{aligned} \tag{2.2.10}$$

and find c_i, $i = 1, ..., I$ satisfying (2.2.2) with the dualities defined in (2.2.9).

Since the bilinear form, $a(\boldsymbol{u}\,,\,\boldsymbol{v}) = \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,dx$ satisfies the Inf-Sup condition (2.1.3) and the right-hand side defines an element of $(\boldsymbol{V}_N^p(\Omega))'$, applying Babuska-Brezzi Theorem, (2.2.10) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$.

Next, the pressure is obtained by using a variant of De Rham's Theorem (see [3, Theorem 2.8]). Indeed, since $\boldsymbol{L}^r(\Omega) \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)$, using the decomposition (2.1.6) the term $\boldsymbol{f} - \Delta\boldsymbol{u}$ belongs to $\boldsymbol{W}^{-1,p}(\Omega)$ and satisfies :

$$\langle \boldsymbol{f} - \Delta\boldsymbol{u}\,,\,\boldsymbol{\psi}\rangle_\Omega = \boldsymbol{0}, \quad \forall\boldsymbol{\psi} \in \mathcal{D}(\Omega), \quad \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in } \Omega.$$

Then there exist $\pi \in L^p(\Omega)$ such that $\boldsymbol{f} - \Delta\boldsymbol{u} = \nabla\pi$.

□

As previously, the regularity of the pressure can be improved by using the fact that π is solution of (2.2.5) and then by choosing an convenient π_0. We write the pressure again in the form $\pi = \lambda + \mu$ with λ and μ satisfying (2.2.6) and (2.2.7).

Theorem 2.2.6. *Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathbf{curl}\,,\Omega))'$ *with* $r \leqslant p$, $\dfrac{1}{r} \leqslant \dfrac{1}{p} + \dfrac{1}{3}$

i) If $\pi_0 \in W^{-1/r^*,r^*}(\Gamma)$, with $r < 3$, then the pressure π belongs to $L^{r^*}(\Omega)$.

ii) If $r \geq 3$, and $\pi_0 \in W^{-1/q,q}(\Gamma)$ with $q > p$, we have $\pi \in L^q(\Omega)$.

iii) If $\pi_0 \in W^{1-1/r,r}(\Gamma)$, we have $\pi \in W^{1,r}(\Omega)$.

Proof. Since $\boldsymbol{f} \in (\boldsymbol{H}_0^{r',p'}(\mathbf{curl}\,,\Omega))'$, $\operatorname{div} \boldsymbol{f} = \operatorname{div} \boldsymbol{F} \in \boldsymbol{W}^{-1,r}(\Omega)$. The solution λ of problem (2.2.6) still belongs to $\boldsymbol{W}^{1,r}(\Omega)$. The regularity of μ depends on the regularity choosen for π_0.

1. If $\pi_0 \in W^{-1/r^*,r^*}(\Gamma)$, with $r < 3$, then μ belongs to $\boldsymbol{L}^{r*}(\Omega)$ since $\lambda \in \boldsymbol{W}^{1,r}(\Omega) \hookrightarrow \boldsymbol{L}^{r*}(\Omega)$, we deduce that $\pi \in \boldsymbol{L}^{r*}(\Omega)$.
2. For the values $r \geqslant 3$, if $\pi_0 \in W^{-\frac{1}{q},q}(\Gamma)$ with $q > p$, we have $\mu \in L^q(\Omega)$. In this case $\lambda \in \boldsymbol{W}^{1,r}(\Omega) \hookrightarrow \boldsymbol{L}^s(\Omega)$ for any finite s if $r = 3$ and $s = \infty$ if $r > 3$.
   we deduce that the pressure π belongs to $L^q(\Omega)$.
3. We use exactly the same argument in Theorem 2.2.3.

□

2.3 Brinkman Forchheimer equations with pressure boundary conditions : $\boldsymbol{L}^2-$Theory

In this section, we are interested to study the regularity of solution for the homogeneous Brinkman Forchheimer problem involving L^2 theory: find $\boldsymbol{u}$, π and $\boldsymbol{c}$ such that:

$$\begin{cases} -\gamma\Delta\,\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\,\boldsymbol{u} + b|\boldsymbol{u}|^\alpha\boldsymbol{u} + \nabla\,q = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 & \text{in}\,\Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad q + \frac{1}{2}|\boldsymbol{u}|^2 = q_0 \;\text{ on } \Gamma_0 \text{ and } q + \frac{1}{2}|\boldsymbol{u}|^2 = q_0 + c_i & \text{on}\,\Gamma_i,\; i = 1,...,I \\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases}$$

Where $\boldsymbol{c} = (c_1, \ldots, c_I) \in \mathbb{R}^I$.

We trun to the non linear term. The fact that we have only the tangential part of the velocity vanishes on the boundary, we have only

$$\int_\Omega (\boldsymbol{u}\cdot\nabla\,)\boldsymbol{u} = \frac{1}{2}\int_\Gamma |\boldsymbol{u}\cdot\boldsymbol{n}|^2(\boldsymbol{u}\cdot\boldsymbol{n}).$$

Since we have no information about the normal part of the velocity, we do not know how to control the last term. To solve this difficulty, we observe that this non linearity can be written using the following identity :

$$\boldsymbol{u}\cdot\nabla\,\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u} + \frac{1}{2}\nabla|\boldsymbol{u}|^2.$$

and to consider the total pressure $\pi = q + \frac{1}{2}|\boldsymbol{u}|^2$ absorbs the additional term $\frac{1}{2}|\boldsymbol{u}|^2$. Then we can rewrite the Brinkman Forchheimer problem as

$$\begin{cases} -\gamma\Delta\,\boldsymbol{u} + \mathbf{curl}\boldsymbol{u}\times\boldsymbol{u} + b|\boldsymbol{u}|^\alpha\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 & \text{in}\,\Omega,\\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \pi = \pi_0 \ \text{ on } \Gamma_0 \quad \text{and} \quad \pi = \pi_0 + c_i & \text{on}\,\Gamma_i,\\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0, \quad 1\leq i\leq I. \end{cases} \tag{2.3.1}$$

The use of this identity facilitate the existence theory. Indeed, in this case, the new non linear term $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u}$ vanishes when one considers energy estimates :

$$\int_\Omega \mathbf{curl}\,(\boldsymbol{u}\times\boldsymbol{u})\cdot\boldsymbol{u}\,dx = 0. \tag{2.3.2}$$

We should point out that this technique was presented in [1], [28] and [7].
The goal of this section is to establish the existence of a weak solution in the hilbertian case for the Brinkman-Forchheimer system (2.3.1). The first, we give a variational formulation.

Theorem 2.3.1. *Let* $\boldsymbol{f}\in(\boldsymbol{H}_0^{6,2}(\mathbf{curl}\,,\Omega))'$, $\pi_0\in H^{-\frac{1}{2}}(\Gamma)$. *Then the following two problems are equivalent :*

1. *Find* $(\boldsymbol{u},\pi)\in\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ *and* c_i *satisfying* (2.3.1).
2. *Find* $\boldsymbol{u}\in\boldsymbol{V}_N(\Omega)$ *such that for all* $\boldsymbol{v}\in\boldsymbol{V}_N(\Omega)$

$$\gamma\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,dx + \int_\Omega(\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u})\cdot\boldsymbol{v}\,dx + b\int_\Omega|\boldsymbol{u}|^\alpha\boldsymbol{u}\cdot\boldsymbol{v}\,dx = \langle\boldsymbol{f},\,\boldsymbol{v}\rangle_\Omega - \langle\pi_0\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma. \tag{2.3.3}$$

and find constants $c=(c_1,...,c_I)$ *satisfying for any* $i=1,...,I$

$$c_i = \langle\boldsymbol{f},\,\nabla q_i^N\rangle_\Omega. \tag{2.3.4}$$

Proof. We begin by proving that if $(\boldsymbol{u},\pi)$ and c are solutions of (2.3.1) then $\boldsymbol{u}$ solutions of (2.3.3). Mutiplying the first equation in (2.3.1) by $\boldsymbol{v}\in\boldsymbol{V}_N(\Omega)$, integrating by parts in Ω, we obtain :

$$\int_\Omega(-\gamma\Delta\boldsymbol{u}+\nabla\pi).\boldsymbol{v}\,dx + \int_\Omega(\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u}).\boldsymbol{v}\,dx + b\int_\Omega|\boldsymbol{u}|^\alpha\boldsymbol{u}.\boldsymbol{v}\,dx = \langle\boldsymbol{f},\boldsymbol{v}\rangle_\Omega. \tag{2.3.5}$$

Observe that : $-\Delta\boldsymbol{u}+\nabla\pi$ belongs to $(\boldsymbol{H}_0^{6,2}(\mathbf{curl}\,,\Omega))'$.
We introduce the space :

$$M(\Omega) = \{(\boldsymbol{v},\theta)\in\boldsymbol{H}_\sigma^1(\Omega)\times L^2(\Omega),\ -\Delta\boldsymbol{v}+\nabla\theta\in(\boldsymbol{H}_0^{6,2}(\mathbf{curl}\,,\Omega))'\} \tag{2.3.6}$$

equipped with the norm :

$$\|(\boldsymbol{v},\theta)\|_{M(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{H}_\sigma^1(\Omega)} + \|\theta\|_{\boldsymbol{L}^2(\Omega)} + \|-\Delta\boldsymbol{v}+\Delta\theta\|_{(\boldsymbol{H}_0^{6,2}(\mathbf{curl}\,,\Omega))'}. \tag{2.3.7}$$

We can verify that $\mathcal{D}_\sigma(\overline{\Omega})\times\mathcal{D}(\overline{\Omega})$ is dense in $M(\Omega)$ (see [8, Lemma 5.5] for a similar proof). Moreover, we have the following Green formula : for any $(\boldsymbol{u},\pi)\in M(\Omega)$ and $\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)$ with $\boldsymbol{v}\times\boldsymbol{n}=0$ on Γ :

$$\int_\Omega(-\gamma\Delta\boldsymbol{u}+\nabla\pi).\boldsymbol{v}\,dx = \int_\Omega\mathbf{curl}\,\boldsymbol{u}.\mathbf{curl}\,\boldsymbol{v}\,dx + \langle\pi\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma. \tag{2.3.8}$$

Using the fact that

$$\forall \boldsymbol{v} \in \boldsymbol{V}_N(\Omega) \ : \ \langle \pi \, , \, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_\Gamma = \langle \pi_0 \, , \, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_\Gamma, \tag{2.3.9}$$

we deduce from (2.3.5) that $\boldsymbol{u}$ is a solution of (2.3.3). It remains to prove the relation (2.3.4). Let $\boldsymbol{v} \in \boldsymbol{H}^1_\sigma(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = 0$ on Γ and set :

$$\boldsymbol{v}_0 = \boldsymbol{v} - \sum_{i=1}^{I} (\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}) \nabla q_i^N,$$

observe that $\boldsymbol{v}_0$ belongs to $\boldsymbol{V}_N(\Omega)$. Multiplying the first equation of (2.3.1) by $\boldsymbol{v}$ and using (2.3.3) with the test function $\boldsymbol{v}_0$, we obtain :

$$\begin{aligned} &\sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \int_\Omega \Big(\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} \Big) . \nabla q_i^N + b \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \int_\Omega |\boldsymbol{u}|^\alpha \boldsymbol{u} . \nabla q_i^N \, dx \\ &\quad = \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \langle \boldsymbol{f} \, , \, \nabla q_i^N \rangle - \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \Big) \Big(c_i + \langle \pi_0 \, , \, \nabla q_i^N \cdot \boldsymbol{n} \rangle \Big). \end{aligned} \tag{2.3.10}$$

Testing with $\boldsymbol{v} = \nabla q_j^N$, we deduce the relation (2.3.4).

Reciprocally, let $\boldsymbol{u} \in \boldsymbol{V}_N(\Omega)$ solution of (2.3.3) and $c_1, ..., c_I$ constants satisfying (2.3.4). In order to check that $\boldsymbol{u}$ satisfies the first equation of the problem (2.3.1), we consider $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$ with $\operatorname{div} \boldsymbol{v} = 0$ as a test function in (2.3.8) and we use the De Rham's Theorem. Moreover, since $\Delta \, \boldsymbol{u} \in \boldsymbol{H}^{-1}(\Omega)$, $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} \in \boldsymbol{L}^{3/2}(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$ and $|\boldsymbol{u}|^\alpha \boldsymbol{u} \in \boldsymbol{L}^q(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$ with $2 \leqslant q \leqslant 3$, we deduce that $\nabla \pi \in \boldsymbol{H}^{-1}(\Omega)$. Due to [3], π belongs to $L^2(\Omega)$. Now, applying divergence operator to the first equations of problem (2.3.1), we obtain:

$$\Delta \, \pi = \operatorname{div} \boldsymbol{f} - \operatorname{div} (\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}) - b |\boldsymbol{u}|^\alpha \boldsymbol{u} \in W^{-1, 6/5}(\Omega). \tag{2.3.11}$$

Since $\pi \in L^2(\Omega)$, we can prove that the trace of π on Γ belongs to $H^{-1/2}(\Gamma)$ (see [5]). Finally to prove the boundary condition on the pressure, we can use the same arguments in [7, Theorem 3.2]. □

Next, we discuss the sovability of the variational problem (2.3.3)-(2.3.4) by means of the Standard Galerkin's method. Nonethless, we state it for completeness.

Step 1: *Construction of approximating solutions.*

$$\boldsymbol{V} = \{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in}\, \Omega, \ \ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \ \text{ on}\, \Gamma \},$$

$$\boldsymbol{Z}_N(\Omega) = \Big\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \ \ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \ \text{ on}\, \Gamma, \ \int_\Omega \boldsymbol{v} \cdot \nabla \, q_i^N = 0, \, 1 \le i \le I \Big\}.$$

In order to apply the Galerkin method, we need a special base of the underlying Hilbert space. While in the classical approach this base is constituted by the eigenfunction of the Stokes operator, in our case, the base will be modified. Indeed, we consider the operator

$$\begin{aligned} \Lambda \quad : \quad \boldsymbol{L}^2(\Omega)\perp \boldsymbol{K}^2_N(\Omega) &\longrightarrow \boldsymbol{Z}_N(\Omega) \longrightarrow \boldsymbol{L}^2(\Omega)\perp \boldsymbol{K}^2_N(\Omega)\\ \boldsymbol{F} &\longmapsto \boldsymbol{z} \ \longmapsto \boldsymbol{z}, \end{aligned}$$

where $\boldsymbol{z}$ is the solution given by Theorem 2.2.2 which satisfies:

$$\int_\Omega \boldsymbol{z}\cdot \mathbf{grad}\, q_i^N = \int_\Omega q_i^N \operatorname{div} \boldsymbol{z} + \sum_{k=0}^{I} \langle \boldsymbol{z}\cdot\boldsymbol{n},\, q_i^N\rangle_{\Gamma_k} = 0, \tag{2.3.12}$$

because $q_i^N|_{\Gamma_0} = 0$, $q_i^N|_{\Gamma_k} = \text{cste}$ and $\langle \boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_k} = 0$ for any $1 \le k \le I$.

We set $\boldsymbol{M} = \boldsymbol{L}^2(\Omega)\perp \boldsymbol{K}^2_N(\Omega)$. The operator Λ is clearly linear and continuous from $\boldsymbol{M}$ into $\boldsymbol{Z}_N(\Omega)$. Since Id is compact from $\boldsymbol{Z}_N(\Omega)$ into $\boldsymbol{M}$, the operator Λ is considered as a compact operator from $\boldsymbol{M}$ into itself. This operator is also self-adjoint as

$$\int_\Omega \Lambda\boldsymbol{F}_1\cdot\boldsymbol{F}_2 = \int_\Omega \mathbf{curl}\,\boldsymbol{z}_1\cdot\boldsymbol{z}_2 = \int_\Omega \boldsymbol{F}_1\cdot\Lambda\boldsymbol{F}_2,$$

when $\Lambda\boldsymbol{F}_i = \boldsymbol{z}_i$, $i = 1, 2$. Hence, this operator Λ possesses a hilbertian basis formed by a sequence of eigenfunctions $\boldsymbol{z}_k$:

$$\Lambda\boldsymbol{z}_k = \lambda_k\boldsymbol{z}_k,\ k\ge 1,\ \lambda_k > 1,\ \lambda_k\to\infty,\ k\to\infty$$

$$\boldsymbol{z}_k \in \boldsymbol{V}^2_N(\Omega), \quad \int_\Omega \mathbf{curl}\,\boldsymbol{z}_k\cdot\mathbf{curl}\,\boldsymbol{v} = \lambda_k\int_\Omega \boldsymbol{z}_k\cdot\boldsymbol{v}, \quad \forall \boldsymbol{v}\in\boldsymbol{V}^2_N(\Omega). \tag{2.3.13}$$

As usual

$$\int_\Omega \boldsymbol{z}_k\cdot\boldsymbol{z}_l = \delta_{kl}, \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{z}_k\cdot\boldsymbol{z}_l = \lambda_k\delta_{kl}.$$

Remark that (2.3.13) is also valid for any $\boldsymbol{v}\in\boldsymbol{V}$. Indeed, let $\boldsymbol{v}$ be in $\boldsymbol{V}$ and we set

$$\widetilde{\boldsymbol{v}} = \boldsymbol{v} - \sum_{i=1}^{I}\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}\nabla q_i^N.$$

Then, we establish

$$\int_\Omega \mathbf{curl}\,\boldsymbol{z}_k\cdot\mathbf{curl}\,\boldsymbol{v} = \int_\Omega \mathbf{curl}\,\boldsymbol{z}_k\cdot\mathbf{curl}\,\widetilde{\boldsymbol{v}} = \lambda_k\int_\Omega \boldsymbol{z}_k\cdot\widetilde{\boldsymbol{v}} = \lambda_k\int_\Omega \boldsymbol{z}_k\cdot\boldsymbol{v},$$

where we have used the fact that (see (2.3.12))

$$\int_\Omega \boldsymbol{z}_k\cdot\mathbf{grad}\, q_i^N = 0, \quad \forall i = 1,\ldots,I.$$

Now, we can interpret (2.3.13) as follows: for each k, there exists $\pi_k \in L^2(\Omega)$ such that:

$$\begin{cases} -\nu\Delta\, \boldsymbol{z}_k + \nabla\, \pi_k = \lambda_k \boldsymbol{z}_k \quad \text{and} \quad \operatorname{div} \boldsymbol{z}_k = 0 & \text{in } \Omega, \\ \boldsymbol{z}_k \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \pi_k = 0 & \text{on } \Gamma, \\ \langle \boldsymbol{z}_k \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \;\; \forall 1 \leq i \leq I. \end{cases}$$

Now, observe that $\boldsymbol{L}^2(\Omega) = \boldsymbol{M} \oplus \boldsymbol{K}^2_N(\Omega)$. We note by $(\boldsymbol{y}_i)_i$ an orthonormal base of $\boldsymbol{K}^2_N(\Omega)$. We know that each $\boldsymbol{y}_i$ is a linear combination of $\nabla\, q^N_1, \ldots, \nabla\, q^N_I$. Then, the sequence $(\boldsymbol{w}_i)_{i\in\mathbb{N}^*}$ defined by

$$\boldsymbol{w}_i = \begin{cases} \boldsymbol{y}_i & \text{if} \quad 1 \leq i \leq I, \\ \boldsymbol{z}_{i-I} & \text{if} \quad i \geq I+1. \end{cases} \tag{2.3.14}$$

is a hilbertian base for the space $\boldsymbol{L}^2(\Omega)$ and for any $i \in \mathbb{N}^*$, $\boldsymbol{w}_i \in \boldsymbol{V}$. Using this base, we introduce the space $\boldsymbol{V}_m = \langle w_1, \ldots, w_m\rangle$ and we can define approximate solution $\boldsymbol{u}_m$ of $(\mathcal{BFP})$ as follows:

$$\boldsymbol{u}_m(t) = \sum_{i=1}^{m} g_{im}(t)\, \boldsymbol{w}_i, \tag{2.3.15}$$

such that:

$$\gamma \int_\Omega \mathbf{curl}\, \boldsymbol{u}_m \cdot \mathbf{curl}\, \boldsymbol{w}_i\, dx + \int_\Omega (\mathbf{curl}\, \boldsymbol{u}_m \times \boldsymbol{u}_m) \cdot\, \boldsymbol{w}_i\, dx + b \int_\Omega |\boldsymbol{u}_m|^\alpha \boldsymbol{u}_m \cdot \boldsymbol{w}_i\, dx = \langle \boldsymbol{f},\, \boldsymbol{w}_i\rangle_\Omega \tag{2.3.16}$$
$$- \langle \pi_0,\; w_i \cdot \boldsymbol{n}\rangle_\Gamma, \quad 1 \leq i \leq m,$$

Step 2: *a priori estimates.*

Let us define the mapping : $\Phi_m \;\; : \;\; \boldsymbol{V}_m \longrightarrow \boldsymbol{V}_m$ as for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_m$:

$$\int_\Omega \Phi_m(\boldsymbol{u})\cdot \boldsymbol{v} = \gamma \int_\Omega \mathbf{curl}\, \boldsymbol{u}\cdot \mathbf{curl} \boldsymbol{v}\, dx + \int_\Omega \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} \cdot\, \boldsymbol{v}\, dx + b \int_\Omega |\boldsymbol{u}|^\alpha \boldsymbol{u}\cdot \boldsymbol{v}\, dx - \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega - \langle \pi_0,\; \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma. \tag{2.3.17}$$

The continuity of the mapping Φ_m is obvious. Using (2.3.2), we get :

$$\begin{aligned} \int_\Omega \Phi_m(\boldsymbol{u}) \cdot \boldsymbol{u} \;\; &= \;\; \|\mathbf{curl}\, \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}\|^{\alpha+2}_{\alpha+2} - \langle \boldsymbol{f},\, \boldsymbol{u}\rangle_\Omega - \langle \pi_0,\; \boldsymbol{u} \cdot \boldsymbol{n}\rangle_\Gamma \\ &\geq C\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{\,1}(\Omega)} - \big(\|\boldsymbol{f}\|_{[\boldsymbol{H}^{\,6,2}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big)\|\boldsymbol{u}\|_{\boldsymbol{H}^{\,1}(\Omega)}, \end{aligned}$$

where we have used both of (1.1.4) and (2.1.15).
Hence, $(\Phi_m(\boldsymbol{u})\,,\, \boldsymbol{u}) \geqslant 0$ for all $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = r$ with $r > C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}^{\,6,2}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big)$. Then the hypothesis of Brower's Theorem (see [43, Lemma 4.3]) is satisfied and there exists a solution $\boldsymbol{u}_m$ of (2.3.16). Moreover, the solution $\boldsymbol{u}_m$ satisfies the estimate :

$$\|\boldsymbol{u}_m\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}^{\,6,2}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\Big). \tag{2.3.18}$$

Step 3: *Passage to the limit.*
It follows from the last estimate that the sequence $(\boldsymbol{u}_m)_m$ is bounded in the space $\boldsymbol{H}^1(\Omega)$, then we can extand a subsequence still denoted $(\boldsymbol{u}_m)_m$ such that:

$$\boldsymbol{u}_m \longrightarrow \boldsymbol{u} \ \text{ in } \ \boldsymbol{H}^1(\Omega) \quad \text{weakly as } m \to \infty.$$

Since the embedding of $\boldsymbol{H}^1(\Omega)$ in $\boldsymbol{L}^4(\Omega)$ is compact, we obtain that:

$$\boldsymbol{u}_m \longrightarrow \boldsymbol{u} \ \text{ in } \ \boldsymbol{L}^4(\Omega) \quad \text{strongly as } m \to \infty.$$

Next, by passing to the limit in (2.3.16); we obtain that $\boldsymbol{u}$ satisfies (2.3.3). Having obtained the velocity, we shall indicate how the pressure is constructed. Since $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, we have $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} \in \boldsymbol{L}^{3/2}(\Omega) \hookrightarrow (\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega))'$ and $|\boldsymbol{u}|^\alpha \boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$ with $2 \leqslant q \leqslant 3$. Then : $-\gamma \Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} + b|\boldsymbol{u}|^\alpha \boldsymbol{u} - \boldsymbol{f} \in \boldsymbol{W}^{-1,p}(\Omega)$. Using De Rham Theorem, there exists a function $\pi \in L^2(\Omega)$ solution of (2.3.1).

We then conclude the following Theorem,

Theorem 2.3.2. *Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega))'$ *and* $\pi_0 \in H^{-1/2}(\Gamma)$, *such that*

$$\Big(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega))'} + \|\pi_0\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \Big) \leqslant r,$$

then the Brinkman Forchheimer problem (2.3.1) *admits at least a solution* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ *and* $\pi \in L^2(\Omega)$ *satisfying the estimate*

$$\| \boldsymbol{u} \|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C \Big(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega))'} + \|\pi_0\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \Big). \tag{2.3.19}$$

2.4 Brinkman-Forchheimer equations with pressure boundary conditions : $\boldsymbol{L}^p$-Theory

In this section, we study the $\boldsymbol{L}^p$ regularity of weak and strong solution of the Brinkman-Forchheimer problem (2.3.1).

2.4.1 General solution in $\boldsymbol{W}^{1,p}(\Omega)$

In the sequel, we suppose that $\boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega))]'$ with r and p satisfying :

$$1 < r < p, \ \frac{1}{r} \leqslant \frac{1}{p} + \frac{1}{3}$$

and we restrict our attention to treat the case $p > 2$, where we use the regularity of the linear Stokes problem.

Theorem 2.4.1. *Let* $p > 2$. *We suppose* $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$ *with* $r \in [\, \frac{3p}{3+p}, \, p\,]$.

(i) if $2 \leq p \leq 3$ and $\pi_0 \in W^{-1/r^*,r^*}(\Gamma)$, then the solution $(\boldsymbol{u},\, \pi)$ given by Theorem 2.3.2 belongs to $\boldsymbol{W}^{1,p}(\Omega) \times L^{r^*}(\Omega)$.

(ii) if $p > 3$, for $\frac{3p}{3+p} \leq r \leq 3$ and $\pi_0 \in W^{-1/r^*,r^*}(\Gamma)$, the pressure still belongs to $L^{r^*}(\Omega)$. For $3 \leq r \leq p$, choosing $\pi_0 \in W^{-1/q,q}(\Gamma)$ with $q > p$ implies that $\pi \in L^q(\Omega)$.

Proof. According to Theorem 2.3.2, there exists a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{H}^{\,1}(\Omega) \times L^2(\Omega)$.

(i) First case: $p \leq 3$. We know that $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} \in \boldsymbol{L}^{3/2}(\Omega)$. Since r varies between $\frac{3p}{3+p}$ and p. Moreover we have $|\boldsymbol{u}|^\alpha \boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$ with $2 \leqslant q \leqslant 3$, we consider two cases and we will use a reasoning based on the Stokes system:

  (a) Case $r \in [\,\frac{3p}{3+p},\, \frac{3}{2}\,]$. Observe that $\boldsymbol{L}^{3/2}(\Omega) \hookrightarrow [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)]'$ and the embedding $\boldsymbol{L}^3(\Omega) \hookrightarrow [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)]'$ then $\boldsymbol{f} - \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} - b|\boldsymbol{u}|^\alpha \boldsymbol{u}$ belongs to $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)]'$. Using the Stokes regularity, we deduce from Theorem 2.2.6 (i), that $(\boldsymbol{u},\, \pi)$ belongs to $\boldsymbol{W}^{\,1,p}(\Omega) \times L^{r^*}(\Omega)$.

  (b) Case $r \in [\,\frac{3}{2},\, \frac{3p}{6-p}]$. Since $\boldsymbol{u} \in \boldsymbol{W}^{\,1,p}(\Omega)$, the term $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}$ belongs to $\boldsymbol{L}^{\frac{3p}{6-p}}(\Omega)$

    i. If $p < 3$, then we have $|\boldsymbol{u}|^\alpha \boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$ such that $q = \frac{3p}{(3-p)(\alpha+1)}$. Because of $q\,(\frac{6-p}{3p}) > 1$ then $\boldsymbol{f} - \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} - b|\boldsymbol{u}|^\alpha \boldsymbol{u} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)]'$. Hence, the Stokes regularity allows us to conclude that $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{\,1,p}(\Omega) \times L^{r^*}(\Omega)$.

    ii. If $p = 3$, hence we obtain $|\boldsymbol{u}|^\alpha \boldsymbol{u} \in \boldsymbol{L}^{\,t}(\Omega)$, for all $t \geqslant 1$, which follows that $\boldsymbol{f} - \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} - b|\boldsymbol{u}|^\alpha \boldsymbol{u} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)]'$. Using again the regularity of Stokes problem, we deduce that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{\,1,p}(\Omega) \times L^{\,r*}(\Omega)$.

(ii) Second case: $p > 3$. We know that $\boldsymbol{u} \in \boldsymbol{W}^{\,1,3}(\Omega)$ and then $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} \in \boldsymbol{L}^{3-\varepsilon}(\Omega)$, with $\varepsilon > 0$ and $|\boldsymbol{u}|^\alpha \boldsymbol{u} \in \boldsymbol{L}^{\,t}(\Omega)$ for $t \in [1, +\infty]$. We consider two cases :

  (a) Case $r \in [\,\frac{3p}{3+p}, 3\,]$. Observe that $\boldsymbol{f} - \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} - b|\boldsymbol{u}|^\alpha \boldsymbol{u} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\, \Omega)]'$. We make use of (i) of Theorem 2.2.6 and we conclude by using the regularity on the Stokes problem.

  (b) Case $r \in [\,3,\, p\,]$. We know from the last case that $\boldsymbol{u} \in \boldsymbol{W}^{\,1,p}(\Omega)$. Then $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}$ belongs to $\boldsymbol{L}^p(\Omega)$ and $|\boldsymbol{u}|^\alpha \boldsymbol{u} \in \boldsymbol{L}^t(\Omega)$ with $t \in [1, \infty]$. For $\pi_0 \in W^{-1/q,q}(\Gamma)$, we apply the result (ii) of Theorem 2.2.6.

$\square$

2.4.2 Strong solutions in $\boldsymbol{W}^{2,p}(\Omega)$

Theorem 2.4.2. *Let* $p \geq 6/5$,

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \pi_0 \in W^{1-1/p,p}(\Gamma).$$

Then, there exist a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ *to problem* (2.3.1) *and satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\big). \tag{2.4.1}$$

Proof. We will use a raisonement based on the Stokes problem. We observe first that:

$$\boldsymbol{L}^{6/5}(\Omega) \hookrightarrow [\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]', \quad \text{and} \quad W^{1-1/p,p}(\Gamma) \hookrightarrow H^{1/2}(\Gamma).$$

Thanks to Theorem 2.3.2, there exists a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times H^1(\Omega)$. We consider three cases:

i) Case $6/5 \leq p \leq 3/2$. Since $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}$ belongs to $\boldsymbol{L}^{3/2}(\Omega)$ and $|\boldsymbol{u}|^\alpha \boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$ where $2 \leqslant q \leqslant 3$, the result follows imediately using the regularity for the stokes problem.

ii) Case $3/2 \leq p \leq 3$. We know that $\boldsymbol{u} \in \boldsymbol{W}^{2,3/2}(\Omega)$ and $\pi \in W^{1,3/2}(\Omega)$. Recall that

$$\boldsymbol{W}^{2,3/2}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p^*}(\Omega) \hookrightarrow \boldsymbol{L}^{p^{**}}(\Omega)$$

where $p^* = 3$ and then $p^{**} \in [1, \infty[$. This implies that $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}$ belongs to $\boldsymbol{L}^q(\Omega)$ for any $q \in [1, \infty[$ and $|\boldsymbol{u}|^\alpha \boldsymbol{u} \in \boldsymbol{L}^k(\Omega)$, for any $k \in [1, \infty[$, however, we should bound q and k according to $p = 3$, so for this reason we have to take $q < 3,\ k < 3$. Next, we apply the regularity for the Stokes problem.

iii) Case $p > 3$. We know that $\boldsymbol{u} \in \boldsymbol{W}^{2,3}(\Omega)$ and $\pi \in W^{1,3}(\Omega)$. Since $\boldsymbol{u} \in \boldsymbol{W}^{2,3}(\Omega) \hookrightarrow \boldsymbol{W}^{1,\infty}(\Omega)$, the non linear term $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$ for any $q \in [1, \infty]$ and $|\boldsymbol{u}|^\alpha \boldsymbol{u}$ belongs to $\boldsymbol{L}^t(\Omega)$ with $t \in [1, \infty]$. Here also, we conclude using the regularity for the Stokes problem.

$\square$

Chapter 3

Approximation of the (DBF) equations with Dirichlet boundary conditions by pseudo-compressibility method

In this chapter, we treat the non stationary incompressible Darcy Brinkman Forchheimer (DBF) equations :

$$\begin{aligned} &\partial_t \boldsymbol{u} - \gamma \Delta \boldsymbol{u} + a\boldsymbol{u} + \beta |\boldsymbol{u}|^{\alpha} \boldsymbol{u} + B(\boldsymbol{u}, \boldsymbol{u}) + \nabla p = \boldsymbol{f} \quad \text{in } \Omega_T, \\ &\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega_T, \end{aligned} \tag{3.0.1}$$

with Dirichlet boundary conditions and the initial datum as follows :

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Sigma_T \quad \text{and} \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{3.0.2}$$

where $\Omega_T = \Omega \times]0, T[$, $\Sigma_T = \Gamma \times]0, T[$ with Ω is a bounded domain of $\mathbb{R}^d$ $(d = 2, 3)$, with suffficiently smooth boundary Γ. Here $\boldsymbol{u}$, p represent respectively fluid velocity and pressure. The constant $\gamma > 0$ defines Brinkman coefficient, $a > 0$ is the Darcy coefficient and $\beta > 0$ is the Forchheimer coefficient. $\alpha \in [1, 2]$ is a given number. The bilinear form $B(\boldsymbol{u}, \boldsymbol{u})$ can be define as $B(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{u}.\nabla)\boldsymbol{u}$.

3.1 Preliminaries

In this section, we introduce some notions and results that we will use in the remainder of this chapter.

We denote by $\|.\|$, $\|.\|_1$ and $\|.\|_2$ respectively the norms in $\boldsymbol{L}^2(\Omega)$, $\boldsymbol{H}^1(\Omega)$ and in $\boldsymbol{H}^2(\Omega)$.

The inner product in $L^2(\Omega)$ will be denoted by $\langle ., . \rangle$.

We consider the space :

$$\boldsymbol{L}^2_\sigma(\Omega) = \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega),\ \mathrm{div}\boldsymbol{w} = 0,\ \text{on},\ \Omega, \quad \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ on } \quad \Gamma\}.$$

Let $\boldsymbol{V}$ be the space :

$$\boldsymbol{V} = \{\boldsymbol{v} \in H^1_0(\Omega), \quad \mathrm{div}\,\boldsymbol{v} = 0\}$$

We define the bilinear form $\widetilde{B}(\boldsymbol{u},\boldsymbol{v})$ as follow:
$\widetilde{B}(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}.\nabla)\boldsymbol{v} + \frac{1}{2}(\nabla\boldsymbol{u})\boldsymbol{v}$.
We recall the property of the monotonicity for any mapping F defined as
$F\ :\ \boldsymbol{x} \longmapsto |\boldsymbol{x}|^\alpha \boldsymbol{x}$:

$$(|\boldsymbol{u}|^\alpha \boldsymbol{u} - |\boldsymbol{v}|^\alpha \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}) \geqslant 0. \tag{3.1.1}$$

The following Sobolev inequality is useful to deal with the nonlinear form F.

$$\|\boldsymbol{v}\|_p \leq \|\nabla \boldsymbol{v}\|, \qquad 1 \leq p \leq 6. \tag{3.1.2}$$

Moreover, under some conditions, the mapping F satisfies an important ineqality which is a key tool used in our work. This result is presented in the next lemma (for the proof see for instance [44]) :

Lemma 3.1.1. *Assume that $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)$ satisfy the estimates* (3.1.2)*. Then*

$$\|F(\boldsymbol{u}) - F(\boldsymbol{v})\| \leqslant C\|\nabla(\boldsymbol{u} - \boldsymbol{v})\|. \tag{3.1.3}$$

Now we give the variational formulation of the system (3.0.1)-(3.0.2). For any $\boldsymbol{v} \in \boldsymbol{V}$:

$$(\boldsymbol{u}_t, \boldsymbol{v}) + \gamma(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + \beta(|\boldsymbol{u}|^\alpha \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \tag{3.1.4}$$

We denote by P the orthogonal projection operator of $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{L}^2_\sigma(\Omega)$. The variational formulation equation (3.1.4) is equivalent to :

$$\begin{aligned} &\boldsymbol{u}_t + \gamma A\boldsymbol{u} + a\boldsymbol{u} + B(\boldsymbol{u},\boldsymbol{u}) + \beta G(\boldsymbol{u}) = \boldsymbol{f} \\ &\boldsymbol{u}(0) = 0, \end{aligned} \tag{3.1.5}$$

where $A\boldsymbol{u} = -P\Delta\boldsymbol{u} = \widetilde{\Delta}\boldsymbol{u}$ is the Stokes operator and $G(u) = PF(\boldsymbol{u})$.
We know that for Γ smooth enough, we have for any $\boldsymbol{v} \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$:

$$\|\boldsymbol{v}\|_2 \leqslant C\|\widetilde{\Delta}\boldsymbol{v}\|. \tag{3.1.6}$$

Moreover, in the following we will use the following algebraic result.

$$(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{v}) = \frac{1}{2}(|\boldsymbol{u}|^2 - |\boldsymbol{v}|^2 - |\boldsymbol{v} - \boldsymbol{u}|^2). \tag{3.1.7}$$

The inequalities below help us to bound one of the nonlinear terms:

$$\widetilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \leqslant \begin{cases} C\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|\|\boldsymbol{w}\|_1, & \forall \boldsymbol{u} \in H^2(\Omega)\cap H_0^1(\Omega), \quad \boldsymbol{v},\boldsymbol{w} \in H_0^1(\Omega) \\ C\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_2\|\boldsymbol{w}\|, & \forall \boldsymbol{v} \in H^2(\Omega)\cap H_0^1(\Omega), \quad \boldsymbol{u},\boldsymbol{w} \in H_0^1(\Omega) \\ C\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|, & \forall \boldsymbol{u} \in H^2(\Omega)\cap H_0^1(\Omega), \quad \boldsymbol{v},\boldsymbol{w} \in H_0^1(\Omega) \\ C\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1, & \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in H_0^1(\Omega) \end{cases} \tag{3.1.8}$$

$$\widetilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \leqslant \|\boldsymbol{u}\|\|\boldsymbol{v}\|\|\boldsymbol{w}\| \quad \forall,\boldsymbol{v},\boldsymbol{w} \in H^1(\Omega)^d, d \leqslant 4. \tag{3.1.9}$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0, \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{3.1.10}$$

3.2 Analysis of the problem

The existence and uniqueness of a solution for the system (3.0.1)-(3.0.2) has been already proved in [44] by combined Faedo Galerkin method.
In the next lemma, we provide the estimate for the velocity's second derivative, the velocity's derivative with respect to time and the pressure.

Lemma 3.2.1. *We assume that the given datum $\boldsymbol{u}_0$ and $\boldsymbol{f}$ satisfy the following regularity*

$$\boldsymbol{u}_0 \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega), \qquad \boldsymbol{f},\ \boldsymbol{f}_t \in \boldsymbol{L}^2(\Omega). \tag{3.2.1}$$

Then there exists $T_1 \leq T$ such that the solution of the system (3.0.1) *satisfies:*

$$\|\boldsymbol{u}(t)\|_2 + \|\boldsymbol{u}_t(t)\| + \|p(t)\|_1 \leqslant C, \quad \forall t \in [0, T_1]. \tag{3.2.2}$$

Proof. Multiply the projection equation (3.1.5) with $\widetilde{\Delta}\boldsymbol{u}$ and integrate the resulting equations to obtain:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}\|^2 + \gamma\|\widetilde{\Delta}\boldsymbol{u}\|^2 + a\|\nabla\boldsymbol{u}\|^2 = -b(\boldsymbol{u},\boldsymbol{u},\widetilde{\Delta}\boldsymbol{u}) - \beta(|\boldsymbol{u}|^\alpha\boldsymbol{u},\widetilde{\Delta}\boldsymbol{u}) + (f,\widetilde{\Delta}\boldsymbol{u}). \tag{3.2.3}$$

Using Gagliardo Nirenberg inequality in combination with Young inequality, the non linear terms can be estimated by

$$b(\boldsymbol{u},\boldsymbol{u},\widetilde{\Delta}\boldsymbol{u}) \leq \frac{\gamma}{6}\|\widetilde{\Delta}\boldsymbol{u}\|^2 + C'\|\nabla\boldsymbol{u}\|^6, \tag{3.2.4}$$

$$\beta(|\boldsymbol{u}|^\alpha\boldsymbol{u},\widetilde{\Delta}\boldsymbol{u}) \leq \frac{\gamma}{6}\|\widetilde{\Delta}\boldsymbol{u}\|^2 + C\|\nabla\boldsymbol{u}\|^{2(\alpha+1)}. \tag{3.2.5}$$

Substituting (3.2.4) and (3.2.5) into (3.2.3) and using the fact that

$$(f,\widetilde{\Delta}\boldsymbol{u}) \leq \frac{\gamma}{6}\|\widetilde{\Delta}\boldsymbol{u}\|^2 + \frac{3}{2\gamma}\|\boldsymbol{f}\|^2,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}\|^2 + \frac{\gamma}{2}\|\widetilde{\Delta}\boldsymbol{u}\|^2 + a\|\nabla \boldsymbol{u}\|^2 \leq C'\|\nabla \boldsymbol{u}\|^6 + \frac{3}{2\gamma}\|\boldsymbol{f}\|^2. \tag{3.2.6}$$

In particular, we have the following differential inequality (by supposing that $C' \geq 1$):

$$\frac{d}{dt}y(t) \leq C'y(t)^3, \quad \text{with} \quad y(t) = \|\nabla \boldsymbol{u}(t)\|^2 + C_1,\ y(0) = \|\nabla \boldsymbol{u}(0)\|^2 + C_1$$

and $C_1 = \frac{3}{2\gamma}\sup\limits_{0\leq t\leq T}\|\boldsymbol{f}\|^2$.

Solving the above inequality, we have a solution defined on $[0,\, T_1]$ where $T_1 \leq \frac{1}{2C'y(0)^2} = \frac{1}{C_0}$ and

$$y(t) \leq \sqrt{2}y(0), \quad t \in [0,\, T_1].$$

So, for all $t \in [0,\, T_1]$ with $T_1 = \min\{T_1,\, \frac{1}{C_0}\}$, we have

$$\|\nabla \boldsymbol{u}(t)\|^2 \leq \sqrt{2}(\|\nabla \boldsymbol{u}(0)\|^2 + C_1) := C_3, \tag{3.2.7}$$

Therefore, by (3.2.6) and (3.2.7), we have:

$$\sup_{0\leq t\leq T_1}\|\nabla \boldsymbol{u}(t)\|^2 + \gamma\int_0^{T_1}\|\widetilde{\Delta}\boldsymbol{u}(t)\|^2\,dt \leq C, \qquad \forall t \in [0,\, T_1]. \tag{3.2.8}$$

Differentiating the first equation of the system (1.3.58) with respect to t, we obtain

$$\boldsymbol{u}_{tt} - \gamma\Delta\boldsymbol{u}_t + a\boldsymbol{u}_t + \beta F'(\boldsymbol{u})\boldsymbol{u}_t + B(\boldsymbol{u}_t, \boldsymbol{u}) + B(\boldsymbol{u}, \boldsymbol{u}_t) + \nabla p_t = f_t. \tag{3.2.9}$$

Taking the inner product of the relation (3.2.9) with $\boldsymbol{u}_t$, we have :

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t\|^2 + \gamma\|\nabla\boldsymbol{u}_t\|^2 + a\|\boldsymbol{u}_t\|^2 = -\beta(F'(\boldsymbol{u})\boldsymbol{u}_t, \boldsymbol{u}_t) - b(\boldsymbol{u}_t, \boldsymbol{u}, \boldsymbol{u}_t) + (f_t, \boldsymbol{u}_t).$$

Since $(F'(\boldsymbol{u})\boldsymbol{u}_t)\cdot\boldsymbol{u}_t$ is positive definite, we have by applying Gagliardo Nirenberg's, Hölder's and Sobolev's inequalities:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t\|^2 + \gamma\|\nabla\boldsymbol{u}_t\|^2 &\leqslant c\|\boldsymbol{u}_t\|\|\boldsymbol{u}\|_1^{\frac{1}{2}}\|\boldsymbol{u}\|_2^{\frac{1}{2}}\|\nabla\boldsymbol{u}_t\| + \frac{1}{4a}\|f_t\|^2,\\ &\leqslant \frac{\gamma}{2}\|\nabla\boldsymbol{u}_t\|^2 + C(\|\boldsymbol{u}\|_1\|\boldsymbol{u}\|_2)\|\boldsymbol{u}_t\|^2 + \frac{1}{4a}\|f_t\|^2.\end{aligned} \tag{3.2.10}$$

Next, by Cauchy-Schwarz inequality, estimates (3.1.6) and (3.2.8), we have:

$$\int_0^{T_1}\|\boldsymbol{u}(s)\|_1\|\boldsymbol{u}(s)\|_2\,ds \leq \Big(\int_0^{T_1}\|\boldsymbol{u}(s)\|_1^2\,ds\Big)^{\frac{1}{2}}\Big(\int_0^{T_1}\|\boldsymbol{u}(s)\|_2^2\,ds\Big)^{\frac{1}{2}} \leq C. \tag{3.2.11}$$

Then, applying Gronwall Lemma to (3.2.10), we obtain:

$$\|\boldsymbol{u}_t\|^2 + \gamma\int_0^{T_1}\|\nabla\boldsymbol{u}_t(s)\|^2ds \leq C, \qquad \forall t \in [0,\, T_1]. \tag{3.2.12}$$

Next, taking the inner product of (3.1.5) with $\widetilde{\Delta}\boldsymbol{u}$, we obtain:

$$\begin{aligned}\gamma\|\widetilde{\Delta}\boldsymbol{u}\|^2 &= -a\|\nabla\boldsymbol{u}\|^2-(\boldsymbol{u}_t,\widetilde{\Delta}\boldsymbol{u})-\beta(|\boldsymbol{u}|^{\alpha}\boldsymbol{u},\widetilde{\Delta}\boldsymbol{u})-b(\boldsymbol{u},\boldsymbol{u},\widetilde{\Delta}\boldsymbol{u})-(\boldsymbol{f},\widetilde{\Delta}\boldsymbol{u})\\ &\leq a\|\nabla\boldsymbol{u}\|^2+\frac{\gamma}{8}\|\widetilde{\Delta}\boldsymbol{u}\|^2+2\gamma\|\boldsymbol{u}_t\|^2+\beta|(|\boldsymbol{u}|^{\alpha}u,\widetilde{\Delta}\boldsymbol{u})|+|b(\boldsymbol{u},\boldsymbol{u},\widetilde{\Delta}\boldsymbol{u})|+\frac{\gamma}{8}\|\widetilde{\Delta}\boldsymbol{u}\|^2+2\gamma\|f\|^2.\end{aligned}$$

The nonlinear terms can be controlled as in (3.2.5) and (3.2.4). So using (3.2.7), (3.2.12) and (3.1.6), we deduce that:

$$\|\boldsymbol{u}\|_2\leq C,\qquad \forall t\in[0,\,T_1]. \tag{3.2.13}$$

In order to get the pressure estimate, we use the first equation of (3.0.1) to derive:

$$\|\nabla p\|\leq\big(\|\boldsymbol{u}_t\|+\gamma\|\nabla\boldsymbol{u}\|+a\|\boldsymbol{u}\|+\beta\||\boldsymbol{u}|^{\alpha+1}\|+\|B(\boldsymbol{u},\boldsymbol{u})\|\big). \tag{3.2.14}$$

Using the Hölder's and Galiardo-Nirenberg's inequality, the estimates (3.2.7) and (3.2.13), since $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^4(\Omega)$, we obtain :

$$\begin{aligned}\|B(\boldsymbol{u},\boldsymbol{u})\|^2 &\leqslant C\|\boldsymbol{u}\|_{\alpha+2}\|\nabla\boldsymbol{u}\|_{\frac{2(\alpha+2)}{\alpha}}\\ &\leqslant C\|\boldsymbol{u}\|_{\alpha+2}\|\Delta\boldsymbol{u}\|_2^{(\frac{10-\alpha}{\alpha+8})}\|\boldsymbol{u}\|_{\alpha+2}^{2(\frac{\alpha-1}{\alpha+8})}\\ &\leqslant C\|\Delta\boldsymbol{u}\|_2^{2(\frac{10-\alpha}{\alpha+8})}\|\boldsymbol{u}\|_{\alpha+2}^{3(\frac{\alpha+2}{\alpha+8})}\leq C.\end{aligned}$$

Collecting the above bounds (3.2.7), (3.2.12) and (3.2.13) to (3.2.14), we obtain the pressure estimate. The proof of Lemma 3.2.1 is finished. □

In the case where the initial data $\boldsymbol{u}(0)$ and $\boldsymbol{f}(0)$ satisfy some nonlocal compatibility conditions, we take $t_0>0$ and we assume that we have an initial data $(\boldsymbol{u}_0,p_0)$ such that

$$\|\boldsymbol{u}_0-\boldsymbol{u}(t_0)\|\leqslant Ck^2,\qquad \|\nabla(\boldsymbol{u}_0-\boldsymbol{u}(t_0)\|+\|\nabla(p_0-p(t_0))\|\leqslant Ck.$$

Lemma 3.2.2. *Under the same assumptions of the lemma 3.2.1, we suppose in addition that*

$$\boldsymbol{f}_t,\ \boldsymbol{f}_{tt}\in C([0,T],\boldsymbol{L}^2(\Omega)). \tag{3.2.15}$$

Then, the solution of the system (1.3.58) *satisfies :*

$$\|\boldsymbol{u}_t(t)\|_2^2+\|\nabla p_t\|+\int_{t_0}^{t}(\|\boldsymbol{u}_{tt}(s)\|_2^2+\|p_{tt}(s)\|_1^2)\,ds\leqslant C,\qquad \forall t\in[t_0,T_1]. \tag{3.2.16}$$

Proof. Differentiating the first equation of the system (3.1.5) with respect to t, we obtain

$$\boldsymbol{u}_{tt}-\gamma\widetilde{\Delta}\boldsymbol{u}_t+a\boldsymbol{u}_t+\beta G'(\boldsymbol{u})\boldsymbol{u}_t+B(\boldsymbol{u}_t,\boldsymbol{u})+B(\boldsymbol{u},\boldsymbol{u}_t)=\boldsymbol{f}_t. \tag{3.2.17}$$

Taking the inner product of the relation (3.2.17) with $\widetilde{\Delta}\boldsymbol{u}_t$, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}_t\|^2+\gamma\|\widetilde{\Delta}\boldsymbol{u}_t\|^2 =&-a\|\nabla\boldsymbol{u}_t\|^2-(\boldsymbol{f}_t,\widetilde{\Delta}\boldsymbol{u}_t)+b(\boldsymbol{u}_t,\boldsymbol{u},\widetilde{\Delta}\boldsymbol{u}_t)+b(\boldsymbol{u},\boldsymbol{u}_t,\widetilde{\Delta}\boldsymbol{u}_t)\\ &+\beta(F'(\boldsymbol{u})\boldsymbol{u}_t,\widetilde{\Delta}\boldsymbol{u}_t)\\ \leq& a\|\nabla\boldsymbol{u}_t\|^2+\frac{\gamma}{8}\|\widetilde{\Delta}\boldsymbol{u}_t\|^2+2\|\boldsymbol{f}_t\|^2+|b(\boldsymbol{u}_t,\boldsymbol{u},\widetilde{\Delta}\boldsymbol{u}_t)|+|b(\boldsymbol{u},\boldsymbol{u}_t,\widetilde{\Delta}\boldsymbol{u}_t)|\\ &+\beta|(F'(\boldsymbol{u})\boldsymbol{u}_t,\widetilde{\Delta}\boldsymbol{u}_t)|\end{aligned} \tag{3.2.18}$$

The nonlinear terms can be controlled with the same arguments as in the proof of Lemma 3.2.1 as follows:

$$|b(\boldsymbol{u}_t, \boldsymbol{u}, \widetilde{\Delta}\boldsymbol{u}_t)| \leqslant \frac{\gamma}{8}\|\widetilde{\Delta}\boldsymbol{u}_t\|^2 + C\|\boldsymbol{u}\|_2^2\|\nabla\boldsymbol{u}_t\|^2 \tag{3.2.19}$$

$$|b(\boldsymbol{u}, \boldsymbol{u}_t, \widetilde{\Delta}\boldsymbol{u}_t)| \leqslant \frac{\gamma}{8}\|\widetilde{\Delta}\boldsymbol{u}_t\|^2 + C\|\boldsymbol{u}\|_2^2\|\nabla\boldsymbol{u}_t\|^2. \tag{3.2.20}$$

Next,

$$\beta|(F'(\boldsymbol{u})\boldsymbol{u}_t, \widetilde{\Delta}\boldsymbol{u}_t)| \leq C(\alpha+1)\|\boldsymbol{u}\|_\infty^\alpha\|\boldsymbol{u}_t\|\|\widetilde{\Delta}\boldsymbol{u}_t\| \leq \frac{\gamma}{8}\|\widetilde{\Delta}\boldsymbol{u}_t\|^2 + C\|\boldsymbol{u}\|_\infty^{2\alpha}\|\|\nabla\boldsymbol{u}_t\|^2 \tag{3.2.21}$$

Gathering the above estimates together and using (3.1.2), (3.2.7) and (3.2.13), we obtain

$$\frac{d}{dt}\|\nabla\boldsymbol{u}_t\|^2 + \gamma\|\widetilde{\Delta}\boldsymbol{u}_t\|^2 \leq 4\|\boldsymbol{f}_t\|^2 + C\|\nabla\boldsymbol{u}_t\|^2. \tag{3.2.22}$$

Thanks to Gronwall's inequality, we have

$$\sup_{t_0\leq t\leq T_1}\|\nabla\boldsymbol{u}_t(t)\|^2 + \gamma\int_{t_0}^t\|\widetilde{\Delta}\boldsymbol{u}_t(s)\|^2 ds \leqslant C, \qquad \forall t\in[t_0, T_1]. \tag{3.2.23}$$

Now, differentiating the equation (3.2.9) with respect to t, we obtain

$$\boldsymbol{u}_{ttt} - \gamma\Delta\boldsymbol{u}_{tt} + a\boldsymbol{u}_{tt} + \beta F''(\boldsymbol{u})|\boldsymbol{u}_t|^2 + \beta F'(\boldsymbol{u})\boldsymbol{u}_{tt} + 2B(\boldsymbol{u}_t, \boldsymbol{u}_t) + B(\boldsymbol{u}_{tt}, \boldsymbol{u}) + B(\boldsymbol{u}, \boldsymbol{u}_{tt}) + \nabla p_{tt} = f_{tt}. \tag{3.2.24}$$

Taking the inner product of the relation (3.2.24) with $\boldsymbol{u}_{tt}$, we obtain :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_{tt}\|^2 + \gamma\|\nabla\boldsymbol{u}_{tt}\|^2 =& -a\|\boldsymbol{u}_{tt}\|^2 - \beta(F''(\boldsymbol{u})|\boldsymbol{u}_t|^2, \boldsymbol{u}_{tt}) - \beta(F'(\boldsymbol{u})\boldsymbol{u}_{tt}, \boldsymbol{u}_{tt})\\ &- 2b(\boldsymbol{u}_t, \boldsymbol{u}_t, \boldsymbol{u}_{tt}) - b(\boldsymbol{u}_{tt}, \boldsymbol{u}, \boldsymbol{u}_{tt}) + (f_{tt}, \boldsymbol{u}_{tt}).\end{aligned}$$

Since $(F'(\boldsymbol{u})\boldsymbol{u}_{tt})\cdot\boldsymbol{u}_{tt}$ is positive definite, using Young's inequality, we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_{tt}\|^2 + \gamma\|\nabla\boldsymbol{u}_{tt}\|^2 \leq& \frac{1}{4a}\|\boldsymbol{f}_{tt}\|^2 + |\beta(F''(\boldsymbol{u})|\boldsymbol{u}_t|^2, \boldsymbol{u}_{tt})| + 2|b(\boldsymbol{u}_t, \boldsymbol{u}_t, \boldsymbol{u}_{tt})|\\ &+ |b(\boldsymbol{u}_{tt}, \boldsymbol{u}, \boldsymbol{u}_{tt})|.\end{aligned} \tag{3.2.25}$$

On the one hand, thanks to (3.1.2), (3.2.7) and (3.2.23), since $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^4(\Omega)$,

$$\begin{aligned}|(F''(\boldsymbol{u})|\boldsymbol{u}_t|^2, \boldsymbol{u}_{tt})| &\leqslant \alpha(\alpha+1)(|\boldsymbol{u}|^{\alpha-1}|\boldsymbol{u}_t|^2, |\boldsymbol{u}_{tt}|)\\ &\leqslant \alpha(\alpha+1)\|\boldsymbol{u}\|_\infty^{\alpha-1}\|\boldsymbol{u}_t\|_4^2\|\boldsymbol{u}_{tt}\|\\ &\leqslant C\|\boldsymbol{u}_{tt}\|^2 + C\|\boldsymbol{u}\|_\infty^{2(\alpha-1)}\|\nabla\boldsymbol{u}_t\|^4 \leq C\|\boldsymbol{u}_{tt}\|^2 + C.\end{aligned} \tag{3.2.26}$$

On the other hand, using Young, Hölder, Gagliardo-Nirenberg inequalities and estimate (3.2.2), we have

$$|b(\boldsymbol{u}_{tt}, \boldsymbol{u}, \boldsymbol{u}_{tt})| \leq \|\boldsymbol{u}_{tt}\|_1\|\boldsymbol{u}\|_2\|\boldsymbol{u}_{tt}\| \leqslant \frac{\gamma}{2}\|\nabla\boldsymbol{u}_{tt}\|^2 + C\|\boldsymbol{u}_{tt}\|^2. \tag{3.2.27}$$

$$|b(\boldsymbol{u}_t, \boldsymbol{u}_t, \boldsymbol{u}_{tt})| \leqslant \|\boldsymbol{u}_t\|_2 \|\boldsymbol{u}_t\|_1 \|\boldsymbol{u}_{tt}\| \leq C\|\nabla \boldsymbol{u}_t\|^2 + \|\boldsymbol{u}_t\|_2^2 \|\boldsymbol{u}_{tt}\|^2. \tag{3.2.28}$$

Then, inserting (3.2.26)-(3.2.28) in (3.2.25), using (3.2.23) and applying Gronwall Lemma yield

$$\|\boldsymbol{u}_{tt}\|^2 + \gamma \int_{t_0}^{T_1} \|\nabla \boldsymbol{u}_{tt}(s)\|^2 \, ds \leq C. \tag{3.2.29}$$

Taking the inner product of (3.1.5) with $\widetilde{\Delta} \boldsymbol{u}_t$, we obtain:

$$\begin{aligned}
\gamma \|\widetilde{\Delta} \boldsymbol{u}_t\|^2 =& (\boldsymbol{u}_{tt}, \widetilde{\Delta} \boldsymbol{u}_t) - a\|\nabla \boldsymbol{u}_t\|^2 - (\boldsymbol{f}_t, \widetilde{\Delta} \boldsymbol{u}_t) - \beta(F'(\boldsymbol{u})\boldsymbol{u}_t, \widetilde{\Delta} \boldsymbol{u}_t) - b(\boldsymbol{u}_t, \boldsymbol{u}, \widetilde{\Delta} \boldsymbol{u}_t) \\
&- b(\boldsymbol{u}, \boldsymbol{u}_t, \widetilde{\Delta} \boldsymbol{u}_t). \\
\leq& \frac{5\gamma}{2} \|\boldsymbol{u}_{tt}\|^2 + \frac{\gamma}{5} \|\widetilde{\Delta} \boldsymbol{u}_t\|^2 + \frac{5\gamma}{2} \|\boldsymbol{f}_t\|^2 + a\|\nabla \boldsymbol{u}_t\|^2 + \beta|(F'(\boldsymbol{u})\boldsymbol{u}_t, \widetilde{\Delta} \boldsymbol{u}_t)| \\
&+ |b(\boldsymbol{u}_t, \boldsymbol{u}, \widetilde{\Delta} \boldsymbol{u}_t)| + |b(\boldsymbol{u}, \boldsymbol{u}_t, \widetilde{\Delta} \boldsymbol{u}_t)|.
\end{aligned}$$

Using the estimates (3.2.19)-(3.2.21) for the nonlinear terms, thanks to (3.1.6), (3.2.29) and (3.2.23), we can deduce the following uniform bound:

$$\|\boldsymbol{u}_t\|_2 \leq C, \qquad \forall t \in [t_0, T_1]. \tag{3.2.30}$$

The verification of the remaining estimates in (3.2.16) can be obtained by repeating the same arguments of Lemma 3.2.1. Notice that (3.2.23) is an analogue of $\int_{t_0}^{t} \|\widetilde{\Delta} \boldsymbol{u}_{tt}(s)\|^2 ds$ for $\boldsymbol{u}_{tt}$ instead of $\boldsymbol{u}_t$.

□

3.3 Study of the perturbed problem

In this section we focus on the existence, the uniqueness and the regularity for the perturbed equations (0.0.14)-(0.0.15) .

3.3.1 Faedo-Galerkin approximation and a priori estimates

Theorem 3.3.1. *For Given $\boldsymbol{u}_0^\epsilon \in \boldsymbol{H}^1(\Omega), p_0^\epsilon \in \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{f} \in \boldsymbol{L}^\infty(0, T, \boldsymbol{L}^2(\Omega))$. Then there exists $K_0 = K_0\left(\gamma, T, \Omega, \boldsymbol{f}, \boldsymbol{u}_0^\epsilon, p_0^\epsilon\right)$ such that for,*

$$T_0 = min\{T \, , \, \frac{\epsilon^2}{K_0}\}$$

The perturbed problem has a unique solution $(\boldsymbol{u}^\epsilon, p^\epsilon) \in \boldsymbol{L}^2(0, T_0, \boldsymbol{H}^2(\Omega)) \cap \boldsymbol{L}^\infty(0, T_0, \boldsymbol{H}^1(\Omega)) \times \boldsymbol{L}^\infty(0, T_0, \boldsymbol{H}^1(\Omega))$

Proof. We first denote K_i to be constants depending on γ, the datum $\boldsymbol{f}$, $\boldsymbol{u}_0^\epsilon, p_0^\epsilon$ and the domain Ω while C_i are the constants depending only on Ω.

The key of the proof is based on some a priori estimates. For this purpose we proceed as follow,

Multiplying $(0.0.14)_1$ with $\boldsymbol{u}^\epsilon$ and $(0.0.14)_2$ with p^ϵ, adding the results obtained, taking into account the relation (3.1.10) and using both of Cauchy-Schwarz's inequality and Young's inequality, we derive

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^\epsilon\|^2+\gamma\|\nabla\boldsymbol{u}^\epsilon\|^2+a\|\boldsymbol{u}^\epsilon\|^2+b\|\boldsymbol{u}^\epsilon\|_{\alpha+2}^{\alpha+2}+\frac{\epsilon}{2}\frac{d}{dt}\|\nabla p^\epsilon\|^2=\langle\boldsymbol{f},\boldsymbol{u}^\epsilon\rangle.$$
$$\leqslant\frac{\gamma}{2}\|\nabla\boldsymbol{u}^\epsilon\|^2+\frac{1}{2\gamma}\|\boldsymbol{f}\|_1^2. \quad (3.3.1)$$

By integrating the last result from 0 to t with $t\in[0,T]$, we obtain :

$$\|\boldsymbol{u}^\epsilon(t)\|^2+\gamma\int_0^t\|\nabla\boldsymbol{u}^\epsilon(s)\|^2\,ds+a\int_0^t\|\boldsymbol{u}^\epsilon(s)\|^2+b\int_0^t\|\boldsymbol{u}^\epsilon(s)\|_{\alpha+2}^{\alpha+2}\,ds+\epsilon\|\nabla p^\epsilon(t)\|^2\leqslant K_0. \quad (3.3.2)$$

Where $K_0=\frac{1}{\gamma}\|\boldsymbol{f}\|_1+\|\boldsymbol{u}_0^\epsilon\|^2+\|\nabla p_0^\epsilon\|^2$.

In order to reach more a priori estimates, we observe the relation $(0.0.14)_1$ as follow

$$\begin{cases}\boldsymbol{u}_t^\epsilon-\gamma\Delta\boldsymbol{u}^\epsilon+a\boldsymbol{u}^\epsilon+b|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon+\widetilde{B}(\boldsymbol{u}^\epsilon,\boldsymbol{u}^\epsilon)=\boldsymbol{f}-\nabla p^\epsilon.\\ \boldsymbol{u}^\epsilon(0)=\boldsymbol{u}_0^\epsilon.\end{cases} \quad (3.3.3)$$

We take the inner product of (3.3.3) with $-\Delta\boldsymbol{u}^\epsilon$, then we obtain :

$$\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}^\epsilon\|^2+\gamma\|\Delta\boldsymbol{u}^\epsilon\|^2+a\|\nabla\boldsymbol{u}^\epsilon\|^2=b(|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon,\Delta\boldsymbol{u}^\epsilon)+\widetilde{b}(\boldsymbol{u}^\epsilon,\boldsymbol{u}^\epsilon,\Delta\boldsymbol{u}^\epsilon)-(\boldsymbol{f}-\nabla p^\epsilon,\Delta\boldsymbol{u}^\epsilon)$$
$$\leqslant\frac{\gamma}{6}\|\Delta\boldsymbol{u}^\epsilon\|^2+\frac{3}{2\gamma}(\|\boldsymbol{f}\|^2+\|\nabla p^\epsilon\|^2). \quad (3.3.4)$$

By using Cauchy Schwarz and Young's inequlaity, we derive

$$b(|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon,\Delta\boldsymbol{u}^\epsilon)\leqslant b\|\boldsymbol{u}^\epsilon\|^{\alpha+1}\|\Delta\boldsymbol{u}^\epsilon\|.$$
$$\leqslant\frac{\gamma}{6}\|\Delta\boldsymbol{u}^\epsilon\|^2+b^2\frac{3}{2\gamma}\|\nabla\boldsymbol{u}^\epsilon\|^{2(\alpha+1)}.$$

Because of $1\leqslant\alpha\leqslant 2$, it infers that,

$$b(|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon,\Delta\boldsymbol{u}^\epsilon)\leqslant\frac{\gamma}{6}\|\Delta\boldsymbol{u}^\epsilon\|^2+C\|\nabla\boldsymbol{u}^\epsilon\|^6. \quad (3.3.5)$$

On the other hand, according to Agmon's inequality, we obtain

$$\widetilde{b}(\boldsymbol{u}^\epsilon,\boldsymbol{u}^\epsilon,\Delta\boldsymbol{u}^\epsilon)\leqslant C\|\boldsymbol{u}^\epsilon\|_\infty\|\boldsymbol{u}^\epsilon\|_1\|\boldsymbol{u}^\epsilon\|_2. \quad (3.3.6)$$
$$\leqslant\frac{\gamma}{6}\|\Delta\boldsymbol{u}^\epsilon\|^2+\frac{C}{\gamma}\|\nabla\boldsymbol{u}^\epsilon\|^6. \quad (3.3.7)$$

Because of (3.3.2) and by collecting the results (3.3.6) and (3.3.5) in relation (3.3.1), we reach

$$\frac{d}{dt}\|\nabla\boldsymbol{u}^\epsilon\|^2+\gamma\|\Delta\boldsymbol{u}^\epsilon\|^2+a\|\nabla\boldsymbol{u}^\epsilon\|^2\leqslant\frac{K_1}{\epsilon}+K_2\|\nabla\boldsymbol{u}^\epsilon\|^6. \quad (3.3.8)$$

We consider $y(t) = \|\nabla \boldsymbol{u}^\epsilon\|^2 + C_1$.
So, we can observe the following differential inequality with ($K' \geqslant 1$),

$$\frac{d}{dt}y(t) \leqslant K'y(t)^3, \text{where } y(t) = \|\nabla \boldsymbol{u}(t)^\epsilon\| + C_1 \quad \text{and } y(0) = \|\nabla \boldsymbol{u}_0^\epsilon\|^2 + C_1. \tag{3.3.9}$$

Due to (3.3.8), we derive a solution defined on t where $0 \leqslant t$ and satisfies,

$$t \leqslant \frac{1}{2K'y(0)^2} = \frac{1}{K_3}.$$

Which leads to :

$$y(t) \leqslant \sqrt{2}y(0).$$

Hence, for a given $t \in [0, T_0]$ with $T_0 = min\{T \, , \, \frac{1}{K_3}\}$. We reach :

$$\|\nabla \boldsymbol{u}^\epsilon(t)\|^2 \leqslant \sqrt{2}(\|\nabla \boldsymbol{u}^\epsilon(0)\|^2 + C_1) := C2.$$

To recap, we rearrange the results (3.3.2) and (3.3.8), then we conclude that,

$$\sup_{0 \leqslant t \leqslant T_0} \|\nabla \boldsymbol{u}^\epsilon(t)\|^2 + \gamma \int_0^{T_0} \|\Delta \boldsymbol{u}^\epsilon(t)\|^2 \, dt + a \int_0^{T_0} \|\nabla \boldsymbol{u}^\epsilon(t)\|^2 \, dt \leqslant C. \quad \forall t \in [0, T_0]. \tag{3.3.10}$$

□

3.3.2 ϵ-independent a priori estimates

In this subsection, we begin with the following result that is necessary for the subsequent investigations and provides ϵ-independent a priori estimates for the solution of (0.0.14) − (0.0.15).

Lemma 3.3.2. *Assume* $(\boldsymbol{u}^\epsilon, \, p^\epsilon)$ *to be the solution of the perturbed problem* (0.0.14)-(0.0.15). *Provided* (3.2.1) *and* (3.2.15) *are satisfied, we have the following a priori estimates:*

$$\|\boldsymbol{u}^\epsilon(t)\|_2^2 + \|p^\epsilon(t)\|_1^2 + \|p_t^\epsilon(t)\|_1^2 \leqslant C, \qquad \forall t \in [t_0, T_0] \tag{3.3.11}$$

$$\|\boldsymbol{u}_t^\epsilon(t) - \boldsymbol{u}_t(t)\|^2 \leqslant C\epsilon, \qquad \forall t \in [t_0, T_0] \tag{3.3.12}$$

Proof. The proof of this lemma requires estimates for the errors $e = \boldsymbol{u} - \boldsymbol{u}^\epsilon$ and $q = p - p^\epsilon$ in different norms. For this, we use the corresponding known results for the system (3.0.1)-(0.0.15) given in Lemma 3.2.1 and Lemma 3.2.2. So, subtracting the perturbed system (0.0.14)-(0.0.15) from the Brinkman Forchheimer equations (3.0.1)-(3.0.2), we get :

$$\begin{aligned}
&\boldsymbol{e}_t - \gamma\Delta \boldsymbol{e} + a\boldsymbol{e} + \nabla q + \widetilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{e}) + \widetilde{B}(\boldsymbol{e}, \boldsymbol{u}) + \beta|\boldsymbol{u}|^\alpha \boldsymbol{u} - \beta|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon = 0,\\
&\text{div}\boldsymbol{e} - \epsilon\Delta q_t = -\epsilon\Delta p_t,\\
&\boldsymbol{e}(t_0) = 0\\
&q(t_0) = 0
\end{aligned} \tag{3.3.13}$$

Multiply $(3.3.13)_1$ by $\boldsymbol{e}$, $(3.3.13)_2$ by q and integrate the resulting equations, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}\|^2+\gamma\|\nabla\boldsymbol{e}\|^2+a\|\boldsymbol{e}\|^2+\frac{\epsilon}{2}\frac{d}{dt}\|\nabla q\|^2=\epsilon(\nabla q,\nabla p_t)-\widetilde{b}(\boldsymbol{e},\boldsymbol{u},\boldsymbol{e})-\beta(|\boldsymbol{u}|^\alpha\boldsymbol{u}-|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon,\boldsymbol{e})$$

Thanks to (3.1.1), Gagliardo Nirenberg's and Young's inequalities, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}\|^2+\gamma\|\nabla\boldsymbol{e}\|^2+a\|\boldsymbol{e}\|^2+\frac{\epsilon}{2}\frac{d}{dt}\|\nabla q\|^2\leq&\frac{\epsilon}{2}\|\nabla p_t\|^2+\frac{\epsilon}{2}\|\nabla q\|^2+\frac{\gamma}{2}\|\nabla\boldsymbol{e}\|^2\\&+C\|\boldsymbol{u}\|_1\|\boldsymbol{u}\|_2\|\boldsymbol{e}\|^2.\end{aligned}\tag{3.3.14}$$

Applying Gronwall Lemma's to the above inequality, using (3.2.2) and (3.2.11), we obtain

$$\|\boldsymbol{e}\|^2+\gamma\int_{t_0}^t\|\nabla\boldsymbol{e}(s)\|^2ds+a\int_{t_0}^t\|\boldsymbol{e}(s)\|^2ds+\epsilon\|\nabla q(t)\|^2\leqslant C\varepsilon\int_{t_0}^t\|\nabla p_t(s)\|^2\,ds\leq C\varepsilon.\tag{3.3.15}$$

Next, proceeding as in Lemma (3.2.1), we show that $\|\nabla\boldsymbol{e}\|$ is uniformly bounded with respect to time. For this purpose, we multiply the equation $(3.3.13)_1$ by $-\Delta\boldsymbol{e}$ and integrate over Ω to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{e}\|^2+\gamma\|\Delta\boldsymbol{e}\|^2+a\|\nabla\boldsymbol{e}\|^2=&(\nabla q,\Delta\boldsymbol{e})+\widetilde{b}(\boldsymbol{u}^\epsilon,\boldsymbol{e},\Delta\boldsymbol{e})+\widetilde{b}(\boldsymbol{e},\boldsymbol{u},\Delta\boldsymbol{e})\\&+\beta(|\boldsymbol{u}|^\alpha\boldsymbol{u}-|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon,\Delta\boldsymbol{e}).\qquad(3.3.16)\\\leqslant&\frac{\gamma}{8}\|\Delta\boldsymbol{e}\|^2+\frac{2}{\gamma}\|\nabla q\|^2+|\widetilde{b}(\boldsymbol{e}-\boldsymbol{u},\boldsymbol{e},\Delta\boldsymbol{e})|+|\widetilde{b}(\boldsymbol{e},\boldsymbol{u},\Delta\boldsymbol{e})|\\&+\beta|(|\boldsymbol{u}|^\alpha\boldsymbol{u}-|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon,\Delta\boldsymbol{e})|.\qquad(3.3.17)\end{aligned}$$

Observe that

$$\widetilde{b}(\boldsymbol{u}^\epsilon,\boldsymbol{e},\Delta\boldsymbol{e})=\widetilde{b}(\boldsymbol{e},\boldsymbol{e},\Delta\boldsymbol{e})-\widetilde{b}(\boldsymbol{u},\boldsymbol{e},\Delta\boldsymbol{e}).$$

Then, we deduce as in (3.2.4)

$$|\widetilde{b}(\boldsymbol{e},\boldsymbol{e},\Delta\boldsymbol{e})|\leqslant C\|\boldsymbol{e}\|_1\|\boldsymbol{e}\|_1^{\frac{1}{2}}|\boldsymbol{e}|_2^{\frac{1}{2}}\|\Delta\boldsymbol{e}\|\leqslant\frac{\gamma}{8}\|\Delta\boldsymbol{e}\|^2+C\|\nabla\boldsymbol{e}\|^6.$$

Next, using the Sobolev inequalities and the uniform bound in the H^2-norm of the velocity given in (3.2.2), we have

$$|\widetilde{b}(\boldsymbol{u},\boldsymbol{e},\Delta\boldsymbol{e})|\leqslant C\|\boldsymbol{u}\|_2\|\boldsymbol{e}\|_1\|\Delta\boldsymbol{e}\|\leqslant\frac{\gamma}{8}\|\Delta\boldsymbol{e}\|^2+C\|\nabla\boldsymbol{e}\|^2,$$

and similarly,

$$|\widetilde{b}(\boldsymbol{e},\boldsymbol{u},\Delta\boldsymbol{e})|\leqslant\frac{\gamma}{8}\|\Delta\boldsymbol{e}\|^2+C\|\nabla\boldsymbol{e}\|^2.$$

Moreover, Using the property (3.1.3), we treat the last nonlinear term as below:

$$|(|\boldsymbol{u}|^\alpha\boldsymbol{u}-|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon,\Delta e)|\leqslant C\|\nabla\boldsymbol{e}\|\|\Delta\boldsymbol{e}\|\leqslant\frac{\gamma}{8}\|\Delta\boldsymbol{e}\|^2+C\|\nabla\boldsymbol{e}\|^2.$$

Substituting the above inequalities in (3.3.16) and using (3.3.15) yield

$$\frac{d}{dt}\|\nabla \boldsymbol{e}\|^2 + \gamma\|\Delta \boldsymbol{e}\|^2 + 2a\|\nabla \boldsymbol{e}\|^2 \leq C_1 + C\|\nabla \boldsymbol{e}\|^2 + C\|\nabla \boldsymbol{e}\|^6. \tag{3.3.18}$$

Then, as in the proof of Lemma 3.2.1, we have a differential inequality

$$\frac{d}{dt}(C_1 + \|\nabla \boldsymbol{e}\|^2) \leqslant C(C_1 + \|\nabla \boldsymbol{e}\|^2)^3,$$

which has a solution defined on $[t_0,\, T_0]$ satisfying:

$$\|\nabla \boldsymbol{e}(t)\|^2 \leq C, \qquad \text{where } T_0 = \min\{T_1,\, \frac{1}{4CC_1^2}\}. \tag{3.3.19}$$

Using this last estimate in (3.3.16), it follows that

$$\int_{t_0}^{t} \|\Delta \boldsymbol{e}(s)\|^2 ds \leqslant C, \quad \forall t \in [t_0,\, T_0]. \tag{3.3.20}$$

Differentiating the first equation of (3.3.13) with respect to t and taking the inner product with $\boldsymbol{e}_t$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}_t\|^2 + \gamma\|\nabla \boldsymbol{e}_t\|^2 + a\|\boldsymbol{e}_t\|^2 + (\nabla q_t,\, \boldsymbol{e}_t) = &- \widetilde{b}(\boldsymbol{e}_t, \boldsymbol{u}, \boldsymbol{e}_t) - \widetilde{b}(\boldsymbol{e}, \boldsymbol{u}_t, \boldsymbol{e}_t) - \widetilde{b}(\boldsymbol{u}_t^\epsilon, \boldsymbol{e}, \boldsymbol{e}_t)\\ &- \beta((|\boldsymbol{u}|^\alpha \boldsymbol{u} - |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon)_t, \boldsymbol{e}_t).\end{aligned} \tag{3.3.21}$$

Differentiating now the second equation of (3.3.13) with respect to t and taking the inner product with q_t, we obtain

$$(\mathrm{div}\boldsymbol{e}_t,\, q_t) + \frac{\epsilon}{2}\frac{d}{dt}\|\nabla q_t\|^2 = \epsilon(\nabla p_{tt}, \nabla q_t). \tag{3.3.22}$$

Summing (3.3.21) and (3.3.22) yields:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}_t\|^2 + \gamma\|\nabla \boldsymbol{e}_t\|^2 + a\|\boldsymbol{e}_t\|^2 + \frac{\epsilon}{2}\frac{d}{dt}\|\nabla q_t\|^2 &= \epsilon(\nabla p_{tt}, \nabla q_t) - \widetilde{b}(\boldsymbol{e}_t, \boldsymbol{u}, \boldsymbol{e}_t) - \widetilde{b}(\boldsymbol{e}, \boldsymbol{u}_t, \boldsymbol{e}_t)\\ &\quad - \widetilde{b}(\boldsymbol{u}_t^\epsilon, \boldsymbol{e}, \boldsymbol{e}_t) - \beta((|\boldsymbol{u}|^\alpha \boldsymbol{u} - |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon)_t, \boldsymbol{e}_t).\\ &\leqslant \frac{\epsilon}{2}\|\nabla p_{tt}\|^2 + \frac{\epsilon}{2}\|\nabla q_t\|^2 + |\widetilde{b}(\boldsymbol{e}_t, \boldsymbol{u}, \boldsymbol{e}_t)| + |\widetilde{b}(\boldsymbol{e}, \boldsymbol{u}_t, \boldsymbol{e}_t)|\\ &\quad + |\widetilde{b}(\boldsymbol{u}_t^\epsilon, \boldsymbol{e}, \boldsymbol{e}_t)| + \beta|((|\boldsymbol{u}|^\alpha \boldsymbol{u} - |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon)_t, \boldsymbol{e}_t)|.\end{aligned} \tag{3.3.23}$$

Invoking Young, Hölder, Gagliardo-Nirenberg inequalities and estimate (3.2.2), the nonlinear terms can be bounded as:

$$|\widetilde{b}(\boldsymbol{e}_t, \boldsymbol{u}, \boldsymbol{e}_t)| \leqslant C\|\boldsymbol{e}_t\|_1\|\boldsymbol{u}\|_2\|\boldsymbol{e}_t\| \leqslant \frac{\gamma}{8}\|\nabla \boldsymbol{e}_t\|^2 + C\|\boldsymbol{e}_t\|^2. \tag{3.3.24}$$

$$|\widetilde{b}(\boldsymbol{e}, \boldsymbol{u}_t, \boldsymbol{e}_t)| \leqslant C\|\boldsymbol{e}\|_1\|\boldsymbol{u}_t\|_1\|\boldsymbol{e}_t\|_1 \leqslant \frac{\gamma}{8}\|\nabla \boldsymbol{e}_t\|^2 + C\|\boldsymbol{e}\|_1^2\|\boldsymbol{u}_t\|_1^2. \tag{3.3.25}$$

$$
\begin{aligned}
|\widetilde{b}(\boldsymbol{u}_t^\epsilon, \boldsymbol{e}, \boldsymbol{e}_t)| &= |\widetilde{b}(\boldsymbol{u}_t, \boldsymbol{e}, \boldsymbol{e}_t) - \widetilde{b}(\boldsymbol{e}_t, \boldsymbol{e}, \boldsymbol{e}_t)| \\
&\leqslant C\|\boldsymbol{u}_t\|_1\|\boldsymbol{e}\|_1\|\boldsymbol{e}_t\|_1 + C\|\boldsymbol{e}_t\|\|\boldsymbol{e}\|_1^{\frac{1}{2}}\|\boldsymbol{e}\|_2^{\frac{1}{2}}\|\nabla \boldsymbol{e}_t\| \\
&\leqslant \frac{\gamma}{8}\|\nabla \boldsymbol{e}_t\|^2 + C\|\boldsymbol{u}_t\|_1^2\|\boldsymbol{e}\|_1^2 + C\|\boldsymbol{e}_t\|^2\|\boldsymbol{e}\|_1\|\boldsymbol{e}\|_2. \qquad (3.3.26)
\end{aligned}
$$

$$
\begin{aligned}
|((|\boldsymbol{u}|^\alpha \boldsymbol{u} - |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon)_t, \boldsymbol{e}_t)| &\le (|F'(\boldsymbol{u})\boldsymbol{u}_t - F'(\boldsymbol{u}^\epsilon)\boldsymbol{u}_t^\epsilon|, |\boldsymbol{e}_t|) \\
&= (|F'(\boldsymbol{u})\boldsymbol{u}_t - F'(\boldsymbol{u}^\epsilon)(\boldsymbol{u}_t - \boldsymbol{e}_t)|, |\boldsymbol{e}_t|) \\
&\le (|(F'(\boldsymbol{u}) - F'(\boldsymbol{u}^\epsilon))\boldsymbol{u}_t|, |\boldsymbol{e}_t|)| + (|F'(\boldsymbol{u}^\epsilon)||\boldsymbol{e}_t|, |\boldsymbol{e}_t|) \\
&= T_1 + T_2.
\end{aligned}
$$

Let us estimate each of the terms T_1, T_2. Since $|F'(\boldsymbol{v})| \le (\alpha+1)|\boldsymbol{v}|^\alpha$, using Hölder and Young inequalities, we have:

$$
\begin{aligned}
T_1 &\le 2(\alpha+1)((|\boldsymbol{u}|^\alpha + |\boldsymbol{u}^\epsilon|^\alpha)|\boldsymbol{u}_t|, |\boldsymbol{e}_t|)| \\
&\le C(\alpha+1)(\|\boldsymbol{u}\|_{3\alpha}^\alpha + \|\boldsymbol{u}^\epsilon\|_{3\alpha}^\alpha)\|\boldsymbol{u}_t\|_6\|\boldsymbol{e}_t\| \le (\alpha+1)^2(\|\boldsymbol{u}\|_{3\alpha}^\alpha + \|\boldsymbol{u}^\epsilon\|_{3\alpha}^\alpha)^2\|\boldsymbol{u}_t\|_6^2 + C\|\boldsymbol{e}_t\|^2.
\end{aligned}
$$

Since $1 \le \alpha \le 2$, we use the Sobolev inequality (3.1.2), estimates (3.3.19), (3.2.7) and (3.2.16). We obtain:

$$
T_1 \le C + C\|\boldsymbol{e}_t\|^2. \qquad (3.3.27)
$$

Similarly,

$$
\begin{aligned}
T_2 \le C(\alpha+1)\|\boldsymbol{u}^\epsilon\|_{3\alpha}^\alpha\|\boldsymbol{e}_t\|_6\|\boldsymbol{e}_t\| &\le \frac{\gamma}{8}\|\nabla \boldsymbol{e}_t\|^2 + C\|\boldsymbol{u}^\epsilon\|_{3\alpha}^{2\alpha}\|\boldsymbol{e}_t\|^2 \\
&\le \frac{\gamma}{8}\|\nabla \boldsymbol{e}_t\|^2 + C\|\boldsymbol{e}_t\|^2. \qquad (3.3.28)
\end{aligned}
$$

Thus, inserting (3.3.24)-(3.3.28) in (3.3.23), using (3.2.16) and applying Gronwall Lemma yield

$$
\|\boldsymbol{e}_t(t)\|^2 + \int_{t_0}^{T_0} (\gamma\|\nabla \boldsymbol{e}_t(s)\|^2 + a\|\boldsymbol{e}_t(s)\|^2)\,ds + \epsilon\|\nabla q_t(t)\|^2 \le C\epsilon, \qquad (3.3.29)
$$

where we have used the following estimate for the nonlinear term in (3.3.26)

$$
\int_{t_0}^{T_0} \|\boldsymbol{e}(s)\|_1\|\boldsymbol{e}(s)\|_2\,ds \le C\Big(\int_{t_0}^{T_0} \|\boldsymbol{e}(s)\|_2^2\,ds\Big)^{\frac{1}{2}} \le C, \qquad (3.3.30)
$$

which follows from the Cauchy-Schwarz inequality, (3.3.19) and (3.3.20). Next, in order to obtain the uniform bound in the H^2-norm of the error on the velocity, we need an ϵ-independent a priori estimate for $\boldsymbol{e}_t$. For this, we start from the equation (3.3.21), we consider the term $(\nabla q_t,\ \boldsymbol{e}_t)$ in the right hand side and we use the ϵ-independent estimate for ∇q_t given in (3.3.29) together with (3.3.24)-(3.3.28). Then, we rewrite the first equation of (3.3.13) as follows:

$$
-\gamma\Delta \boldsymbol{e} + \widetilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{e}) + \widetilde{B}(\boldsymbol{e}, \boldsymbol{u}) + \beta|\boldsymbol{u}|^\alpha \boldsymbol{u} - \beta|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon = -\boldsymbol{e}_t - a\boldsymbol{e} - \nabla q.
$$

Taking the inner product of the above equation with $-\Delta \boldsymbol{e}$, thanks to the above estimates, we conclude that :

$$\|\Delta \boldsymbol{e}(t)\|^2 \leqslant C, \qquad \forall t \in [t_0,, T_0]. \tag{3.3.31}$$

□

Now, we move to check the error estimate for the perturbed system.

3.4 Error analysis

3.4.1 Error estimates for a linearly perturbed problem

Primarily, we will focus on the analysis of the error for the linear case. We follow the strategy proposed in [57].

Let $(\boldsymbol{u}, p)$ be the solution of the system (3.0.1)-(3.0.2) and let $(\boldsymbol{v}^\epsilon, r^\epsilon)$ be the solution of the perturbed linearly problem :

$$\begin{array}{l} \partial_t \boldsymbol{v}^\epsilon - \gamma\Delta \boldsymbol{v}^\epsilon + a\boldsymbol{v}^\epsilon + \nabla r^\epsilon = \boldsymbol{f} - \beta|\boldsymbol{u}|^\alpha \boldsymbol{u} - B(\boldsymbol{u}, \boldsymbol{u}) \quad \text{in } \Omega_T \\ \text{div}\boldsymbol{v}^\epsilon - \epsilon\Delta r^\epsilon_t = 0 \quad \text{in} \quad \Omega_T \\ \boldsymbol{v}^\epsilon = 0 \quad \text{and} \quad \frac{\partial r^\epsilon}{\partial n} = 0 \quad \text{on} \quad \Sigma_T \\ \boldsymbol{v}^\epsilon(t_0) = \boldsymbol{u}(t_0) \quad \text{and} \quad r^\epsilon(t_0) = p(t_0) \quad \text{in } \Omega. \end{array} \tag{3.4.1}$$

The equations for the error $(\boldsymbol{\xi}, \psi) := (\boldsymbol{u} - \boldsymbol{v}^\epsilon, p - r^\epsilon)$ are:

$$\begin{array}{l} \boldsymbol{\xi}_t - \gamma\Delta\boldsymbol{\xi} + a\boldsymbol{\xi} + \nabla\psi = 0 \qquad \text{in } \Omega_T \\ \text{div}\boldsymbol{\xi} - \epsilon\Delta\psi_t = -\epsilon\Delta p_t \quad \text{in } \Omega_T \\ \boldsymbol{\xi} = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial n} = \frac{\partial p}{\partial n} \quad \text{on } \Sigma_T \\ \boldsymbol{\xi}(t_0) = 0 \quad \text{and} \quad \psi(t_0) = 0 \quad \text{in } \Omega. \end{array} \tag{3.4.2}$$

Thanks to the previous study in Lemma 3.3.2 we have the following estimates:

$$\|\boldsymbol{v}^\epsilon\|_2 + \|r^\epsilon(t)\|_1 + \|r^\epsilon_t(t)\|_1 \leqslant C, \quad \forall t \in [t_0, T_0] \tag{3.4.3}$$

$$\|\boldsymbol{\xi}_t(t)\|^2 \leqslant C\epsilon, \quad \forall t \in [t_0, T_0] \tag{3.4.4}$$

The result in the Lemma below describes the behavior of the linear error part.

Lemma 3.4.1. *Assume that the assumptions* (3.2.1) *and* (3.2.15) *are valid. Then, there exists a constant C depending on the given datum such that the following estimate holds true:*

$$\int_{t_0}^t \|\boldsymbol{\xi}(s)\|^2 + \epsilon^{\frac{1}{2}}\|\boldsymbol{\xi}(t)\|^2 + \epsilon(\|\boldsymbol{\xi}(t)\|_1^2 + \|\psi(t)\|^2) \leqslant C\epsilon^2, \quad \forall t \in [t_0, T_0]. \tag{3.4.5}$$

Proof. In order to prove a convergence result for the velocity field in the $L^2(t_0, T_0;\, \boldsymbol{L}^2(\Omega))$-norm, we use a parabolic duality argument: Let $(\boldsymbol{w},\, q)$ be a solution of:

$$\begin{aligned} &\boldsymbol{w}_s + \gamma\Delta\boldsymbol{w} - a\boldsymbol{w} + \nabla q = \boldsymbol{\xi}(s) \quad \text{and} \quad \mathrm{div}\boldsymbol{w} = 0 \quad \text{in } \Omega_T,\ s \in [t_0, t] \\ &\boldsymbol{w} = 0 \quad \text{on} \quad \partial\Omega \quad \text{and} \quad \boldsymbol{w}(t) = 0. \end{aligned} \tag{3.4.6}$$

We can easily verify the following bound for the solution of (3.4.6):

$$\int_{t_0}^{t} (\|\boldsymbol{w}(s)\|_2^2 + \|\ \nabla q(s)\|^2) ds \leqslant C \int_{t_0}^{t} \|\ \boldsymbol{\xi}(s)\|^2 ds. \tag{3.4.7}$$

Taking the inner product of the first equation in (3.4.6) with $\boldsymbol{\xi}(s)$, since $\boldsymbol{\xi}$ is solution of (3.4.2) and div $\boldsymbol{w} = 0$, we obtain:

$$\begin{aligned} \|\boldsymbol{\xi}(s)\|^2 &= \frac{d}{ds}(\boldsymbol{w},\, \boldsymbol{\xi}(s)) - (\boldsymbol{w},\, \boldsymbol{\xi}_s(s)) - \gamma(\nabla\boldsymbol{w},\, \nabla\boldsymbol{\xi}(s)) - a(\boldsymbol{w},\, \boldsymbol{\xi}(s) + (\nabla q,\, \boldsymbol{\xi}(s)) \\ &= \frac{d}{ds}(\boldsymbol{w},\, \boldsymbol{\xi}(s)) + (\nabla\psi,\, \boldsymbol{w}) - (q,\, \mathrm{div}\boldsymbol{\xi}(s)) \\ &= \frac{d}{ds}(\boldsymbol{w},\, \boldsymbol{\xi}(s)) + \epsilon(q,\, \Delta p_s) - \epsilon(q,\, \Delta\psi_s) \\ &= \frac{d}{ds}(\boldsymbol{w},\, \boldsymbol{\xi}(s)) + \epsilon(q,\, \Delta r_s^{\epsilon}) = \frac{d}{ds}(\boldsymbol{w},\, \boldsymbol{\xi}(s)) - \epsilon(\nabla q,\, \nabla r_s^{\epsilon}). \end{aligned}$$

Integrating from t_0 to t, since $\boldsymbol{w}(t) = \mathbf{0}$ and $\boldsymbol{\xi}(t_0) = \mathbf{0}$, using (3.4.7) we have for $\epsilon' > 0$:

$$\begin{aligned} \int_{t_0}^{t} \|\boldsymbol{\xi}(s)\|^2\, \mathrm{d}s &\leq \epsilon \int_{t_0}^{t} \|\nabla q(s)\| \|\nabla r_s^{\epsilon}(s)\|\, \mathrm{d}s \\ &\leq \epsilon' \int_{t_0}^{t} \|\nabla q(s)\|^2\, \mathrm{d}s + \frac{\epsilon^2}{\epsilon'} \int_{t_0}^{t} \|\nabla r_s^{\epsilon}(s)\|^2\, \mathrm{d}s, \\ &\leq C\epsilon' \int_{t_0}^{t} \|\boldsymbol{\xi}(s)\|^2\, \mathrm{d}s + \frac{\epsilon^2}{\epsilon'} \int_{t_0}^{t} \|\nabla r_s^{\epsilon}(s)\|^2\, \mathrm{d}s. \end{aligned}$$

We conclude by choosing $\epsilon' < \frac{1}{C}$ and using (3.4.3) that:

$$\int_{t_0}^{t} \|\boldsymbol{\xi}(s)\|^2 ds \leqslant C\epsilon^2. \tag{3.4.8}$$

Our next goal is to show the error bound for the velocity field in the $L^\infty(t_0, T_0;\, \boldsymbol{H}^1(\Omega))$-norm. For this, we take the inner product of the first equation in the system (3.4.2) with $\boldsymbol{\xi}_t$, we obtain

$$\|\boldsymbol{\xi}_t\|^2 + \frac{\gamma}{2}\frac{d}{dt}\|\nabla\boldsymbol{\xi}\|^2 + \frac{a}{2}\frac{d}{dt}\|\boldsymbol{\xi}\|^2 = (\psi, \mathrm{div}\boldsymbol{\xi}_t). \tag{3.4.9}$$

At the same time, we differentiate the second equation in (3.4.2) with respect to time and test with ψ:

$$\begin{aligned} (\psi, \mathrm{div}\boldsymbol{\xi}_t) &= \epsilon(\psi, \Delta\psi_{tt}) - \epsilon(\psi, \Delta p_{tt}) \\ &= -\epsilon\frac{d}{dt}(\nabla\psi, \nabla\psi_t) + \epsilon\|\nabla\psi_t\|^2 + \epsilon(\nabla\psi, \nabla p_{tt}). \end{aligned} \tag{3.4.10}$$

Substituting (3.4.10) in (3.4.9) and integrating the previous equation from t_0 to t, since $\psi(t_0) = 0$, we deduce :

$$\int_{t_0}^{t} \|\boldsymbol{\xi}_t\|^2 ds + \frac{\gamma}{2}\|\nabla\boldsymbol{\xi}\|^2 + \frac{a}{2}\|\boldsymbol{\xi}\|^2 \leqslant C\epsilon\|\psi(t)\|_1\|\psi_t(t)\|_1 + C\epsilon\int_{t_0}^{t} \|\psi_t(s)\|_1^2 ds$$
$$+C\epsilon\int_{t_0}^{t}(\|p_{tt}(s)\|^2 + \|\psi(s)\|_1^2)ds. \qquad (3.4.11)$$

Thanks to the estimates (3.2.2),(3.2.16) and (3.4.3), we conclude:

$$\int_{t_0}^{t} \|\boldsymbol{\xi}_t\|^2\, ds + \min(\gamma,\, a)\|\boldsymbol{\xi}(t)\|_1^2 \leqslant C\epsilon, \quad \forall t \in [t_0,\, T_0]. \qquad (3.4.12)$$

We consider now the dual problem (3.4.6) with $\boldsymbol{\xi}_s(s)$ instead of $\boldsymbol{\xi}(s)$ in the right hand side:

$$\begin{aligned} &\boldsymbol{w}_s + \gamma\Delta\boldsymbol{w} - a\boldsymbol{w} + \nabla q = \boldsymbol{\xi}_s(s) \quad \text{and} \quad \mathrm{div}\boldsymbol{w} = 0 \quad \text{in } \Omega_T,\ s \in [t_0, t] \\ &\boldsymbol{w} = 0 \quad \text{on} \quad \partial\Omega \quad \text{and} \quad \boldsymbol{w}(t) = 0. \end{aligned} \qquad (3.4.13)$$

The solution $(\boldsymbol{w},\, q)$ of the system (3.4.13) satisfies:

$$\int_{t_0}^{t}(\|\Delta\boldsymbol{w}\|^2 + \|\nabla q\|^2)ds \leqslant C\int_{t_0}^{t} \|\boldsymbol{\xi}_s(s)\|^2 ds. \qquad (3.4.14)$$

By taking the inner product of the first equation in (3.4.13) with $\boldsymbol{\xi}(s)$ and using (3.4.2), we find

$$\frac{1}{2}\frac{d}{ds}\|\boldsymbol{\xi}(s)\|^2 = \frac{d}{ds}(\boldsymbol{\xi}, w) - \epsilon(\nabla q, \nabla r_s^\epsilon).$$

Integrating the above equation from t_0 to t, thanks to the initial conditions for $\boldsymbol{w}$ and $\boldsymbol{\xi}$, estimates (3.4.3) and (3.4.14), we have

$$\|\boldsymbol{\xi}(t)\|^2 \leqslant \epsilon^{1/2}\int_{t_0}^{t} \|\nabla q\|^2 ds + \epsilon^{3/2}\int_{t_0}^{t} \|\nabla r_s^\epsilon\|^2 ds \leq C\epsilon^{3/2}, \quad \forall t \in [t_0, T_0].$$

Finally, we prove the error bound for the pressure. For this, using equations (3.4.2), we have

$$\|\psi(t)\| \leqslant C \sup_{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)} \frac{(\boldsymbol{\xi}_t(t), \boldsymbol{v}) + \gamma(\nabla\boldsymbol{\xi}(t), \nabla\boldsymbol{v}) + a(\boldsymbol{\xi}(t), \boldsymbol{v})}{\|\boldsymbol{v}\|_1}.$$

Next, we have just to use the previous estimates to deduce the following bound:

$$\|\psi(t)\| \leqslant C\epsilon^{1/2}, \qquad (3.4.15)$$

which completes the proof of this lemma.

□

3.4.2 Error estimates for the nonlinear perturbed system

This subsection is devoted to the transfer of the results that have been derived above for the linear case to the nonlinear case.

We use the notation $\boldsymbol{\eta} = \boldsymbol{v}^\epsilon - \boldsymbol{u}^\epsilon$, $\phi = r^\epsilon - p^\epsilon$. Subtracting the perturbed system (0.0.14) from the equations (3.4.1), we obtain :

$$\boldsymbol{\eta}_t - \gamma\Delta\eta + a\boldsymbol{\eta} + \nabla\phi = \widetilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon) - \widetilde{B}(\boldsymbol{u}, \boldsymbol{u}) + \beta|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon - \beta|\boldsymbol{u}|^\alpha \boldsymbol{u} \quad \text{in } \Omega, \tag{3.4.16}$$

with Dirichlet boundary conditions on the function $\boldsymbol{\eta}$ and Neuman boundary conditions on ϕ_t

$$\begin{aligned} &\operatorname{div}\boldsymbol{\eta} - \epsilon\Delta\phi_t = 0, \quad \text{in} \quad \Omega \\ &\boldsymbol{\eta} = 0, \quad \tfrac{\partial\phi}{\partial n} = 0 \quad \text{on} \ \ \Sigma_T \end{aligned} \tag{3.4.17}$$

such that the initial data are given as :

$$\boldsymbol{\eta}(t_0) = 0 \quad \text{and} \quad \phi(t_0) = 0. \tag{3.4.18}$$

In the following lemma, we illustrate the control of the error through the non-linearities.

Lemma 3.4.2. *Assume that assumptions* (3.2.1) *and* (3.2.15) *are valid. Then, there exists a constant C depending on the given data such that the following error bound holds true:*

$$\|\boldsymbol{\eta}(t)\|^2 + \gamma\int_{t_0}^{t} \|\nabla\boldsymbol{\eta}(s)\|^2 ds + \epsilon(\|\nabla\boldsymbol{\eta}(s)\|^2 + \|\nabla\phi(t)\|^2) \leqslant C\epsilon^2, \quad \forall t \in [t_0, T_0] \tag{3.4.19}$$

Proof. Taking the inner product of (3.4.16)-(3.4.17)-(3.4.18) with $(\beta,\, \phi)$ leads to:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|^2 + \gamma\|\nabla\boldsymbol{\eta}\|^2 + \frac{\epsilon}{2}\frac{d}{dt}\|\nabla\phi\|^2 &= -a\|\boldsymbol{\eta}\|^2 + \widetilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{\xi}, \boldsymbol{\eta}) - \widetilde{b}(\boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{u}, \boldsymbol{\eta}) \\ &\quad + \beta(|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon - |\boldsymbol{u}|^\alpha \boldsymbol{u}, \boldsymbol{\eta}), \end{aligned} \tag{3.4.20}$$

where we have used the fact that $\boldsymbol{e} = \boldsymbol{\xi} + \boldsymbol{\eta}$ and

$$\widetilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{\eta}) - \widetilde{b}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\eta}) = \widetilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{\xi}, \boldsymbol{\eta}) - \widetilde{b}(\boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{u}, \boldsymbol{\eta}). \tag{3.4.21}$$

Applying the Gagliardo Nirenberg, Young and Sobolev's inequality, the non linear terms in the right hand side of (3.4.21) can be bounded as

$$|\widetilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{\xi}, \boldsymbol{\eta})| \leqslant \frac{\gamma}{8}\|\nabla\boldsymbol{\eta}\|^2 + C\|\boldsymbol{u}^\epsilon\|_2^2\|\boldsymbol{\xi}\|^2. \tag{3.4.22}$$

$$|\widetilde{b}(\boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{u}, \boldsymbol{\eta})| \leqslant \frac{\gamma}{8}\|\nabla\boldsymbol{\eta}\|^2 + C\|\boldsymbol{u}\|_2^2(\|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\eta}\|^2). \tag{3.4.23}$$

There remains to bound the last term in (3.4.20). On the one hand

$$\beta(|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon - |\boldsymbol{u}|^\alpha \boldsymbol{u}, \boldsymbol{\eta}) = \beta(|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon - |\boldsymbol{v}^\epsilon|^\alpha \boldsymbol{v}^\epsilon, \boldsymbol{\eta}) + \beta(|\boldsymbol{v}^\epsilon|^\alpha \boldsymbol{v}^\epsilon - |\boldsymbol{u}|^\alpha \boldsymbol{u}, \boldsymbol{\eta}) \tag{3.4.24}$$

Thanks to (3.1.1), we have

$$(|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon - |\boldsymbol{v}^\epsilon|^\alpha \boldsymbol{v}^\epsilon, \boldsymbol{\eta}) \leqslant 0.$$

On the other hand, observe that:

$$\beta(|\boldsymbol{v}^\epsilon|^\alpha \boldsymbol{v}^\epsilon - |\boldsymbol{u}|^\alpha \boldsymbol{u}, \boldsymbol{\eta}) = \beta(|\boldsymbol{v}^\epsilon|^\alpha \boldsymbol{v}^\epsilon - |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon, \boldsymbol{\eta}) + \beta(|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon - |\boldsymbol{u}|^\alpha \boldsymbol{u}, \boldsymbol{e} - \boldsymbol{\xi}).$$

Using again (3.1.1), we have:

$$(|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon - |\boldsymbol{u}|^\alpha \boldsymbol{u}, \boldsymbol{e}) \leqslant 0;$$

Then, coming back to (3.4.24), we have:

$$\beta|(|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon - |\boldsymbol{u}|^\alpha \boldsymbol{u}, \boldsymbol{\eta})| \leq \beta|(|\boldsymbol{v}^\epsilon|^\alpha \boldsymbol{v}^\epsilon - |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon, \boldsymbol{\eta})| + \beta|(|\boldsymbol{u}|^\alpha \boldsymbol{u} - |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon, \boldsymbol{\xi})| := A_1 + A_2.$$

Using (3.1.3) and Young inequality, we obtain:

$$A_1 \leq C\|\nabla\boldsymbol{\eta}\|\|\boldsymbol{\eta}\| \leqslant \frac{\gamma}{8}\|\nabla\boldsymbol{\eta}\|^2 + C\|\boldsymbol{\eta}\|^2 \tag{3.4.25}$$

$$\begin{aligned} A_2 &\leq C\|\nabla(\boldsymbol{\xi}+\boldsymbol{\eta})\|\|\boldsymbol{\xi}\| \leq C(\|\nabla\boldsymbol{\eta}\| + \|\nabla\boldsymbol{\xi}\|)\|\boldsymbol{\xi}\| \\ &\leq \frac{\gamma}{8}\|\nabla\boldsymbol{\eta}\|^2 + C\|\boldsymbol{\xi}\|^2 + C\|\nabla\boldsymbol{\xi}\|\|\boldsymbol{\xi}\| \end{aligned} \tag{3.4.26}$$

Substituting (3.4.22),(3.4.23), (3.4.25) and (3.4.26) in (3.4.20), using (3.2.13) and (3.3.11) we obtain:

$$\frac{d}{dt}\|\boldsymbol{\eta}\|^2 + \gamma\|\nabla\boldsymbol{\eta}\|^2 + \epsilon\frac{d}{dt}\|\nabla\phi\|^2 \leq C\|\boldsymbol{\xi}\|^2 + C\|\nabla\boldsymbol{\xi}\|\|\boldsymbol{\xi}\| + C\|\boldsymbol{\eta}\|^2.$$

Using Gronwall's Lemma together with the bound (3.4.5) and the following estimate:

$$\int_{t_0}^t \|\nabla\boldsymbol{\xi}(s)\|\|\boldsymbol{\xi}(s)\|\, ds \leq C\epsilon \int_{t_0}^t \|\boldsymbol{\xi}(s)\| \leq C\epsilon\Big(\int_{t_0}^t \|\boldsymbol{\xi}(s)\|^2\, ds\Big)^{\frac{1}{2}} \leq C\epsilon^2, \qquad \forall t \in [t_0, T_0]$$

leads to

$$\|\boldsymbol{\eta}(t)\|^2 + \gamma\int_{t_0}^t \|\nabla\boldsymbol{\eta}(s)\|^2 ds + \epsilon\|\nabla\phi(t)\|^2 \leqslant C\epsilon^2, \quad \forall t \in [t_0, T_0]. \tag{3.4.27}$$

To end the proof, we are going to verify the estimate $\|\nabla\boldsymbol{\eta}(t)\| \leq C\epsilon$, for any $t \in [t_0, T_0]$. For this purpose, we take the inner product of the equation (3.4.16) with $-\Delta\boldsymbol{\eta}$ and we obtain:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\eta}\|^2 + \gamma\|\Delta\boldsymbol{\eta}\|^2 + a\|\nabla\boldsymbol{\eta}\|^2 = (\nabla\phi, \Delta\boldsymbol{\eta}) + \widetilde{b}(\boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{u}, \Delta\boldsymbol{\eta}) + \widetilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{\xi}+\boldsymbol{\eta}, \Delta\boldsymbol{\eta}) \\ + \beta(|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon - |\boldsymbol{u}|^\alpha \boldsymbol{u}, -\Delta\boldsymbol{\eta}). \end{aligned} \tag{3.4.28}$$

Using the same arguments as in the begining of the proof, we have the following estimates

$$|(\nabla\phi, \Delta\boldsymbol{\eta})\| \leqslant \frac{\gamma}{8}\|\Delta\boldsymbol{\eta}\|^2 + C\|\nabla\phi\|^2, \tag{3.4.29}$$

$$\begin{aligned} |\widetilde{b}(\boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{u}, \Delta\boldsymbol{\eta})| &\leqslant C\|\boldsymbol{\xi}+\boldsymbol{\eta}\|_1\|\boldsymbol{u}\|_2\|\Delta\boldsymbol{\eta}\| \\ &\leqslant \frac{\gamma}{8}\|\Delta\boldsymbol{\eta}\|^2 + C\|\boldsymbol{u}\|_2^2(\|\boldsymbol{\xi}\|_1^2 + \|\boldsymbol{\eta}\|_1^2), \end{aligned} \tag{3.4.30}$$

$$|\widetilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{\xi}+\boldsymbol{\eta}, \Delta\boldsymbol{\eta})| \leqslant \frac{\gamma}{8}\|\Delta\boldsymbol{\eta}\|^2 + C\|\boldsymbol{u}^\epsilon\|_2^2(\|\boldsymbol{\xi}\|_1^2 + \|\boldsymbol{\eta}\|_1^2), \tag{3.4.31}$$

$$\begin{aligned}\beta|(|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon - |\boldsymbol{u}|^\alpha\boldsymbol{u}, -\Delta\boldsymbol{\eta}) &\leqslant C\|\nabla\boldsymbol{e}\|\|\Delta\boldsymbol{\eta}\| \\ &\leqslant \frac{\gamma}{8}\|\Delta\boldsymbol{\eta}\|^2 + C(\|\nabla\boldsymbol{\xi}\|^2 + \|\nabla\boldsymbol{\eta}\|^2).\end{aligned} \tag{3.4.32}$$

Gathering estimates (3.4.29)-(3.4.32), thanks to (3.2.2) and (3.3.11), we get

$$\frac{d}{dt}\|\nabla\boldsymbol{\eta}\|^2 + \gamma\|\Delta\boldsymbol{\eta}\|^2 + 2a\|\nabla\boldsymbol{\eta}\|^2 \leq C\|\nabla\phi\|^2 + C(\|\nabla\boldsymbol{\xi}\|^2 + \|\nabla\boldsymbol{\eta}\|^2),$$

and by in virtue of Gronwall's Lemma and (3.4.27), we get the desired estimate.

□

We are now in position to prove the basic result of this section.

Theorem 3.4.3. *Assume that assumptions* (3.2.1) *and* (3.2.15) *are valid. Then, there exists a constant C depending on the given data such that the following estimate holds true:*

$$\int_{t_0}^t \|\boldsymbol{u}(s) - \boldsymbol{u}^\epsilon(s)\|^2 \quad ds + \epsilon^{\frac{1}{2}}\|\boldsymbol{u}(t) - \boldsymbol{u}^\epsilon(t)\|^2 + \epsilon(\|\boldsymbol{u}(t) - \boldsymbol{u}^\epsilon(t)\|_1^2 + \|p(t) - p^\epsilon(t)\|) \leqslant C\epsilon^2. \tag{3.4.33}$$

Proof. Since $\boldsymbol{e} = \boldsymbol{\xi} + \boldsymbol{\eta}$, it suffices to apply the triangle inequality and to use the estimates (3.4.5) and (3.4.19). □

3.5 Error estimates for time-discretization

In this section, we check the error analysis for the time-discretized Darcy Brinkman Forchheimer problem.

Firstly, we consider $\boldsymbol{u}^{n+\frac{1}{2}} = \dfrac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2}$ and $\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}) = \frac{1}{2}(\boldsymbol{u}(t_{n+1}) + \boldsymbol{u}(t_n))$. Let us define k to be a time step where $t_{n+\frac{1}{2}} = (n+\frac{1}{2})k$.

Let us first consider the projection scheme of the original problem as follow :

$$\begin{aligned}\frac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{k} - \frac{\gamma}{2}\Delta(\widetilde{\boldsymbol{u}}^{n+1} + \boldsymbol{u}^n) + F\Big(\frac{\widetilde{\boldsymbol{u}}^{n+1} + \boldsymbol{u}^n}{2}\Big) + \widetilde{B}\Big(\frac{\widetilde{\boldsymbol{u}}^{n+1} + \boldsymbol{u}^n}{2}, \frac{\widetilde{\boldsymbol{u}}^{n+1} + \boldsymbol{u}^n}{2}\Big) + \nabla p^n = \boldsymbol{f}(t_{n+\frac{1}{2}}).\\ (\widetilde{\boldsymbol{u}}^{n+1} + \boldsymbol{u}^n)|_\Gamma = 0\end{aligned} \tag{3.5.1}$$

$$\begin{aligned}\frac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{k} + \frac{1}{2}\nabla(p^{n+1} + p^n) &= 0,\\ \operatorname{div} u^{n+1} &= 0,\\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n}|_\Gamma &= 0\end{aligned} \tag{3.5.2}$$

We define P to be the projection in $\boldsymbol{L}^2(\Omega)$ onto the divergence free-subspace :

$$M = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \operatorname{div} \boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = 0\}.$$

We derive from (3.5.2) that $\boldsymbol{u}^{n+1} = P\widetilde{\boldsymbol{u}}^{n+1}$. By summing up (3.5.1) at step n and (3.5.2) at step $n-1$. Moreover, by applying the divergence operator to (3.5.2), one can obtain:

$$\begin{aligned}
&\frac{\widetilde{\boldsymbol{u}}^{n+1}-\widetilde{\boldsymbol{u}}^{n}}{k} - \frac{\gamma}{2}\Delta(\widetilde{\boldsymbol{u}}^{n+1} + P\widetilde{\boldsymbol{u}}^{n}) + F(\frac{\widetilde{\boldsymbol{u}}^{n+1} + P\widetilde{\boldsymbol{u}}^{n}}{2}) + \\
&\widetilde{B}(\frac{\widetilde{\boldsymbol{u}}^{n+1} + P\widetilde{\boldsymbol{u}}^{n}}{2}, \frac{\widetilde{\boldsymbol{u}}^{n+1} + P\widetilde{\boldsymbol{u}}^{n}}{2}) + \frac{1}{2}\nabla(3p^n - p^{n-1}) = \boldsymbol{f}(t_{n+\frac{1}{2}}).
\end{aligned} \tag{3.5.3}$$

$$(\widetilde{\boldsymbol{u}}^{n+1} + P\widetilde{\boldsymbol{u}}^{n})|_\Gamma = 0 \tag{3.5.4}$$

$$\operatorname{div} \widetilde{u}^{n+1} - \frac{1}{2}k\Delta(p^{n+1} - p^n) = 0, \quad \frac{\partial p^{n+1}}{\partial \boldsymbol{n}}|_\Gamma = \frac{\partial p^{n}}{\partial \boldsymbol{n}}|_\Gamma$$

The previous system gives a second-error time discretization to the perturbed problem :

$$\begin{aligned}
&\partial_t \boldsymbol{u}^\epsilon - \gamma\Delta\boldsymbol{u}^\epsilon + a\boldsymbol{u}^\epsilon + \widetilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon) + \beta|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon + \nabla p^\epsilon = \boldsymbol{f} \quad \text{in} \quad \Omega_T \\
&\operatorname{div}\boldsymbol{u}^\epsilon - \epsilon\Delta p_t^\epsilon = 0 \quad \text{in} \quad \Omega_T
\end{aligned} \tag{3.5.5}$$

The purpose behind approximate the perturbed problem by time-discretized system is that to establish the second error estimate for the velocity when $\varepsilon \sim \frac{1}{2}k^2$.
We are able to approximate again the perturbed problem by the following scheme :

$$\begin{aligned}
\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n}}{k} - \frac{\gamma}{2}\Delta(\boldsymbol{u}^{n+1} + \boldsymbol{u}^{n}) + a\frac{\boldsymbol{u}^{n+1}+\boldsymbol{u}^{n}}{2} &+ F(\boldsymbol{u}^{n+\frac{1}{2}}) + \widetilde{B}(\boldsymbol{u}^{n+\frac{1}{2}}, \boldsymbol{u}^{n+\frac{1}{2}}) \\
&+\nabla p^n = \boldsymbol{f}(t_{n+\frac{1}{2}}), \quad \text{in} \quad \Omega. \\
\operatorname{div}\boldsymbol{u}^{n+1} - \delta k\Delta(p^{n+1} - p^n) &= 0, \quad \text{in} \quad \Omega. \\
\boldsymbol{u}^{n+1} = 0, \qquad \frac{\partial p^{n+1}}{\partial \boldsymbol{n}} &= \frac{\partial p^{n}}{\partial \boldsymbol{n}}, \qquad \text{on} \quad \Gamma
\end{aligned} \tag{3.5.6}$$

Such that δ define a constant that we search afterward. We assume having the initial data $(\boldsymbol{u}^0, p^0)$ of the approximate system reviewd above to be very close to the initial data $(\boldsymbol{u}(t_0), p(t_0))$ for the Brinkman Forchheimer system (1.3.58) :

$$\|\boldsymbol{u}^0 - \boldsymbol{u}(t_0)\| \leqslant Ck^2, \qquad \|\nabla(\boldsymbol{u}^0 - \boldsymbol{u}(t_0))\| + \|\nabla(p^0 - p(t_0))\| \leqslant Ck. \tag{3.5.7}$$

Moreover, the Lemma 3.7.1 holds if we assume some regularity conditions on the velocity and the pressure which can be characterized by,

$$\int_0^T \left\{\|\boldsymbol{u}_{ttt}\|_{-1}^2 + \|\boldsymbol{u}_{tt}\|^2 + |p_{tt}|\right\} \leqslant M. \tag{3.5.8}$$

Let R^n be the truncation error and R^{n1}, R^{n2} the velocity (resp. pressure) error in R^n such that : $R^n = R^{n1} + R^{n2}$. More precisely we denote by

$$\begin{aligned} R^{n1} =& \frac{\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)}{k} - \frac{\gamma}{2}\Delta(\boldsymbol{u}(t_{n+1}) + \boldsymbol{u}(t_n)) + \frac{a}{2}(\boldsymbol{u}(t_{n+1}) + \boldsymbol{u}(t_n)) \\ &+ B(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}), \boldsymbol{u}(t_{n+\frac{1}{2}})). \\ R^{n2} =& \nabla\widetilde{p}(t_{n+\frac{1}{2}}). \end{aligned} \tag{3.5.9}$$

In the sequel, we will need to use the following algebric relation :

$$\begin{aligned} (\nabla(a^{n+1} - a^{n-1}), \nabla a^n) = & \frac{1}{2}\{\|\nabla a^{n+1}\|^2 - \|\nabla a^{n-1}\|^2\} \\ & + \frac{1}{2}\{\|\nabla(a^n - a^{n-1})\|^2 - \|\nabla(a^{n+1} - a^n)\|^2\} \end{aligned} \tag{3.5.10}$$

We need first to check the stability of the scheme given above, so we establish the following lemma.

Lemma 3.5.1. *If $\delta \geqslant \frac{1}{4}$, then there exists a constant $C > 0$ such that for $1 \leqslant m \leqslant \frac{T}{k} - 1$, we have :*

$$(1 - \frac{1}{4\delta})\|\boldsymbol{u}^{m+1}\|^2 + \frac{\gamma\, k}{4}\sum_{n=1}^{m}\|\nabla(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n)\|^2 + \frac{a\, k}{4}\sum_{n=1}^{m}\|\boldsymbol{u}^{n+1} + \boldsymbol{u}^n\|^2 \tag{3.5.11}$$

$$+\frac{\beta\, k}{2^{\alpha+1}}\sum_{n=1}^{m}\|\boldsymbol{u}^{n+1} + \boldsymbol{u}^n\|^{\alpha+2} \leqslant 2\|\boldsymbol{u}^1\|^2 + \frac{\delta\, k^2}{2}(\|\nabla p^1\|^2 + \|\nabla p^0\|^2) + \|\boldsymbol{f}\|^2. \tag{3.5.12}$$

Proof. By summing the second equation of (3.5.6) with :

$$\operatorname{div} \boldsymbol{u}^n - \delta k\Delta(p^n - p^{n-1}) = 0, \quad \text{in } \Omega$$

It gives :

$$\operatorname{div}(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n) - \delta k\Delta(p^{n+1} - p^{n-1}) = 0 \tag{3.5.13}$$

We take now the inner product of the first equation in (3.5.6) with $k(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n)$ and of (3.5.13) with $k\, p^n$, it follows

$$\begin{aligned} \|\boldsymbol{u}^{n+1}\|^2 - \|\boldsymbol{u}^n\|^2 + \frac{\gamma k}{4}\|\nabla(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n)\|^2 + \frac{ak}{4}\|(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n)\|^2 + \frac{\beta\, k}{2^{\alpha+1}}\|\boldsymbol{u}^{n+1} + \boldsymbol{u}^n\|^{\alpha+2} \\ + \delta\, k^2 2(\nabla(p^{n+1} - p^{n-1}), \nabla p^n) = k(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n, f(t_{n+\frac{1}{2}})) \end{aligned}$$

By adding up the last result for $n = 1$ to m, we obtain :

$$\begin{aligned} \|\boldsymbol{u}^{m+1}\|^2 + \frac{\gamma k}{4}\sum_{n=1}^{m}\|\nabla(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n)\|^2 + \frac{a\, k}{4}\sum_{n=1}^{m}\|\boldsymbol{u}^{n+1} + \boldsymbol{u}^n\|^2 + \frac{\beta\, k}{2^{\alpha+1}}\sum_{n=1}^{m}\|\boldsymbol{u}^{n+1} + \boldsymbol{u}^n\|^{\alpha+2} \\ + \frac{\delta\, k^2}{2}(\|\nabla p^{m+1}\|^2 + \|\nabla p^m\|^2) \leqslant \|\boldsymbol{u}^1\|^2 + \|\boldsymbol{f}\|^2 + \frac{\delta\, k^2}{2}(\|\nabla p^1\|^2 + \|\nabla p^0\|^2) + \frac{\delta\, k^2}{2}\|\nabla(p^{m+1} - p^m)\|^2 \end{aligned} \tag{3.5.14}$$

By using the second equation of (3.5.6), we have :

$$\delta\, k^2\|\nabla(p^{m+1} - p^m)\|^2 \leqslant \frac{1}{\delta}\|\boldsymbol{u}^{m+1}\|^2$$

Moreover, we consider

$$\begin{aligned}\frac{\delta k^2}{2}\|\nabla(p^{m+1}-p^m)\|^2 &\leqslant \frac{1}{4\delta}\|\boldsymbol{u}^{m+1}\|^2+\frac{\delta k^2}{4}\|\nabla(p^{m+1}+p^m)\|^2\\ &\leqslant \frac{1}{4\delta}\|\boldsymbol{u}^{m+1}\|^2+\frac{\delta k^2}{4}(\|\nabla p^{m+1}\|^2+\|\nabla p^m\|^2)\end{aligned}$$

By using the last result and the relation (3.5.14), we reach the desired estimate.

□

The Theorem below provide the fundamental result of this section.

Theorem 3.5.2. *Let $\boldsymbol{u}$ be a solution of problem* (3.5.6) *and $\boldsymbol{u}^n$ solution of the system* (3.0.1)-(3.0.2). *We assume that the initial data $(\boldsymbol{u}^0, p^0)$ satisfies the estimate condition* (3.5.7) *moreover $\boldsymbol{u}^0 \in H^2(\Omega)\cap V$ and $\boldsymbol{f}, \boldsymbol{f}_t, \boldsymbol{f}_{tt} \in C([0,T], L^2(\Omega))$, then there exist $C>0$ satisfies :*

$$\begin{cases}\text{For all} \quad 1\leqslant m\leqslant M=\frac{T-t_0}{k}, \quad \text{we have}\\ k\sum_{n=1}^m\|\boldsymbol{u}(t_n)-\boldsymbol{u}^n\|^2+k^2\|\nabla(\boldsymbol{u}(t_m)-\boldsymbol{u}^m)\|^2+k^2\|(p(t_m)-p^m)\|^2\leqslant Ck^4.\end{cases} \tag{3.5.15}$$

Now we will provide the following auxiliary lemmas which help us to establish the proof of the Theorem 3.5.2.

Lemma 3.5.3. *Under the same assumptions of the Theorem 3.5.2 and the initial conditions* (3.5.7), *we have :*

$$\|\nabla\boldsymbol{u}^{n+1}\|^2+\|\Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\|^2+\|\nabla p^{n+1}\|\leqslant C, \quad \forall\, 0\leqslant n\leqslant N-1. \tag{3.5.16}$$

Proof. Let us define $\theta^n=\boldsymbol{u}(t_n)-\boldsymbol{u}^n$, $\chi^n=p(t_n)-p^n$.
First by subtracting the discrete time Darcy Brinkman Forchheimer equations (3.5.6) from the original problem (1.3.58), we have :

$$\begin{aligned}\frac{\theta^{n+1}-\theta^n}{k}-\frac{\gamma}{2}\Delta(\theta^{n+1}+\theta^n)+\frac{a}{2}(\theta^{n+1}+\theta^n)+\nabla\chi^n=R^n+Q^n.\\ \mathrm{div}\theta^{n+1}-\delta k\Delta(\chi^{n+1}-\chi^n)=-\delta k\Delta(p(t_{n+1})-p(t_n)).\end{aligned} \tag{3.5.17}$$

such that :

$$\begin{aligned}R^n&=\frac{\boldsymbol{u}(t_{n+1})-\boldsymbol{u}(t_n)}{k}-\frac{\gamma}{2}\Delta(\boldsymbol{u}(t_{n+1})+\boldsymbol{u}(t_n))+\frac{a}{2}(\boldsymbol{u}(t_{n+1})-\boldsymbol{u}(t_n))\\ &\quad+\widetilde{B}(\boldsymbol{u}(t_{n+\frac12}),\boldsymbol{u}(t_{n+\frac12}))+F(\boldsymbol{u}(t_{n+\frac12}))+\nabla p(t_n).\\ Q^n&=\widetilde{B}(\boldsymbol{u}^{n+\frac12},\boldsymbol{u}^{n+\frac12})-\widetilde{B}(\boldsymbol{u}(t_{n+\frac12}),\boldsymbol{u}(t_{n+\frac12}))+F(\boldsymbol{u}^{n+\frac12})-F(\boldsymbol{u}(t_{n+\frac12})\\ &=-\widetilde{B}(\boldsymbol{u}^{n+\frac12},\theta^{n+\frac12})-\widetilde{B}(\theta^{n+\frac12},\boldsymbol{u}(t_{n+\frac12}))+F(\boldsymbol{u}^{n+\frac12})-F(\boldsymbol{u}(t_{n+\frac12})).\end{aligned}$$

Taking the inner product of the system (3.5.17) respectively with $k(\theta^{n+1}+\theta^n)$ and $k\chi^n$, then summing the result :

$$\begin{aligned}\|\theta^{n+1}\|^2-\|\theta^n\|^2+\frac{\gamma k}{2}\|\nabla(\theta^{n+1}+\theta^n)\|^2+\frac{ak}{2}\|\theta^{n+1}+\theta^n\|^2+\delta k^2(\nabla(\chi^{n+1}-\chi^{n-1}),\nabla\chi^n)\\ =(R^n+Q^n,2k\theta^{n+\frac12})+\delta k^2(\nabla(p(t_{n+1})-p(t_{n-1})),\nabla\chi^n).\end{aligned}$$

We treat the terms at right hand side as follows :

$$
\begin{aligned}
|k(R^n,\theta^{n+\frac{1}{2}})| &\leqslant Ck\|R_n\|_{-1}^2+\frac{\gamma k}{2}\|\nabla\theta^{n+\frac{1}{2}}\|^2\\
|(Q^n,2k\theta^{n+\frac{1}{2}})| &\leqslant |\widetilde{b}(2k\theta^{n+\frac{1}{2}},u(t_{n+\frac{1}{2}}),2k\theta^{n+\frac{1}{2}})|\\
&+|(F(u(t_{n+\frac{1}{2}}))-F(u^{n+\frac{1}{2}}),2k\theta^{n+\frac{1}{2}})|.\\
&\leqslant Ck\|\theta^{n+\frac{1}{2}}\|_1\|u(t_{n+\frac{1}{2}})\|_2\|\theta^{n+\frac{1}{2}}\|\\
&+Ck\|\nabla\theta^{n+\frac{1}{2}}\|\|\theta^{n+\frac{1}{2}}\|.\\
&\leqslant \frac{\gamma k}{2}\|\nabla\theta^{n+\frac{1}{2}}\|^2+Ck\|\theta^{n+\frac{1}{2}}\|^2\\
|(k.\sqrt{k}\nabla(p(t_{n+1})-p(t_{n-1})),\sqrt{k}\nabla\chi^n)| &\leqslant k^3\|\nabla q^n\|^2+Ck\|\nabla(p(t_{n+1})-p(t_{n-1}))\|^2\\
&\leqslant k^3+Ck^3\|\nabla p_t(t_{n+\frac{1}{2}}))\|^2.
\end{aligned}
$$

We collect the estimates above to our relation, we summing the result from $n=1$ to m, using the equation (3.5.10) and thanks to the initial conditions (3.5.7), the relation (3.2.2), the results (3.7.1), (3.7.32) given in Lemma 3.7.1 and Lemma 3.7.2, we obtain :

$$
\begin{aligned}
&\|\theta^{m+1}\|^2+\frac{\gamma k}{4}\|\nabla(\theta^{n+1}+\theta^n)\|^2+\frac{ak}{2}\|\theta^{n+1}+\theta^n\|^2+\frac{\delta k^2}{2}(\|\nabla\chi^m\|^2+\|\nabla\chi^{m+1}\|^2)\\
&\qquad\leqslant Ck^2+\frac{\delta k^2}{2}\|\nabla(\chi^{m+1}-\chi^m)\|^2)+Ck\sum_{n=1}^{m}(k^2\|\nabla\chi^n\|^2+\|\theta^{n+\frac{1}{2}}\|).
\end{aligned}\tag{3.5.18}
$$

Before applying the Gronwall lemma we need to estimate : $\|\nabla(\chi^{m+1}-\chi^m)\|^2$
So taking the inner product of the second equation in the system (3.5.17) with $\chi^{n+1}-\chi^n$, using the convexity, then we reach the following result :

$$
\begin{aligned}
\delta k\|\nabla(\chi^{m+1}-\chi^m)\|^2 &= (\theta^{m+1}\,,\,\nabla(\chi^{m+1}-\chi^m))+\\
&\quad\delta k\Big(\nabla(p(t_{m+1})-p(t_m))\,,\,\nabla(\chi^{m+1}-\chi^m)\Big).\\
&\leqslant \frac{\delta k}{4}\|\nabla(\chi^{m+1}-\chi^m)\|^2+\frac{1}{4\delta k}\|\theta^{m+1}\|^2+\\
&\quad\frac{\delta k}{4}\|\nabla(\chi^{m+1}-\chi^m)\|^2+\delta k\|\nabla(p(t_{m+1})-p(t_m))\|^2.
\end{aligned}\tag{3.5.19}
$$

Because of $\nabla(p(t_{m+1})-p(t_m))=k\nabla p_t(s_m)$ and by multiplying the last result with k, one can obtain

$$
\frac{\delta k^2}{2}\|\nabla(\chi^{m+1}-\chi^m)\|^2\leqslant\frac{1}{4\delta}\|\theta^{m+1}\|^2+\delta k^4\|\nabla p_t(s_m)\|^2. \tag{3.5.20}
$$

Now, in order to estimate the derivate of the pressure with respect to time, we need first to estimate the second derivate of the velocity function, so we derivate the first equation of Darcy Brinkman Forchheimer system with respect to time then we take the inner product of the result with $\Delta\boldsymbol{u}_t$.

$$
\begin{aligned}
-\gamma\|\Delta\boldsymbol{u}_t\|^2+a\|\nabla\boldsymbol{u}_t\|^2+\beta(\alpha|\boldsymbol{u}|^{\alpha-1}|\boldsymbol{u}_t|\boldsymbol{u}+|\boldsymbol{u}|^{\alpha}\boldsymbol{u}_t,\Delta\boldsymbol{u}_t)+(\boldsymbol{u}_t.\nabla\boldsymbol{u}+\boldsymbol{u}\nabla\boldsymbol{u}_t,\Delta\boldsymbol{u}_t)&\\
+(\nabla p_t,\Delta\boldsymbol{u}_t)=(\boldsymbol{f}_t,\Delta\boldsymbol{u}_t).&
\end{aligned}
$$

Then, using Agmon's inequality, we obtain directly that

$$\begin{aligned}\gamma\|\Delta \boldsymbol{u}_t\|^2 \leqslant & a\|\nabla \boldsymbol{u}_t\|^2 + C\|\boldsymbol{u}_t\|_{\boldsymbol{L}^6(\Omega)}\|\boldsymbol{u}\|_{3\alpha}^{\alpha}\|\Delta \boldsymbol{u}_t\| + (\|\boldsymbol{u}_t\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} + \|\boldsymbol{u}\|_\infty\|\nabla \boldsymbol{u}_t\|)\|\Delta \boldsymbol{u}_t\| \\ & + \|\boldsymbol{f}_t\|\|\Delta \boldsymbol{u}_t\|. \\ & \leqslant a\|\nabla \boldsymbol{u}_t\|^2 + \frac{\gamma}{2}\|\Delta \boldsymbol{u}_t\|^2 + C\|\boldsymbol{u}_t\|_{\boldsymbol{L}^6(\Omega)}^2\|\boldsymbol{u}\|_{3\alpha}^{2\alpha} + C\|\boldsymbol{u}_t\|_{\boldsymbol{L}^3(\Omega)}^2\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}^2 \\ & + C(\|\boldsymbol{u}\|_\infty^2\|\nabla \boldsymbol{u}_t\|^2 + C. \end{aligned} \tag{3.5.21}$$

Hence, due to Lemma 3.2.1 and Lemma 3.2.2, we conclude that

$$\gamma\|\Delta \boldsymbol{u}_t\|^2 \leqslant C. \tag{3.5.22}$$

Based on the first equation in Brinkman Forchheimer problem in (1.3.58) and by taking the derivate with respect to time, we have

$$\nabla p_t(t) = \boldsymbol{f}_t + \gamma\Delta \boldsymbol{u}_t - a\boldsymbol{u}_t - \beta|\boldsymbol{u}|^\alpha \boldsymbol{u}_t - \alpha\beta|\boldsymbol{u}|^{\alpha-1}|\boldsymbol{u}_t|\boldsymbol{u} - \boldsymbol{u}_t\nabla \boldsymbol{u} - \nabla \boldsymbol{u}_t \boldsymbol{u}.$$

Consequently, we obtain,

$$\begin{aligned}\|\nabla p_t\| \leqslant & \|\boldsymbol{f}_t\| + \gamma\|\Delta \boldsymbol{u}_t\| + a\|\boldsymbol{u}_t\| + \beta\|\boldsymbol{u}\|^\alpha\|\boldsymbol{u}_t\| + \alpha\beta\|\boldsymbol{u}\|_{L^3(\Omega)}\|\boldsymbol{u}_t\|_{L^2(\Omega)}\|\boldsymbol{u}\|_{L^6(\Omega)} + \\ & + \|\nabla \boldsymbol{u}_t\|\|\boldsymbol{u}\|.\end{aligned}$$

According again to Lemma 3.2.1, Lemma 3.2.2 and the result (3.5.22), we conclude that

$$\|\nabla p_t\| \leqslant C. \tag{3.5.23}$$

Now, we return to the equation (3.5.18), using the relations (3.5.19), (3.5.20) and (3.5.23), one can obtain,

$$\begin{aligned}(1-\frac{1}{4\beta})\|\theta^{m+1}\|^2 + \frac{\gamma k}{4}\|\nabla(\theta^{n+1}+\theta^n)\|^2 + \frac{ak}{2}\|\theta^{n+1}+\theta^n\|^2 + \frac{\delta k^2}{2}(\|\nabla\chi^m\|^2 + \|\nabla\chi^{m+1}\|^2) \\ \leqslant Ck^2 + +Ck\sum_{n=1}^{m}(k^2\|\nabla\chi^n\|^2 + \|\theta^{n+\frac{1}{2}}\|^2).\end{aligned} \tag{3.5.24}$$

Due to Gronwall Lemma, we have

$$\|\theta^{m+1}\|^2 + k\sum_{n=1}^{m}\|\nabla(\theta^{n+1}+\theta^n)\|^2 + k\|\theta^{n+1}+\theta^n\|^2 + k^2\|\nabla\chi^{m+1}\|^2 \leqslant Ck^2, \quad \forall 1 \leqslant m \leqslant M-1. \tag{3.5.25}$$

Using the above inequality and relation (3.2.2), we obtain :

$$\|\nabla(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\|^2 + \|\nabla p^n\|^2 \leqslant Ck^2, \quad \forall 1 \leqslant m \leqslant M-1, \tag{3.5.26}$$

Taking the inner product of the first equation in the discrete system (3.5.6) with $-2k\Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)$, so we obtain :

$$\begin{aligned}
&2\|\nabla \boldsymbol{u}^{n+1}\|^2 - 2\|\nabla \boldsymbol{u}^n\|^2 + \gamma k\|\Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\|^2 + ak\|\nabla(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\|^2 \\
&= 2k(F(\boldsymbol{u}(t_{n+\frac{1}{2}})) + F(\boldsymbol{u}^{n+\frac{1}{2}}) - F(\boldsymbol{u}(t_{n+\frac{1}{2}})), \Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)) + 2k\tilde{b}(\boldsymbol{u}^{n+\frac{1}{2}}, \boldsymbol{u}^{n+\frac{1}{2}}, 2\Delta\boldsymbol{u}^{n+\frac{1}{2}}) \\
&+ 2k(\nabla p^n, \Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)) \\
&\leqslant Ck\|\nabla(\boldsymbol{u}(t_{n+\frac{1}{2}}) - \boldsymbol{u}^{n+\frac{1}{2}})\|^2 + \frac{\gamma k}{8}\|\Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\|^2 + Ck\|F(\boldsymbol{u}(t_{n+\frac{1}{2}}))\|^2 \\
&+ \frac{\gamma k}{8}\|\Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\|^2 + Ck\|\nabla p^n\|^2 + \frac{\gamma k}{8}\|\Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\|^2 \\
&+ \frac{\gamma k}{8}\|\Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\|^2 + Ck\|\boldsymbol{u}^{n+\frac{1}{2}}\|_1^6
\end{aligned}$$

Hence we can deduce that :

$$\begin{aligned}
&2\|\nabla \boldsymbol{u}^{n+1}\|^2 - 2\|\nabla \boldsymbol{u}^n\|^2 + \frac{\gamma k}{2}\|\Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\|^2 + ak\|\nabla(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\|^2 \\
&\leqslant Ck\|\nabla\theta^{n+\frac{1}{2}}\|^2 + Ck\|F(\boldsymbol{u}(t_{n+\frac{1}{2}}))\|^2 + Ck\|\nabla p^n\|^2 + Ck\|\boldsymbol{u}^{n+\frac{1}{2}}\|_1^6
\end{aligned}$$

Summing up the above from $n = 1$ to m, using the results above (3.5.25) and (3.5.26) we obtain :

$$\|\nabla \boldsymbol{u}^{m+1}\|^2 + k\gamma\sum_{n=1}^{m}\|\Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\|^2 + ak\sum_{n=1}^{m}\|\boldsymbol{u}^{m+1}-\boldsymbol{u}^m\|^2 \leqslant C.$$

Which finish the proof of this Lemma.

□

Lemma 3.5.4. *Under the assumptions of the previous lemma, we have :*

$$\|\theta^{n+1}-\theta^n\| + k\|\nabla(p^{n+1}-p^n)\| \leqslant Ck^2.$$

Proof. First we define :

$$\boldsymbol{\varepsilon}^n = \theta^n - \theta^{n-1}, \qquad \boldsymbol{w}^n = \boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \qquad r^n = \chi^n - \chi^{n-1}$$
$$E_r^n = R^n - R^{n-1}, \quad E_q^n = Q^n - Q^{n-1}, \quad E_p^n = (p(t_{n+1}) - p(t_n)) - (p(t_{n-1}) - p(t_{n-2})).$$

We infer from (3.5.17) that :

$$\begin{aligned}
\frac{\boldsymbol{\varepsilon}^{n+1}-\boldsymbol{\varepsilon}^n}{k} - \frac{\gamma}{2}\Delta(\boldsymbol{\varepsilon}^{n+1}+\boldsymbol{\varepsilon}^n) + \frac{a}{2}(\boldsymbol{\varepsilon}^{n+1}+\boldsymbol{\varepsilon}^n) + \nabla r^n &= E_p^n + E_q^n \\
\operatorname{div}(\boldsymbol{\varepsilon}^{n+1}+\boldsymbol{\varepsilon}^n) - \delta k\Delta(r^{n+1}-r^{n-1}) &= \delta k\Delta E_p^n
\end{aligned} \tag{3.5.27}$$

Taking the first equation in (3.5.27) with $2k\boldsymbol{\varepsilon}^{n+\frac{1}{2}} = k(\boldsymbol{\varepsilon}^{n+1} + \boldsymbol{\varepsilon}^n)$ and the second one with kr^n then we summing the result :

$$\begin{aligned}
\|\boldsymbol{\varepsilon}^{n+1}\|^2 - \|\boldsymbol{\varepsilon}^n\|^2 + 2\gamma k\|\nabla\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|^2 + 2ak\|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|^2 + \delta k^2(\nabla(r^{n+1} - r^{n-1}), \nabla r^n) = \\
2k(E_r^n + E_q^n, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) + \delta k^2(\nabla E_p^n, \nabla r^n) \\
\leqslant \frac{\gamma k}{6}\|\nabla\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|^2 + Ck\|E_r^n\|_1^2 + k\|\nabla E_p^n\|^2 + Ck^3\|\nabla r^n\|^2 + 2k|(E_q^n, \boldsymbol{\varepsilon}^{n+\frac{1}{2}})|
\end{aligned}$$

We rewrite the last term in the right hand as follow :

$$\begin{aligned}
E_q^n = Q^n - Q^{n-1} = &-\widetilde{B}(\boldsymbol{w}^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}}) - \widetilde{B}(\boldsymbol{u}^{n-\frac{1}{2}}, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) \\
&- \widetilde{B}(\boldsymbol{\varepsilon}^{n+\frac{1}{2}}, \boldsymbol{u}(t_{n+\frac{1}{2}})) - \widetilde{B}(\theta^{n-\frac{1}{2}}, \boldsymbol{u}(t_{n+\frac{1}{2}}) - \boldsymbol{u}(t_{n-\frac{1}{2}})) \\
&+ F(\boldsymbol{u}^{n+\frac{1}{2}}) - F(\boldsymbol{u}(t_{n+\frac{1}{2}})) - F(\boldsymbol{u}^{n-\frac{1}{2}}) + F(\boldsymbol{u}(t_{n-\frac{1}{2}})).
\end{aligned}$$

Hence we can observe :

$$\begin{aligned}
(E_q^n, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) = &-\widetilde{b}(\boldsymbol{w}^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}}, \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) - \widetilde{b}(\boldsymbol{\varepsilon}^{n+\frac{1}{2}}, \boldsymbol{u}(t_{n+\frac{1}{2}}), \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) \\
&- \widetilde{b}(\theta^{n-\frac{1}{2}}, \boldsymbol{u}(t_{n+\frac{1}{2}}) - \boldsymbol{u}(t_{n-\frac{1}{2}}), \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) + (F(\boldsymbol{u}^{n+\frac{1}{2}}) - F(\boldsymbol{u}(t_{n+\frac{1}{2}})), \boldsymbol{\varepsilon}^{n+\frac{1}{2}}) \\
&+ (F(\boldsymbol{u}(t_{n-\frac{1}{2}})) - F(\boldsymbol{u}^{n-\frac{1}{2}}), \boldsymbol{\varepsilon}^{n+\frac{1}{2}}).
\end{aligned}$$

Now we can treat the previous term as follow :

$$\begin{aligned}
k|(F(\boldsymbol{u}^{n+\frac{1}{2}}) - F(\boldsymbol{u}(t_{n+\frac{1}{2}})), \boldsymbol{\varepsilon}^{n+\frac{1}{2}})| &\leqslant \frac{\gamma k}{6}\|\nabla\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|^2 + Ck\|\nabla\theta^{n+\frac{1}{2}}\|^2. \\
k|(F(\boldsymbol{u}(t_{n-\frac{1}{2}})) - F(\boldsymbol{u}^{n-\frac{1}{2}}), \boldsymbol{\varepsilon}^{n+\frac{1}{2}})| &\leqslant \frac{\gamma k}{6}\|\nabla\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|^2 + Ck\|\nabla\theta^{n-\frac{1}{2}}\|^2. \\
k|\widetilde{b}(\boldsymbol{w}^{n+\frac{1}{2}}, \theta^{n+\frac{1}{2}}, \boldsymbol{\varepsilon}^{n+\frac{1}{2}})| &\leqslant \frac{\gamma k}{6}\|\nabla\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|^2 + Ck^3\|\nabla\theta^{n+\frac{1}{2}}\|^2 + Ck\|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|^2. \\
k|\widetilde{b}(\boldsymbol{\varepsilon}^{n+\frac{1}{2}}, \boldsymbol{u}(t_{n+\frac{1}{2}}), \boldsymbol{\varepsilon}^{n+\frac{1}{2}})| &\leqslant \frac{\gamma k}{6}\|\nabla\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|^2 + Ck\|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|^2. \\
k|b(\theta^{n-\frac{1}{2}}, \boldsymbol{u}(t_{n+\frac{1}{2}}) - u(t_{n-\frac{1}{2}}), \boldsymbol{\varepsilon}^{n+\frac{1}{2}})| &\leqslant \frac{\gamma k}{6}\|\nabla\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|^2 + Ck^3\|\nabla\theta^{n-\frac{1}{2}}\|^2.
\end{aligned}$$

Using the property (3.5.10), we obtain :

$$\begin{aligned}
\delta k^2(\nabla(r^{n+1} - r^{n-1}), \nabla r^n) = \frac{\delta k^2}{2}\{&\|\nabla r^{n+1}\|^2 - \|\nabla r^{n-1}\|^2 \\
&+ \|\nabla(r^n - r^{n-1})\|^2 - \|\nabla(r^{n+1} - r^{n-1})\|^2\}.
\end{aligned}$$

Now we summing the equation from $n = 2$ to m, thanks to the results above we can conclude that :

$$\begin{aligned}
&\|\varepsilon^{m+1}\|^2+\gamma k\sum_{n=2}^{m}\|\nabla\varepsilon^{n+\frac{1}{2}}\|^2+ak\sum_{n=0}^{m}\|\varepsilon^{n+\frac{1}{2}}\|^2+\frac{\delta k^2}{2}(\|\nabla r^{m+1}\|^2+\|\nabla r^m\|^2)\\
&\leqslant\|\varepsilon^2\|^2+\frac{\delta k^2}{2}(\|\nabla r^2\|^2+\|\nabla r^1\|^2+\|\nabla(r^{m+1}-r^m)\|^2)\\
&+Ck\sum_{n=2}^{m}(\|\varepsilon^{n+\frac{1}{2}}\|^2+\|\nabla\theta^{n+\frac{1}{2}}\|^2+\|\nabla\theta^{n-\frac{1}{2}}\|^2)\\
&+Ck\sum_{n=1}^{m}\{\|E_r^n\|_{-1}^2+\|\nabla E_p^n\|^2\}+Ck^3\sum_{n=2}^{m}\|\nabla r^n\|^2.
\end{aligned}$$

Now using the estimates (3.7.1) and (3.7.32), we can deduce that :

$$\begin{aligned}
&\|\varepsilon^{m+1}\|^2+\frac{\gamma k}{4}\sum_{n=2}^{m}\|\nabla\varepsilon^{n+\frac{1}{2}}\|^2+ak\sum_{n=0}^{m}\|\varepsilon^{n+\frac{1}{2}}\|^2+\frac{\delta k^2}{2}(\|\nabla r^{m+1}\|^2+\|\nabla r^m\|^2)\\
&\leqslant Ck^4+\frac{\delta k^2}{2}\|\nabla(r^{m+1}-r^m)\|^2+Ck\sum_{n=2}^{m}(\|\varepsilon^{n+1}\|^2+k^2\|\nabla r^n\|^2).
\end{aligned}$$

It remains to estimate : $\|\nabla(r^{m+1}-r^m)\|^2$,

$$\begin{aligned}
\|\nabla(r^{m+1}-r^m)\|^2\leqslant&\frac{1+\frac{\gamma}{2}}{1+\gamma}\|\varepsilon^{m+1}\|^2+ck^4\|\nabla p_t\|^2\\
&+\frac{1-\frac{\gamma}{2}}{2}\delta k^2(\|\nabla r^{m+1}\|^2+\|\nabla r^m\|^2).
\end{aligned}$$

Using the above relation and the relations (3.2.16) and (3.7.32), we reach :

$$\begin{aligned}
&\frac{\gamma}{2(\gamma+1)}\|\varepsilon^{m+1}\|^2+\frac{\gamma k}{4}\sum_{n=2}^{m}\|\nabla\varepsilon^{n+\frac{1}{2}}\|^2+ak\sum_{n=0}^{m}\|\varepsilon^{n+\frac{1}{2}}\|^2+\frac{\gamma\delta k^2}{4}(\|\nabla r^{m+1}\|^2+\|\nabla r^m\|^2)\\
&\leqslant Ck^4+Ck\sum_{n=1}^{m}(\|\varepsilon^{n+1}\|^2+k^2\|\nabla r^n\|^2).
\end{aligned}$$

Applying the Gronwall lemma we conclude :

$$\|\varepsilon^{m+1}\|^2+Ck\sum_{n=2}^{m}\|\nabla\varepsilon^{n+\frac{1}{2}}\|^2+ak\sum_{n=0}^{m}\|\varepsilon^{n+\frac{1}{2}}\|^2+k^2(\|\nabla r^{m+1}\|^2+\|\nabla r^m\|^2)\leqslant Ck^4.$$

Which finishes the proof of Lemma.

□

In the next subsection we will analyze the second order error within the original problem (1.3.58) and the linearly Darcy Brinkman Forchheimer.

3.5.1 Error estimate for the linear Brinkman Forchheimer problem

Let us consider $(\boldsymbol{u}(t_{n+1}), p(t_{n+1}))$ to be the solution of the Brinkman Forchheimer equations (1.3.58) at the instant t_{n+1} :

$$\begin{aligned}&\boldsymbol{u}_t(t_{n+1}) - \gamma\Delta\boldsymbol{u}(t_{n+1}) + a\boldsymbol{u}(t_{n+1}) + F(\boldsymbol{u}(t_{n+1})) + \widetilde{B}(\boldsymbol{u}(t_{n+1}), \boldsymbol{u}(t_{n+1})) +\\ &\nabla p(t_{n+1}) = \boldsymbol{f}(t_{n+1}).\\ &\text{div } \boldsymbol{u}(t_{n+1}) = 0.\end{aligned} \tag{3.5.28}$$

We set $(\boldsymbol{v}^n, r^n)$ the solution of the linear Brinkman Forchheimer with $\boldsymbol{v}^0 = \boldsymbol{u}^0$ and $r^0 = p^0$.

$$\begin{aligned}&\frac{\boldsymbol{v}^{n+1}-\boldsymbol{v}^n}{k} - \frac{\gamma}{2}\Delta(\boldsymbol{v}^{n+1}+\boldsymbol{v}^n) + \frac{a}{2}(\boldsymbol{v}^{n+1}+\boldsymbol{v}^n) + \nabla r^n = -F(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})) - \widetilde{B}(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}), \widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})).\\ &\text{div } \boldsymbol{v}^{n+1} - \delta k\Delta(r^{n+1} - r^n) = 0, \quad \text{in } \Omega.\\ &\boldsymbol{v}^{n+1} = 0, \qquad \frac{\partial r^{n+1}}{\partial \boldsymbol{n}} = \frac{\partial r^{\boldsymbol{n}}}{\partial \boldsymbol{n}} \qquad \text{on } \Gamma.\end{aligned} \tag{3.5.29}$$

We summing the second equation in relation (3.5.29) with

$$\text{div } \boldsymbol{v}^n - \delta k\Delta(r^n - r^{n-1}) = 0, \quad \text{in } \Omega,$$

which follows,

$$\text{div } (\boldsymbol{v}^{n+1} + \boldsymbol{v}^n) - \delta k\Delta(r^{n+1} - r^{n-1}) = 0. \tag{3.5.30}$$

Defining $\xi^n = \boldsymbol{u}(t_n) - \boldsymbol{v}^n$ and $\phi^n = p(t_n) - r^n$.
In particular, the auxiliary linear system satisfies the Lemma 3.5.4 and Lemma 3.5.3, so we have,

$$\|\frac{\xi^{m+1}-\xi^m}{k}\|^2 + \|r^{m+1} - r^m\|_1^2 + \|\phi^{m+1} - \phi^m\|_1^2 \leqslant Ck^2, \ \forall\ 1 < m < M-1. \tag{3.5.31}$$

$$\|\nabla\boldsymbol{v}^{m+1}\|^2 + k\|\Delta(\boldsymbol{v}^{m+1} + \boldsymbol{v}^m)\|_2^2 + \|\boldsymbol{r}^{m+1}\|_1^2 \leqslant C, \ \forall 1 \leqslant m \leqslant M-1. \tag{3.5.32}$$

Now subtracting respectively the first equation in the linear Brinkman Forchheimer (3.5.29) and the equation (3.5.30) from the original problem (1.3.58), hence we obtain :

$$\begin{aligned}\frac{\boldsymbol{\xi}^{n+1}-\boldsymbol{\xi}^n}{k} - \frac{\gamma}{2}\Delta(\boldsymbol{\xi}^{n+1}+\boldsymbol{\xi}^n) + \frac{a}{2}(\boldsymbol{\xi}^{n+1}+\boldsymbol{\xi}^n) + \nabla\phi^n = R^n.\\ \text{div}(\boldsymbol{\xi}^{n+1}+\boldsymbol{\xi}^n) + \delta k\Delta(r^{n+1} - r^{n-1}) = 0.\end{aligned} \tag{3.5.33}$$

Where R^n is the truncation error introduced in (3.5.9) :

$$\begin{aligned}R^n = &\frac{\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)}{k} - \frac{\gamma}{2}\Delta(\boldsymbol{u}(t_{n+1}) + \boldsymbol{u}(t_n)) + \frac{a}{2}(\boldsymbol{u}(t_{n+1}) + \boldsymbol{u}(t_n)) + \nabla p(t_n)\\ &+ \widetilde{B}(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}), \widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})) + F(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})).\end{aligned}$$

Lemma 3.5.5. *We assume the same assumptions as in the Theorem 3.5.2, then we have :*

$$\sum_{n=1}^{m} \|\boldsymbol{u}(t_n) - \boldsymbol{v}^n\|^2 \leqslant Ck^3, \quad \forall 1 \leqslant m \leqslant M.$$

Proof. We want to prove that $\sum_{n=1}^{m} \|\boldsymbol{\xi}^n\|^2 \leqslant Ck^3$. For this purpose, we can write : $2\boldsymbol{\xi}^{n+1} = (\boldsymbol{\xi}^{n+1} - \boldsymbol{\xi}^n) + (\boldsymbol{\xi}^{n+1} + \boldsymbol{\xi}^n)$. Using the convexity's inequality :

$$2k\sum_{n=1}^{m} \|\boldsymbol{\xi}^{n+1}\|^2 \leqslant k\sum_{n=1}^{m} \|\boldsymbol{\xi}^{n+1} - \boldsymbol{\xi}^n\|^2 + k\sum_{n=1}^{m} \|\boldsymbol{\xi}^{n+1} + \boldsymbol{\xi}^n\|^2. \tag{3.5.34}$$

Thanks to the estimate (3.5.31), it remains to estimate : $\sum_{n=1}^{m} \|\boldsymbol{\xi}^{n+1} + \boldsymbol{\xi}^n\|^2$.
Here we need to use the following duality problem with the initial condition $\boldsymbol{w}^{N+1} = 0$ where $1 \leqslant N \leqslant M-1$. :

$$\begin{aligned} &\frac{\boldsymbol{w}^{n+1} - \boldsymbol{w}^n}{k} + \frac{\gamma}{2}\Delta(\boldsymbol{w}^{n+1} + \boldsymbol{w}^n) - \frac{a}{2}(\boldsymbol{w}^{n+1}, \boldsymbol{w}^n) + \nabla q^n = \boldsymbol{\xi}^{n+1} + \boldsymbol{\xi}^n. \\ &\operatorname{div} \boldsymbol{w}^n = 0, \qquad \text{in } \Omega. \\ &\boldsymbol{w}^n = 0, \qquad \text{on } \Gamma. \end{aligned} \tag{3.5.35}$$

Taking the inner product of the first equation in the dual system (3.5.35) with $\boldsymbol{\xi}^{n+1} + \boldsymbol{\xi}^n$, we summing the result equation from 1 to N :

$$\begin{aligned} \sum_{n=1}^{N} \|\xi^{n+1} + \xi^n\|^2 = &\frac{2}{k}\{(\boldsymbol{w}^{n+1}, \boldsymbol{\xi}^{n+1}) - (\boldsymbol{w}^1, \boldsymbol{\xi}^1)\} \\ &+ \sum_{n=1}^{N}(\boldsymbol{w}^{n+1} + \boldsymbol{w}^n, (\boldsymbol{R}^{n1} + \boldsymbol{R}^{n2})) - \delta k\sum_{n=1}^{N}(\nabla q^n, \nabla(r^{n+1} - r^{n-1})) \end{aligned} \tag{3.5.36}$$

Since $(\boldsymbol{w}^{n+1} + \boldsymbol{w}^n, \boldsymbol{R}^{n2}) = 0$ and $\boldsymbol{w}^{N+1} = 0$, using the estimates (3.7.1),(3.7.32) and (3.5.31), we obtain :

$$k\sum_{n=1}^{N} \|\boldsymbol{\xi}^{n+1} + \boldsymbol{\xi}^n\|^2 \leqslant \mu\|\boldsymbol{w}^1\|^2 + Ck^2\sum_{n=1}^{N} \|\boldsymbol{w}^{n+1} + \boldsymbol{w}^n\|^2 + \mu k^2\sum_{n=1}^{N} \|\nabla q^n\|^2 + Ck^4.$$

Such that μ is a sufficiently small. For second time, taking the inner product of the same dual problem (3.5.35) with ∇q^n, since $\operatorname{div} \boldsymbol{w}^n = 0 \quad \text{in } \Omega$:

$$k\sum_{n=1}^{N} \|\nabla q^n\|^2 \leqslant Ck\sum_{n=1}^{N} \|\boldsymbol{\xi}^{n+1} + \boldsymbol{\xi}^n\|^2.$$

Now we will move to estimate $\|\boldsymbol{w}^1\|^2$ and $\sum_{n=1}^{N} \|\boldsymbol{w}^{n+1} + \boldsymbol{w}^n\|^2$.
For this purpose taking the inner product of the dual problem (3.5.35) with $\Delta(\boldsymbol{w}^{n+1} + \boldsymbol{w}^n)$ we reach :

$$(\|\nabla \boldsymbol{w}^n\|^2 - \|\nabla \boldsymbol{w}^{n+1}\|^2) + \frac{\gamma k}{4}\|\Delta(\boldsymbol{w}^{n+1} + \boldsymbol{w}^n)\|^2 + \frac{ak}{2}\|\nabla(\boldsymbol{w}^{n+1} + \boldsymbol{w}^n)\|^2 \leqslant \frac{k}{\gamma}\|\boldsymbol{\xi}^{n+1} + \boldsymbol{\xi}^n\|^2.$$

Since the initial condition vanishes ($\boldsymbol{w}^{N+1}=0$) then $\|\boldsymbol{w}^{N+1}\|_1=0$, hence for $n=N$ we can rewrite the last inequality as below :

$$\begin{aligned}\|\nabla \boldsymbol{w}^N\|^2+\frac{k\gamma}{4}\|\Delta \boldsymbol{w}^N\|^2+\frac{a}{2}\|\nabla \boldsymbol{w}^N\|^2 &\leqslant \frac{k}{\gamma}\|\boldsymbol{\xi}^{N+1}+\boldsymbol{\xi}^N\|^2\\ \|\nabla \boldsymbol{w}^N\|^2 &\leqslant Ck\|\boldsymbol{\xi}^{N+1}+\boldsymbol{\xi}^N\|^2.\end{aligned}$$

Now for $n=N-1$ we have :

$$\frac{a}{2}\|\nabla(\boldsymbol{w}^N+\boldsymbol{w}^{N-1})\|^2+\|\nabla \boldsymbol{w}^{N-1}\|^2 \leqslant Ck\sum_{n=N-1}^{N}\|\boldsymbol{\xi}^{n+1}+\boldsymbol{\xi}^n\|^2.$$

So using the same reasoning (successively on n) and thanks to Poincare's inequality we can deduce that :

$$\sum_{n=1}^{N}\|\nabla(\boldsymbol{w}^{n+1}+\boldsymbol{w}^n)\|^2+\|\boldsymbol{w}^1\|^2 \leqslant Ck\sum_{n=1}^{N}\|\boldsymbol{\xi}^{N+1}+\boldsymbol{\xi}^N\|^2.$$

Which completes the proof of Lemma 3.5.5.

□

Lemma 3.5.6. *Under the same hypotheses of the Theorem 3.5.2, we obtain the following estimate :*

$$\|\boldsymbol{u}(t_n)-\boldsymbol{v}^n\|_1^2+\|p(t_n)-r^n\|^2 \leqslant Ck^2, \quad \forall \quad 0\leqslant n\leqslant M.$$

Proof. Firstly we check the estimate for the velocity's first derivate.
For this purpose, taking the inner product of the first equation in (3.5.33) with $\boldsymbol{\xi}^{n+1}-\boldsymbol{\xi}^n$:

$$\begin{aligned}\frac{1}{k}\|\boldsymbol{\xi}^{n+1}-\boldsymbol{\xi}^n\|^2+\frac{\gamma}{2}(\|\nabla\boldsymbol{\xi}^{n+1}\|^2-\|\nabla\boldsymbol{\xi}^n\|^2)+\frac{a}{2}(\|\boldsymbol{\xi}^{n+1}\|^2-\|\boldsymbol{\xi}^n\|^2) &= (\phi^n,\operatorname{div}(\boldsymbol{\xi}^{n+1}-\boldsymbol{\xi}^n))\\ &\quad +(\boldsymbol{R}^n,\boldsymbol{\xi}^{n+1}-\boldsymbol{\xi}^n).\end{aligned} \tag{3.5.37}$$

We summing up the above equation from $n=1$ to m, using the estimates (3.7.1),(3.7.32) and the assumption concerning the initial condition on the velocity (3.5.7), we obtain :

$$\frac{1}{2k}\sum_{n=1}^{m}\|\boldsymbol{\xi}^{n+1}-\boldsymbol{\xi}^n\|^2+\frac{\gamma}{2}\|\nabla\boldsymbol{\xi}^{m+1}\|^2+\frac{a}{2}\|\boldsymbol{\xi}^{m+1}\|^2 \leqslant Ck^2+\sum_{n=1}^{m}(\phi^n,\operatorname{div}(\boldsymbol{\xi}^{n+1}-\boldsymbol{\xi}^n)). \tag{3.5.38}$$

By the definition of the linear Darcy Brinkman Forchheimer system (3.5.29), we can reach :

$$\operatorname{div}(\boldsymbol{\xi}^{n+1}-\boldsymbol{\xi}^n)=-\delta k\Delta(r^{n+1}-2r^n+r^{n-1}).$$

Then we can observe :

$$\begin{aligned}(\phi^n,\operatorname{div}(\boldsymbol{\xi}^{n+1}-\boldsymbol{\xi}^n)) &= \delta k(\nabla\phi^n,\nabla(r^{n+1}-r^n))-\delta k(\nabla\phi^{n-1},\nabla(r^n-r^{n-1}))\\ &\quad -\delta k(\nabla\phi^n-\nabla\phi^{n-1},\nabla(r^n-r^{n-1})).\end{aligned}$$

We summing up the above equation from $n = 1$ to m, then we collect the result to the inequality (3.5.38), using the estimate (3.5.31) :

$$\frac{1}{2k}\sum_{n=1}^{m}\|\boldsymbol{\xi}^{n+1}-\boldsymbol{\xi}^{n}\|^2+\frac{\gamma}{2}\|\nabla\boldsymbol{\xi}^{m+1}\|^2+\frac{a}{2}\|\boldsymbol{\xi}^{m+1}\|^2 \leqslant Ck^2+k^2\{\|\nabla\phi^m\|^2+\|\nabla\phi^0\|^2\}. \quad (3.5.39)$$

Then we need to control the first derivate for the pressure term.
Using the initial assumption for the pressure (3.5.7), the estimate (3.5.31) and because of :

$$\|\nabla\phi^m\|^2=\|\sum_{n=1}^{m}\nabla(\phi^n-\phi^{n-1})+\nabla\phi^0\|^2 \leqslant \sum_{n=1}^{m}\|\nabla(\phi^n-\phi^{n-1})\|^2+\|\nabla\phi^0\|^2$$

We conclude that :

$$\|\nabla\phi^m\|^2 \leqslant Ck^2.$$

Now, we return to the relation (3.5.39), we deduce directly that :

$$\|\nabla\boldsymbol{\xi}^{m+1}\|^2 \leqslant Ck^2.$$

Which completes the velocity part in the proof of lemma 3.5.6.

Next we move to prove the estimate for the pressure $\|\phi^n\|$:
Using Poincaré's inequality :

$$\|\phi^n\| \leqslant \sup_{v\in \boldsymbol{H}_0^1(\Omega)}\frac{(\nabla\phi^n,\boldsymbol{v})}{\|\boldsymbol{v}\|_{\boldsymbol{H}_0^1(\Omega)}}.$$

Thanks to the equation (3.5.33), we obtain :

$$\|\nabla\phi^n\|_{H^{-1}(\Omega)} \leqslant \|\boldsymbol{R}^n\|_{-1}+\|\frac{\boldsymbol{\xi}^{n+1}-\boldsymbol{\xi}^n}{k}\|_{-1}+\frac{\gamma}{2}\|\nabla(\boldsymbol{\xi}^{n+1}+\boldsymbol{\xi}^n)\|+\frac{a}{2}\|\boldsymbol{\xi}^{n+1}+\boldsymbol{\xi}^n\|_{-1}.$$

Thanks to the estimate (3.7.1), (3.5.31) and results above , we conclude that :

$$\|\phi^n\| \leqslant Ck.$$

□

The next paragraph will be concerned with the error estimate for the nonlinear problem. For this purpose let notice $\eta^n = \boldsymbol{v}^n - \boldsymbol{u}^n$ and $\psi^n = r^n - p^n$.

3.5.2 Error estimate for the nonlinear Darcy Brinkman Forchheimer equations

Lemma 3.5.7. *Under the same assumptions of the Theorem 3.5.2 :*

$$\|\boldsymbol{\eta}^{m+1}\|^2+k\|\nabla(\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^n)\|^2+k^2\|\nabla\psi\|^2 \leqslant Ck^4.$$

Proof. We begin by subtracting the equation (3.5.6) from the linear discrete equation (3.5.29), we obtain :

$$\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^{n}}{k}-\frac{\gamma}{2}\Delta(\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n})+\frac{a}{2}(\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n})+\nabla\psi^{n}=-Q^{n}. \tag{3.5.40}$$

Such that :

$$Q^{n}=\widetilde{B}(\boldsymbol{u}(t_{n+\frac{1}{2}}),\boldsymbol{u}(t_{n+\frac{1}{2}}))+F(\boldsymbol{u}(t_{n+\frac{1}{2}}))-\widetilde{B}(\boldsymbol{u}^{n+\frac{1}{2}},\boldsymbol{u}^{n+\frac{1}{2}})-F(\boldsymbol{u}^{n+\frac{1}{2}}).$$

Since $\boldsymbol{e}^{n+\frac{1}{2}}=\boldsymbol{u}(t_{n+\frac{1}{2}})-\boldsymbol{u}^{n+\frac{1}{2}}=\boldsymbol{\xi}^{n+\frac{1}{2}}+\boldsymbol{\eta}^{n+\frac{1}{2}}$, we can rewrite the error as follow :

$$Q^{n}=\widetilde{B}(\boldsymbol{u}^{n+\frac{1}{2}},\boldsymbol{\xi}^{n+\frac{1}{2}}+\boldsymbol{\eta}^{n+\frac{1}{2}})+\widetilde{B}(\boldsymbol{\xi}^{n+\frac{1}{2}}+\boldsymbol{\eta}^{n+\frac{1}{2}},\boldsymbol{u}(t_{n+\frac{1}{2}}))+F(\boldsymbol{u}(t_{n+\frac{1}{2}}))-F(\boldsymbol{u}^{n+\frac{1}{2}}) \tag{3.5.41}$$

We infer from the last equation in the discrete Darcy Brinkman Forchheimer (3.5.6) and the equation (3.5.30) that,

$$\operatorname{div}(\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n})-\delta k\Delta(\psi^{n+1}-\psi^{n-1})=0. \tag{3.5.42}$$

Taking the inner product of equation (3.5.40) and (3.5.42) with respectively $k\dfrac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n}}{2}$ and $k\psi^{n}$, summing the result from $n=1$ to m, we get :

$$\begin{aligned}-k\sum_{n=1}^{m}(Q^{n},\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n}}{2}) =&\frac{1}{2}(\|\boldsymbol{\eta}^{m+1}\|^{2}-\|\boldsymbol{\eta}^{1}\|^{2})+\frac{\gamma k}{4}\sum_{n=1}^{m}\|\nabla(\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n})\|^{2}\\ &+\frac{ak}{4}\sum_{n=1}^{m}\|\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n}\|^{2}\\ &+\frac{\delta k^{2}}{2}\sum_{n=1}^{m}(\nabla\psi^{n},\nabla(\psi^{n+1}-\psi^{n-1})).\end{aligned}$$

We can observe :

$$\begin{aligned}\frac{1}{2}\sum_{n=1}^{m}(\nabla(\psi^{n+1}-\psi^{n-1}),\nabla\psi^{n}) =&\frac{1}{4}\Big(\|\nabla\psi^{m}\|^{2}+\|\nabla\psi^{m+1}\|^{2}-2(\nabla\psi^{1},\nabla\psi^{0})\\ &-\|\nabla(\psi^{m+1}-\psi^{m})\|^{2}\Big).\end{aligned}$$

Furthermore, we have that,

$$\begin{aligned}-k\sum_{n=1}^{m}\Big(Q^{n},\frac{\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^{n}}{2}\Big) &= -k\sum_{n=1}^{m}\Big\{\widetilde{b}(\boldsymbol{u}^{n+\frac{1}{2}},\boldsymbol{\xi}^{n+\frac{1}{2}},\boldsymbol{\eta}^{n+\frac{1}{2}})+(F(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})),\boldsymbol{\eta}^{n+\frac{1}{2}})\Big\}\\ &-k\sum_{n=1}^{m}\Big\{\widetilde{b}(\xi^{n+\frac{1}{2}}+\boldsymbol{\eta}^{n+\frac{1}{2}},\boldsymbol{u}(t_{n+\frac{1}{2}}),\boldsymbol{\eta}^{n+\frac{1}{2}})-(F(\boldsymbol{u}^{n+\frac{1}{2}}),\boldsymbol{\eta}^{n+\frac{1}{2}})\Big\}.\end{aligned}$$

Using young's inequality, it infers :

$$k\widetilde{b}(\boldsymbol{u}^{n+\frac{1}{2}},\boldsymbol{\xi}^{n+\frac{1}{2}},\boldsymbol{\eta}^{n+\frac{1}{2}})\leqslant\frac{k\gamma}{6}\|\nabla\boldsymbol{\eta}^{n+\frac{1}{2}}\|^{2}+Ck\|\boldsymbol{\xi}^{n+\frac{1}{2}}\|^{2}.$$

Using convexity and young's inequality :

$$
\begin{aligned}
k\widetilde{b}(\boldsymbol{\xi}^{n+\frac{1}{2}}+\boldsymbol{\eta}^{n+\frac{1}{2}},\boldsymbol{u}(t_{n+\frac{1}{2}}),\boldsymbol{\eta}^{n+\frac{1}{2}}) &\leqslant \frac{k\gamma}{6}\|\nabla\boldsymbol{\eta}^{n+\frac{1}{2}}\|^2+Ck\Big(\|\boldsymbol{\xi}^{n+\frac{1}{2}}\|^2+\|\boldsymbol{\eta}^{n+\frac{1}{2}}\|^2\Big).\\
k(F(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}-\boldsymbol{u}^{n+\frac{1}{2}})),\boldsymbol{\eta}^{n+\frac{1}{2}}) &\leqslant k\|\nabla(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})-\boldsymbol{u}^{n+\frac{1}{2}})\|\|\boldsymbol{\eta}^{n+\frac{1}{2}}\|\\
&\leqslant \|\nabla(\boldsymbol{\xi}^{n+\frac{1}{2}}+\boldsymbol{\eta}^{n+\frac{1}{2}})\|\|\boldsymbol{\eta}^{n+\frac{1}{2}}\|\\
&\leqslant \frac{k\gamma}{12}\|\nabla(\boldsymbol{\xi}^{n+\frac{1}{2}}+\boldsymbol{\eta}^{n+\frac{1}{2}})\|^2+\frac{3k}{\gamma}\|\boldsymbol{\eta}^{n+\frac{1}{2}}\|^2\\
&\leqslant \frac{k\gamma}{6}\|\nabla\boldsymbol{\eta}^{n+\frac{1}{2}}\|^2+\frac{k\gamma}{6}\|\nabla\boldsymbol{\xi}^{n+\frac{1}{2}}\|^2+\|\boldsymbol{\eta}^{n+\frac{1}{2}}\|^2.
\end{aligned}
$$

Because we considering the initial condition as : $\psi^0=0$, thanks to the relation (3.7.32) and previous lemma we have :

$$
\|\boldsymbol{\eta}^{m+1}\|^2+k\gamma\sum_{n=1}^{m}\|\nabla\boldsymbol{\eta}^{n+\frac{1}{2}}\|^2+2ak\sum_{n=1}^{m}\|\boldsymbol{\eta}^{n+\frac{1}{2}}\|^2+\frac{\delta k^2}{2}(\|\nabla\psi^m\|^2+\|\nabla\psi^{m+1}\|^2)
$$

$$
\leqslant \frac{\delta k^2}{2}\|\nabla(\psi^{m+1}-\psi^m)\|^2+Ck^3.
$$

Subtracting the last equation of the linear Brinkman Forchheimer system (3.5.29) from the last one of the discrete Brinkman Forchheimer then taking the inner product of the result with $(\psi^{m+1}-\psi^m)$, hence we reach :

$$
\delta k^2\|\nabla(\psi^{m+1}-\psi^m)\|^2\leqslant \frac{1}{\delta}\|\boldsymbol{\eta}^{m+1}\|^2.
$$

We consider $\delta=\gamma+\frac{1}{4}$, so we can see that :

$$
\frac{\delta k^2}{2}\|\nabla(\psi^{m+1}-\psi^{m-1})\|^2=\delta k^2\Big(\frac{1+\frac{\gamma}{2}}{4}\Big)\|\nabla(\psi^{m+1}-\psi^{m-1})\|^2+\delta k^2\Big(\frac{1-\frac{\gamma}{2}}{4}\Big)\|\nabla(\psi^{m+1}-\psi^{m-1}\|^2.
$$

Thanks to convexity's inequality :

$$
\frac{\delta k^2}{2}\|\nabla(\psi^{m+1}-\psi^{m-1})\|^2\leqslant\Big(\frac{1+\frac{\gamma}{2}}{4\gamma+1}\Big)\|\boldsymbol{\eta}^{m+1}\|^2+\delta k^2\Big(\frac{1-\frac{\gamma}{2}}{2}\Big)(\|\nabla\psi^{m+1}\|^2+\|\nabla\psi^{m-1}\|^2).
$$

Then :

$$
\frac{7\gamma}{8\gamma+2}\|\boldsymbol{\eta}^{m+1}\|^2+\gamma k\sum_{n=1}^{m}\|\nabla\boldsymbol{\eta}^{n+\frac{1}{2}}\|^2+2ak\sum_{n=1}^{m}\|\boldsymbol{\eta}^{n+\frac{1}{2}}\|^2
$$

$$
+\frac{\delta k^2}{2}(\frac{1+\frac{\gamma}{2}}{2})(\|\nabla\psi^m\|^2+\|\nabla\psi^{m+1}\|^2)\leqslant Ck\sum_{n=1}^{m}\|\boldsymbol{\eta}^{n+\frac{1}{2}}\|^2+Ck^3.
$$

By applying Lemma of Gronwall, we get the result.

□

Proof of Theorem 3.5.6

In order to establish the Theorem 3.5.2, we need to combine the results given in Lemma 3.5.5, Lemma 3.5.6 and Lemma 3.5.7. Moreover, it remains to check the estimate

$$\|\nabla \boldsymbol{\eta}^n\|^2 \leqslant Ck^2, \quad \forall\, 0 \leqslant n \leqslant M. \tag{3.5.43}$$

For this purpose, we take the inner product of the equation (3.5.40) with $(\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n)$,

$$\begin{aligned}
\frac{1}{k}\|\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n\|^2 + \frac{\gamma}{2}(\|\nabla\boldsymbol{\eta}^{n+1}\|^2 - \|\nabla\boldsymbol{\eta}^n\|^2) + \frac{a}{2}(\|\boldsymbol{\eta}^{n+1}\|^2 - \|\boldsymbol{\eta}^n\|^2) = \\
-(\nabla\boldsymbol{\psi}^n + Q^n, \boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n). \qquad (3.5.44)\\
\leqslant \frac{1}{2k}\|\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n\|^2 + \frac{k}{2}\|\nabla\boldsymbol{\psi}^n\|^2 - (Q^n\,,\, \boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n). \qquad (3.5.45)
\end{aligned}$$

Based on the relation (3.5.41), we have

$$\begin{aligned}
(Q^n, \boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n) =& \widetilde{b}(\boldsymbol{u}^{n+\frac{1}{2}}, \boldsymbol{\xi}^{n+\frac{1}{2}} + \boldsymbol{\eta}^{n+\frac{1}{2}}, \boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n)\\
&+ \widetilde{b}(\boldsymbol{\xi}^{n+\frac{1}{2}} + \boldsymbol{\eta}^{n+\frac{1}{2}}, \boldsymbol{u}(t_{n+\frac{1}{2}}), \boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n)\\
&+ (F(\boldsymbol{u}(t_{n+\frac{1}{2}})) - F(\boldsymbol{u}^{n+\frac{1}{2}}), \boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n). \qquad (3.5.46)
\end{aligned}$$

Because of the result given in (3.1.8), we obtain

$$\begin{aligned}
(Q^n, \boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n) \leqslant& C\Big(\|\boldsymbol{u}^{n+\frac{1}{2}}\|_2^2\|\boldsymbol{\xi}^{n+\frac{1}{2}}\|_1 + \|\boldsymbol{\xi}^{n+\frac{1}{2}} + \boldsymbol{\eta}^{n+\frac{1}{2}}\|_1\|\boldsymbol{u}(t_{n+\frac{1}{2}})\|_2^2\Big) \qquad (3.5.47)\\
&+ \|\nabla(\boldsymbol{\xi}^{n+\frac{1}{2}} + \boldsymbol{\eta}^{n+\frac{1}{2}})\|\|\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n\|. \qquad (3.5.48)\\
\leqslant& \frac{1}{2k}\|\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n\|^2 + Ck\Big(\|\boldsymbol{\xi}^{n+\frac{1}{2}}\|_1^2 + \|\boldsymbol{\eta}^{n+\frac{1}{2}}\|_1^2\Big). \qquad (3.5.49)
\end{aligned}$$

By summing up the last result on n from $n = 1$ to m and according to Lemma 3.5.5, Lemma 3.5.6 and Lemma 3.5.7, we derive the relation (3.5.43).
Finally, we will move to the last section, where we will illustrate some numerical results concerning the unsteady Darcy Brinkman Forchheimer problem.

3.6 Numerical results

In this section, we present a numerical experiments related to the approximated (DBF) equations (0.0.14)-(0.0.15) which is in a good agreement with the theoretical results in the previous part of this chapter.

For this purpose, we focus on the spatial discretization which allow us to carry out the finite element approximation of (DBF) equations. We consider a regular triangulation $\mathcal{T}_h$ of the domain Ω, depending on a positive parameter $h > 0$, made up of triangles $\mathcal{T}_h$. Let V_h and

Q_h represent the finite element spaces which approximate the velocity and pressure fields, respectively.

Let V_h consist of $\mathcal{C}^0$ piecewise polynomial functions $\mathcal{P}^m$, ($m = 2$ in the present simulations) over the triangulation $\mathcal{T}_h$ and we define $\mathbf{V_h} = (V_h)^2$ such that $V_h \subset \boldsymbol{H}_0^1(\Omega)$ and for some $m \geq 2$,

$$\inf_{\boldsymbol{v}\in \boldsymbol{V}_h} \{||\boldsymbol{v} - \boldsymbol{v}_h|| + h||\nabla(\boldsymbol{v} - \boldsymbol{v}_h)||\} \leq Ch^m||\boldsymbol{v}||_m, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^m(\Omega), \tag{3.6.1}$$

Let Q_h consist of $\mathcal{C}^0$ piecewise polynomial functions $\mathcal{P}^k$, ($k = 1$ in the present simulations) over the triangulation $\mathcal{T}_h$ such that $Q_h \subset H^1(\Omega) \cap L_0^2(\Omega)$ and for some $k \geq 1$,

$$\inf_{q_h\in Q_h} \{||q - q_h|| + h||\nabla(q - q_h)||\} \leq Ch^k||q||_k, \quad \forall\, q \in L_0^2(\Omega) \cap H^k(\Omega), \tag{3.6.2}$$

We give the variational formulation for the approximated system (3.5.6):
Find $(\boldsymbol{u}_h^{k+1}, p_h^{k+1}) \in \boldsymbol{V}_h \times Q_h$ such that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$:

$$\begin{aligned}
\frac{\langle \boldsymbol{u}^{k+1,h}, \boldsymbol{v}_h\rangle}{k} &+ \frac{\gamma}{2}\langle \nabla \boldsymbol{u}^{k+1,h}, \nabla \boldsymbol{v}_h\rangle + \frac{a}{2}\langle \boldsymbol{u}^{k+1,h}, \boldsymbol{v}_h\rangle \\
&+\langle \boldsymbol{u}^{k+1,h}.\nabla \boldsymbol{u}^{k+1,h}, \boldsymbol{v}_h\rangle + b\langle |\boldsymbol{u}^{k,h}|^\alpha \boldsymbol{u}^{k+1,h}, \boldsymbol{v}_h\rangle \\
&-\langle p^{k,h}, \mathrm{div} \boldsymbol{v}_h\rangle + \langle \mathrm{div} \boldsymbol{u}^{k+1,h}, q_h\rangle + \\
\delta k\langle \nabla p^{k+1,h}, \nabla q_h\rangle &= \frac{\langle \boldsymbol{u}^{k,h}, \boldsymbol{v}_h\rangle}{k} - \frac{\gamma}{2}\langle \nabla \boldsymbol{u}^{k,h}, \nabla \boldsymbol{v}\rangle \\
&-\frac{a}{2}\langle \boldsymbol{u}^{k,h}, \boldsymbol{v}_h\rangle + \delta k\langle \nabla p^{k,h}, \nabla q_h\rangle, \forall q_h \in Q_h
\end{aligned}$$

We implement the above scheme in FreeFem++ which is used to solve partial differential equations using the finite element method. The problem for which we present results involves the lid-driven cavity flow (a widely-used benchmark case for testing Navier-Stokes flow).
Fluid past through a square domain is considered with three stationary sides except the top one moving, what making the fluid in rotation's situation.
We impose having the vanish velocity : $\boldsymbol{u} = 0, \boldsymbol{v} = 0$ at all the boundaries except at the boundary $y = 1$ where we impose $\boldsymbol{u} = 1$.
We run a large number of time steps to ensure that we reach the steady state solution. In all example, we set $\epsilon = 0.000001$, $a = 1$, $\alpha = 0.1$, $\gamma = 1/Reynolds$, $Reynolds = 100$ and $\beta = 0$ (for comparaison with Ghia's data) or $\beta = 1$ (including Forchheimer's term).

We are interested to display the behavior of the velocity at the center of driven cavity's domain, the results obtained will be compared with Ghia's data [32] (when Reynolds number equal to 100), it is found that the results are in good agreement with Ghia's data, which validate the correctness of our numerical code.
Figure (1) illustrate the behavior of the laminar incompressible fluids into the cavity and the figures (2a) and (2b) below, the velocity's profils at the cavity's core.

ß

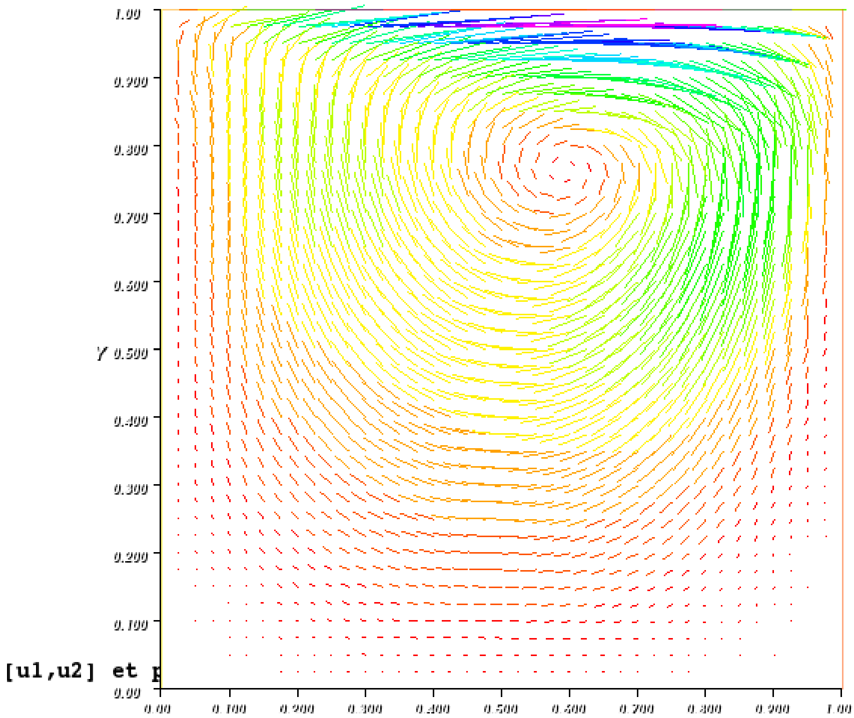
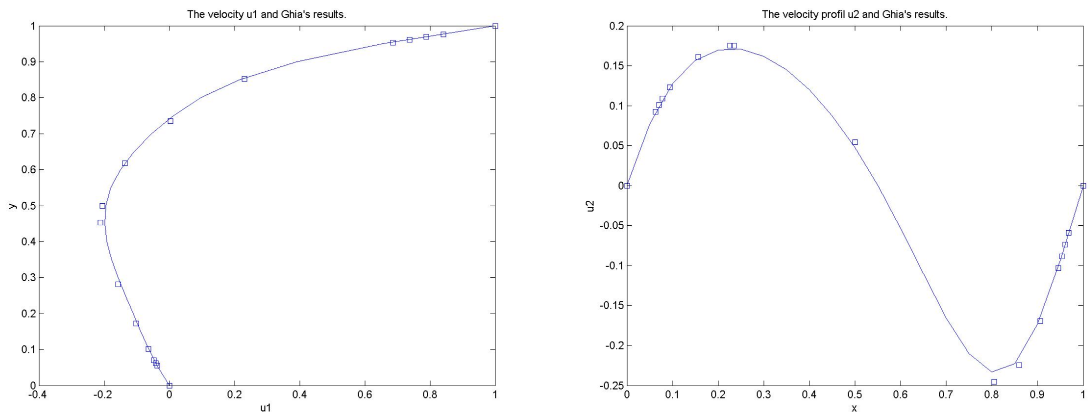
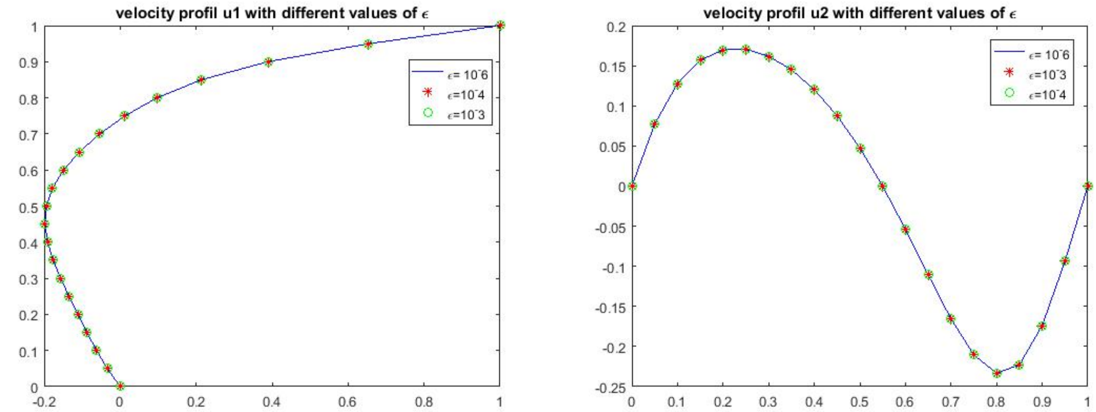


Figure 1: velocity vector in driven cavity.



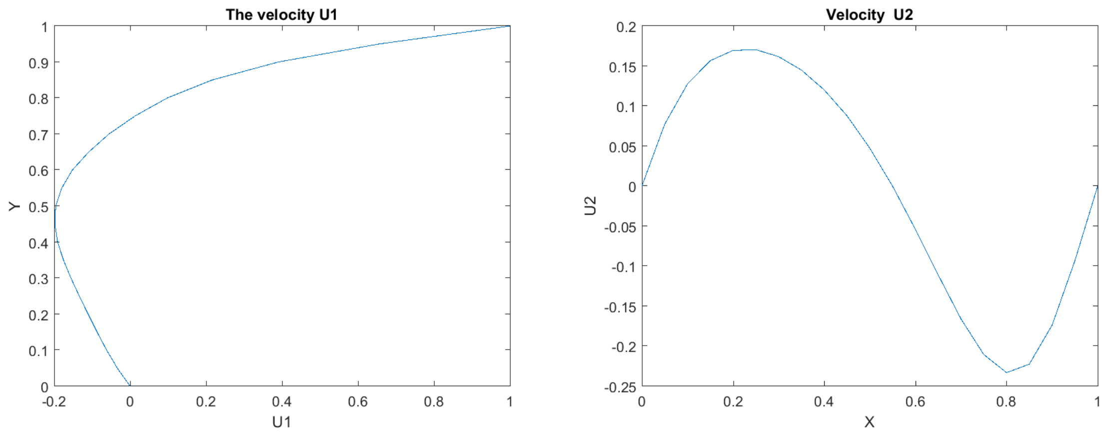
(a) x direction velocity component along the vertical centreline component.

(b) y direction velocity component along the horizontal centreline component.



(a) x direction velocity component along the vertical centreline component.

(b) y direction velocity component along the horizontal centreline component.



(a) Velocity component u_1 with $\beta = 1$.

(b) Velocity component u_2 with $\beta = 1$.

Figure 2: The velocity behavior in the center of domain including Forchheimer's term.

3.6.1 Conclusions

The modeling of the fluid flow in porous media system has been performed using open source software FreeFem++. The results have been compared with Ghia's data. They show a very good agreement.

3.7 Appendix

Lemma 3.7.1. *Under the same assumptions as in the Theorem 3.5.2, we have*

$$k\sum_{n=0}^{\frac{T}{k}-1}\|R^n\|_{-1}^2 \leqslant Ck^4\int_{t_0}^{T}(\|\boldsymbol{u}_{ttt}(s)\|_{-1}^2+\|\boldsymbol{u}_{tt}(s)\|^2+|p_{tt}(s)|^2)ds \leqslant Mk^4. \tag{3.7.1}$$

$$k\sum_{n=2}^{M-1}\|E_p^n\|_1^2 \leqslant Ck^4\int_{t_0}^{T}\|p_{tt}(s)\|_1^2\,ds. \tag{3.7.2}$$

$$\|\boldsymbol{R}^n\| \leqslant Ck^{\frac{3}{2}}\Big(\max_{0\leqslant t\leqslant T}\|\boldsymbol{u}_t(t)\|_2+\max_{0\leqslant t\leqslant T}\|\boldsymbol{u}_{tt}(t)\|+\max_{0\leqslant t\leqslant T}\|p_t(t)\|_1\Big). \tag{3.7.3}$$

Proof. Let us begin by rewriting the first equation in Darcy Brinkman Forchheimer system at time $t=t_{n+\frac{1}{2}}$:

$$\begin{aligned}\boldsymbol{u}_t(t_{n+\frac{1}{2}})-\gamma\Delta\boldsymbol{u}(t_{n+\frac{1}{2}})+a\boldsymbol{u}(t_{n+\frac{1}{2}})+F(\boldsymbol{u}(t_{n+\frac{1}{2}}))+B(\boldsymbol{u}(t_{n+\frac{1}{2}}),\boldsymbol{u}(t_{n+\frac{1}{2}}))\\ +\nabla p(t_{n+\frac{1}{2}})=f(t_{n+\frac{1}{2}})\end{aligned} \tag{3.7.4}$$

We define the truncation error as follow :

$$\begin{aligned}R^n=\frac{\boldsymbol{u}(t_{n+1})-\boldsymbol{u}(t_n)}{k}-\gamma\Delta(\boldsymbol{u}(t_{n+\frac{1}{2}}))+a\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})+\widetilde{B}(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}),\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}))\\ +\nabla\widetilde{p}(t_{n+\frac{1}{2}})+F(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}))-f(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})).\end{aligned}$$

Which can written as :

$$\begin{aligned}R^n=\Big\{\frac{\boldsymbol{u}(t_{n+1})-\boldsymbol{u}(t_n)}{k}-\boldsymbol{u}_t(t_{n+\frac{1}{2}})\Big\}-\gamma\Big\{\Delta\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})-\Delta\boldsymbol{u}(t_{n+\frac{1}{2}})\Big\}\\ +a\Big\{\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})-\boldsymbol{u}(t_{n+\frac{1}{2}})\Big\}+\Big\{(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}).\nabla)\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})-(\boldsymbol{u}(t_{n+\frac{1}{2}}).\nabla)\boldsymbol{u}(t_{n+\frac{1}{2}})\Big\}\\ +F(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}))-F(u(t_{n+\frac{1}{2}}))+(\nabla\widetilde{p}(t_{n+\frac{1}{2}})-\nabla p(t_{n+\frac{1}{2}})).\end{aligned}$$

In order to estimate the truncation error $\|R^n\|$, we will dived it into many terms A_1^n, A_2^n, A_*^n, A_3^n, A_{**}^n, A_4^n, treating one by one as follow :

$$A_1^n=\frac{\boldsymbol{u}(t_{n+1})-\boldsymbol{u}(t_n)}{k}-\boldsymbol{u}_t(t_{n+\frac{1}{2}}).$$

Thanks to the integral formula of series's Taylor for $u(t_{n+1})$ and $u(t_n)$, we infer that :

$$A_1^n=\frac{1}{k}\Big(\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}(t_{n+1}-s)^2\boldsymbol{u}_{ttt}(s)\quad ds-\int_{t_n}^{t_{n+\frac{1}{2}}}(s-t_n)^2\boldsymbol{u}_{ttt}(s)\quad ds\Big). \tag{3.7.5}$$

Using Schwarz and convexity's inequality to the above result, we reach :

$$\begin{aligned}\|A_1^n\|_{-1}^2 &\leqslant \frac{2}{k^2}\int_{t_n}^{t_{n+\frac{1}{2}}}(s-t_n)^4ds\int_{t_n}^{t_{n+\frac{1}{2}}}\|\boldsymbol{u}_{ttt}(s)\|_{-1}^2ds\\ &+\frac{2}{k^2}\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}(t_{n+1}-s)^4ds\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\|\boldsymbol{u}_{ttt}(s)\|_{-1}^2ds\\ &\leqslant Ck^3\int_{t_n}^{t_{n+1}}\|\boldsymbol{u}_{ttt}(s)\|_{-1}^2ds.\end{aligned}$$

Consequently, we deduce that :

$$k\sum_{n=0}^{\frac{T}{K}-1}\|A_1^n\|_{-1}^2 \leqslant Ck^4\int_0^T \|\boldsymbol{u}_{ttt}(s)\|_{-1}^2 ds. \tag{3.7.6}$$

Next we introduce a new variables E_u^n, E_p^n which help us to handle the rest of terms, we define E_u^n, E_p^n as follow :

$$E_u^n = \widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}) - \boldsymbol{u}(t_{n+\frac{1}{2}}), \qquad E_p^n = \widetilde{p}(t_{n+\frac{1}{2}}) - p(t_{n+\frac{1}{2}}).$$

Again, we proceed as before in the estimate of $\|A_1^n\|$:

$$E_u^n = \frac{1}{4}\Big(\int_{t_n}^{t_{n+\frac{1}{2}}}(s-t_n)\boldsymbol{u}''(s)ds + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}}(t_{n+1}-s)\boldsymbol{u}''(s)ds\Big). \tag{3.7.7}$$

$$E_p^n = \frac{1}{4}\Big(\int_{t_n}^{t_{n+\frac{1}{2}}} p''(s)ds + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}}(t_{n+1}-s)p''(s)ds\Big). \tag{3.7.8}$$

We conclude that :

$$\|E_u^n\|^2 \leqslant Ck^3\int_{t_n}^{t_{n+1}} \|\boldsymbol{u}''(s)\|^2 ds, \qquad |E_p^n|^2 \leqslant Ck^3\int_{t_n}^{t_{n+1}} |p''(s)|^2 ds. \tag{3.7.9}$$

$$\tag{3.7.10}$$

We move to the term A_*^n, Let denote : $A_*^n = a(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}) - \boldsymbol{u}(t_{n+\frac{1}{2}}))$.

Using the estimate above of $\|E_u^n\|$ we can conclude directly that :

$$\|A_*^n\|^2 \leqslant Ck^3\int_{t_n}^{t_{n+1}} \|\boldsymbol{u}''(s)\|^2 \quad ds.$$

Now, in order to handle A_{**}^n, we can observe that :

$$\begin{aligned} A_{**}^n &= \beta|\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})|^\alpha\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}) - \beta|u(t_{n+\frac{1}{2}})|^\alpha\boldsymbol{u}(t_{n+\frac{1}{2}}) \\ &= \beta F(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})) - \beta F(\boldsymbol{u}(t_{n+\frac{1}{2}})). \end{aligned}$$

We have :

$$\begin{aligned} \|A_{**}^n\| &= \beta\|F(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})) - F(u(t_{n+\frac{1}{2}}))\| \\ &\leqslant \beta\|\nabla(\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}) - \boldsymbol{u}(t_{n+\frac{1}{2}}))\| \\ &\leqslant \beta\|\nabla E_u^n\|. \end{aligned} \tag{3.7.11}$$

Thanks to the residual integral of Taylor series for E_u^n:

$$\nabla E_u^n = \frac{1}{4}\Big(\int_{t_n}^{t_{n+\frac{1}{2}}}(s-t_n)\nabla\boldsymbol{u}''(s)ds + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}}(t_{n+1}-s)\nabla\boldsymbol{u}''(s)ds\Big). \tag{3.7.12}$$

We obtain :

$$\|\nabla E_u^n\|^2 \leqslant Ck^3 \int_{t_n}^{t_{n+1}} \|\nabla \boldsymbol{u}''(s)\|^2 \, ds. \tag{3.7.13}$$

and conclude that :

$$\|A_{**}^n\|^2 \leqslant \|\nabla E_u^n\|^2 \leqslant Ck^3 \int_{t_n}^{t_{n+1}} \|\nabla \boldsymbol{u}''(s)\|^2 \quad ds. \tag{3.7.14}$$

Let define A_2^n, A_4^n to be :

$$A_2^n = \Delta \widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}) - \Delta \boldsymbol{u}(t_{n+\frac{1}{2}}), \quad A_4^n = \nabla \widetilde{p}(t_{n+\frac{1}{2}}) - \nabla p(t_{n+\frac{1}{2}}).$$

According to the established estimates of $\|E_u^n\|$ and $\|E_p^n\|$:

$$\begin{aligned}
k \sum_{n=0}^{\frac{T}{K}-1} \|A_2^n\|_{-1}^2 &\leqslant C\gamma k \sum_{n=0}^{\frac{T}{K}-1} \|E_u^n\|^2 \leqslant Ck^4 \int_0^T \|\boldsymbol{u}''(s)\|^2 ds. \\
k \sum_{n=0}^{\frac{T}{k}-1} \|A_4^n\|_{-1}^2 &\leqslant Ck \sum_{n=0}^{\frac{T}{k}-1} \max_{w \in H_0^1(\Omega)} \frac{\langle \nabla E_p^n, w \rangle}{\|w\|^2} \\
&\leqslant Ck \sum_{n=0}^{\frac{T}{k}-1} |E_p^n|^2 \leqslant Ck^4 \int_0^T |p''(s)|^2 ds.
\end{aligned}$$

It remains to estimate A_3^n defined by:

$$A_3^n = (E_u^n . \nabla) \widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}) + (\boldsymbol{u}(t_{n+\frac{1}{2}}). \nabla) E_u^n. \tag{3.7.15}$$

Thanks to (3.1.9) with $\|\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})\| \leqslant M$, and $\|\boldsymbol{u}(t_{n+\frac{1}{2}})\| \leqslant M'$, we observe:

$$\begin{aligned}
\langle A_3^n, w \rangle &= b(E_{\boldsymbol{u}}^n, \widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}), w) + b((u_{n+\frac{1}{2}}), E_u^n, w). \\
&\leqslant C\|E_u^n\| \|w\| \|\widetilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})\| + C\|E_u^n\| \|w\| \|\boldsymbol{u}(t_{n+\frac{1}{2}})\|. \\
&\leqslant C\|E_u^n\| \|w\|.
\end{aligned}$$

It follows:

$$\begin{aligned}
k \sum_{n=0}^{\frac{T}{k}-1} \|A_3^n\|_{-1}^2 &\leqslant Ck \sum_{n=0}^{\frac{T}{k}-1} \max_{w \in H_0^1(\Omega)} \frac{(\langle A_3^n, w \rangle)^2}{\|w\|^2}. \\
&\leqslant Ck \sum_{n=0}^{\frac{T}{k}-1} \|E_u^n\|^2 \leqslant Ck^4.
\end{aligned}$$

which gives the result (3.7.1).
Now, let us recall that

$$E_p^n = (p(t_{n+1}) - p(t_n)) - (p(t_{n-1}) - p(t_{n-2})). \tag{3.7.16}$$

By applying Taylor's formula for the second order of the function p, taking into account that $t_{n+1} - t_n = t_{n-1} - t_{n-2} = k$, it gives

$$p(t_{n+1}) - p(t_n) = kp_t(t_n) + \int_{t_n}^{t_{n+1}} p_{tt}(s)(t_{n+1} - s)\, ds. \tag{3.7.17}$$

$$p(t_{n-2}) - p(t_{n-1}) = -kp_t(t_{n-1}) + \int_{t_{n-2}}^{t_{n-1}} p_{tt}(s)(t_{n-1} - s)\, ds. \tag{3.7.18}$$

Based on the previous results (3.7.16), (3.7.17) and (3.7.18), one can observe that

$$E_p^n = k(p_t(t_n) - p_t(t_{n-1})) + \int_{t_n}^{t_{n+1}} p_{tt}(s)(t_{n+1} - s)\, ds + \int_{t_{n-2}}^{t_{n-1}} p_{tt}(s)(t_{n-1} - s)\, ds.$$

$$E_p^n = k\int_{t_{n-1}}^{t_n} p_{tt}(s)\, ds + \int_{t_n}^{t_{n+1}} p_{tt}(s)(t_{n+1} - s)\, ds + \int_{t_{n-2}}^{t_{n-1}} p_{tt}(s)(t_{n-1} - s)\, ds.$$

Due to Cauchy Schwarz's Theorem and by taking the square of the result, we reach on the one hand

$$\begin{aligned}(k\int_{t_{n-1}}^{t_n} p_{tt}(s)\, ds)^2 &\leqslant k^2 \int_{t_{n-1}}^{t_n} \|\nabla p_{tt}(s)\|^2\, ds \int_{t_{n-1}}^{t_n} 1\, ds.\\ &\leqslant k^3 \int_{t_{n-1}}^{t_n} \|\nabla p_{tt}(s)\|^2\, ds. \end{aligned}\tag{3.7.19}$$

On the other hand, if $t_{n-2} \leqslant s \leqslant t_{n-1}$ or $t_n \leqslant s \leqslant t_{n+1}$, then we obtain $(t_{n-1} - s)^2 \in [0\,,\, k^2]$ and $(t_{n+1} - s)^2 \in [0\,,\, k^2]$. So we have

$$\int_{t_{n-2}}^{t_{n-1}} (t_{n-1} - s)^2\, ds \leqslant \frac{k^3}{3}. \tag{3.7.20}$$

$$\int_{t_{n+1}}^{t_n} (t_{n+1} - s)^2\, ds \leqslant \frac{k^3}{3} \tag{3.7.21}$$

By applying again Cauchy Schwarz's inequality and taking the square of the result, using the previous relations (3.7.20) and (3.7.21), it follows:

$$\begin{aligned}\int_{t_n}^{t_{n+1}} p_{tt}(s)(t_{n+1} - s)\, ds + \int_{t_{n-2}}^{t_{n-1}} p_{tt}(s)(t_{n-1} - s)\, ds \leqslant\\ \int_{t_{n+1}}^{t_n} \|\nabla p_{tt}(s)\|^2\, ds \int_{t_n}^{t_{n+1}} (t_{n+1} - s)^2\, ds\\ + \int_{t_{n-2}}^{t_{n-1}} \|\nabla p_{tt}\|^2\, ds \int_{t_{n-2}}^{t_{n-1}} (t_{n-1} - s)^2\, ds. \end{aligned}\tag{3.7.22}$$

$$\leqslant \frac{1}{3}k^3(\int_{t_n}^{t_{n+1}} + \int_{t_{n-2}}^{t_{n-1}})\|\nabla p_{tt}(s)\|^2\, ds. \tag{3.7.23}$$

By summing the results (3.7.19), (3.7.23) and multiplying with k, the result (3.7.2) can be established.

Now, we return to prove the truncation error (3.7.3). So based on the relation (3.7.5) and using Cauchy-Schwarz and the convexity's inequality to the result (3.7.5), we get:

$$\begin{aligned}\|A_1^n\|^2 &\leqslant \frac{2}{k^2}\int_{t_n}^{t_{n+\frac{1}{2}}}(s-t_n)^4 ds \int_{t_n}^{t_{n+\frac{1}{2}}} \|\boldsymbol{u}_{ttt}(s)\|^2\, ds \\ &+ \frac{2}{k^2}\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}(t_{n+1}-s)^4 ds \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \|\boldsymbol{u}_{ttt}(s)\|^2\, ds \\ &\leqslant Ck^3 \max_{0\leqslant t\leqslant T}\|\boldsymbol{u}_{tt}(t)\|^2.\end{aligned}$$

Hence we can deduce :

$$\|A_1^n\| \leqslant Ck^{\frac{3}{2}} \max_{0\leqslant t\leqslant T}\|\boldsymbol{u}_{tt}(t)\|. \tag{3.7.24}$$

Next due to the relation (3.7.7) and (3.7.8), we conclude that :

$$\|E_u^n\| \leqslant Ck^{\frac{3}{2}} \max_{0\leqslant t\leqslant T}\|\boldsymbol{u}_t(t)\|, \qquad |E_p^n|^2 \leqslant Ck^{\frac{3}{2}} \max_{0\leqslant t\leqslant T}|p_t(t)|. \tag{3.7.25}$$

According to the estimate $\|E_u^n\|$, it follows that A_*^n can be handled as,

$$\|A_*^n\| \leqslant Ck^{\frac{3}{2}} \max_{0\leqslant t\leqslant T}\|\boldsymbol{u}_t(t)\|. \tag{3.7.26}$$

Moreover, due to the relations (3.7.11), (3.7.12), (3.7.13) and (3.7.14), we deduce that

$$\|A_{**}^n\| \leqslant Ck^{\frac{3}{2}} \max_{0\leqslant t\leqslant T}\|\nabla\boldsymbol{u}_t(t)\|. \tag{3.7.27}$$

Based on Taylor's formula for $\Delta E_{\boldsymbol{u}}^n$, one can obtain

$$\Delta E_u^n = \frac{1}{4}\Big(\int_{t_n}^{t_{n+\frac{1}{2}}}(s-t_n)\Delta\boldsymbol{u}''(s)\,ds + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}}(t_{n+1}-s)\Delta\boldsymbol{u}''(s)\,ds\Big).$$

Using Cauchy-Schwarz's inequality, we have

$$\|\Delta E_u^n\| \leqslant Ck^{\frac{3}{2}} \max_{0\leqslant t\leqslant T}\|\Delta\boldsymbol{u}_t(t)\|.$$

Consequently, we deduce that

$$\|A_2^n\| \leqslant Ck^{\frac{3}{2}} \max_{0\leqslant t\leqslant T}\|\Delta\boldsymbol{u}_t(t)\|. \tag{3.7.28}$$

By using the definition A_3^n given in relation (3.7.15), we have,

$$\begin{aligned}\|A_3^n\| &\leqslant \|E_u^n\|\|\nabla\widetilde{u}(t_{n+\frac{1}{2}})\| + \|u(t_{n+\frac{1}{2}})\|\|\nabla\boldsymbol{u}_t(t)\|. \\ &\leqslant Ck^{\frac{3}{2}} \max_{0\leqslant t\leqslant T}\|\boldsymbol{u}_t(t)\| + Ck^{\frac{3}{2}} \max_{0\leqslant t\leqslant T}\|\nabla\boldsymbol{u}_t(t)\| \\ &\leqslant Ck^{\frac{3}{2}} \max_{0\leqslant t\leqslant T}\|\boldsymbol{u}_t(t)\|_1.\end{aligned} \tag{3.7.29}$$

Because of $A_4^n = \nabla E_p^n = \nabla\Big(\widetilde{p}(t_{n+\frac{1}{2}}) - p(t_{n+\frac{1}{2}})\Big)$ and using again Taylor's formula, it follows:

$$\nabla E_p^n = \frac{1}{4}\Big(\int_{t_n}^{t_{n+\frac{1}{2}}}(t_{n+\frac{1}{2}} - s)\nabla p''(s)ds + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}}(t_{n+1} - s)\nabla p''(s)ds\Big). \tag{3.7.30}$$

We conclude that :

$$\|A_4^n\| \leqslant Ck^{\frac{3}{2}} \max_{0\leqslant t\leqslant T}\|\nabla p_t(t)\|. \tag{3.7.31}$$

Consequently, the relation (3.7.3) can be established by summing the results (3.7.31), (3.7.29), (3.7.28), (3.7.27), (3.7.26) and (3.7.24). □

Lemma 3.7.2. *Under the same assumptions of the previous lemma, there exist $C > 0$ such that:*

$$\|\boldsymbol{u}(t_j) - \boldsymbol{u}^j\|^2 + k^2\|\nabla(p(t_j) - p^j)\|^2 \leqslant Ck^4, \quad j = 0, ..., m, \tag{3.7.32}$$

where $(\boldsymbol{u}^j, p^j)$ is the solution of Darcy Brinkman Forchheimer's time discretization (3.5.6)

Proof. Taking the inner product of the equations (3.5.17) at first step $n = 0$ respectively with the terms $k(\theta^1 + \theta^0)$ and $k\chi^0$, using the algebraic equation (3.1.7), then summing the result we obtain :

$$\begin{aligned}\|\theta^1\|^2 - \|\theta^0\|^2 + \frac{\gamma k}{2}\|\nabla(\theta^1 + \theta^0)\|^2 + \frac{ak}{2}\|\theta^1 + \theta^0\|^2 + k(\theta^0, \nabla\chi^0) + \frac{\delta k^2}{2}\|\nabla\chi^1\|^2 = \\ \frac{\delta k^2}{2}(\|\nabla\chi^0\|^2 + \|\nabla(\chi^1 - \chi^0)\|^2) + k(R^0, \theta^1 + \theta^0) + k(Q^0, \theta^1 + \theta^0) \\ +\delta k^2(\nabla(kp_t(t_{\frac{1}{2}})), \nabla\chi^0).\end{aligned}$$

Now we need to handle the terms below :

$$\begin{aligned}k|(R^0, \theta^1 + \theta^0)| &\leqslant k^2\|R^0\|^2 + C\|\theta^1 + \theta^0\|^2.\\ k|(Q^0, \theta^1 + \theta^0)| &\leqslant k|\tilde{b(}\theta^{\frac{1}{2}}, u(t_{\frac{1}{2}}), \theta^1 + \theta^0)| + k|(F(u^{\frac{1}{2}}) - F(u(t_{\frac{1}{2}})), \theta^1 + \theta^0)|\\ &\leqslant Ck\|\theta^{\frac{1}{2}}\|_1\|u(t_{\frac{1}{2}})\|_2\|\theta^1 + \theta^0\| + k\|\nabla\theta^{\frac{1}{2}}\|\|\theta^1 + \theta^0\|.\\ &\leqslant Ck^2\|\nabla\theta^{\frac{1}{2}}\|^2 + C\|\theta^0 + \theta^1\|^2.\end{aligned}$$

Moreover, thanks to the result (3.5.20) and the derivate of the pressure's regularity given in the relation (3.5.23), we can observe that :

$$\frac{\delta k^2}{2}\|\nabla(\chi^1 - \chi^0)\|^2 \leqslant \frac{1}{4\delta}\|\theta^1\|^2 + Ck^4.$$

Using the relation (3.7.3) and the initial conditions reviewed in relation (3.5.7), we can conclude that :

$$\|\theta^1\|^2 + \frac{\gamma k}{2}\|\nabla(\theta^1 + \theta^0)\|^2 + \frac{ak}{2}\|\theta^1 + \theta^0\|^2 + \frac{\delta k^2}{2}\|\nabla\chi^1\|^2 \leqslant Ck^4.$$

In order to establish the result (3.7.32), it suffices to repeat the above procedure $m + 1$ times. □

Chapter 4

Approximation of (DBF) equations with Navier's type boundary conditions by pseudocompressibility method

In this chapter, we treat the approximation of the evolution incompressible DBF equations with Navier-type boundary conditions via the pseudo-compressibility method.

This chapter is splitting into three parts, the first one is devoted to the non stationary DBF problem with non standard boundary conditions where we give some regularity results. In section 2, the existence of solution for the perturbed unsteady system is investigated and the convergence to the initial unsteady DBF problem is checked. The last part of this chapter presents the optimal error analysis.

4.1 Study of the time-dependent DBF equations with Navier's type boundary conditions

The mathematical description of the unsteady incompressible fluid flow in saturated porous medium is given by Darcy Brinkamn Forchheimer equations. This model system in a bounded domain $\Omega \subset \mathbb{R}^3$ takes the following form

$$\boldsymbol{u}_t + \gamma \Delta \boldsymbol{u} + a\boldsymbol{u} + |\boldsymbol{u}|^\alpha \boldsymbol{u} + \nabla p = 0, \quad \text{in } \Omega, \tag{4.1.1}$$

With this notations $\boldsymbol{u} = \boldsymbol{u}(x,t)$ denotes the velocity field and $p = p(x,t)$ the scalar pressure function and $\boldsymbol{f} = \boldsymbol{f}(x,t)$ is the given volume force.

Beside the incompressibility condition :

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in}\,\Omega. \tag{4.1.2}$$

Instead of Dirichlet boundary conditions which is classical conditions of adherence on the boundary , we provide here Navier's type boundary conditions,

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\boldsymbol{u}\times\boldsymbol{n} = 0, \quad \text{on}\;\Gamma, \tag{4.1.3}$$

4.1.1 Preliminaries and notations

This section is dedicated to present some basic notations and functional spaces which are needed for this chapter. In what follows, We assume that the boundary Γ is connected and Ω will be considered as an open bounded simply connected domain of $\mathbb{R}^3$ of classe $\mathcal{C}^{1,1}$ and sometimes of classe $\mathcal{C}^{2,1}$. Let us recall that the unit exterior normal vector to the boundary will be denoted by $\boldsymbol{n}$. Note that the vector-valued Laplace operator of a vector field $\boldsymbol{u} = (u_1, u_2, u_3)$ is equivalently defined by:

$$\Delta\boldsymbol{u} = \nabla\operatorname{div}\boldsymbol{u} - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}. \tag{4.1.4}$$

In particular, if $\operatorname{div}\boldsymbol{u} = 0$, thus it gives immediately that :

$$\Delta\boldsymbol{u} = -\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}. \tag{4.1.5}$$

The space $\boldsymbol{L}_\sigma(\Omega)$ defined by:

$$\boldsymbol{L}_\sigma(\Omega) = \{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)\,,\,\operatorname{div}\boldsymbol{v} = 0\;\;\text{in}\,\Omega\}.$$

Because Ω is a bounded and simply-connected domain and thanks to [65], it is well known that there exists a positive constant C where,

$$\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \leqslant C(\|\operatorname{div}\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}), \quad \forall\boldsymbol{v}\in\boldsymbol{X}^2_T(\Omega), \tag{4.1.6}$$

moreover, this vector field satisfies,

$$\|\nabla\boldsymbol{v}\|^2 \leqslant C\left(\|\operatorname{div}\boldsymbol{v}\|^2 + \|\mathbf{curl}\,\boldsymbol{v}\|^2\right) \tag{4.1.7}$$

which implies that $\boldsymbol{v}$ is necessarily belongs to the space $\boldsymbol{H}^1(\Omega)$.
For any mapping F defined by $F:\boldsymbol{u}\longrightarrow|\boldsymbol{u}|^\alpha\boldsymbol{u}$, we have the following monotonocity property:

$$(|\boldsymbol{u}|^\alpha\boldsymbol{u} - |\boldsymbol{v}|^\alpha\boldsymbol{v}\,,\,\boldsymbol{u}-\boldsymbol{v}) \geqslant 0. \tag{4.1.8}$$

Proposition 4.1.1. *For given $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)$, such that $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ, we have*

$$\||\boldsymbol{u}|^\alpha\boldsymbol{u} - |\boldsymbol{v}|^\alpha\boldsymbol{v}\| \leqslant C\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|. \tag{4.1.9}$$

Where $C > 0$ is dependent only on Ω and dimension.

We introduce the following compact Theorem that we need in the sequel of this chapter.

Theorem 4.1.2. *If X_0, X, X_1 are Hilbert spaces with the continuous embeddings $X_0 \hookrightarrow X \hookrightarrow X_1$ and the embedding $X_0 \hookrightarrow X$ is compact.*
For a given $\gamma > 0$, we define the Hilbert space :

$$M^\gamma := M^\gamma(\mathbb{R}; X_0, X_1) := \{\psi \in L^2(\mathbb{R}; X_0), D_t^\gamma \psi \in L^2(\mathbb{R}; X_1)\}$$

with the norm :

$$\|\psi\|_{M^\gamma} = \{\|\psi\|^2_{L^2(\mathbb{R},X_0)} + \left\||\tau|^\gamma \widehat{\psi}(\tau)\right\|^2_{L^2(\mathbb{R},X_1)}\}.$$

Such that $\widehat{\psi}$ denote the Fourier transform. For any set $K \subset \mathbb{R}$, we define

$$M_K^\gamma := M_K^\gamma(\mathbb{R}; X_0, X_1) = \{\psi \in M^\gamma, \text{support}\, \psi \subseteq K\}. \tag{4.1.10}$$

Hence, we obtain the following compact embedding :

$$M_K^\gamma(\mathbb{R}; X_0, X_1) \hookrightarrow L^2(\mathbb{R}; X). \tag{4.1.11}$$

4.1.2 Analysis of the problem

In this section, we focus on the study of the following evolution incompressible Darcy Brinkman Forchheimer equations with Navier's type boundary conditions given as,

$$\begin{cases} \boldsymbol{u}_t - \gamma\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|^\alpha\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{and} \quad \text{div}\ \boldsymbol{u} = 0, & \text{in}\, \Omega \times [0,T] \\ \boldsymbol{u}\cdot\mathbf{n} = 0\,,\ \mathbf{curl}\ \boldsymbol{u} \times \mathbf{n} = \ \mathbf{0}, & \text{on}\, \Gamma \times [0,T] \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0\ \ , p(x,0) = p_0,\ \text{in}\, L^2(\Omega), & \end{cases} \tag{4.1.12}$$

with the following assuptions on the data $\boldsymbol{f}$ and $\boldsymbol{u}_0$:

$$\boldsymbol{f} \in L^2(0,T; (\boldsymbol{H}_0^2(\text{div}\,,\Omega))') \tag{4.1.13}$$

$$\boldsymbol{u}_0 \in \boldsymbol{L}_\sigma(\Omega). \tag{4.1.14}$$

Based on the Green's formulas (1.1.2) and (1.1.1), we prove that any solution

$$(\boldsymbol{u},\, p) \in L^2(0,T; \boldsymbol{X}_T^2(\Omega)) \times L^2(0,T, L^2(\Omega))$$

of the system (4.1.12) is a solution of the following weak variational formulation :
Find $\boldsymbol{u} \in L^2(0,T; \boldsymbol{X}_T^2(\Omega))$ which satisfies that :

$$\begin{cases} \dfrac{d}{dt}\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \gamma\langle \mathbf{curl}\, \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v}\rangle + a\langle \boldsymbol{u}, \boldsymbol{v}\rangle + b\langle |\boldsymbol{u}|^\alpha \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{f}\,, \boldsymbol{v}\rangle_\Omega,\ \forall \boldsymbol{v} \in \boldsymbol{X}_T^2(\Omega). \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{4.1.15}$$

Such that the duality on Ω is given by

$$\langle \cdot, \cdot\rangle_\Omega = \langle \cdot, \cdot\rangle_{\left(\boldsymbol{H}_0^2(\text{div}\,,\Omega)\right)' \times \boldsymbol{H}_0^2(\text{div}\,,\Omega)}.$$

Note that unlike the Brinkman problem with Dirichlet boundary conditions, for any $t \in [0, T]$, the space $\boldsymbol{H}^{-1}(\Omega)$ for $\boldsymbol{f}(t)$ is not suitable to reach the weak solutions. In fact, the test function $\boldsymbol{v}$ belongs to $\boldsymbol{H}_0^2(\mathrm{div}\,,\Omega)$ which implies to assume that the datum $\boldsymbol{f}(t)$ belongs to the dual space $\big(\boldsymbol{H}_0^2(\mathrm{div}\,,\Omega)\big)'$ which is a subspace of $\boldsymbol{H}^{-1}(\Omega)$.

The first main objective of this section is to show that the previous variational problem (4.1.15) has at least one solution. To this end, we apply Faedo-Galerkin's method. We deal with a classic techniques, inspired from Lions [43] and Temam [61].
Currently in the three-dimensional case, the question of the uniqueness of weak solutions as well as the global existence of regular solutions are still open question inspiring a large amount of research.
We will solve the equivalent formulation (4.1.15) on [0,T]. The plan is the following : we introduce a finite dimensional approximate problem, then we prove estimates on the solutions of this approximate problem which are uniform with respect to the approximation parameter. Finally, we use the compactness theorem in order to obtain the strong convergence and justify the limit in the non linear term.

Faedo-Galerkin approximation and a priori estimates We need to describe the Galerkin approximation. Let m be a positive integer. We consider an orthonormal basis of $\boldsymbol{X}_T^2(\Omega)$ constituted of $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m$. We introduce the space $\boldsymbol{V}_m = \langle \boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m\rangle$, we define an approximated solution $\boldsymbol{u}_m$ of (4.1.15) with :

$$\boldsymbol{u}_m(t) = \sum_{i=1}^{m} g_{im}(t)\boldsymbol{w}_i. \tag{4.1.16}$$

and $\mathrm{div}\boldsymbol{u}_m(t) = 0$ where $\boldsymbol{u}_m(t)$ satisfies,

$$\begin{cases} \langle \boldsymbol{u}'_m(t), \boldsymbol{w}_i\rangle + \gamma\langle \mathbf{curl}\,\boldsymbol{u}_m(t), \mathbf{curl}\,\boldsymbol{w}_i\rangle + a\langle \boldsymbol{u}_m(t), \boldsymbol{w}_i\rangle + b\langle |\boldsymbol{u}_m(t)|^{\alpha}\boldsymbol{u}_m(t), \boldsymbol{w}_i\rangle = \langle \boldsymbol{f}(t), \boldsymbol{w}_i\rangle_\Omega \\ \forall i = 1, \dots, m\,,\, t \in [0, t_m], \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m} \end{cases} \tag{4.1.17}$$

Such that $\boldsymbol{u}_{0m}$ is the orthogonal projection of $\boldsymbol{u}_0$ on the space $\boldsymbol{V}_m$.
Observe that the equations (4.1.17) arise a nonlinear differential system for the functions $g_{1,m}, \dots, g_{m,m}$.
Using a classical arguments see [61, Theorem 3.1, page 283-284], the problem (4.1.17) admits at least one solution $\boldsymbol{u}_m$ defined on $[0, t_m]$ with $t_m < T$.

We now move to establish a priori estimates on the approximate function $\boldsymbol{u}_m$ which are independents of m that leads to conclude the existence of weakly convergent subsequences in appropriate spaces and then justify the passage to the limit in the approximate problem (4.1.17). We start with the following Lemma.

Lemma 4.1.3. *For any given* $\boldsymbol{f}$, $\boldsymbol{u}_0$ *satisfying* (4.1.13) *and* (4.1.14), *then the solutions* $\boldsymbol{u}_m$ *satisfy the following a priori estimates:*

$$\sup_{t\in[0,T]} \|\boldsymbol{u}_m(t)\|^2 \leqslant C_1, \tag{4.1.18}$$

$$\|\boldsymbol{u}_m(T)\|^2 + 2\gamma \int_0^T \|\mathbf{curl}\,\boldsymbol{u}_m(t)\|^2 dt + a\int_0^T \|\boldsymbol{u}_m(t)\|^2 dt + 2b\int_0^T \|\boldsymbol{u}_m(t)\|_{2+\alpha}^{2+\alpha} \leqslant C_1, \tag{4.1.19}$$

With

$$C_1 = \|\boldsymbol{u}_0\|^2 + \int_0^T \|\mathbf{f}(t)\|^2_{(\mathbf{H}_0^2(\mathrm{div}\,,\Omega))'} dt. \tag{4.1.20}$$

Proof. Let us first Multiply (4.1.17) by $g_{im}(t)$ and summing these equations for $i = 1, \ldots, m$, using the both Cauchy-Schwarz's inequality and Young's inequality, it follows to obtain :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m(t)\|^2 + \gamma\|\mathbf{curl}\,\boldsymbol{u}_m(t)\|^2 + a\|\boldsymbol{u}_m(t)\|^2 + b\|\boldsymbol{u}_m\|_{\alpha+2}^{\alpha+2} &= \langle \boldsymbol{f}(t), \boldsymbol{u}_m(t)\rangle_\Omega.\\ &\leqslant \frac{1}{2a}\|\boldsymbol{f}(t)\|^2 + \frac{a}{2}\|\boldsymbol{u}_m(t)\|^2.\end{aligned} \tag{4.1.21}$$

Hence, we have

$$\frac{d}{dt}\|\boldsymbol{u}_m(t)\|^2 + 2\gamma\|\mathbf{curl}\,\boldsymbol{u}_m(t)\|^2 + a\|\boldsymbol{u}_m(t)\|^2 + 2b\|\boldsymbol{u}_m(t)\|_{\alpha+2}^{\alpha+2} \leqslant C\|\boldsymbol{f}(t)\|^2_{(\mathbf{H}_0^2(\mathrm{div}\,,\Omega))'} \tag{4.1.22}$$

By integrating (4.1.22) from 0 to t with $t \in [0, t_m]$, we obtain in particular :

$$\sup_{t\in[0,t_m]} (\|\boldsymbol{u}_m(t)\|^2) \leqslant C\int_0^T \|\mathbf{f}(t)\|^2_{(\mathbf{H}_0^2(\mathrm{div}\,,\Omega))'} dt + \|\boldsymbol{u}_{0m}\|^2. \tag{4.1.23}$$

It is clear that the right hand side of the above result (4.1.23) is independent of t_m. so consequently, the estimate (4.1.23) holds with $t_m = T$ which gives the estimate (4.1.18).

Next, by using the simple integration of (4.1.22) over time, from 0 to T, we obtain :

$$\begin{aligned}\|\boldsymbol{u}_m(T)\|^2 + 2\gamma\int_0^T \|\mathbf{curl}\,\boldsymbol{u}_m(t)\|^2\, dt + a\int_0^T \|\boldsymbol{u}_m(t)\|^2\, dt + 2b\int_0^T \|\boldsymbol{u}_m(t)\|_{2+\alpha}^{2+\alpha} \leqslant\\ \frac{1}{a}\int_0^T \|\boldsymbol{f}(t)\|^2_{(\mathbf{H}_0^2(\mathrm{div}\,,\Omega))'}\, dt + \|\boldsymbol{u}_0\|^2.\end{aligned} \tag{4.1.24}$$

Which achives the proof of this Lemma.

□

Passage to the limit On the one hand, due to the a priori estimate (4.1.18), it follows the existence of an element $\boldsymbol{u} \in L^\infty(0,T;L^2_\sigma(\Omega))$ and a subsequence still referred to as $(\boldsymbol{u}_m)_m$ to simplify the notation, satisfying :

$$\boldsymbol{u}_m \rightharpoonup \boldsymbol{u} \quad \text{weak star in } L^\infty(0,T;L^2_\sigma(\Omega)). \tag{4.1.25}$$

On the other hand, the estimate (4.1.19) shows the existence of some $\boldsymbol{u}_*$ in $L^2(0,T;X^2_T(\Omega))$ and some subsequence (still denoted $(\boldsymbol{u}_m)_m$) such that :

$$\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}_* \quad \text{weak in } L^2(0,T;X^2_T(\Omega)). \tag{4.1.26}$$

The both convergences (4.1.25) and (4.1.26) yield to obtain the convergences in the distribution sense. So note that, because the limit of a sequence in the distribution sense is unique, it gives that $\boldsymbol{u}_* = \boldsymbol{u}$.
In order to reach the limit of the nonlinear term, it is necessary to obtain first a strong convergence result. We will closely follow the classical arguments in [61, Chapter 3] to prove the following estimate of the fractional derivative with respect to time of the approximate solution:

$$\int_{\mathbb{R}} |\tau|^{2\gamma} \left\| \widehat{\boldsymbol{u}_m}(\tau) \right\|^2 d\tau \leqslant C, \quad \text{for some } \gamma > 0. \tag{4.1.27}$$

We proceed as follow to prove (4.1.27), we give first the extension of the function $\boldsymbol{u}_m$,

$$\widetilde{\boldsymbol{u}}_m = \begin{cases} \boldsymbol{u}_m(t), & t \in [0,T] \\ 0 & \text{outside,} \end{cases}$$

Note that $\widetilde{\boldsymbol{u}}_m$ the Fourier Transform defined by :

$$\widehat{\boldsymbol{u}}_m(\tau) = \int_{\mathbb{R}} e^{-2i\pi t\tau} \widetilde{\boldsymbol{u}}_m(t)\, dt.$$

Because of the discontiuities on the both moment 0 and T, then we observe the following equations with $\delta_{(T)}$ and $\delta_{(0)}$ defined respectively the dirac distributions at T and 0.,

$$\begin{cases} \dfrac{d}{dt}\langle \widetilde{\boldsymbol{u}}_m, \boldsymbol{\psi}_k\rangle + \gamma\langle \mathbf{curl}\, \widetilde{\boldsymbol{u}}_m, \mathbf{curl}\, \boldsymbol{\psi}_k\rangle + a\langle \widetilde{\boldsymbol{u}}_m, \boldsymbol{\psi}_k\rangle + b\langle |\widetilde{\boldsymbol{u}}_m|^\alpha \widetilde{\boldsymbol{u}}_m, \boldsymbol{\psi}_k\rangle \\ = \langle \widetilde{\boldsymbol{f}}, \boldsymbol{\psi}_k\rangle + \langle \boldsymbol{u}_{0m}, \boldsymbol{\psi}_k\rangle \delta_{(0)} - \langle \boldsymbol{u}_m(T).\boldsymbol{\psi}_k\rangle \delta_{(T)}. \end{cases}$$

The fourier transform of the derivate function $\boldsymbol{u}_m$ with respect to time can be given as follow,

$$\widehat{D_t^\gamma \boldsymbol{u}_m^\epsilon}(t) = (2i\pi\tau)^\gamma \widehat{\boldsymbol{u}}_m^\epsilon(\tau), \quad \text{for some fixed real } \gamma.$$

Which leads to obtain the following system

$$\begin{cases} 2i\pi\tau\langle \widehat{\boldsymbol{u}}_m, \boldsymbol{\psi}_k\rangle + \gamma\langle \mathbf{curl}\, \widehat{\boldsymbol{u}}_m, \mathbf{curl}\, \boldsymbol{\psi}_k\rangle + a\langle \widehat{\boldsymbol{u}}_m\, ,\, \boldsymbol{\psi}_k\rangle + b\langle |\widehat{\boldsymbol{u}}_m|^\alpha \widehat{\boldsymbol{u}}_m\, ,\, \boldsymbol{\psi}_k\rangle + \\ = \langle \widehat{f}, \boldsymbol{\psi}_k\rangle + \langle \boldsymbol{u}_{0m}, \boldsymbol{\psi}_k\rangle - \langle \boldsymbol{u}_m(T), \boldsymbol{\psi}_k\rangle e^{-2i\pi\tau T} \end{cases} \tag{4.1.28}$$

It remains to check that:

$$\boldsymbol{u}_m \in M_K^\gamma(\mathbb{R}, X_T^2(\Omega), \boldsymbol{L}^2(\Omega)).$$

By considering $\boldsymbol{X}_0 := \boldsymbol{X}_T^2(\Omega)$ and $\boldsymbol{X} = \boldsymbol{X}_1 := \boldsymbol{L}^2(\Omega)$, due to Theorem 4.1.2, we conclude that,

$$\boldsymbol{u}_m \in \boldsymbol{L}^2(0, T, \boldsymbol{X}_T^2(\Omega)).$$

We move to prove that the derivate of $\boldsymbol{u}_m$ with respect of time belongs to the space $\boldsymbol{L}^2(0, T, \boldsymbol{L}^2(\Omega))$ to establish the compactness Theorem.

We multiply $(4.1.28)_1$ with $\widehat{g}_{im}$ with $i = 1...m$, so we reach :

$$\begin{aligned} 2i\pi\tau \|\widehat{\boldsymbol{u}}_m\|^2 + \gamma \|\mathbf{curl}\, \widehat{\boldsymbol{u}}_m\|^2 + a\|\widehat{\boldsymbol{u}}_m\|^2 + b\|\widehat{\boldsymbol{u}}_m\|_{\alpha+2}^{\alpha+2} &= \langle \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{u}}_m\rangle + \langle \boldsymbol{u}_{0m}, \widehat{\boldsymbol{u}}_m(\tau)\rangle \\ &\quad - \langle \boldsymbol{u}_m(T), \widehat{\boldsymbol{u}}_m^\epsilon(\tau)\rangle e^{-2i\pi\tau T}. \end{aligned}$$

Due to Cauchy-Schwarz's inequality, we have

$$2i\pi\tau \|\widehat{\boldsymbol{u}}_m\|^2 \leqslant \left\|\widehat{\boldsymbol{f}}\right\| \|\widehat{\boldsymbol{u}}_m\| + \|\boldsymbol{u}_{0m}\| \|\widehat{\boldsymbol{u}}_m(\tau)\| + e^{-2i\pi\tau T} \|\boldsymbol{u}_m(T)\| \|\widehat{\boldsymbol{u}}_m(\tau)\|.$$

Because for any $\mu \in \mathbb{R}$,

$$\|e^{\mu i}\| \leqslant 1.$$

Then it follows strightforward to obtain

$$2\pi\tau \|\widehat{\boldsymbol{u}}_m\|^2 \leqslant \left\|\widehat{\boldsymbol{f}}\right\| \|\widehat{\boldsymbol{u}}_m\| + \|\boldsymbol{u}_{0m}\| \|\widehat{\boldsymbol{u}}_m(\tau)\| + \|\boldsymbol{u}_m(T)\| \|\widehat{\boldsymbol{u}}_m(\tau)\| \tag{4.1.29}$$

For given $\gamma \in]0, \frac{1}{4}[$, we observe that,

$$|\tau|^{2\gamma} \leqslant (2\gamma+1)\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}. \tag{4.1.30}$$

Hence, we reach, with (4.1.30) :

$$\begin{aligned} \int_{\mathbb{R}} |\tau|^{2\gamma} \|\widehat{\boldsymbol{u}}_m(\tau)\|_{\mathbf{L}^2}^2\, d\tau &\leqslant \frac{2\gamma+1}{2} \int_{\mathbb{R}} \frac{\|\widehat{\boldsymbol{u}}_m(\tau)\|_{\mathbf{L}^2}^2}{1+|\tau|^{1-2\gamma}}\, d\tau + \frac{2\gamma+1}{2} \int_{\mathbb{R}} \frac{|\tau|\, \|\widehat{\boldsymbol{u}}_m(\tau)\|^2}{1+|\tau|^{1-2\gamma}}\, d\tau \\ &:= A_1 + A_2. \end{aligned} \tag{4.1.31}$$

Then, let us handle A_1 as follow :

$$\begin{aligned} A_1 &\leqslant \frac{2\gamma+1}{2} \int_{\mathbb{R}} \|\widehat{\boldsymbol{u}}_m(\tau)\|_{\mathbf{L}^2}^2\, d\tau \\ &= \frac{2\gamma+1}{2} \int_0^T \|\boldsymbol{u}_m(t)\|^2\, dt < \infty \end{aligned}$$

Using the inequality (4.1.29) and Cauchy-Schwarz :

$$\begin{aligned}
A_2 &\leqslant \frac{2\gamma+1}{8\pi}\int_{\mathbb{R}}\left\|\widehat{\boldsymbol{f}}(\tau)\right\|^2 d\tau + \frac{(2\gamma+1)\sqrt{C_2}}{2\pi}\int_{\mathbb{R}}\frac{\|\widehat{\boldsymbol{u}}_m(\tau)\|}{1+|\tau|^{1-2\gamma}}d\tau \\
&\leqslant C\int_0^T \|\boldsymbol{f}(\tau)\|^2_{\mathbf{H}_0(\mathrm{div}\,,\Omega)'}\,d\tau \\
&+ C\Big(\int_{\mathbb{R}}\frac{d\tau}{(1+|\tau|^{1-2\gamma})^2}\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}}(\|\widehat{\boldsymbol{u}}_m(\tau)\|^2\,d\tau\Big)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Consequently, we conclude that

$$\int_{\mathbb{R}} |\tau|^{2\gamma}\|\widehat{\boldsymbol{u}}_m\|^2 < \infty. \tag{4.1.32}$$

Based on (4.1.27) and (4.1.26), according to the compactness Theorem 4.1.2 , we can exctract a subsequence of $(\boldsymbol{u}_m)$, still denoted $(\boldsymbol{u}_m)_m$, such that

$$\boldsymbol{u}_m \to \boldsymbol{u} \quad \text{strongly in } L^2(0,T;\boldsymbol{L}^2(\Omega)). \tag{4.1.33}$$

We can now pass to the limit in the weak formulation and show that $\boldsymbol{u}$ satisfies (4.1.15).
Note that to pass to the limit in the non linear term, we can use a general method (see for instance [43, Lemma 1.3]). Hence we drop the proof at this step.

We move now to construct the pressure. We consider $\boldsymbol{v} \in X^2_T(\Omega)$ and we introduce the functional:

$$\begin{aligned}
H(\boldsymbol{v})(t) = &\int_0^T [\langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega - \gamma\langle \mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}\rangle - a\langle \boldsymbol{u}\,,\,\boldsymbol{v}\rangle - \langle |\boldsymbol{u}|^\alpha \boldsymbol{u}, \boldsymbol{v}\rangle]\,ds \\
&- \langle \boldsymbol{u}(t), \boldsymbol{v}\rangle + \langle \boldsymbol{u}_0, \boldsymbol{v}\rangle.
\end{aligned}$$

The functional H is continuous and linear for $\boldsymbol{v}$, which vanishes on $X^2_T(\Omega)$ thanks to variational formulation.
Following, there exists a function $\tilde{p}(t) \in L^2(\Omega)$ and $C > 0$ such as for all $\boldsymbol{v} \in X^2_T(\Omega)$:

$$H(\boldsymbol{v})(t) = \langle \mathrm{div}\,\boldsymbol{v}, \tilde{p}(t)\rangle.$$

We derivate H with respect to time and if we put $p(t) = \tilde{p}_t(t)$, the pressure p is well defined.
We can summarize the results of this section in the following Theorem :

Theorem 4.1.4. *Let $\boldsymbol{f}$ and $\boldsymbol{u}_0$ satisfying (4.1.13) and (4.1.14), then the problem* (4.1.15) *admits at least one solution such as :*

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{X}^2_T(\Omega)) \cap L^\infty(0,T;L^2_\sigma(\Omega))\ \ ,\ \ p \in L^2(0,T;L^2(\Omega))$$

4.1.3 Some regularity results

For the error analysis, we need ϵ-independent a priori estimates. We need also the following regularity results:

Proposition 4.1.5. *Assuming $\boldsymbol{u}_0 \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{X}_T^2(\Omega)$ and $\boldsymbol{f} \in C([0,T]; \boldsymbol{L}^2(\Omega))$. Then, there exists $T_1 \leq T$ such that the solutions of (4.1.12) satisify :*

$$\|\boldsymbol{u}(t)\|_2 + \|\boldsymbol{u}_t(t)\| + \|\nabla p(t)\| \leqslant C, \quad t \in [0,\, T_1]. \tag{4.1.34}$$

Moreover, for any $t_0 \in]0, T_1[$, if , $\boldsymbol{f}_t, \boldsymbol{f}_{tt} \in C([0,T]; \boldsymbol{L}^2(\Omega))$, we have :

$$\|\boldsymbol{u}_t(t)\|_1^2 + \|p_t(t)\|_1^2 + \int_{t_0}^t \Big(\|\boldsymbol{u}_{tt}(s)\|^2_{\boldsymbol{H}^2(\Omega)} + \|p_{tt}(s)\|^2_{\boldsymbol{H}^1(\Omega)}\Big) ds \leqslant C, \quad t \in [t_0,\, T_1]. \tag{4.1.35}$$

Proof. We take the inner product of the equation $(4.1.12)_1$ with $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}$. So consequently, because of $4 \leqslant 2\alpha + 2 \leqslant 6$, we reach :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{curl}\,\boldsymbol{u}\|^2 - \gamma\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\|^2 + a\|\mathbf{curl}\,\boldsymbol{u}\|^2 &\leqslant \frac{\gamma}{2}\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\|^2 + \frac{1}{\gamma}\|\boldsymbol{f}\|^2 + C\|\boldsymbol{u}\|^{2\alpha+2}_{2\alpha+2}.\\ &\leqslant \frac{\gamma}{2}\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\|^2 + \frac{1}{\gamma}\|\boldsymbol{f}\|^2 + C\|\nabla\boldsymbol{u}\|^6.\end{aligned}$$

Due to (4.1.7), we conculde that,

$$\frac{d}{dt}\|\mathbf{curl}\,\boldsymbol{u}\|^2 - \gamma\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\|^2 + 2a\|\mathbf{curl}\,\boldsymbol{u}\|^2 \leqslant C\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\|^6 + C. \tag{4.1.36}$$

Particularly, we consider the following differential inequality,

$$\frac{d}{dt}y(t) \leqslant C_2 y^3(t).$$

We take here $y(t) = \|\mathbf{curl}\,\boldsymbol{u}\|^2 + C_1$ and

$$y(0) = \|\mathbf{curl}\,\boldsymbol{u}(0)\|^2 + C_1.$$

With $C_1 = \frac{1}{\gamma}\sup\limits_{0\leq t\leq T}\|\boldsymbol{f}\|^2$.

By solving the last differential equation, we obtain a solution defined on $[0, T_1]$ such that $T_1 \leqslant \frac{1}{2C_2 y(0)^2} = \frac{1}{C_0}$ and $y(t)$ satisfies

$$y(t) \leqslant \sqrt{2}y(0). \quad t \in [0, T_1]$$

Then, for a given $t \in [0, T_1]$ with $T_1 = \min\{T_1, \frac{1}{C_0}\}$, we have

$$\|\mathbf{curl}\,\boldsymbol{u}(t)\|^2 \leq \sqrt{2}(\|\mathbf{curl}\,\boldsymbol{u}(0)\|^2 + C_1) := C_3, \tag{4.1.37}$$

Therefore, by (4.1.36) and (4.1.37), we have:

$$\sup_{0\leqslant t\leqslant T_1} \|\mathbf{curl}\,\boldsymbol{u}\|^2 + \gamma \int_0^{T_1} \|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\|^2\,dt + 2a\|\mathbf{curl}\,\boldsymbol{u}\|^2 \leqslant C. \quad t \in [0, T_1]. \tag{4.1.38}$$

By differentiating the equation $(4.1.12)_1$ with respect of time t, then we have

$$\boldsymbol{u}_{tt} - \gamma \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t + a\boldsymbol{u}_t + bF'(u)\boldsymbol{u}_t + \nabla p_t = f_t. \tag{4.1.39}$$

Multiplying the previous equation with $\boldsymbol{u}_t$, we obtain :

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t\|^2 + \gamma\|\mathbf{curl}\,\boldsymbol{u}_t\|^2 + a\|\boldsymbol{u}_t\|^2 = (\boldsymbol{f}_t, \boldsymbol{u}_t) - b(F'(\boldsymbol{u})\boldsymbol{u}_t, \boldsymbol{u}_t).$$

Because the term $(F'(\boldsymbol{u})\boldsymbol{u}_t, \boldsymbol{u}_t)$ is positive defined which infers that,

$$\frac{d}{dt}\|\boldsymbol{u}_t\|^2 + 2\gamma\|\mathbf{curl}\,\boldsymbol{u}_t\|^2 + a\|\boldsymbol{u}_t\|^2 \leqslant \frac{1}{a}\|\boldsymbol{f}_t\|^2. \tag{4.1.40}$$

By integrating the previous result, we reach :

$$\|\boldsymbol{u}_t(t)\|^2 + 2\gamma \int_0^T \|\mathbf{curl}\,\boldsymbol{u}_t(t)\|^2\,dt + a\int_0^T \|\boldsymbol{u}_t(t)\|^2\,dt \leqslant C. \quad t \in [0, T]. \tag{4.1.41}$$

Where $C = C'\displaystyle\int_0^T \|\boldsymbol{f}_t(t)\|^2\,dt$.
Now in order to obtain the regularity of the second derivate of the function $\boldsymbol{u}$, we take again the inner product of $(4.1.12)_1$ with $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}$, using the same previous arguments, it gives :

$$\gamma\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\|^2 \leqslant \frac{\gamma}{2}\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\|^2 + \frac{2}{\gamma}\|\boldsymbol{u}_t\|^2 + \frac{2a^2}{\gamma}\|\boldsymbol{u}\|^2 + \frac{2}{\gamma}\|\boldsymbol{u}\|_{2\alpha+2}^{2\alpha+2} + \frac{2}{\gamma}\|\boldsymbol{f}\|^2.$$

Based on the results (4.1.38), (4.1.41), (4.1.7) and Poincaré's inequality, then we obtain,

$$\begin{aligned}\gamma\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\|^2 &\leqslant \frac{4}{\gamma}\|\boldsymbol{u}_t\|^2 + \frac{4a^2}{\gamma}\|\boldsymbol{u}\|^2 + \frac{4}{\gamma}\|\nabla\boldsymbol{u}\|^6 + \frac{2}{\gamma}\|\boldsymbol{f}\|^2 \\ &\leqslant \frac{4a^2}{\gamma}\|\mathbf{curl}\,\boldsymbol{u}\|^2 + \frac{4}{\gamma}\|\mathbf{curl}\,\boldsymbol{u}\|^6 + C. \\ &\leqslant C.\end{aligned}$$

Which leads us to deduce :

$$\|\boldsymbol{u}(t)\|_2 \leqslant C. \quad \forall t \in [0, T_1] \tag{4.1.42}$$

Due to Agmon's inequality, we have :

$$\|\boldsymbol{u}\|_\infty \leqslant C, \quad \forall t \in [0, T_1]. \tag{4.1.43}$$

To conclude the proof, we need to check the pressure's regularity, so we derive from (4.1.12) the following statement,

$$\nabla p = \boldsymbol{f} - \boldsymbol{u}_t + \gamma\Delta\boldsymbol{u} - a\boldsymbol{u} - b|\boldsymbol{u}|^\alpha\boldsymbol{u}.$$

Using (4.1.42), (4.1.41) and (4.1.38), it infers to get

$$\begin{aligned}\|\nabla p\| &\leqslant \|\boldsymbol{f}\| + \|\boldsymbol{u}_t\| + \gamma\|\boldsymbol{u}\|_2 + a\|\boldsymbol{u}\| + b\|\boldsymbol{u}\|^{\alpha+1}\\ &\leqslant C.\end{aligned} \tag{4.1.44}$$

Which is the desired estimate.

We move now to prove the second estimate given in the Proposition 4.1.5, for this purpose we deal as follow. We multiply the equation (4.1.39) with $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t$, we obtain :

$$\begin{aligned}\frac{d}{dt}\|\mathbf{curl}\,\boldsymbol{u}_t\|^2 + \gamma\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t\|^2 \leqslant & \frac{3a^2}{\gamma}\|\mathbf{curl}\,\boldsymbol{u}_t\|^2 + \frac{3}{\gamma}\|\boldsymbol{f}_t\|^2 + \frac{\gamma}{3}\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t\|^2\\ &+ b(|F'(\boldsymbol{u})||\boldsymbol{u}_t|, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t),\end{aligned} \tag{4.1.45}$$

where $F'(u)$ can be handle by $|F'(\boldsymbol{u})| \leqslant (\alpha+1)|\boldsymbol{u}|^{\alpha}$.
Then, we obtain

$$\begin{aligned}((|\boldsymbol{u}|^{\alpha}\boldsymbol{u})_t, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t) &\leqslant (\alpha+1)\|\boldsymbol{u}\|_\infty^{\alpha}\|\boldsymbol{u}_t\|\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t\|.\\ &\leqslant \frac{\gamma}{6}\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t\|^2 + \frac{3}{2\gamma}(\alpha+1)\|\boldsymbol{u}\|_\infty^{2\alpha}\|\boldsymbol{u}_t\|^2.\end{aligned}$$

According to the result given in (4.1.43) and (4.1.41), we obtain

$$b((|\boldsymbol{u}|^{\alpha}\boldsymbol{u})_t, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t) \leqslant \frac{\gamma}{6}\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t\|^2 + C\|\mathbf{curl}\,\boldsymbol{u}_t\|^2. \tag{4.1.46}$$

By collecting the last relation (4.1.46) in the inequality (4.1.3), one can get

$$\frac{d}{dt}\|\mathbf{curl}\,\boldsymbol{u}_t\|^2 + \gamma\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t\|^2 \leqslant C\|\mathbf{curl}\,\boldsymbol{u}_t\|^2 + C_1.$$

Then, we apply Gronwall Lemma to reach,

$$\underset{0\leqslant t\leqslant T_0}{sup}\|\mathbf{curl}\,\boldsymbol{u}_t\|^2 + \gamma\int_0^T \|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t(t)\|^2\,dt \leqslant C, \quad \forall t\in[0,T]. \tag{4.1.47}$$

We derivate again the relation (4.1.39) with respect to time, so we have

$$u_{ttt} - \gamma\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_{tt} + a\boldsymbol{u}_{tt} + bF''(\boldsymbol{u})\boldsymbol{u}_t + bF'(u)u_{tt} + \nabla p_{tt} = f_{tt}. \tag{4.1.48}$$

Next, we take the inner product of the previous result with $\boldsymbol{u}_{tt}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_{tt}\|^2 + \gamma\|\mathbf{curl}\,\boldsymbol{u}_{tt}\|^2 =& - a\|\boldsymbol{u}_{tt}\|^2 - b(F''(\boldsymbol{u})\boldsymbol{u}_t, \boldsymbol{u}_{tt}) - b(F'(\boldsymbol{u})\boldsymbol{u}_{tt}, \boldsymbol{u}_{tt})\\ &- (\nabla p_{tt}, \boldsymbol{u}_{tt}) + (\boldsymbol{f}_{tt}, \boldsymbol{u}_{tt}).\end{aligned} \tag{4.1.49}$$

Using Green's Formula, it is obvious that

$$(\nabla p_{tt}, \boldsymbol{u}_{tt}) = -(p_{tt}, \mathrm{div}\boldsymbol{u}_{tt}) + (p_{tt}, \boldsymbol{u}_{tt}\cdot\boldsymbol{n}) = 0.$$

Furthermore, we observe that $(F'(\boldsymbol{u})\boldsymbol{u}_{tt}, \boldsymbol{u}_{tt})$ is positive definite, then we reach

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_{tt}\|^2 + \gamma\|\mathbf{curl}\,\boldsymbol{u}_{tt}\|^2 \leqslant \|\boldsymbol{f}_{tt}\|^2 + (a+\frac{1}{4})\|\boldsymbol{u}_{tt}\|^2 + b(F''(\boldsymbol{u})\boldsymbol{u}_t, |\boldsymbol{u}_{tt}|). \tag{4.1.50}$$

Based on (4.1.43) and the result (4.1.41), the non linear term can be controlled as follow,

$$\begin{aligned} b(F''(\boldsymbol{u})\boldsymbol{u}_t, |\boldsymbol{u}_{tt}|) &\leqslant \alpha(\alpha+1)\|\boldsymbol{u}\|_\infty^{2(\alpha+1)}\|\boldsymbol{u}_t\|\|\boldsymbol{u}_{tt}\| \\ &\leqslant \frac{1}{4}\|\boldsymbol{u}_{tt}\|^2 + C\|\boldsymbol{u}\|_\infty^{2(\alpha-1)}\|\boldsymbol{u}_t\|^2 \\ &\leqslant \frac{1}{4}\|\boldsymbol{u}_{tt}\|^2 + C. \end{aligned} \tag{4.1.51}$$

We collect the last relation (4.1.3) in the result (4.1.3), then we obtain

$$\frac{d}{dt}\|\boldsymbol{u}_{tt}\|^2 + \gamma\|\mathbf{curl}\,\boldsymbol{u}_{tt}\|^2 \leqslant C\|\boldsymbol{u}_{tt}\|^2 + C_1. \tag{4.1.52}$$

By applying Gronwall Lemma on the previous result, we reach that,

$$\|\boldsymbol{u}_{tt}\|^2 + \gamma\int_{t_0}^{T_1} \|\mathbf{curl}\,\boldsymbol{u}_{tt}(t)\|^2\,dt \leqslant C. \tag{4.1.53}$$

We move now to prove that $\|\boldsymbol{u}_t\|_2 \leqslant C$, we proceed as follow.
Taking the inner product of the result (4.1.39) with $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t$, we obtain :

$$\gamma\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t\|^2 \leqslant \frac{\gamma}{2}\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t\|^2 + \frac{2}{\gamma}\|\boldsymbol{u}_{tt}\|^2 + \frac{2a^2}{\gamma}\|\boldsymbol{u}_t\|^2 + C\|\boldsymbol{u}\|_\infty^{2\alpha}\|\boldsymbol{u}_t\|^2.$$

According to the results (4.1.53), (4.1.6) and (4.1.47), we have

$$\gamma\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_t\|^2 \leqslant C. \tag{4.1.54}$$

Finally to conclude the remainder estimates in (4.1.35), we use a similar proof as the previous arguments while we switch $\boldsymbol{u}_t$ by $\boldsymbol{u}_{tt}$. □

Remark 4.1.6. We claim to obtain the same results when we treat the convective case where the nonlinear term $\boldsymbol{u}.\nabla\boldsymbol{u}$ is replaced by $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u} + \frac{1}{2}\nabla|\boldsymbol{u}|^2$ in the first equation of the system and then we consider teh dynamic pressure $\widetilde{p} = p + \frac{1}{2}|\boldsymbol{u}|^2$.

4.2 Study of the perturbed system

Let us consider the following non stationary perturbed compressible DBF equations with $\epsilon \in]0, 1]$ is a parameter :

$$\begin{cases} \boldsymbol{u}_t^\epsilon - \gamma\Delta\boldsymbol{u}^\epsilon + a\boldsymbol{u}^\epsilon + |\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon + \nabla p^\epsilon = \boldsymbol{f}, & \text{in } \Omega\times[0,T] \\ \text{div }\boldsymbol{u}^\epsilon - \epsilon\Delta p_t^\epsilon = 0, & \text{in } \Omega\times[0,T] \end{cases} \tag{4.2.1}$$

and satisfy the boundary conditions and the initial data :

$$\begin{cases} \boldsymbol{u}^\epsilon \cdot \mathbf{n} = 0\,,\ \mathbf{curl}\ \boldsymbol{u}^\epsilon \times \mathbf{n} = \mathbf{0}, & \text{on}\, \Gamma \times [0,T] \\ \boldsymbol{u}^\epsilon(x,0) = \boldsymbol{u}_0\ \ , p^\epsilon(x,0) = p_0 \in L^2(\Omega) \end{cases} \tag{4.2.2}$$

We need first to write the variational formulation of problem (4.2.1). Then, we deal as in the previous chapter, so we derive some a priori estimates and we reach the existence of solutions by means of Galerkin-Faedo approximation.

Taking the inner product of $(4.2.1)_1$ with a test function $\boldsymbol{v} \in \mathcal{D}(\Omega)$ and $(4.2.1)_2$ with $q \in \mathcal{D}(\Omega)$, integrating by parts over Ω and based on the continuity argument, for any $\boldsymbol{v} \in \boldsymbol{X}^2_T(\Omega)$ and $q \in L^2(\Omega)$, we obtain the following variational formulation :

For $\epsilon > 0$ fixed, the datum $\boldsymbol{f}$, the intitial function $\boldsymbol{u}_0$ given satisfying (4.1.13),(4.1.14) as in Chapter 3 and

$$p_0 \in L^2(\Omega), \tag{4.2.3}$$

Find $\boldsymbol{u}^\epsilon \in L^2(0,T;\boldsymbol{X}^2_T(\Omega))$ and $p^\epsilon \in L^2(0,T;L^2(\Omega))$ such as, for almost all $t \in [0,T]$:

$$\begin{cases} \dfrac{d}{dt}\langle \boldsymbol{u}^\epsilon, \boldsymbol{v}\rangle + \gamma\langle \mathbf{curl}\,\boldsymbol{u}^\epsilon, \mathbf{curl}\,\boldsymbol{v}\rangle + \gamma\langle \operatorname{div}\boldsymbol{u}^\epsilon, \operatorname{div}\boldsymbol{v}\rangle + a\langle \boldsymbol{u}^\epsilon, \boldsymbol{v}\rangle + b(|\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon, \boldsymbol{v}\rangle - \langle p^\epsilon, \operatorname{div}\boldsymbol{v}\rangle \\ = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega, \quad \forall \boldsymbol{v} \in \boldsymbol{X}^2_T(\Omega). \\ \epsilon\dfrac{d}{dt}\langle \nabla p^\epsilon, \nabla q\rangle + \langle \operatorname{div}\boldsymbol{u}^\epsilon, q\rangle_\Omega = 0, \quad \forall q \in L^2(\Omega). \\ \boldsymbol{u}^\epsilon(0) = \boldsymbol{u}_0, \quad p^\epsilon(0) = p_0. \end{cases} \tag{4.2.4}$$

4.2.1 Existence of weak solutions

The main result of this section is the following theorem, (see [43]):

Theorem 4.2.1. *Let ϵ arbitrary in $(0,1]$. Given (4.1.13), (4.1.14) and (4.2.3), the problem* (4.2.4) *admits at least one solution. Furthermore,*

$$\boldsymbol{u}^\epsilon \in L^2(0,T;\boldsymbol{X}^2_T(\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega)\,), \quad p^\epsilon \in L^2(0,T;L^2(\Omega)).$$

Now, in order to prove the Theorem 4.2.1, we use again Faedo-Galerkin method to get a similar a priori estimates with a slight difference that we keep the pressure in the approximate weak formulation.

Proof. ***Faedo-Galerkin approximation:***
we first consider $(\boldsymbol{\psi}_i)$ an orthonormal basis of $\boldsymbol{X}^2_T(\Omega)$ and (r_i) an orthonormal basis of $L^2(\Omega)$. Let $\boldsymbol{V}_m =< \boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_m >$ and $W_m =< r_1, \dots, r_m >$. We denote by:

$$\boldsymbol{u}_m^\epsilon(t) = \sum_{i=1}^m g_{im}(t)\boldsymbol{\psi}_i \qquad \text{and} \qquad p_m^\epsilon(t) = \sum_{j=1}^m \xi_{jm}(t) r_j,$$

the approximated solutions which satisfy :

$$\begin{cases} \dfrac{d}{dt}\langle \boldsymbol{u}_m^\epsilon, \boldsymbol{\psi}_k\rangle + \gamma\langle \mathbf{curl}\, \boldsymbol{u}_m^\epsilon, \mathbf{curl}\, \boldsymbol{\psi}_k\rangle + \gamma\langle \operatorname{div} \boldsymbol{u}_m^\epsilon, \operatorname{div} \boldsymbol{\psi}_k\rangle + a\langle \boldsymbol{u}_m^\epsilon\, \boldsymbol{\psi}_k\rangle + +b\langle |\boldsymbol{u}_m^\epsilon|^\alpha \boldsymbol{u}_m^\epsilon, \boldsymbol{\psi}_k) \\ -\langle p_m^\epsilon, \operatorname{div}\boldsymbol{\psi}_k\rangle = \langle \boldsymbol{f}, \boldsymbol{\psi}_k\rangle_\Omega, \quad k = 1, \dots, m. \\ \epsilon \dfrac{d}{dt}\langle \nabla p_m^\epsilon, \nabla r_l\rangle + \langle \operatorname{div} \boldsymbol{u}_m^\epsilon, r_l\rangle = 0, \quad l = 1, \dots, m. \\ \boldsymbol{u}_m^\epsilon(0) = \boldsymbol{u}_{0m}\ ,\ p_m^\epsilon(0) = p_{0m}, \end{cases} \tag{4.2.5}$$

where $\boldsymbol{u}_{0m}$ is nothing but the orthogonal projection of $\boldsymbol{u}(0)$ on the space $\boldsymbol{V}_m$ and p_{0m} the orthogonal projection of $p(0)$ on the space W_m.

For the functions $g_{1m}, \dots, g_{mm}$ and $\xi_{1m}, \dots, \xi_{mm}$, the system (4.2.5) is a non-linear differential system.
Based on theory of ordinary differential equations, the problem 4.2.5 admits at least one solution on $[0, t_m]$ with $t_m \leqslant T$ and next we will check the fact that $t_m = T$. □

A priori Estimates: Below, in the usual way, we will make some a priori estimates and then pass to the limit. This procedure is standard in the case of the DBF problem with Dirichlet boundary conditions while we give here all details for a complete analysis in the case of the Navier-type boundary conditions. We have the following Theorem :

Theorem 4.2.2. *ϵ arbitrary in $(0, 1]$. Given (4.1.13), (4.1.14) and (4.2.3), the solutions $\boldsymbol{u}_m$ and p_m satisfy the following a priori estimates:*

$$\sup_{t\in[0,T]} \left(\|\boldsymbol{u}_m^\epsilon(t)\|^2 + \epsilon\, \|\nabla p_m^\epsilon(t)\|^2 \right) \leqslant C_2, \tag{4.2.6}$$

$$\gamma \int_0^T \|\mathbf{curl}\, \boldsymbol{u}_m^\epsilon(t)\|^2\ dt + \gamma \int_0^T \|\operatorname{div} \boldsymbol{u}_m^\epsilon(t)\|^2\ dt + a \int_0^T \|\boldsymbol{u}_m^\epsilon(t)\|^2\, dt + b \int_0^T \|\boldsymbol{u}_m^\epsilon(t)\|_{\alpha+2}^{\alpha+2}\, dt \leqslant C_2, \tag{4.2.7}$$

where

$$C_2 = \|\boldsymbol{u}_0^\epsilon\|^2 + \|\nabla p_0^\epsilon\|^2 + \int_0^T \|\boldsymbol{f}(t)\|^2_{\mathbf{H}_0(\operatorname{div},\Omega)'}\ dt. \tag{4.2.8}$$

Proof. We multiply $(4.2.5)_1$ by $g_{km}(t)$ $(k = 1, \dots, m)$, and $(4.2.5)_2$ by $\xi_{lm}(t)$ $(l = 1, \dots, m)$ and then adding these resulting equalities, we obtain:

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\, \|\boldsymbol{u}_m^\epsilon\|^2 + \gamma\, \|\mathbf{curl}\, \boldsymbol{u}_m^\epsilon\|^2 + \gamma\, \|\operatorname{div} \boldsymbol{u}_m^\epsilon\|^2 + a\|\boldsymbol{u}_m^\epsilon\|^2 + b\|\boldsymbol{u}_m^\epsilon\|_{\alpha+2}^{\alpha+2} - \langle p_m^\epsilon, \operatorname{div}\boldsymbol{u}_m^\epsilon\rangle \\ = \langle \boldsymbol{f}, \boldsymbol{u}_m^\epsilon\rangle_\Omega \\ \dfrac{\epsilon}{2}\dfrac{d}{dt}\, \|\nabla p_m^\epsilon\|^2 + \langle \operatorname{div} \boldsymbol{u}_m^\epsilon, p_m^\epsilon\rangle = 0. \end{cases}$$

Now, by adding the both last equations, we have

$$\frac{1}{2}\frac{d}{dt}\,\|\boldsymbol{u}_m^\epsilon\|^2+\gamma\,\|\mathbf{curl}\,\boldsymbol{u}_m^\epsilon\|^2+\gamma\,\|\mathrm{div}\,\boldsymbol{u}_m^\epsilon\|^2+a\|\boldsymbol{u}_m^\epsilon\|^2+b\|\boldsymbol{u}_m^\epsilon\|_{\alpha+2}^{\alpha+2}+\frac{\epsilon}{2}\frac{d}{dt}\,\|\nabla p_m^\epsilon\|^2=\langle \boldsymbol{f}\,,\boldsymbol{u}_m^\epsilon\rangle_\Omega$$

Due to Young's inequality :

$$\begin{aligned}\frac{d}{dt}\,\|\boldsymbol{u}_m^\epsilon\|^2+2\gamma\,\|\mathbf{curl}\,\boldsymbol{u}_m^\epsilon\|^2+2\gamma\,\|\mathrm{div}\,\boldsymbol{u}_m^\epsilon\|^2+a\|\boldsymbol{u}_m^\epsilon\|^2+2b\|\boldsymbol{u}_m^\epsilon\|_{\alpha+2}^{\alpha+2}+\epsilon\frac{d}{dt}\,\|\nabla p_m^\epsilon\|^2\\ \leqslant C\,\|\boldsymbol{f}\|^2_{(\mathbf{H}_0^2(\mathrm{div}\,,\Omega))'}\end{aligned}\tag{4.2.9}$$

By integrating the laste equation from 0 to t with $t\leq t_m$,

$$\sup_{t\in[0,t_m]}\Big(\,\|\boldsymbol{u}_m^\epsilon(t)\|^2+\epsilon\,\|\nabla p_m^\epsilon(t)\|^2\,\Big)\leqslant C\int_0^T\,\|\boldsymbol{f}(t)\|^2_{(\mathbf{H}_0^2(\mathrm{div}\,,\Omega))'}\,dt+\|\boldsymbol{u}_{0m}^\epsilon\|^2+\|\nabla p_{0m}^\epsilon\|^2\tag{4.2.10}$$

Then the estimate (4.2.10) holds with $t_m=T$, which establishes the estimate (4.2.6).

Note, it follows from (4.2.10) that the sequences $(\boldsymbol{u}_m^\epsilon)$ and (p_m^ϵ) still bounded in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$

By integrating again the relation (4.2.9), one can obtain :

$$\begin{aligned}2\gamma\int_0^T\,\|\mathbf{curl}\,\boldsymbol{u}_m^\epsilon(t)\|^2\,dt+a\int_0^T\,\|\boldsymbol{u}_m^\epsilon(t)\|^2\,dt+2\gamma\int_0^T\,\|\mathrm{div}\,\boldsymbol{u}_m^\epsilon(t)\|^2\,dt+2b\int_0^T\,\|\boldsymbol{u}_m^\epsilon(t)\|_{\alpha+2}^{\alpha+2}\,dt\\ \leqslant C\int_0^T\,\|\boldsymbol{f}(t)\|^2_{(\mathbf{H}_0^2(\mathrm{div}\,,\Omega))'}\,dt+\|\boldsymbol{u}_m^\epsilon(0)\|^2\\ +\,\|\nabla p_m^\epsilon(0)\|^2\,,\end{aligned}$$

which is nothing but the estimate (4.2.7), due to the relations (4.1.6)-(4.1.7), we conclude that the sequence $\boldsymbol{u}_m^\epsilon$ remains in a bounded set of $L^2(0,T;X_T^2(\Omega))$. □

Estimates on the fractional derivative with respect to time:

In order to pass to the limit in the nonlinear term in (4.2.5), we need a strong convergence. We proceed as in Section 4.1. Indeed, we are going to bound a fractional derivative in time of the functions $\boldsymbol{u}_m^\epsilon$ by applying the Fourier transform. Let us first extend the both functions $\boldsymbol{u}_m^\epsilon$ and p_m^ϵ as follow :

$$\widetilde{\boldsymbol{u}}_m^\epsilon=\begin{cases}\boldsymbol{u}_m^\epsilon(t), & t\in[0,T]\\ 0 \quad \text{outside,}\end{cases}$$

$$\widetilde{p}_m^\epsilon=\begin{cases}p_m^\epsilon(t), & t\in[0,T]\\ 0 \quad \text{outside.}\end{cases}$$

Note $\widehat{\boldsymbol{u}}_m^\epsilon$ the Fourier transform given by:

$$\widehat{\boldsymbol{u}}_m^\epsilon(\tau)=\int_{\mathbb{R}}e^{-2i\pi t\tau}\widetilde{\boldsymbol{u}}_m^\epsilon(t)\,dt.$$

Thanks to the discontinuities on 0 and T, one can obtain the following system,

$$\begin{cases} \dfrac{d}{dt}\langle \widetilde{\boldsymbol{u}}_m^\epsilon, \boldsymbol{\psi}_k\rangle + \gamma\langle \mathbf{curl}\, \widetilde{\boldsymbol{u}}_m^\epsilon, \mathbf{curl}\, \boldsymbol{\psi}_k\rangle + \gamma\langle \operatorname{div} \widetilde{\boldsymbol{u}}_m^\epsilon, \operatorname{div} \boldsymbol{\psi}_k\rangle + a\langle \boldsymbol{u}_m^\epsilon, \boldsymbol{\psi}_k\rangle + b\langle |\widetilde{\boldsymbol{u}}_m^\epsilon|^\alpha \widetilde{\boldsymbol{u}}_m^\epsilon, \boldsymbol{\psi}_k\rangle \\ + \langle \nabla \widetilde{p}_m^\epsilon, \boldsymbol{\psi}_k\rangle = \langle \widetilde{\boldsymbol{f}}, \boldsymbol{\psi}_k\rangle + \langle \boldsymbol{u}_{0m}, \boldsymbol{\psi}_k\rangle \delta_{(0)} - \langle \boldsymbol{u}_m^\epsilon(T), \boldsymbol{\psi}_k\rangle \delta_{(T)}, \\ \epsilon \dfrac{d}{dt}\langle \nabla \widetilde{p}_m^\epsilon, \nabla r_l\rangle + \langle \operatorname{div} \widetilde{\boldsymbol{u}}_m^\epsilon, r_l\rangle = \epsilon\langle \nabla p_{0m}, \nabla r_l\rangle \delta_{(0)} - \epsilon\langle \nabla p_m^\epsilon(T), \nabla r_k\rangle \delta_{(T)}, \end{cases}$$

where $\delta_{(T)}$ and $\delta_{(0)}$ defined respectively the dirac distributions at T and 0.
Now, we consider the Fourier transform of the derivate function u_m^ϵ with respect of time as follow :

$$\widehat{D_t^\gamma \boldsymbol{u}_m^\epsilon(t)} = (2i\pi\tau)^\gamma \widehat{\boldsymbol{u}}_m^\epsilon(\tau), \quad \text{for some fixed real } \gamma.$$

It yields to obtain :

$$\begin{cases} 2i\pi\tau\langle \widehat{\boldsymbol{u}}_m^\epsilon, \boldsymbol{\psi}_k\rangle + \gamma\langle \mathbf{curl}\, \widehat{\boldsymbol{u}}_m^\epsilon, \mathbf{curl}\, \boldsymbol{\psi}_k\rangle + \gamma\langle \operatorname{div} \widehat{\boldsymbol{u}}_m^\epsilon, \operatorname{div} \boldsymbol{\psi}_k\rangle + a\langle \widehat{\boldsymbol{u}}_m^\epsilon\,,\, \boldsymbol{\psi}_k\rangle \\ + b\langle |\widehat{\boldsymbol{u}}_m^\epsilon|^\alpha \widehat{\boldsymbol{u}}_m^\epsilon\,,\, \boldsymbol{\psi}_k\rangle + \langle \nabla \widehat{p}_m^\epsilon, \boldsymbol{\psi}_k\rangle = \langle \widehat{f}, \boldsymbol{\psi}_k\rangle + \langle \boldsymbol{u}_{0m}, \boldsymbol{\psi}_k\rangle - \langle \boldsymbol{u}_m^\epsilon(T), \boldsymbol{\psi}_k\rangle e^{-2i\pi\tau T} \\ \epsilon 2i\pi\tau\langle \nabla \widehat{p}_m^\epsilon, \nabla r_l\rangle + \langle \operatorname{div} \widehat{\boldsymbol{u}}_m^\epsilon, r_l\rangle = \epsilon\langle \nabla p_{0m}, \nabla r_l\rangle - \epsilon\langle \nabla p_m^\epsilon(T), \nabla r_l\rangle e^{-2i\pi\tau T}. \end{cases} \tag{4.2.11}$$

In order to prove that $\boldsymbol{u}_m^\epsilon \in M_K^\gamma(\mathbb{R}, \boldsymbol{X}_T^2(\Omega), \mathbf{L}^2(\Omega))$, we need to apply the Compactness Theorem 4.1.2, so for this end we take $\boldsymbol{X}_0 = \boldsymbol{X}_T^2(\Omega)$ and $\boldsymbol{X} = \boldsymbol{X}_1 := \boldsymbol{L}^2(\Omega)$, based on (4.2.7), it follows immediately that

$$\boldsymbol{u}_m^\epsilon \in L^2(0, T; \boldsymbol{X}_T^2(\Omega)),$$

Then, it remains only to check that

$$D_t^\gamma \boldsymbol{u}_m^\epsilon \in L^2(0, T; \boldsymbol{L}^2(\Omega)).$$

After extending g_{im} and ξ_{jm} in $\mathbb{R}$ with $\widetilde{g}_{im}$ and $\widetilde{\xi}_{jm}$, we take the inner product of the first equation in (4.2.11) with $\widehat{g}_{im}(t)$ $(i = 1, \ldots, m)$ and the second equation with $\widehat{\xi}_{jm}(t)$ $(j = 1, \ldots, m)$, consequently by summing the resulting equations, one can reach,

$$\begin{aligned} &2i\pi\tau\Big(\|\widehat{\boldsymbol{u}}_m^\epsilon\|^2 + \epsilon\, \|\nabla \widehat{p}_m^\epsilon\|^2 \Big) + \gamma\, \|\mathbf{curl}\, \widehat{\boldsymbol{u}}_m^\epsilon\|^2 + \gamma\, \|\operatorname{div} \widehat{\boldsymbol{u}}_m^\epsilon\|^2 + a\|\widehat{\boldsymbol{u}}_m^\epsilon\|^2 + b\|\widehat{\boldsymbol{u}}_m^\epsilon\|_{\alpha+2}^{\alpha+2} \\ =&\langle \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{u}}_m^\epsilon\rangle + \langle \boldsymbol{u}_{0m}, \widehat{\boldsymbol{u}}_m^\epsilon(\tau)\rangle + \epsilon\langle \nabla p_{0m}, \nabla \widehat{p}_m^\epsilon(\tau)\rangle \\ &- \Big(\langle \boldsymbol{u}_m^\epsilon(T), \widehat{\boldsymbol{u}}_m^\epsilon(\tau)\rangle + \epsilon\langle \nabla p_m^\epsilon(T), \nabla \widehat{p}_m^\epsilon(\tau)\rangle\Big) e^{-2i\pi\tau T}. \end{aligned}$$

According to Cauchy-Schwarz's inequality, we have :

$$\begin{aligned} 2i\pi\tau\, \|\widehat{\boldsymbol{u}}_m^\epsilon\|^2 \leqslant{}& \left\|\widehat{\boldsymbol{f}}\right\| \|\widehat{\boldsymbol{u}}_m^\epsilon\| + \|\boldsymbol{u}_{0m}\|\, \|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\| + \epsilon\, \|\nabla p_{0m}\|\, \|\nabla \widehat{p}_m^\epsilon(\tau)\| + 2i\pi\tau\, \|\boldsymbol{u}_m^\epsilon(T)\|\, \|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\| \\ &+ 2i\pi\tau\epsilon\, \|\nabla p_m^\epsilon(T)\|\, \|\nabla \widehat{p}_m^\epsilon(\tau)\|\,. \end{aligned}$$

Due to relation (4.2.6), Minkowski and Young's inequalities, we obtain :

$$2\pi\tau\,\|\widehat{\boldsymbol{u}}_m^\epsilon\|^2 \leqslant \frac{1}{2}\Big(\Big\|\widehat{\boldsymbol{f}}\Big\|^2 + \|\widehat{\boldsymbol{u}}_m^\epsilon\|^2\Big) + 2\sqrt{C_2}\big(\|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\|^2 + \epsilon\,\|\nabla\widehat{p}_m^\epsilon(\tau)\|^2\big). \tag{4.2.12}$$

In fact, if we have $\gamma \in]0, \frac{1}{4}[$, then we obtain

$$|\tau|^{2\gamma} \leqslant (2\gamma+1)\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}} \tag{4.2.13}$$

Hence, we reach, with (4.2.13) :

$$\begin{aligned}\int_{\mathbb{R}} |\tau|^{2\gamma}\,\|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\|_{\mathbf{L}^2}^2\,d\tau &\leqslant \frac{2\gamma+1}{2}\int_{\mathbb{R}}\frac{\|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\|_{\mathbf{L}^2}^2}{1+|\tau|^{1-2\gamma}}\,d\tau + \frac{2\gamma+1}{2}\int_{\mathbb{R}}\frac{|\tau|\,\|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\|^2}{1+|\tau|^{1-2\gamma}}\,d\tau \\ &:= I_1 + I_2.\end{aligned} \tag{4.2.14}$$

So on the one hand, we have :

$$\begin{aligned}I_1 &\leqslant \frac{2\gamma+1}{2}\int_{\mathbb{R}}\|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\|_{\mathbf{L}^2}^2\,d\tau \\ &= \frac{2\gamma+1}{2}\int_0^T \|\boldsymbol{u}_m^\epsilon(t)\|^2\,dt < \infty\end{aligned}$$

Based on relation (4.2.12) and Cauchy-Schwarz's inequality :

$$\begin{aligned}I_2 &\leqslant \frac{2\gamma+1}{8\pi}\int_{\mathbb{R}}\Big\|\widehat{\boldsymbol{f}}(\tau)\Big\|^2\,d\tau + \frac{(2\gamma+1)\sqrt{C_2}}{2\pi}\int_{\mathbb{R}}\frac{\|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\| + \epsilon\,\|\nabla\widehat{p}_m^\epsilon(\tau)\|}{1+|\tau|^{1-2\gamma}}d\tau \\ &\leqslant C\int_0^T \|\boldsymbol{f}(\tau)\|_{\mathbf{H}_0(\mathrm{div}\,,\Omega)'}^2 \\ &+ C\sqrt{\int_{\mathbb{R}}\frac{d\tau}{(1+|\tau|^{1-2\gamma})^2})}\left(\sqrt{\int_{\mathbb{R}}(\|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\|^2 d\tau} + \epsilon\sqrt{\int_{\mathbb{R}}\|\nabla\widehat{p}_m^\epsilon(\tau)\|^2\,d\tau}\right) < \infty.\end{aligned}$$

Consequently, we recapitulate our results, so we note that

$$\int_{\mathbb{R}} |\tau|^{2\gamma}\|\widehat{\boldsymbol{u}}_m^\epsilon\|^2 < \infty. \tag{4.2.15}$$

We pass now to the limit, so we proceed as follow.
Let us take $\epsilon > 0$ a fixed positive number and we are only concerned with the passage to the limit as $m \to \infty$.

According to (4.2.6) and (4.2.7), then we can derive the existence of a subsquence (still denoted $(\boldsymbol{u}_m)_m$) such that :

$$\boldsymbol{u}_m^\epsilon \rightharpoonup \boldsymbol{u}^\epsilon \quad \text{weak in} \quad L^2(0,T;\boldsymbol{X}_T^2(\Omega)), \tag{4.2.16}$$

$$\boldsymbol{u}_m^\epsilon \rightharpoonup \boldsymbol{u}^\epsilon \quad \text{weak star in} \quad L^\infty(0,T;\boldsymbol{L}^2(\Omega)), \tag{4.2.17}$$

$$|\boldsymbol{u}_m^\epsilon|^\alpha\boldsymbol{u}_m^\epsilon \rightharpoonup \boldsymbol{w}^\epsilon \quad \text{weak star in} \quad L^{\frac{\alpha+2}{\alpha+1}}(0,T;\boldsymbol{L}^{\frac{\alpha+2}{\alpha+1}}(\Omega)), \tag{4.2.18}$$

$$\nabla p_m^\epsilon \rightharpoonup \nabla p^\epsilon \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)). \tag{4.2.19}$$

Moreover, using (4.2.15) and (4.2.16), we can apply the compactness Theorem 4.1.2, to deduce the existence of a subsequence of $(\boldsymbol{u}_m^\epsilon)_m$ (still denoted by $(\boldsymbol{u}_m^\epsilon)_m$) such that

$$\boldsymbol{u}_m^\epsilon \to \boldsymbol{u}^\epsilon \qquad \text{strong in} \quad L^2(0,T;\boldsymbol{L}^2(\Omega)) \tag{4.2.20}$$

Now, we just need to investigate that $(\boldsymbol{u}^\epsilon, p^\epsilon)$ is a solution for the weak formulation (4.2.4).

In order to get this end, we proceed as follow :
Taking the inner product (4.2.5) with $\phi(t) \in C^\infty(0,T)$ such that $\phi(T) = 0$, by integrating from 0 to T, we obtain :

$$\begin{cases} -\displaystyle\int_0^T \langle \boldsymbol{u}_m^\epsilon, \boldsymbol{\psi}_k\rangle \phi'(t)\,dt + \gamma \int_0^T \langle \mathbf{curl}\,\boldsymbol{u}_m^\epsilon, \phi(t)\mathbf{curl}\,\boldsymbol{\psi}_k\rangle\,dt + \gamma \int_0^T \langle \mathrm{div}\,\boldsymbol{u}_m^\epsilon, \phi(t)\mathrm{div}\,\boldsymbol{\psi}_k\rangle\,dt \\ +\, a \displaystyle\int_0^T \langle \boldsymbol{u}_m^\epsilon, \phi(t)\boldsymbol{\psi}_k\rangle\,dt + b\int_0^T \langle |\boldsymbol{u}_m^\epsilon|^\alpha \boldsymbol{u}_m^\epsilon, \phi(t)\boldsymbol{\psi}_k\rangle + \int_0^T \langle \nabla p_m^\epsilon, \psi(t)\boldsymbol{\psi}_k\rangle\,dt \\ = \displaystyle\int_0^T \langle \boldsymbol{f}, \phi(t)\boldsymbol{\psi}_k\rangle\,dt + \langle \boldsymbol{u}_{0m}, \boldsymbol{\psi}_k\rangle\phi(0) \\ \epsilon \displaystyle\int_0^T \langle \nabla p_m^\epsilon, \phi'(t)\nabla r_l\rangle + \int_0^T \langle \mathrm{div}\,\boldsymbol{u}_m^\epsilon, \phi(t) r_l\rangle\,dt = \epsilon\langle \nabla p_{0m}, \nabla r_l\rangle\phi(0). \end{cases} \tag{4.2.21}$$

It is obvious to pass to the limit $m \to \infty$ in the linear terms of (4.2.21). It suffies to use the previous convergence results. We return now to check the convergence of the nonlinear term. We proceed as in [43, Lemma 1.3], so based on (4.2.20), it follows to obtain :

$$|\boldsymbol{u}_m^\epsilon|^\alpha \boldsymbol{u}_m^\epsilon \longrightarrow |\boldsymbol{u}|^\alpha \boldsymbol{u} \quad \text{weak in } L^{\frac{\alpha+2}{\alpha+1}}(0,T;\boldsymbol{L}^{\frac{\alpha+2}{\alpha+1}}(\Omega)). \tag{4.2.22}$$

We now use the result (4.2.18) and the fact of the uniqueness of the limits, it infers that :

$$\boldsymbol{w} = |\boldsymbol{u}|^\alpha \boldsymbol{u}. \tag{4.2.23}$$

Consequently, we are able to pass to the limit of (4.2.21) when $m \longrightarrow \infty$, then we obtain for any $\boldsymbol{\psi} \in \boldsymbol{X}_T^2(\Omega)$ and $r \in L^2(\Omega)$.

$$\begin{cases} -\displaystyle\int_0^T \langle \boldsymbol{u}^\epsilon, \boldsymbol{\psi}\rangle \phi'(t)\,dt + \gamma \int_0^T \langle \mathbf{curl}\,\boldsymbol{u}^\epsilon, \phi(t)\mathbf{curl}\,\boldsymbol{\psi}\rangle\,dt + \gamma \int_0^T \langle \mathrm{div}\,\boldsymbol{u}^\epsilon, \phi(t)\mathrm{div}\,\boldsymbol{\psi}\rangle\,dt \\ +\, a \displaystyle\int_0^T \langle \boldsymbol{u}^\epsilon, \phi(t)\boldsymbol{\psi}\rangle\,dt + b\int_0^T \langle |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon, \phi(t)\boldsymbol{\psi}\rangle + \int_0^T \langle \nabla p^\epsilon, \psi(t)\boldsymbol{\psi}\rangle\,dt \\ = \displaystyle\int_0^T \langle \boldsymbol{f}, \phi(t)\boldsymbol{\psi}\rangle\,dt + \langle \boldsymbol{u}_0, \boldsymbol{\psi}\rangle\phi(0) \\ \epsilon \displaystyle\int_0^T \langle \nabla p^\epsilon, \phi'(t)\nabla r\rangle + \int_0^T \langle \mathrm{div}\,\boldsymbol{u}^\epsilon, \phi(t) r\rangle\,dt = \epsilon\langle \nabla p_0, \nabla r\rangle\phi(0), \end{cases} \tag{4.2.24}$$

which is nothing but the weak formulation (4.2.4) without the initial conditions. Hence it remains only to investigate that the solutions $(\boldsymbol{u}_m^\epsilon, p_m^\epsilon)$ satisfy the intitial conditions $(\boldsymbol{u}^\epsilon(0), p^\epsilon(0))$, for this end, we take a test function $\phi \in C^\infty(0,T)$ with $\phi(T)=0$ and then we exactly use the same arguments in [61] to obtain :

$$\begin{cases} \langle \boldsymbol{u}_0 - \boldsymbol{u}^\epsilon(0), \boldsymbol{\psi}\rangle \phi(0) = 0 \\ \langle p_0 - p^\epsilon(0), r\rangle \phi(0) = 0, \end{cases}$$

with $\phi(0)=1$, then we deduce from the last result that $\boldsymbol{u}_m^\varepsilon(0) = \boldsymbol{u}^\epsilon(0)$ and $p_m^\epsilon(0) = p^\varepsilon(0)$. Then, we summarize our results in the following Theroem,

Theorem 4.2.3. *Let* $\boldsymbol{f}$, $\boldsymbol{u}_0$ *and* p_0 *given functions satisfying (4.1.13), (4.1.14) and (4.2.3). The perturbed system (4.2.1) admits at least one solution such as :*

$$\boldsymbol{u}^\epsilon \in L^2(0,T;\boldsymbol{X}_T^2(\Omega)) \cap L^\infty(0,T;\boldsymbol{L}_\sigma^2(\Omega)\,), \quad \nabla p^\epsilon \in L^2(0,T;L^2(\Omega)).$$

4.2.2 Convergence of solutions of the perturbed system to the initial incompressible DBF system

We check now that the solutions of the evolution compressible (DBF) equations (4.2.1) converge to the solutions of the non stationary incompressible system (DBF). Because we consider here that limit $\epsilon \to 0$, then we take $\epsilon \in]0,1]$. It is useful to establish some a priori estimates for $\boldsymbol{u}^\epsilon$ and p^ϵ, independent of ϵ.

Thanks to estimates (4.2.6), (4.2.7) and (4.1.24), the lower semi-continuity of the norm, we reach the following results for any $\epsilon \in]0,1]$, independent of ϵ :

$$\begin{cases} \|\boldsymbol{u}^\epsilon\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leqslant \liminf\limits_{m\to\infty} \|\boldsymbol{u}_m^\epsilon\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} < \infty. \\ \|\boldsymbol{u}^\epsilon\|_{L^2(0,T;\boldsymbol{X}_T^2(\Omega))} \leqslant \liminf\limits_{m\to\infty} \|\boldsymbol{u}_m^\epsilon\|_{L^2(0,T;\boldsymbol{X}_T^2(\Omega))} < \infty. \\ \sqrt{\epsilon}\,\|\nabla p^\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \liminf\limits_{m\to\infty} \|\nabla p_m^\epsilon\|_{L^\infty(0,T;L^2(\Omega))} < \infty. \end{cases} \tag{4.2.25}$$

As previously, by taking $\theta \in]0, \frac{1}{4}[$ and $\epsilon \in]0,1]$, we have :

$$\int_{\mathbb{R}} |\tau|^{2\theta} \left\|\widehat{\boldsymbol{u}^\epsilon}(\tau)\right\|^2 \, d\tau \leqslant \liminf_{m\to\infty} \int_{\mathbb{R}} |\tau|^{2\theta} \left\|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\right\|^2 \, d\tau < C. \tag{4.2.26}$$

We want to prove the following Theorem :

Theorem 4.2.4. *Under the same assumptions in Theorem 4.2.2, there exists a sequence* $(\epsilon_n)_n \in]0,1]$ *such as* $\epsilon_n \to 0^+$ *as* $n \to \infty$*, and the solutions* $\{\boldsymbol{u}^{\epsilon_n}, p^{\epsilon_n}\}$ *are convergent to* $\{\boldsymbol{u}, p\}$ *where* $\{\boldsymbol{u}, p\}$ *are solutions of initial Navier-Stokes problem with damping* $(\mathcal{DBF})$ *with :*

$$\begin{aligned} &\boldsymbol{u}^\epsilon \rightharpoonup \boldsymbol{u} \quad in \quad L^2(0,T;\boldsymbol{X}_T^2(\Omega))\, weak \\ &\boldsymbol{u}^\epsilon \rightharpoonup \boldsymbol{u} \quad in \quad L^\infty(0,T;\boldsymbol{L}^2(\Omega))\, weak\, star \\ &\boldsymbol{u}^\epsilon \to \boldsymbol{u} \quad in \quad L^2(0,T;\boldsymbol{L}^2(\Omega)) \;\, strong \\ &\nabla p^\epsilon \rightharpoonup \nabla p \quad in \quad L^\infty(0,T;L^2(\Omega))\, weak\, star. \end{aligned} \tag{4.2.27}$$

Proof. According to (4.2.25)-(4.2.26), we have :

$$\begin{aligned}
&\boldsymbol{u}^{\epsilon_n} \rightharpoonup \boldsymbol{u}^* \; in \;\; L^2(0,T;\boldsymbol{X}^2_T(\Omega))\, weak. \\
&\boldsymbol{u}^{\epsilon_n} \rightharpoonup \boldsymbol{u}^* \; in \;\; L^\infty(0,T;\mathbf{L}^2(\Omega))\, weak\, star. \\
&\boldsymbol{u}^{\epsilon_n} \rightarrow \boldsymbol{u}^* \; in \;\; L^2(0,T;\mathbf{L}^2(\Omega)) \;\, strong. \\
&\sqrt{\epsilon_n}\nabla p^{\epsilon_n} \rightharpoonup \chi \;\; in\, L^\infty(0,T;L^2(\Omega))\, weak\, star.
\end{aligned} \tag{4.2.28}$$

We have, in the distributions sense :

$$\sqrt{\epsilon_n}\langle \nabla p_t^{\epsilon_n}, \nabla r\rangle \;\; \rightarrow \;\; \langle \nabla \chi_t ,\, \nabla r\rangle, \quad \forall r \in L^2(\Omega),$$

Then, we have :

$$\epsilon\langle \nabla p_t^{\epsilon_n}, \nabla r\rangle = \sqrt{\epsilon_n}\sqrt{\epsilon_n}\langle \nabla p_t^{\epsilon_n}, \nabla r\rangle \rightarrow 0.$$

This implies that $\langle \operatorname{div} \boldsymbol{u}^* \, , \, r\rangle = 0$ for all $r \in L^2(\Omega)$ and shows that :

$$\operatorname{div} \boldsymbol{u}^* = 0.$$

We now show that $\boldsymbol{u}^*$ verifies the variational formulation (4.1.15). Taking $\boldsymbol{\psi} \in \boldsymbol{X}^2_T(\Omega)$, multiplying in the weak formulation of perturbed problem with $\phi \in C_0^\infty(0,T)$ and integrating from 0 to T, it gives :

$$\begin{aligned}
&-\int_0^T \langle \boldsymbol{u}^{\epsilon_n}, \boldsymbol{\psi}\rangle \phi' \, dt + \gamma \int_0^T \langle \mathbf{curl}\, \boldsymbol{u}^{\epsilon_n}, \phi \mathbf{curl}\, \boldsymbol{\psi}\rangle \, dt \\
&+ \gamma \int_0^T \langle \operatorname{div} \boldsymbol{u}^{\epsilon_n}, \phi \operatorname{div} \boldsymbol{\psi}\rangle \, dt + a \int_0^T \langle \boldsymbol{u}^{\epsilon_n} \, , \, \phi \boldsymbol{\psi}\rangle \, dt + b \int_0^T \langle |\boldsymbol{u}^{\epsilon_n}|^\alpha \boldsymbol{u}^{\epsilon_n} \, , \, \phi \boldsymbol{\psi}\rangle \, dt = \int_0^T \langle \boldsymbol{f}, \phi \boldsymbol{\psi}\rangle \, dt.
\end{aligned}$$

We can easily pass to the limit by using the following convergence for the nonlinear term

$$\int_0^T \langle |\boldsymbol{u}^{\epsilon_n}|^\alpha \boldsymbol{u}^{\epsilon_n} \, , \, \phi \boldsymbol{\psi}\rangle \; dt \rightarrow \int_0^T \langle |\boldsymbol{u}^*|^\alpha \boldsymbol{u}^*, \phi \boldsymbol{\psi}\rangle \, dt = \int_0^T \langle |\boldsymbol{u}^*|^\alpha \boldsymbol{u}^*, \phi \boldsymbol{\psi}\rangle \, dt.$$

Consequently, we reach using furthermore the fact that $\operatorname{div} \boldsymbol{u}^* = 0$:

$$\begin{aligned}
&-\int_0^T \langle \boldsymbol{u}^*, \boldsymbol{\psi}\rangle \phi' \, dt + \int_0^T \langle \mathbf{curl}\, \boldsymbol{u}^*, \phi \mathbf{curl}\, \boldsymbol{\psi}\rangle \, dt + a \int_0^T \langle \boldsymbol{u}^{\epsilon_n} \, , \, \phi \boldsymbol{\psi}\rangle \, dt + b \int_0^T \langle |\boldsymbol{u}^{\epsilon_n}|^\alpha \boldsymbol{u}^{\epsilon_n} \, , \, \phi \boldsymbol{\psi}\rangle \, dt \\
&= \int_0^T \langle \boldsymbol{f}, \phi \boldsymbol{w}\rangle \, dt.
\end{aligned}$$

Observe that the last relation is the same as (4.1.15) in the distribution sense which established the proof. □

4.3 Error analysis

In this section, we derive the error estimates for the pseudocompressibility method using different techniques than those used in [58].

The main result of this section is the following Theorem:

Theorem 4.3.1. *We assume* $\boldsymbol{f} \in C^2([0,T]; \boldsymbol{L}^2(\Omega))$ *and* $\boldsymbol{u}_0 \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{X}_T^2(\Omega)$. *We have :*

$$\int_{t_0}^{t} \|\boldsymbol{u}(s) - \boldsymbol{u}^\epsilon(s)\|^2 \, ds + \epsilon^{\frac{1}{2}} \|\boldsymbol{u}(t) - \boldsymbol{u}^\epsilon(t)\|^2 + \epsilon\Big(\|\boldsymbol{u}(t) - \boldsymbol{u}^\epsilon(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|\nabla(p(t) - p^\epsilon(t))\|^2 \Big) \leqslant C\epsilon^2.$$

4.3.1 ϵ-independent a priori estimates

Let us first define $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}^\epsilon$ and $q = p - p^\epsilon$. We deal here to derive some ϵ-independent a priori estimates.

By subtracting the perturbed (DBF) from the initial (DBF), we have :

$$\begin{cases} \boldsymbol{e}_t - \nu\Delta\boldsymbol{e} + a\boldsymbol{e} + b(|\boldsymbol{u}|^\alpha\boldsymbol{u} - |\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon) + \nabla q = 0 \quad \text{in } \Omega. \\ \operatorname{div} \boldsymbol{e} - \epsilon\Delta q_t = -\epsilon\Delta p_t \quad \text{in}\, \Omega. \end{cases} \tag{4.3.1}$$

and

$$\begin{cases} \dfrac{\partial p_t^\epsilon}{\partial \boldsymbol{n}} = 0, \qquad\qquad \text{on}\, \Gamma \times [t_0, T]. \\ \boldsymbol{e}(t_0) = q(t_0) = 0 \end{cases} \tag{4.3.2}$$

We summarize all the results of this subsection in the following Proposition :

Proposition 4.3.2. *Under the same assumption of Theorem 4.3.1, there exists* $T_0 \in (t_0,\, T_0]$ *such that :*

1. $\|\boldsymbol{u}^\epsilon(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|p^\epsilon(t)\|^2_{H^1(\Omega)} \leqslant C, \quad \forall t \in (t_0,\, T_0]$
2. $\int_{t_0}^{t} \|\boldsymbol{u}_t^\epsilon(s)\|^2_{\boldsymbol{H}^2(\Omega)} \, ds + \|\boldsymbol{u}_t^\epsilon(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|p_t^\epsilon(t)\|^2_{H^1(\Omega)} \leqslant C, \quad \forall t \in (t_0,\, T_0]$
3. $\|\boldsymbol{e}_t(t)\|^2 \leqslant C\epsilon, \quad \forall t \in (t_0,\, T_0],$

where T_0 *is defined in* (4.3.8).

Proof. We take the inner product of $(4.3.1)_1$ with $\boldsymbol{e}$ and $(4.3.1)_2$ with q, then we sum the both results to obtain, with the monotoncity property given in (4.1.8) and (4.1.7) :

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}\|^2 + \nu\,\|\operatorname{div}\boldsymbol{e}\|^2 + \nu\,\|\mathbf{curl}\,\boldsymbol{e}\|^2 + a\|\boldsymbol{e}(t)\|^2 + \frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q\|^2 \\ &= \epsilon(\nabla p_t, \nabla q) - b\langle |\boldsymbol{u}(t)|^\alpha\boldsymbol{u}(t) - |\boldsymbol{u}^\epsilon(t)|^\alpha\boldsymbol{u}^\epsilon\,,\, \boldsymbol{e}(t)\rangle. \\ &\leqslant \frac{\epsilon}{2}\|\nabla p_t\|^2 + \frac{\epsilon}{2}\|\nabla q\|^2. \end{aligned}$$

By applying Gronwall Lemma and using the result (4.1.34), it follows :

$$\begin{aligned} \|\boldsymbol{e}(t)\|^2 + \nu \int_{t_0}^t \Big(\|\mathbf{curl}\, \boldsymbol{e}(s)\|^2 + \|\mathrm{div}\, \boldsymbol{e}(s)\|^2 \Big)\, ds + a \int_{t_0}^t \|e(s)\|^2\, ds + \epsilon \|\nabla q(t)\|^2 \\ \leqslant C\epsilon \int_0^t \|\nabla p_t\|^2 \leqslant C\epsilon. \end{aligned} \tag{4.3.3}$$

Next, in order to estimate $\|\boldsymbol{e}\|^2_{\boldsymbol{H}^1(\Omega)}$, we take the inner product of (4.3.1) with $-\Delta \boldsymbol{e}$, using Young's inequality, it gives :

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\mathrm{div}\, \boldsymbol{e}\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{curl}\, \boldsymbol{e}\|^2 + \nu \|\Delta \boldsymbol{e}\|^2 &= (\nabla q, \Delta \boldsymbol{e}) + a\langle e, \Delta e\rangle + b(|\boldsymbol{u}|^\alpha \boldsymbol{u} - |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon, \Delta \boldsymbol{e}). \\ &\leqslant \frac{\nu}{2}\|\Delta \boldsymbol{e}\|^2 + \frac{1}{2\nu}\|\nabla q\|^2 + \frac{a^2}{2\nu}\|\boldsymbol{e}\|^2 \\ &+ b^2 \| |\boldsymbol{u}|^\alpha \boldsymbol{u} - |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon \|^2 . \end{aligned} \tag{4.3.4}$$

Based on the previous result (4.3.3), we have :

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\mathrm{div}\, \boldsymbol{e}\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{curl}\, \boldsymbol{e}\|^2 + \frac{\nu}{2} \|\Delta \boldsymbol{e}\|^2 \leqslant b^2 \| |\boldsymbol{u}|^\alpha \boldsymbol{u} - |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon \|^2 + C. \tag{4.3.5}$$

Due to relations (4.1.9) and (4.1.7), we bound the nonlinear term, so we obtain :

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|\mathrm{div}\, \boldsymbol{e}\|^2 + \nu \frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{curl}\, \boldsymbol{e}\|^2 + \nu \|\Delta \boldsymbol{e}\|^2 &\leqslant b^2 \|\nabla e\|^2 + C. \\ &\leqslant b^2(\|\mathrm{div}\, \boldsymbol{e}\|^2 + \|\mathbf{curl}\, \boldsymbol{e}\|^2) + C. \end{aligned} \tag{4.3.6}$$

Consequently,

$$\frac{d}{dt}\Big(C + \|\mathrm{div}\, \boldsymbol{e}\|^2 + \|\mathbf{curl}\, \boldsymbol{e}\|^2 \Big) \leqslant C_1(\|\mathrm{div}\, \boldsymbol{e}\|^2 + \|\mathbf{curl}\, \boldsymbol{e}\|^2) + C. \tag{4.3.7}$$

By taking $y(t) = C + \|\mathrm{div}\, \boldsymbol{e}\|^2 + \|\mathbf{curl}\, \boldsymbol{e}\|^2$, we use the Gronwall Lemma, with $t \in [t_0, T_0]$ where

$$T_0 = \min\{T, \frac{1}{4C_2C_1^2}\}, \tag{4.3.8}$$

Then, we can conclude that :

$$\|\mathrm{div}\, \boldsymbol{e}\|^2 + \|\mathbf{curl}\, \boldsymbol{e}\|^2 + \gamma \int_0^t \|\Delta \boldsymbol{e}(s)\|^2\, ds \leqslant C. \tag{4.3.9}$$

Now, we take the time derivative of perturbed system, so we have :

$$\begin{cases} \boldsymbol{e}_{tt} - \nu \Delta \boldsymbol{e}_t + a\boldsymbol{e} + b|\boldsymbol{u}_t|(\alpha + 1|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u} + |\boldsymbol{u}|^\alpha) + \nabla q_t = 0 \\ \mathrm{div}\, \boldsymbol{e}_t - \epsilon \Delta q_{tt} = -\epsilon \Delta p_{tt} \end{cases} \tag{4.3.10}$$

and we obtain :

$$\begin{cases} \boldsymbol{e}_t(t_0) = 0 \\ q_t(t_0) = p_t(t_0) \end{cases} \tag{4.3.11}$$

We multiply $(4.3.10)_1$ with $\boldsymbol{e}_t$ and $(4.3.10)_2$ by q_t, it follows to obtain :

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\boldsymbol{e}_t\|^2+\nu\,\|\mathbf{curl}\,\boldsymbol{e}_t\|^2+\nu\,\|\mathrm{div}\,\boldsymbol{e}_t\|^2+\frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla q_t\|^2\\
&=\epsilon(\nabla p_{tt},\nabla q_t)-a\|\boldsymbol{e}_t\|^2-b\langle|\boldsymbol{u}_t|(\alpha+1|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}+|\boldsymbol{u}|^{\alpha}),\boldsymbol{e}_t\rangle.\\
&\leqslant\frac{\epsilon}{2}\|\nabla p_{tt}\|^2+\frac{\epsilon}{2}\|\nabla q_t\|^2+a\|\boldsymbol{e}_t\|^2+b\langle|\boldsymbol{u}_t|(|\alpha+1|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}+|\boldsymbol{u}|^{\alpha})|,|\boldsymbol{e}_t|\rangle.
\end{aligned} \tag{4.3.12}$$

Let us handle the non linear term with $F'(\boldsymbol{u})=\alpha|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}+|\boldsymbol{u}|^{\alpha}$ and
$F'(\boldsymbol{u}^{\epsilon})=\alpha|\boldsymbol{u}^{\epsilon}|^{\alpha-1}\boldsymbol{u}^{\epsilon}+|\boldsymbol{u}^{\epsilon}|^{\alpha}$ as follow,

$$\begin{aligned}
\langle|\boldsymbol{u}_t|(\alpha+1|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}+|\boldsymbol{u}|^{\alpha}),\boldsymbol{e}_t\rangle&=\langle(|\boldsymbol{u}|^{\alpha}\boldsymbol{u})_t-(|\boldsymbol{u}^{\epsilon}|^{\alpha}\boldsymbol{u}^{\epsilon})_t,\boldsymbol{e}_t\rangle\\
&\leqslant\langle|F'(\boldsymbol{u})\boldsymbol{u}_t-F'(\boldsymbol{u}^{\epsilon})\boldsymbol{u}^{\epsilon}_t|,|\boldsymbol{e}_t|\rangle\\
&\leqslant\langle|F'(\boldsymbol{u})\boldsymbol{u}_t-F'(\boldsymbol{u}^{\epsilon})(\boldsymbol{u}_t+\boldsymbol{e}_t)|,|\boldsymbol{e}_t|\rangle.
\end{aligned}$$

Consequently,

$$\langle|\boldsymbol{u}_t|(\alpha+1|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}+|\boldsymbol{u}|^{\alpha}),\boldsymbol{e}_t\rangle\leqslant\langle|(F'(\boldsymbol{u})-F'(\boldsymbol{u}^{\epsilon}))\boldsymbol{u}_t|,|\boldsymbol{e}_t|\rangle+\langle|F'(\boldsymbol{u}^{\epsilon})||\boldsymbol{e}_t|,|\boldsymbol{e}_t|\rangle=T_1+T_2. \tag{4.3.13}$$

Now, we estimate each terms T_1,T_2, because of $|F'(\boldsymbol{u})|\leqslant(\alpha+1)|\boldsymbol{u}|^{\alpha}$, Then by Hölder and Young's inequalities, we reach :

$$\begin{aligned}
T_1&\leqslant 2(\alpha+1)\langle|\boldsymbol{u}|^{\alpha}+|\boldsymbol{u}^{\epsilon}|^{\alpha})|\boldsymbol{u}^{\alpha}|,|\boldsymbol{e}_t|\rangle.\\
&\leqslant C(\alpha+1)(\|\boldsymbol{u}\|_{3\alpha}^{\alpha}+\|\boldsymbol{u}^{\epsilon}\|_{3\alpha}^{\alpha})\|\boldsymbol{u}_t\|_6\|\boldsymbol{e}_t\|\leqslant(\alpha+1)^2(\|\boldsymbol{u}\|_{3\alpha}^{\alpha}+\|\boldsymbol{u}^{\epsilon}\|_{3\alpha}^{\alpha})^2\|\boldsymbol{u}_t\|_6^2+C\|\boldsymbol{e}_t\|^2.
\end{aligned} \tag{4.3.14}$$

Since $1\leqslant\alpha\leqslant 2$, by applying Sobolev's inequality and the estimates (4.1.34), then we have :

$$T_1\leqslant C+C\|\boldsymbol{e}_t\|^2. \tag{4.3.15}$$

Similarly, by using the result (4.1.7), we obtain

$$\begin{aligned}
T_2\leqslant C(\alpha+1)\|\boldsymbol{u}^{\epsilon}\|_{3\alpha}^{\alpha}\|\boldsymbol{e}_t\|_6\|\boldsymbol{e}_t\|&\leqslant\frac{\gamma}{2}\|\nabla\boldsymbol{e}_t\|^2+C\|\boldsymbol{u}^{\epsilon}\|_{3\alpha}^{2\alpha}\|\boldsymbol{e}_t\|^2.\\
&\leqslant\frac{\gamma}{2}\|\mathbf{curl}\,\boldsymbol{e}_t\|^2+\frac{\gamma}{2}\|\mathrm{div}\boldsymbol{e}_t\|^2+C\|\boldsymbol{u}^{\epsilon}\|_{3\alpha}^{2\alpha}\|\boldsymbol{e}_t\|^2.\\
&\leqslant\frac{\gamma}{2}\|\mathbf{curl}\,\boldsymbol{e}_t\|^2+\frac{\gamma}{2}\|\mathrm{div}\boldsymbol{e}_t\|^2+C\|\boldsymbol{e}_t\|^2.
\end{aligned} \tag{4.3.16}$$

By collecting (4.3.12), the both of results (4.3.15) and (4.3.16), which give :

$$\frac{\mathrm{d}}{\mathrm{dt}}\|\boldsymbol{e}_t\|^2+\nu\,\|\mathbf{curl}\,\boldsymbol{e}_t\|^2+\nu\,\|\mathrm{div}\,\boldsymbol{e}_t\|^2+a\|\boldsymbol{e}_t\|^2+\epsilon\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla q_t\|^2\leqslant\epsilon\|\nabla p_{tt}\|^2+C\|\boldsymbol{e}_t\|^2 \tag{4.3.17}$$

Using the estimate (4.1.35) and Gronwall Lemma, we have :

$$\|\boldsymbol{e}_t\|^2+\nu\Big(\int_{t_0}^{t}\|\mathbf{curl}\,\boldsymbol{e}_t(s)\|^2+\|\mathrm{div}\,\boldsymbol{e}_t(s)\|^2\,ds\Big)+\epsilon\,\|\nabla q_t\|^2\leqslant C\epsilon. \tag{4.3.18}$$

The estimate on the pressure derivative in the Proposition 4.3.2 follows from the previous result (4.3.1) and (4.1.35).

Next, we mutiply the first equation in (4.3.10) with $-\Delta \boldsymbol{e}$, using again the results (4.1.9) and (4.1.7), we get

$$\begin{aligned}\nu\,\|\Delta \boldsymbol{e}\|^2 &= (\boldsymbol{e}_t, \Delta \boldsymbol{e}) + a(\boldsymbol{e}, \Delta \boldsymbol{e}) + b(|\boldsymbol{u}|^\alpha \boldsymbol{u} - |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon, \Delta \boldsymbol{e}) + (\nabla q, \Delta \boldsymbol{e}).\\ &\leqslant \frac{\nu}{2}\,\|\Delta \boldsymbol{e}\|^2 + \frac{4}{\nu}\,\|\boldsymbol{e}_t\|^2 + \frac{4}{\nu}\,\|\nabla q\| + \frac{4a^2}{\nu}\|\boldsymbol{e}\|^2 + C\Big(\|\mathrm{div}\,\boldsymbol{e}\|^2 + \|\mathbf{curl}\,\boldsymbol{e}\|^2\Big).\end{aligned}$$

Based on the estimates (4.3.9), (4.3.18) and (4.3.3), we conclude that :

$$\|\Delta \boldsymbol{e}\|^2 \leqslant C. \tag{4.3.19}$$

In order to achieve the proof of the Proposition 4.3.2, we should check the following statement

$$\|\boldsymbol{e}_t\|^2_{\boldsymbol{H}^1(\Omega)} + \int_{t_0}^{t} \|\boldsymbol{e}_t(s)\|^2_{\boldsymbol{H}^2(\Omega)}\, ds \leqslant C\ ,\ \forall t \in [t_0, T].$$

So for this purpose, we take the inner product of $(4.3.1)_1$ with $-\Delta \boldsymbol{e}_t$, taking into account the result (4.1.34) and we proceed similarly as in (4.3.13) to bound the non linear terms, then we have :

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\,\|\nabla \boldsymbol{e}_t\|^2 + \nu\,\|\Delta \boldsymbol{e}_t\|^2 &= -a\|\nabla \boldsymbol{e}(t)\|^2 + b\langle |\boldsymbol{u}_t|(\alpha + 1|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u} + |\boldsymbol{u}|^\alpha), \Delta \boldsymbol{e}_t\rangle + \langle \nabla q_t, \Delta \boldsymbol{e}_t\rangle\\ &\leqslant \frac{\gamma}{2}\,\|\Delta \boldsymbol{e}_t\|^2 + a\|\nabla \boldsymbol{e}(t)\|^2 + C.\end{aligned} \tag{4.3.20}$$

Hence, it is obvious to obtain :

$$\|\boldsymbol{e}_t\|^2_{\boldsymbol{H}^1(\Omega)} + \gamma\int_{t_0}^{t} \|\boldsymbol{e}_t(s)\|^2_{\boldsymbol{H}^2(\Omega)}\, ds \leqslant C\ ,\ \forall t \in [t_0, T_0].$$

□

4.3.2 Error estimates for a linearly perturbed system

In this subsection, the error analysis associated to the linear perturbed problem was discussed here, so let us first consider the linear perturbed problem:

$$\begin{cases} \boldsymbol{v}_t^\epsilon - \nu\Delta \boldsymbol{v}^\epsilon + a\boldsymbol{v}^\epsilon + \nabla r^\epsilon = \boldsymbol{f} - b|\boldsymbol{u}|^\alpha \boldsymbol{u}. \\ \mathrm{div}\,\boldsymbol{v}^\epsilon - \epsilon\Delta r_t^\epsilon = 0 \end{cases} \tag{4.3.21}$$

with the boundary conditions and the initial data

$$\begin{cases} \boldsymbol{v}^\epsilon\cdot\boldsymbol{n} = 0\,,\ \mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} = 0\,,\ \dfrac{\partial r_t^\epsilon}{\partial \boldsymbol{n}} = 0\ \text{ on }\ \Gamma \\ \boldsymbol{v}^\epsilon(0) = \boldsymbol{u}(0)\ ,\ r^\epsilon(0) = p(0) \end{cases} \tag{4.3.22}$$

Such that $\boldsymbol{u}$ is nothing but the solution of the initial problem and $(\boldsymbol{v}^\epsilon, r^\epsilon)$ defines the solution for the linear perturbed problem.
We denote $\boldsymbol{\xi} = \boldsymbol{u} - \boldsymbol{v}^\epsilon$ and $\psi = p - r^\epsilon$, By substracting this problem from initial system, we have :

$$\begin{cases} \boldsymbol{\xi}_t - \nu\Delta\boldsymbol{\xi} + a\boldsymbol{\xi} + \nabla\psi = 0 \\ \operatorname{div}\boldsymbol{\xi} - \epsilon\Delta\psi_t = -\epsilon\Delta p_t \\ \boldsymbol{\xi}(0) = \psi(0) = 0 \end{cases} \tag{4.3.23}$$

Due to the Proposition 4.3.2, it gives immediately that,

$$\|\boldsymbol{v}^\epsilon\|^2_{\boldsymbol{H}^2(\Omega)} + \|r^\epsilon(t)\|^2_{H^1(\Omega)} + \|r^\epsilon_t\|^2_{H^1(\Omega)} \leqslant C. \tag{4.3.24}$$

Moreover, we have

$$\|\boldsymbol{\xi}_t(t)\|^2 \leqslant C\epsilon. \tag{4.3.25}$$

Now, we should estblish the following Lemma,

Lemma 4.3.3. *Under the same assumptions of Theorem 4.3.1. We have :*

$$\int_0^t \|\boldsymbol{\xi}(s)\|^2\, ds + \epsilon^{\frac{1}{2}} \|\boldsymbol{\xi}(t)\|^2 + \epsilon(\|\boldsymbol{\xi}(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|\nabla\psi(t)\|^2) \leqslant C\epsilon^2. \tag{4.3.26}$$

Proof. Here, we need to employ a parabolic duality argument, given $t \in [t_0, T_0]$.
let $(\boldsymbol{w}, q)$ solution of the dual problem :

$$\begin{cases} \boldsymbol{w}_s + \nu\Delta\boldsymbol{w} - a\boldsymbol{w} + \nabla q = \boldsymbol{\xi}(s) \text{ for } s \in [t_0, t] \\ \operatorname{div}\boldsymbol{w} = 0 \end{cases} \tag{4.3.27}$$

and

$$\begin{cases} \boldsymbol{w}\cdot\mathbf{n} = 0\ , \mathbf{curl}\,\boldsymbol{w} \times \mathbf{n} = 0 \text{ on } \Gamma \times [t_0, t] \\ \boldsymbol{w}(t) = 0. \end{cases} \tag{4.3.28}$$

We need first to prove the following result,

$$\int_0^t (\|\Delta\boldsymbol{w}\|^2 + \|\nabla q\|^2) ds \leqslant C \int_0^t \|\boldsymbol{\xi}(s)\|^2\, ds. \tag{4.3.29}$$

We multiply $(4.3.27)_1$ with $\Delta\boldsymbol{w} = -\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w}$ (since $\operatorname{div} w = 0$), using Young's inequality, then it yields to obtain :

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \|\mathbf{curl}\,\boldsymbol{w}(s)\|^2 + \frac{\nu}{2}\|\Delta\boldsymbol{w}\|^2 + a\|\mathbf{curl}\,\boldsymbol{w}(s)\|^2 \leqslant \frac{1}{2\nu}\|\boldsymbol{\xi}\|^2.$$

Next, by integrating from t_0 to t, we derive :

$$\begin{aligned} \|\mathbf{curl}\,\boldsymbol{w}(t_0)\|^2 + \nu\int_{t_0}^t \|\Delta\boldsymbol{w}(s)\|^2\, ds + 2a\int_{t_0}^t \|\mathbf{curl}\quad \boldsymbol{w}(s)\|^2\, ds &\leqslant C\int_{t_0}^t \|\boldsymbol{\xi}(s)\|^2\, ds. \\ &\leqslant C\epsilon. \end{aligned} \tag{4.3.30}$$

Now, we take the inner product of $(4.3.27)_1$ with $\boldsymbol{w}_s$ to obtain :

$$\|\boldsymbol{w}_s\|^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{curl}\,\boldsymbol{w}\|^2 + \frac{a}{2}\frac{d}{dt}\|\mathbf{curl}\,\boldsymbol{w}\|^2 \leqslant \frac{1}{2}\|\boldsymbol{\xi}\|^2 + \frac{1}{2}\|\boldsymbol{w}_s\|^2 .$$

Then, integrating from t_0 to t, we have the following result :

$$\int_{t_0}^{t} \|\boldsymbol{w}_s(s)\|^2\,ds \leqslant C\int_{t_0}^{t}\|\boldsymbol{\xi}(s)\|^2\,ds \leqslant C\epsilon. \tag{4.3.31}$$

We mutiply $(4.3.27)_1$ with ∇q, because of $\mathrm{div}\boldsymbol{w} = 0$, according to (4.3.30) and (4.3.31), we reach the end of (4.3.29) :

$$\int_{t_0}^{t}\|\nabla q\|^2 \leqslant C\int_{t_0}^{t}\|\boldsymbol{\xi}(s)\|^2\,ds.$$

We move now to take the inner product of $(4.3.27)_1$ with $\boldsymbol{\xi}(s)$, we have :

$$\|\boldsymbol{\xi}(s)\|^2 = \langle \boldsymbol{w}_s, \boldsymbol{\xi}(s)\rangle + \nu\langle\Delta\boldsymbol{w}, \boldsymbol{\xi}(s)\rangle - a\langle\boldsymbol{\xi}, \boldsymbol{w}\rangle + \langle\nabla q, \boldsymbol{\xi}(s)\rangle.$$

Where,

$$\begin{aligned}-\langle \mathrm{div}\boldsymbol{\xi}, q\rangle &= -\epsilon\langle\Delta\psi_t, q\rangle.\\ &= \epsilon\langle\Delta r_t^\epsilon, q\rangle.\\ &= -\epsilon\langle\nabla r_t^\epsilon, \nabla q\rangle.\end{aligned}$$

Then, taking into account the relations on $\boldsymbol{\xi}(s)$ given in (4.3.23) and the fact that $\mathrm{div}\,\boldsymbol{w} = 0$, we obtain :

$$\begin{aligned}\|\boldsymbol{\xi}(s)\|^2 &= \frac{\mathrm{d}}{\mathrm{d}s}\langle\boldsymbol{\xi}, \boldsymbol{w}\rangle - \langle\boldsymbol{\xi}_s, \boldsymbol{w}\rangle + \nu\langle\Delta\boldsymbol{\xi}, \boldsymbol{w}\rangle - a\langle\boldsymbol{w}, \boldsymbol{\xi}\rangle - \epsilon\langle\nabla q, \nabla r_s^\epsilon\rangle.\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\langle\boldsymbol{\xi}, \boldsymbol{w}\rangle + \langle -\boldsymbol{\xi}_s + \nu\Delta\boldsymbol{\xi} - a\boldsymbol{\xi}, \boldsymbol{w}\rangle - \epsilon\langle\nabla q, \nabla r_s^\epsilon\rangle.\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\langle\boldsymbol{\xi}, \boldsymbol{w}\rangle + \langle\nabla\psi, \boldsymbol{w}\rangle - \epsilon\langle\nabla q, \nabla r_s^\epsilon\rangle.\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\langle\boldsymbol{\xi}, \boldsymbol{w}\rangle - \epsilon\langle\nabla q, \nabla r_s^\epsilon\rangle.\end{aligned}$$

We integrate from t_0 to t with δ sufficiently small, so we have :

$$\begin{aligned}\int_{t_0}^{t}\|\boldsymbol{\xi}(s)\|^2\,ds &\leqslant \epsilon\int_{t_0}^{t}\|\nabla q(s)\|\,\|\nabla r_s^\epsilon\|\,ds.\\ &\leqslant \delta\int_{t_0}^{t}\|\nabla q(s)\|^2\,ds + \epsilon^2 C(\delta)\int_{t_0}^{t}\|\nabla r_s^\epsilon(s)\|^2\,ds.\end{aligned}$$

Due to (4.3.24) and (4.3.29),

$$\int_{t_0}^{t}\|\boldsymbol{\xi}(s)\|^2\,ds \leqslant C\epsilon^2\ ,\ \forall t\in[t_0, T] \tag{4.3.32}$$

Let us consider, for given $t \in [t_0, T_0]$, $(\boldsymbol{w}, q)$ solution of the next dual problem :

$$\begin{cases} \boldsymbol{w}_s + \nu\Delta\boldsymbol{w} + \nabla q = \boldsymbol{\xi}_s(s) \text{ for } s \in [t_0, t] \\ \operatorname{div} \boldsymbol{w} = 0 \end{cases} \tag{4.3.33}$$

and satisfy :

$$\begin{cases} \boldsymbol{w} \cdot \mathbf{n} = 0 \ , \ \mathbf{curl}\, \boldsymbol{w} \times \mathbf{n} = 0 \text{ on } \Gamma \times [t_0, t] \\ \boldsymbol{w}(t) = 0 \end{cases} \tag{4.3.34}$$

Similarly to (4.3.29), we have :

$$\int_{t_0}^{t} (\|\Delta\boldsymbol{w}(s)\|^2 + \|\nabla q(s)\|^2) ds \leqslant C \int_{t_0}^{t} \|\boldsymbol{\xi}_s(s)\|^2 \, ds. \tag{4.3.35}$$

We take the time derivative of the second equation in system $(4.3.23)_2$, hence :

$$\operatorname{div} \boldsymbol{\xi}_t - \epsilon\Delta\psi_{tt} = -\epsilon\Delta p_{tt}.$$

Now, mulitplying the first equation of $(4.3.23)_1$ with $\boldsymbol{\xi}_t$ and using the previous equation, we obtain:

$$\begin{aligned} \|\boldsymbol{\xi}_t\|^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{dt}} \|\operatorname{div} \boldsymbol{\xi}\|^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{dt}} \|\mathbf{curl}\, \boldsymbol{\xi}\|^2 + \frac{a}{2}\frac{d}{dt}\|\boldsymbol{\xi}\|^2 &= (\psi, \operatorname{div} \boldsymbol{\xi}_t) \\ &= \epsilon(\nabla\psi, \nabla p_{tt}) - \epsilon(\nabla\psi, \nabla\psi_{tt}) \\ &= \epsilon(\nabla\psi, \nabla p_{tt}) - \epsilon\frac{\mathrm{d}}{\mathrm{dt}}(\nabla\psi, \nabla\psi_t) + \epsilon \|\nabla\psi_t\|^2 . \end{aligned}$$

Now, for given $\forall t \in [t_0, T_0]$, we integrate this equation from t_0 to t and, because of $\psi(t_0) = 0$, we have

$$\begin{aligned} &\int_{t_0}^{t} \|\boldsymbol{\xi}_t(s)\|^2 \, ds + \nu \|\operatorname{div} \boldsymbol{\xi}(t)\|^2 + \nu \|\mathbf{curl}\, \boldsymbol{\xi}(t)\|^2 + a\|\boldsymbol{\xi}(t)\|^2 \\ &\leqslant C\epsilon \int_{t_0}^{t} (\|\nabla p_{tt}(s)\|^2 + \|\nabla\psi(s)\|^2) ds + C\epsilon \|\nabla\psi_t(t)\| \, \|\nabla\psi(t)\| \\ &+ C\epsilon \int_{t_0}^{t} \|\nabla\psi_t(s)\|^2 \, ds. \end{aligned}$$

Using (4.1.35),(4.1.7) and (4.3.24), we obtain in particular :

$$\nu \|\boldsymbol{\xi}(t)\|^2_{\boldsymbol{H}^1(\Omega)} \leqslant C\nu(\|\operatorname{div} \boldsymbol{\xi}(t)\|^2 + \|\mathbf{curl}\, \boldsymbol{\xi}(t)\|^2) \leqslant C\epsilon. \tag{4.3.36}$$

Now, we take the inner product of (4.3.33) with $\boldsymbol{\xi}(s)$ to obtain :

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \|\boldsymbol{\xi}(s)\|^2 = \frac{\mathrm{d}}{\mathrm{ds}}\langle \boldsymbol{\xi}, \boldsymbol{w} \rangle - \epsilon\langle \nabla q, \nabla r_s^\epsilon \rangle.$$

By integrating the previous equation from t_0 to t and due to results (4.3.29), (4.3.24) and (4.3.25), we have for all $t \in [t_0, T_0]$:

$$\begin{aligned}
\|\boldsymbol{\xi}(t)\|^2 &\leqslant \epsilon \int_{t_0}^{t} \|\nabla q(s)\| \, \|\nabla r_s^\epsilon(s)\| \, ds \\
&\leqslant \epsilon^{\frac{1}{2}} \int_{t_0}^{t} \|\nabla q(s)\|^2 \, ds + \epsilon^{\frac{3}{2}} \int_{t_0}^{t} \|\nabla r_s^\epsilon\|^2 \, ds \\
&\leqslant C\epsilon^{\frac{1}{2}} \int_{t_0}^{t} \|\boldsymbol{\xi}_s(s)\|^2 \, ds + C\epsilon^{\frac{3}{2}} \\
&\leqslant C\epsilon^{\frac{3}{2}}.
\end{aligned}$$

We can conclude, with (4.3.25) and (4.3.35), for $t \in [t_0, T_0]$:

$$\begin{aligned}
\|\psi(t)\| &\leqslant C \, \|\nabla \psi\|_{H^{-1}(\Omega)} \, . \\
&\leqslant C \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{\langle \nabla \psi(t), \boldsymbol{v} \rangle}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}}. \\
&\leqslant C \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{\langle \boldsymbol{\xi}_t(t), \boldsymbol{v} \rangle + \langle \nabla \boldsymbol{\xi}, \nabla \boldsymbol{v} \rangle + a\langle \boldsymbol{\xi}, \boldsymbol{v} \rangle}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\|\psi(t)\| &\leqslant C(\|\boldsymbol{\xi}_t(t)\| + \|\nabla \boldsymbol{\xi}(t)\| + \|\boldsymbol{\xi}(t)\|) \\
&\leqslant C\epsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.3.37}$$

□

4.3.3 Error estimates for the nonlinear problem

Let us denote $\boldsymbol{\eta} = \boldsymbol{v}^\epsilon - \boldsymbol{u}^\epsilon$ and $\phi = r^\epsilon - p^\epsilon$. We substract (4.2.1)-(4.2.2) from (4.3.21)-(4.3.22), then we reach :

$$\begin{cases}
\boldsymbol{\eta}_t - \nu \Delta \boldsymbol{\eta} + a\boldsymbol{\eta} + \nabla \phi = \beta |\boldsymbol{u}^\epsilon|^\alpha \boldsymbol{u}^\epsilon - \beta |\boldsymbol{u}|^\alpha \boldsymbol{u} \\
\operatorname{div} \boldsymbol{\eta} - \epsilon \Delta \phi_t = 0
\end{cases} \tag{4.3.38}$$

and

$$\begin{cases}
\boldsymbol{\eta} \cdot \boldsymbol{n} = 0, \quad \dfrac{\partial \phi_t}{\partial \boldsymbol{n}} = 0 \ \text{ on } \ \Gamma \times [t_0, t] \\
\boldsymbol{\eta}(t_0) = 0 \ , \ \phi(t_0) = 0
\end{cases} \tag{4.3.39}$$

The second order error estimate for the non linear problem will be discussed in the following Lemma.

Lemma 4.3.4. *Under the same assumptions of Theorem 4.3.1. We have:*

$$\|\boldsymbol{\eta}(t)\|^2+\epsilon\,\|\boldsymbol{\eta}(t)\|^2_{\boldsymbol{H}^1(\Omega)}+\int_{t_0}^{t}(\|\operatorname{div}\boldsymbol{\eta}(s)\|^2+\|\mathbf{curl}\,\boldsymbol{\eta}(s)\|^2)ds+\epsilon\,\|\nabla\phi(t)\|^2\leqslant C\epsilon^2,\quad \forall t\in[t_0,T_0]$$

Proof. We take the inner product of $(4.3.38)_1$ with $\boldsymbol{\eta}$, due to the equation obtained by taking ϕ as a source term in $(4.3.38)_2$, hence we reach :

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\,\|\boldsymbol{\eta}\|^2+\nu\,\|\operatorname{div}\boldsymbol{\eta}\|^2+\nu\,\|\mathbf{curl}\,\boldsymbol{\eta}\|^2+\frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\,\|\nabla\phi\|^2\\&=\beta\langle|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon-|\boldsymbol{u}|^\alpha\boldsymbol{u},\boldsymbol{\eta}\rangle-a\|\boldsymbol{\eta}\|^2.\end{aligned}$$

So it remains to bound the last non linear term, we have

$$\beta\langle|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon-|\boldsymbol{u}|^\alpha\boldsymbol{u},\boldsymbol{\eta}\rangle=\beta\langle|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon-|\boldsymbol{v}^\epsilon|^\alpha\boldsymbol{v}^\epsilon,\boldsymbol{\eta}\rangle+\langle|\boldsymbol{v}^\epsilon|^\alpha\boldsymbol{v}^\epsilon-|\boldsymbol{u}|^\alpha\boldsymbol{u},\boldsymbol{\eta}\rangle. \tag{4.3.40}$$

because of the property (4.1.8) :

$$\langle|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon-|\boldsymbol{v}^\epsilon|^\alpha\boldsymbol{v}^\epsilon,\boldsymbol{\eta}\rangle\leqslant 0.$$

Since $\boldsymbol{e}=\boldsymbol{\xi}+\boldsymbol{\eta}$, we observe that,

$$\beta\langle|\boldsymbol{v}^\epsilon|^\alpha\boldsymbol{v}^\epsilon-|\boldsymbol{u}|^\alpha\boldsymbol{u},\boldsymbol{\eta}\rangle=\beta\langle|\boldsymbol{v}^\epsilon|^\alpha\boldsymbol{v}^\epsilon-|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon,\boldsymbol{\eta}\rangle+\beta\langle|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon-|\boldsymbol{u}|^\alpha\boldsymbol{u},\boldsymbol{e}-\boldsymbol{\xi}\rangle.$$

Thanks again to (4.1.8), we have,

$$\langle|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon-|\boldsymbol{u}|^\alpha\boldsymbol{u},\boldsymbol{e}\rangle\leqslant 0.$$

To recap, we rearrange the previous results into (4.3.40), then we obtain

$$\beta\langle|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon-|\boldsymbol{u}|^\alpha\boldsymbol{u},\boldsymbol{\eta}\rangle\leqslant\beta\langle|\boldsymbol{v}^\epsilon|^\alpha\boldsymbol{v}^\epsilon-|\boldsymbol{u}^\epsilon|^\alpha\boldsymbol{u}^\epsilon,\boldsymbol{\eta}\rangle+\beta\langle|\boldsymbol{u}|^\alpha\boldsymbol{u}-|\boldsymbol{u}^\epsilon|\boldsymbol{u}^\epsilon,\boldsymbol{\eta}\rangle:=O_1+O_2.$$

Based on (4.1.9) and (4.1.7) , we reach :

$$\begin{aligned}O_1&\leqslant C\|\nabla\boldsymbol{\eta}\|\|\boldsymbol{\eta}\|\\&\leqslant C(\|\operatorname{div}\boldsymbol{\eta}\|+\|\mathbf{curl}\,\boldsymbol{\eta}\|)\|\boldsymbol{\eta}\|\leqslant\frac{\nu}{4}\Big(\|\operatorname{div}\boldsymbol{\eta}\|^2+\|\mathbf{curl}\,\boldsymbol{\eta}\|^2\Big)+C\|\boldsymbol{\eta}\|^2.\end{aligned}$$

Similarly

$$\begin{aligned}O_2&\leqslant C(\|\nabla(\boldsymbol{\xi}+\boldsymbol{\eta})\|)\|\boldsymbol{\xi}\|\\&\leqslant C(\|\operatorname{div}(\boldsymbol{\xi}+\boldsymbol{\eta})\|+\|\mathbf{curl}\,(\boldsymbol{\xi}+\boldsymbol{\eta})\|)\|\boldsymbol{\xi}\|.\\&\leqslant\Big(\|\operatorname{div}\boldsymbol{\xi}\|+\|\mathbf{curl}\,\boldsymbol{\xi}\|+\|\operatorname{div}\boldsymbol{\eta}\|+\|\mathbf{curl}\,\boldsymbol{\eta}\|\Big)\|\boldsymbol{\xi}\|.\\&\leqslant\frac{\nu}{4}\Big(\|\operatorname{div}\boldsymbol{\eta}\|^2+\|\mathbf{curl}\,\boldsymbol{\eta}\|^2\Big)+C\|\boldsymbol{\xi}\|^2+\Big(\|\operatorname{div}\boldsymbol{\xi}\|+\|\mathbf{curl}\,\boldsymbol{\xi}\|\Big)\|\boldsymbol{\xi}\|.\end{aligned}$$

Using Gronwall inequality, we use the regularity of the solution and the following estimate :

$$C\|\boldsymbol{\xi}\|^2+\Big(\|\operatorname{div}\boldsymbol{\xi}\|+\|\mathbf{curl}\,\boldsymbol{\xi}\|\Big)\|\boldsymbol{\xi}\|\leqslant C\epsilon\int_{t_0}^{t}\|\boldsymbol{\xi}(s)\|^2\,ds\leqslant C\epsilon\Big(\int_{t_0}^{t}\|\boldsymbol{\xi}(s)\|^2\,ds\Big)^{\frac{1}{2}}\leqslant C\epsilon^2,\ \forall t\in[t_0,T_0]$$

Hence, we obtain:

$$\|\boldsymbol{\eta}(t)\|^2 + \nu \int_{t_0}^{t} (\|\mathbf{curl}\,\boldsymbol{\eta}(s)\|^2 + \|\mathrm{div}\,\boldsymbol{\eta}(s)\|^2) ds + \epsilon \|\nabla\phi(t)\|^2 \leqslant C\epsilon^2. \tag{4.3.41}$$

In order to check the estimate of $\|\mathrm{div}\,\boldsymbol{\eta}\|^2 + \|\mathbf{curl}\,\boldsymbol{\eta}\|^2$, we proceed as previously then we take the inner product of (4.3.38) with $-\Delta\boldsymbol{\eta}$ to have :

$$\|\boldsymbol{\eta}(t)\|^2_{\boldsymbol{H}^1(\Omega)} \leqslant C\epsilon.$$

□

To achieve the proof of Theorem 4.3.1, we take into account that $\boldsymbol{u} - \boldsymbol{u}^\epsilon = \boldsymbol{\xi} + \boldsymbol{\eta}$ and $p - p^\epsilon = \psi + \phi$ and combine Lemma 4.3.3 and 4.3.4.

Chapter 5

Discontinuous Galerkin method for DBF problem

5.1 Introduction

The main idea in this work is to analyze the discontinuous Galerkin Finite Element methods (DGFEM) for the linear as well as the nonlinear problem. The first introduction of DG method applied to the hyperbolic equations was given by Reed and Hill [51] in 1973.

This chapter can be considered as an extension of the results in [10] where the Discontinuous Galerkin method is employed in order to study the velocity's approximation of the fluid flow through porous medium which is supposed to be governing by Darcy's law (linear elliptic problem), so this work allow us to extand the results given in [10] to the Direct Discontinuous Galerkin method (DDG) established for the nonlinear problem with non homogeneous boundary conditions.
Here, we essentially focus on the famous Discontinuous Galerkin schemes about (SIPG) symmetric interior penalty Galerkin. The objective of this chapter is to establish the error estimates for both the velocity and the pressure of an incompressible flow fluids in porous medium, in particularly Darcy Brinkman Forchheimer model, by using the Discontinuous Galerkin (DG) finite element method to discretize the problem (5.3.1)-(5.3.2)-(5.3.3).
One of the most advantage of DG method is the fact that we are able to treat a discontinuous functions on the boundary of each element (on the triangular surfaces of each mesh).

The organization of this chapter is as follow, in the first Section, we introduce the weak formulation of the considered linearized model, then we provide some notations and results about the DG discretization such as the definitions of the finite dimensional spaces $\boldsymbol{V}_h$ and Q_h, the discretization of the domain Ω into small grids and we recall properties of interpolation theory. After that the DG finite element scheme of the linearized problem is derived. Through the

theorem 5.2.7, the well-posedness of the linearized model's DG approximations is investigated. The Section 5.3 is devoted to study the DG approximation nonlinear problem. By applying Brouwer's fixed point theorem, the existence of the approximate solution $(\boldsymbol{u}_h, p_h)$ is established.
In Section 5.4, we introduce some auxiliary results needed for the subsequent analysis, moreover this Section contains an error estimate of both the velocity and the pressure. The last section presents some numerical results.

5.2 Discontinuous Galerkin method for the linearized problem

Firstly, we introduce the weak formulation of the linearized model which is obtained by modifying the non-linear term in Darcy Brinkman Forchheimer equations. Then, the existence of solution for the linearized problem's (DG) discontinuous Galerkin approximation will be investigated.

5.2.1 Model problem and weak formulation

Let introduce the following linearized Forchheimer problem

$$-\nu\Delta\boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{w}|\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \text{in } \Omega, \tag{5.2.1}$$

$$\operatorname{div}\boldsymbol{u} = 0, \quad \text{in } \Omega, \tag{5.2.2}$$

with a non-standard boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n} = g, \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}, \quad \text{on } \Gamma. \tag{5.2.3}$$

Here $\boldsymbol{u}$ is the velocity, p the pressure, $\boldsymbol{w}$ stands for a convective velocity field, $\boldsymbol{f}$ an external prescribed body force, ν is the kinematic viscosity of the fluid and Ω denotes a Lipschitz bounded domain of $\mathbb{R}^3$ with the boundary $\partial\Omega$. Finally, $\boldsymbol{n}$ denotes the outward unit normal vector to $\partial\Omega$.

We denote by $\boldsymbol{H}^1_g(\Omega)$ and $\boldsymbol{H}^1_\tau(\Omega)$ the spaces:

$$\boldsymbol{H}^1_g(\Omega) = \{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = g \ \text{ on}\,\Gamma\},$$
$$\boldsymbol{H}^1_\tau(\Omega) = \{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on}\,\Gamma\},$$

The variational formulation of (5.2.1)-(5.2.3) is:

$$\begin{cases} \text{Find } (\boldsymbol{u},p)\in\boldsymbol{H}^1_g(\Omega)\times L^2(\Omega). \text{ such that} \\ A(\boldsymbol{u},\ \boldsymbol{v}) + F(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) + B(p,\ \boldsymbol{v}) = l(\boldsymbol{v}), \qquad \forall\boldsymbol{v}\in\boldsymbol{H}^1_\tau(\Omega), \\ B(q,\ \boldsymbol{u}) = 0, \qquad \forall q\in L^2(\Omega), \end{cases} \tag{5.2.4}$$

where A is a bilinear form defined from $\boldsymbol{H}^1_g(\Omega) \times \boldsymbol{H}^1_\tau(\Omega)$ into $\mathbb{R}$, by

$$A(\boldsymbol{u}, \boldsymbol{v}) = \nu \int_\Omega \operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + a \int_\Omega \boldsymbol{u} \; \boldsymbol{v} \; dx.$$

Moreover, the trilinear form F can be defined from $\boldsymbol{L}^{\frac{3}{2}}(\Omega) \times \boldsymbol{H}^1_g(\Omega) \times \boldsymbol{H}^1_\tau(\Omega) \longrightarrow \mathbb{R}$, as

$$F(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = b \int_\Omega |\boldsymbol{w}| \; \boldsymbol{u} \; \boldsymbol{v} \; dx.$$

The last bilinear form $B_h(.,.) : \boldsymbol{L}^2(\Omega) \times \boldsymbol{H}^1_\tau(\Omega) \longrightarrow \mathbb{R}$, satisfies

$$B(p, \boldsymbol{v}) = -\int_\Omega p \operatorname{div} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}.$$

The linear form l_h defined from the space $\boldsymbol{H}^1_\tau(\Omega)$ into $\mathbb{R}$, by

$$l(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \; + \nu \langle \boldsymbol{h} \times \boldsymbol{n}, \; \boldsymbol{v} \rangle_{\boldsymbol{H}^{-1/2}(\Gamma) \times \boldsymbol{H}^{1/2}(\Gamma)}.$$

The well posedness of probelm (5.2.4) can be done by lifting the inhomogeneous boundary condition g to turn into the homogeneous case stated in (1.3.59) and then apply Corollary 1.3.9.

5.2.2 DG discretization

In order to establish the DG method for the linearized problem, we need to introduce some notations and results which are used throughout this chapter.
So let we begin with discontinuous finite element subdivision $\mathcal{T}_h$ of the computational domain $\Omega \subset \mathbb{R}^3$ (i.e $\mathcal{T}_h$ is the family of the regular shape tetrahedra of the polyhedral domain Ω).
For each tetrahedron T, we denote by h_T its diameter, by $|T|$ its volume and let $h = \max_{T \in \mathcal{T}_h} h_T$.
Moreover, the meshes are supposed to be shape-regular i.e, there is a constant $c > 0$, independent of h, such that :

$$\frac{h_T}{\rho_T} \leq C, \; \forall\, h > 0, \quad \forall\, T \in \mathcal{T}_h.$$

Here ρ_T denotes the diameter of the largest sphere inscribed in T.

Γ_h denotes the set of triangular surfaces situated on the boundary $\partial\Omega$. While we denote by $\mathcal{C}_h^{int}$ the set of internal triangular surfaces of $\mathcal{T}_h$.

We introduce the standard definitions of jumps and averages for scalar and vector functions. For any interior triangular surface $\boldsymbol{e}$ shared by two mesh elements T_1 and T_2 i.e $\boldsymbol{e} = \partial T_1 \cap \partial T_2$, having normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ pointing exterior to T_1 and T_2 respectively, then the average $\langle . \rangle$ and jump $[.]$ on $\boldsymbol{e}$ for a vector $\boldsymbol{v}$ and scalar q are defined, respectively, as :

$$\langle \boldsymbol{v}_h \rangle = \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2), \quad [\boldsymbol{v}] = \boldsymbol{v}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{n}_2.$$

$$\langle q\rangle = \frac{1}{2}(q_1+q_2), \quad [q] = q_1\boldsymbol{n} + q_2\boldsymbol{n}_2.$$

Where $\boldsymbol{v}_i = (\boldsymbol{v}|_{T_i})|_e$ and $q_i = (q|_{T_i})|_e$, for $i = 1, 2$.

When $\boldsymbol{e}$ is a triangular surface on the boundary $\partial\Omega$, we define the average and jump of a vector $\boldsymbol{v}$ as follow :

$$\langle \boldsymbol{v}\rangle = \boldsymbol{v}, \quad [\boldsymbol{v}] = \boldsymbol{v}\cdot\boldsymbol{n}.$$

The average and the jump of a scalar q can be noted by

$$\langle q\rangle = q, \quad [q] = q\cdot\boldsymbol{n}.$$

Such that $\boldsymbol{n}$ is nothing but an unit exterior normal vector to the boundary $\partial\Omega$.

We denote $P_m(T)$ the space of polynomial functions of degree less than or equal to m defined on T.

The following trace inequality will be frequently used in the sequel of the work.

For $\boldsymbol{v}\in\boldsymbol{H}^1(T)$ and for any triangular surface of the element T, we have,

$$\sqrt{e}\|\boldsymbol{v}\|_{0,e} \leqslant C(\|\boldsymbol{v}\|_{0,T} + \|\boldsymbol{v}\|_{1,T}). \tag{5.2.5}$$

Let us recall the monotonocity property, for any mapping $G : \boldsymbol{u} \longrightarrow |\boldsymbol{u}|^\alpha\boldsymbol{u}$, then

$$(|\boldsymbol{u}|\boldsymbol{u} - |\boldsymbol{v}|\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}) \geqslant 0. \tag{5.2.6}$$

We search an approximation velocity $\boldsymbol{u}_h$ and pressure p_h in the discontinuous finite dimensional spaces $\boldsymbol{V}_h$ and Q_h

$$\boldsymbol{V}_h = \big\{\boldsymbol{v}_h \in \mathbf{L}^2(\Omega),\ \boldsymbol{v}_h|_T \in \boldsymbol{P}_m,\ \forall T\in\mathcal{T}_h\big\},$$
$$Q_h = \big\{q_h \in L_0^2(\Omega),\ q_h|_T \in P_{m-1},\ \forall T\in\mathcal{T}_h\big\},$$

Moreover, for any function $\boldsymbol{v}_h$ belongs to the discontinuous finite element space V_h, we give the application,

$$\|\boldsymbol{v}_h\|_{DG} = \Big(\sum_{T\in\mathcal{T}_h}\big(\|\mathrm{div}\,\boldsymbol{v}_h\|_{0,T}^2 + \|\mathrm{curl}\,\boldsymbol{v}_h\|_{0,T}^2\big) + a\sum_{T\in\mathcal{T}_h}\|\boldsymbol{v}_h\|_{0,T}^2 + \sigma\big(J_1(\boldsymbol{v}_h,\boldsymbol{v}_h) + J_2(\boldsymbol{v}_h,\boldsymbol{v}_h)\big)\Big)^{\frac{1}{2}}. \tag{5.2.7}$$

Where σ is the penalty parameter. In order to stabilize the scheme, we have added $J_1(.\,,.)$ and $J_2(.\,,.)$.

$$J_1(\boldsymbol{u}_h,\boldsymbol{v}_h) = \sum_{e\in\mathcal{C}_h^{\mathrm{int}}}\frac{1}{|e|}\int_e[\boldsymbol{u}_h\times\boldsymbol{n}_e]\cdot[\boldsymbol{v}_h\times\boldsymbol{n}_e]\,ds, \tag{5.2.8}$$

$$J_2(\boldsymbol{u}_h,\boldsymbol{v}_h) = \sum_{e\in\mathcal{C}_h^{\mathrm{int}}\cup\Gamma_h}\frac{1}{|e|}\int_e[\boldsymbol{u}_h\cdot\boldsymbol{n}_e]\cdot[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]\,ds. \tag{5.2.9}$$

It is well-known that for any $T\in\mathcal{T}_h$ and for any commun surface $\boldsymbol{e}\in\Gamma_h\cup\mathcal{C}_h^{int}$ the continuous orthogonal operator $\boldsymbol{R}_h$ defined from $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{V}_h$ satisfies the following equations, with $m = 1, 2$ or 3 and

$$\int_T q \operatorname{div}(\boldsymbol{R}_h(\boldsymbol{v}) - \boldsymbol{v}) = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \qquad \forall q \in P_{m-1}. \tag{5.2.10}$$

$$\int_T q \operatorname{curl}(\boldsymbol{R}_h(\boldsymbol{v}) - \boldsymbol{v}) = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \qquad \forall q \in P_{m-1}, \tag{5.2.11}$$

$$\int_e q \cdot [\boldsymbol{R}_h(\boldsymbol{v})] = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \quad \forall \boldsymbol{q} \in \boldsymbol{P}_{m-1}, \tag{5.2.12}$$

Moreover, Let r_h be the L^2 continuous projection defined from $L_0^2(\Omega)$ to $Q_h(\Omega)$,

$$\forall p \in L_0^2(\Omega), \quad \forall q \in P_{m-1}, \qquad \int_T q(r_h(p) - p) = 0, \tag{5.2.13}$$

By the usual interpolation theory, the operator $\boldsymbol{R}_h$ satisfies the following approximation property

$$\forall \boldsymbol{v} \in \boldsymbol{H}^{m+1}(\Omega), \qquad \|\boldsymbol{R}_h(\boldsymbol{v}) - \boldsymbol{v}\|_{1,T} \leq Ch^m |\boldsymbol{v}|_{m+1,T}, \tag{5.2.14}$$

Furthermore, the following pressure's interpolation estimate holds:
For any $p \in H^m(\Omega) \cap L_0^2(\Omega)$, we have

$$\|r_h(p) - p\|_{0,T} \leq Ch_T^m \, |p|_{m,T} \, . \tag{5.2.15}$$

The next lemma discusses the fact that the mapping given above in (5.2.7) is a norm on the space V_h.

Lemma 5.2.1. *The mapping $\boldsymbol{v}_h \longrightarrow \|\boldsymbol{v}_h\|_{DG}$ is a norm on $\boldsymbol{V}_h$.*

Proof. For any function $\boldsymbol{v}_h \in \boldsymbol{V}_h$, it is easy to derive that $\|\boldsymbol{v}_h\|_{DG} = 0$ implies $\boldsymbol{v}_h = 0$.
Let we assume $\|\boldsymbol{v}_h\|_{DG}^2 = 0$, then immediately it follows to note that

$$\|\boldsymbol{v}_h\|_{DG}^2 = \sum_{T \in \mathcal{T}_h} \big(\|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 + \|\operatorname{curl} \boldsymbol{v}_h\|_{0,T}^2\big) + a \sum_{T \in \mathcal{T}_h} \|\boldsymbol{v}_h\|_{0,T}^2 + \sigma\big(J_1(\boldsymbol{v}_h, \boldsymbol{v}_h) + J_2(\boldsymbol{v}_h, \boldsymbol{v}_h)\big) = 0. \tag{5.2.16}$$

Due to definition of the stabilization terms $J_1(.\,,.)$, $J_2(.\,,.)$ in (5.2.8), (5.2.9), we observe that the energy norm of any function $\boldsymbol{v}_h$ is the sum of square numbers which equals to zero, this implies that every piece is vanish, then consequently $\boldsymbol{v}_h = 0$. □

5.2.3 DG scheme

Now, based on the variational formulation already given in the previous subsection (5.2.4), we are in front to define the DG finite element scheme in the compact form, corresponding to (5.2.1)-(5.2.2)-(5.2.3) problem which is given by
Find $(\boldsymbol{u}_h,\, q_h) \in \boldsymbol{V}_h \times Q_h$ such that

$$\begin{cases} A_h(\boldsymbol{u}_h,\, \boldsymbol{v}_h) + F_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{v}_h) + B_h(p_h,\, \boldsymbol{v}_h) = l_h(\boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ B_h(q_h,\, \boldsymbol{u}_h) = \boldsymbol{g}_h(q_h) & \forall q_h \in Q_h, \end{cases} \tag{5.2.17}$$

Where the trilinear form F_h can be defined from $\boldsymbol{V}_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h$ into $\mathbb{R}$ as

$$F_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{v}_h) = b \sum_{T \in \mathcal{T}_h} \int_T |\boldsymbol{w}| \boldsymbol{u}_h . \boldsymbol{v}_h \, dx.$$

We rexwrite the bilinear and the linear forms in a useful manner. We begin by $A_h(.,.)$ defined from $\boldsymbol{V}_h \times \boldsymbol{V}_h$ into $\mathbb{R}$,

$$A_h(\cdot,\cdot) = a_0(\cdot,\cdot) + a_1(\cdot,\cdot) + a_2(\cdot,\cdot) + \sigma\big(J_1(\cdot,\cdot) + J_2(\cdot,\cdot)\big).$$

Such that, for any $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h$, we have

$$a_0(\boldsymbol{u}_h, \boldsymbol{v}_h) = \nu \sum_{\mathrm{T} \in \mathcal{T}_h} \Big(\int_{\mathrm{T}} \mathrm{curl} \boldsymbol{u}_h \cdot \mathrm{curl}\, \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} + \int_{\mathrm{T}} \mathrm{div}\, \boldsymbol{u}_h \,\mathrm{div}\, \boldsymbol{v}_h \mathrm{d}\boldsymbol{x} \Big) + a \sum_{\mathrm{T} \in \mathcal{T}_h} \int_{\mathrm{T}} \boldsymbol{u}\, \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}.$$

We give some details on the forms $a_1(.\,,.)$, $a_2(.\,,.)$ as follow :

$$a_1(\boldsymbol{u}_h, \boldsymbol{v}_h) = -\nu \sum_{e \in \mathcal{C}_h^{\mathrm{int}}} \Big(\int_e \langle \mathrm{curl} \boldsymbol{u}_h \rangle [\boldsymbol{v}_h \times \boldsymbol{n}_e] \,\mathrm{d}\, s + \int_e \langle \mathrm{curl} \boldsymbol{v}_h \rangle [\boldsymbol{u}_h \times \boldsymbol{n}_e] \,\mathrm{d}\, s \Big).$$

$$a_2(\boldsymbol{u}_h, \boldsymbol{v}_h) = -\nu \sum_{e \in \mathcal{C}_h^{\mathrm{int}} \cup \Gamma_h} \Big(\int_e \langle \mathrm{div} \boldsymbol{u}_h \rangle [\boldsymbol{v}_h \cdot \boldsymbol{n}_e] \,\mathrm{d}\, s + \int_e \langle \mathrm{div} \boldsymbol{v}_h \rangle [\boldsymbol{u}_h \cdot \boldsymbol{n}_e] \,\mathrm{d}\, s \Big).$$

For given $q_h \in Q_h$ and $\boldsymbol{v}_h \in \boldsymbol{V}_h$, we define the bilinear form $B_h(.,.) : Q_h \times \boldsymbol{V}_h \longrightarrow \mathbb{R}$,

$$B_h(q_h, \boldsymbol{v}_h) = -\sum_{\mathrm{T} \in \mathcal{T}_h} \int_{\mathrm{T}} q_h \mathrm{div} \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} + \sum_{e \in \mathcal{C}_h^{\mathrm{int}}} \int_e \langle q_h \rangle \, [\boldsymbol{v}_h \cdot \boldsymbol{n}_e] \,\mathrm{d}\, s,$$

Now, we return to the linear forms, where l_h is defined from $\boldsymbol{V}_h$ into $\mathbb{R}$, by

$$\begin{aligned} l_h(\boldsymbol{v}_h) = & \sum_{\mathrm{T} \in \mathcal{T}_h} \int_{\mathrm{T}} \boldsymbol{f}\, \boldsymbol{v}_h + \nu \sum_{e \in \Gamma_h} \int_{\boldsymbol{e}} \boldsymbol{h} \times \boldsymbol{n} \cdot \boldsymbol{v}_h \times \boldsymbol{n} \,\mathrm{d}\, s + \sum_{\boldsymbol{e} \in \Gamma_h} \frac{1}{|\boldsymbol{e}|} \int_{\boldsymbol{e}} \boldsymbol{g} .\, \boldsymbol{v}_h . \boldsymbol{n} \,\mathrm{d}\, s \\ & - \gamma \sum_{\boldsymbol{e} \in \Gamma_h} \int_{\boldsymbol{e}} (\mathrm{div}\, \boldsymbol{v}_h) \boldsymbol{g} \,\mathrm{d}\, s. \end{aligned}$$

And the last linear form g_h is defined from the approximate finite element space $Q_h \longrightarrow \mathbb{R}$ as,

$$g_h(q_h) = \sum_{e \in \Gamma_h} \int_e q_h \cdot \boldsymbol{g} \,\mathrm{d}\, s.$$

In the sequel, we focus on the well-posedness of the discrete formulation (DG), so we investigate the coercivity of A_h and the fact that F_h is a positive-definite operator. Moreover the continuity property of the bilinear forms A_h, B_h, the trilinear form F_h and the linear forms l_h, g_h will be established in this section, Furthermore, we check the Inf-Sup condition of B_h.

5.2.4 The well-posedness of DG linearized problem

First, we state some special result useful in the subsequent analysis. In the following, we assume the the following discrete inequality: For any $\boldsymbol{v} \in \boldsymbol{L}^4(\Omega)$, then there exists a constant $c_I > 0$, such that

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)} \leqslant c_I \|\boldsymbol{v}\|_{DG}. \tag{5.2.18}$$

Let us check the coercivity of A_h and treat the trilinear form F_h in the following lemma.

Lemma 5.2.2. *If σ is large enough, there exists a constant $\alpha > 0$ such that*

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geqslant \nu\alpha \|\boldsymbol{u}_h\|_{DG}^2, \quad \forall \boldsymbol{u}_h \in \boldsymbol{V}_h. \tag{5.2.19}$$

Furthermore,

$$F_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{u}_h) \geqslant 0, \quad \forall\ \boldsymbol{u}_h \in \boldsymbol{V}_h. \tag{5.2.20}$$

Proof. Based on the definition of the bilinear form A_h and the energy norm, we have
For any $\boldsymbol{u}_h \in \boldsymbol{V}_h$,

$$a_0(\boldsymbol{u}_h, \boldsymbol{u}_h) + \sigma(J_1(\boldsymbol{u}_h, \boldsymbol{u}_h) + J_2(\boldsymbol{u}_h, \boldsymbol{u}_h)) \geqslant \nu \|\boldsymbol{u}_h\|_{DG}^2 \tag{5.2.21}$$

So, we observe that

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) = a_0(\boldsymbol{u}_h, \boldsymbol{u}_h) + \sigma(J_1(\boldsymbol{u}_h, \boldsymbol{u}_h) + J_2(\boldsymbol{u}_h, \boldsymbol{u}_h)) + a_1(\boldsymbol{u}_h, \boldsymbol{u}_h) + a_2(\boldsymbol{u}_h, \boldsymbol{u}_h) \tag{5.2.22}$$

$$\geqslant \nu \|\boldsymbol{u}_h\|_{DG}^2 + a_1(\boldsymbol{u}_h, \boldsymbol{u}_h) + a_2(\boldsymbol{u}_h, \boldsymbol{u}_h). \tag{5.2.23}$$

Let we recall that

$$a_2(\boldsymbol{u}_h, \boldsymbol{u}_h) = -2\nu \sum_{e \in C_h^{int} \cup \Gamma_h} \int_e \operatorname{div}\ \boldsymbol{u}_h\, [\boldsymbol{u}_h.\boldsymbol{n}_e]\, d\, s. \tag{5.2.24}$$

According to Cauchy-Schwarz and the trace inequality (5.2.5), one obtains

$$\nu \sum_{e \in C_h^{int} \cup \Gamma_h} \int_e \operatorname{div}\ \boldsymbol{u}_h\, [\boldsymbol{u}_h.\boldsymbol{n}_e]\, d\, s \leqslant \nu \Big(\sum_{e \in C_h^{int} \cup \Gamma_h} \int_e |e|\, (\operatorname{div}\ \boldsymbol{u}_h)^2\, d\, s \Big)^{\frac{1}{2}}$$

$$\Big(\sum_{e \in C_h^{int} \cup \Gamma_h} \int_e \frac{1}{|e|} [\boldsymbol{u}_h.\boldsymbol{n}_e] \Big)^{\frac{1}{2}}. \tag{5.2.25}$$

$$\tag{5.2.26}$$

Let $\boldsymbol{e} \in C_h^{int}$ such that $< \boldsymbol{e} >= \partial T_1 \cap \partial T_2 \in \mathcal{T}_h$, we write

$$\sqrt{|\boldsymbol{e}|} \|\operatorname{div} \boldsymbol{u}_h\|_{0,\boldsymbol{e}} \leqslant \frac{\sqrt{|\boldsymbol{e}|}}{2} \sum_{i=1}^{2} \|(\operatorname{div} \boldsymbol{u}_h)_{T_i}\|_{0,\boldsymbol{e}}. \tag{5.2.27}$$

$$\leqslant \frac{c}{2} (\|\operatorname{div} \boldsymbol{u}_h\|_{0,T_1} + \|\operatorname{div} \boldsymbol{u}_h\|_{0,T_2}). \tag{5.2.28}$$

This result holds, even if the triangular surface $\boldsymbol{e}$ belongs to the boundary Γ_h.
Then, we obtain

$$\nu \sum_{e\in C_h^{int}\cup\Gamma_h} \int_e \text{div } \boldsymbol{u}_h\,[\boldsymbol{u}_h.\boldsymbol{n}_e]\,d\,s \leqslant \nu\Big(c\sum_{T\in\mathcal{T}_h}\|\text{div } \boldsymbol{u}_h\|_{0,T}^2\Big)^{\frac{1}{2}}\Big(J_2(\boldsymbol{u}_h,\boldsymbol{u}_h)\Big)^{\frac{1}{2}}.$$

Consequently, we get

$$a_2(\boldsymbol{u}_h,\boldsymbol{u}_h) \geqslant -\nu\Big(\frac{c}{\sigma}\sum_{T\in\mathcal{T}_h}\|\text{div } \boldsymbol{u}_h\|_{0,T}^2\Big)^{\frac{1}{2}}\Big(\sigma J_2(\boldsymbol{u}_h,\boldsymbol{u}_h)\Big)^{\frac{1}{2}}. \tag{5.2.29}$$

$$\geqslant -\nu\frac{c'}{\sqrt{\sigma}}\|\boldsymbol{u}_h\|_{DG}^2. \tag{5.2.30}$$

Using the similar argument, we observe ,

$$a_1(\boldsymbol{u}_h,\boldsymbol{u}_h) \geqslant -\nu\Big(\frac{c}{\sigma}\sum_{T\in\mathcal{T}_h}\|\mathbf{curl}\,\boldsymbol{u}_h\|_{0,T}^2\Big)^{\frac{1}{2}}\Big(\sigma J_1(\boldsymbol{u}_h,\boldsymbol{u}_h)\Big)^{\frac{1}{2}}. \tag{5.2.31}$$

$$\geqslant -\nu\,\frac{c''}{\sqrt{\sigma}}\|\boldsymbol{u}_h\|_{DG}^2. \tag{5.2.32}$$

Hence, by rearrange the relation (5.2.22) and results (5.2.30)-(5.2.32), we reach,

$$A_h(\boldsymbol{u}_h,\boldsymbol{u}_h) \geqslant \nu\|\boldsymbol{u}_h\|_{DG}^2 - \nu\frac{c}{\sqrt{\sigma}}\|\boldsymbol{u}_h\|_{DG}^2. \tag{5.2.33}$$

For large enough σ, the result (5.2.19) is established and we deduce the coercivity of the bilinear form A_h.

By definition of the trilinear form $F_h : \boldsymbol{V}_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h \longrightarrow \mathbb{R}$, we have

$$F_h(w,\boldsymbol{u}_h,\boldsymbol{u}_h) = b\sum_{T\in\mathcal{T}_h}\int_T |\boldsymbol{w}|\boldsymbol{u}_h^2\,dx, \tag{5.2.34}$$

and the trilinear form F_h is an operator definite positive.

□

Now we study the continuity of the bilinear forms $A_h(.\,,\,.)$ and $B_h(.\,,\,.)$.

Lemma 5.2.3. *For any $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h$ there exists $c_A > 0$ and $c_B > 0$ such that*

$$|A_h(\boldsymbol{u}_h,\boldsymbol{v}_h)| \leqslant c_A\|\boldsymbol{u}_h\|_{DG}\|\boldsymbol{v}_h\|_{DG} \tag{5.2.35}$$

$$|B_h(q_h,\boldsymbol{v}_h)| \leqslant c_B\|q_h\|_{L^2(\Omega)}\|\boldsymbol{v}_h\|_{DG}. \tag{5.2.36}$$

Proof. Let $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h$, by applying Cauchy-Schawrz, one can see immediately that $a_0(.\,,\,.)$, can be controled

$$a_0(\boldsymbol{u}_h,\boldsymbol{v}_h) = \nu\sum_{\mathrm{T}\in\mathcal{T}_h}\Big(\int_{\mathrm{T}}\text{curl } \boldsymbol{u}_h \text{ curl } \boldsymbol{v}_h\ \mathrm{d}\boldsymbol{x} + \int_{\mathrm{T}}\text{div } \boldsymbol{u}_h \text{ div } \boldsymbol{v}_h\mathrm{d}\boldsymbol{x}\Big) + a\sum_{\mathrm{T}\in\mathcal{T}_h}\int_{\mathrm{T}}\boldsymbol{u}.\boldsymbol{v}\ \mathrm{d}\boldsymbol{x}.$$

$$\leqslant C\|\boldsymbol{u}_h\|_{DG}\|\boldsymbol{v}_h\|_{DG}. \tag{5.2.37}$$

Moreover, using again Cauchy-Schwarz and the trace inequality, we obtain

$$\nu \sum_{e\in\mathcal{C}_h^{\rm int}\cup\Gamma_h} \int_e \langle \mathrm{div}\boldsymbol{u}_h\rangle[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]\,\mathrm{d}\,s \leqslant (\nu \sum_{e\in\mathcal{C}_h^{int}\cup\Gamma_h} |e|\|\mathrm{div}\boldsymbol{u}_h\|_{0,e}^2)^{\frac{1}{2}}(\nu \sum_{e\in\mathcal{C}_h^{int}\cup\Gamma_h} \frac{1}{|e|}\|\boldsymbol{v}_h.\boldsymbol{n}\|_{0,e}^2)^{\frac{1}{2}}$$
$$\leqslant (C\nu \sum_{T\in\mathcal{T}_h} \|\mathrm{div}\boldsymbol{u}_h\|_{0,e}^2)^{\frac{1}{2}}(J_2(\boldsymbol{u}_h,\boldsymbol{v}_h))^{\frac{1}{2}}.$$
$$\leqslant C\|\boldsymbol{u}_h\|_{DG}\|\boldsymbol{v}_h\|_{DG}.$$

Consequently, we have

$$a_2(\boldsymbol{u}_h,\boldsymbol{v}_h) \leqslant C\|\boldsymbol{u}_h\|_{DG}\|\boldsymbol{v}_h\|_{DG}. \tag{5.2.38}$$

similarly, we have

$$a_1(\boldsymbol{u}_h,\boldsymbol{v}_h) \leqslant C\|\boldsymbol{u}_h\|_{DG}\|\boldsymbol{v}_h\|_{DG}. \tag{5.2.39}$$

the two stabilization terms are controled as:

$$J_1(\boldsymbol{u}_h,\boldsymbol{v}_h) = \nu \sum_{e\in\mathcal{C}_h^{\rm int}} \frac{1}{|e|}\int_e [\boldsymbol{u}_h\times\boldsymbol{n}_e]\cdot[\boldsymbol{v}_h\times\boldsymbol{n}_e]\,ds, \tag{5.2.40}$$

$$\leqslant (\nu \sum_{e\in\mathcal{C}_h^{\rm int}} \frac{1}{|e|}\|u_h\times\boldsymbol{n}_e\|_{0,e}^2)^{\frac{1}{2}}(\nu \sum_{e\in\mathcal{C}_h^{\rm int}} \frac{1}{|e|}\|v_h\times\boldsymbol{n}_e\|_{0,e}^2)^{\frac{1}{2}} \tag{5.2.41}$$

$$\leqslant C\|\boldsymbol{u}_h\|_{DG}\|\boldsymbol{v}_h\|_{DG}, \tag{5.2.42}$$

and

$$J_2(\boldsymbol{u}_h,\boldsymbol{v}_h) = \nu \sum_{e\in\mathcal{C}_h^{\rm int}\cup\Gamma_h} \frac{1}{|e|}\int_e [\boldsymbol{u}_h\cdot\boldsymbol{n}_e]\cdot[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]\,ds. \tag{5.2.43}$$

$$\leqslant C\|\boldsymbol{u}_h\|_{DG}\|\boldsymbol{v}_h\|_{DG}. \tag{5.2.44}$$

This completes the proof of the continuity of the bilinear form A_h.

Next, to obtain the continuity of B_h, it follows for any $(q_h,\boldsymbol{v}_h)\in Q_h\times\boldsymbol{V}_h$:

$$\sum_{e\in\mathcal{C}_h^{\rm int}} \int_e \langle q_h\rangle\,[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]\,\mathrm{d}\,s \leqslant \frac{1}{\sqrt{\nu}}\Big(\sum_{e\in\mathcal{C}_h^{\rm int}} |e|\|q_h\|_{0,e}^2\Big)^{\frac{1}{2}}(J_2(\boldsymbol{v}_h,\boldsymbol{v}_h))^{\frac{1}{2}}. \tag{5.2.45}$$

$$\leqslant \frac{C}{\sqrt{\nu}}\|q_h\|_{0,T}(J_2(\boldsymbol{v}_h,\boldsymbol{v}_h))^{\frac{1}{2}} \tag{5.2.46}$$

$$\leqslant C_B\|q_h\|_{L^2(\Omega)}\|\boldsymbol{v}_h\|_{DG}, \tag{5.2.47}$$

which states the continuity property of the bilinear form B_h.

□

We shall derive the continuity of the trilinear form F_h.

Lemma 5.2.4. *Let $\boldsymbol{u}_h$, $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\boldsymbol{w} \in \boldsymbol{V}_h$, then there exists a positive constant $C_F > 0$ such that*

$$|F_h(\boldsymbol{w}_1, \boldsymbol{u}_h, \boldsymbol{v}_h) - F_h(\boldsymbol{w}_2, \boldsymbol{u}_h, \boldsymbol{v}_h)| \leqslant C_F \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{DG} \|\boldsymbol{u}_h\|_{DG} \|\boldsymbol{v}_h\|_{DG}. \tag{5.2.48}$$

Proof. For given $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{u}_h$, $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and based on the definition of the trilinear form F_h, we obtain

$$|F_h(\boldsymbol{w}_1, \boldsymbol{u}_h, \boldsymbol{v}_h) - F_h(\boldsymbol{w}_2, \boldsymbol{u}_h, \boldsymbol{v}_h)| = |b \sum_{T \in \mathcal{T}_h} \int_T |\boldsymbol{w}_1|.\boldsymbol{u}_h.\boldsymbol{v}_h \, dx - b \sum_{T \in \mathcal{T}_h} \int_T |\boldsymbol{w}_2| \boldsymbol{u}_h.\boldsymbol{v}_h \, dx| \tag{5.2.49}$$

$$= |b \sum_{T \in \mathcal{T}_h} \int_T (|\boldsymbol{w}_1| - |\boldsymbol{w}_2|) \boldsymbol{u}_h.\boldsymbol{v}_h \, dx| \tag{5.2.50}$$

$$\leqslant b \sum_{T \in \mathcal{T}_h} \int_T |\boldsymbol{w}_1 - \boldsymbol{w}_2| |\boldsymbol{u}_h| |\boldsymbol{v}_h| \, dx. \tag{5.2.51}$$

$$\leqslant b \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}_h\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{v}_h\|_{L^4(\Omega)}. \tag{5.2.52}$$

Because of the result (5.2.18), we have

$$|F_h(\boldsymbol{w}_1, \boldsymbol{u}_h, \boldsymbol{v}_h) - F_h(\boldsymbol{w}_2, \boldsymbol{u}_h, \boldsymbol{v}_h)| \leqslant b \, c_I^2 \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{DG} \, \|\boldsymbol{u}_h\|_{DG} \, \|\boldsymbol{v}_h\|_{DG}. \tag{5.2.53}$$

The proof is completes by taking $C_F = b \, c_I^2$. □

The next lemma treats the Inf-Sup condition of B_h.

Lemma 5.2.5. *There exists $\beta > 0$, independent of the mesh size h and ν, such that*

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{B(q_h, \boldsymbol{v}_h)}{\|q_h\|_{L^2(\Omega)} \|\boldsymbol{v}_h\|_{DG}} \geqslant \beta. \tag{5.2.54}$$

Proof. The approach is rather classical. With any $q_h \in Q_h$, we shall associate $\boldsymbol{v}_h \in \boldsymbol{V}_h$, such that

$$B(q_h, \boldsymbol{v}_h) = \|q_h\|_{0,\Omega}^2 \tag{5.2.55}$$

$$\|\boldsymbol{v}_h\|_{DG} \leq c \|q_h\|_{0,\Omega}. \tag{5.2.56}$$

For this purpose, we make use of the continuous inf-sup condition for the linearized problem. Thus, let $q_h \in Q_h \subset L_0^2(\Omega)$ and let $\boldsymbol{z} \in \boldsymbol{H}_0^1(\Omega)$ such that:

$$\begin{cases} \operatorname{div} \boldsymbol{z} = q_h \\ \|\boldsymbol{z}\|_{1,\Omega} \leq C \|q_h\|_{0,\Omega}. \end{cases}$$

Then we put $\boldsymbol{v}_h = \boldsymbol{R}_h(\boldsymbol{z}) \in \boldsymbol{V}_h$. By construction, we have according to (5.2.10) and (5.2.12) on every $T \in \mathcal{T}_h$ and every surface $e \in \mathcal{C}_h^{int}$:

$$\int_T q_h \operatorname{div} \boldsymbol{v}_h dx = \int_T q \operatorname{div} \boldsymbol{z} = \|q_h\|_{0,\Omega}^2.$$

$$\int_e \langle q_h \rangle [\boldsymbol{v}_h \cdot \boldsymbol{n}_e] ds = 0.$$

Consequently $B_h(q_h, \boldsymbol{v}_h) = \|q_h\|^2_{0,\Omega}$. Next, since $[\boldsymbol{z} \cdot \boldsymbol{n}_e] = 0$ we have

$$\begin{aligned} J_1(\boldsymbol{v}_h, \boldsymbol{v}_h) &= J_1(\boldsymbol{z} - \boldsymbol{R}_h \boldsymbol{z}, \boldsymbol{z} - \boldsymbol{R}_h \boldsymbol{z}) \\ &= \sum_{e \in \mathcal{C}_h^{int}} \frac{1}{|e|} \|[(\boldsymbol{z} - \boldsymbol{R}_h \boldsymbol{z}) \times \boldsymbol{n}_e]\|^2_{0,e} \\ &\leq C |\boldsymbol{z}|^2_{1,\Omega}. \end{aligned}$$

The last inequality is obtained in a classical way by using (5.2.14) after passing to reference element and by making use of the trace inequality. Similarly, we can check that

$$J_2(\boldsymbol{v}_h, \boldsymbol{v}_h) \leq C' |\boldsymbol{z}|^2_{1,\Omega}.$$

Therefore, by using the interpolation estimate (5.2.14), we get

$$\begin{aligned} \|\boldsymbol{v}_h\|^2_{DG} = &\sum_{T \in \mathcal{T}_h} \|\mathrm{div}\,(\boldsymbol{R}_h \boldsymbol{z})\|^2_{0,\Omega} + \sum_{T \in \mathcal{T}_h} \|\mathrm{curl}\,(\boldsymbol{R}_h \boldsymbol{z})\|^2_{0,\Omega} + a \|\boldsymbol{v}_h\|^2_{\boldsymbol{L}^2(\Omega)} \\ &+ \sigma \big(J_1(\boldsymbol{v}_h, \boldsymbol{v}_h) + J_2(\boldsymbol{v}_h, \boldsymbol{v}_h) \big) \\ &\leq C \|\boldsymbol{z}\|^2_{1,\Omega}. \end{aligned}$$

Hence,

$$\|\boldsymbol{v}_h\|_{DG} \leq C \|q_h\|_{0,\Omega},$$

which yields to

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{B_h(q_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{DG}} \geq \frac{B_h(q_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{DG}} \geq \frac{\|q_h\|^2_{0,\Omega}}{C\sqrt{\nu} \|q_h\|_{0,\Omega}} \geq \beta \|q_h\|_{0,\Omega}.$$

□

Finally, it remains to discuss the continuity of the linear forms l_h and g_h.

Lemma 5.2.6. *We assume that $\boldsymbol{v}_h \in \boldsymbol{V}_h$, then there exists $c_l > 0$ and $c_g > 0$ such that*

$$|l_h(\boldsymbol{v}_h)| \leqslant c_l \|\boldsymbol{v}_h\|_{DG}. \tag{5.2.57}$$

$$|g_h(\boldsymbol{v}_h)| \leqslant c_g \|\boldsymbol{v}_h\| DG. \tag{5.2.58}$$

Proof. Let us begin by proving the continuity of l_h. Using Cauchy-Schwarz inequality, we note that

$$\int_\Omega \boldsymbol{f} . \boldsymbol{v}\, dx \leqslant \Big(\sum_{T \in \mathcal{T}_h} \int_T \boldsymbol{f}^2\, dx \Big)^{\frac{1}{2}} \Big(\sum_{T \in \mathcal{T}_h} \int_T \boldsymbol{v}_h^2\, dx \Big)^{\frac{1}{2}} \tag{5.2.59}$$

$$\leqslant \Big(\sum_{T \in \mathcal{T}_h} \|\boldsymbol{f}\|^2_{0,T} \Big)^{\frac{1}{2}} \|\boldsymbol{v}_h\|_{DG}. \tag{5.2.60}$$

Thanks again to Cauchy-Schwarz inequality, we have

$$\nu \sum_{e\in\Gamma_h} \int_{\boldsymbol{e}} \boldsymbol{h} \times \boldsymbol{n}.\boldsymbol{v}_h \times \boldsymbol{n}\,\mathrm{d}\,s \leqslant \nu\Big(\sum_{e\in\Gamma_h}\int_e |e|(\,\boldsymbol{h}\times\boldsymbol{n})^2\,ds\Big)^{\frac{1}{2}}\Big(\sum_{e\in\Gamma_h}\frac{1}{|e|}(\boldsymbol{v}_h\times\boldsymbol{n})^2\,ds\Big)^{\frac{1}{2}} \quad (5.2.61)$$

$$\leqslant \nu\sqrt{|e|}(\sum_{e\in\Gamma_h}\|\boldsymbol{h}\times\boldsymbol{n}\|_{0,e}^2)^{\frac{1}{2}}(J_1(\boldsymbol{v}_h,\boldsymbol{v}_h))^{\frac{1}{2}}. \quad (5.2.62)$$

Due to the trace inequality (5.2.5), it yields

$$\nu \sum_{e\in\Gamma_h} \int_{\boldsymbol{e}} \boldsymbol{h} \times \boldsymbol{n}.\boldsymbol{v}_h \times \boldsymbol{n}\,\mathrm{d}\,s \leqslant c\Big(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{h}\times\boldsymbol{n}\|_{0,T}^2\Big)^{\frac{1}{2}}\|\boldsymbol{v}_h\|_{DG}. \quad (5.2.63)$$

Now in order to control the third term in the linear form, we proceed with a similar argument, so we have,

$$\sum_{\boldsymbol{e}\in\Gamma_h}\frac{1}{|\boldsymbol{e}|}\int_{\boldsymbol{e}} \boldsymbol{g}.\,\boldsymbol{v}_h.\boldsymbol{n}\,\mathrm{d}\,s \leqslant \Big(\sum_{e\in\Gamma_h}\int_e\frac{1}{|e|}\,(\,\boldsymbol{g}\,)^2\,ds\Big)^{\frac{1}{2}}\Big(\sum_{e\in\Gamma_h}\frac{1}{|e|}\,\|\boldsymbol{v}_h.\boldsymbol{n}_e\|_{0,e}^2\Big)^{\frac{1}{2}}. \quad (5.2.64)$$

Based again on the trace inequality (5.2.5), we obtain

$$\sum_{\boldsymbol{e}\in\Gamma_h}\frac{1}{|\boldsymbol{e}|}\int_{\boldsymbol{e}} \boldsymbol{g}.\,\boldsymbol{v}_h.\boldsymbol{n}\,\mathrm{d}\,s \leqslant \frac{c}{\sqrt{|e|}}\Big(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{g}\|_{0,T}^2\Big)^{\frac{1}{2}}\|\boldsymbol{v}_h\|_{DG}. \quad (5.2.65)$$

The last term in l_h can be handled by using the same approach, we reach,

$$\gamma\sum_{e\in\Gamma_h}\int_e \operatorname{div}\boldsymbol{v}_h.\,\boldsymbol{g}\,dx \leqslant c\Big(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{g}\|_{0,T}^2\Big)^{\frac{1}{2}}\|v_h\|_{DG}. \quad (5.2.66)$$

Similarly, we obtain inequality (5.2.58) and then the continuity of g_h is established. □

Theorem 5.2.7. *We assume that σ is large enough then there exists at least a solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ for the linear problem*(5.2.4).

Proof. Based on Babuska-Brezzi's theorem and using the lemma 5.2.2, forms continuity proven in lemma 5.2.3, lemma 5.2.4, lemma 5.2.6 and the condition inf-sup in lemma 5.2.5 then there exists $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ solution of the linearized problem (5.2.4). □

5.3 Discontinuous Galerkin method for the non linear problem

In this section, we give the DG discretization of DBF problem, in order to establish an error estimate between the continuous solution $(\boldsymbol{u}, p)$ associate to (5.3.1)-(5.3.2)-(5.3.3) equations and the approximation solution $(\boldsymbol{u}_h, p_h)$ of the discrete problem (5.3.6).

5.3.1 Model problem

Let we have the incompressible DBF problem in an open bounded domain Ω.

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + a\boldsymbol{u} + b|\boldsymbol{u}|\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \quad \text{in } \Omega, && (5.3.1)\\ \operatorname{div} \boldsymbol{u} &= 0, \quad \text{in } \Omega, && (5.3.2) \end{aligned}$$

which satisfy the following boundary conditions :

$$\boldsymbol{u}\cdot\boldsymbol{n} = g, \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}. \quad \text{on } \Gamma. \tag{5.3.3}$$

A weak formulation of the previous system (5.3.1)-(5.3.2)-(5.3.3) can be discribed as follow

$$\begin{cases} \text{Find } (\boldsymbol{u}, p) \in \boldsymbol{H}^1_g(\Omega)\times L^2(\Omega). \text{ such that} \\ A(\boldsymbol{u},\, \boldsymbol{v}) + F(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + B(p,\, \boldsymbol{v}) = l(\boldsymbol{v}), & \forall \boldsymbol{v} \in \boldsymbol{H}^1_\tau(\Omega), \\ B(q,\, \boldsymbol{u}) = 0, & \forall q \in L^2(\Omega), \end{cases} \tag{5.3.4}$$

Where the trilinear form $F(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})$ is given by

$$F(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = b\int_\Omega |\boldsymbol{u}|\boldsymbol{u}\,.\,\boldsymbol{v}\;dx. \tag{5.3.5}$$

Here, the well posedness of probelm (5.3.4) can be also done by lifting the inhomogeneous boundary condition g to turn into the homogeneous case.

5.3.2 DG scheme

The DG finite element approximation for (5.3.1)-(5.3.2)-(5.3.3) equations can be defined as Find $(\boldsymbol{u}_h,\, q_h) \in \boldsymbol{V}_h \times Q_h$

$$\begin{cases} A_h(\boldsymbol{u}_h,\, \boldsymbol{v}_h) + F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + B_h(p_h,\, \boldsymbol{v}_h) = l_h(\boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ B_h(q_h,\, \boldsymbol{u}_h) = \boldsymbol{g}_h(q_h) & \forall q_h \in Q_h, \end{cases} \tag{5.3.6}$$

Such that $(\boldsymbol{u}_h, p_h)$ is the approximate solution of the problem above (5.3.6).

Now, we will apply the fixed point's theorem (Brouwer's theorem) in order to check that $(\boldsymbol{u}_h, p_h)$ is the unique discret solution for problem (5.3.6) .
The next theorem gives one of the main result of this chapter, where we discuss the existence and uniqueness of the approximation solution.

Theorem 5.3.1. *Let we have*

$$\theta = \frac{1}{\nu^2\alpha^2}\, c_F\; c_l\; c_I^2 < 1. \tag{5.3.7}$$

Then, there exists a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ *for the discontinuous Galerkin method* (5.3.6)*, which satisfies the estimate :*

$$\|\boldsymbol{u}_h\|_{DG} \leqslant \frac{c_l}{\nu\,\alpha} \tag{5.3.8}$$

$$\|p_h\|_{L^2(\Omega)} \leqslant (\frac{1}{\nu\alpha} c_A\, c_l + \frac{1}{\nu^2\alpha^2}\, c_F\, c_l^2 + c_l). \tag{5.3.9}$$

Moreover, we have

$$F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h) \leqslant \frac{1}{2\nu\alpha}\, c_l^2. \tag{5.3.10}$$

Proof. Let introduce the following space,

$$K_h = \{\boldsymbol{v}_h \in \boldsymbol{V}_h,\ B_h(\boldsymbol{v}_h, q_h) = 0,\ q_h \in Q_h\}. \tag{5.3.11}$$

Which allows us to neglect the pressure, so we obtain

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = l_h(\boldsymbol{v}_h). \tag{5.3.12}$$

We proceed with an operator S which characterizes a contraction on a sphere K_h whose only fixed point is the velocity $\boldsymbol{u}_h$.
To obtain the pressure properties, we use the identity:

$$B_h(\boldsymbol{v}_h, p_h) = l_h(\boldsymbol{v}_h) - A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h/K_h. \tag{5.3.13}$$

Step 1: A construction of the operator S
Let S to be an operator defined from K_h into itself, such that for any $w \in K_h$, $S(w) = \boldsymbol{u}_h$, which is the solution of the following variational formulation.

Find $\boldsymbol{u}_h \in K_h$, where

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + F_h(w, \boldsymbol{u}_h, \boldsymbol{v}) = l_h(\boldsymbol{v}_h). \tag{5.3.14}$$

Thanks to the previous theorem, the problem (5.3.14) has a unique solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$.
Now, by applying the coercivity of the form A_h from lemma 5.2.3 and the continuity of l_h from lemma 5.2.6,

$$\nu\alpha\|\boldsymbol{u}_h\|_{DG}^2 \leqslant A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) + F_h(w, \boldsymbol{u}_h, \boldsymbol{u}_h) = l_h(v_h) \tag{5.3.15}$$

$$\leqslant c_l\|\boldsymbol{u}_h\|_{DG} \tag{5.3.16}$$

Then the solution $\boldsymbol{u}_h$ satisfies the estimate,

$$\|\boldsymbol{u}_h\|_{DG} \leqslant \frac{c_l}{\nu\alpha}, \tag{5.3.17}$$

which infers that the operator S is well defined from $\boldsymbol{Z}_h$ into $\boldsymbol{Z}_h$ with

$$\boldsymbol{Z}_h = \{\boldsymbol{v}_h \in \boldsymbol{K}_h, \|\boldsymbol{v}_h\|_{DG} \leqslant \frac{c_l}{\nu\alpha}\} \tag{5.3.18}$$

Step 2: The operator S characterizes a contraction
Now, taking into account the smallness condition in the expresion of θ, we prove that S is a contraction.
In order to check that, let $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ be in $\boldsymbol{Z}_h$, moreover we put $\boldsymbol{u}_h^1 = S(\boldsymbol{w}_1)$ and $\boldsymbol{u}_h^2 = S(\boldsymbol{w}_2)$. For the sake of simple notations, we set $\boldsymbol{u}_h^1 = \boldsymbol{u}_1$, $\boldsymbol{u}_h^2 = \boldsymbol{u}_2$.

Now, due to the coercivity of the bilinear form A_h in the lemma 5.2.2

$$A_h(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \geqslant \nu\alpha \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{DG}^2. \tag{5.3.19}$$

Because, for any $\boldsymbol{v}_h \in \boldsymbol{K}_h$, we have

$$A_h(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{v}_h) = -F_h(\boldsymbol{w}_1, \boldsymbol{u}_1, \boldsymbol{v}_h) + F_h(\boldsymbol{w}_2, \boldsymbol{u}_2, \boldsymbol{v}_h). \tag{5.3.20}$$

Next, putting $\boldsymbol{v}_h = \boldsymbol{u}_1 - \boldsymbol{u}_2$, it gives that

$$\nu\alpha \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{DG}^2 \leqslant - F_h(\boldsymbol{w}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \tag{5.3.21}$$
$$+ F_h(\boldsymbol{w}_2, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - F_h(\boldsymbol{w}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) \tag{5.3.22}$$

In the one hand, using the fact that the form F_h is a positive definite operator in lemma 5.2.2,

$$-F_h(\boldsymbol{w}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leqslant 0. \tag{5.3.23}$$

In the other hand, according to the continuity property of the form F_h in lemma 5.2.4, we get

$$+F_h(\boldsymbol{w}_2, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - F_h(\boldsymbol{w}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leqslant c_F \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{L^2(\Omega)} \|\boldsymbol{u}_1\|_{L^4(\Omega)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^4(\Omega)}. \tag{5.3.24}$$

Using the relation (5.3.17) and because of (5.2.18), we obtain

$$F_h(\boldsymbol{w}_2, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - F_h(\boldsymbol{w}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leqslant \frac{1}{\nu\alpha} c_F\, c_l\, c_I^2 \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{DG} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{DG}. \tag{5.3.25}$$

Consequently, we conclude

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{DG} \leqslant \frac{1}{\nu^2\alpha^2} c_F\, c_l\, c_I^2 \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{DG}. \tag{5.3.26}$$

Hence, the operator S is a contraction if $\theta = \dfrac{1}{\nu^2\alpha^2} c_F\ c_l\ c_I^2$ satisfies the estimate (5.3.7).

Remark 5.3.2. Thanks to the contraction S defined from $\boldsymbol{V}_h$ into itself with the smallness condition $\theta < 1$, we have $S(\boldsymbol{u}_h) = \boldsymbol{u}_h$ is the fixed point which satisfies (5.3.6).

Step 3: The existence and uniqueness of the pressure $p_h \in Q_h$
To study the pressure p_h case, we note that for any $\boldsymbol{v}_h \in \boldsymbol{V}_h/\boldsymbol{K}_h$,

$$B_h(\boldsymbol{v}_h, p_h) = l_h(\boldsymbol{v}_h) - A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - F_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{v}_h). \tag{5.3.27}$$

According to the continuity property of the form l_h, A_h and F_h in lemma 5.2.6, lemma 5.2.3 and lemma 5.2.4, we infer that

$$l_h(\boldsymbol{v}_h) - A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - F_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{v}_h) \leqslant (c_l + c_A\|\boldsymbol{u}_h\|_{DG})\|\boldsymbol{v}_h\|_{DG} + c_F\|\boldsymbol{w}\|_{L^2(\Omega)}\|\boldsymbol{u}_h\|_{L^4(\Omega)}\|\boldsymbol{v}_h\|_{L^4(\Omega)}. \tag{5.3.28}$$

Because of the result (5.2.18), for any $\boldsymbol{v}_h \in \boldsymbol{V}_h/\boldsymbol{K}_h$ and using the relation (5.3.17),

$$l_h(\boldsymbol{v}_h) - A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - F_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{v}_h) \leqslant (c_l + \frac{1}{\nu\alpha}c_A\ c_l)\|\boldsymbol{v}_h\|_{DG} + \frac{1}{\nu\alpha}c_F c_I^2\ c_l\|\boldsymbol{w}\|_{DG}\|\boldsymbol{v}_h\|_{DG}. \tag{5.3.29}$$

We conclude that

$$l_h(\boldsymbol{v}_h) - A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - F_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{v}_h) \leqslant c_l(1 + \frac{1}{\nu\alpha}c_A + \frac{1}{\nu\alpha}c_F\, c_I^2\,\|\boldsymbol{w}\|_{DG})\|\boldsymbol{v}_h\|_{DG}. \tag{5.3.30}$$

Taking $\widetilde{\alpha} = c_l(1 + \frac{1}{\nu\alpha}c_A + \frac{1}{\nu\alpha}c_F\, c_I^2\,\|\boldsymbol{w}\|_{DG})$, we obtain the continuity of B_h. In addition to the Inf-sup condition in lemma 5.2.5 which gives the existence of a unique solution $p_h \in Q_h$ to the problem (5.3.27) [33, Theorem 1.4].

To recap, the couple $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h/\boldsymbol{K}_h \times Q_h$ is a unique solution to the discrete Galerkin method in (5.3.6).

Step 4: The stability bound for $\boldsymbol{u}_h$ and p_h

Because of $\boldsymbol{u}_h \in \boldsymbol{Z}_h$, so immediately the bound of $\boldsymbol{u}_h$ in (5.3.17) is obtained.
Now, in order to bound the form F_h, we use the same arguments as in step 1, i.e taking into account the coercivity property of A_h in lemma 5.2.2 and the fact that l_h is a continuous form (see lemma 5.2.6) with $\boldsymbol{v}_h = \boldsymbol{u}_h$ and by using Hölder's inequality,

$$\nu\alpha\|\boldsymbol{u}_h\|_{DG}^2 + F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h) \leqslant c_l\|\boldsymbol{u}_h\|_{DG}. \tag{5.3.31}$$

$$\leqslant \frac{1}{2}\frac{c_l^2}{\nu\alpha} + \frac{\nu\alpha}{2}\|\boldsymbol{u}_h\|_{DG}^2. \tag{5.3.32}$$

Consequently, it gives

$$F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h) \leqslant \frac{1}{2}\frac{c_l^2}{\nu\alpha}. \tag{5.3.33}$$

Next, we move to bound the pressure for any $p_h \in Q_h$.

$$B_h(\boldsymbol{v}_h, p_h) = -A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + l_h(\boldsymbol{v}_h). \quad \forall \boldsymbol{u}_h,\ \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{5.3.34}$$

Due to the continuity of A_h, F_h and the linear form l_h, one can obtain,

$$B_h(\boldsymbol{v}_h, p_h) \leqslant c_A\|\boldsymbol{u}_h\|_{DG}\|\boldsymbol{v}_h\|_{DG} + c_F\|\boldsymbol{u}_h\|_{DG}^2\|\boldsymbol{v}_h\|_{DG} + c_l\|\boldsymbol{v}_h\|_{DG}. \tag{5.3.35}$$

Using the result

$$B_h(\boldsymbol{v}_h, p_h) \leqslant \frac{1}{\nu\,\alpha}\, c_A\, c_l \|\boldsymbol{v}_h\|_{DG} + \frac{1}{\nu^2\alpha^2} c_F\, c_l^2 \|\boldsymbol{v}_h\|_{DG} + c_l \|\boldsymbol{v}_h\|_{DG}, \tag{5.3.36}$$

and thanks to the Inf-sup condition of the bilinear form B_h, we have

$$\|p_h\|_{L^2(\Omega)} \leqslant (\frac{1}{\nu\alpha} c_A\, c_l + \frac{1}{\nu^2\alpha^2}\, c_F\, c_l^2 + c_l). \tag{5.3.37}$$

Which finished the proof.

□

5.4 A priori error estimates

In this section, we study the error estimate for both velocity and pressure related to the replacement of $\boldsymbol{H}^1(\Omega)$ by the finite dimensional subspace $\boldsymbol{V}_h$. For this purpose, we give firstly some auxiliary results.

In the next lemma, we discuss the error analysis between the velocity $\boldsymbol{u}$ and the velocity obtained after applying the orthogonal projection $R_h(\boldsymbol{u})$, which is resided in the bilinear form A_h.

Lemma 5.4.1. *Let $\boldsymbol{u} \in \boldsymbol{H}^{m+1}$, then there exists a positive constant c independent of mesh size h such that*
For any $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$|A_h(\boldsymbol{u} - R_h(\boldsymbol{u}), \boldsymbol{v}_h)| \leqslant ch^m \|\boldsymbol{v}_h\|_{DG} |\boldsymbol{u}|_{\boldsymbol{H}^{m+1}}. \tag{5.4.1}$$

Where $c = max(a, \nu)$.

Proof. Let us recall that

$$A_h(.\ ,\ .) = a_0(\ .,\ .) + a_1(.\ ,\ .) + a_2(.\ ,\ .) + J_1(.\ ,\ .) + J_2(.\ ,\ .). \tag{5.4.2}$$

First, we restrict our attention to estimate $a_0(.,.)$ based on the relation (5.2.3), it follows

$$\begin{aligned} a_0(\boldsymbol{u} - R_h(\boldsymbol{u}), \boldsymbol{v}_h) \leqslant & \nu\Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\,(\boldsymbol{u} - R_h(\boldsymbol{u}))\|_{0,T}^2\Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\,\boldsymbol{v}_h\|_{0,T}^2\Big)^{\frac{1}{2}} \\ & + \nu\Big(\sum_{T\in\mathcal{T}_h} \|\mathrm{div}\,(\boldsymbol{u} - R_h(\boldsymbol{u}))\|_{0,T}^2\Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \|\mathrm{div}\,\boldsymbol{v}_h\|_{0,T}^2\Big)^{\frac{1}{2}} \\ & a\Big(\sum_{T\in\mathcal{T}_h} \|\boldsymbol{u} - R_h(\boldsymbol{u})\|_{0,T}^2\Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \|\boldsymbol{v}_h\|_{0,T}^2\Big)^{\frac{1}{2}}. \end{aligned} \tag{5.4.3}$$

By the definition of the norm on $\boldsymbol{V}_h$, we obtain,

$$a_0(\boldsymbol{u}-R_h(\boldsymbol{u}),\boldsymbol{v}_h)\leqslant\Big\{\nu\Big((\sum_{T\in\mathcal{T}_h}\|\mathbf{curl}\,(\boldsymbol{u}-R_h(\boldsymbol{u}))\|^2_{0,T})^{\frac12}+(\sum_{T\in\mathcal{T}_h}\|\mathrm{div}\,(\boldsymbol{u}-R_h(\boldsymbol{u}))\|^2_{0,T})^{\frac12}\Big) \tag{5.4.4}$$

$$+\,a(\sum_{T\in\mathcal{T}_h}\|(\boldsymbol{u}-R_h(\boldsymbol{u}))\|^2_{0,T})^{\frac12}\Big\}\sum_{T\in\mathcal{T}_h}\|\boldsymbol{v}_h\|_{0,T}. \tag{5.4.5}$$

Now, due to relation (5.2.14), it follows

$$a_0(\boldsymbol{u}-R_h(\boldsymbol{u}),\boldsymbol{v}_h)\leqslant c\,h^m|\boldsymbol{u}|_{H^{m+1}}\|\boldsymbol{v}_h\|_{DG}. \tag{5.4.6}$$

Such that $c=max(\nu,a)$.

Using Cauchy-Schwarz inequality in the first jump term, we obtain

$$J_1(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u},\boldsymbol{v}_h)\ \ \leq\ \ c\left(\sum_{e\in\mathcal{C}_h^{int}}\frac{1}{|e|}\|[\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}]\|^2_{0,e}\right)^{1/2}\|\boldsymbol{v}_h\|_{DG}.$$

Let e be an internal surface common to the tetrahedrons T_1 and T_2. The proof is completely similar for a boundary triangular surface. using the Trace inequality:

$$\frac{1}{\sqrt{|e|}}\|[(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\cdot\boldsymbol{n}_e]\|_{0,e}\leq\quad c\quad\Big(\frac{1}{h_{T_1}}\|\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}\|_{0,T_1}+\frac{1}{h_{T_2}}\|\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}\|_{0,T_2}+|\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}|_{1,T_1}$$
$$+\quad|\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}|_{1,T_1}\Big) \tag{5.4.7}$$

Thanks to (5.2.14), it follows

$$J_1(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u},\boldsymbol{v}_h)\leq ch^m\left(\sum_{e\in\mathcal{C}_h^{int}}\frac{1}{|e|}\|[\boldsymbol{v}_h]\|^2_{0,e}\right)^{1/2}|\boldsymbol{u}|_{m+1,\Omega}\leq ch^m\|\boldsymbol{v}_h\|_{DG}\,|\boldsymbol{u}|_{m+1,\Omega}.$$

We can easily check that the same bound follows for the second jump term $J_2(\cdot,\cdot)$.
On the one hand, the property (5.2.11) of the interpolation operator $\boldsymbol{R}_h$ gives

$$\sum_{e\in\mathcal{C}_h^{int}}\int_e\langle\mathrm{curl}\,\boldsymbol{v}_h\rangle\cdot[(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\times\boldsymbol{n}_e]=0$$

And on the other hand

$$\sum_{e\in\mathcal{C}_h^{int}\cup\Gamma_h}\int_e\langle\mathrm{div}\,\boldsymbol{v}_h\rangle\cdot[(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\cdot\boldsymbol{n}_e]=0$$

Because of $\langle\mathrm{curl}\boldsymbol{v}_h\rangle$ and $\langle\mathrm{div}\boldsymbol{v}_h\rangle$ belong to $\boldsymbol{P}_{m-1}$ on every surface.
So, we only have to bound the remainning parts of the terms a_1 and a_2 respectively

$$\nu\sum_{e\in\mathcal{C}_h^{int}}\int_e\langle\mathrm{curl}(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\rangle\cdot[\boldsymbol{v}_h\times\boldsymbol{n}_e] \tag{5.4.8}$$

and

$$\nu \sum_{e\in\mathcal{C}_h^{int}\cup\Gamma_h} \int_e \langle \mathrm{div}(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\rangle \cdot [\boldsymbol{v}_h\cdot\boldsymbol{n}_e]. \tag{5.4.9}$$

For this purpose, let us introduce the classical Lagrange interpolation operator of polynomial degree m, denoted by $\boldsymbol{L}_h$, and let us insert it in the two last terms. Then, we can write that

$$\begin{aligned}
\int_e \langle \mathrm{curl}\,(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\rangle \cdot [\boldsymbol{v}_h\times\boldsymbol{n}_e]ds &= \int_e \langle \mathrm{curl}\,(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\rangle \cdot [\boldsymbol{v}_h\times\boldsymbol{n}_e]ds\\
&+ \int_e \langle \mathrm{curl}\,(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\rangle \cdot [\boldsymbol{v}_h\times\boldsymbol{n}_e]ds\\
&\le \|[\boldsymbol{v}_h\times\boldsymbol{n}_e]\|_{0,e}\|\langle \mathrm{curl}\,(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\rangle\|_{0,e}\\
&+ \|[\boldsymbol{v}_h\times\boldsymbol{n}_e]\|_{0,e}\|\langle \mathrm{curl}(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\rangle\|_{0,e}.
\end{aligned}$$

Thanks to the same trace inequality as in (5.4.7), one obtain

$$\frac{1}{\sqrt{|e|}}\|\langle \mathrm{curl}\,(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\rangle\|_{0,e} \le ch^{m-1}|\boldsymbol{u}|_{m+1,T_1\cup T_2}.$$

Then,

$$\begin{aligned}
\sum_{e\in\mathcal{C}_h^{int}} \int_e \langle \mathrm{curl}\,(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\rangle \cdot [\boldsymbol{v}_h\times\boldsymbol{n}_e]ds &\le ch^m \left(\sum_{e\in\mathcal{C}_h^{int}} \frac{1}{|e|}\|[\boldsymbol{v}_h\times\boldsymbol{n}_e]\|_{0,e}^2\right)^{1/2} |\boldsymbol{u}|_{m+1,\Omega}\\
&\le c\,h^m\|\boldsymbol{v}_h\|_{DG}\,|\boldsymbol{u}|_{m+1,\Omega}.
\end{aligned}$$

Next, using that $\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}$ is a piecewise polynomial, we obtain by means of a scaling argument that :

$$\frac{1}{\sqrt{|e|}}\|\{\mathrm{curl}\,(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\}\|_{0,e} \le c\left(\frac{1}{h_{T_1}}\|\mathrm{curl}\,(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\|_{0,T_1} + \frac{1}{h_{T_2}}\|\mathrm{curl}\,(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\|_{0,T_2}\right).$$

By means of the triangle inequality, we can deduce that on any tetrahedral T,

$$\begin{aligned}
\|\mathrm{curl}(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\|_{0,T} &\le |\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{u}|_{1,T} + |\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}|_{1,T}\\
&\le ch_T^m|\boldsymbol{u}|_{m+1,\Delta_T}
\end{aligned}$$

So finally,

$$\sum_{e\in\mathcal{C}_h^{int}} \int_e \langle \mathrm{curl}\,(\boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\rangle \cdot [\boldsymbol{v}_h\times\boldsymbol{n}_e]ds \le c\,h^m(J_1(\boldsymbol{v}_h,\boldsymbol{v}_h))^{\frac{1}{2}}\,|\boldsymbol{u}|_{m+1,\Omega}.$$

The term in (5.4.9) has the same structure as the one in (5.4.8), thus it satisfies the bound

$$\sum_{e\in\mathcal{C}_h^{int}\cup\Gamma_h} \int_e \langle \mathrm{div}\,(\boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\rangle \cdot [\boldsymbol{v}_h\cdot\boldsymbol{n}_e]ds \le c\,h^m(J_2(\boldsymbol{v}_h,\boldsymbol{v}_h))^{\frac{1}{2}}\,|\boldsymbol{u}|_{m+1,\Omega}.$$

It is now sufficient the rearrangement of the previous estimates in order to end the proof. □

Lemma 5.4.2. *We assume that $p \in H^m(\Omega)$, then there exists a constant $c > 0$ independent of h, then*

$$\text{For any } \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad B_h(p - r_h(p), \boldsymbol{v}_h) \leqslant c\, h^m \|\boldsymbol{v}_h\| |p|_{H^m}. \tag{5.4.10}$$

Proof. We recall that

$$\begin{aligned} B_h(p - r_h p, \boldsymbol{v}_h) &= -\sum_{T \in \mathcal{T}_h} \int_T (p - r_h p) \operatorname{div} \boldsymbol{v}_h \, dx + \sum_{e \in \mathcal{C}_h^{int}} \int_e \langle p - r_h p \rangle \, [\boldsymbol{v}_h \cdot \boldsymbol{n}_e] \, ds \\ &= \sum_{e \in \mathcal{C}_h^{int}} \int_e \langle p - r_h p \rangle \, [\boldsymbol{v}_h \cdot \boldsymbol{n}_e] \, ds. \end{aligned}$$

Then,

$$|B_h(p - r_h p, \boldsymbol{v}_h)| \leqslant c \left(\sum_{e \in \mathcal{C}_h^{int}} |e| \, \|\langle p - r_h p \rangle\|_{0,e}^2 \right)^{1/2} \|\boldsymbol{v}_h\|_{DG}.$$

On each triangular surface $e \in \mathcal{C}_h^{int}$ such that $e \subset \partial T$, we bound the term $\|\langle p - r_h p \rangle\|_{0,e}$ as in the proof of the previous Lemma. Denoting by L_h the Lagrange interpolation operator on P_{m-1} for $m = 2$ or 3, we obtain:

$$\begin{aligned} \sqrt{|e|} \|p - r_h p\|_{0,\Omega} &\leq \sqrt{|e|} (\|p - L_h p\|_{0,\Omega} + \|L_h p - r_h p\|_{0,\Omega}) \\ &\leq c \left(\|p - L_h p\|_{0,T} + h_T |p - L_h p|_{1,T} + \|L_h p - r_h p\|_{0,T} \right) \\ &\leq c h^m |p|_{m,T}. \end{aligned}$$

For $m = 1$, we directly have:

$$\sqrt{|e|} \|p - r_h p\|_{0,\Omega} \leq c \left(\|p - r_h p\|_{0,T} + h_T |p - r_h p|_{1,T} \right) \leq c h |p|_{1,T}.$$

So the announced result holds. □

Now, we move to analyze the error estimate of both the approximation velocity $\boldsymbol{u}_h$ and the pressure p_h.

Theorem 5.4.3. *Assume that $(\boldsymbol{u}, p) \in \boldsymbol{H}^{m+1}(\Omega) \times H^m(\Omega)$ is a solution for (5.3.1) problem, then there exists a positive constant c such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{DG}^2 \leqslant c h^{2m} |\boldsymbol{u}|_{m+1}^2 + c h^{2m} |p|_m^2 \tag{5.4.11}$$

Proof. Let us begin by putting $\widetilde{\boldsymbol{u}} = \boldsymbol{R}_h(\boldsymbol{u})$ and $\widetilde{p} = r_h(p)$.
We denote that

$$\chi = \boldsymbol{u}_h - \widetilde{\boldsymbol{u}}, \qquad \varepsilon = p_h - \widetilde{p} \tag{5.4.12}$$

$$\xi = \widetilde{\boldsymbol{u}} - \boldsymbol{u}, \qquad \eta = \widetilde{p} - p. \tag{5.4.13}$$

Because of ξ and η have been already controlled by the interpolation operator (5.2.14) and (5.2.15), then we need only to estimate the remainder terms.

The error equations give,

$$\begin{cases} A_h(\boldsymbol{u}_h-\boldsymbol{u},\,\boldsymbol{v}_h)+F_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h)-F_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_h)+B_h(p_h-p,\boldsymbol{v}_h)=0, & \forall \boldsymbol{v}_h\in \boldsymbol{V}_h,\\ B_h(q_h,\,\boldsymbol{u}-\boldsymbol{u}_h)=\boldsymbol{g}_h(q_h)=0 & \forall q_h\in Q_h, \end{cases} \tag{5.4.14}$$

Because for any $q_h\in Q_h$, we have $B_h(q_h,\,\boldsymbol{u}-\boldsymbol{u}_h)=0$.
Moreover, due to the orthogonality of the operator $\boldsymbol{R}_h$, we have $B_h(q_h,\,\widetilde{\boldsymbol{u}}-\boldsymbol{u})=0$.
Then, it infers immediately $B_h(q_h,\boldsymbol{u}_h-\widetilde{\boldsymbol{u}})=0$
Since $\varepsilon=p_h-\widetilde{p}\in Q_h$ which implies that $B_h(\boldsymbol{v}_h,\varepsilon)=0$, we obtain the error equations,

$$\begin{cases} A_h(\chi,\,\boldsymbol{v}_h)+F_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h)-F_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_h)=A_h(\xi\,,\,\boldsymbol{v}_h)-B_h(\boldsymbol{v}_h,\eta), & \forall \boldsymbol{v}_h\in \boldsymbol{V}_h,\\ B_h(q_h,\,\chi)=B_h(q_h,\,\xi)=0 & \forall q_h\in Q_h, \end{cases} \tag{5.4.15}$$

Now, we set $\boldsymbol{v}_h=\chi$, the error equations becomes

$$A_h(\chi\,,\,\chi)=-F_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\chi)+F_h(\boldsymbol{u},\boldsymbol{u},\chi)-A_h(\xi\,,\,\chi)-B_h(\chi\,,\,\eta) \tag{5.4.16}$$

By applying the coercivity of A_h and using the lemma 5.4.1 and lemma 5.4.2,

$$\nu\alpha\|\chi\|_{DG}^2\leqslant F_h(\boldsymbol{u},\boldsymbol{u},\chi)-F_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\chi)+ch^m\|\chi\|_{DG}|\boldsymbol{u}|_{m+1,\Omega}+c\,h^m\|\chi\|_{DG}|p|_{m,\Omega}. \tag{5.4.17}$$

Due to Hölder's inequality,

$$\begin{aligned}\nu\alpha\|\chi\|_{DG}^2 &\leqslant F_h(\boldsymbol{u},\boldsymbol{u},\chi)-F_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\chi)+\frac{\nu\alpha}{8}\|\chi\|_{DG}^2+ch^{2m}|\boldsymbol{u}|_{m+1}^2+\frac{\nu\alpha}{8}\|\chi\|_{DG}^2+c\,h^{2m}|p|_{m,\Omega}^2.\\ &\leqslant \frac{\nu\alpha}{4}\|\chi\|_{DG}^2+c\,h^{2m}|\boldsymbol{u}|_{m+1,\Omega}^2+c\,h^{2m}|p|_{m,\Omega}^2+F_h(\boldsymbol{u},\boldsymbol{u},\chi)-F_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\chi).\end{aligned} \tag{5.4.18}$$

Then, it remains to estimate $F_h(\boldsymbol{u},\boldsymbol{u},\chi)-F_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\chi)$, Obviously,

$$F_h(\boldsymbol{u},\boldsymbol{u},\chi)-F_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\chi)=(|\boldsymbol{u}|^\alpha\boldsymbol{u}-|\boldsymbol{u}_h|^\alpha\boldsymbol{u}_h,\chi). \tag{5.4.19}$$

$$=(|\boldsymbol{u}|^\alpha\boldsymbol{u}-|\boldsymbol{u}_h|^\alpha\boldsymbol{u}_h,\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}+\boldsymbol{u}-\boldsymbol{u}). \tag{5.4.20}$$

$$=(|\boldsymbol{u}|^\alpha\boldsymbol{u}-|\boldsymbol{u}_h|^\alpha\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{u})+(|\boldsymbol{u}|^\alpha\boldsymbol{u}-|\boldsymbol{u}_h|^\alpha\boldsymbol{u}_h,\boldsymbol{u}-\widetilde{\boldsymbol{u}}). \tag{5.4.21}$$

Because of the monotonocity relation (5.2.6), it infers

$$(|\boldsymbol{u}|^\alpha\boldsymbol{u}-|\boldsymbol{u}_h|^\alpha\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{u})\leqslant 0. \tag{5.4.22}$$

Hence, we should only treat the term $(|\boldsymbol{u}|^{\alpha}\boldsymbol{u} - |\boldsymbol{u}_h|^{\alpha}\boldsymbol{u}_h, -\xi)$, so for this purpose, we proceed as follow

$$\begin{aligned}(|\boldsymbol{u}|^{\alpha}\boldsymbol{u} - |\boldsymbol{u}_h|^{\alpha}\boldsymbol{u}_h, -\xi) &= F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \xi) - F_h(\boldsymbol{u}, \boldsymbol{u}, \xi). \\ &= F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \xi) - F_h(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h + \boldsymbol{u}_h, \xi). \\ &= -F_h(\boldsymbol{u}, \chi + \xi, \xi) + F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \xi) - F_h(\boldsymbol{u}, \boldsymbol{u}_h, \xi). \qquad (5.4.23)\end{aligned}$$

Now, by using the continuity of the form F_h, we control the first term

$$F_h(\boldsymbol{u}, \chi + \xi, \xi) \leqslant c_F \|\boldsymbol{u}\|_{DG} \|\chi + \xi\|_{DG} \|\xi\|_{DG}. \qquad (5.4.24)$$

Thanks to the triangle inequality, and using the approximation property (5.2.14), one can see

$$\begin{aligned}F_h(\boldsymbol{u}, \chi + \xi, \xi) &\leqslant c_F \|\boldsymbol{u}\|_{DG} (\|\chi\|_{DG} + \|\xi\|_{DG}) \|\xi\|_{DG}. \\ &\leqslant c_F C\, ch^m |\boldsymbol{u}|_{m+1} \|\chi\|_{DG} + c_F\, C\, ch^{2m} |\boldsymbol{u}|^2_{m+1}. \qquad (5.4.25)\end{aligned}$$

By applying Hölder's inequality,

$$F_h(\boldsymbol{u}, \chi + \xi, \xi) \leqslant \frac{\nu\alpha}{8} \|\chi\|^2 + \mu_1 h^{2m} |\boldsymbol{u}|^2_{m+1}. \qquad (5.4.26)$$

Where $\mu_1 = \frac{1}{\nu\alpha} 2\, C\, c_F\, c$.

Finally, let us control the estimate $F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \xi) - F_h(\boldsymbol{u}, \boldsymbol{u}_h, \xi)$.
One can observe that

$$\begin{aligned}F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \xi) - F_h(\boldsymbol{u}, \boldsymbol{u}_h, \xi) &= \langle (|\boldsymbol{u}_h| - |\boldsymbol{u}|).\boldsymbol{u}_h\, ,\, \xi \rangle. \qquad (5.4.27) \\ &\leqslant \langle |\boldsymbol{u}_h - \boldsymbol{u}| \boldsymbol{u}_h\, ,\, \xi \rangle. \qquad (5.4.28) \\ &= F_h(\,\chi + \xi,\ \boldsymbol{u}_h,\ \xi). \qquad (5.4.29)\end{aligned}$$

Using again the trivial inequality and according to the continuity of the form F_h in lemma 5.2.4, it gives

$$F_h(\,\chi + \xi,\ \boldsymbol{u}_h,\ \xi) \leqslant c_F (\|\chi\|_{DG} + \|\xi\|_{DG}) \|\boldsymbol{u}_h\|_{DG} \|\xi\|_{DG}. \qquad (5.4.30)$$

Based on the bound of the approximation velocity in relation (5.3.8) and the approximation inequality (5.2.14),

$$\begin{aligned}F_h(\,\chi + \xi,\ \boldsymbol{u}_h,\ \xi) &\leqslant \frac{1}{\nu\alpha} c_F\, c_l\, (\|\chi\|_{DG} + ch^m |\boldsymbol{u}|_{m+1}) ch^m |\boldsymbol{u}|_{m+1} \qquad (5.4.31) \\ &\leqslant \frac{1}{\nu\alpha} c_F\, c_l\, ch^m |\boldsymbol{u}|_{m+1} \|\chi\|_{DG} + \frac{1}{\nu\alpha} c_F\, c_l\, ch^{2m} |\boldsymbol{u}|^2_{m+1}. \qquad (5.4.32)\end{aligned}$$

Due to Hölder's inequality, we obtain

$$F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \xi) - F_h(\boldsymbol{u}, \boldsymbol{u}_h, \xi) \leqslant \frac{\nu\alpha}{8} \|\chi\|^2_{DG} + \mu_2 h^{2m} |\boldsymbol{u}|^2_{m+1}. \qquad (5.4.33)$$

Such that $\mu_2 = \max(\frac{4}{\nu^3\alpha^3}\, c_F^2\, c_l^2, \frac{1}{\nu\alpha} c_F\, c_l)$.
To recap, using relation (5.4.18), (5.4.22), (5.4.26) and according to results (5.4.23) and (5.4.33), we reach

$$\|\chi\|_{DG}^2 \leqslant \nu^{-1}\alpha^{-1}(ch^{2m}|\boldsymbol{u}|_{m+1}^2 + ch^{2m}|p|_m^2). \tag{5.4.34}$$

Which completes the proof. □

Theorem 5.4.4. *Under the same assumption of the previous theorem, then there exists a constant $C > 0$ such that,*

$$\|p - p_h\|_{Q_h} \leqslant C\, h^m(|\boldsymbol{u}|_{m+1} + |p|_m). \tag{5.4.35}$$

Proof. Based on the inf-sup condition

$$\|p_h - \widetilde{p}\|_{Q_h} \leqslant \frac{1}{\beta} \sup_{\substack{\boldsymbol{v}\neq 0\\ \boldsymbol{v}\in \boldsymbol{V}_h}} \frac{B_h(v_h, p_h - \widetilde{p})}{\|\boldsymbol{v}_h\|_{DG}} \tag{5.4.36}$$

Due to the error equation, it gives that, for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$.

$$B_h(\boldsymbol{v}_h, p_h - \widetilde{p}) = -\, A_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) + F_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) - F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) \tag{5.4.37}$$

$$+\, B_h(\boldsymbol{v}_h, p - \widetilde{p}). \tag{5.4.38}$$

Thanks to lemma 5.4.1, lemma 5.4.2 and the continuity of A_h, it gives

$$B_h(\boldsymbol{v}_h, p_h - \widetilde{p}) \leqslant c\, h^m \|\boldsymbol{v}_h\|_{DG} |\boldsymbol{u}|_{m+1} + c_A \|\boldsymbol{v}_h\|_{DG} \|\chi\|_{DG} + c\, h^m \|\boldsymbol{v}_h\|_{DG} |p|_m \tag{5.4.39}$$

$$+\, F_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) - F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h). \tag{5.4.40}$$

It remains to treat the term $F_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) - F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h)$, so for this purpose, let

$$\begin{aligned} F_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) - F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) &= F_h(\boldsymbol{u}, \chi + \xi, \boldsymbol{v}_h) + F_h(\boldsymbol{u}, \boldsymbol{u}_h, \boldsymbol{v}_h) - F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h). \\ &\leqslant F_h(\boldsymbol{u}, \chi + \xi, \boldsymbol{v}_h) + F_h(\chi + \xi, \boldsymbol{u}_h, \boldsymbol{v}_h). \\ &\leqslant c_F \|\boldsymbol{u}\|_{DG} \|\chi + \xi\|_{DG} \|\boldsymbol{v}_h\|_{DG} + c_F \|\chi + \xi\|_{DG} \|\boldsymbol{u}_h\|_{DG} \|\boldsymbol{v}_h\|_{DG}. \end{aligned} \tag{5.4.41}$$

According to the previous theorem, the bound of the approximation velocity (5.3.8), it gives

$$F_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) - F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) \leqslant c_F\, C\, (ch^{2m}|\boldsymbol{u}|_{m+1}^2 + ch^{2m}|p|_m^2)^{\frac{1}{2}} \|\boldsymbol{v}_h\|_{DG} \tag{5.4.42}$$

$$+\, \frac{1}{\nu\alpha} c_F c_l (ch^{2m}|\boldsymbol{u}|_{m+1}^2 + ch^{2m}|p|_m^2)^{\frac{1}{2}} \|\boldsymbol{v}_h\|_{DG}. \tag{5.4.43}$$

Consequently,

$$F_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) - F_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) \leqslant ch^m(|\boldsymbol{u}|_{m+1}^2 + |p|_m^2)^{\frac{1}{2}} \|\boldsymbol{v}_h\|_{DG}. \tag{5.4.44}$$

Hence, we obtain

$$\|p_h - \widetilde{p}\|_{Q_h} \leqslant \frac{1}{\beta}\, c\, h^m |\boldsymbol{u}|_{m+1} + \frac{1}{\beta}\, c_A \nu^{-\frac{1}{2}} \alpha^{-\frac{1}{2}} c h^m (|\boldsymbol{u}|^2_{m+1} + |p|^2_m)^{\frac{1}{2}} \tag{5.4.45}$$

$$+ \frac{1}{\beta} c h^m |p|_m + \frac{1}{\beta}\, c\, h^m (|\boldsymbol{u}|^2_{m+1} + |p|^2_m)^{\frac{1}{2}}. \tag{5.4.46}$$

$$\leqslant c h^m (|\boldsymbol{u}|_{m+1} + |p|_m) + c h^m (|\boldsymbol{u}|^2_{m+1} + |p|^2_m)^{\frac{1}{2}}. \tag{5.4.47}$$

We conclude that the announced result in lemma (5.4.35) holds.

□

5.5 Numerical tests

In this section, we present a several numerical experiments with the Discontinuous Galerkin (DG) method implemented with FreeFem++ software to prove that the method is well suited for the construction of robust high-order numerical schemes on unstructured grids for a variety of problems.

FreeFem++ is a partial differential equation solver that has an advanced automatic mesh generator, it uses fast algorithms such as the multi-frontal method UMFPACK, Super LU. Several triangular finite elements, including discontinuous elements.

FreeFem++ provides tools to define discontinuous Galerkin finite element formulations. The known types of finite element are:

- `P1dc` piecewise linear discontinuous,
- `P2dc` piecewise quadratic discontinuous,
- `P3dc` piecewise cubic discontinuous (need `load` "Element_P3dc"),
- `P4dc` piecewise quartic discontinuous (need `load` "Element_P4dc"),

and keywords: `jump`, `mean`, `intalledges`, `nTonEdge`, `lenEdge`.

From the viewpoint of the user, this lead to a more readable code:

```
fespace Vh(Th,P2dc);      // Discontinous P2 finite element
intalledges(Th)(...)//  loop on all  edge of all triangle
nTonEdge // gives the number of adjacent triangle of the current edge
            =1 on border edge and =2 on internal
jump((N.x*v1+N.y*v2) // N.x,N.y is the normal to the edge
lenEdg // gives the length of the current edge
```

5.5.1 Example 1 : Application on the current work in L shape geometry

Let us consider a nonconvex L-shaped domain $\Omega = [-1,1]^2/]0,1[^2$. The forcing term is $f = (x_2, 0)$ and the following boundary conditions.

$$\mathbf{curl}\, \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{u}_0 \text{ on } \Gamma \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} = \begin{cases} x_2^2 - 1, & \text{if} \quad x_1 = -1,\ -1 \leqslant x_2 \leqslant 1, \\ -8x_2(1+x_2) & \text{if} \quad x_1 = 1,\ -1 \leqslant x_2 \leqslant 0, \\ 0 & \text{elsewhere on } \Gamma. \end{cases} \tag{5.5.1}$$

We focus first on the behavior of the scheme (5.3.6) which corresponding to the problem (5.3.4). We set the parameters values: $\gamma = 0.01$, $a = 1$, $\sigma = 50$ and $b = 1$. The approximation solution $\boldsymbol{u}_h$ illustrated in the Figure 1 is computed by using refined mesh (758 triangles and 424 vertices).

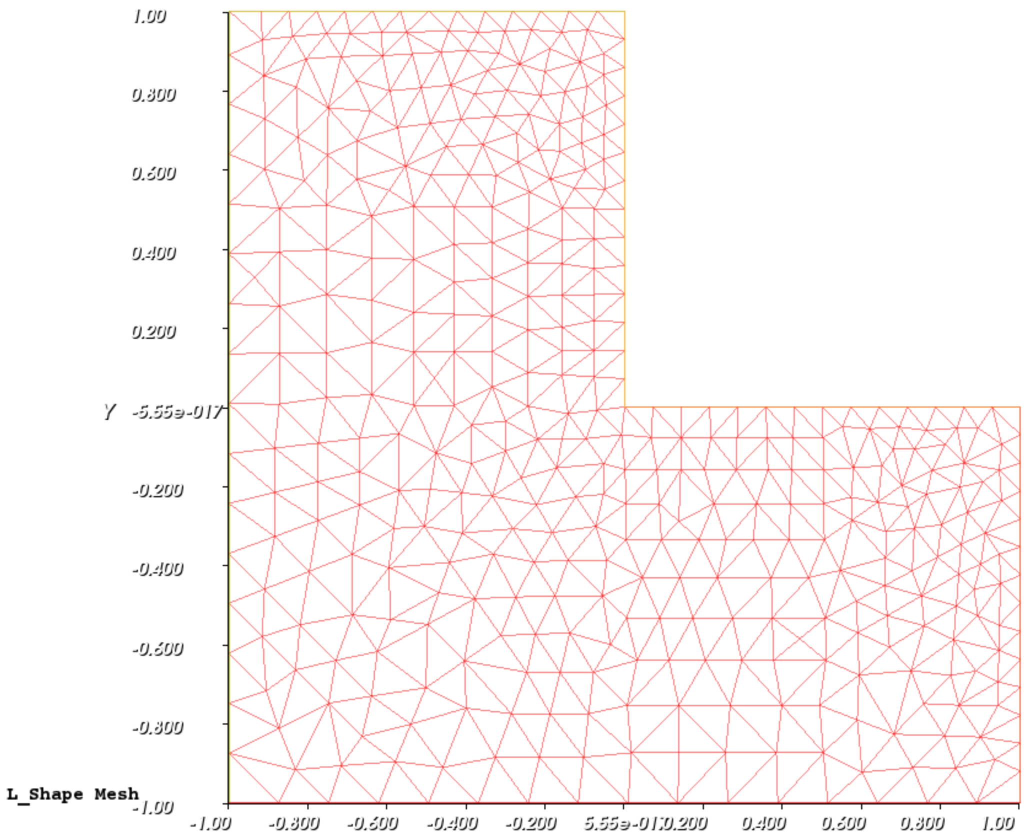


Figure 1: L shape geometry.

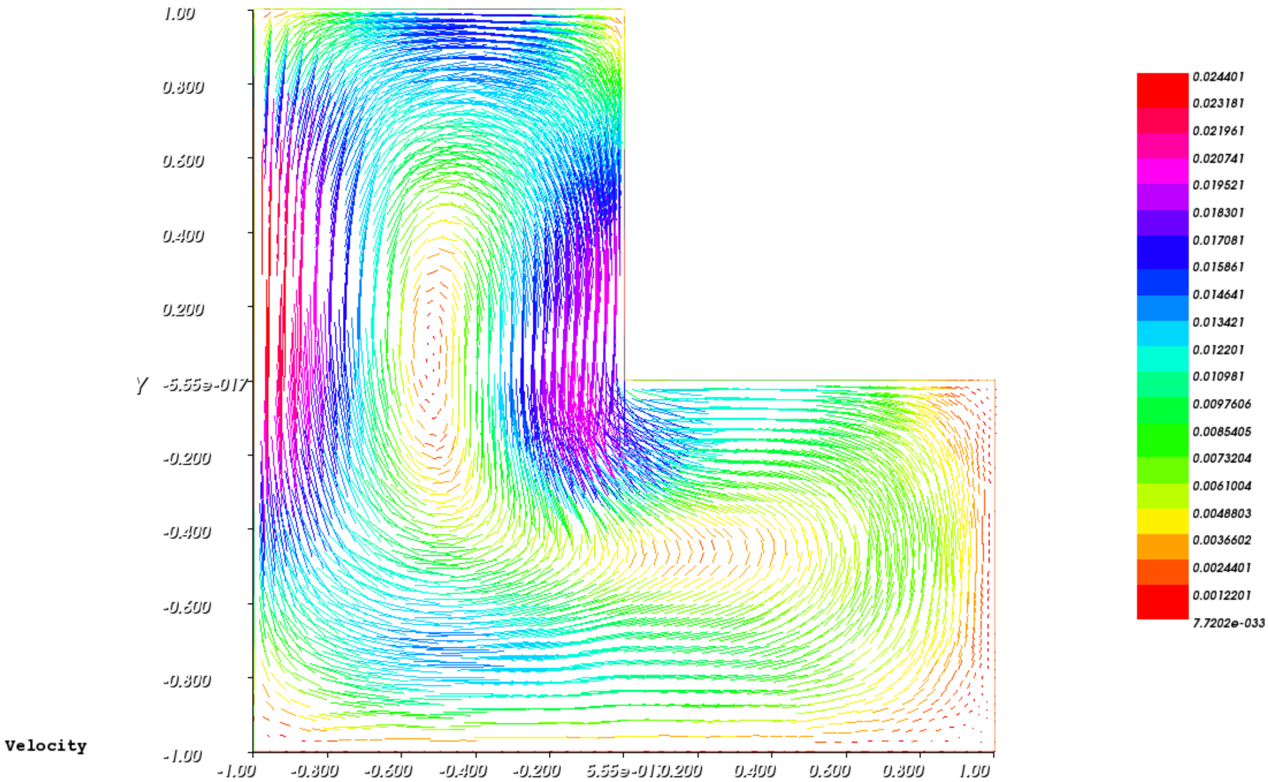


Figure 2: Contour plots of the approximated velocity components.

5.5.2 Example 2 : Flow fluid through a bifurcate pipe

Let consider a bidimensional pipe network with T shape geometry. The boundary is splitted into Γ_1 and Γ_2. Γ_1 composed of three components. The bottom of T is defined by Γ_{11} the inflow part while Γ_{12} and Γ_{13} introduce the outflow which are located the two extremes of the horizental branch. Γ_2 is constitued of the laterals surfaces of the pipes. The objectif of this example is to illustrate the approximating velocity $\boldsymbol{u}_h$ while the exacte solution $\boldsymbol{u}$ correspondaing to the non linear BF problem (2.3.1) given in chapter 2 which modelize the flows in a network of pipes.

To solve numerically the problem (2.3.1), we set the phsical coefficients $\gamma = 0.025$, $a = 1$, $b = 1$, $\sigma = 50$ and the data $\boldsymbol{f} = \boldsymbol{0}$, $\pi_0 = C_i$ on Γ_{2i}, $i = 1, 2, 3$.

In order to get the Figures below we should use a refined mesh with 1800 elements and 871 nodes.

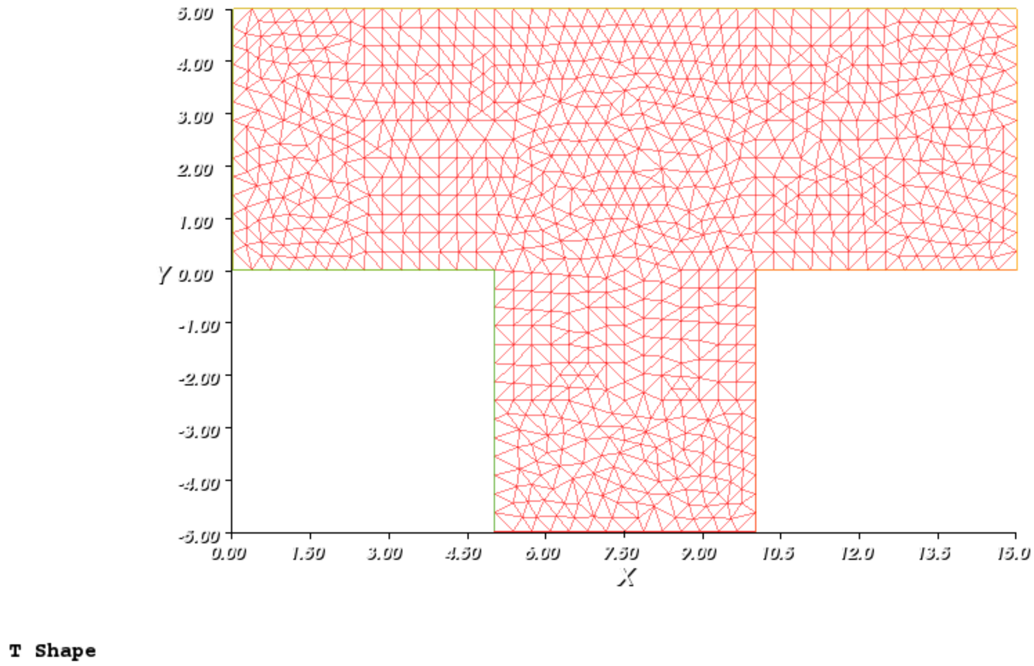


Figure 3: Mesh fot the T-shaped geometry.

5.5.3 Example 3 : Bercovier-Engelman test

Let us first introduce the vorticity $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u}$. We are interested to treat the following problem :

$$\gamma\mathbf{curl}\,\boldsymbol{w} + a\boldsymbol{u} + b|\boldsymbol{u}|\boldsymbol{u} + \nabla\pi = 0 \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega,$$

with the boundary conditions :

$$\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{u}_0\cdot\boldsymbol{n}, \qquad \boldsymbol{w} = \boldsymbol{w}_0 \qquad \text{on } \Gamma.$$

In this example, we solve the following scheme :

$$\begin{cases} A_h(\boldsymbol{u}_h,\,\boldsymbol{v}_h) + F_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h) + B_h(p_h,\,\boldsymbol{v}_h) = l_h(\boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ B_h(q_h,\,\boldsymbol{u}_h) = \boldsymbol{g}_h(q_h) & \forall q_h \in Q_h, \end{cases} \tag{5.5.2}$$

Le us consider $\Omega =]0,1[\times]0,1[$. The vorticity $\boldsymbol{w}$ and $\boldsymbol{u}$ are given on the boundary. We take data as : $\gamma = 0.1$, $a = 1$, $b = 1$, $\sigma = 50$. To realize this implementation we use 3800 elements with 1981 vertices.

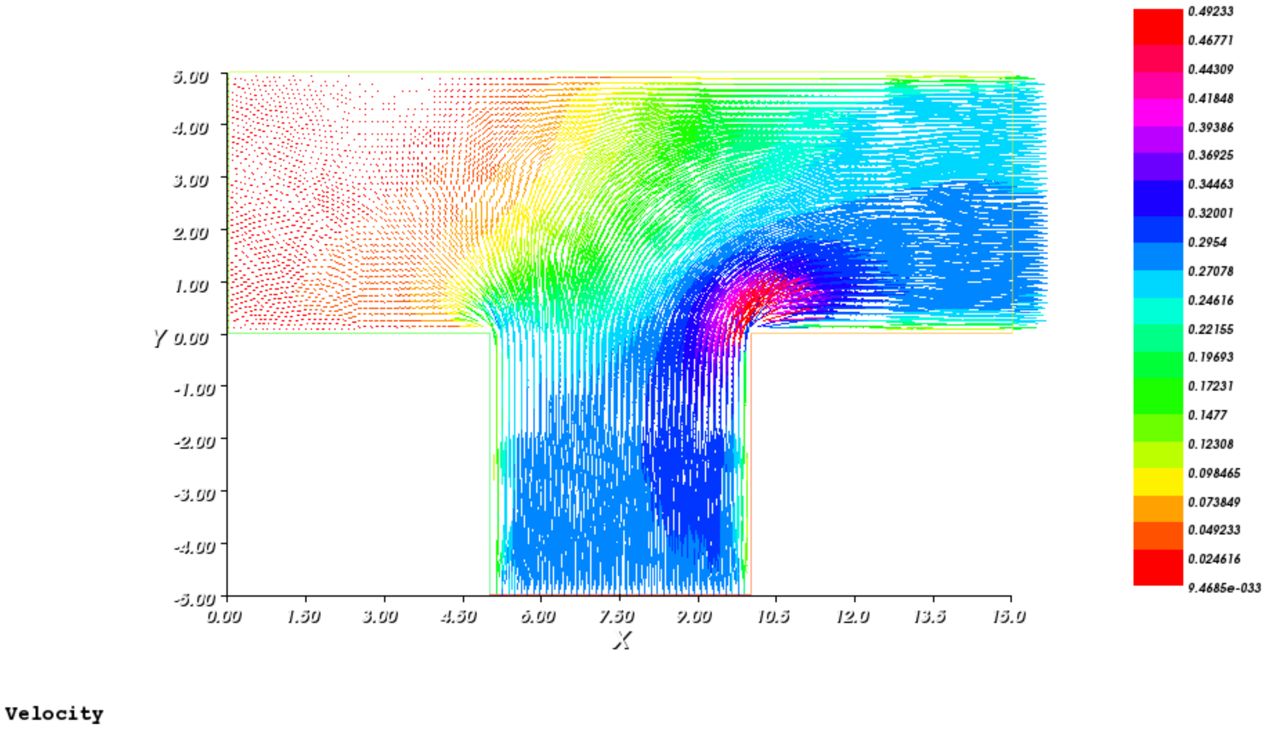
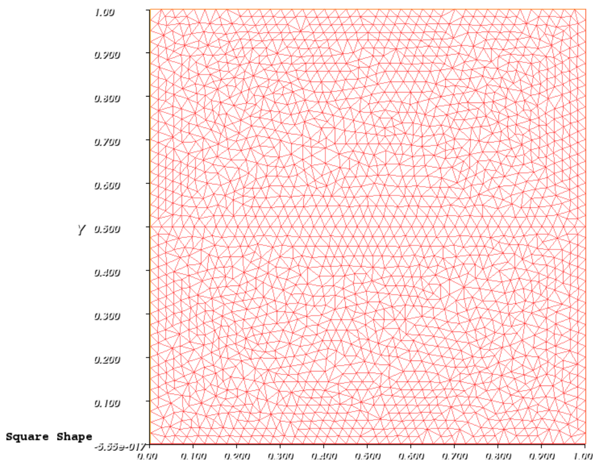
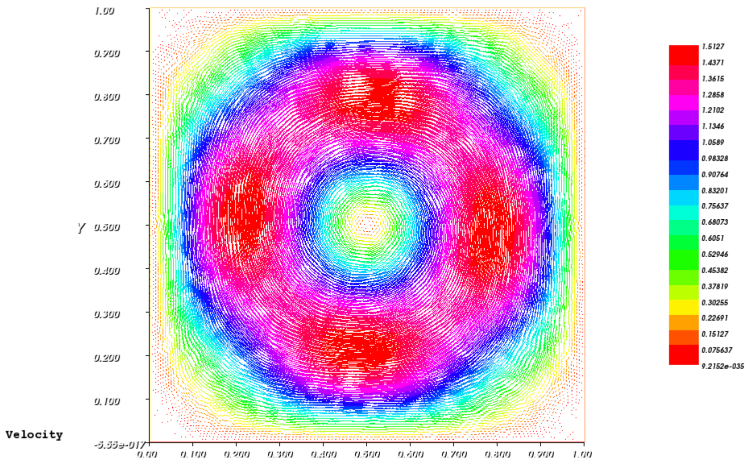


Figure 4: Velocity in a two-dimensional T-shaped bifurcation with $Re = 40$ and the imposed pressure on the inlet and outlet boundaries.



(a)

Figure 5: Mesh for the square cavity.



(a)

Figure 6: Computed velocity.

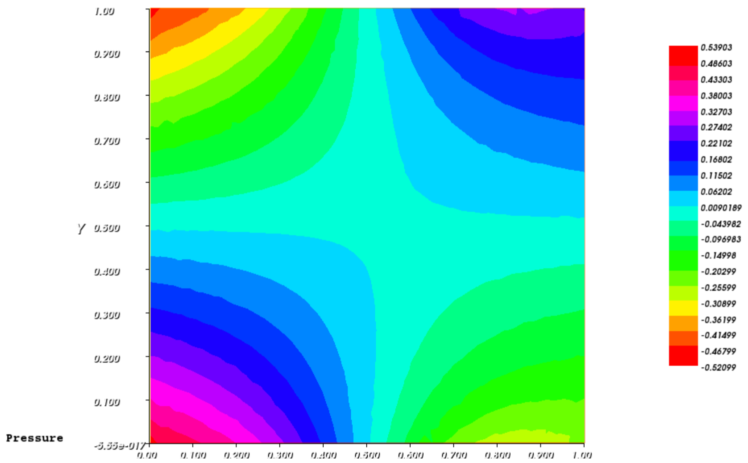


Figure 7: Camputed Pressure.

5.5.4 Conclusions

Numerical validation was provided by testing the method against a number of well-known problems. In all cases, the robustness and the accuracy of the method seems to be present with respect to other methods.

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Analyse mathématique et approximation numérique de modèles d'écoulements en milieux Poreux

Résumé : Cette thèse est consacrée à l'étude des équations du Darcy Brinkman Forchheimer (DBF) avec des conditions aux limites non standards. Nous montrons d'abord l'existence de différents type de solutions (faible et forte) correspondant au problème DBF stationnaire dans un domaine simplement connexe avec des conditions portants sur la composante normale du champ de vitesse et la composante tangentielle du tourbillon. Ensuite, nous considérons le système Brinkman Forchheimer (BF) avec des conditions sur la pression dans un domaine non simplement connexe. Nous prouvons que ce problème est bien posé ainsi que l'existence de la solution forte. Nous établissons la régularité de la solution dans les espaces $\boldsymbol{L}^p$ pour $p \geqslant 2$.
L'approximation du problème DBF non stationnaire est basée sur une approche pseudo-compressibilité. Une estimation d'erreur d'ordre deux est établie dans le cas où les conditions aux limites sont de types Dirichlet ou Navier.
Enfin, une méthode d'éléments finis Galerkin Discontinue est proposée et la convergence établie concernant le problème DBF linéarisé et le système DBF non linéaire avec des conditions aux limites non standard.

Mots clés : Équations de Darcy Brinkman Forchheimer, Conditions aux Limites, Pseudo-Compressibilité, Éléments Finis, Méthode de Galerkin Discontinue.

Mathematical analysis and numerical approximation of models for flow in porous media

Abstract : This thesis is devoted to Darcy Brinkman Forchheimer (DBF) equations with a non standard boundary conditions. We prove first the existence of different type of solutions (weak and strong) of the stationary DBF problem in a simply connected domain with boundary conditions on the normal component of the velocity field and the tangential component of the vorticity. Next, we consider Brinkman Forchheimer (BF) system with boundary conditions on the pressure in a non simply connected domain. We prove the well-posedness and the existence of a strong solution of this problem. We establish the regularity of the solution in the $\boldsymbol{L}^p$ spaces, for $p \geqslant 2$.
The approximation of the non stationary DBF problem is based on the pseudo-compressibility approach. The second order's error estimate is established in the case where the boundary conditions are of type Dirichlet or Navier. Finally, the finite elements Galerkin Discontinous method is proposed and the convergence is settled concerning the linearized DBF problem and the non linear DBF system with a non standard boundary conditions.

Keywords : Darcy Brinkman Forchheimer equations, Boundary Conditions, Pseudo-compressibility, Finite Elements, Discontinuous Galerkin Method.